NULL-CONTROLLABILITY FOR ONE-DIMENSIONAL HEAT EQUATIONS WITH POWER GROWTH POTENTIALS FROM MEASURABLE SETS

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ABSTRACT. We study. . .

CONTENTS

1. INTRODUCTION

Consider the 1D heat equation,

heat

$$\begin{cases} \partial_t u - \partial_x^2 u + V(x)u = h(t,x)\mathbb{1}_\Omega, & x \in \mathbb{R},\ t>0 \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}), \end{cases} \tag{1}$$

where the potential V is a real-valued continuous funtion, $h(t,x) \in L^2\left((0,T)\times\mathbb{R}\right)$, and Ω is a given measurable set. The equation (1) is said to be *null-controllable* from the set Ω in time $T>0$ if, for any intial datum $u_0 \in L^2(\mathbb{R})$, there exists $h(t,x) \in L^2\left((0,T)\times\Omega\right)$ such that the mild solution to (1) satisfies $u(T)=0$.

In this article, we study the null-controllability of heat equation (1) with a potential satisfying the following assumption:

assump1

Assumption 1.1. $V(x) \in C(\mathbb{R})$ is a continuous real-valued potential and there exist constants $c_1>0$, $c_2>0$, c_3 and $\beta_2 \geq \beta_1 > 0$ such that

$$c_1|x|^{\beta_1} \leq V(x)+c_3 \leq c_2|x|^{\beta_2}, \quad \forall x \in \mathbb{R}.$$

Before stating our result, we first give some definitions to describe the observable set Ω we concerned.

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def:1 **Definition 1.2.** Let $L > 0$ and $s \geq 0$, we define the sequence $\{x_n\}_{n\in\mathbb{Z}}$ of real numbers as the following: set $x_0 = 0, x_1 = L$, define x_n for each $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$ using the recurrence formula

$$x_{n+1} = x_n + L\left(\frac{1}{x_n}\right)^s, \quad \forall n \in \mathbb{N}$$

and define x_{-n} for each $n \in \mathbb{N}$ by

$$x_{-n} = -x_n.$$

Define

$$I_n := I_{1n} := [x_n, x_{n+1}], \quad \forall n \in \mathbb{N}$$

and

$$I_n := -I_{|n|}, \quad \forall n \in -\mathbb{N}.$$

Definition 1.3. Given the sequence as in Definition 1.2 and $\gamma > 0$. We call a measurable set $\Omega \subset \mathbb{R}$ is γ-thick of type (L, s) if

thick
$$|\omega_n| \geq \gamma|I_n|, \tag{2}$$

where $\omega_n := \Omega \cap I_n$.

Remark 1.4. For $s = 0$, we obtain $x_n = |n|L$ and the thick condition (2) becomes

$$|\Omega \cap [nL, n(L+1)] \geq \gamma L, \quad \forall n \in \mathbb{Z},$$

and this is just the usual definition of γ-thick set.

Now we can state our result as the following:

main **Theorem 1.5.** *Let Ω be a γ-thick set of type (L, s), V be the potential under Assumption 1.1 and $s \geq \beta_2$. Then the equation (1) is exactly null-controllable from Ω in any time $T > 0$.*

By the Hilbert uniqueness method, the null-controllability of (1) from the set Ω in time $T > 0$ is equivalent to the inequality

obs
$$\|e^{-Ht}\|^2_{L^2(\mathbb{R})} \leq C(T, V, \Omega) \int_0^T \|e^{-Ht}u_0\|^2_{L^2(\Omega)}\,\mathrm{d}t, \quad \forall u_0 \in L^2(\mathbb{R}),$$

where $C(T, V, \Omega)$ is a constant which depends only on T, V and Ω.

