Assignment 2: Solutions

Problem 1 (10 points)

Consider an individual whose end of period wealth will be $W = Y + \epsilon$ where Y is his labor income and ϵ is an insurable risk (e.g., car accident). Assume both are jointly normally distributed with $E[Y] = \mu_Y$, $E[\epsilon] = -\mu_\epsilon$, $V[Y] = \sigma_Y^2$, $V[\epsilon] = \sigma_\epsilon^2$ and $cor[Y, \epsilon] = \rho$.

Assume that the investor is risk-averse with negative exponential utility given by:

$$u(W) = -e^{-aW}, \quad a > 0. \tag{1}$$

(a) Derive the coefficients of relative risk aversion and absolute risk aversion of this agent

(b) Compute the maximum insurance premium π the individual would be willing to pay to insure against the risk ϵ. That is, find π such that

$$E[u(Y + \epsilon)] = E[u(Y - \pi)].$$

(c) How does π change with $\mu_Y, \sigma_\epsilon, \sigma_Y, \rho, a$. Interpret these findings. Are they reasonable?

Solution

(a) We have

$$\begin{aligned} ARA(W) &= -\frac{u''(W)}{u'(W)} = a \\ RRA(W) &= -W\frac{u''(W)}{u'(W)} = aW \end{aligned}$$

(b) Define $X = (Y \quad \epsilon)'$. X follows a multivariate normal distribution with mean vector

$\boldsymbol{\mu} = \begin{pmatrix} \mu_Y & -\mu_\epsilon \end{pmatrix}'$, and covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_Y^2 & \rho\sigma_Y\sigma_\epsilon \\ \rho\sigma_Y\sigma_\epsilon & \sigma_\epsilon^2 \end{pmatrix}$. Let $\mathbf{1} = \begin{pmatrix} 1 & 1 \end{pmatrix}'$.

We have:

$$\begin{aligned} -E[e^{-a\mathbf{1}'X}] &= -E[e^{-a(Y-\pi)}] \\ E[e^{(-a\mathbf{1})'X}] &= e^{a\pi}E[e^{-aY}] \\ e^{-a\mathbf{1}'\boldsymbol{\mu}+\frac{1}{2}a^2\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}} &= e^{a\pi}e^{-a\mu_Y+\frac{1}{2}a^2\sigma_Y^2} \\ -\mathbf{1}'\boldsymbol{\mu}+\frac{1}{2}a\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1} &= \pi-\mu_Y+\frac{1}{2}a\sigma_Y^2 \\ -\mu_Y+\mu_\epsilon+\frac{1}{2}a\left(\sigma_Y^2+\sigma_\epsilon^2+2\rho\sigma_Y\sigma_\epsilon\right) &= \pi-\mu_Y+\frac{1}{2}a\sigma_Y^2, \end{aligned}$$

which gives

$$\pi = \mu_\epsilon + \frac{a\sigma_\epsilon}{2}\left(\sigma_\epsilon + 2\rho\sigma_Y\right).$$

(c) We note the following results.

(i) $\frac{\partial\pi}{\partial\mu_Y} = 0$.

The insurance premium, π, does not depend on the expected labor income, μ_Y. The reason is that the individual has exponential utility, which is characterized by constant absolute risk aversion. Thus, the expected level of wealth of the investor has no impact on how much he is willing to pay to insure against the risk ϵ. Usually, we think that wealthier individuals are hurt less by a shock to their wealth of a given absolute size, and therefore are willing to pay a smaller insurance premium to insure against the risk (this assumption is called "DARA" – decreasing absolute risk aversion). Thus, this result is not really reasonable.

(ii) $\frac{\partial\pi}{\partial\sigma_\epsilon} = a(\sigma_\epsilon + \rho\sigma_Y) \quad \Longrightarrow \quad \frac{\partial\pi}{\partial\sigma_\epsilon} > 0$ if $\frac{\sigma_\epsilon}{\sigma_Y} > -\rho$ and $\frac{\partial\pi}{\partial\sigma_\epsilon} \leq 0$ otherwise.

