Appendix. Distributionally Robust Stochastic Dual Dynamic Programming

Proof of Proposition 1: We proceed by forming the duals of (11) and (15). To facilitate our argument, we introduce auxiliary variables for all $j \in [n_{t+1}]$ and $\kappa \in \mathcal{K}_{t+1}^{j,k}$:

$$\alpha_t^{j,\kappa} = \sum_{i \in [n_{t+1}]} z_{t,\kappa}^{i,j}.$$

Hence, we can write the dual of (11):

$$\begin{aligned}
\mathfrak{Q}_t^k(x_{t-1},\xi_t) = \max_{\pi_t, z_t, \alpha_t} \quad & \pi_t\left(B_t x_{t-1} + \xi_t\right) + \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} g_{t,\kappa}^j \alpha_t^{j,\kappa} && \text{(23a)}\\
\text{s.t.} \quad & \pi_t A_t - \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} G_{t,\kappa}^j \alpha_t^{j,\kappa} \le c_t, && \text{(23b)}\\
& \sum_{i \in [n_{t+1}]} \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} d_{t+1}^{i,j} z_{t,\kappa}^{i,j} \le r, && \text{(23c)}\\
& \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} z_{t,\kappa}^{i,j} = q_{t+1}^i \quad \forall i \in [n_{t+1}], && \text{(23d)}\\
& \sum_{i \in [n_{t+1}]} z_{t,\kappa}^{i,j} - \alpha_t^{j,\kappa} = 0 \quad \forall j \in [n_{t+1}], \kappa \in \mathcal{K}_{t+1}^{j,k}, && \text{(23e)}\\
& z_t \ge 0. && \text{(23f)}
\end{aligned}$$

To write the dual of (15), let π_t, λ_t^κ, and $\alpha_t^{j,\kappa}$ be the the dual variables for constraints (15b)–(15d), respectively. Hence, we can write the dual problem of (15):

$$\begin{aligned}
\underline{\mathfrak{Q}}_t^k(x_{t-1},\xi_t) = \max_{\pi_t, z_t, \alpha_t} \quad & \pi_t\left(B_t x_{t-1} + \xi_t\right) + \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} g_{t,\kappa}^j \alpha_t^{j,\kappa} && \text{(24a)}\\
\text{s.t.} \quad & \pi_t A_t - \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} G_{t,\kappa}^j \alpha_t^{j,\kappa} \le c_t, && \text{(24b)}\\
& \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} \alpha_t^{j,\kappa} - \sum_{\kappa \in [k]} p_{t+1}^{j,\kappa} \lambda_t^\kappa = 0 \qquad \forall j \in [n_{t+1}], && \text{(24c)}\\
& \sum_{\kappa \in [k]} \lambda_t^\kappa = 1, && \text{(24d)}\\
& \alpha_t, \lambda_t \ge 0. && \text{(24e)}
\end{aligned}$$

The projection of the set defined by constraints (5b)-(5e) onto $p_{t+1} = (p_{t+1}^j : j \in [n_{t+1}])$ characterizes the uncertainty set, $\mathcal{P}_{t+1}$. Constraints (23c)–(23f) play the same role in characterizing $\mathcal{P}_{t+1}$, where $z_t^{i,j} = \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} z_{t,\kappa}^{i,j}$ and where p_t^j is defined by equation (13). The parameters, $p_{t+1}^{j,\kappa}$, $j \in [n_{t+1}], \kappa \in \mathcal{K}_{t+1}^{j,k}$, in subproblem (24) are computed at each iteration by optimizing an auxiliary problem over $\mathcal{P}_{t+1}$. After k iterations, in the stage-t subproblem we have $(p_{t+1})^\kappa, \kappa = 1, 2, \ldots, k$, and the constraints (24c)–(24e) characterize the convex hull of these k vectors, i.e., they describe only a subset of $\mathcal{P}_{t+1}$. The feasible region of (23) captures all of $\mathcal{P}_{t+1}$, and that of (24) only captures a subset, and hence we have $\mathfrak{Q}_t^k(x_{t-1},\xi_t) \ge \underline{\mathfrak{Q}}_t^k(x_{t-1},\xi_t)$. Q.E.D.

