THE LOWER BOUND OF NUMBER OF EDGES

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Consider a Graph $G := G(V, E)$ with M vertices, and assume that given any k vertices in G there are two adjacent vertices, then we can define

$$\mathcal{S}(k, M, G) := \mathcal{S}(k, M) := \text{ number of edges in } G.$$

Then we have the following:

Theorem 1. *Given the definition above, we have*

$$\mathcal{S}(k, M) \geq \frac{M^2 - 4k^2 M}{(2k)^4(2k-1)}. \tag{1}$$

Proof. First for simplicity, we prove it for $M = k^m$ and $k = 2^l$ with $k, l \in \mathbb{N}$. In this case it is sufficient to prove

$$S(k, k^m) \geq \frac{k^m(k^m - 1)}{k^2(k-1)}. \tag{2}$$

We prove (2) by induction on m. For $m = 1$, it is true obviously. Now assume (2) is true for $M = k^m = 2^{lm}$, we need to prove the case $M = k^{m+1} = 2^{l(m+1)}$. Divide $M = k^{m+1}$ into $k = 2^l$ parts $M_1, M_2, \cdots, M_k$ such that each part has $k^m = 2^{lm}$ vertices. For convenience, we use both k and 2^l, although they are in essential the same thing. Then we have

$$\mathcal{S}(k, M) = \sum_{j=1}^{k} S(k, M_j) + \mathcal{L} \geq \frac{k^m(k^m - 1)}{k(k-1)} + \mathcal{L}$$

where L is defined to be the number of edges whose endpoints belong to distinct M_j's.

To estimate $\mathcal{L}$, we denote the vertices in each M_j by

$$a_{j,l}, \quad , l = 1, 2, \cdots, k^m.$$

First consider the following k subgraphs of G Define the permutation σ of set $\{1, 2, \cdots, k^m\}$ by

$$\sigma(i) = \begin{cases} i + 1, & 1 \leq i \leq k^m - 1, \\ 1, & i = k^m. \end{cases} \tag{3}$$

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Now, we define new subgraphs $N_{i,j}$ by

$$N_{i,j} := \text{ the subgraph who contains vertices}$$
$$\left\{a_{1,i}, a_{2,\sigma^{1j}(i)}, a_{3,\sigma^{2j}(i)}, \cdots, a_{s,\sigma^{(s-1)j}(i)}, \cdots, a_{k-1,\sigma^{(k-2)j}(i)}, a_{k,\sigma^{(k-1)j}(i)}\right\}$$

where $i \in \{1, 2, \cdots, k^m\}$ and $j \in \{1, 2, \cdots, k^{m-1}\}$. We claim that for distinct $N_{i,j}$ and $N_{i',j'}$, they can have at most 1 common point. Indeed, if they have two common points, then there exist $1 \leq s < s' \leq k-1$ such that

$$a_{s,\sigma^{(s-1)j}(i)} = a_{s,\sigma^{(s-1)j'}(i')}$$
$$a_{s',\sigma^{(s'-1)j}(i)} = a_{s',\sigma^{(s'-1)j'}(i')}.$$

The above equations imply

$$\begin{aligned} \sigma^{(s-1)j}(i) &= \sigma^{(s-1)j'}(i') \\ \sigma^{(s'-1)j}(i) &= \sigma^{(s'-1)j'}(i'). \end{aligned} \tag{4}$$

Let $i' = \sigma^{\lambda}(i), \lambda \in \{1, 2, \cdots, k^m\}$. Then we obtain

$$(s-1)(j-j') - \lambda \equiv 0 \pmod{k^m},$$
$$(s'-1)(j-j') - \lambda \equiv 0 \pmod{k^m}.$$

Extracting the second one from the first one, we obtain

$$k^m | (s-s')(j-j').$$

Remember $|s - s'| \leq k - 1 = 2^l - 1$, we must have $k^{m-1} | j - j'$. Again remember that $|j - j'| \leq k^{m-1} - 1$, hence we can only have $j = j'$, and then we have $i = i'$ by (4). Hence any two of $N_{i,j}$s cannot have more than one common vertices.

Now we can estimate $\mathcal{L}$: there are k^{2m-1} distinct N_{ij}s, hence

$$\mathcal{L} \geq k^{2m-1}.$$

Then we have

$$\mathcal{S}(k, M) \geq \frac{k^m(k^m - 1)}{k(k-1)} + k^{2m-1} = \frac{k^{m+1}(k^{m+1} - 1)}{k^2(k-1)}.$$

Hence we complete the proof for the regular case $M = k^m$ and $k = 2^l$.

For a general integer $k \geq 3$, there exists a unique integer l such that $2^{l-1} \leq k < 2^l$ (or equivalently $l := \lceil \log_2 k \rceil$). We define $k' = 2^l$. Similarly, for a general integer M, there exists a unique integer m such that $k'^m \leq M < k'^{m+1}$. We define $M' = k'^m$. Obviously we have

$$\mathcal{S}(k, M) \geq \mathcal{S}(k', M'),$$

and for the later one, we have proved that

$$\mathcal{S}(k', M') \geq \frac{k'^m(k'^m - 1)}{k'^2(k'-1)}.$$

Since $k'^{m+1} > M$ and $k' \leq 2k$, we have $k'^m > \dfrac{M}{k'} \geq \dfrac{M}{2k}$. Then we obtain

$$\mathcal{S}(k', M') \geq \frac{\frac{M}{2k}\left(\frac{M}{2k} - 1\right)}{(2k)^2(2k-1)} = \frac{M^2 - 4k^2 M}{(2k)^4(2k-1)},$$

and complete the proof. $\square$