

# Hardness of circuit and monotone diameters of polytopes

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## Abstract

The *Circuit diameter* of polytopes was introduced by Borgwardt, Finhold and Hemmecke [BFH15] as a fundamental tool for the study of circuit augmentation schemes for linear programming and for estimating combinatorial diameters. Determining the complexity of computing the circuit diameter of polytopes was posed as an open problem by Sanità [San20] as well as by Kafer [Kaf22], and was recently reiterated by Borgwardt, Grewe, Kafer, Lee and Sanità [BGKLS24]. In this paper, we solve this problem by showing that computing the circuit diameter of a polytope given in halfspace-description is strongly NP-hard. To prove this result, we show that computing the combinatorial diameter of the *perfect matching polytope* of a bipartite graph is NP-hard. This complements a result by Sanità (FOCS 2018, [San18]) on the NP-hardness of computing the diameter of fractional matching polytopes and implies the new result that computing the diameter of a  $\{0, 1\}$ -polytope is strongly NP-hard, which may be of independent interest. In our second main result, we give a precise graph-theoretic description of the *monotone diameter* of perfect matching polytopes and use this description to prove that computing the monotone (circuit) diameter of a given input polytope is strongly NP-hard as well.

## 1 Introduction

One of the most central open problems in the theory of mathematical optimization is *Smale's 9th problem*, included in his list of open problems for the 21st century [Sma98]. It asks for a strongly polynomial time algorithm for the linear programming problem, i.e., the algorithmic problem of optimizing a linear functional subject to linear inequality and equality constraints.

One of the canonical candidates that still holds potential for a positive resolution of Smale's problem is the famous *simplex method*, whose basic form was invented by George Dantzig around 1950. Roughly speaking, to solve a given linear program, the simplex method updates an extreme point of the (polyhedral) feasible region while moving along its edges in such a way that the objective value is successively improved. In particular, the sequence of extreme points visited by the simplex method describes a path on the 1-skeleton of the constraint-polyhedron. While efficient in practice, the theoretical complexity of the simplex method is a well-known open problem. In particular, the existence of a pivot rule that would make the simplex algorithm run in strongly polynomial time remains unknown.

Suppose we would like to minimize a linear functional  $\mathbf{c}^T \mathbf{x}$  over a polytope  $P \subseteq \mathbb{R}^d$ . Then the number of steps taken by an execution of the simplex algorithm using *any* pivot-rule is lower-bounded by the minimum length of a path in the 1-skeleton of  $P$  connecting the starting vertex to the optimal solution. Since, by varying  $\mathbf{c}$ , every vertex of  $P$  can be made the (unique) optimal solution of the corresponding linear program, and since the starting vertex in the simplex method can also be chosen arbitrarily, this shows that a lower bound for the complexity of the

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simplex method is given by the *diameter*<sup>1</sup>  $\text{diam}(P)$  of  $P$ . In fact, since the simplex algorithm always follows a *c-monotone* path on the polytope, i.e. a path along which the objective value  $\mathbf{c}^T \mathbf{x}$  is non-increasing, a stronger lower bound holds, which is called the *monotone diameter*<sup>2</sup>  $\text{mdiam}(P)$  of the polytope  $P$ .

In consequence, a necessary condition for the existence of a strongly polynomial pivot rule is that the (monotone) diameter of every polytope  $P \subseteq \mathbb{R}^d$  with  $n$  facets is bounded by a polynomial function in  $n$  and  $d$ . However, even this relaxed problem, known as the *polynomial Hirsch conjecture*, remains unsolved. The classical *Hirsch conjecture* stated that every  $d$ -dimensional polytope  $P$  with  $n$  facets satisfies  $\text{diam}(P) \leq n - d$ , but in 2012 Santos constructed a family of counterexamples to this conjecture [San12]. Interestingly however, the Hirsch conjecture could still survive for a natural variant of the combinatorial diameter, called the *circuit diameter* of polytopes. This concept was introduced by Borgwardt, Finhold and Hemmecke [BFH15] and will be a main focus of this paper. The study of the circuit diameter is motivated by the recently popular concepts of *circuit moves* and *circuit augmentation schemes* which, as the simplex method, provide a framework for designing algorithms for linear programming. Roughly speaking, circuit moves extend the simplex paradigm of traversing the edges of a polytope, by allowing to follow additional directions through the polytope, called *circuits*. Several pivot-rules and algorithms for linear and combinatorial optimization based on circuit moves have been proposed and studied recently, see e.g. [DHL15; BV20; BBFK21; DKS22; DKNV22; BM23]. The following gives formal definitions of the fundamental notions related to circuits.

**Definition 1** (cf. Definitions 1–4 in [DKS22]). *Consider a polyhedron*

$$P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} = \mathbf{b}, B\mathbf{x} \leq \mathbf{d}\}.$$

- (i) *A circuit of  $P$  is a vector  $\mathbf{g} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$  such that*
  - $A\mathbf{g} = \mathbf{0}$ , and
  - $B\mathbf{g}$  is inclusion-wise support-minimal in the collection  $\{B\mathbf{y} \mid A\mathbf{y} = \mathbf{0}, \mathbf{y} \neq \mathbf{0}\}$ .
- (ii) *Given a point  $\mathbf{x} \in P$ , a circuit move at  $\mathbf{x}$  consists of selecting a circuit  $\mathbf{g}$  of  $P$  and moving to a new point  $\mathbf{x}' = \mathbf{x} + \alpha\mathbf{g}$ , where  $\alpha > 0$  is maximal w.r.t.  $\mathbf{x} + \alpha\mathbf{g} \in P$ .*
- (iii) *A circuit walk of length  $k$  is a sequence  $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k$  of points in  $P$  such that for every  $i = 1, \dots, k$ , we have that  $\mathbf{x}_i$  is obtained from  $\mathbf{x}_{i-1}$  by a circuit move.*
- (iv) *Given two points  $\mathbf{x}, \mathbf{x}' \in P$ , the circuit distance  $\text{cdist}(\mathbf{x}, \mathbf{x}')$  is defined as the minimum length of a circuit walk that starts in  $\mathbf{x}$  and ends in  $\mathbf{x}'$ .*

We refer to [Fin15] for a detailed discussion and analysis of these concepts. With the above notions in place, we may now give a definition of the circuit diameter of polytopes.

**Definition 2** (cf. Definition 5 in [DKS22]). *The circuit diameter of a polytope  $P$ , denoted  $\text{cdiam}(P)$ , is the maximum circuit distance among all pairs of vertices of  $P$ .*

In the same way as the ordinary diameter of a polytope  $P$  forms a lower bound on the run-time of the simplex-method for linear programs with feasible region  $P$ , we have that the circuit diameter  $\text{cdiam}(P)$  forms a lower bound on the time complexity of circuit augmentation schemes for linear optimization over  $P$ . This motivates bounding the circuit diameter of  $d$ -dimensional polytopes with  $n$  facets by a polynomial function in  $n$  and  $d$ . While this remains open, DeLoera, Kafer and Sanità [DKS22] could recently prove the remarkable result that the circuit diameter of a polytope is polynomially bounded in terms of  $n$ ,  $d$  and the maximum encoding-length among the coefficients in its description. As alluded to above, the analogue of the classical Hirsch conjecture for circuit diameters, proposed in 2015 by Borgwardt, Finhold and Hemmecke [BFH15] has still neither been proved nor disproved.

<sup>1</sup>The diameter of a polytope  $P$  is defined as the diameter of the graph formed by the vertices and edges of  $P$ .

<sup>2</sup>We postpone the formal definition of the monotone diameter to Definition 5 further below.

**Conjecture 3** ([BFH15]). *Every polytope  $P \subseteq \mathbb{R}^d$  with  $n$  facets satisfies  $\text{cdiam}(P) \leq n - d$ .*

Many more results on the circuit diameter, including general bounds as well as for special classes of polytopes, have been obtained recently, see [BFH15; SY15; BSY18; KPS19; DKNV22; BBB23], providing positive evidence towards Conjecture 3.

## Our results

**Complexity of the circuit diameter.** A long-standing open question in linear programming and discrete geometry is to determine the complexity of computing the diameter of a given input polytope. A classical result in this direction is due to Frieze and Teng [FT94] from 1994, who showed that the problem of computing the diameter of a polytope  $P$ , given in halfspace-description, is *weakly* NP-hard. In a more recent breakthrough, Sanità [San18] strengthened this result by showing that the same problem is in fact *strongly* NP-hard. In this paper, we address the natural analogous problem for the circuit diameter, as follows.

### CIRCUIT DIAMETER

**Input:** A matrix  $A \in \mathbb{Q}^{m \times d}$  and a vector  $\mathbf{b} \in \mathbb{Q}^m$ , defining the polytope

$$P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}.$$

**Output:**  $\text{cdiam}(P)$ .

Determining the computational complexity of CIRCUIT DIAMETER was raised as an open problem by Sanità [San20]) as well as by Kafer [Kaf22]. Very recently, the problem was reiterated by Borgwardt et al. [BGKLS24]. In our first main result, we solve this problem by showing that CIRCUIT DIAMETER is likely computationally intractable.

**Theorem 4.** CIRCUIT DIAMETER *is strongly* NP-hard.

**Complexity of the monotone (circuit) diameter.** Our second result addresses the complexity of computing the monotone versions of the diameter and circuit diameter of polytopes. We need the following formal definitions.

