Assignment 12

1. **Problem 1 (10 points)** A stock is trading at \$24.50. The market consensus expectation is that it will pay a dividend of \$0.50 in two months' time. No other payouts are expected on the stock over the next three months. Assume interest rates are constant at 6% for all maturities. You enter into a long position to buy 10,000 shares of stock in three months' time.

   (a) What is the arbitrage-free price of the three-month forward contract?

   (b) After one month, the stock is trading at \$23.50. What is the value of your forward contract?

   (c) Now suppose that at this point, the company unexpectedly announces that dividends will be \$1.00 per share due to larger-than-expected earnings. Buoyed by the good news, the share price jumps up to \$24.50. What is now the value of your forward contract?

   Solution:

   (a) To find the arbitrage-free forward price, consider a replication strategy for the short forward contract (delivering the stock in the future). The general strategy is to borrow the money, buy the stock, and hold it until the maturity of the forward contract.

   A simple way to fund the purchase is to take a loan for three months. In this case, however, the outcome of the strategy will depend on the future value of the interest rate as we will invest the dividend in 2 months for 1 month.

   To have a completely static replication one should fund the purchase using two separate loans (x\$ for two months and y\$ for three months) and to repay the first loan using the dividend. To be able to exactly repay the first loan using the dividend we must borrow $y = e^{-0.06\times 2/12}0.5 = 0.4950$. We need to raise 24.50 so that $x+y = 24.5$ and $x = 24.0049$. The cashflows at maturity are $-xe^{0.06\times 3/12}+F$ where F is the forward price, so that

$$F = (24.5 - 0.5e^{-0.06\times 2/12}) \times e^{0.06\times 3/12} = 24.368$$

(b) In one month the contract maturity is 2 months. Given that the new spot is 23.50 the forward price is now

$$F = (23.5 - 0.5e^{-0.06\times 1/12}) \times e^{0.06\times 2/12} = 23.234.$$

So the marked-to-market value of the original contract with delivery price $K =$ 24.638 is

$$PV(F - K) = (23.234 - 24.368)e^{-0.06\times 2/12} = -1.1227,$$

i.e. a loss of \$1.1227 per share.

(c) If the dividend changes to 1.00 the forward price is

$$F' = (24.5 - 1.00e^{-0.06\times 1/12}) \times e^{0.06\times 2/12} = 23.741.$$

Given this forward price, the value of the original contract is

$$PV(F' - K) = -0.6208,$$

or a loss of \$0.6208 per share.

2. **Problem 2 (10 points)** The SPX index is currently trading at a value of \$1,265, and the Dow Jones EuroSTOXX Index of 50 stocks, referred to from here on as "STOXX", is trading at EUR 3,671. The dollar interest rate is 3%, and the euro interest rate is 5%. The exchange rate is \$1.28/EUR. The six-month futures on the STOXX is quoted at EUR 3,782. All interest rates are continuously compounded. There are no borrowing costs for securities.

   (a) Compute the correct six-month forward prices of the SPX and STOXX, as well as the forward price between the dollar and the euro.

   (b) Is the futures on the STOXX correctly priced? If not, show how to undertake an arbitrage strategy assuming you are not allowed to undertake borrowing or lending transactions in either currency.

Solution:

(a) The fair FX forward price is $F = 1.28 \times e^{(r_{USD} - r_{EUR})/2} = 1.2673$. Fair prices for the index forwards are $F_{SPX} = 1265 e^{r_{USD}/2} = 1284.12$ and $F_{STOXX} = 2671 e^{r_{EUR}/2} = 3763.93$.

(b) Since $3782 > 3763.93$ the STOXX futures is overpriced and hence it is profitable to establish the short position in the futures offset by a long position in the underlying stocks. The latter requires capital which can be obtained by establishing a reverse position in the S&P, namely going long the futures contract and shorting the stocks in the index.

To summarize, the actions are

- short-sell SPX stocks
- convert the proceeds into EUR at the spot rate
- buy stocks in the STOXX index
- go short STOXX futures
- go long SPX futures to make the position neutral w.r.t. SPX
- establish a currency forward to sell EUR at 1.2673 in 1/2 years

What remains is to figure out the amounts and the profit from the trade. But before doing that, two remarks are in order. First, since interest rates are constant, forward prices and futures prices coincide in this exercise. So, going long futures will be treated as going long a forward contract. Second, we will neglect from margin requirements on futures and short sales. This is why the investor can generate USD cash by selling the stocks in the SPX index. In the real-world, it is more likely that the proceeds from the short sale would be posted as collateral to the lenders of the SPX shares. Instead, the investor would probably try to borrow EUR directly by using his long position in stocks of the STOXX index as collateral to obtain a EUR loan from a bank. In addition, futures positions would too require some (small) initial margin. We abstract from these considerations in this exercise.

