Chapter 1

Measures

Exercise 1.1 *A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is called a ring if it is closed under finite unions and differences (i.e., if $E_1, E_2, \cdots, E_n \in \mathcal{R}$, then $\bigcup_1^n E_j \in \mathcal{R}$, and if $E, F \in \mathcal{R}$, then $E \backslash F \in \mathcal{R}$). A ring that is closed under countable unions is called a σ-ring.*

a. *Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.*

b. *If $\mathcal{R}$ is a ring (resp. σ-ring), then R is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.*

c. *If R is a σ-ring, then $\{E \subset X : E \in \mathcal{R}$ or $E^c \in R\}$ is a σ-algebra.*

d. *If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R}$ for all $F \in \mathcal{R}\}$ is a σ-algebra.*

Solution:

a. $E \cap F = E \backslash (E \backslash F)$.

b. This is obvious.

c. Let
$$\mathcal{G} = \{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}.$$
Choose a set $E \in \mathcal{G}$, then $X = E \cup E^c \in \mathcal{G}$. Let $E_1, E_2, \cdots, E_n, \cdots \in \mathcal{G}$. Then $E_i \in \mathcal{R}$ or $E_i^c \in \mathcal{R}$ for every $i \in \mathbb{N}$. Assume $E_i \in \mathcal{R}$ for $i \in I$ and $E_i^c \in \mathcal{R}$ for $i \in \mathbb{N} \backslash I$. Then $\bigcup_{i \in \mathbb{N}} E_i = A \backslash B \in \mathcal{R} \subset \mathcal{G}$ since $A = \bigcup_{i \in I} E_i \in \mathcal{R}$ and $B = \bigcup_{i \in \mathbb{N} \backslash I} E_i \in \mathcal{R}$. It is easy to verify $E - F \in \mathcal{G}$ for $E, F \in \mathcal{G}$.

d. Let
$$\mathcal{G} = \{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}.$$
Obviously, $X \in \mathcal{G}$. Let $E_i \in \mathcal{G}, i \in \mathbb{N}$, then $E_i \cap F \in \mathcal{R}$ for all $F \in \mathcal{R}$. $(\bigcup_{i=1}^\infty E_i) \cap F = \bigcup_{i=1}^\infty (E_i \cap F) \in \mathcal{R}$ since $E_i \cap F \in \mathcal{R}$ for all $F \in \mathcal{R}$. Hence $\bigcup_{i=1}^\infty E_i \in \mathcal{G}$. Let $A, B \in \mathcal{G}$, $(A \backslash B) \cap F = (A \cap F) \setminus (B \cap F) \in \mathcal{R}$ for all $F \in \mathcal{R}$ since $A \cap F \in \mathcal{R}, B \cap F \in \mathcal{R}$ and $\mathcal{R}$ is a σ-ring.

$\square$

Exercise 1.2 *Complete the proof of Proposition 1.2.*

Solution:

a. From the definition of the $\mathcal{B}_{\mathbb{R}}$ directly.

b. $(a,b) = \bigcup_{n=1}^{\infty} \left[a+\frac{1}{n}, b-\frac{1}{n}\right]$.

c. $(a,b) = \bigcup_{n=1}^{\infty} \left(a, b-\frac{1}{n}\right] = \bigcup_{n=1}^{\infty} \left[a+\frac{1}{n}, b\right)$.

d. If $(a,\infty) \in \mathcal{M}(\mathcal{E}_5)$, then $(-\infty, a] \in \mathcal{M}(\mathcal{E}_5)$. Hence $(a,b] = (a,\infty) \cap (-\infty, b] \in \mathcal{M}(\mathcal{E}_5)$. We can get $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{R}_5)$ by c. The other one is similar.

e. Same as d.

□

Exercise 1.3 *Let $\mathcal{M}$ be an infinite σ-algebra.*

a. *$\mathcal{M}$ contains an infinite sequence of disjoint sets.*

b. *$card(\mathcal{M}) \geq \mathfrak{c}$.*

Solution:

a. Since $\mathcal{M}$ is a σ-algebra, $X \in \mathcal{M}$. Let $E_1 = X$. Since $\mathcal{M}$ is infinite, $\mathcal{M} \backslash \{X\}$ is not empty. There is a nonempty set $A_2 \subsetneq E_1 = X$. Consider $A_2 \cap \mathcal{M}$ and $A_2^c \cap \mathcal{M}$, at least one of them is an infinite σ-algebra. Let E_2 be the set whose intersection with $\mathcal{M}$ is an infinite σ-algebra. Repeat this step, we can get an infinite sequence of sets $\cdots \subsetneq E_3 \subsetneq E_2 \subsetneq E_1$. Let $F_k = E_k \backslash E_{k+1}$, we get an infinite sequence of disjoint sets $\{F_k\}_{k=1}^{\infty}$.

