ALGEBRAIC GEOMETRY
PART I

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ABSTRACT. It is a note when I study algebraic geometry myself in YouTube from the channel ICTP Math, the speaker of the videos is Lothar Göttsche.

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Proposition 1. *Same as an affine space, in a projective space we have the following propositions:*

1. $X \subset Y \subset \mathbb{P}^n$ *are projective algebraic sets, then* $I(X) \supset I(Y)$*;*
2. $X \subset \mathbb{P}^n$ *is a projective algebraic set, then* $Z(I(X)) = X$*;*
3. $\mathfrak{a} \subset k[x_0, \ldots, x_n]$ *is a homogeneous ideal, then* $I(Z(\mathfrak{a})) \supset \mathfrak{a}$*;*
4. *If* $S \subset k[x_0, \ldots, x_n]$ *is a set of homogeneous polynomials, then* $Z(S) = Z(\langle S \rangle)$*;*
5. *For a family* $\{S_\alpha\}$ *of sets of homogeneous polynomials,* $Z(\bigcup\limits_{\alpha} S_\alpha) = \bigcap\limits_{a} lpha Z(S_\alpha)$*;*
6. *If* $T, S \subset k[x_0, \ldots, x_n]$ *are sets of homogeneous polynomials, then* $Z(ST) = Z(S) \cup Z(T)$*.*

Remark. From the proposition (5) and (6) we know that arbitrary intersections and finite unions of projective algebraic sets are projective algebraic sets, then we can define a topology through these two propositions.

Definition 1. The Zariski topology on $\mathbb{P}^n$ is the topology whose closed sets are the projective algebraic sets.

If $X \subset \mathbb{P}^n$ is a subset, we give it the induced topology, called Zariski topology on X.

Definition 2. A quasi-projective algebraic set is an open subset of a projective algebraic set. Fro example, let U and V be closed subsets, then $Y = U \backslash V \neq \emptyset$ is a quasi-projective algebraic set.

Proposition 2. *We jnow* $k[x_0, \ldots, x_n]$ *is noetherian, then follows the same proof as in affine case shows that* $\mathbb{P}^n$ *is a noetherian topological sapce.*

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Remark. Every subspace of $\mathbb{P}^n$ is noetherian. In particular, quasi-projective algebraic sets are noetherian, hence have unique decompositions into irreducible components.

Definition 3. A quasi-projective variety is an irreducible quasi-projective algebraic set.

Remark. If we use the identification $\mathbb{A}^n = U_0 \subset \mathbb{P}^n$, then $\mathbb{A}^n$ is an open set $\mathbb{A}^n = \mathbb{P}^n \backslash Z(x_0)$, i.e. $\mathbb{A}^n$ is a quasi-projective variety.

Definition 4. A nonempty algebraic set $X \subset \mathbb{A}^{n+1}$ is called a cone if for all $p = (a_0, \dots, a_n) \in X$ and all $\lambda \in k$, we have $(\lambda a_0, \dots, \lambda a_n) = \lambda p \in X$.

If $X \subset \mathbb{P}^n$ is a projective algebraic set, its affine cone is

$$C(X) := \{(a_0, \dots, a_n) \in \mathbb{A}^{n+1} | [a_0, \dots, a_n] \in X\} \cup \{0\} \tag{0.1}$$

Lemma 1. *Let $X \neq \emptyset$ be a projective algebraic set, then :*

1. $X = Z_p(\mathfrak{a})$*, for* $\mathfrak{a} \subset k[x_0, \dots, x_n]$ *a homogeneous ideal* $\Rightarrow C(X) = Z_a(\mathfrak{a}) \subset \mathbb{A}^{n+1}$*;*
2. $I_a(C(X)) = I_H(X)$*.*

Theorem 1 (Projective Nullstellensatz)**.** *Let $\mathfrak{a} \subset k[x_0, \dots, x_n]$ be a homogeneous ideal:*

1. $Z_p(\mathfrak{a}) = \emptyset \Leftrightarrow \mathfrak{a}$ *contains all homogeneous polynomials of degree N for some $N \in \mathbb{N}$;*
2. *If $Z_p(\mathfrak{a}) \neq \emptyset$, then $I_p(Z_p(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof. Let $X = Z_p(\mathfrak{a})$.

(1) $X = \emptyset \Leftrightarrow C(X) = \{0\}$. Since $C(X) = Z_a(\mathfrak{a}) \cup \{0\}$, we get

$$X = \emptyset \Leftrightarrow Z_a(\mathfrak{a}) = \emptyset \text{ or } Z_a(\mathfrak{a}) = \{0\}.$$

By affine Nullstellensatz, we get

$$\sqrt{\mathfrak{a}} = k[x_0, \dots, x_n] \text{ or } \sqrt{\mathfrak{a}} = \langle x_0, \dots, x_n \rangle.$$

So $\sqrt{\mathfrak{a}} \supset \langle x_0, \dots, x_n \rangle$. Thus for any $i = 0, \dots, n$, $\exists m_i$ s.t. $x_i^{m_i} \in \mathfrak{a}$. Let $N = m_1 + \cdots + m_n$, then any monomial of degree N in $k[x_0, \dots, x_n]$ lies in $\mathfrak{a}$.

(2)Let $X = Z_p(\mathfrak{a} \neq \emptyset$, then

$$I_H(X) = I_a(C(X)) = I_a(Z_a(\mathfrak{a})) = \sqrt{\mathfrak{a}}. \tag{0.2}$$

□

Remark. $\langle x_0, \dots, x_n \rangle$ is called the irrelevant ideal, an ideal different from $\langle x_0, \dots, x_n \rangle$ is called relevant.

Corollary 1. *There is a one-to-one correspondence between homogeneous relevant radical ideals and projective algebraic sets:*

Z_p: homogeneous relevant radical ideals in $k[x_0, \dots, x_n] \to$ projective algebraic sets in $\mathbb{P}^n$

I_H: projective algebraic sets in $\mathbb{P}^n \to$ homogeneous relevant radical ideals in $k[x_0, \dots, x_n]$.

Remark. We use subscripts to recognize affine spaces and projective spaces, such as $Z_p(\mathfrak{a}), Z_a(\mathfrak{a})$. Sometimes we can infer the difference from the context, so we usually write briefly as $Z(\mathfrak{a})$.

Proposition 3. (1) *A projective algebraic set $X \neq \emptyset \subset \mathbb{P}^n$ is irreducible if and only if $I = I_H(X)$ is a homogeneous prime ideal;*
(2) *If $f \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial and irreducible, then $Z_p(f)$ is irreducible.*

Proof. (1) $\Leftarrow$: Assume X reducible, then $X = X_1 \cup X_2$,$X_1, X_2 \subsetneqq X$b are closed subsets. Then we get $C(X) = C(X_1) \cup C(X_2)$, $C(X_1) \subsetneqq C(X)$,$C(X_2) \subsetneqq C(X)$ are closed, hence $C(X)$ is reducible, $I_H(X) = I(C(X))$ is not prime.

$\Rightarrow$: Assume $I_H(X)$ not prime, it means $\exists f, g \in k[x_0, \ldots, x_n]$, $fg \in I_H(X)$ and $f, g \notin I_H(X)$. Let $i, j \in \mathbb{Z} \geq 0$ be minimal such that $f^{(i)} \notin I$ and $g^{(j)} \notin I$. Subtract homogeneous components of lower degrees from f and g, we can assume f starts in degree i and g starts in degree j. Thus $f^{(i)}g^{(j)}$ is homogeneous component of minimal degree in $fg \in I$. Because I is homogeneous, we get $f^{(i)}g^{(j)} \in I$. Let $X_1 := Z(I) \cap Z(f^(i))$ and $X_2 := Z(I) \cap Z(g^{(j)})$, then $X_1, X_2 \subsetneqq X$, $X = X_1 \cup X_2$, thus X is reducible.