This result is inspired by the recent work of Su, Sun and Yuan [1],who concerned the observability inequality of 1D Schrödinger equation for $V(x) \in C(\mathbb{R})$ and bounded, and proved for Ω being γ-thick. Here we call a set Ω is γ-thick if there exists a positive constant L and γ such that

gma-thick
$$|\Omega \cap [x, x + L]| \geq \gamma L, \quad \forall x \in \mathbb{R}..$$

To prove it, they establish the spectral inequality for the Schrödinger operator $H = -\partial_x^2 + V(x)$ with $V(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. By the well-known Lebeau-Robbiano method, our result is also reduced to the proof of a spectral inequality.

2. Notation and Conventions

Let $u(t)$ be a solution of (1) and $\theta > 0$, then $U(t) := e^{-\theta t}u$ is a solution of

$$\partial_t U - \partial_x^2 U + V(x)U + \theta U = 0, \quad U|_{t=0} = u_0 \in L^2(\mathbb{R}).$$

By this transform we can reduce Assumption 1.1 in Theorem 1.5 to the following:

assump2

Assumption 2.1. $V(x) \in C(\mathbb{R})$ is a continuous real-valued potential and there exist constants $c_1 > 0$, $c_2 > 0$ and $\beta_2 \geq \beta_1 > 0$ such that

$$1 \leq c_1 \langle x \rangle^{\beta_1} \leq V(x) \leq c_2 \langle x \rangle^{\beta_2}.$$

Here we use the Japanese bracket $\langle x \rangle := (1+|x|^2)^{\frac{1}{2}}$. Hence we only need to prove 1.5 under Assumption 2.1.

We denote the Schrödinger operator

$$Hf(x) := H_V f(x) := -\partial_x^2 f(x) + V(x)f(x), \quad \forall f \in D(H)$$

where $D(H)$ denotes the domain of the operator H:

$$D(H) = \left\{ f \in L^2(\mathbb{R}) : \partial_x f \in L^2(\mathbb{R}) \text{ and } Vf \in L^2(\mathbb{R}) \right\}.$$

The space of Schwartz functions, denoted by $\mathcal{S}(\mathbb{R})$, is contained in $D(H)$.

Note that under Assumption 2.1, the potential V satisfies

$$\lim_{|x| \to \infty} V(x) = \infty.$$

This implies that the inverse operator H^{-1} is compact in $L^2(\mathbb{R})$ and therefore the spectrum of H are discrete and unbounded. Precisely speaking, there exists a sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ with $0 < \lambda_0 \leq \lambda_1 \leq \cdots$ and $\lambda_k \to \infty$, and a normal basis $\{\phi_k\}_{k \in \mathbb{N}}$ of $L^2(\mathbb{R})$, such that

$$H\phi_k = \lambda_k \phi_k, \quad \forall k \in \mathbb{N}.$$

We will write $\mathcal{E}_\lambda(H)$ for the spectral set associated to H, that is,

$$\mathcal{E}_\lambda(H) := \operatorname{span} \{\phi_k : k \text{ such that } \lambda_k \leq \lambda\}.$$

For future reference, we define

$$I_{2n} := [x_n - |I_n|, x_{n+1} + |I_n|], \quad \forall n \in \mathbb{Z}$$

and

$$I_{3n} := [x_n - 2|I_n|, x_{n+1} + 2|I_n|], \quad \forall n \in \mathbb{Z}.$$

Define

$$D_{1n} := I_{1n} \times \left[-\frac{I_n}{2}, \frac{I_n}{2} \right], \; D_{2n} := I_{2n} \times \left[-\frac{3I_n}{2}, \frac{3I_n}{2} \right] \text{ and } D_{3n} := I_{3n} \times \left[-\frac{5I_n}{2}, \frac{5I_n}{2} \right].$$

We also define

$$I_1 := [0,1], \; I_2 := [-1,2], \; I_3 = [-2,3]$$

and

$$D_1 := I_1 \times \left[-\frac{I_{1n}}{2}, \frac{I_{1n}}{2} \right], \; D_2 := I_2 \times \left[-\frac{3}{2}, \frac{3}{2} \right] \; D_3 := I_3 \times \left[-\frac{5}{2}, \frac{5}{2} \right].$$

Without loss of generality, we can also assume that $L = 1$ in Theorem 1.5. Indeed, we may do the linear transform $x = ay$ and denote $y_n := x_n/a$, then by choosing $a = L^{\frac{1}{1+s}}$ the new parameter becomes $L' = 1$.