b. $\text{card}(\mathbb{N}) = \text{card}(\{F_k\}_{k=1}^{\infty}) \to \mathfrak{c} = \text{card}(\mathbb{R}) \leq \text{card}(\mathcal{M})$.

□

Exercise 1.4 *An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e., if $\{E_j\}_1^{\infty} \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\bigcup_1^{\infty} E_j \in \mathcal{A}$).*

Solution: If $\mathcal{A}$ is a σ-algebra, then $\mathcal{A}$ is closed under countable increasing unions. If $\mathcal{A}$ is closed under countable increasing unions, we need to prove that $\bigcup_1^{\infty} E_j \in \mathcal{A}$ for arbitrary sequence $\{E_j\}_1^{\infty}$ contained in $\mathcal{A}$. Let $F_k = \bigcup_{j=1}^{k} E_j$, then $F_k \in \mathcal{A}$ since $\mathcal{A}$ is an algebra. Since $\mathcal{A}$ is closed under countable unions, we have $\bigcup_{k=1}^{\infty} F_k \in \mathcal{A}$. Hence $\bigcup_{i=1}^{\infty} E_i = \bigcup_{k=1}^{\infty} F_k \in \mathcal{A}$. This implies $\mathcal{A}$ is a σ-algebra. □

Exercise 1.5 *If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (Hint: Show that the latter object is a σ-algebra.)*

Solution: We need to prove

$$\mathcal{M}(E) = \bigcup_{\mathcal{F}\text{: countable subsets of } \mathcal{E}} \mathcal{M}(\mathcal{F}).$$

Obviously, we have

$$\mathcal{M}(\mathcal{E}) \supset \bigcup_{\mathcal{F}\text{: countable subsets of } \mathcal{E}} \mathcal{M}(\mathcal{F}).$$

The converse inclusion is right only if we can show the latter object is a σ-algebra. Let $\{E_i\}_{i=1}^{\infty}$ is a sequnce of the latter object. There is a countable subset of $\mathcal{E}$ $\mathcal{F}_i$ such that $E_i \in \mathcal{F}_i$. Hence $E_i = \bigcup_{j=1}^{\infty} F_{ij}$ where $F_{ij} \in \mathcal{F}_i$. $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1,j=1}^{\infty} F_{ij} \in \mathcal{F}_\omega$ where $\mathcal{F}_\omega$ is a σ-algebra generated by F_{ij}, hence $\bigcup_{i=1}^{\infty}$ is in the latter object. The closed property of the difference is obvious. $\square$

Exercise 1.6 *Complete the proof of Theorem 1.9.*

Solution: If there is another measure ν on $\overline{\mathcal{M}}$ that extends μ. Let $E \in \mathcal{M}$ and $F \subset N$ for some $N \in \mathcal{N}$. Then

$$\nu(E \cup F) \leq \nu(E \cup N) = \mu(E \cup N) = \mu(E) = \nu(E).$$

But $\nu(E) \leq \nu(E \cup F)$. Hence we get

$$\nu(E \cup F) = \mu(E) = \overline{\mu}(E \cup F)..$$

This means ν is exactly the same as $\overline{\mu}$. $\square$

Exercise 1.7 *If $\mu_1, \mu_2, \cdots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, a_2, \cdots, a_n \in [0, \infty)$, then $\sum_1^n a_j\mu_j$ is a meaure on $(X, \mathcal{M})$.*

Solution:

$$\mu(\emptyset) = \sum_1^n a_j\mu_j(\emptyset) = 0.$$

Let E_i be disjoint subets of X,

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \sum_{j=1}^{n} a_j\mu_j\left(\bigcup_{i=1}^{\infty} E_i\right) \\
&= \sum_{j=1}^{n} a_j\left(\sum_{i=1}^{\infty} \mu_j(E_i)\right) \\
&= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} a_j\mu_j(E_i)\right) \\
&= \sum_{i=1}^{\infty} \mu(E_i).
\end{aligned}$$

