F. L. Nazarov's paper
Local Estimates of Exponential Polynomials and Their Applications to Inequalities of Uncertainty Pinciple Type

Notes taken by 89hao

January 17, 2020

Abstract

This is a learning note about Nazarov's paper(see [2]).

Contents

Definition 1. *An exponential polynomial is*

$$p(t)=\sum_{k=1}^{n} c_k e^{\lambda_k t} \quad (c_k \in \mathbb{C}, \lambda_k \in \mathbb{C}).$$

The main purpose of the first part of the paper is to establish the following inequality

$$\sup_{t\in I} |p(t)| \le \left(\frac{A\mu(I)}{\mu(E)}\right)^{n-1} \sup_{t\in E} |p(t)|, \tag{1}$$

where $I \subset \mathbb{R}$ is an interval, $E \subset I$ is a measurable set of positive Lebesgue measure and A is an absolute constant.

1 The Turan lemma: original form

The following lemma was derived by Turan (see [3]).

Theorem 1. *Let $z_1, \cdots, z_n$ be complex numbers, $|z_j| \geq 1, j = 1, \cdots, n$. Let*

$$b_1, \cdots, b_n \in \mathbb{C}, \quad S_m \stackrel{\text{def}}{=} \sum_{k=1}^{n} b_k z_k^m \quad (m \in \mathbb{Z}).$$

Then

$$|S_0| \leq n \left(\frac{2e(m+n-1)}{n}^{n-1} \right) \max_{k=m+1}^{m+n} |S_k| \leq \left(\frac{4e(m+n-1)}{n} \right)^{n-1} \max_{k=m+1}^{m+n} |S_k| \quad (2)$$

for all $m \in \mathbb{Z}_+$.

Proof. To prove the lemma, we need to construct a polynomial $q(z) = 1 + \sum_{k=1}^{n} \gamma_k z^{m+k}$ such that

(1) $q(z_j) = 0$ for each $= 1, \cdots, n$ and

(2) $\sum_{k=1}^{n} |\gamma_k| \leq n \left(\frac{2e(m+n-1)}{n} \right)^{n-1}$.

Let

$$q(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right) \sigma_m(z),$$

where $\sigma_m(z)$ is the m-th partial sum of the series $\prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right)^{-1} = \sum_{k=0}^{\infty} \beta_k z^k$, i.e.

$$\sigma_m(z) = \sum_{k=1}^{m} \beta_k z^k.$$

By definition, we have

$$1 = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right) \sum_{k=0}^{\infty} \beta_k z^k.$$

This identity implies that the s-th coefficient in the expansion of the right side depends only on $\beta_{s-n}, \cdots, \beta_s$. Hence the coefficients at the powers $z, z^2, \cdots, z^m$

of $q(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right) \sigma_m(z)$ all vanish (since they only depend on $\sigma_m(z)$). Recalling the Taylor expansion

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \frac{(k+n-1)!}{k!(n-1)!} z^k,$$

hence we have (by using the condition $|z_j| \geq 1$ and assuming $|z| < 1$)

$$\left| \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right)^{-1} \right| \leq (1-|z|)^{-n} = \sum_{k=0}^{\infty} \frac{(k+n-1)!}{k!(n-1)!} |z|^k.$$

Thus, all coefficients of $\sigma_m(z)$ do not exceed[1]

$$\frac{(m+n-1)!}{m!(n-1)!} \leq \left(\frac{e(m+n-1)}{n} \right)^{n-1}.$$

Then we get the extimates

$$|\gamma_k| \leq \left(\frac{e(m+n-1)}{n} \right)^{n-1} \sum_{s=k}^{n} \binom{n}{s}$$

and

$$\sum_{k=1}^{n} |\gamma_k| = \frac{1}{2} \sum_{k=1}^{n} (|\gamma_k| + |\gamma_{n+1-k}|) \leq 2^{n-1} n \left(\frac{e(m+n-1)}{n} \right)^{n-1}.$$

Now we've constructed the desired polynomial $q(z)$.

Since

$$\begin{aligned} S_0 =& b_1 + b_2 + \cdots + b_n \\ =& \sum_{j=1}^{n} b_j \cdot 1 \\ =& \sum_{j=1}^{n} \left(- \sum_{k=1}^{n} \gamma_k z_j^{m+k} \right) \\ =& - \sum_{k=1}^{n} \gamma_k S_{m+k}. \end{aligned} \tag{3}$$

[1]Here needs some estimates: we need to prove

$$\binom{n}{k} \leq \left(\frac{en}{k+1} \right)^k.$$

This inequality can be proved by induction.

Hence the estimates above and (3) complete the proof. $\square$

Recalling the definition of an exponential polynomial

$$p(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t},$$

now let $t_k = t_0 + k\delta$, we have

$$p(t_k) = \sum_{j=1}^{n} c_j e^{i\lambda_j (t_0+k\delta)} = \sum_{j=1}^{n} b_j \left(e^{i\lambda_j \delta}\right)^k = \sum_{j=1}^{n} b_j z_j^k,$$

where $z_j = e^{i\lambda_k \delta}$ and $b_j = c_j e^{i\lambda_j t_0}$. Then we can use the lemmma directly and get

$$|p(t_0)| \leq \left\{\frac{4e(m+n-1)}{n}\right\}^{n-1} \max_{k=m+1}^{m+n} |p(t_k)|\,. \tag{4}$$

Now the inequality (1) for the case where E is an interval can be derived in an almost immediate way (with the constant A=4e).

Using the same idea in

Theorem 2. *Let I be an interval, let $E \subset I$ be a measurable set of positive Lebesgue measure. Then*

$$\max_{t\in I} |p(t)| \leq 2^n \left(\frac{\mu(I)}{\mu(E)}\right)^{2n^2} \max_{t\in E} |p(t)|\,. \tag{5}$$

Proof. By (4), the following inequality

$$\max_{t\in I} |p(t)| \leq 2^n \max_{t\in E} |p(t)| \tag{6}$$

holds if t_0 is the first term of the arithmetic progression $t_k = t_0 + k\delta\,(k = 0, \cdots, n)$ with all other terms belonging to E. The point of the proof is to find a set E_1 that is "close" to E and we can choose a δ such that all t_k's belongs to E.

Step 1.Let $J \subset I$ is an open interval and

$$\mu\left(E \cap J\right) > \left(1 - \frac{1}{2n}\right) \mu(J).$$

Let $t_0 \in J$ be any fixed point. Such a point t_0 splits the interval J into two subintervals J_- and J_+. At least one of them, let's say J_+ has the property

$$\mu\left(E \cap J_+\right) > \left(1 - \frac{1}{2n}\right) \mu(J).$$

Let $\varphi(t)=\chi(t)$ be the characteristic function of $J_+\backslash E$, then by applying the lattice averaging lemma we see that the average number of points $t_k=t_0+k\delta(k\in\mathbb{N})$ belonging to $J_+\backslash E$ as δ runs over the interval $\left(\frac{\mu(J_+)}{2n},\frac{\mu(J_+)}{n}\right)$ is (here we write $\frac{\mu(J_+)}{2n}$ as s)

$$\begin{aligned}\frac{\int_s^{2s}\sum_{k\in\mathbb{Z}\backslash\{0\}}\varphi(k\delta)\,\mathrm{d}\delta}{\int_s^{2s}\mathrm{d}\delta} &= \frac{1}{s}\int_1^2\sum_{k\in Z\backslash\{0\}}\varphi(ks\frac{\delta}{s})s\mathrm{d}\left(\frac{\delta}{s}\right)\\ &=\int_1^2\sum_{k\in\mathbb{Z}\backslash\{0\}}\varphi(ksv)\mathrm{d}v\\ &\le\frac{1}{s}\int_{\mathbb{R}}\varphi(t)\mathrm{d}t\\ &=\frac{2n}{\mu(J_+)}\mu\left(J_+\setminus E\right)\\ &<1.\end{aligned}\tag{7}$$

Hence there exists a positive $\delta<\frac{\mu(J_+)}{n}$ such that none of the points $t_1,\cdots,t_n$ belongs to $J_+\setminus E$. Since $k\delta<\frac{k\mu(J_+)}{n}\le 1$ and t_0 is the endpoint of J_+, all these points lie in J_+ and, consequently, in E. Since the choise of $t_0\in J$ is arbitrary, any points in J have the property that $t_k\in E$ for each $k=1,\cdots,n$.

