Assignment 5

1. **Mean-variance portfolio choice and leverage constraints (30 points).** Consider an economy with $N = 3$ risky assets R_1, R_2, R_3 and one risk-free asset R_0. The expected return vector is $\mu = [0.09; 0.12; 0.14]$ and standard deviation $\sigma = [0.15; 0.25; 0.30]$. The pair-wise correlation between any two returns is 0.2. There is a risk-free rate $R_0 = 0.05$. We want to solve the problem of a mean-variance investor who faces leverage constraints and cannot borrow more than 20% of his wealth. The investor seeks the portfolio R_P such that $\max E[R_P] - \frac{a}{2}V[R_p]$ subject to $w'\mathbf{1} \leq m$ where $m = 1.2$ and w is the vector of weights invested in the risky assets.

   - Determine the tangency portfolio w_t, its mean, variance and Sharpe ratio.

     Solution: The weights w_t of the tangency portfolio are given by

     $$w_t = \frac{\Sigma^{-1}(\mu - R_0)}{\mathbf{1}\Sigma^{-1}(\mu - R_0)} = (0.4407, 0.2886, 0.2707)',$$

     where Σ is the covariance matrix of the returns.

     Its expected return is given by

     $$\mu_t = \mu' w_t = 0.1122.$$

     Its variance is given by

     $$\sigma_t^2 = w_t' \Sigma w_t = 0.0226.$$

     Its Sharpe Ratio is given by

     $$SR_t = \frac{\mu_t - R_0}{\sigma_t} = 0.414.$$

   - Determine the zero beta portfolio w_z (which has zero correlation with the tangency portfolio), its mean, variance and Sharpe ratio.

     Solution:

     Using the expression for w_t and $w_z' \Sigma w_t = 0$ gives $w_z' \mu = \mu_z = R_0 = 0.05$.

Since it is a minimum-variance portfolio, the zero-beta portfolios can be written as

$$w_z = \lambda\Sigma^{-1}\mathbf{1} + \gamma\Sigma^{-1}\mu,$$

where

$$\lambda = \frac{C - \mu_z B}{\Delta} \qquad \gamma = \frac{\mu_z A - B}{\Delta}$$

and $A = \mathbf{1}'\Sigma^{-1}\mathbf{1}$, $B = \mathbf{1}'\Sigma^{-1}\mu$, $C = \mu'\Sigma^{-1}\mu$, $\Delta = AB - B^2$. Substituting the parameter values and $\mu_z = 0.05$, we get

$$w_z = (1.9099, -0.2748, -0.6351)'.$$

The variance of the zero-beta portfolio is then

$$\sigma_z^2 = w_z \Sigma w_z = 0.0986.$$

Its Sharpe Ratio is evidently equal to 0.

- Prove that the that the investor will optimally choose to invest in a combination of a risky-asset-only mean-variance efficient portfolio and the risk-free rate. Prove further that this implies that we can restrict his optimal portfolio choice to portfolios with returns of the form $R_P = R_0 + x_t(R_t - R_0) + x_z(R_z - R_0)$. Setup the Lagrangian of the agent's problem, derive the first-order condition, and compute the optimal portfolio in terms of x_t, x_z the holdings of tangency and zero-beta portfolio. Then given the portfolio in terms of the underlying securities w_0, w_1, w_2, w_3.

  Solution: For the first statement, see slide 44 in lecture 3. The optimal portfolio is a combination of the tangency portfolio, the minimum variance portfolio and the risk-free asset. Since the minimum variance frontier can be spanned by any two portfolios on it, it is enough to invest in the risk-free asset, w_t and w_z.

