The Low-Degree Hardness of Finding Large Independent Sets in Sparse Random Hypergraphs

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Abstract

We study the algorithmic task of finding large independent sets in sparse Erdős–Rényi random r-uniform hypergraphs on n vertices having average degree d. Krivelevich and Sudakov showed that the maximum independent set has density $\left(\frac{r}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ in the double limit $n\to\infty$ followed by $d\to\infty$. We show that the class of low-degree polynomial algorithms can find independent sets of density $\left(\frac{1}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ but no larger. This extends and generalizes earlier results of Gamarnik and Sudan, Rahman and Virág, and Wein on graphs, and answers a question of Bal and Bennett. We conjecture that this statistical-computational gap of a multiplicative factor of $r^{-1/(r-1)}$ indeed holds for this problem.

Additionally, we explore the universality of this gap by examining r-partite hypergraphs. A hypergraph H=(V,E) is r-partite if there is a partition $V=V_1\cup\cdots\cup V_r$ such that each edge contains exactly one vertex from each set V_i . We consider the problem of finding large balanced independent sets (independent sets containing the same number of vertices in each partition) in random r-uniform r-partite hypergraphs with n vertices in each partition and average degree d. We prove that the maximum balanced independent set has density $\left(\frac{r}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ asymptotically, matching that of independent sets in ordinary hypergraphs.

Furthermore, we prove an analogous computational threshold of $\left(\frac{1}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ for low-degree polynomial algorithms, answering a question of the first author. We prove more general statements regarding γ -balanced independent sets (where we specify the proportion of vertices of the independent set contained within each partition). Our results recover and generalize recent work of Perkins and the second author on bipartite graphs.

While the graph case has been extensively studied, this work is the first to consider statistical-computational gaps of optimization problems on random hypergraphs. Our results suggest that these gaps persist for larger uniformities as well as across many models. A somewhat surprising aspect of the gap for balanced independent sets is that the algorithm achieving the lower bound is a simple degree-1 polynomial.

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Contents

1	Inti	roduction	3
	1.1	Relation to Prior Work	6
	1.2	The Low-Degree Framework	8
	1.3	Main Results	9
	1.4	Proof Overview	11
	1.5	Concluding Remarks	13
2	\mathbf{Pre}	eliminaries	13
	2.1	Notation	13
	2.2	Local Algorithms on Hypergraphs	14
	2.3	Interpolating Paths and Stable Algorithms	15
3	Independent Sets in $\mathcal{H}_r(n,p)$		
	3.1	Proof of the Achievability Result	16
		3.1.1 Local Algorithms: from T_{Δ}^r to \mathbb{T}_d^{GW}	16
		3.1.2 From Local to Low-Degree	19
	3.2	Forbidden Structures	25
	3.3	Proof of the Intractability Result in $\mathcal{H}_r(n,p)$	29
4	Balanced Independent Sets in $\mathcal{H}(r, n, p)$		32
	4.1	Statistical Threshold	32
	4.2	Proof of the Achievability Result	38
	4.3	Forbidden Structures	40
	4.4	Proof of the Intractability Result	43

1 Introduction

A hypergraph H is an ordered pair (V, E) where E is a subset of the power set of V. The elements in V and E are referred to as the vertices and edges of H, respectively. If every edge $e \in E$ is a set of size r, then we say H is r-uniform (for r = 2, H is a graph). An independent set I of H is a subset of V such that for each $e \in E$, we have $e \not\subseteq I$, i.e., the hypergraph induced by the set I contains no edges. We say an independent set $I \subseteq V$ has density |I|/|V|.

Independent Sets in Graphs and Hypergraphs

The problem of finding a maximum size independent set in hypergraphs is of fundamental interest, both in practical and theoretical aspects. It arises in various applications in data mining, image processing, database design, and parallel computing, to name a few. It is well-known that finding the maximum independent set in graphs is an NP-hard problem [Kar10]. In fact, it is hard to approximate the maximum independent set in a graph to within $n^{1-\varepsilon}$ for any $\varepsilon > 0$ [Has96; Kho01; Zuc06]. As hypergraphs are a generalization of graphs, these hardness results clearly extend to this setting. In order to better understand the hardness of maximum independent set, a natural line of research is to analyze the structure of the "hard" instances. We consider a somewhat related question: is maximum independent set tractable on "typical" instances? In particular, we consider the r-uniform Erdős–Rényi hypergraphs $\mathcal{H}_r(n,p)$, where each potential edge in an n-vertex r-uniform hypergraph is included independently with probability p (see Definition 1.4 for a formal description).

The study of finding large independent sets in random graphs dates back to Matula [Mat76] who showed that an Erdős–Rényi graph does not contain an independent set of size at least $(2 + o(1)) \log_{1/(1-p)} n$ with high probability for a large range of p. A simple algorithm can construct an independent set of size $\sim \log_{1/(1-p)} n$, however, finding a larger one seems to be hard. In 1976, Karp conjectured that no polynomial time algorithm can compute an independent set of size at least $(1 + \varepsilon) \log_2 n$ for any $\varepsilon > 0$ with high probability for p = 1/2 [Kar76]. This conjecture has spurred a large research effort in order to determine the so-called *statistical-computational gap* for a number of "hard" problems on random graphs (see for example [Jer92; DM15; GJW20; Wei22; BH22]).

We are interested in the sparse regime, i.e., $p = d/\binom{n-1}{r-1}$, where d is a constant. Note that for our choice of p, the expected degree of a vertex is precisely d. For r=2, Frieze showed that with high probability as $n \to \infty$ the maximum independent set of $\mathcal{H}_2(n,p)$ has density $(2 \pm o_d(1)) \log d/d$ for sufficiently large constant d [Fri90]. It is conjectured that no polynomial time algorithm can find an independent set of density $(1+\varepsilon) \log d/d$ with high probability for any $\varepsilon > 0$, i.e., there is a statistical-computational gap of a multiplicative factor of 1/2. Proving this conjecture is equivalent to proving a statement stronger than P = NP and so researchers have focused on providing evidence of intractability through a variety of methods. One such method is to prove intractability for restricted classes of algorithms (see §1.1 for examples). Gamarnik and Sudan [GS14a] first proved a gap of a multiplicative factor of $(1/2 + 1/2\sqrt{2})$ for local algorithms and pioneered the

framework of the *Overlap Gap Property*, which subsequently led to proving the conjectured gap of a multiplicative factor of 1/2 for both local algorithms [RV17] and low-degree algorithms [Wei22]. In our work we focus on *low-degree algorithms* (see §1.2 for an overview of the framework).

Moving beyond graphs to hypergraphs, Krivelevich and Sudakov showed that with high probability the maximum independent set of $\mathcal{H}_r(n,p)$ has density $(1\pm o(1))\left(\frac{r}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ [KS98], which matches Frieze's result when r=2. It is natural to ask whether this phenomenon of a statistical-computational gap for finding large independent sets extends to the hypergraph setting. Additionally, determining the size of this gap is of interest as well. To the best of our knowledge, this paper is the first to consider the computational threshold for independent sets in random hypergraphs.

Theorem (Informal version of Theorems 1.5 and 1.6). Let $\varepsilon > 0$ and let $r \ge 2$. For d, n sufficiently large in terms of ε and r, the following hold for $p = d/\binom{n-1}{r-1}$.

- There is a low-degree algorithm that with high probability computes an independent set of density $(1-\varepsilon)\left(\frac{1}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ in $\mathcal{H}_r(n,p)$.
- There is no low-degree algorithm that with high probability finds an independent set of density $(1+\varepsilon)\left(\frac{1}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ in $\mathcal{H}_r(n,p)$.

As local algorithms are a subclass of low-degree algorithms, our results answer a question of Bal and Bennett [BB23, §6.2]. In light of this result and the aforementioned statistical-computational gap conjecture of independent sets in random graphs, we make the following analogous conjecture for hypergraphs.

Conjecture 1. For any fixed $\varepsilon > 0$ and integer $r \geq 2$, there are $d, n \in \mathbb{N}$ sufficiently large such that there is no polynomial-time algorithm that finds an independent set in $\mathcal{H}_r\left(n,d/\binom{n-1}{r-1}\right)$ of density at least $(1+\varepsilon)\left(\frac{1}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ with high probability.

Balanced Independent Sets in Multipartite Hypergraphs

In recent work, Perkins and the second author explored the universality of the statistical-computational gap of independent sets in graphs [PW24]. They considered the following question: in what other random graph models (and for what kinds of independent sets) do we see such a gap? Specifically, they studied independent sets of the Erdős–Rényi bipartite graph with n vertices in each partition. When considering bipartite graphs, there is a trivial independent set of density at least 1/2, namely, the larger partition. Furthermore, there exists a max-flow based polynomial time algorithm to find the maximum independent set in bipartite graphs. As observed by Perkins and the second author, imposing global constraints on the independent set I can introduce computational intractability.

When considering bipartite graphs, a natural global constraint to consider is balancedness. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. We say an independent set $I \subseteq V(G)$ is balanced if $|I \cap V_1| = |I|/2$, i.e., I contains an equal number of vertices from each partition. There has been extensive research in determining the density of the maximum balanced independent set in both deterministic and random bipartite graphs [FMO93; Axe+21; Cha23]. Perkins and the second author showed that the Erdős–Rényi bipartite graph with n vertices in each partition has a balanced independent set with density $(2 \pm o(1)) \log d/d$ with high probability. They also showed that the low-degree computational threshold of the density of the maximum balanced independent set is $\log d/d$. In particular, balanced independent sets in bipartite graphs exhibit the same behavior as independent sets in graphs. We remark that Perkins and the second author prove a more general statement for so-called γ -balanced independent sets (see Definition 1.7).

We extend these results to r-uniform r-partite hypergraphs. We say H = (V, E) is r-partite if there exists a partition $V = V_1 \cup \cdots \cup V_r$ such that each edge $e \in E$ contains precisely one vertex from each set V_i . Furthermore, if $|V_1| = \cdots = |V_r|$, we call such hypergraphs balanced. Multipartite hypergraphs find a wide array of applications in satisfiability problems, Steiner triple systems, and particle tracking in physics, to name a few. Note that the union of any r-1 partitions forms an independent set in such a hypergraph, and so the problem of finding an independent set of density at least 1-1/r is trivial. As this class of hypergraphs is a natural generalization of bipartite graphs, we may extend the definition of balanced independent sets to this setting. In particular, we say an independent set I in an r-uniform r-partite hypergraph $H = (V_1 \cup \cdots \cup V_r, E)$ is balanced if $|I \cap V_i| = |I|/r$ for each $1 \le i \le r$ (I contains an equal number of vertices from each partition). Such a notion was introduced in recent work of the first author [Dha23a], where he considered the size of the largest balanced independent set in deterministic r-partite hypergraphs.

We consider the Erdős–Rényi r-uniform r-partite balanced hypergraph $\mathcal{H}(r,n,p)$ with n vertices in each partition (note the notational distinction between $\mathcal{H}(r,n,p)$ and $\mathcal{H}_r(n,p)$). The second main result of this paper concerns the statistical-computational gap of finding large balanced independent sets in $\mathcal{H}(r,n,p)$ for $p=d/n^{r-1}$. We first prove a high probability bound on the asymptotic density of the largest balanced independent set in $\mathcal{H}(r,n,p)$. Moreover, we describe a simple, efficient algorithm that can find a balanced independent set within a multiplicative factor of $r^{-1/(r-1)}$ of the statistical threshold. This begs the following question: can one do better or is there a universality of the statistical–computational gap for finding large independent sets in hypergraphs as well? We provide evidence toward the latter.

Theorem (Informal version of Theorems 1.8 and 1.9). Let $\varepsilon > 0$ and let $r \ge 2$. For d, n sufficiently large in terms of ε and r, the following hold for $p = d/n^{r-1}$.

- The largest balanced independent set in $\mathcal{H}(r,n,p)$ has density $(1 \pm \varepsilon) \left(\frac{r}{r-1} \cdot \frac{\log d}{d}\right)^{1/(r-1)}$ with high probability.
- There is a low-degree algorithm that with high probability finds a balanced independent set of density $(1-\varepsilon)\left(\frac{1}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ in $\mathcal{H}(r,n,p)$.
- There is no low-degree algorithm that with high probability finds a balanced independent set of density $(1+\varepsilon)\left(\frac{1}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ in $\mathcal{H}(r,n,p)$.

Our results answer a question of the first author, and for r=2, we recover the results of Perkins and the second author. We also conjecture that this statistical-computational gap persists for polynomial-time algorithms (a version of this conjecture for r=2 appeared in [PW24]).

Conjecture 2. For any $\varepsilon > 0$ and integer $r \geq 2$, there are $d, n \in \mathbb{N}$ sufficiently large such that there is no polynomial-time algorithm that finds a balanced independent set in $\mathcal{H}\left(r,n,d/n^{r-1}\right)$ of density at least $(1+\varepsilon)\left(\frac{1}{r-1}\cdot\frac{\log d}{d}\right)^{1/(r-1)}$ with high probability.

We do in fact prove more general statements for γ -balanced independent sets (see Definition 1.7 for a formal definition). Here, rather than considering an independent set I with an equal proportion of vertices in each partition, we specify the proportion of vertices in I within each partition in a vector $\gamma \in (0,1)^r$. For $\gamma = (1/r, \ldots, 1/r)$, we recover the original notion of balanced independent sets.

We remark that while multipartite hypergraphs are a natural extension of bipartite graphs to larger uniformities, certain properties do not extend. For example, bipartite graphs are triangle-free, while r-partite hypergraphs can contain triangles for $r \geq 3$ (the same holds for odd cycles). This makes our results all the more interesting as we recover the results for r = 2.

1.1 Relation to Prior Work

Statistical-computational gaps. Inference problems with conjectured statistical-computational gaps are ubiquitous throughout statistics and computer science. Some classical examples include the planted clique problem [Jer92; DM15; MPW15; Bar+19] and community detection in random graphs [Dec+11; AV14; HS17], structured principal component analysis of matrices [BR13; LKZ15] and tensors [HSS15; Hop+17], and solving or refuting constraint satisfaction problems [AC08; Kot+17].

While proving statistical-computational gaps of the above problems is considered a daunting task, it has spurred a large research effort toward providing rigorous evidence of hardness for average-case instances. As mentioned earlier, this involves proving hardness for certain restricted classes of algorithms. These techniques include Markov chain Monte Carlo methods [Jer92; DFJ02], local algorithms [GS14a; RV17; Che+19], belief propagation and approximate message passing algorithms [Dec+11; LKZ15], reductions from the presumed hard planted clique problem [HWX15; BB19], and statistical query models [Kea98; FPV15], to name a few.

Within the random graph framework, there are three primary problems of interest: optimization, testing, and estimation. This work focuses on the optimization problem, i.e., finding a large (balanced) independent set in a random hypergraph. One can analogously consider the testing and estimation problems by analyzing so-called "planted" models. The above techniques have been applied to a wide variety of such problems including the densest k-subgraph problem [FS97; Bha+10], planted dense subgraph problem [Jon+23; MWZ23; MWZ24], and hypergraph versions [Chl+18; Cor+22; DMW23].

Low-degree algorithms. The main focus of this work is to provide sharp thresholds for the tractability of low-degree polynomial algorithms (to be formally defined in the next section). Here, each vertex's membership in the (balanced) independent set is determined by a multivariate polynomial in the edge indicator variables of the hypergraph. This is a powerful class of algorithms as it includes the class of local algorithms as well as the algorithmic paradigms of approximate message passing and power iteration (see the discussion in [GJW24, Appendix A]). Furthermore, it has been observed that low-degree algorithms are as powerful as the best known polynomial-time algorithms for a number of problems in high-dimensional statistics including those mentioned in the preceding paragraphs.

There is by now a standard method for proving low-degree optimization bounds based on the Overlap Gap Property, which we describe in $\S1.4$ (see the survey [Gam21] for a more extensive overview). This technique has been employed in a number of inference problems including independent sets in random graphs [GJW20; Wei22], random k-SAT [BH22], discrepancy of random matrices [Ven22], and random CSPs [CHM23]. Our work is the first to apply this strategy to random hypergraphs.

Statistical inference on random hypergraphs. Random graph inference has been a central research area for decades, while the hypergraph extension is less well-studied. In general, the hypergraph setting is believed to be considerably harder. In recent years, however, there has been an increase in both theoretical and applied interest in this setting. These include applying spectral methods to solve testing and estimation problems [LZ22; Jon+23] and determining statistical and computational thresholds for planted problems [YS21a; YS21b; DMW23].

While the maximum independent set problem is NP-hard, there has been progress for restricted classes of algorithms on structurally constrained hypergraphs. In particular, [HL09; GS11] consider bounded-degree hypergraphs, [Hal+16] consider streaming algorithms for sparse hypergraphs, [KLP21] consider semi-random hypergraphs, and [Hal02; AHL13] explore SDP based methods of solving maximum independent set. To the best of our knowledge, this is the first paper to consider the statistical-computational gap of finding independent sets in Erdős–Rényi hypergraphs.

In contrast, multipartite hypergraphs have been less extensively studied from a computational standpoint, however, there are a number of theoretical results which may be of interest to the reader [KSV17; Dha23b; BM24]. In [GSS15], the authors consider the minimum vertex cover problem on such hypergraphs. A somewhat surprising result is that while the problem is tractable for r = 2, it is NP-hard for $r \geq 3$. This makes our results all the more interesting as we show that the behavior of maximum balanced independent set does not exhibit such a distinction based on r. A similar observation was made in [BWZ12] who consider the appearance of a k-core (a subhypergraph of minimum degree k) in certain random r-partite hypergraph models.

1.2 The Low-Degree Framework

We now define the framework of low-degree polynomial algorithms. We say a function $f: \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial of degree (at most) D if it can be written as $f(A) = (f_1(A), \ldots, f_n(A))$, where each $f_i: \mathbb{R}^m \to \mathbb{R}$ is a multivariate polynomial of degree at most D. For a probability space (Ω, P_ω) , we say $f: \mathbb{R}^m \times \Omega \to \mathbb{R}^n$ is a random polynomial if $f(\cdot, \omega)$ is a degree D polynomial for each $\omega \in \Omega$, i.e., the coefficients of f are random but do not depend on f. For our purposes, the input of f is an indicator vector f is an indicator vector f is encoding the edges of an f in f in f in f in f in f is an indicator vector f in f

We must formally define what it means for such a polynomial to find an independent set in a hypergraph. We follow the notation and definitions from [Wei22], in which a rounding procedure is used to produce an independent set of $\mathcal{H}_2(n,p)$ based on the output f(A). For brevity, we just state the following definition in this section for ordinary hypergraphs (the multipartite analog can be inferred as it is nearly identical).

