Algebraic Topology

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Abstract

This is a learning note about Zhang Yitang's lectures on modular forms.

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before 1900 Euler formula $V - E + F = 2$
Winding number

1900 H. Poincaré introduce Homology, Fundamental Group

Aimed to study "spaces"
Topological spaces and continuous mappings
Invariants
$X \in \{$ Topological spaces $\} \Rightarrow$ e.g. $G(X) \in \{$ abelian groups$\}$
If $X \to G(X)$, $Y \to G(Y)$ and $f : X \to Y$, we wish to get $G(f)$:

$$\begin{array}{ccc} X & \longrightarrow & G(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(f)} \\ Y & \longrightarrow & G(Y) \end{array}$$

and let $G(f)$ be a homomorphism of groups.

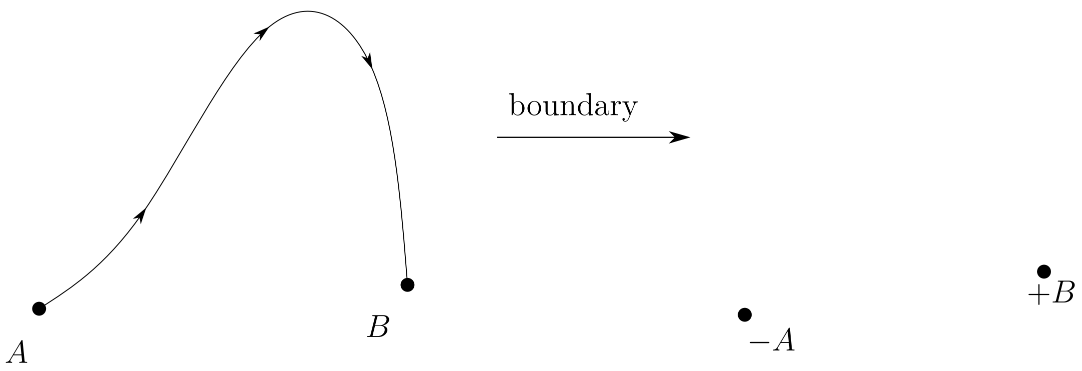

Figure 1: boundary of a segment

Boundary of the boundary $+C$ is 0 in Figure 2.

0 Categories, Functors, and Natural Transformations

Definition 0.1 (categories). A category $\mathcal{C}$ consists of

a. (objects) $\mathrm{Ob}(\mathcal{C})$ consists of the class of objects in $\mathcal{C}$.

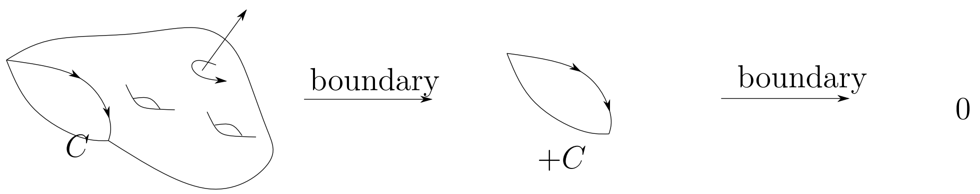

Figure 2: boundary of a surface

b. (morphisms) $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, we have a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ s.t. $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X',Y')$ iff $X = X', Y = Y'$.

c. (composition law) $\forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, we have a map:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) &\xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X,Z) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & Z \end{tikzcd}$$

which satisfy the following two axioms:

(1) (Associativity) $X \xrightarrow{f} Y, Y \xrightarrow{g} Z, Z \xrightarrow{h} W$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(2) (Identity) $\forall X \in \mathrm{Ob}(\mathcal{C})$, $\exists X \xrightarrow{1_X} X$ s.t.

$$h \circ 1_X = H, 1_X \circ k = k$$

$\forall X \xrightarrow{h} H, K \xrightarrow{k} X$.

