Assignment 6

1. **APT (20 points)**

   Consider the following model of returns for stock returns R_i with $i = 1, \ldots, N$:

$$R_i = \alpha_i + \sum_{k=1}^{K} B_{ik} F_k + \epsilon_i$$

   where the factor exposure coefficient B_{ik} are known constants, the F_k $\forall k = 1, \ldots, K$ are returns of specific stock portfolios which are uncorrelated with each other and have a normal distribution $F_k \sim N(m_k, \sigma_k^2)$ and ϵ_i are independent normal random variables with $\epsilon_i \sim N(0, \sigma^2)$ $\forall i$. In addition assume there is a risk-free rate R_0.

   (a) According to the APT what should be expected stock return $E[R_i]$ for stocks $i = 1, \ldots, N$? What restriction does it imply for α_i?

   Solution:

   The APT specifies

$$E(R_i) = R_0 + \sum_{k=1}^{K} B_{ik}(E(F_k) - R_0). \tag{1}$$

   This implies

$$\alpha_i = R_0 - \sum_{k=1}^{K} B_{ik} R_0. \tag{2}$$

   (b) Show that if the market portfolio is spanned by the factors in the sense that there exists some weights w_k such that $R_M = \sum_{k=1}^{K} w_k F_k$, then the CAPM holds if and only if the APT holds. Find how w_k is related to factor risk-premia and volatility.

   Solution: As a mean-variance efficient portfolio, the market portfolio can be used to price the individual factors:

$$E(F_k) - R_0 = \frac{cov(F_k, R_M)}{Var(R_M)}(E(R_M) - R_0). \tag{3}$$

   We now have to show that for any asset i

$$E(R_i) - R_0 = \beta_i(E(R_M) - R_0) \tag{4}$$

if and only if

$$E(R_i) - R_0 = \sum_{k=1}^{K} B_{ik}(E(F_k) - R_0). \tag{5}$$

This can be done by showing that the right-hand sides of (4) and (6) are equal to each other for any asset i:

$$\begin{aligned}
& \beta_i(E(R_M) - R_0) \\
= & \frac{\sum_{k=1}^{K} B_{ik} w_k \sigma_k^2}{Var(R_M)}(E(R_M) - R_0) \\
= & \frac{\sum_{k=1}^{K} B_{ik} cov(F_k, R_M)}{Var(R_M)}(E(R_M) - R_0) \\
= & \sum_{k=1}^{K} B_{ik}(E(F_k) - R_0).
\end{aligned}$$

In the last calculation, the step from the first to the second line used the factor structure of returns and the assumption on the market portfolio to calculate β_i. The step between the third and fourth line used (3).

Since the market portfolio is efficient, the vector $(w_1, ..., w_K)$ must be proportional to $\left(\frac{1}{\sigma_1^2}(E(F_1) - R_0), ..., \frac{1}{\sigma_K^2}(E(F_K) - R_0)\right)$.

(c) Now suppose that

$$R_i = R_0 + \sum_{k=1}^{K} B_{ik}\lambda_k + \epsilon_i$$

where the $\lambda_k > 0$ are constants and we assume that the stock exposures are all bounded away from zero $B_{ik} \geq b > 0 \ \forall i, k$. So unlike in the previous question the factors F_k are not random. So stock returns are only affected by the ϵ_i shocks that are iid normal random variables. Is this model consistent with the APT? If not, find an asymptotic arbitrage portfolio that as the number of stocks N grows arbitrarily large will have zero risk and strictly positive profits.

Solution: This model is not consistent with the APT, since there is an asymptotic arbitrage. Consider the following equal-weighted portfolio:

$$R_p = \frac{1}{N}\sum_{i=1}^{N} R_i = R_0 + \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{k=1}^{K} B_{ik}\lambda_k + \epsilon_i\right). \tag{6}$$

As $N \to \infty$, one has

$$\begin{aligned} R_p &\to R_0 + \frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{K} B_{ik}\lambda_k \qquad \text{a.s.} \\ &\geq R_0 + \sum_{k=1}^{K} b\lambda_k \\ &> R_0. \end{aligned}$$

Financing this portfolio by borrowing at the risk-free rate, one obtains a positive return that does not require any investment, i.e. one has an asymptotic arbitrage.

2. **Beta and expected returns (40 points).** In this exercise we will test the CAPM using portfolio sorted based on beta.

   (a) Download monthly stock returns from CRSP for all common stocks (share codes (shcd) 10 and 11) traded on NYSE (exchange codes (excd) 1 and 2) from 2000 to December 31,2019. Also download a risk-free rate and the value-weighted CRSP market return. For the risk-free rate, use the same data that you also used in Problem Set 3. Use `select date, vwretd from crsp.msi` to obtain data on the CRSP value-weighted index return. In order to get the exchange codes you can for instance access CRSP's stock event file. You can do this by using a query of the following form:

```
select a.permno, a.date, b.shrcd, b.exchcd, ...  (etc.)
from crsp.msf as a left join crsp.msenames as b
on a.permno=b.permno and b.namedt<=a.date and a.date<=b.nameendt (etc.)
```

   Delete data of all stocks for which you have less than 240 observations on returns.

This should leave you with 639 stocks that have been traded every single month from the beginning of 2000 to the end of 2019.

(b) Using the full sample, estimate the market beta for each stock. One way of doing this is to use `df.groupby()` to calculate the relevant moments for each stock,[1] merge these data with the original dataset (left join) and then create a new column in the original dataset that contains the market betas (you are however free to choose any procedure that works). For each month, sort stocks by beta into 10 decile portfolios. For each portfolio compute the equal-weighted average average return. Compute the beta of the portfolio excess returns with respect to the market excess return for the full sample. Plot the 10 average portfolio returns for the full sample[2] versus the portfolios' betas. If you fit a line through these points, how does the slope of that line compare to the average market excess return for the sample?

(c) Notice that the previous results are forward looking in the sense that the strategy could not have been implemented in real time, since we used the full-sample to estimate the betas. Instead, we would like to have a test that does not suffer from look ahead bias. To that end, compute market betas using the period from 2000 to December 31, 2010. Then, starting in 2010, form 10 portfolios as in point b), but using the betas based on the period from 2000 to 2010. Then do the following:

- Compute (equally weighted) average returns of those portfolios for the second sample (2010-2019)
- Plot those returns against the portfolios' betas from the first sample period, based on which you formed the porfolios (2000-2010).
- Also compute the portfolios' betas in the second part of the sample (2010-2019) and plot them against the corresponding betas in the first sample (2000-2010).

How are betas in the first sample period related to the average returns in the second sample period? How are betas in the first sample period related to the betas in the second sample period?

[1] With the .goupby() method you can perform operations on subsets of your data separately, e.g. calculate covariances for each PERMNO.

[2] "Full sample" refers to the period from 2000 to 2019. You should still exclude all stocks with less than 240 observations.