1D

- Solución aproximada del modelo de almacenamiento térmico con intercambio de calor:

El modelo de Schumann (1929) es una aproximación sencilla para el intercambio de calor entre un flujo que atraviesa un medio poroso a una determinada temperatura:

$$\begin{cases} \varepsilon\left(\dfrac{\partial T_f}{\partial t} + u_0 \dfrac{\partial T_f}{\partial x}\right) = \varepsilon \alpha_f \dfrac{\partial^2 T_f}{\partial x^2} - \dfrac{h_v}{\rho_f C_{p_f}}(T_f - T_s) \\ (1-\varepsilon)\dfrac{\partial T_s}{\partial t} = (1-\varepsilon)\alpha_s \dfrac{\partial^2 T_s}{\partial x^2} + \dfrac{h_v}{\rho_s C_{p_s}}(T_f - T_s) \end{cases}$$

Con las condiciones de contorno:

$$T_f(x=0,t) = T_{in}(t) \qquad \frac{\partial T_s}{\partial x}(x=0,t) = \frac{\partial T_s}{\partial x}(x=1,t) = 0$$

$$\frac{\partial T_f}{\partial x}(x=0,t) = 0$$

y la condición inicial:

$$T_f(x,t=0) = T_s(x,t=0) = T_0$$

En el sistema eqs.,

$\alpha \rightarrow$ difusividad térmica (m^2/s)

$h_v \rightarrow$ coeff. de intercambio de calor $\left(\frac{W}{m^2 K}\right)$

$\varepsilon \rightarrow$ Porosidad

$u_0 \rightarrow$ Velocidad del flujo.

$T_f, T_s \rightarrow$ Temperaturas del fluido y del sólido.

Adimensionalización:

$$\eta = x/L \,, \quad \tau = t/t_c \,, \quad \theta_f = \frac{T_f - T_0}{T_0}$$

(1) $$\epsilon\left(\frac{1}{t_c}\frac{\partial \theta_f}{\partial \tau} + \frac{u_0}{L}\frac{\partial \theta_f}{\partial \eta}\right) = \epsilon \frac{\alpha_f}{L^2}\frac{\partial^2 \theta_f}{\partial \eta^2} - \frac{h_v}{\rho_f C_{pf}}(\theta_f - \theta_s)$$

(2) $$(1-\epsilon)\frac{1}{t_c}\frac{\partial \theta_s}{\partial \tau} = (1-\epsilon)\frac{\alpha_s}{L^2}\frac{\partial^2 \theta_s}{\partial \eta^2} + \frac{h_v}{\rho_s C_{ps}}(\theta_f - \theta_s)$$

$$\theta_f(0,\tau) = \frac{T_{in}}{T_0} - 1$$

$$\frac{\partial \theta_f}{\partial \eta}(1,\tau) = 0, \quad \frac{\partial \theta_s}{\partial \eta}(0,\tau) = \frac{\partial \theta_s}{\partial \eta}(1,\tau) = 0.$$

$$\theta_f(\eta,0) = \theta_s(\eta,0) = 0$$

$$\frac{h_v L}{\rho_f C_{pf} u_0} \equiv \Lambda_f \,, \quad \frac{h_v L}{\rho_s C_{ps} u_0} \equiv \Lambda_s \,.$$

El tiempo característico lo tomo para que sea del orden del término convectivo.

$$t_c = \frac{L}{u_0}$$

$$\epsilon\left(\frac{\partial \theta_f}{\partial \tau} + \frac{\partial \theta_f}{\partial \eta}\right) = \epsilon \frac{\alpha_f L}{L^2 u_0}\frac{\partial^2 \theta_f}{\partial \eta^2} - \Lambda_f(\theta_f - \theta_s)$$

$$(1-\epsilon)\frac{\partial \theta_s}{\partial \tau} = (1-\epsilon)\frac{\alpha_s L}{L^2 u_0}\frac{\partial^2 \theta_s}{\partial \eta^2} + \Lambda_s(\theta_f - \theta_s)$$

$$\frac{\alpha_f}{L u_0} \sim O(10^{-4}), \quad \frac{\alpha_s}{L u_0} \sim O(10^{-4})$$

$$\Lambda_f \sim O(10^2), \quad \Lambda_s \sim O(10^{-1})$$

En primera aprox. tomamos $\frac{\alpha_f}{L u_0} \sim \frac{\alpha_s}{L u_0} \sim \epsilon \ll 1$

sería un problema singular de perturbac. (capa límite)

Buscamos una solución para $\epsilon \to 0$,

$$\left.\begin{aligned} (1)\quad & \frac{\partial \theta_f}{\partial \tau} + \frac{\partial \theta_f}{\partial \eta} = -\frac{\Lambda_f}{\varepsilon}(\theta_f - \theta_s) \\ (2)\quad & \frac{\partial \theta_s}{\partial \tau} = \frac{\Lambda_s}{1-\varepsilon}(\theta_f - \theta_s) \end{aligned}\right\}$$

Renombro: $\Lambda_f \leftarrow \dfrac{\Lambda_f}{\varepsilon}$, $\Lambda_s \leftarrow \dfrac{\Lambda_s}{1-\varepsilon}$.

