Parameterized Complexity of Submodular Minimization under Uncertainty

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Abstract

This paper studies the computational complexity of a robust variant of a two-stage submodular minimization problem that we call ROBUST SUBMODULAR MINIMIZER. In this problem, we are given k submodular functions f_1, \ldots, f_k over a set family 2^V , which represent k possible scenarios in the future when we will need to find an optimal solution for one of these scenarios, i.e., a minimizer for one of the functions. The present task is to find a set $X \subseteq V$ that is close to *some* optimal solution for each f_i in the sense that some minimizer of f_i can be obtained from X by adding/removing at most d elements for a given integer $d \in \mathbb{N}$. The main contribution of this paper is to provide a complete computational map of this problem with respect to parameters k and d, which reveals a tight complexity threshold for both parameters:

- ROBUST SUBMODULAR MINIMIZER can be solved in polynomial time when $k \leq 2$, but is NP-hard if k is a constant with $k \geq 3$.
- ROBUST SUBMODULAR MINIMIZER can be solved in polynomial time when d=0, but is NP-hard if d is a constant with $d \ge 1$.
- ROBUST SUBMODULAR MINIMIZER is fixed-parameter tractable when parameterized by (k, d). We also show that if some submodular function f_i has a polynomial number of minimizers, then the problem becomes fixed-parameter tractable when parameterized by d. We remark that all our hardness results hold even if each submodular function is given by a cut function of a directed graph.

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1 Introduction

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This paper proposes a two-stage robust optimization problem under uncertainty. Suppose that we want to find a minimum cut on a directed graph under uncertainty. The uncertainty here is represented by k directed graphs G_1, \ldots, G_k on the same vertex set $V \cup \{s, t\}$. That is, we have k possible scenarios of graph realizations in the future. At the moment, we want to choose an (s,t)-cut in advance, so that after the graph is revealed, we will be able to obtain a minimum (s,t)-cut in the graph with small modification. Therefore, our aim is to find an (s,t)-cut that is close to some minimum (s,t)-cut in each graph G_i for $i=1,\ldots,k$.

Let us formalize this problem. For a vertex set X in a directed graph $G = (V \cup \{s, t\}, E)$, the cut function $f: 2^V \to \mathbb{Z}$ is the number of out-going edges from X. Let us denote the family of minimum (s,t)-cuts in G by $\mathcal{C}_{s,t}(G)$, that is, $\mathcal{C}_{s,t}(G) = \{Y \subseteq V : f(Y) \leq f(Y') \ \forall Y' \subseteq V\}$. Given directed graphs G_1, \ldots, G_k over a common vertex set $V \cup \{s, t\}$, we want to find a subset $X \subseteq V$ and sets $Y_i \in \mathcal{C}_{s,t}(G_i)$ for each $i \in [k]$ that minimizes $\max_{i \in [k]} |X \triangle Y_i|$ where \triangle stands for symmetric difference and [k] denotes $\{1,\ldots,k\}$ for any positive integer k.

We study a natural generalization of this problem where, instead of the cut functions of directed graphs which are known to be submodular [27], we consider arbitrary submodular set functions over some non-empty finite set V. Let $f_1, \ldots, f_k \colon 2^V \to \mathbb{R}$ be k submodular functions. Let $\arg \min f_i = \{Y \subseteq V : f_i(Y) \le f_i(Y') \ \forall Y' \subseteq V\}$ refer to the set of minimizers of f_i . We want to find a subset $X \subseteq V$ and sets $Y_i \in \arg\min f_i$ for all $i \in [k]$ that

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\max_{i \in [k]} |X \triangle Y_i|.
minimize
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We call the decision version of this problem ROBUST SUBMODULAR MINIMIZER.

ROBUST SUBMODULAR MINIMIZER:

A finite set V, submodular functions $f_1, \ldots, f_k : 2^V \to \mathbb{R}$, and an integer $d \in \mathbb{N}$. Input: Find a set $X \subseteq V$ such that for each $i \in [k]$ there exists $Y_i \in \arg\min f_i$ with Task: $|X \triangle Y_i| \leq d$, or detect if no such set exists.

We remark that the min-sum variant of the problem, that is, the problem obtained by replacing the condition $\max_{i \in [k]} |X \triangle Y_i| \le d$ with $\sum_{i \in [k]} |X \triangle Y_i| \le d$, was introduced by Kakimura et al. [16], who showed that it can be solved in polynomial time.

1.1 **Our Contributions and Techniques**

Our contribution is to reveal the complete computational complexity of ROBUST SUBMODU-LAR MINIMIZER with respect to the parameters k and d. We also provide an algorithm for the case when one of the submodular functions has only polynomially many minimizers. Our results are as follows:

- 1. Robust Submodular Minimizer can be solved in polynomial time when $k \leq 2$ (Theorem 6), but is NP-hard if k is a constant with $k \geq 3$ (Corollary 24).
- 2. Robust Submodular Minimizer can be solved in polynomial time when d=0(Observation 4), but is NP-hard if d is a constant with $d \geq 1$ (Theorem 20).
- 3. ROBUST SUBMODULAR MINIMIZER is fixed-parameter tractable when parameterized by (k,d).
- 4. ROBUST SUBMODULAR MINIMIZER is fixed-parameter tractable when parameterized by d, if the size of arg min f_i for some $i \in [k]$ is polynomially bounded.

When k = 1, Robust Submodular Minimizer is equivalent to the efficiently solvable submodular function minimization problem [19], in which we are given a single submodular function $f: 2^V \to \mathbb{R}$ and want to find a set $X \subseteq V$ in arg min f. It is not difficult to observe that Robust Submodular Minimizer for d=0 can also be solved in polynomial time by computing a minimizer of the submodular function $\sum_{i=1}^k f_i$; see Section 3.1.

The rest of our positive results are based on Birkhoff's representation theorem on distributive lattices [1] that allows us to maintain the family of minimizers of a submodular function in a compact way. Specifically, even though the number of minimizers may be exponential in the input size, we can represent all minimizers as a family of cuts in a directed acyclic graph with polynomial size. As we show in Section 3.1, we can use this representation to solve an instance I of ROBUST SUBMODULAR MINIMIZER with k=2 by constructing a directed graph with two distinct vertices, s and t, in which a minimum (s,t)-cut yields a solution for I. More generally, Birkhoff's compact representation allows us to reduce ROBUST SUBMODULAR MINIMIZER for arbitrary k to the so-called MULTI-BUDGETED DIRECTED CUT problem, solvable by an algorithm due to Kratsch et al. [17], leading to a fixed-parameter tractable algorithm for the parameter (k,d). We note that a similar construction was used to show that the min-sum variant of the problem is polynomial-time solvable [16].

In Section 3.3, we consider the case when one of the k submodular functions has only polynomially many minimizers. As mentioned in [16], ROBUST SUBMODULAR MINIMIZER is NP-hard even when each submodular function f_i has a unique minimizer. In fact, the problem is equivalent to the CLOSEST STRING problem over a binary alphabet, shown to be NP-hard under the name MINIMUM RADIUS by Frances and Litman [11]. For the case when $|\arg\min f_i|$ is polynomially bounded for some $i \in [k]$, we present a fixed-parameter tractable algorithm parameterized only by d. Our algorithm guesses a set in $\arg\min f_i$ and uses it as an "anchor," then solves the problem recursively by the bounded search-tree technique.

Section 4 contains our NP-hardness results for the cases when either d is a constant at least 1, or k is a constant at least 3. We present reductions from an intermediate problem that may be of independent interest: in this problem, we are given k set families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ over a universe V containing two distinguished elements, s and t, with each \mathcal{F}_i containing pairwise disjoint subsets of V; the task is to find a set $X \subseteq V$ containing s but not t that has a bounded distance from each family \mathcal{F}_i for a specific distance measure.

1.2 Related Work

ROBUST SUBMODULAR MINIMIZER is related to the robust recoverable combinatorial optimization problem, introduced by Liebchen et al. [21]. It is a framework of mathematical optimization that allows recourse in decision-making to deal with uncertainty. In this framework, we are given a problem instance with some scenarios and a recovery bound d, and the task is to find a feasible solution X (the first-stage solution) to the instance that can be transformed to a feasible solution Y_i (the second-stage solutions) in each scenario i respecting the recovery bound (e.g., $|X \triangle Y_i| \leq d$ for each i). The cost of the solution is usually evaluated by the sum of the cost of X and the sum of the costs of Y_i 's. Robust recoverable versions have been studied for a variety of standard combinatorial optimization problems. Examples include the shortest path problem [5], the assignment problem [10], the travelling salesman problem [12], and others [14, 18, 20]. The setting was originally motivated from the situation where the source of uncertainty was the cost function which changes in the second stage. We consider another situation dealing with structural uncertainty, where some unknown set of input elements can be interdicted [8, 15]. Recently, a variant of robust recoverable problems has been studied where certain operations are allowed in the second stage [13].

Reoptimization is another concept related to ROBUST SUBMODULAR MINIMIZER. In general reoptimization, we are given an instance I of a combinatorial optimization problem and an optimal solution X for I. Then, for a slightly modified instance I' of the problem, we need to make a small change to X so that the resulting solution X' is an optimal (or a good approximate) solution to the modified instance I'. Reoptimization has been studied for several combinatorial optimization problems such as the minimum spanning tree problem [4], the traveling salesman problem [22], and the Steiner tree problem [2].

4 Parameterized Complexity of Submodular Minimization under Uncertainty

2 Preliminaries

Graphs and Cuts

Given a directed graph G = (V, E), we write uv for an edge pointing from u to v. For a subset $X \subseteq V$ of vertices in G, let $\delta_G(X)$ denote the set of edges leaving X. If G is an undirected graph, then $\delta_G(X)$ for some set X of vertices denotes the set of edges with exactly one endpoint in X. We may simply write $\delta(X)$ if the graph is clear from the context.

For two vertices s and t in a directed or undirected graph G = (V, E), an (s, t)-cut is a set X of vertices such that $s \in X$ but $t \notin X$. A minimum (s, t)-cut in G is an (s, t)-cut X that minimizes $|\delta(X)|$. Given a cost function $c: E \to \mathbb{R}_+ \cup \{+\infty\}$ on the edges of G where \mathbb{R}_+ is the set of all non-negative real numbers, the (weighted) cut function $\kappa_G: 2^V \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\kappa_G(X) = \sum_{e \in \delta(X)} c(e). \tag{1}$$

A minimum-cost (s,t)-cut is an (s,t)-cut X that minimizes $\kappa_G(X)$.

Distributive Lattices

In this paper, we will make use of properties of finite distributive lattices on a ground set V. A distributive lattice is a set family $\mathcal{L} \subseteq 2^V$ that is closed under union and intersection, that is, $X, Y \in \mathcal{L}$ implies $X \cup Y \in \mathcal{L}$ and $X \cap Y \in \mathcal{L}$. Then \mathcal{L} is a partially ordered set with respect to set inclusion \subseteq , and has a unique minimal element and a unique maximal element. Birkhoff's representation theorem is a useful tool for studying distributive lattices.

- ▶ **Theorem 1** (Birkhoff's representation theorem [1]). Let $\mathcal{L} \subseteq 2^V$ be a distributive lattice. Then there exists a partition of V into $U_0, U_1, \ldots, U_b, U_\infty$, where U_0 and U_∞ can possibly be empty, such that the following hold:
- (1) Every set in \mathcal{L} contains U_0 .
- (2) Every set in \mathcal{L} is disjoint from U_{∞} .
- (3) For every set $X \in \mathcal{L}$, there exists a set $J \subseteq [b]$ of indices such that $X = U_0 \cup \bigcup_{j \in J} U_j$.
- (4) There exists a directed acyclic graph $G(\mathcal{L})$ that has the following properties.
 - (a) The vertex set is $\{U_0, U_1, \dots, U_b\}$.
 - (b) U_0 is a unique $sink^1$ of $G(\mathcal{L})$.
 - (c) For a non-empty set Z of vertices in $G(\mathcal{L})$, Z has no out-going edge if and only if $\bigcup_{U_j \in Z} U_j \in \mathcal{L}$.

For a distributive lattice $\mathcal{L} \subseteq 2^V$, we call the directed acyclic graph $G(\mathcal{L})$ above a *compact* representation of \mathcal{L} . Note that the size of $G(\mathcal{L})$ is $O(|V|^2)$ while $|\mathcal{L}|$ can be as large as $2^{|V|}$.

