SPECTRAL INEQUALITY WITH EXPLICIT DEPENDENCE ON THE SCALE

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Abstract. In this note, under more restrictive assumption of the potential, we give the spectral inequality with explicit dependence on the scale $l > 0$ in the region Ω who satisfies some density conditions.

Contents

1. Assumption and notation

In this note, we always assume the potential

$$V(x) = c|x|^{\beta}, \quad x \in \mathbb{R}^n \tag{1}$$

and $\beta > 1$ for simplicity.

In [1], it has the following general assumption: Assume $V \in L^{\infty}_{\text{loc}}(\mathbb{R}^n)$ satisfies the following two conditions:

(i) There exist positive constant c_1 and β_1 such that for all $x \in \mathbb{R}^n$,

$$c_1 \left(|x| - 1\right)^{\beta_1} \leq V(x), \tag{2}$$

(ii) We can write $V = V_1 + V_2$ such that there exist positive constants c_2 and β_2 such that

$$|V_1(x)| + |DV_1(x)| + |V_2(x)|^{\frac{4}{3}} \leq c_2 \left(|x| + 1\right)^{\beta_2}. \tag{3}$$

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Given $L > 0$, we denote

$$\Lambda_L := \left[-\frac{L}{2}, \frac{L}{2}\right]^n.$$

Denote $\mathcal{B}_r(x) \subset \mathbb{R}^n$ be a ball with radius r and center x. Denote $\mathbb{B}_r(x)$ be a ball in $\mathbb{R}^{n+1}$.

There two kinds of sensor sets Ω which we should be careful about:

(i) let $l > 0, \gamma \in (0,1)$ and $\sigma > 0$, the set Ω is said to be (l, γ, σ)-distributed if there exists a set of points $\{z_k : k \in \mathbb{Z}^n\}$ such that

$$\Omega \cap (lk + \Lambda_l) \supset \mathcal{B}_{\gamma^{1+|k|^\sigma} l}(z_k) \tag{Type 1}$$

(ii) sometimes the density condition (Type 1) can be modified to the following form

$$\Omega \cap (lk + \Lambda_l) \supset \mathcal{B}_{\gamma^{1+|lk|^\sigma} l}(z_k). \tag{Type 2}$$

In this note, we always use the type 1 density condition, i.e., the sensor set Ω who satisfies (Type 1).

2. Scaling

We consider the solutions of

$$-\Delta v + Vv = 0, \text{ in } \mathbb{R}^{n+1}. \tag{4}$$

Define the scaling function

$$g(x) = lx, \quad x \in \mathbb{R}^{n+1}.$$

Then for any solution v of (4) we define

$$\widetilde{v}(x) := (v \circ g_{n+1})(x) = v(lx). \tag{5}$$

Then

$$\begin{aligned} -(\Delta v) \circ g + (Vv) \circ g &= 0, \\ -\frac{1}{l^2}\Delta(v \circ g) + (V \circ g)(v \circ g) &= 0, \end{aligned}$$

$$-\Delta \widetilde{v} + \widetilde{V}\widetilde{v} = 0. \tag{6}$$

here we denote $\widetilde{V}(x) = l^2 V(lx)$. Then

$$\widetilde{V}(x) = cl^{2+\beta}|x|^\beta \tag{7}$$

by assumption.

Let $\delta \in (0, \frac{1}{2})$, $b = (0, \cdots, 0, -b_{n+1})$ and $b_{n+1} = \frac{\delta}{100}$. Define

$$W_1 = \left\{ y \in \mathbb{R}_+^{n+1} \middle| |y - b| \le \frac{1}{4}\delta \right\},$$
$$W_2 = \left\{ y \in \mathbb{R}_+^{n+1} \middle| |y - b| \le \frac{1}{2}\delta \right\},$$
$$W_3 = \left\{ y \in \mathbb{R}_+^{n+1} \middle| |y - b| \le \frac{2}{3}\delta \right\}.$$

