Flight Dynamics 4
Revision Notes

The Stability of Dynamic Systems

The stability of any dynamic system is obtained by consideration of its free (unforced) motion and its response to an input (or control) is established from its forced motion. In the context of mechanical systems the simplest example to consider might be the free vibration of a mass/spring/damper system, Figure1. The equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{1}$$

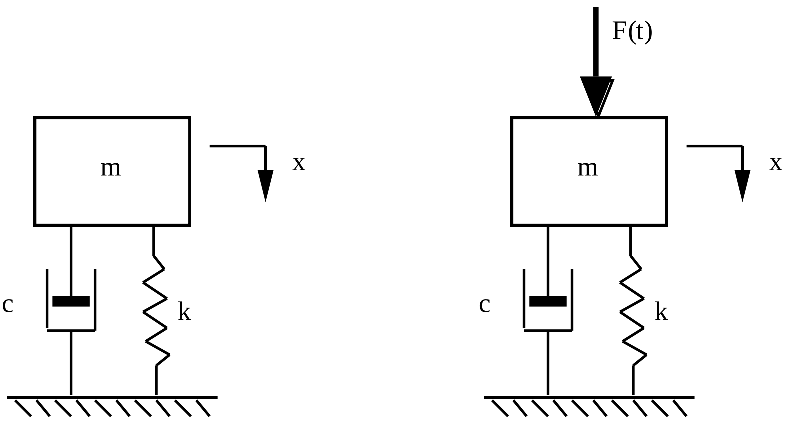
which for unforced motion gives $m\ddot{x} + c\dot{x} + kx = 0$

or

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{2}$$

where ω_n = the undamped natural frequency of the system (rad/s) = $\sqrt{\frac{k}{m}}$ and

ζ = the damping ratio of the system = $\frac{c}{2\sqrt{km}}$.



Free-vibration Forced-vibration

Figure 1: Simple Mass/Spring/Damper System

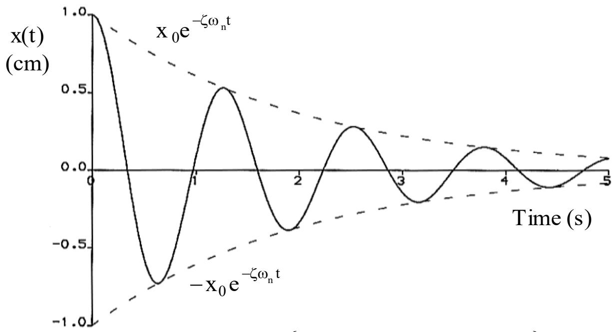
The system's stability is determined from equation (2) directly by solution for free vibration, for example, when the system is underdamped (ie $\zeta < 1$) and is given an initial displacement x_0 its response in free vibration is obtained by solving equation (2) with initial conditions x(0) = x_0, and $\dot{x}(0) = 0$ giving :

$$x(t) = x_0 e^{-\zeta\omega_n t}\left(\sin\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\cos\omega_d t\right) \tag{3}$$

where ω_d = the damped natural frequency = $\omega_n\sqrt{1-\zeta^2}$, shown in Figure 2

By inspection this gives an oscillation with angular frequency ω_d, and decaying amplitude and we can deduce that the characteristic motion of the system will be stable (any disturbance tending to decay with time).

For a more complex system with further degrees of freedom, inspection of the poles of the system or use of the Routh-Horowitz Criteria would be a more appropriate approach.



$$x(t) = x_0 e^{-\zeta\omega_n t}\left\{\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right\}$$

Figure 2: Free Vibration of a Simple Mass/Spring/Damper System

The system's response to an input is obtained by setting $F(t)$ to an appropriate value and solving, for example if $F(t) = F_0$ then:

$$x(t) = \frac{F_0}{m}\left[1 - x_0 e^{-\zeta\omega_n t}\left(\sin\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)\right] \tag{4}$$

giving the response shown in Figure 2.3.

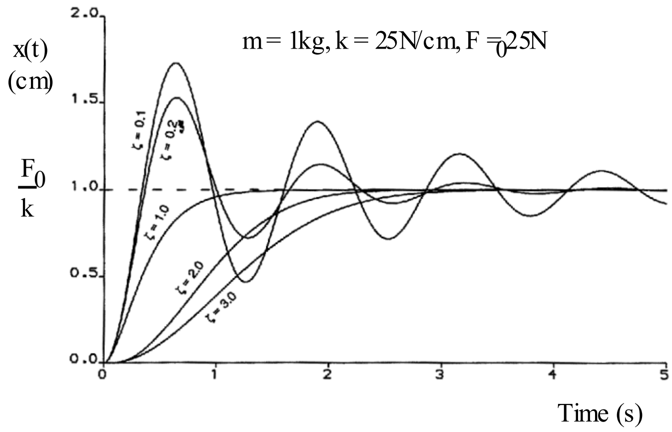


Figure 2.3: Forced Vibration of a Simple Mass/Spring/Damper System

1. <u>Determining Stability - Eigenvalues</u>

Consider firstly the single degree of freedom system with equation of motion given by (2). Assuming the general solution:

$$x = x_0 e^{\lambda t} \tag{5}$$

such that: $\dot{x} = \lambda x_0 e^{\lambda t}$ and $\ddot{x} = \lambda^2 x_0 e^{\lambda t}$, then substituting in (2) (and dividing by $x_0 e^{\lambda t}$) gives