Submodular Function Minimization

Let V be a non-empty finite set. A function $f : 2^V \to \mathbb{R}$ is submodular if $f(X) + f(Y) \ge f(X \cup Y) + f(X \cap Y)$ for all $X, Y \subseteq V$. A typical example of submodular functions is the cut function κ_G of a directed (or undirected) edge-weighted graph G as defined in (1). If the graph $G = (V \cup \{s, t\}, E)$ contains two distinct vertices, s and t, then we can restrict the cut function to the domain of (s, t)-cuts in the following sense: each $X \subseteq V$ corresponds to an (s, t)-cut

¹ A *sink* is a vertex of out-degree zero.

 $X \cup \{s\}$ in G; then the function $\lambda_G : 2^V \to \mathbb{R}_+ \cup \{+\infty\}$ defined by $\lambda_G(X) = \kappa_G(X \cup \{s\})$ is submodular.

When we discuss computations on a submodular function $f: 2^V \to \mathbb{R}$, we assume that we are given a value oracle of f. A value oracle takes $X \subseteq V$ as an input, and returns the value f(X). Assuming that we are given a value oracle, we can minimize a submodular function in polynomial time. The currently fastest algorithm for submodular function minimization was given by Lee et al. [19] and runs in $\tilde{O}(n^3 \text{EO} + n^4)$ time, where n = |V| and EO is the query time of a value oracle.

- Let $f: 2^V \to \mathbb{R}$ be a submodular function. A subset $Y \subseteq V$ is a *minimizer* of the function f if $f(Y) \leq f(Y')$ for all $Y' \subseteq V$. The set of minimizers of f is denoted by $\arg \min f$. The following is a well-known fact on submodular functions.
- ▶ Lemma 2 (See e.g., [27]). Let $f: 2^V \to \mathbb{R}$ be a submodular function. Then $\arg \min f$ forms a distributive lattice.

A compact representation of the distributive lattice $\arg \min f$ can be constructed in $\tilde{O}(n^5 \text{EO} + n^6)$ time via Orlin's submodular function minimization algorithm [24]. See [23, Notes 10.11–10.12]. Unless otherwise stated, we will assume that the submodular functions given in our problem instances are given via their compact representation.

As a special case, consider minimum (s,t)-cuts in a directed graph $G = (V \cup \{s,t\}, E)$ with a positive cost function c on its edges. By Lemma 2, the family of minimum (s,t)-cuts forms a distributive lattice. A compact representation for this lattice can be constructed from a maximum flow in the (s,t)-network in linear time [25]. Thus the running time is dominated by the maximum flow computation, and this can be done in $|E|^{1+o(1)}$ time [6].

Parameterized Complexity

In parameterized complexity, each input instance I of a parameterized problem Q is associated with a parameter k, usually an integer or a tuple of integers, and we consider the running time of any algorithm solving Q as not only a function of the input length |I|, but also as a function of the parameter k. An algorithm for Q is fixed-parameter tractable or FPT, if it runs in time $g(k)|I|^{O(1)}$ for some computable function g. Such an algorithm can be efficient in practice if the parameter is small. See the books [7, 9] for more background.

3 Algorithms for Robust Submodular Minimizer

In this section, we present algorithms for Robust Submodular Minimizer. We start with a construction that we will use in most of our algorithms. Let $I_{RSM} = (V, f_1, \dots, f_k, d)$ be our input instance.

For each $i \in [k]$, let $\mathcal{L}_i = \arg \min f_i$ denote the set of minimizers. By Lemma 2, using Birkhoff's representation theorem we may assume that f_i is given through a compact representation $G(\mathcal{L}_i)$ of \mathcal{L}_i , whose vertex set is $\{U_0^i, U_1^i, \dots, U_{b_i}^i\}$ with $U_\infty^i = V \setminus \bigcup_{i=0}^{b_i} U_j^i$.

We then construct a directed graph G^i from $G(\mathcal{L}_i)$ by expanding each vertex in $G(\mathcal{L}_i)$ to a complete graph. More precisely, G^i has vertex set $V^i \cup \{s,t\}$ where $V^i = \{v^i : v \in V\}$ is a copy of V, and its edge set E^i is defined as follows.

- $u^i v^i \in E^i \text{ if } u, v \in U_i^i \text{ for some } j \in \{0, 1, \dots, b_i, \infty\}.$
- $u^i v^i \in E^i$ for any $u \in U^i_j$ and $v \in U^i_{j'}$ if $G(\mathcal{L}_i)$ has an edge from U^i_j to $U^i_{j'}$.
- $u^i s \in E^i \text{ and } su^i \in E^i \text{ if } u \in U_0^i.$
- $u^i t \in E^i \text{ and } tu^i \in E^i \text{ if } u \in U^i_{\infty}.$

Recall that G^i can be computed in $O(|V|^5 EO_i + |V|^6)$ time where EO_i denotes the query time of a value oracle for function f_i .

We define the function $\lambda_i: 2^{V^i} \to \mathbb{Z}_+$ so that $\lambda_i(X) = |\delta_{G^i}(X \cup \{s\})|$ for a subset $X \subseteq V^i$. Then it is observed below that each subset $X \subseteq V^i$ with $\lambda_i(X) = 0$ corresponds to a minimizer of f_i .

▶ Lemma 3 ([16, Lemma 3.2]). Let X be a subset in V, and $X^i = \{v^i \in V^i : v \in X\}$ its copy in G^i . Then $\lambda_i(X^i) = 0$ if and only if $X \in \mathcal{L}_i$.

The rest of the section is organized as follows. In Section 3.1 we present polynomial-time algorithms for the cases d=0 and k=2. In Section 3.2 we give an FPT algorithm for the combined parameter (k, d). Section 3.3 deals with the case when some function f_i has only polynomially many minimizers, allowing for an FPT algorithm with parameter d.

3.1 Polynomial-time algorithms

We start by observing that the case d=0 is efficiently solvable by computing a minimizer for the function $\sum_{i \in [k]} f_i$ which is also submodular; for this result, it will be convenient to use value oracles.

ightharpoonup Observation 4. Robust Submodular Minimizer for d=0 can be solved in time $O(|V|^3 \sum_{i=1}^k EO_i + |V|^4)$ where EO_i is the query time of a value oracle for f_i for $i \in [k]$.

Proof. Let $I = (V, f_1, \dots, f_k, 0)$ be our input. Since the addition of submodular functions yields a submodular function, the function $f = \sum_{i \in [k]} f_i$ is also submodular. Clearly, if some set X is contained in $\arg \min f_i$ for each $i \in [k]$, then $X \in \arg \min f$. Conversely, if some X is contained in $\arg \min f$ but $X \notin \arg \min f_i$ for some $i \in [k]$, then there is no common minimizer Y of the functions f_1, \ldots, f_k . Indeed, such a set Y would also necessarily minimize f, implying

$$f(Y) = f_i(Y) + \sum_{j \in [k], j \neq i} f_j(Y) < f_i(X) + \sum_{j \in [k], j \neq i} f_j(Y) = f(X)$$

where the inequality follows from our assumption that X is not a minimizer for f_i . However, this means that $X \notin \arg \min f$, a contradiction.

Therefore, to solve our input instance I, it suffices to compute a minimizer X for the function f, which can be done in polynomial time, due to the submodularity of f. Then we check whether $X \in \arg \min f_i$ for each $i \in [k]$ (by computing any minimizer for each function f_i). If $X \in \arg \min f_i$ holds for each f_i , then we return X; otherwise we output

The claimed running time follows from the fact that the value of f on some subset of V can be computed in $\sum_{i=1}^k EO_i$ time, and thus we can minimize f in time $O(|V|^3 \sum_{i=1}^k EO_i + |V|^4)$ using the algorithm by Lee at al. [19].

From now on, we will assume that the functions f_1, \ldots, f_k are given through a compact representation. Under this condition, we next show that the problem is polynomial-time solvable when k=2. We will need the following intuitive fact.

▶ Proposition 5. Let Y_1 , Y_2 be two subsets of a set V. Then $|Y_1 \triangle Y_2| \leq 2d$ if and only if there exists a set $X \subseteq V$ such that $|X \triangle Y_i| \le d$ for each $i \in \{1, 2\}$.

Proof. To prove the sufficiency, suppose that $|Y_1 \triangle Y_2| > 2d$. Then for any set $X \subseteq V$,

$$2d < |Y_1 \triangle Y_2| \le |Y_1 \triangle X| + |X \triangle Y_2|.$$

Hence at least one of $|Y_1 \triangle X|$ and $|X \triangle Y_2|$ is more than d. Thus no subset X satisfies $|X \triangle Y_i| \le d$ for both i = 1, 2.

For the necessity, suppose that $|Y_1 \triangle Y_2| \le 2d$. We denote $d_1 = |Y_1 \setminus Y_2|$ and $d_2 = |Y_2 \setminus Y_1|$. We choose arbitrary subsets $Z_1 \subseteq Y_1 \setminus Y_2$ with size $\lfloor d_1/2 \rfloor$ and $Z_2 \subseteq Y_2 \setminus Y_1$ with size $\lfloor d_2/2 \rfloor$. Note that $|Y_1 \triangle Y_2| = d_1 + d_2 \le 2d$.

Define $X = (Y_1 \cap Y_2) \cup Z_1 \cup Z_2$. We claim that X satisfies that $|X \triangle Y_i| \leq d$ for both i = 1, 2. In fact, it holds that

$$|X \triangle Y_1| = |(Y_1 \setminus Y_2) \setminus Z_1| + |Z_2| = (d_1 - |d_1/2|) + |d_2/2| = \lceil d_1/2 \rceil + |d_2/2|. \tag{2}$$

If d_1 is even, then we get $|X \triangle Y_1| + d_1/2 + \lfloor d_2/2 \rfloor \le (d_1 + d_2)/2 \le d$. Suppose that d_1 is odd and d_2 is odd. Then (2) implies that

$$|X \triangle Y_1| = (d_1 + 1)/2 + (d_2 - 1)/2 = (d_1 + d_2)/2 < d.$$

Finally, suppose that d_1 is odd and d_2 is even. Then, since $d_1 + d_2 \le 2d$, we see that $d_1 + d_2 \le 2d - 1$ also holds. Hence, by (2) we get

$$|X \triangle Y_1| = (d_1 + 1)/2 + d_2/2 = (d_1 + d_2 + 1)/2 \le d.$$

Thus $|X \triangle Y_1|$, and by symmetry this shows also that $|X \triangle Y_i| \le d$ for each $i \in \{1, 2\}$.

▶ **Theorem 6.** ROBUST SUBMODULAR MINIMIZER for k = 2 can be solved in polynomial time via a maximum flow computation.

Proof. Let our instance be $I_{\text{RSM}} = (V, f_1, f_2, d)$. Using the method explained at the beginning of Section 3, we construct the directed graphs $G^1 = (V^1 \cup \{s,t\}, E^2)$ and $G^2 = (V^2 \cup \{s,t\}, E^2)$ for f_1 and f_2 . We then construct a directed graph $\tilde{G} = (\tilde{V}, \tilde{E})$ by identifying s, as well as t, in G^1 and G^2 , and then connecting the corresponding copies of each vertex with a bidirected edge. That is, $\tilde{V} = V^1 \cup V^2 \cup \{s,t\}$ and $\tilde{E} = E^1 \cup E^2 \cup E'$ where $E' = \{v^1 v^2, v^2 v^1 : v \in V\}$. We set $c(e) = +\infty$ for all edges $e \in E^1 \cup E^2$, and we set c(e) = 1 for all edges $e \in E'$.

We next compute a minimum-cost (s,t)-cut Z in the graph \tilde{G} with cost function c using standard flow techniques. Let $\kappa_{\tilde{G}}$ denote the cut function in this graph. We will show below that $\kappa_{\tilde{G}}(Z) \leq 2d$ if and only if the answer is "yes".