Definition 1.1. Let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a random polynomial, with $m = \binom{n}{r}$. For $A \in \{0,1\}^m$ indicating the edges of a hypergraph on n vertices and $\eta > 0$, let $V_f^{\eta}(A,\omega)$ be the independent set constructed as follows. Let

$$I = \left\{ i \in [n] : f_i(A, \omega) \ge 1 \right\}$$

$$\tilde{I} = \left\{ i \in I : \forall e \ni i, e \not\subseteq I \right\}, \quad and$$

$$J = \left\{ i \in [n] : f_i(A, \omega) \in \left(\frac{1}{2}, 1\right) \right\}.$$

Then define

$$V_f^{\eta}(A,\omega) = \begin{cases} \tilde{I} & \text{if } |I \setminus \tilde{I}| + |J| \leq \eta n; \\ \varnothing & \text{otherwise.} \end{cases}$$

In essence, a vertex i is in the independent set if the output of the corresponding polynomial f_i is at least 1, and it is not in the independent set if the output of f_i is at most 1/2. We allow up to ηn "errors", i.e., vertices i which satisfy one of the following:

- either $f_i(A) \in (1/2, 1)$, or
- there is some edge e containing i such that $e \subseteq I$ (i violates the independence constraint).

The error tolerance of (1/2, 1) is important for our impossibility results, as this ensures that a small change in $f(A, \omega)$ cannot cause a large change in the resulting independent set without encountering the failure event \varnothing . With the mapping V_f^{η} in hand, let us formally define how a random polynomial finds an independent set of a certain size.

Definition 1.2. For parameters k > 0, $\delta \in [0,1]$, and $\xi \geq 1$, a random function $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to (k, δ, ξ, η) -optimize the independent set problem in $\mathcal{H}_r(n, p)$ if the following are satisfied when $A \sim \mathcal{H}_r(n, p)$:

- $\mathbb{E}_{A,\omega}\left[\|f(A,\omega)\|^2\right] \leq \xi k$ and
- $\mathbb{P}_{A,\omega}\left[|V_f^{\eta}(A,\omega)| \ge k\right] \ge 1 \delta.$

Here, k is the desired size of the independent set, δ is the failure probability of the algorithm, ξ is a normalization parameter, and η is the error tolerance parameter of the rounding procedure. Similarly, we define how a random polynomial finds a balanced independent set of a certain size.

Definition 1.3. For parameters $k_1, \ldots, k_r > 0$, $\delta \in [0,1]$, and $\xi \geq 1$, a random function $f : \mathbb{R}^m \to \mathbb{R}^m$ is said to $(k_1, \ldots, k_r, \delta, \xi, \eta)$ -optimize the balanced independent set problem in $\mathcal{H}(r, n, p)$ if the following are satisfied when $A \sim \mathcal{H}(r, n, p)$:

- $\mathbb{E}_{A,\omega}\left[\|f(A,\omega)\|^2\right] \leq \xi(k_1 + \cdots + k_r)$ and
- $\mathbb{P}_{A,\omega}\left[\forall i \in [r], |V_f^{\eta}(A,\omega) \cap V_i| \ge k_i\right] \ge 1 \delta,$

where V_1, \ldots, V_r are the vertex partitions of the r-partite hypergraph $\mathcal{H}(r, n, p)$.

Here, k_1, \ldots, k_r are the desired sizes of the intersections of the independent set with each partition, while the other parameters are the same as in Definition 1.2. We note that for balanced independent sets, we would have $k_1 = \cdots = k_r$, however, as we will see in the next section, we consider the more general γ -balanced independent sets in which we allow different values of k_i .

1.3 Main Results

We are now ready to state our main results. First, we define the random models we consider.

Definition 1.4 (The Random Hypergraph Models). Let $n, r \in \mathbb{N}$ such that $n \geq r \geq 2$.

- We construct the hypergraph $H \sim \mathcal{H}_r(n,p)$ on vertex set [n] by including each $e \subseteq [n]$ of size r in E(H) independently with probability p.
- We construct the hypergraph $H \sim \mathcal{H}(r, n, p)$ on vertex set $[n] \times [r]$ by including each

$$e \in V_1 \times \cdots \times V_r$$

in E(H) independently with probability p. Here, $V_i = [n] \times \{i\}$.

Our first results are on the low-degree computational threshold for constructing large independent sets in Erdős–Rényi hypergraphs.

Theorem 1.5 (Achievability result for $\mathcal{H}_r(n,p)$). Let $\varepsilon > 0$ and $r \in \mathbb{N}$ such that $r \geq 2$. There exists $d^* > 0$ such that for any $d \geq d^*$ and $\eta > 0$, there exist n^* , C, D > 0 and $\xi \geq 1$ such that the following holds for all $n \geq n^*$. For

$$k = n (1 - \varepsilon) \left(\frac{1}{r - 1} \cdot \frac{\log d}{d} \right)^{1/(r - 1)}, \quad and \quad \delta = \exp\left(-Cn^{1/3} \right),$$

there exists a degree-D polynomial that (k, δ, ξ, η) -optimizes the independent set problem in $\mathcal{H}_r(n, p)$ for $p = d/\binom{n-1}{r-1}$.

Theorem 1.6 (Impossibility result for $\mathcal{H}_r(n,p)$). Let $\varepsilon > 0$ and $r \in \mathbb{N}$ such that $r \geq 2$. There exists $d^* > 0$ such that for any $d \geq d^*$ there exist η , C_1 , C_2 , $n^* > 0$ such that the following holds for any $n \geq n^*$, $\xi \geq 1$, $1 \leq D \leq \frac{C_1 n}{\xi \log n}$, and $\delta \leq \exp(-C_2 \xi D \log n)$. If $k \geq n (1+\varepsilon) \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}}$, there is no degree-D polynomial that (k, δ, ξ, η) -optimizes the independent set problem in $\mathcal{H}_r(n, p)$.

Before we state our results for multipartite hypergraphs, we define γ -balanced independent sets.

Definition 1.7 (Balanced independent sets). Let $H = (V_1 \cup \cdots \cup V_r, E)$ be an r-uniform r-partite hypergraph for $r \geq 2$, and let $\gamma = (\gamma_1, \ldots, \gamma_r)$ be such that $\gamma_i \in (0,1)$ and $\sum_{i=1}^r \gamma_i = 1$. An independent set $I \subseteq V(H)$ is γ -balanced if $|I \cap V_i| = \gamma_i |I|$ for each $i \in [r]$. We let $\alpha_{\gamma}(H)$ denote the size of the largest γ -balanced independent set in H.

Our first result in this setting establishes the statistical threshold for $\alpha_{\gamma}(H)$ when $H \sim \mathcal{H}(r, n, p)$ (see §4.1 for the proof).

Theorem 1.8 (Statistical threshold for γ -balanced independent sets). Let $\varepsilon > 0$ and $r \in \mathbb{N}$ such that $r \geq 2$, and let $\gamma = (\gamma_1, \ldots, \gamma_r)$ be such that $\gamma_i \in (0,1)$ and $\sum_{i=1}^r \gamma_i = 1$. There exists $d^* > 0$ such that for any $d \geq d^*$, there exists $n^* > 0$ such that for any $n \geq n^*$ and $p = d/n^{r-1}$, the hypergraph $H \sim \mathcal{H}(r, n, p)$ satisfies

$$(1-\varepsilon)\left(\frac{\log d}{d\,r^{r-1}(r-1)\prod_{i}\gamma_{i}}\right)^{\frac{1}{r-1}} \leq \frac{\alpha_{\gamma}(H)}{rn} \leq (1+\varepsilon)\left(\frac{\log d}{d\,r^{r-1}(r-1)\prod_{i}\gamma_{i}}\right)^{\frac{1}{r-1}},$$

with probability $1 - \exp(-\Omega(n))$.

Note that setting $\gamma_i = 1/r$ for each $1 \le i \le r$ implies the maximum balanced independent set has density $(1 \pm \varepsilon) \left(\frac{r}{r-1} \cdot \frac{\log d}{d}\right)^{1/(r-1)}$ with high probability (as we claimed in §1). Finally, we prove the low-degree computational threshold for finding large γ -balanced independent sets.

Theorem 1.9 (Low-degree threshold for γ -balanced independent sets). Let $\varepsilon > 0$ and $r \in \mathbb{N}$ such that $r \geq 2$, and let $\gamma = (\gamma_1, \ldots, \gamma_r)$ be such that $\gamma_i \in (0,1)$ and $\sum_{i=1}^r \gamma_i = 1$. There exists $d^* > 0$ such that for any $d \geq d^*$, the following hold.

• There exist $n^* > 0$ and $\xi \ge 1$ such that for any $n \ge n^*$ and

$$k_j \le \gamma_j r n(1 - \varepsilon) \left(\frac{\log d}{d r^r(r-1) \prod_{i=1}^r \gamma_i} \right)^{1/(r-1)} \quad for \quad 1 \le j \le r,$$

there is a degree-1 polynomial that $(k_1, \ldots, k_r, o_n(1), \xi, 0)$ -optimizes the balanced independent set problem in $\mathcal{H}(r, n, p)$ for $p = d/n^{r-1}$.

• There exist η , C_1 , C_2 , $n^* > 0$ such that the following holds for any $n \ge n^*$, $\xi \ge 1$, $1 \le D \le \frac{C_1 n}{\xi \log n}$, and $\delta \le \exp(-C_2 \xi D \log n)$. If

$$k_j \ge \gamma_j r n(1+\varepsilon) \left(\frac{\log d}{d r^r(r-1) \prod_{i=1}^r \gamma_i}\right)^{1/(r-1)} \quad for \quad 1 \le j \le r,$$

then there is no random degree-D polynomial that $(k_1, \ldots, k_r, \delta, \xi, \eta)$ -optimizes the balanced independent set problem in $\mathcal{H}(r, n, p)$ for $p = d/n^{r-1}$.

1.4 Proof Overview

We will now provide an overview of our proof techniques. We will consider each model separately.

Independent Sets in $\mathcal{H}_r(n,p)$. Let us first discuss the achievability result (Theorem 1.5). As observed by Wein in [Wei22], sparse random graphs are locally tree-like. In particular, the sneighborhood of a vertex in $\mathcal{H}_2(n,d/n)$ can be well approximated by the Poisson Galton-Watson tree, a random graph model for generating trees. In [PZ21], Pal and Zhu generalized this model to the hypergraph setting, describing the r-uniform Poisson Galton-Watson Hypertree \mathbb{T}_d^{GW} (see §2.2 for a formal description). We observe that for $p = d/\binom{n-1}{r-1}$, the hypergraph $\mathcal{H}_r(n,p)$ is locally \mathbb{T}_d^{GW} -like, which implies that the performance of a local algorithm on $\mathcal{H}_r(n,p)$ is determined, up to the first order, by the expectation of the corresponding algorithm evaluated at the root of \mathbb{T}_d^{GW} . Our proof now follows in two steps: first, we describe a local algorithm for finding large independent sets in \mathbb{T}_d^{GW} (which can be adapted to one on $\mathcal{H}_r(n,p)$ by the aforementioned observation); next, we show how a local algorithm on $\mathcal{H}_r(n,p)$ can be well-approximated by a low-degree one, completing the proof. For the first step, we adapt a local algorithm of Nie and Verstraëte on the Δ -regular r-uniform hypertree T_Δ^r [NV21] to one on \mathbb{T}_d^{GW} for an appropriate $\Delta := \Delta(d) > 0$. The second step follows a similar approach to that of Wein on graphs.

The proof of our impossibility result (Theorem 1.6) falls into a line of work initiated by Gamarnik and Sudan [GJW20], who studied local algorithms for independent sets of $\mathcal{H}_2(n,p)$. The proof of their impossibility result relies on the so-called Overlap Gap Property (OGP). In particular, they show the following: if I_1 and I_2 are "large" independent sets, then either $I_1 \cap I_2$ is a "large" independent set or it is "small", i.e., either I_1 and I_2 contain a lot of common vertices or are nearly disjoint. They then show that if a local algorithm succeeds in finding a large independent set, it can be used to construct two independent sets violating the OGP condition. Rahman and Virág [RV17] improved upon their results by considering a more intricate "forbidden" structure involving many independent sets as opposed to just two. Their approach inspired further works in a number of different areas (see for example [GS14b; CHH17; Che+19]). For low-degree polynomial algorithms, Wein applied this "ensemble" variant of the OGP in the proof of his impossibility result for independent sets in $\mathcal{H}_2(n,p)$ [Wei22]. In our proof, we employ the ensemble-OGP in a similar fashion by considering a forbidden structure that involves many independent sets across many correlated random hypergraphs. We show that with high probability this structure does not exist,

and any low-degree algorithm that can construct a large independent set can also be used to construct an instance of this structure, leading to a contradiction.

Balanced Independent sets in $\mathcal{H}(r,n,p)$. Let us first discuss the statistical threshold (Theorem 1.8). The proof of the upper bound follows by a standard application of the first moment method. For the lower bound, our approach is inspired by that of Frieze in [Fri90] where he proves a high probability bound on the asymptotic size of the maximum independent set in $\mathcal{H}_2(n,p)$ (Krivelevich and Sudakov used a similar approach in [KS98] for $\mathcal{H}_r(n,p)$). Here, we fix a partition P of the vertex set and we say an independent set I is P-independent if I contains at most 1 vertex from each set in P. For an appropriate partition P, we show that there exists a γ -balanced P-independent set of the desired size with high probability. A similar approach was employed by Perkins and the second author in [PW24] where they consider the case when r = 2. For $r \geq 3$, however, certain combinatorial arguments no longer hold, requiring a more extensive analysis. The details are provided in §4.1.

The low-degree algorithm we describe for our achievability result in Theorem 1.9 is inspired by recent work of the first author who considered balanced independent sets of deterministic r-partite hypergraphs [Dha23a]. While their primary interest was on the existence of large balanced independent sets, the proof is constructive and yields a simple degree-1 algorithm. The goal is to adapt this algorithm to construct γ -balanced independent sets in $\mathcal{H}(r,n,p)$. We remark that for r=2, a similar approach was used by Perkins and the second author [PW24], where they describe an algorithm inspired by work of Chakraborty [Cha23]. In their setting, they note that it must be the case that $\gamma_i \leq 1/2$ for some $i \in \{1,2\}$. This allows them to apply Chakraborty's procedure directly. In order for us to apply the first author's procedure directly, we would require $\gamma_i \leq 1/r$ for at least r-1 indices $i \in [r]$ (which need not be the case). Therefore, certain modifications are necessary. We describe this modification in §4.2 describing a degree-1 polynomial algorithm in order to prove our achievability result.

The proof of our impossibility result follows a similar approach to that of Theorem 1.6. We once again employ the ensemble-OGP and are able to arrive at the desired contradiction. There is an additional layer of complexity due to the structural constraints on the hypergraph and on the independent sets considered (being r-partite and γ -balanced, respectively). Therefore, while the overall strategy of the proof follows identically to that of Theorem 1.6, the analysis differs greatly. For r = 2, Perkins and the second author were able to reduce their argument to the case when $\gamma_1 = \gamma_2 = 1/2$. This is due to a similar observation stated in the previous paragraph, i.e., one of the γ_i 's is at least 1/2. For $r \geq 3$, we would require at least r - 1 of the γ_i 's to be at least 1/r (which need not be the case). Therefore, we cannot apply a similar reduction. Furthermore, as in the case of the proof of Theorem 1.8, the combinatorial arguments are more complex for $r \geq 3$ and so the overall analysis is much more involved.

1.5 Concluding Remarks

In this work, we consider the computational hardness of finding large (balanced) independent sets in random hypergraphs. We focus on low-degree polynomials, a powerful class of algorithms considered to be a useful proxy for computationally efficient algorithms. For the Erdős–Rényi hypergraph $\mathcal{H}_r(n,p)$, we determine the threshold for success of such algorithms in constructing independent sets, recovering and generalizing results of Wein [Wei22]. For the Erdős–Rényi r-partite hypergraph $\mathcal{H}(r,n,p)$, we prove a matching threshold for balanced independent sets, generalizing results of Perkins and the second author [PW24]. In both problems, our results indicate a statistical-computational gap of a multiplicative factor of $r^{-1/(r-1)}$ and we conjecture this to be true.

A natural extension of our results would be to consider denser hypergraphs, i.e., larger values of the probability parameter p. For $\mathcal{H}_2(n,p)$ where $p = \Theta(1)$, Matula and Karp's results [Mat76; Kar76] indicate a statistical-computational gap of a multiplicative factor of 1/2 for independent sets. It is not clear how to describe Karp's algorithm as a low-degree polynomial and so the question of the low-degree threshold in this regime still remains open. Furthermore, to the best of our knowledge, hypergraph versions of these results have not yet appeared in the literature, which begs the following question: is there a statistical-computational gap of a multiplicative factor of $r^{-1/(r-1)}$ for independent sets in $\mathcal{H}_r(n,p)$ when $p = \Theta(1)$?

By now, OGP based methods have become standard in proving low-degree thresholds for optimization problems on graphs (see for example [Wei22; KM22; DGH23; GJW24; PW24]). Our work is the first to apply the low-degree framework and the OGP based proof strategy to optimization problems on random hypergraphs. A potential future line of inquiry would be to adapt this strategy to other optimization problems in this more general setting.

We conclude this section with an observation regarding Theorem 1.9. What is most surprising about our result is that the algorithm achieving the low-degree threshold is a simple degree-1 polynomial. While Conjecture 1 poses a long-standing challenge, the simplicity of our algorithm seems to indicate that finding large balanced independent sets may be easier (which would imply Conjecture 2 is false).

2 Preliminaries

2.1 Notation

Let \mathbb{N} denote the set of nonnegative integers. For $n \in \mathbb{N}$ such that $n \geq 1$, we let $[n] := \{1, \ldots, n\}$. Throughout this work, we consider $n \to \infty$ and use the asymptotic notation $O(\cdot)$, $O(\cdot)$, $o(\cdot)$, etc.

For any fixed $r \geq 2$, we let K_n^r denote the n-vertex complete r-uniform hypergraph and $K_{r \times n}$ denote the (rn)-vertex complete balanced r-uniform r-partite hypergraph. For a hypergraph H, we let V(H) denote its vertex set and E(H) denote its edge set. The degree of a vertex $v \in V(H)$ (denoted $\deg_H(v)$) is the number of edges containing v. We say H is d-regular if all vertices have degree d. If H is r-partite, we let $V_1(H), \ldots, V_r(H)$ denote each vertex partition. We say H is l-inear if for any $e, f \in E(H), |e \cap f| \leq 1$.

A path in a hypergraph H of length k is a sequence $(v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1})$ of distinct vertices and edges such that $v_i, v_{i+1} \in e_i$ for each $i \in [k]$. A cycle in a hypergraph H of length k is a path of length k with $v_{k+1} = v_1$. The girth of a hypergraph H is the length of the shortest cycle in H. We say a hypergraph H is connected if every pair of vertices $u, v \in V(H)$ can be connected by a path, and acyclic if it contains no cycles. A hyperforest is an acyclic graph and a hypertree is a connected hyperforest (as analogously defined for graphs). It is not too difficult to see that hyperforests are linear as well.

For a distribution \mathcal{P} , we let $\mathbb{E}_{\mathcal{P}}[\cdot]$ and $\mathsf{Var}_{\mathcal{P}}(\cdot)$ denote the expectation and variance, respectively. We drop the subscript when \mathcal{P} is clear. We let $\mathsf{Unif}[0,1]$ denote the uniform distribution over the closed interval [0,1], and $\mathsf{Unif}[0,1]^n$ denote the n-dimensional vector whose entries are i.i.d $\mathsf{Unif}[0,1]$. Let $\mathsf{Ber}(p)$ denote the Bernoulli distribution with parameter $p \in [0,1]$, and $\mathsf{Pois}(d)$ denote the Poisson distribution with parameter d > 0.

