ALGEBRAIC GEOMETRY

RIEUNITY

ABSTRACT. These notes was written when I studied algebraic geometry myself in YouTube from the channel ICTP Math, the speaker of the videos is Lothar Göttsche. The notes include first 19 lectures of the course.

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1. ZARISKI TOPOLOGY, NOETHERIAN AND IRREDUCIBILITY

Definition 1.1 (Zariski Topology)**.** Let S be a set of arbitrary polynomials of $k[x_1, \cdots, x_n]$, k be an algebraic closed field. We define the common zero set of polynomials in S as a closed subset of k^n. The topology defined by this is called Zariski topology and denote the space as $\mathbb{A}^n$. The closed sets are called affine algebraic sets.

Remark. In all the conditions of the notes, we always take k to be an algebraic closed field.

Definition 1.2. Let X be a set of $\mathbb{A}^n$ and S be a set of $k[x_1, \cdots, x_n]$, we define

$$I(X) := \{f \in k[x_1, \cdots, x_n] | f(p) = 0 \text{ for all } p \in \mathbb{A}^n\}$$

and

$$Z(S) := \{p \in \mathbb{A}^n | f(p) = 0 \text{ for all } f \in S\}.$$

Proposition 1.3.

$$Z(I(X)) = \bar{X}.$$

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Proof. First, it is obvious that $X \subset Z(I(X))$. To show the inverse inclusion, assume a closed set Y who satisfies $X \subset Y$, then we have an ideal $\mathfrak{a}$ that satisfies $Y = Z(\mathfrak{a})$. Then $Z(\mathfrak{a}) \supset X$, hence $\mathfrak{a} \subset I(Z(\mathfrak{a})) \subset I(X)$, hence $Y = Z(\mathfrak{a}) \supset Z(I(X))$. □

Proposition 1.4 (Noetherian Ring). *Let R be a ring, the following are equivalent:*

(1) *Every ideal $I \subset R$ is finitely generated;*
(2) *R satisfies the ascending chain condition: if $I_1 \subset I_2 \subset \cdots$ is a chain of ideals, this chain becomes stationary, i.e.,*

$$\exists N, \ \textit{s.t.}\ I_N = I_{N+1} = \cdots .$$

If R fullfils these properties, it is called noetherian.

Proof. $\Rightarrow$: Let $I_1 \subset I_2 \subset \cdots$ be a chain of ideals. Let $I = \cup_{i>0} I_i$, I is an ideal. So by (1) I is finitely generated:

$$I = \langle f_1, f_2, \cdots, f_s \rangle.$$

where f_j contained in I_{k_j}. Let $N = \max_j k_j$, we have $I \subset I_N \subset I$, so the chain is stationary. $\Leftarrow$: Assume a ring $I \subset R$ is not finitely generated, choose an element $f_1 \in I$, $f_2 \in I \backslash \langle f_1 \rangle$, $f_3 \in I \backslash \langle f_1, f_2 \rangle, \cdots$. Then we have a chain who is not stationary:

$$\langle f_1 \rangle \subsetneqq \langle f_1, f_2 \rangle \subsetneqq \cdots .$$

□

Theorem 1.5 (Hilbert Base Theorem). *R is a noetherian ring $\Rightarrow$ The polynomial ring $R[x_1, x_2, \cdots, x_n]$ is a noetherian ring.*

Proof. Since $R[x_1, x_2, \cdots, x_n] = R[x_1, x_2, \cdots, x_{n-1}][x_n]$, we only need to prove $R[x]$ is a noetherian ring for a noetherian ring R. Assume $R[x]$ is not noetherian, let I be an ideal which is not finitely generated. Choose $f_1 \in I \backslash \{0\}$, $f_2 \in I \backslash \langle f_1 \rangle, \cdots, f_{i+1} \in I \backslash \langle f_1, \cdots, f_i \rangle$, s.t. the degree of $f_i \in I \backslash \langle f_1, \cdots, f_{i-1} \rangle$ is minimal. Let $n_i := deg(f_i)$, a_i the leading coefficient of f_i. Then we have $n_1 \leq n_2 \leq \cdots$ and an ascending chain