If we scale the trade by 1 unit of STOXX index, we then obtain the following flows:

	$t=0$				$t=T$			
	EUR	USD	STOXX	SPX	EUR	USD	STOXX	SPX
buy STOXX	-3671		1					
go short fwd STOXX					3782		-1	
short-sell SPX		4,698.88		-3.71				
go long fwd SPX						-4,769.89		3.71
convert EUR into USD spot	3671	-4,698.88						
go short EUR-USD fwd					-3782	4792.79		
total	0	0	1	-3.71	0	22.90	-1	3.71

At day 0 we need 3671 EUR to buy the European stocks. Hence we need to short $3671 \times 1.28 = 4698.88$ worth of SPX stocks and go long $4698.88/1265 = 3.71$ SPX futures contracts. In six months we close the STOXX futures position by delivering the constituent stocks and are paid 3782 EUR. We convert those to $3782 \times 1.2673 = 4792.79$\$ dollars using the currency forward and pay $3.71 \times 1284.12 = 4769.89$ to buy back the SPX stocks at the futures price. Hence the profit is 22.90\$.

3. **Problem 3 (5 points)** Triangular arbitrage is a strategy that exploits an arbitrage opportunity resulting from a pricing discrepancy among three different currencies in the foreign exchange market. A triangular arbitrage strategy involves three trades, exchanging the initial currency for a second, the second currency for a third, and the third currency for the initial. Consider three exchange rate, dollar/euro, yen/euro and yen/dollar. Provided below are their spot FX rates and one-year interest rates (continuously compounded):

   Spot FX rates: dollar/euro = 1.2822, yen/euro = 146.15, and yen/dollar = 113.98.

   Interest rates: dollar = 3%, euro = 5%, and yen = 1%.

   (a) Check whether triangular arbitrage exists in the spot FX market.

   (b) Suppose, as it is the case in practice, that the FX trading desk of a larger bank makes markets in FX forwards for all major currencies. Assume further that the bank employs one trader for each currency pair, and that the traders do not talk to each other. If each trader sets prices using the arbitrage-free pricing approach (assume zero bid-offer spread) does triangular arbitrage exist in the one-year forward FX market?

Solution:

(a) Changing 1 euro into dollars directly yields \$1.2822. Converting 1 euro to yen yields 146.15 yen, which can be converted to $146.15 \times (113.98)^{-1} = 1.2822\$$. Thus, there is no triangular arbitrage in the spot market.

(b) We first compute fair forwards prices. The logic is the following. Consider, for instance, the dollar/euro pair. Under no arbitrage investing at the risk-free rate in euros should be equivalent to changing euros to dollars on the spot, investing at the risk-free rate in dollars and converting dollars back to euros using a one-forward contract. If it is not the case and one strategy is more profitable, there is clearly an arbitrage opportunity.

Numerically, we have $Xe^{r_{EUR}} = Se^{r_{USD}}$ where S is the spot dollar/euro rate and X is the corresponding rate. Hence $X = Se^{r_{USD}-r_{EUR}} = 1.2568$ and by the same logic the fair forward prices for the other pairs are 140.42 yen/euro and 111.72 yen/dollar. Selling forward 1 euro into dollars directly yields 1.2568\$. Selling forward 1 euro into yen yields 140.42 yen, which can be sold forward into dollars to obtain $140.42/111.72 \approx 1.2568\$$. Thus, there is no triangular arbitrage in the forward market.

4. **Problem 4 (5 points)** Suppose you are given the following information:

- The current price of copper is \$83.55 per 100 lbs.
- The term-structure of interest rates is 5%, i.e., the risk-free interest rate for borrowing/investment is 5% for all maturities in continuously-compounded and annualized terms.
- You can take long and short positions in copper costlessly.
- There are no costs of storing or holding copper.

Consider a forward contract in which the short position has to make *two* deliveries: 10,000 lbs of copper in one month, and 10,000 lbs in two months. The common delivery price in the contract for both deliveries is P, that is, the short position receives P upon making the one-month delivery and P upon making the two-month delivery. What is the arbitrage-free value of P?

Solution: Let Q denote the quantity delivered each month (i.e., $Q = 10000$ lbs). To replicate this contract, we need to buy 2Q units of copper today and store it. After one month, we deliver the first Q units, and after one more month, the second Q units. The cost of this replication strategy is the current spot price of $2Q$ units, which is $2 \times 100 \times 83.55$. This must equal the present value of the cash outflows on the forward strategy, which is

$$Pe^{-0.05 \times 1/12} + Pe^{0.05 \times 2/12} = P \times (e^{0.05 \times 1/12} + e^{0.05 \times 2/12})$$

Equating these, we can solve for P:

$$P = \frac{2 \times 8,355}{\exp(-0.05 \times 1/12) + \exp(-0.05 \times 2/12)} = 8,407.40$$

This is the arbitrage free value of P.