**Definition 5** (cf. Definitions 1.1, 1.2 in [BDL21] and [BBB23], page 2). *Let  $P \subseteq \mathbb{R}^d$  be a polytope.*

- (i) *Let  $\mathbf{c} \in \mathbb{R}^d$  be an objective direction. A sequence of points  $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k$  of  $P$  is called  $\mathbf{c}$ -monotone if  $\mathbf{c}^T \mathbf{x}_{i+1} \leq \mathbf{c}^T \mathbf{x}_i$  for every  $i \in \{0, \dots, k-1\}$ . We say that a path on the skeleton of  $P$  or a circuit-walk in  $P$  is  $\mathbf{c}$ -monotone if it corresponds to a  $\mathbf{c}$ -monotone sequence.*
- (ii) *Given  $\mathbf{c} \in \mathbb{R}^d$ , the  $\mathbf{c}$ -monotone diameter  $\text{mdiam}(P, \mathbf{c})$  is defined as the maximum, taken over all vertices  $\mathbf{x}$  of  $P$ , of the length of a shortest  $\mathbf{c}$ -monotone path from  $\mathbf{x}$  to a  $\mathbf{c}$ -optimal vertex<sup>3</sup> of  $P$ .*
- (iii) *Given  $\mathbf{c} \in \mathbb{R}^d$ , the  $\mathbf{c}$ -monotone circuit diameter  $\text{mcdiam}(P, \mathbf{c})$  is defined as the maximum, taken over all vertices  $\mathbf{x}$  of  $P$ , of the length of a shortest  $\mathbf{c}$ -monotone circuit walk from  $\mathbf{x}$  to a  $\mathbf{c}$ -optimal vertex of  $P$ .*
- (iv) *Finally, the monotone diameter and the monotone circuit diameter of  $P$  are defined as*

$$\text{mdiam}(P) := \max_{\mathbf{c} \in \mathbb{R}^d} \text{mdiam}(P, \mathbf{c}), \quad \text{mcdiam}(P) := \max_{\mathbf{c} \in \mathbb{R}^d} \text{mcdiam}(P, \mathbf{c}).$$

Concretely, we are interested in the following two algorithmic problems.

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<sup>3</sup>A vertex  $v$  of  $P$  is called  $\mathbf{c}$ -optimal, if  $\mathbf{c}^T v \leq \mathbf{c}^T w$  for every vertex  $w$  of  $P$ .

**MONOTONE DIAMETER**

**Input:** A matrix  $A \in \mathbb{Q}^{m \times d}$  and a vector  $\mathbf{b} \in \mathbb{Q}^m$ , defining the polytope

$$P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}.$$

**Output:**  $\text{mdiam}(P)$ .

**MONOTONE CIRCUIT DIAMETER**

**Input:** A matrix  $A \in \mathbb{Q}^{m \times d}$  and a vector  $\mathbf{b} \in \mathbb{Q}^m$ , defining the polytope

$$P = \{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}.$$

**Output:**  $\text{mcdiam}(P)$ .

As our second main contribution, we show the novel result that also these monotone versions of the diameter and circuit diameter are likely computationally intractable.

**Theorem 6.** *MONOTONE DIAMETER and MONOTONE CIRCUIT DIAMETER are both strongly NP-hard.*

**Perfect matching polytopes.** In order to prove [Theorems 4](#) and [6](#) we show the hardness of determining the diameter and the monotone diameter within a famous class of graph-based  $\{0, 1\}$ -polytopes, called *perfect matching polytopes*, defined as follows.

**Definition 7.** *Let  $G = (V, E)$  be a bipartite graph. For every perfect matching  $M$  of  $G$ , define the vector  $\chi^M \in \mathbb{R}^E$  that has a 1-entry for every  $e \in M$  and all other entries equal to 0. The perfect matching polytope  $P_G$  associated with  $G$  is the polytope in  $\mathbb{R}^E$  defined as the convex hull*

$$P_G = \text{conv}\{\chi^M \mid M \text{ perfect matching of } G\}.$$

It is well-known (see, e.g. Chapter 18 in [\[Sch+03\]](#)) that for every bipartite graph  $G = (V, E)$ , the polytope  $P_G$  also admits a compact halfspace-encoding. Namely, an edge-indexed vector  $(x_e)_{e \in E} \in \mathbb{R}^E$  belongs to  $P_G$  if and only if the following hold.

$$\begin{aligned} \sum_{e \ni v} x_e &= 1, & (\forall v \in V) \\ x_e &\geq 0, & (\forall e \in E). \end{aligned}$$

Our main motivation for considering the diameter of perfect matching polytopes is the following simple connection between its circuit moves and its edge moves:

**Theorem** (cf. Lemma 2 in [\[CS23\]](#)). *Let  $G$  be a bipartite graph, let  $\mathbf{x}$  be a vertex of  $P_G$ , and let  $\mathbf{x}' \in P_G$ . Then  $\mathbf{x}'$  can be obtained from  $\mathbf{x}$  by a circuit move if and only if  $\mathbf{x}'$  is also a vertex of  $P_G$  and adjacent to  $\mathbf{x}$  on the skeleton of  $P_G$ .*

In particular, the preceding result implies that any circuit walk starting at a vertex of a perfect matching polytope will always move along edges to other vertices, and thus form a walk on the skeleton of the polytope. This directly implies that for every perfect matching polytope  $P$  of a bipartite graph, we have  $\text{diam}(P) = \text{cdiam}(P)$  as well as  $\text{mdiam}(P) = \text{mcdiam}(P)$ . Together with the fact that all coefficients in the halfspace-description of  $P_G$  are either 0 or 1, this shows that the following two new results imply [Theorems 4](#) and [6](#), respectively.

**Theorem 8.** *The following problem is NP-hard: Given as input a bipartite graph  $G$ , determine the diameter of the associated perfect matching polytope  $P_G$ .*

**Theorem 9.** *The following problem is NP-hard: Given as input a bipartite graph  $G$ , determine the monotone diameter of the associated perfect matching polytope  $P_G$ .*

Besides establishing Theorems 4 and 6, the Theorems 8 and 9 and their proofs have further interesting consequences:

- (1) Theorem 8 provides an alternative proof for the strong NP-hardness of computing the diameter of polytopes, shown first by Sanità [San18]. Interestingly, the proof of this result in [San18] also used a special family of combinatorial polytopes that is closely related to perfect matching polytopes. Namely, Sanità proved that computing the diameter of the *fractional matching polytope* of a given input graph is NP-hard. Theorem 8 nicely complements this prior result by proving hardness for another well-studied class of combinatorial polytopes relating to matchings in graphs.
- (2) To prove Theorem 9, we use a precise combinatorial description (Lemma 21) of the monotone diameter of a perfect matching polytope  $P_G$  of a bipartite graph  $G$  in terms of a restricted cycle-packing invariant in  $G$ . A direct consequence of this description is that  $\text{mdiam}(P_{K_{n,n}}) = \lfloor \frac{n}{2} \rfloor$  for every number  $n$ . The polytope  $P_{K_{n,n}}$  is also known as the *assignment polytope*, and its monotone diameter was previously determined by Rispoli [Ris92].
- (3) Since perfect matching polytopes of bipartite graphs form  $\{0, 1\}$ -polytopes with a totally unimodular constraint matrix, Theorems 8 and 9 also have the following consequence, that appears to be novel and possibly of independent interest.

**Corollary 10.** *Determining the diameter and the monotone diameter of  $\{0, 1\}$ -polytopes are both strongly NP-hard problems, even when restricted to polytopes with a totally unimodular constraint matrix.*

**Related work.** Cardinal and the second author [CS23] proved that (monotone) circuit distances in perfect matching polytopes cannot be approximated to within any constant factor, unless  $P = NP$ . In a similar direction, Borgwardt et al. [BGKLS24] recently proved that computing exact circuit distances is NP-hard for 0/1-network flow polytopes. While conceptually related to our Theorems 8 and 9, the results are incomparable. In particular, one major difference between the approaches in [CS23; BGKLS24] and the diameter problem considered here is that the reductions in [CS23; BGKLS24] are based on *very* short paths, namely of length two, while our reduction here naturally needs to deal with *maximal distances* and thus with rather long paths in the perfect matching polytope. Strengthening Theorems 8 and 9 to an inapproximability result for the diameter would be quite interesting, but this seems to be outside the reach of the methods used in our proof.

## 2 Proof of Theorem 4

In this section, we present the proof of our first main result. We start by stating the simple characterization of adjacency on perfect matching polytopes of bipartite graphs, which reduces the analysis of the diameter of these polytopes to a purely graph-theoretic issue.

**Lemma** (cf. [Chv75], [Iwa02]). *Let  $G = (V, E)$  be a bipartite graph. Consider two vertices  $\mathbf{x} = \chi^M$  and  $\mathbf{y} = \chi^{M'}$  of  $P_G$  corresponding to perfect matchings  $M$  and  $M'$  in  $G$ . Then  $\mathbf{x}$  and  $\mathbf{y}$  are adjacent in the skeleton of  $P_G$  if and only if the symmetric difference  $M \Delta M'$  consists of the edge-set of exactly one cycle in  $G$ .*

From the above we can see that the problem of determining the (combinatorial) diameter of the perfect matching polytope of  $G$  boils down to determining the maximal distance of two

matchings in  $G$ , when one is allowed to flip<sup>4</sup> a single alternating cycle at a time. As we will work with this iterative flipping of cycles from now on it will be useful to introduce the following definition.

