# EXACT CONTROL FOR SCHRÖDINGER EQUATIONS ON TORI

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Abstract. We prove an observability inequality for free Schrödinger equations on tori $\mathbb{T}^1$ and give an exact controllability constant.

## 1. Introduction

We consider the following system

$$\begin{cases} u_t + iu_{xx} = 0 & \text{in } \mathbb{R}\times(0,2\pi) \\ u(t,0)=u(t,2\pi) & \text{for } t\in\mathbb{R} \\ u_x(t,0)=u_x(t,2\pi) & \text{for } t\in\mathbb{R} \\ u(0,x)=u_0(x) & \text{for } x\in(0,2\pi). \end{cases} \tag{1}$$

Setting $L^2 := L^2(0,2\pi)$ for brevity and introducing the Sobolev space

$$H_p^2 := \left\{ v\in H^2(0,2\pi) : v(0)=v(1) \text{ and } v_x(0)=v_x(1) \right\},$$

for each intial datum $u_0 \in H_p^2$ there is a unique weak solution

$$u\in C(\mathbb{R},H_p^2)\cap C^1(\mathbb{R},L^2).$$

Furthermore, $u$ has a Fourier series representation

$$u(t,x)=\sum_{k\in\mathbb{Z}} c_k e^{i\left(k^2t+kx\right)} \tag{2}$$

where the $c_k$'s are the Fourier coefficients of $u_0$ :

$$u_0(x)=\sum_{k\in\mathbb{Z}} c_k e^{ikx}.$$

Note that $u(t,x)$ periodical in $t$, we consider $u(t,x)$ in torus $\mathbb{T}^2=\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z}$. We want to prove

**Theorem 1.1.** *Fix $(t_1,x_1)\in\mathbb{T}^2$, $a\in\mathbb{R}$ and $T>0$ arbitrarily, and consider the solutions of (1).*

(i) *The inequality*

$$\int_0^T |u(t_1+t,x_1+at)|^2 \,\mathrm{d}t \le C_2 \sum_{k\in\mathbb{Z}} |c_k|^2 \tag{3}$$

*always holds.*

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(ii) *If $a \notin \mathbb{Z}$, then the inequality*

$$C_1 \sum_{k\in\mathbb{Z}} |c_k|^2 \leq \int_0^T |u(t_1+t, x_1+at)|^2 \,\mathrm{d}t \leq C_2 \sum_{k\in\mathbb{Z}} |c_k|^2 \tag{4}$$

*also holds and $C_1^{-1}$ can be chosen as $C_1 e^{\frac{C_2}{T}}$.*

**Remark.** Theorem 1.1 has been proven in [2] **EXCEPT** the estimate of constants $C_1$ and $C_2$.

## 2. Ingham Inequality

**Definition 2.1.** A set $\Lambda$ of a real numbers is called *uniformly separated* if

$$\gamma(\Lambda) := \inf\{|\lambda_1 - \lambda_2| : \lambda_1, \lambda_2 \in \Lambda \text{ and } \lambda_1 \neq \lambda_2\} > 0. \tag{5}$$

Then $\gamma(\Lambda)$ is called the *uniform gap* of $\Lambda$.

The following three theorems are from [3].

**Theorem 2.2.** *Let $f(t) = \sum_{k\in\mathbb{Z}} a_k e^{i\lambda_k t}$ where $\Lambda := \{\lambda_n\}_{n\in\mathbb{Z}}$ is a uniformly separated set with $\gamma(\Lambda) = \gamma_0 > 0$. Let $I = [0,T]$ with $T > 0$. Then, there exists a positive constant $C_1^0 = C_1^0(|I|, \gamma_0) = \frac{2(|I|\gamma_0 - 2\pi)(|I|\gamma_0 + 2\pi)}{\pi(I\gamma_0)^2}|I|$ such that for all $\{\lambda_n\}_{n\in\mathbb{Z}} \in \ell^2$,*

$$C_1^0 \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_I |f(t)|^2 \,\mathrm{d}t. \tag{6}$$

*Proof.* s □

**Theorem 2.3.** *Let $f(t) = \sum_{k\in\mathbb{Z}} a_k e^{i\lambda_k t}$ where $\Lambda := \{\lambda_n\}_{n\in\mathbb{Z}}$ is a uniformly separated set with $\gamma(\Lambda) = \gamma_0 > 0$. Let $I = [0,T]$ with $T > 0$. Then, there exists a positive constant $C_2^0 = C_2^0(\gamma_0, |I|) = \frac{10}{\min\{\pi, |I|\gamma_0\}}|I|$ such that for all $\{a_n\}_{n\in\mathbb{Z}} \in \ell^2$,*

$$\int_I |f(t)|^2 \,\mathrm{d}t \leq C_2^0 \sum_{n\geq 1} |a_n^2|. \tag{7}$$

**Theorem 2.4.** *Let $f(t) = \sum_{k\in\mathbb{Z}} a_k e^{i\lambda_k t}$ where $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ is a uniformly separated set with $\gamma := \gamma(\Lambda) > 0$. We assume that there exist $N \geq 1$ and $\gamma_\infty > 0$ such that*

$$\gamma(\Lambda \backslash \{\lambda_1, \cdots, \lambda_N\} \geq \gamma_\infty. \tag{8}$$

