Null-controllability of the Generalized Baouendi-Grushin heat like equations

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Abstract

In this article, we study Zhu-Zhuge's spectral inequality for Schrödinger operators with power growth potentials, and give a precised form of it, i.e., the spectral inequality with explicit form of the constant. Based on this precised form of the spectral inequality, we study a generalized form of Baouendi-Grushin operators and prove the exact null-controllability results associated with it.

Contents

1 Introduction

1.1 Spectral inequalities for Schrödinger operators

In control theory, a spectral inequality for a nonnegative selfadjoint operator H in $L^2(\mathbb{R}^n)$ take the form

$$\|\phi\|^2_{L^2(\mathbb{R}^n)} \leq d_0 e^{d_1\lambda^\zeta}\|\phi\|^2_{L^2(\omega)}, \quad \forall \phi \in \mathcal{E}_H(\lambda), \lambda \geq 0, \tag{1.1}$$ 1a

where ω is a measurable subset of $\mathbb{R}^n$, $\mathcal{E}_H(\lambda) = \mathbb{1}_{(-\infty,\lambda)}(H)$ is the resolution of identity associated to H, and d_0, d_1, ζ are constants. Such an inequality is a quantitative version of a unique continuation property (i.e., $f = 0$ on ω implies $f = 0$ on $\mathbb{R}^n$). Due to the famous Lebeau-Robbiano method introduced in [LR95] (see also [TT11, BPS18, NTTV20, GST20]), it can be applied to observability or null-controllability results from ω of the abstract Cauchy problem

$$\partial_t u + Hu = 0. \tag{1.2}$$ 1.2a

In particular, for the Schröidnger operator

$$H = -\Delta_x + V(x), \quad x \in \mathbb{R}^n,$$

a wealth of results have been established under several assumptions on V and ω. For the special case $V = 0$, the spectral inequality from thick sets (see [WWZZ19, Kov01] for the definition of thick sets) was proven in [WWZZ19] and independently in [EV18]. For $V = |x|^2$, H is the harmonic oscillator and spectral inequalities from different kinds of ω were established, see [BPS18, BJPS21, ES21, MPS22, DSV23]. For $V = |x|^{2k}$, the spectral inequalities were established in [AS23, Alp20, MPS22].

The results we mentioned above consider different kinds of restrictions on ω. Here we introduce one restriction, wich will be useful in presenting our main results:

Definition 1.1. Let $l > 0$, $\gamma \in (0, 1)$ and $\sigma > 0$, the set ω is said to be (l, γ, σ)-distributed if there exists a set of points $\{z_k | k \in \mathbb{Z}^n\}$ such that

$$\omega \cap (lk + \Lambda_l) \supset \mathcal{B}_{\gamma^{1+|lk|^\sigma}}(z_k), \tag{1.3}$$

where $\Lambda_l := [-l/2, l/2]^n$ denotes a cube of length l centered on the origin and $\mathcal{B}_r(z)$ denote a ball with radius r centered on z.

If $\sigma = 0$, the set ω satisfying (1.3) is called (l, γ)-equidistributed set or simply equidistributed set, which was introduced in [RMV13].

Recently, Dicke, Seelmann and Veselić [DSV22] consider the Schrödinger operator with power growth potentials and ω who is $(1, \gamma, \sigma)$-distributed. Precisely speaking, they establish the spectral inequality (1.1) for $\zeta = \frac{\sigma}{\beta_1} + \frac{2\beta_2}{3\beta_1}$ with suitable power growth potentials $V \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$. Shortly after, Zhu and Zhuge [ZZ23] optimize the exponent of λ in (1.1) to $\zeta = \frac{\sigma}{\beta_1} + \frac{\beta_2}{\beta_1}$ under more general assumption, which can be stated as the following:

(i) there exist positive constants c_1 and β_1 such that for all $x \in \mathbb{R}^n$,

$$c_1(|x| - 1)_+^{\beta_1} \leq V(x). \tag{1.4}$$

where $(a)_+ := \max\{a, 0\}$;

(ii) we can write $V = V_1 + V_2$ such that there exists positive constants c_2 and β_2 such that

$$|V_1(x)| + |DV_1(x)| + |V_2(x)|^{\frac{4}{3}} \leq c_2(|x| + 1)^{\beta_2}. \tag{1.5}$$

However, both results do not give the explicit form of the constant d_0 and d_1 with respect to c_1 and c_2, which is necessary for our main results in this article.

1.2 Generalized Baouendi-Grushin operator

We consider the evolution equation

$$\partial_t u + Hu = 0 \tag{1.6}$$

either for a generalized form of Baouendi-Grushin operator

$$H_G = -\Delta_x - V(x)\Delta_y, \quad x \in \mathbb{R}^n, y \in \mathbb{R}^m \tag{1.7}$$

or a generalized form of Baouendi-Grushin operator

$$H_{G_p} = -\Delta_x - V(x)\Delta_y, \quad x \in \mathbb{R}^n, y \in \mathbb{T}^m \tag{1.8}$$

where $\alpha > 0$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and the potential V satisfies some general assumptions stated later. These two kinds of operators generalize the Grushin operator

$$\Delta_G := \Delta_x + |x|^2\Delta_y$$

and the Baouendi-Grushin operator

$$\Delta_k := \Delta_x + |x|^{2k}\Delta_y.$$

In this article, we consider general potentials V in (1.7) and (1.8) satisfy the following assumption.

A1 **Assumption A1.** *There exist positive constants $c_1, c_2, \beta_1, \beta_2$ such that $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ satisfies*

$$c_1|x|^{\beta_1} \leq V(x) \ \text{ and } \ |V(x)| + |DV(x)| \leq c_2|x|^{\beta_2}. \tag{1.9}$$

We also consider the following more general assumption.

A2 **Assumption A2.** *Assume $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ satisfies the following two conditions:*

(i) *There exist positive constants c_1 and β_1 such that for all $x \in \mathbb{R}^n$,*

$$c_1|x|^{\beta_1} \leq V(x). \tag{1.10}$$ 1.1

(ii) *We can write $V = V_1 + V_2$ such that there exists positive constants c_2 and β_2 such that*

$$|V_1(x)| + |DV_1(x)| + |V_2(x)|^{\frac{4}{3}} \leq c_2(|x|+1)^{\beta_2}. \tag{1.11}$$

Remark 1.1. It should be noted that in [ZZ23] the original assumption is different from this one only in (1.10), in which the lower bound is $c_1\left(|x|-1\right)_+^{\beta_1}$ instead of $c_1|x|^{\beta_1}$. Here we use the later for simple calculating of the lower bound of the first eigenvalue (see Subsection 2.1 in detail). Indeed, it gives a relation of the lowest eigenvalue $\lambda_0 \geq c^{\frac{2}{\beta_1+2}}\lambda_*$, which is vital for us (see the proofs in Section 4 for more details).

r1.2 **Remark 1.2.** The standard case $V = |x|^\beta$ satisfies Assumption A1 only for $\beta \geq 1$. However, the case $V = |x|^\beta$ with $\beta > 0$ is included in Assumption A2. Indeed, for $\beta \geq 1$ it is obvious. For $0 < \beta < 1$, we can choose a smooth cut-off function such that $\eta = 1$ in $\mathcal{B}_1$ and $\eta = 0$ in $\mathbb{R}^n \backslash \mathcal{B}_2$. Then we just choose $V_1 = |x|^\beta(1-\eta(x))$ and $V_2(x) = |x|^\beta \eta(x)$ (see [ZZ23, Corollary 1]).

1.3 State of the main results

In this article, we give a precised form of the spectral inequality in [ZZ23, Theorem 1]. Based on it, we obtain our main results, Theorem 1.1 and Theorem 1.2 under Assumption A1, Theorem 1.4 and Theorem 1.5 under Assumption A2, which give the exactly null-controllability of two kinds of the evolution equations.

Firstly, consider the fractional heat-like evolution equation associated with the generalized Baouendi-Grushin operator for $y \in \mathbb{R}^m$

$$\begin{cases} \partial_t u(t,x,y) + H_G^s u(t,x,y) = h(t,x,y)\mathbb{1}_\omega(x,y), & t>0,\ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m, \\ u(0,\cdot,\cdot) = u_0 \in L^2(\mathbb{R}^{n+m}). \end{cases} \tag{$E_{G,s}$}$$ eg

Now we present our results about the exactly null-controllability of equation ($E_{G,s}$).

thm1.3 **Theorem 1.1.** *Assume that V satisfies Assumption A1 with $\beta_1 = \beta_2 = \beta > 0$. Let $\gamma \in (0, 1/2)$, $\sigma = 0$. Then*

(i) *for all $s > (\beta+2)/4$, the equation ($E_{G,s}$) is exactly null-controllable in every positive time $T > 0$ and from every control support of the form $\omega \times \mathbb{R}^m$ with a $(1,\gamma)$-equidistributed set $\omega \subset \mathbb{R}^n$;*

(ii) *for $s = (\beta+2)/4$ and all $(1,\gamma)$-equidistributed sets $\omega \subset \mathbb{R}^n$, there exists a positive constant K depending only on n, β, c_1 and c_2, such that the equation ($E_{G,s}$) is exactly null-controllable from the control support $\omega \times \mathbb{R}^m$ in every positive time $T \geq K\log(1/\gamma)$.*

Secondly, we consider the fractional heat-like evolution equation associated with the generalized Bauendi-Grushin operator for $y \in \mathbb{T}^m$

$$\begin{cases} \partial_t u(t,x,y) + H_{G_p}^s u(t,x,y) = h(t,x,y)\mathbb{1}_\omega(x,y), & t>0,\ (x,y) \in \mathbb{R}^n \times \mathbb{T}^m, \\ u(0,\cdot,\cdot) = u_0 \in L^2(\mathbb{R}^n \times \mathbb{T}^m). \end{cases} \tag{$E_{G_p,s}$}$$ egb

We are ready to present our second exactly null-controllability result.

thm1.4 **Theorem 1.2.** *Assume that V satisfies Assumption A1. Let $\gamma \in (0, 1/2)$, $\sigma = 0$ and ω be a $(1, \gamma)$-equidistributed set. Then*

(i) *for all $s > (\beta_1 + 2)/4$, the equation ($E_{G_p,s}$) is exactly null-controllable from the control support $\omega \times \mathbb{T}^m$ in every positive time $T > 0$;*

(ii) *for $s = (\beta_1 + 2)/4$, there exists a positive constant K depending only on n, β_1, β_2, c_1 and c_2, such that the equation ($E_{G_p,s}$) is exactly null-controllable from the control support $\omega \times \mathbb{T}^m$ in every positive time $T \geq K \log(1/\gamma)$.*

Both Theorem 1.1 and Theorem 1.2 are under the condition $\sigma = 0$. This means that the set ω is equidistributed. The reason for $\sigma = 0$ is the appearance (or an approximate form) of the semigroup $e^{t\Delta}$ when we do the partial Fourier transform with respect to the variable y. To release the condition $\sigma = 0$ to $\sigma \geq 0$, a simple way is to change H_G to

$$H'_G := -\Delta_x + V(x)\left(-\Delta_y + 1\right), \quad y \in \mathbb{R}^m$$

in ($E_{G,s}$) and change H_{G_p} to

$$H'_{G_p} := -\Delta_x + V(x)\left(-\Delta_y + 1\right), \quad y \in \mathbb{T}^m$$

in ($E_{G_p,s}$).