Step 2. Let $E_1=\bigcup\left\{J:J\subset I\text{ is open},\mu(E\cap J)>\left(1-\frac{1}{2n}\right)\mu(J)\right\}$. Since E_1 is the union of open sets, E_1 itself is also open, hence, the union of disjoint open intervals. Let Q be one constituent interval of E_1, if

$$\mu\left(E\cap Q\right)>\left(1-\frac{1}{2n}\right)\mu(Q)$$

holds, then we can find a larger open interval Q' such that $Q'\subset Q\subset E_1$, this contradicts the chosen of Q. Hence all the cons constituent intervals of E_1 satisfy the relation

$$\mu(E\cap Q)\le\left(1-\frac{1}{2n}\right)\mu(Q).$$

Thus, the set E_1 has the following two properties

$$\sup_{t\in E_1}|p(t)|\le 2^n\sup_{t\in E}|p(t)|,\tag{8}$$

$$\mu(E_1)\ge\left(1-\frac{1}{2n}\right)^{-1}\mu(E)\ge e^{\frac{1}{2n}}\mu(E)\text{ or }E_1=I.\text{[2]}\tag{9}$$

[2]Here we use the inequality $\frac{1}{e}\ge\left(1-\frac{1}{2n}\right)^{2n}$.

Step 3. Iterating this procedure we obtain a sequence of sets $E_1, E_2, \cdots$ such that

$$\sup_{t\in E_k}|p(t)| \le 2^{nk}\sup_{t\in E}|p(t)|, \tag{10}$$

$$\mu(E_k) \ge e^{\frac{k}{2n}}\mu(E) \text{ or } E_k = I. \tag{11}$$

If $k > 2n\log\frac{\mu(I)}{\mu(E)}$, then the first case of (11) cannot occur. Therefore we obtain

$$E_{\left[2n\log\frac{\mu(I)}{\mu(E)}+1\right]} = I,$$

whence

$$\sup_{t\in I}|p(t)| \le 2^{\left(2n\log\frac{\mu(I)}{\mu(E)}+1\right)n}\sup_{t\in E}|p(t)| \le 2^n\left(\frac{\mu(I)}{\mu(E)}\right)^{2n^2}\sup_{t\in E}|p(t)|.$$

□

Remark. The proof of Theorem 2 is based on Theorem 1. We can regard Theorem 1 is a discrete version of Theorem 2. From the discrete version to Lebesgue measurable sets, the simplest thought is to find the discrete points which Theorem 1 can be used to. If there exists, then our problem can be solved easily. But unfortunately the arithmetic progression t_k may not exists in E for any point in I. To overcome this difficulty, we need to find an interval close to E (here the sense of "close" has exact meaning in the proof), and any point fixed t_0 in this interval satisfy the condition $t_k \in E$ for each $k = 1, \cdots, n$. Finally, by iterating the procedure, the chosen set becomes strictly larger, and finaly equals to I.

2 Two usefull lemmas

Lemma 1. *If $P(z)$ is an algebraic polynomial of degree n, then*

$$\mu\left(\left\{x\in\mathbb{R} : \left|\frac{\mathrm{d}}{\mathrm{d}x}\log P(x)\right| > y\right\}\right) \le \frac{8n}{y}$$

and

$$\mu\left(\left\{z\in\mathbb{T} : \left|\frac{\mathrm{d}}{\mathrm{d}z}\log P(z)\right| > y\right\}\right) \le \frac{8n}{\pi y}.$$

Proof. A First we shall prove the inequality for the real line. Let $z_1, \cdots, z_{n_1}$ and $\zeta_1, \cdots, \zeta_{n_2}$ $(n_1 + n_2 = n)$ be complex zeros of the polynomial P enumerated in such a way that $\mathrm{Im} z_j \le 0$ $(j = 1, \cdots, n_1)$ and $\mathrm{Im}\zeta_j > 0$ $(j = 1, \cdots, n_2)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}z}\log P(z) = \sum_{j=1}^{n_1}\frac{1}{z-z_j} + \sum_{j=1}^{n_2}\frac{1}{z-\zeta_j} = \sum_1(z) + \sum_2(z).$$

The function $\sum_1(z)$ is analytic in the upper half-plane $\mathbb{H}$, and

$$\operatorname{Im}\sum_1(z) = \sum_{j=1}^{n_1} \frac{\operatorname{Im} z_j - \operatorname{Im} z}{|z - z_j|^2} < 0$$

for all $z \in \mathbb{H}$.

Let $h(\xi)$ be the harmonic measure of the set $\mathbb{R}\setminus[-y, y]$ with respect to the upper-half plane and a point $\xi \in \mathbb{H}$. We put $u(z) \stackrel{\text{def}}{=} h\left(-\sum_1(z)\right)$, it is harmonic in $\mathbb{H}$.

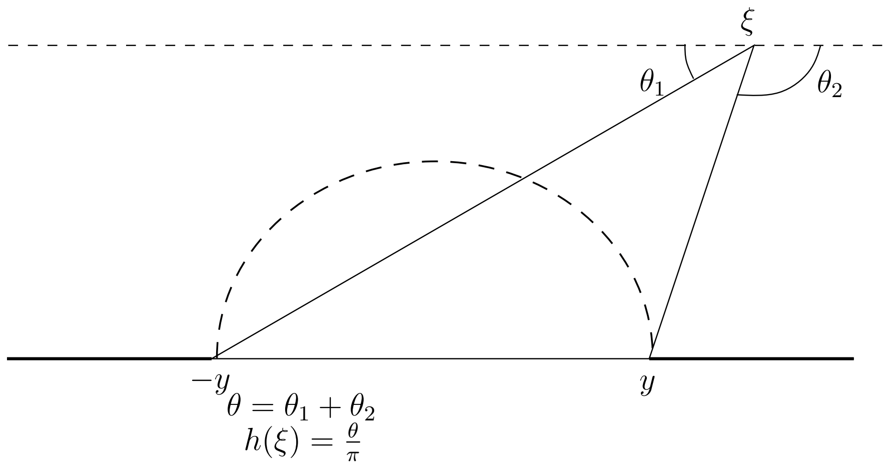

Figure 1: Harmonic function $h(\xi)$

If $t \to +\infty$, then $-\sum_1(it) \to i0^+$ and $u(it) \to 0$. If $|\sum_1(z)| \geq y$, then $u(z) \geq \frac{1}{2}$.

Hence, we have

$$\lim_{t\to+\infty} \pi t u(it) = \int_{\mathbb{R}} u(x)\mathrm{d}x \geq \frac{1}{2}\mu\left(\left\{x \in \mathbb{R} : \left|\sum_1(x)\right| > y\right\}\right).$$

On the other hand,

$$\begin{aligned}\lim_{t\to+\infty} \pi t u(it) &= \lim_{t\to+\infty} \pi t h\left(\frac{in_1}{t} + \mathcal{O}\left(\frac{1}{t^2}\right)\right) \\ &= \lim_{t\to+\infty} \pi t\left(2\arctan\left(\frac{n_1}{ty}\right)/\pi\right) \\ &= \frac{2n_1}{y}.\end{aligned}$$

Hence

$$\mu\left(\left\{x\in\mathbb{R}:\left|\sum_1(x)\right|>y\right\}\right)\leq\frac{4n_1}{y}.$$

Similarly

$$\mu\left(\left\{x\in\mathbb{R}:\left|\sum_2(x)\right|>y\right\}\right)\leq\frac{4n_2}{y}.$$

Combining these estimates, we obtain

$$\begin{aligned}\mu\left(\left\{x\in\mathbb{R}:\left|\sum(x)\right|>y\right\}\right)\leq&\mu\left(\left\{x\in R:\left|\sum_1\right|>\frac{n_1}{n}y\right\}\right)\\ &+\mu\left(\left\{x\in\mathbb{R}:\left|\sum_1\right|>\frac{n_2}{n}y\right\}\right)\\ \leq&\frac{8n}{y}.\end{aligned}$$

Now we pass to the case of the circumference. As above, we split the zeros of $P(z)$ into two groups $z_1,\cdots,z_{n_1}\in\mathbb{D}$ and $\zeta_1,\cdots,\zeta_{n_2}\in\mathbb{C}\backslash\mathbb{D}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}z}\log P(z)=\frac{1}{z}\left(\sum_{j=1}^{n_1}\frac{z}{z-z_j}+\sum_{j=1}^{n_2}\frac{z}{z-\zeta_j}\right)=\frac{1}{z}\left(\sum_1(z)+\sum_2(z)\right).$$

The factor $\frac{1}{z}$ can be disregarded since its absolute value is equal to 1. The estimate for $\sum_1(z)$ is essentially the same as above: having established the inequality

$$\mathrm{Re}\sum_1(z)=\sum_{j=1}^{n_1}\frac{|z|^2-\mathrm{Re}z\overline{z}_j}{|z-z_j|^2}\geq 0$$

for $z\in\mathbb{C}\backslash\mathbb{D}$, we consider the function $u(z)\stackrel{\text{def}}{=}h\left(i\sum_1(z)\right)$, which is harmonic outside the unit disk, and derive the estimate

$$u(\infty)=\frac{2\arctan\frac{n_1}{y}}{\pi}=\int_{\mathbb{T}}u(z)\mathrm{d}\mu(z)\geq\frac{1}{2}\mu\left(\left\{z\in\mathbb{T}:\left|\sum_1(z)\right|>y\right\}\right),$$

which implies

$$\mu\left(\left\{\left|\sum_1(z)\right|>y\right\}\right)\leq\frac{4}{\pi}\arctan\frac{n_1}{y}\leq\frac{4}{\pi}\frac{n_1}{y}.$$

