1 Canonical Quantization of the Klein-Gordon Field

1.1 Quantization

In quantum mechanics

$$[q_a, p_b] = i\delta_{ab}.$$
$$[q_a, q_b] = 0.$$
$$[p_a, p_b] = 0.$$

Similarly, quantize the Klein-Gordon field as following

$$[\varphi(\mathbf{x}, \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$$
$$[\varphi(\mathbf{x}, \varphi(\mathbf{y}] = 0$$
$$[\pi(\mathbf{x}), \pi(\mathbf{y})] = 0.$$

In classical field theory, the coefficients $a(\mathbf{k})$ and $a^*(\mathbf{k})$ are numbers, after quantization, they are changed into operators

$$a(\mathbf{k}) \to a_{\mathbf{k}}$$
$$a^*(\mathbf{k}) \to a^\dagger_{\mathbf{k}}.$$

$$\varphi(\mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_{\mathbf{k}}} \left[a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \right].$$

$$\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{p}}\right] = (2\pi)^3 2E_{\mathbf{k}} \delta^{(3)}(\mathbf{k} - \mathbf{p})$$
$$[a_{\mathbf{k}}, a_{\mathbf{p}}] = 0$$
$$\left[a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{p}}\right] = 0.$$

The Hamiltonian is

$$H = \frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_{\mathbf{k}}} E_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}} a^\dagger_{\mathbf{k}} \right)$$

$$H = \frac{1}{4} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}} a^\dagger_{\mathbf{k}} \right).$$

1.2 States

Vacuum state $|0\rangle$

$$a_{\mathbf{k}}|0\rangle = 0$$
$$\langle 0|0\rangle = 1.$$

$$\begin{aligned} H|0\rangle =& E_0|0\rangle \\ =& \frac{1}{4} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} a_{\mathbf{k}} a^\dagger_{\mathbf{k}} |0\rangle \\ =& \frac{1}{2} \int \mathrm{d}^3 k E_{\mathbf{k}} \delta^{(3)}(\mathbf{k} - \mathbf{k})|0\rangle \\ =& \infty |0\rangle. \end{aligned}$$

The vacuum energy is infinite.

1.3 IR-regulate

IR-regulate by putting theory in a box of size L.

$$\begin{aligned}(2\pi)^3\delta^{(3)}(\mathbf{0}) &= \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\Big|_{\mathbf{p}=0} \\ &= \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}^3x \\ &= \lim_{L\to\infty} V.\end{aligned}$$

$$\rho_0 = \frac{E_0}{V} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2}E_{\mathbf{k}}.$$

Total energy diverges if V diverges unless $\rho_0 = 0$. This is a UV divergence.

Normal Hamiltonian is

$$:H:= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}} E_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}.$$

1.4 One Particle States

Let

$$|\mathbf{k}\rangle = a_{\mathbf{k}}^\dagger|0\rangle.$$

$|\mathbf{k}\rangle$ has definite momentum and energy, sometimes also be denoted by $|k\rangle$.

$$\begin{aligned}\langle\mathbf{p}|\mathbf{k}\rangle =&\langle 0|a_{\mathbf{p}}a_{\mathbf{k}}^\dagger|0\rangle \\ =&(2\pi)^3 2E_{\mathbf{k}}\delta^{(3)}(\mathbf{p}-\mathbf{k}).\end{aligned}$$

This is Lorentz invariant.

$\varphi(\mathbf{x})|0\rangle$ is an one-particle state localized at $\mathbf{x}$.

$$N = \int \frac{\mathrm{d}^3k}{(2\pi^3)2E_{\mathbf{k}}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$$

$$\begin{aligned}Na_{\mathbf{p}}^\dagger|0\rangle &= a_p^\dagger|0\rangle \\ N\varphi(\mathbf{x})|0\rangle &= \varphi(\mathbf{x})|0\rangle \\ \langle\mathbf{k}|\varphi(\mathbf{x})|0\rangle &= e^{-i\mathbf{k}\cdot\mathbf{x}}.\end{aligned}$$

This formula is similar to $\langle\mathbf{k}|\mathbf{x}\rangle = e^{-i\mathbf{k}\cdot\mathbf{x}}$ in quantum mechanics.

