MICROLOCAL ANALYSIS BY P. HINTZ

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Contents

1. Lecture 1

Part I pseudodifferential operators (ps.d.o.s) on $\mathbb{R}^n$

$$\Delta_{\mathbb{R}^n} := \sum_{j=1}^{n} D_{x_j}^2, \quad D_{x_j} := \frac{1}{i}\frac{\partial V}{\partial x_j}.$$

$$L := \Delta_{\mathbb{R}^n} + 1,$$

$L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is inverse. What is inverse:

- $L \circ L^{-1} = I \in Diff^0(\mathbb{R}^n)$.
- $L^{-1} \in \Psi^{-2}(\mathbb{R}^n)$.

Can write

$$\left(L^{-1}u\right)(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi}\left(1+|\xi|^2\right)^{-1} u(y)\,\mathrm{d}y\mathrm{d}\xi \sim \mathcal{F}^{-1}\left(\left(1+|\xi|^2\right)^{-1}(\mathcal{F}u)(\xi)\right).$$

Define $\Psi^m(\mathbb{R}^n)$ by such integrals with $(1+|\xi|^{-1})$ replaced by more general "symbols" $a(x,\xi) \sim |\xi|^m$. E.g.,

$$\Psi^m(\mathbb{R}^n) \circ \Psi^{m'}(\mathbb{R}^n) \subset \Psi^{m+m'}(\mathbb{R}^n).$$

Highlight: Elliptic operators $L : C^\infty(M) \to C^\infty(M)$ with M compact without boundary are Fredholm: $\dim ker L < \infty, \dim coker L < oo, Ran(L)$ is closed.

Part II Microlocalization. Main notion: $\mathrm{WF}(u)-$ wave front set of distribution, $u \in \mathscr{D}'(\mathbb{R}^n)$. $\mathrm{WF}(u) \subset \mathbb{R}^n_x \times \left(\mathbb{R}^n_\xi \setminus \{0\}\right)$ or $\mathbb{R}^n_x \times \mathbb{S}^{n-1}_\xi$. Measure where($x$) and in what "direction"(ξ) u is singular. Ex. $u = \delta_0$, $\mathrm{F}(\delta) = \{(x,\xi) : x = 0, \xi \neq 0\}$. Ex. $u = 1_\Omega$, Ω =smoothly bounded domain, $\mathrm{WF}(u) = \{(x,\xi) : x \in \partial\Omega, \xi \in N^*_x\partial\Omega\}$. Application to PDE. $L \in \mathrm{Diff}^2(\mathbb{R}^n)$, say $L = -D_t^2 + D_x^2$ on

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$\mathbb{R}^2_{t,x}$, if $u \in \mathscr{D}'(\mathbb{R}^2)$ solves $Lu = 0$, then $\mathrm{WF}(u) \subset \mathrm{char}(L) := \{(t, x, \sigma, \xi) : \sigma(L)(t, x, \sigma, \xi) = -\sigma^2 + \xi^2 = 0\}$
Theorem: $\mathrm{WF}(u) =$ union of integral curves of the Hamiltonian vector field of the principal symbol of L. Proof uses ps.d.o.s as tools.

Part III Applications to long time wave asymptotics de Sitter space (General Relativity).

Definition 1.1. The space $\mathcal{S}(\mathbb{R}^n)$ consists of all $\varphi \in C^\infty(\mathbb{R}^n)$ s.t. for all $k \in \mathbb{N}_0$,

$$\|\varphi\|_k := \sup_{|\alpha|+|\beta|\le k} \left|x^\alpha D^\beta\right| < \infty.$$

Here

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n,$$
$$|\alpha| = \sum \alpha_j,$$
$$x^\alpha = \prod x_j^{\alpha_j},$$
$$D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} \in \mathrm{Diff}^{|\alpha|}(\mathbb{R}^n).$$

Fact: $(\mathcal{S}(\mathbb{R}^n), \{\|\cdot\|_0, \|\cdot\|_1, \cdots\})$ is a Fréchet space.