3. Spectral inequality

3.1. **Localization property of eigenfunctions.** Let V be a real-valued nonnegative function on $\mathbb{R}$ with $V \in L^\infty_{\text{loc}}(\mathbb{R})$ such that

$$\lim_{|x|\to\infty} V(x) = \infty$$

and consider the associated Schrödinger operator

$$Hf(x) := H_V f(x) := -\Delta f(x) + V(x)f(x).$$

localization **Proposition 3.1.** *Assume that $V \in L^\infty_{\text{loc}}(\mathbb{R})$ satisfies $V(x) \geq c|x|^\beta$ with $\beta > 0$. Then there exists a constant $C := C(c,\beta)$, depending only on c and β, such that for all $\lambda > 0$ and $\phi \in \mathcal{E}_\lambda(H)$, we have*

$$\|\phi\|_{L^2(\mathbb{R})} \leq 2\|\phi\|_{L^2(I_\lambda)}$$

where $I_\lambda := [-C\lambda^{1/\beta}, C\lambda^{1/\beta}]$.

3.2. **Propagation of smallness.** In this section, we introduce the L^2-propagation of smallness for H^2_{loc} solution of the following 2D elliptic equation in nondivergence form

2d-elliptic
$$-\Delta\Phi(z) + V(x)\Phi(z) = 0 \;\text{ with }\; \partial_y\Phi|_{y=0}. \tag{3}$$

propagation-prp **Proposition 3.2.** *Let $C_0 > 0$ be a positive constant. Then for any measurablet set $\omega \subset I$ with $|\omega| > 0$, any potential $V \in C(I_3)$ with $0 < V(x) \leq C_0$, and any real-valued H^2_{loc} solution Φ of (3) in D_3, we have*

uniform-propagation
$$\|\Phi\|_{L^2(D_1)} \leq C\|\Phi\|^\alpha_{L^2(\omega)}\left(\sup_{D_2}|\Phi|^{1-\alpha}\right), \tag{4}$$

where $\alpha = \alpha(C_0,|\omega|) \in (0,1)$ and $C = C(C_0,|\omega|) > 0$ depend only on C_0 and $|\omega|$.

propagation-crc **Corollary 3.3.** *Let $\gamma > 0$ and $V \in C(\mathbb{R})$ be the potential as in Assumption 2.1. Let Ω be a measurable subset of $\mathbb{R}$ such that $|\omega_n| \geq \gamma|I_n|$ for every $n \in \mathbb{Z}$. Then for any real-valued H^2_{loc} solution Φ of (3) in $\mathbb{R}^2$, we have*

$$\|\Phi\|_{L^2(D_{1n})} \leq C\|\Phi\|^\alpha_{L^2(\omega_n)}\left(\sup_{D_{2n}}|\Phi|^{1-\alpha}\right), \quad \forall n \in \mathbb{Z},$$

where $\alpha = \alpha(c_2,\beta_2,\gamma) \in (0,1)$ and $C = C(c_2,\beta_2,\gamma)$ depend only on c_2, β_2 and γ.