(2) If $I \subset k[x_0, \ldots, x_n]$ is homogeneous and prime with $Z(I) \neq \emptyset$, then follow the result from (1) we know $Z(f)$ is irreducible. $\square$

1. Functions And Morphisms

Definition 5. Let $X \subset \mathbb{A}^n$ be an affine algebraic set, the affine coordinate ring of X is

$$A(X) := k[x_0, \ldots, x_n]/I(X). \tag{1.1}$$

It is a ring, also a k-algebra.

Definition 6. A polynomial function on X is a function $f : X \to k$ s.t. $f = F|_X$ for $F \in k[x_0, \ldots, x_n]$. This is the ring with pointwise addition and multiplication:

$$(f + g)(p) = f(p) + g(p), fg(p) = f(p)g(p), \forall p \in X.$$

There is a ring homomorphism:

$$\begin{gathered} k[x_0, \ldots, x_n] \to \{\text{polynomial functions on } X\} \\ F \to F|_X \end{gathered}$$

It is surjective and its kernel is $I(X)$. Thus we have the isomorphism:

$$A(X) \cong \{\text{polynomial functions on } X\}.$$

We will not distinguish them.

Remark. The zero set of a polynomial function is closed.Let X be an affine algebraic set, $f \in A(X)$, then

$$Z(f) = \{p \in X | f(p) = 0\} \tag{1.2}$$

is closed in X. $f \in A(X)$ means $f = F|_X$ for some $F \in k[x_1, \ldots, x_n]$, then

$$Z(f) = \{p \in X | F(p) = 0\} = X \cap Z(F) \tag{1.3}$$

so it is closed.

Definition 7. Let X be an affine variety, then $I(X)$ is prime, then $A(X)$ is integral. The quotient field $Q(A(X))$ is a field of rational functions on X and denoted by $K(X)$. Let $V \subset X$ be a quasi-affine variety, since $I(V) = I(X)$, we can denote its field of rational functions by $K(V) := K(X)$.

Definition 8. Let $p \in V$, the local ring of V at p is

$$\mathcal{O}_{V,p} := \{h \in K(V) | \exists f, g \in A(V), \text{ s.t. } h = \frac{f}{g} \text{ and } g(p) \neq 0\} \tag{1.4}$$

For simplicity in future we can write this:

$$\mathcal{O}_{V,p} = \{\frac{f}{g} \in K(V) | g(p) \neq 0\}. \tag{1.5}$$

If $U \subset V$ is an open subset, the regular functions on U are defined by

$$\mathcal{O}_V(U) = \underset{V,p}{\cap} \subset K(V). \tag{1.6}$$

Proposition 4. *We have an injective ring homomorphism:*

$$\mathcal{O}_V(U) \to \{\text{functions from } U \text{ to } k\}.$$

For $h \in \mathcal{O}_V(U), p \in U$, there exists an open subset W and $p \in W \subset U$, s.t. $h = \frac{f}{g}$ with $g(p) \neq 0$. We define the homomorphism by setting $h(p) = \frac{f(p)}{g(p)}$, the homomorphism is

$$h \in \mathcal{O}_V(U) \to h(p) = \tfrac{f(p)}{g(p)}, p \in U.$$

Proof. It is well defined: if $h = \frac{f}{g} = \frac{f'}{g'}$ with $g(p) \neq 0, g'(p) \neq 0$.Then $fg' = f'g \Rightarrow f(p)g'(p) = f'(p)g(p) \Rightarrow \frac{f(p)}{g(p)} = \frac{f'(p)}{g'(p)}$.

Injective: Let $h, h' \in \mathcal{O}_V(U)$ such that $h(p) = h'(p) \forall p \in U$. Define $l = h - h' \in \mathcal{O}_V(U)$, then $l(p) = 0, \forall p \in U$. There exists an open subset W, s.t. $l = \frac{f}{g}$ with $g(p) \neq 0 \forall p \in W$. For $p \in W$, $l(p) = \frac{f(p)}{g(p)} = 0 \Rightarrow f(p) = 0 \forall p \in W$.As zero set $Z(f)$ of f is closed, we get $f = 0 \in A(V)$, then $l = 0$ and hence $h = h'$. □

Remark. We had called $\mathcal{O}_{V,p}$ a local ring of V at p. The maximal ideal at p is $\mathfrak{m}(p) := \{h \in \mathcal{O}_{V,p} | h(p) = 0\}$, this is a maximal ideal in $\mathcal{O}_{V,p}$. It is easy to verify that the local ring of a variety is alocal ring.

Proposition 5. *For an affine variety X, functions which are regular functions everywhere are polynomial functions, i.e., $\mathcal{O}_X(X) = A(X)$.*

Proof. Obviously, $A(X) \subset \mathcal{O}_X(X)$. We have to show the other inclusion. Let $h \in \mathcal{O}_X(X)$, $\forall p \in X$, $\exists F_p, G_p \in k[x_1, \ldots, x_n]$ s.t. $h = \frac{[F_p]}{[G_p]}$ and $G_p(p) \neq 0$. It is equivalent to: $\forall p \in X$, $\exists G_p \in k[x_1, \ldots, x_n]$ s.t. $h \cdot [G_p] \in A(X)$ and $[G_p(p)] \neq 0$. Let

$$\mathcal{G} := \{G \in k[x_1, \ldots, x_n] | h \cdot [G_p] \in A(X)\} \tag{1.7}$$

$\mathcal{G}$ is an ideal and $\mathcal{G} \supset I(X)$, so $Z(\mathcal{G}) \subset X$. But $Z(\mathcal{G}) \cap X = \emptyset$, so $Z(\mathcal{G}) = \emptyset$. By Nullstellensatz $1 \in \mathcal{G}$, so $h = h \cdot 1 \in A(X)$. □

Definition 9. Let $X \subset \mathbb{P}^n$ be a projective algebraic set. The homogeneous coordinate ring of X is defined as

$$S(X) := k[x_0, \ldots, x_n]/I_H(X) \tag{1.8}$$

If X is irreducible, then $S(X)$ is an integral domain, $Q(S(X))$ is its quotient field.

Remark. $X \subset \mathbb{P}^n$ is a quasi-projective variety, then polynomial $F \in k[x_0, \ldots, x_n]$ will not define a function $X \to k$. But we can take quotients of homogeneous polynomials of the same degree and get a well defined function.

Definition 10. Let $f = [F] \in S(X), F \in k[x_0, \dots, x_n]$. The homogeneous part $f^{(d)}$ of f is $[F^{(d)}] \in S(X)$, and $S^{(d)}(X) = \{f^{(d)} \in S(X)\}$.

Definition 11. X is a quasi-projective variety, the field of rational functions on X(on $V \subset X$ open subset) is $K(V) := K(X) := \{\frac{f}{g} \in Q(S(X)) | f, g \text{ both in } S^{(d)}(X) \text{ for some d}\}$. Elements of $K(X)(K(V))$ are called rational functions on X(on V).

Definition 12. Let $p \in V \subset \mathbb{P}^n$, the local ring of V at p is

$$\mathcal{O}_{V,p} := \{\frac{f}{g} \in K(V) | g(p) \neq 0\}. \tag{1.9}$$

If $U \subset V$ is open, the ring of regular functions on U is

$$\mathcal{O}_V(U) := \bigcap_{p \in U} \mathcal{O}_{V,p}. \tag{1.10}$$

Proposition 6. (1) *(k-algebra)Constant functions $a \in k$ are regular on U. If $f, g \in \mathcal{O}_V(U)$, then $f + g$ and fg are regular on U, and if g has no zero in U, then $\frac{f}{g} \in \mathcal{O}_V(U)$.*

(2) *(Local)Let (U_i) be a open cover of U. A function $f : U \to k$ is regular if and only if $f|_{U_i}$ is regular for all i.*

(3) *Regular functions are continuous. i.e., let $h \in \mathcal{O}_V(U)$, then $h : U \to k = \mathbb{A}^1$ is continuous($k = \mathbb{A}^1$ is given Zariski topology).*

Proof. (1) By definition, $\mathcal{O}_V(U) = \bigcap_{p \in U} \mathcal{O}_{V,p}$, thus enough to show if $f, g \in \mathcal{O}_{V,p}$, then $f + g, fg \in \mathcal{O}_{V,p}$, and it is obvious. Assume g has no zero on U, then $g\frac{1}{g} \in \mathcal{O}_V(U)$, then $\frac{f}{g} \in \mathcal{O}_V(U)$.

