# Lanchester Equations

The English engineer Frederick William Lanchester in 1916 and Russian military officer Mikhail Osipov in 1915 independently proposed a model of mutual attrition of two fighting forces by a set of ordinary differential equations. The Lanchester equations can model attrition of forces in combat, they set the casualty rate for an army as a function of the size of each army. The state variables represent fighting entities in the battlefield, and each of the differential equations capture the rate of decrease in a certain state variable as a function of the other state variables. These models were inspired by air-combat scenarios in World War I. These models are set up here for RED and BLUE forces.

Lanchester’s Second Linear Law (, see below) is used for unaimed fires, where a military indiscriminately shells large swathes of land rather than specifically targeting enemy forces. In this case, the casualty rate scales with the number of firers and the number of targets. Lanchester’s Square Law (, see below) is used for modern militaries that are concentrating their forces and targeting specific enemies. For this case, the casualty rate scales with the number of firers.

## Square law model of modern warfare with aimed fire

The aimed-fire model represents a combat situation where each combatant on the BLUE (RED) side effectively reduces the force on the RED (BLUE) side at a certain fixed attrition rate This model describes modern warfare with aimed fire, where all shooters can concentrate their fire on all targets.

Variables of aimed fire model: size of BLUE forces , size of RED forces , attrition coefficient of killing BLUE for RED , attrition coefficient of killing RED for BLUE :

Solving the equations and explicitly:

Dividing the second equation by the first and then separating variables, lead to the quadratic state equation:

Since this quantity never changes sign, only one of and can ever be zero. If the initial value of is positive, for example, only can equal zero, and so only BLUE can win. This quantity is the key in determining who will win the battle, and a side’s fighting strength and varies as the units’ fighting effectiveness times the square of their numbers.

### Quality vs. Quantity

Suppose RED begins with twice as many units as BLUE, , but the BLUE units are three times as effective, , then RED wins – to draw with RED, BLUE needs :

### Dividing Forces

What would have happened had BLUE been able to divide RED into two equal forces and engage them sequentially? At the end of the first engagement, between and , remain:

And in the second engagement, BLUE will win with left:

With an N-fold division of RED, after the simple resulting iteration, BLUE now wins with a final number of units remaining, so that the N-fold division has reduced red’s fighting strength N-fold:

Again, this is a classic military maxim: you should (almost) never divide your forces. The model assumes that the battle is a homogeneous mixing of forces and in such engagements the tactical conclusion is simple: if your strength is in numbers, then you need to fight in this way, bringing all your units to engage with the enemy as rapidly as possible. Conversely, if your units are fewer in number but more effective, then you need to pick-off opponents, and preventing the enemy from bringing all his units to bear – like Hannibal at Cannae.

The Lanchester model’s main point is to distinguish the importance of numbers from that of fighting effectiveness. In case that we take BLUE’s basic unit to be troops, then the number and effectiveness of the BLUE forces scale accordingly, and the course and outcome of the battle do not change:

## First linear law model of ancient warfare

Lanchester’s first linear law model of ancient battle assumes that the battle comprises warfare with one-on-one encounters, typical to battles in early history. In such a battle, there is no meaning for concentration of forces and the attrition is fixed.

Variables of ancient warfare model: size of BLUE forces , size of RED forces , attrition coefficient of killing BLUE for RED , attrition coefficient of killing RED for BLUE :

Dividing the second equation by the first and then separating variables lead to the linear state equation:

## Second linear law model with unaimed area fire

Lanchester considered a model with unaimed fire as (ancient) hand-to-hand combat without firearms, while a modern model with this property would be that of artillery fire or bombardment, in which, if RED guns fire with effectiveness at random into an area in which there are known to be BLUE units (which therefore have density per unit area), leading to:

The second Linear Law describes unaimed fire, where the effect of one’s fire does not only depend on the size of its own surviving force but also on the density of the targets at the opposing force. As the fire is not aimed, the probability of acquiring a target on the other side depends on the number of such targets in each area:

Variables of area fire model: size of BLUE forces , size of RED forces , attrition coefficient of killing BLUE for RED , attrition coefficient of killing RED for BLUE :

Dividing the second equation by the first and then separating variables lead to the linear state equation:

With and , a solution of Lanchester’s linear law is given by:

Using the state equation :

Sketch of finding the solution:

Test solution with differential equations:

And in case of equivalent forces with

### Quality vs. Quantity

Suppose RED begins with twice as many units as BLUE, , but the BLUE units are two times as effective, , then neither RED nor BLUE wins:

### Dividing Forces

What would have happened had BLUE been able to divide RED into two equal forces and engage them sequentially? At the end of the first engagement, between and , remain:

And in the second engagement, there will be a draw between BLUE and RED with no one left:

With an N-fold division of RED, after the simple resulting iteration, BLUE now wins with a final number of units remaining, so that the N-fold division has reduced red’s fighting strength N-fold:

The linear law does not give a predominant role to the size of the forces with respect to his combat power. In the case of the square law, a lack of combat expertise could be easily balanced by rising the number of soldiers by a small amount, whereas in the case of the linear law this is no longer true.

## Fitting Warfare with generalized model

Attempts in the literature to fit either the aimed or the unaimed-fire model to data have only been partly successful. Bracken’s model generalizes to intermediate grades of aimed and unaimed fire.

Variables of model: size of BLUE forces , size of RED forces , attrition coefficient of killing BLUE for RED , attrition coefficient of killing RED for BLUE :

The Lanchester square model of modern warfare with aimed fire corresponds to , and thus to , the unaimed-fire model to and thus . If then numbers matter more than effectiveness, and conversely. It might be natural to assume that (and thus ), so that one’s rate of loss scales more quickly with the enemy’s numbers than with one’s own.

## References

Frederick William Lanchester (1916). Aircraft in Warfare: The Dawn of the Fourth Arm. Constable, London.

J. Bracken (1995). Lanchester models of the Ardennes campaign. Nav. Res. Log. 42, 559-577.

## Further Literature

Wikipedia: Lanchester's laws. <https://en.wikipedia.org/wiki/Lanchester%27s_laws>

Michael J. Artelli, Richard F. Deckro. Modeling the Lanchester Laws with System Dynamics (2008). The Journal of Defense Modeling and Simulation 5(1):1-20. <https://www.academia.edu/9956784/Modeling_the_Lanchester_Laws_with_System_Dynamics>

N. Cangiotti, M. Capolli, and M. Sensi (2023). A generalization of unaimed fire Lanchester’s model in multi-battle warfare. Oper Res Int J 23, 38 (2023). <https://doi.org/10.1007/s12351-023-00776-8>

M. McCartney (2023). Battling with Lanchester’s equations in the classroom. International Journal of Mathematical Education in Science and Technology, 54:3, 451-461. <https://doi.org/10.1080/0020739X.2021.2022230>

Niall MacKay (2005). Lanchester combat models. <https://arxiv.org/abs/math/0606300> <https://www.researchgate.net/publication/2128695_Lanchester_combat_models>

# Mixed models in asymmetric engagements

## Defending narrow passages

An asymmetric combat situation is manifested in attacks typically conducted in narrow passages such as in mountainous regions or bridges. The defending RED force is effectively deployed in an area dominating the passage so that it can concentrate its fire on the approaching attacking BLUE force, which moves in a single column because of the topographical constraints. While RED can apply direct fire from all its units, BLUE can only fire from its front weapon.

