SUBHARMONIC FUNCTIONS AND HARMONIC MEASURES

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Contents

1. Subharmonic Functions

Definition 1.1 (Semi-continuous functions)**.** A real function $f(x)$ defined in a set $E \subset \mathbb{R}^m$ is said to be upper semi-continuous (u.s.c) on E if

(i) $-\infty \leq f(x) < \infty, \forall x \in E$.
(ii) The sets $\{x : x \in E, f(x) < a\}$ are open in E for $-\infty < a < \infty$.

A function $f(x)$ is said to be lower semi-continuous (l.s.c.) in E if $-f(x)$ is u.s.c in E. If $f(x)$ is both u.s.c. and l.s.c. then $f(x)$ is a continuous function.

Theorem 1.1. *If $f_n(x)$ is a decreasing sequence of u.s.c. functions defined on a set E, then $f(x) = \lim_{n\to\infty} f(x)$ is u.s.c. on E.*

Theorem 1.2. *If $f(x)$ is u.s.c. on a set E then there exists a decreasing sequence $f_n(x)$ of functions continuous on E such that*

$$f_n(x) \to f(x) \quad as \quad n \to \infty.$$

Definition 1.2. If $u \in C^2(D)$ and $\nabla^2 u = 0$ in D, then u is said to be harmonic in D.

Definition 1.3 (Subharmonic functions)**.** A function $u(x)$ defined in a domain $D \subset \mathbb{R}^m$ is said to be subharmonic (s.h.) in D if

(i) $-\infty \leq u(x) < \infty, \forall x \in D$.

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(ii) $u(x)$ is u.s.c. in D.

(iii) Given any point $x_0 \in D$, there exist arbitrarily small positive values of r such that[1]

$$u(x_0) \le \frac{1}{c_m r^{m-1}} \int_{S(x_0,r)} u(x)\,\mathrm{d}\sigma(x)$$

where $\mathrm{d}\sigma(x)$ denotes surface area on $S(x_0, r)$.

2. The Maximum Principle

Lemma 2.1. *If $u(x)$ is s.h. and $u(x) \le 0$ in $D(x_0, r)$ and $u(x_0) = 0$, then $u(x) \equiv 0$ in $S(x_0, \rho)$ for some arbitrarily small ρ.*

Proof. It follows from property (iii) of Definition 1.3 that we can find ρ as small as we please such that

$$u(x_0) = 0 \le \frac{1}{c_m \rho^{m-1}} \int_{S(x_0,\rho)} u(x)\,\mathrm{d}\sigma(x).$$

Since $u(x) \le 0$, we deduce that

$$\int_{S(x_0,\rho)} u(x)\,\mathrm{d}\sigma(x) = 0.$$

Suppose that there exists x_1 in $S(x_0, \rho)$ such that $u(x_1) < 0$. Then by (ii) of Definition 1.3 we can find a neighborhood N_1 of x_1 such that $u(x) < -\eta$ in N_1, where $\eta > 0$. If N_2 is the intersection of N_1 and $S(x_0, \rho)$ and E_2 is the complement of N_2 on $S(x_0, \rho)$ then

$$\int_{S(x_0,\rho)} u(x)\,\mathrm{d}\sigma(x) = \int_{N_2} + \int_{E_2} \le \int_{N_2} u(x)\,\mathrm{d}\sigma(x) \le -\eta \int_{N_2} \mathrm{d}\sigma(x) < 0,$$

giving a contradiction. Thus $u(x) \equiv 0$ in $S(x_0, \rho)$. □

Theorem 2.2. *Suppose that $u(x)$ is s.h. in a domain D of $\mathbb{R}^m$ and that, if ξ is any boundary point of D and $\varepsilon > 0$, we can find a neighbourhood N of ξ such that*

$$u(x) < \varepsilon \quad \textit{in} \quad N \cap D. \tag{2.1}$$

Then $u(x) < 0$ in D or $u(x) \equiv 0$. If D is unbounded we consider $\xi = \infty$ to be a boundary point of D and assume that (2.1) holds when N is the exterior of some hyperball $|x| > R$.

Proof. Let

$$M = \sup_{x \in D} u(x).$$

[1] Here and subsequently $c_m = 2\pi^{m/2}/\Gamma(m/2)$.

