Solutions to HW9

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate. The solution to problem 6.2.1 is mine.

Problem 6.1.2 •

Let X_1 and X_2 denote a sequence of independent samples of a random variable X with variance $\text{Var}[X]$.

(a) What is $E[X_1 - X_2]$, the expected difference between two outcomes?

(b) What is $\text{Var}[X_1 - X_2]$, the variance of the difference between two outcomes?

Problem 6.1.2 Solution

Let $Y = X_1 - X_2$.

(a) Since $Y = X_1 + (-X_2)$, Theorem 6.1 says that the expected value of the difference is

$$E[Y] = E[X_1] + E[-X_2] = E[X] - E[X] = 0 \tag{1}$$

(b) By Theorem 6.2, the variance of the difference is

$$\text{Var}[Y] = \text{Var}[X_1] + \text{Var}[-X_2] = 2\,\text{Var}[X] \tag{2}$$

Problem 6.1.4 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 2 & x \geq 0, y \geq 0, x + y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the variance of $W = X + Y$?

Problem 6.1.4 Solution

We can solve this problem using Theorem 6.2 which says that

$$\text{Var}[W] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X, Y] \tag{1}$$

The first two moments of X are

$$E[X] = \int_0^1 \int_0^{1-x} 2x\,dy\,dx = \int_0^1 2x(1-x)\,dx = 1/3 \tag{2}$$

$$E\left[X^2\right] = \int_0^1 \int_0^{1-x} 2x^2\,dy\,dx = \int_0^1 2x^2(1-x)\,dx = 1/6 \tag{3}$$

$$\tag{4}$$

Thus the variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/18$. By symmetry, it should be apparent that $E[Y] = E[X] = 1/3$ and $\text{Var}[Y] = \text{Var}[X] = 1/18$. To find the covariance, we first find the correlation

$$E[XY] = \int_0^1 \int_0^{1-x} 2xy\, dy\, dx = \int_0^1 x(1-x)^2\, dx = 1/12 \tag{5}$$

The covariance is

$$\text{Cov}[X, Y] = E[XY] - E[X]E[Y] = 1/12 - (1/3)^2 = -1/36 \tag{6}$$

Finally, the variance of the sum $W = X + Y$ is

$$\text{Var}[W] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X, Y] = 2/18 - 2/36 = 1/18 \tag{7}$$

For this specific problem, it's arguable whether it would easier to find $\text{Var}[W]$ by first deriving the CDF and PDF of W. In particular, for $0 \leq w \leq 1$,

$$F_W(w) = P[X + Y \leq w] = \int_0^w \int_0^{w-x} 2\, dy\, dx = \int_0^w 2(w - x)\, dx = w^2 \tag{8}$$

Hence, by taking the derivative of the CDF, the PDF of W is

$$f_W(w) = \begin{cases} 2w & 0 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

From the PDF, the first and second moments of W are

$$E[W] = \int_0^1 2w^2\, dw = 2/3 \qquad E\left[W^2\right] = \int_0^1 2w^3\, dw = 1/2 \tag{10}$$

The variance of W is $\text{Var}[W] = E[W^2] - (E[W])^2 = 1/18$. Not surprisingly, we get the same answer both ways.

Problem 6.2.1 ■

Find the PDF of $W = X + Y$ when X and Y have the joint PDF

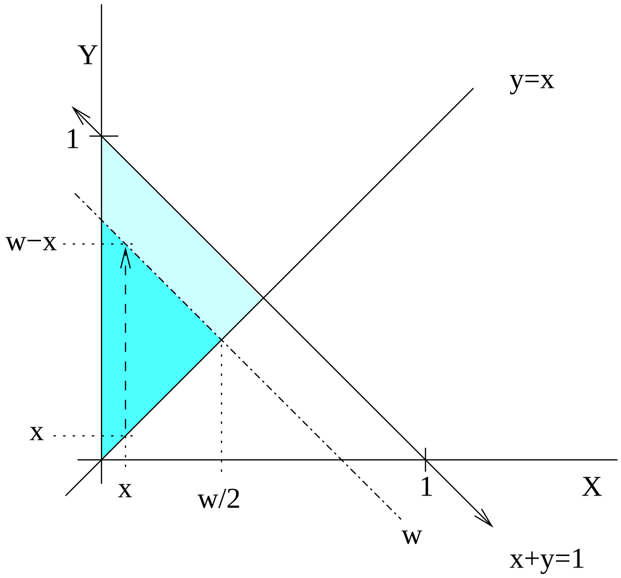
$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 \leq x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 6.2.1 Solution