Th function $\operatorname{Re}\sum_2(z)$ may change sign. Therefore we use another inequality

$$\operatorname{Re}\sum_2(z) = \sum_j^{n_2} \frac{\operatorname{Re} z(\overline{z} - \overline{\zeta}_j)}{|z-\zeta_j|^2} = n_2 - \sum_{j=1}^{n_2} \frac{|\zeta_j|^2 - \operatorname{Re} z\overline{\zeta}_j}{|z-\zeta_j|^2} \le n_2 \qquad (z \in \mathbb{D}).$$

This time we choose the function h to be harmonic in $\mathbb{H} - in_2$. In order to obtain the estimate $\mu\left(|\sum_2| > y\right) \le \frac{4}{\pi}\frac{n_1}{y}$, we can restrict ourselves to values $y > n_2$. Let $h(\xi)$ be harmonic measure of $(\mathbb{R} - in_2)\setminus I$ (where I is the interval cut off from the line $\mathbb{R} - in_2$ by the circle centered at 0 and of radius y) with respct to the half-plane $\mathbb{H} - in_2$ and the point $\xi \in \mathbb{H} - in_2$. One can easily check that the

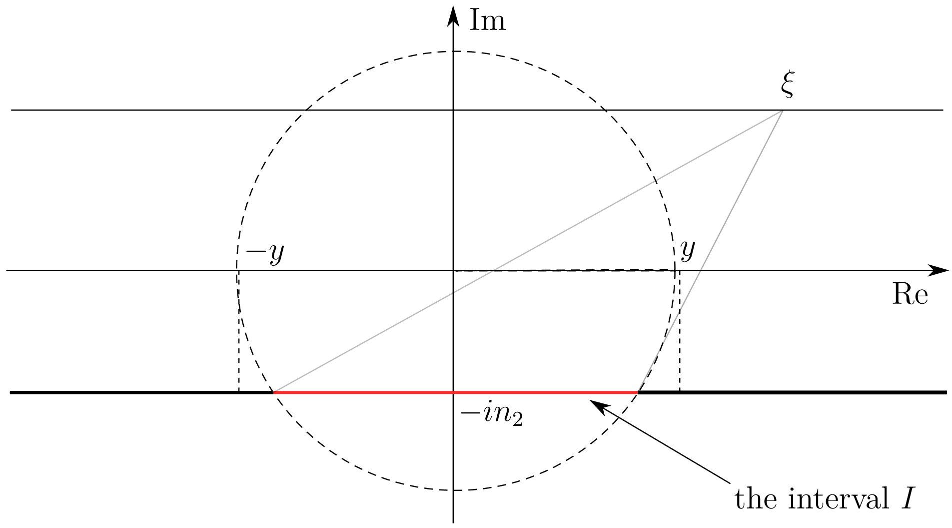

Figure 2: Domain of the harmonic function

function $u(z) \stackrel{\text{def}}{=} h\left(-i\sum_2(z)\right)$ is harmonic in $\mathbb{D}$, $u(z) > \frac{\pi - \arccos\frac{n_2}{y}}{\pi}$ if $|\sum_2(z)| > y$, and $u(0) = h(0) = \frac{2\arcsin\frac{n_2}{y}}{\pi}$. Therefore

$$\mu\left(\left\{\left|\sum_2\right| > y\right\}\right) \le \frac{2\arcsin\frac{n_2}{y}}{\pi - \arccos\frac{n_2}{y}} = \frac{2\arcsin\frac{n_2}{y}}{\pi + \arcsin\frac{n_2}{y}}.$$

Now, to get the desired estimate it suffices to verify that $\frac{2\theta}{\theta/2+\theta} \le \frac{4}{\pi}\sin\theta$ for each $\theta \in \left[0, \frac{\pi}{2}\right]$. The last inequality is equivalent to $\frac{\sin\theta}{\theta} + \frac{2}{\pi}\sin\theta \ge 1$. Taking into account that $\sin\theta \ge \theta - \frac{1}{6}\theta^3$ for every $\theta > 0$ and $\sin\theta \ge \frac{2}{\pi}\theta$ for every $\theta \in \left[0, \frac{\pi}{2}\right]$, we have

$$\frac{\sin\theta}{\theta} + \frac{2}{\pi}\sin\theta \ge 1 - \frac{1}{6}\theta^2 + \frac{4}{\pi^2}\theta = 1 + \theta\left(\frac{4}{\pi^2} - \frac{1}{6}\theta\right) \ge 1 + \theta\left(\frac{4}{\pi^2} - \frac{\pi}{12}\right)$$

and it remains to notice that $\pi^3 \leq 48$.

As above, the estimat of $\sum(z)$ results from the estimates of $\sum_1(z)$ and $\sum_2(z)$. Lemma 1 is proved. $\square$

Lemma 2 (Langer Lemma). *Let* $p(z) = \sum_{k=1}^{n} c_k e^{i\lambda_k z} (0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda)$ *be an exponential polynomial not vanishing identically. Then the number of complex zeros of* $p(z)$ *in an open vertical strip* $x_0 < Re z < x_0 + \Delta$ *of width* Δ *does not exceed* $(n-1) + \frac{\lambda\Delta}{2\pi}$.

Proof. Without loss of generality we assume that the coefficients c_1 and c_2 do not vanish and the boundary of the strip $x_0 < \mathrm{Re} z < x_0 + \Delta$ is free of zeros of the exponential polynomial $p(z)$. We make use of the argument principle to estimate the number of zeros of $p(z)$ in the rectangle $Q = \{z : x_0 < \mathrm{Re} z < x_0 + \Delta, |\mathrm{Im} z| \leq y\}$, as $y \to +\infty$.

On the upper edge of Q we have $p(z) = c_1 + \mathcal{O}\left(e^{-\lambda_2 y}\right)$ (recall $\lambda_1 = 0$ and $\lambda_i < \lambda_{i+1}$). Therefore, the argument increment along this edge tends to 0 as $y \to +\infty$. Similarly, the representation $p(z) = c_n e^{i\lambda z}\left(1 + \mathcal{O}\left(e^{-(\lambda-\lambda_{n-1})y}\right)\right)$, which is valid on the lower edge of Q, implies that the argument increment along the lower edge tends to $\lambda\Delta$ as $y \to +\infty$.

We show that the argument increment along any vertical segment

$$\{z = x + it : t \in [\alpha, \beta]\}$$

free of zeros of $p(z)$ does not exceed $\pi(n-1)$.

Here we construct a real exponential polynomial out of $p(z)$. Let

$$\xi \stackrel{\mathrm{def}}{=} e^{i \arg p(x_0 + i\alpha)}.$$

The function

$$q(t) \stackrel{\mathrm{def}}{=} \mathrm{Im}\left(\bar{\xi} p(x_0 + it)\right) = \sum_{k=1}^{n} a_k e^{-\lambda_k t} \qquad \left(a_k = \mathrm{Im}\left(\bar{\xi} c_k e^{i\lambda_k x_0} \in \mathbb{R}\right)\right)$$

is a real exponential polynomial. Actually, xi is used to rotate $p(x_0 + it)$ to make the imaginary part of $p(x_0 + i\alpha)$ be 0, i.e., $q(\alpha) = 0$.

Since we have assumed that there are no zeros of $p(x + it)$ among $t \in [\alpha, \beta]$, $p(x+it)$ cannot pass through 0. If $q \equiv 0$, along with $p(x+i\alpha) \in \mathbb{R}$ and $\bar{\xi} p(x+i\alpha) > 0$ by definition, then all values $p(x_0 + it$ for $t \in [\alpha, \beta]$ lie on the ray $\{\xi y : y > 0\}$. Therefore $\Delta_{[\alpha,\beta]} \arg p(x_0 + it) = 0$. Otherwise, real zeros of $q(t)$ split the segment $[\alpha, \beta]$ into at most $n-1$ intervals I_j (it is well known that a real exponential polynomial of order n has at most $n-1$ zeros). Within each of intervals I_j ($q(\alpha) = 0$, hence there are $n-1$ intervals not n intervals), the values $p(x_0 +$

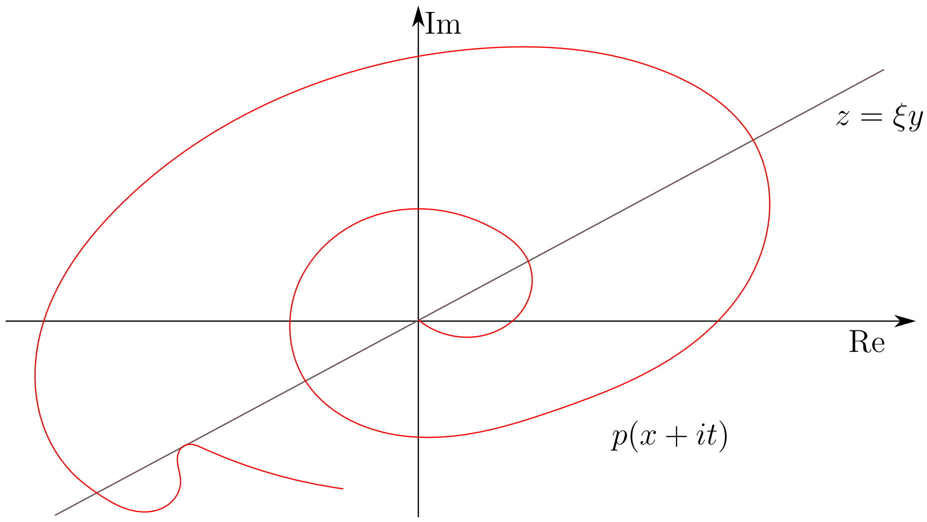

Figure 3: Argument of $p(x+it)$

$it)$ lie in one of the two half-planes generated by the line $\{\xi y : y \in \mathbb{R}\}$, whence $\left|\Delta_{I_j} \arg p(x_0+it)\right| \le \pi$.