(2) $h : U \to k$ is regular $\Leftrightarrow h \in \mathcal{O}_{V,p} \forall p \in U \Leftrightarrow h \in \mathcal{O}_{V,p} \forall p \in U_i \forall i$.

(3) $h : U \to k$ is continuous $\Leftrightarrow h|_{U_i}$ is continuous for all U_i of an open cover of U. We just replace U by a suitable U_i and show h is continuous in U_i. From the definition of regular functions, we can simply assume $h = \frac{f}{g}, f, g \in k[x_0, \dots, x_n]$ are homogeneous of the same degree, and g has no zero on U_i. Zariski topology on $\mathbb{A}^1$ has closed subsets $\emptyset, k$ and finite points subsets. Thus we only have to show $h^{-1}(a)$ is closed in U_i for all a in k,

$$h^{-1}(a) = \{p \in U_i | h(p) = a\} = \{p \in U_i | (f - ag)(p) = 0\}. \tag{1.11}$$

This is the zero set $Z(f - ag) \cap U$, hence the inverse of the closed sets are closed, hence h is continuous in $U_i \forall i$, hence continuous in U. $\square$

Definition 13 (Polynomial Map)**.** Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be affine algebraic sets. A map

$$(F_1, \dots, F_m) : X \to Y, p \to (F_1(p), \dots, F_m(p)), F_1, \dots, F_m \in k[x_1, \dots, x_n]$$

is called a polynomial map. A surjective polynomial map whose inverse is also a polynomial map is an isomorphism.

Example 1. (1) If X is an affine algebraic set, the polynomial map $f : X \to k$ is the polynomial function in $A(X)$.

(2) Let $X = \mathbb{A}^1$, $Y = Z(y - x^2) \subset \mathbb{A}^2$, the polynomial map

$$(t, t^2) : \mathbb{A}^2 \to Y$$

is isomorphism.

Definition 14. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine algebraic sets. Let

$$\varphi : X \to Y$$

be a polynomial map. The pull back of $h \in A(Y)$ is $\varphi^* h := h \circ \varphi \in A(X)$. If $h = H|_Y, H \in k[y_1, \dots, y_m]$, $\varphi = (F_1, \dots, F_m)$, then

$$\varphi^* h(a_1, \dots, a_n) = h(F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)).$$

i.e.,

$$\varphi^* h = H(F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n))|_X \in A(X).$$

The pull back $\varphi^* : A(Y) \to A(X)$ is obviously a ring homomorphism. If $\varphi : X \to Y$ is an isomorphism, then $\varphi^* : A(Y) \to A(X)$ is an isomorphism of k-algebra.

Definition 15. Let X, Y be varieties, a map $\varphi : X \to Y$ is a morphism(regular map) if :

(1) φ is continuous;
(2) for all open subsets $U \in Y$, all regular functions $f \in \mathcal{O}_Y(U)$, we have

$$\varphi^* := f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U)).$$

Remark. Thus for each open subset $U \in Y$,

$$\varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$$

is a k-algebra homomorphism. φ is called an isomorphism if φ is bijective and φ^{-1} is also a morphism.

(1) id_X is a morphism form X itself.
(2) If $\varphi : X \to Y, \psi : Y \to Z$ are morphisms, then

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$$

.
(3) If $\varphi : X \to Y$ is isomorphism, then $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism for all $U \subset Y$.

Proposition 7. (1) *Let $\varphi : X \to Y$ and $(U_i)_{i \in I}$ be an open cover of X s.t. $\varphi|_{U_i} : U_i \to Y$ is a morphism. Then φ is a morphism.*
(2) *Let $Z \subset X, W \subset Y$ be varieties, let $\varphi : X \to Y$ be a morphism with $\varphi(Z) \subset W$. Then $\varphi|_Z : Z \to W$ is a morphism.*

Proof. (1) Let $W \subset Y$ be open, then we can write $\varphi^{-1}(W) = \bigcup_{i \in I} (\varphi|_{U_i}^{-1}(W))$, it is open so φ is continuous. Let $h \in \mathcal{O}_Y(W)$ then the pull back of regular functions h from $\mathcal{O}_Y(W)$ to $\mathcal{O}_X(U_i \cap \varphi^{-1}(W))$ is $\varphi|_{U_i}^* h = \varphi^* h|_{U_i \cap \varphi^{-1}(W)}$, since $\varphi|_{U_i}$ is a morphism we get that $U_i \cap \varphi^{-1}(W)$ is open. Then

$$\varphi^{-1}(W) = \bigcup_{i \in I} U_i \cap \varphi^{-1}(W) \tag{1.12}$$

and $(U_i \cap \varphi^{-1}(W))_{i \in I}$ is an open cover of $\varphi^{-1}(W)$, then we can get the conclusion that φ is a morphism by proposition 6.

(2) First, $\varphi|_Z$ is continuous as a restriction of a continuous map. Let $U \subset W$ be open, let $h \in \mathcal{O}_W(U)$. Replace if necessary U by a smaller open subset sucht that we can assume $h = \frac{F}{G}$. This quotient also defines a regular function H on open subset $\tilde{U} \subset Y$ s.t. $U \subset \tilde{U}$, then $\varphi^* H \in \mathcal{O}_X(\varphi^{-1}(\tilde{U}))$ is regular. Then $\varphi^* h = \varphi^* H|_{\varphi^{-1}(U) \cap Z}$ is regular on $\varphi^{-1}(U) \cap Z$. □

Definition 16. An affine variety is a variety which is isomorphis to irreducible closed subset of some $\mathbb{A}^n$.

Theorem 2. *Let X, Y be subvarieties, assume $Y \subset \mathbb{A}^n$. A map $\varphi : X \to Y$ is a morphism if and only if $\exists f_1, \dots, f_n \in \mathcal{O}_X(X)$ s.t.*

$$\varphi(p) = (f_1(p), \dots, f_n(p)), \forall p \in X. \tag{1.13}$$

We can write $\varphi = (f_1, \dots, f_n)$.

Proof. $\Rightarrow$: Let $\varphi : X \to Y$ be a morphism. Let $y_1, \dots, y_n \in \mathcal{O}_Y(Y)$ be restrictions of the coordinates on $\mathbb{A}^n$ to Y, i.e., if $q = (a_1, \dots, a_n) \in Y$, then $a_i = y_i(q)$. The pull back of y_i is

$$f_i := \varphi^* y_i = y_i \circ \varphi \in \mathcal{O}_X(X). \tag{1.14}$$

Let $p \in X$, $\varphi(p) = (b_1, \dots, b_n)$, $b_i = y_i(\varphi(p)) = f_i(p)$, thus

$$\varphi = (f_1, \dots, f_n)$$

where $f_i \in \mathcal{O}_X(X)$.