We are given that $W = X + Y$ and that the joint PDF of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We are asked to find the PDF of W. The first step is to find the CDF of W, $F_W(w)$. Note that we must integrate over different shaped areas for different values of w.



To distinguish between the random variables and their values, I have been careful here to use capital letters for the random variable names and lower case letters for the specific values they take.
For values of W in the range $0 \leq w \leq 1$, we integrate over the shaded area in the figure to the left. A particular value w (indicated by the dotted and dashed line) of the random variable W corresponds to a set of pairs of X and Y values. For this value of w, we integrate from $Y = w - x$ to $Y = w$. To integrate over all values of the random variable W up to the value w, we then integrate with respect to X. As the value of the random variable W goes from 0 to w, the value of the random variable X goes from 0 to $w/2$. The lightly shaded area represents the region in which $0 \leq w \leq 1$ and $f_{X,Y}(x, y) > 0$. The darker shaded area represents the region corresponding to the limits of integration.

Here is the integration.

$$\begin{aligned}
F_W(w) &= \int_0^{\frac{w}{2}} \int_x^{w-x} 2\, dy\, dx && (2)\\
&= \int_0^{\frac{w}{2}} 2(w - x - x)\, dx && (3)\\
&= \int_0^{\frac{w}{2}} (2w - 4x)\, dx && (4)\\
&= 2wx - \frac{4x^2}{2}\bigg|_0^{\frac{w}{2}} && (5)\\
&= w^2 - w^2/2 && (6)\\
&= \frac{w^2}{2} && (7)
\end{aligned}$$

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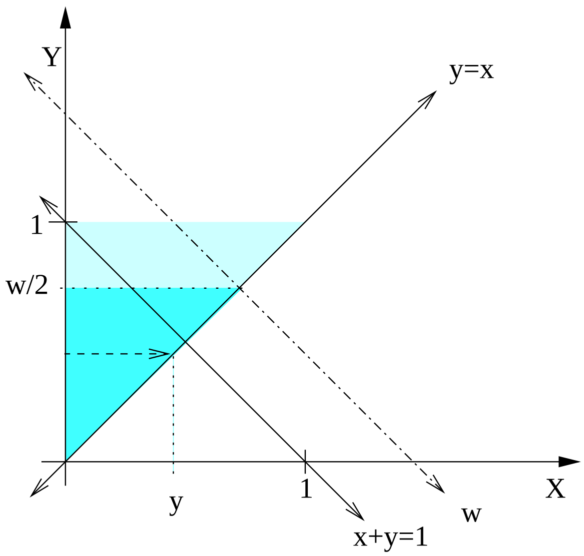
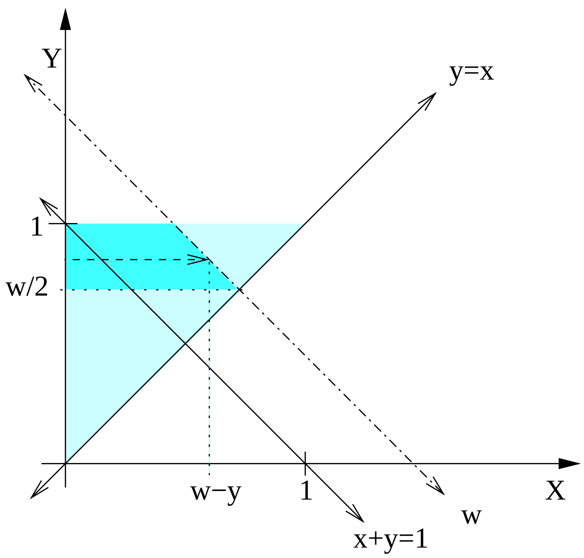


