# LOGVINENKO-SEREDA THEOREM ON MANIFOLDS UNDER SOME CIRCUMSTANCES

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Abstract. By giving some conditions in heat kernel $p(t,x,y)$, we can get Logvinenko-Sereda Theorem on manifolds.

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## 1. Poisson kernel

Given an arbitrary smooth connected Riemannian manifold $M$, the Laplace-Beltrami operator is defined by

$$\Delta_g := \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right)$$

where $g := \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$.

From now on, we further assume $M$ is compact, then the spectrum of the Laplacian is discrete and there is a sequence of eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

and an orthonormal basis $\{\phi_i\}_{i\in\mathbb{N}}$ of smooth real eigenfunctions of the Laplacian, i.e.,

$$\Delta_g \phi_i = -\lambda_i \phi_i.$$

Let

$$p(t,x,y) = \sum_{i=1}^{\infty} e^{-\sqrt{\lambda_i} t} \phi_i(x) \phi_i(y),$$

this is called the Poisson kernel.

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We considering the following spaces of $L^2(M)$

$$E_L := \left\{ f \in L^2(M) : f = \sum_{i=1}^{n} \beta_i \phi_i, \lambda_i \leq L, \forall i = 1, 2, \cdots, n \right\}$$

and define $k_L := \dim E_L$.

## 2. Jensen's Inequality

We define the Poisson transform

$$\mathrm{d}\Pi_{t,x} := p(t,x,y)\,\mathrm{d}y \tag{2.1}$$

and

$$P(f)(x) := \int_M p(t,x,y) f(y)\,\mathrm{d}y = \int_M f\,\mathrm{d}\Pi_{t,x}, \quad \forall f \in L^2(M).$$

Without loss of generality, we can choose a volume form such that

$$\int_M \mathrm{d}x = 1. \tag{2.2}$$

**Definition 2.1.** Given a fixed $t > 0$, we call a set $S \subset M$ $\gamma$-thick (at time $t$) if for some constant $\gamma > 0$ it satisfy

$$\int_S \mathrm{d}\Pi_{t,x} = \int_S p(t,x,y)\,\mathrm{d}y \geq \gamma, \quad \forall x \in M. \tag{2.3}$$

**Definition 2.2.** Given a fixed $t > 0$, if there exists constants $a := a(t,L)$ and $b := b(t,L)$ such that for any $L \geq 1$, the inequality

$$\log |P(f)(x)| \leq aP(\log|f|) + b, \quad \forall f \in E_L, \tag{2.4}$$

holds, we call $P$ satisfy the Jensen's condition w.r.t. $a$ and $b$ at time $t$, and (2.4) the Jensen's inequality w.r.t. $a$ and $b$ at time $t$.

**Theorem 2.1.** *Given a fixed $t > 0$, a $\gamma$-thick set $S$, if the Poisson transform $P$ defined in (2.1) satisfies (2.3) and (2.4), then we have*

$$\|f\|_{L^2}^2 \leq \left(e^{a\log 2 + 2b}\right)^{\frac{1}{1-a+a\gamma}} \left(\int_S |f(x)|^2\,\mathrm{d}x\right)^{\frac{a\gamma}{1-a+a\gamma}}, \quad \forall f \in E_L, \forall L \in \mathbb{N}. \tag{2.5}$$

*Proof.* It is known that

$$p(t,x,y) > 0, \quad ,\forall (t,x,y) \in (0,\infty) \times M \times M.$$

For simplicity in the following proof, we define

$$k := \int_S \mathrm{d}\Pi_{t,x}, \quad k' := \int_{S'} \mathrm{d}\Pi_{t,x}$$

and

$$\mathrm{d}\lambda := \frac{1}{k}\,\mathrm{d}\Pi_{t,x}, \quad \mathrm{d}\lambda' := \frac{1}{k'}\,\mathrm{d}\Pi_{t,x}.$$

Then we have $k+k' = \int_M \mathrm{d}\Pi_{t,x} = \int_M p(t,x,y)\,\mathrm{d}y = 1$. For a possible generalization, we assume