$\square$

Exercise 1.8 *If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^{\infty} \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\bigcup_1^{\infty} E_j) < \infty$.*

Solution:

$$\liminf E_j = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k.$$

Let $F_n = \bigcap_{k\geq n} E_k$, then it is an increasing sequnece, by continuity from below we get

$$\begin{aligned}\mu\left(\liminf E_j\right) =& \mu\left(\bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k\right) \\ =& \mu\left(\bigcup_{n=1}^{\infty} F_n\right) \\ =& \lim_{n\to\infty} \mu\left(F_n\right).\end{aligned}$$

Since $F_n \subset E_k$ for all $k \in \mathbb{N}$ and $k \geq n$, $\mu\left(F_n\right) \leq \mu\left(E_k\right)$ for all $k \in \mathbb{N}$ and $k \geq n$. Hence $\mu\left(F_n\right) = \inf_{k\geq n} \mu\left(E_k\right)$. Combine these two formulas we get

$$\mu\left(\liminf E_j\right) \leq \lim_{n\to\infty} \inf_{k\geq n} \mu\left(E_k\right) = \liminf \mu\left(E_j\right).$$

$$\limsup E_j = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k.$$

Let $G_n = \bigcup_{k\geq n} E_k$, then it is an decreasing sequence, and $\mu\left(G_1\right) < \infty$ by the condition. By continuity from above we get

$$\begin{aligned}\mu\left(\limsup E_j\right) =& \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k\right) \\ =& \mu\left(\bigcap_{n=1}^{\infty} G_n\right) \\ =& \lim_{n\to\infty} \mu\left(G_n\right).\end{aligned}$$

Since $G_n \supset E_k$ for all $k \in \mathbb{N}$ and $k \geq n$, $\mu\left(G_n\right) \geq \mu\left(E_k\right)$ for all $k \in \mathbb{N}$ and $k \geq n$. Hence $\mu\left(G_n\right) \geq \sup_{k\geq n} \mu\left(E_k\right)$. Combine these two formulas we get

$$\mu\left(\limsup E_j\right) \geq \lim_{n\to\infty} \sup_{k\geq n} \mu\left(E_k\right) = \limsup \mu\left(E_j\right).$$

$\square$

Exercise 1.9 *If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.*

Solution:

$$\mu\left(E\right) = \mu\left(E\backslash F\right) + \mu\left(E \cap F\right).$$
$$\mu\left(F\right) = \mu\left(F\backslash E\right) + \mu\left(E \cap F\right).$$
$$\mu\left(E \cup F\right) = \mu\left(E\backslash F\right) + \mu\left(F\backslash E\right) + \mu\left(E \cap F\right).$$

Then we can get

$$\mu\left(E\right) + \mu\left(F\right) = \mu\left(E \cup F\right) + \mu\left(E \cap F\right).$$

$\square$

Exercise 1.10 *Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Then μ_E is a measure.*

Solution: Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of disjoint sets in $\mathcal{M}$.

$$\begin{aligned}
\mu_E\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(\left(\bigcup_{i=1}^{\infty} E_i\right) \cap E\right) \\
&= \mu\left(\bigcup_{i=1}^{\infty} (E_i \cap E)\right) \\
&= \sum_{i=1}^{\infty} \mu(E_i \cap E) \\
&= \sum_{i=1}^{\infty} \mu_E(E_i).
\end{aligned}$$

□

Exercise 1.11 *A finitely additive measure μ is a measure iff it is continuous from below as in Theorem 1.8c. If $\mu(X) < \infty$, μ is a measure iff it is continuous from above as in Theorem 1.8d.*

Solution: Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of disjoint sets in $\mathcal{M}$. If μ is continuous from below, we set $F_k = \bigcup_{i=1}^{k} E_i$. By continuity from below, we have

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \\
&= \lim_{k\to\infty} \mu(F_k) \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \mu(E_i) \\
&= \sum_{i=1}^{\infty} \mu(E_i).
\end{aligned}$$