Then we obtain the following result directly from the proof of Theorem 1.1 and Theorem 1.2.

crc1.5 **Corollary 1.3.** *Assume that V satisfies Assumption A1, $\gamma \in (0, 1/2)$, $\sigma > 0$ and ω is a $(1, \gamma, \sigma)$-distributed set. Then*

(i) *for all $s > \frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}$ and $s > (\beta_1 + 2)/4$, the modified equations ($E_{G,s}$) (replace H_G with H'_G) and ($E_{G_p,s}$) (replace H_{G_p} with H'_{G_p}) are exactly null-controllabile from the control support $\omega \times \mathbb{T}^m$ in every positive time $T > 0$;*

(ii) *for all $s > \frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}$ and $s = (\beta_1 + 2)/4$, there exists a positive constant K depending only on n, β_1, β_2, c_1 and c_2 such that the modified equations ($E_{G,s}$) (replace H_G with H'_G) and ($E_{G_p,s}$) (replace H_{G_p} with H'_{G_p}) are exactly null-controllable from the control support $\omega \times \mathbb{T}^m$ in every positive time $T \geq K \log(1/\gamma)$.*

As we have mentioned in Remark 1.2, the standard case $V = |x|^\beta$ with $\beta \in (0, 1)$ does not satisfy Assumption A1. With the same strategy, we obtain the following two results under Assumption A2 with some loss of accuracy in the critical value of s.

thm1.4g **Theorem 1.4.** *Assume that V satisfies Assumption A2 with $\beta_1 = \beta_2 = \beta > 0$. Let $\gamma \in (0, 1/2)$, $\sigma = 0$. Then*

(i) *for all $s > (\beta + 2)/3$, the equation ($E_{G,s}$) is exactly null-controllable in every positive time $T > 0$ and from every control support of the form $\omega \times \mathbb{R}^m$ with a $(1, \gamma)$-equidistributed set $\omega \subset \mathbb{R}^n$;*

(ii) *for all $s = (\beta + 2)/3$ and all $(1, \gamma)$-equidistributed sets $\omega \subset \mathbb{R}^n$, there exists a positive constant K depending only on n, β, c_1 and c_2, such that the equation ($E_{G,s}$) is exactly null-controllable from the control support $\omega \times \mathbb{R}^m$ in every positive time $T \geq K \log(1/\gamma)$.*

thm1.5g **Theorem 1.5.** *Assume that V satisfies Assumption A2. Let $\gamma \in (0, 1/2)$, $\sigma = 0$ and ω be a $(1, \gamma)$-equidistributed set. Then*

(i) *for all $s > \frac{\beta_2}{2\beta_1}$ and $s > (\beta_1 + 2)/3$, the equation ($E_{G_p,s}$) is exactly null-controllable from the control support $\omega \times \mathbb{T}^m$ in every positive time $T > 0$;*

(ii) *for all $s > \frac{\beta_2}{2\beta_1}$ and $s = (\beta_1 + 2)/3$, there exists a positive constant K depending only on n, β_1, β_2, c_1 and c_2, such that the equation ($E_{G_p,s}$) is exactly null-controllable from the control support $\omega \times \mathbb{T}^m$ in every positive time $T \geq K \log(1/\gamma)$.*

Now we go back to the standard case ($V = |x|^\beta, \beta > 0$)

$$-\Delta_{G,\beta} := -\Delta_x - |x|^\beta \Delta_y$$

where $y \in \mathbb{R}^m$ or $\mathbb{T}^m$. As we have mentioned in Remark 1.2, only Assumption A2 includes the case for $0 < \beta < 1$.

Consider the evolution equation

$$\begin{cases} \partial_t u(t,x,y) + (-\Delta_{G,\beta})^s u(t,x,y) = h(t,x,y)\mathbb{1}_\omega(x,y), \quad t>0,\ x \in \mathbb{R}^n, y \in \mathbb{R}^m \text{ or } \mathbb{T}^m, \\ u(0,\cdot,\cdot) = u_0 \in L^2(\mathbb{R}^{n+m}). \end{cases} \tag{$E_{\beta,s}$}$$

Then by Theorem 1.1, 1.2, 1.4 and 1.5, we may obtain the null-controllability results associated to equation ($E_{\beta,s}$), see Table 1 for details.

<table>
<tr><td></td><td>$0 < \beta < 1$</td><td>$\beta \geq 1$</td></tr>
<tr><td rowspan="2">$s > (\beta+2)/4$</td><td>$s > (\beta+2)/3$, exactly null-controllable for any $T > 0$</td><td rowspan="3">exactly null-controllable for any $T > 0$</td></tr>
<tr><td>$s = (\beta+2)/3$, exactly null-controllable for $T \geq T^*$</td></tr>
<tr><td></td><td rowspan="3">not known under Assumption A2
not known even for the standard case</td></tr>
<tr><td>$s = (\beta+2)/4$</td><td>exactly null-controllable for $T \geq T^*$</td></tr>
<tr><td>$s < (\beta+2)/4$</td><td>not known under Assumption A1
see Remark 1.4 for the standard case</td></tr>
</table>

Table 1: The exactly null-controllability results of equation ($E_{\beta,s}$) obtained. In general, the results in middle column can be achieved under Assumption A2 with $\beta = \beta_1$, the results in the right column can be achieved under Assumption A1 with $\beta = \beta_1$. For the case $y \in \mathbb{R}^m$, we need to assume $\beta = \beta_1 = \beta_2$.

Remark 1.3. Compaired with exactly null-controllability results in [AS23], we allow more general potentials, i.e., do not need to restrict $V = |x|^\beta$ with $\beta = 2k, k \in \mathbb{N}$ anymore, and assume a bit more restrictive support condtion.

Remark 1.4. For the standard case $-\Delta_{G,k} := -\Delta_x - |x|^{2k}\Delta_y$ defined in $\mathbb{R}^n \times \mathbb{T}^m$ and $k \in \mathbb{N}$, Alphonse and Seelmann [AS23, Theorem 2.17] shows that the evolution equation is never exactly null-controllable from any control support $\omega \subset \mathbb{R}^n \times \mathbb{T}^m$ satisfying $\overline{\omega} \cap \{x = 0\} = \emptyset$. This implies $s = (\beta+2)/4$ is the critical value for $\beta = 2k \geq 2$. Based on our results, it is reasonable to conjecture that it is the critical value for all $\beta \geq 1$ under Assumption A1 and $\beta = \beta_1 = \beta_2$.

For the case $0 < \beta < 1$, it seems that the critical value may be a little worse (we only obtain exactly null-controllability results up to $s = (\beta+2)/3$, which is strictly larger than $(\beta+2)/4$). One of the possible reason may be the wild behavior of $V = |x|^\beta$ around 0.

Remark 1.5. Though we follow the strategy of Alphonse and Seelmann's work in [AS23] to prove the main theorems, there are two key differences here:

(i) We use a different kind of spectral inequality, which we call Zhu-Zhuge's inequality given in [ZZ23, Theorem 1]. However, the original form cannot be used directly, since the loss of the explicit dependence relation between the cost constant and the parameters. This forces us to establish a more detailed decaying property of eigenfunctions, see Proposition 2.3 in Subsection 2.2. Finally, we give the explicit form of the cost constant in the spectral inequality, which is vital for our proofs of exactly null-controllability results.

(ii) The lowest eigenvalue of the Schrödinger operator is easily obtained in the case $V = |x|^{2k}, k \in \mathbb{N}$ by the rescaling approach. It does not work for our general potentials under Assumption A1and A2. To overcome this difficulty, we do not calculate the exact number of the lowest eigenvalue, instead we just calculate a lower bound which satisfies our needs. This is a more general way if we would like to replace the standard case $V = |x|^{2k}$ with a general potential.

1.4 Outline of the work

In Section 2, we prove the lowest bound of eigenvalues and our assumptions and decaying properties of eigenfunctions for the Schrödinger operators. In Section 3, we follow the method in [ZZ23] to prove the spectral inequality with the precised form of the constant. In this section, we also give Corollary 3.6 and 3.7, which can be directly used later. In Section 4, we prove the exact null-controllability results, i.e., Theorem 1.1, 1.2, 1.3, 1.4 and 1.5.

2 Eigenfunctions of the Schrödinger operator

In this section, we first try to find a lower bound of the lowest eigenvalue for a given potential $V(x) = c|x|^\beta$, named $\widetilde{\lambda}_0$. Then we give a detailed decaying property of eigenfunctions for the Schrödinger operator with potential V under Assumption A1.

We denote $\{\phi_k\}_{k\in\mathbb{N}}$ the set of eigenfunctions of the Schrödinger operator, namely,

$$-\Delta_x\phi_k + V(x)\phi_k = \lambda_k\phi_k, \quad x\in\mathbb{R}^n, \tag{2.1}$$

where λ_k is the eigenvalue of ϕ_k.

2.1 Lower bound of eigenvalues

Proposition 2.1. *Let $V \geq c|x|^\beta$ and $\lambda_0(V)$ be the lowest eigenvalue of the operator $H = -\Delta_x + V(x)$. Then we have*

$$\lambda_0(V) \geq c^{\frac{2}{\beta+2}}\lambda_* \tag{2.2}$$

where λ_* depends only β and n.

We denote $\lambda_0(V)$ the lowest eigenvalue of the operator $H = -\Delta_x + V(x)$. For all $a > 0$, we define

$$I_V(a) = \int_{\mathbb{R}^n} e^{-aV(x)}\,\mathrm{d}x.$$

Theorem 2.2 ([BBL76])**.** *Under the condition for every $a > 0$ such that $I_V(a) < +\infty$. Then we have*

$$\lambda_0(V) \geq \sup_{t>0} t\left[n + \frac{n}{2}\ln\frac{\pi}{t} - \ln I_V\left(\frac{1}{t}\right)\right].$$

Now we finish the proof of Proposition 2.1.