Then $W_1 \subset W_2 \subset W_3 \subset \mathbb{B}_\delta$. Define

$$W_j(z_i) := (z_i, 0) + W_j, \quad j = 1, 2, 3,$$

with $Q_L := \Lambda_L \cap \mathbb{Z}^n$, and

$$P_j(L) = \bigcup_{i \in Q_L} W_j(z_i) \text{ and } D_\delta(L) = \bigcup_{i \in Q_L} \mathcal{B}_\delta(z_i).$$

Lemma 1. *Let $\delta \in (0, \frac{1}{2})$. Let v be the solution of (4) with $v(y) = 0$ on the hyperplane $\{y \,|y_{n+1} = 0\}$. There exist $0 < \alpha < 1$ and $C > 0$, depending only on n such that*

$$\|v\|_{H^1(P_1(L))} \le \delta^{-\alpha} \exp\left(C\left(1 + \mathcal{G}(V_1, V_2, 9\sqrt{n}L)\right)\right) \|v\|^{\alpha}_{H^1(P_3(L))} \|\frac{\partial v}{\partial y_{n+1}}\|^{1-\alpha}_{L^2(D_\delta(L))}, \tag{8}$$

where

$$\mathcal{G}(V_1, V_2, L) = \|V_1\|^{\frac{1}{2}}_{W^{1,\infty}(\Lambda_L)} + \|V_2\|^{\frac{2}{3}}_{L^\infty(\Lambda_L)}. \tag{9}$$

Since $\widetilde{v}$ satisfy (9), we substitute $\widetilde{v}$ into (8) and get

$$\|\widetilde{v}\|_{H^1(P_1(L))} \le \delta^{-\alpha} \exp\left(C\left(1 + \mathcal{G}(\widetilde{V}_1, \widetilde{V}_2, 9\sqrt{n}L\right)\right) \|\widetilde{v}\|^{\alpha}_{H^1(P_3(L))} \|\frac{\partial \widetilde{v}}{\partial y_{n+1}}\|^{1-\alpha}_{L^2(D_\delta(L))} \tag{10}$$

with

$$\mathcal{G}(\widetilde{V}_1, \widetilde{V}_2, 9\sqrt{n}L) = \|\widetilde{V}_1\|^{\frac{1}{2}}_{W^{1,\infty}(\Lambda_{9\sqrt{n}L})} + \|\widetilde{V}_2\|^{\frac{2}{3}}_{\Lambda_{9\sqrt{n}L}}. \tag{11}$$

Now we compute each term in (10):

$$
\begin{aligned}
\|\widetilde{v}\|^2_{H^1(P_1(L))} &= \int_{P_1(L)} |\widetilde{v}|^2 \,\mathrm{d}x + \int_{P_1(L)} |D_x\widetilde{v}|^2 \,\mathrm{d}x \\
&= \int_{P_1(L)} |v(lx)|^2 \,\mathrm{d}x + \int_{P_1(L)} |l(D_x v)(lx)|^2 \,\mathrm{d}x \\
&= \frac{1}{l^{n+1}} \int_{lP_1(L)} |v(x)|^2 \,\mathrm{d}x + \frac{1}{l^{n-1}} \int_{lP_1(L)} |D_x v|^2 \,\mathrm{d}x \\
&\boxed{= \frac{1}{l^{n+1}} \|v\|^2_{L^2(lP_1(L))} + \frac{1}{l^{n-1}} \|Dv\|^2_{L^2(lP_1(L))}},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\|\widetilde{v}\|^2_{H^1(P_3(L))} &= \frac{1}{l^{n+1}} \int_{lP_1(L)} |v(x)|^2 \,\mathrm{d}x + \frac{1}{l^{n-1}} \int_{lP_3(L)} |D_x v|^2 \,\mathrm{d}x \\
&\boxed{= \frac{1}{l^{n+1}} \|v\|^2_{L^2(lP_1(L))} + \frac{1}{l^{n-1}} \|Dv\|^2_{L^2(lP_3(L))}},
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\|\frac{\partial \widetilde{v}}{\partial y_{n+1}}\|^2_{L^2(D_\delta(L))} &= \int_{D_\delta(L)} \left|\frac{\partial \widetilde{v}}{\partial y_{n+1}}\right|^2 \,\mathrm{d}x \\
&= \int_{D_\delta(L)} l^2 \left|\frac{\partial v}{\partial y_{n+1}}(lx)\right|^2 \,\mathrm{d}x \\
&= \frac{1}{l^{n-1}} \int_{lD_\delta(L)} \left|\frac{\partial v}{\partial y_{n+1}}(x)\right|^2 \,\mathrm{d}x \\
&\boxed{= \frac{1}{l^{n-1}} \|\frac{\partial v}{\partial y_{n+1}}\|^2_{L^2(lD_\delta(L))}}.
\end{aligned} \tag{14}
$$