The converse is obvious. The second condition can be proven the same way. □

Exercise 1.12 *Let $(X, \mathcal{M}, \mu)$ be a finite measure space.*

a. *If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.*

b. *Say that $E \sim F$ if $\mu(E \Delta F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.*

c. *For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.*

Solution:

a. Since $E \backslash F, F \backslash E \subset E \Delta F$, we have $\mu(E \backslash F) = \mu(F \backslash E) \leq \mu(E \Delta F) = 0$. Hence

$$\mu(E) = \mu(E \cap F) + \mu(E \backslash F) = \mu(E \cap F) + \mu(F \backslash E) = \mu(F).$$

b. Let $E \sim F$ and $F \sim G$, what we need to prove is $E \sim G$. Since $E\backslash G = ((E \cap F)\backslash G) \cup ((E\backslash F)\backslash G)$, we have

$$\mu(E\backslash G) = \mu((E \cap F)\backslash G) + \mu((E\backslash F)\backslash G).$$

There are relations

$$((E \cap F)\backslash G) \subset F\backslash G$$

and

$$((E\backslash F)\backslash G) \subset E\backslash F.$$

Hence

$$\mu((E \cap F)\backslash G) = \mu((E\backslash F)\backslash G) = 0.$$

This implies

$$\mu(E\backslash G) = 0.$$

We can also deompose $G\backslash E$ into two disjoint parts $(G \cap F)\backslash E$ and $(G\backslash F)\backslash E$ and get $\mu(G\backslash E) = 0$ the same way. Then

$$\mu(E\Delta G) = \mu(E\backslash G) + \mu(G\backslash E) = 0.$$

c.

$$\begin{aligned}\mu(E\Delta G) =& \mu(E\backslash G) + \mu(G\backslash E) \\ \leq& \mu(F\backslash G) + \mu(E\backslash F) \\ &+ \mu(F\backslash E) + \mu(G\backslash F) \\ =& \mu(E\Delta F) + \mu(F\Delta G).\end{aligned}$$

The second line uses the conclusion of item b. □

Exercise 1.13 *Every σ-finite measure is semifinite.*

Solution: By definition, μ is called semifinite if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$. Let $X = \bigcup_{i=1}^{\infty} E_i$ with $\mu(E_i) < \infty$, $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets. Then $E = X \cap E = \bigcup_{i=1}^{\infty}(E_i \cap E)$. Since $\mu(E) \neq 0$, $\mu(E_i \cap E)$ cannot be 0 simultaneously. Hence ther exists i such that $0 < \mu(E_i \cap E) \leq \mu(E_i) < \infty$. Let $F = E_i \cap F$ and we complete the proof. □

Exercise 1.14 *If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.*

Solution: If not, there exists a constant $C_0 > 0$, for any $F \subset E$ either $\mu(F) \leq C_0$ or $\mu(F) = \infty$. Let

$$\mathcal{G} = \{F \subset E : \mu(F) \leq C_0\}.$$

Let $C_1 = \sup_{F \in \mathcal{G}} \mu(F)$. Since μ is semifinite, there is always a set $F_1 \subset \mathcal{G}$ such that $\mu(F_1) > 0$. This implies $C_1 > 0$. Now we prove that there exists a set $F_0 \in \mathcal{G}$ such that $\mu(F_0) = C_0$. We can choose a sequence $\{E_i\}_{i=1}^{\infty}$ such that $\mu(E_i) \to C_0$. Let $F_k = \bigcup_{i=1}^{k} E_i$. Then $\mu(F_k) \leq C_0$ since $F_k \subset E$ and

$$\mu(F_k) \leq \sum_{i=1}^{k} \mu(E_i) < \infty.$$

Let $F = \lim_{k\to\infty} F_k = \bigcup_{i=1}^{\infty} E_i$, then $\mu(F) \le C_0$. On the other hand

$$\mu(F) \ge \mu(F_k) \ge \mu(E_k).$$

Taking $k \to \infty$ we get $\mu(F) \ge C_0$. Hence $\mu(F) = C_0$. Since $\mu(E) = \infty$ we get $\mu(E - F) = \infty$. Since μ is semifinite we must have a subset $W \subset E$ and $0 < \mu(W) < \infty$. But this implies

$$0 < \mu(F \cup W) = C_0 + \mu(W) < \infty.$$

This contradicts the assumption. □

Exercise 1.15 *Given a measure μ on $(X, \mathcal{M})$, define μ_0 on $\mathcal{M}$ by $\mu_0 = \sup\{\mu(F) : F \subset E$ and $\mu(F) < \infty\}$.*

a. *μ_0 is a semifinite measure. It is called the **semifinite part** of μ.*

b. *If μ is semifinite, then $\mu = \mu_0$. (Use Exercise 14.)*

c. *There is a measure ν on $\mathcal{M}$ (in general, not unique) which assumes only the values 0 and ∞ such that $\mu = \mu_0 + \nu$*

Solution:

a. If $\mu_0(E) = \infty$, there must exists F such that $F \subset E$ and $\mu(F) < \infty$ by definition of μ_0. Hence μ_0 must be a semifinite measure if μ_0 is a measure. What we need to do now is to check the countable additivity of μ_0.