Adding these inequalities, we obtain argument increment along each of the lateral edges of Q does not exceed $\pi(n-1)$. So the total argument increment of $p(z)$ along the boundary of Q traced counter clockwise can be estimated from above by a quantity tending to $2\pi\left(\frac{\Delta\lambda}{2\pi} + (n-1)\right)$ as $y \to +\infty$, whence Lemma 2 follows. $\square$

3 The Turan lemma for polynomials on the unit circumference

Here we shall prove inequality (1) for the case of a 1-periordic exponential polynomial $p(t) = \sum_{k=1}^n c_k e^{2\pi i m_k t}$, where $c_k \in \mathbb{C}$, $m_1 < \cdots m_n \in \mathbb{Z}$, and for the segment $I = [0,1]$.

Theorem 3. *Let $p(z) = \sum_{k=1}^n c_k z^{m_k}$ $(c_k \in \mathbb{C}, m_1 < \cdots m_n \in \mathbb{Z})$ be a trignometric plynomial on the unit circumference T, and let E be a measurable subset of $\mathbb{T}$. Then*

$$\|p\|_W \stackrel{\text{def}}{=} \sum_{k=1}^n |c_k| \le \left(\frac{16e}{\pi}\frac{1}{\mu(E)}\right)^{n-1} \sup_{z\in E} |p(z)| \le \left(\frac{14}{\mu(E)}\right)^{n-1} \sup_{z\in E} |p(z)|\,. \tag{12}$$

Proof.

Step 1. We shall construct by induction a sequence of polynonials $p_n, \cdots, p_1$ such that

(1) $p_n = p$;

(2) $\operatorname{ord} p_k = k \ (k = 1, \cdots, n)$;

(3) $\|p_{k-1}\|_W \geq \frac{\pi}{16}\|p_k\|_W$;

(4) the ratio $\varphi_k \overset{\text{def}}{=} \left|\frac{p_{k-1}}{p_k}\right|$ admits the weak type estimate

$$\mu\left(\{z \in \mathbb{T} : \varphi_k(z) > t\}\right) \leq \frac{1}{t}$$

for all $t > 0$.

The construction is as follows. Let $p_n = p$. The polynomial $p_k(z) = \sum_{s=1}^{k} d_s z^{r_s}$ ($r_1 < r_2 < \cdots < r_k \in \mathbb{Z}$ being chosen, we introduce two polynomials

$$\underline{q} \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-r_1} p_k(z)\right)$$

and

$$\overline{q} \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-r_k} p_k(z)\right).$$

Obviously, $\operatorname{ord} \underline{q} = \operatorname{ord} \overline{q} = k - 1$. We have

$$\|\underline{q}\|_W = \sum_{s=1}^{k} |d_s| \left(r_s - r_1\right), \quad \|\overline{q}\|_W = \sum_{s=1}^{k} |d_s| \left(r_k - r_s\right),$$

whence

$$\|\underline{q}\|_W + \|\overline{q}\|_W = (r_k - r_1) \sum_{s=1}^{k} |d_s| = r\|p_k\|_W,$$

where $r \overset{\text{def}}{=} r_k - r_1$. Hence at least one of the norms larger than or equal to $\frac{r}{2}\|p_k\|_W$. We assume $\|\overline{q}\|_W \geq \frac{r}{2}\|p_k\|_W$ (the other case is similar). Put $p_{k-1}(z) = \frac{\pi}{8r}\overline{q}(z)$, then conditions (2) and (3) are satisfied. It remains to check condition (4). Since $r_1 < r_2 < \cdots < r_k \in \mathbb{Z}$, let $g(\frac{1}{z}) = z^{-r_k} p_k(z)$, then $g(z)$ is an algebraic polynomial of degree r. Then

$$\overline{q}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-r_k} p_k(z)\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(g\left(\frac{1}{z}\right)\right) = -\frac{1}{z^2} g'\left(\frac{1}{z}\right).$$

Since $g\left(\frac{1}{z}\right)$ is an algebrail polynomial of degree r, we can use Lemma 1 and get [3]

$$\mu\left(\{z\in\mathbb{T}:\varphi_k(z)>t\}\right)=\mu\left(\left\{z\in\mathbb{T}:\left|\frac{g'(1/z)}{g(1/z)}\right|>\frac{8r}{\pi}t\right\}\right)\leq\frac{1}{t}$$

since
$$\left|\frac{p_{k-1}}{p_k}=\frac{\pi}{8r}\frac{\overline{q}(z)}{p_k}\right|=\left|\frac{\pi}{8r}\frac{g'(1/z)(-1/z^2)}{g(1/z)z^{r_k}}\right|=\frac{\pi}{8r}\left|\frac{g'(1/z)}{g(1/z)}\right|.$$

The above inequality also explains how the weird coefficient $\frac{\pi}{16}$ of condition (3) chooses.

Step 2. Before proving the theorem, we first illustrate what the step 2 does. By step 1, we have constructed a sequence of polynomials and they have the relation

$$\|p_{k-1}\|_W\geq\frac{\pi}{16}\|p_k\|_W.$$

Hence we can get
$$\left(\frac{\pi}{16}\right)^{n-1}\|p\|_W\leq\|p_1\|_W.$$

Since $\operatorname{ord}p_1=1$, the norm of p_1 is equivalent to any $|p_1(z)|$. We want to get the inequality (12), that means we may need to establish the inequality between $|p_1(z)|$ and $|p(z)|$ for $z\in E$. More precisely, we want to find some point $z_0\in E$ such that
$$\left|\frac{p_1(z_0)}{p(z_0)}\right|<\text{ some large number.}\tag{13}$$

The constant can be chosen large enough so that the measure of points which don't satisfy condition (13) is less than $\mu(E)$, hence cannot cover all points of E, i.e., the point $z_0\in E$ satisfies the condition exists.

Now we estimate the measure of the set of all points $z\in\mathbb{T}$ for which $|p_1(z)|$ is essentially greater than $|p_n(z)|=|p(z)|$ (the meaning of "essentially greater" would be clear later). We have

$$\left|\frac{p_1(z)}{p_n(z)}\right|=\prod_{k=2}^n\varphi_k(z)\leq\exp\left(\sum_{k=2}^n\psi_k(z)\right),$$

where $\psi_k(z)\overset{\text{def}}{=}\log_+\varphi_k(z)$ ($\log_+x$ means $\log_+x=0$ if $\log x<0$). The weak type esimate of φ_k gives the inequality

$$\mu\left(\psi_k>t\right)\leq e^{-t}$$

[3]In Lemma 1, the term $\left|\frac{P'(z)}{P(z)}\right|$ can be changed into $\left|\frac{P'(1/z)}{P(1/z)}\right|$ since the substitution $z\mapsto 1/z$ preserves Lebesgue measure on the unit circumference.

for all $t > 0$. Let $\alpha > 0$, we decompose $\psi_k(z)$ into the sum of $\eta_k(z) \overset{\text{def}}{=} \min(\psi_k(z), \alpha)$ and $\omega_k(z) \overset{\text{def}}{=} \psi_k(z) - \eta_k(z)$. Then $\sum_{k=2}^{n} \eta_k(z) \leq \alpha(n-1)$ for all $z \in \mathbb{T}$. Since for a nonnegative measurable function in measure space $(X, \mathcal{M}, \mu)$ we have

$$\int f(x)\mathrm{d}\mu(x) = \int_0^\infty \mu(f(x) > t)\mathrm{d}t,$$

we obtain

$$\int_{\mathbb{T}} \omega_k(z)\mathrm{d}\mu(z) = \int_\alpha^\infty \mu(\psi_k > t)\mathrm{d}t \leq \int_\alpha^\infty e^{-t}\mathrm{d}t = e^{-\alpha}.$$

Therefore,

$$\int_{\mathbb{T}} \left(\sum_{k=2}^{n} \omega_k(z)\right) \mathrm{d}\mu(z) \leq e^{-\alpha}(n-1). \tag{14}$$

