Assignment 9

1. **Mean-variance investing with fixed, linear-proportional, and quadratic transaction costs (15 points)**

   Suppose there is only one risky asset with $E[R] = \mu$ and $Var[R] = \sigma^2$. In addition, there is a risk-free asset which pays an interest rate $R_f = 0\%$.

   Assume the investor starts with some initial dollar position X_0 and seeks the terminal position X_1 so as to maximize the mean-variance objective function

$$\max_{X_1} X_1\mu - \frac{\gamma}{2}X_1^2\sigma^2 - \{\mathbf{1}_{X_1 \neq X_0} b_0 + |X_1 - X_0| b_1 + \frac{1}{2}\lambda(X_1 - X_0)^2\}$$

   The total transaction costs paid for trading $X_1 - X_0$ dollars of the risky asset are

$$TC = \mathbf{1}_{X_1 \neq X_0} b_0 + |X_1 - X_0| b + \frac{1}{2}\lambda(X_1 - X_0)^2,$$

   which include a fixed cost, a linear-proportional bid-ask spread component, and a quadratic price impact component.

   (a) Solve for the optimal trading strategy. Show in particular, that the optimal strategy can be described by a no-trade region $NT = [\underline{X}, \overline{X}]$, such that if $X_0 \in NT$ then the optimal $X_1 = X_0$ and that if $X_0 \notin NT$ then it is optimal to trade towards a particular aim portfolio at a specific trading speed, that is $X_1 = \tau\, aim + (1 - \tau)\, X_0$.

   You should characterize the no-trade region, the trading speedand the aim portfolio in terms of the parameters of the model.

   (b) Explain how the optimal strategy changes:

   - when you turn-off fixed costs, i.e., if $b_0 = 0$.
   - when you turn-off linear-proporitional costs, i.e., if $b_1 = 0$.
   - when you turn-off fixed and linear-proporitional costs, i.e., if $b_0 = b_1 = 0$.
   - when you turn-off quadratic costs, i.e., if $\lambda = 0$.

Solution:

(a) The first-order condition gives

$$X_1 = \begin{cases} \frac{\mu - b_1 + \lambda X_0}{\gamma\sigma^2 + \lambda} & \text{if } X_1 > X_0 \\ \frac{\mu + b_1 + \lambda X_0}{\gamma\sigma^2 + \lambda} & \text{if } X_1 < X_0 \end{cases}.$$

As shown in the lecture notes, the last two conditions imply that trading is optimal only if $X_0 \notin [\underline{X}^1, \overline{X}^1]$ for some interval $X_0 \notin [\underline{X}^1, \overline{X}^1]$. Due to the fixed cost b_0, trading is only optimal if the benefits of trading are sufficiently large compared to not trading at all, which implies that $X_0 \notin [\underline{X}^2, \overline{X}^2]$ for some interval $[\underline{X}^2, \overline{X}^2]$ if trading is optimal. The no-trade region is thus given by $[\underline{X}^1, \overline{X}^1] \cup [\underline{X}^2, \overline{X}^2]$, which is an interval, since both $[\underline{X}^1, \overline{X}^1]$ and $[\underline{X}^2, \overline{X}^2]$ contain $\frac{\mu}{\gamma\sigma^2}$.

Moreover, one has

$$\tau = \frac{\gamma\sigma^2}{\gamma\sigma^2 + \lambda}$$

and

$$aim = \frac{\mu - b_1 sign(X_1 - X_0)}{\gamma\sigma^2}.$$

(b) When turning off the fixed cost, the trading strategy, conditional on the case that trading is optimal, does not change. However, the no-trade region may shrink and thus the investor may trade in more situations. When turning off the linear-proportional cost, both the no-trade region and the trading strategy, conditional on the case that trading is optimal, may change. The investor is trading more aggressively. When turning off the quadratic cost, the investor trades more aggressively. Also the no-trade region may change, since the gain from trading increases. It is thus more likely for any given X_0 that trading makes the investor better off than not trading.

2. **Problem 2 (20 points)**

Consider N stocks with $E[R_i] = \mu_i$ and $Var[R_i] = \sigma_i^2$ and correlation ρ_{ij}. In addition there is a risk-free rate R_f. Assume the investor starts with some initial dollar position vector X_0 and seeks the vector of terminal position X_1, so as to maximize the mean-variance objective function

$$\max_{X_1} R_f + X_1^\top(\mu - R_f) - \frac{\gamma}{2} X_1^\top \Sigma X_1 - |X_1 - X_0|^\top b$$

where b is a linear proportional transaction cost vector. We assume there are no transaction costs for trading the risk-free asset. The risk-free rate is equal to 2%. The asset-specific parameters are given in the table below.