$\Leftarrow$ Let $\varphi := (f_1, \dots, f_n), f_i \in \mathcal{O}_X(X)$. First we show φ is continuous. Let $B \in Y$ be closed, it is equivalent to $B = Y \cap Z(G_1, \dots, G_m)$ and $G_i \in k[x_1, \dots, x_n]$. Since $G_i \circ \varphi = G(f_1, \dots, f_n) \in \mathcal{O}_X(X)$, we get $\varphi^{-1}(B) = Z(G_1 \circ \varphi, \dots, G_m \circ \varphi)$ and it is closed in X. So φ is continuous. Let $h \in \mathcal{O}_Y(U)$, write $W = \varphi^{-1}(U) \subset Y$. we need to show $h \circ \varphi \in \mathcal{O}_X(W)$. We can always make U smaller and assume $h(q) = \frac{F(q)}{G(q)}, \forall q \in U$, F and G are some polynomials and G has no zero on U. Then we have

$$h \circ \varphi = \frac{F \circ \varphi}{G \circ \varphi} = \frac{F(f_1, \dots, f_n)}{G(f_1, \dots, f_n)} \tag{1.15}$$

where $F(f_1, \dots, f_n)$ and $G(f_1, \dots, f_n)$ are regular on $\mathcal{O}_X(W)$. Since $\varphi(W) = U$ and G has no zero on U, $G(f_1, \dots, f_n)$ also has no zero on W, i.e., $h \circ \varphi \in \mathcal{O}_X(W)$. $\square$

Remark. The regular functions on a variety X are the same as the morphisms $X \to \mathbb{A}^1$.

Corollary 2. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be closed subvarieties. The morphisms*

$$\varphi : X \to Y$$

are precisely the polynomial map.

Proof. From theorem 2 we know $\varphi = (f_1, \dots, f_m)$ and $f_i \in \mathcal{O}_X(X) \forall i$. From theorem 5 we know $f_i \in A(X)$, so φ is a polynomial map. $\square$

Theorem 3. *Let X, Y be varieties, assume $Y \subset \mathbb{A}^m$ be a closed affine variety. Then there is a bijection between morphisms $X \to Y$ and k-algebra homomorphisms $A(Y) \to \mathcal{O}_X(X)$:*

$$\begin{array}{ccc} \{\textit{morphisms } X \to Y\} & \xrightarrow{\textit{bijection}} & \{\textit{homomorphisms } A(Y) \to \mathcal{O}_X(X)\} \\ \varphi & \longrightarrow & \varphi^* \end{array}$$

Proof. $\Rightarrow$: Let $\varphi : X \to Y$ be a morphism, then $\varphi^* : A(Y) \to \mathcal{O}_X(X)$ is a k-algebra homomorphism by definition 15.

$\Leftarrow$: Let $\phi : A(Y) \to \mathcal{O}_X(X)$ be a k-algebraic homomorphism, let $y_1, \dots, y_n \in A(Y)$ be the coordinate functions. We set

$$f_i = \phi(y_i) \in \mathcal{O}_X(X).$$

Let $\varphi = (f_1, \ldots, f_m) : X \to \mathbb{A}^m$. This is a morphism from X to Y. To see it is a morphism we have to show $\varphi(X) \subset Y$. Let $h \in I(Y)$, $h \circ \varphi = h(f_1, \ldots, f_m) = h(\phi(y_1), \ldots, \phi(y_m)) = \phi(h(y_1, \ldots, y_m))$. The second equality is based on the homomorphic property of ϕ, for example, if $h(x_1, x_2) = x_1^2 - x_2^3$, then $h(\phi(y_1), \phi(y_2)) = \phi(y_1)^2 - \phi(y_2)^3 = \phi(y_1^2) - \phi(y_2^3) = \phi(y_1^2 - y_2^3) = \phi(h(y_1, y_2))$. So $h(y_1, \ldots, y_m) \in A(Y)$, we choose an arbitrary element $p = (a_1, \ldots, a_m) \in Y$, then $h(y_1, \ldots, y_m)(p) = h(a_1, \ldots, a_m) = 0$ because $h \in I(Y)$. So for arbitrary $h \in I(Y)$, we get $h \circ \varphi = 0$, it implies $\varphi(X) \subset \cap_{h \in I(Y)} Z(h) = Y$. □

Example 2. A bijective polynomial map need not to be an isomorphism. For example, let$X = \mathbb{A}^1$, $Y = Z(x_2^2 - x_1^3) \subset \mathbb{A}^2$. Then

$$\varphi = (t^2, t^3) : X \to Y$$

is a morphism and bijective and the inverse is

$$\varphi^{-1}(a, b) = \begin{cases} \frac{b}{a} & \text{if } a \neq 0 \\ 0 & \text{if } (a, b) = 0 \end{cases}$$

φ is not an isomorphism(φ^{-1} is not a morphism). To show this we see the pull back:

$$\varphi^* : A(Y) \to \mathcal{O}_X(X)$$

where $A(Y) = k[x_1, x_2]/\langle x_2^2 - x_1^3 \rangle$ and $A(X) = k[t]$. φ^* makes $x_1 \to t^2$ and $x_2 \to t^3$. Since φ^* is not surjective(there is no element maps into t), φ^* is not an isomorphism. By theorem 3 we know φ is not an isomorphism. So bijective morphism is not necessary to be an isomorphism.

Definition 17. Let $X \subset \mathbb{A}^n$ be a closed variety, $F \in k[x_1, \ldots, x_n] \backslash I(X)$. The principal open defined by F is $X_F := X \backslash Z(F)$.

Proposition 8. *X_F is an affine variety.*

Proof. Let $Z := Z(\langle I(X), F \cdot x_{n+1} - 1 \rangle) \subset \mathbb{A}^{n+1}$. We need to prove Z is a closed subvariety of $\mathbb{A}^{n+1}$ isomorphic to X_F. Let $\varphi : (x_1, \ldots, x_n, \frac{1}{F}) : X_F \to \mathbb{A}^{n+1}$, it is a bijective morphism and $\varphi(X_F) = Z$. As X_F is irreducible, Z is also irreducible. So Z is closed variety of $\mathbb{A}^{n+1}$. On the other hand, the inverse of φ is

$$\varphi^{-1} = (x_1, \ldots, x_n) : Z \to X_F$$

is a morphism, so φ is an isomorphism. □

2. Morphisms of Quasi-projective varieties

Definition 18. Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be quasi-projective algebraic sets. A map $\varphi : X \to Y$ is called a polynomial map if there exists homogeneous polynomials $F_0, \ldots, F_m \in k[x_0, \ldots, x_n]$ of the same degree with no common zero on X s.t. $\varphi(p) = [F_0(p), \ldots, F_m(p)]$, $\forall p \in X$, write $\varphi = [F_0, \ldots, F_m]$.

Definition 19. The homogenization of $F \in k[x_0, \ldots, x_n]$ is:

$$F_a := F(1, x_1, \ldots, x_n).$$

Theorem 4. $\varphi_i = (\frac{x_0}{x_i}, \ldots, \hat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}) : U_i \to \mathbb{A}^n$ *is an isomorphism.*

Proof. We can assume $i = 0$, $\varphi := \varphi_0$, $U := U_0$, then $\varphi = (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$. $\frac{x_i}{x_0}$ is a regular function in $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$, so φ is a morphism. We need to show that $u = \varphi^{-1}(x_1, \ldots, x_n) = [1, x_1, \ldots, x_n]$ is a morphism.

(a) $u = \varphi^{-1}$ is continuous. Let $W = Z(F_1, \ldots, F_m) \cap U$ be closed in U, $F_i \in k[x_0, \ldots, x_n]$ are homogeneous, then

$$\begin{aligned} u^{-1}(W) = & \quad \{(a_1, \ldots, a_n) \in \mathbb{A}^n | [1, a_1, \ldots, a_n] \in W\} \\ = & \quad \{(a_1, \ldots, a_n) \in \mathbb{A}^n | F_i(1, a_1 \ldots, a_n) = 0, \forall i = 1, \ldots, m\} \\ = & \quad Z(F_{1a}, \ldots, F_{ma}) \end{aligned}$$

where F_{ia} is homogenization of F_i, it shows that $u^{-1}(W)$ is closed in $\mathbb{A}^n$.