1.5 Multiparticle States

$$|\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n\rangle = a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2}^\dagger \cdots a_{\mathbf{k}_n}|0\rangle$$

The operators in the right hand are commutative → bosons.

$$[:H:, N] = 0.$$

The state space is Fock space $\mathcal{F} = \oplus_{n=0}^\infty \mathcal{H}_n$.

1.6 Heisenberg Picture

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt}.$$
$$a_{\mathbf{p}}(t) = e^{iHt}a_{\mathbf{p}}e^{-iHt}.$$

Using $e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \cdots$ we get

$$[H, a_{\mathbf{p}}] = -E_{\mathbf{p}}a_{\mathbf{p}}$$

and

$$a_{\mathbf{p}}(t) = e^{-iE_{\mathbf{p}}t}a_{\mathbf{p}}$$
$$a^{\dagger}_{\mathbf{p}}(t) = e^{iE_{\mathbf{p}}t}a^{\dagger}_{\mathbf{p}}.$$

$$\varphi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}}\left[a_{\mathbf{k}}e^{-ik\cdot x} + a^{\dagger}_{\mathbf{k}}e^{ik\cdot x}\right].$$

$$\begin{aligned}[:H:, \varphi] =& \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\left[E_{\mathbf{k}}a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}, a_{\mathbf{p}}e^{-ip\cdot x} + a^{\dagger}_{\mathbf{p}}e^{ip*x}\right] \\ =& \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}}\left(-E_{\mathbf{k}}a_{\mathbf{k}}e^{-ik\cdot x} + E_{\mathbf{k}}a^{\dagger}_{k}e^{ik\cdot x}\right) \\ =& - i\partial_t\varphi(t, \mathbf{x}).\end{aligned}$$

The interaction field can be written as

$$\Phi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}\left[b_{\mathbf{p}}(t)e^{-ip\cdot x} + b^{\dagger}_{\mathbf{p}}(t)e^{ip\cdot x}\right].$$

At any fixed time $b^{\dagger}_{\mathbf{p}}(t)$ and $b_{\mathbf{p}}(t)$ satisfy the same algebra as free theory.

Propagator

$$\begin{aligned}D(x-y) =& \langle 0|\varphi(x)\varphi(y)|0\rangle \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}\frac{\mathrm{d}^3k}{2E_{\mathbf{k}}}e^{-ip\cdot x + ip\cdot y}\langle 0|a_{\mathbf{p}}a^{\dagger}_{\mathbf{k}}|0\rangle \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}.\end{aligned}$$

Space like: $x^0 = y^0, \mathbf{x} - \mathbf{y} = \mathbf{r} \neq 0$

$$\begin{aligned}D(x-y) =& \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}e^{\mathbf{ip}\cdot r\mathbf{p}} \\ \sim& e^{-mr} \neq 0.\end{aligned}$$

If $\Delta(x, y) = [\varphi(x), \varphi(y)] = 0$, then the measurement at x cannot affect y.

$$[\varphi(x), \varphi(y)] = D(x-y) - D(y-x) = \langle 0|\left[\varphi(x), \varphi(y)\right]|0\rangle.$$

These two amplitude eliminate with each other when $(x-y)^2 < 0$.

For time like separation, assume $x^0 > y^0$

$$\begin{aligned}\Delta(x,y) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \\ &= \int \frac{\mathrm{d}^3 p}{2E_{\mathbf{p}}} \left(\frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \Big|_{p^0=E_{\mathbf{p}}} + \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot x - y} \Big|_{p^0=-E_{\mathbf{p}}} \right) \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int_{C_R} \frac{\mathrm{d}p^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip\cdot(x-y)}.\end{aligned}$$

$$\Delta_R(x-y) = \Theta(x^0 - y^0) \langle 0 | \left[\varphi(x), \varphi(y) \right] | 0 \rangle = \int_{C'_R} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}.$$