HEAT TRANSFER: A LIQUID FLOWING THROUGH A POROUS PRISM.

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A METHOD often employed in practice for heating or cooling a fluid is to pass it through a mass of crushed material. Vice versa, a mass of crushed material may be heated or cooled by passing a stream of fluid through it.

It is therefore of some importance to formulate the laws governing the rate of heat transfer in such a case, and if possible, to obtain mathematical expressions for the temperature distribution throughout such a system.

A problem of this type which permits of exact mathematical treatment is the following:

Given a right porous prism (or cylinder) consisting of crushed material at a uniform temperature; a fluid, initially at some higher uniform temperature, is allowed to pass lengthwise through the prism at a uniform rate of flow, the sides of the prism being impervious to the fluid. The problem is, to find the distribution of temperature in the prism and in the fluid for all time, assuming that

(*a*) The lumps constituting the prism are so small or have such high thermal diffusivity that any given lump may be considered as being at a uniform temperature at any instant;

(*b*) Compared to the transfer of heat from fluid to solid, the transfer of heat by conduction in the fluid itself or in the solid itself is small and may be neglected;

(*c*) The rate of heat transfer from fluid to solid at any point is proportional to the average difference in temperature between fluid and solid at that point;

(*d*) Change in volume of fluid and solid due to change in temperature may be neglected; and

(*e*) The thermal constants are independent of the temperature.

Under what conditions and to what extent these assumptions are justified can be determined only by experiment. An inquiry into the experimental applications of the theory will not be made in the present paper, the main purpose being to present the mathematical treatment of the problem. For this purpose the following symbols will be employed:

x = distance traversed by the fluid in the prism
t = time
T_0 = initial temperature of the fluid
T_g = temperature of the fluid
T_s = temperature of the solid
h_g = heat capacity per unit volume of the fluid
h_s = heat capacity per unit volume of the solid
f = fractional porosity of the solid, i.e., free space per unit volume
W = volume of fluid passing unit cross-section in unit time
v = W/f = average linear velocity of the fluid
k = constant of heat transfer.

The initial temperature of the solid is taken as zero.

Let us consider the transfer of heat to an element of the fluid of unit cross-sectional area contained between x and $x + dx$. The heat imparted to this element by the solid in time dt

= − constant × difference in temperature × the width of the element × dt

$$= -k(T_g - T_s)dxdt.$$

The heat carried in by the flowing fluid

$$= -\frac{\partial T_g}{\partial x} h_g v f dxdt.$$

The sum of these must be = rise in temperature in time dt

× the heat capacity of the element

$$= \frac{\partial T_g}{\partial t} h_g f dxdt.$$

The resulting equation is

$$\frac{\partial T_g}{\partial t} + v\,\frac{\partial T_g}{\partial x} = -\,\frac{k}{h_g f}\,(T_g - T_s) \qquad (1)$$
$$= -\,k_2(T_g - T_s).$$

In the same way, by considering an element of the solid, we find

$$\frac{\partial T_s}{\partial t} = k_1(T_g - T_s), \qquad (2)$$

where

$$k_1 = k/h_s(1 - f) \qquad (3)$$

and

$$k_2 = k/h_g f. \qquad (4)$$

The significance of the constants k_1 and k_2 will be subsequently discussed.

(1) and (2) are the two equations which determine the transfer of heat. In addition to these there are certain boundary conditions which have to be satisfied if the problem is to be solved completely.

Consider the temperature at $x = 0$.

Here

$$T_g = T_0 \text{ for all time}, \qquad (5)$$
$$T_s = 0 \text{ at } t = 0,$$

and from equation (2) it follows that at $x = 0$

$$T_s = T_0(1 - e^{-k_1 t}). \qquad (6)$$

If we imagine the prism to extend to infinity in the positive x direction then we can find the temperatures at the farthest point reached by the fluid. At

$$x = vt,$$
$$T_s = 0, \qquad (7)$$
$$T_g = T_0 e^{-k_2 t}. \qquad (8)$$

This latter equation is found by following the frontmost element of fluid, which is always in contact with a part of the solid still at zero temperature. If we therefore consider the heat transfer from this element to the solid we arrive at the equation

$$\frac{dT_g}{dt} = -k_2T_g \text{ at } x = vt.$$

Integrated, this equation leads to equation (8).