	μ_i	σ_i	ρ_{ij}	b_i
Asset 1	5%	15%	50%	3%
Asset 2	15%	25%		3%

(a) Solve for the optimal portfolio when there is one single risky asset and the risk-free asset. Derive an explicit solution for the no-trade region, that is two numbers $[X_L, X_H]$ such that when $X_L \leq X_0 \leq X_H$ it is optimal not to trade.

(b) Now solve for the optimal portfolio in the case where there are two risky assets in addition to the risk-free asset. Characterize the no-trade region. Plot the optimal trading regions (no trade, buy 1/sell 2, buy 1/buy 2, ...) on a graph with x-axis X_{10} and y-axis X_{20}, that is the initial positions held in both assets.

(c) How does the shape of the no-trade region change as you increase the correlation coefficient ρ between the two assets? As you make asset 2 riskier than asset 1?

Solution:

(a) Suppose it is optimal to *buy* more of the risky asset. The optimal position is obtained by solving

$$\max_{X_1} \; f(X_1) = R_f + X_1(\mu - R_f) - \frac{\gamma}{2}\Sigma X_1^2 - (X_1 - X_0)b$$

The first-order condition yields

$$\begin{aligned} 0 &= \mu - R_f - b - \gamma \Sigma X_1 \\ X_1^* &= (\gamma\Sigma)^{-1}(\mu - R_f - b). \end{aligned}$$

It is optimal to increase the position to X_1^* as long as $X_0 < X_1^*$, i.e. as long as

$$X_0 < (\gamma\Sigma)^{-1}(\mu - R_f - b) \equiv X_L.$$

Now suppose it is optimal to *sell* some of the risky asset. The optimal position is obtained by solving

$$\max_{X_1} \; f(X_1) = R_f + X_1(\mu - R_f) - \frac{\gamma}{2}\Sigma X_1^2 - (X_0 - X_1)b$$

The first-order condition yields

$$\begin{aligned} 0 &= \mu - R_f + b - \gamma\Sigma X_1 \\ X_1^* &= (\gamma\Sigma)^{-1}(\mu - R_f + b). \end{aligned}$$

This optimal to reduce the position to X_1^* as long as $X_0 > X_1^*$, i.e. as long as

$$X_0 > (\gamma\Sigma)^{-1}(\mu - R_f + b) \equiv X_H.$$

To summarize, the optimal position is

$$X_1^* = \begin{cases} (\gamma\Sigma)^{-1}(\mu - R_f - b) & \text{if } X_0 < X_L \\ (\gamma\Sigma)^{-1}(\mu - R_f + b) & \text{if } X_0 > X_H \\ X_0 & \text{else} \end{cases}$$

The *no-trade region* is the *interval*

$$(\gamma\Sigma)^{-1}(\mu - R_f - b) \equiv X_L \le X_0 \le X_H \equiv (\gamma\Sigma)^{-1}(\mu - R_f + b).$$

(b) With three possible actions (buy, sell, no trade) for each risky asset and two risky assets, there are a total of $3^2 = 9$ regions: 8 trading regions, plus the no-trade region. We are going to characterize each region in turn.

Let's start with the regions where the investor trades *both* assets. Let a_i, $i = 1, 2$, denote the optimal transaction "type" for asset i:

$$a_i = \begin{cases} 1 & \text{if the investor buys asset } i \\ -1 & \text{if the investor sells asset } i \end{cases}$$

Further, let the vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ denote the optimal transaction "type" for both assets. *Conditional* on the transaction being of "type" $\mathbf{a}$, the optimal trade is obtained by solving

$$\max_{X_1} R_f + X_1^\top(\mu - R_f) - \frac{\gamma}{2} X_1^\top \Sigma X_1 - (X_1 - X_0)^\top (b \circ \mathbf{a}),$$

where $\circ$ denotes the Hadamard product. The first-order condition yields

$$\begin{aligned} 0 &= \mu - R_f - \gamma \Sigma X_1 - b \circ \mathbf{a} \\ X_1^*(\mathbf{a}) &= (\gamma\Sigma)^{-1} (\mu - R_f - b \circ \mathbf{a}) . \end{aligned}$$