*Let $I = (0,T) \subset \mathbb{R}$ be a finite interval with $|I| > \frac{2\pi}{\gamma_\infty}$. Then there exist two positive constants $C_1, C_2 > 0$ such that for all $(a_n)_{n\in\mathbb{Z}} \in \ell^2$,*

$$C_1 \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_I |f(t)|^2 \,\mathrm{d}t \leq C_2 \sum_{k\in\mathbb{Z}} |a_k|^2. \tag{9}$$

*More precisely $C_2 = C_2(\gamma) = \frac{10|I|}{\min\{\pi, |I|\gamma\}}$ and $C_1 = C_1(N)$ is given by the following recurrent formula:*

$$C_1(j) = \left[\left(\frac{2C_2(r_j)}{|I|} + 1\right)\frac{4}{C_1(j-1)p_j} + \frac{2}{|I|}\right]^{-1}, \quad 1 \leq j \leq N. \tag{10}$$

*where* $r_j = \min\{|\lambda-\lambda'| : \lambda, \lambda' \in \Lambda\backslash\{\lambda_1, \cdots, \lambda_{N-j}\}\}$, $p_j = \min\left\{1, \frac{4r_j^2}{\pi^4}\left(|I| - \frac{2\pi}{\gamma_\infty}\right)^2\right\}$ *with* $0 \le j \le N$.

## 3. Estimates

Consider $f(t) = u(t_1+t, x_1+at) = \sum_{k\in\mathbb{Z}} c_k e^{ik^2t_1+kx_1} e^{i\left(k^2+ak\right)t} = \sum_{k\in\mathbb{Z}} a_k e^{i\left(k^2+ak\right)t}$, where we use the notaions $a_k = e^{ik^2t_1+kx_1}$, $k\in\mathbb{Z}$.

(ii) If $a \notin \mathbb{Z}$, then the set $\{k^2+ak : k\in\mathbb{Z}\}$ itself is uniformly separated. Indeed, if $k$ and $m$ are different integers, then

$$\left|(k^2+ak)-(m^2+am)\right| = |k-m|\,|k+m+a| \ge d(a,\mathbb{Z}) := \max\left(a-[a], [a]+1-a\right)$$

where $[a]$ is the integer part of $a$. For some positive integer $N$, $k\neq m$ and $k,m \notin \{-N,\cdots,N\}$, then, using again the identity

$$(k^2+ak)-(m^2+am) = (k-m)(k+m+a),$$

For simplicity, we assume $0<a<1$, then we have

$$\left|(k^2+ak)+(m^2+am)\right| \ge \begin{cases} 2N+a & \text{if } k>m\ge N,\\ 2N-a & \text{if } k<m\le -N,\\ 2Nd(a,\mathbb{Z}) & \text{if } km<0.\end{cases}$$

It follows that

$$\gamma(\Lambda\backslash\{-N,\cdots,N\}) \ge Nd(a,\mathbb{Z})$$

for all integers.

In order to apply Theorem 2.4, we set $\lambda_1=0, \lambda_2=1, \lambda_3=-1, \cdots, \lambda_{2N}=N, \lambda_{2N+1}=-N$. Then we have

$$\begin{aligned}
&r_0 = \gamma\left(\Lambda\backslash\{-N,\cdots,N\}\right) \ge Nd(a,\mathbb{Z}),\\
&r_1 = \gamma\left(\Lambda\backslash\{-N+1,\cdots,-N\}\right) \ge (N-1)d(a,\mathbb{Z}),\\
&r_2 = \gamma\left(\Lambda\backslash\{-N+1,\cdots,N-1\}\right) \ge (N-1)d(a,\mathbb{Z}),\\
&\cdots\\
&r_{2N-2} = \gamma\left(\Lambda\backslash\{-1,0,1\}\right) \ge d(a,\mathbb{Z}),\\
&r_{2N-1} = \gamma\left(\Lambda\backslash\{0,1\}\right) \ge d(a,\mathbb{Z}),\\
&r_{2N} = \gamma(\Lambda\backslash\{0\} \ge d(a,\mathbb{Z}),\\
&r_{2N+1} = \gamma\left(\Lambda\right) = d(a,\mathbb{Z}).
\end{aligned}$$

