F. L. Nazarov's paper
Local Estimates of Exponential Polynomials and Their Applications to Inequalities of Uncertainty Pinciple Type

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Abstract

This is a learning note about Nazarov's paper(see [1]).

Contents

Definition 1. *An exponential polynomial is*

$$p(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t} \quad (c_k \in \mathbb{C}, \lambda_k \in \mathbb{C}) \, .$$

The main purpose of the first part of the paper is to establish the following inequality

$$\sup_{t \in I} |p\,(t)| \leq \left(\frac{A\mu(I)}{\mu(E)} \right)^{n-1} \sup_{t \in E} |p(t)| \, , \tag{1}$$

where $I \subset \mathbb{R}$ is an interval, $E \subset I$ is a measurable set of positive Lebesgue measure and A is an absolute constant.

1 The Turan lemma: original form

The following lemma was derived by Turan (see [2]).

Theorem 1. *Let $z_1, \cdots, z_n$ be complex numbers, $|z_j| \geq 1, j = 1, \cdots, n$. Let*

$$b_1, \cdots, b_n \in \mathbb{C}, \quad S_m \stackrel{\text{def}}{=} \sum_{k=1}^{n} b_k z_k^m \quad (m \in \mathbb{Z}).$$

Then

$$|S_0| \leq n \left(\frac{2e(m+n-1)}{n} \right)^{n-1} \max_{k=m+1}^{m+n} |S_k| \leq \left(\frac{4e(m+n-1)}{n} \right)^{n-1} \max_{k=m+1}^{m+n} |S_k| \tag{2}$$

for all $m \in \mathbb{Z}_+$.

Proof. To prove the lemma, we need to construct a polynomial $q(z) = 1 + \sum_{k=1}^{n} \gamma_k z^{m+k}$ such that

(1) $q(z_j) = 0$ for each $= 1, \cdots, n$ and

(2) $\sum_{k=1}^{n} |\gamma_k| \leq n \left(\frac{2e(m+n-1)}{n} \right)^{n-1}$.

Let

$$q(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k} \right) \sigma_m(z),$$

where $\sigma_m(z)$ is the m-th partial sum of the series $\prod_{k=1}^{n} \left(1 - \frac{z}{z_k} \right)^{-1} = \sum_{k=0}^{\infty} \beta_k z^k$, i.e.

$$\sigma_m(z) = \sum_{k=1}^{m} \beta_k z^k.$$

By definition, we have

$$1 = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k} \right) \sum_{k=0}^{\infty} \beta_k z^k.$$

This identity implies that the s-th coefficient in the expansion of the right side depends only on $\beta_{s-n}, \cdots, \beta_s$. Hence the coefficients at the powers $z, z^2, \cdots, z^m$ of $q(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k} \right) \sigma_m(z)$ all vanish (since they only depend on $\sigma_m(z)$). Recalling the Taylor expansion

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \frac{(k+n-1)!}{k!(n-1)!} z^k,$$

hence we have (by using the condition $|z_j| \geq 1$ and assuming $|z| < 1$)

$$\left| \prod_{k=1}^{n} \left(1 - \frac{z}{z_k} \right)^{-1} \right| \leq (1 - |z|)^{-n} = \sum_{k=0}^{\infty} \frac{(k+n-1)!}{k!(n-1)!} |z|^k.$$

Thus, all coefficients of $\sigma_m(z)$ do not exceed[1]

$$\frac{(m+n-1)!}{m!(n-1)!} \le \left(\frac{e(m+n-1)}{n}\right)^{n-1}.$$

Then we get the extimates

$$|\gamma_k| \le \left(\frac{e(m+n-1)}{n}\right)^{n-1} \sum_{s=k}^{n} \binom{n}{s}$$

and

$$\sum_{k=1}^{n} |\gamma_k| = \frac{1}{2}\sum_{k=1}^{n} (|\gamma_k| + |\gamma_{n+1-k}|) \le 2^{n-1} n \left(\frac{e(m+n-1)}{n}\right)^{n-1}.$$

Now we've constructed the desired polynomial $q(z)$.