$$\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \cdots \tag{1.1}$$

of ideals in R. Since it is stationary, then for some k we have

$$\langle a_1, \cdots, a_k \rangle = \langle a_1, \cdots, a_k, a_{k+1} \rangle$$

This implies $a_{k+1} \in \langle a_1, \cdots, a_k \rangle$. So we can write

$$a_{k+1} = \sum_{i=1}^{k} b_i a_i, \quad b_i \in R$$

Let $g := f_{k+1} - \sum_{i=1}^{k} b_i x^{n_{k+1} - n_i} f_i$, then$g \in I \backslash \langle f_1, \cdots, f_k \rangle$(otherwise $f_{k+1} = g + \sum_{i+1}^{k} b_1 x^{n_{k+1} - n_i} f_i$ would be in $\langle f_1, \cdots, f_k \rangle$). The sum of the leading term in the right hand side is

$$a_{k+1} - \sum_{i=1}^{k} b_i a_i = 0.$$

It shows that $deg(g) < n_{k+1}$, which contradicts to the choose of f_i, so the chain (1.1) is not stationary, i.e., the ring R is not noetherian. □

Corollary 1.6. *Every affine algebraic set $X \subset \mathbb{A}^n$ is the zero set of finite algebraic polynomials.*

Proof. Every affine algebraic set is the zero set of some polynomial set S, i.e. $Z(S)$. Since $Z(S) = Z(\langle S \rangle)$, it is a zero set of an ideal, we choose the generators of the ideal, name T, then $Z(S) = Z(T)$. $\square$

Definition 1.7. A topological space X is *reducible* if $X = X_1 \cup X_2$, where X_1,X_2 are closed subsets and $X_1 \subsetneqq X_2, X_2 \subsetneqq X_1$. X is called *irreducible* if it is not reducible,i.e., if $X = X_1 \cup X_2, X_i \subset X$ is closed for $i = 1, 2$, then we have $X = X_1$ or $X = X_2$.

Remark. When we talk about whether a subset of a topological space is irreducible, it refers to its induced topology from the space where the set is on.

Proposition 1.8. *Let X be irreducible, $\emptyset \neq U \subset X$, U is an open subset of X, then*

1. *U is dense in X. Because if it is not dense, we can write $X = (X \backslash U) \cup \overline{U}$, so X is not irreducible.*
2. *$U \subset X$ itself is also irreducible.*

Definition 1.9. A topological space is called *noetherian* if every descending chain:$X \supset X_1 \supset X_2 \supset \cdots$ of closed subsets is stationary(i.e., $X_N = X_{N+1} = \cdots$ for some $N \in \mathbb{N}^+$).

Proposition 1.10. *Any subspace Y of noetherian topological space X is noetherian.*

Proof. Assume $Y \supset Y_1 \supset Y_2 \supset \cdots$ a chain of closed subsets. Then $\forall i, Y_i = Y \cap X_i, X_i \subset X$ is closed. Let $X'_i = \cap_{1 \leq j \leq i} X_j$, $X'_i \cap Y = Y_i$. Then $X \supset X'_1 \supset X'_2 \supset \cdots$ is a descending chain. Since X is noetherian, $\exists N$ s.t. $X'_N = X'_{N+1} = \cdots$. It follows $Y_N = Y_{N+1} = \cdots$. Thus $Y \supset Y_1 \supset Y_2 \supset \cdots$ is stationary. $\square$

Proposition 1.11. *$\mathbb{A}^n$ is noetherian topological space.*

Proof. Let $\mathbb{A}^n = X \supset X_1 \supset X_2 \supset \cdots$ be a chain of closed subsets. Then we have $I(X_1) \subset I(X_2) \subset \cdots$. Since $k[x_1, x_2, \cdots, x_n]$ is noetherian, $\exists N, I(X_N) = I(X_{N+1}) = \cdots$. Note that$X_i = Z(I(X_i))$, we get $X_N = X_{N+1} = \cdots$. It shows that $\mathbb{A}^n$ is a noetherian topological space. $\square$

Theorem 1.12. *Let X be a noetherian topological space.*

1. *X is a union of finitely many irreducible closed subsets: $X = X_1 \cup \cdots \cup X_r$;*
2. *If we require $X_i \not\subset X_j$ for $i \neq j$, then this decomposition is unique.*

Proof. (1) Assume X does not have a decomposition with finitely many closed subsets. In particular, X is reducible: $X = X_1 \cup Y_1$, X_1, Y_1 are closed subsets. so one of the two sets does not have decomposition, say X_1. Repeat the argument we ge a descending chain

$$X \supsetneqq X_1 \supsetneqq X_2 \supsetneqq \cdots$$

which is not stationary, it contradicts our existing condition.