Variables of model: size of BLUE forces , size of RED forces , attrition coefficient of killing BLUE for RED , attrition coefficient of killing RED for BLUE :

## Guerilla warfare

In an asymmetric warfare situation between regular and guerilla forces, the guerilla forces are somewhat aware of the location and whereabouts of the conventional forces. They use aimed fire at the regular forces, who are fully exposed to the guerrillas. On the other hand, the guerilla forces are small in numbers and their location is often only known to be within a general region. The guerrillas might be well hidden in an ambush or mixed in the civilian population. The regular forces can only apply unaimed area fire on the guerrillas, so that their effectiveness depends on the density of the guerrillas’ combatants. As the number of guerrillas decreases with attrition, it is harder to acquire a target. In this model of guerrilla warfare by RED, or an attack by BLUE over open ground against concealed RED defenders, RED forces can aim but the BLUE ones cannot.

Variables of model: size of BLUE forces , size of RED forces , attrition coefficient of killing BLUE for RED , attrition coefficient of killing RED for BLUE :

Dividing the second equation by the first and then separating variables:

Suppose RED begins with twice as many units as BLUE, , how effective must BLUE be to achieve a draw with RED:

## Variable target visibility

While the square law model of modern warfare with aimed fire model () assumes perfect visibility of targets on both sides, the second linear model with unaimed area fire () assumes none, their effectiveness depends on the density of targets in the area. A rather simple model combines aimed as well as unaimed fire for both sides.

Variables of model: size of BLUE forces , size of RED forces , attrition coefficient of killing BLUE for RED with unaimed fire and aimed fire , attrition coefficient of killing RED for BLUE with unaimed fire and aimed fire , the rate of unaimed fire for RED and BLUE , the area of effect of unaimed fire for RED and BLUE , the area of spreading for RED and BLUE :

A model for a situation somewhere in between, where some portion of the force is visible to the other side while the rest of the force remains concealed, and where the level of situational awareness regarding the opponent’s targets affects the outcome of the battle, introduces visibility parameters and for BLUE and Red. The parameter for unaimed fire has necessarily a different dimension than one for aimed fire. Rather than introducing a single parameter for unaimed fire the model retains a parameter for aimed fire but introduces new fixed parameters and with the same dimensions as and , which parametrizes the density effect of unaimed fire.

Variables of model: size of BLUE forces , size of RED forces , attrition coefficient of killing BLUE for RED , attrition coefficient of killing RED for BLUE , visibility parameters and density effect of unaimed fire , for RED and BLUE:

This expression interpolates between Lanchester linear and square law. Dividing the second equation by the first, separating variables, and computing partial fractions:

Using:

With:

The linear law with unaimed fire is the limit:

The square law of aimed fire uses the Taylor expansion of logarithm in the limit:

## References

Howard Brackney (1959). The dynamics of military combat. Operations Research, 7, 30-44.

S. J. Deitchman (1962). A Lanchester model of guerrilla warfare. Operations Research 10, 818-827.

## Further Literature

Vikram Mittal (20.09.2022). Basic Attrition Models Provide Insight into Russian Woes in Russia-Ukraine War. Forbes. <https://www.forbes.com/sites/vikrammittal/2022/09/20/basic-attrition-models-provide-insight-into-russian-woes-in-russia-ukraine-war/>

Moshe Kress (07.05.2020). Lanchester Models for Irregular Warfare. Mathematics 2020, 8(5), 737; <https://doi.org/10.3390/math8050737>

Kress, Moshe; MacKay, Niall J. (13.08.2013). Bits or Shots in Combat? The Generalized Deitchman Model of Guerrilla Warfare. <https://apps.dtic.mil/sti/citations/ADA588025>

Schramm, Harrison C. (2012). Lanchester Models with Discontinuities: An Application to Networked Forces. Naval Postgraduate School, Monterey, California. <https://hdl.handle.net/10945/37797>

Wikipedia: Salvo combat model. <https://en.wikipedia.org/wiki/Salvo_combat_model>

# Causal-Loop Diagrams

## Square law model



Figure 1: Causal-loop diagram of Lanchester’s model with targeting fire.

size of BLUE forces=INTEG(-RED killing BLUE, initial size of BLUE forces), Units: Strength

size of RED forces=INTEG(-BLUE killing RED, initial size of RED forces), Units: Strength