Similarly, we have

$$
\|\widetilde{V}_1\|_{W^{1,\infty}\left(\Lambda_{9\sqrt{n}L}\right)} = \|V_1\|_{L^\infty(\Lambda_{9\sqrt{n}lL})} + l\|DV_1\|_{L^\infty\left(\Lambda_{9\sqrt{n}lL}\right)} \tag{15}
$$

and

$$
\|\widetilde{V}_2\|_{L^\infty(\Lambda_{9\sqrt{n}L})} = \|V_2\|_{L^\infty\left(\Lambda_{9\sqrt{n}lL}\right)}. \tag{16}
$$

Let $\delta \in (0, \frac{1}{2})$ and

$$R_1 = \frac{1}{16}\delta \text{ and } r_1 = \frac{1}{32}\delta,$$
$$R_2 = 3\sqrt{n} \text{ and } r_2 = \frac{1}{2},$$
$$R_3 = 9\sqrt{n} \text{ and } r_3 = 6\sqrt{n}.$$

Choose $R = 2R_3 = 18\sqrt{n}$. Define

$$X_1 = \Lambda_L \times [-1, 1] \text{ and } \widetilde{X}_{R_3} = \Lambda_{L+R_3} \times [-R_3, R_3]. \tag{17}$$

Lemma 2. *Let $\delta \in (0, \frac{1}{2})$. Let v be the solution of (4) which is odd with repect to y_{n+1}. There exist $C > 0$ depending only on n, $0 < \alpha < 1$ depending on δ and n such that*

$$\|v\|_{H^1(X_1)} \leq \delta^{-2\alpha_1} \exp\left(C\left(1 + \mathcal{G}\left(V_1, V_2, 9\sqrt{n}L\right)\right)\right) \|v\|^{1-\alpha_1}_{H^1\left(\widetilde{X}_{R_3}\right)} \|v\|^{\alpha_1}_{H^1(P_1(L))}, \tag{18}$$

where $\mathcal{G}(V_1, V_2, L)$ is given by (11).

To explicitly denote α_1, we define $\psi(\widehat{r}) = e^{-s\widehat{r}}$ and then

$$\kappa_1 = \log\frac{96\sqrt{n}}{\delta} \text{ and } \kappa_2 = \psi\left(\frac{r_0}{3}\right) - \psi\left(\frac{r_0}{6}\right). \tag{19}$$

Now we have

$$0 < \alpha_1 = \frac{|\kappa_2|}{|\log\delta| + \log(96\sqrt{n}) + |\kappa_2|} < 1. \tag{20}$$

The same as in Lemma 1, substitute $\widetilde{v}$ into (18) and get

$$\|\widetilde{v}\|_{H^1(X_1)} \leq \delta^{-2\alpha_1} \exp\left(C\left(1 + \mathcal{G}\left(\widetilde{V}_1, \widetilde{V}_2, 9\sqrt{n}L\right)\right)\right) \|\widetilde{v}\|^{1-\alpha_1}_{H^1\left(\widetilde{X}_{R_3}\right)} \|\widetilde{v}\|^{\alpha_1}_{H^1(P_1(L))}. \tag{21}$$