Proof. For each $n \in \mathbb{Z}$, we first reduce the inequality (4) to uniformly bounded potential: define

$$f(z) := \Phi\left(\frac{z}{a_n}\right), \quad \forall z \in a_n D_{3n}, \forall n \in \mathbb{Z},$$

and substitute this into (3), then we obtain

reduced-2d-elliptic
$$-\Delta f(z) + \widetilde{V}(x)f(z) = 0, \quad \forall z \in a_n D_{3n}, \forall n \in \mathbb{Z}, \tag{5}$$

where $\widetilde{V}(x) := \dfrac{1}{a_n^2}V\left(\dfrac{x}{a_n}\right)$. By Assumption 2.1, the new potential satisfy the condition

$$c_1'\left(\frac{1}{a_n}\right)^{2+\beta_1}\langle x\rangle^{\beta_1} \leq \widetilde{V}(x) \leq c_2'\left(\frac{1}{a_n}\right)^{2+\beta_2}\langle x\rangle^{\beta_2}$$

for all $x \in a_n I_{3n}$ and for all $n \in \mathbb{Z}$, where $c_1' := c_1'(c_1, \beta_1)$ and $c_2' := c_2'(c_2, \beta_2)$ are two new constants.

Now we choose

$$a_n = |x_n|^s$$

for each $n \in \mathbb{Z} \setminus \{0\}$ and $a_0 = 1$. Then by the assumption

$$s > \frac{2}{3}\beta_2 \Longrightarrow s > \frac{\beta_2}{\beta_2 + 2},$$

we obtain that for any $x \in I_{3n}$, there exists a constant C' such that

$$c_2' \left(\frac{1}{a_n}\right)^{2+\beta_2} \langle x \rangle^{\beta_2} \leq C'$$

hold uniformly for all $n \in \mathbb{Z}$ and $C' = C'(c_2, \beta_2)$ depends only on c_2 and β_2.

Now we can use Proposition 3.2 in (5), then for any $n \in \mathbb{Z}$ and real-valued H^2_{loc} solution f of (5) in $a_n D_{3n}$, we have

aux-propagation

$$\|f\|_{L^2(a_n D_{1n})} \leq C \|f\|^{\alpha}_{L^2(a_n \omega_n)} \left(\sup_{a_n D_{2n}} |f|^{1-\alpha} \right), \tag{6}$$

where $C := C(c_2, \beta_2, \gamma) > 0$ and $\alpha := \alpha(c_2, \beta_2, \gamma) \in (0,1)$ depend only on c_2, β_2 and γ. Note that

$$\|f\|_{L^2(a_n D_{1n})} = a_n^{\frac{1}{2}} \|\phi\|_{L^2(D_{1n})},$$

$$\|f\|^{\alpha}_{L^2(a_n \omega_n)} = a_n^{\frac{\alpha}{2}} \|\phi\|^{\alpha}_{L^2(\omega_n)},$$

and

$$\sup_{a_n D_{2n}} |f|^{1-\alpha} = \sup_{D_{2n}} |\phi|^{1-\alpha}.$$

Take the above three equations into (6), then we obtain

a-1

$$\|\phi\|_{L^2(D_{1n})} \leq C a_n^{\frac{\alpha-1}{2}} \|\phi\|^{\alpha}_{L^2(\omega_n)} \left(\sup_{D_{2n}} |\phi|^{1-\alpha} \right). \tag{7}$$

Since $\alpha \in (0,1)$ and $a_n \geq 1$, we have

$$a_n^{\frac{\alpha-1}{2}} \leq 1,$$

then combine this with (7), we obtain the desired inequality. □

3.3. **Spectral inequality.**

spectral-inequality

Lemma 3.4. *Let Ω be a γ-thick set of type $(1, s)$, V be a potential under Assumption 2.1. Then there exists a constant $C = C(c_1, \beta_1, c_2, \beta_2, \gamma) > 0$, depending only on $c_1, \beta_1, c_2, \beta_2$ and γ, such that for any $\lambda > 0$ and any $\phi \in \mathcal{E}_\lambda(H)$, we have*

$$\|\phi\|_{L^2(\mathbb{R})} \leq C e^{C\lambda} \|\phi\|_{L^2(\Omega)}.$$

Proof. Without loss of generality, we assume $\lambda > 1$. For any $\lambda > 1$, $z = (x, y) \in \mathbb{R} \times \left[-\frac{5}{2}, \frac{5}{2}\right]$, we assume $\phi \in \mathcal{E}_\lambda(H)$ and

$$\phi = \sum_{\lambda_k \leq \lambda} a_k \phi_k, \quad a_k \in \mathbb{C}.$$

We set

a-2
$$\Phi(x,y) := \sum_{\lambda_k \le \lambda} a_k \cosh(\lambda_k y)\phi_k(x). \tag{8}$$