For values of w in the region $1 \leq w \leq 2$ we must compute the CDF as the sum of the integrals over two regions. The first region is shown at left and the second below. The first region is triangular but this time we integrate first with respect to X and then with respect to Y. We need to integrate with respect to X from 0 to y so that as Y goes from 0 to $w/2$, we cover the darker shaded triangular region. As before, the lightly shaded area represents the portion of the region in which $0 \leq w \leq 1$ and $f_{X,Y}(x, y) > 0$.



Next we consider the remainder of the region over which we must integrate to find the CDF of W. The extent of the remaining region in the X direction is limited by the value of w. The integral over this remaining region is captured in the second integral below, corresponding to the darkly shaded trapezoid at left. In the inner integration, the random variable X takes values from 0 to $w - y$ as indicated by the dashed line. Then in the outer integration, the random variable Y takes values from $w/2$ to 1.

Here's the integration.

$$\begin{aligned}
F_W(w) &= \int_0^{\frac{w}{2}} \int_0^{y} 2\,dx\,dy + \int_{\frac{w}{2}}^{1} \int_0^{w-y} 2\,dx\,dy && (8)\\
&= \int_0^{\frac{w}{2}} 2y\,dy + \int_{\frac{w}{2}}^{1} 2(w-y)\,dy && (9)\\
&= \frac{w^2}{4} + \left(2wy - y^2\right)\Big|_{\frac{w}{2}}^{1} && (10)\\
&= \frac{w^2}{4} + 2w - 1 - \left(w^2 - \frac{w^2}{4}\right) && (11)\\
&= 2w - 1 - \frac{w^2}{2} && (12)
\end{aligned}$$

Calculating the CDF of W over the various ranges of values w was the first step. The second step is to assemble the parts of the CDF $F_W(w)$ calculated above, and, by taking the derivative, calculate the PDF $f_W(w)$.

$$F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{w^2}{2} & 0 \le w \le 1 \\ 2w - 1 - \frac{w^2}{2} & 1 \le w \le 2 \\ 1 & w > 2 \end{cases} \qquad f_W(w) = \begin{cases} w & 0 \le w \le 1 \\ 2 - w & 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Problem 6.2.4 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

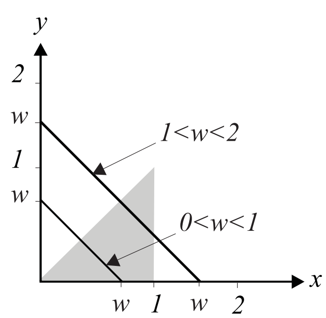
What is the PDF of $W = X + Y$?

Problem 6.2.4 Solution

In this problem, X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can find the PDF of W using Theorem 6.4: $f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)\,dx$. The only tricky part remaining is to determine the limits of the integration. First, for $w < 0$, $f_W(w) = 0$. The two remaining cases are shown in the accompanying figure. The shaded area shows where the joint PDF $f_{X,Y}(x,y)$ is nonzero. The diagonal lines depict $y = w - x$ as a function of x. The intersection of the diagonal line and the shaded area define our limits of integration.



For $0 \le w \le 1$,

$$f_W(w) = \int_{w/2}^{w} 8x(w-x)\,dx \tag{2}$$

$$= 4wx^2 - 8x^3/3\Big|_{w/2}^{w} = 2w^3/3 \tag{3}$$

For $1 \le w \le 2$,

$$f_W(w) = \int_{w/2}^{1} 8x(w-x)\,dx \tag{4}$$

$$= 4wx^2 - 8x^3/3\Big|_{w/2}^{1} \tag{5}$$

$$= 4w - 8/3 - 2w^3/3 \tag{6}$$

Since $X + Y \le 2$, $f_W(w) = 0$ for $w > 2$. Hence the complete expression for the PDF of W is

$$f_W(w) = \begin{cases} 2w^3/3 & 0 \le w \le 1 \\ 4w - 8/3 - 2w^3/3 & 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 6.3.1 •
For a constant $a > 0$, a Laplace random variable X has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|}, \quad -\infty < x < \infty.$$

Calculate the moment generating function $\phi_X(s)$.