Since

$$\sum_{k=2}^{n} \omega_k(z) = \sum_{k=2}^{n} \psi_k(z) - \sum_{k=2}^{n} \eta_k(z)$$

and $\sum_{k=2}^{n} \eta_k(z) \leq \alpha(n-1)$, we have

$$\mu\left(\left\{z \in \mathbb{T} : \sum_{k=2}^{n} \psi_k(z) > (\alpha+1)(n-1)\right\}\right) \leq \mu\left(\left\{z \in \mathbb{T} : \sum_{k=2}^{n} \omega_k(z) > n-1\right\}\right).$$

Let $F \overset{\text{def}}{=} \{z \in \mathbb{T} : \sum_{k=2}^{n} \omega_k(z) > n-1\}$, then we have

$$\mu(F) < \frac{1}{n-1} \int_F \sum_{k=2}^{n} \omega_k(z)\mathrm{d}\mu(z) \leq e^{-\alpha}$$

by using (14). Hence

$$\mu\left(\left\{z \in \mathbb{T} : \sum_{k=2}^{n} \psi_k(z) > (\alpha+1)(n-1)\right\}\right) < e^{-\alpha}. \tag{15}$$

Let $\alpha = \log \frac{1}{\mu(E)}$, then $e^{-\alpha} = \mu(E)$. Substitute this into (refexists) then this inequality implies that there exists a point $z_0 \in E$ for which $\sum_{k=2}^{n} \psi_k(z_0) \leq$

$(\alpha+1)(n-1)$. Now we have

$$\begin{aligned}\left(\frac{\pi}{16}\right)^{n-1}\|p\|_W \leq &\|p_1\|_W \overset{(\operatorname{ord} p_1 = 1!)}{=} \|p_1(z_0)\| \\ \leq &\exp\left(\left(1+\log\frac{1}{\mu(E)}\right)(n-1)\right)|p(z_0)| \\ = &\left(\frac{e}{\mu(E)}\right)^{n-1}|p(z_0)| \\ \leq &\left(\frac{e}{\mu(E)}\right)^{n-1}\sup_{z\in E}|p(z)|,\end{aligned}$$

and the theorem is proved. □

Remark. We first construct the polynomial sequence $p_n = p, p_{n-1}, \cdots, p_2, p_1$, and they satisfy $\|p_{k-1}\|_W \geq \frac{\pi}{16}\|p_k\|_W$, $\operatorname{ord} p_k = k$ and so on. Then we can get

$$\left(\frac{\pi}{16}\right)^{n-1}\|p\|_W \leq \|p_1\|_W.$$

This means we transform the question into the proof of the certain inequality between $\|p_1\|_W = |p_1(z)| \forall z \in \mathbb{T}$ and $p = p_n$. Then we need to find a point $z_0 \in \mathbb{T}$ such that $|p_1(z_0)| \leq \exp\left(\left(1+\log\frac{1}{\mu(E)}\right)(n-1)\right)|p(z_0)|$, this step needs to estimate the amount or measure of the points that have large function values. If the measure of these points are smaller tham $\mu(E)$, then we can get a point $z_0 \in E$ that satisfies the condition.

4 The Turan lemma in general form

Theorem 4. *Let $p(t) = \sum_{k=1}^n c_k e^{i\lambda_k t}$ where $c_k \in \mathbb{C}$ and $\lambda_1 < \cdots \lambda_n \in \mathbb{R}$. If E is a measurable subset of the segment $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, then*

$$\sup_{t\in I}|p(t)| \leq \left(\frac{316}{\mu(E)}\right)^{n-1}\sup_{t\in E}|p(t)|.$$

Before proving Theorem 4, we first introduce a weak type estimate:

Lemma 3. *Let $g(t) = \sum_{k=1}^n c_k e^{i\lambda_k t}$, $(c_k \in \mathbb{C},\ 0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda)$. If $\lambda \geq n-1$, then*

$$\mu\left(\left\{t \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \left|\frac{\mathrm{d}}{\mathrm{d}t}\log g(t)\right| > y\right\}\right) \leq \frac{29\lambda}{y}$$

for all $y > 0$.

Proof. Let z_j be the complex zeros of $g(z)$ enumerated in the order of increase of $|\mathrm{Re} z_j|$. The Langer lemma yields $|\mathrm{Re} z_j| \geq \pi \frac{j-(n-1)}{(n-1)} \geq \frac{\pi}{\lambda}(j-(n-1))$ (otherwise, there are j zeros in the strip of width $\Delta < 2\pi \frac{j-(n-1)}{(n-1)}$, but by Langer lemma the number of zero points in the strip is less than $\frac{\lambda\Delta}{2\pi} + n - 1 < \frac{\lambda(j-(n-1))}{(n-1)} + (n-1) = \left(\frac{j}{n-1} - 1\right)\lambda + (n-1) \leq \left(\frac{j}{n-1} - 1\right)(n-1) + (n-1) = j$). We write the Hadamard factorization

$$g(z) = ce^{az} \prod_{j \leq 2\lambda} (z - z_j) \prod_{j > 2\lambda} \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j}} \stackrel{\text{def}}{=} ce^{az} Q(z) R(z).$$

The estimate for $\left|\frac{\mathrm{d}}{\mathrm{d}z} \log R(z)\right|$

Notice that $|\mathrm{Re} z_j| \geq \pi$ if $j > 2\lambda$. Let $|\mathrm{Re} z| < \frac{\pi}{2}$, then

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}z} \log R(z)\right| \leq |z| \sum_{j>2\lambda} \frac{1}{|\mathrm{Re} z_j| \left(|\mathrm{Re} z_j| - \pi/2\right)} \\ \leq & 2|z| \sum_{j>2\lambda} \frac{1}{|\mathrm{Re} z_j|^2} \\ \leq & 2|z| \sum_{j>2\lambda} \frac{\lambda^2}{\pi^2} \frac{1}{(j-(n-1))^2} \\ \leq & 2\frac{\lambda^2}{\pi^2} |z| \sum_{j>2\lambda} \int_{j-(n-1)-1/2}^{j-(n-1)+1/2} \frac{\mathrm{d}t}{t^2}.\end{aligned}$$

But if $j > 2\lambda > 2(n-1)$, then $j \geq 2n-1$, and $j-(n-1)-1/2 \geq j/2 \geq \lambda$. Therefore

$$\sum_{j>2\lambda} \int_{j-(n-1)-1/2}^{j-(n-1)+1/2} \frac{\mathrm{d}t}{t^2} \leq \int_{\lambda}^{\infty} \frac{\mathrm{d}t}{t^2} = \frac{1}{\lambda}$$

and $\left|\frac{\mathrm{d}}{\mathrm{d}z} \log R(z)\right| \leq \frac{2|z|\lambda}{\pi^2}$ if $|\mathrm{Re} z| < \pi/2$. In particular, $\left|\frac{\mathrm{d}}{\mathrm{d}z} \log R(z)\right| \leq \frac{\lambda}{\pi^2}$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

The estimate for $|a|$

It can be estimated by considering the argument increment of $g(z)$ along segment $\left[-i\omega\frac{\bar{a}}{|a|}, i\omega\frac{\bar{a}}{|a|}\right]$, similar to the proof of Case 1 Theorem 4 below. Here we use another approach. Consider an exponential polynomial $\tilde{g}(t) \stackrel{\text{def}}{=} e^{\lambda t} g\left(-\frac{\bar{a}}{|a|}t\right)$ on the interval $t \in \left[-\frac{3}{2}, \frac{3}{2}\right]$, then

$$\tilde{g}(t) = e^{\lambda t} \sum_{k=1}^{n} c_k e^{-i\lambda_k \frac{\bar{a}}{|a|} t} = \sum_{k=1}^{n} c_k e^{\left(\lambda - i\lambda_k \frac{\bar{a}}{|a|}\right)t}.$$

Its remarkable property is that the real parts of exponent in its terms are nonnegative (Re $\left(\lambda - i\lambda_k \frac{\overline{a}}{|a|}\right) \geq 0$, then it satisfies the condition $|z_j| \geq 1$ of Theorem 1). The reasoning of the first half of Section 1 ensures the estimate

$$\sup_{t\in\left[-\frac{3}{2},-\frac{1}{2}\right]} |\tilde{g}(t)| \leq \sup_{t\in\left[-\frac{3}{2},\frac{3}{2}\right]} |\tilde{g}(t)| \leq (12e)^{n-1} \sup_{t\in\left[\frac{1}{2},\frac{3}{2}\right]} |\tilde{g}(t)| \leq (12e)^{\lambda} \sup_{t\in\left[\frac{1}{2},\frac{3}{2}\right]} |\tilde{g}(t)| .$$