Taking the derivative of (8) with respect to y twice and using the fact that $(\cosh s)'' = \cosh s$, we obtain

$$\partial_y^2 \Phi = H\Phi = \sum_{\lambda_k \le \lambda} \lambda_k a_k \cosh(\lambda_k y)\phi_k.$$

It follows that

$$-\Delta\Phi + V(x)\Phi = -\partial_y^2\Phi + H\Phi = 0.$$

On the other hand, for the case of $y = 0$, using the fact that $(\cosh s)' = \sinh s$, we obtain

$$\partial_y \Phi|_{y=0} = \sum_{\lambda_k \le \lambda} \lambda_k a_k \sinh(\lambda_k 0)\phi_k = 0.$$

Hence the function Φ is a H^2_{loc} solution for (3) with $\Phi(x,0) = \phi(x)$ on $\mathbb{R}$. Applying Corollary 3.3 to Φ, we obtain

$$\|\Phi\|_{L^2(D_{1n})} \le C\|\phi\|^{\alpha}_{L^2(\omega_n)} \left(\sup_{D_{2n}} |\Phi|^{1-\alpha}\right), \quad \forall n \in \mathbb{Z},$$

where $C = C(c_2, \beta_2, \gamma) > 0$ depends only on c_2, β_2 and γ. From Young's inequality for products, *i.e.*, $ab \le \alpha a^{\frac{1}{\alpha}} + (1-\alpha)b^{\frac{1}{1-\alpha}}$ for any $a, b \ge 0$, we have for all $n \in \mathbb{Z}$

a-3
$$\|\Phi\|^2_{L^2(D_{1n})} \le \frac{C_1\alpha}{\varepsilon}\|\phi\|^2_{L^2(\omega_n)} + C_1\varepsilon^{\frac{\alpha}{1-\alpha}}\|\Phi\|^2_{L^\infty(D_{2n})}, \tag{9}$$

where $C_1 = C_1(c_2, \beta_2, \gamma)$ depends only on c_2, β_2 and γ.

Now we define a cut-off function $\chi : \mathbb{R}^2 \to \mathbb{R}$: it is a C^2 function such that

$$\chi \equiv 1 \ \text{ on } \left[-\frac{3}{2}, \frac{3}{2}\right]^2 \ \text{ and } \ \text{supp}\chi \subset \left[-\frac{5}{2}, \frac{5}{2}\right]^2.$$

For any $n \in \mathbb{Z}$, we set

$$\chi_n(x,y) := \chi\left(a_n(x - x_n) + \frac{1}{2}, a_n y\right).$$

By this setting, for any $n \in \mathbb{Z}$ we have

$$\chi_n \equiv 1 \ \text{ on } D_{2n} \ \text{ and } \ \text{supp}\chi_n \subset D_{3n}.$$

There exists a positive constant $C_2 > 0$ which depends only on the choice of χ, such that

$$|D\chi|_{L^\infty(\mathbb{R}^2)} \le C_2 \ \text{ and } \ |\text{Hess}\chi|_{L^\infty(\mathbb{R}^2)} \le C_2.$$

Then after the rescaling, we have

$$|D\chi|_{L^\infty(\mathbb{R}^2)} \le C_2 a_n \ \text{ and } \ |\text{Hess}|_{L^\infty(\mathbb{R}^2)} \le C_2 a_n^2,$$

where C_2 still depends only on the choice of χ.