  His portfolio of risky assets is $w = x_t w_t + x_z w_z$. The total exposure to risky assets is then

$$\mathbf{1}^\top w = \mathbf{1}^\top(x_t w_t + x_z w_z) = (x_t \underbrace{\mathbf{1}^\top w_t}_{=1} + x_z \underbrace{\mathbf{1}^\top w_z}_{=1}) = x_t + x_z,$$

  since the tangency and zero-beta portfolios consist of risky assets only. The leverage constraint thus rewrites: $x_t + x_z \leq m$. We then have the maximization program:

$$\max_{x_t, x_z} E[R_p] - \frac{a}{2} V[R_p] \qquad \text{s.t.}$$

$$\begin{aligned} R_p &= R_0 + x_t(R_t - R_0) + x_z(R_z - R_0) \\ x_t + x_z &\leq m \end{aligned}$$

Note that

$$V[R_p] = x_t^2 \sigma_t^2 + x_z^2 \sigma_z^2.$$

To solve this problem, set up the Lagrangian

$$L = R_0 + x_t(\mu_t - R_0) + x_z(\mu_z - R_0) - \frac{a}{2}(x_t^2\sigma_t^2 + x_z^2\sigma_z^2) + \lambda\left(m - x_t - x_z\right).$$

We have the following Kuhn-Tucker conditions for optimality

$$\begin{aligned} \frac{\partial L}{\partial x_t} &= \mu_t - R_0 - a x_t \sigma_t^2 - \lambda = 0 \\ \frac{\partial L}{\partial x_z} &= \mu_z - R_0 - a x_z \sigma_z^2 - \lambda = 0 \\ \lambda &\geq 0. \\ \lambda(m - x_t - x_z) &= 0. \end{aligned}$$

From the first two equations, we obtain

$$\begin{aligned} x_t &= \frac{1}{a}\left(\frac{\mu_t - R_0}{\sigma_t^2} - \frac{\lambda}{\sigma_t^2}\right) = \frac{1}{a\sigma_t}\left(SR_t - \frac{\lambda}{\sigma_t}\right) \\ x_z &= \frac{1}{a}\left(\frac{\mu_z - R_0}{\sigma_z^2} - \frac{\lambda}{\sigma_z^2}\right) = -\frac{1}{a\sigma_z}\frac{\lambda}{\sigma_z} \end{aligned}$$

If $m - x_t - x_z > 0$, then $\lambda = 0$. But in this case $x_t = \frac{1}{a\sigma_t} SR_t$ and $x_z = 0$. Then, consistency requires that

$$\begin{aligned} m &> \frac{1}{a\sigma_t} SR_t \\ a &> \frac{SR_t}{m\sigma_t} \equiv a^*. \end{aligned}$$

Substituting the values, we have: $a^* = 2.2963$. If $a > a^*$, then the agent is unconstrained. He does not hold the zero-beta portfolio: $x_z = 0$. Using the fact that $\frac{SR_t}{\sigma_t} = B - AR_0$, his portfolio of risky assets is

$$\begin{aligned} w_u &= x_t w_t \\ &= \frac{1}{a}(B - AR_0)\frac{\Sigma^{-1}(\mu - R_0\mathbf{1})}{B - AR_0} \\ &= \frac{1}{a}\Sigma^{-1}(\mu - R_0\mathbf{1}), \end{aligned}$$

as we should expect. The formulas for the expected return, variance and Sharpe Ratio are as usual (see Jupyter document). If, on the other hand, $a < a^*$, then the investor will short-sell the zero-beta portfolio. Using the fact that the leverage constraint is binding, we can compute the optimal portfolio explicitly

$$\begin{aligned} \mu_z - R_0 - a(m - x_t)\sigma_z^2 &= \mu_t - R_0 - a x_t \sigma_t^2 \\ a x_t(\sigma_z^2 + \sigma_t^2) &= \mu_t - \mu_z + a m \sigma_z^2, \end{aligned}$$

from which

$$\begin{aligned} x_t &= \frac{1}{a}\frac{\mu_t - \mu_z}{\sigma_t^2 + \sigma_z^2} + m\frac{\sigma_z^2}{\sigma_t^2 + \sigma_z^2} \\ x_z &= -\frac{1}{a}\frac{\mu_t - \mu_z}{\sigma_t^2 + \sigma_z^2} + m\frac{\sigma_t^2}{\sigma_t^2 + \sigma_z^2} \end{aligned}$$