By introducing two new independent variables

$$y = k_2x/v, \tag{9}$$

$$z = k_1(t - x/v), \tag{10}$$

equations (1) and (2) are reduced to the simpler forms,

$$\frac{\partial T_s}{\partial z} = T_g - T_s, \tag{11}$$

$$\frac{\partial T_g}{\partial y} = T_s - T_g. \tag{12}$$

These can be further simplified by introducing the two new dependent variables U and V, where

$$T_s = T_0(U - V)e^{-y-z}, \tag{13}$$

$$T_g = T_0(U + V)e^{-y-z}. \tag{14}$$

Substituting these values in (11) and (12), we find that

$$\frac{\partial U}{\partial z} - \frac{\partial V}{\partial z} = U + V, \tag{15}$$

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial y} = U - V, \tag{16}$$

and by further differentiation

$$\frac{\partial^2 V}{\partial y \partial z} = V. \tag{17}$$

The boundary conditions for U and V will obviously be:
(a) when $y = 0$

$$U = e^z - \tfrac{1}{2}, \tag{18}$$

$$V = \tfrac{1}{2}; \tag{19}$$

(*b*) when $z = 0$

$$U = \tfrac{1}{2}, \tag{20}$$

$$V = \tfrac{1}{2}. \tag{21}$$

Our first step will be to solve equation (17) subject to the boundary conditions (19) and (21).

Put

$$\varphi^2 = -4yz \tag{22}$$

then equation (17) reduces to

$$\frac{d^2V}{d\varphi^2} + \frac{1}{\varphi}\frac{dV}{d\varphi} + V = 0, \tag{23}$$

which is a form of Bessel's equation, of which a well-known solution is

$$V = AJ_0(\varphi), \tag{24}$$

where A is a constant and $J_0(\varphi)$ is a Bessel Function of the first kind and of order zero.

The boundary conditions (19) and (21) are both satisfied if $A = \frac{1}{2}$; hence the final solution is

$$\begin{aligned} V &= \tfrac{1}{2}J_0(\varphi) \\ &= \tfrac{1}{2}J_0(2i\sqrt{zy}) \\ &= \tfrac{1}{2}M_0(yz)\,(\text{say}), \end{aligned} \tag{25}$$

where we introduce the function M_0 which is defined thus:

$$\begin{aligned} M_0(a) &= J_0(2i\sqrt{a}) \\ &= 1 + a + \frac{a^2}{(2!)^2} + \frac{a^3}{(3!)^2} + \cdots. \end{aligned} \tag{26}$$

Having obtained the value of V, the next step is to find a solution for U which will satisfy equations (15) and (16) subject to the given boundary conditions.

Integrating equation (15) as an ordinary linear differential equation, we find

$$\begin{aligned} U &= e^z\int e^{-z}\left(V + \frac{\partial V}{\partial z}\right)dz + e^z f(y) \\ &= V + 2e^z\int e^{-z}V\,dz + f(y)e^z, \end{aligned} \tag{27}$$

$f(y)$ being a function of y. But by means of successive partial integration we find that

$$
\begin{aligned}
2e^{z}\int e^{-z}Vdz &= -2\left(V+\frac{\partial V}{\partial z}+\frac{\partial^2 V}{\partial z^2}+\cdots\right) \\
&= -\sum_{n=0}^{\infty} y^n M_n(yz),
\end{aligned}
\tag{28}
$$

where the M functions are thus defined:

$$
\begin{aligned}
M_0(a) &= J_0(2i\sqrt{a}), \\
M_n(a) &= \frac{d^n M_0(a)}{da^n}.
\end{aligned}
\tag{29}
$$

