SPECTRAL INEQUALITIES OF THE GENERALIZED GRUSHIN OPERATOR

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Contents

1. Introduction

We define the generalized Grushin operator as

$$H_G := -\Delta_x - V(x)\Delta_y, \quad x \in \mathbb{R}^n, y \in \mathbb{R}^n, \tag{1.1}$$

where the potential V satisfy the following assumption:

assump **Assumption 1.** Assume $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ satisfies the following two conditions:

(i) There exist positive constants c_1 and β_1 such that for all $x \in \mathbb{R}^n$,

$$c_1|x|^{\beta_1} \leq V(x). \tag{1.2}$$

(ii) We can write $V = V_1 + V_2$ such that there exists positive constants c_2 and β_2 such that

$$|V_1(x)| + |DV_1(x)| + |V_2(x)|^{\frac{4}{3}} \leq c_2(|x| + 1)^{\beta_2}. \tag{1.3}$$

Consider the Schrödinger operator

$$H := -\Delta + V(x). \tag{1.4}$$

Let $\phi_k, k \in \mathbb{N}$ be the eigenfunctions of H, i.e.,

$$H\phi_k = \lambda_k\phi_k. \tag{1.5}$$

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Denote by $\Lambda_L = \left(-\frac{L}{2}, \frac{L}{2}\right)^n$ the cube with side length $L > 0$. Let $d > 0, \gamma \in (0,1)$ and $\sigma \in [0,1)$. We consider the measurable sensor sets $\omega \subset \mathbb{R}^n$ satisfying the following property: there exists an equidistributed sequence $\{z_k : k \in \mathbb{Z}^n\}$ such that

{eqn:1.6}
$$\omega \cap (dk + \Lambda_d) \supset \mathcal{B}_{\gamma^{1+|kd|^\sigma} d}(z_k) \tag{1.6}$$

for all $k \in \mathbb{Z}^n$. We call it (d, γ, σ)-*distributed.*

Theorem 1.1 ([3, Theorem 1]). *Let $H = -\Delta + V(x)$. Assume that V satisfies Assumption 1 and Ω satisfies (1.6) with $d = 1, \sigma \in [0, \infty)$ and $\gamma \in (0, \frac{1}{2})$. Then there exists a constant C depending only on $\beta_1, \beta_2, c_1, c_2, \sigma$ and n such that*

$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{1}{\gamma}\right)^{C\lambda^{\frac{\sigma}{\beta_1}+\frac{\beta_2}{2\beta_1}}} \|\phi\|_{L^2(\Omega)}, \quad \forall \phi \in Ran\left(P_\lambda(H)\right). \tag{1.7}$$

2. Uniform lower bound of lowest eigenvalue

We denote $\lambda_0(V)$ the lowest eigenvalue of the operator H.

For all $a > 0$, we define

$$I_V(a) = \int_{\mathbb{R}^n} e^{-aV(x)}\,\mathrm{d}x. \tag{2.1}$$

Theorem 2.1 ([1]). *Under the condition for every $a > 0$ such that $I_V(a) < +\infty$. Then we have*

$$\lambda_0(V) \geq \sup_{t>0} t\left[n + \frac{n}{2}\ln\frac{\pi}{t} - \ln I_V\left(\frac{1}{t}\right)\right]. \tag{2.2}$$

In the particular case of $V(x) = c|x|^\beta$, a change into polar coordinates and then a chenge of variable $s = ar^{\beta_1}$ shows that

$$I_{c|x|^\beta}(a) = \int_{\mathbb{R}^n} e^{-ac|x|^\beta}\,\mathrm{d}x = \frac{\sigma_n}{\beta(ac)^{n/\beta}}\Gamma\left(\frac{n}{\beta}\right)$$

where $\sigma_n = \dfrac{2\pi^{n/2}}{\Gamma\left(n/2\right)}$ is the surface measure of the unit ball in $\mathbb{R}^n$. It follows that

$$\lambda_0\left(c|x|^\beta\right) \geq \sup_{t>0} t\left[n - \ln\frac{2}{\beta}\frac{\Gamma\left(\frac{n}{\beta}\right)}{\Gamma\left(\frac{n}{2}\right)} - n\left(\frac{1}{\beta}+\frac{1}{2}\right)\ln t + \frac{n}{\beta}\ln c\right].$$