$$c_{t,x} := \int_M p(t,x,y)\,\mathrm{d}y. \tag{2.6}$$

$$\begin{aligned}
&2\log|P(f)(x)| \le 2aP(\log|f|) + 2b\\
=&2a\left(\int_S \log|f(y)|\,\mathrm{d}\Pi_{t,x} + \int_{S'} \log|f(y)|\,\mathrm{d}\Pi_{t,x}\right) + 2b\\
=&a\left(k\int_S \log|f(y)|^2\,\mathrm{d}\lambda + k'\int_{S'} \log|f(y)|^2\,\mathrm{d}\lambda'\right) + 2b\\
\le&a\left(k\log\int_S |f(y)|^2\,\mathrm{d}\lambda + k'\log_{S'}|f(y)|^2\,\mathrm{d}\lambda'\right) + 2b\\
=&a\left(k\log\frac{1}{k} + k'\log\frac{1}{k'} + k\log\int_S |f(y)|^2\,\mathrm{d}\Pi_{t,x} + k'\log\int_{S'} |f(y)|^2\,\mathrm{d}\Pi_{t,x}\right) + 2b\\
\le&a\Bigg(c_{t,x}\log\frac{2}{c_{t,x}} + \gamma\log\int_S |f(y)|^2\,\mathrm{d}\Pi_{t,x} + (k-\gamma)\log\int_S |f(y)|^2\,\mathrm{d}\Pi_{t,x}\\
&\quad + k'\log\int_{S'} |f(y)|^2\,\mathrm{d}\Pi_{t,x}\Bigg) + 2b\\
\le&a\left(c_{t,x}\log\frac{2}{c_{t,x}} + \gamma\log\int_S |f(y)|^2\,\mathrm{d}\Pi_{t,x} + (c_{t,x}-\gamma)\log\int_M |f(y)|^2\,\mathrm{d}\Pi_{t,x}\right) + 2b.
\end{aligned}$$

This implies

$$|P(f)(x)|^2 \le e^{ac_{t,x}\log\frac{2}{c_{t,x}}+2b}\left(\int_S |f(y)|^2\,\mathrm{d}\Pi_{t,x}\right)^{a\gamma}\left(\int_M |f(y)|^2\,\mathrm{d}\Pi_{t,x}\right)^{a(c_{t,x}-\gamma)} \tag{2.7}$$

Then we integrate both sides and use Hölder's inequality with index $\dfrac{1}{c_{t,x}/\gamma}+\dfrac{1}{c_{t,x}/\left(c_{t,x}-\gamma\right)}=1$

$$\begin{aligned} &\int_M |P(f)(x)|^2\,\mathrm{d}x \\ \le & e^{ac_{t,x}\log\frac{2}{c_{t,x}}+2b}\left(\int_M\left(\int_S |f(y)|^2\,\mathrm{d}\Pi_{t,x}\right)^{ac_{t,x}}\,\mathrm{d}x\right)^{\frac{\gamma}{c_{t,x}}} \\ &\times\left(\int_M\left(\int_M |f(y)|^2\,\mathrm{d}\Pi_{t,x}\right)^{ac_{t,x}}\,\mathrm{d}x\right)^{\frac{c_{t,x}-\gamma}{c_{t,x}}}. \end{aligned} \tag{2.8}$$

Remember that $c_{t,x}=1$ for all $(t,x)\in(0,+\infty)\times M$, we have

$$\int_M |P(f)(x)|^2\,\mathrm{d}x \le e^{a\log 2+2b}\left(\int_M\left(\int_S |f(y)|^2\,\mathrm{d}\Pi_{t,x}\right)^{a}\,\mathrm{d}x\right)^{\gamma}\left(\int_M\left(\int_M |f(y)|^2\,\mathrm{d}\Pi_{t,x}\right)^{a}\,\mathrm{d}x\right)^{1-\gamma}. \tag{2.9}$$

If $a\le 1$, by assumption $\int_M \mathrm{d}x=1$ we have

$$\|f\|_{L^a}\le\|f\|_{L^1},\quad \forall f\in L^1(M). \tag{2.10}$$

Since $\displaystyle\int_S |f(y)|^2\,\mathrm{d}\Pi_{t,x}=\sum_{i=1}^{k_L}\phi_i(x)\int_S |f(y)|^2\phi_i(y)\,\mathrm{d}y\in E_L$ for arbitrary $S\subset M$, we obtain

$$\|P(f)\|_{L^2}^2\le e^{a\log 2+2b}\left(\int_S |f(x)|^2\,\mathrm{d}x\right)^{a\gamma}\|f\|_{L^2}^{2a(1-\gamma)}. \tag{2.11}$$

Since

$$f=\sum_{i=1}^{k_L}e^{\sqrt{\lambda_i}t}P(f),\quad \forall f\in E_L,$$

we obtain

$$\|f\|_{L^2}^2\le\left(e^{a\log 2+2b}\right)^{\frac{1}{1-a+a\gamma}}\left(\int_S |f(x)|^2\,\mathrm{d}x\right)^{\frac{a\gamma}{1-a+a\gamma}}.$$

$\square$

## References