BLUE killing RED=attrition coefficient of RED for BLUE\*size of BLUE forces, Units: Strength/Year

RED killing BLUE=attrition coefficient of BLUE for RED\*size of RED forces, Units: Strength/Year

initial size of BLUE forces=1, Units: Strength

initial size of RED forces=2, Units: Strength

attrition coefficient of BLUE for RED=0.25, Units: 1/Year

attrition coefficient of RED for BLUE=0.75, Units: 1/Year

INITIAL TIME=0, FINAL TIME=2, TIME STEP=0.125, Units: Year

Figure 2: Behaviour over time of levels and phase space for Lanchester’s model with targeting fire with and .

## Second linear law model



Figure 3: Causal-loop diagram of Lanchester’s model with unaimed fire.

size of BLUE forces=INTEG(-RED killing BLUE, initial size of BLUE forces), Units: Strength

size of RED forces=INTEG(-BLUE killing RED, initial size of RED forces), Units: Strength

BLUE killing RED=attrition coefficient of RED for BLUE\*size of BLUE forces\*size of RED forces, Units: Strength/Year

RED killing BLUE=attrition coefficient of BLUE for RED\*size of RED forces\*size of BLUE forces, Units: Strength/Year

initial size of BLUE forces=1, Units: Strength

initial size of RED forces=2, Units: Strength

attrition coefficient of BLUE for RED=0.25, Units: 1/(Strength\*Year)

attrition coefficient of RED for BLUE=0.5, Units: 1/(Strength\*Year)

INITIAL TIME=0, FINAL TIME=10, TIME STEP=0.125, Units: Year

Figure 4: Behaviour over time of levels and phase space for Lanchester’s model with unaimed fire with and

## Mixed model of guerilla warfare



Figure 5: Causal-loop diagram of mixed Lanchester’s model.

size of RED forces= INTEG (-Ukrainians killing RED, initial size of RED forces), Units: Strength

size of BLUE forces= INTEG (-RED killing BLUE, initial size of BLUE forces), Units: Strength

RED killing BLUE=attrition coefficient for RED\*size of RED forces\*size of BLUE forces, Units: Strength/Year

BLUE killing RED=attrition coefficient for BLUE\*size of BLUE forces, Units: Strength/Year

initial size of RED forces=2, Units: Strength

initial size of BLUE forces=1, Units: Strength

attrition coefficient of BLUE for RED=0.1, Units: 1/(Strength\*Year)

attrition coefficient of RED for BLUE=0.5, Units: 1/Year

INITIAL TIME=0, FINAL TIME=4, TIME STEP=0.125, Units: Year

Figure 6: Behaviour over time of levels and phase space for mixed Lanchester’s model.

## Variable target visibility

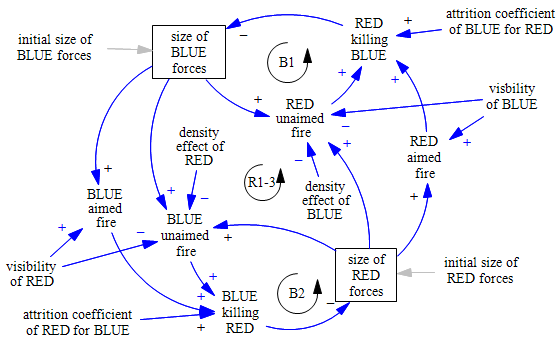


Figure 7: Causal-loop diagram of model with target visibility.

size of BLUE forces=INTEG(-RED killing BLUE, initial size of BLUE forces), Units: Strength

size of RED forces=INTEG(-BLUE killing RED, initial size of RED forces), Units: Strength

BLUE killing RED=attrition coefficient of RED for BLUE\*(BLUE aimed fire+BLUE unaimed fire), Units: Strength/Year

RED killing BLUE=attrition coefficient of BLUE for RED\*(RED aimed fire+RED unaimed fire), Units: Strength/Year