Suppose it is optimal to buy more of each risky asset. The optimal position is given by

$$X_{BB} = X_1^* \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) .$$

It is optimal to increase the position to X_{BB} as long as $X_0 \leq X_{BB}$, i.e. as long as $X_{10} \leq X_{1BB}$ and $X_{20} \leq X_{2BB}$. This gives us the first trading region:

$$\mathcal{BB} = \{X_0 \in \mathbb{R}^2 | X_{10} \leq X_{1BB} \text{ and } X_{20} \leq X_{2BB}\}.$$

Suppose it is optimal to buy more of asset 1 and to sell some of asset 2. In this case, the optimal position is given by

$$X_{BS} = X_1^* \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) .$$

It is optimal to adjust the position to X_{BS} as long as $X_{10} \leq X_{1SS}$ and $X_{20} \geq X_{2SS}$. Thus:

$$\mathcal{BS} = \{X_0 \in \mathbb{R}^2 | X_{10} \leq X_{1BS} \text{ and } X_{20} \geq X_{2BS}\}.$$

Following the same argument, we can derive the trading region where the investor sells both assets, and the region where the investor optimally sells asset 1 and buys asset 2:

$$X_{SS} = X_1^*\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) \qquad\qquad X_{SB} = X_1^*\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

so that

$$\begin{aligned} \mathcal{SS} &= \{X_0 \in \mathbb{R}^2 | X_{10} \geq X_{1SS} \text{ and } X_{20} \geq X_{2SS}\} \\ \mathcal{SB} &= \{X_0 \in \mathbb{R}^2 | X_{10} \geq X_{1BS} \text{ and } X_{20} \leq X_{2BS}\}. \end{aligned}$$

We now analyse the regions where the investor trades *only one* asset. For example, suppose now that it is optimal to trade asset 1, but not asset 2. Assume the investor optimally buys more of asset 1. Setting $X_{21} = X_{20}$ (the investor does not trade asset 2), the optimal trade is obtained by solving

$$\max_{X_{11}} R_f + X_{11}(\mu_1 - R_f - b) + X_{20}(\mu_1 - R_f) - \frac{\gamma}{2}\left(\sigma_1^2 X_{11}^2 + \sigma_2^2 X_{20}^2 + 2\rho\sigma_1\sigma_2 X_{11} X_{20}\right)$$

where I used the usual notation for the individual assets' volatilities,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

The first-order condition yields

$$\begin{aligned} 0 &= \mu_1 - R_f - b - \gamma\sigma_1^2 X_{11} - \rho\sigma_1\sigma_2 X_{20} \\ X_{1BN}(X_{20}) &= \frac{\mu_1 - R_f - b}{\gamma\sigma_1^2} - \frac{\rho\sigma_2}{\gamma\sigma_1} X_{20}. \end{aligned} \tag{1}$$

As long as $\rho \neq 0$, the optimal trade in asset 1 is a (linear) function of the current position in asset 2. Moreover, by construction it must be the case that

$$\begin{aligned} X_{1BN}(X_{2BB}) &= X_{1BB} \\ X_{1BN}(X_{2BS}) &= X_{1BS}. \end{aligned}$$

And since for $X_{20} < X_{2BB}$ ($X_{20} > X_{2BS}$) it is optimal to buy (sell) asset 2, it is clear that eq. (**??**) is only valid for $X_{2BB} \leq X_{20} \leq X_{2BS}$. This yields the trading region

$$\mathcal{BN} \;=\; \{X_0 \in \mathbb{R}^2 | X_{2BB} \leq X_{20} \leq X_{2BS} \text{ and } X_{10} \leq X_{1BN}(X_{20})\}.$$

If, instead, the investor optimally sells some of asset 1, the optimal trade is given by

$$X_{1SN}(X_{20}) \;=\; \frac{\mu_1 - R_f + b}{\gamma\sigma_1^2} - \frac{\rho\sigma_2}{\gamma\sigma_1}X_{20}. \tag{2}$$

This yields the trading region

$$\mathcal{SN} \;=\; \{X_0 \in \mathbb{R}^2 | X_{2SB} \leq X_{20} \leq X_{2SS} \text{ and } X_{10} \geq X_{1SN}(X_{20})\}.$$