(2)Let $X = X_1 \cup \cdots \cup X_t = Y_1 \cup \cdots \cup Y_s$. Then we have $X_i = \bigcup_{j=1}^{s} (X_i \cap Y_j)$. Since X_i is irreducible, $\exists j, X_i = X_i \cap Y_j$, thus $X_i \subset Y_j$.Similarly, we can get $Y_j \subset X_k$ for some k. Then we have $X_i \subset X_k$, it implies $i = k$ and thus $X_i = Y_j$. So we get the conclusion: each X_i is equal to some Y_j and each Y_j is equal to some X_i. So $r = s$ and the Y_j's are permutations of X_i's. □

Definition 1.13. An irreducible affine algebraic set is called an *affine variety.*

Proposition 1.14. *$X \subset \mathbb{A}^n$ is an affine algebraic set. Then we have the following equivalent relations:*

(1) *X is irreducible;*
(2) *$I(X)$ is a prime ideal.*

Proof. (1)$\Rightarrow$ (2): let X be irreducible, f, g some polynomials s.t. $fg \in I(X)$. Then we have $X \subset Z(fg) = Z(f) \cup Z(g)$, hence $X = (X \cap Z(f)) \cup (X \cap Z(g))$. Since X is irreducible, we get $X = X \cap Z(f)$ or $X = X \cap Z(g)$, so $X \subset Z(f)$ or $X \subset Z(g)$, i.e. $f \in I(X)$ or $g \in I(X)$.

(2)$\Leftarrow$ (1): Assume X is reducible, then we have $X = X_1 \cup X_2$ and $X_i \subsetneqq X$ are closed subsets. Since $Z(I(X_i)) = X_i \subsetneqq X = Z(I(X))$, we get $I(X_i) \supsetneqq I(X)$. Let $f \in I(X_1) \backslash I(X)$, $g \in I(X_2) \backslash I(X_2)$, fg vanishes on $X_1 \cup X_2 = X$, then $fg \in I(X)$, i.e., $I(X)$ is not prime. □

Example 1.15. $\mathbb{A}^n$ is irreducible since $I(\mathbb{A}^n) = \{0\}$ is a prime ideal.

Definition 1.16. Let $X \neq \emptyset$ be an irreducible topological space. The *dimension* of X is the largest $n \in \mathbb{Z}$ s.t. there is an ascending chain

$$\emptyset \neq X_0 \subsetneqq X_1 \subsetneqq X_2 \subsetneqq \cdots \subsetneqq X_n = X$$

of irreducible closed subsets. If X is a noetherian topological space then

$$\dim X = \text{ maximum of dimension of irreducible components of } X.$$

Remark. (1) The point $p \in \mathbb{A}^n$ has dimension 0;
(2) $\mathbb{A}^1$ has dimension 1;
(3) In the moment, we still cannot prove but true is $\dim \mathbb{A}^n = n$

It is easy to verify $\dim \mathbb{A}^n \geq n$ because we have a chain:

$$\{(0, 0, \cdots, 0)\} \subsetneqq Z(x_2, x_3, \cdots, x_n) \subsetneqq Z(x_3, \cdots, x_n) \subsetneqq \cdots \subsetneqq Z(x_n) \subsetneqq \mathbb{A}^n.$$

2. Nullstallensatz

Theorem 2.1 (The Weak Form Hilbert's Nullstellensatz)**.** *Let $\mathfrak{a} \subsetneqq k[x_1, \cdots, x_n]$ be a proper ideal, then $Z(\mathfrak{a}) \neq \emptyset$*

Remark. We usually use the following form:

$$\mathfrak{a} \subset k[x_1, \cdots, x_n] \text{ and } Z(\mathfrak{a}) = \emptyset \Rightarrow 1 \in I.$$

It is true when k is algebraically closed, otherwise the theorem 2.1 is wrong:

$$\mathfrak{a} = \langle x^2 + 1 \rangle \in \mathbb{R}[x], Z(\mathfrak{a}) = \emptyset.$$

Definition 2.2. Let $\mathfrak{a}$ be an ideal in a ring R. The *radical* of $\mathfrak{a}$ is

$$\sqrt{\mathfrak{a}} = \{r \in R | \exists n > 0, r^n \in \mathfrak{a}\}.$$

$\sqrt{\mathfrak{a}}$ is an ideal in R. $\mathfrak{a}$ is called radical ideal if $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

Remark. If $X \subset \mathbb{A}^n$ is an affine algebraic set, then $I(X)$ is a radical ideal.

