Schemes and Sheaves

Based on notes by Andreas Gathmann

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October 5, 2021

The learning notes are based on Andreas Gathmann's *Algebraic Geometry*.

Contents

1 Scheme

Definition 1.1. Let R be a ring. The set of all prime ideals of R is called the *spectrum* of R or the *affine scheme* associated to R. We denote it by $\operatorname{Spec} R$.

Definition 1.2. Let R be a ring, and let $P \in \operatorname{Spec} R$ be a point in the corresponding affine scheme, i.e. aprime ideal $P \trianglelefteq R$.

a. We denote by $K(P)$ the quotient field of the integral R/P. It is called the *residue field* of $\operatorname{Spec} R$ at P.

b. For any $f \in R$ we define the *value* of f at P, written as $f(P)$, to be the image of f under the composite ring homomorphism $R \to R/P \to K(P)$. In particular, we have $f(P) = 0$ if and only if $f \in P$.

Definition 1.3. Let R be a ring.

a. For a subset $S \subset R$, we define the *zero locus* of S to be the set

$$V(S) := \{P \in \operatorname{Spec} R : f(P) = 0 \text{ for all } f \in S\} \subset \operatorname{Spec} R.$$

As usual, if $S = \{f_1, \cdots, f_k\}$ is a finite set, we will write $V(S)$ also as $V(f_1, \cdots, f_k)$.

b. For a subset $X \subset \operatorname{Spec} R$, we define the *ideal* of X to be

$$I(X) := \{f \in R : f(P) = 0 \text{ for all } P \in X\} \trianglelefteq R.$$

Definition 1.4. We define the *Zariski topology* on an affine scheme $\operatorname{Spec} R$ to be the topology whose closed sets are exactly the sets of the form $V(S) = \{P \in \operatorname{Spec} R : P \supset S\}$ for some $S \subset R$.

Remark. Compare to the case of affine varieties, points are not necessarily closed in affime schemes. In fact, for a point P in an affine scheme $\operatorname{Spec} R$ we have

$$\overline{\{P\}} = V(P) = \{Q \in \operatorname{Spec} R : Q \supset P\},$$

so that $\{P\}$ is closed if and only if P is a maximal ideal.
For an affine scheme $\operatorname{Spec} A(X)$ associated to an affine variety X, this means that the closed points of $\operatorname{Spec} A(X)$ correspond exactly to the minimal subvarieties of X, i.e. to the points of the variety X in the usual sense. The other non-closed points of $\operatorname{Spec} A(X)$ are of the form $I(Y) \in \operatorname{Spec} A(X)$is usually called the *generic* or *general point* of Y. One motivation for this name is that evaluation at Y takes values in the function field $K(Y)$ of Y, which encodes *rational* functions on Y, i.e. regular functions that functions that are not necessarily defined on all of Y, but only at a "general point" of Y.

Proposition 1.5. *Let R be a ring.*

a. *For any closed subset $X \subset \operatorname{Spec} R$ we have $V(I(X)) = X$.*

b. *For any ideal $J \trianglelefteq R$ we have $I(V(J)) = \sqrt{J}$.*

c. *For any two ideals J_1, J_2 in a ring R we have*
$$V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$$
and
$$V(J_1) \cap V(J_2) = V(J_1 + J_2)$$
in $\operatorname{Spec} R$.

d. *For any two closed subsets X_1, X_2 of* $\operatorname{Spec} R$ *we have*
$$I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$$
and
$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}.$$

Definition 1.6. For a ring R and an element $f \in R$, we call
$$D(f) := \operatorname{Spec} R \backslash V(f) = \{P \in \operatorname{Spec} R : f \notin P\}$$
the *distinguished open subset* of f in $\operatorname{Spec} R$.

Remark. The distinguished open subsets form a basis of the topology of an affine scheme $\operatorname{Spec} R$:
$$U = \operatorname{Spec} R \backslash \bigcap_{f \in S} V(f) = \bigcup_{f \in S} (\operatorname{Spec} R \backslash V(f)) = \bigcup_{f \in S} D(f).$$

Definition 1.7. Let R be a ring, and let U be an open subset of the affine scheme $\operatorname{Spec} R$. A *regular function* on U is a family $\varphi = (\varphi_P)_{P \in U}$ with $\varphi_P \in R_P$ for all $P \in U$, such that the following propertiy holds: For every $P \in U$ there are $f, g \in R$ with $f \notin Q$ and
$$\varphi_Q = \frac{g}{f} \in R_Q$$
for all Q in an open subset U_P with $P \in U_P \subset U$.
The set of all such regular functions on U is clearly a ring; we will denote it by $\mathcal{O}_{\operatorname{Spec} R}(U)$. $\mathcal{O}_{\operatorname{Spec} R}$ is a sheaf and called the *structure sheaf* of $\operatorname{Spec} R$.

Remark. For a prime ideal P in a ring R, the quotient R_P/P_P of the local ring R_P by its maximal ideal P_P is just the residue field $K(P)$. Hence any regular function $\varphi \in \mathcal{O}_{\operatorname{Spec} R}(U)$ has a well-defined valued $\varphi(P) \in K(P)$ for all $P \in U$. However, in contrast to the case of affine varieties, a regular function on an affine scheme is not determined by its values.

Lemma 1.8. *Let R be a ring. Then for any point $P \in \operatorname{Spec} R$ the stalk $\mathcal{O}_{\operatorname{Spec} R,P}$ of the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ at P is isomorphic to the localization R_P.*

Proposition 1.9. *Let R be a ring and $f \in R$. Then $\mathcal{O}_{\operatorname{Spec} R}(D(f))$ is isomorphic to the localization R_f.*

Definition 1.10. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that each stalk $\mathcal{O}_{X,P}$ for $P \in X$ is a local ring.

Definition 1.11. A *morphism* of locally ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is given by the following data:

a. a continuous map $f : X \to Y$;

b. for every open subset $U \subset Y$ a ring homomorphism $f_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X\left(f^{-1}(U)\right)$ called *pullback* on U ;

such that the following two conditions hold

a. The pull-back maps are compatible with restrictions, i.e. we have $f_U^*(\varphi|_U) = (f_V^*\varphi)\,|_{f^{-1}(U)}$ for all $U \subset V \subset Y$ and $\varphi \in \mathcal{O}_Y(V)$. In particular, this implies that there are induced ring homomorphisms $f_P^* : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ on the stalks for all $P \in X$.

b. For all $P \in X$, we have $(f_P^*)^{-1}(I_P) = I_{f(P)}$, where I_P and $I_{f(P)}$ denote the maximal ideals in the local rings $\mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,f(P)}$,respectively.

Proposition 1.12. *For any two rings R and S there is a bijection*

$$\{\textit{morphisms } \operatorname{Spec} R \to \operatorname{Spec} S\} \overset{1:1}{\longleftrightarrow} \{\textit{ring homorphisms } S \to R\}$$
$$f \mapsto f^*.$$

In particular, this means that there is a natural bijection

$$\{\textit{affine schemes}\}\,/\textit{isomorphisms} \overset{1:1}{\longleftrightarrow} \{\textit{rings}\}\,/\textit{isomorphisms.}$$

Proposition 1.13. *Let R be a ring, and let $f \in R$. Then the distinguished open subset $D(f) \subset \operatorname{Spec} R$is isomorphic to the affine scheme $\operatorname{Spec} R_f$.*

Definition 1.14. A *scheme* is a locally ringed space that has an open cover by affine schemes. Morphisms of schemes are just morphisms as locally ringed spaces.