The maximum is attained when

$$n - \ln\frac{2}{\beta}\frac{\Gamma\left(\frac{n}{\beta}\right)}{\Gamma\left(\frac{n}{2}\right)} - n\left(\frac{1}{\beta}+\frac{1}{2}\right)\ln t + \frac{n}{\beta}\ln c = n\left(\frac{1}{\beta}+\frac{1}{2}\right)$$

so that

$$\lambda_0(c|x|^\beta) \geq n\frac{\beta+2}{2\beta}\exp\left(\frac{\beta-2}{\beta+2}\right)\left(\frac{\beta}{2}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{\beta}\right)}\right)^{\frac{2\beta}{n(\beta+2)}} c^{\frac{\beta+2}{\beta}} := \widetilde{\lambda}_0 > 0.$$

Geiven $V(x) \geq c|x|^\beta$, it is obvious that $I_V(a) \leq I_{c_1|x|^\beta}(a)$. Hence we obtain

$$\lambda_0(V) \geq \lambda_0(c|x|^\beta) = \widetilde{\lambda}_0 > 0.$$

This implies that given any $V(x) \geq c|x|^\beta$, their first eigenvalues have a uniform lower bound $\widetilde{\lambda}_0 > 0$.

3. Generalized Grushin operator

In this section, due to some techinical reasons we need more restrictive assumptions:

assump2 **Assumption 2.** Assume $V \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$ satisfies the following two conditions

(i) There exist positive constants c_3, c_4 and $\beta > 1$ such that for all $x \in \mathbb{R}^n$

$$c_3|x|^\beta \leq V(x) \leq c_4|x|^\beta. \tag{3.1}$$

(ii) There exist a positive constant c_5 such that for all $x \in \mathbb{R}^n$

$$|DV(x)| \leq c_5|x|^{\beta-1}. \tag{3.2}$$

It can be verified that Assumption 2 satisfies Assumption 1 by setting $\beta_1 = \beta_2 = \beta$, $c_1 = c_3$, $c_2 = c_4 + c_5$ and $V_2 = 0$.

Applying the partial Fourier transform with respect to the $y \in \mathbb{R}^n$ variable, the Grushin operator H_G is transformed to

$$H_\eta := -\Delta_x + |\eta|^2 V(x), \tag{3.3}$$

where $\eta \in \mathbb{R}^d$ denotes the dual variable of $y \in \mathbb{R}^n$. Then we have

$$\left(e^{-tH_G}g\right)(x,y) = \int_{\mathbb{R}^n} e^{iy\cdot\eta}\left(e^{-tH_\eta}g_\eta\right)(x)\,\mathrm{d}\eta, \quad g \in L^2(\mathbb{R}^{2n}), (x,y) \in \mathbb{R}^{2n}, \tag{3.4}$$

where

$$g_\eta := \int_{\mathbb{R}^d} e^{-iy\cdot\eta} g(\cdot, y)\,\mathrm{d}y.$$

Indeed, we can consider more general operator

$$H_\beta := -\Delta_x + V(x)\left(-\Delta_y\right)^{\frac{\frac{2}{\beta}+1}{2}} \tag{3.5}$$

and its partial Fourier transform

$$H_{\beta,\eta} := -\Delta_x + |\eta|^{\frac{2}{\beta}+1} V(x). \tag{3.6}$$

Let us introduce for $\eta \in \mathbb{R}^n \backslash \{0\}$ the isometry M_η on $L^2(\mathbb{R}^n)$ by

$$M_\eta g = |\eta|^{\frac{n}{2\beta}} f\left(|\eta|^{\frac{1}{\beta}} \cdot\right), \quad \forall f \in L^2(\mathbb{R}^n).$$