Reversing the role of asset 1 and asset 2 gives the last two trading regions:

$$X_{2NB}(X_{10}) \;=\; \frac{\mu_2 - R_f - b}{\gamma\sigma_2^2} - \frac{\rho\sigma_1}{\gamma\sigma_2}X_{10} \tag{3}$$

$$X_{2NS}(X_{10}) \;=\; \frac{\mu_2 - R_f + b}{\gamma\sigma_2^2} - \frac{\rho\sigma_1}{\gamma\sigma_2}X_{10} \tag{4}$$

so that

$$\begin{aligned} \mathcal{NB} &= \{X_0 \in \mathbb{R}^2 | X_{1BB} \leq X_{10} \leq X_{1SB} \text{ and } X_{20} \leq X_{2NB}(X_{10})\} \\ \mathcal{NS} &= \{X_0 \in \mathbb{R}^2 | X_{1BS} \leq X_{10} \leq X_{1SS} \text{ and } X_{20} \geq X_{2NS}(X_{10})\}. \end{aligned}$$

Finally, the *no-trade region* is just the remaining part of the plan

$$\mathcal{NN} = \{X_0 \in \mathbb{R}^2 \backslash (\mathcal{BB} \cup \mathcal{BS} \cup \mathcal{SB} \cup \mathcal{SS} \cup \mathcal{BN} \cup \mathcal{SN} \cup \mathcal{NB} \cup \mathcal{NS}\}.$$

How does the *no-trade region* look like then? Notice that the graphs of the functions $X_{1SN}(\cdot)$ and $X_{1BN}(\cdot)$ are two parallel lines since for all x:

$$X_{1SN}(x) - X_{1BN}(x) = \frac{2b}{\gamma\sigma_1^2}.$$

The same is true for the functions $X_{2NB}(\cdot)$ and $X_{2NS}(\cdot)$, since $X_{2NS}(x)-X_{2NB}(x) = \frac{2b}{\gamma\sigma_2^2}$. Moreover, by construction the four points X_{BB}, X_{BS}, X_{SB} and X_{SS} correspond to the intersection points of the four lines traced by the graphs of the functions $X_{1BN}(\cdot)$, $X_{1SN}(\cdot)$, $X_{2NB}(\cdot)$ and $X_{2NS}(\cdot)$. Thus, the *no-trade region* is a *parallelogram.*

(c) Note first that when $\rho = 0$, $X_{1BN}(\cdot)$ and $X_{1SN}(\cdot)$ are constant functions, since the value that they take does not depend on X_{20}. Similarly, $X_{2NB}(\cdot)$ and $X_{2NS}(\cdot)$ are constant functions, since the value that they take does not depend on X_{10}. Therefore, we have

$$\begin{aligned} X_{1BB} &= X_{1BS} \equiv X_{1L} \qquad & X_{1SB} &= X_{1SS} \equiv X_{1H} \\ X_{2BB} &= X_{2SB} \equiv X_{2L} \qquad & X_{2BS} &= X_{2SS} \equiv X_{2H} \end{aligned}$$

The no-trade region is a *rectangle.* In this case, the optimal strategy in the 2-asset model is very similar to the optimal strategy in the one-asset model. For each asset i, when the current position is sufficiently small, the investor increases his position (to X_{iL}); when the current position is sufficiently large, the investor decreases his position (to X_{iH}). When $X_{i0} \in [X_{iL}, X_{iH}]$, it is optimal not to trade asset i. In particular, the optimal strategy regarding when and how much to trade, say, asset 1 does not depend on the investor's current position in asset 2!

This is no longer true when $\rho \neq 0$. The reason is that the marginal increase in portfolio variance associated with an increase in the position in asset 1 — which must be balanced against the marginal increase in the portfolio's expected return— depends on the current position on asset 2. In particular, the marginal increase in portfolio variance is higher, the higher the current position on asset 2, the higher the volatility of asset 2, and the higher the correlation between both assets.

For example, consider your decision whether or not to trade asset 1. As your position in asset 2 increases, the marginal impact of an increase asset 1 on portfolio variance increases. This incentivizes you to *decrease* the minimal position in asset 1 below which you stop reducing your position in asset 1. Similarly, you now require an smaller position in asset 1 before you start increasing your position in asset 1. And these adjustments are more pronounced —the edges of the parallelogram are more downward-sloping— the higher ρ and σ_2.

