LOGVINENKO-SEREDA THEOREM ON HIGHER DIMENSIONS

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Contents

1. Poisson kernel for higher dimensions

Consider the steady-state heat equation in the upper half-space $\left\{x \in \mathbb{R}^d, y > 0\right\}$

$$\sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y} = 0$$

with the Dirichlet boundary condition $u(x, 0) = f(x)$. A solutioon to this problem is given by the convolution

$$u(x, y) = \left(f * P_y^{(d)}(x)\right) \tag{1.1}$$

where $P_y^{(d)}(x)$ is the d-dimensional Poisson kernel

$$P_y^{(d)}(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} e^{-2\pi|\xi| y} \,\mathrm{d}\xi. \tag{1.2}$$

By using the subordination principle and the d-dimensional heat kernel, we obtain

$$P_y^{(d)}(x) = \frac{\Gamma\left(\left(d+1\right)/2\right)}{\pi^{(d+1)/2}} \frac{y}{\left(|x|^2 + y^2\right)^{(d+1)/2}}. \tag{1.3}$$

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2. Poisson-Jensen Formula in general form

Definition 2.1. The function $g(x, \xi, D)$ is said to be a Green's function of x with respect to the bounded domain D in $\mathbb{R}^m$ and the point ξ of D, if

(i) g is a harmonic function of x in D except at the point $x = \xi$;
(ii) g is continuous in $\overline{D}$ except at $x = \xi$ and $g = 0$ on the boundary of D;
(iii) $g + \log|x - \xi|$ remains harmonic at $x = \xi$ if $m = 2$, $g - |x - \xi|^{2-m}$ remains harmonic at $x = \xi$ if $m > 2$..

Theorem 2.1. *Suppose that D is a bounded regular domain in $\mathbb{R}^m$ with boundary ∂D. Then for every x in D and Borel set e on ∂D there exists a number $\omega(x, e)$ with the following properties*

(i) *For fixed $x \in D$, $\omega(x, e)$ is a Borel measure on ∂D and $\omega(x, \partial D) = 1$.*
(ii) *For fixed $e \subset \partial D$, $\omega(x, e)$ is a harmonic function of x in D.*
(iii) *If $f(\xi)$ is a semi-continuous function defined on ∂D then*

$$u(x) = \int_{\partial D} f(\xi)\, \mathrm{d}\omega(x, e_\xi) \tag{2.1}$$

is the harmonic extension of $f(\xi)$ to D. The measure $\omega(x, e) = \omega(x, e, D)$ will be called the harmonic measure of e at x with respect to D.

Theorem 2.2. *Suppose that D is a bounded regular domain in $\mathbb{R}^m$ whose boundary ∂D has zero m-dimensional Lebesgue measure, and that $u(x)$ is s.h. and not identically $-\infty$ on $D \cup \partial D$. Then we have for $x \in D$*

$$u(x) = \int_{\partial D} u(\xi)\, \mathrm{d}\omega(x, e_\xi) - \int_D g(x, \xi, D)\, \mathrm{d}\mu e_\xi, \tag{2.2}$$

where $\omega(x, e)$ is the harmonic measure of e at x, $g(x, \xi, D)$ is the Green's function of D and $\mathrm{d}\mu$ is the Riesz measure of u in D.

In particular, we can set D as the upper half-space

$$\mathbb{H}^{d+1} := \left\{x \in \mathbb{R}^d, y > 0\right\}.$$

The boundary is

$$\partial\mathbb{H}^{d+1} = \left\{x \in \mathbb{R}^d, y = 0\right\} = \mathbb{R}^d.$$

And the harmonic measure $P_y^{(d)} := \omega(x, y, e_\xi)$ is

$$P_y^{(d)}(x) = c_d \frac{y}{\left(|x|^2 + y^2\right)^{(d+1)/2}} \tag{2.3}$$

with $c_d = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$.

3. Logvinenko-Sereda Theorem in higher dimension spaces

Theorem 3.1 (Logvinenko-Sereda). *Given any function $f \in L^2\left(\mathbb{R}^d\right), d \geq 2$ and $\operatorname{spec} f \subset [-R, R]^d$, then we have*

$$\int_{\mathbb{R}^d} |f|^2 \,\mathrm{d}x \leq C \int_S |f|^2 \,\mathrm{d}x \tag{3.1}$$

where S is a γ-thick set and C depends only on γ, R and d.

Proof. From now on, we always consider functions f of the space

$$H^2(\mathbb{R}^d) := \left\{ f(x) := \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} F(\xi)\,\mathrm{d}\xi : \operatorname{supp} F(\xi) \subset [0,\infty)^d \text{ and } \|F\|_{L^2(\mathbb{R}^d)} < \infty \right\}.$$

To prove the Logvinenko-Sereda Theorem in $\mathbb{R}^d, d \geq 2$, we can view the function f as a multivariable holomorphic function $f(z_1, z_2, \cdots, z_d)$. For example, in 2-dimensional case, we consider function $f(z_1, z_2)$ in domain $\mathbb{H} \times \mathbb{H}$, or a function in domain $\mathbb{D} \times \mathbb{D}$ by conformal transformation. Indeed, the key step in the proof of the theorem in $\mathbb{R}$ is the following inequality for all $f \in H^2(\mathbb{R})$

$$\log |(P_1^{(1)} * f)(x)| \leq \left(P_1^{(1)} * \log|f|\right)(x). \tag{3.2}$$