Now we compute each term in (21):

$$\boxed{\|\widetilde{v}\|^2_{H^1(X_1)} = \frac{1}{l^{n+1}}\|v\|^2_{L^2(lX_1)} + \frac{1}{l^{n-1}}\|Dv\|^2_{L^2(lX_1)},} \tag{22}$$

$$\boxed{\|\widetilde{v}\|^2_{H^1\left(\widetilde{X}_{R_3}\right)} = \frac{1}{l^{n+1}}\|v\|^2_{L^2(l\widetilde{X}_{R_3})} + \frac{1}{l^{n-1}}\|v\|^2_{L^2\left(l\widetilde{X}_{R_3}\right)},} \tag{23}$$

and

$$\boxed{\|\widetilde{v}\|^2_{H^1(P_1(L))} = \frac{1}{l^{n+1}}\|v\|^2_{L^2(lP_1(L))} + \frac{1}{l^{n-1}}\|v\|^2_{L^2(lP_1(L))}.} \tag{24}$$

3. Ghost dimension

We denote $H = -\Delta + V$ and let $\phi \in \mathcal{E}_\lambda(H)$ satisfying

$$\phi = \sum_{\lambda_k \le \lambda} \alpha_k \phi_k. \tag{25}$$

Define

$$\Phi(x, x_{n+1}) = \sum_{0<\alpha_k \le \lambda} \alpha_k \phi_k(x) \frac{\sinh\left(\sqrt{\lambda_k} x_{n+1}\right)}{\sqrt{\lambda_k}}. \tag{26}$$

Then $\Phi(x, x_{n+1})$ satisfies (4).

Proposition 3. *There exists a constant $\widehat{C}$ depending on $\beta_1, c_1, \beta_2, c_2$ such that for all $\lambda \ge \lambda_1$ and $\phi \in \mathcal{E}_\lambda(H)$, we have*

$$\|\phi\|^2_{H^1\left(\mathbb{R}^n \setminus \mathcal{B}_{\widehat{C}\lambda^{1/\beta_1}}\right)} \le \frac{1}{2} \|\phi\|_{L^2(\mathbb{R}^n)}. \tag{27}$$

According to this proposition, we have

$$\|\Phi\|^2_{H^1(\mathbb{R}^n \times (-l,l))} \le 2\|\Phi\|^2_{H^1\left(\mathcal{B}_{\widehat{C}\lambda^{1/\beta_1}} \times (-l,l)\right)}. \tag{28}$$

Lemma 4. *Let $\phi \in \mathcal{E}_\lambda(H)$ and Φ be given in (26). For any $\rho > 0$, we have*

$$2\rho \|\phi\|^2_{L^2(\mathbb{R}^n)} \le \|\Phi\|^2_{H^1(\mathbb{R}^n \times (-\rho,\rho))} \le 2\rho \left(1 + \frac{\rho^2}{3}(1+\lambda)\right) e^{2\rho\sqrt{\lambda}} \|\phi\|^2_{L^2(\mathbb{R}^n)}. \tag{29}$$

4. Large scale

In this section, we try to prove the spectral inequality in the large scale, i.e., $l \ge 1$. From (12) and (13), we obtain

$$\|\widetilde{v}\|^2_{H^1(P_1(L))} \ge \frac{1}{l^{n+1}} \|v\|^2_{H^1(lP_1(L))} \tag{30}$$

and

$$\|\widetilde{v}\|^2_{H^1(P_3(L))} \le \frac{1}{l^{n-1}} \|v\|^2_{H^1(P_3(L))}. \tag{31}$$

Substituting (30), (31) and (14) into (10), we obtain

$$\|v\|_{H^1(lP_1(L))} \le l\delta^{-\alpha} \exp\left(C\left(1 + \mathcal{G}\left(\widetilde{V}_1, \widetilde{V}_2, 9\sqrt{n}L\right)\right)\right) \|v\|^\alpha_{H^1(lP_3(L))} \|\frac{\partial v}{\partial y_{n+1}}\|^{1-\alpha}_{L^2(lD_\delta(L))}. \tag{32}$$

