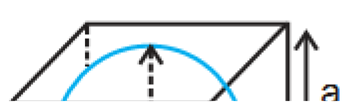
Geometry and Equations Final

1. The largest possible sphere that can be chiseled out from a cube of side "a" cm.



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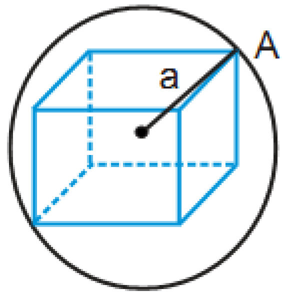
Diagonal of the sphere is a, so radius = a/2.

Remaining empty space in the cube = $a^3 - \frac{\pi a^3}{6}$

2. The largest possible cube that can be chiseled out from a sphere of radius "a" cm



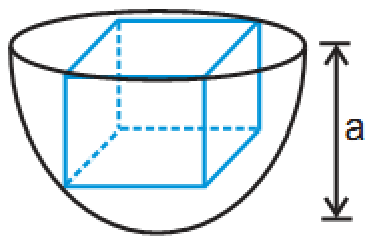
Here OA = radius of the sphere. So diameter of the sphere = 2a.

Let the side of the square = x, then the diagonal of the cube = $\sqrt{3}x$

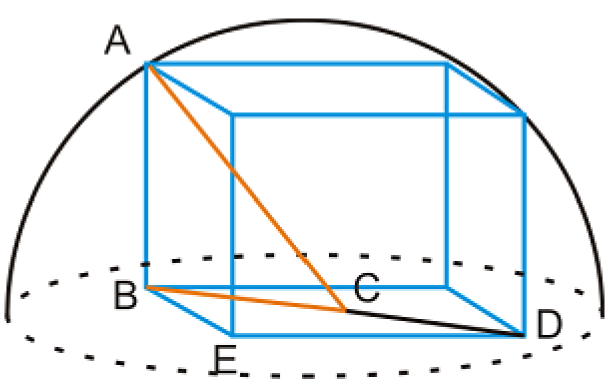
$\Rightarrow \sqrt{3}x = 2a \Rightarrow x = \frac{2a}{\sqrt{3}}$

Therefore side of the square = $\frac{2a}{\sqrt{3}}$

3. The largest possible cube that can be chiseled out from a hemisphere of radius 'a' cm.



Sol:



Given, the radius of the hemi sphere AC = a. Let the side of the cube is x.

From the above diagram, $BE^2 + ED^2 = BD^2$

$$\Rightarrow BC = \frac{\sqrt{2}x}{2} = \frac{x}{\sqrt{2}}$$

From ΔABC, $AC^2 = AB^2 + BC^2$

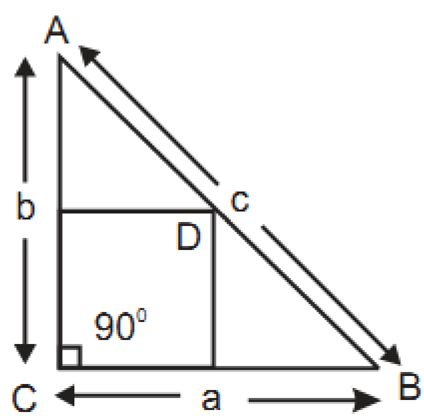
$$\Rightarrow a^2 = x^2 + \left(\frac{x}{\sqrt{2}}\right)^2$$

