Category Theory

Based on notes by Tom Leinster

Notes taken by Rieunity

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The learning notes are a collection of some notions and important theorems about category theory. I learned it from the notes *Basic Category Theory* written by Tom Leinster.

0 Basic notions

Definition 0.1. A functor $F : \mathscr{A} \to \mathscr{B}$ is *faithful* (respectively, *full*) if for each $A, A' \in \mathscr{A}$, the function

$$\begin{aligned} \mathrm{Mor}(A, A') &\longrightarrow \mathrm{Mor}(F(A), F(A')) \\ f &\longmapsto F(f). \end{aligned}$$

is injective (respectively, surjective).

1 Natural transformations

Definition 1.1. Let $\mathscr{A}$ and $\mathscr{B}$ be categories and let $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ be functors. A *natural transformation* $\alpha : F \to G$ is a family $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A \in \mathscr{A}}$ of morphisms in $\mathscr{B}$ for every map $A \xrightarrow{f} A'$ in $\mathscr{A}$, the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd} \tag{1}$$

commutes. The morphisms α_A are called the components of α. We also write

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd}$$

to mean that α is a natural transformation from F to G.

Given natural transformations

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathscr{B} \end{tikzcd}$$

There is a composite natural transformation

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathscr{B} \end{tikzcd}$$

defined by $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ for all $A \in \mathscr{A}$.

There is also an identity natural transformation

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow 1_F"] & \mathscr{B} \end{tikzcd}$$

on any functor F, defined by $(1_F)_A = 1_{F(A)}$.

Definition 1.2. For any two categories $\mathscr{A}$ and $\mathscr{B}$, there is a category whose objects are the functors between $\mathscr{A}$ and $\mathscr{B}$ and whose morphisms are the natural transformation between thenm. The composition law and identity morphism are defined and shown above. This is called the *functor category* from $\mathscr{A}$ t $\mathscr{B}$ and written as $[\mathscr{A}, \mathscr{B}]$.

Definition 1.3. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A *natural isomorphism* between functors from $\mathscr{A}$ to $\mathscr{B}$ is an isomorphism in $[\mathscr{A}, \mathscr{B}]$. In other words, let α be a natural transformation from F to G where F and G are functors from $\mathscr{A}$ to $\mathscr{B}$, then α is a natural isomorphism if and only if $\alpha_A : F(A) \to G(A)$ is an isomorphicsm for all $A \in \mathscr{A}$.

Definition 1.4. Let F, G be two functors from $\mathscr{A}$ to $\mathscr{B}$, we say that

$$F(A) \cong G(A) \text{ naturally in } A$$

if F and G are naturally isomorphic.

Definition 1.5. An *equivalence* between categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair of functiors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ such that

$$G \circ F \cong 1_{\mathscr{A}} \text{ and } F \circ G \cong 1_{\mathscr{B}}.$$

We say that $\mathscr{A}$ and $\mathscr{B}$ are equivalent if there is an equivalence between them and write $\mathscr{A} \simeq \mathscr{B}$. The functors F and G are equivalences.

Definition 1.6. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor, we say F is *essentially surjective on objects* if for all $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $F(A) \cong B$.

Proposition 1.7. *A functor $F : \mathscr{A} \to \mathscr{B}$ is an equivalence if and only if it is full, faithfull and essentially surjective on objects.*

Proof. First assume two natural isomorphisms

$$\eta : G \circ F \to 1_{\mathscr{A}}, \quad \varepsilon : F \circ G \to 1_{\mathscr{B}}.$$

Let $f, f' : A \to A'$ and $F(f) = F(f') : F(A) \to F(A')$, then $G \circ F(f) = G(F(f)) = G(F(f')) = G \circ F(f') : G(F(A)) \to G(F(A'))$. Then $\eta \circ (G \circ F(f)) = \eta \circ (G \circ F(f')) \Rightarrow f = f'$. Hence F is faithuful. Let $g \in \mathrm{Mor}(F(A), F(A'))$, then $g = (F \circ G) \circ (\varepsilon(g))$. Then there exists $f = G \circ \varepsilon(g)$ s.t. $F(f) = g$, hence F is full. Given any $B \in \mathscr{B}$, let $A = G(B)$, then $F(A) = F \circ G(B) \cong B$. The converse is to construct natural isomorphisms η and ε by reversing the deduction above. $\square$