**Definition 11.** Let  $G = (V, E)$  be a graph and let  $M_1, M_2$  be two perfect matchings in  $G$ . We call a collection of cycles  $\mathcal{C} = (C_1, \dots, C_\ell)$  in  $G$  a flip sequence of length  $\ell$  from  $M_1$  to  $M_2$  if

- (i) For every  $i \in \{1, \dots, \ell\}$  the cycle  $C_i$  is  $M_1 \Delta C_1 \Delta C_2 \Delta \dots \Delta C_{i-1}$ -alternating, and
- (ii)  $M_2 = M_1 \Delta C_1 \Delta C_2 \Delta \dots \Delta C_\ell$ .

We will reduce the HAMILTONIAN CYCLE PROBLEM to the problem of determining the diameter of the perfect matching polytope  $P_G$  of a given graph  $G$ . Recall that the Hamiltonian cycle problem is the algorithmic decision problem of deciding whether or not an input graph contains a Hamiltonian cycle, and that this problem is well-known to be NP-complete [GJ90].

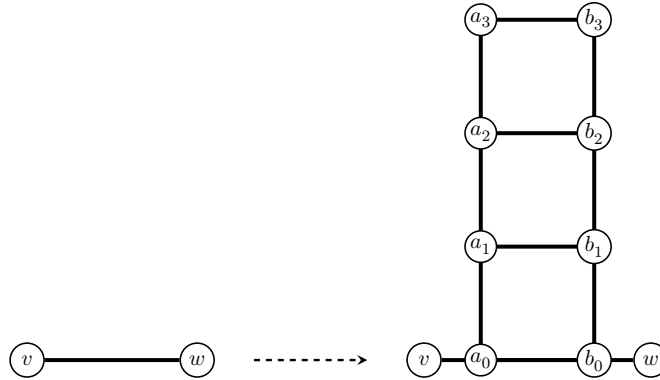
Let an input graph  $H = (V_H, E_H)$  be given, for which we would like to determine whether it contains a Hamiltonian cycle. Set  $n = |V_H|$ . In the following we will describe the construction of a bipartite graph  $G_H$  based on  $H$  such that we can determine whether or not  $H$  is Hamiltonian just based on knowing the value of the diameter of the perfect matching polytope  $P_{G_H}$  associated with  $G_H$ .

Before we can state the final construction, it will be convenient to first introduce an auxiliary gadget, which we will call a *tower* in the following, and to analyze some of its basic properties with respect to flip-sequences of alternating cycles.

For the definition of a tower, consider any edge  $e$  between vertices  $v$  and  $w$  in some graph. A tower  $T$  of height  $h$  on  $e$  is obtained by removing the edge  $\{v, w\}$  and replacing it by  $2h + 2$  vertices  $a_i, b_i$  for  $i \in \{0, \dots, h\}$  together with edges

$$E_T = \{\{a_i, b_i\} : i \in \{0, \dots, h\}\} \cup \{\{a_i, a_{i+1}\}, \{b_i, b_{i+1}\} : i \in \{0, \dots, h-1\}\} \cup \{\{v, a_0\}, \{b_h, w\}\}.$$

An example of a tower of height 3 can be found in Figure 1. If the tower we consider is clear from the context we use  $a_i$  and  $b_i$  as above, if there are multiple towers under consideration then we may add a superscript and use  $a_i^T$  and  $b_i^T$ , respectively, to reference the vertices of a tower  $T$ .

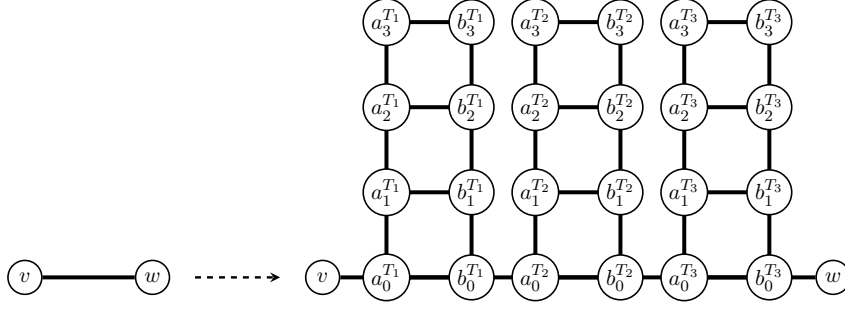


**Figure 1:** A tower of height 3 on the edge  $\{v, w\}$

Next we also introduce a gadget to combine multiple towers. This can be done by iterating the above construction. In order to build  $t$  towers of height  $h$  on an edge  $e$  we first use the above construction to build a single tower  $T_1$ . Once we constructed a total of  $k-1$  towers  $T_1, \dots, T_{k-1}$  on the edge  $e$  we proceed by building a tower gadget on the edge  $\{b_0^{T_{k-1}}, w\}$ . Figure 2 shows an example of three towers of height 3 constructed on an edge.

<sup>4</sup>If  $C$  is a cycle in  $G$  that is alternating w.r.t. the current perfect matching  $M$ , then flipping  $C$  means exchanging matching and non-matching edges along  $C$ , i.e., moving from  $M$  to the new perfect matching given by the symmetric difference  $M \Delta C$ .





**Figure 2:** Three towers  $T_1, T_2$  and  $T_3$  of height 3 on the edge  $\{v, w\}$ .

We now construct the auxiliary graph  $G_H$  from  $H$ . For this let  $h > |V_H|$  and  $t = 4h$  be numbers we choose precisely later. As a first step we make the graph  $H$  bipartite. To do so duplicate all vertices into pairs, to obtain the following set of new vertices:

$$V = \{v_1 : v \in V_H\} \cup \{v_2 : v \in V_H\}.$$

We will also use the notation  $V_1$  for vertices from the first set and  $V_2$  for vertices of the second set. Next we duplicate and redirect all edges of  $H$  and add edges between the two copies of every vertex, such that we obtain the following new set of edges:

$$E = \{\{v_1, v_2\} : v \in V_H\} \cup \{\{v_1, w_2\}, \{v_2, w_1\} : \{v, w\} \in E_H\}.$$

Altogether, this yields a new graph  $G'_H = (V, E)$  on  $2n$  vertices. To go from here to the final graph  $G_H$ , we take a second step.

Namely, for every edge in  $G'_H$  of the form  $\{v_1, v_2\}$ , i.e., an edge between two copies of the same vertex  $v$  of  $H$ , we add  $t$  new tower gadgets of height  $h$  onto it, as described above, and call the resulting graph  $G_H$ . A visualization of this construction can be found in [Figure 3](#).

We will obtain the claimed hardness result in [Theorem 4](#) by showing the following bounds on the diameter of the perfect matching polytope of  $G_H$ .

**Theorem 12.** *If  $H$  is Hamiltonian, then  $\text{diam}(P_{G_H}) \leq 2h + 4n$ .*

**Theorem 13.** *If the graph  $H$  has at least 3 vertices, then  $\text{diam}(P_{G_H}) \geq \frac{n}{n-1}(2h - 2)$ .*

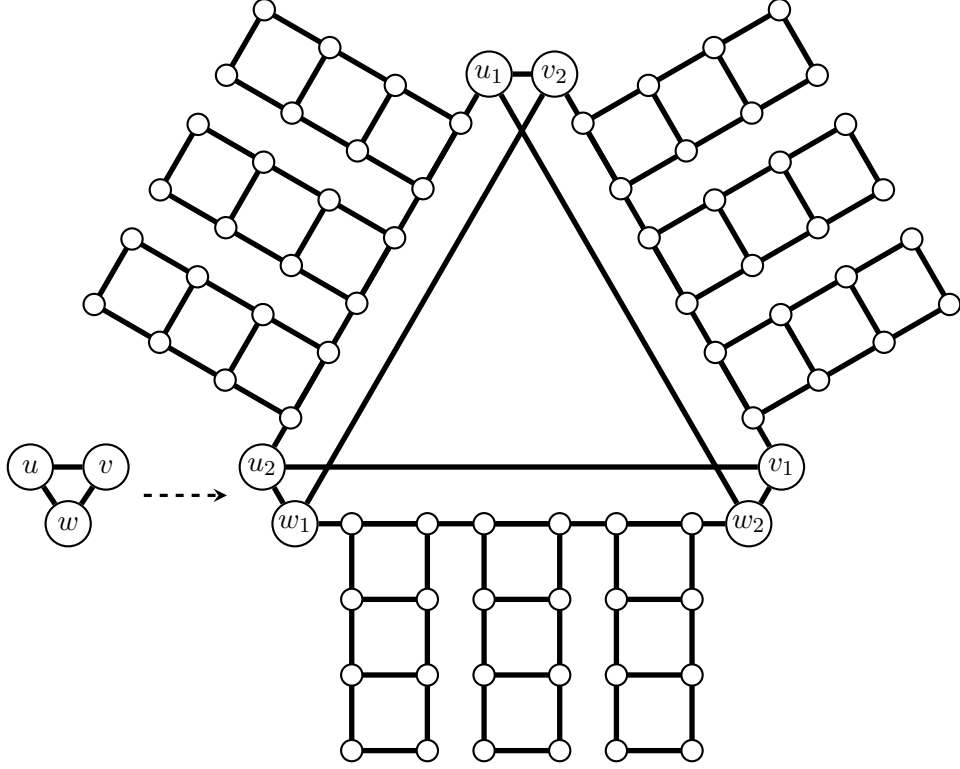
Assume for now that the two theorems hold, and let us deduce [Theorem 4](#) from it.