First suppose that $\kappa_{\tilde{G}}(Z) \leq 2d$. Let $Y_1 = \{v \in V : v^1 \in Z\}$ and $Y_2 = \{v \in V : v^2 \in Z\}$. Since $\delta_{\tilde{G}}(Z)$ has no edges in $E^1 \cup E^2$, we see that $\lambda_i(\{v^i \in V^i : v \in Y_i\}) = 0$ for both i = 1, 2, and therefore the set Y_i is in \mathcal{L}_i by Lemma 3. Since $|Y_1 \triangle Y_2| = \kappa_{\tilde{G}}(Z) \leq 2d$, we can compute a set X such that $|X \triangle Y_i| \leq d$ for both i = 1, 2 by Proposition 5.

Conversely, let $X \subseteq V$ and $Y_i \in \mathcal{L}_i$ for each i = 1, 2 such that $|X \triangle Y_i| \leq d$. Define $Z = \{s\} \cup \{v^1 \in V^1 : v \in Y_1\} \cup \{v^2 \in V^2 : v \in Y_2\}$. Due to Lemma 3 we know that $\lambda_i(\{v^i \in V^i : v \in Y_i\}) = 0$ for both i = 1, 2. This implies $\kappa_{\tilde{G}}(Z) = |Y_1 \triangle Y_2| \leq 2d$ where the inequality follows from Proposition 5.

3.2 FPT algorithm for parameter (k, d)

We propose a fixed-parameter tractable algorithm for Robust Submodular Minimizer parameterized by k and d; let $I_{\text{RSM}} = (V, f_1, \dots, f_k, d)$ denote our instance.

▶ **Theorem 7.** ROBUST SUBMODULAR MINIMIZER can be solved in FPT time when parameterized by (k, d).

To this end, we reduce our problem to the MULTI-BUDGETED DIRECTED CUT problem [17], defined as follows. We are given a directed graph D = (V, E) with distinct vertices s and t, together with pairwise disjoint edge sets A_1, \ldots, A_k , and positive integers d_1, \ldots, d_k . The task is to decide whether D has an (s, t)-cut X such that $|\delta(X) \cap A_i| \leq d_i$ for each $i \in [k]$.

▶ **Proposition 8** (Kratsch et al. [17]). The MULTI-BUDGETED DIRECTED CUT problem can be solved in FPT time when the parameter is $\sum_{i=1}^{k} d_i$.

In fact, we will need to use forbidden edges, so let us define the MULTI-BUDGETED DIRECTED CUT WITH FORBIDDEN EDGES problem as follows. Given an instance $I_{\rm MBC}$ of MULTI-BUDGETED DIRECTED CUT and a set F of forbidden edges, find a solution X for $I_{\rm MBC}$ such that $\delta(X)$ is disjoint from F. It is straightforward to solve this problem using the results by Kratsch et al. [17], after replacing each forbidden edge with an appropriate number of parallel edges. Hence, we get the following.

▶ Proposition 9. The MULTI-BUDGETED DIRECTED CUT WITH FORBIDDEN EDGES problem can be solved in FPT time when the parameter is $\sum_{i=1}^{k} d_i$.

Proof. Let $I=(D,s,t,A_1,\ldots,A_k,d_1,\ldots,d_k)$ be our instance with a set F of forbidden edges in D. First, we create a set $A_{k+1}=F\setminus\bigcup_{i=1}^kA_i$ and let $d_{k+1}=1$. Next, for each $i\in[k+1]$ and each forbidden edge $f\in F\cap A_i$, we replace f with d_i+1 parallel copies of f in D. Let D' be the resulting directed graph, and consider the instance $I'=(D',s,t,A_1,\ldots,A_{k+1},d_1,\ldots,d_{k+1})$ of MULTI-BUDGETED DIRECTED CUT. Clearly, if a set X of vertices is a solution for I', then $\delta(X)$ cannot contain a forbidden edge or a copy of a forbidden edge, as then it would have to contain all copies of that forbidden edge, and thus would violate $|\delta(X)\cap A_i|\leq d_i$ for some $i\in[k+1]$. Thus, a set X of vertices in D is a solution for our original instance (I,F) of MULTI-BUDGETED DIRECTED CUT WITH FORBIDDEN EDGES if and only if X is a solution for the instance I' of MULTI-BUDGETED DIRECTED CUT. Note that this construction increases the parameter by exactly one. Hence, the algorithm provided by Proposition 8 can be used to solve MULTI-BUDGETED DIRECTED CUT WITH FORBIDDEN EDGES in FPT time.

Reduction to Multi-Budgeted Directed Cut with Forbidden Edges

Compute the graph G^i for each $i \in [k]$, as described at the beginning of Section 3. We construct a large directed graph $\tilde{G} = (\tilde{V}, \tilde{E})$ as follows. We identify all vertices s (and t, respectively) in the graphs G^i into a single vertex s (and t, respectively). We further prepare another copy of V, which is denoted by $V^* = \{v^* : v \in V\}$. Thus the vertex set of \tilde{G} is defined by $\tilde{V} = \bigcup_{i=1}^k V^i \cup V^* \cup \{s,t\}$. The edge set of \tilde{G} consists of E^i and bidirected edges connecting v^* and the copy v^i of v in G^i , for each $i \in [k]$. That is,

$$\tilde{E} = \bigcup_{i=1}^{k} (E^i \cup A^i)$$
 where $A^i = \{v^*v^i, v^iv^* : v \in V\}$.

To avoid using parallel edges, one can alternatively replace each edge $f = uv \in F \cap A_i$ with a set of $d_i + 1$ paths of length two, i.e., with vertices $p_f^1, p_f^2, \dots, p_f^{d_i+1}$ and edges $\{up_f^j, p_f^j v : j \in [d_i+1]\}$.

We also set $d_i = d$ for each $i \in [k]$. Consider the instance $I_{\text{MBC}} = (\tilde{G}, s, t, \{A^i\}_{i=1}^k, \{d_i\}_{i=1}^k)$ of MULTI-BUDGETED DIRECTED CUT with $F = \bigcup_{i=1}^k E^i$ as forbidden edges; note that its parameter is $k \cdot d$. Theorem 7 immediately follows from Proposition 9 and Lemma 10 below.

▶ **Lemma 10.** There exists a solution for I_{RSM} if and only if there exists a solution for the instance (I_{MBC}, F) of MULTI-BUDGETED DIRECTED CUT WITH FORBIDDEN EDGES.

Proof. Suppose that (I_{MBC}, F) admits a solution. That is, there exists a subset X of \tilde{V} containing s but not t such that $\delta_{\tilde{G}}(X)$ is disjoint from F and satisfies $|\delta_{\tilde{G}}(X) \cap A^i| \leq d_i$ for each $i \in [k]$. Define $Y^i = X \cap V^i$ for $i = 1, \ldots, k$. Observe that all edges within G^i leaving $Y^i \cup \{s\}$ also leave X in \tilde{G} , since $s \in X$ but $t \notin X$. Since all edges in E^i are forbidden edges, we see that $\lambda_i(Y^i) = 0$. Let $Y_i = \{v \in V : v^i \in Y^i\}$, so that Y^i contains the copy of each vertex of Y_i in G^i . Then Y_i is in \mathcal{L}_i by Lemma 3.

Define the subset $X^* = \{v : v^* \in X\}$ of V. Observe that

$$\delta_{\tilde{G}}(X) \cap A^i = \{v^*v^i : v \in X^*, v \notin Y_i\} \cup \{v^iv^* : v \notin X^*, v \in Y_i\}.$$

Therefore, we get that $|X^* \triangle Y_i| = |\delta_{\tilde{G}}(X) \cap A^i| \le d_i = d$ for each $i \in [k]$ as required, so X^* is a solution to our instance I_{RSM} of ROBUST SUBMODULAR MINIMIZER.

Conversely, let $X \subseteq V$ and $Y_i \in \mathcal{L}_i$ for each $i \in [k]$ such that $|X \triangle Y_i| \leq d$. Define $X^* = \{v^* \in V^* : v \in X\}$ and $Y^i = \{v^i \in V^i : v \in Y_i\}$. Then the set $\tilde{X} = \{s\} \cup X^* \cup \bigcup_{i=1}^k Y^i$ is an (s,t)-cut of \tilde{G} such that

$$\begin{split} \delta_{\tilde{G}}(\tilde{X}) \cap A^i &= \{v^*v^i : v^* \in X^*, v \not\in Y^i\} \cup \{v^iv^* : v^* \not\in X^*, v^i \in Y^i\} \\ &= \{v^*v^i : v \in X, v \not\in Y_i\} \cup \{v^iv^* : v \not\in X, v \in Y_i\} = X \bigtriangleup Y_i. \end{split}$$

Hence we obtain $|\delta_{\tilde{G}}(\tilde{X}) \cap A^i| = |X \triangle Y_i| \le d = d_i$ for each $i \in [k]$. Since Y_i is in \mathcal{L}_i , by Lemma 3 we know $\lambda_i(Y^i) = 0$ for each $i \in [k]$. Thus $\delta_{\tilde{G}}(\tilde{X})$ is disjoint from the set F of forbidden edges, and therefore \tilde{X} is indeed a solution to our instance (I_{MBC}, F) of MULTI-BUDGETED DIRECTED CUT WITH FORBIDDEN EDGES.

3.3 Polynomially many minimizers: FPT algorithm parameterized by d

In this section, we present a fixed-parameter tractable algorithm for the case when our threshold d is small, assuming that $|\mathcal{L}_1|$ can bounded by a polynomial of the input size. Note that even with a much stronger assumption, ROBUST SUBMODULAR MINIMIZER remains intractable (see also [16]):

▶ **Observation 11.** ROBUST SUBMODULAR MINIMIZER is NP-hard even if $|\mathcal{L}_i| = 1$ for each $i \in [k]$.

Proof. If $|\mathcal{L}_i| = 1$ for each $i \in [k]$, then there is a unique minimizer $Y_i \subseteq V$ for each f_i , and the problem is equivalent with finding a set $X \subseteq V$ whose symmetric difference is at most d from each of the sets Y_i , $i \in [k]$. This is the Closest String problem over a binary alphabet, shown to be NP-hard under the name MINIMUM RADIUS by Frances and Litman [11].

▶ **Theorem 12.** ROBUST SUBMODULAR MINIMIZER can be solved in $|\mathcal{L}_1|g(d)n^c$ time where c is a constant and g is a computable function.

Let us consider a slightly more general version of ROBUST SUBMODULAR MINIMIZER which we call Anchored Submodular Minimizer. In this problem, in addition to an

instance $I_{RSM} = (V, f_1, \dots, f_k, d)$ of Robust Submodular Minimizer, we are given a set $Y_0 \subseteq V$ and integer $d_0 \leq d$, and we aim to find a subset X such that

$$|X \triangle Y_0| \le d_0$$
 and (3)

$$|X \triangle Y_i| \le d$$
 for some $Y_i \in \mathcal{L}_i$, for each $i \in [k]$. (4)

Observe that we can solve our instance $I_{RSM} = (V, f_1, \dots, f_k, d)$ by solving the instance $(V, f_2, \ldots, f_k, d, Y_0, d_0)$ of Anchored Submodular Minimizer for each $Y_0 \in \mathcal{L}_1$ and $d_0 = d$. Hence, Theorem 12 follows from Theorem 13 below.

▶ Theorem 13. Anchored Submodular Minimizer can be solved in FPT time when parameterized by d.

To prove Theorem 13, we will use the technique of bounded search-trees. Given an instance $I = (V, f_1, \dots, f_k, d, Y_0, d_0)$, after checking whether Y_0 itself is a solution, we search for a minimizer $Y_i \in \mathcal{L}_i$ for which $d < |Y_0 \triangle Y_i| \le d + d_0$. It is not hard to see the following.

▶ **Observation 14.** If X is a solution for an instance $I = (V, f_1, ..., f_k, d, Y_0, d_0)$ of AN-CHORED SUBMODULAR MINIMIZER, and $Y_i \in \mathcal{L}_i$ fulfills $|X \triangle Y_i| \leq d$, then for all $T \subseteq Y_0 \triangle Y_i$ with |T| > d it holds that there exists some $v \in T$ with $v \in X \triangle Y_0$.