BLUE aimed fire=visibility of RED\*size of BLUE forces, Units: Strength

BLUE unaimed fire=(1-visibility of RED)\*size of BLUE forces\*size of RED forces/density effect of RED, Units: Strength

RED aimed fire=visbility of BLUE\*size of RED forces, Units: Strength

RED unaimed fire=(1-visbility of BLUE)\*size of RED forces\*size of BLUE forces/density effect of BLUE, Units: Strength

initial size of BLUE forces=1, Units: Strength

initial size of RED forces=2, Units: Strength

attrition coefficient of BLUE for RED=0.25, Units: 1/Year

attrition coefficient of RED for BLUE=0.5, Units: 1/Year

density effect of BLUE=1, Units: Strength

density effect of RED=1, Units: Strength

visbility of BLUE=0.5, Units: Dmnl

visibility of RED=0.5, Units: Dmnl

INITIAL TIME=0, TIME STEP=0.25, FINAL TIME=4, Units: Year

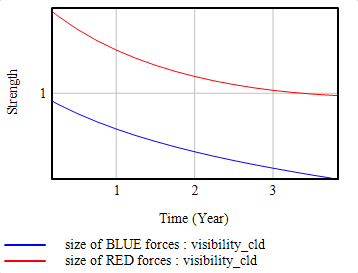
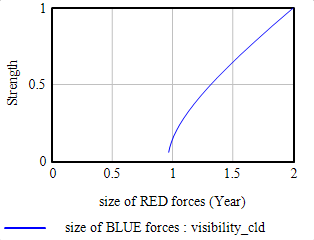
 

Figure 8: Behaviour over time of levels and phase space for model with target visibility.

# Lanchester’s Square Law

An agent-based model to replicate basic Lanchester’s Square Law.

## What Is It

This section gives a general understanding of what the model is trying to show or explain.

Lanchester (1916) presented a now classic deterministic model of outcomes of ranged combat based on the fighting effectiveness and the troop strengths (i.e., number of soldiers) of two opposing forces. The system of differential equations that came to be known as the Lanchester Square Law for targeted fire is given by:

where and are the fighting effectiveness coefficients of the red and blue forces, respectively, and and are the troop strengths of the blue and red forces, respectively.

The Lanchester Linear Law describes untargeted area fire:

## How It Works

This section explains what rules the agents use to create the overall behavior of the model.

In the basic model, the agents are placed randomly, targeting occurs without regard for range, and all agents are homogeneous.

To setup, all variables and agents are cleared, and then the square space is filled with blue agent and red agents. To go, in random order, each agent kills exactly one enemy, if an enemy remains to kill.

### Scenario 1: Accuracy as basic heterogeneity

Suppose a force's weaponry will kill an enemy with probability of given accuracy, at a range of 1 unit, with that probability decreasing exponentially with range. Each force unit will choose the closest target.

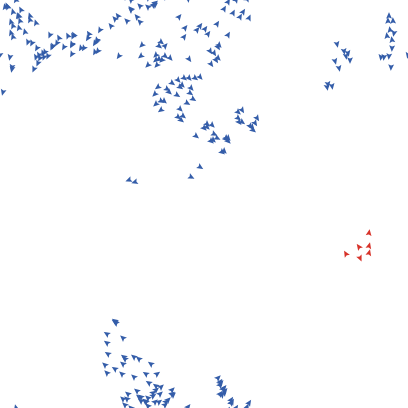
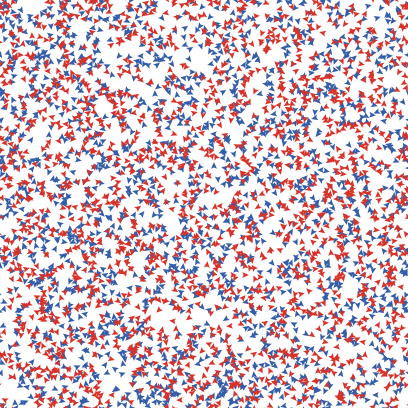


Figure 9: Scenario one with targeted fire at start (left) and after 5 ticks (right).