Example 0.1. a. $\mathcal{C} =$ (set), (Ab),(Mod$_R$)(R is a ring), (Top),(TopGp).

b. $\mathcal{C}^{\text{op}}$ (the opposite of $\mathcal{C}$):

$$\text{Ob}(\mathcal{C}^{\text{op}}) := \text{Ob}(\mathcal{C})$$
$$\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) := \text{Hom}_{\mathcal{C}}(Y, X)$$

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) \times \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z) \overset{\circ_{\mathcal{C}^{\text{op}}}}{\rightarrow} \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Z)$$
$$(f, g) \rightarrow g \circ_{\mathcal{C}^{\text{op}}} f$$
$$X \overset{f}{\leftarrow} Y, Y \overset{g}{\leftarrow} Z \quad X \overset{f \circ g}{\leftarrow} Z.$$

Definition 0.2. $X, X' \in \text{Ob}(\mathcal{C}), X \overset{f}{\mathcal{C}} X'$, f is an *isomorphism* $\Leftrightarrow \exists X' \overset{\widetilde{f}}{\rightarrow} X$ s.t.

$$\widetilde{f} \circ f = 1_X$$
$$f \circ \widetilde{f} = 1_{X'}.$$

Definition 0.3 (Functors). $\mathcal{C}, \mathcal{C}'$: categories. A covariant(contravariant) functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ ($\mathcal{C} \overset{F}{\rightarrow} \mathcal{C}'$) consists of

- a rule of associating to each $X \in \text{Ob}(\mathcal{C})$ an object $F(X) \in \text{Ob}(\mathcal{C}')$.
- A map $\text{Hom}_{\mathcal{C}}(X, Y) \overset{F}{\rightarrow} \text{Hom}_{\mathcal{C}'}(F(X), F(Y))$ ($\text{Hom}_{\mathcal{C}'}(F(Y), F(X))$) for each pair $X, Y \in \text{Ob}(\mathcal{C})$ s.t. $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$($F(g \circ f) = F(f) \circ F(g)$) i.e.

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g \circ f"] & & Z \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & F(Y) \arrow[dr, "F(g)"] & \\ F(X) \arrow[ur, "F(f)"] \arrow[rr, "F(g \circ f)"] & & F(Z) \end{tikzcd}$$

Example 0.2.

(1) $\mathcal{C} \overset{\text{op}}{\rightarrow} \mathcal{C}^{\text{op}}$, $X^{\text{op}} := X$

(2) $\forall X \in \mathcal{O}\lfloor(\mathcal{C})$, $h_X : \mathcal{C} \rightarrow$ (set),

$$h_X(Y) := \text{Hom}_{\mathcal{C}}(Y, X), \forall Y \in \text{Ob}(\mathcal{C})$$

$$h_X(f) := \circ f : h_X(Y) \rightarrow h_X(Y'), \forall Y' \overset{f}{\rightarrow} Y (\rightarrow X)$$

h_X is contravariant.

Definition 0.4 (Natural Transformations). $\mathcal{C} \underset{F_2}{\overset{F_1}{\rightrightarrows}} \mathcal{C}'$ two functors of the same variance.

a. A *natural transformation* T form F_1 to F_2 (denoted as $F_1 \xrightarrow{T} F_2$) is a rule of associating to each $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $F_1(X) \xrightarrow[\mathcal{C}']{T(X)} F_2(X)$ s.t. for each $X \xrightarrow[\mathcal{C}]{f} Y$ we have :

$$\begin{tikzcd} F_1(X) \arrow[r, "T(X)"] \arrow[d, "F_1(f)"'] & F_2(X) \arrow[d, "F_2(f)"] \\ F_1(Y) \arrow[r, "T(Y)"'] & F_2(Y) \end{tikzcd}$$

b. A natural transformation $F_1 \xrightarrow{T} F_2$ is called a *natural equivalence* if $F_1(X) \xrightarrow{T(X)} F_2(X)$ is an isomorphism for each $X \in \mathrm{Ob}(\mathcal{C})$.