2.2 Local Algorithms on Hypergraphs

In this section, we define some terminology pertaining to local algorithms on hypergraphs, which will be important in the proof of Theorem 1.5. We will consider locally finite hypergraphs H, i.e. each vertex has a finite number of edges containing it.

For any integer $s \geq 0$ and $v \in V(H)$, let $N_s(H,v)$ denote the the rooted hypergraph with root v and its depth-s neighborhood in H, and let $|N_s(H,v)|$ denote the number of hyeredges in $N_s(H,v)$. An s-local algorithm for finding an independent set in a hypergraph H is defined by a measurable s-local function $g = g(H, v, \mathbf{X})$. Here, the input is a rooted hypergraph H with root v and depth at most s, and a vector $\mathbf{X} \in [0,1]^{|V(H)|}$. The output of the function is 0 ('out') or 1 ('in') and is invariant under labeled isomorphisms of $N_s(H,v)$. We apply an s-local function g to a hypergraph H by assigning each vertex v a label $\mathbf{X}_v \sim \mathsf{Unif}[0,1]$ independently, and then evaluating $g(\cdot)$ on $N_s(H,v)$ and the restriction of the labels \mathbf{X} to $V(N_s(H,v))$. For any s-local function g, hypergraph H, and $\mathbf{X} \in [0,1]^{|V(H)|}$, we require $\{v: g(H,v,\mathbf{X})=1\}$ is an independent set with probability 1. A local algorithm is an s-local function for some constant s.

We measure the performance of local algorithms for independent sets in random hypergraphs by the typical size or density of an independent set returned by the algorithm, with high probability over both the random graph and the random labels \mathbf{X} .

As mentioned in §1.4, we will consider two kinds of hypertrees in our proof of Theorem 1.5. First, we describe the r-uniform Poisson(d) Galton-Watson Hypertree (denoted \mathbb{T}_d^{GW}) introduced by Pal and Zhu in [PZ21]. The process generates a rooted r-uniform hypertree (T, o) as follows:

- start with a root vertex o at level 0.
- For $k = 0, 1, 2, \ldots$, each vertex at level k independently spawns $\mathsf{Pois}(d)$ offspring edges that pairwise intersect only at the parent vertex. Note that for each vertex at level k, we have $(r-1)\mathsf{Pois}(d)$ children vertices at level k+1.

As sparse random hypergraphs are locally hypertree-like, the performance of a local algorithm

on $\mathcal{H}_r(n,p)$ is determined, up to the first order, by the expectation of the corresponding local function g evaluated at the root of \mathbb{T}_d^{GW} . More precisely, we have the following lemma:

Lemma 2.1. Suppose g is an s-local function for independent sets. Let I be the random independent set obtained by applying g to $H \sim \mathcal{H}_r(n,p)$. Then for $T \sim \mathbb{T}_d^{GW}$ with high probability over the randomness in H and the random label $\mathbf{X} \sim \mathsf{Unif}[0,1]^n$,

$$\frac{|I|}{n} = \mathbb{E}[g(T, o, \mathbf{X})] + o(1).$$

Proof. The expectation of |I| is $\mathbb{E}[g(T, o, \mathbf{X})]n + o(n)$. Furthermore, the variance of |I| is O(n) as there are at most O(n) pairs of vertices of u, v such that $g(H, u, \mathbf{X})$ and $g(H, v, \mathbf{X})$ are not independent. The lemma follows by Chebyshev's Inequality.

To assist with our proofs, we define

$$\alpha_r(d) := \left(\frac{d}{\log d}\right)^{\frac{1}{r-1}} \sup \left\{ \mathbb{E}[g(T, o, \mathbf{X})] : s \ge 0, g \text{ is an } s\text{-local function} \right\}. \tag{1}$$

By Lemma 2.1, $\alpha_r(d) \left(\frac{\log d}{d}\right)^{\frac{1}{r-1}}$ is the optimal density (up to the first order) of an independent set that a local algorithm can find in $\mathcal{H}_r(n,p)$ with high probability.

The second hypertree we consider is the the rooted r-uniform Δ -regular hypertree (denoted T_{Δ}^{r}) with root o. Observe that when applying an s-local algorithm h to T_{Δ}^{r} with vertex labels \mathbf{X} , the distribution of $h(T_{\Delta}^{r}, v, \mathbf{X})$ does not depend on the vertex v. Hence we can define the density of the independent set I obtained by applying h to T_{Δ}^{r} as $density(I) = \mathbb{P}[h(T_{\Delta}^{r}, o, \mathbf{X}) = 1] = \mathbb{E}[h(T_{\Delta}^{r}, o, \mathbf{X})]$. Therefore, we can define the following parameter similar to (1):

$$\alpha(r, \Delta) := \left(\frac{\Delta}{\log \Delta}\right)^{\frac{1}{r-1}} \sup \left\{ \mathbb{E}[h(T_{\Delta}^r, o, \mathbf{X})] : s \ge 0, h \text{ is an } s\text{-local function} \right\}. \tag{2}$$

2.3 Interpolating Paths and Stable Algorithms

We write $A \sim \mathcal{H}_r(n,p)$ (resp. $A \sim \mathcal{H}(r,n,p)$) to denote a random vector $A \in \{0,1\}^m$ for $m = \binom{n}{r}$ (resp. $m = n^r$), with i.i.d $\mathsf{Ber}(p)$ entries. As mentioned in §1.4, the proof of our impossibility results will involve analyzing a sequence of correlated hypergraphs, which we will refer to as interpolation paths. Let us define these for each of the random hypergraph models we consider.

Definition 2.2 (Interpolation Paths for $\mathcal{H}_r(n,p)$ and $\mathcal{H}(r,n,p)$). Let $T \in \mathbb{N}$ and let $m = \binom{n}{r}$ (resp. $m = n^r$). We construct a length T interpolation path $A^{(0)}, \ldots, A^{(T)}$ as follows: $A^{(0)} \sim \mathcal{H}_r(n,p)$ (resp. $\mathcal{H}(r,n,p)$); then for each $1 \leq t \leq T$, $A^{(t)}$ is obtained by resampling the coordinate $\sigma(t) \in [m]$ from $Ber\left(d/\binom{n-1}{r-1}\right)$ (resp. $Ber\left(d/n^{r-1}\right)$). Here $\sigma(t) = t - k_t m$ where k_t is the unique integer making $1 \leq \sigma(t) \leq m$.

Consider the hypercube Q_m with vertex set $\{0,1\}^m$ and edges of the form uv where u and v differ in exactly one coordinate. Interpolation paths can therefore be viewed as random walks on

 Q_m (for the appropriate value of m). We define (D, Γ, c) -stable algorithms with respect to this random walk (the distribution \mathcal{P} below is either $\mathcal{H}_r(n, p)$ or $\mathcal{H}(r, n, p)$).

Definition 2.3. Let $f:\{0,1\}^m \to \mathbb{R}^n$ and c>0. An edge uv of the hypercube Q_m is c-bad for f if

$$||f(u) - f(v)||^2 \ge c \cdot \mathbb{E}_{A \sim \mathcal{P}} [||f(A)||^2]$$

Definition 2.4. Let $\Gamma \in \mathbb{N}$ and c > 0. Let $A^{(0)}, \ldots, A^{(T)}$ be the interpolation path from Definition 2.2 of length $T = \Gamma m$, with appropriate $p \leq 1/2$. A degree-D polynomial $f : \{0,1\}^m \to \mathbb{R}^n$ is (D,Γ,c) -stable if

 $\mathbb{P}\left[\text{no edge of }A^{(0)},\ldots,A^{(T)}\text{ is c-bad for }f\right]\geq p^{4\Gamma D/c}.$

The following lemma will be crucial to proving our impossibility results. The proof follows an identical argument to that of [Wei22, Lemmas 2.7 and 2.8] on graphs and so we omit it here.

Lemma 2.5. For any constants $\Gamma \in \mathbb{N}$ and c > 0, any degree-D polynomial f on $A \sim \mathcal{H}_r(n, p)$ or $A \sim \mathcal{H}(r, n, p)$ is (D, Γ, c) -stable.

3 Independent Sets in $\mathcal{H}_r(n,p)$

In this section, we will consider the random r-uniform hypergraph $\mathcal{H}_r(n,p)$ for $p = d/\binom{n-1}{r-1}$ (see Definition 1.4). We split this section into three subsections. First, we prove the achievability result (Theorem 1.5). In the second subsection, we prove a version of the *Overlap Gap Property* for independent sets in $\mathcal{H}_r(n,p)$. In the final subsection, we apply this result to prove intractability of low-degree algorithms for independent sets in the stated regime.

3.1 Proof of the Achievability Result

In this section we prove Theorem 1.5, which shows that there exist low-degree algorithms that can find independent sets of density $(1-\varepsilon)\left(\frac{1}{r-1}\cdot\frac{\log d}{d}\right)^{\frac{1}{r-1}}$ in $\mathcal{H}_r(n,p)$ for $p=d/\binom{n-1}{r-1}$. We further split this subsection into two parts. First, we show how local algorithms for finding independent sets in T_{Δ}^r can be adapted to find independent sets in \mathbb{T}_d^{GW} . Next, we show how to describe a low-degree algorithm for finding independent sets in $\mathcal{H}_r(n,p)$ from a local algorithm for finding independent sets in \mathbb{T}_d^{GW} .

3.1.1 Local Algorithms: from T_{Δ}^{r} to \mathbb{T}_{d}^{GW}

In this section, we prove that there exists a local algorithm that outputs an independent set of the desired density on \mathbb{T}_d^{GW} , which is crucial for the construction of our low-degree polynomial in the sequel. First, we prove the following theorem.

Theorem 3.1. For any $\varepsilon > 0$ and any sufficiently large $d = d(\varepsilon) > 0$, there exists $s := s(\varepsilon, d)$ and an s-local function g such that for $(T, o) \sim \mathbb{T}_d^{GW}$ and i.i.d $\mathsf{Unif}[0, 1]$ labels \mathbf{X} over its vertices, $\mathbb{E}[g(T, o, \mathbf{X})] \geq (1 - \varepsilon) \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}}$.

We start by describing the following random greedy algorithm analyzed in [NV21], which generates an independent set in an r-uniform Δ -regular hypergraph H with large girth (which implies the case of T_{Δ}^{r}):

- Equip each vertex v with i.i.d. labels $\mathbf{X}_v \sim \mathsf{Unif}[0,1]$.
- Iteratively select the vertex with the largest label of the remaining vertices of H, and add it to the independent set. Next, remove any remaining vertices that form an edge with the selected vertices, and repeat until no vertices remain.

We remark that while this algorithm as described is not local, it can be simulated by one that is (this can be inferred by the discussion in [NV21, §2]). Nie and Verstraëte [NV21, Theorem 4] showed that the random greedy algorithm outputs an independent set of density $(1 - \varepsilon) \left(\frac{1}{r-1} \cdot \frac{\log \Delta}{\Delta}\right)^{1/r-1}$ on T_{Δ}^{r} .

Following an idea of Rahman and Virág in [RV17], we will show that a local algorithm finding an independent set in T_{Δ}^r can be adapted to construct an independent set in \mathbb{T}_d^{GW} as well. We remark that the parameter d we choose will not be equal to Δ , however, it will be chosen such that $\log d/d = (1 \pm o_d(1)) \log \Delta/\Delta$. In particular, the independent set output by our local algorithm on \mathbb{T}_d^{GW} will have the same density as the one in T_{Δ}^r for d sufficiently large. Let $E(d, \Delta)$ denote the event that the root of \mathbb{T}_d^{GW} and all of its neighbors have degree at most Δ . With this definition in hand, we prove the following proposition:

Proposition 3.2. Given a local algorithm h to find an independent set in T_{Δ}^r , there exists a local algorithm g satisfying the following:

$$\mathbb{E}_{\mathbf{Y}}[h(T_{\Delta}^r, o, \mathbf{Y})] \, \mathbb{P}[E(d, \Delta)] \, \leq \, \mathbb{E}[g(T, o, \mathbf{X})] \, \leq \, \mathbb{E}_{\mathbf{Y}}[h(T_{\Delta}^r, o, \mathbf{Y})].$$

for any $T \sim \mathbb{T}_d^{GW}$ and i.i.d $\mathsf{Unif}[0,1]$ vertex labels $\mathbf{X},\,\mathbf{Y}.$

Proof. Let I be an independent set generated by h on T_{Δ}^{r} and some i.i.d $\mathsf{Unif}[0,1]$ vertex labels, and let $T \sim \mathbb{T}_{d}^{GW}$. The local algorithm g, that can generate the independent set J of the desired density in T, is described explicitly through three steps.

Step 1: Edge removal. We remove edges from T to define a hyperforest S where all vertices have degree at most Δ . Begin with a random labelling \mathbf{X} of the vertices of T. Consider a vertex v satisfying $\deg_T(v) > \Delta$. For each edge $e \ni v$, let $X_e := \max\{\mathbf{X}_u : u \in e \setminus \{v\}\}$. Order the edges containing v by X_e and remove the edges with the $\deg_T(v) - \Delta$ largest values (break ties arbitrarily). It is not difficult to see that S is a disjoint collection of countably many r-uniform hypertrees.

Step 2: Regularization. We describe how to construct a Δ -regular hyperforest T' from S such that $S \subseteq T'$. Define the $(\Delta - 1)$ -ary hypertree to be the rooted hypertree in which every vertex has exactly $\Delta - 1$ offspring edges. Consider r - 1 such $(\Delta - 1)$ -ary hypertrees with roots

 $u_1, u_2, \ldots, u_{r-1}$ and create an edge $\{v, u_1, \ldots, u_{r-1}\}$. Repeat this process $\Delta - \deg_S(v)$ many times for each $v \in V(S)$. Randomly label T' by a new labeling \mathbf{X}' independent of \mathbf{X} .

Step 3: Inclusion. Since T' is a disjoint collection of r-uniform Δ -regular hypertrees, we can use h with input \mathbf{X}' to construct an independent set I' of T' with the same density I. However, due to the removal of edges from T, I' may no longer be independent in T. We construct the desired set J from I' by including all vertices $v \in I'$ such that no edges containing v were removed during Step 1.

By design, h is a local algorithm on \mathbb{T}_d^{GW} . For $e \in E(T)$, if $e \subseteq J$, then $e \subseteq I'$ as well. Since I' is an independent set in I', e must have been removed during Step 1. By construction, none of the vertices in e would have been added to J. Therefore, J is indeed an independent set in T.

Let us now prove the desired bounds on the density of J. Clearly, $\mathsf{density}(J) \leq \mathsf{density}(I')$ since $J \subseteq I'$. Furthermore, for any $v \in I'$, if v and all of it's neighbors in T have degree at most Δ then none of the edges containing v will be removed, which implies that $v \in J$. Hence the lower bound follows by the previous observation.

Lemma 3.3. If
$$d = \Delta - \Delta^u$$
 for any $1/2 < u < 1$ then $\mathbb{P}[E(d, \Delta)] \to 1$ as $\Delta \to \infty$.

Proof. Let X denote the degree of the root of a \mathbb{T}_d^{GW} Tree. Let $\{Z_{i,j}\}_{i\in[X],j\in[r-1]}$ denote the number of offspring edges adjacent to the j-th vertex in the i-th offspring edge of the root. Recall that X has distribution $\mathsf{Pois}(d)$ and that conditioned on X all $Z_{i,j}$ are i.i.d with distribution $\mathsf{Pois}(d)$. Let $p(d,\Delta) := \mathbb{P}[\mathsf{Pois}(d) > \Delta]$. We have

$$\mathbb{P}[E(d,\Delta)] = \mathbb{E}\left[\mathbbm{1}_{X \leq \Delta} \cdot \prod_{i=1}^{X} \prod_{j=1}^{r-1} \mathbbm{1}_{Z_{i,j} \leq \Delta - 1}\right]$$

$$= \mathbb{E}\left[\mathbbm{1}_{X \leq \Delta} \cdot \mathbb{E}\left[\prod_{i=1}^{X} \prod_{j=1}^{r-1} \mathbbm{1}_{Z_{i,j} \leq \Delta - 1} \middle| X\right]\right]$$

$$= \mathbb{E}\left[\mathbbm{1}_{X \leq \Delta} \cdot \left(1 - p(d,\Delta - 1)\right)^{(r-1)X}\right]$$

$$= \mathbb{E}\left[\left(1 - p(d,\Delta - 1)\right)^{(r-1)X}\right] - \mathbb{E}\left[\mathbbm{1}_{X > \Delta} \cdot \left(1 - p(d,\Delta - 1)\right)^{(r-1)X}\right].$$

Using the moment generating function of a Pois(d) random variable we have

$$\mathbb{E}\left[\left(1 - p(d, \Delta - 1)\right)^{(r-1)X}\right] = \mathbb{E}\left[\exp\left((r - 1)X\log\left(1 - p(d, \Delta - 1)\right)\right)\right]$$
$$= \exp\left(d\left(e^{(r-1)\log\left(1 - p(d, \Delta - 1)\right)} - 1\right)\right)$$
$$= \exp\left(d\left(\left(1 - p(d, \Delta - 1)\right)^{(r-1)} - 1\right)\right)$$
$$\geq \exp\left(d\left(1 - (r - 1)p(d, \Delta - 1) - 1\right)\right)$$

$$= \exp\left(-d(r-1)p(d, \Delta - 1)\right)$$

Additionally,

$$\mathbb{E}\left[\mathbb{1}_{X>\Delta}\cdot\left(1-p(d,\Delta-1)\right)^{(r-1)X}\right] = \sum_{x>\Delta}\left(1-p(d,\Delta-1)\right)^{(r-1)x}\cdot\mathbb{P}[X=x]$$

$$\leq \left(1-p(d,\Delta-1)\right)^{\Delta(r-1)}\sum_{x>\Delta}\mathbb{P}[X=x]$$

$$= \left(1-p(d,\Delta-1)\right)^{\Delta(r-1)}\cdot p(d,\Delta)$$

$$\leq p(d,\Delta) \leq p(d,\Delta-1)$$

Putting the above together, we have

$$\mathbb{P}[E(d,\Delta)] \ge \exp\left(-d(r-1)p(d,\Delta-1)\right) - p(d,\Delta-1).$$

From here, we may complete the proof by an exponential moment argument to bound $p(d, \Delta - 1)$. The proof follows identically to that in [RV17, Lemma 4.6] and so we omit the details.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Given d, let $\Delta := \lceil d + d^{3/4} \rceil$. Recall the definitions of $\alpha_r(d)$ and $\alpha(r, \Delta)$ in (1) and (2), respectively. By the conclusion of Proposition 3.2, we have

$$\alpha_r(d) \left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} \ge \alpha(r, \Delta) \left(\frac{\log \Delta}{\Delta}\right)^{\frac{1}{r-1}} \cdot \mathbb{P}[E(d, \Delta)].$$

The random greedy algorithm discussed at the beginning of this section implies that

$$\liminf_{\Delta \to \infty} \alpha(r, \Delta) \ge \left(\frac{1}{r - 1}\right)^{\frac{1}{r - 1}}.$$

By the choice of Δ (as a function of d), we have $(\log \Delta/\Delta)/(\log d/d) \to 1$ as $d \to \infty$. Combining the above with Lemma 3.3, we may conclude

$$\liminf_{d \to \infty} \alpha_r(d) \ge \left(\frac{1}{r-1}\right)^{\frac{1}{r-1}},$$

which completes the proof.