$$\Rightarrow a^2 = \frac{3x^2}{2}$$

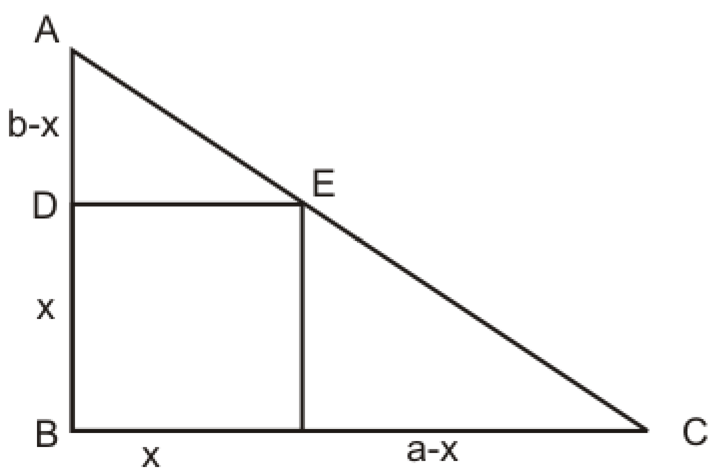
$$\Rightarrow x = \sqrt{\frac{2}{3}}a$$

The edge of the cube = $a\sqrt{\frac{2}{3}}$

4. The largest square that can be inscribed in a right angled triangle ABC when one of its vertices coincide with the vertex of right of the triangle.



Solution:



let the side of the square = x

DE // BC, therefore, ΔADE and ΔABC are similar.

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$b \qquad a$

$\Rightarrow 1-\frac{x}{b}=\frac{x}{a}$

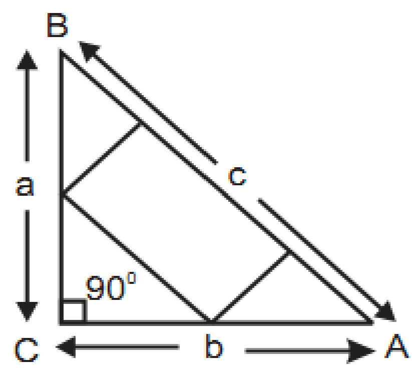
$\Rightarrow 1=\frac{x}{b}+\frac{x}{a}$

$\Rightarrow 1=x\left(\frac{a+b}{ab}\right)$

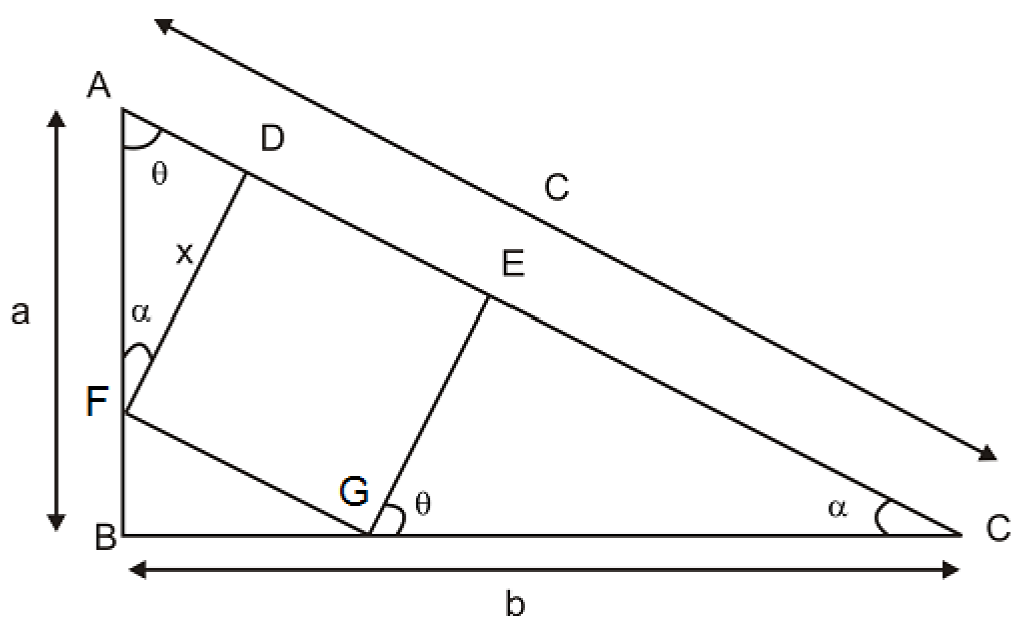
$\Rightarrow x=\left(\frac{ab}{a+b}\right)$

Side of the square = $\frac{ab}{a+b}$ and area of the square = $\left(\frac{ab}{a+b}\right)^2$

5. The largest square that can be inscribed in a right angled triangle ABC when one of its vertices lies on the hypotenuse of the triangle



Solution 1:



From the above diagram, ΔABC and ΔAFD are similar.