Since

$$\begin{aligned} S_0 =& b_1 + b_2 + \cdots + b_n \\ =& \sum_{j=1}^{n} b_j \cdot 1 \\ =& \sum_{j=1}^{n} \left(-\sum_{k=1}^{n} \gamma_k z_j^{m+k}\right) \\ =& -\sum_{k=1}^{n} \gamma_k S_{m+k}. \end{aligned} \tag{3}$$

Hence the estimates above and (3) complete the proof. □

Recalling the definition of an exponential polynomial

$$p(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t},$$

now let $t_k = t_0 + k\delta$, we have

$$p(t_k) = \sum_{j=1}^{n} c_j e^{i\lambda_j (t_0 + k\delta)} = \sum_{j=1}^{n} b_j \left(e^{i\lambda_j \delta}\right)^k = \sum_{j=1}^{n} b_j z_j^k,$$

where $z_j = e^{i\lambda_k \delta}$ and $b_j = c_j e^{i\lambda_j t_0}$. Then we can use the lemmma directly and get

$$|p(t_0)| \le \left\{\frac{4e(m+n-1)}{n}\right\}^{n-1} \max_{k=m+1}^{m+n} |p(t_k)|. \tag{4}$$

Now the inequality (1) for the case where E is an interval can be derived in an almost immediate way (with the constant A=4e).

Using the same idea in

[1]Here needs some estimates: we need to prove

$$\binom{n}{k} \le \left(\frac{en}{k+1}\right)^k.$$

This inequality can be proved by induction.

Theorem 2. *Let I be an interval, let $E \subset I$ be a measurable set of positive Lebesgue measure. Then*

$$\max_{t\in I}|p(t)| \leq 2^n \left(\frac{\mu(I)}{\mu(E)}\right)^{2n^2} \max_{t\in E}|p(t)|. \tag{5}$$

Proof. By (4), the following inequality

$$\max_{t\in I}|p(t)| \leq 2^n \max_{t\in E}|p(t)| \tag{6}$$

holds if t_0 is the first term of the arithmetic progression $t_k = t_0 + k\delta\,(k = 0, \cdots, n)$ with all other terms belonging to E. The point of the proof is to find a set E_1 that is "close" to E and we can choose a δ such that all t_k's belongs to E.

Step 1.Let $J \subset I$ is an open interval and

$$\mu\left(E\cap J\right) > \left(1-\frac{1}{2n}\right)\mu(J).$$

Let $t_0 \in J$ be any fixed point. Such a point t_0 splits the interval J into two subintervals J_- and J_+. At least one of them, let's say J_+ has the property

$$\mu\left(E\cap J_+\right) > \left(1-\frac{1}{2n}\right)\mu(J).$$

Let $\varphi(t) = \chi\,(t)$ be the characteristic function of $J_+ \setminus E$, then by applying the lattice averaging lemma we see that the average number of points $t_k = t_0 + k\delta(k \in \mathbb{N})$ belonging to $J_+\backslash E$ as δ runs over the interval $\left(\frac{\mu(J_+)}{2n}, \frac{\mu(J_+)}{n}\right)$ is (here we write $\frac{\mu(J_+)}{2n}$ as s)

$$\begin{aligned}
\frac{\int_s^{2s}\sum_{k\in\mathbb{Z}\backslash\{0\}}\varphi\left(k\delta\right)\mathrm{d}\delta}{\int_s^{2s}\mathrm{d}\delta} =&\frac{1}{s}\int_1^2\sum_{k\in Z\backslash\{0\}}\varphi(ks\frac{\delta}{s})s\mathrm{d}\left(\frac{\delta}{s}\right)\\
=&\int_1^2\sum_{k\in\mathbb{Z}\backslash\{0\}}\varphi(ksv)\mathrm{d}v\\
\leq&\frac{1}{s}\int_{\mathbb{R}}\varphi(t)\mathrm{d}t\\
=&\frac{2n}{\mu(J_+)}\mu\left(J_+\setminus E\right)\\
<&1.
\end{aligned} \tag{7}$$

Hence there exists a positive $\delta < \frac{\mu(J_+)}{n}$ such that none of the points $t_1, \cdots, t_n$ belongs to $J_+ \setminus E$. Since $k\delta < \frac{k\mu(J_+)}{n} \leq 1$ and t_0 is the endpoint of J_+, all these points lie in J_+ and, consequently, in E. Since the choise of $t_0 \in J$ is arbitrary, any points in J have the property that $t_k \in E$ for each $k = 1, \cdots, n$.

