ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
LECTURE 14

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Theorem 1 (Noether Normalization). (1) *Let $Z(F) \subset \mathbb{A}^n$ be a hyperplane, then there exists a finite surjective morphism*

$$\Pi : Z(F) \to \mathbb{A}^{n-1}.$$

(2) *If $X \neq \emptyset$ is an affine variety, then there exists a finite surjective morphism*

$$\Pi : X \to \mathbb{A}^k$$

for some positive integer k.

Lemma 1. *Let F be a nonzero polynomial in $k[x_1, \ldots, x_n]$, then there exists a point $p \in \mathbb{A}^n$ s.t. $f(p) \neq 0$.*

Proof. Prove it by induction. For $n = 0$ and $n = 1$, it is obvious. Now assume $n-1$ is true , for $f \in k[x_1, \ldots, x_n]$ we can write $f = \sum_i f_i x_n^i$ with $f_i \in k[x_1, \ldots, x_{n-1}]$. There exists j such that $f_j \neq 0$, by induction on n, there exists $(b_1, \ldots, b_{n-1}) \in \mathbb{A}^{n-1}$ such that $f_j(b_1, \ldots, b_{n-1}) \neq 0$. Then we get $g(X) := f(b_1, \ldots, b_n, x)$ in $k[x]\{0\}$. Of course there exists b_n such that $g(b_n) \neq 0$, i.e. $f(b_1, \ldots, b_n) \neq 0$. □

Proof of Theorem 1. (1) Let $F^{(d)}$ be the homogeneous part of F with the top degree., then $F^{(d)}(x_1, \ldots, x_{n-1}) \neq 0$. Thus there exists $(b_1, \ldots, b_{n-1}) \in \mathbb{A}^{n-1}$ such that $F^{(d)}(b_1, \ldots, b_{n-1}, 1) \neq 0$. By change of coordinates and multiplying F by a constant, we can get $F^{(d)}(0, \ldots, 0, 1) = 1$, it is equivalent to say the coefficient of x_n^d in F is 1. Let $\Pi = (x_1, \ldots, x_{n-1}) : Z(F) \to \mathbb{A}^{n-1}$ and $w_n \in A(Z(F))$ be the class of the last variable x_n. Then we have

$$A(Z(F)) = \Pi^*(k[x_1, \ldots, x_{n-1}])[w_n].$$

Since $F = x_n^d + \sum_{i=1}^{d-1} a_i x_n^i$ with $a_i \in k[x_1, \ldots, x_{n-1}]$, in $A(Z(F))$ we can get

$$0 = w_n^d + \sum_{i=1}^{d-1} \Pi^*(a_i) w_n^i.$$

Thus $A(Z(F))$ is finite over $\Pi^*(k[x_1, \ldots, x_{n-1}])$, i.e $\Pi : Z(F) \to \mathbb{A}^{n-1}$ is finite. Let $b = (b_1, \ldots, b_{n-1}) \in \mathbb{A}^{n-1}$, to see $\Pi^{-1}(b) \neq \emptyset$. Put $g(x) := F(b_1, \ldots, b_{n-1}, x) \in k[x]$, the coefficient of x_n of F is 1, then $g(x)$ is not constant. Hence g has a zero $b_n \in k$,

$$\Pi^{-1}(b) = \{(b_1, \ldots, b_{n-1}, b_n) | F(b_1, \ldots, b_{n-1}, b_n) = 0\} \neq \emptyset.$$

So the morphism is surjective.

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(2) If $X = \mathbb{A}^n$, then it is clear. Assume $\emptyset \neq X \subsetneqq \mathbb{A}^n$ is a closed subvariety, we prove the statement by induction on n. Let $F \in I(X)\backslash\{0\}$ be irreducible. By (1) there exists a finite surjective morphism

$$\Pi : Z(F) \to \mathbb{A}^{n-1}$$

where $X \subset Z(F)$ is closed. The embedding of $i : X \to Z(F)$ is finite, so $\tilde{\Pi} = \Pi \circ i : X \to \mathbb{A}^{n-1}$ is a finite morphism. Let $Y \subset \mathbb{A}^{n-1}$ be the image of X. By induction on n there is a finite surjective morphism $\varphi : Y \to \mathbb{A}^k$ for some k, then $\varphi \circ \Pi \circ i$ is a finite surjective morphism from $X \to \mathbb{A}^k$ for some k.

□

Lemma 2. *Let $\varphi : X \to Y$ be a finite surjective morphism, let Z, W be closed subvarieties of X and $Z \subsetneqq W$, then $\varphi(Z) \subsetneqq \varphi(W)$.*

Proof. We can assume $X = W$ and $Y = f(W)$, thus the lemma is equivalent to : if $Z \subsetneqq X$ is a closed subvariety, then $f(Z) \subsetneqq Y$. Let $g \in A(X)\backslash\{0\}$ such that $g|Z = 0$, since φ is finite, g satisfies a monic equation

$$g^n + \sum_{i=0}^{n-1} \varphi^*(a_i) g^i = 0$$

with $a_i \in Y$. Take the one with the smallest degree n, then $\varphi^*(a_0) \neq 0$(otherwise divide by g), then we get

$$0 \neq \varphi^*(a_0) = -g(g^{n-1} + \sum_{i=1}^{n-1} \varphi^*(a_i) g^{i-1}).$$