Proof of Lema 2: Let $\hat{\pi}_T$ be a dual feasible extreme point to problem (11) for $t = T$, and let $\mathcal{K}_T^{j,k}$ denote the corresponding index set. (Note that at stage T there are no cuts, and therefore dual variables only involve π_T.) For any $j \in [n_T]$ and any feasible x_{T-1}, we have:

$$\begin{aligned} G_{T-1}^j x_{T-1} + g_{T-1}^j &= \hat{\pi}_T B_T x_{T-1} + \hat{\pi}_T \xi_T^j \\ &\leq \max_{\pi_T} \{ \pi_T (B_T x_{T-1} + \xi_T^j) : \pi_T A_T \leq c_T \} \\ &= \min_{x_T} \{ c_T x_T : A_T x_T = B_T x_{T-1} + \xi_T^j ; x_T \geq 0 \} \\ &= Q_T(x_{T-1}, \xi_T^j), \end{aligned}$$

where the first equality holds by equation (12), and the penultimate equality holds by Assumption (A.2). Thus, the cut is valid for stage $T-1$.

Proceeding by induction, now consider problem (11) for $t \leq T-1$ with input $(\hat{x}_{t-1}, \xi_t^j)$; assume that the cuts on the right-hand side of inequality (11c) are valid; and, let $(\hat{\pi}_t, \hat{z}_t)$ be a dual feasible solution. Then, by equation (12):

$$G_{t-1}^j = \hat{\pi}_t B_t, \text{ and} \tag{25a}$$

$$\begin{aligned} g_{t-1}^j &= \mathfrak{Q}_t^k(\hat{x}_{t-1}, \xi_t^j) - \hat{\pi}_t B_t \hat{x}_{t-1} \\ &= \hat{\pi}_t \xi_t^j + \sum_{i \in [n_{t+1}]} \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} g_{t,\kappa}^j \hat{z}_{t,\kappa}^{i,j}. \end{aligned} \tag{25b}$$

If problem (11) had cuts in (11c) indexed by $\mathcal{K}_{t+1}^j$, $j \in [n_{t+1}]$, rather than $\mathcal{K}_{t+1}^{j,k}$, $j \in [n_{t+1}]$, then by duality its optimal value would be identical to that of problem (9). Let $\Pi_t = \{(\pi_t, z_t) : (\pi_t, z_t) \text{ satisfies (9b)–(9e)}\}$, and let Π_t^k denote the analogous set when the cuts indexed by $\mathcal{K}_{t+1}^j$, $j \in [n_{t+1}]$, are replaced by those of $\mathcal{K}_{t+1}^{j,k}$, $j \in [n_{t+1}]$. We then have that:

$$G_{t-1}^j x_{t-1} + g_{t-1}^j = \hat{\pi}_t (B_t x_{t-1} + \xi_t^j) + \sum_{i \in [n_{t+1}]} \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_t^{j,k}} g_{t,\kappa}^j \hat{z}_{t,\kappa}^{i,j} \tag{26}$$

$$\leq \max_{(\pi_t, z_t) \in \Pi_t^k} \pi_t \left(B_t x_{t-1} + \xi_t^j \right) + \sum_{i \in [n_{t+1}]} \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_{t+1}^{j,k}} g_{t,\kappa}^j z_{t,\kappa}^{i,j} \tag{27}$$

$$\leq \max_{(\pi_t, z_t) \in \Pi_t} \pi_t \left(B_t x_{t-1} + \xi_t^j \right) + \sum_{i \in [n_{t+1}]} \sum_{j \in [n_{t+1}]} \sum_{\kappa \in \mathcal{K}_{t+1}^{j}} g_{t,\kappa}^j z_{t,\kappa}^{i,j} \tag{28}$$

$$= Q_t(x_{t-1}, \xi_t^j). \tag{29}$$

Equation (26) simply applies equations (25), which define the cut gradient and intercept, and inequality (27) holds because $(\hat{\pi}_t, \hat{z}_t) \in \Pi_t^k$. Inequality (28) holds because $\mathcal{K}_{t+1}^j$ indexes cuts which define $Q_{t+1}(x_t, \xi_t^j)$ while by hypothesis $\mathcal{K}_{t+1}^{j,k}$ indexes valid, i.e., lower bounding, cuts. Equation (29) follows from strong duality of problems (8) and (9). This establishes the first claim that all cuts are valid, and hence $\mathfrak{Q}_t^k(x_{t-1}, \xi_t) \leq Q_t(x_{t-1}, \xi_t)$ follows immediately. Q.E.D.