Now consider any function $f \in H^2(\mathbb{R}^2)$, then we have the multivariable holomorphic function

$$f(z_1, z_2) = \int_{\mathbb{R}^2} e^{2\pi i z_1 \xi_1} e^{2\pi i z_2 \xi_2} F(\xi_1, \xi_2).$$

Then by using (3.2) twice we obtain

$$\begin{aligned}\log \left| \left(P_1^{(1)}(x_2) * \left(P_1^{(1)}(x_1) * f\right)\right)(x)\right| &\leq \left(P_1^{(1)}(x_2) * \left(\log |P_1^{(1)}(x_1) * f|\right)\right)(x) \\ &\leq \left(P_1^{(1)}(x_2) * \left(P_1^{(1)}(x_1) * \log|f|\right)\right)(x).\end{aligned}$$

Here, the term $P(f)(x) := \left(P_1^{(1)}(x_2) * \left(P_1^{(1)}(x_1) * f\right)\right)(x)$ can be written as

$$P(f)(x) = \frac{1}{\pi^2} \int_{\mathbb{R}} \frac{\mathrm{d}t_2}{1 + (x_2 - t_2)^2} \int_{\mathbb{R}} f(t_1, t_2) \frac{\mathrm{d}t_1}{1 + (x_1 - t_1)^2}. \tag{3.3}$$

Then we have the similar inequality

$$\log |P(f)(x)| \leq \left(P(\log|f|)\right)(x). \tag{3.4}$$

And the rest is to prove that

$$S \text{ is a } \gamma\text{-thick set if and only if } P(\chi_S)(x) \geq \gamma' > 0 \text{ for any } x \in \mathbb{R}^d.$$

The proof is exactly the same as 1-dimensional case.

□

We have mentioned that in 1-dimension, we need the inequality

$$\log|P(f)(x)| \leq (P(\log|f|))(x). \tag{3.5}$$

This inequality can be derived from the Poisson-Jensen's Formula, since $u(z) = \log|f(z)|$ is a subharmonic function given $f(z)$ in hardy space. However, the key step (3.5) of the proof of the Logvinenko-Sereda Theorem for $d \geq 2$ is established by reducing the inequality to 1-dimensional case.

If we can prove that the harmonic extension u of any $f \in H^2(\mathbb{R}^d)$ is log-subharmonic (i.e., $\log|u|$ is subharmonic), then we can establish the inequality

$$\log\left|\left(P_y^{(d)} * f\right)\right|(x) \leq \left(P_y^{(d)} * \log|f|\right)(x)$$

directly through the Poisson-Jensen's Formula. Hence the L-S Theorem can be achieved if the following conjecture is true:

Conjecture. The harmonic extension of any $f \in H^2(\mathbb{R}^d)$ is log-subharmonic.

4. Logvinenko-Sereda Theorem in manifolds

To generalize the proof of the L-S Theorem in a Riemannian manifold (M, g), we need to

(i) Define the Paley-Wiener space. For example, in closed manifolds, use $X_L := \text{span}\{\phi_i : \Delta_g\phi_i = -\lambda_i\phi_i, \lambda_i \leq L\}$ to replace $H^2(\mathbb{R}^d)$.

(ii) Define the harmonic extension u of $f \in X_L$ in manifold $N := M \times \mathbb{R}_{\geq 0}$, $\partial N = M$. For closed manifolds, let $k_L = \dim X_L$ and $f = \sum_{i=1}^{k_L} \beta_i\phi_i$, then we can define the harmonic extension

$$u(x,t) = \sum_{i=1}^{k_L} \beta_i\phi_i e^{-\sqrt{\lambda_i}t}. \tag{4.1}$$

Then we need to give a similar Poisson-Jensen's Formula

$$u(x) = \int_M u(\xi)\,\mathrm{d}\omega(x, e_\xi) - \int_N g(x,\xi,N)\,\mathrm{d}\mu e_\xi \tag{4.2}$$

for any subharmonic function u.

(iii) Prove that the harmonic extension u of f is log-subharmonic. This step, together with the Poisson-Jensen's Formula, is not necessary if we can get the inequality (3.5) by other ways.

(iv) Give the definition of L-S thick set.

Remark 4.1. (1) The symbol $P(f)$ in (3.5) now represents the harmonic extension of f with respect to some fixed point. (2) The condition of bounded spectrum is necessary for the theorem since we need $\|f\|_{L^2(M)} \leq C\|P(f)\|_{L^2(M)}$.

For closed manifolds, the paper [2] gives the definition of L-S thick set and L-S Theorem. However, the proof of the theorem is based on a contradiction argument, hence cannot give an explicit constant.

In particular, for sphere $\mathbb{S}^d$, the paper [1] gives an explicit constant based on the fact that the eigenfunctions are spherical polynomials.

References

[1] Alexander Dicke and Ivan Veselic. Spherical logvinenko-sereda-kovrijkine type inequality and null-controllability of the heat equation on the sphere. *arXiv preprint arXiv:2207.01369*, 2022.

[2] Joaquim Ortega-Cerdà and Bharti Pridhnani. Carleson measures and logvinenko–sereda sets on compact manifolds. In *Forum Mathematicum*, volume 25, pages 151–172. De Gruyter, 2013.