ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
LECTURE 15

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Remark. In (2) of theorem 4we can drop the assumption that F on X is irreducible.

Corollary 1. *Every affine variety is finite dimensional.*

Proposition 1. *Let $X \subset \mathbb{A}^N$ be an affine variety of dimension n and $F \in k[x_1, \ldots, x_N] \backslash I(X)$. If $Z(F) \cap X \neq \emptyset$, then dim$(Z(F) \cap X) = n-1$.($Z(F) \cap X$ may not be irreducible).*

Proof. We need to show for all irreducible components Y_i of $Z(F) \cap X$, dim$Y_i \leq n-1$ and there exists a component Y_j with dim$Y_j = n-1$(later we will show that all irreducible components have dimension $n-1$). By Noether normalization theorem, there is a finite surjective morphism $\Pi : X \to \mathbb{A}^n$. Identify $k[x_1, \ldots, x_n]$ with $\Pi^*(k[x_1, \ldots, x_n]) \subset A(X)$. Let $\bar{F}$ be the class of F in $A(X)$, there exists a nonzero polynomial

$$H = x_{n+1}^d + \sum_{i=0}^{d-1} a_i x_{n+1}^i$$

with $a_i \in k[x_1, \ldots, x_n]$ such that $H(x_1, \ldots, x_n, \bar{F}) = 0$. Replacing H by an irreducible factor if necessary, we can assume H is irreducible. Let $\varphi = (\Pi, F) : X \to \mathbb{A}^{n+1}$, $\Pi = (x_1, \ldots, x_n) \circ \varphi$ is finite, thus φ is finite. By definition $\varphi(X) \subset Z(H)$, then $\varphi(X)$ is a closed subvariety of dimension n in $Z(H)$. Thus $\varphi(X) = Z(H)$, $\varphi : X \to Z(H)$ is a finite surjective morphism. By definition, $Z(F) \cap X = \varphi^{-1}(Z(H, x_{n+1})) = \varphi^{-1}(Z(a_0) \times \{0\})$, thus dim$(Z(F) \cap X) = $ dim$Z(a_0)$ where $a_0 \in k[x_1, \ldots, x_n]$. If a_0 is constant, then $Z(F) \cap X = \emptyset$, contradict with the condition, so drop it. Now we know a_0 is nonconstant polynomial , hence dim$Z(a_0) = n-1$. □

Theorem 1. *Let X be a variety, $\emptyset \neq U \subset X$, U is an open subset of X. Then dim$U$ = dim$X$.*

Proof. Let $U_0 \subsetneqq U_1 \subsetneqq \cdots \subsetneqq U_n = U$ be a chain in U, let $X_i = \bar{U}$ the closure of U_i in X. By definition $U_i = U \cap X_i$, thus

$$X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n = X$$

is a chain in X, thus dim$U \leq$ dim$X$.

Let $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n = X$ be a chain of largest length in X and $X_0 = \{x_0\}$ be a point, let $W \subset X$ be an open subset with $x_0 \in W$. Then we set $W_i = X_i \cap W$ for all i. Since W_{i+1} is dense in X_{i+1}, we have $W_{i+1} \supsetneqq W_i$ for all i. Thus $W_0 = \{x_0\} \subsetneqq W_1 \subsetneqq \cdots \subsetneqq W_n$ is a chain in W, we get dim$X$ = dim$W$. Thus we can replace X by W and U by $W \cap U$. Now we reduce to the case X is affine.

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(1) If $X = \mathbb{A}^n$, let x_0 be a point in U, X_i be affine linear subspaces containing X_{i-1} for all i. Put $U_i = X_i \cap U$, $U_0 \subsetneqq U_1 \subsetneqq \cdots \subsetneqq U_n$ is a chain in U, then $\dim U = n = \dim X$.

(2) If X is affine, there exists a finite surjective morphism $\varphi : X \to \mathbb{A}^n$. $\varphi(X\backslash U) \subsetneqq \mathbb{A}^n$ is closed, let $f \in I(\varphi(X\backslash U))$ and $V = \mathbb{A}^n \backslash Z(f)$, V is open and dense in $\mathbb{A}^n$, $\dim V = n$. Let $W = \varphi^{-1}(V) \subset X$, then $\varphi|_W : W \to V$ is surjective and closed, thus $\dim W \geq \dim V = n$, but $U \supset W$, hence $\dim U \geq \dim W \geq n$.

□

Corollary 2. *All varieties are finite dimensional.*

Corollary 3. *If X and Y are birational, then $dim = dim Y$.*

Corollary 4. (1) *$dim \mathbb{P}^n = n$.*

(2) *If $F \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial of positive degree, then $dim Z(F) = n - 1$.*

(3) *If $X \subset \mathbb{P}^n$ is a closed subvariety of dimension $n-1$, then $X = Z(F)$ for some homogeneous polynomial $F \in k[x_0, \ldots, x_n]$.*

Proof. (1) It is obvious since $U_i \simeq \mathbb{A}^n$ is open dense in $\mathbb{P}^n$.

(2) By projective transformation we can set $Z(F) \not\subset H_\infty$, then $Z(F) \cap \mathbb{A}^n = Z(F(1, x_1, \ldots, x_n))$. It has dimension $n-1$ and is open in $Z(F)$, so $\dim Z(F) = n - 1$.

(3) Same as the affine condition in theorem 4.

□

Theorem 2. *Let $X \subset \mathbb{A}^n$ be an affine variety, $F \in k[x_1, \ldots, x_n] \backslash I(X)$, then every irreducible component(if there is any) of $Z(F) \cap X$ has dimension $dim X - 1$.*

Proof. Let Z be a irreducible component of $Z(F) \cap X$. Take W be the union of all the other irreducible components of $Z(F) \cap X$. Take $g \in I(W) \backslash I(Z)$ and $U := X \backslash Z(g)$, then U can be viewed as an affine variety in $\mathbb{A}^{n+1}$. Since $Z(g) \supset W$, we get $U \subset Z$. Hence $U \cap Z(F) = U \cap Z$. Viewing F as a polynomial function on U(since $U = X \backslash Z(g)$ is open and dense in X, F is not zero in U, otherwise it is zero in the whole set X, contradicts with $F \notin I(X)$), then we get $\dim Z = \dim(Z \cap U) = \dim U - 1 = \dim X - 1$. The second equality $\dim(Z \cap U) = \dim U - 1$ is from proposition 1 by viewing it in $\mathbb{A}^{n+1}$ □

Proposition 2. *Let $\varphi : X \to Y$ be a morphism of varietise. Assume there exists a nonempty open subset $U \subset Y$ such that for all $p \in U$, $dim(\varphi^{-1}(p)) = n$, then we have*

$$dim X = dim Y + n.$$

Theorem 3. *Let $\varphi : X \to Y$ be a surjective morphism, assume $dim X = dim Y + n$, then*

(1) *for all points $p \in X$, $dim(\varphi^{-1}(p)) \geq n$;*

(2) *there is a nonempty open subset $U \subset Y$ such that for all $p \in U$, $dim \varphi^{-1}(p) = U$.*

1. Conclusions We Need From Previous Lectures

In lecture 14:

Theorem 4. *(1) $dim\mathbb{A}^n = n$.*

(2) Let $F \in k[x_1, \ldots, x_n] \backslash k$ be a irreducible polynomial, then $dim Z(F) = n - 1$.

(3) Conversely any subvariety $X \subset \mathbb{A}^n$ of dimension $n-1$ is a hypersurface, i.e. $X = Z(F)$ with F irreducible.

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