Proof of Proposition 2.1. In the case of $V(x) = c|x|^\beta$, a change into polar coordinates and then a chenge of variable $s = acr^{\beta_1}$ shows that

$$I_{c|x|^\beta}(a) = \int_{\mathbb{R}^n} e^{-ac|x|^\beta}\,\mathrm{d}x = \frac{\sigma_n}{\beta(ac)^{n/\beta}}\Gamma\left(\frac{n}{\beta}\right)$$

where $\sigma_n = \dfrac{2\pi^{n/2}}{\Gamma\left(n/2\right)}$ is the surface measure of the unit ball in $\mathbb{R}^n$, and $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t$ is the gamma function. It follows that

$$\lambda_0\left(c|x|^\beta\right) \geq \sup_{t>0} t\left[n - \ln\frac{2}{\beta}\frac{\Gamma\left(\frac{n}{\beta}\right)}{\Gamma\left(\frac{n}{2}\right)} - n\left(\frac{1}{\beta}+\frac{1}{2}\right)\ln t + \frac{n}{\beta}\ln c\right].$$

The maximum is attained when

$$n - \ln\frac{2\pi^{\frac{n}{2}}}{\beta}\frac{\Gamma\left(\frac{n}{\beta}\right)}{\Gamma\left(\frac{n}{2}\right)} - n\left(\frac{1}{\beta}+\frac{1}{2}\right)\ln t + \frac{n}{\beta}\ln c = n\left(\frac{1}{\beta}+\frac{1}{2}\right)$$

so that

$$\lambda_0(c|x|^\beta) \geq n\frac{\beta+2}{2\beta}\exp\left(\frac{\beta-2}{\beta+2}\right)\left(\frac{\beta}{2\pi^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{2\pi^{\frac{n}{2}}}\right)}{\Gamma\left(\frac{n}{\beta}\right)}\right)^{\frac{2\beta}{n(\beta+2)}} c^{\frac{2}{\beta+2}} := \widetilde{\lambda}_0 > 0.$$

Geiven $V(x) \geq c|x|^\beta$, it is obvious that $I_V(a) \leq I_{c_1|x|^\beta}(a)$. Hence we obtain

$$\lambda_0(V) \geq \lambda_0(c|x|^\beta) \geq \widetilde{\lambda}_0 > 0.$$

This implies that given any $V(x) \geq c|x|^\beta$, their first eigenvalues have a uniform lower bound $\widetilde{\lambda}_0 > 0$. Let λ_* denote the value of $\widetilde{\lambda}_0$ when $c = 1$, then $\lambda_0(V) \geq c^{\frac{2}{\beta+2}}\lambda_*$. By definition, $\lambda_* > 0$ and depends only on β and n. $\square$

subsec2.2h

2.2 Decaying property of eigenfunctions

In this section we prove the following decay property of eigenfunctions:

prp2.1 **Proposition 2.3.** *There exists a constant $\hat{C}$ depending only on n, such that*

$$\|\phi\|_{H^1(\mathbb{R}^n)} \leq 2\|\phi\|_{L^2(\mathcal{B}_r(0))}$$

with

$$r = \hat{C}\left(\frac{n+4}{2\beta_1}\log_+\frac{\lambda+1}{c_1} + \left(\frac{\lambda+2}{c_1}\right)^{1/\beta_1} + 1\right).$$

Before the proof of Proposition prp2.1 2.3, we give several lemmas.

lma2.1 **Lemma 2.4** ([DSV22, Proposition 2.3])**.** *Let $R^{\beta_1} = \max\{(\lambda_k+2)/c_1, 1\}$, then we have*

$$\|e^{|\cdot|/2}\phi_k\|^2_{L^2(\mathbb{R}^n)} \leq 7e^{R+1}\|\phi_k\|^2_{L^2(\mathbb{R}^n)}. \tag{2.3}$$ 2.1

lma2.2 **Lemma 2.5.** *Let $R^{\beta_1} = \max\{(\lambda_k+2)/c_1, 1\}$, then there exists a constatn C depending only on n such that*

$$\|e^{|\cdot|/2}\nabla\phi_k\|^2_{L^2(\mathbb{R}^n)} \leq Ce^{3R}\|\phi_k\|^2_{L^2(\mathcal{B}_2(z))}. \tag{2.4}$$ 2.2

To prove Lemma lma2.2 2.5, we first prove a local Caccioppoli inequality:

Lemma 2.6. *There exists a constant $C(r)$ depending on r such that*

$$\|\nabla\phi_k\|_{\mathcal{B}_r(z)} \leq C(r)\left(1+\lambda_k\right)\|\phi_k\|^2_{L^2(\mathcal{B}_{2r}(z))}. \tag{2.5}$$

Proof. Choose the cutoff function $\eta \in C_c^\infty\left(\mathcal{B}_{2r}(z)\right)$ and $\eta = 1$ in $\mathcal{B}_r(z)$ and $|\nabla\eta| < \frac{2}{r}$. Let $\psi_k = \eta^2\phi_k$, then

$$\begin{aligned}
\int_{\mathcal{B}_{2r}(z)} \nabla\phi_k \cdot \nabla\psi_k &= -\int_{\mathcal{B}_{2r}(z)} \psi_k\Delta\phi_k, \\
\int_{\mathcal{B}_{2r}(z)} \nabla\phi_k \cdot \nabla\psi_k &= -\int_{\mathcal{B}_{2r}(z)} \psi_k\left(V-\lambda_k\right)\phi_k, \\
\int_{\mathcal{B}_{2r}(z)} \eta^2\nabla\phi_k \cdot \nabla\phi_k &= -\int_{\mathcal{B}_{2r}(z)} 2\eta(\nabla\eta\cdot\nabla\phi_k)\phi_k - \int_{\mathcal{B}_{2r}(z)} \eta^2\left(V-\lambda_k\right)\phi_k^2, \\
\int_{\mathcal{B}_{2r}(z)} |\eta\nabla\phi_k|^2 &\leq \int_{\mathcal{B}_{2r}(z)} \frac{4}{r}|\eta\nabla\phi_k|\cdot|\phi_k| - \int_{\mathcal{B}_{2r}(z)} V|\eta\phi_k|^2 + \lambda_k\|\phi_k\|^2_{L^2(\mathcal{B}_{2r}(z))},
\end{aligned}$$

Note that

$$\int_{\mathcal{B}_{2r}(z)} \frac{4}{r}|\eta\nabla\phi_k|\cdot|\phi_k| \leq \frac{1}{2}\int_{\mathcal{B}_{2r}(z)} |\eta\nabla\phi_k|^2 + \frac{32}{r^2}\int_{\mathcal{B}_{2r}(z)} |\phi_k|^2,$$

we obtain

$$\begin{aligned}
\int_{\mathcal{B}_{2r}(z)} |\eta\nabla\phi_k|^2 &\le \frac{1}{2}\int_{\mathcal{B}_{2r}(z)} |\eta\nabla\phi_k|^2 + \frac{32}{r^2}\int_{\mathcal{B}_{2r}(z)} |\phi_k|^2 + \lambda_k\|\phi_k\|^2_{L^2(\mathcal{B}_{2r}(z))},\\
\int_{\mathcal{B}_{r}(z)} (1+V)|\nabla\phi_k|^2 &\le \frac{1}{2}\int_{\mathcal{B}_{2r}(z)} |\eta\nabla\phi_k|^2 + \frac{32}{r^2}\int_{\mathcal{B}_{2r}(z)} |\phi_k|^2 + \lambda_k\|\phi_k\|^2_{L^2(\mathcal{B}_{2r}(z))},\\
\|\nabla\phi_k\|^2_{L^2(\mathcal{B}_r(z))} &\le C(r)(1+\lambda_k)\|\phi_k\|^2_{L^2(\mathcal{B}_{2r}(z))},
\end{aligned}$$

where $C(r) := \max\left\{64/r^2, 2\right\}$. This completes the proof. $\square$

Now we go back to the proof of Lemma 2.5.

Proof of Lemma 2.5. Taking $r = 1$ and using Lemma 2.4, we obtain

$$\|e^{|\cdot|/2}\nabla\phi_k\|^2_{L^2(\mathcal{B}_1(z))} \le C(1+\lambda_k)e^{2R}\|\phi_k\|^2_{L^2(\mathcal{B}_2(z))} \le Ce^{3R}\|\phi_k\|^2_{L^2(\mathcal{B}_2(z))}. \tag{2.6}$$

Covering $\mathbb{R}^n$ by $\{\mathcal{B}_1(z_i)\}_{i\in\mathbb{N}}$ with finite overlaps, summing over z_i we obtain (2.4). $\square$

Define

$$N(\lambda) := \#\left\{\lambda_k | \lambda_k \le \lambda\right\}. \tag{2.7}$$

Note that

$$N(\lambda) \le \sum_{k=1}^{N(\lambda)} (\lambda + 1 - \lambda_k) \tag{2.8}$$

and the lower bound $V(x) \ge c_1|x|^{\beta_1}$ on the potential of Assumption A1, the right hand side can be estimated explicitly by means of the classic Lieb-Thirring bound from [LT01, Theorem 1]. More precisely, for $\lambda > 0$ we have

$$\begin{aligned}
\sum_{k=1}^{N(\lambda)} (\lambda+1-\lambda_k) &\lesssim_n \int_{\mathbb{R}^n} \max\left\{\lambda+1-V(x), 0\right\}^{n/2+1}\,\mathrm{d}x\\
&\le \int_{\mathcal{B}_0\left(((\lambda+1)/c_1)^{1/\beta_1}\right)} |x|^{n/2+1}\,\mathrm{d}x\\
&\lesssim_n \left(\frac{\lambda+1}{c_1}\right)^{\frac{n+4}{2\beta_1}}.
\end{aligned} \tag{2.9}$$

2.8

We are now in position to prove the main result of this section.

Proof of Proposition 2.3. For every $r > 0$, we have

$$\begin{aligned}
\|\phi\|^2_{H^1(\mathbb{R}^n\setminus\mathcal{B}_0(r))} &= \|\phi\|^2_{L^2(\mathbb{R}^n\setminus\mathcal{B}_0(r))} + \|\nabla\phi\|^2_{L^2(\mathbb{R}^n\setminus\mathcal{B}_0(r))}\\
&\le e^{-r}\left(\|e^{|\cdot|/2}\phi\|^2_{L^2(\mathbb{R}^n)} + \|e^{|\cdot|/2}\nabla\phi\|^2_{L^2(\mathbb{R}^n)}\right).
\end{aligned} \tag{2.10}$$

2.8b

Moreover, using the expansion (2.9) and H Cauchy-Schwartz inequality, we obtain

$$\|e^{|\cdot|/2}\phi\|^2_{L^2(\mathbb{R}^n)} \le \left(\sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2}\phi_k\|_{L^2(\mathbb{R}^n)}\right)^2 \le N(\lambda)\sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2}\phi_k\|^2_{L^2(\mathbb{R}^n)}. \tag{2.11}$$

2.8a

Similarly,

$$\|e^{|\cdot|/2}\nabla\phi\|^2_{L^2(\mathbb{R}^n)} \le N(\lambda)\sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2}\nabla\phi_k\|^2_{L^2(\mathbb{R}^n)}. \tag{2.12}$$

2.9

Taking (2.11) and (2.12) into (2.10), we obtan

$$\|\phi\|^2_{H^1(\mathbb{R}^n\setminus\mathcal{B}_0(r))} \le e^{-r} N(\lambda) \left(\sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2}\phi_k\|^2_{L^2(\mathbb{R}^n)} + \sum_{k=1}^{N(\lambda)} \|e^{|\cdot|}\nabla\phi_k\|^2_{L^2(\mathbb{R}^n)} \right) \tag{2.13}$$

Taking (2.3), (2.4) and (2.9) into (2.13) we obtain

$$\|\phi\|^2_{H^1(\mathbb{R}^n\setminus\mathcal{B}_0(r))} \le C e^{-r} \left(\frac{\lambda+1}{c_1}\right)^{\frac{n+4}{2\beta_1}} e^{3R} \|\phi\|^2_{L^2(\mathbb{R}^n)} \tag{2.14}$$

where C is a constant depending only on n. Choose r so that it satisfies

$$r \ge \log 2 + \log C + \frac{n+4}{2\beta_1} \log \frac{\lambda+1}{c_1} + 3R. \tag{2.15}$$

In particular, we choose

$$r = \hat{C} \left(\frac{n+4}{2\beta_1} \log_+ \frac{\lambda+1}{c_1} + \left(\frac{\lambda+2}{c_1}\right)^{1/\beta_1} + 1 \right) \tag{2.16}$$

with $\hat{C}$ large enough so that (2.15) is satisfied, here $\hat{C}$ is dependent only on n. □

3 Spectral inequality for the Schrödinger operator

In this section, equipped with Proposition 2.3, we follow the approach in [ZZ23] step by step to obtain the spectral inequality whose constant in the exponent has explicit dependence on c_1 and c_2.