Proof. Indeed, assuming that the claim does not hold, we have that $T \cap (Y_0 \setminus Y_i) \subseteq X$ and that $(T \cap (Y_i \setminus Y_0)) \cap X = \emptyset$. From the former, $T \cap (Y_0 \setminus Y_i) \subseteq X \setminus Y_i$ follows, while the latter implies $T \cap (Y_i \setminus Y_0) \subseteq Y_i \setminus X$. Thus,

$$X \triangle Y_i = (X \setminus Y_i) \cup (Y_i \setminus X) \supseteq (T \cap (Y_0 \setminus Y_i)) \cup (T \cap (Y_i \setminus Y_0)) = T \cap (Y_0 \triangle Y_i) = T.$$

Hence, $|X \triangle Y_i| \ge |T| > d$, contradicting our assumption that X is a solution for I.

Our algorithm will compute in $O^*(2^d)$ time³ a set $T \subseteq Y_0 \setminus Y_i$ of size $d < |T| \le d + d_0$ that contains some element v fulfilling the above conditions. Then, by setting $Y_0 \leftarrow Y_0 \triangle \{v\}$ and reducing the value of d_0 by one, we obtain an equivalent instance I' of Anchored SUBMODULAR MINIMIZER which we solve by applying recursion.

Description of our algorithm.

Our algorithm will make "guesses"; nevertheless, it is a deterministic one, where guessing a value from a given set U is interpreted as branching into |U| branches. We continue the computations in each branch, and whenever a branch returns a solution for the given instance, we return it; if all branches reject the instance (by outputting "No"), we also reject it. See Algorithm ASM for a pseudo-code description.

We start by checking whether Y_0 is a solution for our instance $I = (V, f_1, \dots, f_k, d, Y_0, d_0)$, that is, whether it satisfies (4). This can be done in polynomial time, since the set function $\gamma_i(Z) = \min\{|Z \triangle Y_i| : Y_i \in \mathcal{L}_i\}$ is known to be submodular and can be computed via a maximum flow computation [16]. If Y_0 satisfies (4), i.e., $\gamma_i(Y_0) \leq d$ for each $i \in [k]$, then we output Y_0 ; note that (3) is obviously satisfied by Y_0 , so Y_0 is a solution for I.

Otherwise, if $d_0 = 0$, then we output "No" as in this case the only possible solution could be Y_0 . We proceed by fixing an index $i \in [k]$ such that $\gamma_i(Y_0) > d$, that is, $|Y_0 \triangle Y| > d$ for all minimizers $Y \in \mathcal{L}_i$.

³ The $O^*()$ notation hides polynomial factors.

▶ **Observation 15.** If X is a solution for I that satisfies $|X \triangle Y_i| \le d$ for some $Y_i \in \mathcal{L}_i$, then $|Y_i \triangle Y_0| \le d + d_0$.

Proof. Since X is a solution for I, we have $|X \triangle Y_0| \le d_0$, and thus the triangle inequality implies $|Y_i \triangle Y_0| \le |X \triangle Y_i| + |X \triangle Y_0| \le d + d_0$.

By our choice of i and Observation 15, we know that $d < |Y_0 \triangle Y_i| \le d + d_0$. We are going to compute a set $T \subseteq Y_0 \triangle Y_i$ with the same bounds on its cardinality, i.e., $d < |T| \le d + d_0$.

To this end, we compute a compact representation $G(\mathcal{L}_i)$ of the distributive lattice \mathcal{L}_i ; let $\mathcal{P} = \{U_0, U_1, \dots, U_b, U_\infty\}$ be the partition of V in this representation.

Next, we proceed with an iterative procedure which also involves a set of guesses. We start by setting $Y = Y_0$ and $T = \emptyset$. We will maintain a family of *fixed sets* from \mathcal{P} for which we already know whether they are in Y_i or not (according to our guesses); initially, no set from \mathcal{P} is fixed.

After this initialization, we start an iteration where at each step we check whether $Y \in \mathcal{L}_i$ or |T| > d. If yes, then we stop the iteration. If not, then it can be shown that one of the following conditions holds:

Condition 1: there exists a set $S \in \mathcal{P}$ such that $S \cap Y \neq \emptyset$ and $S \setminus Y \neq \emptyset$;

Condition 2: there exists an edge (S, S') in $G(\mathcal{L}_i)$ for which $S \subseteq Y$ but $S' \cap Y = \emptyset$.

If Condition 1 holds for some set $S \in \mathcal{P}$, then we guess whether S is contained in Y_i . If $S \subseteq Y_i$ according to our guesses, then we add $S \setminus Y$ to T; otherwise, we add $S \cap Y$ to T. In either case, we declare S as fixed, and proceed with the next iteration.

By contrast, if Condition 1 fails, but Condition 2 holds for some edge (S, S') in $G(\mathcal{L}_i)$ with endpoints $S, S' \in \mathcal{P}$, then we proceed as follows. If both S and S' are fixed, then we stop and reject the current set of guesses. If S is fixed but S' is not, then we add all elements of S' to T. If S' is fixed but S is not, then we add S to S' is fixed then we guess whether S is contained in S' or not, and in the former case we add S' to S' while in the latter case we add S' to S' and S' as fixed; in the last case declare only S as fixed.

Next, we modify Y to reflect the current value of T by updating Y to $Y_0 \triangle T$. If $|T| > d + d_0$, then we reject the current branch. If $d < |T| \le d + d_0$, then we finish the iteration; otherwise, we proceed with the next iteration.

Finally, when the iteration stops, we guess a vertex $v \in T$, define $Y'_{0,v} = Y_0 \triangle \{v\}$ and call the algorithm recursively on the instance $I'_v := (V, f_1, \dots, f_k, d, Y'_{0,v}, d_0 - 1)$.

Proof of Theorem 13. We first prove the correctness of the algorithm. Clearly, for $d_0 = 0$, the algorithm either correctly outputs the solution Y_0 , or rejects the instance. Hence, we can apply induction on d_0 , and assume that the algorithm works correctly when called for an instance with a smaller value for d_0 .

We show that any set X returned by the algorithm is a solution for I. First, this is clear if $X = Y_0$, as the algorithm explicitly checks whether $\gamma_i(Y_0) \leq d$ holds for each $i \in [k]$; second, if X was returned by a recursive call on some instance I'_v , then by our induction hypothesis we know that X is a solution for $I'_v = (V, f_1, \ldots, f_k, d, Y'_{0,v}, d_0 - 1)$. Hence, X satisfies (4); moreover, by $|X \triangle Y'_{0,v}| \leq d_0 - 1$, it also satisfies $|X \triangle Y_0| \leq d_0$, because $|Y_0 \triangle Y'_{0,v}| = 1$.

Let us now prove that if I admits a solution X, then the algorithm correctly returns a solution for I. Let $Y_i \in \mathcal{L}_i$ be a minimizer such that $|X \triangle Y_i| \leq d$ where i is the index fixed for which $\gamma_i(Y_0) > d$.

 \triangleright Claim 16. Assuming that all guesses made by the algorithm are correct, in the iterative process of modifying T and Y it will always hold that

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Algorithm ASM Solving Anchored Submodular Minimizer on I = (V, f_1, \dots, f_k, d, Y_0, d_0).
```

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1: for all j \in [k] do compute the value \gamma_j = \min\{|Y_0 \triangle Y| : Y \in \arg\min f_j\}.
 2: if \gamma_j \leq d for each j \in [k] then return Y_0.
 3: if d_0 = 0 then return "No".
 4: Fix an index i \in [k] such that \gamma_i > d.
 5: Compute the graph G(\mathcal{L}_i), and let \mathcal{P} be its vertex set.
 6: Set T := \emptyset and Y := Y_0, and fixed(S) := false for each S \in \mathcal{P}.
 7: while Y \notin \mathcal{L}_i and |T| \leq d do
         if \exists S \in \mathcal{P} : S \cap Y_0 \neq \emptyset, S \setminus Y_0 \neq \emptyset then
 8:
             Guess contained(S) from {false, true}.
 9:
10:
             if contained(S) = true then set T := T \cup (S \setminus Y).
11:
             else set T := T \cup (S \cap Y).
             Set fixed(S) := true.
12:
         else Find an edge (S, S') \in G(\mathcal{L}_i) such that S \subseteq Y and S' \cap Y = \emptyset.
13:
14:
             if fixed(S) = true then
                 if fixed(S') = true then return "No".
15:
                 else set T := T \cup S' and fixed(S') := true.
16:
             else
                                                                                           \triangleright \mathsf{fixed}(S) = \mathsf{false}.
17:
18:
                 if fixed(S') = true then set T := T \cup S and fixed(S) := true.
                 else guess contained(S) from {false, true}.
19:
                      if contained(S) = true then set T := T \cup S', fixed(S) := fixed(S') := true.
20:
                      else set T := T \cup S and fixed(S) := \text{true}.
21:
         Set Y := Y_0 \triangle T.
22:
         if |T| > d + d_0 then return "No".
23:
24: Guess a vertex v from T.
25: Set Y'_{0,v} = Y_0 \triangle \{v\} and I'_v = (V, f_1, \dots, f_k, d, Y'_{0,v}, d_0 - 1).
26: return ASM(I'_v).
```

- (i) $T \subseteq Y_i \triangle Y_0$, and
- (ii) for each $S \in \mathcal{P}$:
 - (a) if S is fixed, then $S \subseteq Y \iff S \subseteq Y_i$, and $S \cap Y = \emptyset \iff S \cap Y_i = \emptyset$, and
 - (b) if $v \in S$ and S is not fixed, then $v \in Y \iff v \in Y_0$.

Proof. The claim clearly holds initially, when $T = \emptyset$, $Y = Y_0$ and no set is fixed. Consider now the j-th run of the iteration for some j > 0. Let T and Y be as at the beginning of the iteration. Note that $Y = Y_0 \triangle T$ due to the last step of the (j-1)-st run of the iteration on line 22.

Suppose that Condition 1 holds for some set $S \in \mathcal{P}$. Using Birkhoff's representation theorem, we know that either $S \subseteq Y_i$ or $S \cap Y_i = \emptyset$. By Condition 1 and claim (ii/a) of our induction hypothesis (\mathcal{IH} for short), S is not fixed. If $S \subseteq Y_i$, then by claim (ii/b) of our \mathcal{IH} (applicable as S is not fixed), we know $S \setminus Y = S \setminus Y_0 \subseteq Y_i \triangle Y_0$. Similarly, if $S \cap Y_i = \emptyset$, then we get $S \cap Y = S \cap Y_0 \subseteq Y_i \triangle Y_0$. Assuming that the algorithm correctly guesses which among these two cases holds, the elements added to T on line 10 or 11 are indeed contained in $Y_i \triangle Y_0$, proving that (i) remains true in this case.

Suppose now that Condition 2 holds for some edge (S, S') in $G(\mathcal{L}_i)$. By our inductive hypothesis, S and S' cannot both be fixed (assuming correct guesses), since $S \subseteq Y_i$ and

 $S' \cap Y_i = \emptyset$ would then contradict Birkhoff's representation theorem, as the edge (S, S') would leave the closed set $\mathcal{P}(Y_i) := \{S : S \in \mathcal{P}, S \subseteq Y_i\}$ in $G(\mathcal{L}_i)$.

Now, if S is fixed, then by claim (ii/a) of our \mathcal{IH} , $S \subseteq Y$ implies $S \subseteq Y_i$. This means that $S' \subseteq Y_i$ must hold as well, since $\mathcal{P}(Y_i)$ is closed. Moreover, if S' is not fixed, then $S' \cap Y = \emptyset$ implies $S' \cap Y_0 = \emptyset$ due to claim (ii/b) of the \mathcal{IH} ; hence $S' \subseteq Y_i \triangle Y_0$. The same arguments hold for the case when initially neither S nor S' is fixed, and the algorithm correctly guesses that S is contained in Y_i . We obtain that the vertices added to T on line 16 or 20 are indeed contained in $Y_i \triangle Y_0$.

By contrast, if S is not fixed, then claim (ii/b) of our \mathcal{IH} and $S \subseteq Y$ imply $S \subseteq Y_0$. Now, if S' is fixed, then claim (ii/a) of our \mathcal{IH} means that $S' \cap Y_i = \emptyset$; hence we get $S \notin \mathcal{P}(Y_i)$, because $\mathcal{P}(Y_i)$ is closed in $G(\mathcal{L}_i)$. Then $S \subseteq Y_i \triangle Y_0$ follows, so the set added to T on line 18 is indeed contained in $Y_i \triangle Y_0$. The only remaining case is when neither S nor S' is fixed initially, and the algorithm guesses $S \not\subseteq Y_i$. Recall that by claim (ii/b) of the \mathcal{IH} , we have $S \subseteq Y_0$, so a correct guess immediately yields $S \subseteq Y_i \triangle Y_0$, and thus the set added to T on line 21 is in $Y_i \triangle Y_0$ as well. This proves that (i) remains true at the end of the j-th run of the algorithm as well.