To do so, we choose  $h$  as  $2n^2 - n + 1 = O(n^2)$  and  $t = 4h = 8n^2 - 4n + 4$  in our reduction (which in particular ensures that  $G_H$  has polynomial size in terms of  $n$ , namely its size is in  $O(n^5)$ ). It can easily be checked that with this choice of  $h$ , we have  $2h + 4n < \frac{n}{n-1}(2h - 2)$ . It then follows from [Theorem 12](#) and [Theorem 13](#) that  $H$  is Hamiltonian if and only if the diameter of  $P_H$  is at most  $2h + 2n$ .

Therefore, we obtain a polynomial reduction from the Hamiltonian cycle problem to the problem of determining the diameter of the perfect matching polytope of a bipartite graph. This then yields the NP-hardness of the problem of computing the diameter of the perfect matching polytope of a given bipartite graph, and thus the statement of [Theorem 8](#). As argued in the introduction, [Theorem 8](#) then directly also implies our first main result, [Theorem 4](#).

Thus what remains to show are [Theorems 12](#) and [13](#), which we will establish in the following two sections. Before we proceed we need two more definitions, that will enable us to closely analyze matchings within one tower.

**Definition 14.** *Let  $G = (V, E)$  be a graph and let  $e = \{v, w\} \in E$  be an edge of  $G$ . Consider the graph  $G_T$  obtained from  $G$  by constructing a tower  $T$  on  $e$ . We say that a cycle  $C$  in  $G_T$  touches the tower  $T$ , if  $C$  contains the edges  $\{v, a_0\}$  and  $\{b_0, w\}$ .*



**Figure 3:** Illustration of the construction performed in the proof of [Theorem 4](#). We start with the triangle graph on the vertices  $\{u, v, w\}$  given on the left. After splitting the nodes, duplicating and redirecting the edges and finally adding the tower gadgets we arrive at the graph  $G_H$  on the right. For the sake of presentation the height of the towers as well as their number on each edge is reduced compared to the actual construction.

In particular a cycle that touches the tower  $T$  can be separated into a  $v$ – $w$  path through the vertices of  $T$  and a  $w$ – $v$  path in  $G \setminus e$ .

**Definition 15.** Let  $G = (V, E)$  be a graph and let  $e = \{v, w\} \in E$  be an edge of  $G$ . Consider the graph  $G_T$  obtained from  $G$  by constructing a tower  $T$  on  $e$ . Let  $M$  be a matching in  $G_T$ . Define  $\mathcal{H}(M) = \{i \in \{0, \dots, h\} : \{a_i, b_i\} \in M\}$ , which we call the horizontal indices of  $M$ . Furthermore set  $d(M) = \min(\mathcal{H}(M))$  which we call the depth of  $M$ .

As a final bit of preparation we need the following statement on the structural changes of  $M$  when augmenting along a  $v$ – $w$  path:

**Lemma 16.** Let  $G = (V, E)$  be a graph and let  $e = \{v, w\} \in E$  be an edge of  $G$ . Consider the graph  $G_T$  obtained from  $G$  by constructing a tower  $T$  of height  $h$  on  $e$ . Let  $M$  be a perfect matching in  $G_T$  and let  $C$  be an  $M$ –alternating cycle in  $G_T$  that touches  $T$ . Finally set  $M' = M \Delta C$  to be the matching we obtain from  $M$  by flipping along  $C$ . Then

- (i)  $|\mathcal{H}(M) \Delta \mathcal{H}(M')| = 1$ , and
- (ii)  $\mathcal{H}(M) \setminus \mathcal{H}(M') \subseteq \{d(M)\}$ .

So, in words, flipping the edges of  $C$  the number of horizontal edges changes by exactly one and if we remove a horizontal edge of  $M$ , then it must be the edge  $\{a_{d(M)}, b_{d(M)}\}$ .

*Proof.* To prove the first part, let  $k$  be the maximum number for which  $C$  contains the edge  $\{a_{k-1}, a_k\}$ . Then  $C$  must contain the edge  $\{a_k, b_k\}$  as well. As  $C$  touches  $T$  its restriction to  $T$  is a  $v$ – $w$  path. It follows from the structure of  $T$  that this restriction consists of the edges  $\{v, a_0\}$ ,  $\{b_0, w\}$ ,  $\{a_k, b_k\}$  together with the edges  $\{a_{i-1}, a_i\}$  and  $\{b_i, b_{i-1}\}$  for  $i \in \{1, \dots, k\}$ . In particular we have  $\mathcal{H}(M) \Delta \mathcal{H}(M') = \{k\}$ , proving the first statement.



To prove the second statement, and using the notation from above, it suffices to show that  $k \leq d(M)$ . Assume  $k \geq d(M)$ , so  $C$  contains the vertex  $a_{d(M)}$ . As  $C$  is  $M$ -alternating it must contain the matching edge incident to  $a_{d(M)}$ , which is the edge  $\{a_{d(M)}, b_{d(M)}\}$ . So  $k \geq d(M)$  implies  $k = d(M)$  and  $k \leq d(M)$  holds as claimed.  $\square$

## 2.1 Proof of Theorem 12

In this section we want to show, that if  $H$  is Hamiltonian, then the diameter of  $P_{G_H}$  is at most  $2h + 4n$ . By the discussion from the first section this is equivalent to showing that given any two matchings in  $G_H$  we can transform one to the other by flipping along at most  $2h + 4n$  many cycles. In order to do so, we need to consider how we can extend flip sequences over a tower. This is summarized in the following technical lemma.

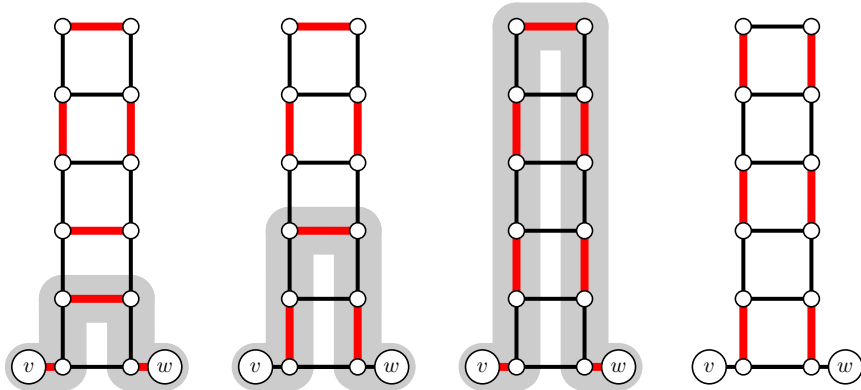
**Lemma 17.** *Let  $G = (V, E)$  be a graph,  $e = \{v, w\} \in E$  be an edge of  $G$  and  $\widetilde{M}_1$  and  $\widetilde{M}_2$  be two perfect matchings in  $G$  that contain  $e$ . Let  $h \in \mathbb{Z}_{\geq 1}$ . Furthermore assume we are given a flip sequence  $\mathcal{C} = (C_1, \dots, C_{2h})$  of length  $2h$  from  $\widetilde{M}_1$  to  $\widetilde{M}_2$ , such that  $e \in C_i$  for all  $i \in \{1, \dots, 2h\}$ . Consider the graph  $G_T$  obtained by constructing a tower  $T$  of height  $h$  on  $e$ . Let  $M_1$  and  $M_2$  be two perfect matchings in  $G_T$  such that for  $i \in \{1, 2\}$  the matching  $M_i$  agrees with  $\widetilde{M}_i$  outside of the tower gadget  $T$ . In particular the matching  $M_i$  contains the edges  $\{v, a_0\}$  and  $\{b_0, w\}$ .*

*Then there exists a flip sequence  $\mathcal{C}' = (C'_1, \dots, C'_{2h})$  of length  $2h$  from  $M_1$  to  $M_2$ .*

*Proof.* The idea is to first construct a local transformation with  $v$ - $w$  paths in the tower  $T$  to transform the restriction of  $M_1$  to  $T$  to the restriction of  $M_2$  to  $T$ . These can afterwards be combined with the cycles of  $\mathcal{C}$  to obtain  $\mathcal{C}'$ .

As noted in the proof of Lemma 16, every  $v$ - $w$  path  $P$  in the tower  $T$  is uniquely defined by a number  $k$  such that  $\{a_k, b_k\} \in E(P)$ . Let  $P_k$  denote the unique  $v$ - $w$  path with  $\{a_k, b_k\} \in E(P_k)$ , i.e.  $P_k$  is the path with vertex sequence  $(v, a_0, a_1, \dots, a_k, b_k, \dots, b_1, b_0, w)$ . Pause to note that for any perfect matching  $N$  in  $G_T$  the path  $P_{d(N)}$  is  $N$ -alternating; by definition of  $d(N)$ .