$AD \qquad AB$

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$\Rightarrow AD = \frac{xa}{b}$ - - - - - (1)

Also, ΔABC and ΔEGC are similar.

$\Rightarrow Tan\alpha = \frac{GE}{EC} = \frac{a}{b}$

$\Rightarrow \frac{x}{EC} = \frac{a}{b}$

$\Rightarrow EC = \frac{xb}{a}$ - - - - - (2)

We know that c= AD + x + EC

$\Rightarrow c = \frac{xa}{b} + x + \frac{xb}{a}$

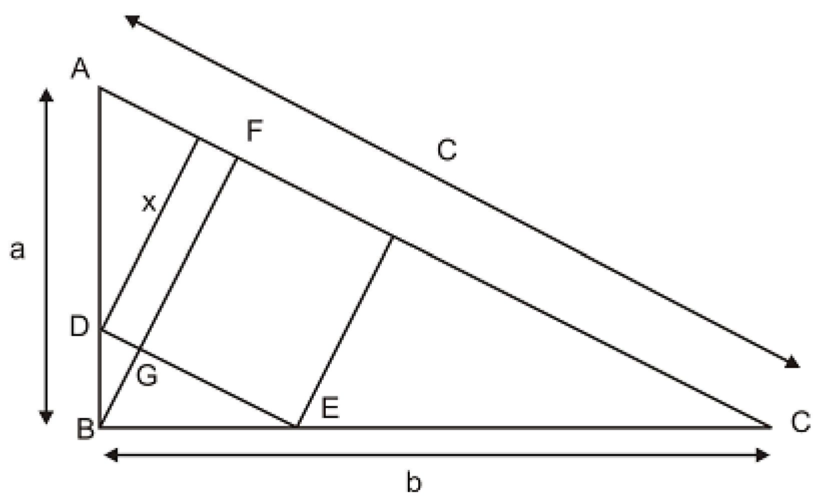
$\Rightarrow c = x\left(\frac{a}{b} + 1 + \frac{b}{a}\right)$

$\Rightarrow c = x\left(\frac{a^2 + ab + b^2}{ab}\right)$

$\Rightarrow x = \left(\frac{abc}{a^2 + ab + b^2}\right)$

Side of the square = $\frac{abc}{a^2 + b^2 + ab}$

Solution 2:



From the above diagram, drop a perpendicular to AC from vertex B.

Area of ΔABC = $\frac{1}{2} \times a \times b = \frac{1}{2} \times BF \times c$

Geometry and Equations Final

$$\Rightarrow \frac{BG}{BF} = \frac{DE}{AC}$$

$$\Rightarrow \frac{\frac{ab}{c} - x}{\frac{ab}{c}} = \frac{x}{c}$$

$$\Rightarrow 1 - \frac{xc}{ab} = \frac{x}{c}$$

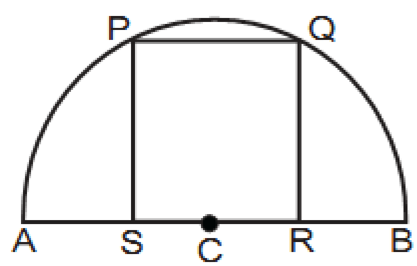
$$\Rightarrow 1 = \frac{xc}{ab} + \frac{x}{c}$$

$$\Rightarrow 1 = x\left(\frac{c}{ab} + \frac{1}{c}\right)$$

$$\Rightarrow 1 = x\left(\frac{c^2 + ab}{abc}\right)$$

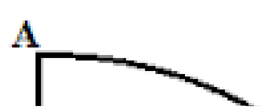
$$\Rightarrow x = \frac{abc}{c^2 + ab} = \frac{abc}{a^2 + b^2 + ab}$$