Theorem 2.3 (Nullstallensatz)**.** *Let $\mathfrak{a} \subset k[x_1, \cdots, x_n]$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Definition 2.4. R is an integral domain, the quotient field $Q(R)$ is the set of equivalent classes of pairs $(f,g), f,g \in R, g \neq 0$, which satisfy the equivalent relation

$$(f,g) \cong (f',g') \Leftrightarrow fg' - f'g = 0.$$

We denote it by $\frac{f}{g}$.

Remark. $Q(R)$ is a field. We always identify $r \in R$ with $\frac{r}{1} \in Q(R)$, then we can say R is the subring of $Q(R)$. $Q(k[x_1,\cdots,x_n]) := k(x_1,\cdots,x_n)$ is called the *field of rational functions* in $x_1, x_2, \cdots, x_n$.

Now we prove the Nullstellensatz:

Proof of Nullstellensatz. Let $\mathfrak{a} = \langle f_1,\cdots,f_r\rangle, f_i \in \mathfrak{a}$. Then $I(Z(\mathfrak{a}))$ is a radical ideal containing $\mathfrak{a}$, so we get

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

Let$f \in I(Z(\mathfrak{a}))$. To show $\exists N > 0$, s.t. $f^N \in \mathfrak{a}$, we use the weak Nullstellensatz in $k[x_1,\cdots,x_n]$.

Let

$$\mathfrak{b} := \langle f_1,\cdots,f_r, f\cdot t - 1\rangle \subset k[x_1,\cdots,x_n,t] \tag{2.1}$$

Let $(p,a) \in \mathbb{A}^{n+1}, p \in \mathbb{A}^n, a \in k$.

$$(p,a) \in Z(\mathfrak{b}) \Leftrightarrow f_1(p) = \cdots = f_r(p) = 0 \text{ and } f(p)\cdot a = 1.$$

But $f(p) = 0$, so we know $Z(\mathfrak{b}) = \emptyset$. By the weak Nullstellensatz, $1 \in \mathfrak{b}$, we can write

$$1 = g_0 \cdot (ft-1) + \sum_{i=1}^{r} g_i \cdot f_i \tag{2.2}$$

Back to $k[x_1,\cdots,x_n]$ in $k(x_1,\cdots,x_n)$, define homomorphism:

$$\begin{aligned} \varphi : k[x_1,\cdots,x_n,t] &\to k(x_1,\cdots,x_n)\\ g(x_1,\cdots,x_n,t) &\to g(x_1,\cdots,x_n,\tfrac{1}{f}) \end{aligned}$$

Use φ to equation (2.2) we get

$$1 = \sum_{i=1}^{r} \varphi(g_i)\cdot f_i \tag{2.3}$$

where $\varphi(g_i) = \frac{G_i}{f^{n_i}}, G_i \in k[x_1,\cdots,x_n]$. Let $N := \max\limits_{1\le i\le r} n_i$, multiply equation 2.3 by f^N:

$$f^N = \sum_{i=1}^{r} G_i f^{N-n_i}\cdot f_i \in \mathfrak{a} \tag{2.4}$$

□

Corollary 2.5.
(1) *If $\mathfrak{a} \subset k[x_1,\cdots,x_n]$ is a prime ideal, then $Z(\mathfrak{a})$ is irreducible;*
(2) *If $f \in k[x_1,\cdots,x_n]$ is irreducible, then $Z(f)$ is irreducible.*

Proof. (1) Set $X = Z(\mathfrak{a})$. Prime ideals are radical, so we get $I(X) = \mathfrak{a}$ and $\mathfrak{a}$ is prime, use proposition 1.14 we know that X is irreducible.