Observe that whenever the algorithm puts some element $v \in V$ into Y, it also declares the set in \mathcal{P} containing v fixed; from this, it immediately follows that claim (ii/b) remains true. To see that claim (ii/a) is maintained well, note first that the algorithm never removes or adds vertices of a fixed set from Y or to Y, respectively. Therefore, we only need to check those sets that we declared fixed during this iteration. By the above arguments, it is not hard to verify that whenever we declare some set S as fixed, then we also ensure $S \subseteq Y \iff S \subseteq Y_i$ when updating Y to $Y_0 \triangle T$. This finishes the proof of the claim.

Next, we show that in each run of the iteration, Condition 1 or Condition 2 holds. Indeed, if neither holds, then (1) $Y = \bigcup_{U \in \mathcal{P}'} U$ for some $\mathcal{P}' \subseteq \mathcal{P}$, and (2) no edge leaves \mathcal{P}' in $G(\mathcal{L}_i)$. Hence, $Y \in \mathcal{L}_i$ by Birkhoff's representation theorem. However, since $|T| \leq d$ holds at the beginning of each iteration, $|Y \triangle Y_0| = |T| \leq d$ follows, contradicting our choice of i.

Therefore, in each run of the iteration, at least one element of V is put into T. Thus, the iteration stops after at most d+1 runs, at which point the obtained set T has size greater than d. Using now statement (i) of Claim 16, Observation 14 yields that T contains at least one vertex from $X \triangle Y_0$. Assuming that the algorithm guesses such a vertex v correctly, it is clear that our solution X for I will also be a solution for the instance I'_v . Using our inductive hypothesis, we obtain that the recursive call returns a correct solution for I'_v which, as discussed already, will be a solution for I as well. Hence, our algorithm is correct.

Finally, let us bound the running time. Consider the search tree \mathcal{T} where each node corresponds to a call of Algorithm ASM. Note that the value of d_0 decreases by one in each recursive call, and the algorithm stops when $d_0 = 0$. Hence \mathcal{T} has depth at most d_0 . Consider the guesses made during the execution of a single call of the algorithm (without taking into account the guesses in the recursive calls): we make at most one guess in each iteration on line 9 or on line 19, leading to at most 2^{d+1} possibilities. Then the algorithm further guesses a vertex from T, leading to a total of at most $2^{d+1}|T| \leq 2^{d+1}(d+d_0) = 2^{O(d)}$ possibilities; recall that $d_0 \leq d$. We get that the number of nodes in our search tree is $2^{d_0O(d)}$. Since all computations for a fixed series of guesses take polynomial time, we obtain that the running time is indeed fixed-parameter tractable with parameter d.

4 Hardness Results

We first introduce a separation problem that we will use as an intermediary problem in our hardness proofs. Given a subset $X \subseteq V$ of some universe V that contains two distinguished elements, s and t, and a family Π of pairwise disjoint subsets of V, we define the distance of the set X from Π as $\sum_{S \in \Pi} \mathsf{dist}_{s,t}(X,S)$ where

$$\mathsf{dist}_{s,t}(X,S) = \left\{ \begin{array}{ll} \min\{|S \setminus X|, |S \cap X|\} & \text{ if } s \notin S, t \notin S; \\ |S \setminus X| & \text{ if } s \in S, t \notin S; \\ |S \cap X| & \text{ if } s \notin S, t \in S; \\ +\infty & \text{ if } s \in S, t \in S. \end{array} \right.$$

Given a collection of set families Π_1, \ldots, Π_k , the goal is to find a set $X \subseteq V$ that separates s from t in the sense that $s \in X$ but $t \notin X$, and subject to this constraint, minimizes the maximum distance of X from the given set families. Formally, the problem is:

ROBUST SEPARATION:

Input: A finite set V with two elements $s, t \in V$, set families Π_1, \ldots, Π_k where each Π_i

is a collection of pairwise disjoint subsets of V, and an integer $d \in \mathbb{N}$.

Task: Find a set $X \subseteq V$ containing s but not t such that for each $i \in [k]$

$$\sum_{S \in \Pi_i} \mathsf{dist}_{s,t}(X,S) \le d,\tag{5}$$

or output "No" if no such set X exists.

Given an instance $(V, s, t, \Pi_1, \dots, \Pi_k, d)$ of ROBUST SEPARATION, the reduction proving Lemma 17 below constructs a graph G_i over V for each $i \in [k]$ in which each set in Π_i forms a clique, and defines a submodular function f_i based on the cut function of G_i .

▶ Lemma 17. Robust Separation can be reduced to Robust Submodular Minimizer in polynomial time via a reduction that preserves the values of both k and d.

Proof. Let us be given an instance $I_{\text{RSep}} = (V, s, t, \Pi_1, \dots, \Pi_k, d)$ of ROBUST SEPARATION. Clearly, we may assume that no set family Π_i contains a set S that contains both s and t, as that would imply $\mathsf{dist}_{s,t}(X,S) = +\infty$ for all sets $X \subseteq V$, in which case we can clearly answer "No".

Let $\tilde{V} = V \setminus \{s, t\}$. For each $i \in [k]$ we construct a directed graph $G_i = (V, E_i)$ from Π_i by setting

$$E_i = \{uv : \exists S \in \Pi_i, u, v \in S\}.$$

We set the cost of every edge in E_i to be $+\infty$. Observe that an (s,t)-cut $X \subseteq V$ is a minimum (s,t)-cut in G_i if and only if for each $S \in \Pi_i$ either X contains S, or is disjoint from S. Let κ_i denote the (weighted) cut function for G_i , and define $f_i : 2^{\tilde{V}} \to \mathbb{R}$ such that $f_i(X) = \kappa_i(X \cup \{s\})$ for each $X \subseteq \tilde{V}$. Then f_i is submodular.

We claim that for each set $X \subseteq \tilde{V}$, the set $X \cup \{s\}$ is a solution for our instance I_{RSep} of Robust Separation if and only if X is a solution for the instance $I_{\text{RSubMin}} = (\tilde{V}, f_1, \dots, f_k, d)$ of Robust Submodular Minimizer.

To see this, first assume that $X \subseteq V$ contains s but not t, and fulfills (5) for each $i \in [k]$. For an element $v \in V$, let $\Pi_i(v)$ be the unique set in Π_i containing v, if there is such a set, otherwise set $\Pi_i(v) = \emptyset$. By our assumption that no set in Π_i contains both s and t, we know that $\Pi_i(s) \neq \Pi_i(t)$ unless both of them are empty. Define

$$Y_i = X \cup \Pi_i(s) \setminus \Pi_i(t) \setminus \left(\bigcup \left\{ S \in \Pi_i : |S \cap X| < |S \setminus X|, s \notin S \right\} \right)$$
$$\cup \left(\bigcup \left\{ S \in \Pi_i : |S \cap X| \ge |S \setminus X|, t \notin S \right\} \right).$$

Notice that when constructing Y_i , each set in Π_i is either fully added to Y_i , or fully removed from it. Moreover, it is an (s,t)-cut, since no set containing s is ever removed from X, and similarly, no set containing t is ever added to X. Therefore, Y_i is a minimum (s,t)-cut in G_i by the observations above.

Since both X and Y_i are (s,t)-cuts, let us define $\tilde{X} = X \setminus \{s\}$ and $\tilde{Y}_i = Y_i \setminus \{s\}$; since Y_i is a minimum (s,t)-cut, we have $\tilde{Y}_i \in \arg \min f_i$. Moreover,

$$\begin{split} |\tilde{X} \bigtriangleup \tilde{Y}_i| &= |X \bigtriangleup Y_i| = |X \backslash Y_i| + |Y_i \backslash X| \\ &= |\Pi_i(t) \cap X| + \left| \bigcup \{S \cap X : S \in \Pi_i, |S \cap X| < |S \backslash X|, s \not \in S, t \not \in S\} \right| \\ &+ |\Pi_i(s) \backslash X| + \left| \bigcup \{S \backslash X : S \in \Pi_i, |S \cap X| \ge |S \backslash X|, s \not \in S, t \not \in S\} \right| \\ &= \mathsf{dist}_{s,t}(X,\Pi_i(s)) + \mathsf{dist}_{s,t}(X,\Pi_i(t)) + \sum_{S \in \Pi_i, s \not \in S, t \not \in S} \min\{|S \backslash X|, |S \cap X|\} \\ &= \sum_{S \in \Pi_i} \mathsf{dist}_{s,t}(X,S) \le d \end{split}$$

where the last inequality holds because X satisfies (5). Hence, \tilde{X} is indeed a solution for I_{RSubMin} .

For the other direction, let $X\subseteq \tilde{V}$ be a solution for I_{RSubMin} . Then for each $i\in [k]$ there exists a set $Y_i\in \arg\min f_i$ that satisfies $|X\bigtriangleup Y_i|\le d$; define Y_i among all such sets so that $|X\bigtriangleup Y_i|$ is minimized. Recall that $Y_i\in \arg\min f_i$ means that $Y_i\cup \{s\}$ is a minimum (s,t)-cut in G_i ; let us use the notation $Y_i'=Y_i\cup \{s\}$. Note that if some element $v\in \tilde{V}$ is not contained in any set of Π_i , then adding it to, or removing it from, a minimum (s,t)-cut in G_i also yields a minimum (s,t)-cut. This implies that $Y_i\cap (V\setminus\bigcup_{S\in\Pi_i}S)=X\cap (V\setminus\bigcup\Pi_i)$, as otherwise we could replace $Y_i\cap (V\setminus\bigcup_{S\in\Pi_i}S)$ with $X\cap (V\setminus\bigcup_{S\in\Pi_i}S)$ in Y_i to obtain another minimizer for f_i that is closer to X than Y_i ; we refer to this as fact (f1).

Recall also that since Y_i' is a minimum (s,t)-cut in G_i , for each $S \in \Pi_i$ it either contains S or is disjoint from it. We refer to this observation as fact (f2).

Let us define $X' = X \cup \{s\}$. We now prove that X' is a solution for the instance I_{RSep} . Applying facts (f1) and (f2), for each $i \in [k]$ we get

$$\begin{split} d \geq |X \bigtriangleup Y_i| &\stackrel{(f1)}{=} |\{v: v \in S, S \in \Pi_i, v \in X \bigtriangleup Y_i\}| \\ &\stackrel{(f2)}{=} |\{v: v \in S, S \in \Pi_i, S \subseteq Y_i', v \in X \bigtriangleup Y_i\}| \\ &+ |\{v: v \in S, S \in \Pi_i, S \cap Y_i' = \emptyset, v \in X \bigtriangleup Y_i\}| \\ &= |\{v: v \in S, S \in \Pi_i, S \subseteq Y_i', v \in (S \cap Y_i) \setminus X\}| \\ &+ |\{v: v \in S, S \in \Pi_i, S \cap Y_i' = \emptyset, v \in S \cap X\}| \\ &= |(\Pi_i(s) \setminus \{s\}) \setminus X| + \left|\bigcup \{S \setminus X: s \notin S \in \Pi_i, S \subseteq Y_i\}\right| \\ &+ \left|\bigcup \{S \cap X: S \in \Pi_i, S \cap Y_i' = \emptyset\}\right| \end{split}$$

$$= \left|\Pi_i(s) \setminus X'\right| + \left|\bigcup\{S \setminus X': s \notin S \in \Pi_i, S \subseteq Y_i\}\right| \\ + \left|\bigcup\{S \cap X': S \in \Pi_i, S \cap Y_i' = \emptyset\}\right| \\ \geq \operatorname{dist}_{s,t}(X', \Pi_i(s)) + \sum_{S \in \Pi_i, s \notin S, t \notin S} \min\{|S \setminus X|, |S \cap X|\} + \operatorname{dist}_{s,t}(X', \Pi_i(t)) \\ = \sum_{S \in \Pi_i} \operatorname{dist}_{s,t}(X', S).$$

Hence, X' is indeed a solution for the instance I_{RSep} of ROBUST SEPARATION, as promised. This proves the correctness of our reduction.