In fact, for $N$ large enough, we have $\gamma_0 \sim N$. By (10), we have

$$\begin{aligned}
(C_1(N))^{-1} &\le \frac{2C_2(N)}{T}\frac{4}{p_N}\left(C_1(N-1)\right)^{-1} \le \cdots\\
&\le \left(\frac{8}{T}\right)^N\left(\prod_{1\le n\le N}\frac{C_2(n-1)}{p_n}\right)(C_1(0))^{-1},
\end{aligned}$$

where $C_1(0) := C_1^0$ and $C_2(0) := C_2^0$ in Theorem 2.2 and 2.3 with $\gamma_0 = \gamma_\infty$. Besides, we have

$$C_2(n) = C_2(r_n) = \frac{10T}{\min\{\pi, Tr_n\}} \leq 10\left(\frac{T}{\pi} + \frac{1}{r_n}\right) \leq 10\frac{Tr_n + \pi}{\pi r_n}, \tag{11}$$

and

$$p_n = \min\left\{1, \frac{4r_n^2}{\pi^4}\left(T - \frac{2\pi}{\gamma_\infty}\right)\right\}. \tag{12}$$

In our case, we observe that

$$r_0 > r_2 > \cdots > r_{2N}.$$

Choose the smallest $N$ such that $T > 2\frac{2\pi}{r_0}$, then we have

$$2\frac{2\pi}{r_0} < T \leq 2\frac{2\pi}{r_1}. \tag{13}$$

This implies $T \sim \frac{1}{N}$. We now use $C_1(N)$ to denote the previous $2N+1$'th term and $C_1(k)$ to denote the $2k+1$'s term, and other notations the same way. Then we have

$$(C_1(N)^{-1} \leq \left(\frac{8}{T}\right)^{2N}\left(\prod_{1\leq n\leq N}\frac{C_2(n-1)}{p_n}\right)^2 (C_1(0))^{-1}.$$

Since $C_2(n) \leq 10\frac{Tr_n+\pi}{\pi r_n} \leq 10\frac{Tr_0+\frac{Tr_0}{4}}{\pi r_n} = \frac{25}{2\pi}\frac{Tr_0}{r_n}$ and $p_n = \frac{4r_n^2}{\pi^4}\left(T - \frac{2\pi}{r_0}\right) \geq \frac{8}{\pi^3}\frac{r_n^2}{r_0}$ for $r_n \leq c\sqrt{N}$ where $c$ is a numerical constant, we have

$$\begin{aligned}
&(C_1(N))^{-1} \\
\leq &\left(\frac{8}{T}\right)^{2N}\left(\frac{25T}{2\pi}\right)^{2N} r_0^{2N}\left(\prod_{1\leq n\leq N}\frac{1}{r_n^2}\right)\left(\prod_{1\leq n\leq c\sqrt{N}}\frac{1}{r_n^2}\right)^2 r_0^{2c\sqrt{N}}\left(\frac{\pi^3}{8}\right)^{2c\sqrt{N}}\frac{\pi T}{2\left(T-\frac{2\pi}{r_0}\right)\left(T+\frac{2\pi}{r_0}\right)} \\
\lesssim &\left(\frac{100}{\pi}\right)^{2N} N^{2N}\left(\frac{e^{2N}}{N^{2N}2\pi N}\right)\left(\frac{e^{2c\sqrt{N}}}{\left(c\sqrt{N}\right)^{2c\sqrt{N}}2\pi c\sqrt{N}}\right)^2\left(\frac{\pi^3}{8}\right)^{2c\sqrt{N}} N^{2c\sqrt{N}}\frac{\pi T}{\left(\frac{4\pi}{r_0}\right)^2} \\
\lesssim &C_1 e^{\frac{C_2}{T}},
\end{aligned}$$

where $C_1, C_2$ are two numerical constants depending on the choice of $a$. For a ball $B_r(t_1, x_1)$ with $r$ small enough, choose $a = \frac{\sqrt{2}}{2}$, then we have (imagine an oblique inscribed square with slope $a$)

$$\int_0^{2\pi} |u_0|^2 \,\mathrm{d}x \leq \frac{C_1' e^{\frac{C_2'}{T}}}{T}\int_{B_r} |u|^2 \,\mathrm{d}x\mathrm{d}t. \tag{14}$$

**Theorem 3.1.** *Let $u(t,x)$ be the solution of (1), and $\Omega = [t_0, t_0+T] \times [x_0, x_0+l]$ for any fixed $(t_0, x_0) \in \mathbb{T}^2$ and any $T, L > 0$. Then we have*

$$\int_0^{2\pi} |u_0|^2 \,\mathrm{d}x \leq \frac{C_1' e^{\frac{C_2'}{L}}}{T} \int_\Omega |u|^2 \,\mathrm{d}x\mathrm{d}t. \tag{15}$$

## References


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