Here we need to esitmate $\mathcal{G}\left(\widetilde{V}_1, \widetilde{V}_2, 9\sqrt{n}L\right)$, by (15) and (16) we obtain

$$\mathcal{G}\left(\widetilde{V}_1, \widetilde{V}_2, 9\sqrt{n}L\right) \le l^{\frac{1}{2}} \left(\|V_1\|^{\frac{1}{2}}_{W^{1,\infty}\left(\Lambda_{9\sqrt{n}lL}\right)} + \|V_2\|^{\frac{2}{3}}_{L^\infty\left(\Lambda_{9\sqrt{n}lL}\right)}\right). \tag{33}$$

Similarly, we can derive the following from (21)

$$\|v\|_{H^1(lX_1)} \leq l\delta^{-2\alpha_1} \exp\left(C\left(1+\mathcal{G}\left(\widetilde{V}_1, \widetilde{V}_2, 9\sqrt{n}L\right)\right)\right) \|v\|^{1-\alpha_1}_{H^1\left(l\widetilde{X}_{R_3}\right)} \|v\|^{\alpha_1}_{H^1(lP_1(L))}. \tag{34}$$

We assume the potential V satisfy (1), $V_1 = V$ and $V_2 = 0$.

Proof of the case $l > 1$. Let $lL = 2\left\lceil \widehat{C}\lambda^{1/\beta}\right\rceil + 1$, then it is easy to see that $\mathcal{B}_{\widehat{C}\lambda^{1/\beta_1}} \subset \Lambda_{lL} = l\Lambda_L$. Now we decompose Λ_{lL} as

$$\Lambda_{lL} = \bigcup_{k\in\Lambda_L\cap\mathbb{Z}^n} \left(lk + \left[-\frac{l}{2}, \frac{l}{2}\right]\right). \tag{35}$$

Observe that $l|k| \leq \sqrt{n}\left\lceil \widehat{C}\lambda^{1/\beta}\right\rceil$ for each $k \in \Lambda_L \cap \mathbb{Z}^n$. Let $\gamma \in (0, \frac{1}{2})$ be as in (Type 1) and

$$\delta := \gamma^{1+\left(\frac{1}{l}\sqrt{n}\left\lceil \widehat{C}\lambda^{\frac{1}{\beta}}\right\rceil\right)^{\sigma}} \leq \gamma^{1+|k|^{\sigma}} \text{ for all } k \in \Lambda_L \cap \mathbb{Z}^n. \tag{36}$$

Now we show an interpolation inequality using (32, 33, 34)

(37)

$$\begin{aligned}
\|\Phi\|_{H^1(lX_1)} \leq& \delta^{-2\alpha_1} \exp\left(C\left(1+l^{\frac{1}{2}}\|V\|^{\frac{1}{2}}_{W^{1,\infty}\left(\Lambda_{9\sqrt{n}lL}\right)}\right)\right) \|\Phi\|^{1-\alpha_1}_{H^1\left(l\widetilde{X}_{R_3}\right)} \|\Phi\|^{\alpha_1}_{H^1(lP_1(L))} \\
\leq& \delta^{-2\alpha_1-\alpha\alpha_1} \exp\left(C\left(1+l^{\frac{1}{2}}\|V\|^{\frac{1}{2}}_{W^{1,\infty}(\Lambda_{9\sqrt{n}lL})}\right)\right) \|\Phi\|^{\alpha\alpha_1}_{H^1(lP_3(L))} \left\|\frac{\partial\Phi}{\partial y_{n+1}}\right\|^{\alpha_1(1-\alpha)}_{L^2(lD_\delta(L))} \|\Phi\|^{1-\alpha_1}_{H^1\left(l\widetilde{X}_{R_3}\right)} \\
\leq& \delta^{-3\alpha_1} \exp\left(C\left(1+l^{\frac{1}{2}}\|V\|^{\frac{1}{2}}_{W^{1,\infty}\left(\Lambda_{9\sqrt{n}lL}\right)}\right)\right) \|\phi\|^{\widehat{\alpha}}_{L^2(lD_\delta(L))} \|\Phi\|^{1-\widehat{\alpha}}_{H^1\left(l\widetilde{X}_{R_3}\right)}
\end{aligned}$$