Let  $\mathcal{H}(M_1) = \{h_1, \dots, h_r\}$  with  $h_1 < h_2 < \dots < h_r$  denote the horizontal indices of  $M_1$ . Note that  $h_1 > 0$ , as  $M_1$  has to match  $v$  with  $a_0$  and  $w$  with  $b_0$ . Consider the collection  $\mathcal{P}_1 = (P_{h_1}, \dots, P_{h_r})$ . By definition we have  $h_1 = d(M_1)$  and  $P_{h_1}$  is  $M_1$ -alternating. Thus  $M_1 \Delta P_{h_1}$  is a matching. Observe that  $d(M_1 \Delta P_{h_1}) = h_2$ , and hence  $P_{h_2}$  is  $M_1 \Delta P_{h_1}$ -alternating. Iterating this argument gives  $d(M_1 \Delta P_{h_1} \Delta \dots \Delta P_{h_{i-1}}) = h_i$  and thus  $P_{h_i}$  is  $M_1 \Delta P_{h_1} \Delta \dots \Delta P_{h_{i-1}}$ -alternating.



**Figure 4:** Example of the transformation from Lemma 17. We transform an arbitrary matching in a tower  $T$  to the unique matching with no horizontal edges. In each step we mark the alternating  $v$ - $w$  path we use.

Altogether we get that the restriction  $N$  of  $M_1 \Delta P_{h_1} \Delta \dots \Delta P_{h_r}$  to  $T$  is a matching covering the vertices  $a_i$  and  $b_i$  for all  $i \in \{0, \dots, h\}$  that contains no horizontal edge. In particular  $N$  does not contain  $\{a_h, b_h\}$ . Thus, it has to match  $a_h$  and  $b_h$  to  $a_{h-1}$  and  $b_{h-1}$ , respectively. Repeating

this argument shows that every such matching has to match  $a_{h-2i}$  and  $b_{h-2i}$  to  $a_{h-2i-1}$  and  $b_{h-2i-1}$ , respectively, for  $i \in \{0, \dots, \lfloor \frac{h}{2} \rfloor\}$ .

Doing likewise for  $M_2$  gives rise to a collection  $\mathcal{P}_2$  of paths, which transforms  $M_2$  to a perfect matching whose restriction to  $T$  contains no horizontal edges. By the above argument, this property uniquely defines the restriction of perfect matchings to  $T$ , and thus this restriction must coincide with  $N$ .

Let  $\mathcal{P}$  be the collection of paths consisting of the paths of  $\mathcal{P}_1$ , followed by the paths of  $\mathcal{P}_2$  in reverse order. Note that  $|\mathcal{P}| = |\mathcal{H}(M_1)| + |\mathcal{H}(M_2)| \leq 2h$ . Furthermore when flipping all the paths of  $\mathcal{P}$  the restriction of  $M_1$  to  $T$  is transformed to the restriction of  $M_2$  to  $T$ . In order to get the correct number of paths, we append to  $\mathcal{P}$  the path  $P = (v, a_0, b_0, w)$  a number of times such that we end up with  $2h$  paths in total. First we need to check that the path  $P$  is alternating with respect to the restriction of  $M_2$  to  $T$ . This is due to the fact that  $M_2$  contains the edges  $\{v, a_0\}$  and  $\{b_0, w\}$ . After flipping the edges of an alternating path this path remains alternating, so appending the copies of  $P$  still gives a sequence of alternating paths.

In order to get the desired collection  $\mathcal{C}'$  we define  $\mathcal{C}'_i$  to be the cycle in  $G_T$  we obtain from  $\mathcal{C}_i$  by replacing the edge  $e$  by the path  $P_i$ . By considering  $\mathcal{C}'$  separately inside of  $T$  and in the rest of  $G_T$  we can see that it indeed transforms  $M_1$  into  $M_2$ . Inside of  $T$  this follows from how we constructed  $\mathcal{P}$ , outside of  $T$  this follows from the assumptions on  $\mathcal{C}$ . Importantly, every cycle in  $\mathcal{C}'$  (by assumption on  $\mathcal{C}$  and by construction) contains both the edges  $\{v, a_0\}$  and  $\{w, b_0\}$ . Since  $\mathcal{C}'$  has even size, these edges get flipped an even number of times and thus stay in the matching when transforming  $M_1$  using the flip-sequence  $\mathcal{C}'$ .  $\square$

**Remark 18.** One can strengthen the result above slightly and show, that  $2h - 2$  cycles suffice, matching the lower bound we will show later in [Lemma 19](#).

Equipped with the above lemma we are set to finish the proof of [Theorem 12](#).

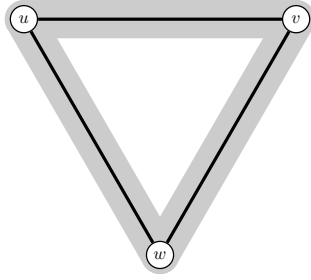
*Proof of Theorem 12.* Let  $M_1$  and  $M_2$  be two matchings in  $G_H$ .

First we want to transform  $M_1$  and  $M_2$  to matchings on which we can apply [Lemma 17](#). To do so consider an auxiliary matching  $M$  as follows. On every edge  $\{v_1, v_2\}$ , which we subdivided  $2t$ -times when constructing the towers, every other edge is part of  $M$ , including the edges incident to  $v_1$  and  $v_2$ . So if the towers on  $\{v_1, v_2\}$  are  $(T_1, \dots, T_t)$  in this order, then the edges  $\{v_1, a_0^{T_1}\}$ ,  $\{b_0^{T_t}, w\}$  as well as the edges  $\{b_0^{T_i}, a_0^{T_{i+1}}\}$  for  $i \in \{1, \dots, t-1\}$  are part of  $M$ . We extend  $M$  to a perfect matching by an arbitrary matching in every tower  $T$ , e.g. by matching  $a_i^T$  with  $b_i^T$  for all  $i \in \{1, \dots, h\}$ .

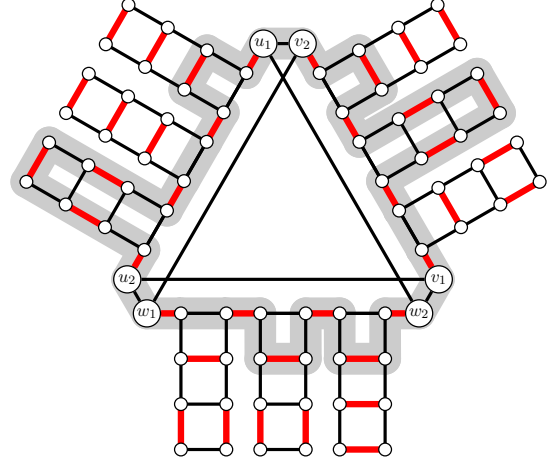
Now consider the symmetric difference  $M_1 \Delta M$ . It can be written as a union  $\mathcal{C}_{\text{all}}$  of  $M_1$ -alternating cycles. Let  $\mathcal{C}$  be the set of cycles of  $\mathcal{C}_{\text{all}}$ , that contain a vertex of  $V_1$  or  $V_2$ . Then we have  $|\mathcal{C}| \leq |V_1| + |V_2| = 2n$ . Let  $N_1$  be the matching we obtain from  $M_1$  by flipping the cycles of  $\mathcal{C}$ . Then  $N_1$  matches the vertices of  $V_1$  and  $V_2$  in the same way as  $M$  does. In particular considering an edge  $\{v_1, v_2\}$  on which we added  $t$  towers  $(T_1, \dots, T_t)$  in this order, then the edges  $\{v_1, a_0^{T_1}\}$ ,  $\{b_0^{T_t}, w\}$  are part of  $N_1$ . Using the structure of the towers, in particular that they contain an even number of vertices, we can deduce that  $N_1$  contains the edges  $\{a_0^{T_i}, b_0^{T_{i+1}}\}$  for  $i \in \{1, \dots, t-1\}$ .

In the same way as described above for  $M_1$ , we can also find a perfect matching  $N_2$  that is obtained from  $M_2$  by flipping at most  $2n$  alternating cycles and such that  $N_2$  matches the vertices in  $V_1 \cup V_2$  as well as all the vertices  $a_0^T, b_0^T$  for all towers  $T$  in the same way that  $M$  does.

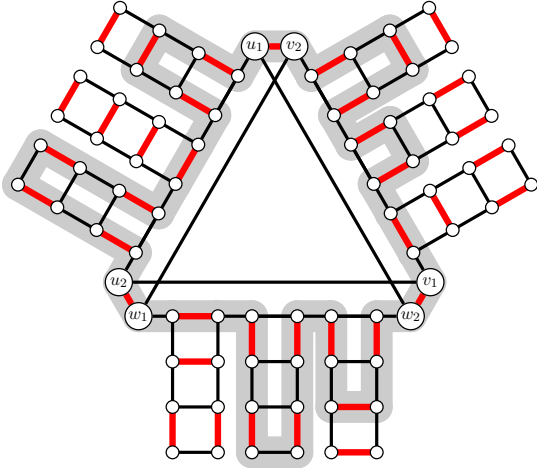
Let  $C = (v_1, \dots, v_n, v_1)$  be the vertex order of a Hamiltonian cycle in  $H$ . Consider the graph  $G'_H$ , which we recall was obtained from  $H$  by splitting every node into two copies. In  $G'_H$  we define the cycle  $C' = (v_1^1, v_1^2, v_2^1, v_2^2, \dots, v_n^2, v_1^1)$  and consider the cycle collection  $\mathcal{C}$  consisting of  $2h$  copies of  $C'$ .