Problem 6.3.1 Solution
For a constant $a > 0$, a zero mean Laplace random variable X has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|} \quad -\infty < x < \infty \tag{1}$$

The moment generating function of X is

$$\begin{aligned}
\phi_X(s) = E\left[e^{sX}\right] &= \frac{a}{2}\int_{-\infty}^{0} e^{sx} e^{ax}\, dx + \frac{a}{2}\int_{0}^{\infty} e^{sx} e^{-ax}\, dx & (2)\\
&= \frac{a}{2} \frac{e^{(s+a)x}}{s+a}\bigg|_{-\infty}^{0} + \frac{a}{2} \frac{e^{(s-a)x}}{s-a}\bigg|_{0}^{\infty} & (3)\\
&= \frac{a}{2}\left(\frac{1}{s+a} - \frac{1}{s-a}\right) & (4)\\
&= \frac{a^2}{a^2 - s^2} & (5)
\end{aligned}$$

Problem 6.3.3 ■
Continuous random variable X has a uniform distribution over $[a, b]$. Find the MGF $\phi_X(s)$. Use the MGF to calculate the first and second moments of X.

Problem 6.3.3 Solution
We find the MGF by calculating $E[e^{sX}]$ from the PDF $f_X(x)$.

$$\phi_X(s) = E\left[e^{sX}\right] = \int_a^b e^{sX} \frac{1}{b-a}\, dx = \frac{e^{bs} - e^{as}}{s(b-a)} \tag{1}$$

Now to find the first moment, we evaluate the derivative of $\phi_X(s)$ at $s = 0$.

$$E[X] = \frac{d\phi_X(s)}{ds}\bigg|_{s=0} = \frac{s\left[be^{bs} - ae^{as}\right] - \left[e^{bs} - e^{as}\right]}{(b-a)s^2}\Bigg|_{s=0} \tag{2}$$

Direct evaluation of the above expression at $s = 0$ yields $0/0$ so we must apply l'Hôpital's rule and differentiate the numerator and denominator.

$$\begin{aligned}
E[X] &= \lim_{s\to 0} \frac{be^{bs} - ae^{as} + s\left[b^2 e^{bs} - a^2 e^{as}\right] - \left[be^{bs} - ae^{as}\right]}{2(b-a)s} & (3)\\
&= \lim_{s\to 0} \frac{b^2 e^{bs} - a^2 e^{as}}{2(b-a)} = \frac{b+a}{2} & (4)
\end{aligned}$$

To find the second moment of X, we first find that the second derivative of $\phi_X(s)$ is

$$\frac{d^2\phi_X(s)}{ds^2} = \frac{s^2\left[b^2e^{bs} - a^2e^{as}\right] - 2s\left[be^{bs} - ae^{as}\right] + 2\left[be^{bs} - ae^{as}\right]}{(b-a)s^3} \tag{5}$$

Substituting $s = 0$ will yield 0/0 so once again we apply l'Hôpital's rule and differentiate the numerator and denominator.

$$E\left[X^2\right] = \lim_{s\to 0}\frac{d^2\phi_X(s)}{ds^2} = \lim_{s\to 0}\frac{s^2\left[b^3e^{bs} - a^3e^{as}\right]}{3(b-a)s^2} \tag{6}$$

$$= \frac{b^3 - a^3}{3(b-a)} = (b^2 + ab + a^2)/3 \tag{7}$$

In this case, it is probably simpler to find these moments without using the MGF.

Problem 6.4.1 •

N is a binomial $(n = 100, p = 0.4)$ random variable. M is a binomial $(n = 50, p = 0.4)$ random variable. Given that M and N are independent, what is the PMF of $L = M + N$?