The series obtained above can be shown to be converging so that the expansion is valid. It follows that

$$U = \frac{1}{2} M_0(yz) + f(y)e^z - \sum_0^{\infty} y^n M_n(yz). \tag{30}$$

Now, when

$$
\begin{aligned}
z &= 0, \\
V &= \tfrac{1}{2}, \\
U &= \tfrac{1}{2},
\end{aligned}
$$

and

$$\sum_0^{\infty} y^n M_n(yz) = e^y.$$

Therefore

$$\tfrac{1}{2} = \tfrac{1}{2} + f(y) - e^y,$$

i.e.

$$f(y) = e^y$$

and equation (30) becomes

$$U = \frac{1}{2} M_0(yz) + e^{y+z} - \sum_0^{\infty} y^n M_n(yz). \tag{31}$$

This solution satisfies both the boundary conditions, but it must still be demonstrated that it also satisfies equation (16), which states that

$$\frac{\partial U}{\partial y} - U = -\frac{\partial V}{\partial y} - V. \tag{16}$$

From equations (27) and (28) it is evident that the expression for U can be written in the form

$$U = V + e^{y+z} - 2\left(V + \frac{\partial V}{\partial z} + \frac{\partial^2 V}{\partial z^2} + \cdots\right).$$

Hence

$$\frac{\partial U}{\partial y} = \frac{\partial V}{\partial y} + e^{y+z} - 2\left(\frac{\partial V}{\partial y} + \frac{\partial^2 V}{\partial y \partial z} + \frac{\partial^3 V}{\partial y \partial z^2} + \cdots\right)$$

$$= e^{y+z} - \frac{\partial V}{\partial y} - 2\left(V + \frac{\partial V}{\partial z} + \frac{\partial^2 V}{\partial z^2} + \cdots\right), \qquad (32)$$

since $\dfrac{\partial^2 V}{\partial y \partial z} = V.$

Substituting these values for U and $\dfrac{\partial U}{\partial y}$ in (16), it is evident that (16) is satisfied. Equation (31) therefore gives the value of U which satisfies the conditions of the problem completely.

It can be shown that there exists between the last term in equation (31) and the expression $\sum\limits_{0}^{\infty} z^n M_n(yz)$ a certain very simple relation.

To demonstrate this, put

$$B = \sum_{0}^{\infty} (y^n + z^n) M_n(yz)$$

$$= -2e^y \int e^{-y} V dy - 2e^z \int e^{-z} V dz. \qquad (32)$$

Now,

$$\frac{\partial B}{\partial y} = -2V - 2e^y \int e^{-y} V dy + 2\frac{\partial}{\partial y}\left(V + \frac{\partial V}{\partial z} + \frac{\partial^2 V}{\partial z^2} + \cdots\right)$$

$$= -2V - 2e^y \int e^{-y} V dy + 2\frac{\partial V}{\partial y} - 2e^z \int e^{-z} V dz$$

$$= B - 2V + 2\frac{\partial V}{\partial y}$$

i.e.

$$\frac{\partial (B - 2V)}{\partial y} = B - 2V. \qquad (33)$$

Similarly

$$\frac{\partial(B - 2V)}{\partial z} = B - 2V. \tag{34}$$

Hence

$$B - 2V = e^{y+z}$$

or

$$\sum_0^\infty (y^n + z^n)M_n(yz) = e^{y+z} + M_0(yz), \tag{35}$$

a fact which may be readily verified by expansion of the expressions and proper rearrangement of the terms.