where $\widehat{\alpha} = \alpha_1(1-\alpha)$ and we have used the facts $lP_3(L) \subset l\widetilde{X}_{R_3}$ and $\frac{\partial\Phi}{\partial y_{n+1}}(\cdot, 0) = \phi$, and absorbed the polynomial factor l by taking C large enough (which is not dependent on l). Since $\alpha_1 \approx \widehat{\alpha} \approx \frac{1}{|\log\delta|}$ for any $\delta \in (0, \frac{1}{2})$ and then $\delta^{-3\alpha_1} \leq C$, we obtain

$$\|\Phi\|_{H^1(lX_1)} \leq \exp\left(C\left(1+l^{\frac{1}{2}}\|V\|^{\frac{1}{2}}_{W^{1,\infty}\left(\Lambda_{9\sqrt{n}lL}\right)}\right)\right) \|\phi\|^{\widehat{\alpha}}_{L^2(\Omega\cap\Lambda_{lL})} \|\Phi\|^{1-\widehat{\alpha}}_{H^1\left(l\widetilde{X}_{R_3}\right)} \tag{38}$$

where we have also used the fact $lD_\delta(L) \subset \Omega \cap \Lambda_{lL}$.

Remember we assume $V = c|x|^\beta$, hence we have

$$C\left(1+l^{\frac{1}{2}}\|V\|^{\frac{1}{2}}_{W^{1,\infty}\left(\Lambda_{9\sqrt{n}lL}\right)}\right) \leq C\left(1+l^{\frac{1}{2}}\left(9\sqrt{n}lL\right)^{\frac{\beta}{2}}\right) \leq C_* l^{\frac{1}{2}}\lambda^{\frac{1}{2}}. \tag{39}$$

If we do not assume $V = c|x|^\beta$ and just use the general assumption, then the above estimate should be derived from (33) and becomes

$$\mathcal{G}\left(\widetilde{V}_1, \widetilde{V}_2, 9\sqrt{n}L\right) \leq C_* l^{\frac{1}{2}} \lambda^{\frac{\beta_2}{\beta_1}}. \tag{40}$$

But the constant C_* now depends on l also. The reason is we do not give the explicit dependence relation of $\widehat{C}$ with respect to $c_1, c_2, \beta_1, \beta_2$.

Then we have

$$\|\Phi\|_{H^1(lX_1)} \leq \exp\left(C_* l^{\frac{1}{2}} \lambda^{\frac{1}{2}}\right) \|\phi\|^{\widehat{\alpha}}_{L^2(\Omega\cap\Lambda_{lL})} \|\Phi\|^{1-\widehat{\alpha}}_{H^1\left(l\widetilde{X}_{R_3}\right)}. \tag{41}$$

Applying $\rho = lR_3$ and $\rho = l$ in Lemma 4 for upper and lower bounds, respectively, we obtain

$$\frac{\|\Phi\|^2_{H^1(\mathbb{R}^n\times(-lR_3,lR_3))}}{\|\Phi\|^2_{H^1(\mathbb{R}^n\times(-l,l))}} \leq \frac{2lR_3\left(1+\frac{(lR_3)^2}{3}(1+\lambda)\right)\exp(2lR_3\sqrt{\lambda})}{2l} \leq \exp\left(C_2 l\sqrt{\lambda}\right). \tag{42}$$

With the aid of (28) and the fact $\mathcal{B}_{\widehat{C}\lambda^{1/\beta}} \subset \Lambda_{lL}$, we get

$$\|\Phi\|_{H^1(\mathbb{R}^n\times(-lR_3,lR_3))} \leq \exp\left(\frac{1}{2}C_2\sqrt{\lambda}\right)\|\Phi\|_{H^1(\mathbb{R}^n\times(-l,l))} \leq \sqrt{2}\exp\left(\frac{1}{2}C_2\sqrt{\lambda}\right)\|\Phi\|_{H^1(\Lambda_{lL}\times(-l,l))}. \tag{43}$$