3. **Optimal Dynamic trading of a single asset with linear-proportional price impact (quadratic transaction costs) (40 points)**

   Suppose you hold n_{-1} shares of a stock with current price P_0. The price process is as follows for all $t \geq 0$:

   $$P_{t+1} = P_t + \mu + \sigma\epsilon_{t+1}$$

   ϵ_t is an iid shock with zero mean and variance 1. The risk-free rate is equal to 0. When you trade u_t shares of the stock you pay a trading cost of $\frac{\lambda}{2}u_t$ per share traded.

   Suppose that you want to maximize your total discounted wealth at time T net of trading costs and of a risk-penalty, using discount rate ρ. Specifically you want to maximize:

   $$E\left[\sum_{t=0}^{T}\rho^t\left\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2\right\}\right]$$

   (a) Define the value function at time $k \in [0, T]$ to be

   $$V(k, n_{k-1}) = \max_{n_t, t\geq k} E\left[\sum_{t=k}^{T}\rho^{t-k}\left\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2\right\}\right].$$

   Solve for $V(T, n_{T-1})$ and for the optimal trade at the terminal date $n^*(T, n_{T-1})$. In particular, show that $V(T, n) = -\frac{1}{2}n^2Q_T + nq_T + c_T$ where Q_T, q_T, c_T are constants for you to determine.

   (b) Now assume that at any time $t < T$ we have determined that $V(t+1, n) = -\frac{1}{2}n^2Q_{t+1} + nq_{t+1} + c_{t+1}$ for some known constants $Q_{t+1}, q_{t+1}, c_{t+1}$, then show that the value function at time t is also of the same form, namely $V(t, n) = -\frac{1}{2}n^2Q_t + nq_t + c_t$, where the parameters Q_t, q_t, c_t satisfy a set of recursions for you to determine. Recall that the value function satisfies the Bellman equation of optimality:

   $$V(t, n_{t-1}) = \max_{n_t}\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2 + \rho E_t[V(t+1, n_t)]\}$$

(c) Show that the optimal trading rule is of the form

$$n_{t+1} = (1 - \tau_t)n_t + \tau_t aim_t$$

where the aim_t portfolio has the property that it maximizes the value function at any time t.

(d) Now we want to see what the optimal trading looks like. We will consider this model to be one of 'optimal liquidation' of a position within one day. To that effect we set $\mu = 0$. Explain why this implies that at T the investor will want to hold zero shares of the asset. The stock has an annual volatility of 30%. She holds $n_{-1} = 1,000,000$ shares that have an initial price of $P_0 - \$100$. The price impact cost have been estimated to be fairly significant $\lambda = 1bps$ and there is no discounting, i.e. $\rho = 1$. So to sell all the shares in one trade would cost as much as the entire value of the shares (it would effectively be impossible to push these shares through in one trade). Suppose you start trading at 9:30am and you aim to be finished trading by 4pm. Assume that you will send out a trade every 30 minutes. Calibrate the model parameters to this trading interval and plot the optimal trading schedule that you would follow based on your model. Plot the expected cost of trading.

Now assume that you will be trading every 10 minutes. Plot the optimal trading schedule and your expected cost of trading.

(e) Note that the model predicts that your optimal trading schedule does not depend on the realized price shocks, that is if the stock price goes up over the trading day or if it goes down does not affect your optimal trading rule. Do you think this is reasonable? Can you explain why, given the model assumption made above, this is actually the optimal thing to do? What assumptions might you want to change that would change your optimal trading rule?

(f) An important stylized fact about stock liquidity is that it is much lower during the middle of the day (e.g., the traders' 'lunch break'). We can model this by assuming that λ_t is equal to 1 bps during the whole day except during 12noon and 2pm where it jumps to 2 bps. Explain how you could solve the model to account for such a stylized fact. If you have time, solve the model and compare the new solution you obtain to the previous one.