We present Zhu-Zhuge's spectral inequality in the following.

Theorem 3.1. *Assume that V satisfies Assumption A1 and ω satisfies (1.3) with $l = 1, \sigma \in [0,\infty)$ and $\gamma \in (0, 1/2)$. Then there exists a constant C depending only on n such that*

$$\|\phi\|_{L^2(\mathbb{R}^n)} \le \left(\frac{1}{\gamma}\right)^{C\mathcal{J}} \|\phi\|_{L^2(\omega)}, \quad \forall \phi \in \mathcal{E}_\lambda(H),\ \lambda \ge 0, \tag{3.1}$$

where $\mathcal{J} := \mathcal{J}(c_1, c_2, \lambda) := \mathcal{J}_1^{2\sigma/\beta_2} \left(\lambda^{1/2} + c_2^{\frac{1}{2}} \mathcal{J}_1\right)$ and

$$\mathcal{J}_1(c_1,\lambda) := \left(\frac{n+4}{2\beta_1} \log_+ \frac{\lambda+1}{c_1} + \left(\frac{\lambda+2}{c_1}\right)^{1/\beta_1} + 1 \right)^{\frac{\beta_2}{2}}. \tag{3.2}$$

Here we use the notation $\log_+ u := \max\{u, 0\}$.

There are two main difference between this theorem and [ZZ23, Theorem 1]:

(i) we replace the lower bound $V(x) \ge (|x|-1)_+^{\beta_1}$ in [ZZ23] to $V(x) \ge |x|^{\beta_1}$;

(ii) we give an explicit form of the constant in the exponent of $1/\gamma$.

3.1 Precised form of Zhu-Zhuge's spectral inequality

We consider the solutions of

$$-\Delta v + V v = 0, \quad x \in \mathbb{R}^{n+1}, \tag{3.3}$$

and give two kinds of 3-ball inequalities from [ZZ23]. Then, we follow the strategy in [DSV22, ZZ23] to prove the spectral inequality.

Given $L > 0$, we denote

$$\Lambda_L := \left[-\frac{L}{2}, \frac{L}{2}\right]^n.$$

Denote $\mathcal{B}_r(x) \subset \mathbb{R}^n$ be a ball with radius r and center x. Denote $\mathbb{B}_r(x)$ be a ball in $\mathbb{R}^{n+1}$.

Let $\delta \in (0, \frac{1}{2})$, $b = (0, \cdots, 0, -b_{n+1})$ and $b_{n+1} = \frac{\delta}{100}$. Define

$$W_1 = \left\{ y \in \mathbb{R}^{n+1}_+ \big| |y - b| \le \frac{1}{4}\delta \right\},$$
$$W_2 = \left\{ y \in \mathbb{R}^{n+1}_+ \big| |y - b| \le \frac{2}{3}\delta \right\}.$$

Then $W_1 \subset W_2 \subset \mathbb{B}_\delta$. Define

$$W_j(z_i) := (z_i, 0) + W_j, \quad j = 1, 2,$$

with $Q_L := \Lambda_L \cap \mathbb{Z}^n$, and

$$P_j(L) = \bigcup_{i \in Q_L} W_j(z_i) \text{ and } D_\delta(L) = \bigcup_{i \in Q_L} \mathcal{B}_\delta(z_i), \quad j = 1, 2.$$

Define $R = 9\sqrt{n}$ and

$$X_1 = \Lambda_L \times [-1, 1] \text{ and } \widetilde{X}_R = \Lambda_{L+R} \times [-R, R]. \tag{3.4}$$

Now we present two kinds of 3-ball inequalities.

Lemma 3.2 ([ZZ23, Lemma 1]). *Let $\delta \in (0, \frac{1}{2})$. Let v be the solution of (3.3) with $v(y) = 0$ on the hyperplane $\{y \,|y_{n+1} = 0\}$. There exist $0 < \alpha < 1$ and $C > 0$, depending only on n such that*

$$\|v\|_{H^1(P_1(L))} \le \delta^{-\alpha} \exp\left(C\left(1 + \mathcal{G}(V_1, V_2, 9\sqrt{n}L)\right)\right) \|v\|^{\alpha}_{H^1(P_2(L))} \|\frac{\partial v}{\partial y_{n+1}}\|^{1-\alpha}_{L^2(D_\delta(L))}, \tag{3.5}$$

where

$$\mathcal{G}(V_1, V_2, L) = \|V_1\|^{\frac{1}{2}}_{W^{1,\infty}(\Lambda_L)} + \|V_2\|^{\frac{2}{3}}_{L^\infty(\Lambda_L)}. \tag{3.6}$$

Lemma 3.3 ([ZZ23, Lemma 2]). *Let $\delta \in (0, \frac{1}{2})$. Let v be the solution of (3.3) which is odd with repect to y_{n+1}. There exist $C > 0$ depending only on n, $0 < \alpha < 1$ depending on δ and n such that*

$$\|v\|_{H^1(X_1)} \le \delta^{-2\alpha_1} \exp\left(C\left(1 + \mathcal{G}\left(V_1, V_2, 9\sqrt{n}L\right)\right)\right) \|v\|^{1-\alpha_1}_{H^1(\widetilde{X}_R)} \|v\|^{\alpha_1}_{H^1(P_1(L))}, \tag{3.7}$$

where $\mathcal{G}(V_1, V_2, L)$ is given by (3.6). Indeed, α_1 can be given in the form

$$0 < \alpha_1 = \frac{\epsilon_1}{|\log \delta| + \epsilon_2} < 1 \tag{3.8}$$

with positive constants ϵ_1 and ϵ_2 depending only on n.

Let $\phi \in \mathcal{E}_\lambda(H)$ as before, define

$$\Phi(x, x_{n+1}) = \sum_{0 < \lambda_k \le \lambda} \alpha_k \phi_k(x) \frac{\sinh(\sqrt{\lambda_k} x_{n+1})}{\sqrt{\lambda_k}}. \tag{3.9}$$

Then $\Phi(x, x_{n+1})$ satisfies the equation

$$-\Delta\Phi + V(x)\Phi = 0, \quad (x, x_{n+1}) \in \mathbb{R}^{n+1}. \tag{3.10}$$

We need to mention that $\Delta = \sum_{j=1}^{n+1} D_j^2$ in (3.10), which is not the same as $\Delta_x = \sum_{j=1}^{n} D_j^2$. It is easy to check that $D_{n+1}\Phi(x, 0) = \phi(x)$ and $\Phi(x, 0) = 0$ where $\Delta\Phi = \sum_{j=1}^{n+1} D_j^2 \Phi$.

The following estimate for Φ is standard and can be found in [ZZ23, Lemma 3].

lma3.3 **Lemma 3.4.** *Let $\phi \in \mathcal{E}_\lambda(H)$ and Φ be given in (3.10). For any $\rho > 0$, we have*

$$2\rho\|\phi\|^2_{L^2(\mathbb{R}^n)} \le \|\Phi\|^2_{H^1(\mathbb{R}^n\times(-\rho,\rho)} \le 2\rho\left(1+\frac{\rho^2}{3}(1+\lambda)e^{2\rho\sqrt{\lambda}}\right)\|\phi\|^2_{L^2(\mathbb{R}^n)}. \tag{3.11}$$

To use Proposition 2.3, we need to extend it from ϕ to Φ. Indeed, we have the following corollary:

crc3.4 **Corollary 3.5.** *Given the same condition as Proposition 2.3, we have*

$$\|\Phi\|^2_{H^1(\mathbb{R}^n\times(-1,1))} \le 2\|\Phi\|^2_{H^1(\mathcal{B}_r\times(-1,1))}. \tag{3.12}$$ 3.11

Proof. Since $\Phi(\cdot, x_{n+1}) \in \mathcal{E}_\lambda(H)$, by Proposition 2.3 we obtain

$$\|\Phi(\cdot,x_{n+1})\|^2_{H^1(\mathbb{R}^n)} \le 2\|\Phi(\cdot,x_{n+1})\|^2_{L^2(\mathcal{B}_r(0))} \le 2\|\Phi(\cdot,x_{n+1})\|^2_{H^1(\mathcal{B}_r(0))}. \tag{3.13}$$ 3.12

Since $D_{n+1}\Phi(\cdot,x_{n+1}) \in \mathcal{E}_\lambda(H)$, we obtain

$$\|D_{n+1}\Phi(\cdot,x_{n+1})\|^2_{L^2(\mathbb{R}^n)} \le \|D_{n+1}\Phi(\cdot,x_{n+1})\|^2_{H^1(\mathbb{R}^n)} \le 2\|D_{n+1}\Phi\|^2_{L^2(\mathcal{B}_r(0))}. \tag{3.14}$$ 3.13

Then we have

$$\begin{aligned}
&\|\Phi\|^2_{H^1(\mathbb{R}^n\times(-1,1))}\\
=&\int_{-1}^1\int_{\mathbb{R}^n}|\Phi|^2\,\mathrm{d}x\mathrm{d}x_{n+1} + \int_{-1}^1\int_{\mathbb{R}^n}\sum_{j=1}^{n+1}|D_j\Phi|^2\,\mathrm{d}x\mathrm{d}x_{n+1}\\
=&\int_{-1}^1\|\Phi(\cdot,x_{n+1})\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}x_{n+1} + \int_{-1}^1\sum_{j=1}^{n}\|D_j\Phi(\cdot,x_{n+1})\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}x_{n+1}\\
&+\int_{-1}^1\|D_{n+1}\Phi(\cdot,x_{n+1})\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}x_{n+1}\\
\overset{(3.14)}{\le}&\int_{-1}^1\|\Phi(\cdot,x_{n+1})\|^2_{H^1(\mathbb{R}^n)}\,\mathrm{d}x_{n+1} + \int_{-1}^1 2\|D_{n+1}\Phi(\cdot,x_{n+1})\|^2_{L^2(\mathcal{B}_r(0))}\,\mathrm{d}x_{n+1}\\
\overset{(3.13)}{\le}&\int_1^1 2\|\Phi(\cdot,x_{n+1}\|^2_{H^1(\mathcal{B}_r(0))}\,\mathrm{d}x_{n+1} + \int_{-1}^1 2\|D_{n+1}\Phi(\cdot,x_{n+1})\|^2_{L^2(\mathcal{B}_r(0))}\,\mathrm{d}x_{n+1}\\
=&\|\Phi\|^2_{H^1(\mathcal{B}_r(0))}.
\end{aligned} \tag{3.15}$$