6. The largest square that can be inscribed in a semi circle of radius 'r' units



Area of the square = $\frac{3}{5}r^2$

7. The largest square that can be inscribed in a quadrant of radius 'r' cm.



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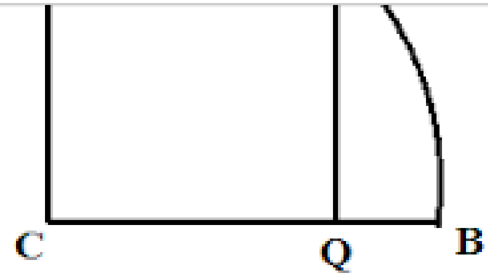
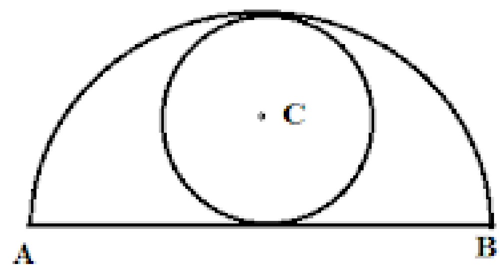


Side of the square = $\frac{r}{\sqrt{2}}$, and area of the square = $\frac{r^2}{2}$

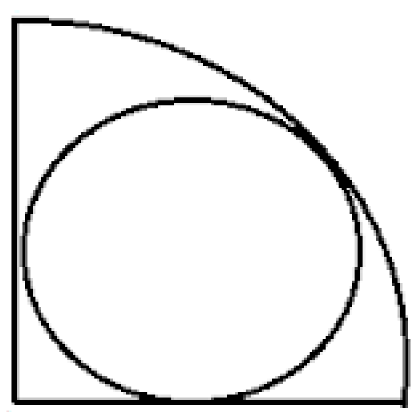
8. The largest circle that can be inscribed in the semi circle of radius 'r' cm is



Inscribed circle area = $\frac{\pi r^2}{4}$

(Rememeber: Inscribed circle area is half of the semi circle area)

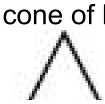
9. The largest circle that can be inscribed in a quadrant of radius 'r' cm is



Area of the circle = $\frac{\pi r^2}{3+2\sqrt{2}}$

Level - 2:

10. The largest cube that can be chiseled out from a cone of height 'h' cm and radius of 'r' cm



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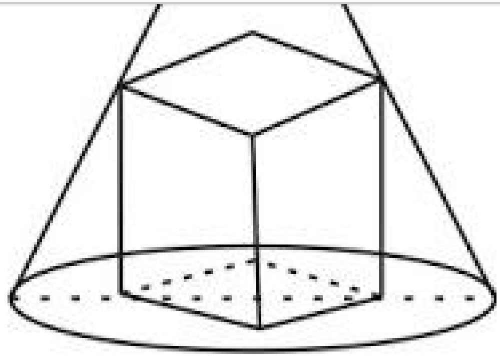
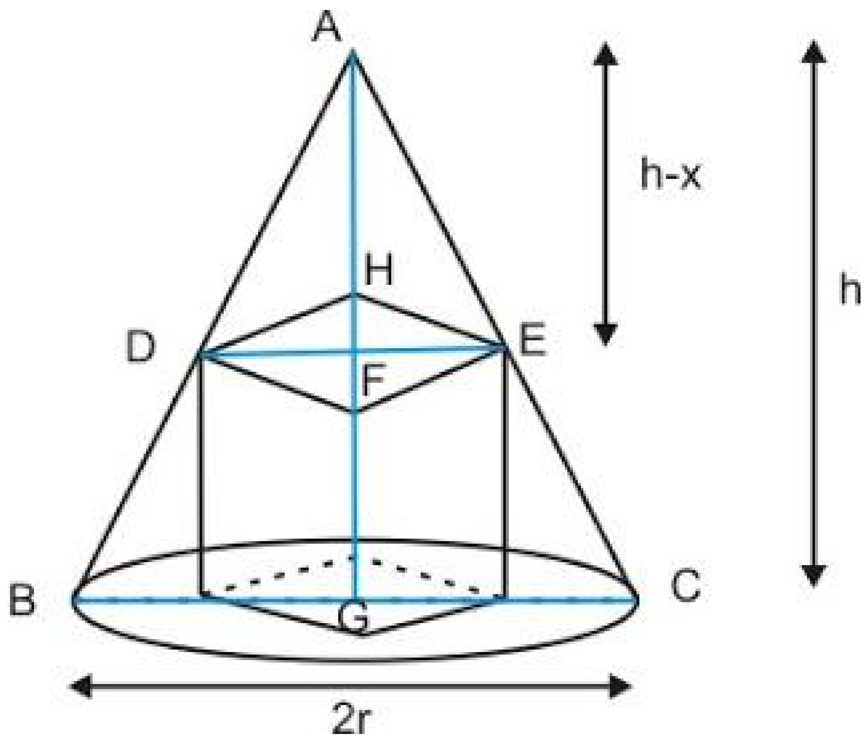