If $M < 0$ there is nothing to prove. Suppose $M > 0$, let x_n be a sequence of points in D such that

$$u(x_n) \to M.$$

By taking a subsequence if necessary, we may assume that $x_n \to \xi$. Since $M > 0$ this contradicts our basis hypothesis with $\varepsilon = M/2$ if $\xi \in \partial D$. Thus ξ is a point of D. Also since $u(x)$ is u.s.c. we obtain a contradiction of $u(\xi) < M$. Hence we must have $u(\xi) = M$.

Thus if E is the set of all points of D for which $u(x) = M$, we see that E is not empty. If $M = 0$ and $u < 0$ in D there is again nothing to prove. So we may assume in all cases that $M \geq 0$ and the set E where $u(x) = M$ is not empty. Since u is u.s.c., E is closed. We proceed to prove that E contains the whole of D.

Suppose that x_1, x_2 are points of D such that $u(x_1) < M = u(x_2)$. Then we can join x_1, x_2 by a polygonal path $x_1 = \xi_1, \xi_2, \ldots, \xi_n = x_2$ in D, so that each straight line segment $\xi_j\xi_{j+1} \in D$ for $j = 1$ to $n - 1$. Let j be the last integer so that $u(\xi_j) < M$. Then $u(\xi_{j+1}) = M$. Let

$$x(t) = (1 - t)\xi_j + t\xi_{j+1}$$

and let t_0 be the lower bound of all t in $0 < t < 1$ such that $x(t) \in E$. Since E is closed $x_0 = x(t_0) \in E$. We now apply Lemma 2.1 to $u(x) - M$ and deduce that there exists ρ, such that $0 < \rho < |x_0 - \xi_j|$ and $S(x_0, \rho) \subset E$. Also $S(x_0, \rho)$ meets the segment $[\xi_j, x_0]$, whch contradicts the definition of t_0. Thus E contains the whole of D and $u(x) \equiv M$ in D.

If D is bounded, ∂D contains at least one finite point ξ and if $M > 0$, we obtain a contradiction. If D is unbounded, D contains the point $\xi = \infty$, and we again obtain a contradiction. Thus $M \leq 0$, and $u \equiv M$ in D or $u < M$ in D. □

We can deduce immediately the following:

Theorem 2.3. *Suppose that $u(x)$ is s.h. and $v(x)$ is harmonic in a bounded domain D and that*

$$\overline{\lim_{x \to \xi}}\,(u(x) - v(x)) \leq 0$$

as x approaches any point of ∂D from inside D. Then $u(x) < v(x)$ in D or $u(x) \equiv v(x)$ in D.

3. Boundary Behavior and Regular Domain

Definition 3.1. Let D be a domain in $\mathbb{R}^m$ and $f(\xi)$ be a bounded function defined on ∂D. We define the class $U(f)$ of functions u with the following properties

(i) u is s.h. in D.
(ii) $\overline{\lim}\, u(x) \leq f(\xi)$ as x approaches any point ξ of S from inside D.

We define

$$v(x) := \sup_{u \in U(f)} u(x). \tag{3.1}$$

Lemma 3.1. *The function $v(x)$ is harmonic in D and if $m \le f(\xi) \le M$ on ∂D, then we have $m \le v(x) \le M$ in D.*

Proof. Suppose that $m \le f(\xi) \le M$. Then $u = m$ is in $U(f)$ and so

$$v(x) \ge u(x) = m.$$

Again supoose that $u \in U(f)$. Then it follows from the maximaum principle $\square$

4. Harmonic Extensions

Definition 4.1. Suppose that D is a bounded regular domain in $\mathbb{R}^m$. Let $f(\zeta)$ be a continuous function defined on the boundary ∂D of D. Then if $u(x)$ is continuous in $\overline{D}$, harmonic in D and $u(\zeta) = f(\zeta)$ in ∂D, we say that $u(x)$ is the harmonic extension of f from ∂D into D.

Theorem 4.1. *Suppose that $f(\zeta)$ is u.s.c. in ∂D, $-\infty \le f < \infty$ and that $f_n(\zeta)$ is a sequence of continuous functions, monotonically decreasing to $f(\zeta)$ as $n \to \infty$ for each ζ in ∂D. Let $u_n(x)$ be the harmonic extension of f_n from ∂D into D. Then $u_n(x)$ decreases to a limit $u(x)$ as $n \to \infty$ which is independent of the choice of the sequence f_n and is either harmonic or identically $-\infty$ in D.*

Definition 4.2. The function $u(x)$ in the above theorem will be called the harmonic extension of f from ∂D into D. If f is l.s.c., the harmonic extension of f from ∂D into D is defined to be $u(x)$, where $-u(x)$ is the harmonic extension of $-f(x)$ from ∂D into D.