Proof of Lemma 3: The solution $(x_{T-1}^k, \gamma_{T-1}^k, \nu_{T-1}^k)$ satisfies constraints (11b) and (11d), and hence also satisfies the identical constraints (8b) and (8d). Examining constraint (8c), we are in one of the following two cases:

$$d_T^{i,j} \gamma_{T-1}^k + \nu_{T-1}^{i,k} \geq \max_{\kappa \in \mathcal{K}_T^j} \left\{ G_{T-1,\kappa}^j x_{T-1}^k + g_{T-1,\kappa}^j \right\} \quad \forall i, j \in [n_T] \tag{30a}$$

$$\exists\ i,j \in [n_T] \text{ such that } d_T^{i,j}\gamma_{T-1}^k + \nu_{T-1}^{i,k} < \max_{\kappa\in\mathcal{K}_T^j}\left\{G_{T-1,\kappa}^j x_{T-1}^k + g_{T-1,\kappa}^j\right\}. \tag{30b}$$

Suppose case (30a) holds. By Lemma 2 for any feasible x_{T-1} and $j \in [n_T]$

$$\max_{\kappa\in\mathcal{K}_T^j}\left\{G_{T-1,\kappa}^j x_{T-1} + g_{T-1,\kappa}^j\right\} \geq \max_{\kappa\in\mathcal{K}_T^{j,k}}\left\{G_{T-1,\kappa}^j x_{T-1} + g_{T-1,\kappa}^j\right\}.$$

As a result, (ii) holds because problem (11) is a relaxation of problem (8), and an optimal solution to the former is feasible to the latter. In addition, by equations (12) and (30a) for each $i,j \in [n_T]$:

$$\begin{aligned} d_T^{i,j}\gamma_{T-1}^k + \nu_{T-1}^{i,k} &\geq \max_{\kappa\in\mathcal{K}_T^j}\{\pi_{T,\kappa}(B_T x_{T-1}^k + \xi_T^j)\} \\ &\geq \hat{\pi}_T(B_T x_{T-1}^k + \xi_T^j) \quad \forall \hat{\pi}_T \in \{\pi_T : \pi_T A_T \leq c_T\}, \end{aligned}$$

and so (i) does not hold.

Suppose case (30b) holds. Solution $(x_{T-1}^k, \gamma_{T-1}^k, \nu_{T-1}^k)$ violates at least one constraint of form (8c), and hence (ii) does not hold. Using equations (12) and (30b) there exists $i,j \in [n_T]$ and an extreme point $\hat{\pi}_T \in \{\pi_T : \pi_T A_T \leq c_T\}$ with $G_{T-1,\kappa}^j = \hat{\pi}_T B_T$ and $g_{T-1,\kappa}^j = \hat{\pi}_T \xi_T^j$ such that (i) holds. Q.E.D.

Proof of Lemma 4: The proof proceeds in a fashion parallel to that of Lemma 3: Problems (8) and (11) are identical except for constraints (8c) and (11c). By Lemma 2 problem (11) is a relaxation of problem (8). Solution $(x_t^k, \gamma_t^k, \nu_t^k)$ either satisfies constraint (8c), i.e., satisfies the inequality for all $i,j \in [n_{t+1}]$ and $\kappa \in \mathcal{K}_{t+1}^j$, or the constraint is violated for some $i,j \in [n_{t+1}]$ and $\kappa \in \mathcal{K}_{t+1}^j$. In the former case, it is immediate that (ii) holds and (i) does not. In the latter case, again analogous to the proof of Lemma 3, it is immediate that (ii) does not hold. It remains to show that (i) holds in the latter case. The descendant subproblems (11) can be written:

$$\begin{aligned} \mathfrak{Q}_{t+1}^k(x_t^k, \xi_{t+1}^j) = \min_{x_{t+1},\gamma_{t+1},\nu_{t+1}} \quad & c_{t+1}x_{t+1} + r\gamma_{t+1} + \sum_{i'\in[n_{t+2}]} q_{t+2}^{i'}\nu_{t+1}^{i'} \\ \text{s.t.} \quad & A_{t+1}x_{t+1} = B_{t+1}x_t^k + \xi_{t+1}^j, \\ & d_{t+2}^{i',j'}\gamma_{t+1} + \nu_{t+1}^{i'} \geq G_{t+1,\kappa}^{j'}x_{t+1} + g_{t+1,\kappa}^{j'} \ \forall i',j' \in [n_{t+2}], \kappa \in \mathcal{K}_{t+2}^{j',k}, \\ & x_{t+1}, \gamma_{t+1} \geq 0, \end{aligned}$$