Then

$$\begin{aligned}
M_\eta^* H_{\beta,\eta} M_\eta f(x) =& M_\eta^* H_\eta \left(|\eta|^{\frac{n}{2\beta_1}} f\left(|\eta|^{\frac{1}{\beta_1}} x\right)\right)\\
=& M_\eta^* \left(-\Delta_x + |\eta|^{\frac{2}{\beta}+1} V(x)\right)\left(|\eta|^{\frac{n}{2\beta_1}} f\left(|\eta|^{\frac{1}{\beta_1}} x\right)\right)\\
=& |\eta|^{\frac{n}{2\beta}} M_\eta^* \left(-|\eta|^{\frac{2}{\beta}} f''\left(|\eta|^{\frac{1}{\beta}} x\right) + |\eta|^{\frac{2}{\beta}+1} V(x) f\left(|\eta|^{\frac{1}{\beta}} x\right)\right)\\
=& -|\eta|^{\frac{2}{\beta}} \Delta_x f(x) + |\eta|^{\frac{2}{\beta}+1} V\left(|\eta|^{-\frac{1}{\beta}} x\right) f(x)\\
=& |\eta|^{\frac{2}{\beta}} \left(-\Delta_x + |\eta| V\left(|\eta|^{-\frac{1}{\beta}} x\right)\right) f(x)\\
=& |\eta|^{\frac{2}{\beta}} \widetilde{H}_{\beta,\eta} f(x).
\end{aligned}$$

Here we define the new operator

$$\widetilde{H}_{\beta,\eta} := -\Delta_x + V_\eta(x)$$

with

$$V_\eta(x) = |\eta| V\left(|\eta|^{-\frac{1}{\beta}} x\right).$$

Note that the new potential $V_\eta(x)$ still satisfies $V_\eta(x) \geq c|x|^\beta$ if $V(x)$ does. This ensures that for any $\eta \in \mathbb{R}^n$, the new potentials have a uniform lower bound of first eigenvalues.

In general, we can assume

$$H_\beta := -\Delta_x + V(x)\left(-\Delta_y\right)^{\frac{\alpha}{2}} \tag{3.7}$$

with $\alpha > \frac{2}{\beta}$, then

$$\widetilde{H}_{\beta,\eta} := -\Delta_x + V_\eta(x)$$

with

$$V_\eta = |\eta|^{\alpha - \frac{2}{\beta}} V\left(|\eta|^{-\frac{1}{\beta}} x\right)$$

, this generalization would be useful in $y \in \mathbb{T}^n$, since in this case $|\eta| > 1$.

4. Spectral inequality for the Schrödinger operator

In this section, we consider

{A.1}
$$-\Delta v + V v = 0 \quad \text{in } \mathbb{R}^{n+1}. \tag{4.1}$$

Define the sets

$$W_1 = \left\{ y \in \mathbb{R}_+^{n+1} | |y-b| \le \frac{1}{4}\delta \right\},$$
$$W_2 = \left\{ y \in \mathbb{R}_+^{n+1} | |y-b| \le \frac{1}{2}\delta \right\},$$
$$W_3 = \left\{ y \in \mathbb{R}_+^{n+1} | |y-b| \le \frac{2}{3}\delta \right\},$$

where $0 < \delta < \frac{1}{2}$. Note that $W_1 \subset W_2 \subset \mathbb{B}_\delta \subset \mathbb{R}^{n+1}$.

lma4.1 **Lemma 4.1** ([3, Lemma 1])**.** *Let $\delta \in \left(0, \frac{1}{2}\right)$. Let v be the solution of (4.1) with $v(y) = 0$ on the hyperplane $\{y | y_{n+1} = 0\}$. There exist $0 < \alpha < 1$ and $C > 0$, depending only on n such that*