Step 2. Let $E_1 = \bigcup\left\{J : J \subset I \text{ is open}, \mu(E\cap J) > \left(1-\frac{1}{2n}\right)\mu(J)\right\}$. Since E_1 is the union of open sets, E_1 itself is also open, hence, the union of disjoint open intervals. Let Q be one constituent interval of E_1, if

$$\mu\left(E\cap Q\right) > \left(1-\frac{1}{2n}\right)\mu(Q)$$

holds, then we can find a larger open interval Q' such that $Q' \subset Q \subset E_1$, this contradicts the chosen of Q. Hence all the cons constituent intervals of E_1 satisfy the relation

$$\mu(E \cap Q) \leq \left(1 - \frac{1}{2n}\right) \mu(Q).$$

Thus, the set E_1 has the following two properties

$$\sup_{t \in E_1} |p(t)| \leq 2^n \sup_{t \in E} |p(t)|, \tag{8}$$

$$\mu(E_1) \geq \left(1 - \frac{1}{2n}\right)^{-1} \mu(E) \geq e^{\frac{1}{2n}} \mu(E) \text{ or } E_1 = I.^{2} \tag{9}$$

Step 3. Iterating this procedure we obtain a sequence of sets $E_1, E_2, \cdots$ such that

$$\sup_{t \in E_k} |p(t)| \leq 2^{nk} \sup_{t \in E} |p(t)|, \tag{10}$$

$$\mu(E_k) \geq e^{\frac{k}{2n}} \mu(E) \text{ or } E_k = I. \tag{11}$$

If $k > 2n \log \frac{\mu(I)}{\mu(E)}$, then the first case of (11) cannot occur. Therefore we obtain

$$E_{\left[2n \log \frac{\mu(I)}{\mu(E)} + 1\right]} = I,$$

whence

$$\sup_{t \in I} |p(t)| \leq 2^{\left(2n \log \frac{\mu(I)}{\mu(E)} + 1\right) n} \sup_{t \in E} |p(t)| \leq 2^n \left(\frac{\mu(I)}{\mu(E)}\right)^{2n^2} \sup_{t \in E} |p(t)|.$$

$\square$

Remark. The proof of Theorem 2 is based on Theorem 1. We can regard Theorem 1 is a discrete version of Theorem 2. From the discrete version to Lebesgue measurable sets, the simplest thought is to find the discrete points which Theorem 1 can be used to. If there exists, then our problem can be solved easily. But unfortunately the arithmetic progression t_k may not exists in E for any point in I. To overcome this difficulty, we need to find an interval close to E (here the sense of "close" has exact meaning in the proof), and any point fixed t_0 in this interval satisfy the condition $t_k \in E$ for each $k = 1, \cdots, n$. Finally, by iterating the procedure, the chosen set becomes strictly larger, and finaly equals to I.

2 Two Usefull Lemmas

Lemma 1. *If $P(z)$ is an algebraic polynomial of degree n, then*

$$\mu\left(\left\{x \in \mathbb{R} : \left|\frac{\mathrm{d}}{\mathrm{d}x} \log P(x)\right| > y\right\}\right) \leq \frac{8n}{y}$$

and

$$\mu\left(\left\{z \in \mathbb{T} : \left|\frac{\mathrm{d}}{\mathrm{d}z} \log P(z)\right| > y\right\}\right) \leq \frac{8n}{\pi y}.$$

Lemma 2 (Langer Lemma). *Let $p(z) = \sum_{k=1}^{n} c_k e^{i\lambda_k z} (0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda)$ be an exponential polynomial not vanishing identically. Then the number of complex zeros of $p(z)$ in an open vertical strip $x_0 < Rez < x_0 + \Delta$ of width Δ does not exceed $(n-1) + \frac{\lambda\Delta}{2\pi}$.*

[2]Here we use the inequality $\frac{1}{e} \geq (1 - \frac{1}{2n})^{2n}$.