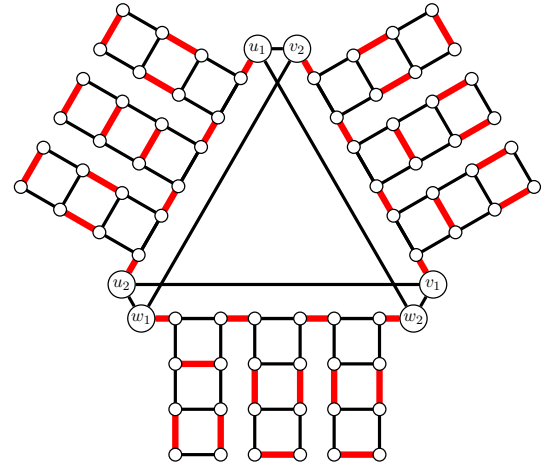
(a) The graph  $H$  with a Hamiltonian cycle  $C$  marked.



(b) A matching  $M$  in  $G_H$ . Using  $C$ , one can find a cycle  $C_1$ , marked in gray, that goes through every tower.



(c) Flipping  $C_1$  we get a new matching,  $M_1$ , that may differ from  $M$  in every tower. Using  $C$  again we can find a new cycle  $C_2$ , again marked in gray.



(d) Flipping  $C_2$  we can again modify  $M_1$  in every tower.

**Figure 5:** Visualization of the idea used in the construction in the proof of [Theorem 12](#). The number and height of the towers, the matchings and the augmenting paths may differ from the actual values used in the proof. The graph  $H$  and the auxiliary graph  $G_H$  coincide with the graphs in [Figure 3](#). The Hamiltonian cycle  $C$  in  $H$ , see [Figure 5a](#), can be used to modify a matching in all towers simultaneously. Note, that the matchings in [Figure 5b](#) and [Figure 5d](#) contain every second edge on the subdivisions of the edges  $v_1, v_2$  for  $v \in V_H$ , while the matching in [Figure 5c](#) instead uses the edges  $\{v_1, w_2\}$  for  $\{v, w\} \in C$ .

Let us assume we obtained  $G_H$  from  $G'_H$  by adding the towers  $T_1, \dots, T_k$  in this order. Let  $N_1^j$  and  $N_2^j$  be the matchings we obtain from  $N_1$  and  $N_2$ , respectively, by replacing the towers  $T_{j+1}, \dots, T_k$  by a single matching edge. Due to the construction above this gives perfect matchings in the graph we obtain from  $G'_H$  by constructing the towers  $T_1, \dots, T_j$ .

Inductively we will show, that there is a flip sequence of length  $\ell$  from  $N_1^j$  to  $N_2^j$  of length  $2h$ . For  $j = 0$  the matchings  $N_1^0$  and  $N_2^0$  agree. Namely, by construction of the auxiliary matching  $M$  they are both the matching in  $G'_H$  that consists of the edges  $\{v^1, v^2\}$  for  $v \in H$ . To conclude assume that we have a flip sequence  $\mathcal{C}_j$  from  $N_1^j$  to  $N_2^j$ . Then we can apply [Lemma 17](#) to the tower  $T_j$  to extend it to a flip sequence  $\mathcal{C}_{j+1}$  from  $N_1^{j+1}$  to  $N_2^{j+1}$ .

In particular combining this with the flip sequence from  $M_1$  to  $N_1$  and from  $M_2$  to  $N_2$  (in reverse) we proved that we can reach  $M_2$  from  $N_1$  by flipping  $2h + 4n$  many cycles. As  $M_1$  and  $M_2$  were arbitrary perfect matchings this shows that any two vertices of  $P_H$  have distance at most  $2h + 4n$ , finishing the proof of [Theorem 12](#).  $\square$

## 2.2 Proof of [Theorem 13](#)

In order to prove [Theorem 13](#), we have to prove that if  $H$  is not Hamiltonian, then the diameter of  $P_{G_H}$  is relatively large. We will first consider a single tower  $T$ . In  $T$  we will construct two matchings, such that every transformation from one to the other using  $v$ - $w$  paths only, requires many paths. We will use this to finally construct two matchings in  $G_H$ , such that every flip sequence from one to the other has length at least  $\frac{n}{n-1}(2h-2)$ . This is collected in the following technical lemma.

**Lemma 19.** *Let  $G = (V, E)$  be a graph,  $e = \{v, w\} \in E$  be an edge of  $G$  and  $\widetilde{M}_1$  and  $\widetilde{M}_2$  be two perfect matchings in  $G$  that both contain  $e$ . Consider the graph  $G_T$  obtained by constructing a tower  $T$  of height  $h$  on  $e$ . Then there exist two perfect matchings  $M_1$  and  $M_2$  in  $G_T$ , such that  $M_i$  agrees with  $\widetilde{M}_i$  outside of the tower  $T$ , with the following property: For every flip sequence  $\mathcal{C} = (C_1, \dots, C_\ell)$  from  $M_1$  to  $M_2$ , we have that  $\mathcal{C}$  contains a cycle that is fully contained in  $T$  or  $\mathcal{C}$  contains at least  $2h - 2$  cycles touching the tower  $T$ .*

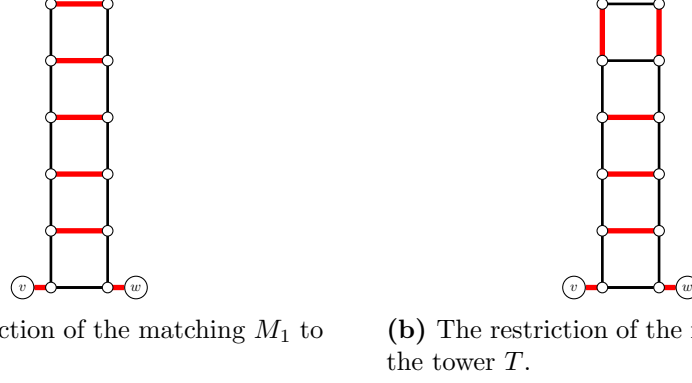
*Proof.* We will explicitly construct two such matchings  $M_1$  and  $M_2$ . For all edges  $f$  of  $G_T$  that are not connected to the tower  $T$  (these are the edges in  $E \setminus \{e\}$ ) by assumption the matchings have to agree with  $\widetilde{M}_1$  and  $\widetilde{M}_2$ , respectively. Within the tower  $T$  let  $M_1$  contain the edges  $\{a_i, b_i\}$  for  $i \in \{1, \dots, h\}$ , together with the edges  $\{v, a_0\}$  and  $\{b_0, w\}$ .

Let the second matching  $M_2$  coincide with  $M_1$  within the tower  $T$ , except that it contains the edges  $\{a_{h-1}, a_h\}$  and  $\{b_{h-1}, b_h\}$  instead of  $\{a_{h-1}, b_{h-1}\}$  and  $\{a_h, b_h\}$ , so

$$\begin{aligned} M_1 &= (\widetilde{M}_1 \setminus \{e\}) \cup \{\{v, a_0\}, \{b_0, w\}\} \cup \{\{a_i, b_i\} : i \in [h]\} \\ M_2 &= (\widetilde{M}_2 \setminus \{e\}) \cup \{\{v, a_0\}, \{b_0, w\}\} \cup \{\{a_i, b_i\} : i \in [h-2]\} \cup \{\{b_{h-1}, b_h\}, \{a_h, a_{h-1}\}\}. \end{aligned}$$

This defines perfect matchings in  $G_T$ , as we assumed  $\widetilde{M}_1$  and  $\widetilde{M}_2$  to contain the edge  $e$ . A visualization of  $M_1$  and  $M_2$  for a tower of height 5 can be seen in [Figure 6a](#) and [Figure 6b](#).

Let  $\mathcal{C} = (C_1, \dots, C_\ell)$  be a flip sequence from  $M_1$  to  $M_2$ . Furthermore assume that none of these cycles is completely contained in the tower  $T$ , else we are done. Hence each  $C_i$  is either disjoint from  $T$  or it touches  $T$ . Let  $a_1 < a_2 < \dots < a_r$  be the indices of the cycles that touch  $T$ . For  $i \in \{1, \dots, r\}$  set  $N_i = M_1 \Delta C_1 \Delta \dots \Delta C_{a_i}$ . As  $\{a_h, b_h\} \in M_1 \setminus M_2$  there is a smallest index  $k$ , such that  $\{a_h, b_h\} \notin N_k$ . Let  $N' = M_1 \Delta C_1 \Delta \dots \Delta C_{a_{k-1}}$  be the matching before flipping  $C_{a_k}$ . Applying the second statement of [Lemma 16](#), we must have  $d(N') = h$ , so  $\mathcal{H}(N_k) = \emptyset$ . By the first part of [Lemma 16](#) the number of horizontal edges changes by at most one when flipping along a cycle of  $\mathcal{C}$ . Finally observe that the cycles that do not touch  $T$  do not change the horizontal edges when we flip them. As we start with  $h$  horizontal edges in  $M_1$ , reach 0 horizontal edges in  $N_k$  and then go up to  $h - 2$  horizontal edges in  $M_2$ , we have  $k \geq h$  and  $r - k \geq h - 2$ , so  $r = k + (r - k) \geq 2h - 2$ , finishing the proof.  $\square$



**Figure 6:** The extension of the matchings  $\widetilde{M}_1$  and  $\widetilde{M}_2$  to a tower of height 5, as constructed in the proof of Lemma 19.