□

Now we give the proof of Theorem 3.1

Proof of Theorem 3.1. Let $L = 2\lceil r\rceil + 1$, where r is given in Proposition 2.3 and $\lceil a\rceil$ means the largest integer smaller than $a+1$. Then we have $\mathcal{B}_r(0) \subset \Lambda_L$ with $\Lambda_L = [-L/2, L/2]$. Moreover, we can decompose Λ_L as

$$\Lambda_L = \bigcup_{k\in\Lambda_L\cap\mathbb{Z}^n}\left(k+\left[-\frac{1}{2},\frac{1}{2}\right]^n\right). \tag{3.16}$$

For each $k \in \Lambda_L\cap\mathbb{Z}^n$, we have $|k| \le \sqrt{n}\lceil r\rceil$. Let $\gamma \in (0, \frac{1}{2})$ be as in the theorem and

$$\delta := \gamma^{1+\left(\sqrt{n}\lceil r\rceil\right)^\sigma} \le \gamma^{1+|k|^\sigma}, \quad \forall k \in \Lambda_L\cap\mathbb{Z}^n. \tag{3.17}$$

Now we show an interpolation inequality. We replace v in (3.3) by Φ in (3.9). Note that Φ is odd in x_{n+1}, we combine (3.5) in Lemma 3.2 and (3.7) in Lemma 3.3 with δ and L defined above to get

$$\begin{aligned}
&\|\Phi\|_{H^1(X_1)} \le \delta^{-2\alpha_1}\exp\left(C\left(1+\mathcal{G}\left(V_1,V_2,9\sqrt{n}L\right)\right)\right)\|\Phi\|^{1-\alpha_1}_{H^1(\widetilde{X}_R)}\|\Phi\|^{\alpha_1}_{H^1(P_1(L))}\\
&\le \delta^{-2\alpha_1-\alpha\alpha_1}\exp\left(C\left(1+\mathcal{G}\left(V_1,V_2,9\sqrt{n}L\right)\right)\right)\|\Phi\|^{\alpha\alpha_1}_{H^1(P_2(L))}\|\frac{\partial\Phi}{\partial y_{n+1}}\|^{\alpha_1(1-\alpha)}_{L^2(D_\delta(L))}\|\Phi\|^{1-\alpha_1}_{H^1(\widetilde{X}_R)}\\
&\le \delta^{-3\alpha_1}\exp\left(C\left(1+\mathcal{G}\left(V_1,V_2,9\sqrt{n}L\right)\right)\right)\|\phi\|^{\hat{\alpha}}_{L^2(D_\delta(L))}\|\Phi\|^{1-\hat{\alpha}}_{H^1(\widetilde{X}_R)},
\end{aligned} \tag{3.18}$$

where $\hat{\alpha} = \alpha_1(1-\alpha)$ and we have used the facts $P_2(L) \subset \widetilde{X}_R$ and $\frac{\partial \Phi}{\partial y_{n+1}}(\cdot, 0) = \phi$. Here and below, the symbol C may represent different positive constants depending on n.

Recall α_1 in (3.8), we have $\alpha_1 \approx \hat{\alpha} \approx \frac{1}{|\log \delta|}$ for any $\delta \in (0, \frac{1}{2})$. Hence $\delta^{-3\alpha_1} \leq C$ and then

$$\|\Phi\|_{H^1(X_1)} \leq \exp\left(C\left(1 + \mathcal{G}\left(V_1, V_2, 9\sqrt{n}L\right)\right)\right) \|\phi\|^{\hat{\alpha}}_{L^2(\omega\cap\Lambda_L)} \|\Phi\|^{1-\hat{\alpha}}_{H^1(\widetilde{X}_R)}, \tag{3.19}$$

3.18

where we have also used the fact $D_\delta(L) \subset \omega \cap \Lambda_L$.

Substituting $L = 2\lceil r \rceil + 1$ and (2.16) into $\mathcal{G}(V_1, V_2, L)$ and by Assumption A1, we have

$$\mathcal{G}(V_1, V_2, 9\sqrt{n}L) \leq c_2^{\frac{1}{2}} \left(2\lceil r \rceil + 2\right)^{\frac{\beta_2}{2}} \leq C c_2^{\frac{1}{2}} \left(\frac{n+4}{2\beta_1} \log \frac{\lambda+1}{c_1} + \left(\frac{\lambda+2}{c_1}\right)^{1/\beta_1} + 1\right)^{\frac{\beta_2}{2}}. \tag{3.20}$$

Define

$$\mathcal{J}_1(c_1, \lambda) = \left(\frac{n+4}{2\beta_1} \log_+ \frac{\lambda+1}{c_1} + \left(\frac{\lambda+2}{c_1}\right)^{1/\beta_1} + 1\right)^{\frac{\beta_2}{2}}, \tag{3.21}$$

j.1

then we can write (3.19) as

$$\|\Phi\|_{H^1(X_1)} \leq \exp\left(C c_2^{\frac{1}{2}} \mathcal{J}_1(c_1, c_2, \lambda)\right) \|\phi\|^{\hat{\alpha}}_{L^2(\omega\cap\Lambda_L)} \|\Phi\|^{1-\hat{\alpha}}_{H^1(\widetilde{X}_R)}. \tag{3.22}$$

3.20

Applying $\rho = R$ and $\rho = 1$ in Lemma 3.4 for upper and lower bounds, respectively, we obtain

$$\frac{\|\Phi\|^2_{H^1(\mathbb{R}^n\times(-R,R))}}{\|\Phi\|^2_{H^1(\mathbb{R}^n\times(-1,1))}} \leq R\left(1 + \frac{R^2}{3}(1+\lambda)\right) \exp\left(2R\sqrt{\lambda}\right) \leq \exp\left(C_2\sqrt{\lambda}\right). \tag{3.23}$$

With the aid of (3.12) and the fact $\mathcal{B}_r(0) \subset \Lambda_L$, we get

$$\begin{aligned} \|\Phi\|_{H^1(\mathbb{R}^n\times(-R,R))} &\leq \exp\left(\frac{1}{2}C_2\sqrt{\lambda}\right) \|\Phi\|_{H^1(\mathbb{R}^n\times(-1,1))} \\ &\leq \sqrt{2}\exp\left(\frac{1}{2}C_2\sqrt{\lambda}\right) \|\Phi\|_{H^1(\Lambda_L\times(-1,1))}. \end{aligned} \tag{3.24}$$

3.22

Recall that $X_1 = \Lambda_L \times (-1, 1)$, substituting (3.24) into (3.22) we obtain

$$\|\Phi\|_{H^1(\mathbb{R}^n\times(-R,R))} \leq \exp\left(C_3 \mathcal{J}_2(c_1, c_2, \lambda)\right) \|\phi\|^{\hat{\alpha}}_{L^2(\omega\cap\Lambda_L)} \|\Phi\|^{1-\hat{\alpha}}_{H^1(\widetilde{X}_R)} \tag{3.25}$$

with

$$\mathcal{J}_2(c_1, c_2, \lambda) = \lambda^{\frac{1}{2}} + c_2^{\frac{1}{2}} \mathcal{J}_1(c_1, c_2, \lambda). \tag{3.26}$$

j.2

Since $\widetilde{X}_R \subset \mathbb{R}^n \times (-R, R)$, it follows that

$$\|\Phi\|_{H^1(\mathbb{R}^n\times(-R,R))} \leq \exp\left(\hat{\alpha}^{-1} C_3 \mathcal{J}_2\left(c_1, c_2, \lambda\right)\right) \|\phi\|_{L^2(\omega\cap\Lambda_L)}. \tag{3.27}$$

Recall that

$$\hat{\alpha}^{-1} \approx \alpha_1^{-1} \approx |\log \delta| \approx |\log \gamma| \mathcal{J}_1^{\frac{2\sigma}{\beta_2}} \tag{3.28}$$

we obtain

$$\|\Phi\|_{H^1(\mathbb{R}^n\times(-R,R))} \leq \left(\frac{1}{\gamma}\right)^{C\mathcal{J}_1^{\frac{2\sigma}{\beta_2}}\mathcal{J}_2} \|\phi\|_{L^2(\omega\cap\Lambda_L)}. \tag{3.29}$$

Finally, using the lower bound in Lemma 3.4 with $\rho = R$, we obtain

$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{1}{2R}\right)^{\frac{1}{2}} \|\Phi\|_{H^1(\mathbb{R}^n\times(-R,R))} \leq \left(\frac{1}{\gamma}\right)^{C\mathcal{J}_1^{\frac{2\sigma}{\beta_2}}\mathcal{J}_2} \|\phi\|_{L^2(\omega)} \tag{3.30}$$

with positive constant C depending only on n. $\square$

3.2 Changing of the constant

To make use of Theorem 3.1, we consider the new Schrödinger operator

$$H_r = -\Delta_x + rV(x), \quad r > 0 \tag{3.31}$$

and observe the influence of r in $\mathcal{J}$.

First, we assume that V in (3.31) satisfies Assumption A1. Then the new potential rV satisfies Assumption A1 by replacing c_1, c_2 with rc_1, rc_2 respectively. This implies $\mathcal{J}(c_1, c_2, \lambda)$ changes to $\mathcal{J}(rc_1, rc_2, \lambda)$. Now fix the constant $\beta_1, \beta_2, c_1, c_2, \sigma$ and we get the following corollary.

Corollary 3.6. *Assume that V satisfies Assumption A1 and ω satisfies (1.3) with $l = 1, \sigma \in [0, \infty)$ and $\gamma \in (0, 1/2)$. Then the term $\mathcal{J}(rc_1, rc_2, \lambda)$, denote by $\mathcal{J}_r$, associated with the potential rV in (3.1) and any $\lambda > 0$, have following relations:*

(i) *if $r \geq 1$,*

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} \lambda^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{\sigma}{\beta_2} + \frac{1}{2}}. \tag{3.32}$$

(ii) *if $0 < r < 1$ and $\beta_1 = \beta_2 = \beta$,*

$$\mathcal{J}_r \lesssim_{n,\beta,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta}} \lambda^{\frac{\sigma}{\beta} + \frac{1}{2}} + r^{-\frac{\sigma}{\beta}}. \tag{3.33}$$

(iii) *if $0 < r < 1$ and $\beta_1 < \beta_2$,*

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{s_1} \lambda^{s_2} + r^{s_3} \tag{3.34}$$

where s_1, s_2 and s_3 are constants depending on $n, \beta_1, \beta_2, c_1, c_2, \sigma$ with $s_3 < 0$.