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{6}$$

This is the <u>Characteristic Equation</u> of the system and its roots are known as the system's <u>Eigenvalues</u>. Note that if we had simply taken the Laplace Transform of the equation with zero initial conditions we would get the characteristic equation in terms of the Laplace operator, s:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{7}$$

Clearly equations (6) and (7) are equivalent and so we can deduce that <u>the poles and eigenvalues of a system are equivalent</u>. Further we may deduce that as the poles of a system give information on the system's stability characteristics then so do its eigenvalues.

Solving (6), the eigenvalues are given by :

$$\lambda = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

or

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{8}$$

We therefore see that a system's eigenvalues determine its stability and the eigenvalue itself is a function of damping, ζ. This is explored in the following section.

2. <u>Relationship Between Eigenvalue, Damping and Stability</u>

When <u>$\zeta > 1$</u> (an overdamped system) we can express λ as shown in (8), and the eigenvalue will be a <u>real</u> number. From (5) the response will be in the form:

$$x = x_0 e^{\left(-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}\right)t} \quad \text{or} \quad x = x_0 e^{\left(-\zeta \pm_n \sqrt{\zeta^2-1}\right)\omega_n t}$$

This is consistent with what we already know of overdamped systems – they have a characteristic exponential motion. Now, if the eigenvalue is negative (which will always occur provided ζ is positive) then the response will be an exponential decay and the system is then *Statically Stable*. If the eigenvalue is positive (implying that ζ is negative) the response will be an exponential growth and the system is *Statically Unstable*.

When <u>$\zeta < 1$</u> (an underdamped system) we can write (8) in the form:

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm i\omega_d$$

and hence we see that the eigenvalue is in fact an imaginary number. The response is obtained by substituting this into equation (5):

$$x = x_0 e^{(-\zeta\omega_n \pm i\omega_d)t} = x_0 e^{-\zeta\omega_n t} e^{\pm\zeta\omega_d t}$$

or

$$x = x_0 e^{-\zeta\omega_n t}\left(\cos\omega_d t \pm i\sin\omega_d t\right)$$

This is consistent with what we already know of underdamped systems – they have a characteristic oscillatory motion. As before we find that the response is an oscillation with angular frequency $\omega_{\square}$ (the real part of the eigenvalue). The amplitude of the oscillation will decrease provided the term "$\zeta\omega_n$" is positive (which is always the case for mechanical systems – damping cannot be negative). Hence, for the system to exhibit oscillations of decaying amplitude the real part of the eigenvalue must be negative and this system is said to be *Dynamically Stable*. Conversely should the term "$\zeta\omega_n$" be negative (implying that ζ is negative and the real part of the eigenvalue is positive), then the amplitude of the oscillations will increase and the system will be *Dynamically Unstable*. This situation of negative damping can only occur if energy is added to the system and occurs in aircraft flight where there is an interchange between kinetic and potential energy and thrust.

<u>SUMMARY</u>

λ Real and Negative :	This gives an exponential decay or a "Convergence". This situation is indicative of <u>static stability</u>.
λ Real and Positive :	This gives an exponential growth or a "Divergence". This situation is indicative of <u>static instability</u>.
λ Complex -ve Real Part :	This gives an oscillatory motion with a decreasing amplitude. This situation is indicative of <u>dynamic stability</u>.

λ Complex +ve Real Part : This gives an oscillatory motion with an increasing amplitude. This situation is indicative of dynamic instability.

These response types are shown in Figure 4 and the locations of eigenvalues on a complex plane (root locus plot) in Figure 5.