5. Harmonic Measure

Theorem 5.1. *Suppose that D is a bounded regular domain in $\mathbb{R}^m$. Then for every x in D and Borel set $e \subset \partial D$ there exists a number $\omega(x,e)$ with the following properties*

(i) *For fixed $x \in D$, $\omega(x,e)$ is a Borel measure on ∂D and $\omega(x,\partial D) = 1$.*
(ii) *For fixed $e \subset \partial D$, $\omega(x,e)$ is a harmonic function of x in D.*
(iii) *If $f(\xi)$ is a semi-continuous function defined on ∂D, then*

$$u(x) = \int_{\partial D} f(\xi)\, \mathrm{d}\omega(x, e_\xi) \tag{5.1}$$

is the harmonic extension of $f(\xi)$ to D.

The measure $\omega(x,e) = \omega(x,e,D)$ will be called the harmonic measure of e at x with respect to D.

Proof. We define $L_x(f)$ as the harmonic extension of f to the point x in D. Then for fixed x, $L_x(f)$ is a positive linear functional on the class of continuous functions f on ∂D. Hence by Riesz Representation Theorem there exists a measure $\omega(x, e)$ uniquely determined by ∂D, x and D such that (iii) holds for continuous f. By Theorem 4.1 and the property

$$L(f_n) \to L(f) \quad \text{as} \quad n \to \infty$$

of linear functionals it follows that $u(x) = L_x(f)$ is still the harmonic extension of f to D when f is semi-continuous. Also for any bounded Borel measurable function f on ∂D we define the harmonic extension of f from ∂D onto D to be given by (5.1).

Let $f = \chi_{\partial D}$ then use (5.1) we obtain $\omega(x, \partial D) = 1$, which proves (i).

It remains to show that (ii) holds. We prove more generally that when f is bounded and Borel measurable, $u(x)$ defined by (5.1) is harmonic in x. Then for any Borel set $e \subset \partial D$ we may then take $f = \chi_e$ and deduce (ii).

If f is upper semi-continuous it follows from Theorem 4.1 that $u(x)$ is harmonic or identically $-\infty$ and similarly if f is lower semi-continuous $u(x)$ is harmonic or identically $+\infty$. The infinite case is excluded if f is bounded since in this case $u(x)$ also lies between the same bounds for each x. Suppose finally that f is bounded and Borel measurable. Let x_0 be a fixed point of D. Then f is integrable with respect to $\omega(x_0, e)$ and so we can find u.s.c. functions $f_n(\xi)$ on ∂D such that

$$L_{x_0}(f_n) > L_{x_0}(f) - \frac{1}{n} \tag{5.2}$$

and $f_n \leq f$ on E. We may suppose further that the sequence f_n is monotone increasing since otherwise we may replace f_n by

$$g_n = \max_{\nu=1 \text{ to } n} f_\nu.$$

Let u_n be the harmonic extension of f_n from ∂D to D. Since f_n is u.s.c., it follows from Theorem 4.1 that $u_n(x)$ is harmonic in x and, since $u_n(x) = L_x(f_n)$ is a positive linear functional, $u_n(x)$ increases with n for each fixed x in D. Thus by Harnack's Theorem $u_n(x)$ converges to a harmonic function $u(x)$ in D and in view of (5.2) we have

$$L_{x_0}(f) = u(x_0).$$

Also since $f_n \leq f$ on ∂D it follows that for $x \in D$

$$u_n(x) = L_x(f_n) \leq L_x(f)$$

and so

$$u(x) \leq L_x(f), \quad x \in D$$

Similarly, we choose a decreasing sequence of lower semi-continuous functions $g_n(\xi)$, such that $g_n(\xi) \geq f(\xi)$ on ∂D and that

$$L_{x_0}(g_n) \geq f(\xi)$$

on ∂D and that

$$L_{x_0}(g_n) \to L_{x_0}(f) \quad \text{as} \quad n \to \infty.$$

Then if $v_n(x)$ is the harmonic extension of g_n into D it follows that $v_n(x)$ decreases to a harmonic limit $v(x)$ in D, such that $v(x_0) = L_{x_0}(f)$ and

$$v(x) \geq L_x(f), \quad x \in D.$$

Thus

$$v(x) - u(x) \geq 0$$

in D with equality at $x = x_0$ and hence we deduce from the maximum principle that $v(x) = u(x)$ in D and so that

$$u(x) = L_x(f) = v(x)$$

so that $L_x(f)$ is harmonic in D as a function of x. □