A NOTE ON LOGVINENKO-SEREDA THEOREM

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ABSTRACT. This note is an introduction of Logvinenko-Sereda Theorem.

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1. INTRODUCTION

Before we introduce the Logvinenko-Sereda Theorem, we first give some definitions.

Definition 1.1. Let Σ be a measurable set and $\mathscr{E}$ be a subspace of $L^2(\mathbb{R})$. A measurable set S is called $\mathscr{E}$-determining if there is a positive number c such that

$$f \in \mathscr{E} \Rightarrow c\|f\|_2^2 \leq \int_S |f|^2 \,\mathrm{d}x. \tag{1.1}$$

Furthermore, if S is $\mathscr{E}(\widehat{\Sigma})$-determining for any bounded $\Sigma \subset \mathbb{R}$, we say S is determining. Here $\mathscr{E}(\widehat{\Sigma})$ denotes the subspace of all functions with spectrum contained in Σ.

Definition 1.2. Let S be a measurable set on $\mathbb{R}$. We say S is γ-thick if there exists an interval $K = [-L, L]$ and a constant $\gamma > 0$ such that:

$$\forall h \in \mathbb{R} \implies |S \cap (K + h)| \geq \gamma.$$

We call S thick if the exact value of γ is not concerned.

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Now we can state the theorem as following:

Theorem 1.1. *Let S be a measurable set on $\mathbb{R}$. Then the following two assertions are equivalent:*

(a) *S is determining;*
(b) *S is thick.*

The proof of (a) $\Rightarrow$ (b) is much easier than the converse.
Proof of (a) $\Rightarrow$ (b). Given bounded set Σ, then $\mathscr{E}(\widehat{\Sigma})$ is a shift invariant subspace of $L^2(\mathbb{R})$. Hence we only need to prove: If $\mathscr{E}$ is a shift invariant non-trivial subspace of $L^2(\mathbb{R})$, then any $\mathscr{E}$-determining set is thick. Assume $f \in \mathscr{E}$ and define $\omega_f(\delta) := \sup\big\{\int_e |f|^2 \,\mathrm{d}x : |e| \le \delta\big\}$. Let $K = [-L, L]$ large enough so that

$$\int_{K'} |f|^2 \,\mathrm{d}x \le \frac{c}{2},$$

where c is the constant from (1.1). Set $f_h(x) := f(x-h), \forall h \in \mathbb{R}$ and we have

$$\int_{(K+h)'} |f_h|^2 \,\mathrm{d}x \le \frac{c}{2}.$$

Then

$$\begin{aligned} c = c\int |f_h|^2 \,\mathrm{d}x &\le \int_S |f_h|^2 \,\mathrm{d}x = \Big(\int_{S\cap(K+h)} |f_h|^2 \,\mathrm{d}x + \int_{S\cap(K+h)'} |f_h|^2 \,\mathrm{d}x\Big) \\ &\le \omega_f\big(|S\cap(K+h)|\big) + \frac{c}{2}. \end{aligned}$$

Then we obtain $\omega_f\big(|S\cap(K+h)|\big) \ge \frac{c}{2}$, and this implies $|S\cap(K+h)|$ is bounded off zero for any $h \in \mathbb{R}$. $\square$

Definition 1.3. The Poisson measure is defined by

$$\Pi := \frac{1}{\pi(1+x^2)} m$$

where m is the Lebesgue measure on $\mathbb{R}$.

Furthermore, we define the probability measure with respect to x

$$\Pi_x(A) := \Pi(x - A) = \frac{1}{\pi}\int_A \frac{1}{1+(x-t)^2}\,\mathrm{d}t, \quad \forall A \subset \mathbb{R}.$$

Then we have the following property:

Proposition 1.2. *Let S be a measurable set on $\mathbb{R}$. Then S is thick if and only if $\inf_{x\in\mathbb{R}} \Pi_x(S) > 0$.*

Proof. $\Rightarrow$: Assume S is γ-thick for some $\gamma > 0$ and $K = [-L, L]$ defined in Definition 1.2. Then for all $x \in \mathbb{R}$, we obtain

$$\begin{aligned}\Pi_x(S) &= \int_S \mathrm{d}\Pi_x \geq \int_{(K+x)\cap S} \mathrm{d}\Pi_x = \frac{1}{\pi}\int_{x-L}^{x+L} \chi_S(t)\frac{\mathrm{d}t}{1+(x-t)^2} \\ &= \frac{1}{\pi}\int_L^L \chi_S(t+x)\frac{\mathrm{d}t}{1+t^2} \geq \frac{1}{\pi}\left(1+L^2\right)^{-1}|S\cap(K+x)| \geq \pi^{-1}\left(1+L^2\right)^{-1}\gamma.\end{aligned}$$