(2) Since $k[x_1,\cdots,x_n]$ is a UFD, we get

$$f \in k[x_1, \cdots, x_n] \text{ is irreducible} \Rightarrow \langle f \rangle \text{ is a prime ideal.}$$

So $Z(f) = Z(\langle f \rangle)$ is irreducible. $\square$

3. Projective algebraic sets and varieties

Definition 3.1. Define an equivalence relation $\sim$ in $k^{n+1} \backslash \{0\}$ as the following:

$$(a_0, \cdots, a_n) \sim (b_0, \cdots, b_n) \Leftrightarrow \exists \lambda \in k \backslash \{0\} \text{ s.t.} (a_0, \cdots, a_n) = (\lambda b_0, \cdots, \lambda b_n).$$

Then we call $k^{n+1} \backslash \{0\}$ with this relation the *projective n-space* and write it as $(k^{n+1} \backslash \{0\}) / \sim =: \mathbb{P}^n$.

Definition 3.2. Let $U_i := \{[a_0, \cdots, a_n] \in \mathbb{P}^n | a_i \neq 0\}$. $\varphi_i : U_i \to \mathbb{A}^n$, $[a_0, \cdots, a_n] \to (\frac{a_0}{a_i}, \cdots, \frac{\hat{a_i}}{a_i}, \cdots, \frac{a_n}{a_i})$ is a projection, write inverse $u_i : \mathbb{A}^n \to U_i$, $(b_0, \cdots, \hat{b_i}, \cdots, b_n) \to [b_0, \cdots, 1, \cdots, b_n]$.

Usually we fix $i = 0$, view $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$ by identify the point $(a_1, \cdots, a_n) \in \mathbb{A}^n$ with $[1, a_1, \cdots, a_n] \in \mathbb{P}^n$. With this identification we have

$$\mathbb{P}^n = \mathbb{A}^n \cup H_\infty \tag{3.1}$$

where $H_\infty := \{[a_0, \cdots, a_n] \in \mathbb{P}^n | a_0 = 0\}$ is called hyperplane at infinity.

Remark. Define projective algebraic sets are zero sets of polynomials in $k[x_0, \cdots, x_n]$, but $f \in k[x_0, \cdots, x_n]$ does not define a function on $\mathbb{P}^n$:

$$f(a_0, \cdots, a_n) \neq f(\lambda a_0, \cdots, \lambda a_n). \tag{3.2}$$

However if f is homogeneous we can still see whether $p \in \mathbb{P}^n$ is a zero point of f or not. f is homogeneous if

$$f(\lambda a_0, \cdots, \lambda a_n) = \lambda^d f(a_0, \cdots, a_n). \tag{3.3}$$

Thus whether $f = 0$ is decided only on $[a_0, \cdots, a_n]$.

Definition 3.3. Let $g \in k[x_0, \cdots, x_n]$ be homogeneous, a point $p = [a_0, \cdots, a_n]$ is a zero point of g if $g(a_0, \cdots, a_n) = 0$. Let $S \subset k[x_0, \cdots, x_n]$,

$$Z(S) := \{p \in \mathbb{P}^n | f(p) = 0, \forall f \in S\}. \tag{3.4}$$

A subset of $\mathbb{P}^n$ of the form $Z(S)$ is called a *projective algebraic set*. Further more, if it is irreducible, it is called the *projective variety*.

Example 3.4. (1) $\emptyset = Z(1)$, $\mathbb{P}^n = Z(\emptyset)$;

(2) Any point $p = [a_0, \cdots, a_n] \in \mathbb{P}^n$ is a projective algebraic set

$$\begin{aligned} \{p\} = \quad & Z(a_1x_0 - a_0x_1, a_2x_0 - a_0x_2, \cdots, a_nx_0 - a_0x_n, \\ & a_2x_1 - a_1x_2, \cdots, a_nx_1 - a_1x_n, \\ & \cdots). \end{aligned}$$

Definition 3.5. A polynomial $f \in k[x_0, \cdots, x_n]$ cab be written uniquely as $f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$, with $f^{(i)}$ homoegeneous of degree i. $f^{(i)}$ is called *homogeneous component* of f. An ideal $\mathfrak{a} \subset k[x_0, \cdots, x_n]$ is called *homogeneous* if for every $f \in \mathfrak{a}$ all homogeneous components $f^{(i)}$ are in $\mathfrak{a}$.

Proposition 3.6. *An ideal $\mathfrak{a} \subset k[x_0, \cdots, x_n]$ is homogeneous $\Leftrightarrow$ It is generated by the homogeneous polynomials.*

Proof. $\Rightarrow$: Assume I homogeneous, let $\{f_\alpha\}_\alpha$ be a set of generators, then $\{f_\alpha^{(i)}\}_{\alpha,i}$ is a set of homogeneous generators.