(b) Let $V \subset U$ be open, $h \in \mathcal{O}_U(V)$, we need to show $u^*h \in \mathcal{O}_{\mathbb{A}^n}(u^{-1}(V))$. Making V smaller necessary, we can assume $h = \frac{F}{G}$, $F, G \in k[x_0, \ldots, x_n]$ are homogeneous polynomials of the same degree.

$$u^*h = h \circ u = \frac{F \circ u}{G \circ u} = \frac{F(1, x_1, \ldots, x_n)}{G(1, x_1, \ldots, x_n)}.$$

Thus $u^*h \in \mathcal{O}_{\mathbb{A}^n}(u^{-1}(V))$, $phi : \mathbb{A}^n \to u$ is an isomorphism. □

Remark. From theorem 4 we find that if we identify $\mathbb{A}^n$ with $u_0 \subset \mathbb{P}^n$, the Zariski topology on $\mathbb{A}^n$ is equivalent to the induced topology of u_0 from $\mathbb{P}^n$.

Corollary 3. (1) *Every variety is isomorphic to a quasi-projective variety.*
(2) *Every variety has an open cover by affine varieties.*

Proof. (1) Let X be a variety, if X is locally closed in $\mathbb{P}^n$, then it is a quasi-projective variety, so we only need to consider the condition in $\mathbb{A}^n$. Assume X be locally closed in $\mathbb{A}^n$. $Y = \varphi_0^{-1}(X) \subset \mathbb{P}^n$ is locally closed subvariety and $\varphi_0^{-1} : X \to Y$ is an isomorphism.

(2) For varieties in $\mathbb{A}^n$, it is trivial. Let $X \subset \mathbb{P}^n$ be a quasi-projective variety, then $X = \bigcup_{i=0}^{n} X \cap U_i$. $X \cap U_i$ is isomorphic to locally closed subvariety in $\mathbb{A}^n$. We can regard $X \cap U_i$ simply as $X \subset \mathbb{A}^n$, where X is locally closed. It is equivalent to prove:

For every point $p \in X$, there exists a neighborhood $U \subset X$ and U is an affine variety.

Since X is locally closed, there exist $Y, Z \subset \mathbb{A}^n$ closed in $\mathbb{A}^n$ s.t. $X = Y \backslash Z$. For any point $p \in X$, $\exists F_p \in I(Z)$ with $F_p(p) \neq 0$. Then we have $Y_{F_p} = Y \backslash Z(F_p) \subset X$. According to proposition 8, Y_{F_p} is an affine variety. □

Theorem 5. *Let $X \subset \mathbb{P}^m$, $Y \subset \mathbb{P}^n$ be quasi-projective varieties. Let $\varphi : X \to Y$ be a map. The following conditions are equivalent:*

(1) *φ is a morphism;*
(2) *φ is locally given by regular functions, i.e., for all $p \in X$, there exists a neighborhood $U \subset X$, $h_0, \ldots, h_n \in \mathcal{O}_X(U)$ with no common zero on U, s.t.*

$$\varphi(q) = [h_0(q), \ldots, h_n(q)], \quad \forall q \in U.$$

We write $\varphi = [h_0, \ldots, h_n]$ on U;
(3) *φ is locally a polynomial map, i.e.:*

$\forall p \in X$, $\exists$ open neighborhood $U \subset X$, $F_0, \ldots, F_n \in k[x_0, \ldots, x_,]$ homogeneous of the same degree with no common zero s.t.

$$\varphi(q) = [F_(q), \ldots, F_n(q)] \quad \forall q \in U.$$

We write $\varphi = [F_0, \ldots, F_n]$ on U.

Proof. (1) $\Rightarrow$ (2): If $\varphi : X \to \mathbb{P}^n$ is amorphism, then $\forall p\ inX$, $\exists i$, s.t. $\varphi(p) \in U_i$. Assume $i = 0$ and then $\varphi(p) \in U_0$. Let U be an open neighborhood of$p$ in X s.t. $\varphi(U) \subset U_0$. Then $\varphi_0 \circ \varphi : U \to \mathbb{A}^n$ is a morphism, so $\varphi_0 \circ \varphi = (h_1, \ldots, h_n)$ with $h_i \in \mathcal{O}_X(U)$. Since the inverse of φ_0 is u_0 we get

$$\varphi = u_0 \circ \varphi_0 \circ \varphi = [1, h_1, \ldots, h_n]. \tag{2.1}$$

(2) $\Rightarrow$ (3): Assume $\varphi = [h_0, \ldots, h_n]$ on $U \subset X$, where $h_i \in \mathcal{O}_X(U)$ with no common zeros on U. By making U possibly smaller we can further assume $h_i = \frac{F_i}{G_i}$, $F_i, G_i \in k[x_0, \ldots, x_m]$ are homogeneous of the same degree(F_i and G_i are of the same degree, it is not necessary that F_i and G_j are of the same degree for $i \neq j$), G_i has no zeros on U. Let $L_i = F_i \cdot G_0 \cdot \hat{G}_i \cdot G_n$, L_i are homogeneous of the same degree, we get

$$\varphi = [h_0, \ldots, h_n] = [L_0, \ldots, L_n]. \tag{2.2}$$

(3) $\Rightarrow$ (1): Let $\varphi|_U = [L_0, \ldots, L_n]$, $L_i \in k[x_0, \ldots, x_m]$ are homogeneous of the same degree with no common zero. Making U smaller, we can assume one of L_i(say L_0) has no zero in U. Then for $i = 1, \ldots, n$, let $h_i = \frac{L_i}{L_0} \in \mathcal{O}_X(U)$. Rewrite the map as

$$\varphi = [1, h_1, \ldots, h_n] \tag{2.3}$$

$$\Rightarrow \varphi_0 \circ \varphi = (h_1, \ldots, h_n). \tag{2.4}$$

So$\varphi_0 \circ \varphi$ is amorphism, then $\varphi = u_0 \circ \varphi_0 \circ \varphi$ is a morphism. □

Definition 20 (Projective Transformation)**.** Let

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{2.5}$$

be a $(n+1) \times (n+1)$ matrix in k, then we can construct a map from $\mathbb{P}^n \to \mathbb{P}^n$:

$$[A] : [b_0, \ldots, b_n] \to [b_0, \ldots, b_n] \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix}^T.$$

It is called a projective transformation. This is a morphism and if A is inverse then it is an isomorphism.

Remark. All automorphisms of $\mathbb{P}^n$ are projective transformations. It is not so easy to prove.

Definition 21 (Projection)**.** Let $X \subset \mathbb{P}^n$ be a variety, $W \subset \mathbb{P}^n$ be a projective subspace of $\mathbb{P}^n$ of $\dim W = k$. Assume $X \cap W = \emptyset$ and there exist linear forms $H_0, \ldots, H_{n-k-1}$ such that $W = Z(H_0, \ldots, H_{n-k-1})$. The projection from W is

$$\Pi_W = [H_0, \ldots, H_{n-k-1}] : X \to \mathbb{P}^{n-k-1}.$$

This is a morphism(H_i have no common zero on X because $W \cap X = \emptyset$).

Remark. Π_W depends on $H_0, \ldots, H_{n-k-1}$, but if we have another relation $W = Z(L_0, \ldots, L_{n-k-1})$, then there exists a projection transformation $[A] : \mathbb{P}^{n-k-1} \to \mathbb{P}^{n-k-1}$. In particular, if $p \in \mathbb{P}^n \backslash X$, for example, $p = [0, \ldots, 0, 1]$, then $\Pi_p = [x_0, \ldots, x_{n_1}] : X \to \mathbb{P}^{n-1}$.

3. Products of Varieties

Theorem 6 (Products of Affine Varieties)**.** *If $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ are closed subvarieties, then $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ is a closed subvariety.*

Before we prove it, we need to prove a conclusion in topology.

Lemma 2. *Let X, Y be irreducible topological spaces. Assume we have a topology on the product $X \times Y$ s.t.:*

$$\begin{array}{lll} y_p : & Y \to X \times Y, & q \to (p,q) \textit{ is continuous } \forall p \in X; \\ l_q : & X \to X \times Y, & p \to (p,q) \textit{is continuous } \forall q \in Y. \end{array}$$

Then $X \times Y$ is irreducible.