$\Leftarrow$: Assume $\Pi_x(S) \geq \sigma > 0$ for all $x \in \mathbb{R}$. Let $K = [-L, L]$ and L is defined later. We have

$$\begin{aligned}\pi\sigma \leq& \pi\Pi_x(S) = \int_{\mathbb{R}} \chi_S(t)\frac{1}{1+(x-t)^2}\,\mathrm{d}t \\ =& \int_{x-L}^{x+L} \chi_S(t)\frac{1}{1+(x-t)^2}\,\mathrm{d}t + \int_{|x-t|>L} \chi_S(t)\frac{1}{1+(x-t)^2}\,\mathrm{d}t \\ \leq& |S\cap(K+x)| + 2\int_L^\infty \frac{1}{1+t^2}\,\mathrm{d}t \\ \leq& |S\cap(K+x)| + 2\left(\frac{\pi}{2} - \arctan L\right).\end{aligned}$$

Choose L large enough so that $\gamma = \pi\sigma - 2\left(\frac{\pi}{2} - \arctan L\right) > 0$, then we have $|S\cap(K+x)| \geq \gamma > 0$. $\square$

2. Jensen's Inequalities and Poisson Operator

Definition 2.1. We call a distribution $f \in S'(\mathbb{R})$ a plus-function (resp. a minus-function) if $\widehat{f}|_{(-\infty,0)} = 0$ (resp. $\widehat{f}|_{(0,\infty)} = 0$). A distribution $f \in S'(\mathbb{T})(= D'(\mathbb{T})$ is called a plus-function (resp. a minus-function) if $\widehat{f}|_{\mathbb{Z}\cap(-\infty,0)} = 0$ (resp. $\widehat{f}|_{\mathbb{Z}\cap[0,\infty)} = 0$).

Definition 2.2. The set of all plus-functions $f \in L^p(X, m)$ where $p \in [1, \infty)$ is denoted by the symbol $H^p(X)$ and is called the Hardy class (with the index p).

2.1. Jensen's Inequality for Plus-Functions on $\mathbb{T}$.

For Jensen's inequality, we are most familiar with

$$\exp\int \log|f|\,\mathrm{d}\mu \leq \int |f|\,\mathrm{d}\mu,$$

where μ is any probability measure and $f \in L^1(\mu)$. Now we introduce another Jensen's inequality:

$$f \in H^1(\mathbb{T}) \Rightarrow \log\left|\widehat{f}(0)\right| \leq \int_{\mathbb{T}} \log|f|\,\mathrm{d}m. \tag{$J_{\mathbb{T}}$}$$

Here m is the normalized Lebesgue measure on $\mathbb{T}$. Compaired with the trivial estimate $\left|\int f\,\mathrm{d}m\right| \leq \int |f|\,\mathrm{d}m$, ($J_{\mathbb{T}}$) is an essential refinement.

Let $C_+ := H^1(\mathbb{T}) \cap C(\mathbb{T})$. We first have the following lemma:

Lemma 2.1. *Any function in C_+ satisfies ($J_{\mathbb{T}}$).*

Proof. Take $f \in C_+$ and $\varepsilon > 0$. By density of trigonometric polynomials in $C(\mathbb{T})$, there is a real trigonometric polynomial t such that

$$t(\zeta) - \varepsilon < \log\left(|f(\zeta)| + \varepsilon\right) < t(\zeta) + \varepsilon, \quad \forall \zeta \in \mathbb{T}.$$

Put $s := t + i\widetilde{t}$, where $\widetilde{t} := \frac{1}{i}\sum_{k\in\mathbb{Z}} \operatorname{sgn} k \cdot \widehat{t}(k) z^k$ and $\widetilde{t}$ is also real. Then $s \in C_+$. Then we obtain

$$\left|f(\zeta)e^{-s(\zeta)}\right| = |f(\zeta)|e^{-t(\zeta)} < e^{\log(|f(\zeta)|+\varepsilon)-t(\zeta)} < e^{\varepsilon}, \quad \forall \zeta \in \mathbb{T}.$$