Proof of Corollary 3.6. Remember

$$\mathcal{J} = \mathcal{J}(c_1, c_2, \lambda) = \mathcal{J}_1^{\frac{2\sigma}{\beta_2}} \mathcal{J}_2. \tag{3.35}$$

By the definition of $\mathcal{J}_1$, we obtain

$$\mathcal{J}_1(c_1, c_2, \lambda) \lesssim_{n,\beta_1,\beta_2} \left(\frac{\lambda + 2}{c_1}\right)^{\frac{\beta_2}{2\beta_1}} + 1 \tag{3.36}$$

Then we have

$$\mathcal{J} \lesssim_{n,\beta_1,\beta_2,\sigma} \left(\left(\frac{\lambda + 2}{c_1}\right)^{\frac{\beta_2}{2\beta_1}} + 1\right)^{\frac{2\sigma}{\beta_2}} \cdot \left(\lambda^{\frac{1}{2}} + c_2^{\frac{1}{2}} \left(\left(\frac{\lambda + 2}{c_1}\right)^{\frac{\beta_2}{2\beta_1}} + 1\right)\right). \tag{3.37}$$

Here and below we frequently use the fact that given a fixed number $\beta > 0$, we have

$$(\lambda + 1)^{\beta} \lesssim_{\beta} \lambda^{\beta} + 1, \quad \forall \lambda > 0. \tag{3.38}$$

We simplify (3.37) to the following

$$\mathcal{J} \lesssim_{n,\beta_1,\beta_2,\sigma} c_1^{-\frac{\sigma}{\beta_1}} \left(\lambda^{\frac{\sigma}{\beta_1}} + c_1^{\frac{\sigma}{\beta_1}} + 1\right) \cdot \left(\lambda^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_1^{-\frac{\beta_2}{2\beta_1}} \lambda^{\frac{\beta_2}{2\beta_1}} + c_2^{\frac{1}{2}} + c_2^{\frac{1}{2}} c_1^{-\frac{\beta_2}{2\beta_1}}\right). \tag{3.39}$$

Recall $\mathcal{J}_r := \mathcal{J}(rc_1, rc_2, \lambda)$. From now on, we fix the constant c_1 and c_2, and consider the change of $\mathcal{J}_r$ with respect to any positive real number r under the condition $\lambda \geq r^{\frac{2}{\beta_1 + 2}} \widetilde{\lambda}_0$.

To capture the change of the lower bound of λ, we define the rescaled value of λ

$$\mu := \frac{\lambda}{r^{2/(\beta_1 + 2)} \widetilde{\lambda}_0}. \tag{3.40}$$

Then $\mu \geq 1$. Substituting (3.40) into (3.39) and replacing c_1 and c_2 by rc_1 and rc_2 respectively, we obtain

$$\begin{aligned} \mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} & r^{-\frac{\sigma}{\beta_1}} \left(r^{\frac{2\sigma}{\beta_1(\beta_1+2)}} \mu^{\frac{\sigma}{\beta_1}} + r^{\frac{\sigma}{\beta_1}} + 1 \right) \\ & \times \left(r^{\frac{1}{\beta_1+2}} \mu^{\frac{1}{2}} + r^{\frac{1}{2} - \frac{\beta_2}{2\beta_1} + \frac{\beta_2}{\beta_1(\beta_1+2)}} \mu^{\frac{\beta_2}{2\beta_1}} + r^{\frac{1}{2}} + r^{\frac{1}{2} - \frac{\beta_2}{2\beta_1}} \right) \end{aligned} \tag{3.41}$$

Now we study the relation between $\mathcal{J}_r$ and r in three cases.

Case $r \geq 1$. Note that the exponent in (3.41) has the relation

$$\frac{1}{\beta_1 + 2} \geq \frac{1}{2} - \frac{\beta_2}{2\beta_1} + \frac{\beta_2}{\beta_1(\beta_1+2)} \tag{3.42}$$

provided the condition $\beta_2 \geq \beta_1 > 0$. Then we obtan

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1}} \left(r^{\frac{2\sigma}{\beta_1(\beta_1+2)}} \mu^{\frac{\sigma}{\beta_1}} + r^{\frac{\sigma}{\beta_1}} \right) \cdot \left(r^{\frac{1}{\beta_1+2}} \mu^{\frac{\beta_2}{2\beta_1}} + r^{\frac{1}{2}} \right) \tag{3.43}$$

where we used the condition $r \geq 1$ to absorb lower order terms. It can be simplified further to

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1} + \frac{2\sigma}{\beta_1(\beta_1+2)} + \frac{1}{\beta_1+2}} \mu^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{1}{2}}. \tag{3.44}$$

Now we can recover the term λ and obtain

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1} - \frac{\beta_2 - \beta_1}{\beta_1(\beta_1+2)}} \lambda^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{1}{2}}. \tag{3.45}$$

If we do not care about the nonpositive power, we obtain

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} \lambda^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{1}{2}}. \tag{3.46}$$

Case $0 < r < 1$ and $\beta_1 = \beta_2 = \beta$. In this case, we observe (3.41) and obtain

$$\mathcal{J}_r \lesssim_{n,\beta,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta}} \left(\left(r^{\frac{2}{\beta+2}} \mu \right)^{\frac{\sigma}{\beta}} + 1 \right) \cdot \left(\left(r^{\frac{2}{\beta+2}} \mu \right)^{\frac{1}{2}} + 1 \right). \tag{3.47}$$

Decompose it and we obtain Let $s = r^{\frac{2}{\beta+2}} \mu$, then

$$\mathcal{J}_r \lesssim_{n,\beta,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta}} \left(s^{\frac{\sigma}{\beta} + \frac{1}{2}} + s^{\frac{\sigma}{\beta}} + s^{\frac{1}{2}} + 1 \right). \tag{3.48}$$

There is a competition between s and 1,

$$\mathcal{J}_r \lesssim_{n,\beta,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta}} \times \begin{cases} 1, & s \leq 1, \\ s^{\frac{\sigma}{\beta} + \frac{1}{2}} + 1, & s > 1. \end{cases} \tag{3.49}$$

Hence we always have $\mathcal{J}_r \lesssim_{n,\beta,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta}} s^{\frac{1}{2} + \frac{\sigma}{\beta}} + r^{-\frac{\sigma}{\beta}}$, then we recover the original term and get

$$\mathcal{J}_r \lesssim_{n,\beta,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta}} \lambda^{\frac{\sigma}{\beta} + \frac{1}{2}} + r^{-\frac{\sigma}{\beta}}. \tag{3.50}$$

Case $0 < r < 1$ and $\beta_1 < \beta_2$. In this case, we observe (3.41) and obtain

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1}} \left(r^{\frac{2\sigma}{\beta_1(\beta_1+2)}} \mu^{\frac{\sigma}{\beta_1}} + 1 \right) \cdot \left(r^{\frac{1}{2} - \frac{\beta_2}{2\beta_1} + \frac{\beta_2}{\beta_1(\beta_1+2)}} \mu^{\frac{\beta_2}{2\beta_1}} + r^{\frac{1}{2} - \frac{\beta_2}{2\beta_1}} \right). \tag{3.51}$$

Note that there exists a term $r^{-\frac{\sigma}{\beta_1} + \frac{1}{2} - \frac{\beta_2}{2\beta_1}}$ with the exponent

$$-\frac{\sigma}{\beta_1} + \frac{1}{2} - \frac{\beta_2}{2\beta_1} < 0.$$

Hence we can write it as

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{s_1} \lambda^{s_2} + r^{s_3}$$

with s_1, s_2 and s_3 depending on $n, \beta_1, \beta_2, c_1, c_2, \sigma$ and $s_3 < 0$. $\square$

Then we assume that V in (3.31) satisfies Assumption A2. For $r \geq 1$, the new potential rV satisfies Assumption A2 by replacing c_1, c_2 with $rc_1, r^{4/3}c_2$ respectively. This implies $\mathcal{J}(c_1, c_2, \lambda)$ changes to

$$\mathcal{J}_r := \begin{cases} \mathcal{J}(rc_1, r^{\frac{4}{3}}c_2, \lambda), & r \geq 1, \\ \mathcal{J}(rc_1, rc_2, \lambda), & 0 < r < 1. \end{cases} \tag{3.52}$$

Now fix the constant $\beta_1, \beta_2, c_1, c_2, \sigma$ and we get the following corollary.

crc1.3h **Corollary 3.7.** *Assume that V satisfies Assumption A2 and ω satisfies (1.3) with $l = 1, \sigma \in [0, \infty)$ and $\gamma \in (0, 1/2)$. Then the term $\mathcal{J}_r$ associated with the potential rV in (3.1) and any $\lambda > 0$ have the following relations:*

(i) *if $r \geq 1$,*

$$\mathcal{J}_r \lesssim \begin{cases} r^{-\frac{\sigma}{\beta_1} + \frac{2}{3} - \frac{\beta_2}{2\beta_1}} \lambda^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}}, & \beta_2 \leq (4\beta_1 + 2)/3, \\ \lambda^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}}, & \beta_2 > (4\beta_1 + 2)/3. \end{cases} \tag{3.53}$$

(ii) *if $0 < r < 1$, the situation is the same as* (ii) *and* (iii) *in Corollary 3.6.*

Proof. We only need to consider the case $r \geq 1$ since $\mathcal{J}_r$ has the same form under the case $0 < r < 1$. Then $\mathcal{J}_r = \mathcal{J}(rc_1, r^{4/3}c_2, \lambda)$. Replacing c_1 and c_2 by rc_1 and $r^{4/3}c_2$ respectivel, and substituting (3.40) into (3.39), we obtain

$$\begin{aligned} \mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} & r^{-\frac{\sigma}{\beta_1}} \left(r^{\frac{2\sigma}{\beta_1(\beta_1+2)}} \mu^{\frac{\sigma}{\beta_1}} + r^{\frac{\sigma}{\beta_1}} + 1 \right) \\ & \times \left(r^{\frac{1}{\beta_1+2}} \mu^{\frac{1}{2}} + r^{\frac{2}{3} - \frac{\beta_2}{2\beta_1} + \frac{\beta_2}{\beta_1(\beta_1+2)}} \mu^{\frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}} + r^{\frac{2}{3} - \frac{\beta_2}{2\beta_1}} \right) \end{aligned} \tag{3.54}$$ 3.53g

Now we need to compare the exponent $\frac{1}{\beta_1+2}$ and $\frac{2}{3} - \frac{\beta_2}{2\beta_1} + \frac{\beta_2}{\beta_1(\beta_1+2)}$:

Case $\beta_2 \leq (4\beta_1 + 2)/3$**.** Since

$$\beta_2 \leq \frac{4\beta_1 + 2}{3} \iff \frac{1}{\beta_1 + 2} \leq \frac{2}{3} - \frac{\beta_2}{2\beta_1} + \frac{\beta_2}{\beta_1(\beta_1 + 2)}, \tag{3.55}$$ 3.54g

hence we obtain from (3.54)