NP-hardness for a constant $d \ge 1$ 4.1

In this section, we prove that ROBUST SUBMODULAR MINIMIZER is NP-hard for each constant d > 1. To this end, we first prove the NP-hardness of ROBUST SEPARATION in the case d=1, and then extend this result to hold for any constant $d\geq 1$.

For the case d=1, we present a reduction from the 1-IN-3 SAT problem. In this problem, we are given a set V of variables and a set \mathcal{C} of clauses, with each clause $C \in \mathcal{C}$ containing exactly three distinct literals; here, a literal is either a variable $v \in V$ or its negation \overline{v} . Given a truth assignment $\phi: V \to \{\text{true}, \text{false}\}\$, we automatically extend it to the set $\overline{V} = \{\overline{v} : v \in V\}$ of negative literals by setting $\phi(\overline{v}) = \text{true}$ if and only if $\phi(v) = \text{false}$. We say that a truth assignment is valid, if it maps exactly one literal in each clause to true. The task in the 1-IN-3 SAT problem is to decide whether a valid truth assignment exists. This problem is NP-complete [26].

▶ **Theorem 18.** ROBUST SEPARATION is NP-hard even when d = 1.

Proof. Suppose that we are given an instance of the 1-IN-3 SAT problem with variable set Vand clause set $C = \{C_1, \ldots, C_m\}$. We construct an instance I_{RS} of ROBUST SEPARATION as follows. In addition to the set V of variables and the set $\overline{V} = \{\overline{v} : v \in V\}$ of negative literals, we introduce our two distinguished elements, s and t. We further introduce a set $R_j = \{r_{j,1}, r_{j,2}, r_{j,3}\}$ together with an extra element z_j for each clause $C_j \in \mathcal{C}$ to form our universe U. We let $R = R_1 \cup \cdots \cup R_m$ and $Z = \{z_1, \ldots, z_m\}$, so that

$$U = V \cup \overline{V} \cup \{s, t\} \cup \bigcup_{j \in [m]} (R_j \cup \{z_j\}) = V \cup \overline{V} \cup \{s, t\} \cup R \cup Z.$$

Next, for each variable, we introduce two set families, Π_v and $\Pi_{\overline{v}}$, where

$$\Pi_v = \{ \{s, v, \overline{v}\} \cup R \}$$
 and $\Pi_{\overline{v}} = \{ \{v, \overline{v}, t\} \}.$

For simplicity, we write $\Pi(V) = \langle \Pi_v, \Pi_{\overline{v}} : v \in V \rangle$ to denote the 2|V|-tuple formed by these set families. For each clause $C_j \in \mathcal{C}$, we fix an arbitrary ordering of its literals, and we denote the first, second, and third literals in C_j as $\ell_{j,1}, \ell_{j,2}$ and $\ell_{j,3}$. We define three set families (see also Figure 1):

$$\begin{split} \Pi_{C_j} &= \{S_j\} & \text{where} \quad S_j &= C_j \cup \{t\} = \{\ell_{j,1}, \ell_{j,2}, \ell_{j,3}, t\}, \\ \Pi_{C_j}^{\alpha} &= \{S_j^{\alpha,1}, S_j^{\alpha,2}\} & \text{where} \quad S_j^{\alpha,1} &= \{\ell_{j,1}, z_j\}, \\ & S_j^{\alpha,2} &= \{\ell_{j,2}, r_{j,2}\}; \\ \Pi_{C_j}^{\beta} &= \{S_j^{\beta,1}, S_j^{\beta,2}\} & \text{where} \quad S_j^{\beta,1} &= \{r_{j,1}, z_j\}, \\ & S_j^{\beta,2} &= \{\ell_{j,3}, r_{j,3}\}. \end{split}$$

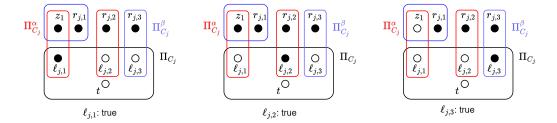


Figure 1 The set families Π_{C_j} , $\Pi_{C_j}^{\alpha}$, and $\Pi_{C_j}^{\beta}$ defined for some clause C_j . The three figures depict the construction of the set X depending on which of the literals takes **true** value; elements of the set X are shown as black circles.

We also write $\Pi(\mathcal{C}) = \langle \Pi_C, \Pi_C^{\alpha}, \Pi_C^{\beta} : C \in \mathcal{C} \rangle$ to denote the $3|\mathcal{C}|$ -tuple formed by these set families in an arbitrarily fixed ordering. We set our threshold as d = 1. Thus, our instance of ROBUST SEPARATION is $I_{RS} = (U, s, t, \Pi(V), \Pi(\mathcal{C}), 1)$.

We will show that the constructed instance I_{RS} has a solution if and only if our instance (V, \mathcal{C}) of the 1-IN-3 SAT problem is solvable.

First suppose that there is a valid truth assignment ϕ for (V, \mathcal{C}) . Consider the set

$$X = \{s\} \cup R \cup \{\ell : \ell \in V \cup \overline{V}, \phi(\ell) = \mathtt{true}\} \cup \{z_j : z_j \in Z, \phi(\ell_{j,3}) = \mathtt{false}\}.$$

Note that X contains s, but not t (see again Figure 1 for an illustration). We are going to show that X is a solution for I_{RS} . Since ϕ maps exactly one literal in $\{v, \overline{v}\}$ to true for each $v \in V$, by $R \cup \{s\} \subseteq X$ we get that

$$\begin{split} \sum_{S \in \Pi_v} \mathsf{dist}_{s,t}(X,S) &= |(\{s,v,\overline{v}\} \cup R) \setminus X| = |\{v,\overline{v}\} \setminus X| = 1 \\ \sum_{S \in \Pi_{\overline{v}}} \mathsf{dist}_{s,t}(X,S) &= |(\{v,\overline{v}\}) \cap X| = |\{v,\overline{v}\} \cap X| = 1. \end{split}$$
 and

For the distance of X from the set families associated with some clause $C_j \in \mathcal{C}$, by the validity of ϕ we obtain

$$\begin{split} \sum_{S \in \Pi_{C_j}} \operatorname{dist}_{s,t}(X,S) &= |(C_j \cup \{t\}) \cap X| = 1; \\ \sum_{S \in \Pi_{C_j}^{\alpha}} \operatorname{dist}_{s,t}(X,S) &= \min\{|S_j^{\alpha,1} \setminus X|, |S_j^{\alpha,1} \cap X|\} + \min\{|S_j^{\alpha,2} \setminus X|, |S_j^{\alpha,2} \cap X|\} \\ &= \min\{|\{\ell_{j,1}, z_j\} \setminus X|, |\{\ell_{j,1}, z_j\} \cap X|\} \\ &+ \min\{|\{\ell_{j,2}, r_{j,2}\} \setminus X|, |\{\ell_{j,2}, r_{j,2}\} \cap X|\} \\ &= \left\{ \begin{array}{ll} \min\{0, 2\} + \min\{1, 1\} = 1 & \text{if } \phi(\ell_{j,1}) = \text{true} \\ \min\{1, 1\} + \min\{0, 2\} = 1 & \text{if } \phi(\ell_{j,2}) = \text{true} \\ \min\{2, 0\} + \min\{1, 1\} = 1 & \text{if } \phi(\ell_{j,3}) = \text{true} \end{array} \right\} \\ &= \sum_{S \in \Pi_{C_j}^{\beta}} \operatorname{dist}_{s,t}(X,S) = \min\{|S_j^{\beta,1} \setminus X|, |S_j^{\beta,1} \cap X|\} + \min\{|S_j^{\beta,2} \setminus X|, |S_j^{\beta,2} \cap X|\} \\ &= \min\{|\{r_{j,1}, z_j\} \setminus X|, |\{r_{j,1}, z_j\} \cap X|\} \\ &+ \min\{|\{\ell_{j,3}, r_{j,3}\} \setminus X|, |\{\ell_{j,3}, r_{j,3}\} \cap X|\} \end{split}$$

$$= \left\{ \begin{array}{ll} \min\{0,2\} + \min\{1,1\} = 1 & \text{ if } \phi(\ell_{j,1}) = \mathtt{true} \\ \min\{0,2\} + \min\{1,1\} = 1 & \text{ if } \phi(\ell_{j,2}) = \mathtt{true} \\ \min\{1,1\} + \min\{0,2\} = 1 & \text{ if } \phi(\ell_{j,3}) = \mathtt{true} \end{array} \right\} = 1.$$

Hence, X satisfies constraint (5) for each set family, and thus is a solution for I_{RS} .

For the other direction, suppose that there exists a subset $X \subseteq U$, containing s but not t, whose distance from each set family in I_{RS} is at most 1. Let us define a truth assignment ϕ by setting $\phi(v) = \text{true}$ if and only if $v \in X$ for each variable $v \in V$; we are going to show that ϕ is valid.

To this end, let us first observe that

$$1 = d \ge \sum_{S \in \Pi_{\overline{v}}} \mathsf{dist}_{s,t}(X,S) = |(\{s,v,\overline{v}\} \cup R) \setminus X| \quad \text{ and }$$

$$1 = d \ge \sum_{S \in \Pi_{\overline{v}}} \mathsf{dist}_{s,t}(X,S) = |\{v,\overline{v},t\} \cap X|.$$

$$(6)$$

This implies that

$$|\{v, \overline{v}\} \setminus X| \le 1$$
 and $|\{v, \overline{v}\} \cap X| \le 1$. (7)

By $|\{v, \overline{v}\} \setminus X| + |\{v, \overline{v}\} \cap X| = |\{v, \overline{v}\}| = 2$, we obtain that all inequalities in (7), and hence also in (6), must hold with equality. That is, for each variable $v \in V$, exactly one of the literals v and \overline{v} is contained in X. In other words, a literal $\ell \in V \cup \overline{V}$ is set to **true** by ϕ if and only if $\ell \in X$. Moreover, we must also have $R \subseteq X$ due to the equality in (6).

Consider now the set family Π_C for some $C \in \mathcal{C}$, and recall that $t \notin X$:

$$1 \geq d = \sum_{S \in \Pi_C} \mathsf{dist}_{s,t}(X,S) = \mathsf{dist}_{s,t}(X,S_j) = |(C \cup \{t\}) \cap X| = |C \cap X|.$$

This means that ϕ sets at most one literal to true in each clause $C \in \mathcal{C}$. To prove that ϕ is valid, let us assume for the sake of contradiction that $C_j \cap X = \emptyset$ for some $C_j \in \mathcal{C}$. In this case we have $\{\ell_{j,1}, \ell_{j,2}, \ell_{j,3}\} \cap X = \emptyset$; keep in mind also that $R \subseteq X$. On the one hand, if $z_j \in X$, then we get

$$\begin{split} &1 \geq \sum_{S \in \Pi_{C_j}^{\alpha}} \mathsf{dist}_{s,t}(X,S) = \min\{|S_j^{\alpha,1} \setminus X|, |S_j^{\alpha,1} \cap X|\} + \min\{|S_j^{\alpha,2} \setminus X|, |S_j^{\alpha,2} \cap X|\} \\ &= \min\{|\{\ell_{j,1}, z_j\} \setminus X|, |\{\ell_{j,1}, z_j\} \cap X|\} + \min\{|\{\ell_{j,2}, r_{j,2}\} \setminus X|, |\{\ell_{j,2}, r_{j,2}\} \cap X|\} \\ &= \min\{1, 1\} + \min\{1, 1\} = 2, \end{split}$$

a contradiction. On the other hand, if $z_j \notin X$, then we get

$$\begin{split} 1 &\geq \sum_{S \in \Pi_{C_j}^{\beta}} \mathsf{dist}_{s,t}(X,S) = \min\{|S_j^{\beta,1} \setminus X|, |S_j^{\beta,1} \cap X|\} + \min\{|S_j^{\beta,2} \setminus X|, |S_j^{\beta,2} \cap X|\} \\ &= \min\{|\{r_{j,1}, z_j\} \setminus X|, |\{r_{j,1}, z_j\} \cap X|\} + \min\{|\{\ell_{j,3}, r_{j,3}\} \setminus X|, |\{\ell_{j,3}, r_{j,3}\} \cap X|\} \\ &= \min\{1, 1\} + \min\{1, 1\} = 2, \end{split}$$

which is again a contradiction. This proves that ϕ is indeed a valid truth assignment for our instance of 1-IN-3 SAT, and so the claim holds.

