Assignment 3

1. **Efficient portfolios (20 points).**

   Consider an investor who wants to invest in N risky assets with return $R_i \;\; \forall i = 1, \ldots, N$ with expected return $E[R_i] = \mu_i$ and variance $V[R_i] = \sigma_i^2$, and in a risk-free asset with return R_f. The investor seeks a N-risky asset portfolio weight vector w (and a weight $1 - w^\top \mathbf{1}$ in the risk-free asset), such that her portfolio return $R_p = R_f + w^\top (R - R_f \mathbf{1})$ maximizes her mean-variance objective function $U(w) = E[R_p] - \frac{\gamma}{2} V[R_p]$.

   - Show that an optimal portfolio weight vector w is such that for the corresponding mean-variance efficient portfolio return R_p we have

     $$\mu_i - R_f = \gamma \, cov[R_i, R_p] \;\; \forall i = 1, \ldots N$$

   - Show that for any such mean-variance efficient portfolio, we have

     $$\mu_i - R_f = \beta_{i,P}(\mu_P - R_f)$$

     where $\beta_{i,P} = \frac{cov(R_i, R_P)}{\sigma_P^2}$ is the linear regression coefficient of return R_i on the mean-variance efficient portfolio return R_P.

   - In turn, show that this implies that, if R_p is the return to a mean-variance efficient portfolio, then for any return i we have

     $$R_i = R_f + \beta_i (R_P - R_f) + \epsilon_i$$

     where $cov(R_P, \epsilon_i) = 0$ and $E(\epsilon_i) = 0$.

     Hint: use the definition of a linear regression

   - Show that all mean-variance efficient portfolios have the same Sharpe ratio where we define its Sharpe ratio as $SR_p = \frac{\mu_P - R_f}{\sigma_p}$.

   Solution

   (a) Denoting the risk-free rate R_f by R_0, the investor's optimization problem is due to the mean-variance preferences given by

$$\max_{\omega_0,\ldots,\omega_N} \left\{ \mathbb{E}\left(\sum_{i=0}^{N} \omega_i R_i\right) - \frac{\gamma}{2}\mathbb{V}\left(\sum_{i=0}^{N} \omega_i R_i\right) \right\}, \tag{1}$$

$$\text{s.t. } \sum_{i=0}^{N} \omega_i = 1. \tag{2}$$

Since the risk-free asset has a constant return, one has

$$\mathbb{V}\left(\sum_{i=0}^{N} \omega_i R_i\right) = \mathbb{V}\left(\sum_{i=1}^{N} \omega_i R_i\right).$$

Now, we can plug in the constraint (2) into the term to be maximized in (1) to rewrite the investor's maximization problem as

$$\max_{\omega_1,\ldots,\omega_N} \left\{ R_f + \mathbb{E}\left(\sum_{i=1}^{N} \omega_i (R_i - R_f)\right) - \frac{\gamma}{2}\mathbb{V}\left(\sum_{i=1}^{N} \omega_i R_i\right) \right\} \tag{3}$$

or (due to the linearity of the expectations operator and the normality of $R_1, ..., R_N$)

$$\max_{\omega_1,\ldots,\omega_N} \left\{ R_f + \sum_{i=1}^{N} \omega_i (\mu_i - R_f) - \frac{\gamma}{2} \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i \omega_j \Omega_{ij} \right\}, \tag{4}$$

where Ω_{ij} denotes the $(i, j) - th$ element of the covariance matrix Ω.