3 The Turan lemma for polynomials on the unit circumference

Here we shall prove inequality (1) for the case of a 1-periordic exponential polynomial $p(t) = \sum_{k=1}^{n} c_k e^{2\pi i m_k t}$, where $c_k \in \mathbb{C}$, $m_1 < \cdots m_n \in \mathbb{Z}$, and for the segment $I = [0,1]$.

Theorem 3. *Let $p(z) = \sum_{k=1}^{n} c_k z^{m_k}$ $(c_k \in \mathbb{C}, m_1 < \cdots m_n \in \mathbb{Z})$ be a trignometric plynomial on the unit circumference T, and let E be a measurable subset of $\mathbb{T}$. Then*

$$\|p\|_W \stackrel{\text{def}}{=} \sum_{k=1}^{n} |c_k| \leq \left(\frac{16e}{\pi}\frac{1}{\mu(E)}\right)^{n-1} \sup_{z\in E}|p(z)| \leq \left(\frac{14}{\mu(E)}\right)^{n-1} \sup_{z\in E}|p(z)|. \tag{12}$$

Proof.

Step 1. We shall construct by induction a sequence of polynonials $p_n, p_{n-1}, \cdots, p_1$ such that

(1) $p_n = p$;

(2) $\operatorname{ord} p_k = k$ $(k = 1, \cdots, n)$;

(3) $\|p_{k-1}\|_W \geq \frac{\pi}{16}\|p_k\|_W$;

(4) the ratio $\varphi_k \stackrel{\text{def}}{=} \left|\frac{p_{k-1}}{p_k}\right|$ admits the weak type estimate $\mu\left(\{z \in \mathbb{T} : \varphi_k(z) > t\} \leq \frac{1}{t}\right)$ for all $t > 0$.

The construction is as follows. Let $p_n = p$. The polynomial $p_k(z) = \sum_{s=1}^{k} d_s z^{r_s}$ $(r_1 < r_2 < \cdots < r_k \in \mathbb{Z}$ being chosen, we introduce two polynomials

$$\underline{q} \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-r_1} p_k(z)\right)$$

and

$$\overline{q} \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-r_k} p_k(z)\right).$$

Obviously, $\operatorname{ord}\underline{q} = \operatorname{ord}\overline{q} = k-1$. We have

$$\|\underline{q}\|_W = \sum_{s=1}^{k} |d_s|(r_s - r_1), \quad \|\overline{q}\|_W = \sum_{s=1}^{k} |d_s|(r_k - r_s),$$

whence

$$\|\underline{q}\|_W + \|\overline{q}\|_W = (r_k - r_1)\sum_{s=1}^{k}|d_s| = r\|p_k\|_W,$$

where $r \stackrel{\text{def}}{=} r_k - r_1$. Hence at least one of the norms larger than or equal to $\frac{r}{2}\|p_k\|_W$. We assume $\|\overline{q}\|_W \geq \frac{r}{2}\|p_k\|_W$ (the other case is similar). Put $p_{k-1}(z) = \frac{\pi}{8r}\overline{q}(z)$, then conditions (2) and (3) are satisfied. It remains to check condition (4). Since $r_1 < r_2 < \cdots < r_k \in \mathbb{Z}$, let $g(\frac{1}{z}) = z^{-r_k}p_k(z)$, then $g(z)$ is an algebraic polynomial of degree r. Then

$$\overline{q}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-r_k}p_k(z)\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(g\left(\frac{1}{z}\right)\right) = -\frac{1}{z^2}g'\left(\frac{1}{z}\right).$$