Proof. Assume $X \times Y = S_1 \cup S_2$, $S_i \subsetneqq X \times Y$ are closed. For $i = 1, 2$, set $T_i = \bigcap\limits_{q \in Y} l_q^{-1}(S_i) = \{p \in X | (p,q) \in S_i \quad \forall q \in Y\}$. It is the same as $T_i = \{p \in X | \{p\} \times Y \subset S_i\}$. Since y_p is continuous and Y is irreducible, we get $y_p(Y) = \{p\} \times Y$ is irreducible. So we get $\{p\} \times Y \subset S_1$ or $\{p\} \times Y \subset S_2 \quad \forall p \in X$(it implies $T_1 \cap T_2 = \emptyset$ and $T_i \subsetneqq X$). Hence $X = T_1 \cup T_2$. Since l_q is continuous, T_i are closed, then X is reducible. □

Proof of Theorem 6. Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be closed subvarieties, the product of X and Y is just

$$X \times Y = \{(p,q) \in \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m} | p \in X \text{ and } q \in Y\}.$$

Let $x_1, \ldots, x_n$ be coordinates in $\mathbb{A}^n$ and $y_1, \ldots, y_m$ be coordinates in $\mathbb{A}^m$, we can assume $X = Z(F_1, \ldots, F_k)$ and $Y = Z(G_1, \ldots, G_l)$ where $F_i \in k[x_1, \ldots, x_n], G_j \in k[y_1, \ldots, y_m]$. Then

$$X \times Y = Z(F_1, \ldots, F_k, G_1, \ldots, G_l) \subset \mathbb{A}^{n+m} \tag{3.1}$$

is a closed subset. By lemma 2 we only need to check $\forall q \in Y$, $l_q : X \to Y$ is continuous. Write $q = (b_1, \ldots, b_m)$, then $l_q = (x_1, \ldots, x_n, b_1, \ldots, b_m)$. It is a morphism, so it is continuous, thus we finish the whole proof.

Proposition 9 (Universal Property)**.** *Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be varieties, then*

(1) *The projections*

$$\begin{array}{ll} p_1 & = (x_1, \ldots, x_n) : X \times Y \to X \\ p_2 & = (y_1, \ldots, y_m) : X \times Y \to Y \end{array}$$

are morphisms.

(2) *Let Z be a variety. The morphism $\varphi : Z \to X \times Y$ are precisely the*

$$(f, g) : Z \to X \times Y, \quad p \to (f(p), g(p)) \quad \forall p \in Z$$

where $f : Z \to X$ and $g : Z \to Y$ are morphisms. In other words, $\varphi : Z \to X \times Y$ is a morphism if and only if both $p_1 \circ \varphi$ and $p_2 \circ \varphi$ are morphisms.

Proof. The first is obvious, we only check the second.

$\Rightarrow$: Let $\varphi : Z \to X \times Y$ be a morphism, then $f = p_1 \circ \varphi$ and $g = p_2 \circ \varphi$ are morphisms and $\varphi = (f, g)$.

$\Leftarrow$: Assume $f : Z \to X$ and $g : Z \to Y$ are both morphisms. then there exist $f_1, \dots, f_n \in \mathcal{O}_Z(Z)$ and $g_1, \dots, g_m \in \mathcal{O}_Z(Z)$ s.t. $f = (f_1, \dots, f_n), \quad g = (g_1, \dots, g_m)$. Then $(f, g) = (f_1, \dots, f_n, g_1, \dots, g_m)$ is a morphism. □

Remark. Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be subvarieties, $X \times Y$ does not lie rationally in some projective space. Thus we need to find an embedding $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ to denote the products of quasi-projective varieties.

Definition 22 ([Segre Embedding)**.**] We put $N := (n + 1) \cdot (m + 1) - 1$, let $x_0, \dots, x_n$ be coordinates on $\mathbb{P}^n$, $y_0, \dots, y_m$ be coordinates on $\mathbb{P}^m$. Let $z_{ij}, i = 0, \dots, n, j = 0, \dots, m$ be coordinates on $\mathbb{P}^N$. Define a map

$$\begin{aligned} \sigma : \mathbb{P}^n \times \mathbb{P}^m &\to \mathbb{P}^N \\ ([x_0, \dots, x_n], [y_0, \dots, y_m]) &\to [z_{ij}] = [x_i y_j] \end{aligned}$$

σ is called the Segre embedding.

Definition 23. We define the image of σ as

$$\Sigma := \sigma(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N.$$

For $i = 0, \dots, n$, put

$$U_i := \{[x_0, \dots, x_n] \in \mathbb{P}^n | x_i \neq 0\}.$$

For $j = 0, \dots, m$, put

$$U_j := \{[y_0, \dots, y_m] \in \mathbb{P}^m | y_j \neq 0\}.$$

And for $i = 0, \dots, n, j = 0, \dots, m$, put

$$U_{ij} := \{[z_{kl}] \in \mathbb{P}^N | z_{ij} \neq = 0\}.$$

there are isomorphisms:

$$\begin{aligned} \mathbb{A}^n &\underset{\varphi_i}{\overset{u_i}{\rightleftarrows}} U_i \\ \mathbb{A}^m &\underset{\varphi_j}{\overset{u_j}{\rightleftarrows}} U_j \\ \mathbb{A}^N &\underset{\varphi_{ij}}{\overset{u_{ij}}{\rightleftarrows}} U_{ij}. \end{aligned}$$

Since $\mathbb{P}^N = \cup_{i,j} U_{ij}$, we get $\Sigma = \cup_{i,j}(\Sigma \cap U_{ij})$, define

$$\Sigma^{ij} = \Sigma \cap U_{ij}.$$

Define the map σ^{ij}

$$\begin{aligned} \sigma^{ij} : \mathbb{A}^{n+m} &\to U_{ij} \\ (p, q) &\to \sigma(u_i(p), u_j(q)). \end{aligned}$$

By definition we know $\sigma^{ij}(\mathbb{A}^{n+m}) = \Sigma^{ij}$.

Theorem 7. (1) *$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ is injective and Σ is closed in $\mathbb{P}^N$:*

$$\Sigma = Z\left(\left\{z_{ij}z_{kl} - z_{il}z_{kj}\Big|\begin{matrix} i,k & = 0,\dots,n \\ j,l & = 0,\dots,m\end{matrix}\right\}\right). \tag{3.2}$$

(2) *$\sigma^{ij} : \mathbb{A}^{n+m} \to \Sigma^{ij}$ is an isomorphism.*

(3) *$\forall q \in \mathbb{P}^m$, the map*

$$\begin{aligned} \bar{i_q} : \mathbb{P}^n & \to \mathbb{P}^N \\ p & \to \sigma(p,q) \end{aligned}$$

is a morphism. Similarly, $j_p = \sigma(p,q) : \mathbb{P}^m \to \mathbb{P}^N$ is a morphism.

(4) *Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be quasi-projective varieties, then $\sigma(X \times Y) \subset \mathbb{P}^N$ is also a quasi-projective variety. What's more, if X and Y are both projective varieties, then $\sigma(X \times Y)$ is a projective variety.*

Proof. (1) If $\sigma([a_0,\dots,a_n],[b_0,\dots,b_m]) = \sigma([a'_0,\dots,a'_n],[b'_0,\dots,b'_m])$, then $\exists\lambda \in k\backslash\{0\}$, s.t. $\lambda a'_i b'_j = \lambda a_i b_j\ \forall i,j$. Choose i_0, j_0 s.t. $a_{i_0}b_{j_0} \neq 0$, then $\forall i = 0,\dots,n$, $a_i b_{j_0} = \lambda a'_i b'_{j_0} \Rightarrow a_i = \left(\frac{\lambda b'_{j_0}}{b_{j_0}}\right) a'_i \Rightarrow [a_0,\dots,a_n] = [a_0{}',\dots,a_n{}']$. The same way can be used to prove $[b_0,\dots,b_m] = [b_0{}',\dots,b_m{}']$. Let W be the zero set on the right hand side of the equation 3.2, clearly we have the relation $\Sigma \subset W$. Now let $[a_{ij}] \in W$, choose i_0, j_0 s.t. $a_{i_0j_0} \neq 0$, then we get $[a_{ij}] = [a_{i_0j_0}a_{ij}] = [a_{i_0j}a_{ij_0}] = [a_{ij_0}a_{i_0j}] = \sigma([a_{0j_0},\dots,a_{nj_0}],[a_{i_00},\dots,a_{i_0m}]) \subset \Sigma$.