3.1.2 From Local to Low-Degree

Before we prove the main result (Theorem 1.5), we state two useful lemmas.

Let \mathcal{T} be a set of rooted r-uniform hypertrees consisting of one representative from each isomorphism class of rooted hypertrees of depth at most 2s, and let $\mathcal{T}_q \subseteq \mathcal{T}$ contain only those hypertrees

with at most q edges. For $H \sim \mathcal{H}_r\left(n, d/\binom{n-1}{r-1}\right)$ and for $T \in \mathcal{T}$, let n_T denote the number of occurrences of T in a local neighborhood of H, that is

$$n_T = |\{v \in [n] : N_{2s}(H, v) \cong T\}|.$$

(Here, \cong denotes the isomorphism of rooted hypergraphs). The following lemma shows that n_T is well-concentrated.

Lemma 3.4. There is a constant c := c(r) > 0 such that for any $t \ge (2e)^{3/2} c \sqrt{n} (2rd)^{2s}$, we have

$$\mathbb{P}\left[\left|n_T - \mathbb{E}[n_T]\right| \ge t\right] \le \exp\left(-\frac{3t^{2/3}}{2ec^{2/3}n^{1/3}(2rd)^{2s/3}}\right).$$

Proof. The proof follows by combining Markov's inequality with a hypergraph version of Proposition 12.3 [Bar+19] (for which the proof is identical, $mutatis\ mutandis$).

Let p_T denote the probability that T appears as the 2s-neighborhood of a rooted \mathbb{T}_d^{GW} , i.e.,

$$p_T = \mathbb{P}_{(W,o) \sim \mathbb{T}_d^{GW}}[N_{2s}(W,o) \cong T].$$

The following lemma shows that the local neighborhood of $\mathcal{H}_r(n,p)$ converges to \mathbb{T}_d^{GW} .

Lemma 3.5. There is a constant c := c(r) > 0 such that the following holds for n sufficient large:

$$\left| \mathbb{E}[n_T] - p_T \cdot n \right| \le c n^{3/4} \log n.$$

Proof. The proof follows by a hypergraph version of [Bar+19, Lemma 12.4] (for which the proof is identical, *mutatis mutandis*), and linearity of expectation.

Corollary 3.6. For any $\tau > 0$ and n sufficiently large in terms of d, s, r, and τ , there is a constant $C := C(d, s, r, \tau) > 0$ such that

$$\mathbb{P}\left[\left|n_T - p_T \cdot n\right| \ge \tau n\right] \le \exp(-Cn^{1/3}).$$

Proof. Follows directly by combining Lemmas 3.4 and 3.5.

Before we prove the main result, we make an analogous observation to the one in [Wei22, Lemma 2.11] on graphs. In particular, a random polynomial can be converted to a deterministic polynomial that works almost as well.

Lemma 3.7. Suppose f is a random degree-D polynomial that (k, δ, ξ, η) -optimizes the independent set problem in $\mathcal{H}_r(n,p)$. Then for any c > 2, there exists a deterministic degree-D polynomial that $(k, c\delta, c\xi, \eta)$ -optimizes the independent set problem in $\mathcal{H}_r(n,p)$.

Proof. By Definition 1.2, we have

$$\mathbb{E}_{A,\omega}[\|f(A,\omega)\|^2] \le \xi k$$
, and $\mathbb{P}_{A,\omega}[|V_f^{\eta}(A,\omega)| < k] \le \delta$.

By Markov's inequality,

$$\mathbb{P}_{\omega}\left[\mathbb{E}_{A}[\|f(A,\omega)\|^{2}] \geq c\xi k\right] \leq \frac{1}{c} < \frac{1}{2}, \quad \text{and} \quad \mathbb{P}_{\omega}\left[\mathbb{P}_{A}[|V_{f}^{\eta}(A,\omega)| < k] \geq c\delta\right] \leq \frac{1}{c} < \frac{1}{2}.$$

Thus, there exists a $\omega^* \in \Omega$ such that the deterministic degree-D polynomial $f^*(\cdot) = f(\cdot, \omega)$ satisfies

$$\mathbb{E}_A[\|f(A)\|^2] \le c\xi k$$
 and $\mathbb{P}_A[|V_f^{\eta}(A)| < k] \le c\delta$

as desired.
$$\Box$$

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Given $\varepsilon > 0$, let d and s be sufficiently large in order to apply Theorem 3.1. As a result, we obtain an s-local function g that outputs independent sets of density

$$\mathbb{E}[g(T, o, \mathbf{X})] \ge (1 - \varepsilon/5) \left(\frac{1}{r - 1} \frac{\log d}{d}\right)^{\frac{1}{r - 1}},$$

when $(T,o) \sim \mathbb{T}_d^{GW}$ and \mathbf{X} encodes i.i.d Unif[0,1] labels on the vertices of T. By Lemma 3.7, it is sufficient to prove the results for a random polynomial instead of deterministic one (up to a change in the constants ξ , C). We will construct a random polynomial $f: \mathbb{R}^{\binom{n}{r}} \to \mathbb{R}^n$ where the input A to f represents a hypergraph on vertex set [n], and the internal randomness of f samples vertex labels \mathbf{X} i.i.d from Unif[0,1]. Let $\mathbb{H}_{v,s,q}$ be the collection of hypergraphs H on vertex set [n] for which $|E(H)| \leq q$ and every non-isolated vertex is reachable from v by a path of length at most s. In other words, $\mathbb{H}_{v,s,q}$ consists of all possible s-neighborhoods of v of size at most q. Let

$$f_v(A, \mathbf{X}) = \sum_{H \in \mathbb{H}_{v,s,q}} \alpha(H, v, \mathbf{X}) \prod_{e \in E(H)} A_e,$$
(3)

where $\alpha(H, v, \mathbf{X})$ is defined recursively by

$$\alpha(H, v, \mathbf{X}) = g(H, v, \mathbf{X}) - \sum_{\substack{H' \in \mathbb{H}_{v, s, q} \\ E(H') \subseteq E(H)}} \alpha(H', v, \mathbf{X}). \tag{4}$$

Observe that by this construction, we have the following for any $v \in [n]$: if $N_s(A, v)$ is a hypertree with $|N_s(A, v)| \leq q$, then $f_v(A, \mathbf{X}) = g(H, v, \mathbf{X})$, where $q = q(\varepsilon, d, \eta)$ will be chosen later.

We first show that, under this construction, the rounding procedure $V_f^{\eta}(A, \mathbf{X})$ will not fail. Fix any vertex v, if $N_{2s}(A, v)$ is a hypertree with $|N_{2s}(A, v)| \leq q$, then for all $u \in V(N_1(A, v))$, $N_s(A, u)$ is a hypertree with with $|N_s(A, u)| \leq q$. Hence $f_u(A, \mathbf{X}) = g(A, u, \mathbf{X})$ by construction. Since the

output of g is an independent set, v will not be in the "bad" set $(I \setminus \tilde{I}) \cup J$ (see Definition 1.1). Therefore, $(I \setminus \tilde{I}) \cup J$ is disjoint from the following set:

$$V_q := \bigcup_{T \in \mathcal{T}_q} \{ v \in [n] : N_{2s}(A, v) \cong T \}.$$

By Corollary 3.6, we have that $|n_T - p_T \cdot n| \le \eta n/(2|\mathcal{T}_q|)$ with probability $1 - \exp(-\Omega(n^{1/3}))$, where $\Omega(\cdot)$ hides a constant depending on $\varepsilon, d, r, s, q, \eta$. Choose q sufficiently large such that

$$\sum_{T \in \mathcal{T}_q} p_T \ge 1 - \eta/2.$$

Then we have

$$|I \setminus \tilde{I}| + |J| = |(I \setminus \tilde{I}) \cup J| \le n - |V_q| \le n - \sum_{T \in T_q} \left(p_T \cdot n - \frac{\eta n}{2|\mathcal{T}_q|} \right) = \left(1 - \sum_{T \in \mathcal{T}_q} p_T \right) + \frac{\eta n}{2} \le \eta n,$$

which concludes that the rounding procedure succeeds with probability at least $1 - \exp(-\Omega(n^{1/3}))$.

Next we show that the independent set $I_f := V_f^{\eta}(A, \mathbf{X})$ has the desired density with high probability. By our choice of the local function g, we have

$$\left(1 - \frac{\varepsilon}{5}\right) \left(\frac{1}{r - 1} \cdot \frac{\log d}{d}\right)^{1/r - 1} \le \mathbb{E}_{(T, o) \sim \mathbb{T}_d^{GW}}[g(T, o, \mathbf{X})] = \sum_{T \in \mathcal{T}} p_T \phi_T,$$

where ϕ_T is defined as the probability over **X** that $g(A, v, \mathbf{X}) = 1$ conditioned on the event $N_{2s}(A, v) \cong T$. Let q be sufficiently large such that

$$\sum_{T \in \mathcal{T}_q} p_T \ge 1 - \frac{\varepsilon}{5} \left(\frac{1}{r-1} \cdot \frac{\log d}{d} \right)^{1/r-1}.$$

Given that $\phi_T \in [0,1]$ for any T, we have

$$\frac{\varepsilon}{5} \left(\frac{1}{r-1} \cdot \frac{\log d}{d} \right)^{1/r-1} \ge 1 - \sum_{T \in \mathcal{T}_q} p_T = \sum_{T \in \mathcal{T}} p_T - \sum_{T \in \mathcal{T}_q} p_T$$

$$\ge \sum_{T \in \mathcal{T}} p_T \phi_T - \sum_{T \in \mathcal{T}_q} p_T \phi_T$$

$$\ge \left(1 - \frac{\varepsilon}{5} \right) \left(\frac{1}{r-1} \cdot \frac{\log d}{d} \right)^{1/r-1} - \sum_{T \in \mathcal{T}_q} p_T \phi_T,$$

which implies

$$\sum_{T \in \mathcal{T}_q} p_T \phi_T \ge \left(1 - \frac{2\varepsilon}{5}\right) \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{1/r-1}.$$

Using Corollary 3.6 once again, with probability at least $1 - \exp(-\Omega(n^{1/3}))$ over $A \sim \mathcal{H}_r(n, p)$, we have

$$\sum_{T \in \mathcal{T}_q} n_T \phi_T \ge \sum_{T \in \mathcal{T}_q} \left(p_T n - \frac{\varepsilon}{5|\mathcal{T}_q|} \left(\frac{1}{r-1} \cdot \frac{\log d}{d} \right)^{1/r-1} n \right) \phi_T$$

$$= \left(\sum_{T \in \mathcal{T}_q} p_T \phi_T \right) n - \frac{\varepsilon}{5} \left(\frac{1}{r-1} \cdot \frac{\log d}{d} \right)^{1/r-1} n$$

$$\ge n \left(1 - \frac{3\varepsilon}{5} \right) \left(\frac{1}{r-1} \cdot \frac{\log d}{d} \right)^{1/r-1}.$$

Now fix A satisfying the inequality above and consider the randomness over the vertex label **X**. As we showed above, V_q is disjoint from the bad set $(I \setminus \tilde{I}) \cup J$. Therefore, if $N_{2s}(A, v) \cong T$ for some $v \in V(H)$ and $T \in \mathcal{T}_q$, then $v \in I_f$ if and only if $g(A, v, \mathbf{X}) = 1$, which happens with probability $\phi_v = \phi_T$.

The idea now is to partition V_q into subsets $U_1, \ldots, U_{(r-1)q+1}$ such that for each U_i , the vertices in U_i have disjoint s-neighborhoods. In particular, the random variables $\{\mathbbm{1}_{v\in I_f}\}_{v\in U_i}$ are independent conditioned on the outcome A. Each vertex $v\in V_q$ has at most (r-1)q vertices in $N_{2s}(A,v)$, which implies there are at most (r-1)q vertices $u\in V_q$ such that $u\neq v$ and $N_s(A,v)\cap N_s(A,u)\neq\varnothing$. We can greedily form the partition $\{U_i\}$ while maintaining the disjointness property (given that there are (r-1)q+1 many such subsets available). For such a partition $\{U_i\}$, we apply Chernoff's bound for each i to obtain the following:

$$\mathbb{P}_{\mathbf{X}}\left[\sum_{v \in U_i} \mathbb{1}_{v \in I_f} \le \left(1 - \frac{\varepsilon}{5}\right) \mu_i\right] \le \exp\left(-\frac{\varepsilon^2}{50} \mu_i\right) \tag{5}$$

where

$$\mu_i = \mathbb{E}_{\mathbf{X}} \left[\sum_{v \in U_i} \mathbb{1}_{v \in I_f} \right] = \sum_{v \in U_i} \phi_v.$$

We say a subset U_i is "large" if the corresponding value μ_i satisfies

$$\mu_i \ge \frac{\varepsilon}{5((r-1)q+1)} \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{1/r-1} n,$$

and "small" otherwise. By (5) and a union bound over i, we have that every large subset U_i satisfies $\sum_{v \in U_i} \mathbb{1}_{v \in I_f} \ge \left(1 - \frac{\varepsilon}{5}\right) \mu_i$ with probability $1 - \exp(-\Omega(n))$. With this in hand, we have

$$\begin{split} |I_f| &\geq \sum_{v \in V_q} \mathbbm{1}_{v \in I_f} \geq \sum_{i: U_i \text{ large}} \sum_{v \in U_i} \mathbbm{1}_{v \in I_f} \geq \sum_{i: U_i \text{ large}} \left(1 - \frac{\varepsilon}{5}\right) \mu_i \\ &= \left(1 - \frac{\varepsilon}{5}\right) \left[\sum_i \mu_i - \sum_{i: U_i \text{ small}} \mu_i\right] \end{split}$$

$$\geq \left(1 - \frac{\varepsilon}{5}\right) \left[\sum_{i} \mu_{i} - \sum_{i:U_{i} \text{ small}} \frac{\varepsilon}{5((r-1)q+1)} \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}} n \right]$$

$$\geq \left(1 - \frac{\varepsilon}{5}\right) \left[\sum_{i} \mu_{i} - \frac{\varepsilon}{5} \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}} n \right]$$

$$= \left(1 - \frac{\varepsilon}{5}\right) \left[\sum_{v \in V_{q}} \phi_{v} - \frac{\varepsilon}{5} \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}} n \right]$$

$$= \left(1 - \frac{\varepsilon}{5}\right) \left[\sum_{T \in \mathcal{T}_{q}} n_{T} \phi_{T} - \frac{\varepsilon}{5} \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}} n \right]$$

$$\geq \left(1 - \frac{\varepsilon}{5}\right) \left[\left(1 - \frac{3\varepsilon}{5}\right) \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}} n - \frac{\varepsilon}{5} \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}} n \right]$$

$$= \left(1 - \frac{\varepsilon}{5}\right) \left(1 - \frac{4\varepsilon}{5}\right) \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}} n$$

$$\geq (1 - \varepsilon) \left(\frac{1}{r-1} \frac{\log d}{d}\right)^{\frac{1}{r-1}} n.$$

Thus, the random polynomial f outputs an independent set of density at least $(1 - \varepsilon) \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}}$ with probability $1 - \exp(-\Omega(n^{1/3}))$ over both A and X.

Finally, we need to verify the normalization condition in Definition 1.2:

$$\mathbb{E}_{A,\mathbf{X}}[\|f(A,\mathbf{X})\|^2] \le \xi (1-\varepsilon) \left(\frac{1}{r-1} \frac{\log d}{d}\right)^{\frac{1}{r-1}} n, \quad \text{for some constant} \quad \xi \ge 1.$$

It suffices to show that $\mathbb{E}_{A,\mathbf{X}}[f_v(A,\mathbf{X})^2] = O(1)$ for all vertices v. Fix a vertex v and let $N = |N_s(A,v)|$. By (3), for each $H \in \mathbb{H}_{v,s,q}$ the corresponding term is not 0 only if H is a subgraph of $N_s(A,v)$. This implies that number of of nonzero terms is at most

$$\binom{N}{\leq q} = \sum_{i=0}^{q} \binom{N}{i} \leq \sum_{i=1}^{q} N^i \leq (N+1)^q.$$

Observe that by (4), $|\alpha(H, v, \mathbf{X})| \leq \chi$ for some $\chi = \chi(r, s, q)$ independent of v and \mathbf{X} . Therefore, we may conclude that

$$f_v(A, \mathbf{X})^2 \le [\chi(N+1)^q]^2 = \chi^2(N+1)^{2q}.$$
 (6)

In order to bound the $\mathbb{E}_{A,\mathbf{X}}[(f_v(A,\mathbf{X})^2]]$, it is sufficient to prove a tail bound for N. Starting from $m_0 = 1$, let m_i be the number of vertices whose distance from v in A is exactly i. Conditioned on m_i , m_{i+1} is stochastically dominated by $\text{Bin}\left(r\binom{n-1}{r-1}m_i,d/\binom{n-1}{r-1}\right)$. By a Chernoff bound, for fixed

 m_i and any $\delta \geq 1$, we have

$$\mathbb{P}[m_{i+1} \ge (1+\delta)rdm_i)] \le \exp\left(-\frac{\delta rdm_i}{3}\right) \le \exp\left(-\frac{\delta rd}{3}\right).$$

Taking a union bound over all $i \in [s]$,

$$\mathbb{P}\left[m_i < \left[(1+\delta)rd\right]^i, \forall i \in [s]\right] \ge 1 - s \cdot \exp\left(-\frac{\delta rd}{3}\right).$$

In particular, we conclude

$$N < \sum_{i=1}^{s} [(1+\delta)rd]^{i} \le [(1+\delta)rd + 1]^{s} \le (2\delta rd + 1)^{s} \le (3\delta rd)^{s}.$$

Alternatively,

$$\mathbb{P}[N \geq (3\delta rd)^s] \leq s \cdot \exp\left(-\frac{\delta rd}{3}\right), \quad \forall \delta \geq 1.$$

For $\delta' = (3\delta rd)^s$, we have an exponentially small tail bound on N:

$$\mathbb{P}[N \ge t] \le s \cdot \exp\left(-\frac{t^{1/s}}{9}\right), \quad \forall t \ge (3rd)^s.$$

Putting everything together, we have

$$\mathbb{E}_{A,\mathbf{X}}[(f_v(A,\mathbf{X})^2] \le \sum_{i=0}^{\infty} \chi^2(t+1)^{2q} \mathbb{P}[N=t]$$

$$\le \sum_{i=0}^{\lceil (3rd)^s \rceil} \chi^2(i+1)^{2q} + \sum_{i=\lceil (3rd)^s \rceil+1}^{\infty} \chi^2(i+1)^{2q} s \exp\left(-\frac{i^{1/s}}{9}\right),$$

which is a finite constant independent of n.