Before the check, it is useful to notice that $\mu_0(E) \le \mu(E)$ for all $E \in \mathcal{M}$ and $\mu_0(E) = \mu(E)$ if $\mu(E) < \infty$. We can also find easily that $\mu_0(E) \le \mu_0(F)$ if $E \subset F$.

Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of disjoint sets. If one of them, lets say, E_i, has $\mu_0(E_i) = \infty$, then the countable additivity is obvious

$$\mu_0\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu_0(E_i).$$

Now assume $\mu_0(E_i) < \infty$ for all $i \in \mathbb{N}$. There exist F_i such that $\mu_0(E_i) < \mu(F_i) + \frac{\epsilon}{2^i}$ for any $\epsilon > 0$. Then we get

$$\sum_{i=1}^{n} \mu_0(E_i) < \sum_{i=1}^{n} \mu(F_i) + \epsilon = \mu\left(\bigcup_{i=1}^{n} F_i\right) + \epsilon \le \mu_0\left(\bigcup_{i=1}^{\infty} E_i\right) + \epsilon.$$

We get the third inequality because $\mu\left(\bigcup_{i=1}^{n} F_i\right) < \infty$ and $\bigcup_{i=1}^{n} F_i \subset \bigcup_{i=1}^{\infty} E_i$. Since n and ϵ is arbitrary, we get

$$\sum_{i=1}^{\infty} \mu_0(E_i) \le \mu_0\left(\bigcup_{i=1}^{\infty} E_i\right).$$

Next we prove that the converse of the inequality also holds. If $\sum_{i=1}^{\infty} \mu_0(E_i) = \infty$, by the previous inequality we get

$$\sum_{i=1}^{\infty} \mu_0(E_i) = \mu_0\left(\bigcup_{i=1}^{\infty} E_i\right)$$

directly. Now consider $\sum_{i=1}^{\infty} \mu_0 (E_i) < \infty$. Let F be any subset of $\bigcup_{i=1}^{\infty} E_i$ and $\mu (F) < \infty$. Let $F_i = E_i \cap F$. It is easy to verify that $F_i \subset E_i$ and $\mu (F_i) < \infty$. Then we have

$$\sum_{i=1}^{\infty} \mu_0 (E_i) \geq \sum_{i=1}^{\infty} \mu (F_i) = \mu \left(\bigcup_{i=1}^{\infty} F_i \right).$$

By definition $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (E_i \cap F) = \bigcup_{i=1}^{\infty} F$. Hence

$$\sum_{i=1}^{\infty} \mu_0 (E_i) \geq \mu (F).$$

Since $F \subset \bigcup_{i=1}^{\infty} E_i$ and $\mu (F) < \infty$, we get

$$\sum_{i=1}^{\infty} \mu (E_i) \geq \mu_0 \left(\bigcup_{i=1}^{\infty} E_i \right).$$

b. Let $E \in \mathcal{M}$, $\mu (E) = \mu_0 (E)$ occurs only if $\mu (E) = \infty$. But μ is semifinite, for any $C > 0$ we can choose a set F such that $C < \mu (F) < \infty$ by using Exercise 14. This means $\mu_0 (E) = \infty$. Hence $\mu = \mu_0$.

c. Define $\nu = \mu - \mu_0$ and we are done.

$\square$

Exercise 1.16 *Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called **locally measurable** if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu (A) < \infty$. Let $\tilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \tilde{\mathcal{M}}$; if $\mathcal{M} = \tilde{\mathcal{M}}$, then μ is called **saturated**.*

a. *If μ is σ-finite, then μ is saturated.*

b. *$\tilde{\mathcal{M}}$ is a σ-algebra.*

c. *Define $\tilde{\mu}$ on $\tilde{\mathcal{M}}$ by $\tilde{\mu} (E) = \mu (E)$ if $E \in \mathcal{M}$ and $\tilde{\mathcal{M}} (E) = \infty$ otherwise. Then $\tilde{\mu}$ is a saturated measure on $\tilde{\mathcal{M}}$, called the **saturation** of μ.*