$F_1 \xrightarrow{T} F_2, F_2 \xrightarrow{S} F_3 \rightsquigarrow S \circ T$.

1 Singular Homolgy Groups

Definition 1.1 (Standard simplexes). $k \in \mathbb{N} \cup \{0\}$,

$$\Delta_k := \left\{ (t_0, \cdots, t_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} t_i = 1, t_i \geq 0, i = 0, \cdots, k \right\}.$$

Definition 1.2 (The i-th face inclusion). $i \leq k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \Delta_k &\xrightarrow{l_i} \Delta_{k+1} \\ (t_0, \cdots, t_k) &\mapsto (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_k). \end{aligned}$$

Definition 1.3 (Singular complexes, ude to Lefschetz-Eilenberg). X : topological space, $k \in \mathbb{N} \cup \{0\}$. A (singular) k-simplex in X is a continuous map $\sigma : \Delta_k \to X$.

Definition 1.4 (Faces of a singular simplex). $\sigma : \Delta_k \to X$ continuous, $\sigma_i := \sigma \circ l_i$ where $l_i : (t_0, \cdots, t_{k-1}) \mapsto (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{k-1}), i = 0, \cdots, k$.

Definition 1.5 (Singular chain groups). $k \in \mathbb{Z}$,

$$\begin{aligned} S_k(X) :=& \text{ the free abelian group generated by all singular } k\text{-simplexes in } X \\ &= \oplus_{\sigma:\Delta_k \to X} \mathbb{Z}\sigma, k \geq 0 \\ &= \{0\}, k < 0. \end{aligned}$$

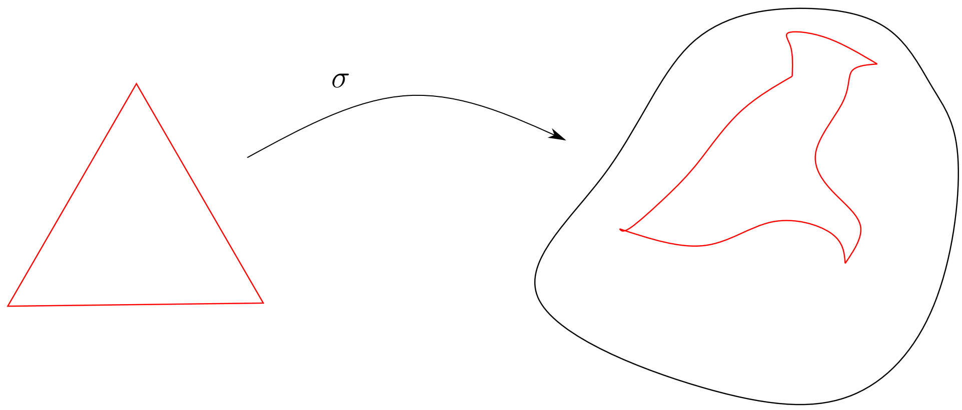

Figure 3: singular complexes

$X \xrightarrow[\text{continuous}]{f} Y$, we can define a functor $S_k : S_k(X) \xrightarrow{S_k(f)=f_\#} S_k(Y)$:

$$\sigma : \Delta_k \to X \mapsto \begin{tikzcd} \Delta_k \arrow[rr, "f\circ\sigma"] \arrow[dr, "\sigma"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd} .$$

$S_k : (\mathrm{Top}) \to (\mathrm{Ab})$ is a covariant functor.