Using Theorem 18, it is not hard to show that Robust Separation remains NP-hard for any constant $d \geq 1$.

▶ **Lemma 19.** Robust Separation is NP-hard for each constant $d \ge 1$.

Proof. We present a reduction from ROBUST SEPARATION with d=1 which is NP-hard by Theorem 18. Let $I=(V,s,t,\Pi_1,\ldots,\Pi_k,1)$ be our input instance. We compute a modified instance I' of ROBUST SEPARATION with threshold $d\geq 1$ as follows. First, we introduce a set D of dummy 2d items and add them to our universe. Next, we add two new set families, each containing a single set: namely, we let $\Pi_{k+1}=\{\{D\cup\{s\}\}\}\}$ and $\Pi_{k+2}=\{\{D\cup\{t\}\}\}\}$. Furthermore, let \hat{D} be an arbitrarily fixed subset of D of size 2d-1. For each $i\in[k]$ we define $\Pi_i'=\Pi_i\cup\{\hat{D}\}$, that is, we add the set \hat{D} to Π_i . Let $I'=(V\cup D,s,t,\Pi_1',\ldots,\Pi_k',\Pi_{k+1},\Pi_{k+2},d)$ be our constructed instance.

We claim I admits a solution if and only if I' admits one.

First, let X be a solution for I. Let us fix some $D' \subseteq \hat{D}$ with |D'| = d, and define $X' = X \cup D'$. Then for each $i \in [k]$ and $S \in \Pi_i$, since $S \cap D = \emptyset$ we get $S \setminus X' = S \setminus X$ and $S \cap X' = S \cap X$; this yields $\mathsf{dist}_{s,t}(X',S) = \mathsf{dist}_{s,t}(X,S)$. Hence, for each $i \in [k]$ we obtain

$$\begin{split} \sum_{S \in \Pi'_i} \mathsf{dist}_{s,t}(X',S) &= \sum_{S \in \Pi_i} \mathsf{dist}_{s,t}(X',S) + \mathsf{dist}_{s,t}(X',\hat{D}) \\ &= \sum_{S \in \Pi_i} \mathsf{dist}_{s,t}(X,S) + \mathsf{dist}_{s,t}(X',\hat{D}) \\ &\leq 1 + \min\{|\hat{D} \setminus X'|, |\hat{D} \cap X'|\} = 1 + \min\{d-1,d\} = d \end{split}$$

where the inequality holds by our assumption on X. Moreover, since $s \in X \subseteq X'$, the distance of X' from Π_{k+1} is

$$\sum_{S \in \Pi_{k+1}} \mathsf{dist}_{s,t}(X',S) = \mathsf{dist}_{s,t}(X',D \cup \{s\}) = |(D \cup \{s\}) \setminus X'| = |D \setminus D'| = d$$

and similarly, since $t \notin X$ implies $t \notin X'$, the distance of X' from Π_{k+2} is

$$\sum_{S \in \Pi_{k+2}} \mathsf{dist}_{s,t}(X',S) = \mathsf{dist}_{s,t}(X',D \cup \{t\}) = |(D \cup \{t\}) \cap X'| = |D \cap D'| = d.$$

Hence, X' is indeed a solution for I'.

For the other direction, let X' be a solution for I'. Since X' contains s but not t, we have

$$d \geq \sum_{S \in \Pi_{k+1}} \mathsf{dist}_{s,t}(X',S) = \mathsf{dist}_{s,t}(X',D \cup \{s\}) = |(D \cup \{s\}) \setminus X'| = |D \setminus X'| \qquad \text{and} \qquad |(D \cup \{s\}) \setminus X'| = |D \setminus X'|$$

$$d \geq \sum_{S \in \Pi_{k+2}} \mathsf{dist}_{s,t}(X',S) = \mathsf{dist}_{s,t}(X',D \cup \{t\}) = |(D \cup \{t\}) \cap X'| = |D \cap X'|.$$

However, using $|D \setminus X'| + |D \cap X'| = |D| = 2d$, we obtain that both of the above inequalities hold with equality. Therefore, X' contains exactly d dummy elements from D. Recall that $|\hat{D}| = 2d - 1$. This implies that $|\hat{D} \cap X'| \in \{d - 1, d\}$ and $|\hat{D} \setminus X'| \in \{d - 1, d\}$. From this, it follows that $\mathsf{dist}_{s,t}(X', \hat{D}) = \min\{|\hat{D} \setminus X'|, |\hat{D} \cap X'|\} \ge d - 1$. We obtain that for each $i \in [k]$

$$d \geq \sum_{S \in \Pi_i'} \mathsf{dist}_{s,t}(X',S) = \sum_{S \in \Pi_i} \mathsf{dist}_{s,t}(X',S) + \mathsf{dist}_{s,t}(X',\hat{D}) \geq \sum_{S \in \Pi_i} \mathsf{dist}_{s,t}(X',S) + d - 1$$

which implies that the distance of X' from Π_i is $\sum_{S \in \Pi_i} \mathsf{dist}_{s,t}(X',S) \leq 1$.

Let $X = X' \cap V$. Again, recall that for each $i \in [k]$ and $S \in \Pi_i$ we know $S \subseteq V$, since S contains no dummies. Thus $S \cap X' = S \cap X$, and $S \setminus X' = S \setminus X$, from which $\mathsf{dist}_{s,t}(X',S) = \mathsf{dist}_{s,t}(X,S)$ follows. Hence, we obtain that $\sum_{S \in \Pi_i} \mathsf{dist}_{s,t}(X,S) \leq 1$ holds for each $i \in [k]$, and so X is a solution for the original instance I.

▶ Corollary 20. ROBUST SUBMODULAR MINIMIZER is NP-hard for each constant $d \ge 1$.

4.2 NP-hardness for a constant $k \ge 3$

In this section we prove that ROBUST SEPARATION, and hence, ROBUST SUBMODULAR MINIMIZER is NP-hard even for k=3. To this end, we are going to define another intermediary problem. First consider the MOST BALANCED MINIMUM CUT problem, proved to be NP-complete by Bonsma [3]. The input of this problem is an undirected graph G=(V,E) with two distinguished vertices, s and t, and a parameter ℓ . The task is to decide whether there exists a minimum (s,t)-cut $X\subseteq V$ in G such that $\min\{|X|,|V\setminus X|\}\geq \ell$; recall that a set of vertices $X\subseteq V$ is a minimum (s,t)-cut in the undirected graph G if $s\in X, t\notin X$ and subject to this, the value $|\delta(X)|$, i.e., the number of edges between X and $V\setminus X$, is minimized.

Instead of the Most Balanced Minimum Cut problem, it will be more convenient to use a variant that we call Perfectly Balanced Minimum Cut where we seek a minimum (s,t)-cut that contains exactly half of the vertices. Formally, its input is an undirected graph G=(V,E) with two distinguished vertices, s and t, and its task is to find a minimum (s,t)-cut X with |X|=|V|/2. Since Most Balanced Minimum Cut can be reduced to Perfectly Balanced Minimum Cut by simply adding a sufficient number of isolated vertices, we obtain the following.

▶ Lemma 21. Perfectly Balanced Minimum Cut is NP-complete.

Proof. We give a reduction from Most Balanced Minimum Cut. Let $I = (G, s, t, \ell)$ with G = (V, E) be our input instance. We may assume $\ell \leq |V|/2$, as otherwise I is clearly a "no"-instance. We construct an instance I' = (G', s, t) of Perfectly Balanced Minimum Cut by simply adding $|V| - 2\ell$ isolated vertices to G, so that the number of vertices in G' is $2|V| - 2\ell$. Clearly, if there is a minimum (s, t)-cut X such that $\min\{|X|, |V| - |X|\} \geq \ell$, then adding $|V| - \ell - |X| \geq 0$ vertices yields a minimum (s, t)-cut in G' that contains exactly half of the vertices of G'. Conversely, if X' is a minimum (s, t)-cut in G' that contains half of the vertices, i.e., $|V| - \ell$ vertices, then removing the newly added isolated vertices from it yields a minimum (s, t)-cut X in G with $|V| - \ell - (|V| - 2\ell) = \ell \leq |X| \leq |V| - \ell$.

▶ **Theorem 22** (*). ROBUST SEPARATION is NP-hard even when k = 3.

Proof. We present a reduction from the PERFECTLY BALANCED MINIMUM CUT problem. Let I = (G, s, t) be our input instance where G = (V, E). Clearly, we may assume that |V| is even, as otherwise I is trivially a "no"-instance. First we compute the number of edges in a minimum (s, t)-cut using standard flow techniques; let δ^* denote this value, that is, $\delta^* = \min_{Y: s \in Y \subset V \setminus \{t\}} |\delta(Y)|$.

Second, we modify G in order to ensure that there are at least $2\delta^* + 2$ vertices in the graph; if this holds already for G, then we set G' = G. Otherwise, we construct a new graph G' = (V', E') by adding two sets of vertices, A_s and A_t , to the graph with $|A_s| = |A_t| = \lceil (2\delta^* + 2 - |V|)/2 \rceil$, and connecting each vertex in A_s to s, as well as each vertex in A_t to t, with an edge. Observe that all minimum (s,t)-cuts in G' contain A_s and are disjoint from A_t . Moreover, any minimum (s,t)-cut X in G corresponds to a minimum (s,t)-cut $X \cup A_S$ in G' and vice versa. Thus, I' = (G',s,t) is an instance of PERFECTLY BALANCED MINIMUM CUT equivalent with I. Let 2n+2 denote the number of vertices in G', so that $\tilde{V} := V' \setminus \{s,t\}$ has 2n vertices. By our choice of $|A_s| = |A_t|$, we know that the number of vertices in G' is $|V'| = 2n+2 \ge |V| + (2\delta^* + 2 - |V|) = 2\delta^* + 2$, as promised.

Let us construct an instance J of ROBUST SEPARATION. We define our universe U as follows. For each $v \in V'$ we introduce a set $P(v) = \{\hat{v}\} \cup \{v^u : uv \in E\}$, and we

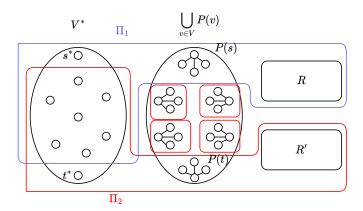


Figure 2 Set families Π_1 and Π_2 defined in the proof of Theorem 22. The figure depicts each set P(v) as a star with its center corresponding to \hat{v} ; however, note that the connections within P(v) are used only for presentational purposes, and are not part of the construction itself (as the constructed instance involves set families and no graphs).

additionally define a copy $V^* = \{v^* : v \in V\}$ of V, a set R of size $|R| = n - \delta^*$, and a copy $R' = \{r' : r \in R\}$ of R. Thus, we have

$$U = \bigcup_{v \in V'} P(v) \cup V^* \cup R \cup R'.$$

We set s^* and t^* , both in V^* , as our two distinguished vertices.

We define our three families for J as follows:

$$\begin{split} \Pi_1 &= \{S_1\} & \text{where } S_1 = V^* \setminus \{t^*\} \cup R \cup P(s); \\ \Pi_2 &= \{S_2\} \cup \{S_2^v : v \in \tilde{V}\} \\ & \text{where } S_2 = V^* \setminus \{s^*\} \cup R' \cup P(t), \\ & S_2^v = P(v) \quad \forall v \in \tilde{V}; \\ \Pi_3 &= \{S_3^v : v \in \tilde{V}\} \cup \{S_3^e : e \in E'\} \cup \{S_3^r : r \in R\} \\ & \text{where } S_3^v = \{\hat{v}, v^*\} \quad \forall v \in \tilde{V}, \\ & S_3^e = \{u^v, v^u\} \quad \forall e = uv \in E', \\ & S_3^r = \{r, r'\} \quad \forall r \in R. \end{split}$$

Thus, Π_1 contains only a single set, Π_2 contains $|\tilde{V}| + 1$ pairwise disjoint sets, and Π_3 contains $|\tilde{V}| + |E'| + |R|$ pairwise disjoint sets. See Figures 2 and 3 for better understanding. We finish the definition of our instance J by setting d = n as our threshold, so that $J = (U, s^*, t^*, \Pi_1, \Pi_2, \Pi_3, n)$.