Problem 6.4.1 Solution

N is a binomial $(n = 100, p = 0.4)$ random variable. M is a binomial $(n = 50, p = 0.4)$ random variable. Thus N is the sum of 100 independent Bernoulli $(p = 0.4)$ and M is the sum of 50 independent Bernoulli $(p = 0.4)$ random variables. Since M and N are independent, $L = M + N$ is the sum of 150 independent Bernoulli $(p = 0.4)$ random variables. Hence L is a binomial $n = 150, p = 0.4)$ and has PMF

$$P_L(l) = \binom{150}{l}(0.4)^l(0.6)^{150-l}. \tag{1}$$

Problem 6.4.2 •

Random variable Y has the moment generating function $\phi_Y(s) = 1/(1-s)$. Random variable V has the moment generating function $\phi_V(s) = 1/(1-s)^4$. Y and V are independent. $W = Y + V$.

(a) What are $E[Y]$, $E[Y^2]$, and $E[Y^3]$?

(b) What is $E[W^2]$?

Problem 6.4.2 Solution

Random variable Y has the moment generating function $\phi_Y(s) = 1/(1-s)$. Random variable V has the moment generating function $\phi_V(s) = 1/(1-s)^4$. Y and V are independent. $W = Y + V$.

(a) From Table 6.1, Y is an exponential $(\lambda = 1)$ random variable. For an exponential (λ) random variable, Example 6.5 derives the moments of the exponential random variable. For $\lambda = 1$, the moments of Y are

$$E[Y] = 1, \qquad E\left[Y^2\right] = 2, \qquad E\left[Y^3\right] = 3! = 6. \tag{1}$$

(b) Since Y and V are independent, $W = Y + V$ has MGF

$$\phi_W(s) = \phi_Y(s)\phi_V(s) = \left(\frac{1}{1-s}\right)\left(\frac{1}{1-s}\right)^4 = \left(\frac{1}{1-s}\right)^5. \tag{2}$$

W is the sum of five independent exponential ($\lambda = 1$) random variables $X_1, \ldots, X_5$. (That is, W is an Erlang ($n = 5, \lambda = 1$) random variable.) Each X_i has expected value $E[X] = 1$ and variance $\mathrm{Var}[X] = 1$. From Theorem 6.1 and Theorem 6.3,

$$E[W] = 5E[X] = 5, \qquad \mathrm{Var}[W] = 5\,\mathrm{Var}[X] = 5. \tag{3}$$

It follows that

$$E\left[W^2\right] = \mathrm{Var}[W] + (E[W])^2 = 5 + 25 = 30. \tag{4}$$

Problem 6.4.5 •

At time $t = 0$, you begin counting the arrivals of buses at a depot. The number of buses K_i that arrive between time $i-1$ minutes and time i minutes, has the Poisson PMF

$$P_{K_i}(k) = \begin{cases} 2^k e^{-2}/k! & k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

and $K_1, K_2, \ldots$ are an iid random sequence. Let $R_i = K_1 + K_2 + \cdots + K_i$ denote the number of buses arriving in the first i minutes.

(a) What is the moment generating function $\phi_{K_i}(s)$?

(b) Find the MGF $\phi_{R_i}(s)$.

(c) Find the PMF $P_{R_i}(r)$. Hint: Compare $\phi_{R_i}(s)$ and $\phi_{K_i}(s)$.

(d) Find $E[R_i]$ and $\mathrm{Var}[R_i]$.

Problem 6.4.5 Solution

$$P_{K_i}(k) = \begin{cases} 2^k e^{-2}/k! & k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

And let $R_i = K_1 + K_2 + \ldots + K_i$

(a) From Table 6.1, we find that the Poisson ($\alpha = 2$) random variable K has MGF $\phi_K(s) = e^{2(e^s - 1)}$.

(b) The MGF of R_i is the product of the MGFs of the K_i's.

$$\phi_{R_i}(s) = \prod_{n=1}^{i} \phi_K(s) = e^{2i(e^s - 1)} \tag{2}$$

(c) Since the MGF of R_i is of the same form as that of the Poisson with parameter, $\alpha = 2i$. Therefore we can conclude that R_i is in fact a Poisson random variable with parameter $\alpha = 2i$. That is,

$$P_{R_i}(r) = \begin{cases} (2i)^r e^{-2i}/r! & r = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(d) Because R_i is a Poisson random variable with parameter $\alpha = 2i$, the mean and variance of R_i are then both $2i$.