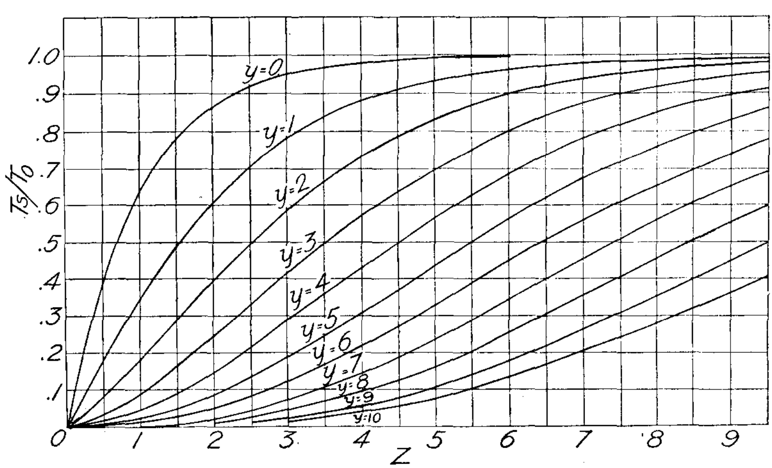
By substituting the values found for U and V in equations (13) and (14), the final solution is obtained

$$T_s/T_0 = 1 - e^{-y-z}\sum_{n=0}^\infty y^n M_n(yz) = e^{-y-z}\sum_{n=1}^\infty z^n M_n(yz), \tag{36}$$

$$T_g/T_0 = 1 - e^{-y-z}\sum_{n=1}^\infty y^n M_n(yz) = e^{-y-z}\sum_{n=0}^\infty z^n M_n(yz), \tag{37}$$

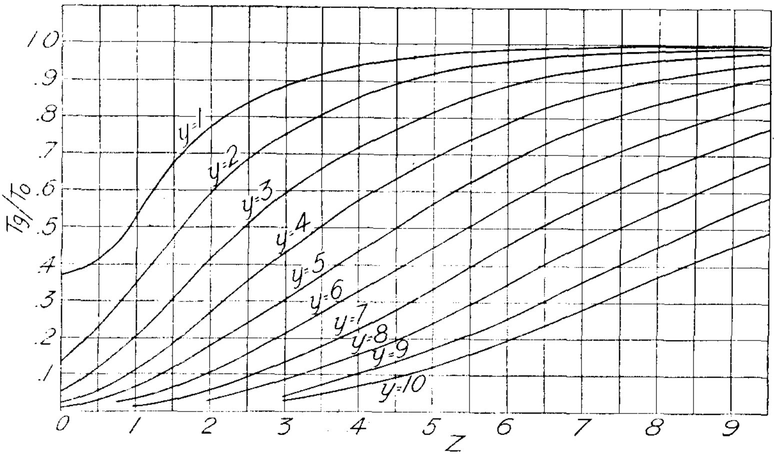
where y, z and the M functions have been previously defined. The numerical values of T_s/T_0 and T_g/T_0 were calculated from equations (36) and (37) for values of y and z ranging from zero to 10, and the results are graphically represented in Figures 1 and 2.

FIG. 1.



Curves showing the temperature in the solid.

FIG. 2.



Curves showing the temperature in the liquid.

From equations (1) and (2) it is evident that k_1 and k_2 both have the physical dimensions of an inverse time, and it follows from equations (9) and (10), which define y and z, that both these quantities are dimensionless. Thus, in the above-mentioned figures, all the quantities involved, namely, T_s/T_0, T_g/T_0, y and z are dimensionless. The charts are therefore of universal application to all problems of this type no matter what the actual constants involved may be.

There remains the discussion of k as used in the above analysis. For a given value of v, k is equal to the amount of heat transferred in unit time from fluid to solid in unit volume of the prism for unit difference of temperature.

From experiment [1] we know that k is proportional to some power, say the mth power, of the linear velocity and to the area of common surface between fluid and solid. If we denote by s the volume of a particle or lump divided by the area of its surface, then the area of the surface in unit volume of the prism will be $(1 - f)/s$.

Hence

$$k = \frac{K(1 - f)v^m}{s} = \frac{K(1 - f)W^m}{sf^m}, \tag{38}$$

[1] See e.g. Gröber, "Wärmeübertragung," p. 87.

where K is defined as the amount of heat transferred across unit area in unit time for unit temperature difference, the linear rate of flow being equal to unity.