3.2 Forbidden Structures

In this section, we will define two "forbidden structures" in order to assist with our proof of Theorem 1.6. Before we define these structures and prove the corresponding results, we state the following theorem due to Krivelevich and Sudakov on the size of the maximum independent set in $\mathcal{H}_r(n,p)$ (this result wasn't explicitly stated, but can inferred from Corollary 1 in their paper) [KS98].

Theorem 3.8. Let $\varepsilon > 0$ and $r \in \mathbb{N}$ such that $r \geq 2$. There is $d^* > 0$ such that for any $d \geq d^*$, there exists $n^* > 0$ such that for any $n \geq n^*$ and $p = d/\binom{n-1}{r-1}$, the graph $H \sim \mathcal{H}_r(n,p)$ satisfies

$$(1-\varepsilon)\left(\frac{r}{r-1}\frac{\log d}{d}\right)^{\frac{1}{r-1}} \leq \frac{\alpha(H)}{n} \leq (1+\varepsilon)\left(\frac{r}{r-1}\frac{\log d}{d}\right)^{\frac{1}{r-1}},$$

with probability at least $1 - \exp(-\Omega(n))$.

First, we will define a sequence of correlated random graphs and a corresponding forbidden structure. Recall that we represent an r-uniform hypergraph $H \sim \mathcal{H}_r(n,p)$ by its edge indicator vector $A = \{0,1\}^m$ where $m = \binom{n}{r}$. Additionally, recall the interpolation path for $\mathcal{H}_r(n,p)$ from Definition 2.2. It is easy to see that the marginal distribution of $A^{(t)}$ is $\mathcal{H}_r(n,p)$. Furthermore, $A^{(t+m)}$ is independent of $A^{(0)}, \ldots, A^{(t)}$ as all edges are resampled by then. We will now show that a certain structure of independent sets across the correlated sequence of random hypergraphs exists only with exponentially small probability.

Proposition 3.9. Fix constants $\varepsilon > 0$ and $K \in \mathbb{N}$ with $K \ge \left\lceil \frac{2^{r+1}}{\varepsilon^r} \right\rceil + 1$. Consider the interpolation path $A^{(0)}, \dots, A^{(T)}$ of any length $T = n^{O(r)}$. If $d = d(\varepsilon, K) > 0$ is sufficiently large, then with probability $1 - \exp\left(-\Omega(n)\right)$, there does not exist a sequence of sets $S_1, \dots, S_K \subseteq [n]$ satisfying the following properties

(H1) For each $k \in [K]$ there exist $0 \le t_k \le T$ such that S_k is an Independent set in $A^{(t_k)}$,

$$(H2) |S_k| \ge (1+\varepsilon) \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}} n \text{ for all } k \in [K],$$

$$(H3) |S_k \setminus (\bigcup_{\ell < k} S_\ell)| \in \left[\frac{\varepsilon}{4} \left(\frac{1}{(r-1)} \cdot \frac{\log d}{d} \right)^{\frac{1}{r-1}} n, \frac{\varepsilon}{2} \left(\frac{1}{(r-1)} \cdot \frac{\log d}{d} \right)^{\frac{1}{r-1}} n \right] \text{ for all } 2 \le k \le K.$$

Proof. Let N be the number of the sequences (S_1, \ldots, S_K) satisfying the properties (H1)–(H3). We will compute $\mathbb{E}[N]$ and show that it is exponentially small, then the assertion follows by Markov's Inequality.

Define $\Phi := \left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} n$ and let a_k, b_k and c be defined as follows:

$$a_k = \frac{|S_k|}{\Phi}, \quad b_k = \frac{|S_k \setminus \bigcup_{l < k} S_\ell|}{\Phi}, \quad \text{and} \quad c_k = \frac{|\bigcup_k S_k|}{\Phi}.$$

By (H2), (H3), and Theorem 3.8, we have

$$a_k \in \left[(1+\varepsilon) \left(\frac{1}{r-1} \right)^{\frac{1}{r-1}}, (1+\varepsilon) \left(\frac{r}{r-1} \right)^{\frac{1}{r-1}} \right],$$

$$b_k \in \left[\frac{\varepsilon}{4} \left(\frac{1}{(r-1)} \right)^{\frac{1}{r-1}}, \frac{\varepsilon}{2} \left(\frac{1}{(r-1)} \right)^{\frac{1}{r-1}} \right], \text{ and }$$

$$c \le a_1 + \frac{\varepsilon(K-1)}{2} \left(\frac{1}{(r-1)} \right)^{\frac{1}{r-1}}$$

Thus, c is bounded from above by a constant $C(\varepsilon, K)$ that does not depend on d.

First, note that there are at most n^{2K} choices for $\{a_k\}$ and $\{b_k\}$. Once $\{a_k\}$ and $\{b_k\}$ are fixed, the number of choices for $\{S_k\}$ is at most

$$\binom{n}{a_1 \Phi} \prod_{k=2}^K \binom{n}{b_k \Phi} \binom{c \Phi}{(a_k - b_k) \Phi}$$

$$\leq \left(\frac{en}{a_1\Phi}\right)^{a_1\Phi} \prod_{k=2}^K \left(\frac{en}{b_k\Phi}\right)^{b_k\Phi} \left(\frac{en}{(a_k-b_k)}\right)^{(a_k-b_k)\Phi}$$

$$= \exp\left(a_1\Phi\log\left(\frac{e}{a_1}\left(\frac{d}{\log d}\right)^{\frac{1}{r-1}}\right) + \sum_{k=2}^K b_k\Phi\log\left(\frac{e}{b_k}\left(\frac{d}{\log d}\right)^{\frac{1}{r-1}}\right) + (a_k-b_k)\Phi\log\left(\frac{ec}{a_k-b_k}\right)\right)$$

$$= \exp\left(\frac{1}{r-1}\Phi\log d\left(a_1 + \sum_{k=2}^K b_k + o_d(1)\right)\right)$$

where we used the bounds on a_k , b_k , and c computed earlier.

Now for a fixed $\{S_k\}$ satisfying (H2) and (H3), we take union bound over all $(T+1)^K$ possible choices of $\{t_k\}$ and compute an upper bound on the probability that (H1) is satisfied. Let E be the number of edges e for which there exists k such that all vertices of e lie inside of S_k . We compute a lower bound on E as follows:

$$E = {\binom{a_1 \Phi}{r}} + \sum_{k=2}^{K} {\binom{a_k \Phi}{r}} - {\binom{(a_k - b_k)\Phi}{r}}$$

$$= \frac{\Phi^r}{r!} \left(a_1^r + \sum_{k=2}^{K} a_k^r - (a_k - b_k)^r + o(1) \right)$$

$$= \frac{\Phi^r}{r!} \left(a_1^r + \sum_{k=2}^{K} b_k \left((a_k - b_k)^{r-1} + a_k (a_k - b_k)^{r-2} + \dots + a_k^{r-1} \right) + o(1) \right)$$

$$\geq \frac{\Phi^r}{r!} \left(a_1^r + \sum_{k=2}^{K} r b_k (a_k - b_k)^{r-1} + o(1) \right)$$

where the first term counts edges within S_1 and the kth term counts edges crossing S_k and $\bigcup_{\ell < k} S_{\ell}$. For fixed $\{S_k\}$ and $\{t_k\}$, (H1) occurs when at least E independent non-edges occur in the sampling of $\{A^{(t)}\}$; this happens with probability at most

$$\left(1 - \frac{d}{\binom{n-1}{r-1}}\right)^E \le \exp\left\{-\frac{Ed}{\binom{n-1}{r-1}}\right\}.$$

We compute the following supremum which will be used to upper bound $\mathbb{E}[N]$,

$$\sup_{\{a_k\},\{b_k\}} \exp\left(\frac{1}{r-1}\Phi \log d\left(a_1 + \sum_{k=2}^K b_k + o_d(1)\right)\right) \cdot \exp\left(-\frac{d}{\binom{n-1}{r-1}}E\right)$$

$$\leq \sup_{\{a_k\},\{b_k\}} \exp\left(\frac{1}{r-1}\Phi \log d\left(a_1 + \sum_{k=2}^K b_k + o_d(1)\right) - \frac{1}{r}\Phi \log d\left(a_1^r + \sum_{k=2}^K rb_k(a_k - b_k)^{r-1} + o(1)\right)\right)$$

$$\leq \sup_{\{a_k\},\{b_k\}} \exp\left(\Phi \log d\left(\frac{a_1}{r-1} - \frac{a_1^r}{r} + \sum_{k=2}^K \frac{b_k}{r-1} - b_k(a_k - b_k)^{r-1} + o_d(1) + o(1)\right)\right)$$

$$= \sup_{\{a_k\},\{b_k\}} \exp\left(\Phi \log d \left(\frac{a_1}{r-1} - \frac{a_1^r}{r} - \sum_{k=2}^K b_k \left((a_k - b_k)^{r-1} - \frac{1}{r-1}\right) + o_d(1) + o(1)\right)\right)$$

From a basic calculus argument, we may conclude the following:

$$\sup_{a \in \mathbb{R}} \frac{a}{r-1} - \frac{a^r}{r} = \frac{1}{r} \left(\frac{1}{r-1} \right)^{\frac{1}{r-1}}.$$

In addition, by the bounds on a_k and b_k , we can bound the other term as follows:

$$b_k \left((a_k - b_k)^{r-1} - \frac{1}{r-1} \right)$$

$$\geq \frac{\varepsilon}{4} \left(\frac{1}{r-1} \right)^{\frac{1}{r-1}} \left[\left((1+\varepsilon) \left(\frac{1}{r-1} \right)^{\frac{1}{r-1}} - \frac{\varepsilon}{2} \left(\frac{1}{r-1} \right)^{\frac{1}{r-1}} \right)^{r-1} - \frac{1}{r-1} \right]$$

$$= \frac{\varepsilon}{4} \left(\frac{1}{r-1} \right)^{\frac{r}{r-1}} \left[\left(1 + \frac{\varepsilon}{2} \right)^{r-1} - 1 \right]$$

$$\geq \frac{\varepsilon}{4} \left(\frac{1}{r-1} \right)^{\frac{r}{r-1}} \left[1 + \left(\frac{\varepsilon}{2} \right)^{r-1} - 1 \right]$$

$$\geq \frac{\varepsilon^r}{2^{r+1}} \left(\frac{1}{r-1} \right)^{\frac{r}{r-1}}$$

Combing the above with the assumption that $K \ge \left\lceil \frac{2^{r+1}}{\varepsilon^r} \right\rceil + 1$, we have

$$\frac{a_1}{r-1} - \frac{a_1^r}{r} - \sum_{k=2}^K b_k \left((a_k - b_k)^{r-1} - \frac{1}{r-1} \right)$$

$$\leq \frac{1}{r} \left(\frac{1}{r-1} \right)^{\frac{1}{r-1}} - \sum_{k=2}^K \frac{\varepsilon^r}{2^{r+1}} \left(\frac{1}{r-1} \right)^{\frac{r}{r-1}}$$

$$= \frac{1}{r} \left(\frac{1}{r-1} \right)^{\frac{1}{r-1}} - (K-1) \frac{\varepsilon^r}{2^{r+1}} \left(\frac{1}{r-1} \right)^{\frac{r}{r-1}}$$

$$\leq -\frac{1}{r(r-1)} \left(\frac{1}{r-1} \right)^{\frac{1}{r-1}}.$$

Put everything together, we have

$$\mathbb{E}[N] = n^{2K} (T+1)^K \sup_{\{a_k\}, \{b_k\}} \exp\left\{\frac{1}{r-1} \Phi \log d \left(a_1 + \sum_{k=2}^K b_k + o_d(1)\right)\right\} \cdot \exp\left(-\frac{d}{\binom{n-1}{r-1}} E\right)$$

$$\leq n^{2K} (T+1)^K \exp\left(\Phi \log d \left(-\frac{1}{r(r-1)} \left(\frac{1}{r-1}\right)^{\frac{1}{r-1}} + o_d(1) + o(1)\right)\right)$$

$$= \exp\left(-\Omega(n)\right)$$

for sufficiently large d, as desired.

Next, we show that no independent set in $\mathcal{H}_r(n,p)$ has a large intersection with some fixed set of vertices.

Lemma 3.10. Let $\varepsilon > 0$, $r \ge 2$, and a > 0 be constants. Fix $S \subseteq [n]$ with $|S| \le a \left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} n$. If d is sufficiently large in terms of ε , r, and a, then with probability $1 - \exp(\Omega(n))$ there is no independent set T in $\mathcal{H}_r(n,p)$ such that $|T \cap S| \ge \varepsilon \left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} n$.

Proof. As usual, we let $\Phi := \left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} n$. Let N be the number of subsets S' of S such that $|S'| = \lceil \varepsilon \Phi \rceil := b\Phi$ and S' is an independent set in $\mathcal{H}_r(n,p)$. It remains to show that N = 0 with high probability. To this end, we have

$$\mathbb{E}[N] = {|S| \choose b\Phi} \left(1 - \frac{d}{{n-1 \choose r-1}}\right)^{{b\Phi \choose r}}$$

$$\leq \left(\frac{ea}{b}\right)^{b\Phi} \exp\left(-\frac{d}{{n-1 \choose r-1}} \cdot {b\Phi \choose r}\right)$$

$$\leq \exp\left(b\Phi \log\left(\frac{ea}{b}\right) - \Phi \log d \cdot \frac{b^r}{r} + O(1)\right)$$

$$= \exp\left(\Phi \log d \left(-\frac{b^r}{r} + o(1) + o_d(1)\right)\right)$$

$$= \exp\left(-\Omega(n)\right)$$

For sufficiently large d. The result follows by Markov's inequality.

3.3 Proof of the Intractability Result in $\mathcal{H}_r(n,p)$

We are now ready to prove Theorem 1.6. We will show that no (D, Γ, c) -stable algorithm can find a large independent set in $\mathcal{H}_r(n, p)$ in the sense of Definition 1.2. As a result of Lemma 2.5, this would complete the proof of the impossibility result.

Proposition 3.11. For any $r \geq 2$ and $\varepsilon > 0$, there exist K, $d_0 > 0$ such that for any $d \geq d_0$ there exists $n_0, \eta, C_1, C_2 > 0$ such that for any $n \geq n_0$, $\xi \geq 1$, $1 \leq D \leq \frac{C_1 n}{\xi \log n}$, and $\delta \leq \exp(-C_2 \xi D \log n)$, if

$$k \ge n (1 + \varepsilon) \left(\frac{1}{r - 1} \cdot \frac{\log d}{d} \right)^{1/(r - 1)},$$

there is no $\left(D, K-1, \frac{\varepsilon}{32\xi(1+\varepsilon)}\right)$ -stable function that (k, δ, ξ, η) -optimizes the independent set problem in $\mathcal{H}_r(n, p)$ for $p = d/\binom{n-1}{r-1}$.

Proof. Fix $\epsilon > 0$, and let $K = \left\lceil \frac{2^{r+1}}{\varepsilon^r} \right\rceil + 1$, T = (K-1)m, $c = \frac{\varepsilon}{32\xi(1+\varepsilon)}$ and $\eta = \frac{\varepsilon}{16} \left(\frac{\log d}{d}\right)^{\frac{1}{r-1}}$. Let $d_0 = d_0(\varepsilon)$ large enough to apply Theorem 3.8, Proposition 3.9, and Lemma 3.10. As usual, we let $\Phi := \left(\frac{\log d}{d}\right)^{\frac{1}{r-1}} n$.

Assume for a contradiction that such a degree-D polynomial f indeed exists. Sample the interpolation path $A^{(0)}, \ldots, A^{(T)}$ as in Definition 2.2, and let $V_t = V_f^{\eta}\left(A^{(t)}\right)$ be the resulting independent sets. We construct a sequence of sets S_1, \ldots, S_K based on $\{V_t\}$ as follows: let $S_1 = V_0$, and for $k \geq 2$, let S_k be the first V_t such that $|V_t \setminus \bigcup_{l < k} S_l| \geq \frac{\varepsilon}{4} \left(\frac{\log d}{d(r-1)}\right)^{\frac{1}{r-1}} n$; if no such t exists then the process fails. We define the following three events:

- (E1) $|V_t| \ge (1+\varepsilon)\Phi$ for all $t \in [T]$, and the process of constructing S_1, \ldots, S_K succeeds.
- (E2) No edge of the interpolation path is c-bad for f.
- (E3) The forbidden structure of Proposition 3.9 does not exist.

In the following claim, we show that if (E1) and (E2) occur, then the forbidden structure will be produced, which implies that (E1), (E2), and (E3) cannot occur concurrently.

Claim 3.12. If (E1) and (E2) both occur, then the sequence of sets S_1, \ldots, S_k satisfies the properties of the forbidden structure of Proposition 3.9.

Proof. First we show that $|V_t \triangle V_{t-1}| \leq \frac{\varepsilon}{4} \left(\frac{1}{r-1} \cdot \frac{\log d}{d}\right)^{\frac{1}{r-1}} n$. From (E1) we know that the failure event setting $V_f^{\eta} = \emptyset$ does not occur for all $t \in [T]$. By Definition 1.1, $i \in V_t \triangle V_{t-1}$ if one of two events occur:

- $i \in (I \setminus \tilde{I}) \cup J$ for either $V_f^{\eta}\left(A^{(t)}\right)$ or $V_f^{\eta}\left(A^{(t-1)}\right)$, or
- one of $f_i\left(A^{(t)}\right)$ and $f_i\left(A^{(t-1)}\right)$ is ≥ 1 and the other is $\leq \frac{1}{2}$.

Notice that the first case occurs for at most $2\eta n$ coordinates as $|(I \setminus \tilde{I}) \cup J| \leq \eta n$ in Definition 1.1. Hence the second case occurs for at least $|V_t \triangle V_{t-1}| - 2\eta n$ coordinates $i \in [n]$ and for those coordinates i we have $|f_i(A^{(t)}) - f_i(A^{(t-1)})| \geq \frac{1}{2}$. Together with event (E2), this implies

$$\frac{1}{4} (|V_t \triangle V_{t-1}| - 2\eta n) \le ||f(A^{(t)}) - f(A^{(t-1)})||^2 \le c \cdot \mathbb{E}_{A \sim \mathcal{H}_r(n,p)} [||f(A)||^2].$$

By assumption, the independent set problem is (k, δ, ξ, η) -optimized by the function f, and so we have

$$\mathbb{E}_{A \sim \mathcal{H}_r(n,p)} [\|f(A)\|^2] \le \xi (1+\varepsilon) \Phi$$

Putting the above two inequalities together, we have

$$|V_t \triangle V_{t-1}| \le 4c\xi(1+\varepsilon)\Phi + 2\eta n = 4 \cdot \frac{\varepsilon}{32\xi(1+\varepsilon)} \cdot \xi(1+\varepsilon)\Phi + 2 \cdot \frac{\varepsilon}{16}\Phi = \frac{\varepsilon}{4}\Phi.$$

Recall that S_k is the first V_t for which $|V_t \setminus \bigcup_{l \le k} S_l| \ge \frac{\varepsilon}{4} \Phi$. In particular, we must have $|V_{t-1} \setminus \bigcup_{l \le k} S_l| < \frac{\varepsilon}{4} \Phi$. With this observation in hand, we have

$$|S_k \setminus \cup_{l \le k} S_l| = |V_t \setminus \cup_{l \le k} S_l| = |(V_t \cap V_{t-1}) \setminus \cup_{l \le k} S_l| + |(V_t \setminus V_{t-1}) \setminus \cup_{l \le k} S_l|$$

$$\leq |V_{t-1} \setminus \bigcup_{l \leq k} S_l| + |V_t \triangle V_{t-1}| \leq \frac{\varepsilon}{2} \Phi.$$

Thus, $|S_k \setminus \bigcup_{l \le k} S_l| \in [\frac{\varepsilon}{4}\Phi, \frac{\varepsilon}{2}\Phi]$. for all $2 \le k \le K$. By construction, S_k is an independent set in $A^{(t)}$ for some t, and so we conclude that S_1, \dots, S_k satisfies the properties of the forbidden structure in Proposition 3.9.