Since $g\left(\frac{1}{z}\right)$ is an algebrail polynomial of degree r, we can use Lemma 1 and get [3]

$$\mu\left(\{z\in\mathbb{T}:\varphi_k(z)>t\}\right)=\mu\left(\{z\in\mathbb{T}:\left|\frac{g'(1/z)}{g(1/z)}\right|>\frac{8r}{\pi}t\}\right)\le\frac{1}{t}$$

since

$$\left|\frac{p_{k-1}}{p_k}=\frac{\pi}{8r}\frac{\overline{q}(z)}{p_k}\right|=\left|\frac{\pi}{8r}\frac{g'(1/z)(-1/z^2)}{g(1/z)z^{r_k}}\right|=\frac{\pi}{8r}\left|\frac{g'(1/z)}{g(1/z)}\right|.$$

The above inequality also explains how the weird coefficient $\frac{\pi}{16}$ of condition (3) chooses.

Step 2. Before proving the theorem, we first illustrate what the step 2 does. By step 1, we have constructed a sequence of polynomials and they have the relation

$$\|p_{k-1}\|_W\ge\frac{\pi}{16}\|p_k\|_W.$$

Hence we can get

$$\left(\frac{\pi}{16}\right)^{n-1}\|p\|_W\le\|p_1\|_W.$$

Since $\operatorname{ord}p_1=1$, the norm of p_1 is equivalent to any $|p_1(z)|$. We want to get the inequality (12), that means we may need to establish the inequality between $|p_1(z)|$ and $|p(z)|$ for $z\in E$. More precisely, we want to find some point $z_0\in E$ such that

$$\left|\frac{p_1(z_0)}{p(z_0)}\right|<\text{ some large number.}\tag{13}$$

The constant can be chosen large enough so that the measure of points which don't satisfy condition (13) is less than $\mu(E)$, hence cannot cover all points of E, i.e., the point $z_0\in E$ satisfies the condition exists.

Now we estimate the measure of the set of all points $z\in\mathbb{T}$ for which $|p_1(z)|$ is essentially greater than $|p_n(z)|=|p(z)|$ (the meaning of "essentially greater" would be clear later). We have

$$\left|\frac{p_1(z)}{p_n(z)}\right|=\prod_{k=2}^{n}\varphi_k(z)\le\exp\left(\sum_{k=2}^{n}\psi_k(z)\right),$$

where $\psi_k(z)\overset{\text{def}}{=}\log_+\varphi_k(z)$ ($\log_+x$ means $\log_+x=0$ if $\log x<0$). The weak type esimate of φ_k gives the inequality

$$\mu\left(\psi_k>t\right)\le e^{-t}$$

for all $t>0$. Let $\alpha>0$, we decompose $\psi_k(z)$ into the sum of $\eta_k(z)\overset{\text{def}}{=}\min\left(\psi_k(z),\alpha\right)$ and $\omega_k(z)\overset{\text{def}}{=}\psi_k(z)-\eta_k(z)$. Then $\sum_{k=2}^{n}\eta_k(z)\le\alpha(n-1)$ for all $z\in\mathbb{T}$. Since for a nonnegative measurable function in measure space $(X,\mathcal{M},\mu)$ we have $\int f(x)\mathrm{d}\mu(x)=\int_0^\infty\mu(f(x)>t)\mathrm{d}t$, we obtain

$$\int_{\mathbb{T}}\omega_k(z)\mathrm{d}\mu(z)=\int_\alpha^\infty\mu(\psi_k>t)\mathrm{d}t\le\int_\alpha^\infty e^{-t}\mathrm{d}t=e^{-\alpha}.$$

Therefore,

$$\int_{\mathbb{T}}\left(\sum_{k=2}^{n}\omega_k(z)\right)\mathrm{d}\mu(z)\le e^{-\alpha}(n-1).\tag{14}$$

[3]In Lemma 1, the term $\left|\frac{P'(z)}{P(z)}\right|$ can be changed into $\left|\frac{P'(1/z)}{P(1/z)}\right|$ since the substitution $z\mapsto 1/z$ preserves Lebesgue measure on the unit circumference.