His portfolio of risky assets is $w_c = x_t w_t + x_z w_z$. The full expression is a bit messy. Nevertheless, by substituting the expressions we obtained for the tangency and zero-beta portfolios, we easily obtain the expected return, variance and Sharpe Ratio of the optimal constrained portfolio.

$$\begin{aligned} \mu_c &= R_0 + x_t(\mu_t - R_0) \\ \sigma_c^2 &= x_t^2\sigma_t^2 + x_z^2\sigma_z^2 \\ SR_c &= \frac{x_t(\mu_t - R_0)}{\sqrt{x_t^2\sigma_t^2 + x_z^2\sigma_z^2}}. \end{aligned}$$

Finally, the optimal portfolio is simply obtained by pasting these two cases together:

$$w_p = w_u \times \mathbb{I}_{\{a \geq a^*\}} + w_c \times \mathbb{I}_{\{a < a^*\}}.$$

The portfolio expected return, variance and Sharpe Ratio are obtained in the same way.

- Prove that there exists a risk-aversion level a^* so that if $a > a^*$ then the agent is unconstrained and does not hold the zero-beta portfolio. Instead, if $a < a^*$ then the agent will also invest in the zero-beta portfolio.

  Solution: See last bullet point.

- Plot the Sharpe ratio on the optimal portfolio as a function of the risk-aversion level. What happens to the Sharpe ratio of the optimal portfolio as a falls below a^*? Interpret the finding.

  Solution: See Jupyter notebook.

2. **APT (10 points).** Assume that stock market returns have the market index as a common factor, and that all stocks in the economy have a beta of 1 on the market index. Firm-specific returns all have a standard deviation of 20%. Suppose that an analyst studies 10 stocks, and finds that one-half have an alpha of 1%, and the other half an alpha of -1%. Suppose the analyst buys \$1 million of an equally weighted portfolio of the positive alpha stocks, and shorts \$1 million of an equally weighted portfolio of the negative alpha stocks.

   (a) What is the expectation (in dollars) and standard deviation of the analyst's profit?

   Solution: The expected profit is only determined by the alpha, since the long-short strategy has no exposure to the common factor. In expectation, the analyst gains 1% on each of the two positions, i.e. 20,000 \$. Since the risk is firm specific and there are 10 stocks in total, the standard deviation is $0.2 \times \frac{2{,}000{,}000\$}{\sqrt{10}} \approx$ $\$0.126mn$

   (b) How does your answer change if the analyst examines 40 stocks instead of 10 stocks? 100 stocks?

Solution: The expected return does not change, but the standard deviations become $0.2 \times \frac{2{,}000{,}000\$}{\sqrt{40}} \approx \$0.063mn$ and $0.2 \times \frac{2{,}000{,}000\$}{\sqrt{100}} \approx \$0.04mn$, respectively.

3. **Understanding Warren Buffett's performance (30 points).** Warren Buffett is widely considered as one of the most successful investor of the last 50 years. He is the chairman, CEO and largest shareholder of Berkshire Hathaway (BRK). His outstanding performance is illustrated by Figures 1 and 2. Figure 1 shows the annualized Information Ratio (IR) of BRK vs. all actively managed equity funds on the CRSP mutual fund database with at least 30 years of return history.[1] Similarly, Figure 2 shows the annualized IR of BRK vs. all common stock on the CRSP database with at least 30 years of return history. In both cases, he is clearly an outlier. In this exercise you will examine the extent to which Buffett's performance is due to exposure to systematic risk factors.

   (a) Download from CRSP the historical time series of the BRK (permno=17778) monthly stock returns from 1976 to 2019.