**Remark 20.** Consider the case  $\widetilde{M}_1 = \widetilde{M}_2$ . Then the symmetric difference of the matchings  $M_1$  and  $M_2$  constructed in Lemma 19 is a single alternating cycle. So when allowing to flip arbitrary cycles one can easily transform  $M_1$  to  $M_2$ . The important difference in the above consideration is, that this cycle can only change the matchings in the given tower. On the other hand, a cycle touching  $T$  may also touch other towers, allowing us to use a single cycle to modify many towers.

With the above considerations we are ready to finish the proof of Theorem 13.

*Proof of Theorem 13.* The idea of the proof is to construct two specific matchings in  $G_H$  with the following property: If one can transform one of matchings to the other by flipping less than  $\frac{n}{n-1}(2h-2)$  cycles, then one can use one of these cycles to construct a Hamiltonian cycle in  $H$ .

We start with the perfect matching  $M = \{\{v_1, v_2\} : v \in V(H)\}$  in  $G'_H$ . We extend  $M$  to the towers one after the other by using the construction of Lemma 19. For this assume we obtained  $G_H$  from  $G'_H$  by adding the towers  $(T_1, \dots, T_r)$  in this order. Let  $M_1^0 = M_2^0 = M$ . Assume we constructed two matchings  $M_1^i$  and  $M_2^i$  in the graph obtained from  $G'_H$  by adding the towers  $(T_1, \dots, T_i)$ . Now we can apply Lemma 19 to extend these matchings further to  $T_{i+1}$  giving rise to two matchings  $M_1^{i+1}$  and  $M_2^{i+1}$  in the graph obtained from  $G'_H$  by adding the towers  $(T_1, \dots, T_{i+1})$ . Finally set  $M_1 = M_1^r$  and  $M_2 = M_2^r$ .

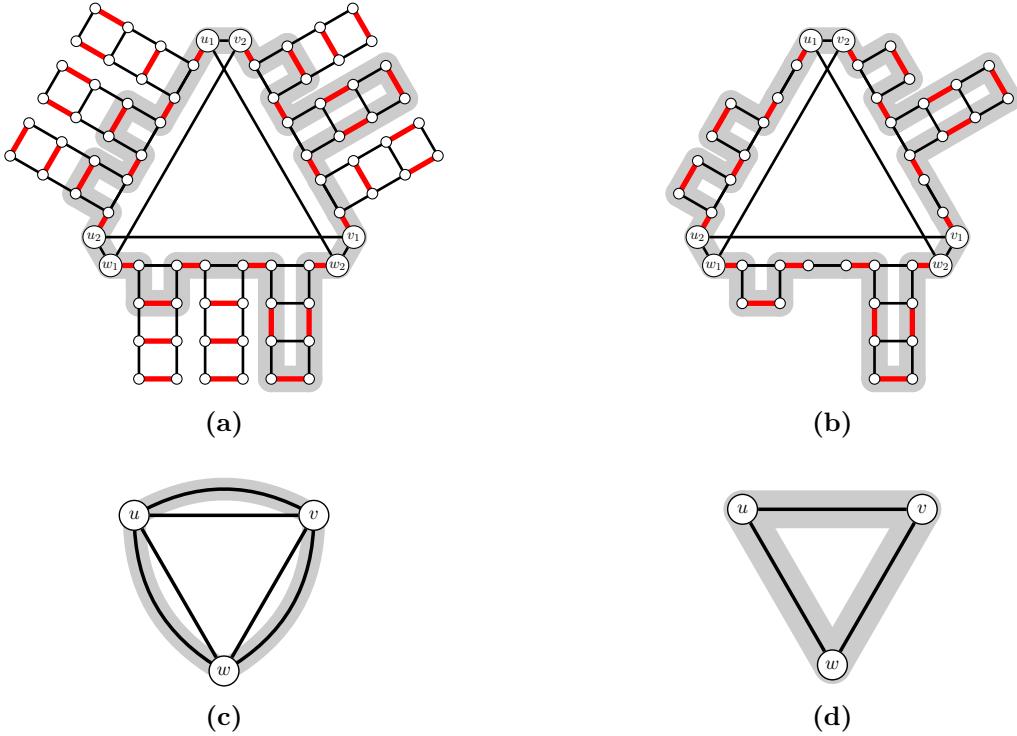
Let us make the following observation that will be useful later: Consider a tower  $T$  and let  $G/T$  be the graph we obtain from  $G_H$  when replacing the tower  $T$  by a single edge. Then  $M_1$  and  $M_2$  give rise to two matchings  $\widetilde{M}_1^T$  and  $\widetilde{M}_2^T$  in  $G/T$ , such that applying Lemma 19 to  $\widetilde{M}_1^T$  and  $\widetilde{M}_2^T$  gives us  $M_1$  and  $M_2$ , respectively. So we can use the conclusion of Lemma 19 for every tower  $T$  simultaneously, although we constructed the matchings one tower at the time.

Suppose that  $M_2$  was reachable from  $M_1$  by flipping the cycles  $\mathcal{C} = (C_1, \dots, C_m)$  with  $m < \frac{n}{n-1}(2h-2)$ , i.e.  $M_2 = M_1 \Delta C_1 \Delta C_2 \Delta \dots \Delta C_m$ . First we consider the set  $\mathcal{C}_s \subseteq \mathcal{C}$  of cycles, for which there is a tower completely containing this cycle. We have  $|\mathcal{C}_s| \leq |\mathcal{C}| < 2h = t$ . For every  $v \in V_H$ , we first introduced the edge  $e_v = \{v_1, v_2\}$ , on which we constructed  $t$  towers. In particular for one of these towers, say  $T_v$ , no cycle of  $\mathcal{C}$  is completely contained in  $T_v$ .

For every  $v \in V_H$  define  $\mathcal{C}_v$  to be the sub-collection of  $\mathcal{C}$  consisting of the cycles that touch  $T_v$ . By construction of the matchings  $M_1$  and  $M_2$ , using Lemma 19, we obtain that  $|\mathcal{C}_v| \geq 2h-2$  for all  $v \in V$ .

We claim that every cycle of  $\mathcal{C}$  is part of  $\mathcal{C}_v$  for at most  $n-1$  vertices  $v$  of  $H$ . Assume the opposite, so there is a cycle  $C \in \mathcal{C}$  such that  $C \in \mathcal{C}_v$  for all  $v \in V(H)$ . We will show, that we can then construct a Hamiltonian cycle in  $H$ , which contradicts the assumptions of Theorem 13. The cycle  $C$  contains edges from  $T_v$  for every edge  $e \in E_H$ . In particular, for every vertex  $v \in V_H$ ,  $C$  contains an  $v_1$ - $v_2$  path through the tower gadgets on the edge  $\{v_1, v_2\}$ . Now delete

all vertices in  $V_H \setminus V(C)$ . By the above observation this does not remove any vertex of  $V_1$  or  $V_2$ . Contracting parts of a cycle still gives a cycle, so if we contract the  $v_1$ - $v_2$  path along  $C$  in both  $C$  and  $G_H$  for every  $v \in V_H$ , we still have a cycle. By the above consideration this contracts a path through the tower gadgets of the edge  $\{v_1, v_2\}$ . In particular contracting all these paths in  $G_H$  gives rise to a multi-graph  $\tilde{G}$  with  $|V_H|$  vertices, namely one for each contraction of the path between  $v_1$  and  $v_2$  on  $C$ , for every  $v \in V_H$ . Furthermore, the contraction  $\tilde{C}$  of the cycle  $C$  visits all these vertices. The only edges we did not contract or delete and thus remain in  $\tilde{G}$  are the edges of the form  $\{v_1, w_2\}, \{v_2, w_1\}$  for all  $\{v, w\} \in E_H$ . From this one can see that  $\tilde{G}$  is isomorphic to the graph obtained from  $H$  by replacing every edge with a parallel pair of edges. It is easy to see that, as long as both graphs have at least 3 vertices, any Hamiltonian cycle in  $\tilde{G}$  also induces a Hamiltonian cycle in  $H$ . So in particular the contracted cycle  $\tilde{C}$  in  $\tilde{G}$  gives rise to a Hamiltonian cycle in  $H$ , a contradiction. A visualization of this idea can be seen in Figure 7.



**Figure 7:** Visualization of the idea used in the proof of Theorem 13. For the sake of simplicity, the number and height of the towers, the matchings and the augmenting paths may differ from the actual choices done in the proof. We start with a cycle  $C$  in the graph  $G_H$  that touches towers on every subdivided edge in  $G_H$  (Figure 7a). As a first step we remove all vertices that do not lie on  $C$  (Figure 7b). Next we contract the subpath of  $C$  along the subdivision of the edge  $\{v_1, v_2\}$  for all  $v \in V_H$  (Figure 7c). This yields a Hamiltonian cycle  $\tilde{C}$  in  $\tilde{G}$ . Finally omitting the duplicate edges we end up with a Hamiltonian cycle in  $H$  (Figure 7d).

So we conclude, that every cycle of  $\mathcal{C}$  is part of  $\mathcal{C}_v$  for at most  $n - 1$  vertices  $v$ . To finish the proof observe that  $(n - 1)|\mathcal{C}| \geq \sum_{v \in V_H} |\mathcal{C}_v| \geq n(2h - 2)$ , so  $|\mathcal{C}| \geq \frac{n}{n-1}(2h - 2)$ .