Note that $lX_1 = \Lambda_{lL}\times(-l,l)$, by (41) we obtain

$$\|\Phi\|_{H^1(\mathbb{R}^n\times(-lR_3,lR_3))} \leq \exp\left(C_3 l^{\frac{1}{2}}\lambda^{\frac{1}{2}}\right)\|\phi\|^{\widehat{\alpha}}_{L^2(\Omega\cap\Lambda_{lL})}\|\Phi\|^{1-\widehat{\alpha}}_{H^1\left(l\widetilde{X}_{R_3}\right)}. \tag{44}$$

Since $l\widetilde{X}_{R_3} \subset \mathbb{R}^n\times(-lR_3,lR_3)$, it follows that

$$\|\Phi\|_{\mathbb{R}^n\times(-lR_3,lR_3)} \leq \exp\left(\widehat{\alpha}^{-1}C_3 l^{\frac{1}{2}}\lambda^{\frac{1}{2}}\right). \tag{45}$$

Recall $\widehat{\alpha}^{-1} \approx \alpha_1^{-1} \approx |\log\delta| \approx \frac{1}{l^\sigma}|\log\gamma|\lambda^{\frac{\sigma}{\beta}}$. It follows that

$$\|\Phi\|_{H^1(\mathbb{R}^n\times(-lR_3,lR_3))} \leq \left(\frac{1}{\gamma}\right)^{Cl^{\frac{1}{2}}\lambda^{\frac{\sigma}{\beta}+\frac{1}{2}}}\|\phi\|_{L^2(\Omega\cap\Lambda_{lL})}. \tag{46}$$

Finally, using the lower bound in Lemma 4 with $\rho = lR_3$, we obtain

$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{1}{2lR_3}\right)^{\frac{1}{2}}\|\Phi\|_{H^1(\mathbb{R}^n\times(-lR_3,lR_3))} \leq \left(\frac{1}{\gamma}\right)^{Cl^{\frac{1}{2}}\lambda^{\frac{\sigma}{\beta}+\frac{1}{2}}}\|\phi\|_{L^2(\Omega\cap\Lambda_{lL})}. \tag{47}$$

This completes the proof of case $l \geq 1$. $\square$

5. Small scale

In this section, we try to prove the spectral inequality in the small scale, i.e., $l < 1$.

From (12) and (13), we obtain

$$\|\widetilde{v}\|^2_{H^1(P_1(L))} \geq \frac{1}{l^{n-1}}\|v\|^2_{H^1(lP_1(L))} \tag{48}$$

and

$$\|\widetilde{v}\|^2_{H^1(P_3(L))} \leq \frac{1}{l^{n+1}}\|v\|^2_{H^1(lP_3(L))}. \tag{49}$$

Substituting (48), (49) and (14) into (10), we obtain

$$\|v\|_{H^1(lP_1(L))} \leq \frac{1}{l^{1-\alpha}}\delta^{-\alpha}\exp\left(C\left(1+\mathcal{G}\left(\widetilde{V}_1,\widetilde{V}_2,9\sqrt{n}L\right)\right)\right)\|v\|^{\alpha}_{H^1(lP_3(L))}\|\frac{\partial v}{\partial y_{n+1}}\|^{1-\alpha}_{L^2(lD_\delta(L))}. \tag{50}$$

Similarly, we obtain the variance of (21)

$$\|\widetilde{v}\|_{H_1(lX_1)} \leq \frac{1}{l}\cdots . \tag{51}$$

Remark 1. These two inequalities with the polynomial factors of $\frac{1}{l}$ in the cost constants are bad for our final purpose. The reason is that we choose the density condition (Type 1). This implies that we can only consider the Generalized Grushin operator under for $y \in \mathbb{T}^m$ case.

The same situation appears if we use the density condition (Type 2). In my opinion, it is due to the cutoff property which Carleman estimates used. More precisely, if $l \to 0$, then the cutoff function must be sharper and leads to a large deviation, hence the carleman estimates fails.

Hence for $y \in \mathbb{R}^m$ case, we must change the method of the proof.

References

[1] Jiuyi Zhu and Jinping Zhuge. Spectral inequality for schrödinger equations with power growth potentials. *arXiv preprint arXiv:2301.12338*, 2023.