Solution:

(a) The value function at time T is:

$$\begin{aligned} V(T, n_{T-1}) &= \max_{n_T} \left\{ n_T\mu - \frac{\lambda}{2}(n_T - n_{T-1})^2 - \frac{\gamma}{2} n_T^2 \sigma^2 \right\} \\ &= \max_{n_T} \left\{ n_T\mu - \frac{\lambda + \gamma\sigma^2}{2} n_T^2 - \frac{\lambda}{2}(n_{T-1}^2 - 2 n_T n_{T-1}) \right\}. \end{aligned}$$

The FOC is

$$\mu - (\lambda + \gamma\sigma^2) n_T + \lambda n_{T-1} = 0$$

so that

$$n_T^* = \frac{\mu + \lambda n_{T-1}}{\gamma\sigma^2 + \lambda}$$

Thus, substituting back into the value function yields

$$\begin{aligned} V(T, n_{T-1}) &= \Big(\mu + \overbrace{\mu - (\lambda + \gamma\sigma^2) n_T^*}^{= -\lambda n_{T-1} \quad \text{by FOC}} \Big) \frac{n_T^*}{2} - \frac{\lambda}{2}(n_{T-1}^2 - 2 n_T^* n_{T-1}) \\ &= \frac{\mu}{2} n_T^* - \frac{\lambda}{2} n_{T-1} n_T^* - \frac{\lambda}{2} n_{T-1}^2 + \lambda n_T^* n_{T-1} \\ &= \frac{\mu}{2} \frac{\mu + \lambda n_{T-1}}{\gamma\sigma^2 + \lambda} + \frac{\lambda}{2} n_{T-1} \frac{\mu + \lambda n_{T-1}}{\gamma\sigma^2 + \lambda} - \frac{\lambda}{2} n_{T-1}^2 \\ &= -\frac{1}{2}\left(\lambda - \frac{\lambda^2}{\gamma\sigma^2 + \lambda}\right) n_{T-1}^2 + \frac{\mu\lambda}{\gamma\sigma^2 + \lambda} n_{T-1} + \frac{1}{2}\frac{\mu^2}{\gamma\sigma^2 + \lambda} \\ &= -\frac{1}{2} Q_T n_{T-1}^2 + q_T n_{T-1} + C_T, \end{aligned}$$

where

$$\begin{aligned} Q_T &= \lambda\left(1 - \frac{\lambda}{\gamma\sigma^2 + \lambda}\right) \\ q_T &= \frac{\mu\lambda}{\gamma\sigma^2 + \lambda} \\ C_T &= \frac{1}{2}\frac{\mu^2}{\gamma\sigma^2 + \lambda}. \end{aligned}$$

(b) At any time t,

$$V(t, n_{t-1}) \quad = \quad \max_{n_t} \left\{ n_t \mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2} n_t^2 \sigma^2 + \rho \left(-\frac{1}{2} Q_{t+1} n_t^2 + q_{t+1} n_t + C_{t+1} \right) \right\}$$

The FOC is

$$\mu - \lambda(n_t - n_{t-1}) - \gamma\sigma^2 n_t - \rho Q_{t+1} n_t + \rho q_{t+1} = 0$$

so that

$$n_t^* = \frac{\rho q_{t+1} + \mu + \lambda n_{t-1}}{\rho Q_{t+1} + \gamma\sigma^2 + \lambda}.$$

Thus, substituting back into the value function yields

$$\begin{aligned}
V(t, n_{t-1}) &= \left(2\mu - (\lambda + \gamma\sigma^2) n_t^*\right) \frac{n_t^*}{2} - \frac{\lambda}{2}(n_{t-1}^2 - 2 n_t^* n_{t-1}) - \frac{1}{2}\rho Q_{t+1} (n_t^*)^2 + \rho q_{t+1} n_t^* + \rho C_{t+1} \\
&= \left(\mu + \rho q_{t+1} + \overbrace{\mu - (\rho Q_{t+1} + \lambda + \gamma\sigma^2) n_t^* + \rho q_{t+1}}^{= -\lambda n_{t-1} \quad \text{by FOC}} \right) \frac{n_t^*}{2} - \frac{\lambda}{2}(n_{t-1}^2 - 2 n_t^* n_{t-1}) + \rho C_{t+1} \\
&= \frac{\mu + \rho q_{t+1}}{2} n_t^* + \frac{\lambda}{2} n_{t-1} n_t^* - \frac{\lambda}{2} n_{t-1}^2 + \rho C_{t+1} \\
&= -\frac{1}{2}\left(\lambda - \frac{\lambda^2}{\rho Q_{t+1} + \gamma\sigma^2 + \lambda}\right) n_{t-1}^2 + \frac{(\rho q_{t+1} + \mu)\lambda}{\rho Q_{t+1} + \gamma\sigma^2 + \lambda} n_{t-1} \\
&+ \left(\rho C_{t+1} \frac{1}{2} \frac{(\rho q_{t+1} + \mu)^2}{\rho Q_{t+1} + \gamma\sigma^2 + \lambda} \right) \\
&= -\frac{1}{2} Q_t n_{t-1}^2 + q_t n_{t-1} + C_t,
\end{aligned}$$