Next, we will show that with positive probability the events (E1), (E2), and (E3) occur simultaneously. This would complete the proof as by Claim 3.12, this implies a contradiction and so no such f can exist.

We first bound the probability that event (E1) occurs. Since the independent set problem is (k, δ, ξ, η) -optimized by f, we have

$$\mathbb{P}_{A \sim \mathcal{H}_r(n,p)} \left[\left| V_f^{\eta}(A) \right| \ge (1+\varepsilon)\Phi \right] \ge 1 - \delta.$$

Combining with Lemma ??, we have $(1+\varepsilon)\Phi \leq |V_t\cap L| \leq \left(r^{\frac{1}{r-1}}+\varepsilon\right)\Phi$ with probability at least $1-\delta-\exp\{-\Omega(n)\}$. Now suppose that for some $0\leq T'\leq T-m$, we have sampled $A^{(0)},\ dots, A^{(T')}$ and $S_1=V_0,\cdots,S_{K'}=V_{t_{k'}}$ have been successfully selected. As mentioned previously, $A^{(T'+m)}$ is independent from $A^{(0)},\ldots,A^{(T')}$. So, provided $|S_k|\leq \left(r^{\frac{1}{r-1}}+\varepsilon\right)\Phi$ for $1\leq k\leq K'$, we may apply Lemma 3.10 with $S=\cup_{k\leq K'}S_k$ and $a=\left(r^{\frac{1}{r-1}}+\varepsilon\right)K'$. As a result, $|V_{T'+m}\cap(\cup_{k\leq K'}S_k)|\leq \varepsilon\Phi$ with probability $1-\exp\{-\Omega(n)\}$. Therefore, we have

$$|V_{T'+m} \setminus \bigcup_{k \le K'} S_k| = |V_{T'+m}| - |V_{T'+m} \cap (\bigcup_{k \le K'} S_k)| \ge \Phi \ge \frac{\varepsilon}{4} \Phi.$$

This implies that we can find $S_{K'+1} = V_t$ for some $t \leq T' + m$ and thus by induction the process succeeds by timestep T = (K-1)m. By a union bound over t, (E1) holds with probability at least $1 - \delta(T+1) - \exp\{-\Omega(n)\}$.

By Definition 2.4 and the stability of f, (E2) holds with probability at least $\left(d/\binom{n-1}{r-1}\right)^{4(K-1)D/c}$. By Proposition 3.9, (E3) holds with probability $1 - \exp\{-\Omega(n)\}$. We claim that it suffices to have

$$\left(\frac{d}{\binom{n-1}{r-1}}\right)^{4(K-1)D/c} \ge 2\exp\{-Cn\}, \quad \text{and} \quad \left(\frac{d}{\binom{n-1}{r-1}}\right)^{4(K-1)D/c} \ge 2\delta K n^r \tag{7}$$

for some constant $C = C(\varepsilon, d)$ to conclude that all three events happen simultaneously with non-zero probability, since (7) implies

$$\left(\frac{d}{\binom{n-1}{r-1}}\right)^{4(K-1)D/c} \ge \exp\{-Cn\} + \delta K n^r$$

$$> \exp\{-\Omega(n)\} + \delta K m$$

$$> \exp\{-\Omega(n)\} + \delta (T+1).$$

For d > 1, the first inequality in (7) is equivalent to $D \le \frac{c(Cn - \log 2)}{4(K-1)(\log \binom{n-1}{r-1} - \log d)}$ which is implied by

$$D \le \frac{c(Cn - \log 2)}{4(K - 1)(r - 1)\log n} = \frac{\varepsilon(Cn - \log 2)}{128(1 + \varepsilon)(K - 1)(r - 1)\xi \log n}.$$

And this is implied by $D < \frac{C_1 n}{\xi \log n}$, for large enough n and some constant $C_1 = C_1(\varepsilon, d) > 0$. The second inequality is implied by $\delta \le \exp\left(-\frac{4(K-1)(r-1)D}{c}\log n - r\log n - \log(2K)\right)$. For large enough n, given $\xi, D \ge 1$, there exists another constant $C_2 = C_2(\varepsilon, d) > 0$ such that the prior inequality is implied by $\delta \le \exp\left(-C_2\xi D\log n\right)$.

4 Balanced Independent Sets in $\mathcal{H}(r, n, p)$

In this section, we will consider the random r-uniform r-partite hypergraph $\mathcal{H}(r,n,p)$ for $p=d/n^{r-1}$ (see Definition 1.4). We split this section into four subsections. First, we prove the statistical threshold stated in Theorem 1.8. In the second subsection, we describe a degree-1 algorithm in order to prove the tractability result in Theorem 1.9. In the final two subsections, we prove the impossibility result of Theorem 1.9 by first proving a version of the Overlap Gap Property for balanced independent sets in $\mathcal{H}(r,n,p)$ and then applying this result to prove intractability of low-degree algorithms for balanced independent sets in the stated regime.

4.1 Statistical Threshold

In this section, we will prove Theorem 1.8. Let n, d, p, γ , H be as in the statement of the theorem. To assist with our proof, we define the following parameters:

$$c_{\gamma} = \frac{1}{r^{r-1}(r-1)\prod_{i}\gamma_{i}}, \text{ and } f \coloneqq \left(\frac{c_{\gamma}\log d}{d}\right)^{1/(r-1)}.$$

To prove the upper bound, we will show that

$$\mathbb{P}[\alpha_{\gamma}(H) \le (1+\varepsilon)rnf] = 1 - o(1), \quad \text{as } n \to \infty, \tag{8}$$

for any $\varepsilon > 0$. To this end, we note the following by a union bound over all possible γ -balanced independent sets of size $(1 + \varepsilon)rnf$:

$$\begin{split} & \mathbb{P}[\alpha_{\gamma}(H) \geq (1+\varepsilon)rnf] \\ & \leq \left(\prod_{i=1}^{r} \binom{n}{\gamma_{i} (1+\varepsilon)rnf}\right) \left(1 - \frac{d}{n^{r-1}}\right)^{((1+\varepsilon)rnf)^{r} \prod_{i=1}^{r} \gamma_{i}} \\ & \leq \left(\prod_{i=1}^{r} \left(\frac{en}{\gamma_{i} (1+\varepsilon)rnf}\right)^{\gamma_{i} (1+\varepsilon)rnf}\right) \exp\left(-(1+\varepsilon)^{r} r^{r} n f^{r} d \prod_{i=1}^{r} \gamma_{i}\right) \end{split}$$

$$\leq \exp\left(\left(1+\varepsilon\right)r\,n\,f\left(1-\log r-\log f-\sum_{i=1}^{r}\gamma_{i}\log\gamma_{i}\right)-\left(1+\varepsilon\right)^{r}\,r^{r}\,n\,f^{r}d\prod_{i=1}^{r}\gamma_{i}\right)$$

$$=\exp\left(\left(1+\varepsilon\right)r\,n\,f\left(1-\log r-\log f-\sum_{i=1}^{r}\gamma_{i}\log\gamma_{i}-\left(1+\varepsilon\right)^{r-1}r^{r-1}\,c_{\gamma}\,\log d\prod_{i=1}^{r}\gamma_{i}\right)\right).$$

Note that $-\sum_{i=1}^{r} \gamma_i \log \gamma_i$ subject to the constraint $\sum_i \gamma_i = 1$ is maximized at $\gamma_i = 1/r$ for all i. With this in hand, plugging in the value for c_{γ} , we may conclude that

$$\mathbb{P}[\alpha_{\gamma}(H) \ge (1+\varepsilon)rnf] \le \exp\left((1+\varepsilon)rnf\left(1-\log f - \frac{(1+\varepsilon)^{r-1}\log d}{r-1}\right)\right)$$

$$= \exp\left((1+\varepsilon)rnf\left(1 - \frac{\log c_{\gamma}}{r-1} - \frac{((1+\varepsilon)^{r-1}-1)\log d}{r-1} - \frac{\log\log d}{r-1}\right)\right)$$

$$\le \exp\left((1+\varepsilon)rnf\left(1 - \frac{\log c_{\gamma}}{r-1} - \varepsilon\log d - \frac{\log\log d}{r-1}\right)\right)$$

$$= \exp\left(-\Omega(n)\right),$$

for d large enough, completing the proof of (8).

Now, to prove the lower bound, we would like to show that

$$\mathbb{P}[\alpha_{\gamma}(H) \ge (1 - \varepsilon)rnf] = 1 - o(1), \quad \text{as } n \to \infty.$$
 (9)

To this end, we will apply the approach of [PW24] for bipartite graphs, which was inspired by the proof of [Fri90] for ordinary graphs. For each $i \in [r]$, denote the vertices of V_i as $\{1_i, \ldots, n_i\}$. Let $m = \left(\frac{d}{(\log d)^r}\right)^{\frac{1}{r-1}}$, n' = n/m, and let $P_{i,j} = \{((j-1)m+1)_i, \ldots, (jm)_i\}$ for each $i \in [r]$ and $j \in [n']$. We say I is a γ -balanced P-independent set if I is γ -balanced and $|X \cap P_{i,j}| \leq 1$ for each $i \in [r]$ and $j \in [n']$. Let $\beta(H)$ denote the size of the largest γ -balanced P-independent set in a hypergraph $H \sim \mathcal{H}(r, n, p)$ and let Z_k denote the number of such independent sets of size k.

Let us first show that $\beta(H)$ is concentrated about its mean.

Lemma 4.1. Let $H \sim \mathcal{H}(r, n, p)$ and let $\overline{\beta} = \mathbb{E}[\beta(H)]$. Then

$$\mathbb{P}\left[|\beta(H) - \overline{\beta}| \ge \lambda\right] \le 2 \exp\left(-\frac{\lambda^2 m \min_{i \in [r]} \gamma_i^2}{2rn}\right).$$

We will apply McDiarmid's inequality to prove this lemma.

Theorem 4.2 (McDiarmid's inequality). Let $X = f(\vec{Z})$, where $\vec{Z} = (Z_1, ..., Z_t)$ and the Z_i are independent random variables. Assume the function f has the property that whenever \vec{z} , \vec{w} differ in only one coordinate we have $|f(\vec{z}) - f(\vec{w})| \le c$. Then, for all $\lambda > 0$ we have

$$\mathbb{P}[|X - \mathbb{E}[X]| \ge \lambda] \le 2 \exp\left(-\frac{\lambda^2}{2c^2t}\right).$$

Proof of Lemma 4.1. We will employ the so-called "vertex-based" approach toward applying Theorem 4.2 to problems in random hypergraphs. First, we partition the edges of $K_{r\times n}$ into rn' groups $\{E_{ij}: 1 \leq i \leq r, 1 \leq j \leq n'\}$. We include $(v_1, \ldots, v_r) \in E_{ij}$ if and only if the following hold for $1 \leq l \leq r$:

- $v_l \in P_{li_l}$ for some $j_l > j$ if l < i, and
- $v_l \in P_{lj_l}$ for some $j_l \ge j$ if $l \ge i$.

Alternatively, an edge (v_1, \ldots, v_r) is assigned to E_{ij} where j is the minimum "partition number" of a vertex $v \in e$, and i is the minimum value l such that v_l is in P_{lj} .

For each $1 \leq i \leq r$ and $1 \leq j \leq n'$, we let Z_{ij} denote the outcomes of the edges in E_{ij} . Clearly, the variables Z_{ij} are independent. Furthermore, changing the outcomes of the edges in a single E_{ij} can change the value $\beta(\cdot)$ by at most $1/\gamma_i$. Therefore, we may apply Lemma 4.2 with $c = 1/\min_{i \in [r]} \gamma_i$ and t = rn' to get

$$\mathbb{P}\left[|\beta(H) - \overline{\beta}| \ge \lambda\right] \le 2 \exp\left(-\frac{\lambda^2 \min_{i \in [r]} \gamma_i^2}{2rn'}\right),\,$$

as claimed. \Box

Next, we show there exists a large γ -balanced P-independent set in $H \sim \mathcal{H}(r, n, p)$ with positive probability.

Lemma 4.3. Let $k = (1 - \rho)rnf$ for some $\rho > 0$ sufficiently small. Then,

$$\mathbb{P}[Z_k > 0] \ge \exp\left(-\frac{50n\sqrt{r(c_{\gamma}\log d)^{\frac{1}{r-1}}}}{d^{\frac{1-\rho/4}{r-1}}}\right).$$

Proof. We will use the inequality $\mathbb{P}[Z_k > 0] \ge \mathbb{E}[Z_k]^2 / \mathbb{E}[Z_k^2]$. To this end, let us compute the mean of Z_k . We have

$$\mathbb{E}[Z_k] = \left(\prod_{i=1}^r \binom{n'}{\gamma_i k} m^{\gamma_i k}\right) \left(1 - \frac{d}{n^{r-1}}\right)^{k^r \prod_{i=1}^r \gamma_i} = \left(\prod_{i=1}^r \binom{n'}{\gamma_i k}\right) m^k \left(1 - \frac{d}{n^{r-1}}\right)^{k^r \prod_{i=1}^r \gamma_i}.$$

To compute $\mathbb{E}[Z_k^2]$, we define \mathcal{I}_k to be the set of all possible γ -balanced P-independent sets of size k in $\mathcal{H}(r,n,p)$. For brevity, we let A_I denote the event that I is independent in H, where $I \in \mathcal{I}_k$. With this in hand, we have

$$\begin{split} \mathbb{E}[Z_k^2] &= \sum_{I,I' \in \mathcal{I}_k} \mathbb{P}[A_I, \, A_{I'}] \\ &= \sum_{I \in \mathcal{I}_k} \mathbb{P}[A_I] \sum_{I' \in \mathcal{I}_k} \mathbb{P}[A_{I'} \mid A_I] \end{split}$$

$$\begin{split} &= \sum_{I \in \mathcal{I}_k} \mathbb{P}[A_I] \sum_{I' \in \mathcal{I}_k} \left(1 - \frac{d}{n^{r-1}}\right)^{k^r \prod_{i=1}^r \gamma_i - \prod_{i=1}^r |I \cap I' \cap V_i|} \\ &= \sum_{I \in \mathcal{I}_k} \mathbb{P}[A_I] \sum_{\ell_1 = 0}^{\gamma_1 k} \cdots \sum_{\ell_r = 0}^{\gamma_r k} \sum_{\substack{I' \in \mathcal{I}_k, \\ |I \cap I' \cap V_i| = \ell_i}} \left(1 - \frac{d}{n^{r-1}}\right)^{k^r \prod_{i=1}^r \gamma_i - \prod_{i=1}^r \ell_i} \\ &\leq \sum_{I \in \mathcal{I}_k} \mathbb{P}[A_I] \sum_{\ell_1 = 0}^{\gamma_1 k} \cdots \sum_{\ell_r = 0}^{\gamma_r k} \left(\prod_{i=1}^r \binom{\gamma_i k}{\ell_i} \binom{n' - \ell_i}{\gamma_i k - \ell_i} m^{\gamma_i k - \ell_i}\right) \left(1 - \frac{d}{n^{r-1}}\right)^{k^r \prod_{i=1}^r \gamma_i - \prod_{i=1}^r \ell_i} \\ &= \mathbb{E}[Z_k] \, m^k \sum_{\ell_1 = 0}^{\gamma_1 k} \cdots \sum_{\ell_r = 0}^{\gamma_r k} m^{-\sum_{i=1}^r \ell_i} \left(\prod_{i=1}^r \binom{\gamma_i k}{\ell_i} \binom{n' - \ell_i}{\gamma_i k - \ell_i}\right) \left(1 - \frac{d}{n^{r-1}}\right)^{k^r \prod_{i=1}^r \gamma_i - \prod_{i=1}^r \ell_i}. \end{split}$$

From here, we may simplify further as follows:

$$\frac{\mathbb{E}[Z_k^2]}{\mathbb{E}[Z_k]^2} \leq \sum_{\ell_1=0}^{\gamma_1 k} \cdots \sum_{\ell_r=0}^{\gamma_r k} \frac{m^{-\sum_{i=1}^r \ell_i} \left(\prod_{i=1}^r \binom{\gamma_i k}{\ell_i} \binom{n'-\ell_i}{\gamma_i k-\ell_i}\right) \left(1 - \frac{d}{n^{r-1}}\right)^{k^r \prod_{i=1}^r \gamma_i - \prod_{i=1}^r \ell_i}}{\left(\prod_{i=1}^r \binom{n'}{\gamma_i k}\right) \left(1 - \frac{d}{n^{r-1}}\right)^{k^r \prod_{i=1}^r \gamma_i}} \\
= \sum_{\ell_1=0}^{\gamma_1 k} \cdots \sum_{\ell_r=0}^{\gamma_r k} m^{-\sum_{i=1}^r \ell_i} \left(1 - \frac{d}{n^{r-1}}\right)^{-\prod_{i=1}^r \ell_i} \left(\prod_{i=1}^r \frac{\binom{\gamma_i k}{\gamma_i k} \binom{n'-\ell_i}{\gamma_i k}}{\binom{n'}{\gamma_i k} \ell_i}\right).$$

Note the following for any $i \in [r]$:

$$\frac{\binom{n'-\ell_i}{\gamma_i k-\ell_i}}{\binom{n'}{\gamma_i k}} = \frac{\frac{(n'-\ell_i)!}{(\gamma_i k-\ell_i)!(n'-\gamma_i k)!}}{\frac{(n')!}{(n'-\gamma_i k)!(\gamma_i k)!}} = \frac{(n'-\ell_i)!}{(n')!} \frac{(\gamma_i k)!}{(\gamma_i k-\ell_i)!} \le \left(\frac{\gamma_i k}{n'}\right)^{\ell_i}.$$

Plugging this in above, we get

$$\frac{\mathbb{E}[Z_k^2]}{\mathbb{E}[Z_k]^2} \leq \sum_{\ell_1=0}^{\gamma_1 k} \cdots \sum_{\ell_r=0}^{\gamma_r k} \left(\frac{k}{n'm}\right)^{\sum_{i=1}^r \ell_i} \left(1 - \frac{d}{n^{r-1}}\right)^{-\prod_{i=1}^r \ell_i} \left(\prod_{i=1}^r {\gamma_i k \choose \ell_i} \gamma_i^{\ell_i}\right) \\
\leq \sum_{\ell=0}^k \left(\frac{k}{n}\right)^{\ell} \left(1 - \frac{d}{n^{r-1}}\right)^{-m(\ell)} \sum_{\substack{\ell_1, \dots, \ell_r \\ \sum_{i=1}^r \ell_i = \ell}} \prod_{i=1}^r {\gamma_i k \choose \ell_i},$$

where we use the fact that $\gamma_i \leq 1$, n'm = n and define $m(\ell)$ to be the maximum value of $\prod_{i=1}^r \ell_i$ subject to the constraints $\sum_{i=1}^r \ell_i = \ell$ and $\ell_i \leq \gamma_i k$ for all i. Note that since $\sum_{i=1}^r \gamma_i = 1$, the final term is just $\binom{k}{\ell}$ and so, we have

$$\frac{\mathbb{E}[Z_k^2]}{\mathbb{E}[Z_k]^2} \le \sum_{\ell=0}^k \binom{k}{\ell} \left(\frac{k}{n}\right)^\ell \left(1 - \frac{d}{n^{r-1}}\right)^{-m(\ell)}.$$

For n large enough in terms of d, we conclude the following for $\delta = 1 - \frac{(1-\rho)^{r-1}}{1-\rho/2} \approx r\rho/2$:

$$\frac{\mathbb{E}[Z_k^2]}{\mathbb{E}[Z_k]^2} \le \sum_{\ell=0}^k \binom{k}{\ell} \left(\frac{k}{n}\right)^\ell \exp\left(\frac{d \, m(\ell)}{(1-\delta)n^{r-1}}\right).$$

Let u_{ℓ} denote the parameter inside the sum. The goal now is to provide an upper bound on $\sum_{\ell=0}^{k} u_{\ell}$. To this end, we define the following parameter:

$$\beta := \left(r^r \prod_{i=1}^r \gamma_i\right)^{\frac{1}{r-1}}.$$

Note that $\beta \leq 1$ as $\prod_{i=1}^{r} \gamma_i$ is largest when $\gamma_i = 1/r$ for all i. For A > 0, the following fact will be useful for our computations and can be verified by a simple calculus argument:

$$\max_{x>0} \left(\frac{A}{x^s}\right)^x = \exp\left(\frac{sA^{1/s}}{e}\right). \tag{10}$$

We will split into cases depending on the value of ℓ .

