Computational Science - ITB

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1 Summary - Euler Method

Euler method is first order single-point method for solving ordinary differential equations (ODEs) with a given initial value (IV). It is the most bacic explicit method for numerical integration of ODEs and the simplest Runge-Kutta method. Although this method is too inaccurate to be of much practical value, it's useful to illustrate many concepts relevant to the finite difference solution of initial-value ODEs.
Consider the general nonlinear first-order ODE:

$$\overline{y}' = f(t, \overline{y}) \qquad\qquad \overline{y}(t_0) = \overline{y}_0 \tag{1.1}$$

From Taylor series:

$$\overline{y}_{n+1} = \overline{y}_n + \Delta t \overline{f}_n + \frac{1}{2}\overline{y}''(\tau_n)\Delta t^2 = \overline{y}_n + \Delta t \overline{f}_n + O(\Delta t^2) \tag{1.2}$$

Truncating the remainder term, which is $O(\Delta t^2)$, and solving for y_{n+1} yields the *explicit Euler* finite difference equation (FDE):

$$y_{n+1} = y_n + \Delta t f_n \qquad\qquad O(\Delta t^2) \tag{1.3}$$

where the $O(\Delta t^2)$ term is included as a reminder of the order of the local truncation error.
Several features of above equation are summarized below:

1. The FDE is explicit, since f_n does not depend on y_{n+1}.
2. The FDE requires only known point. Hence, it is a single point method.
3. The FDE requires only one derivative function evaluation [i.e. $f(t, y)$] per step.
4. The error in calculating y_{n+1} for a single step, the local truncation error, is $O(\Delta t^2)$.
5. The global (i.e. total) error accumulated after N steps is $O(\Delta t)$.

At the final point t_N:

$$y_N = y_0 + \sum (y_{n+1} - yn) = y_0 + \sum \Delta y_{n+1} \tag{1.4}$$

and the total truncation error is given by,

$$Error = \sum \frac{1}{2} y''(\tau_n) \Delta t^2 = N \frac{1}{2} y''(\tau) \Delta t^2, \tag{1.5}$$

where $t_0 \leq \tau \leq t_N$, so the number of steps N is related to the step size Δt as follows:

$$N = \frac{t_N - t_0}{\Delta t} \tag{1.6}$$

Substituting (1.6) to (1.5) yields:

$$Error = \frac{1}{2}(t_N - t_0) y''(\tau) \Delta t = O\Delta t \tag{1.7}$$

Consequently, the global (i.e. total) error of the *explicit Euler* FDE is $O\Delta t$.

The other first-order single-point method for FDEs is *implicit Euler* method. There are many second-order method from Euler method which is generally called the modified Euler method. Runge-Kutta is fourth-order single-point method, and there are many higher order by modifying this method.

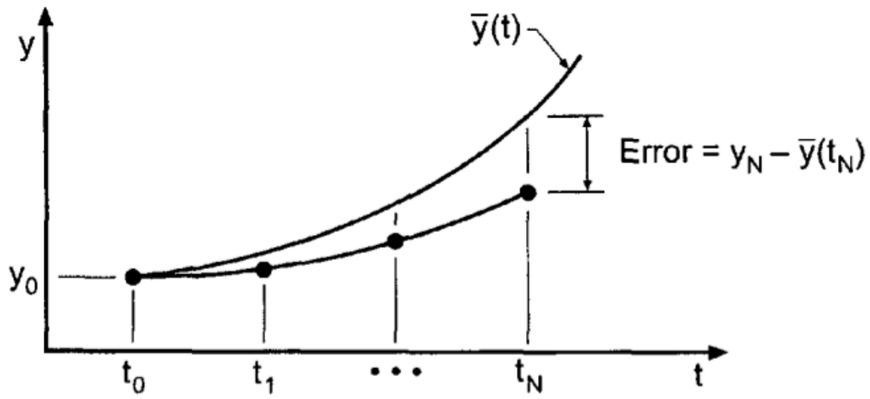

Figure 1.1: Repetitive application of the explicit Euler method.