Using the 2D Sobolev embedding theorem, we obtain

$$\begin{aligned}\|\Phi\|^2_{L^\infty(D_{2n})} &\le \pi \|\chi_n \Phi\|^2_{H^2(\mathbb{R}^2)}\\ &\le \pi C_2^2 a_n^4 \|\Phi\|^2_{H^2(D_{3n})}\\ &\le C_3 a_n^4 (1+\lambda^4)\|\Phi\|^2_{L^2(D_{3n})},\end{aligned}$$

where $C_3 > 0$ depends only c_2, β_2 and the choice of χ. Combining this and (9), we have

a-4 $$\|\Phi\|^2_{L^2(D_{1n})} \le \frac{C_1 \alpha}{\varepsilon}\|\phi\|^2_{L^2(\omega_n)} + C_1 C_3 \varepsilon^{\frac{\alpha}{1-\alpha}} a_n^4 (1+\lambda^4)\|\Phi\|^2_{L^2(D_{3n})}. \tag{10}$$

Thanks to Proposition 3.1, we only need to consider the value of ϕ in I_λ. Define

$$\mathcal{J} := \{n \in \mathbb{Z} : I_n \cap I_\lambda \neq \varnothing\}.$$

Then we have

a-5 $$a_n = |x_n|^s \le C_4 \langle \lambda \rangle^{\frac{s}{\beta_1}}, \tag{11}$$

where $C_4 = C_4(c_1, \beta_1)$ depends only on c_1 and β_1. Summing over $n \in \mathcal{J}$ for (10) and using (11), we obtain

a-8 $$\sum_{n\in\mathcal{J}} \|\Phi\|^2_{L^2(D_{1n})} \le \frac{C_5}{\varepsilon}\sum_{n\in\mathcal{J}} \|\phi\|^2_{L^2(\omega)} + C_5 \varepsilon^{\frac{\alpha}{1-\alpha}} \langle\lambda\rangle^{4+\frac{4s}{\beta_1}} \sum_{n\in\mathcal{J}} \|\Phi\|^2_{L^2(D_{3n})}, \tag{12}$$

where $C_5 = C_5(c_1, \beta_1, c_2, \beta_2, \gamma)$ depends $c_1, \beta_1, c_2, \beta_2$ and γ.

On the other hand,

a-6 $$\begin{aligned}\sum_{n\in\mathcal{J}} \|\Phi\|^2_{L^2(D_{1n})} &\ge \int_{-|I_n|/2}^{|I_n|/2}\int_{I_\lambda} |\Phi(x,y)|^2 \,\mathrm{d}x\mathrm{d}y\\ &\ge \frac{1}{2}\int_{-|I_n|/2}^{|I_n|/2}\int_{\mathbb{R}} |\Phi(x,y)|^2 \,\mathrm{d}x\mathrm{d}y\\ &= \int_0^{|I_n|/2}\int_{\mathbb{R}} \Big|\sum_{\lambda_k\le\lambda} a_k \cosh(\lambda_k y)\phi_k(x)\Big|^2 \,\mathrm{d}x\mathrm{d}y\\ &= \int_0^{|I_n|/2}\int_{\mathbb{R}} \sum_{\lambda_k\le\lambda} |a_k|^2 \cosh(\lambda_k y)^2 |\phi_k(x)|^2 \,\mathrm{d}x\mathrm{d}y\\ &\ge \frac{|I_n|}{2}\|\phi\|^2_{L^2(\mathbb{R})} \ge \frac{1}{C_4}\langle\lambda\rangle^{-\frac{s}{\beta_1}}\|\phi\|^2_{L^2(\mathbb{R})},\end{aligned} \tag{13}$$

where the second line uses Proposition 3.1, the fourth line uses the orthogonality of the basis and the last line uses the relation $a_n = |I_n|^{-1}$. We also have

a-7 $$\begin{aligned}\sum_{n\in\mathcal{J}} \|\Phi\|^2_{L^2(D_{3n})} &\le \int_{-\frac{5}{2}}^{\frac{5}{2}}\int_{\mathbb{R}} \Big|\sum_{\lambda_k\le\lambda} a_k \cosh(\lambda_k y)\phi_k(x)\Big|^2 \,\mathrm{d}x\mathrm{d}y\\ &\le \int_{-\frac{5}{2}}^{\frac{5}{2}}\int_{\mathbb{R}} \sum_{\lambda_k\le\lambda} |a_k|^2 \cosh(\lambda_k y)^2 |\phi_k(x)|^2 \,\mathrm{d}x\mathrm{d}y\\ &\le 5e^{5\lambda}\|\phi\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{14}$$