Since

$$\sum_{k=2}^{n} \omega_k(z) = \sum_{k=2}^{n} \psi_k(z) - \sum_{k=2}^{n} \eta_k(z)$$

and $\sum_{k=2}^{n} \eta_k(z) \leq \alpha(n-1)$, we have

$$\mu\left(\left\{z \in \mathbb{T} : \sum_{k=2}^{n} \psi_k(z) > (\alpha+1)(n-1)\right\}\right) \leq \mu\left(\left\{z \in \mathbb{T} : \sum_{k=2}^{n} \omega_k(z) > n-1\right\}\right).$$

Let $F \stackrel{\text{def}}{=} \{z \in \mathbb{T} : \sum_{k=2}^{n} \omega_k(z) > n-1\}$, then we have

$$\mu(F) < \frac{1}{n-1} \int_F \sum_{k=2}^{n} \omega_k(z) \mathrm{d}\mu(z) \leq e^{-\alpha}$$

by using (14). Hence

$$\mu\left(\left\{z \in \mathbb{T} : \sum_{k=2}^{n} \psi_k(z) > (\alpha+1)(n-1)\right\}\right) < e^{-\alpha}. \tag{15}$$

Let $\alpha = \log \frac{1}{\mu(E)}$, then $e^{-\alpha} = \mu(E)$. Substitute this into (refexists) then this inequality implies that there exists a point $z_0 \in E$ for which $\sum_{k=2}^{n} \psi_k(z_0) \leq (\alpha+1)(n-1)$. Now we have

$$\begin{aligned}
\left(\frac{\pi}{16}\right)^{n-1} \|p\|_W \leq & \|p_1\|_W \stackrel{(\operatorname{ord} p_1 = 1!)}{=} \|p_1(z_0)\| \\
\leq & \exp\left(\left(1 + \log \frac{1}{\mu(E)}\right)(n-1)\right) |p(z_0)| \\
= & \left(\frac{e}{\mu(E)}\right)^{n-1} |p(z_0)| \\
\leq & \left(\frac{e}{\mu(E)}\right)^{n-1} \sup_{z \in E} |p(z)|,
\end{aligned}$$

and the theorem is proved. □

Remark. We first construct the polynomial sequence $p_n = p, p_{n-1}, \cdots, p_2, p_1$, and they satisfy $\|p_{k-1}\|_W \geq \frac{\pi}{16}\|p_k\|_W$, $\operatorname{ord} p_k = k$ and so on. Then we can get

$$\left(\frac{\pi}{16}\right)^{n-1} \|p\|_W \leq \|p_1\|_W.$$

This means we transform the question into the proof of the certain inequality between $\|p_1\|_W = |p_1(z)| \forall z \in \mathbb{T}$ and $p = p_n$. Then we need to find a point $z_0 \in \mathbb{T}$ such that $|p_1(z_0)| \leq \exp\left(\left(1 + \log \frac{1}{\mu(E)}\right)(n-1)\right) |p(z_0)|$, this step needs to estimate the amount or measure of the points that have large function values. If the measure of these points are smaller tham $\mu(E)$, then we can get a point $z_0 \in E$ that satisfies the condition.

4 The Turan lemma in general form

Theorem 4. *Let* $p(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$ *where* $c_k \in \mathbb{C}$ *and* $\lambda_1 < \cdots \lambda_n \in \mathbb{R}$. *If* E *is a measurable subset of the segment* $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, *then*

$$\sup_{t\in I} |p(t)| \leq \left(\frac{316}{\mu(E)}\right)^{n-1} \sup_{t\in E} |p(t)| .$$

Lemma 3. *Let* $g(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$, $(c_k \in \mathbb{C},\ 0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda)$. *If* $\lambda \geq n-1$, *then*

$$\mu\left(\left\{t \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \left|\frac{\mathrm{d}}{\mathrm{d}t} \log g(t)\right| > y\right\}\right) \leq \frac{29\lambda}{y}$$

for all $y > 0$.

