JENSEN'S INEQUALITY FOR HIGHER DIMENSIONS

YUNLEI WANG

Let $D^n := \left\{\sum_{j=1}^n |z_j|^2 \leq 1 : z_1, z_2, \cdots, z_n \in \mathbb{C}\right\}$ be a unit ball in $\mathbb{C}^n$. We also write $z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$. By Poisson-Jensen's Formula, we know that given any subharmonic function $u(z)$ defined on D^n, we have for $z \in D^n$

$$u(z) \leq \int_{\partial D^n} u(\xi)\, \mathrm{d}\omega(z, e_\xi)$$

where $\omega(z, e)$ is the harmonic measure of $e \subset \partial D^n$ at z.

The harmonic measure $\omega(z, e)$ on D^n is

$$\omega(z, e) = \int_e \frac{1 - |z|^2}{|z - \xi|^2} \frac{\mathrm{d}m(e_\xi)}{\sigma_{2n-1}}$$

where $m(e)$ denotes the surface measure on ∂D^n and $\sigma_{2n-1} = m(\partial D^{n-1})$.

Let $z = 0$, we obtain

$$u(0) \leq \int_{\partial D^n} u(\xi) \frac{\mathrm{d}m(e_\xi)}{\sigma_{2n-1}}.$$

Consider holomorphic function $f(z_1, z_2, \cdots, z_n) \in D^n$, then $\log|f| = \mathrm{Re}\log f$ is harmonic for $f(z) \neq 0$, hence $\log|f|$ is subharmonic. Let $u = \log|f|$, we obtain

$$\log|f(0)| \leq \int_{\partial D^n} u(\xi) \frac{\mathrm{d}m\,(e_\xi)}{\sigma_{2n-1}}.$$

Now we change $m(e)$ to the normalized measure for simplicity, then the above inequality is equivalent to

$$\log\left|\int_{\partial D^n} f(\xi)\, \mathrm{d}m(e_\xi)\right| \leq \int_{\partial D^n} \log|f(\xi)|\, \mathrm{d}m(e_\xi).$$

Denote $z \in \partial D^n = S^{2n-1}$ by spherical coordinate

$$z = (\varphi, \Theta)$$

E-mail address: `wcghdwyl@gmail.com`.

with $\varphi \in [0, \pi)$ and $\Theta \in S^{2n-2}$. Then the stereographic map from $\mathbb{R}^{2n-1}$ (we use polar coordinates (r, θ)) to S^{2n-1} is

$$\begin{cases} \varphi = 2 \arctan \frac{1}{r}, \\ \Theta = \theta. \end{cases}$$

Rewrite the integral (we omit the exact constant in the integral)

$$\begin{aligned} \int_{\partial D^n} = \int_{S^{2n-1}} &= \int_0^\pi \mathrm{d}\varphi \, (\sin\varphi)^{2n-2} \int_{S^{2n-2}} \mathrm{d}\Theta \\ &= \int_{\mathbb{R}^{2n-1}} \frac{1}{(1+r^2)^{2n-1}} r^{2n-2} \, \mathrm{d}r\mathrm{d}\theta = \int_{\mathbb{R}^{2n-1}} \frac{1}{(1+r^2)^{2n-1}} \, \mathrm{d}x. \end{aligned}$$

Here $r = \sum_{j=1}^{n-1} |z_j|^2 + |x_n|^2$. Indeed, by this transformation, our inequality becomes

$$\log \left| \int_{\mathbb{R}^{2n-1}} f(y) P_1^{2n-1}(y) dy \right| \le \int_{\mathbb{R}^{2n-1}} \log |f(y)| P_1^{2n-1}(y) \, \mathrm{d}y.$$

Thus we finish the proof of Jensen's inequality on $\mathbb{R}^{2n-1}$ by using holomorphic property of function f.

For Jensen's inequality on $\mathbb{R}^{2n}$, we can not do this becasue $\sum_{j=1}^{2n+1} |x_j|^2 \le 1, x \in \mathbb{R}^{2n+1}$ can not be viewed as a complex domain.