Assignment 11

1. **Bayesian updating (20 points)** Consider a return $R = \mu + \epsilon$ where $\epsilon \sim N(0, \sigma_\epsilon^2)$. Further, we assume we are not sure what the value for μ is. Instead we think that $\mu = \mu_0 + \nu_0$ where $\nu_0 \sim N(0, \upsilon_0^2)$. In addition, we receive a signal that $\mu + \nu_1 = \mu_1$ where $\nu_1 \sim N(0, \upsilon_1^2)$.

   (a) Using the Gaussian projection theorem, show that our posterior estimate of μ given the additional signal μ_1 is normally distributed, $N(\hat{\mu}, \hat{\upsilon})$ with posterior mean $\hat{\mu} = E[\mu \,|\, \mu + \nu_1 = \mu_1] = \mu_0 + \beta(\mu_1 - \mu_0)$ and with posterior variance $\hat{\upsilon}^2 = V[\mu \,|\, \mu + \nu_1 = \mu_1] = \upsilon_0^2 - \beta^2(\upsilon_0^2 + \upsilon_1^2)$, where β is for you to determine.

   (b) Prove that the posterior mean and variance can be rewritten as:

   $$\hat{\mu} = \frac{\frac{1}{\upsilon_0^2}\mu_0 + \frac{1}{\upsilon_1^2}\mu_1}{\frac{1}{\upsilon_0^2} + \frac{1}{\upsilon_1^2}}$$
   $$\hat{\upsilon}^2 = \frac{1}{\frac{1}{\upsilon_0^2} + \frac{1}{\upsilon_1^2}}$$

   Interpret this formula. Note, in particular, that the prior and signal act symmetrically on the posterior distribution.

   (c) Conclude that if you have N signals of the form $\mu + \nu_i = \mu_i$ where $\nu_i \sim N(0, \upsilon_i^2)$, $\forall i = 0, \ldots, n$, and with all ν_i independent from each other, then the posterior distribution of μ is normal $N(\hat{\mu}, \hat{\upsilon}^2)$ with

   $$\hat{\mu} = \frac{\sum_{i=0}^{n} \frac{1}{\upsilon_i^2}\mu_i}{\sum_{i=0}^{n} \frac{1}{\upsilon_i^2}}$$
   $$\hat{\upsilon}^2 = \frac{1}{\sum_{i=0}^{n} \frac{1}{\upsilon_i^2}}$$

   What happens in the limit when you get a very large number of signals, i.e., $n \to \infty$? Interpret. *(hint: use an inductive argument and your previous results.)*

2. **Black Litterman (40 points)**

   We will replicate the results of He-Litterman (1992) to better understand how to apply the Black-Litterman formula. We are considering the optimal asset allocation to seven country equity index returns with correlation matrix given on table 1 page 21 of the lecture notes and with volatility and relative market capitalization weights given in table 2 of page 21 of the lecture notes.

(a) Assume an investor has a risk-aversion coefficient $\gamma = 3$ and no uncertainty about his estimate of the mean vector μ_0. Compute the expected return vector μ_0 that would have him hold a portfolio equal to the market portfolio with weights w_{eq} given in table 2.

(b) Assume another investor with risk-aversion $\gamma = 2.5$ views returns as $R = \mu + \epsilon$ where $\epsilon \sim N(0, \Sigma)$. He starts with a prior that $\mu \sim N(\mu_0, \tau\Sigma)$, where Σ is the empirical covariance matrix of returns. Suppose that $\tau = 0.05$. Derive his optimal portfolio w_0 and compare how it deviates from the equilibrium market weights w_{eq}.

(c) Assume that same investor obtains two additional views on the relative performance of different country returns from two different analysts. The first analyst thinks that Germany will outperform a market value weighted basket of France and UK equities by 6%. The investor's confidence in this view is $\Omega_{11} = 0.021 \times \tau$. The second analyst thinks that the canadian equity market will outperform the US market by 2% on average. The investor's confidence in that view is $\Omega_{22} = 0.017 \times \tau$. He considers both signal to be independent as he obtained them from different analysts. Using the Black-Litterman formula, derive the posterior distribution of the mean return $\mu \sim N(\bar{\mu}, \bar{\Omega})$ as a function of the prior and the views. Verify numerically that the two sets of equations for $\bar{\mu}$ and $\bar{\Omega}$ on page 11 of the lecture notes indeed give the same answers.

(d) Given his signals the investor sees returns as $R = \mu + \epsilon$ where $\epsilon \sim N(0, \Sigma)$ and $\mu \sim N(\bar{\mu}, \bar{\Omega})$. Derive his optimal unconstrained mean-variance portfolio w^*. Compare it to his prior portfolio w_0 and to the market weights w_{eq}.

(e) Show that the optimal portfolio w^* can be decomposed into the prior portfolio and an 'overlay' of view portfolios. That is we can rewrite $w^* = w_0 + \lambda_1 P_1^\top + \lambda_2 P_2^\top$ where P_i denotes the i^th row of the view portfolio matrix P. Find the view-weights λ_1, λ_2.

(f) In addition the investor has an absolute view that the Japanese stock market will outperform the equilibrium view. In particular he thinks that the Japanese market equity return will be 4.5%. His uncertainty about the view is $\Omega_{33}/\tau = 0.03$. Derive the new optimal portfolio and the weights on the three views $\lambda_1, \lambda_2, \lambda_3$. Discuss how the portfolio and the weights change as his uncertainty becomes smaller, e.g., $\Omega_{33}/\tau = 0.01$.