Taking (13)and (14) into (12), then we obtain

$$\|\phi\|^2_{L^2(\mathbb{R})} \le \frac{C_5 C_4}{\varepsilon}\langle\lambda\rangle^{\frac{s}{\beta_1}}\|\phi\|^2_{L^2(\omega)} + C_5 \varepsilon^{\frac{\alpha}{1-\alpha}}\langle\lambda\rangle^{4+\frac{3s}{\beta_1}} 5e^{5\lambda}\|\phi\|_{L^2(\mathbb{R})}.$$

Finally, we complete the proof of Lemma 3.4 by taking ε small enough. $\square$

Appendix A. Growing estimate of the sequence

In this appendix, we consider the recurrence formula

recur (15) $$x_{n+1} = x_n + \frac{1}{x_n^s}, \quad x_0 = 1,\, n \in \mathbb{N},$$

where $s > 0$.

Lemma A.1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence given by (15), then for all $n \in \mathbb{N}$, we have*

$$(n+1)^{\frac{1}{s+1}} \le x_n \le (s+1)(n+1)^{\frac{1}{s+1}} + 1 - s, \quad n \in \mathbb{N}.$$

Proof. Rewrite (15) to

$$x_n^s(x_{n+1} - x_n) = 1, \quad \forall n \in \mathbb{N}.$$

Summing it up to $n \in \mathbb{N}_+$, we obtain

$$\sum_{k=0}^{n} \left(x_k^s x_{k+1} - x_k^{s+1} \right) = n + 1.$$

Note that $x_k^{s+1} \ge x_{k-1}^s x_k$ since $x_k \ge x_{k-1}$ for all $k \in \mathbb{N}$ and $k \ge 1$, we obtain

$$\begin{aligned} n+1 \le& \sum_{k=1}^{n} \left(x_k^s x_{k+1} - x_{k-1}^s x_k \right) + x_0^s x_1 - x_0^{s+1} \\ =& x_n^s x_{n+1} - 1 \le x_{n+1}^{s+1} - 1. \end{aligned}$$

This implies

$$x_{n+1} \ge (n+2)^{\frac{1}{s+1}}, \quad \forall n \in \mathbb{N}_+.$$

This combines with $x_0 = 1$ and $x_1 = 2$, we obtain

$$x_n \ge (n+1)^{\frac{1}{s+1}}, \quad \forall n \in \mathbb{N}.$$

On the other hand,

$$x_{n+1} = x_n + \frac{1}{x_n^s} \le x_n + \frac{1}{(n+1)^{\frac{s}{s+1}}}, \quad n \in \mathbb{N}.$$

Then we have

$$x_{k+1} - x_k \le \frac{1}{(k+1)^{\frac{s}{s+1}}}, \quad k = 0, 1, \cdots, n.$$

Summing it up we obtain for all $n \in \mathbb{N}$

$$\begin{aligned} x_{n+1} - 1 =& \sum_{k=0}^{n} (x_{k+1} - x_k) \le \sum_{k=0}^{n} \frac{1}{(k+1)^{\frac{s}{s+1}}} \\ \le& 1 + \int_1^{n+1} \frac{1}{x^{\frac{s}{s+1}}} \,\mathrm{d}x = (s+1)(n+1)^{\frac{1}{s+1}} - s. \end{aligned}$$

This combinning with $x_0 = 1, x_1 = 2$ finishes the proof of the upper bound. $\square$

References

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