Proof. We procedd like we did in Case 1. $\square$

Proof of Theorem 4. Let $\lambda \stackrel{\text{def}}{=} \lambda_n - \lambda_1$, we prove the theorem separately in two cases.

Case $\lambda \leq n-1$. If $n = 1$, the statement is obvious. Let $n > 1$, without loss of generality, we assume that $0 = \lambda_1 < \cdots \lambda_n = \lambda_n = \lambda \leq n-1$. By virtue of the Langer lemma, complex zeros of the exponential polynomial $p(z)$ are well separated, i.e., each verical strip of width Δ contains at most $\frac{\Delta\lambda}{2\pi} + (n-1) \leq \left(1 + \frac{\Delta}{2\pi}\right)(n-1)$ zeros.

Lets enumerate z_j in the order of increase of $|\mathrm{Re} z_j|$. For every $j \in \mathbb{N}$, the inequality $|\mathrm{Re} z_j| \geq \pi \frac{j-(n-1)}{(n-1)}$ holds.

Case $\lambda > n-1$. We shall reduce this case to Case 1 in the same way as in Section 3. This is why we need Lemma 3. We can finish the proof by constructing a sequence of exponential polynomials $p_n, p_{n-1}, \cdots, p_s (s \geq 1)$ such that

(1) $p_n = p$;

(2) $\operatorname{ord} p_k = k\ (k = s, \cdots, n)$;

(3) $\|p_{k-1}\|_\infty \geq \frac{1}{58} \|p_k\|_\infty\ (k = s+1, \cdots, n)$;

(4) the ratio $\varphi_k \stackrel{\text{def}}{=} \left|\frac{p_{k-1}}{p_k}\right|$ satisfies the weak type estimate $\mu\left(\left\{x \in \left[-\frac{1}{2}, \frac{1}{2} : \varphi_k(x) > t\right]\right\}\right) \leq \frac{1}{t}$ for $t > 0$;

(5) the difference between the greatest and the smallest exponent of p_s does not exceed $s-1$ (i.e., p_s meets the condition of Case 1 investigated above).

The construction is almost the same as in Section 3. The difference is that, firstly, we make use of the identity $\underline{q}(t) - \overline{q}(t) = i\left(\rho_k - \rho_1\right) p_k(t)$, where

$$p_k(t) \stackrel{\text{def}}{=} \sum_{m=1}^{k} d_m e^{i\rho_m t} \quad (\rho_1 < \cdots \rho_k \in \mathbb{R}) ,$$

$$\underline{q}(t) \stackrel{\text{def}}{=} e^{i\rho_1 t} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-i\rho_1 t} p_k(t)\right) ,$$

$$\overline{q}(t) \stackrel{\text{def}}{=} e^{i\rho_k t} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-i\rho_k t} p_k(t)\right)$$

to estimate the sum of norms $\|\underline{q}\|_\infty + \|\overline{q}\|_\infty$ from below, and, secondly, we stop the sequence at the polynomial p_s satisfying the condition of Case 1, i.e., at that very moment when we cannot apply Lemma 3 to estimate φ_s once more.

Since $\|p_{k-1}\|_\infty \geq \left(\frac{1}{58}\right)\|p_k\|_\infty$, we obtain

$$\left(\frac{1}{58}\right)^{n-s}\|p\|_\infty \leq \|p_s\|_\infty. \tag{16}$$

By the construction procedure, p_s satisfies the condition of Case 1, hance for a measurable set F we have

$$\|p_s\|_\infty \leq \left(\frac{154}{\mu(F)}\right)^{s-1}\sup_{t\in F}|p_s(t)|\,. \tag{17}$$