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1}} \left(r^{\frac{2\sigma}{\beta_1(\beta_1+2)}} \mu^{\frac{\sigma}{\beta_1}} + r^{\frac{\sigma}{\beta_1}} \right) \cdot \left(r^{\frac{2}{3} - \frac{\beta_2}{2\beta_1} + \frac{\beta_2}{\beta_1(\beta_1+2)}} \mu^{\frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}} \right) \tag{3.56}$$

where we used the condition $r \geq 1$ to absorb lower order terms. It can be simplified further to

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1} + \frac{2\sigma}{\beta_1(\beta_1+2)} + \frac{2}{3} - \frac{\beta_2}{2\beta_1} + \frac{\beta_2}{\beta_1(\beta_1+2)}} \mu^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}}. \tag{3.57}$$

Recover the term λ by (3.40) we obtain

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1} + \frac{2}{3} - \frac{\beta_2}{2\beta_1}} \lambda^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}}. \tag{3.58}$$

Case $\beta_2 > (4\beta_1 + 2)/3$**.** Now we have the reverse inequality of (3.55)

$$\frac{1}{\beta_1 + 2} > \frac{2}{3} - \frac{\beta_2}{2\beta_1} + \frac{\beta_2}{\beta_1(\beta_1 + 2)}. \tag{3.59}$$

Then we obtain from (3.54)

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1}} \left(r^{\frac{2\sigma}{\beta_1(\beta_1+2)}} \mu^{\frac{\sigma}{\beta_1}} + r^{\frac{\sigma}{\beta_1}} \right) \cdot \left(r^{\frac{1}{\beta_1+2}} \mu^{\frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}} \right). \tag{3.60}$$

It can be simplified to

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1} + \frac{2\sigma}{\beta_1(\beta_1+2)} + \frac{1}{\beta_1+2}} \mu^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}}. \tag{3.61}$$

Recover the term λ by (3.40) we obtain

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} r^{-\frac{\sigma}{\beta_1}+\frac{1}{\beta_1+2}-\frac{\beta_2}{\beta_1(\beta_1+2)}} \lambda^{\frac{\sigma}{\beta_1}+\frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}}. \tag{3.62}$$

Note that the exponential of the first r is negative, we can obsorb it and obtain

$$\mathcal{J}_r \lesssim_{n,\beta_1,\beta_2,\sigma,c_1,c_2} \lambda^{\frac{\sigma}{\beta_1}+\frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}}. \tag{3.63}$$

The proof is finished. □

4 Proof of the exact controllability results

sec2d

In this section, we prove the exact null-controllability results for the evolution equations ($E_{G,s}$) and ($E_{G_p,s}$), i.e., prove Theorem 1.1, 1.2, 1.4 and 1.5. The proof is motivated by [AS23].

4.1 Exact observability

Note that the operators H_G^s and $H_{G_p}^s$ are selfadjoint in $L^2(\mathbb{R}^{n+m})$ and $L^2(\mathbb{R}^n \times \mathbb{T}^m)$, respectively, then the Hilbert Uniqueness Method implies that the exact null-controllability of ($E_{G,s}$) and ($E_{G_p,s}$) is equivalent to the exact observability of the associated semigroups $(e^{-tH_G^s})_{t\geq 0}$ and $(e^{-tH_{G_p}^s})_{t\geq 0}$. The latter is defined as follows.

Definition 4.1 (Exact observability)**.** Let $T>0$, and let $\Omega \subset \mathbb{R}^n$ and $\omega \subset \Omega$ be measurable. A strongly continuous semigroup $(S(t))_{t\geq 0}$ on $L^2(\Omega)$ is said to be exactly observable from the set ω in time T if there exists a positive constant $C_{\omega,T}>0$ such that for all $g \in L^2(\Omega)$, we have

$$\|S(T)g\|^2_{L^2(\Omega)} \leq C_{\omega,T} \int_0^T \|S(t)g\|^2_{L^2(\omega)}\,\mathrm{d}t.$$

The strategy of the proof relies on the exact observability with explicit form of the constant $C_{\omega,T}$, which is based on the following quatitative result.

thm2.1d **Theorem 4.1** ([NTTV20, Theorem 2.8])**.** *Let A be a non-negative selfadjoint operator on $L^2(\mathbb{R}^n)$, and let $\omega \subset \mathbb{R}^n$ be measurable. Suppose that there are $d_0, d_1 \geq 0$ and $\zeta \in (0,1)$ such that for all $\lambda \geq 0$ and $\phi \in \mathcal{E}_\lambda(A)$,*

$$\|\phi\|^2_{L^2(\mathbb{R}^n)} \leq d_0 e^{d_1 \lambda^\zeta} \|\phi\|^2_{L^2(\omega)}. \tag{4.1}$$ 2.1c

Then there exist positve constants $\kappa_1, \kappa_2, \kappa_3 > 0$ only depending on ζ, such that for all $T>0$ and $g \in L^2(\mathbb{R}^n)$ we have the observability estimate

$$\|e^{-tA}g\|^2_{L^2(\mathbb{R}^n)} \leq \frac{C_{\mathrm{obs}}}{T} \int_0^T \|e^{-tA}g\|^2_{L^2(\omega)}\,\mathrm{d}t,$$

where the positive constant $C_{\mathrm{obs}}>0$ is given by

$$C_{\mathrm{obs}} = \kappa_1 d_0 (2d_0+1)^{\kappa_2} \exp\left(\kappa_3 \left(\frac{d_1}{T^\zeta}\right)^{\frac{1}{1-\zeta}}\right). \tag{4.2}$$ 2.2c

Applying the partial Fourier transformation with respect to $y \in \mathbb{R}^m$, the operator H_G is transformed to

$$H_\eta := -\Delta_x + |\eta|^2 V(x)$$

where $\eta \in \mathbb{R}^m$ denotes the dual variable of $y \in \mathbb{R}^m$. Then we have

$$(e^{-tH_G}g)(x,y) = \int_{\mathbb{R}^n} \left(e^{-tH_\eta} g_\eta\right)(x)\,\mathrm{d}\eta, \quad g \in L^2\left(\mathbb{R}^{n+m}\right), (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \tag{4.3}$$ 2.2b

where
$$g_\eta(x) = \int_{\mathbb{R}^m} e^{-iy\cdot\eta} g(x,y)\,\mathrm{d}y.$$

By Proposition 2.1, we have $\lambda_0(rV) \geq r^{\frac{2}{\beta+2}}\widetilde{\lambda}_0$. Hence for the operator $H_r = -\Delta + rV$ we have the following decaying property
$$\|e^{-tH_r}g\|_{L^2(\mathbb{R}^n)} \leq e^{-t|r|^{\frac{2}{\beta_1+2}}\widetilde{\lambda}_0}\|g\|_{L^2(\mathbb{R}^n)}. \tag{4.4}$$

4.2 Proof of main theorems

Now we prove an exact observability estimate for the semigroups generated by H_r^s

Proposition 4.2. *Let H_r be given in (3.31) with V satisfying Assumption A1 and $\sigma = 0$. Then there exists a constant $K > 0$ depending only on n, β, c_1, c_2 such that for all $(1, \gamma, 0)$-distributed sets $\omega \subset \mathbb{R}^n$, $r > 0$, $T > 0$, and $g \in L^2(\mathbb{R}^n)$ we have*
$$\|e^{-TH_r^s}g\|^2_{L^2(\mathbb{R}^n)} \leq \frac{C_{\mathrm{obs}}}{T}\int_0^T \|e^{-tH_r^s}g\|^2_{L^2(\omega)}\,\mathrm{d}t$$
where the positive constant C_{obs} is given by
$$C_{\mathrm{obs}} = K\left(\exp\left(K\log(1/\gamma)r^{\frac{1}{2}} + \log^{\frac{2s}{2s-1}}(1/\gamma)/T^{\frac{1}{2s-1}}\right)\right). \tag{4.5}$$

Proof. Given $\beta_1 = \beta_2 = \beta > 0, \sigma = 0$ and $\gamma \in (0, 1/2)$, Corollary 3.6 (i)(ii) implies the spectral inequality for the operator H_r defined in (3.31)
$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq C_1\left(\frac{1}{\gamma}\right)^{C_2\lambda^{\frac{1}{2}}+r^{\frac{1}{2}}}\|\phi\|^2_{L^2(\omega)}, \quad \forall\phi \in \mathcal{E}_{H_r}(\lambda),$$
where C_1 and C_2 depend on β, c_1, c_2 and not on r. For the fractional operator H_G^s, by the transformation formula for spectral measures (see [Sch12, Proposition 4.24]), for all $s > 0$ and $\lambda \geq 0$ we have
$$\mathcal{E}_\lambda(H_r^s) = \mathcal{E}_{\lambda^{\frac{1}{s}}}(H_r). \tag{4.6}$$
This implies that the spectral inequality for H_r^s can be achieved from the spectral inequality for H_r by simply replacing λ by$\lambda^{1/s}$. Then we obtain the spectral inequality for the operator H_r^s
$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq C_1\left(\frac{1}{\gamma}\right)^{C_2\lambda^{\frac{1}{2s}}+r^{\frac{1}{2}}}, \quad \forall\phi \in \mathcal{E}_{H_r^s}(\lambda).$$

Now we write the constant the same form as in (4.1), i.e.,
$$d_0 = C_1 e^{\log(\frac{1}{\gamma})r^{\frac{1}{2}}},\ d_1 = C_2\log\left(\frac{1}{\gamma}\right),\ \zeta = \frac{1}{2s}.$$
Then we estimate the observability constant C_{obs} given in (4.2),
$$\kappa_1 d_0(2d_0+1)^{\kappa_2}\exp\left(\kappa_3\left(\frac{d_1}{T^\zeta}\right)^{\frac{1}{1-\zeta}}\right) \leq K\exp\left(K\log\left(1/\gamma\right)r^{\frac{1}{2}} + \log^{\frac{2s}{2s-1}}\left(1/\gamma\right)/T^{\frac{1}{2s-1}}\right)$$
where the constant K depends only on n, β, c_1, c_2. This ends the proof. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. We have to show that whenever $T>0$, there exsits a constant $C_{\omega,T}>0$ such that for all $g\in L^2(\mathbb{R}^{n+m})$ we have

$$\|e^{-TH_G^s}g\|^2_{L^2(\mathbb{R}^{n+m})}\le C_{\omega,T}\int_0^T\|e^{-tH_G^s}g\|^2_{L^2(\omega\times\mathbb{R}^m)}\,\mathrm{d}t. \tag{4.7}$$

Observe from (4.3) and Fubini's theorem that for every measurable set $\Omega\subset\mathbb{R}^n$ and all $t>0$ and $g\in L^2\left(\mathbb{R}^{n+m}\right)$ we have

$$\|e^{-tH_G}g\|^2_{L^2(\Omega\times\mathbb{R}^m)}=\int_{\mathbb{R}^m}\|e^{-tH_\eta}g_\eta\|^2_{L^2(\Omega)}\,\mathrm{d}\eta.$$