**This modification alters the outcome of the battle**: the blue forces can overcome the numerically larger red forces with their technological superiority.

Figure 10: Remaining forces over time.

### Scenario 2: Further deviation from the Lanchester model

The red force is attacked from two sides by a blue force with only 90% their troop strength. Blue force weapons have mean accuracy of 0.62 at 1 unit distance, whereas red force weapons have mean accuracy of 0.60 at the same range, with individuals’ accuracy varying according to a normal distribution with a standard deviation of 0.1. Blue forces are short on ammunition, so they have orders to only fire from a maximum range of 5 units. The red forces, meanwhile, can fire at any range. If a soldier fires, they can move only 25% the distance that they could move otherwise.

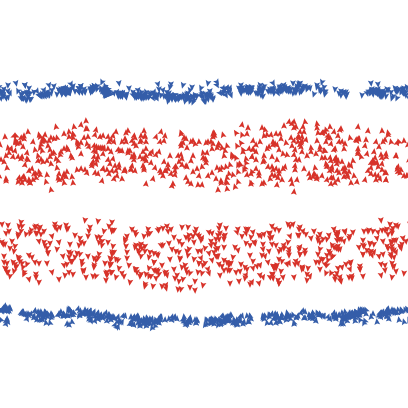
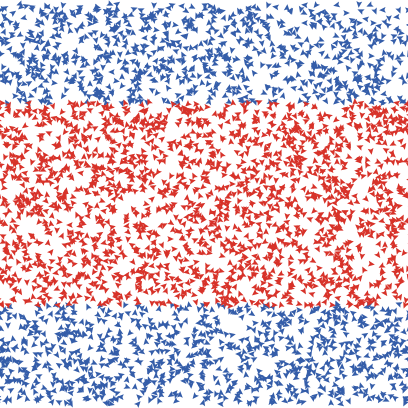


Figure 11: Scenario two with targeted fire at start (left) and after 10 ticks (right).

**This more complex model yields a slight advantage to the blue forces**: over 100 replicates, 68 result in a blue victory. This result is statistically significant ().

Figure 12: Remaining forces over time.

## How To Use It

This section explains how to use the model, including a description of each of the items in the interface tab.

Running many iterations of this simulation, the mean number of agents remaining once one side has been killed is equal to that predicted by Lanchester’s Square Law.

The **setup** button starts a new setup based on the interface and global variables.

The **go** button starts the simulation.

The **step** button executes a single tick (step) only.

The **startup** button prepares a default scenario.

Press one of the buttons **scenario-one** or **scenario-two** to set up the corresponding scenario.

## Credits and References

Christopher W. Weimer, J. O. Miller, Raymond R. Hill. Agent-Based Modeling: An Introduction and Primer. Proceedings of the 2016 Winter Simulation Conference, T. M. K. Roeder, P. I. Frazier, R. Szechtman, E. Zhou, T. Huschka, and S. E. Chick, eds.

<http://simulation.su/uploads/files/default/2016-weimer-miller-hill.pdf>

**breed**[force-units force-unit]  
  
**force-units-own**[  
 force-range  
 accuracy  
]  
  
**globals** [  
 separated-forces?  
 accuracy-deviation  
 free-move  
 battle-move  
]  
  
**to** setup-globals  
 set separated-forces? **false**  
 set accuracy-deviation 0  
 set free-move 0  
 set battle-move 0  
**end**  
  
**to** startup  
 set initial-blue-forces 2000  
 set initial-red-forces 2050  
 set blue-accuracy 1  
 set red-accuracy 1  
 set blue-range max-range + 1  
 set red-range max-range + 1  
 set blue-area-fire? **false**  
 set red-area-fire? **false**  
 setup  
**end**  
  