Hence

$$\left|\widehat{fe^{-s}}(0)\right| = \left|\int_{\mathbb{T}} fe^{-s}\,\mathrm{d}m\right| \le e^{\varepsilon}.$$

On the other hand, $\left|\widehat{fe^{-s}}(0)\right| = |\widehat{f}(0)|\left|e^{-\widehat{s}(0)}\right| = |\widehat{f}(0)|e^{-\widehat{t}(0)}$, hence

$$|\widehat{f}(0)| \le e^{\varepsilon+\widehat{t}(0)} = e^{\varepsilon+\int t\,\mathrm{d}m} \le e^{2\varepsilon+\int \log(|f|+\varepsilon)\,\mathrm{d}m}.$$

The right hand side tends to $\exp\int \log|f|\,\mathrm{d}m$ as $\varepsilon \to 0$, then we obtain ($J_{\mathbb{T}}$). ∎

Now we can give the proof of ($J_{\mathbb{T}}$) for any $f \in H^1(\mathbb{T})$.
Proof of ($J_{\mathbb{T}}$). If $f \in H^1(\mathbb{T})$ m then $f = \lim_{j\to\infty} f_j$ (in $L^1(\mathbb{T})$) for a sequence (f_j) of plus-polynomials. Thus, for any $\varepsilon > 0$,

$$\int_{\mathbb{T}} \log(|f| + \varepsilon)\,\mathrm{d}m = \lim_{j\to\infty} \int_{\mathbb{T}} \log(|f_j| + \varepsilon)\,\mathrm{d}m,$$

since $\left|\log(|f| + \varepsilon) - \log(|f_j| + \varepsilon)\right| \le \frac{1}{\varepsilon}\left||f| - |f_j|\right|$. Since ($J_{\mathbb{T}}$) is valid for $f \in C_+$, we obtain

$$\log|\widehat{f_j}(0)| \le \int_{\mathbb{T}} \log(|f_j| + \varepsilon)\,\mathrm{d}m,$$

and then passing to the limit we obtain

$$\log|\widehat{f}(0)| \le \int_{\mathbb{T}} \log(|f| + \varepsilon)\,\mathrm{d}m.$$

Now using the Monotone Convergence Theorem we obtain

$$\lim_{\varepsilon\to+0} \int_{\mathbb{T}} \log(|f| + \varepsilon)\,\mathrm{d}m = \int_{\mathbb{T}} \log|f|\,\mathrm{d}m.$$

∎

2.2. **Jensen's Inequality for Plus-Functions on $\mathbb{R}$.** We have an analogue of $(J_{\mathbb{T}})$ for plus-functions on $\mathbb{R}$:

$$f \in H^1(\mathbb{R}, \Pi) \Rightarrow \log\left|\int_{\mathbb{R}} f \,\mathrm{d}\Pi\right| \le \int_{\mathbb{R}} \log|f| \,\mathrm{d}\Pi. \tag{$J_{\mathbb{R}}$}$$

Here we define $H^p(\mathbb{R}, \Pi) := \{f \in L^p(\mathbb{R}, \Pi) : f \text{ is a plus-function}\}$ for $p \in [1, \infty)$.

Lemma 2.2. *Let $f \in H^1(\mathbb{R}, m)$, $\omega := \dfrac{x-i}{x+i}$. Then*

$$\int_{\mathbb{R}} f\omega^n \,\mathrm{d}\Pi = 0 \tag{2.1}$$

holds for all $n \in \mathbb{N}_+$

Lemma 2.3. *Let $f \in H^1(\mathbb{R}, \Pi)$. Then (2.1) still holds for all $n \in \mathbb{N}_+$.*

Proof of $(J_{\mathbb{R}})$. Suppose $f \in H^1(\mathbb{R}, \Pi)$. Define

$$F(e^{i\theta}) := f\left(-\cot\frac{\theta}{2}\right), \quad \theta \in (0, 2\pi).$$

The variables θ and $x = -\cot\frac{\theta}{2}$ are connected by the following equalities:

$$x = -i\frac{e^{i\theta}+1}{e^{i\theta}-1}, \quad \mathrm{d}\theta = \frac{2\mathrm{d}x}{1+x^2}.$$

Hence

$$\int_0^{2\pi} \left|F(e^{i\theta})\right| \mathrm{d}\theta = 2\int_{\mathbb{R}} |f(t)| \frac{1}{1+t^2} \,\mathrm{d}t < \infty.$$