Inserting the latter into both sides of (4.7), once with $\Omega=\mathbb{R}^n$ and $t=T$ and once with $\Omega=\omega$, we obtain

$$\int_{\mathbb{R}^m}\|e^{-TH_\eta^s}g_\eta\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}\eta\le C_{\omega,T}\int_{\mathbb{R}^m}\left(\int_0^T\|e^{-tH_\eta^s}g_\eta\|^2_{L^2(\omega)}\,\mathrm{d}t\right)\mathrm{d}\eta$$

Then by Fubini's theorem again, it suffices to show that

$$\|e^{-TH_\eta^s}g\|^2_{L^2(\mathbb{R}^n)}\le C_{\omega,T}\int_0^T\|e^{-tH_\eta^s}g\|^2_{L^2(\omega)}\,\mathrm{d}t,\quad g\in L^2(\mathbb{R}^n),\ r>0.$$

On the one hand, we have by (4.4) that

$$\|e^{-TH_\eta^s}g\|^2_{L^2(\mathbb{R}^n)}\le e^{-T|\eta|^{\frac{4s}{\beta+2}}\widetilde{\lambda}_0}\|e^{-(T/2)H_\eta^s}g\|^2_{L^2(\omega)},\quad g\in L^2(\mathbb{R}^n),\ |\eta|>0. \tag{4.8}$$

On the other hand, by Proposition 4.2 we obtan that for all $r>0$ and $g\in L^2(\mathbb{R}^n)$

$$\|e^{-(T/2)H_\eta^s}g\|^2_{L^2(\mathbb{R}^n)}\le\frac{2C_{\mathrm{obs}}}{T}\int_0^{T/2}\|e^{-tH_\eta^s}g\|^2_{L^2(\omega)}\,\mathrm{d}t \tag{4.9}$$

where $C_{\mathrm{obs}}=C_{\mathrm{obs}}(\omega,T/2,|\eta|^2)$ is given by (4.5) with T replaced by $T/2$ and r replaced by $|\eta|^2$. Combining (4.8) and (4.9) we obtain that for all $r>0$ and $g\in L^2(\mathbb{R}^n)$

$$\begin{aligned}\|e^{-TH_\eta^s}g\|^2_{L^2(\mathbb{R}^n)}&\le\exp\left(-T\widetilde{\lambda}_0|\eta|^{\frac{4s}{\beta+2}}\right)\frac{2C_{\mathrm{obs}}}{T}\int_0^{T/2}\|e^{-tH_\eta^s}g\|^2_{L^2(\omega)}\,\mathrm{d}t\\&=\exp\left(-T\widetilde{\lambda}_0|\eta|^{\frac{4s}{\beta+2}}+K\log(1/\gamma)|\eta|\right)\exp\left(\frac{\log^{\frac{2s}{2s-1}}(1/\gamma)}{(T/2)^{\frac{1}{2s-1}}}\right)\frac{2}{T}\int_0^{T/2}\|e^{-tH_\eta^s}g\|^2_{L^2(\omega)}\,\mathrm{d}t.\end{aligned}$$

Now we consider it in two cases

(i) $\dfrac{4s}{\beta+2}>1$, i.e., $s>\dfrac{\beta+2}{4}$. Then for any fixed $T>0$, we have

$$\sup_{|\eta|>0}\exp\left(-T\widetilde{\lambda}_0|\eta|^{\frac{4s}{\beta+2}}+K\log(1/\gamma)|\eta|\right)<\infty.$$

(ii) $\dfrac{4s}{\beta+2}=1$, i.e., $s=\dfrac{\beta+2}{4}$. Then

$$\sup_{|\eta|>0}\exp\left(-T\widetilde{\lambda}_0|\eta|^{\frac{4s}{\beta+2}}+K\log(1/\gamma)|\eta|\right)<\infty\quad\Longleftrightarrow\quad T\ge T^*:=\frac{K\log(1/\gamma)}{\widetilde{\lambda}_0}.$$

Replace $K/\widetilde{\lambda}_0$ with new constant K and the proof for the operator H_G^s is finished. $\square$

Now we consider $y \in \mathbb{T}^m$ case. In this case, We also do the partial Fourier transformation with respect to $y \in \mathbb{T}^m$, the operator H_{G_p} is transformed to

$$H_k := -\Delta_x + |k|^2 V(x)$$

where $k \in \mathbb{Z}^m$ denotes the dual variable of $y \in \mathbb{T}^m$. This implies that $|k| \geq 1$ if $k \neq 0$. Similar to Proposition 4.2, we obtain an exact observability estimate for the semigroups generated by H_r^s with $r \geq 1$.

prp2.4e **Proposition 4.3.** *Let H_r be given in (3.31) with V satisfying Assumption A1 and $\sigma/\beta_1 + \beta_2/2\beta_1 < s$. Then there exists a constant $K > 0$ depending only on $n, \beta_1, \beta_2, \sigma, c_1, c_2$ such that for all $(1, \gamma, \sigma)$-distributed sets $\omega \subset \mathbb{R}^n$, $r \geq 1$, $T > 0$, and $g \in L^2(\mathbb{R}^n)$ we have*

$$\|e^{-TH_r^s} g\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{C_{\text{obs}}}{T} \int_0^T \|e^{-tH_r^s} g\|_{L^2(\omega)}^2 \, \mathrm{d}t \tag{4.10}$$

where the positive constant C_{obs} is given by

$$C_{\text{obs}} = K \left(\exp \left(K \log(1/\gamma) r^{\frac{1}{2}} + \left(\log(1/\gamma)/T^{\zeta} \right)^{\frac{1}{1-\zeta}} \right) \right), \quad \zeta = \frac{1}{s} \left(\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1} \right) < 1. \tag{4.11}$$

Proof. Different from Proposition 4.2, we only need $r \geq 1$ here. Then applying Corollary 3.6 (i) we obtain the spectral inequality for the operator H_r defined in (3.31) with $r \geq 1$

$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq C_1 \left(\frac{1}{\gamma} \right)^{C_2 \lambda^{\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1}} + r^{\frac{1}{2}}} \|\phi\|_{L^2(\omega)}^2, \quad \forall \phi \in \mathcal{E}_{H_r}(\lambda), \tag{4.12}$$

where C_1 and C_2 depend on $\beta_1, \beta_2, c_1, c_2$ and not on r. Then use (4.6) again we obtain the spectral inequality for the operator H_r^s with $r \geq 1$

$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq C_1 \left(\frac{1}{\gamma} \right)^{C_2 \lambda^{\zeta} + r^{\frac{1}{2}}} \|\phi\|_{L^2(\omega)}^2, \quad \forall \phi \in \mathcal{E}_{H_r^s}(\lambda)$$

where $\zeta = \frac{1}{s} \left(\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1} \right) < 1$.

Now we write the constant the same form as in (4.1), i.e.,

$$d_0 = C_1 e^{\log(\frac{1}{\gamma}) r^{\frac{1}{2}}}, \, d_1 = C_2 \log \left(\frac{1}{\gamma} \right), \, \zeta = \frac{1}{s} \left(\frac{\sigma}{\beta_1} + \frac{\beta_2}{2\beta_1} \right). \tag{4.13}$$

Then the same as in the proof of Proposition 4.2, we estimate the observability constant C_{obs} given in (4.2) and obtain the desired result. $\square$

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. The method is the same as in the proof of Theorem 1.1. The key difference is that Fourier series replaces the Fourier transform. Then we need to prove

$$\sum_{k \in \mathbb{Z}^m} \|e^{-TH_{k,\alpha}} g\|_{L^2(\mathbb{R}^n)}^2 \leq C_{\omega,T} \sum_{k \in \mathbb{Z}^m} \int_0^T \|e^{-tH_{k,\alpha}} g\|_{L^2(\omega)}^2 \, \mathrm{d}t. \tag{4.14}$$

It is sufficient to prove

$$\|e^{-TH_{k,\alpha}} g\|_{L^2(\mathbb{R}^n)}^2 \leq C_{\omega,T} \int_0^T \|e^{-tH_{k,\alpha}} g\|_{L^2(\omega)}^2 \, \mathrm{d}t \tag{4.15}$$

for all $k \in \mathbb{Z}^m$.

For $k \neq 0$, using the same strategy as the proof of Theorem 1.1 and equipped with Proposition 4.3 we obtan

$$
\begin{aligned}
\|e^{-TH_{k,\alpha}}g\|^2_{L^2(\mathbb{R}^n)} &\le \exp\left(-T\widetilde{\lambda}_0|k|^{\frac{2\alpha}{\beta_1+2}}\right)\frac{2C_{\mathrm{obs}}}{T}\int_0^{T/2}\|e^{-tH_{k,\alpha}}g\|^2_{L^2(\omega)}\,\mathrm{d}t \\
&= \exp\left(-T\widetilde{\lambda}_0|k|^{\frac{2\alpha}{\beta_1+2}} + K\log(1/\gamma)|k|^{\zeta}\right)\exp\left(\left(\frac{\log(1/\gamma)}{(T/2)^{\zeta}}\right)^{\frac{1}{1-\zeta}}\right)\frac{2}{T}\int_0^{T/2}\|e^{-tH_{k,\alpha}}g\|^2_{L^2(\omega)}\,\mathrm{d}t.
\end{aligned}
\tag{4.16}
$$

However, we also need to consider $k = 0$, which we can omit in $y \in \mathbb{R}^m$ case. This particular case is equivalent to the observability of heat equation

$$
\partial_t u(t,x) - \Delta_x u(t,x) = 0, \quad x \in \mathbb{R}^n. \tag{4.17}
$$

This implies the sensor sets ω must be (γ, L)-thick sets with $\gamma > 0$ and $L > 0$, see [WWZZ19, Theorem 1.1]. Hence it demands $\sigma = 0$, which illustrates appearance of this condition in Theorem 1.2.

Under the condition $s = 1$, $\sigma = 0$ and $\omega \subset \mathbb{R}^n$ being $(1, \gamma, 0)$-distributed, the rest of the proof and the more general case for $s \ge \frac{\beta_2}{2\beta_1}$ is the same as in Theorem 1.1, hence we do not repeat it here. □

Proof of Corollary 1.3. If we modify the operator H_G to H'_G and H_{G_p} to H'_{G_p}, then we have the lower bound $r \ge 1$ in rV after the partial Fourier transformation. Hence under this assumption, we only need to care about $r \ge 1$, i.e., $r \ge 1$ case in Corollary 3.6. Then Corollary 1.3 can be derived from Proposition 4.3 by repeating the same strategy. □

The method to prove Theorem 1.4 and Theorem 1.5 is exactly the same as in the proof of Theorem 1.1 and Theorem 1.2. We first need to prove the observability estimates like Proposition 4.2 and 4.3. The only change is from $r^{\frac{1}{2}}$ to $r^{\frac{2}{3}}$ in the exponent of the constant in the spectral inequalities (see the difference between Corollary 3.6 and Corollary 3.7). Except this difference, the proofs are same hence we omit here.

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