(2) Assume $i = j = 0$, then

$$\begin{aligned} \varphi_{00} \circ \sigma^{00}(a_1,\dots,a_n,b_1,\dots,b_m) &= \varphi_{00}(\sigma([1,a_1,\dots,a_n],[1,b_1,\dots,b_m])) \\ &= (z_{ij})_{(i,j)\neq(0,0)} \end{aligned}$$

where $z_{i0} = a_i$ for $i = 1,\dots,n$, $z_{0j} = b_j$ for $j = 1,\dots,m$, $z_{ij} = a_i b_j$ for $i,j \geq 1$. These are all regular functions, so $\varphi_{00} \circ \sigma^{00}$ is a morphism, so σ^{00} is a morphism. Finally, σ^{00} is an isomorphism because the inverse map is

$$(\sigma^{00})^{-1} = \left(\frac{z_{10}}{z_{00}},\dots,\frac{z_{n0}}{z_{00}},\frac{z_{01}}{z_{00}},\dots,\frac{z_{0m}}{z_{00}}\right).$$

Remark. In fact, Σ^{ij} is a quasi-projective variety. Because $\mathbb{A}^{n+m}$ is irreducible, Σ^{ij} is irreducible, hence a quasi-projective variety.

(3) Let $q = [b_0,\dots,b_m]$, then $i_q = [x_i b_j]$, $x_i b_j$'s are homogeneous polynomials, so by theorem 5 we know it is a morphism.

(4) Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be projective varieties. We can decompose the map into the following:

$$\begin{aligned} \sigma(X \times Y) &= \bigcup_{i,j} \sigma(X \times Y) \cap U_{ij} \\ &= \bigcup_{i,j} \sigma^{ij}(\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j)) \end{aligned}$$

$\varphi_i(X \cap U_i)$ and $\varphi_j(Y \cap U_j)$ are closed subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively. By the theorem 6 $\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j)$ is closed in $\mathbb{A}^{n+m}$. Since σ^{ij} is an isomorphism, then $\sigma^{ij}(\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j))$ is closed in $\Sigma^{ij} = \Sigma \cap U_{ij}$. So $\sigma(X \times Y)$ is closed in Σ, hence closed in $\mathbb{P}^N$ because Σ itself is closed. To show its irreducible, we use the lemma 2. Since σ is injective we can endow $\mathbb{P}^n \times \mathbb{P}^m$ with the topological structure of $\mathbb{P}^N$, hence we can identify $\mathbb{P}^n \times \mathbb{P}^m$ with Σ provided with the topology induced from $\mathbb{P}^N$. Now we can use the lemma 2, we have known i_q and j_p are continuous, so $\sigma(X \times Y)$ is irreducible. For quasi-projective conditions ,we just get the conclusion by simply difference two projective varieties. $\square$

Remark. For $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ we can now identify $X \times Y$ with $\sigma(X \times Y) \subset \mathbb{P}^N$. In particular we can identify $\mathbb{P}^n \times \mathbb{P}^m$ with Σ.

From this perspective, part (2) of the theorem just says $U_i \times U_j \subset \mathbb{P}^n \times \mathbb{P}^m$ is open and $\varphi_i \times \varphi_j : U_i \times U_j \to \mathbb{A}^{n+m}$ is an isomorphism.

Proposition 10 (Universal Property)**.** *Let X, Y be quasi-projective varieties, then*

(1) *The projections*

$$\begin{aligned} p_1 &= (x_1, \ldots, x_n) : X \times Y \to X \\ p_2 &= (y_1, \ldots, y_m) : X \times Y \to Y \end{aligned}$$

are morphisms.

(2) *Let Z be a variety. The morphism $\varphi : Z \to X \times Y$ are precisely the*

$$(f, g) : Z \to X \times Y, \quad p \to (f(p), g(p)) \quad \forall p \in Z$$

where $f : Z \to X$ and $g : Z \to Y$ are morphisms. In other words, $\varphi : Z \to X \times Y$ is a morphism if and only if both $p_1 \circ \varphi$ and $p_2 \circ \varphi$ are morphisms.

Proof. (1) It is enough to show $p_1|_{U_i \times U_j}$ is a morphism from $U_i \times U_j$ to U_i. Identify $U_i \times u_j$ with $\mathbb{A}^{n+m}$ and U_i with $\mathbb{A}^n$, then we can see that p_1 is the same as the projection defined by the proposition 9, so it is a morphism.

(2) $\Rightarrow$: Let $\varphi : Z \to X \times Y$ be a morphism. Then $f := p_1 \circ \varphi$ and $g := p_2 \circ \varphi$ are morphisms.

$\Leftarrow$: Let $f : Z \to X$ and $g : Z \to Y$ be morphisms. Define

$$Z^{ij} := f^{-1}(U_i) \cap g^{-1}(U_j).$$

Then (f, g) is a morphism $\Leftrightarrow (f, g)|_{Z^{ij}}$ is a morphism for $i = 1, \ldots, n, j = 1, \ldots, m$. Consider the following mapping chain

$$Z^{ij} \xrightarrow{(f,g)} (X \times Y) \cap (U_i \times U_j) \xrightarrow{\varphi_i \times \varphi_j} \varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j) \subset \mathbb{A}^{n+m}.$$

the whole chain $(\varphi_i \circ f, \varphi_j \circ g) : Z^{ij} \to \mathbb{A}^{n+m}$ is a morphism, so (f, g) is a morphism. $\square$

Corollary 4. *Let X_1, X_2, Y_1, Y_2 be varieties. If $f : X_1 \to Y_1$ and $X_2 \to Y_2$ are morphisms, then the map:*

$$\begin{aligned} f \times g : X_1 \times X_2 &\to Y_1 \times Y_2 \\ (p, q) \to (f(p), g(q)) & \end{aligned}$$

is a morphism. In particular, if X_1 is isomorphic to Y_1 and X_2 is isomorphic to Y_2, then $X_1 \times X_2$ is isomorphic to $Y_1 \times Y_2$

Proof. We can write $f \times g$ as $f \circ p_1$ and $g \circ p_2$, both $f \circ p_1$ and $g \circ p_2$ are morphisms, so $f \times g = (f \circ p_1, g \circ p_2)$ is a morphism. $\square$

Lemma 3. *The closed subset in $\mathbb{P}^n \times \mathbb{P}^m$ is the zero set of sets of polynomials of $f_k(x_0, \ldots, x_n, y_0, \ldots, y_m)$ for $k = 1, \ldots, r$ which are homogeneous in x_i and y_j, and the degree in x_i is equal to the degree in y_j, we called it behomogeneous.*

Proof. Let $W \subset \mathbb{P}^n \times \mathbb{P}^m$ be closed. $W = \sigma^{-1}(A)$, for $A \subset \mathbb{P}^N$ closed. Then A is the zero set of homogeneous polynomials in z_{ij}, write it as $A = (f_1(z_{ij}), \ldots, f_r(z_{ij})$. Then we get $W = (f_1(x_i y_j), \ldots, f_r(x_i y_j))$. For $k = 1, \ldots, r$, $f_k(x_i y_j)$ are bihomogeneous. Conversely, assume

$$W = Z(g_1(x_0, \ldots, x_n, y_0, \ldots, y_m), \ldots, g_l(x_0, \ldots, x_n, y_0, \ldots, y_m))$$

where g_k are bihomogeneous. Then
$(\varphi_i \times \varphi_j)(W \cap (U_i \times U_j)) = Z(g_1(x_0, \ldots, x_i = 1, \ldots, x_n, y_0, \ldots, y_j = 1, \ldots, y_m), \ldots, g_l(x_0, \ldots, x_i = 1, \ldots, x_n, y_0, \ldots$
are closed in $\mathbb{A}^{n+m}$. So $W \cap (U_i \times U_j)$ are closed in $U_i \times U_j$. $U_i \times U_j$ form a finite open cover of $\mathbb{P}^n \times \mathbb{P}^m$, so W is closed. □