Hence from equations (3), (4), (9) and (10)

$$y = \frac{Kx(1 - f)W^{m-1}}{sh_g f}, \tag{39}$$

$$z = \frac{KtW^m}{sh_s f^m} - \frac{KxW^{m-1}}{sh_s f^{m-1}}. \tag{40}$$

The temperature distributions in fluid and solid are thus expressed in quantities which are known or which can easily be found by experiment, and the problem is completely solved from the theoretical standpoint. It need hardly be pointed out that the theory also applies to the case where the prism is initially at a higher temperature than the liquid.

As will be readily recognized, the above analysis also applies to the turbulent flow of a liquid through an adiabatically shielded thin-walled pipe.

From the practical point of view, it would be of great value if the same problem could be solved for the transfer of heat from a flowing gas to a prism of crushed material, taking due consideration of the fact that a gas expands with increasing temperature. The mathematical difficulties, however, seem to be insuperable. Nevertheless, there remains a method of at least partially solving this very interesting problem, namely the method of dimensional analysis. The particular method we adopt for this purpose is to write down the differential equations which determine the transfer of heat and from these to draw certain conclusions concerning the dimensional similarity between different systems.

If it be assumed that the gas in question obeys Charles' Law, the differential equations corresponding to equations (1) and (2) can be shown to be

$$\frac{\partial T_s}{\partial t} = \frac{Kv^m}{sh_s}(T_g - T_s), \tag{41}$$

$$\frac{\partial T_g}{\partial t} + v\frac{\partial T_g}{\partial x} = -\frac{K(1 - f)v^m T_g}{sh_0 T_0 f}(T_g - T_s), \tag{42}$$

where now

$$v = \frac{WT_g}{fT_0} + T_g \int_0^x \frac{1}{T_g^2} \frac{\partial T_g}{\partial t}\, dx$$

$$= \frac{WT_g}{fT_0}, \qquad (43)$$

since the second term is so small as to be insignificant in all practical cases. The method of derivation of the above expressions is quite straightforward and will not be given here. In equations (41)–(43) all the temperatures should be expressed in degrees absolute, and W = volume of the gas flowing through unit cross section in unit time measured at temperature T_0

h_0 = thermal capacity of unit volume of the gas at temperature T_0.

Examining equations (41) and (42), it will be found that they remain unaltered if

(*a*)	W is multiplied by a factor	M,
	t " " " " "	M^{-m},
	x " " " " "	M^{1-m};
(*b*)	s " " " " "	M,
	t " " " " "	M,
	x " " " " "	M.

The conclusions to be drawn from these statements are:

(*a*) If the mass of gas passing through the prism be increased M times the temperature history of a point x will be repeated at a point xM^{1-m} in $1/M^m$ of the original time. In other words if an experiment has been performed on a given system at a given rate of flow, and the temperature distribution has been found throughout the prism for all times, then for any other rate of flow the temperature distribution may be predicted.

(*b*) In the same way, if the linear dimensions s of the solid particles constituting the prism be increased M times the original temperature history at a point x will be reproduced at a point Mx in M times the time. This means that having experimented with one size of particle, the theory allows us to predict the temperature distribution for a prism consisting of particles of any other size, provided all the other factors remain the same. In the same way some further conclusions may be drawn, but these two examples of dimensional similarity will suffice to illustrate the method.

SUMMARY.

If a liquid, initially at uniform temperature, passes lengthwise through a right, porous prism, initially at some other uniform temperature, the sides of the prism being adiabatic and impervious to the liquid, then the temperatures of both liquid and solid will be functions of the time and of a distance. The problem of finding the temperature distributions has been solved assuming the well-known laws governing the transfer of heat from a liquid in turbulent motion to a solid. The solution as presented involves the use of some interesting mathematical functions which are related to the well-known Bessel Functions. If a gas is used instead of a liquid, the problem is much more complicated, but a dimensional method of treating the problem leads to results which may be very useful in practice.

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