(Case 1) $0 \le \ell \le \beta k$. When considering $m(\ell)$, by relaxing the constraints $\ell_i \le \gamma_i k$ we may conclude that $m(\ell) \le (\ell/r)^r$. With this in hand, we have:

$$\exp\left(\frac{d \, m(\ell)}{(1-\delta)n^{r-1}}\right) \le \exp\left(\frac{d \, \ell^r}{(1-\delta)n^{r-1}r^r}\right)$$

$$\le \exp\left(\frac{d \, (\beta \, k)^{r-1}\ell}{(1-\delta)n^{r-1}r^r}\right)$$

$$= \exp\left(\left(\frac{1-\rho/2}{r-1}\right)\ell \, \log d\right)$$

In particular, applying (10) for s = 1 we have

$$u_{\ell} \le \left(\frac{k e}{\ell} \frac{k}{n} d^{\frac{1-\rho/2}{r-1}}\right)^{\ell} \le \exp\left(r f k d^{\frac{1-\rho/2}{r-1}}\right) \le \exp\left(\frac{r^2 n \left(c_{\gamma} \log d\right)^{2/(r-1)}}{d^{\frac{1+\rho/2}{r-1}}}\right).$$

(Case 2) $\beta k < \ell \le k$. Consider $\ell < k$. We have the following:

$$\frac{u_{\ell}}{u_{\ell+1}} = \frac{(\ell+1)}{(k-\ell)} \frac{n}{k} \exp\left(-\frac{d}{n^{r-1}(1-\delta)} \left(m(\ell+1) - m(\ell)\right)\right)$$
$$\leq \frac{kn}{(k-\ell)\ell} \exp\left(-\frac{d}{n^{r-1}(1-\delta)} \left(m(\ell+1) - m(\ell)\right)\right).$$

From here, we may conclude that

$$u_{\ell} \leq \frac{(kn)^{k-\ell}}{(k-\ell)!\ell(\ell+1)\cdots(k-1)} \exp\left(-\frac{d}{n^{r-1}(1-\delta)}\left(m(k)-m(\ell)\right)\right) u_{k}$$

$$= \frac{(\ell-1)!(kn)^{k-\ell}}{(k-\ell)!(k-1)!} \exp\left(-\frac{d}{n^{r-1}(1-\delta)} (m(k) - m(\ell))\right) u_k.$$

Note that

$$\binom{k-1}{\ell-1} \ge 1 \implies \frac{(k-1)!}{(\ell-1)!} \ge (k-\ell)!.$$

In particular, we have

$$u_{\ell} \le \frac{(kn)^{k-\ell}}{(k-\ell)!^2} \exp\left(-\frac{d}{n^{r-1}(1-\delta)} \left(m(k) - m(\ell)\right)\right) u_k$$

$$\le \left(\frac{e^2 kn}{(k-\ell)^2}\right)^{k-\ell} \exp\left(-\frac{d}{n^{r-1}(1-\delta)} \left(m(k) - m(\ell)\right)\right) u_k.$$

Let ℓ_1, \ldots, ℓ_r be the parameters that maximize $m(\ell)$. There exists $i \in [r]$ such that $\ell_i \leq \gamma_i \ell$ (if not, we would violate the constraint $\sum_{i=1}^r \ell_i = \ell$). For $j \neq i$, we also have $\ell_j \leq \gamma_j k$. Furthermore, it is easy to see that $m(k) = k^r \prod_{i=1}^r \gamma_i$. Therefore, we have

$$m(k) - m(\ell) = k^r \prod_{i=1}^r \gamma_i - \prod_{i=1}^r \ell_i \ge k^r \prod_{i=1}^r \gamma_i - k^{r-1} \ell \prod_{i=1}^r \gamma_i = (k-\ell)k^{r-1} \prod_{i=1}^r \gamma_i.$$

In particular,

$$\exp\left(-\frac{d}{n^{r-1}(1-\delta)}\left(m(k)-m(\ell)\right)\right) \le \exp\left(-\frac{d}{n^{r-1}(1-\delta)}\left((k-\ell)k^{r-1}\prod_{i=1}^r \gamma_i\right)\right)$$
$$= \exp\left(-\left(\frac{1-\rho/2}{r-1}\right)(k-\ell)\log d\right).$$

Finally, we may bound u_k as follows:

$$u_k = \left(\frac{k}{n}\right)^k \exp\left(\frac{dk^r \prod_{i=1}^r \gamma_i}{(1-\delta)n^{r-1}}\right)$$

$$\leq \left(r f \exp\left(\left(\frac{1-\rho/2}{r-1}\right) \log d\right)\right)^k$$

$$= \left(\frac{r \left(c_\gamma \log d\right)^{\frac{1}{r-1}}}{d^{\frac{\rho/2}{r-1}}}\right)^k \leq 1,$$

where the last step follows for d large enough. Putting together all the pieces, we may conclude the following by applying (10) with s = 2:

$$u_{\ell} \le \left(\frac{e^2 k n}{(k-\ell)^2 d^{\frac{1-\rho/2}{r-1}}}\right)^{k-\ell} \le \exp\left(2\sqrt{\frac{k n}{d^{\frac{1-\rho/2}{r-1}}}}\right) = \exp\left(\frac{2n\sqrt{r(c_{\gamma}\log d)^{\frac{1}{r-1}}}}{d^{\frac{1-\rho/4}{r-1}}}\right).$$

Putting together both cases, we have:

$$\sum_{\ell=0}^{k} u_{\ell} \le k \left(\exp\left(\frac{r^{2} n \left(c_{\gamma} \log d\right)^{2/(r-1)}}{d^{\frac{1+\rho/2}{r-1}}}\right) + \exp\left(\frac{2n\sqrt{r(c_{\gamma} \log d)^{\frac{1}{r-1}}}}{d^{\frac{1-\rho/4}{r-1}}}\right) \right)$$

$$\le \exp\left(\frac{50n\sqrt{r(c_{\gamma} \log d)^{\frac{1}{r-1}}}}{d^{\frac{1-\rho/4}{r-1}}}\right),$$

for d large enough, completing the proof.

We are now ready to prove (9). Setting $\lambda = \lambda_0 := \frac{\varepsilon n}{20d^{1/(r-1)}}$ in Lemma 4.1 and comparing with Lemma 4.3 for $\rho = \varepsilon/5$ we see that $\overline{\beta} \ge k - \lambda_0$. Applying Lemma 4.1 with $\lambda = \lambda_0$ again provides the desired result.

4.2 Proof of the Achievability Result

In this section, we will construct a degree-1 polynomial to prove the first part of Theorem 1.9. In particular, we will construct a polynomial $f: \{0,1\}^{n^r} \to \mathbb{R}^{rn}$ that $(k_1,\ldots,k_r,\delta,\xi,0)$ -optimizes the maximum γ -balanced independent set problem in H for some $\xi := \xi(d,r,\gamma) \ge 1$ and $\delta = o_n(1)$. As mentioned in §1.4, the algorithm we describe is adapted from that of [Dha23a] for deterministic hypergraphs.

Without loss of generality, assume

$$k_i = \gamma_i r n (1 - \varepsilon) \left(\frac{\log d}{d r^r (r - 1) \prod_{i=1}^r \gamma_i} \right)^{1/(r-1)}, \text{ for each } 1 \le i \le r.$$

If some k_i is smaller, we can zero out some coordinates to achieve this. Furthermore, assume $\varepsilon \leq 1/(100r)$ and $\gamma_r = \max_{i \in [r]} \gamma_i$. In particular, $\gamma_r \geq 1/r$.

For each $1 \le i < r$, fix $L_i \subseteq V_i$ such that $|L_i| = k_i$ and for vertices $v \in V_i$, let $f_v(A) = \mathbf{1}\{v \in L_i\}$. In particular, f_v is a degree-0 polynomial for each $v \in V_1 \cup \cdots \cup V_{r-1}$. For $v \in V_r$, we define

$$f_v(A) = 1 - \sum_{\substack{v_1, \dots, v_{r-1}, \\ v_i \in L_i}} A_{(v_1, \dots, v_{r-1}, v)},$$

which is a degree-1 polynomial. Let $L_r = \{v \in V_r : f(v, A) = 1\}$. By definition of $f_v(A)$, the set $L_1 \cup \cdots \cup L_r$ is an independent set in H. Additionally, if $f_v(A) \neq 1$, then $f_v(A) \leq 0$ and so we may set $\eta = 0$, as desired.

It remains to show the conditions of Definition 1.3 are satisfied for some $\xi \geq 1$ and $\delta = o_n(1)$. To this end, consider a vertex $v \in V_r$. For n large enough and ε small enough, we have the following:

$$\mathbb{P}[f_v(A) = 1] = \left(1 - \frac{d}{n^{r-1}}\right)^{\prod_{i=1}^{r-1} k_i}$$

$$\geq \exp\left(-\frac{(1-\varepsilon)^{r-1}}{(1-r\varepsilon/2)} \cdot \frac{1}{r(r-1)\gamma_r} \cdot \log d\right)$$
$$\geq d^{-\left(\frac{1-\varepsilon}{r-1}\right)},$$

where we use the fact that $\gamma_r r \geq 1$ and let $\tilde{\varepsilon} = r\varepsilon/4$. Let $\mu := \mathbb{E}\left[\sum_{v \in V_r} f_v(A)\right]$. In particular, we may conclude that

$$\frac{k_r}{\mu} \le r \, \gamma_r \, \left(\frac{\log d}{d^{\tilde{\varepsilon}} \, r^r \, (r-1) \, \prod_{i=1}^r \gamma_i} \right)^{1/(r-1)} \le \frac{1}{2},$$

for d large enough. As the events $\{f_v(A) = 1\}$ and $\{f_u(A) = 1\}$ are independent for $u, v \in V_r$ and $u \neq v$, by a simple Chernoff bound, we may conclude that

$$\mathbb{P}\left[\sum_{v \in V_r} f_v(A) < k_r\right] \le \mathbb{P}\left[\sum_{v \in V_r} f_v(A) \le \mu/2\right] = o_n(1),$$

as desired. As $|V_f^0(A) \cap V_i|$ is defined deterministically for $1 \le i < r$, this completes the proof of the second condition of Definition 1.3.

Let us now consider the first condition. We have

$$\mathbb{E}[f_v(A)^2] = \sum_{N=1-\prod_{i=1}^{r-1} k_i}^{1} N^2 \mathbb{P}[f_v(A) = N]$$

$$= \mathbb{P}[f_v(A) = 1] + \sum_{N=2}^{\prod_{i=1}^{r-1} k_i} (N-1)^2 \mathbb{P}[\deg_{L_1 \times \dots \times L_{r-1}}(v) = N],$$

where $\deg_{L_1 \times \cdots \times L_{r-1}}(v)$ is the number of edges incident on v containing vertices from $L_1 \cup \cdots \cup L_{r-1}$ (apart from v itself, of course). From here, we may conclude that

$$\mathbb{E}[f_v(A)^2] \le 1 + \sum_{N=2}^{\infty} (N-1)^2 \mathbb{P}[\deg_{L_1 \times \dots \times L_{r-1}}(v) = N]$$

$$\le 1 + \sum_{N=0}^{\infty} N^2 \mathbb{P}[\deg_{L_1 \times \dots \times L_{r-1}}(v) = N]$$

$$= 1 + \mathbb{E}[\deg_{L_1 \times \dots \times L_{r-1}}^2(v)]$$

$$\le 1 + \mathbb{E}[\deg_H^2(v)] = O(d^2).$$

It follows that

$$\mathbb{E}[\|f\|^2] = \sum_{i=1}^r \sum_{v \in V_i} \mathbb{E}[f_v(A)^2] \le O(nd^2) + \sum_{i=1}^{r-1} k_i \le \xi \sum_{i=1}^r k_i,$$

for some $\xi := \xi(d, r, \gamma)$ large enough. This proves the first condition in Definition 1.3 and therefore, completes the proof of the achievability result in Theorem 1.9.

4.3 Forbidden Structures

Here we define a sequence of correlated random graphs and a corresponding forbidden structure. Recall that we represent an r-uniform r-partite hypergraph $H \sim \mathcal{H}(r, n, p)$ by its edge indicator vector $A = \{0, 1\}^m$ where $m = n^r$. Additionally, we let V_1, \ldots, V_r denote the partition of the vertex set V of H.

Recall the interpolation path for $\mathcal{H}(r,n,p)$ from Definition 2.2. It is easy to see that the marginal distribution of $A^{(t)}$ is $\mathcal{H}(r,n,p)$. Furthermore, $A^{(t+m)}$ is independent of $A^{(0)},\ldots,A^{(t)}$ as all edges are resampled by then. We will now show that a certain structure of independent sets across the correlated sequence of random hypergraphs exists only with exponentially small probability.

Proposition 4.4. Fix $r \geq 2$, $\varepsilon > 0$, and $\gamma = (\gamma_1, \dots, \gamma_r)$ such that $\gamma_i \in (0,1)$ and $\sum_{i=1}^r \gamma_i = 1$. Let $K \in \mathbb{N}$ be a sufficiently large constant in terms of r, ε , and γ . Consider an interpolation path $A^{(0)}, \dots, A^{(T)}$ of length $T = n^{O(r)}$. If $d := d(\varepsilon, r, \gamma, K)$ is sufficiently large, then with probability $1 - \exp(-\Omega(n))$ there does not exist a sequence of subsets S_1, \dots, S_K of V satisfying the following:

- (B1) for each $k \in [K]$, there exists $t_k \in [T]$ such that S_k is an independent set in $A^{(t_k)}$,
- (B2) for each $k \in [K]$ and $i \in [r]$, we have $|S_k \cap V_i| \ge (1+\varepsilon)\gamma_i r n \left(\frac{\log d}{d r^r (r-1) \prod_{i=1}^r \gamma_i}\right)^{1/(r-1)}$.
- (B3) for $2 \le k \le K$ and $\tilde{\varepsilon} = \varepsilon \min_{i \in [r]} \gamma_i$, we have

$$|S_k \setminus \bigcup_{j \le k} S_j| \in \left[\frac{\tilde{\varepsilon}}{4} n \left(\frac{\log d}{d \, r^r \, (r-1) \prod_{i=1}^r \gamma_i} \right)^{1/(r-1)}, \frac{\tilde{\varepsilon}}{2} n \left(\frac{\log d}{d \, r^r \, (r-1) \prod_{i=1}^r \gamma_i} \right)^{1/(r-1)} \right].$$

Proof. Let N denote the number of tuples (S_1, \ldots, S_K) satisfying conditions (B1)–(B3). We will show that $\mathbb{E}[N]$ is exponentially small (the result then follows by Markov's inequality). To this end, let

$$\Phi := n \left(\frac{\log d}{d \, r^r \, (r-1) \prod_{i=1}^r \gamma_i} \right)^{1/(r-1)},$$

and for each $i \in [r]$ and $k \in [K]$, we define:

$$a_{i,k} = \frac{|S_k \cap V_i|}{\Phi}, \quad b_{i,k} = \frac{|(S_k \setminus \bigcup_{j < k} S_j) \cap V_i|}{\Phi}, \quad \text{and} \quad b_k = \sum_{i=1}^k b_{i,k}.$$

Conditions (B2) and (B3) imply that $a_{i,k} \geq (1+\varepsilon)\gamma_i r$ and $b_k \in [\tilde{\varepsilon}/4, \tilde{\varepsilon}/2]$. Then for $c = |\cup_k S_k|/\Phi$, we have $c \leq \sum_i a_{i,k} + (K-1)\tilde{\varepsilon}/2$. As a result of the upper bound in Theorem 1.8, we may assume that $\sum_i a_{i,k} \leq (1+\varepsilon)r^{r/(r-1)}$. In particular, $c = C(r,\varepsilon,K)$ is a bounded constant independent of d. There are at most n^{2rK} choices for $\{a_{i,k}\}$ and $\{b_{i,k}\}$ ($\{b_k\}$ can be determined from $\{b_{i,k}\}$). Once these are fixed, the number of tuples (S_1,\ldots,S_K) is at most

$$\left(\prod_{i=1}^r \binom{n}{a_{i,1}\Phi}\right) \prod_{k=2}^K \prod_{i=1}^r \binom{n}{b_{i,k}\Phi} \binom{c\Phi}{(a_{i,k}-b_{i,k})\Phi}$$

$$\leq \left(\prod_{i=1}^r \left(\frac{en}{a_{i,1}\Phi}\right)^{a_{i,1}\Phi}\right) \prod_{k=2}^K \prod_{i=1}^r \left(\frac{en}{b_{i,k}\Phi}\right)^{b_{i,k}\Phi} \left(\frac{ec}{(a_{i,k}-b_{i,k})}\right)^{(a_{i,k}-b_{i,k})\Phi}$$

$$= \exp\left(\sum_{i=1}^r a_{i,1}\Phi \log\left(\frac{en}{a_{i,1}\Phi}\right) + \sum_{k=2}^K \sum_{i=1}^r \left(b_{i,k}\Phi \log\left(\frac{en}{b_{i,k}\Phi}\right) + (a_{i,k}-b_{i,k})\Phi \log\left(\frac{ec}{(a_{i,k}-b_{i,k})}\right)\right)\right).$$

Plugging in the value for Φ wherever n/Φ appears, the above is at most

$$\exp\left(\frac{\Phi \log d}{r-1} \left(\sum_{i=1}^{r} a_{i,1} + \sum_{k=2}^{K} b_k + o(1)\right)\right).$$

Now, for a fixed $\{S_k\}$, there are at most $(T+1)^k$ choices for $\{t_k\}$. Let E be the edges in $K_{r\times n}$ for which there is some $k \in [K]$ such that $e \subseteq S_k$. For fixed $\{S_k\}$ and $\{t_k\}$, condition (B1) is satisfied if and only if a certain collection of at least |E| independent non-edges occur in the sampling of $\{A^{(t)}\}$. This occurs with probability at most

$$\left(1 - \frac{d}{n^{r-1}}\right)^{|E|} \le \exp\left(-\frac{|E| d}{n^{r-1}}\right).$$

Furthermore, we have

$$|E| \ge \Phi^r \prod_{i=1}^r a_{i,1} + \Phi^r \sum_{k=2}^K \left(\prod_{i=1}^r a_{i,k} - \prod_{i=1}^r (a_{i,k} - b_{i,k}) \right)$$

$$= \frac{\Phi n^{r-1} \log d}{d(r-1)} \left(\prod_{i=1}^r \frac{a_{i,1}}{r\gamma_i} + \sum_{k=2}^K \left(\prod_{i=1}^r \frac{a_{i,k}}{r\gamma_i} - \prod_{i=1}^r \frac{(a_{i,k} - b_{i,k})}{r\gamma_i} \right) \right).$$

Let $\tilde{a}_{i,k} = \frac{a_{i,k}}{r\gamma_i}$ and $\tilde{b}_{i,k} = \frac{b_{i,k}}{r\gamma_i}$. Note that $\tilde{a}_{i,k} > 1$. It can be verified by a simple calculus argument that

$$\max_{x_i > 1} \sum_{i=1}^{r} x_i - \prod_{i=1}^{r} x_i \le r - 1.$$

In particular, we have

$$\prod_{i=1}^{r} \tilde{a}_{i,k} \ge \sum_{i=1}^{r} \tilde{a}_{i,k} - r + 1.$$

The following simple claim will assist with our proof.