Then by Lemma 2.3 we have

$$\widehat{F}(-n) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta})e^{in\theta} \,\mathrm{d}\theta = \int_{\mathbb{R}} f\omega^n \,\mathrm{d}\Pi = 0.$$

Thus $F \in H^1(\mathbb{T})$ and hence satisfies $(J_{\mathbb{T}})$, which is equivalent to $(J_{\mathbb{R}})$. $\square$

Remark 2.1. Indeed, the transformation $F(e^{i\theta}) = f\left(-\cot\dfrac{\theta}{2}\right)$ comes from conformal maping between the upper half plane and the disk:

$$\eta = -i\frac{z+1}{z-1}.$$

So $(J_{\mathbb{R}})$ and $(J_{\mathbb{T}})$ are essentially the same inequality.

2.3. **Poisson Operator.** We define an operator on $L^1(\mathbb{R},\Pi)$ by

$$P(f)(x) := \int_{\mathbb{R}} f \,\mathrm{d}\Pi_x = \int_{\mathbb{R}} f(x+t)\,\mathrm{d}\Pi(t), \quad x \in \mathbb{R}.$$

The operator P is called Poisson transformation.

If $f \in H^1(\mathbb{R},\Pi)$, then the same is true for the function $t \mapsto f(x+t)$. Hence we can rewrite the inequality in $(J_{\mathbb{R}})$ as

$$\log|P(f)(x)| \le P(\log|f|)(x). \tag{$J'_{\mathbb{R}}$}$$

If $f \ge 0$, then

$$\int_{\mathbb{R}} P(f)(x)\,\mathrm{d}x = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x+t)\,\mathrm{d}x\right] \mathrm{d}\Pi(t) = \int_{\mathbb{R}} f(x)\,\mathrm{d}x. \tag{2.2}$$

In particular, for any measurable $S \subset \mathbb{R}$, we have

$$\int_{\mathbb{R}} \left(\int_S f\,\mathrm{d}\Pi_x\right) \mathrm{d}x = \int_{\mathbb{R}} P(\chi_S f)(x)\,\mathrm{d}x = \int_S f(x)\,\mathrm{d}x. \tag{2.3}$$

Lemma 2.4. *Let $p \in [1,+\infty)$, $\gamma > 0$. Let $S \subset \mathbb{R}$ be a measurable set and $\Pi_x(S) \ge \gamma$ for all $x \in \mathbb{R}$ and a constant $\gamma > 0$. If $f \in H^p(\mathbb{R})$, then*

$$\int_{\mathbb{R}} |P(f)|^p\,\mathrm{d}x \le 2\left(\int_S |f|^p\,\mathrm{d}x\right)^{\gamma} \|f\|_p^{p(1-\gamma)}. \tag{2.4}$$

Proof. Fix $x \in \mathbb{R}$ and define

$$k := \Pi_x(S), k' := \Pi_x(S'),$$
$$\lambda(A) := k^{-1}\Pi_x(A\cap S), \lambda'(A) := (k')^{-1}\Pi_x(A \cap S'), \forall A \subset \mathbb{R}.$$

Then we obtain

$$\begin{aligned}
&p\log|P(f)(x)| \le pP(\log|f|)(x)\\
=&p\int_S \log|f|\,\mathrm{d}\Pi_x + p\int_{S'} \log|f|\,\mathrm{d}\Pi_x\\
\le&k\log\Big(\int_S |f|^p\,\mathrm{d}\lambda\Big) + k'\log\Big(\int_{S'} |f|^p\,\mathrm{d}\lambda\Big)\\
=&k\log\frac{1}{k} + k'\log\frac{1}{k'} + k\log\int_S |f|^p\,\mathrm{d}\Pi_x + k'\log\int_{S'} |f|^p\,\mathrm{d}\Pi_x\\
\le&\log 2 + \gamma\log\int_S |f|^p\,\mathrm{d}\Pi_x + (k-\gamma)\log\int_S |f|^p\,\mathrm{d}\Pi_x + k'\log\int_{S'} |f|^p\,\mathrm{d}\Pi_x\\
\le&\log 2 + \gamma\log\int_S |f|^p\,\mathrm{d}\Pi_x + (k-\gamma+k')\log\int_{\mathbb{R}} |f|^p\,\mathrm{d}\Pi_x\\
=&\log 2 + \gamma\log\int_S |f|^p\,\mathrm{d}\Pi_x + (1-\gamma)\log\int_{\mathbb{R}} |f|^p\,\mathrm{d}\Pi_x.
\end{aligned}$$