where

$$\begin{aligned}
Q_t &= \lambda - \frac{\lambda^2}{\rho Q_{t+1} + \gamma\sigma^2 + \lambda} \\
q_t &= \frac{(\rho q_{t+1} + \mu)\lambda}{\rho Q_{t+1} + \gamma\sigma^2 + \lambda} \\
C_t &= \rho C_{t+1} \frac{1}{2} \frac{(\rho q_{t+1} + \mu)^2}{\rho Q_{t+1} + \gamma\sigma^2 + \lambda}.
\end{aligned}$$

We can solve recursively from $t = T, T-1, ..., 1$ by setting $Q_{T+1} = q_{T+1} = C_{T+1} = 0$.

(c) The aim_t portfolio has the property that it maximizes the value function at any time t. Setting the derivative of the value function, $V(t-1, n_t)$, equal to zero, yields:

$$0 = -Q_t aim_t + q_t \qquad \Rightarrow aim_t = \frac{q_t}{Q_t}.$$

Since $\frac{Q_t}{q_t} = \frac{\rho Q_{t+1}+\gamma\sigma^2+\lambda}{\rho q_{t+1}+\mu} - \frac{\lambda}{\rho q_{t+1}+\mu} = \frac{\rho Q_{t+1}+\gamma\sigma^2}{\rho q_{t+1}+\mu}$, we have:

$$aim_t = \frac{\rho q_{t+1} + \mu}{\rho Q_{t+1} + \gamma\sigma^2}.$$

Then, the optimal trading rule can be written as

$$\begin{aligned} n_t^* &= \frac{\rho Q_{t+1} + \gamma\sigma^2}{\rho Q_{t+1} + \gamma\sigma^2 + \lambda} \frac{\rho q_{t+1} + \mu}{\rho Q_{t+1} + \gamma\sigma^2} + \frac{\lambda}{\rho Q_{t+1} + \gamma\sigma^2 + \lambda} n_{t-1} \\ &= \tau_t aim_t + (1 - \tau_t) n_{t-1}, \end{aligned}$$

where we defined $\tau_t := \frac{\rho Q_{t+1}+\gamma\sigma^2}{\rho Q_{t+1}+\gamma\sigma^2+\lambda}$.

(d) See Python file.

(e) The mean-return trade-off at time t only depends on the position chosen at time t (and carried into period $t-1$). That is, the price history does not influence the way that the investor balances the stock's the expected return against its risk. The only impact of the investor's previous position n_{t-1} on the investor's objective function is through the t-costs, but these are symmetrical: the investor pays the same t-costs to buy $n_t - n_{t-1}$ shares as he would to sell $n_{t-1} - n_t$ shares. As a result, the optimal trading schedule does not depend on the realized price shocks.

There are at least two ways to break this independence. First, if the risk-aversion of the investor depends on his level of wealth. Suppose the investor is more risk averse when he is poor. In this case, if he initially owns more shares than his target position, he would be ready to sell more aggressively towards his target position following an initial drop in the share price. Another instance where this independence would break down, is if the t-costs are state-dependent. For instance, liquidity could dry up in a falling market, so that the investor must pay higher t-costs if the stock price goes down. Alternatively, the t-costs could depend on the notional traded, rather than on the number of shares traded. In this case, the investor would incur higher t-costs if the stock price goes up.

(f) Liquidity costs are now modelled as a *function of time* $\lambda_t = \lambda(t)$ instead of a constant. Yet, the model can still be solved in the same recursive fashion. All we need to do is to set the liquidity at time t to λ_t.

To interpret the results, it is useful to plot the *difference* in the number of shares held at each point of the day. When liquidity is lower during "lunch", the investor sells more aggressively in the morning and thus holds few shares at noon. He then sells less aggressively during the lunch break in order to avoid paying excessively high transactions costs. As a result, he owns more shares towards the end of the lunch break (compared to the situation where liquidity is constant throughout the day). He then "catches up" by selling more aggressively in the afternoon once the liquidity has improved.

See the Python file for some numerical examples.