Solution:



Let the side of the square = x

DHE is a right angle triangle. Therefore, $DE^2 = DH^2 + HE^2$

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$$\Rightarrow \frac{AH}{AG} = \frac{DE}{BC}$$

$$\Rightarrow \frac{h-x}{h} = \frac{\sqrt{2}x}{2r}$$

$$\Rightarrow 1 - \frac{x}{h} = \frac{x}{\sqrt{2}r}$$

$$\Rightarrow \frac{x}{\sqrt{2}r} + \frac{x}{h} = 1$$

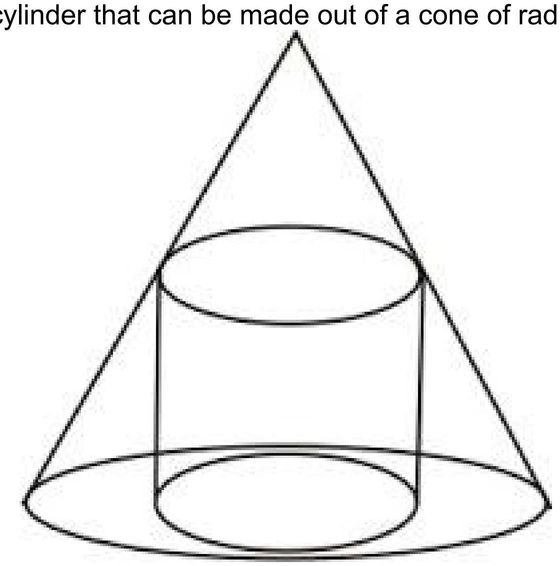
$$\Rightarrow x\left(\frac{1}{\sqrt{2}r} + \frac{1}{h}\right) = 1$$

$$\Rightarrow x\left(\frac{h+\sqrt{2}r}{\sqrt{2}rh}\right) = 1$$

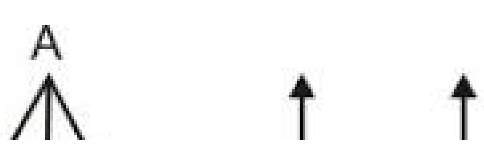
$$\Rightarrow x = \frac{\sqrt{2}rh}{h+\sqrt{2}r}$$

Square side = $\frac{\sqrt{2}rh}{h+\sqrt{2}r}$

11. Find the maximum volume of cylinder that can be made out of a cone of radius 'r' and height 'h'



Sol:
Let the radius of the cylinder = xx and height = yy. We have to maximize the volume of the cylinder V = πx2yπx2y. To maximize the volume we have to change one of the given variable in the equation into "r"and "h" and differentiate the equation with respect to the other variable.