The function $Q\left(-\frac{\overline{a}}{|a|}t\right)$ is an algebraic polynomial of degree at most 2λ, consequently, it is a limit of exponential polynomials of order at most $2\lambda + 1$ with purely imaginary exponents[4]. Applying the Turan lemma again, we obtain the inequalities

$$\begin{aligned}\sup_{t\in\left[-\frac{3}{2},-\frac{1}{2}\right]} \left|Q\left(-\frac{\overline{a}}{|a|}t\right)\right| &\geq (12e)^{-2\lambda} \sup_{t\in\left[-\frac{3}{2},\frac{3}{2}\right]} \left|Q\left(-\frac{\overline{a}}{|a|}t\right)\right| \\ &\geq (12e)^{-2\lambda} \sup_{t\in\left[\frac{1}{2},\frac{3}{2}\right]} \left|Q\left(-\frac{\overline{a}}{|a|}t\right)\right|\end{aligned}$$

and

$$\begin{aligned}&\inf_{t\in\left[-\frac{3}{2},-\frac{1}{2}\right]} \left|R\left(-\frac{\overline{a}}{|a|}t\right)\right| \\ \geq &\exp\left(-\int_{-3/2}^{3/2} \left|\frac{\mathrm{d}}{\mathrm{d}t} \log R\left(-\frac{\overline{a}}{|a|}t\right)\right| \mathrm{d}t\right) \sup_{t\in\left[-\frac{3}{2},\frac{3}{2}\right]} \left|R\left(-\frac{\overline{a}}{|a|}t\right)\right| \\ \geq &e^{-\lambda/2} \sup_{t\in\left[\frac{1}{2},\frac{3}{2}\right]} \left|R\left(-\frac{\overline{a}}{|a|}t\right)\right| .\end{aligned}$$

If $|a| > \lambda$, then $\inf_{t\in\left[-\frac{3}{2},-\frac{1}{2}\right]} \left|ce^{(\lambda-|a|)t}\right| \geq e^{|a|-\lambda} \sup_{t\in\left[\frac{1}{2},\frac{3}{2}\right]} \left|ce^{(\lambda-|a|)t}\right|$. Since

$$\tilde{g}(t) = ce^{(\lambda-|a|)t} Q\left(-\frac{\overline{a}}{|a|}t\right) R\left(-\frac{\overline{a}}{|a|}t\right),$$

we have

$$\sup_{t\in\left[-\frac{3}{2},-\frac{1}{2}\right]} |\tilde{g}(t)| \geq \sup_{t\in\left[-\frac{3}{2},-\frac{1}{2}\right]} \left|Q\left(-\frac{\overline{a}}{|a|}t\right)\right| \inf_{t\in\left[-\frac{3}{2},-\frac{1}{2}\right]} \left|R\left(-\frac{\overline{a}}{|a|}t\right)\right| \inf_{t\in\left[-\frac{3}{2},-\frac{1}{2}\right]} \left|ce^{(\lambda-|a|)t}\right| .$$

[4]Consider $g_\lambda(x) = \frac{e^{\lambda x}}{\lambda} - 1$, it is easy to check that

$$\sup_{x\in[0,1]} |g_\lambda - x| \leq \frac{1}{2}\lambda.$$

These inequalities may hold simultaneously only if $|a| \leq \lambda\left(3\log(12e)+1/2+1\right) \leq \frac{25}{2}\lambda$.

The polynomial $Q(z)$

By Lemma 1, the polynomial $Q(z)$ satisfies the weak type estimate

$$\mu\left(\left\{t \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \left|\frac{\mathrm{d}}{\mathrm{d}t}\log Q(t)\right| > y\right\}\right) \leq \frac{16\lambda}{y}$$

on the segment $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Combine all the above estimates and make use of the inequality $\left|\frac{\mathrm{d}}{\mathrm{d}t}\log g(t)\right| \leq |a| + \left|\frac{\mathrm{d}}{\mathrm{d}t}\log R(t)\right| + \left|\frac{\mathrm{d}}{\mathrm{d}t}\log Q(t)\right|$, we obtain

$$\begin{aligned}
&\mu\left(\left\{t \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \left|\frac{\mathrm{d}}{\mathrm{d}t}\log g(t)\right| > y\right\}\right) \\
\leq &\mu\left(\left\{t \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \left|\frac{\mathrm{d}}{\mathrm{d}t}\log Q(t)\right| > y - 13\lambda\right\}\right) \\
\leq &\frac{16\lambda}{y-13\lambda} \leq \frac{29\lambda}{y}
\end{aligned}$$

for $y \geq 29\lambda$. But if $y < 29\lambda$, then the corresponding estimate becomes trivial because $\frac{29\lambda}{y} \geq 1 = \mu\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$. Lemma 2 is proved. □

Now we go back to the proof of Theorem 4.

Proof of Theorem 4. Let $\lambda \overset{\text{def}}{=} \lambda_n - \lambda_1$, we prove the theorem separately in two cases.

Case $\lambda \leq n-1$. If $n = 1$, the statement is obvious. Let $n > 1$, without loss of generality, we assume that $0 = \lambda_1 < \cdots \lambda_n = \lambda_n = \lambda \leq n-1$. By virtue of the Langer lemma, complex zeros of the exponential polynomial $p(z)$ are well separated, i.e., each verical strip of width Δ contains at most $\frac{\Delta\lambda}{2\pi} + (n-1) \leq \left(1 + \frac{\Delta}{2\pi}\right)(n-1)$ zeros.

Lets enumerate z_j in the order of increase of $|\mathrm{Re} z_j|$. For every $j \in \mathbb{N}$, the inequality $|\mathrm{Re} z_j| \geq \pi\frac{j-(n-1)}{(n-1)}$ holds (otherwise the zeross $z_1, \cdots, z_j$ would lie in a vertical strip of width $\Delta < 2\pi\frac{j-(n-1)}{(n-1)}$, but this strip can contain at most $\left(1+\frac{\Delta}{2\pi}\right)(n-1) < j$ zeros). Now we write the Hadamard factorization of $p(z)$:

$$p(z) = ce^{az}\prod_{j=1}^{2(n-1)}(z - z_j)\prod_{j>2(n-1)}\left(1 - \frac{z}{z_j}\right)e^{z/z_j} \overset{\text{def}}{=} ce^{az}Q(z)R(z).$$

We shall examine the behavior of each of the above three factors separately.

The canonical product $R(z) = \prod_{j>2(n-1)}\left(1 - \frac{z}{z_j}\right)e^{z/z_j}$

First of all, we notice that $|\mathrm{Re} z_j| \geq \pi$ if $j > 2(n-1)$ (since $|\mathrm{Re} z_j| \geq \pi \frac{j-(n-1)}{(n-1)} > \pi$). We have (since $|\mathrm{Re} z| \leq 1/2 < \pi$):

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\log R(z)\right| = \left|\sum_{j>2(n-1)}\left(\frac{1}{z_j}+\frac{1}{z-z_j}\right)\right| \leq |z| \sum_{j>2(n-1)} \frac{1}{|z_j|\,|z-z_j|}$$
$$\leq |z| \sum_{j>2(n-1)} \frac{1}{|\mathrm{Re} z_j|\,(|\mathrm{Re} z_j| - |\mathrm{Re} z|)}.$$

whence it follows, since $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, that

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\log R(z)\right| \leq |z| \sum_{j>2(n-1)} \frac{1}{|\mathrm{Re} z_j|\left(\left|\mathrm{Re} z_j - \frac{1}{2}\right|\right)}$$
$$\leq 2\,|z| \sum_{j=1}^{\infty} \frac{1}{\left|\mathrm{Re} z_{2(n-1)+j}\right|^2}$$
$$\leq 2\,|z| \sum_{j=1}^{\infty} \frac{1}{\left(\pi + \frac{\pi j}{n-1}\right)^2}$$
$$\leq \frac{2(n-1)}{\pi}\,|z| \int_{\pi}^{\infty} \frac{\mathrm{d}t}{t^2}$$
$$= \frac{2\,|z|}{\pi^2}(n-1).$$

Now,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\mathrm{d}}{\mathrm{d}z}\log R(z)\right| \mathrm{d}z \leq \frac{2(n-1)}{\pi^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |z|\,\mathrm{d}z = \frac{n-1}{2\pi^2},$$

and, therefore

$$\max_{z\in\left[-\frac{1}{2},\frac{1}{2}\right]} |R(z)| \leq \exp\left(\frac{n-1}{2\pi^2}\right) \min_{z\in\left[-\frac{1}{2},\frac{1}{2}\right]} |R(z)|.$$

The factor ce^{az}

The simplest way to estimate $|\mathrm{Re} a|$ is to consider the argument increment of $p(z)$ along a segment $[-i\omega, i\omega]\,(\omega > 0)$. It follows from the proof of the Langer lemma that $\left|\Delta_{[-i\omega,i\omega]}\arg p\right| \leq \pi(n-1)$. The argument increment brought in by each of the zeros of $Q(z)$ does not exceed π. So, we have

$$\left|\Delta_{[-i\omega,i\omega]}\arg Q\right| \leq 2\pi(n-1),$$

$$\left|\Delta_{[-i\omega,i\omega]}\arg R\right| \leq \int_{\omega}^{\omega} \left|\frac{\mathrm{d}}{\mathrm{d}z}\log R(it)\right| \mathrm{d}t \leq \frac{n-1}{\pi^2}\int_{-\omega}^{\omega} |t|\,\mathrm{d}t = \frac{n-1}{\pi^2}\omega^2,$$

and

$$\Delta_{[-i\omega,i\omega]}\arg(ce^{az}) = 2\omega\mathrm{Re}a.$$

The identity

$$\Delta_{[-i\omega,i\omega]}\arg = 2\omega\mathrm{Re}a + \Delta_{[-i\omega,i\omega]}\arg Q + \Delta_{[-i\omega,i\omega]}\arg R$$

implies

$$|\mathrm{Re}a| \leq \min_{\omega>0}\left(\frac{3\pi}{2\omega} + \frac{\omega}{2\pi^2}\right)(n-1) = \sqrt{\frac{3}{\pi}}(n-1).$$

It remains to examine

The behavior of the polynomial $Q(z)$.