**to** setup  
 clear-all  
 setup-globals  
 setup-forces  
 reset-ticks  
**end**  
  
**to** go  
 if battle-finished? [ stop ]  
 go-forces  
 tick  
**end**  
  
**to-report** max-range  
 report ceiling sqrt ( max-pxcor \* max-pxcor + max-pycor \* max-pycor )  
**end**  
  
**to** setup-forces  
 spwan-blue-forces  
 spawn-red-forces  
**end**  
  
**to** setup-accuracy [accuracy-mean accuracy-standard-deviation]  
 ifelse accuracy-standard-deviation > 0 [  
 set accuracy random-normal accuracy-mean accuracy-standard-deviation  
 ] [  
 set accuracy accuracy-mean  
 ]  
 set accuracy min list 1 accuracy  
 set accuracy max list 0 accuracy  
**end**  
  
**to** spwan-blue-forces  
 create-force-units initial-blue-forces [  
 set color blue  
 ifelse separated-forces? [  
 set xcor random-xcor  
 set ycor (max-pycor / 2) + random-float (max-pycor / 2)  
 if random 2 = 1 [ set ycor (- ycor) ]  
 ] [  
 setxy random-xcor random-ycor  
 ]  
 setup-accuracy blue-accuracy accuracy-deviation  
 set force-range blue-range  
 ]  
**end**  
  
**to** spawn-red-forces  
 create-force-units initial-red-forces [  
 set color red  
 ifelse separated-forces? [  
 set xcor random-xcor  
 set ycor (random-ycor / 2)  
 ] [  
 setxy random-xcor random-ycor  
 ]  
 setup-accuracy red-accuracy accuracy-deviation  
 set force-range red-range  
 ]  
**end**  
  
**to** go-forces  
 ask force-units [ force-unit-attack ]  
**end**  
  
**to-report** hostile-force-units  
 report force-units with [color != [color] of myself]  
**end**  
  
**to-report** force-unit-select-target  
 let target nobody  
 let area-fire? ((color = blue and blue-area-fire?)  
 or (color = red and red-area-fire?))  
 ifelse area-fire? [  
 let target-patch one-of patches  
 if force-range < max-range [  
 set target-patch one-of patches in-radius force-range  
 ]  
 ask target-patch [  
 if any? force-units-here [ set target one-of force-units-here ]  
 ]  
 ] [  
 let targets hostile-force-units  
 if force-range < max-range [ set targets targets in-radius force-range ]  
 set target min-one-of targets [distance myself]  
 ]  
 report target  
**end**  
  
**to** force-unit-attack  
 let target force-unit-select-target  
 ifelse target != nobody [  
 face target  
 if random-float 1 < (accuracy ^ (distance target)) [ ask target [ die ] ]  
 force-unit-move battle-move  
 ] [  
 force-unit-move free-move  
 ]  
**end**  
  
**to** force-unit-move [ steps ]  
 if steps > 0 [  
 if any? hostile-force-units [  
 let target min-one-of hostile-force-units [distance myself]  
 face target  
 ]  
 forward steps  
 ]  
**end**  
  
**to-report** battle-finished?  
 report not any? turtles with [color = red] or not any? turtles with [color = blue]  
**end**  
  
**to** scenario-one  
 set initial-blue-forces 2000  
 set initial-red-forces 2050  
 set blue-accuracy 0.95  
 set red-accuracy 0.9  
 set blue-range max-range + 1  
 set red-range max-range + 1  
 setup  
**end**  
  
**to** scenario-two  
 set initial-blue-forces 1800  
 set blue-accuracy 0.62  
 set blue-range 5  
 set initial-red-forces 2000  
 set red-accuracy 0.6  
 set red-range max-range + 1  
 clear-all  
 set separated-forces? **true**  
 set accuracy-deviation 0.1  
 set free-move 1  
 set battle-move 0.25  
 setup-forces  
 reset-ticks  
**end**