The first-order conditions for the problem in (4) are

$$\mu_i - R_f - \gamma \omega_i^* \Omega_{ii} - \frac{\gamma}{2} \sum_{j \neq i} \omega_j^* \Omega_{ij} - \frac{\gamma}{2} \sum_{j \neq i} \omega_j^* \Omega_{ji} = 0 \qquad (i = 1, ..., N).$$

Since Ω is symmetric($\Omega_{ij} = \Omega_{ji}$), the last line is equivalent to

$$\mu_i - R_f - \gamma \sum_{j=1}^{N} \omega_j^* \Omega_{ij} = 0 \qquad (i = 1, ..., N).$$

The claim now follows from realizing that

$$Cov(R_i, R_p) = Cov\left(R_i, \sum_{j=1}^{N} \omega_i^* R_j\right) = \sum_{j=1}^{N} \omega_j^* \Omega_{ij}.$$

(b) Let $\omega := (\omega_1, ..., \omega_N)'$, where $\omega_i = \omega_i^*$ from above. Similarly, let $\mu := (\mu_1, ..., \mu_N)'$.

Let $\mathbb{1}$ denote the $N \times 1$ vector of ones.

From the previous bullet point we have

$$\mu - R_f \mathbb{1} = \gamma \Omega \omega. \tag{5}$$

Using $\omega' \Omega \omega = \sigma_P^2$, premultiplying both sides of (5) by ω gives

$$\mu_P - R_f = \gamma \sigma_P^2. \tag{6}$$

Dividing both sides of (5) by the respective sides of (6) gives

$$(\mu - R_f \mathbb{1}) \frac{1}{\mu_P - R_f} = \frac{\gamma}{\gamma \sigma_P^2} \Omega \omega.$$

The claim now follows from rearranging the last equation.

Note: For any return $\mu_P > R_f$, one can find a unique value of $\gamma > 0$, such that a mean-variance investor with risk-aversion γ wants to hold the mean-variance efficient portfolio with mean μ_P. This can be seen as follows: From (5) one gets

$$w = \frac{1}{\gamma} \Omega^{-1} (\mu - R_f \mathbb{1}).$$

Thus, the mean return of the investor's optimal portfolio is given by

$$\mu_P := R_f + (\mu' - R_f \mathbb{1}') \omega = R_f + \frac{1}{\gamma} (\mu' - R_f \mathbb{1}') \Omega^{-1} (\mu - R_f \mathbb{1}).$$

The last line defines a bijective map $\Psi : (0, \infty) \to (R_f, \infty)$ with $\Psi(\gamma) = \mu_P$ whenever Ω is positive definite, i.e. Ω^{-1} is positive definite and therefore

$$(\mu' - R_f \mathbb{1}') \Omega^{-1} (\mu - R_f \mathbb{1}) > 0.$$

(c) A linear OLS regression of R_i on R_P decomposes the return of asset i as follows:

$$R_i = \beta_0 + \beta_1 (R_P - R_f) + \epsilon_i, \tag{7}$$

where $\mathbb{E}(\epsilon_i) = 0$ and $Cov(R_P - R_f, \epsilon_i) = Cov(R_P, \epsilon_i) = 0$. The regression coefficient β_1 is given by

$$\beta_1 = \frac{Cov(R_i, R_p - R_f)}{Var(R_P - R_f)} = \frac{Cov(R_i, R_p)}{\sigma_P^2}.$$

It is left to show that $\beta_0 = R_f$. Taking expectations of (7) and subtracting R_f gives

$$\mu_i - R_f = \beta_0 - R_f + \frac{Cov(R_i, R_p)}{\sigma_P^2}(\mu_P - R_f). \tag{8}$$

As shown in the previous bullet point, we also have

$$\mu_i - R_f = \frac{Cov(R_i, R_p)}{\sigma_P^2}\left(\mu_P - R_f\right). \tag{9}$$

Together, (8) and (9) imply that

$$\beta_0 - R_f + \frac{Cov(R_i, R_p)}{\sigma_P^2}(\mu_P - R_f) = \frac{Cov(R_i, R_p)}{\sigma_P^2}\left(\mu_P - R_f\right).$$

The last line implies $\beta_0 = R_f$ and the claim follows.

