JENSEN'S INEQUALITY ON POLYDISCS

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Given a polydisc $\mathbb{D}^n$, its distinguished boundary is $\mathbb{T}^n$.

Definition 0.1. We say a u.s.c. function $-\infty \leq f < \infty$ defined in $\mathbb{D}^n$ is n-subharmonic if f is subharmonc in each variable separately.

Using the subharmonicity successively in each variable, we obtain

$$
\begin{aligned}
f(z_1, z_2) &\leq \int_{\mathbb{T}} P(z_1, w_1) f(w_1, z_2) \, \mathrm{d}m_1(w_1) \\
&\leq \int_{\mathbb{T}} P(z_1, w_1) \left(\int_{\mathbb{T}} P(z_2, w_2) f(w_1, w_2) \, \mathrm{d}m_2(w_2) \right) \mathrm{d}m_1(w_1) \\
&= \int_{\mathbb{T}^2} P^{(2)}(z, w) f(w) \, \mathrm{d}m.
\end{aligned} \tag{0.1}
$$

Here $P(z,w)$ is the poisson kernel in $\mathbb{D}$ and $P^{(2)}$ is the Poisson kernel in $\mathbb{D}^2$ w.r.t. the distinguished boudary $\mathbb{T}^2$. Thus in general, we have the inequality in $\mathbb{D}^n$

$$
f(z) \leq \int_{\mathbb{T}^n} P(z, w) f(w) \, \mathrm{d}m \tag{0.2}
$$

for any n-subharmonic function $f \in \mathbb{D}^n$.

Now, given a holomorphic function $f \in \mathbb{D}^n$, since $\log |f|$ is n-subharmonic, we obtain

$$
\log |f(z)| \leq \int_{\mathbb{T}^n} P(z, w) \log |f(w)| \, \mathrm{d}m. \tag{0.3}
$$

Choose $z = 0$ and observe that $f(0) = \int_{\mathbb{T}^n} f(w) \, \mathrm{d}m$ we obtain

$$
\log \left| \int_{\mathbb{T}^n} f(w) \, \mathrm{d}m \right| \leq \int_{\mathbb{T}^n} \log |f(w)| \, \mathrm{d}m. \tag{0.4}
$$

To transform the integral from $\mathbb{T}^n$ to $\mathbb{R}^n$, we need to do the transform from $\mathbb{D}^n$ to $\mathbb{H}^n$. Define $H^1(\mathbb{R}, m)$ the class of functions $f \in L^1(\mathbb{R}, m)$ and $\operatorname{supp} \widehat{f} \subset (0, \infty)$.

In the following, we define

$$
\mathrm{d}\Pi = \frac{1}{\pi(1 + x^2)} \, \mathrm{d}x
$$

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in $\mathbb{R}$ and the higher dimension is just the product of one dimension and uses the same notation.

Lemma 0.1. *Let $f \in H^1(\mathbb{R}, m)$, $\omega := \frac{x-i}{x+i}$. Then*

$$\int_{\mathbb{R}} f\omega^n \, \mathrm{d}\Pi = 0 \tag{0.5}$$

holds for all $n \in \mathbb{N}_+$.

Lemma 0.2. *Let $f \in H^1(\mathbb{R}, \Pi)$. Then (0.5) still holds for all $n \in \mathbb{N}_+$.*

Now we can transform the polydisc case to $\mathbb{H}^n$ (here $\mathbb{H}^n := \mathbb{H} \times \cdots \times \mathbb{H}$) case. Define

$$F(e^{i\theta}) := F(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n}) = f\left(-\cot\frac{\theta_1}{2}, -\cot\frac{\theta_2}{2}, \cdots, -\cot\frac{\theta_n}{2}\right)$$

for all $\theta_i \in (0, 2\pi), i = 1, 2, \cdots, n$. The variable θ_i and $x_i = -\cot\frac{\theta_i}{2}$ are connected by the following formula

$$x_i = -i\frac{e^{i\theta_i}+1}{e^{i\theta_i}-1}, \qquad \mathrm{d}\theta_i = \frac{2\,\mathrm{d}x_i}{1+x_i^2}.$$

Then by the preceding lemma we obtain

$$\widehat{F}(-n_i) = \frac{1}{2\pi}\int_{\mathbb{T}} F(e^{i\theta_i})e^{in\theta_i}\,\mathrm{d}\theta_i = \int_{\mathbb{R}} f\left(x_1, x_2, \cdots, x_i\right)\left(\frac{x_i - i}{x_i + i}\right)^n \mathrm{d}\Pi_i = 0.$$

This implies that F is holomorphic in $\mathbb{D}^n$. Hence we have

$$\log\left|\int_{\mathbb{T}^n} F(w)\,\mathrm{d}m\right| \le \int_{\mathbb{T}^n} \log|F(w)|\,\mathrm{d}m.$$

This is equivalent to

$$\log\left|\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\Pi\right| \le \int_{\mathbb{R}^n} \log|f(x)|\,\mathrm{d}\Pi. \tag{0.6}$$

In general , I think this proof is very rigid and cannot be modified, because it is relied on conformal transformation and hence the relation (0.5). The reason we first consider in polydiscs not in $\mathbb{H}^n$ directly is that Poisson-Jensen's Formula is always established in bounded domain. Though in principle, we can give the assumption that the formula can be established in unbounded domain directly, for example in $\mathbb{H}^n$ domain. However, we finally need to use the property of subharmonicity of $\log|f|$ which f is holomorphic.