Suppose $\rho > 0$. The higher the variance of the insurable risk ϵ, the higher the insurance premium which the individual is ready to pay to insure against the risk. This is reasonable.

(We note that if: (1) the insurable risk is negatively correlated with the individual's labor income; and (2) the variance of the insurable risk is not too high, then the insurable risk offers "hedging benefits" to the investor. Good realizations of the insurable risk tends to offset bad realization of the labor income. In this case,

$\pi < \mu_\epsilon$: the investor pays a premium which is smaller than the expected losses to insure against the risk, since he values its hedging benefits. Moreover, the higher the variance of the insurable risk ϵ, the lower the insurance premium which the investor is ready to pay to insure against it. This is because the higher variability of ϵ makes it a more effective hedging tool.)

(iii) $\frac{\partial \pi}{\partial \sigma_Y} = a\rho\sigma_\epsilon \qquad \Longrightarrow \qquad \frac{\partial \pi}{\partial \sigma_Y} > 0$ if $\rho > 0$, and $\frac{\partial \pi}{\partial \sigma_Y} \leq 0$ otherwise.

Suppose $\rho > 0$. When the income of the individual is more volatile, he is willing to pay a higher insurance premium to get rid of the additional variability to his wealth created by the insurable risk ϵ. This is reasonable.

(If $\rho < 0$, then a more volatile labor income makes the invididual more keen to acquire assets with hedging properties. This reduces the risk premium.)

(iv) $\frac{\partial \pi}{\partial \rho} = a\sigma_\epsilon\sigma_Y > 0$

The higher the correlation between ϵ and Y, the higher the insurance premium he is willing to pay to insure against the risk. Suppose $\rho > 0$. Then, a higher correlation means that bad realizations of the insurable risk tend to occur when the individual is already hurt by a low labor income. As a result, the individual is willing to pay a higher insurance premium to insure against ϵ. This is reasonable.

(If $\rho < 0$, then a higher correlation —i.e., a correlation close to zero— means that the hedging properties of ϵ are less effective.)

(v) $\frac{\partial \pi}{\partial a} = \frac{\sigma_\epsilon}{2}\left(\sigma_\epsilon + 2\rho\sigma_Y\right) \qquad \Longrightarrow \qquad \frac{\partial \pi}{\partial \sigma_\epsilon} > 0$ if $\frac{\sigma_\epsilon}{\sigma_Y} > -2\rho$ and $\frac{\partial \pi}{\partial \sigma_\epsilon} \leq 0$ otherwise.

Suppose $\rho > 0$. Then, the higher the risk aversion, the higher the insurance premium he is willing to pay to insure against the risk. The insurable risk ϵ increases the variability of the individual's wealth, and this hurts more risk-averse individuals to a greater extent than less risk-averse individuals. This is reasonable.

(If $\frac{\sigma_\epsilon}{\sigma_Y} < -2\rho$, then $\pi < \mu_\epsilon$: the individual pays an insurance premium which is smaller than the expected losses to insure against the risk, since he values its hedging benefits. In this case, the risk premium —which is negative if $\mu_\epsilon = 0$— decreases with the individual's risk aversion.)

Problem 2: Value at Risk, Expected Shortfall, and Expected Utility

Consider the following asset return distribution:

Probability	return
.025	-.40
.05	-.20
.10	-.10
.10	0
.10	0.05
.20	0.1
.20	0.15
.20	0.20
.025	0.30

(a) Compute the mean, standard deviation, skewness, kurtosis of this asset return.

(b) Compute the 5% and 1% Value at Risk of an investor who has a $100 million investment in that asset.

(c) Compute the the 5% and 1% Conditional expected shortfall of an investor who has a $100 million investment in that asset.