{4.2} (4.2)

$$\|v\|_{H^1(P_1(L))} \le \delta^{-\alpha} \exp\left(C\left(1 + \mathcal{G}\left(V_1, V_2, 9\sqrt{n}L\right)\right)\right) \|v\|^{\alpha}_{H^1(P_3(L))} \|\frac{\partial v}{\partial y_{n+1}}\|^{1-\alpha}_{L^2(D_\delta(L))},$$

where

$$\mathcal{G}\left(V_1, V_2, L\right) = \|V_1\|^{\frac{1}{2}}_{W^{1,\infty}(\Lambda_L)} + \|V_2\|^{\frac{2}{3}}_{L^\infty(\Lambda_L)}. \tag{4.3}$$

Now we want to rescale it to the case of $d\delta$, with $d > 0$ a scaling constant. To do it, assume $\widetilde{v}$ satisfies Lemma 4.1 and define

$$v(x) = \widetilde{v}(dx),$$

then we have

$$\widetilde{v}(x) = v\left(\frac{x}{d}\right).$$

Substituting $\widetilde{v} = v\left(\frac{x}{d}\right)$ into (4.2) and we get

5. Observability inequality for the Generalized Grushin operator

Definition 1 (Exact observability)**.** Let $\tau > 0$, and let $\Omega \subset \mathbb{R}^n$ and $\omega \subset \Omega$ be measurable. A strongly continuous semigroup $(T(t))_{t \ge 0}$ on $L^2(\Omega)$ is said to be exactly observable from the set ω in time τ if there exists a positive constant $C_{\omega,\tau} > 0$ such that for all $g \in L^2(\Omega)$, we have

$$\|T(\tau)g\|^2_{L^2(\Omega)} \le C_{\omega,\tau} \int_0^\tau \|T(t)g\|^2_{L^2(\omega)}\, \mathrm{d}t.$$

Theorem 5.1 ([2, Theorem 2.8])**.** *Let A be a non-negative selfadjoint operator on $L^2(\mathbb{R}^b)$, and $\omega \subset \mathbb{R}^n$ be measurable. Suppose that there are $d_0 > 0$ and $d_1 \ge 0$, and $\zeta \in (0,1)$ such that for all $\lambda \ge 0$ and $f \in Ran(P_\lambda(A))$,*

$$\|f\|^2_{L^2(\mathbb{R}^n)} \le d_0 e^{d_1 \lambda^\zeta} \|f\|^2_{L^2(\omega)}.$$

Then, there exist positive constants $c_1, c_2, c_3 > 0$, only depending on ζ, such that for all $T > 0$ and $g \in L^2(\mathbb{R}^n)$ we have the observability estimate

$$\|e^{-tA}g\|^2_{L^2(\mathbb{R}^n)} \leq \frac{C_{\mathrm{obs}}}{T}\int_0^T \|e^{-tA}g\|^2_{L^2(\omega)}\,\mathrm{d}t,$$

where the positive constant $C_{\mathrm{obs}} > 0$ is given by

$$C_{\mathrm{obs}} = c_1 d_0 (2d_0+1)^{c_2} \exp\left(c_3 \left(\frac{d_1}{T^{\zeta}}\right)^{\frac{1}{1-\zeta}}\right). \tag{5.1}$$

Proposition 5.2. *There exists a constant $K > 0$, depending only on $\beta_1, \beta_2, c_1, c_2, \sigma$ and n such that for all $(1, \gamma, \sigma)$-distributed sets $\omega \subset \mathbb{R}^n$, $r > 0, T > 0$, and $g \in L^2(\mathbb{R}^n)$ we have*

$$\|e^{-TH_r}g\|^2_{L^2(\mathbb{R}^n)} \leq \frac{C_{\mathrm{obs}}}{T}\int_0^T \|e^{-tH_r}g\|^2_{L^2(\omega)}\,\mathrm{d}t, \tag{5.2}$$

where the positive constant $C_{\mathrm{obs}} > 0$ is given by

$$C_{\mathrm{obs}} = \tag{5.3}$$

Proof. Recall □

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