Hence we obtain

$$|P(f)(x)|^p \le 2\left(\int_S |f|^p \,\mathrm{d}\Pi_x\right)^{\gamma}\left(\int_{\mathbb{R}} |f|^p \,\mathrm{d}\Pi_x\right)^{1-\gamma}.$$

Integrating both sides together with (2.2) and (2.3) we obtain (2.4). □

3. End of the Proof

Let $f \in L^p(\mathbb{R})$, Poissson transformation $P(f)$ can be rewritten as convolution

$$P(f) = f * k,$$

where $k := \dfrac{1}{\pi(1+x^2)}$. Since $\widehat{k}(\xi) = e^{-|\xi|}/2\pi, \xi \in \mathbb{R}$, we have $\widehat{P(f)}(\xi) = e^{-|\xi|}\widehat{f}(\xi)$.

We introduce a lemma below, which is used in L^p-version ($p \in [1,\infty)$) of the Logvinenko-Sereda Theorem:

Lemma 3.1. *Let $p \in [1,\infty], f \in L^p(\mathbb{R})$, then $P(f) \in L^p(\mathbb{R})$ and*

$$\|P(f)\|_p \le \|f\|_p.$$

Proof. The case $p = 1$ or ∞ is simple, hence we suppose $1 < p < \infty$ and $q := p/(p-1)$. Then

$$|P(f)(x)|^p \le \left(\int (k(x-t))^{1/p}\,|f(t)|\,(k(x-t))^{1/q}\,\mathrm{d}t\right)^p$$

$$\le \left(\int k(x-t)|f(t)|^p\,\mathrm{d}t\right)\cdot\left(\int k(x-t)\,\mathrm{d}t\right)^{p/q} = P(|f|^p)(x).$$

Integrating this estimate with respect to x and use the equality $\int P(|f|^p) = \int |f|^p$, we obtain the desired result. □

Lemma 3.2. *If $f \in H^2(\mathbb{R})$ and $\operatorname{ess\,spec} f \subset (0,l)$, then*

$$\|f\|_2 \le e^l\|P(f)\|_2. \tag{3.1}$$

Proof. Since $f \in H^2(\mathbb{R})$, we have $\widehat{P(f)}(\xi) = e^{-\xi}\widehat{f}(\xi)$. Hence

$$|\widehat{f}(\xi)| = e^{\xi}|\widehat{P(f)}(\xi)| \le e^l|\widehat{P(f)}(\xi)|, \quad \xi \in \mathbb{R}. \tag{3.2}$$

Then use the Plancherel Theorem we obtain the desired inequality. □

Now we finish the proof of Theorem 1.1.

Proof of (b) $\Rightarrow$ (a). Suppose $f \in L^2(\mathbb{R})$ and $\operatorname{ess\,spec} f \subset (a,b), l = b-a$. We transform f into a plus-function by

$$\varphi := fe^{-iax}.$$

Then $\varphi \in H^2(\mathbb{R})$ and $\operatorname{ess\,spec}\varphi \subset (0, l)$, $|\varphi| \equiv |f|$. By Lemma 3.2,

$$\int_{\mathbb{R}} |f|^2 \, \mathrm{d}x = \int_{\mathbb{R}} |\varphi|^2 \, \mathrm{d}x \le e^{2l} \int_{\mathbb{R}} |P(\varphi)|^2. \tag{3.3}$$

Then by Proposition 1.2 we know $\Pi_x(S) \ge \sigma$ for any $x \in \mathbb{R}$, where $\sigma > 0$ does not depend on x. Hence we can apply Lemma 2.4 and obtain

$$\|P(\varphi)\|_2^2 \le 2 \left(\int_S |\varphi|^2 \right)^{\sigma} \|\varphi\|_2^{2(1-\sigma)} = 2 \left(\int_S |f|^2 \right)^{\sigma} \|f\|_2^{2(1-\sigma)}.$$

This estimate combined with (3.3) gives

$$\|f\|_2^{2\sigma} \le 2e^{2l} \left(\int_S |f|^2 \right)^{\sigma} \Rightarrow \|f\|_2^2 \le (2e^{2l})^{1/\sigma} \int_S |f|^2 \, \mathrm{d}x.$$

$\square$