Definition 1.6 (Boundary operation). $S_k(X) \xrightarrow{\partial_k} S_{k-1}(X)$

$$\partial_k \sigma := \sum_{i=0}^{k} (-1)^i \sigma_i$$

Exercise 1.1. The following two diagrams are commutative:

$$\begin{tikzcd} S_k(X) \arrow[r, "f_\#"] \arrow[d, "\partial_k"'] & S_k(Y) \arrow[d, "\partial_\#"] \\ S_{k-1}(X) \arrow[r, "f_\#"'] & S_{k-1}(Y) \end{tikzcd}$$

$$\begin{tikzcd}
\Delta_k \arrow[r, "l_j"] \arrow[d, "l_{i-1}"'] & \Delta_{k-1} \arrow[d, "l_i"] \\
\Delta_{k-1} \arrow[r, "l_j"'] & \Delta_k
\end{tikzcd}$$

if $1 \leq j+1 \leq i \leq k, k \geq 2$.

Definition 1.7 (Singular chain complexes). $\sigma : \Delta_k \to X$: a singular k-simplex in X,

$$\begin{aligned}
\partial_{k-1}(\partial_k \sigma) &= \sum_{j=0}^{k-1}\sum_{i=0}^{k}(-1)^{i+j}(\sigma_i)_j \\
&= \sum_{k-1 \geq j \geq i \geq 0}(-1)^{i+j}\sigma \circ l_i \circ l_j + \sum_{1 \leq j+1 \leq i \leq k}(-1)^{i+j}\sigma \circ l_i \circ l_j \\
&= \sum_{k-1 \geq j \geq i \geq 0}(-1)^{i+j}\sigma \circ l_i \circ l_j + \sum_{1 \leq j+1 \leq i \leq k}\sigma \circ l_j \circ l_{i-1} \\
&=0.
\end{aligned}$$

Then, we have the chain complex:

$$\cdots \xrightarrow{\partial_{k+2}} S_{k+1}(X) \xrightarrow{\partial_{k+1}} S_k(X) \xrightarrow{\partial_k} S_{k-1} \xrightarrow{\partial_{k-1}} S_{k-2} \cdots .$$

Let X be a topological space $k \in \mathbb{Z}$. Recall

$$S_k(X) = \oplus_{\sigma:\Delta_k \to X}\mathbb{Z}\sigma, \quad k \geq 0.$$

$\forall$ set S, define $\mathbb{Z}^{\oplus S} := \{\phi : S \to \mathbb{Z} | \phi(s) \neq 0 \text{ for only finitely many } s \in S\}$, it is an abelian group. Write $\phi = \sum_{s \in S}\phi(s)s$, define the map

$$\begin{aligned}
&S \to \mathbb{Z}^{\oplus S} \\
&s \mapsto e_s : s' \mapsto \begin{cases} 1, s' = s \\ 0, s' \neq s \end{cases}
\end{aligned}$$

Universal property: consider any map ϕ and any abelian group A, we have

$$\begin{tikzcd}
S \arrow[rr, "\forall \text{ map } \phi"] \arrow[dr, "e"'] & & A \\
& \mathbb{Z}^{\oplus S} \arrow[ur, "\exists! \Phi \text{ group homomorphism}"'] &
\end{tikzcd}$$

Example 1.1. Consider $\mathrm{Hom}_{(\mathrm{Top})}(\Delta_k, X) = \{\text{all singular } k\text{-simplexes in } X\}$, then we can define $S_k(X)$ in another way

$$S_k(X) := \mathbb{Z}^{\oplus \mathrm{Hom}_{(\mathrm{top})}(\Delta_k, X)}.$$

Consider the map

$$\begin{aligned} \mathbb{Z}^{\oplus} : (\mathrm{Set}) &\to (\mathrm{Ab}) \\ S &\mapsto \mathbb{Z}^{\oplus S}. \end{aligned}$$

$$\begin{tikzcd} S \arrow[d, "u"] & \longmapsto & \mathbb{Z}^{\oplus S} \arrow[d] & e_s \arrow[d, mapsto, "\text{extend linearly}"] \\ T & & Z^{\oplus T} & e_{u(s)} \end{tikzcd}$$

Hence we can view S_k as

$$S_k = \mathbb{Z}^{\oplus} \circ \mathrm{Hom}_{(\mathrm{Top})}(\Delta_k, \cdot) = \mathbb{Z}^{\oplus} \circ {}_{\Delta_k}h$$

where ${}_{\Delta_k}h$ is a covariant functor.

Consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}_{(\mathrm{top})}(\Delta_k, X) \arrow[rr, "\partial_{k-1}\circ\partial_k"] \arrow[dr, "e"] & & S_{k-2(X)} \\ & S_k(X) \arrow[ur, "\partial_{k-1}\circ\partial_k\circ e=0"'] & \end{tikzcd}$$

This diagram explains why $\partial_{k-1} \circ \partial_k \sigma = 0 \to \partial_{k-1} \circ \partial_k = 0$ through universal property.

Definition 1.8 (Singular homology groups). Let X be a toplogical space, we have a singular chain complexes

$$\cdots \xrightarrow{\partial_{k+2}} S_{k+1}(X) \xrightarrow{\partial_{k+1}} S_k(X) \xrightarrow{\partial_k} S_{k-1} \xrightarrow{\partial_{k-1}} S_{k-2} \cdots.$$

a. $S_k(X)$: the group of (singular) k-chains in X.

b. $Z_k(X) := \ker\partial_k(X)$ the group of k-cycles in X.

c. $B_k(X) := \mathrm{im}\partial_{k+1}$ the group of k-boundaries in X.

d. $H_k(X) := Z_k(X) \big/ B_k(X)$ the k-th sinular homology group of X.

2 Chain complexes

Definition 2.1 (Chain complexes and chain maps). A chain complex of abelian groups is a sequence of abelian groups linked by homomorphisms

$$C_\cdot : \cdots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots$$

such that $\partial_{k-1} \circ \partial_k = 0, \forall k \in \mathbb{Z}$.

A chain map of $f_\cdot : C_\cdot \to C'_\cdot$ consists of homomorphisms $f_k : C_k \to C'_k, k \in \mathbb{Z}$ such that the following diagram commute:

$$\begin{tikzcd} C_k \arrow[r, "\partial_k"] \arrow[d, "f_k"] & C_{k-1} \arrow[d, "f_{k-1}"] \\ C'_k \arrow[r, "\partial'_k"] & C'_{k-1} \end{tikzcd}$$

Definition 2.2 (Composition of chain maps). Given two chain maps $f_\cdot \xrightarrow{f_*} C'_*$ and $C'_\cdot \xrightarrow{f'_\cdot} C''_\cdot$, the composition of chain maps $(f' \circ f)_\cdot : C_\cdot \to C''_\cdot$ is defined by

$$(f' \circ f)_k := f'_k \circ f_k, k \in \mathbb{Z}.$$

All chain complexes and chain maps form a category, we use "(cKom)" to denote it.

Definition 2.3. Let $f_\cdot : C_\cdot \to C'_\cdot$be a chain map, then we have

$$f_k(Z_k(C_\cdot)) \subset Z_k(C'_\cdot) \quad \text{and} \quad f_k(B_k(C_\cdot)) \subset B_k(C_\cdot).$$

It induces a group homomorphism

Exercise 2.1.

$$\begin{aligned} H_k : (\text{cKom}) &\to (\text{Ab}) \\ C_\cdot &\to H_k(C_\cdot) \\ C_\cdot \xrightarrow{f_\cdot} C'_\cdot &\mapsto H_k(C_\cdot) \xrightarrow{f_{*k}} H_k(C'_\cdot) \end{aligned}$$

is a covariant functor.