   (b) Compute the annualized mean and standard deviation of BRK excess returns as well as the annualized Sharpe ratio. Contrast with that of the three Fama-French factor portfolio returns ($R_m^e = R_m - r_f$, SMB, HML) and the momentum (MOM) as well as the risk-free rate which you may download from Ken French's data web site:

   $https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html$

   (c) Run the following regressions:

$$R_t^e = \alpha + \beta_1 R_{mt}^e + \varepsilon_t \tag{1}$$

$$R_t^e = \alpha + \beta_1 R_{mt}^e + \beta_2 SMB_t + \beta_3 HML_t + \varepsilon_t \tag{2}$$

$$R_t^e = \alpha + \beta_1 R_{mt}^e + \beta_2 SMB_t + \beta_3 HML_t + \beta_4 MOM_t + \varepsilon_t \tag{3}$$

$$R_t^e = \alpha + \beta_1 R_{mt}^e + \beta_2 SMB_t + \beta_3 HML_t + \beta_4 MOM_t + \beta_5 RMW_t + \beta_5 CMA_t + \varepsilon_t \tag{4}$$

   The first regression controls for market exposure.

[1] The IR is defined as the intercept in a regression of monthly excess returns divided by the standard deviation of the residuals. The explanatory variable in the regression is the monthly excess returns of the CRSP value-weighted market portfolio.

The second regression controls for standard factors that capture the effects of size and value (Fama and French (1993)). The third regression controls for momentum (Carhart (1997), Jegadeesh and Titman (1993)). The momentum factor (MOM) is a strategy of buying recent "winner" stocks and shorting recent "loser" stocks. The fourth regression also controls for two additional factors identified by Fama-French in their recent paper "A 5-factor asset pricing model" (posted on moodle). They are called Robust-minus-Weak and Conservative-minus-Aggressive.

For each regression report the estimated α and βs along with the associated t-statistics. Also report the R^2 in the regression and the information ratio of BRK.

(d) Based on your reading of the FF 5-factor paper, explain how RMW and CMA are constructed and why these factor portfolios may capture sources of priced return.

(e) Interpret what the β-estimates say about Warren Buffett's investment strategy.[2]

(f) Does exposure to common risk factors explain Warren Buffett's performance (what happens to the α and information ratio)?

(g) If you can choose your optimal mean-variance efficient portfolio that combines all 6 factor portfolios ($R_M^e, SMB, UMD, MOM, RMW, CMA$), the risk-free rate ($R_f$ proxied by the average short-term T-Bill rate over the same period) and BRK what would be the optimal portfolio weight vector if you target a volatility of 20% for your portfolio?

(h) Rerun all the regressions for data until 1995 and compare with the full-sample results. How does Buffett's performance compare in the first half of the sample and in the full sample (or second half)? What do you think can explain the difference in performance between the two sub-periods?

[2]See also `http://www.forbes.com/sites/phildemuth/2013/06/27/the-mysterious-factor-p-charlie-munger-robert-novy-marx-and-the-profitability-factor`.