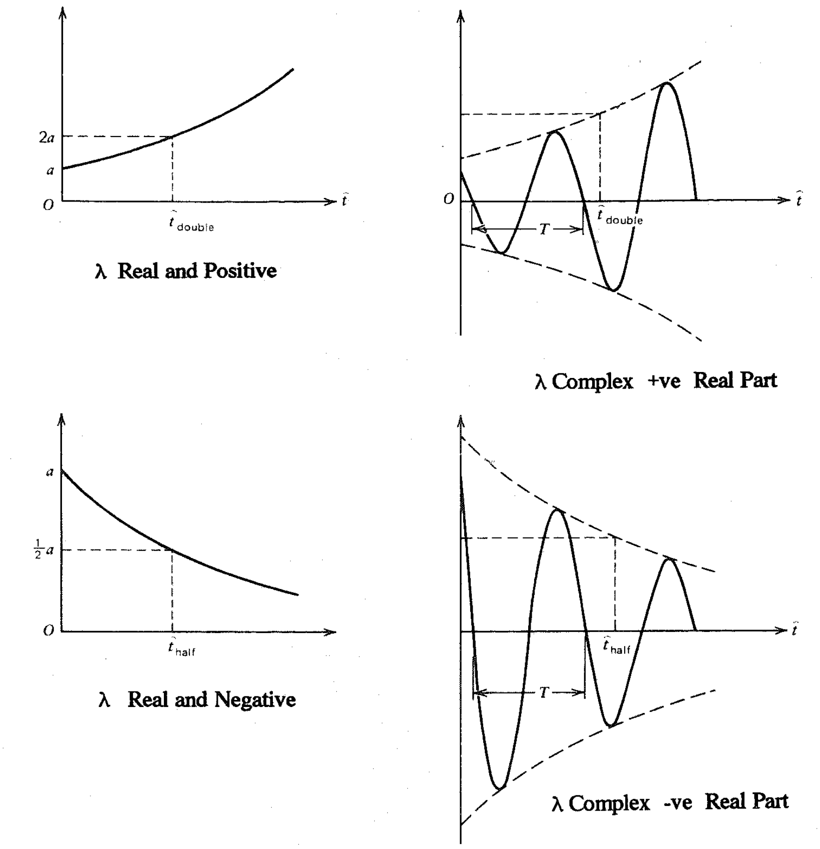


Figure 4: Characteristic Response Types

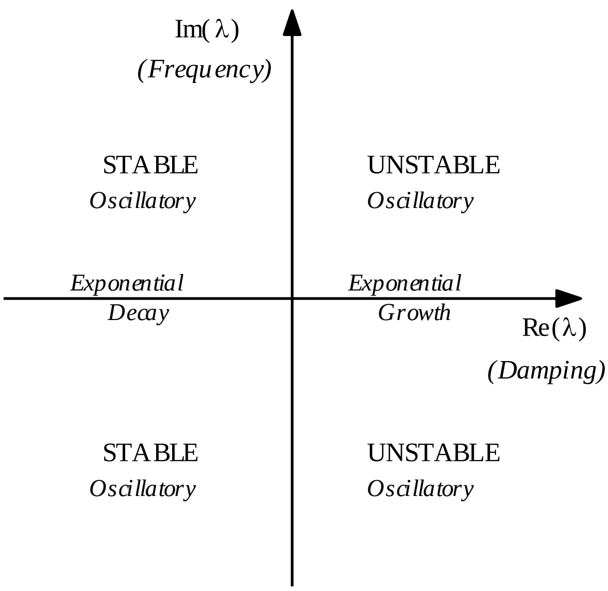


Figure 5: Stability Presented on Complex Plane

3. Period and Time to Half Amplitude

Consider the general eigenvalue: $\lambda = \mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda)$

which gives the response from (5):

$$x = x_0 e^{\lambda t} = x_0 e^{(\mathrm{Re}(\lambda)+i\,\mathrm{Im}(\lambda))t} = x_0 e^{\mathrm{Re}(\lambda)t} e^{i\,\mathrm{Im}(\lambda)t}$$

or

$$x = x_0 e^{\mathrm{Re}(\lambda)t}\left(\cos(\mathrm{Im}(\lambda)t) \pm i \sin(\mathrm{Im}(\lambda)t)\right)$$

Thus, for oscillatory modes the angular frequency (in rad/s) of the oscillation is given by imaginary part of the eigenvalue, Im(λ), so that its period, *T*, is:

$$T = \frac{2\pi}{\mathrm{Im}(\lambda)} \tag{9}$$

and the frequency is:

$$f = \frac{1}{T}\,\mathrm{Hz}$$

The rate of decay of the oscillation (i.e. the damping) clearly comes from the real part of the eigenvalue. Damping of an aircraft mode is usually measured by the time to half amplitude in the case of a convergent mode or time to double amplitude in the case of a divergent mode. The decay of successive amplitudes comes from:

$$x = x_0 e^{\mathrm{Re}(\lambda)t}$$

For stable systems the time to half amplitude, $t_{\frac{1}{2}}$, occurs when $x = 0.5x_0$

i.e.

$$0.5x_0 = x_0 e^{\mathrm{Re}(\lambda)t_{\frac{1}{2}}}$$

or

$$\ln(0.5) = \mathrm{Re}(\lambda)t_{\frac{1}{2}}$$

giving

$$t_{\frac{1}{2}} = \frac{0.69}{|\mathrm{Re}(\lambda)|} \tag{10}$$

As time to half amplitude is only calculated for a stable system where the eigenvalue will be negative, the magnitude of its value is taken to give a positive time. The time to double amplitude, t_d, of an unstable system is determined in the same way:

i.e.

$$2x_0 = x_0 e^{\mathrm{Re}(\lambda)t_d}$$

or

$$\ln(2) = \mathrm{Re}(\lambda)t_d$$

giving

$$t_d = \frac{0.69}{\mathrm{Re}(\lambda)} \tag{11}$$

Note that ln(2) = -0.69, again the magnitude is taken to give a positive time.