Assignment 9

1. **Mean-variance investing with fixed, linear-proportional, and quadratic transaction costs (15 points)**

   Suppose there is only one risky asset with $E[R] = \mu$ and $Var[R] = \sigma^2$. In addition, there is a risk-free asset which pays an interest rate $R_f = 0\%$.

   Assume the investor starts with some initial dollar position X_0 and seeks the terminal position X_1 so as to maximize the mean-variance objective function

   $$\max_{X_1} X_1\mu - \frac{\gamma}{2}X_1^2\sigma^2 - \{\mathbf{1}_{X_1 \neq X_0} b_0 + |X_1 - X_0| b_1 + \frac{1}{2}\lambda(X_1 - X_0)^2\}$$

   The total transaction costs paid for trading $X_1 - X_0$ dollars of the risky asset are

   $$TC = \mathbf{1}_{X_1 \neq X_0} b_0 + |X_1 - X_0| b + \frac{1}{2}\lambda(X_1 - X_0)^2,$$

   which include a fixed cost, a linear-proportional bid-ask spread component, and a quadratic price impact component.

   (a) Solve for the optimal trading strategy. Show in particular, that the optimal strategy can be described by a no-trade region $NT = [\underline{X}, \overline{X}]$, such that if $X_0 \in NT$ then the optimal $X_1 = X_0$ and that if $X_0 \notin NT$ then it is optimal to trade towards a particular aim portfolio at a specific trading speed, that is $X_1 = \tau\, aim + (1-\tau)\, X_0$.

   You should characterize the no-trade region, the trading speedand the aim portfolio in terms of the parameters of the model.

   (b) Explain how the optimal strategy changes:

   - when you turn-off fixed costs, i.e., if $b_0 = 0$.
   - when you turn-off linear-proporitional costs, i.e., if $b_1 = 0$.
   - when you turn-off fixed and linear-proporitional costs, i.e., if $b_0 = b_1 = 0$.
   - when you turn-off quadratic costs, i.e., if $\lambda = 0$.

2. **Problem 2 (20 points)**

   Consider N stocks with $E[R_i] = \mu_i$ and $Var[R_i] = \sigma_i^2$ and correlation ρ_{ij}. In addition there is a risk-free rate R_f. Assume the investor starts with some initial dollar position vector X_0 and seeks the vector of terminal position X_1, so as to maximize the mean-variance objective function

$$\max_{X_1} R_f + X_1^\top(\mu - R_f) - \frac{\gamma}{2} X_1^\top \Sigma X_1 - |X_1 - X_0|^\top b$$

where b is a linear proportional transaction cost vector. We assume there are no transaction costs for trading the risk-free asset. The risk-free rate is equal to 2%. The asset-specific parameters are given in the table below.

	μ_i	σ_i	ρ_{ij}	b_i
Asset 1	5%	15%	50%	3%
Asset 2	15%	25%		3%

(a) Solve for the optimal portfolio when there is one single risky asset and the risk-free asset. Derive an explicit solution for the no-trade region, that is two numbers $[X_L, X_H]$ such that when $X_L \leq X_0 \leq X_H$ it is optimal not to trade.

(b) Now solve for the optimal portfolio in the case where there are two risky assets in addition to the risk-free asset. Characterize the no-trade region. Plot the optimal trading regions (no trade, buy 1/sell 2, buy 1/buy 2, ...) on a graph with x-axis X_{10} and y-axis X_{20}, that is the initial positions held in both assets.

(c) How does the shape of the no-trade region change as you increase the correlation coefficient ρ between the two assets? As you make asset 2 riskier than asset 1?

3. **Optimal Dynamic trading of a single asset with linear-proportional price impact (quadratic transaction costs) (40 points)**

Suppose you hold n_{-1} shares of a stock with current price P_0. The price process is as follows for all $t \geq 0$:

$$P_{t+1} = P_t + \mu + \sigma \epsilon_{t+1}$$

ϵ_t is an iid shock with zero mean and variance 1. The risk-free rate is equal to 0. When you trade u_t shares of the stock you pay a trading cost of $\frac{\lambda}{2} u_t$ per share traded.

Suppose that you want to maximize your total discounted wealth at time T net of trading costs and of a risk-penalty, using discount rate ρ. Specifically you want to maximize:

$$E\left[\sum_{t=0}^{T}\rho^t\left\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2\right\}\right]$$

(a) Define the value function at time $k \in [0, T]$ to be

$$V(k, n_{k-1}) = \max_{n_t, t\geq k} E\left[\sum_{t=k}^{T}\rho^{t-k}\left\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2\right\}\right].$$

Solve for $V(T, n_{T-1})$ and for the optimal trade at the terminal date $n^*(T, n_{T-1})$. In particular, show that $V(T, n) = -\frac{1}{2}n^2Q_T + nq_T + c_T$ where Q_T, q_T, c_T are constants for you to determine.

(b) Now assume that at any time $t < T$ we have determined that $V(t+1, n) = -\frac{1}{2}n^2Q_{t+1} + nq_{t+1} + c_{t+1}$ for some known constants $Q_{t+1}, q_{t+1}, c_{t+1}$, then show that the value function at time t is also of the same form, namely $V(t, n) = -\frac{1}{2}n^2Q_t + nq_t + c_t$, where the parameters Q_t, q_t, c_t satisfy a set of recursions for you to determine. Recall that the value function satisfies the Bellman equation of optimality:

$$V(t, n_{t-1}) = \max_{n_t}\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2 + \rho E_t[V(t+1, n_t)]\}$$

(c) Show that the optimal trading rule is of the form

$$n_{t+1} = (1 - \tau_t)n_t + \tau_t aim_t$$

where the aim_t portfolio has the property that it maximizes the value function at any time t.

(d) Now we want to see what the optimal trading looks like. We will consider this model to be one of 'optimal liquidation' of a position within one day. To that effect we set $\mu = 0$. Explain why this implies that at T the investor will want to hold zero shares of the asset. The stock has an annual volatility of 30%. She holds $n_{-1} = 1,000,000$ shares that have an initial price of $P_0 - \$100$. The price impact cost have been estimated to be fairly significant $\lambda = 1bps$ and there is no discounting, i.e. $\rho = 1$. So to sell all the shares in one trade would cost as much as the entire value of the shares (it would effectively be impossible to push these shares through in one trade). Suppose you start trading at 9:30am and you aim to be finished trading by 4pm. Assume that you will send out a trade every 30 minutes. Calibrate the model parameters to this trading interval and plot the optimal trading schedule that you would follow based on your model. Plot the expected cost of trading.

Now assume that you will be trading every 10 minutes. Plot the optimal trading schedule and your expected cost of trading.

(e) Note that the model predicts that your optimal trading schedule does not depend on the realized price shocks, that is if the stock price goes up over the trading day or if it goes down does not affect your optimal trading rule. Do you think this is reasonable? Can you explain why, given the model assumption made above, this is actually the optimal thing to do? What assumptions might you want to change that would change your optimal trading rule?

(f) An important stylized fact about stock liquidity is that it is much lower during the middle of the day (e.g., the traders' 'lunch break'). We can model this by assuming that λ_t is equal to 1 bps during the whole day except during 12noon and 2pm where it jumps to 2 bps. Explain how you could solve the model to account for such a stylized fact. If you have time, solve the model and compare the new solution you obtain to the previous one.