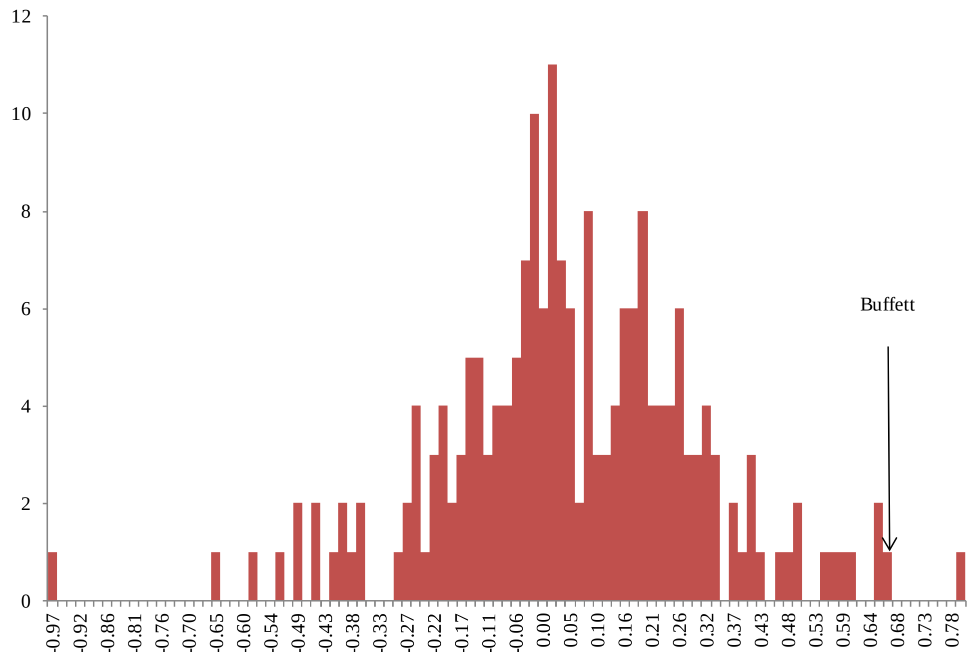

Figure 1: Berkshire vs. mutual funds

This figure shows the distribution of annualized Information Ratios of all actively managed equity funds on the CRSP mutual fund database with at least 30 years of return history. Information ratio is defined as the intercept in a regression of monthly excess returns divided by the standard deviation of the residuals. The explanatory variable in the regression is the monthly excess returns of the CRSP value-weighted market portfolio. The vertical line shows the Information ratio of Berkshire Hathaway.

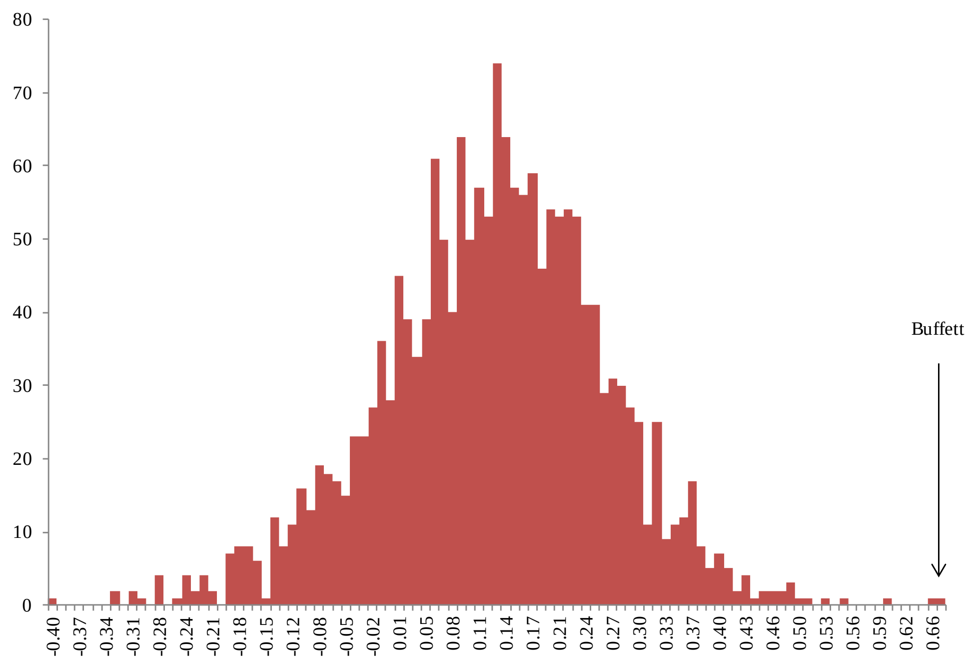

Figure 2: Berkshire vs. common stocks

This figure shows the distribution of annualized Information Ratios of all common stock on the CRSP database with at least 30 years of return history. Information ratio is defined as the intercept in a regression of monthly excess returns divided by the standard deviation of the residuals. The explanatory variable in the regression is the monthly excess returns of the CRSP value-weighted market portfolio. The vertical line shows the Information ratio of Berkshire Hathaway.

Figure 3: Berkshire