Let $0 < h < \frac{1}{8}$. We shall carry out the Cartan lemma construction. Let n_1 be the maximal integer for which there exists a disk D_1 of radius $\frac{n_1}{n-1}h$ containing at least n_1 zeros of the polynomial Q. It is clear that D_1 contains exactly n_1 zeros of Q because otherwise n_1 could be enlarged (the strip of width $\frac{h}{n-1}$ contains at most 1 point according to Langer lemma). Let n_2 be the maximal integer for which there exists a disk D_2 of radius $\frac{n_2}{n-1}h$ containing at least n_2 zeros of Q among those not lying in D_1, and so on, till all the zeros of Q are covered. Putting $D'_k = 2D_k$ (i.e., the disk centered at the same point and of double radius), we obtain the corresponding sequence of integers $n_1 \geq \cdots \geq n_s$ with the sum $n_1 + \cdots n_s = 2(n-1)$ and the corresponding sequencedisks $D'_1, \cdots, D'_s$ with the sum of radii equal to $4h$. We fix a point

$$z \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \bigcup_{k=1}^{s} D'_k$$

and enumerate the zeros of Q in the order of increase of $|z - z_j|$. Following Cartan, we shall show that $|z - z_j| \geq \frac{j}{n-1}h$. Indeed, if this is not the case, then the disk D centered at z and of radius $\frac{j}{n-1}h$ contains at least j zeros of Q. Choose an $m \in \{1, \cdots, s\}$ such that $n_1 \geq \cdots \geq n_m \geq j > n_{m+1} \geq \cdots n_s$. For every $z \notin \bigcup_{k=1}^{s} D'_k$ and $k \leq m$, we have $z_j \notin D'_{n_k}$, hence the distance between z and the center of D_k is at least

$$\frac{2n_k}{n-1}h \geq \frac{n_k}{n-1}h + \frac{j}{n-1}h.$$

Hence D does not intersect any of the disks $D_1, \cdots, D_m$. But if this were true, the disk D (or a disk with larger number of zeros) would have been taken instead of D_{m+1} at the $m+1$-th step. This contradiction proves the claim.

Besides, the Langer lemma implies the inequality $|z - z_j| \geq \pi\frac{j-(n-1)}{(n-1)}$ (otherwise the zeros $z_1, \cdots, z_j$ would lie in a disk of radius strictly less than $\pi\frac{j-(n-1)}{(n-1)}$, and, consequently, in the strip of width $\Delta < 2\pi\frac{j-(n-1)}{(n-1)}$). Thus, we have

$$
\begin{aligned}
\frac{|Q(z)|}{\max\left\{|Q(t)| : t\in\left[-\frac{1}{2},\frac{1}{2}\right]\right\}} &\geq \prod_{j=1}^{2(n-1)} \frac{|z-z_j|}{\max\left\{|t-z_j| : t\in\left[-\frac{1}{2},\frac{1}{2}\right]\right\}} \\
&\geq \prod_{j=1}^{2(n-1)} \frac{|z-z_j|}{1+|z-z_j|} \\
&= \prod_{j=1}^{n-1} \frac{|z-z_j|}{1+|z-z_j|} \times \prod_{j=1}^{n-1} \frac{|z-z_{n-1+j}|}{1+|z-z_{n-1+j}|} \\
&\geq \prod_{j=1}^{n-1} \frac{\frac{j}{n-1}h}{1+\frac{j}{n-1}h} \times \prod_{j=1}^{n-1} \frac{\frac{\pi j}{n-1}}{1+\frac{\pi j}{n-1}} \\
&\geq \prod_{j=1}^{n-1} \frac{\frac{j}{n-1}h}{1+\frac{j}{n-1}\frac{1}{8}} \times \prod_{j=1}^{n-1} \frac{\frac{\pi j}{n-1}}{1+\frac{\pi j}{n-1}} \\
&\geq (8h)^{n-1} \times \prod_{j=1}^{n-1} \frac{\frac{j}{n-1}\frac{1}{8}}{1+\frac{j}{n-1}\frac{1}{8}} \times \prod_{j=1}^{n-1} \frac{\frac{3j}{n-1}}{1+\frac{3j}{n-1}}.
\end{aligned}
$$

But for each $\theta > 0$ we have

$$
\prod_{j=1}^{n-1} \frac{\frac{j}{n-1}\theta}{1+\frac{j}{n-1}\theta} \geq \exp\left((n-1)\int_0^1 \log\frac{\theta t}{1+\theta t}\mathrm{d}t\right) = \left(\frac{\theta}{(1+\theta)^{1+\frac{1}{\theta}}}\right)^{n-1},
$$

whence it follows that

$$
\frac{|Q(z)|}{\max\left\{|Q(t)| : t\in\left[-\frac{1}{2},\frac{1}{2}\right]\right\}} \geq (8h)^{n-1}\left(8\times\left(\frac{9}{8}\right)^9 \times \frac{4\sqrt[3]{4}}{3}\right)^{-(n-1)} \geq \left(\frac{8h}{32\sqrt[3]{4}}\right)^{n-1}.
$$

Observe that the measure of the exceptional set $\left[-\frac{1}{2},\frac{1}{2}\right] \cap \left(\bigcup_{k=1}^{s} D'_k\right)$ is at most

$8h$, we can set $z \in E$ if $h = \mu(E)/8$. Combining all these estimates, we find

$$\begin{aligned}
\sup_{t\in I}|p(t)| &\le \sup_{t\in I}\left|ce^{at}\right| \times \sup_{t\in I}|R(t)| \times \sup_{t\in I}|Q(t)| \\
&\le \left| c\exp\left(\sqrt{\frac{3}{\pi}}(n-1)\right)\right| \times \exp\left(\frac{n-1}{2\pi^2}\right)\min_{t\in I}|R(t)| \times \sup_{t\in I}|Q(t)| \\
&\le |c| \times \exp\left(\left(\sqrt{\frac{3}{\pi}} + \frac{1}{2\pi^2}\right)(n-1)\right)\min_{t\in I}|R(t)| \times \sup_{t\in I}|Q(t)| \\
&\le |c|\, 3^{n-1}\min_{t\in I}|R(t)| \times \left(\frac{32\sqrt[3]{4}}{8h}\right)^{n-1}|Q(z)| \\
&\le \left(\frac{154}{\mu(E)}\right)^{n-1}\sup_{t\in E}|p(t)|\,.
\end{aligned}$$

Case $\lambda > n-1$. We shall reduce this case to Case 1 in the same way as in Section 3. This is why we need Lemma 3. We can finish the proof by constructing a sequence of exponential polynomials $p_n, p_{n-1}, \cdots, p_s (s \ge 1)$ such that

(1) $p_n = p$;

(2) $\operatorname{ord} p_k = k$ $(k = s, \cdots, n)$;

(3) $\|p_{k-1}\|_\infty \ge \frac{1}{58}\|p_k\|_\infty$ $(k = s+1, \cdots, n)$;

(4) the ratio $\varphi_k \stackrel{\text{def}}{=} \left|\frac{p_{k-1}}{p_k}\right|$ satisfies the weak type estimate

$$\mu\left(\left\{x \in \left[-\frac{1}{2}, \frac{1}{2} : \varphi_k(x) > t\right]\right\}\right) \le \frac{1}{t}$$

for $t > 0$;

(5) the difference between the greatest and the smallest exponent of p_s does not exceed $s-1$ (i.e., p_s meets the condition of Case 1 investigated above).