(d) By the fund separation theorem (see e.g. slide 37 in lecture 2), all mean-variance efficient portfolios are a linear combination of the risk-free asset and the tangency portfolio. Thus, the correlation between two efficient portfolios (with a mean return greater than R_f) have a correlation of $\rho = 1$.
With the second bullet point, one now gets for two mean-variance efficient portfolios ω_P and ω_Q (with the usual notation) that

$$\frac{\mu_P - R_f}{\sigma_P} = \frac{\frac{Cov(R_P, R_Q)}{\sigma_Q^2}(\mu_Q - R_f)}{\sigma_P} = \frac{\mu_Q - R_f}{\sigma_Q}\frac{Cov(R_P, R_Q)}{\sigma_P \sigma_Q} = \frac{\mu_Q - R_f}{\sigma_Q}\rho = \frac{\mu_Q - R_f}{\sigma_Q}.$$

Thus all mean-variance efficient portfolios (those portfolios that maximize the mean return for a given variance) have the same Sharpe ratio.

2. **Portfolio math (10 points).**

- Show that any risky-asset only minimum variance frontier portfolio w can be rewritten as a convex combination of any two arbitrary minimum variance frontier portfolios w_a, w_b in the sense that $w = \alpha w_a + (1 - \alpha) w_b$.

- Let R_{min} denote the return on the global minimum-variance portfolio of risky assets. Let R be the return on any risky asset or portfolio of risky assets, efficient or not. Show that $Cov(R, R_{min}) = Var(R_{min})$. *Hint*: Consider a portfolio consisting of a fraction w in this risky asset. and a fraction $(1 - w)$ in the global minimum-variance portfolio. Compute the variance of the return on this portfolio and realize that the variance has to be minimized for $w = 0$.

Solution

(a) We know that the minimum variance frontier consists of affine combinations of the global minimum variance portfolio w_{min} and the tangency portfolio w_{tan}. It needs to be shown that any portfolio $w = \alpha w_a + (1 - \alpha) w_b$ can be expressed as $w = \beta w_{min} + (1-\beta) w_{tan}$ and vice versa. We know that w_a and w_b are combinations of w_{min} and w_{tan}. Now, we have

$$
\begin{aligned}
w &= \alpha w_a + (1 - \alpha) w_b \\
&= \alpha(\beta^a w_{min} + (1 - \beta^a) w_{tan}) + (1 - \alpha)(\beta^b w_{min} + (1 - \beta^b) w_{tan})) \\
&= (\alpha\beta^a + (1 - \alpha)\beta^b) w_{min} + (\alpha(1 - \beta^a) + (1 - \alpha)(1 - \beta^b)) w_{tan} \\
&= \beta w_{min} + (1 - \beta) w_{tan}
\end{aligned}
$$

where

$$\beta = \alpha\beta^a w_{min} + (1 - \alpha)\beta^b,$$

since one can verify that indeed

$$
\begin{aligned}
& (\alpha\beta^a + (1 - \alpha)\beta^b) + (\alpha(1 - \beta^a) + (1 - \alpha)(1 - \beta^b)) \\
= & \alpha(\beta^a - \beta^a - \beta^b + \beta^b + 1 - 1) + \beta^b - \beta^b + 1 \\
= & 1.
\end{aligned}
$$

Thus, w is indeed an affine combination of w_a and w_b.

(b) Following the hint, we have

$$Var(wR+(1-w)R_{min}) = w^2Var(R)+2w(1-w)Cov(R,R_{min})+(1-w)^2Var(R_{min}).$$

For the global minimum variance portfolio, we must have

$$\frac{\partial}{\partial w}Var(wR+(1-w)R_{min}) \Big|_{w=0} = 0,$$

i.e.

$$\begin{aligned}
& (2wVar(R) + 2(1-2w))Cov(R,R_{min}) - 2(1-w)Var(R_{min})) \Big|_{w=0} \\
= & \quad 2Cov(R,R_{min}) - 2Var(R_{min}) \\
= & \quad 0.
\end{aligned}$$

It follows that $Cov(R,R_{min}) = Var(R_{min})$.