We claim that G' admits a minimum (s,t)-cut containing exactly n+1 vertices if and only if J is a "yes"-instance of ROBUST SEPARATION.

Direction "\Longrightarrow". Let us first suppose that X is a minimum (s,t)-cut in G' with |X|=n+1; recall that there must be exactly δ^* edges between X and $V'\setminus X$. Define

$$Z = \bigcup_{v \in X} (P(v) \cup \{v^*\}) \cup R.$$

Since $s \in X$ but $t \notin X$, we know $P(s) \subseteq Z$ and $P(t) \cap Z = \emptyset$; in particular, $s^* \in Z$ but $t^* \notin Z$. By $s^* \in S_1$ we get

$$\mathsf{dist}_{s^* \ t^*}(Z, S_1) = |S_1 \setminus Z| = |\{v^* : v \in V' \setminus \{t\} \setminus X\}| = 2n + 1 - (n+1) = n,$$

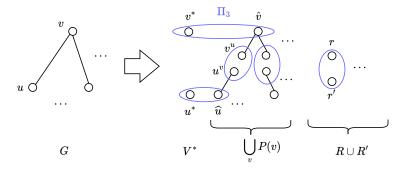


Figure 3 Illustration of the set family Π_3 , using the notation of Figure 2. The figure assumes that uv is an edge in the input graph G.

implying that

$$\sum_{S \in \Pi_1} \mathsf{dist}_{s^*, t^*}(Z, S) = n. \tag{8}$$

Similarly, by $t^* \in S_2$ we get

$$\mathsf{dist}_{s^*,t^*}(Z,S_2) = |S_2 \cap Z| = |\{v^* : v \in (V' \setminus \{s\}) \cap X\}| = |X \setminus \{s\}| = n.$$

Observe also that for each $v \in \tilde{V}$, the set $S_2^v = P(v)$ is either contained in Z, or is disjoint from it. Taking into account that P(v) contains neither s^* nor t^* , we get that for each $v \in \tilde{V}$

$$\mathsf{dist}_{s^*,t^*}(Z, S_2^v) = \min\{|S_2^v \setminus Z|, |S_2^v \cap Z|\} = 0,$$

and hence

$$\sum_{S \in \Pi_2} \mathsf{dist}_{s^*,t^*}(Z,S) = \mathsf{dist}_{s^*,t^*}(Z,S_2) + \sum_{v \in \tilde{V}} \mathsf{dist}_{s^*,t^*}(Z,S_2^v) = n. \tag{9}$$

Consider now the set family Π_3 . First, note that for each $v \in \tilde{V}$, the set $S_3^v = \{\hat{v}, v^*\} \subseteq P(v) \cup \{v^*\}$ is either contained in Z, or is disjoint from it. Hence,

$$\operatorname{dist}_{s^*,t^*}(Z, S_3^v) = \min\{|S_3^v \setminus Z|, |S_3^v \cap Z|\} = \min\{|\{\hat{v}, v^*\} \setminus Z|, |\{\hat{v}, v^*\} \cap Z|\} = 0. \tag{10}$$

Second, for each edge $e = uv \in E'$ we know that $|\{u^v, v^u\} \cap Z| = 1$ if and only if exactly one among u and v is contained in X. This implies

$$\begin{aligned} \operatorname{dist}_{s^*,t^*}(Z,S_3^e) &= \min\{|S_3^e \setminus Z|, |S_3^e \cap Z|\} = \min\{|\{u^v,v^u\} \setminus Z|, |\{u^v,v^u\} \cap Z|\} \\ &= \begin{cases} 1 & \text{if } e \text{ runs between } X \text{ and } V' \setminus X \text{ in } G', \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{11}$$

Third, observe that for each $r \in R$, we have $r \in Z$ but $r' \notin Z$, so

$$\operatorname{dist}_{s^*,t^*}(Z,S_3^r) = \min\{|S_3^r \setminus Z|, |S_3^r \cap Z|\} = \min\{|\{r,r'\} \setminus Z|, |\{r,r'\} \cap Z|\} = 1. \tag{12}$$

Using the bounds (10), (11), and (12), we get that the distance of Z from Π_3 is

$$\begin{split} \sum_{S \in \Pi_3} \mathrm{dist}_{s^*,t^*}(Z,S) &= \sum_{v \in \tilde{V}} \mathrm{dist}_{s^*,t^*}(Z,S_3^v) + \sum_{e \in E'} \mathrm{dist}_{s^*,t^*}(Z,S_3^e) + \sum_{r \in R} \mathrm{dist}_{s^*,t^*}(Z,S_3^r) \\ &= \delta^* + |R| = \delta^* + (n-\delta^*) = n. \end{split}$$

Taking into account (8) and (9) as well, this implies that Z has distance at most n=d from each of the set families Π_1 , Π_2 , and Π_3 . Thus, Z is indeed a solution for the instance J of ROBUST SEPARATION.

Direction "\Leftarrow". For the other direction, assume now that Z is a solution for J; we are going to construct a minimum (s,t)-cut for G' that contains exactly half of its vertices, i.e., n+1 vertices.

To this end, let us first observe that by $s^* \in S_1$ and $t^* \in S_2$ we have

$$n = d \ge \sum_{S \in \Pi_1} \mathsf{dist}_{s^*, t^*}(Z, S) = \mathsf{dist}_{s^*, t^*}(Z, S_1) = |S_1 \setminus Z|$$
 and (13)

$$n = d \ge \sum_{S \in \Pi_2} \mathsf{dist}_{s^*, t^*}(Z, S) \ge \mathsf{dist}_{s^*, t^*}(Z, S_2) = |S_2 \cap Z| \tag{14}$$

Since $V^* \setminus \{s^*, t^*\} \setminus Z \subseteq S_1 \setminus Z$ and $V^* \setminus \{s^*, t^*\} \cap Z \subseteq S_2 \cap Z$, the bounds in (13) and in (14), respectively, imply that

$$n \ge |V^* \setminus \{s^*, t^*\} \setminus Z|$$
 and $n \ge |V^* \setminus \{s^*, t^*\} \cap Z|$. (15)

However,

$$n + n \ge |V^* \setminus \{s^*, t^*\} \setminus Z| + |V^* \setminus \{s^*, t^*\} \cap Z| = |V^* \setminus \{s^*, t^*\}| = 2n$$

and consequently, all inequalities in (13), (14) and (15) must hold with equality; moreover, we must also have $S_1 \setminus Z = V^* \setminus \{s^*, t^*\} \setminus Z$ and $S_2 \cap Z = V^* \setminus \{s^*, t^*\} \cap Z$. This means that

$$R \cup P(s) \subseteq Z$$
 and $(R' \cup P(t)) \cap Z = \emptyset.$ (16)

Let us define a set X as follows:

$$X = \{ v \in V' : v^* \in Z \}.$$

Since Z contains s^* but not t^* , we know that X contains s but not t; in other words, X is an (s,t)-cut. Moreover, since both inequalities in (15) hold with equality, it follows that |X| = n + 1, that is, it contains half of the vertices of G'. It remains to show that X is a minimum (s,t)-cut.

To this end, first recall that $R \subseteq Z$ but $R' \cap Z = \emptyset$ by fact (16). This yields

$$\sum_{r \in R} \mathsf{dist}_{s^*, t^*}(Z, S_3^r) = \sum_{r \in R} \min\{|\{r, r'\} \setminus Z|, |\{r, r'\} \cap Z|\} = |R| = n - \delta^*. \tag{17}$$

Recall now that all inequalities in (14) hold with equality, which implies that for each $v \in \tilde{V}$ we must have $\mathsf{dist}_{s,t}(Z,S_2^v) = 0$, i.e., the set $S_2^v = P(v)$ is either contained in Z, or is disjoint from Z. Let us define the set $\hat{X} = \{v : \hat{v} \in Z\}$; we are going to prove that $X = \hat{X}$. Note that $s \in \hat{X}$ but $t \notin \hat{X}$, due to fact (16). Hence, \hat{X} is an (s,t)-cut in G'.

Now, for each edge $e = uv \in E'$, the set $S_3^e = \{u^v, v^u\}$ is either contained in Z, is disjoint from Z, or e = uv runs between \hat{X} and $V' \setminus \hat{X}$ in G'. Thus, we have that

$$\sum_{e \in E'} \mathsf{dist}_{s^*, t^*}(Z, S_3^e) = \sum_{e = uv \in E'} \min\{|\{u^v, v^u\} \setminus Z|, |\{u^v, v^u\} \cap Z|\}$$

$$= |\{uv \in E' : |\{u, v\} \cap \hat{X}| = 1\}| = |\delta(\hat{X})| \ge \delta^*, \tag{18}$$

by the definition of δ^* , and since \hat{X} is an (s,t)-cut. Taking into account (17) we obtain that

$$\begin{split} n &\geq \sum_{S \in \Pi_3} \mathsf{dist}_{s^*,t^*}(Z,S) = \sum_{v \in \tilde{V}} \mathsf{dist}_{s^*,t^*}(Z,S_3^v) + \sum_{e \in E'} \mathsf{dist}_{s^*,t^*}(Z,S_3^e) + \sum_{r \in R} \mathsf{dist}_{s^*,t^*}(Z,S_3^r) \\ &\geq \sum_{v \in \tilde{V}} \mathsf{dist}_{s^*,t^*}(Z,S_3^v) + \delta^* + (n-\delta^*) = \sum_{v \in \tilde{V}} \mathsf{dist}_{s^*,t^*}(Z,S_3^v) + n. \end{split}$$

$$\mathsf{dist}_{s^*,t^*}(Z,S_3^v) = \min\{|\{\hat{v},v^*\} \setminus Z|, |\{\hat{v},v^*\} \cap Z|\} = 0,$$

and thus either both \hat{v} and v^* are contained in Z, or neither of them is contained in Z. In other words, $v \in X$ if and only if $v \in \hat{X}$ holds for each $v \in \tilde{V}$, proving $X = \hat{X}$. Therefore, X is a minimum (s,t)-cut as well, as promised. This proves the correctness of our reduction.

Clearly, we can increase the value of parameter k without changing the solution set of our instance of Robust Separation by repeatedly adding a copy of, say, the first set family Π_1 . Using also Lemma 17, we have the following easy consequences of Theorem 22:

- ▶ Corollary 23. ROBUST SEPARATION is NP-hard for each constant $k \ge 3$.
- ▶ Corollary 24. ROBUST SUBMODULAR MINIMIZER is NP-hard for each constant $k \geq 3$.

5 Conclusion

In this paper, we studied the computational complexity of ROBUST SUBMODULAR MINIMIZER, and provided a complete computational map of the problem with respect to the parameters k and d, offering dichotomies for the case when one of these parameters is a constant, and giving an FPT algorithm for the combined parameter (k,d). Regarding the case when one of the functions f_i has only polynomially bounded minimizers, there are a few questions left open: First, what is the computational complexity of this variant when parameterized by k? Second, is there an algorithm for this case with running time $2^{O(d)}|I|^{O(1)}$ on some instance I instead of the running time $2^{O(d^2)}|I|^{O(1)}$ we obtained based on the algorithm for Theorem 13?

We remark that our algorithmic results can be adapted in a straightforward way to a slightly generalized problem: given k submodular functions f_1, \ldots, f_k with non-negative integers d_1, \ldots, d_k , we aim to find a set X such that, for each $i \in [k]$, there exists some set $Y_i \in \arg\min f_i$ with $|X \triangle Y_i| \leq d_i$ for each $i \in [k]$. As mentioned in Section 1.2, ROBUST SUBMODULAR MINIMIZER is related to recoverable robustness. We can consider the robust recoverable variant of submodular minimization: given submodular functions f_0, f_1, \ldots, f_k , we aim to find a set X that minimizes

$$f_0(X) + \max_{i \in [k]} \min_{Y_i : |Y_i \triangle X| \le d} f_i(X_i).$$

The optimal value is lower-bounded by $f_0(Y_0) + \max_{i \in [k]} f_i(Y_i)$ where $Y_i \in \arg \min f_i$ for each $i \in \{0, 1, ..., k\}$. Our results imply that we can decide efficiently whether the optimal value attains this lower bound or not, when d and k are parameters, or when f_0 has polynomially many minimizers.

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