The construction is almost the same as in Section 3. The difference is that, firstly, we make use of the identity $\underline{q}(t) - \overline{q}(t) = i\,(\rho_k - \rho_1)\,p_k(t)$, where

$$p_k(t) \stackrel{\text{def}}{=} \sum_{m=1}^{k} d_m e^{i\rho_m t} \quad (\rho_1 < \cdots \rho_k \in \mathbb{R})\,,$$

$$\underline{q}(t) \stackrel{\text{def}}{=} e^{i\rho_1 t}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-i\rho_1 t}p_k(t)\right),$$

$$\overline{q}(t) \stackrel{\text{def}}{=} e^{i\rho_k t}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-i\rho_k t}p_k(t)\right)$$

to estimate the sum of norms $\|\underline{q}\|_\infty + \|\overline{q}\|_\infty$ from below, and, secondly, we stop the sequence at the polynomial p_s satisfying the condition of Case 1, i.e., at that very moment when we cannot apply Lemma 3 to estimate φ_s once more.

Since $\|p_{k-1}\|_\infty \geq \left(\frac{1}{58}\right)\|p_k\|_\infty$, we obtain

$$\left(\frac{1}{58}\right)^{n-s}\|p\|_\infty \leq \|p_s\|_\infty. \tag{16}$$

By the construction procedure, p_s satisfies the condition of Case 1, hance for a measurable set F we have

$$\|p_s\|_\infty \leq \left(\frac{154}{\mu(F)}\right)^{s-1}\sup_{t\in F}|p_s(t)|\,. \tag{17}$$

Now we use the same reasoning as in Section 3 to establish $\left|\frac{p_s(t)}{p_n(t)}\right| \leq \left(\frac{2e}{\mu(E)}\right)^{n-s}$ outside an exceptional set E' of measure $\mu(E') \leq \mu(E)/2$. We have

$$\left|\frac{p_s(x)}{p_n(x)}\right| = \prod_{k=s+1}^{n}\varphi_k(z) \leq \exp\left(\sum_{k=s+1}^{n}\psi_k(x)\right),$$

where $\psi_k(x) \stackrel{\text{def}}{=} \log_+\varphi_k(x)$. The weak type estimate of φ_k gives the inequality $\mu(\psi_k > t) \leq e^{-t}$ for all $t > 0$. Let $\alpha > 0$, we decompose $\psi_k(x)$ into the sum of $\eta_k(x) \stackrel{\text{def}}{=} \min(\psi_k(x), \alpha)$ and $\omega_k(x) \stackrel{\text{def}}{=} \psi_k(x) - \eta_k(x)$. Then $\sum_{k=s+1}^{n}\eta_k(x) \leq \alpha(n-s)$ for all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We also have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\omega_k(x)\mathrm{d}x = \int_\alpha^\infty \mu(\psi_k > t)\,\mathrm{d}t \leq \int_\alpha^\infty e^{-t}\mathrm{d}t = e^{-\alpha}.$$

Therefore,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sum_{k=s+1}^{n}\omega_k(z)\right)\mathrm{d}\mu(z) \leq e^{-\alpha}(n-s).$$

Since

$$\sum_{k=s+1}^{n}\omega_k(x) = \sum_{k=s+1}^{n}\psi_k(x) - \sum_{k=s+1}^{n}\eta_k(x)$$

and $\sum_{k=s+1}^{n} \eta_k(x) \leq \alpha(n-s)$, we have

$$\begin{aligned} &\mu\left(\left\{x \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \sum_{k=s+1}^{n} \psi_k(x) > (\alpha+1)(n-s)\right\}\right) \\ \leq &\mu\left(\left\{x \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \sum_{k=s+1}^{n} \omega_k(x) > n-s\right\}\right) < e^{-\alpha}. \end{aligned}$$

Let $\alpha = \log\left(\frac{2}{\mu(E)}\right)$, then we have

$$\mu\left(\left\{x \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \sum_{k=s+1}^{n} \psi_k(x) > (\alpha+1)(n-s)\right\}\right) < \frac{\mu(E)}{2}.$$

Thus the measure of the set $E' = \left\{x \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \left|\frac{p_s(x)}{p_n(x)}\right| > \left(\frac{2e}{\mu(E)}\right)^{n-s}\right\}$ satisfies

$$\mu(E') < \frac{\mu(E)}{2}$$

and hence

$$\mu(E \backslash E') \geq \frac{\mu(E)}{2}. \tag{18}$$

By definition of the set E', we know $\left|\frac{p_s(x)}{p_n(x)}\right| \leq \left(\frac{2e}{\mu(E)}\right)^{n-s}$ for each $x \in E\backslash E'$. By using (17) (let $F = E\backslash E'$), (16) and (18) we obtain

$$\begin{aligned} \left(\frac{1}{58}\right)^{n-s} \|p\|_\infty \leq &\|p_s\|_\infty \leq \left(\frac{154}{\mu\left(E\backslash E'\right)}\right)^{s-1} \sup_{t \in E\backslash E'} |p_s(t)| \\ &\leq \left(\frac{308}{\mu(E)}\right)^{s-1} \left(\frac{2e}{\mu(E)}\right)^{n-s} \sup_{t\in E} |p(t)|. \end{aligned}$$

Now Theorem (4) easily follows if we take into account the inequality $116e < 316$. □

5 Summary: Two important techniques used

a. Construct a sequence of polynomials like $p_k, p_{k-1}, \cdots, p_1$ to decrease the order of p_k. In this note, the order is the $\operatorname{ord} p_k$ of exponential polynomials, it may have different meaning when we solve other problems.

b. Weak type estiamtes allow us to get an upper bound of a measure of a set A that satisfies some property P, then compaire it to the measure of a given set B . If the latter is strictly larger than the former, then there must be some point in B which does not meet the property P.

A Harmonic measure

Let $\mathbb{H}$ be the upper half-plane. Suppose $a < b$ are real. Then the function

$$\theta = \theta(z) = \arg\left(\frac{z-b}{z-a}\right) = \operatorname{Im}\log\left(\frac{z-b}{z-a}\right)$$

is harmonic on $\mathbb{H}$, and $\theta = \pi$ on (a,b) and $\theta = 0$ on $\mathbb{R}\backslash[a,b]$.

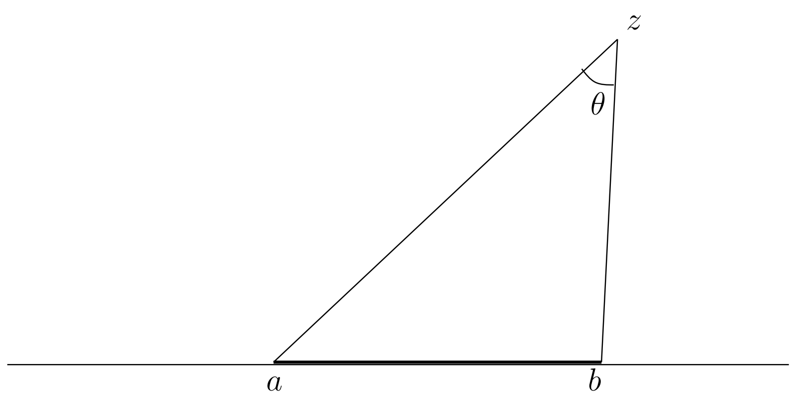

Figure 4: Harmonic funcion $\theta(z)$

Viewed geometrically, $\theta(z) = \mathrm{Re}\varphi(z)$ where $\varphi(z)$ is any conformal mapping from $\mathbb{H}$ to the strip $\{0 < \mathrm{Re}z < \pi\}$ which maps (a,b) onto $\{z : \mathrm{Re}z = \pi\}$ and $\mathbb{R}\backslash[a,b]$ into $\{z : \mathrm{Re}z = 0\}$.

Definition 2. *Let $E \subset \mathbb{R}$ be a finite union of open intervals and write $E = \bigcup_{j=1}^{n}(a_j, b_j)$ with $b_{j-1} < a_j < b_j$. Set*

$$\theta_j = \theta_j(z) = \arg\left(\frac{z-b_j}{z-a_j}\right).$$

Then the harmonic measure of E at $z \in \mathbb{H}$ is

$$\omega(z, E, \mathbb{H}) \stackrel{\text{def}}{=} \sum_{j=1}^{n} \frac{\theta_j}{\pi}. \tag{19}$$

It satisfies the following properties:

a. $0 < \omega(z, E, \mathbb{H}) < 1$ *for* $z \in \mathbb{H}$,

b. $\omega(z, E, \mathbb{H}) \to 1$ *as* $z \to E$, *and*

c. $\omega(z, E, \mathbb{H}) \to 0$ *as* $z \to \mathbb{R}\backslash\overline{E}$.

The function $\omega(z, E, \mathbb{H})$ is the unique harmonic function on $\mathbb{H}$ that satisfies a,b and c. The uniqueness of $\omega(z, E, \mathbb{H})$ is a consequence of Lindelöf's maximum principle (see [1, p. 2]).

References

[1] John B Garnett and Donald E Marshall. *Harmonic measure*. Vol. 2. Cambridge University Press, 2005.

[2] FL Nazarov. "Local estimates of exponential polynomials and their applications to inequalities of uncertainty priciple type". In: *St Petersburg Mathematical Journal* 5.4 (1994), pp. 663–718.

[3] Paul Turán. *Eine neue Methode in der Analysis und deren Anwendungen*. Akadémiai Kiadó, 1953.