This shows that  $M_1$  and  $M_2$  correspond to vertices of the perfect matching polytope of  $G_H$  at distance at least  $\frac{n}{n-1}(2h - 2)$ , and thus  $\text{diam}(P_{G_H}) \geq \frac{n}{n-1}(2h - 2)$ , as claimed.  $\square$

### 3 Proof of Theorem 6

The remainder of this article is dedicated to the proof our second main result, Theorem 6. To do so, as mentioned before, it suffices to prove Theorem 9. We start by obtaining a precise



description of the monotone diameter of the perfect matching polytope of a bipartite graph in graph theoretical terms. In the special case of complete (bipartite) graphs, this description was already observed by Rispoli [Ris92].

**Lemma 21.** *Let  $G$  be a bipartite graph with perfect matching polytope  $P_G$ . Then the monotone diameter of  $P_G$  agrees with the maximum number of cycles in the symmetric difference of two perfect matchings in  $G$ .*

*Proof.* First let  $\mathbf{x}$  be a vertex of  $P_G$  corresponding to the perfect matching  $M$ , and let  $\mathbf{c} \in \mathbb{R}^E$  be a cost function. Let  $M^*$  be a minimum cost perfect matching in  $G$  with respect to the costs  $c$ , corresponding to the  $c$ -optimal vertex  $y$ . Consider the cycles  $\mathcal{C}$  of the symmetric difference  $M \Delta M^*$ . Flipping the edges of any cycle of  $\mathcal{C}$  in  $M$  reduces the costs with respect to  $c$ , as  $M^*$  is a minimum cost matching. So in particular flipping the cycles of  $\mathcal{C}$  one after another gives rise to a monotone walk of length  $|\mathcal{C}|$  from  $x$  to the optimal vertex  $y$ .

As  $\mathbf{x}$  and  $\mathbf{c}$  were arbitrary, this proves that the monotone diameter of  $P_G$  is bounded from above by the maximum number of cycles in the symmetric difference of two perfect matchings.

In order to prove the matching lower bound it is enough to show the following: Given any two perfect matchings  $M$  and  $M^*$ , corresponding to the vertices  $\chi^M$  and  $\chi^{M^*}$  of  $G$ , respectively, with a collection of cycles  $\mathcal{C}$  in the symmetric difference  $M \Delta M^*$ . Then there exists a cost function  $c \in \mathbb{R}^E$  such that  $\chi^{M^*}$  is  $c$ -minimal and such that the shortest  $c$ -monotone walk from  $\chi^M$  to  $\chi^{M^*}$  has length  $|\mathcal{C}|$ .

We will show that defining  $c$  as

$$c(e) = \begin{cases} 0 & \text{if } e \in M^*, \\ 1 & \text{if } e \in M \setminus M^*, \\ |V| & \text{if } e \in E \setminus \{M \cup M^*\}, \end{cases}$$

fulfills the above requirements. First, as  $c$  is non-negative and  $c(M^*) = 0$ ,  $M^*$  is indeed  $c$ -minimal. Next observe that  $c(M) \leq |M| < |V|$ . So if we consider an arbitrary monotone walk from  $\chi^M$  to  $\chi^{M^*}$ , corresponding to flipping a sequence of negative alternating cycles, we may not use an edge of  $E \setminus \{M \cup M^*\}$ . If we did, then one of the intermediate vertices would correspond to a matching containing an edge of this set. Then the cost of that vertex is at least  $|V|$ , contradicting the monotonicity of the walk.

Hence the cycles corresponding to the moves of any monotone walk from  $\chi^M$  to  $\chi^{M^*}$  may only use the edges of  $M \Delta M^*$ . So every monotone walk from  $\chi^M$  to  $\chi^{M^*}$  has to flip the cycles of  $M \Delta M^*$  one after another. In particular every monotone walk has length  $|\mathcal{C}|$ , finishing the proof.  $\square$

Lemma 21 shows that determining the monotone diameter of  $G$  corresponds one to one to a specific cycle packing problem. A set of pairwise disjoint cycles is the symmetric difference of two matchings if and only the complement of the cycles contains a perfect matching. So given a bipartite graph  $G$ , we are interested in packing the maximum number of vertex-disjoint cycles such that the subgraph of  $G$  induced by the vertices that are not covered by any of the cycles still contains a perfect matching. When omitting the matching condition strong NP-hardness of the problem was established in [KP11] by a reduction from 3-dimensional matching. In order to capture the additional matching constraint we will provide a related but extended reduction from 4-dimensional matching. The latter is the following decision problem:

**4-DIMENSIONAL MATCHING**

**Input:** Four disjoint sets  $W, X, Y, Z$  and a subset  $E \subseteq W \times X \times Y \times Z$ .

**Decision:** Is there a subset  $M \subseteq E$  such that every element of  $W, X, Y$  and  $Z$  is part of exactly one element of  $M$ ?

In order to reduce 4-DIMENSIONAL MATCHING to the problem of computing the monotone diameter of perfect matching polytopes, we observe the following consequence of Lemma 21:

**Corollary 22.** *Let  $G = (V, E)$  be a bipartite graph. Then we have  $\text{mdiam}(P_G) = \frac{|V|}{4}$  if and only if there exists a collection of 4-cycles in  $G$  that covers every vertex in  $V$  precisely once.*

*Proof.* By Lemma 21 we have that  $\text{mdiam}(P_G)$  equals the maximum number of cycles in the symmetric difference of two perfect matchings of  $G$ . Since all the cycles in  $G$  have length at least 4, for any pair of perfect matchings, there can be at most  $\frac{|V|}{4}$  such cycles, and thus we always have  $\text{mdiam}(P_G) \leq \frac{|V|}{4}$ . On the other hand, if  $\text{mdiam}(P_G) = \frac{|V|}{4}$ , then there must exist a collection of  $\frac{|V|}{4}$  vertex-disjoint cycles in  $G$ . Necessarily, every cycle in this collection must then have length 4 and all vertices of  $G$  must be covered by at least one cycle, proving the first direction of the stated equivalence. Vice-versa, suppose there exists a collection  $\mathcal{C}$  of 4-cycles in  $G$  that covers every vertex in  $G$  exactly once. Then clearly,  $|\mathcal{C}| = \frac{|V|}{4}$ . Let  $M_1$  and  $M_2$  be two perfect matchings of  $G$ , defined in such a way that each 4-cycle  $C \in \mathcal{C}$  contains two opposing edges from  $M_1$ , and two opposing edges from  $M_2$ . Then clearly, the cycles in  $M_1 \Delta M_2$  are exactly the cycles in  $\mathcal{C}$ , and using Lemma 21 it follows that  $\text{mdiam}(P_G) \geq |\mathcal{C}| = \frac{|V|}{4}$ , and thus  $\text{mdiam}(P_G) = \frac{|V|}{4}$ , as desired. This concludes the proof.  $\square$

Corollary 22 implies that the following decision problem reduces to computing the monotone diameter of perfect matching polytopes of bipartite graphs:

VERTEX-DISJOINT 4-CYCLE COVER

**Input:** A bipartite graph  $G = (V, E)$ .

**Decision:** Is there a collection  $\mathcal{C}$  of cycles of length 4 such that every  $v \in V$  is part of exactly one cycle of  $\mathcal{C}$ ?

So to finish the proof of Theorem 6, it suffices to prove the following.

**Lemma 23.** *The VERTEX-DISJOINT 4-CYCLE COVER problem is NP-complete.*

*Proof.* As mentioned above we will reduce from the 4-DIMENSIONAL MATCHING problem. Karp [Kar10] showed that 3-DIMENSIONAL MATCHING matching is NP-hard. It is straightforward to reduce 3-DIMENSIONAL MATCHING to 4-DIMENSIONAL MATCHING, and thus also 4-DIMENSIONAL MATCHING is NP-hard.

So let  $W, X, Y, Z$  and  $E$  be an instance of 4-DIMENSIONAL MATCHING. We will construct an instance of VERTEX-DISJOINT 4-CYCLE COVER, such that the answers to both problems agree. To do so we construct a bipartite graph  $G$  in the following way: For every element  $a \in W \cup X \cup Y \cup Z$  we add a vertex to  $G$ . These vertices will be called the *exterior vertices* of  $G$ . Furthermore, for every element  $e \in E$  we add twelve vertices and 28 edges forming a gadget as shown in Figure 8. More precisely, for every hyperedge  $e = (w, x, y, z) \in E$  we add vertices  $a_i^e$  for  $a \in e$  and  $i \in \{1, 2, 3\}$ , which we will call the *auxiliary vertices* of  $G$  in the following. Additionally we add the edges of

$$E_{\text{ext}} = \{\{a, a_1^e\}, \{a_1^e, a_2^e\}, \{a_2^e, a_3^e\}, \{a_3^e, a\} : a \in \{w, x, y, z\}\}$$

and

$$E_{\text{int}} = \{\{w_i^e, x_i^e\}, \{x_i^e, y_i^e\}, \{y_i^e, z_i^e\}, \{z_i^e, w_i^e\} : i \in \{1, 2, 3\}\}.$$

Observe that the edges of  $E_{\text{ext}}$  form four vertex-disjoint cycles of length four, marked in black and solid in Figure 8. The edges of  $E_{\text{int}}$  form three vertex-disjoint cycles of length four, depicted in red and dashed in Figure 8. Furthermore both the first and the second set of cycles cover all auxiliary vertices. The first set additionally covers the external vertices.



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