Problem 6.5.1 ■

Let $X_1, X_2, \ldots$ be a sequence of iid random variables each with exponential PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\phi_X(s)$.

(b) Let K be a geometric random variable with PMF

$$P_K(k) = \begin{cases} (1-q)q^{k-1} & k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Find the MGF and PDF of $V = X_1 + \cdots + X_K$.

Problem 6.5.1 Solution

(a) From Table 6.1, we see that the exponential random variable X has MGF

$$\phi_X(s) = \frac{\lambda}{\lambda - s} \tag{1}$$

(b) Note that K is a geometric random variable identical to the geometric random variable X in Table 6.1 with parameter $p = 1 - q$. From Table 6.1, we know that random variable K has MGF

$$\phi_K(s) = \frac{(1-q)e^s}{1 - qe^s} \tag{2}$$

Since K is independent of each X_i, $V = X_1 + \cdots + X_K$ is a random sum of random variables. From Theorem 6.12,

$$\phi_V(s) = \phi_K(\ln \phi_X(s)) = \frac{(1-q)\frac{\lambda}{\lambda - s}}{1 - q\frac{\lambda}{\lambda - s}} = \frac{(1-q)\lambda}{(1-q)\lambda - s} \tag{3}$$

We see that the MGF of V is that of an exponential random variable with parameter $(1-q)\lambda$. The PDF of V is

$$f_V(v) = \begin{cases} (1-q)\lambda e^{-(1-q)\lambda v} & v \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 6.6.1 •

The waiting time W for accessing one record from a computer database is a random variable uniformly distributed between 0 and 10 milliseconds. The read time R (for moving the information from the disk to main memory) is 3 milliseconds. The random variable X milliseconds is the total access time (waiting time + read time) to get one block of information from the disk. Before performing a certain task, the computer must access 12 different blocks of information from the disk. (Access times for different blocks are independent of one another.) The total access time for all the information is a random variable A milliseconds.

(a) What is $E[X]$, the expected value of the access time?

(b) What is $\text{Var}[X]$, the variance of the access time?

(c) What is $E[A]$, the expected value of the total access time?

(d) What is σ_A, the standard deviation of the total access time?

(e) Use the central limit theorem to estimate $P[A > 116\text{ms}]$, the probability that the total access time exceeds 116 ms.

(f) Use the central limit theorem to estimate $P[A < 86\text{ms}]$, the probability that the total access time is less than 86 ms.

Problem 6.6.1 Solution

We know that the waiting time, W is uniformly distributed on [0,10] and therefore has the following PDF.

$$f_W(w) = \begin{cases} 1/10 & 0 \le w \le 10 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We also know that the total time is 3 milliseconds plus the waiting time, that is $X = W + 3$, *and that the random variable A represents the total access time for accessing 12 blocks of information, also that the access times for different blocks are independent, in other words that the X_n for consecutive blocks are iid.*

(a) *By Theorem 3.5b,* the expected value of X is $E[X] = E[W + 3] = E[W] + 3 = 5 + 3 = 8$.

(b) *By Theorem 3.5d,* the variance of X is $\text{Var}[X] = \text{Var}[W + 3] = \text{Var}[W] = 25/3$.

(c) $A = \sum_{n=1}^{12} X_n$ *and the X_n are iid so* $\text{Var}[X_n] = \text{Var}[X]$ *for all n, and by Theorem 6.1,* the expected value of A is $E[A] = 12E[X] = 96$.

(d) *Independence implies, by Theorem 4.27c uncorrelatedness, then by Theorem 6.3, the variance of A is the sum of the variances of the X_i. Thus* the standard deviation of A is $\sigma_A = \sqrt{\text{Var}[A]} = \sqrt{12(25/3)} = 10$.

(e) $P[A > 116] = 1 - \Phi(\frac{116-96}{10}) = 1 - \Phi(2) = 0.02275$.