Now we use the same reasoning as in Section 3 to establish $\left|\frac{p_s(t)}{p_n(t)}\right| \leq \left(\frac{2e}{\mu(E)}\right)^{n-s}$ outside an exceptional set E' of measure $\mu(E') \leq \mu\left(E\right)/2$. We have

$$\left|\frac{p_s(x)}{p_n(x)}\right| = \prod_{k=s+1}^{n}\varphi_k(z) \leq \exp\left(\sum_{k=s+1}^{n}\psi_k(x)\right),$$

where $\psi_k(x) \stackrel{\text{def}}{=} \log_+\varphi_k(x)$. The weak type estimate of φ_k gives the inequality $\mu\left(\psi_k > t\right) \leq e^{-t}$ for all $t > 0$. Let $\alpha > 0$, we decompose $\psi_k(x)$ into the sum of $\eta_k(x) \stackrel{\text{def}}{=} \min\left(\psi_k(x), \alpha\right)$ and $\omega_k(x) \stackrel{\text{def}}{=} \psi_k(x) - \eta_k(x)$. Then $\sum_{k=s+1}^{n}\eta_k(x) \leq \alpha(n-s)$ for all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We also have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\omega_k(x)\mathrm{d}x = \int_{\alpha}^{\infty}\mu\left(\psi_k > t\right)\mathrm{d}t \leq \int_{\alpha}^{\infty}e^{-t}\mathrm{d}t = e^{-\alpha}.$$

Therefore,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sum_{k=s+1}^{n}\omega_k(z)\right)\mathrm{d}\mu(z) \leq e^{-\alpha}(n-s).$$

Since

$$\sum_{k=s+1}^{n}\omega_k(x) = \sum_{k=s+1}^{n}\psi_k(x) - \sum_{k=s+1}^{n}\eta_k(x)$$

and $\sum_{k=s+1}^{n}\eta_k(x) \leq \alpha(n-s)$, we have

$$\begin{aligned}&\mu\left(\left\{x \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \sum_{k=s+1}^{n}\psi_k(x) > (\alpha+1)(n-s)\right\}\right)\\ \leq &\mu\left(\left\{x \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \sum_{k=s+1}^{n}\omega_k(x) > n-s\right\}\right) < e^{-\alpha}.\end{aligned}$$

Let $\alpha = \log\left(\frac{2}{\mu(E)}\right)$, then we have

$$\mu\left(\left\{x \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \sum_{k=s+1}^{n}\psi_k(x) > (\alpha+1)(n-s)\right\}\right) < \frac{\mu(E)}{2}.$$

Thus the measure of the set $E' = \left\{ x \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \left|\frac{p_s(x)}{p_n(x)}\right| > \left(\frac{2e}{\mu(E)}\right)^{n-s} \right\}$ satisfies

$$\mu(E') < \frac{\mu(E)}{2}$$

and hence

$$\mu(E \backslash E') \geq \frac{\mu(E)}{2}. \tag{18}$$

By definition of the set E', we know $\left|\frac{p_s(x)}{p_n(x)}\right| \leq \left(\frac{2e}{\mu(E)}\right)^{n-s}$ for each $x \in E\backslash E'$. By using (17) (let $F = E\backslash E'$), (16) and (18) we obtain

$$\begin{aligned} \left(\frac{1}{58}\right)^{n-s} \|p\|_\infty \leq & \|p_s\|_\infty \leq \left(\frac{154}{\mu\left(E\backslash E'\right)}\right)^{s-1} \sup_{t\in E\backslash E'} |p_s(t)| \\ & \leq \left(\frac{308}{\mu(E)}\right)^{s-1} \left(\frac{2e}{\mu(E)}\right)^{n-s} \sup_{t\in E} |p(t)| \,. \end{aligned}$$

Now Theorem (4) easily follows if we take into account the inequality $116e < 316$. $\square$

5 Summary: Two important techniques used

a. Construct a sequence of polynomials like $p_k, p_{k-1}, \cdots, p_1$ to decrease the order of p_k. In this note, the order is the $\operatorname{ord} p_k$ of exponential polynomials, it may have different meaning when we solve other problems.

b. Weak type estiamtes allow us to get an upper bound of a measure of a set A that satisfies some property P, then compaire it to the measure of a given set B . If the latter is strictly larger than the former, then there must be some point in B which does not meet the property P.

References

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