Definition 24. Let X be a variety, the diagonal is

$$\Delta_X := \{(p, p) \in X \times X | p \in X\} \subset X \times X.$$

The diagonal morphism is

$$\begin{aligned} \delta_X : X \quad &\to \Delta_X \subset X \times X \\ p \quad &\to (p, p). \end{aligned}$$

Lemma 4. *Δ_X is closed in $X \times X$ and $\delta_X : X \to \Delta_X$ is an isomorphism.*

Proof. Any variety X is isomorphic to a locally closed subvariety of some projective space, so we can assume $X \subset \mathbb{P}^n$ is locally closed, then

$$\Delta_X = \Delta_{\mathbb{P}^n} \cap (X \times X).$$

Thus we know if $\Delta_{\mathbb{P}^n}$ is closed then Δ_X is closed in $X \times X$. In fact $\Delta_{\mathbb{P}^n} = Z\left(\{x_i y_j - x_j y_i | i, j = 0, \ldots, n\}\right)$ is closed.

$\delta_X : X \to \Delta_X$ is isomorphic because $p_1 : \Delta_X \to X$ is its inverse morphism. □

Remark. The fact that $\Delta_X \subset X \times X$ is closed replaces for us the Hausdorff property in topology.

Definition 25. A variety is called separated if $\Delta_X \subset X \times X$ is closed. By the lemma 4 all varieties are separated.

Corollary 5. *Let $\varphi, \psi : X \to Y$ be morphisms of varieties, then $W = \{p \in X | \varphi(p) = \psi(p)\}$ is closed in X. In particular, if $\varphi|_U = \psi|_U$ for an open subset of X, then $\varphi = \psi$.*

Proof. See the following chain

$$X \xrightarrow{\delta_X} \Delta_X \xrightarrow{\varphi \times \psi} Y \times Y.$$

So $W = \delta^{-1}((\varphi \times \psi)^{-1}(\Delta_Y))$ is closed. Because varieties are irreducible, the open set U is dense in X, let $\omega = \varphi - \psi$ and we get $l(x) = 0$ in U, hence $l = 0$ in X because of the continuity of l, hence $\varphi = \psi$. □

Definition 26. Let $\varphi : X \to Y$ be a morphism of varieties. The graph of φ is defined as

$$\Gamma_\varphi := \{(p, \varphi(p)) | p \in X\} \subset X \times Y. \tag{3.3}$$

Corollary 6. *Γ_φ is closed in $X \times Y$.*

Proof. Define the map

$$\begin{aligned} \varphi \times \mathrm{id}_Y : X \times Y \quad &\to Y \times Y \\ (p, q) \quad &\to (\varphi(p), q). \end{aligned}$$

Then we have $\Gamma_\varphi = (\varphi \times \mathrm{id}_Y)^{-1}(\Delta Y)$, so it is closed. In fact Γ_φ is isomorphic to X. □

Definition 27. A map $\varphi : X \to Y$ of topological spaces is called closed if $\varphi(Z)$ is closed in Y for all closed subsets $Z \subset X$.

Definition 28. A variety complete if the projection $p_2 : X \times Y \to Y$ is a closed map for all varieties Y.

Remark. Completeness replaces for us compactness in topology.

Example 3. $\mathbb{A}^1$ is not complete. Let $Z = Z(x_1y_1 - 1) \subset \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$, then $p_2(Z) = \mathbb{A}^1 \backslash \{0\}$ is not closed in $\mathbb{A}^1$.

Proposition 11. *Let X be a complete variety, $\varphi : X \to Y$ be a morphism of varieties. Then $\varphi(X)$ is closed in Y.*

Proof. Since $\Gamma_\varphi \subset X \times Y$ is closed and $\varphi(X) = p_2(\Gamma_\varphi)$, thus if X is complete, $\varphi(X)$ is closed in Y. □

Theorem 8. *All projective varieties are complete.*

Proof. We finish the proof by two steps.

(1) Main step to show $p_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed. Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be closed, we can write it as

$$X = Z(f_1(x, y), \dots, f_r(x, y))$$

where f_i isbihomogeneous, $x = (x_0, \dots, x_n), y = (y_0, \dots, y_m)$. We can assume all f_i have the same degree d in y. If f_j has a lower degree l, we can replace it by polynomials $y_0^{d-l} f_j, y_1^{d-l} f_j, \dots, y_n^{d-l} f_j$. Fix a point $q \in \mathbb{P}^m$, then $q \in p_2(X) \Leftrightarrow Z(f_1(x, q), \dots, f_r(x, q)) \neq \emptyset$. By the projective Nullstellensatz, this is equivalent to:

$$\forall s > 0,\ (*)\ \mathfrak{a} := \langle f_1(x, q), \dots, f_r(x, q)\rangle \text{ does not contain all monomials of degree } s \text{ in } x.$$

It is trival for $s < d$, so it is enough to show:

$$\forall s \geq d, \text{ the set } X_s := \{q \in \mathbb{P}^m | q \text{ satisfies the condition } (*)\} \text{ is closed in } \mathbb{P}^m. \text{ Hence } p_2(X) = \bigcap_{s \geq d} X_s \text{ is closed in } \mathbb{P}^m.$$

Denote monomials in x of degree s with $M_i(x)$, $i = 1, \dots, \binom{n+s}{n}$. Denote monomials in x of degree $s - d$ with $N_j(x)$, $j = 1, \dots, \binom{n+s-d}{n}$. The elements of degree s in $\mathfrak{a}$ are the linear span of $\{N_i(x) f_j(x, q) | i = 1, \dots, \binom{n+s-d}{n}, j = 0, \dots, r\}$. Define all $\{N_i(x) f_j(x, y)\}$ by $\{G_k(x, y), k = 1, \dots, t\}$. The condition $(*)$ is equivalent to:

$$\{G_k(x, q)\} \text{ does not equal to the whole space of degree } s \text{ in } x.$$

We can write $G_k(x, y) = \sum_{i=1}^{\binom{n+s}{n}} A_{ik}(y) M_i(x)$. The dimension of the linear span of $\{G_k(x, q), k = 1, \dots, t\}$ is the rank of the matrix $A := (A_{ik}(q))$. Thus the condition $(*)$ is equivalent to $\operatorname{rank}(A) < \binom{n+s}{n}$. Thus

$$\{q \in \mathbb{P}^m | q \text{ satisfies the condition } (*)\} = \text{ zero set of all } \binom{n+s}{n} \times \binom{n+s}{n} \text{ minors of } A.$$

Thus $p_2(X)$ is closed in $\mathbb{P}^m$.

(2) General case. First show $\mathbb{P}^n$ is completed. Let Y be a variety, we can assume $Y \subset \mathbb{P}^m$ is locally closed subvariety. Let $Z \subset \mathbb{P}^n \times Y$ be closed in $\mathbb{P} \times Y$, $\bar{Z}$ be the closure of Z in $\mathbb{P}^n \times \mathbb{P}^m$. Then $p_2(\bar{Z})$ is closed in $\mathbb{P}^m$, hence $p_2(Z) = p_2(\bar{Z} \cap (\mathbb{P}^n \times Y)) = p_2(\bar{Z}) \cap Y$ is closed in Y. Finally, let $X \subset \mathbb{P}^n$ be closed

subvariety, $Z \subset X \times Y$ be closed, it follows that Z is also closed in $\mathbb{P}^n \times Y$, therefore by trival step $p_2(Z)$ is closed in Y. $\square$

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