(f) $P[A < 86] = \Phi(\frac{86-96}{10}) = \Phi(-1) = 1 - \Phi(1) = 0.1587$

Problem 6.6.2 •
Telephone calls can be classified as voice (V) if someone is speaking, or data (D) if there is a modem or fax transmission. Based on a lot of observations taken by the telephone company, we have the following probability model: $P[V] = 0.8$, $P[D] = 0.2$. Data calls and voice calls occur independently of one another. The random variable K_n is the number of data calls in a collection of n phone calls.

(a) What is $E[K_{100}]$, the expected number of voice calls in a set of 100 calls?

(b) What is $\sigma_{K_{100}}$, the standard deviation of the number of voice calls in a set of 100 calls?

(c) Use the central limit theorem to estimate $P[K_{100} \geq 18]$, the probability of at least 18 voice calls in a set of 100 calls.

(d) Use the central limit theorem to estimate $P[16 \leq K_{100} \leq 24]$, the probability of between 16 and 24 voice calls in a set of 100 calls.

Problem 6.6.2 Solution
In the preamble to the problem statement, K_n is defined to be the number of data calls in a collection of n phone calls. In (a) K_{100} is defined as the number of voice calls in a set of 100 phone calls. Obviously these are inconsistent. The textbook authors give the solution for the data option. I will give the solution below for the voice option. I will use L_{100} for the number of voice calls in 100 phone calls.

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable D_i as the number of data calls in a single telephone call. It is obvious that for any i there are only two possible values for D_i, namely 0 and 1. Furthermore for all i the D_i's are independent and identically distributed with the following PMF.

$$P_D(d) = \begin{cases} 0.8 & d = 0 \\ 0.2 & d = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the above we can determine that

$$E[D] = 0.2 \qquad \text{Var}[D] = 0.2 - 0.04 = 0.16 \tag{2}$$

With these facts, we can answer the questions posed by the problem.

(a) $E[K_{100}] = 100E[D] = 20$

(b) $\text{Var}[K_{100}] = \sqrt{100\,\text{Var}[D]} = \sqrt{16} = 4$

(c) $P[K_{100} \geq 18] = 1 - \Phi\left(\frac{18-20}{4}\right) = 1 - \Phi(-1/2) = \Phi(1/2) = 0.6915$

(d) $P[16 \leq K_{100} \leq 24] = \Phi(\frac{24-20}{4}) - \Phi(\frac{16-20}{4}) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826$

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable V_i as the number of voice calls in a single telephone call. It is obvious that for any i there are only two possible values for V_i, namely 0 and 1. Furthermore for all i the V_i's are independent and identically distributed with the following PMF.

$$P_V(v) = \begin{cases} 0.8 & v = 0 \\ 0.2 & v = 1 \\ 0 & \textit{otherwise} \end{cases} \tag{3}$$

From the above we can determine that

$$E[V] = 0.8 \qquad \mathrm{Var}[D] = 0.8 - 0.64 = 0.16 \tag{4}$$

With these facts, we can answer the questions posed by the problem. Let L_n be the number of voice calls in n phone calls.

(a) $E[L_{100}] = 100E[V] = 80$

(b) $\mathrm{Var}[L_{100}] = \sqrt{100\,\mathrm{Var}[V]} = \sqrt{16} = 4$

(c) $P[L_{100} \geq 18] = 1 - \Phi\left(\frac{18-80}{4}\right) = 1 - \Phi(-31/2) = \Phi(31/2) \cong 1$ *(Our tables don't include* $\Phi(31/2)$. *We'd consult a book of mathematical or statistical tables if we needed a precise value.)*

(d) $P[16 \leq L_{100} \leq 24] = \Phi(\frac{24-80}{4}) - \Phi(\frac{16-80}{4}) = \Phi(-14) - \Phi(-16) = (1 - \Phi(14)) - (1 - \Phi(16)) = \Phi(16) - \Phi(14) =$ *a very small number. (Again, our tables don't include such values. We'd consult a book of mathematical or statistical tables if we needed a precise value.)*