and hypothesis (16) allows for equivalently indexing the cut constraints over $\mathcal{K}_{t+2}^{j',k}$ or over $\mathcal{K}_{t+2}^{j'}$. As a result, $\mathfrak{Q}_{t+1}^k(x_t^k, \xi_{t+1}^j) = Q_{t+1}(x_t^k, \xi_{t+1}^j)$. With corresponding dual variables, $(\hat{\pi}_{t+1}, \hat{z}_{t+1})$, and the equations for $\hat{G}_t$ and $\hat{g}_t$ indicated in (i) we therefore have, by strong duality, that

$$\begin{aligned} & \hat{\pi}_{t+1}B_{t+1}x_t^k + \hat{\pi}_{t+1}\xi_{t+1}^j + \sum_{i',j'\in[n_{t+2}]}\sum_{\kappa\in\mathcal{K}_{t+2}^{j',k}} g_{t+1,\kappa}^{j'}\hat{z}_{t+1,\kappa}^{i',j'} \\ &= \hat{G}_t x_t^k + \hat{g}_t \\ &= Q_{t+1}(x_t^k, \xi_{t+1}^j) \\ &= \max_{\kappa\in\mathcal{K}_{t+1}^j}[G_{t,\kappa}^j x_t^k + g_{t,\kappa}^j] \\ &> d_{t+1}^{i,j}\gamma_t^k + \nu_t^{i,k}, \end{aligned}$$

where the inequality holds for some $i,j \in [n_{t+1}]$ by the constraint-violation hypothesis. Thus (i) holds in the latter case. Q.E.D.

The remainder of the proof relies on a line of argument developed in Philpott and Guan Philpott and Guan (2008).

Proof of Lemma 5: By Lemma 1, we have that for all $t=2,\ldots,T$, $Q_t(\cdot,\xi_t)$ is a piecewise linear convex function in x_{t-1} with a finite number of pieces. For any sequence of stage $T-1$ solutions x^k_{T-1} generated by the deterministic variant of Algorithm 1, the fact that the backward pass uses extreme-point dual solutions implies there exists $\bar{k}_{T-1}$ such that no new cuts are generated for stage $T-1$ for $k>\bar{k}_{T-1}$. Lemma 3, coupled with the consistent primal tie-breaking Assumption (A.3), then implies that $\mathfrak{Q}^k_T(x^k_{T-1},\xi^j_T)=Q_T(x^k_{T-1},\xi^j_T)$ for all $j\in[n_T]$ and all $k>\bar{k}_{T-1}$.

We now similarly employ Lemmas 1 and 4 in inductive fashion to argue that after a finite number of iterations, $\bar{k}_t$, we have $\mathfrak{Q}^k_{t+1}(x^k_t,\xi^j_{t+1})=Q_{t+1}(x^k_t,\xi^j_{t+1})$ for all $j\in[n_{t+1}]$ and $k>\bar{k}_t$ for $t=T-2,T-3,\ldots,1$. Thus for $k>\bar{k}_t$, $(x^k_t,\gamma^k_t,\nu^k_t)$ solves problem (8), given its input, (x^k_{t-1},ξ^j_t). In particular, $(x^k_1,\gamma^k_1,\nu^k_1)$ solves problem (8) for $t=1$, given its input (x^k_0,ξ_1) for all $k>\bar{k}_1$. The desired result follows given that under (A.1)-(A.2), problem (8) is an equivalent reformulation of models (1) for $t=1$ and (2) for $t=2,\ldots,T-1$. Q.E.D.

Proof of Theorem 1: First, we note that Lemma 5 again holds if the stage $t=1,2,\ldots,T-1$ problems (11) are initialized with any set of valid lower-bounding cuts, even if the specific values of $\bar{k}_t$, $t=1,2,\ldots,T-1$, may differ. We know by Lemma 2 that the algorithm produces such cuts.

Given Assumptions (A.1) and (A.3) there are finitely many values of (x^k_{t-1},ξ^j_t) that problems (11) take as input at each stage. With probability one, in the course of running the algorithm there is an iteration after which no more cuts are added.

By Lemma 5, if Algorithm 1 follows a prespecified deterministic order of the $\prod_{t=2}^T n_t$ scenarios a sufficient number of times, we obtain an optimal solution. By the second Borel-Cantelli lemma (e.g., Grimmett and Stirzaker (1992)), Algorithm 1, under the sampling procedure of Algorithm 2 with $\beta<1$ and Assumption (A.1), will almost surely sample this sequence after the last iteration in which a cut has been added. Hence, we obtain the desired result. Q.E.D.