The right hand side of the equation is in $\langle g \rangle$, thus $\varphi^*(a_0)|Z = 0 \Rightarrow a_0|_{\varphi(Z)} = 0$ $\varphi(Z) \subsetneqq Y$(if $\varphi(Z) = Y$, then $a_0 = 0 \in A(Y) \Rightarrow \varphi^*(a_0) =$, it makes a contradiction). □

Corollary 1. *Let $\varphi : X \to Y$ be a finite surjective morphism, then all the fibres of φ are finite.*

Proof. It is enough to show that every irreducible component Z of $\varphi^{-1}(y)$ is a point. Let $z \in Z$ be a point, then $\varphi(z) = y = \varphi(Z)$, by lemma 2 we get $\{0\} = Z$. □

Definition 1 (Dimension of Varieties)**.** Let X be a variety, $\emptyset \neq X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n = X$ be a chain of irreducible closed subsets on X, we call it a chain in X, n is called length of the chain. The dimension of X is the maximal n such that there exists a chain of length n in X or ∞ if this maximum does not exist.

Lemma 3. (1) *Let $Y \subset X$ be a closed subvariety, then $dimY \leq dimX$. If $Y \subsetneqq X$ and $dimY < \infty$, then $dimY < dimX$.*
(2) *Let $f : X \to Y$ be a surjective closd morphism, then $dimX \geq dimY$.*

Proof. (1) Let $Y_0 \subsetneqq Y_1 \subsetneqq \cdots \subsetneqq Y_k$ be a chain in Y, it is also a chain in X, thus $\dim X \supset \dim Y$. If $Y \subsetneqq X$, then $Y_0 \subsetneqq Y_1 \subsetneqq \cdots \subsetneqq Y_k \subsetneqq X$ is a chain in X, hence if the dimension of Y is finite, we get $\dim Y < \dim X$.
(2) Let $Y_0 \subsetneqq \cdots \subsetneqq Y_n$ be a chain in Y, we need to show that there exists a chain $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n$ in X such that $\varphi(X_i) = Y_i$ for all i. Use induction on n, it is obvious for $n = 0$. Let $Z_1, \ldots, Z_r$ be irreducible components

of $f^{-1}(Y_{n-1})$, then $\overset{r}{\underset{i=1}{\cup}} \varphi(Z_i) = Y_{n-1}, f(Z_i)$ are closed, Y_{n-1} is irreducible. Thus one of the $f(Z_i)$ is equal to Y_{n-1}. Since $\varphi : Z_i \to Y_{n-1}$ is a surjective closed morphism, by induction we get a chain $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_{n-1} = Z_i$ in X with $f(X_i) = Y_i$ for $i = 0, \ldots, n-1$, then $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_{n-1} \subsetneqq X_n = X$ is a chian with $f(X_i) = Y_i$ for all i.

□

Theorem 2. *Let $\varphi : X \to Y$ be a finite surjective morphism of varieties, then $dimX = dimY$.*

Proof. We already know $\dim X \geq \dim Y$ because φ is surjective and closed. To show $\dim Y \geq \dim X$, let $X_0 \subsetneqq \ldots, \subsetneqq X_n$ be a chain in X, for i let $Y_i = f(X_i)$, then by lemma 2 $Y_0 \subsetneqq Y_1 \subsetneqq \cdots \subsetneqq Y_n$ is also a chain in Y. □

Theorem 3. (1) *$dim\mathbb{A}^n = n$.*

(2) *Let $F \in k[x_1, \ldots, x_n] \backslash k$ be a irreducible polynomial, then $dimZ(F) = n-1$.*
(3) *Conversely any subvariety $X \subset \mathbb{A}^n$ of dimension $n-1$ is a hypersurface, i.e. $X = Z(F)$ with F irreducible.*

Proof. We first prove $\dim Z(F) = \dim \mathbb{A}^{n-1}$ for $F \in k[x_1, \ldots, x_n] \backslash k$. By theorem 1 we know therre exists a surjective finite morphism from $Z(F)$ to $\mathbb{A}^{n-1}$, thus $\dim Z(F) = \dim \mathbb{A}^{n-1}$.

(1) Let $Z_i = Z(x_{i+1}, \ldots, x_n) \subset \mathbb{A}^n$, then $Z_i \simeq \mathbb{A}^i$ and thus
$$Z_0 \subsetneqq Z_1 \subsetneqq Z_2 \subsetneqq \cdots \subsetneqq Z_n = \mathbb{A}^n$$
is a chain in $\mathbb{A}^n$ of length n, it implies $\dim \mathbb{A}^n \geq n$. Now we prove the opposite inequality by induction on n. For $n = 0$, it is true, let $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_{k-1} = X \subsetneqq \mathbb{A}^n$ be a chain in $\mathbb{A}^n$. Then $X \subsetneqq \mathbb{A}^n$ is a closed subvariety, we can choose $F \in I(X)$ and F is irreducible, then $X \subset Z(F)$. Thus $k - 1 \leq \dim Z(F) = \dim \mathbb{A}^{n-1} = n-1$ by induction. Since the chain we choose is arbitrary, we get $\dim \mathbb{A}^n \leq n$. Hence $\dim \mathbb{A}^n = n$.
(2) It follows from (1) immediately.
(3) Let $\emptyset \neq X \subsetneqq \mathbb{A}^n$ and $\dim X = n-1$, then there exists $F \in I(X) \backslash k$ being irreducible, thus $X \subset Z(F)$, X and $Z(F)$ are both irreducible of the same dimension, hence $X = Z(F)$.

□

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