Fubini's Theorem

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Definition 1. *Let μ and ν be outer measures on the non-empty sets X and Y respectively. We define the product measure of μ and ν on the product set $X \times Y$ as, for $E \subset X \times Y$,*

$$\begin{aligned}&(\mu \times \nu)(E)\\ =& \inf\left\{\sum_{j=1}^{\infty}\mu(A_j)\nu(B_j) : E \subset \bigcup_{j=1}^{\infty} A_j \times B_j, A_j \ \mu\text{-measurable}, B_j \ \nu\text{-measurable}\right\}.\end{aligned}$$

To evaluate $\mu \times \nu$ in terms of μ and ν, we introduce the following notations:

$$\begin{aligned}\mathcal{P}_0 &= \{A \times B : A \ \mu\text{-measurable and } B \ \nu\text{-measurable}\}\\ \mathcal{P}_1 &= \left\{R : R = \bigcup_{j=1}^{n} A_j \times B_j, 1 \le n \le \infty, A_j \times B_j \in \mathcal{P}_0\right\}\\ \mathcal{P}_2 &= \left\{R : R = \bigcap_{j=1}^{n} R_j, 1 \le n \le \infty, R_j \in \mathcal{P}_1\right\}.\end{aligned}$$

Elements in $\mathcal{P}_0$ are called measurable rectangles. We also set

$$\begin{aligned}\mathcal{F} = \{R :\ & \text{For } \nu\text{-a.e. } y, x \mapsto \chi_R(x,y) \text{ is } \mu\text{-measurable and}\\ & y \mapsto \int \chi_R(x,y)\mathrm{d}\mu(x) \text{ is } \nu\text{-measurable}\Big\}\end{aligned}$$

Note that the map

$$y \mapsto \int \chi_R(x,y)\mathrm{d}\mu(x)$$

is defined almost everywhere in Y.

For $R \in \mathcal{F}$, we can define

$$\rho(R) = \int_Y \left(\int_X \chi_R(x,y)\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y).$$

The following lemmas show that $\mathcal{P}_0, \mathcal{P}_1$ and $\mathcal{P}_2 \subset \mathcal{F}$ and they are $\mu\times\nu$-measurable. Moreover,

$$(\mu \times \nu)(R) = \rho(R),$$

for $R \in \mathcal{P}_1$ or $R \in \mathcal{P}_2$ provided in the latter R satisfies $\rho(R) < \infty$.

Lemma 1. *$\mathcal{P}_0 \subset \mathcal{F}$ and*

$$\rho(A \times B) = \mu(A)\nu(B), A \times B \in \mathcal{P}_0.$$

Lemma 2. *$\mathcal{P}_1 \subset \mathcal{F}$ and*

$$\rho(R) = \sum_1^\infty \mu(A_j)\nu(B_j), \ \ \textit{whenever } R = \overset{\circ}{\bigcup} A_j \times B_j, A_j \times B_j \in \mathcal{P}_0.$$

We have put a circle on top of the union sign to indicate that this is a union of pairwise disjoint sets.

Lemma 3. *For $E \subset X \times Y$,*

$$(\mu \times \nu)(E) = \inf \left\{ \rho(R) : E \subset R, R \in \mathcal{P}_1 \right\}.$$

In particular, for $A \times B \in \mathcal{P}_0$,

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B) = \rho(A \times B).$$

Lemma 4. *$\mathcal{P}_1$ and $\mathcal{P}_2$ consist of $\mu \times \nu$-measurable sets. For $R \in \mathcal{P}_1$,*

$$(\mu \times \nu)(R) = \sum_j \mu(A_j)\nu(B_j) = \rho(R).$$

Lemma 5. *Let $R \in \mathcal{P}_2$. Suppose that $R = \bigcap_{j=1}^\infty R_j$, $R_j \in \mathcal{P}_1$, and $\rho(R_1) < \infty$. Then $R \in \mathcal{F}$ and*

$$(\mu \times \nu)(R) = \rho(R).$$

Lemma 6. *For $E \subset X \times Y$, $\exists R \in \mathcal{P}_2, E \subset R$ such that*

$$(\mu \times \nu)(E) = (\mu \times \nu)(R).$$

Theorem 1 (Fubini's Theorem). *Let μ and ν be σ-finite outer measurable on X and Y respectively.*

a. *For any non-negative $\mu \times \nu$-measurable function f,*

   $x \mapsto f(x, y)$ is μ-measurable for ν-a.e. y, and

   $y \mapsto \int_X f(x, y) \mathrm{d}\mu(x)$ is ν-measurable.

b. (a) *holds for $f \in L^1(\mu \times \nu)$.*

Part (b) was first formulated by Tobelli and is also called Tonelli's theorem.