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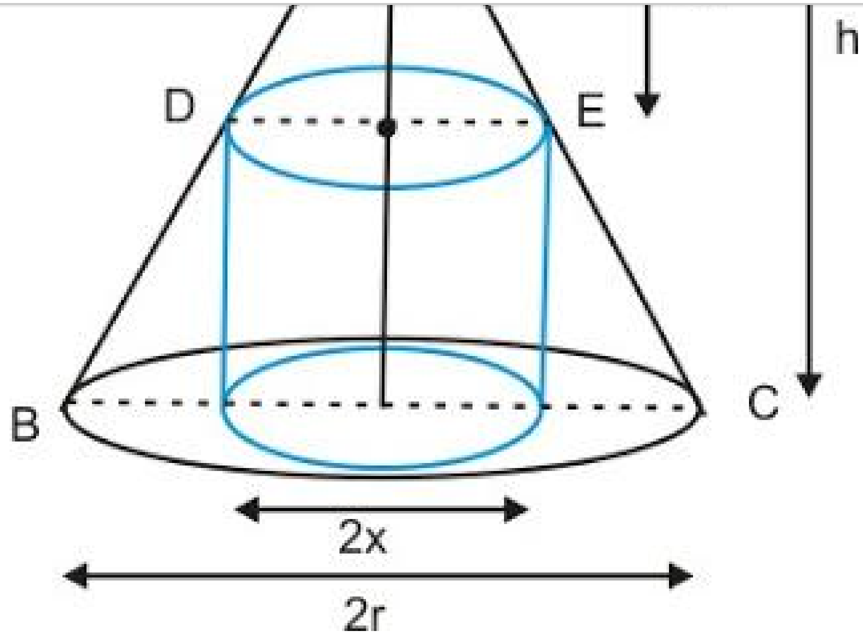


From the above, ΔADE and ΔABC are similar.

So, $\Rightarrow \frac{h-y}{h} = \frac{2x}{2r}$ - - - - - (1)

Geometry and Equations Final

Substituting the above result in the volume we get,

$V = \pi x^2 \left(h - \frac{hx}{r}\right)$

$V = \pi h x^2 - \frac{\pi h x^3}{r}$

Differentiation the above equation w.r.t x,

$V^1(x) = 2\pi h x - \frac{3\pi h x^2}{r}$

By equating the above equation to zero, we can find the value of x where the above equation becomes maximum.

$2\pi h x - \frac{3\pi h x^2}{r} = 0$

$\Rightarrow 2\pi h x \left(2 - \frac{3x}{r}\right) = 0$

h, x cannot be zero. So $2 - \frac{3x}{r} = 0$

$\Rightarrow x = \frac{2r}{3}$ - - - - - (2)

Substituting the above result in equation (1),

$\Rightarrow \frac{h-y}{h} = \frac{2\left(\frac{2r}{3}\right)}{2r}$

$\Rightarrow \frac{h-y}{h} = \frac{2}{3}$

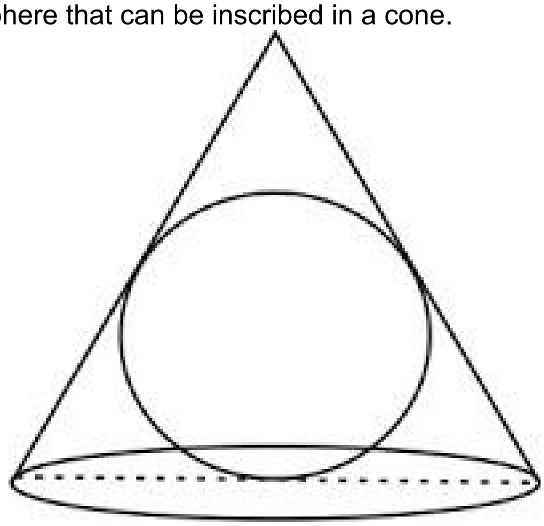
$\Rightarrow h - y = \frac{2}{3}h$

$\Rightarrow y = \frac{h}{3}$ - - - - - (3)

