Homework - Introduction to Frontiers of Computational Science

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1 Problem 1

Solve the following equations:

$$\frac{d}{dt}x_1(t) = \dot{x}_1 = \mu(-x_1(t) + x_2(t)) \tag{1.1}$$

$$\frac{d}{dt}x_2(t) = \dot{x}_2 = \mu(x_1(t) - 2x_2(t) + x_3(t)) \tag{1.2}$$

$$\frac{d}{dt}x_3(t) = \dot{x}_3 = \mu(x_2(t) - x_3(t)) \tag{1.3}$$

where $x_i(0) = x_i^0$ $(i = 1, 2, 3)$.

Answer
if we sum all the equations we obtain:

$$\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0 \tag{1.4}$$

then if we substract (1.1) with (1.3), we obtain:

$$\dot{x}_1 - \dot{x}_3 = -\mu(x_1 - x_3) \tag{1.5}$$

also, we can sum (1.1) and (1.2) then substract it with two times (1.2), and we will get:

$$\dot{x}_1 - 2\dot{x}_2 + \dot{x}_3 = -3\mu(x_1 - 2x_3 + x_3) \tag{1.6}$$

the solution:

$$(x_1 + x_2 + x_3)_{(t)} = (x_1 + x_2 + x_3)_{(0)} \tag{1.7}$$

$$(x_1 - x_3)_{(t)} = (x_1 - x_3)_{(0)}e^{-\mu t} \tag{1.8}$$

$$(x_1 - 2x_2 + x_3)_{(t)} = (x_1 - 2x_2 + x_3)_{(0)}e^{-3\mu t} \tag{1.9}$$

if we add two times (1.7) with (1.9):

$$(x_1 + x_3)_{(t)} = \frac{1}{3}x_1^{(0)}(2 + e^{-3\mu t}) + \frac{2}{3}x_2^{(0)}(1 - e^{-3\mu t}) + \frac{1}{3}x_3^{(0)}(2 + e^{-3\mu t}) \tag{1.10}$$

operate with (1.8):

$$x_1^{(t)} = \frac{1}{6}x_1^{(0)}(2 + e^{-3\mu t} + e^{-\mu t}) + \frac{1}{3}x_2^{(0)}(1 - e^{-3\mu t}) + \frac{1}{6}x_3^{(0)}(2 + e^{-3\mu t} - e^{-\mu t}) \tag{1.11}$$

$$x_3^{(t)} = \frac{1}{6}x_1^{(0)}(2 + e^{-3\mu t} - e^{-\mu t}) + \frac{1}{3}x_2^{(0)}(1 - e^{-3\mu t}) + \frac{1}{6}x_3^{(0)}(2 + e^{-3\mu t} + e^{-\mu t}) \tag{1.12}$$

if we substract (1.7) with (1.9):

$$x_2^{(t)} = \frac{1}{3}x_1^{(0)}(1 - e^{-3\mu t}) + \frac{1}{3}x_2^{(0)}(1 + 2e^{-3\mu t}) + \frac{1}{3}x_3^{(0)}(1 - e^{-3\mu t}) \tag{1.13}$$

so we get the solutions for this problem in equation (1.11), (1.12), and (1.13). We also can use eigenvalue problem to get the same result.

2 Problem 2

Check if $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\0\\-1\end{pmatrix}$, and $\begin{pmatrix}1\\-2\\1\end{pmatrix}$ are orthogonal to each others. Normalize and check the unitary condition.

Answer:
for orthogonality we can use dot product to each pair of vector: $v_i \cdot v_j = 0$ for $i \neq j$, easily we can prove that all combination resulting zero dot product. For example:

$$v_1 \cdot v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix} = 1 \cdot 1 + 1 \cdot 0 + 1 \cdot -1 = 0$$

Normalize vector would be (e_1, e_2, e_3):

$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$, and $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$

Checking unitary relation

$$Q = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

if we inverse it using $Q^{-1} = \frac{1}{\det(Q)}(adj(Q))$, this matrix will become:

$$Q^{-1} = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

and the unitary relation is proven $Q^{-1} = Q^T$