$\Lambda_f \sim O(10^3)$, $\Lambda_s \sim O(10^{-1})$

De (1), $\theta_s = \dfrac{1}{\Lambda_f}\left[\dfrac{\partial \theta_f}{\partial \tau} + \dfrac{\partial \theta_f}{\partial \eta}\right] + \theta_f$

De (2), $\theta_f = \dfrac{1}{\Lambda_s}\dfrac{\partial \theta_s}{\partial \tau} + \theta_s$

Combinando para aislar θ_f,

$$\left(\frac{\partial \theta_s}{\partial \tau} = \frac{1}{\Lambda_f}\left[\frac{\partial^2 \theta_f}{\partial \tau^2} + \frac{\partial^2 \theta_f}{\partial \tau \partial \eta}\right] + \frac{\partial \theta_f}{\partial \tau}\right)$$

$$\cancel{\theta_f} = \frac{1}{\cancel{\Lambda_s}}\frac{1}{\cancel{\Lambda_f}}\left[\frac{\partial^2 \theta_f}{\partial \tau^2} + \frac{\partial^2 \theta_f}{\partial \tau \partial \eta}\right] + \frac{\Lambda_f}{\cancel{\Lambda_s}}\frac{\partial \theta_f}{\partial \tau} +$$

$$+ \frac{\Lambda_s}{\cancel{\Lambda_f}}\left[\frac{\partial \theta_f}{\partial \tau} + \frac{\partial \theta_f}{\partial \eta}\right] + \cancel{\theta_f}$$

$$0 = \frac{\partial^2 \theta_f}{\partial \tau^2} + \frac{\partial^2 \theta_f}{\partial \tau \partial \eta} + (\Lambda_f + \Lambda_s)\frac{\partial \theta_f}{\partial \tau} +$$

$$+ \frac{\partial \theta_f}{\partial \eta}$$

Resolver para θ_f y luego obtener θ_s

$\to$ Sep. Variables.

$$\Theta_g = X(\eta)\, T(\tau)$$

$$0 = X T'' + X'T' + (\underbrace{\Lambda_g + \Lambda_s}_{\sim \Lambda_g}) X T' + X'T$$

$$0 = \frac{T''}{T} + \frac{X'T'}{XT} + \Lambda_g \frac{T'}{T} + \frac{X'}{X}$$

$$\frac{T''}{T} + \Lambda_g \frac{T'}{T} + \frac{X'T'}{XT} = -\frac{X'}{X}$$

$$\frac{T''}{T} + \frac{T'}{T}\left(\Lambda_g + \frac{X'}{X}\right) = -\frac{X'}{X}$$

$$\frac{d}{d\tau}\left(\frac{T''}{T}\right) + \left(\Lambda_g + \frac{X'}{X}\right)\frac{d}{d\tau}\left(\frac{T'}{T}\right) = 0.$$

$$\frac{T'''T - T''T'}{\cancel{T^2}} + \left(\Lambda_g + \frac{X'}{X}\right)\frac{T''T - T'^2}{\cancel{T^2}} = 0.$$

$$T'''T - T''T' = -\left(\Lambda_g + \frac{X'}{X}\right)\left(T''T - T'^2\right)$$

$$\frac{T'''T - T''T'}{T''T - T'^2} = -\left(\Lambda_g + \frac{X'}{X}\right) = -\lambda.$$

$$\frac{X'}{X} = \underbrace{\lambda - \Lambda_g}_{} \equiv \mu \Rightarrow X = c_1 e^{\mu\eta}$$

$$X' = \mu c_1 e^{\mu\eta}$$

$$\frac{X'}{X} = \mu$$

$$\frac{T''}{T} + \Lambda_g \frac{T'}{T} + \mu \frac{T'}{T} = -\mu$$

$$T'' + T'(\underbrace{\Lambda_g + \mu}_{\lambda}) + T\mu = 0.$$

$$T = c_1 e^{-\frac{1}{2}\tau\left(\sqrt{\lambda^2 - 4\mu} + \lambda\right)} + c_2 e^{\frac{1}{2}\tau\left(\sqrt{\lambda^2 - 4\mu} - \lambda\right)}$$

La solución se recupera de

$$\theta_f = X T = e^{\mu\eta}\left(C_1 e^{-\frac{1}{2}\tau\left(\sqrt{\lambda^2-4\mu}-\lambda\right)} + C_2 e^{\frac{1}{2}\tau\left(\sqrt{\lambda^2-4\mu}+\lambda\right)} \right)$$

$$\theta_f(\eta, 0) = 0 = \cancel{e^{\mu\eta}}\,(C_1 + C_2) \Rightarrow \boxed{C_1 = -C_2}$$

$$\theta_f(0,\tau) = \frac{T_{in}}{T_0} - 1 = C_1\left(e^{-\frac{1}{2}\tau\left(\sqrt{\lambda^2-4\mu}-\lambda\right)} - e^{\frac{1}{2}\tau\left(\sqrt{\lambda^2-4\mu}+\lambda\right)} \right)$$

$$\boxed{C_1 = \left(1 - \frac{T_{in}}{T_0}\right) \frac{e^{\frac{1}{2}\tau\left(-\lambda+\sqrt{\lambda^2-4\mu}\right)}}{e^{\tau\sqrt{\lambda^2-4\mu}} - 1}}$$

$$\frac{\partial \theta_f}{\partial \eta}(1,\tau) = 0 \Rightarrow \mu e^{\mu}\cancel{(\quad)} = 0$$

$$\boxed{\mu = 0 \Rightarrow \lambda = \Lambda_f}$$

$$C_1 = \left(1 - \frac{T_{in}}{T_0}\right) \frac{1}{e^{\tau\Lambda_f} - 1}$$

$$\theta_f = \left(1 - \frac{T_{in}}{T_0}\right) \frac{1}{e^{\tau\Lambda_f} - 1}\left(1 - e^{\Lambda_f \tau}\right)$$