Let $\mathcal{C}$ be a category. Giving a functor $\mathcal{C} \xrightarrow{K} (\text{cKom})$ is equivalent to giving functor $\mathcal{C} \xrightarrow{K_k} (\text{Ab})$ and natural transformations

$$K_k \xrightarrow{D_k} K_{k-1}, \quad k \in \mathbb{Z}$$

such that

$$D_{k-1} \circ D_k = 0, \quad \forall k \in \mathbb{Z}.$$

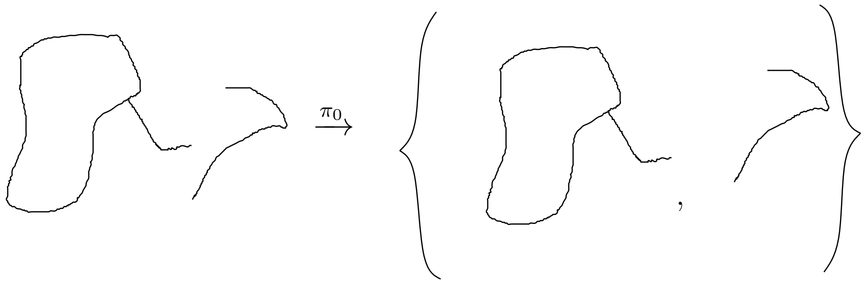

Figure 4: $\pi_0 : X \to$ the set of path-connected components of X

Definition 2.4. Let X be a topological space,

$$\pi_0(X) := \text{ the set of path-connected components of } X.$$

π_0 is a functor:

$$X \mapsto \pi_0(X)$$

$$X \xrightarrow{f} Y \mapsto \pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y)$$

where $\pi_0(f)c :=$ the path-connected components containing $f(c)$ for any $c \in \pi_0(X)$.

Exercise 2.2. For every topological space X, establish a group isomorphism

$$\mathbb{Z}^{\oplus \pi_0(X)} \xrightarrow{T(X)} H_0(X)$$

and show that your T is a natural transformation between $\mathbb{Z}^{\oplus} \circ \pi_0$ and H_0.

3 The singular homology of a star-shaped

Definition 3.1 (Star-shaped set). Let $0 \in X \subset \mathbb{R}^n$ s.t.

$$\forall p \in X \Rightarrow tp \in X \text{ for any } t \in [0, 1].$$

Then we call X a star-shaped set in $\mathbb{R}^n$.

Let $\sigma : \Delta_k \to X$ be a singular k-simplex in X, then we define

$$H_\sigma : \Delta_{k+1} \to X$$
$$(t_0, \cdots, t_n) \mapsto \begin{cases} 0, & (t_0, \cdots, t_{k+1}) = (1, 0, \cdots, 0) \\ (1-t_0)\sigma\left(\dfrac{t_1}{t_1 + \cdots t_{k+1}}, \cdots, \dfrac{t_{k+1}}{t_1 + \cdots t_{k+1}}\right), & \text{others.} \end{cases}$$

It is easy to verify that

$$(H\sigma)_0 = \sigma, (H\sigma)_i = H(\sigma_{i-1}), \quad i = 1, \cdots, k+1.$$

Calculate for $k \geq 1$

$$\begin{aligned} \partial_{k+1}(H_k\sigma) =& \sum_{i=0}^{k+1} (-1)^i (H_k\sigma)_i \\ =& \sigma + \sum_{i=1}^{k+1} (-1)^i H_{k-1}(\sigma_{i-1}) \\ =& \sigma - \sum_{j=0}^{k} (-1)^j H_{k-1}(\sigma_j) \\ =& \sigma - H_{k-1}\partial_k \sigma. \end{aligned}$$

Hence

$$\sigma = (\partial H + H\partial)\, \sigma.$$

$$\sigma \in Z_k(X) \Leftrightarrow \partial\sigma = 0 \Rightarrow \sigma = \partial(H\sigma) \in B_k(X) \Rightarrow H_k(X) = 0, k \geq 1.$$

For $k = 0$, $H_0(X) = \mathbb{Z}$ since a star-shaped set has only one component.

4 Chain homotopy vs. Homotopy

5 Acyclic models theorem

6 Subdivision

7 Homology exact sequence

8 Mayer-Vietoris Sequences

9 Some variants of singular homology