Claim 4.5. For $r \geq 2$ and $\varepsilon > 0$, let $x, y \in \mathbb{R}^r$ be such that for each $i \in [r]$, we have $x_i \geq x_i - y_i \geq 1 + \varepsilon$. Then,

$$\prod_{i=1}^{r} x_i - \prod_{i=1}^{r} (x_i - y_i) \ge (1 + \varepsilon)^{r-1} \sum_{j=1}^{r} y_j.$$

Proof. As a base case, let r = 2. We have

$$x_1x_2 - (x_1 - y_1)(x_2 - y_2) = y_1(x_2 - y_2) + y_2x_1$$

$$\geq (1+\varepsilon)(y_1+y_2),$$

as desired. Assume the claim holds for r < n and consider r = n. We have

$$\prod_{i=1}^{n} x_i - \prod_{i=1}^{n} (x_i - y_i) = (x_1 - y_1) \prod_{i=2}^{n} x_i - (x_1 - y_1) \prod_{i=2}^{n} (x_i - y_i) + y_1 \prod_{i=2}^{n} x_i$$

$$= (x_1 - y_1) \left(\prod_{i=2}^{n} x_i - \prod_{i=2}^{n} (x_i - y_i) \right) + y_1 \prod_{i=2}^{n} x_i$$

$$\ge (1 + \varepsilon) (1 + \varepsilon)^{n-2} \sum_{i=2}^{n} y_i + (1 + \varepsilon)^{n-1} y_1$$

$$= (1 + \varepsilon)^{n-1} \sum_{i=1}^{n} y_i,$$

as desired.

The goal is to apply the above claim with $x_i = \tilde{a}_{i,k}$ and $y_i = \tilde{b}_{i,k}$. Note that $\tilde{b}_{i,k} \leq \frac{b_k}{r \min_i \gamma_i} \leq \varepsilon/(2r)$ and so we may apply Claim 4.5 with $\varepsilon/2$ to get

$$|E| \ge \frac{\Phi n^{r-1} \log d}{d(r-1)} \left(\sum_{i=1}^r \tilde{a}_{i,k} - r + 1 + \sum_{k=2}^K (1 + \varepsilon/2)^{r-1} \sum_{i=1}^r \tilde{b}_{i,k} \right).$$

Putting everything together, we have

$$\mathbb{E}[N] \leq n^{2rK} (T+1)^k \sup_{\{a_{i,k}\}, \{b_{i,k}\}} \exp\left(\frac{\Phi \log d}{r-1} \left(\sum_{i=1}^r a_{i,1} + \sum_{k=2}^K b_k + o_{n,d}(1)\right)\right)$$

$$\exp\left(-\frac{\Phi \log d}{r-1} \left(\sum_{i=1}^r \tilde{a}_{i,k} - r + 1 + \sum_{k=2}^K (1+\varepsilon/2)^{r-1} \sum_{i=1}^r \tilde{b}_{i,k}\right)\right)$$

$$\leq n^{2rK} (T+1)^k \sup_{\{a_{i,k}\}, \{b_{i,k}\}} \exp\left(\frac{\Phi \log d}{r-1} \left(r - 1 + \sum_{i=1}^r (a_{i,1} - \tilde{a}_{i,1})\right)\right)$$

$$\exp\left(\frac{\Phi \log d}{r-1} \left(\sum_{k=2}^K \left(b_k - (1+\varepsilon/2)^{r-1} \sum_{i=1}^r \tilde{b}_{i,k}\right) + o_{n,d}(1)\right)\right).$$

Note the following:

$$\tilde{b}_{i,k} = \frac{b_{i,k}}{r\gamma_i} \ge \frac{b_{i,k}}{r \max_i \gamma_i}.$$

In particular, for ε small enough, we have the following for some constant $\alpha := \alpha(\varepsilon, \gamma) > 0$:

$$\mathbb{E}[N] \le n^{2rK} (T+1)^k \sup_{\{a_{i,k}\}, \{b_{i,k}\}} \exp\left(\frac{\Phi \log d}{r-1} \left(r-1 + \sum_{i=1}^r a_{i,1} - \alpha \sum_{k=2}^K b_k + o_{n,d}(1)\right)\right)$$

Once again, as a result of Theorem 1.8, we may assume that $\sum_{i=1}^{r} a_{i,1} \leq (1+\varepsilon)r^{r/(r-1)}$. As

 $b_k \geq \tilde{\varepsilon}/4$, for K sufficiently large in terms of r, ε , and γ , we have

$$\mathbb{E}[N] \le n^{2rK} (T+1)^k \exp\left(-\frac{\Phi \log d}{r-1}\right) = \exp\left(-\Omega(n)\right),$$

as desired. \Box

Next, we show that no independent set in $\mathcal{H}(r, n, p)$ has a large intersection with some fixed set of vertices in all partitions.

Lemma 4.6. Fix $\varepsilon, s > 0$ and $\gamma = (\gamma_1, \dots, \gamma_r)$ such that $\gamma_i \in (0,1)$ and $\sum_{i=1}^r \gamma_i = 1$. Let $d = d(\varepsilon, s, \gamma) > 0$ be sufficiently large, and for each $i \in [r]$ fix any $S_i \subseteq V_i$ such that

$$|S_i| \le s \gamma_i r n \left(\frac{\log d}{d r^r (r-1) \prod_{i=1}^r \gamma_i}\right)^{1/(r-1)}.$$

Then for $p = d/n^{r-1}$ there is no independent set S' in $\mathcal{H}(r, n, p)$ satisfying

$$|S' \cap S_i| \ge \varepsilon \gamma_i r n \left(\frac{\log d}{d r^r (r-1) \prod_{i=1}^r \gamma_i} \right)^{1/(r-1)}, \text{ for each } 1 \le i \le r,$$

with probability $1 - \exp(-\Omega(n))$.

Proof. Let Φ be as defined in the proof of Proposition 4.4. We will compute the expected number of such sets S' and show it is exponentially small (the proof then follows by Markov's inequality). Let N denote the number of possible S'. We have

$$\mathbb{E}[N] \le \left(\prod_{i=1}^{r} {|S_i| \choose \varepsilon \gamma_i r \Phi}\right) \left(1 - \frac{d}{n^{r-1}}\right)^{\varepsilon^r r^r \Phi^r \prod_{i=1}^{r} \gamma_i}$$

$$\le \left(\prod_{i=1}^{r} {s \gamma_i r \Phi \choose \varepsilon \gamma_i r \Phi}\right) \exp\left(-\frac{d}{n^{r-1}} \varepsilon^r r^r \Phi^r \prod_{i=1}^{r} \gamma_i\right)$$

$$\le \left(\prod_{i=1}^{r} \left(\frac{es}{\varepsilon}\right)^{\varepsilon \gamma_i r \Phi}\right) \exp\left(-\frac{\varepsilon^r \Phi \log d}{r-1}\right)$$

$$= \exp\left(-\Phi\left(\frac{\varepsilon^r \log d}{r-1} - \varepsilon r \log\left(\frac{es}{\varepsilon}\right)\right)\right)$$

$$= \exp\left(-\Omega(n)\right),$$

as claimed.

4.4 Proof of the Intractability Result

We will show that no (D, Γ, c) -stable algorithm can find a large γ -balanced independent set in $\mathcal{H}(r, n, p)$ in the sense of Definition 1.3. As a result of Lemma 2.5, this would complete the proof of the impossibility result.

Proposition 4.7. For any $r \geq 2$, $\varepsilon > 0$, and $\gamma = (\gamma_1, \dots, \gamma_r)$ such that $\gamma_i \in (0,1)$ and $\sum_{i=1}^r \gamma_i = 1$, there exist K, $d_0 > 0$ such that for any $d \geq d_0$ there exists $n_0, \eta, C_1, C_2 > 0$ such that for any $n \geq n_0$, $\xi \geq 1$, $1 \leq D \leq \frac{C_1 n}{\xi \log n}$, and $\delta \leq \exp(-C_2 \xi D \log n)$, if

$$k_i \geq (1+\varepsilon)\gamma_i r n \left(\frac{\log d}{d r^r (r-1) \prod_{i=1}^r \gamma_i}\right)^{1/(r-1)}, \text{ for each } 1 \leq i \leq r,$$

there is no $\left(D, K-1, \frac{\tilde{\varepsilon}}{32r\xi(1+\varepsilon)}\right)$ -stable function that $(k_1, \ldots, k_r, \delta, \xi, \eta)$ -optimizes the γ -independent set problem in $\mathcal{H}(r, n, p)$ for $p = d/n^{r-1}$.

Proof. Fix r, ε , and γ as described in the statement of the proposition. We define the following parameters:

$$\Phi = \left(\frac{\log d}{d\,r^r\,(r-1)\prod_{i=1}^r\gamma_i}\right)^{1/(r-1)}, \qquad \tilde{\varepsilon} = \varepsilon \min_{i \in [r]}\gamma_i.$$

Let K be sufficiently large in terms of r, ε, γ as described in Proposition 4.4 and let

$$T = (K-1)m, \quad c = \frac{\tilde{\varepsilon}}{32r\xi(1+\varepsilon)}, \quad \text{and} \quad \eta = \frac{\tilde{\varepsilon}\Phi}{20r}.$$

Let d_0 and n_0 be large enough such that Theorem 1.8, Proposition 4.4, and Lemma 4.6 can be applied.

Suppose for contradiction there exists a (D, K-1, c)-stable algorithm f. Sample the interpolation path $A^{(0)}, \ldots, A^{(T)}$ as in Definition 2.2 and let $I_t = V_f^{\eta}(A^{(t)})$ be the resulting independent sets for $0 \le t \le T$. We construct a sequence of sets S_1, \ldots, S_K as follows: let $S_1 = I_0$, and for $k \ge 2$, let S_k be the first I_t such that $|I_t \setminus \bigcup_{j \le k} S_j| \ge \tilde{\varepsilon} \, n \, \Phi/4$ (if no such t exists, we say the process fails). We define the following events:

- (EB1) For each $i \in [r]$ and $t \in [T]$, we have $|I_t \cap V_i| \ge (1+\varepsilon)\gamma_i r n \Phi$, and the process of constructing S_1, \ldots, S_K succeeds.
- (EB2) No edge of the interpolation path is c-bad for f.
- (EB3) The forbidden structure of Proposition 4.4 does not exist.

The following claim shows that if the events (EB1) and (EB2) occur, then so does (EB3).

Claim 4.8. If the events (EB1) and (EB2) occur, then the sequence S_1, \ldots, S_K constructed satisfy the properties forbidden by Proposition 4.4.

Proof. Let us show that $|I_t \triangle I_{t-1}| \le \tilde{\varepsilon} n \Phi/4$. As a result of (EB1), we know that the failure event setting $V_f^{\eta}(A^{(t)}) = \emptyset$ does not occur. Therefore, if $i \in I_t \triangle I_{t-1}$, we must have one of the following:

- $i \in (I \setminus \tilde{I}) \cup J$ for either $V_f^{\eta}(A^{(t)})$ or $V_f^{\eta}(A^{(t-1)})$, or
- one of $f_i(A^{(t)})$ or $f_i(A^{(t-1)})$ is ≥ 1 and the other is $\leq 1/2$.

By Definition 1.1, the first case above occurs for at most $2\eta rn$ coordinates. It follows that for at least $|I_t \triangle I_{t-1}| - 2\eta rn$ coordinates i, we have $|f_i(A^{(t)}) - f_i(A^{(t-1)})| \ge 1/2$. Therefore, as a result of event (EB2), we may conclude the following:

$$\frac{1}{4}(|I_t \triangle I_{t-1}| - 2\eta \, rn) \le \|f(A^{(t)}) - f(A^{(t-1)})\|_2^2 \le c \, \mathbb{E}[\|f(A)\|^2].$$

As f is assumed to $(k_1, \ldots, k_r, \delta, \xi, \eta)$ -optimize the γ -independent set problem in $\mathcal{H}(r, n, p)$, we have

$$\mathbb{E}[\|f(A)\|^2] \le r\xi(1+\varepsilon)n\Phi.$$

Putting the above inequalities together, we have

$$|I_t \triangle I_{t-1}| \le 2\eta \, rn + 4 \, c \, r \, \xi (1+\varepsilon) n \Phi \le \frac{\tilde{\varepsilon} \, n \, \Phi}{4},$$

as desired. Let $2 \le k \le K$ and let $t \in [T]$ be such that $S_k = I_t$. By definition of S_k , we have

$$|I_t \setminus \bigcup_{j < k} S_j| \ge \frac{\tilde{\varepsilon} n \Phi}{4}$$
, and $|I_{t-1} \setminus \bigcup_{j < k} S_j| \le \frac{\tilde{\varepsilon} n \Phi}{4}$.

From here, we may conclude that

$$|S_k \setminus \bigcup_{j < k} S_j| = |I_t \setminus \bigcup_{j < k} S_j| = |(I_t \cap I_{t-1}) \setminus \bigcup_{j < k} S_j| + |(I_t \setminus I_{t-1}) \setminus \bigcup_{j < k} S_j|$$

$$\leq |I_{t-1} \setminus \bigcup_{j < k} S_j| + |I_t \triangle I_{t-1}|$$

$$\leq \frac{\tilde{\varepsilon} n \Phi}{2}.$$

It follows that the sequence of sets S_1, \ldots, S_K satisfy the conditions in Proposition 4.4, i.e., such a forbidden structure exists as claimed.

We will now show that with positive probability all three events (EB1), (EB2), and (EB3) occur. As this contradicts Claim 4.8, this would imply no such f can exist, completing the proof.

By Theorem 1.8 and Definition 1.3, we may conclude that

$$(1+\varepsilon)\gamma_i r n \Phi \leq |I_t \cap V_i| \leq (1+\varepsilon) r^{r/(r-1)} n \Phi,$$
 for all $0 \leq t \leq T$, $1 \leq i \leq r$,

with probability at least

$$1 - (T+1)(\delta + \exp(-\Omega(n))).$$

Conditioned on such an outcome, we will show that the procedure to define S_k succeeds with high probability. Suppose for some $0 \le T' \le T - m$, we have sampled $A^{(0)}, \ldots, A^{(T')}$ and successfully selected $S_1 = I_0, \ldots, S_{K'} = I_{t_{K'}}$ for some $1 \le K' < K$. Let $S = \bigcup_{k \le K'} S_k$ and note that $A^{(T'+m)}$ is

independent of $A^{(T')}$. Applying Lemma 4.6 with

$$S_i = S \cap V_i$$
, and $s = \frac{(1+\varepsilon) r^{r/(r-1)} K'}{\min_i \gamma_i}$,

we may conclude that

$$\exists j \in [r], \quad |I_{T'+m} \cap S \cap V_j| \le \varepsilon \, r \, \gamma_j \, n\Phi,$$

with probability at least $1 - \exp(-\Omega(n))$. Fix such a j. We have:

$$|I_{T'+m} \setminus S| \ge |(I_{T'+m} \setminus S) \cap V_j| = |I_{T'+m} \cap V_j| - |I_{T'+m} \cap S \cap V_j|$$
$$\ge \frac{\tilde{\varepsilon} n \Phi}{4}.$$

In particular, $S_{K'+1} = I_t$ for some $T' < t \le T' + m$ and thus the process succeeds by timestep T = (K-1)m with high probability. We may conclude by a union bound over t that (EB1) occurs with probability at least $1 - (T+1)\delta - \exp(-\Omega(n))$.

Since f is assumed to be (D, K-1, c)-stable, (EB2) occurs with probability at least $p^{4(K-1)D/c}$, where $p = d/n^{r-1}$. By Proposition 4.4, we may conclude that at least 1 of the events (EB1)–(EB3) occurs with probability at most

$$1 - \left(\frac{d}{n^{r-1}}\right)^{4(K-1)D/c} + (T+1)\delta + \exp\left(-\Omega(n)\right). \tag{11}$$

It is now enough to show that the above is < 1, which is implied by the following for some constant C > 0 sufficiently large:

$$\left(\frac{d}{n^{r-1}}\right)^{4(K-1)D/c} \geq 2\exp\left(-Cn\right), \quad \text{and} \quad \left(\frac{d}{n^{r-1}}\right)^{4(K-1)D/c} \geq 2\delta Km.$$

The first inequality follows by the upper bound on D for $C_1 := C_1(\varepsilon, K, d) > 0$ sufficiently large, and the second follows by the upper bound on δ for $C_2 := C_2(\varepsilon, K, d) > 0$ sufficiently large. Plugging this in above, we have that (11) is at most

$$1 - \exp(-Cn) - \delta Km + (T+1)\delta + \exp(-\Omega(n))$$

$$\leq 1 - \exp(-\Omega(n)) + \delta(T+1-Km)$$

$$= 1 - \exp(-\Omega(n)) - \delta(m-1) < 1,$$

for C sufficiently large, completing the proof.

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