(d) Suppose an investor with a utility $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ with coefficient of relative risk-aversion $\gamma = 2$ is considering investing in this asset. The investor has total initial wealth of $W_0 = 100$. She can also invest in a risk-free asset with fixed return r_f. Suppose that the investor has to choose to invest 100% of her wealth either in the risky asset or in the risk-free rate. What is the level of risk-free rate above which she would choose to invest all her wealth in the risk-free asset? How does this cut-off rate change with the risk-aversion coefficient?

(e) Instead suppose she can invest a fraction of her wealth in the risk-free asset. What is the risk-free rate where she would choose to invest nothing in the risk-free asset? Is this rate different (higher or lower) than that identified in the previous question? Why?

Solution

- moments:

  - mean: $\mu = 0.0725$
  - standard deviation: $\sigma = 0.136907816$
  - skewness: $\mathbb{E}\left(\frac{(R-\mu)^3}{\sigma^3}\right) = -1.323119362$
  - variance: $\mathbb{E}\left(\frac{(R-\mu)^4}{\sigma^4}\right) = 4.953390265$

- VaR

  - 5% VaR: $\inf\{L \in \mathbb{R} : P(loss > L) \leq 0.05\} = \$100mn \cdot 0.2 = \$20mn$
  - 1% VaR: $\inf\{L \in \mathbb{R} : P(loss > L) \leq 0.01\} = \$100mn \cdot 0.4 = \$40mn$

- ES

  - 5% ES: $\frac{1}{0.05}\int_0^{0.05} VaR_\gamma \,\mathrm{d}\gamma = \frac{1}{0.05}\int_{05}^{0.025} 40mn \,\mathrm{d}\gamma + \frac{1}{0.05}\int_{0.025}^{0.05} 20mn \,\mathrm{d}\gamma = 30mn$
  - 1% ES: $\frac{1}{0.01}\int_0^{0.01} VaR_\gamma \,\mathrm{d}\gamma = \frac{1}{0.05}\int_0^{0.01} 40mn \,\mathrm{d}\gamma = 40mn$

- Optimal portfolio weights. Suppose, the investor can choose the fraction ω invested in the risky asset optimally. The first-order condition to the problem

  $$\max_\omega \left[u(W_0\omega(1+R) + W_0(1-\omega)(1+r_f))\right]$$

  is

  $$0 = \mathbb{E}\left(u'(W_0\omega(1+R) + W_0(1-\omega)(1+r_f))W_0(R-r_f)\right).$$

  If the investor optimally invests everything in the risky asset ($\omega = 1$), the FOC becomes

  $$0 = \mathbb{E}\left(u'(W_0(1+R))W_0(R-r_f)\right).$$

  Solving for r_f gives

  $$r_f = \frac{\mathbb{E}\left(u'(W_0(1+R))R\right)}{\mathbb{E}\left(u'(W_0(1+R))\right)} \approx 0.021692183.$$

If the investor optimally invests everything in the risk-free asset, the FOC becomes (since r_f is constant and $u' > 0$)

$$0 = \mathbb{E}\left(R - r_f\right).$$

Solving for r_f gives

$$r_f = \mathbb{E}\left(R\right).$$

If the investor has to choose between investing 100% of the wealth either in the risky asset or in the risk-free asset, the investor optimally invests in the risk-ree asset if

$$u(W_0(1 + r_r)) > \mathbb{E}\left(u(W_0(1 + R))\right).$$

With $\gamma = 2$, the last inequality means

$$r_f = \frac{u^{-1}\left(\mathbb{E}(u(W_0(1 + R)))\right)}{W_0} - 1 \approx 0.050260716$$

This risk-free rate is higher than the one at which the investor would choose $\omega = 1$ above. Intuitively, the investor is indifferent at that rate between investing everything in the risk-free asset and investing everything in the risky asset. If the investor optimally chooses $\omega = 1$, there is a strict preference for the risky asset, which must be associated with a lower risk-free rate (compared to the case in which the investor is indifferent). Analogously, the risk-free rate at which the investor is indifferent is lower than the rate at which the investor would choose $\omega = 0$.