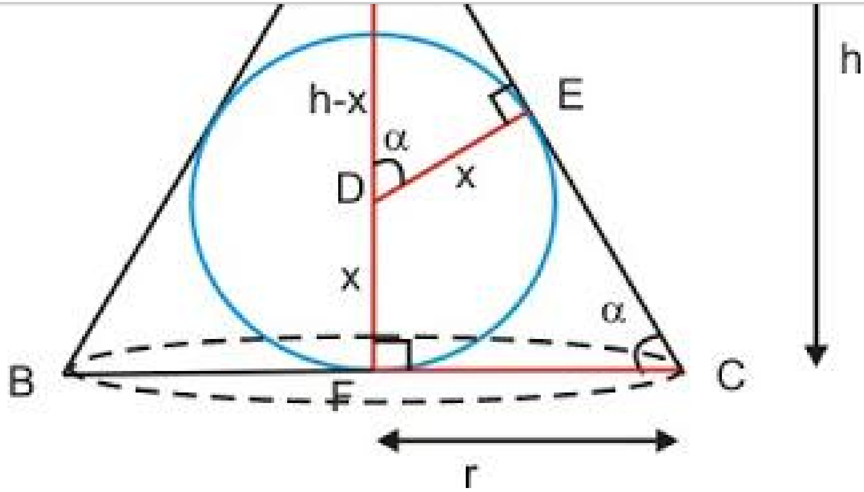
Therefore, maximum volume of the cylinder = $\pi\left(\frac{2r}{3}\right)^2\left(\frac{h}{3}\right)$

$\Rightarrow \frac{4}{27}\pi r^2 h$

12. Find the maximum volume of sphere that can be inscribed in a cone.



Sol:



Let the cone height = h and radius = r. Also let the radius of the sphere inside the cone = x

We have to maximize the volume of the sphere = $V = \frac{4}{3}\pi x^3$

From the above diagram, ΔAFC and ΔADE are similar. ($\because$ AAA rule)

Therefore, $\frac{CF}{DE} = \frac{AC}{AD}$

From Pythagoras rule, $AC = \sqrt{AF^2 + FC^2} = \sqrt{h^2 + r^2}$

Now, $\frac{r}{x} = \frac{\sqrt{r^2 + h^2}}{h - x}$

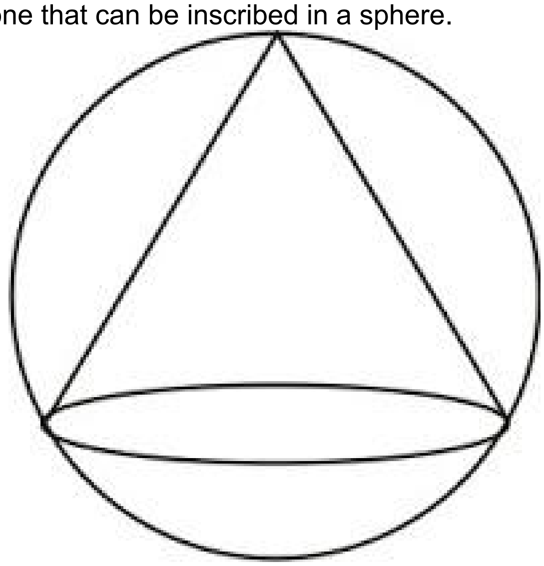
$\Rightarrow rh - rx = x\sqrt{r^2 + h^2}$

$\Rightarrow rh = rx + x\sqrt{r^2 + h^2}$

$\Rightarrow rh = x\left(r + \sqrt{r^2 + h^2}\right)$

$\Rightarrow x = \frac{rh}{r + \sqrt{r^2 + h^2}}$

13. Find the maximum volume of cone that can be inscribed in a sphere.



Sol:

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