$\Leftarrow$: Let $\mathfrak{a} = \langle g_i \rangle$ and g_i be homogeneous. Let $f \in \mathfrak{a}$, then we can write

$$f = \sum_i a_i g_i. \tag{3.5}$$

Note g_i is homogeneous, thus the homogeneous part of $a_i g_i$ of degree d is just $a_i^{(d-deg(g_i))} g_i$, so

$$f^{(d)} = \sum_i a_i^{(d-deg(g_i))} g_i. \tag{3.6}$$

Since $g_i \in \mathfrak{a}$ we get $f^{(d)} \in \mathfrak{a}$. $\square$

Definition 3.7. Let $\mathfrak{a} \subset k[x_0, \cdots, x_n]$ be a homogeneous ideal, the *zero set of* $\mathfrak{a}$ is written as

$$Z(\mathfrak{a}) := \{p \in \mathbb{P}^n | f(p) = 0 \text{ for all homogeneous elements } f \in \mathfrak{a}\}. \tag{3.7}$$

For a subset $X \subset \mathbb{P}^n$, the *homogeneous ideal of* X is

$$\begin{aligned} I(X) := \ & \text{ideal generated by } \{f \in k[x_0, \cdots, x_n] | f \\ & \text{be homogeneous and } f(p) = 0 \forall p \in X\} \end{aligned} \tag{3.8}$$

By definition this is a homogeneous ideal.

Remark. If $f \in k[x_0, \cdots, x_n]$ is not homogeneous, we can define

$$Z(f) := \{p \in \mathbb{P}^n | f(a_0, \cdots, a_n) = 0 \text{ for all representative } (a_0, \cdots, a_n) \text{ of } p\} \tag{3.9}$$

In fact, if $f = f^{(0)} + f^{(1)} \cdots + f^{(d)}$, then we have

$$Z(f) = \bigcap_{i=0}^{d} Z(f^{(i)}) \tag{3.10}$$

With this property, if $\mathfrak{a} \subset k[x_0, \cdots, x_n]$ is a homogeneous ideal then formula 3.7 can be written as

$$Z(\mathfrak{a}) = \{p \in \mathbb{P}^n | f(p) = 0 \forall f \in \mathfrak{a}\} \tag{3.11}$$

and formula 3.8 can be written as

$$I(X) = \{f \in k[x_0, \cdots, x_n] | f(p) = 0 \forall p \in X\} \tag{3.12}$$

Proposition 3.8. *Same as an affine space, in a projective space we have the following propositions:*

1. *$X \subset Y \subset \mathbb{P}^n$ are projective algebraic sets, then $I(X) \supset I(Y)$;*
2. *$X \subset \mathbb{P}^n$ is a projective algebraic set, then $Z(I(X)) = X$;*
3. *$\mathfrak{a} \subset k[x_0, \cdots, x_n]$ is a homogeneous ideal, then $I(Z(\mathfrak{a})) \supset \mathfrak{a}$;*
4. *If $S \subset k[x_0, \cdots, x_n]$ is a set of homogeneous polynomials, then $Z(S) = Z(\langle S \rangle)$;*
5. *For a family $\{S_\alpha\}$ of sets of homogeneous polynomials, $Z(\bigcup_\alpha S_\alpha) = \bigcap_\alpha Z(S_\alpha)$;*
6. *If $T, S \subset k[x_0, \cdots, x_n]$ are sets of homogeneous polynomials, then $Z(ST) = Z(S) \cup Z(T)$.*

Remark. From the proposition (5) and (6) we know that arbitrary intersections and finite unions of projective algebraic sets are projective algebraic sets, then we can define a topology through these two propositions.

Definition 3.9. The *Zariski topology on* $\mathbb{P}^n$ is the topology whose closed sets are the projective algebraic sets.

If $X \subset \mathbb{P}^n$ is a subset, we give it the induced topology, called the *Zariski topology on* X.

4. Projective Nullstellensatz

Definition 4.1. A *quasi-projective algebraic set* is an open subset of a projective algebraic set. Fro example, let U and V be closed subsets, then $Y = U \backslash V \neq \emptyset$ is a quasi-projective algebraic set.

Proposition 4.2. *We know $k[x_0, \cdots, x_n]$ is noetherian, then follows the same proof as in affine case shows that $\mathbb{P}^n$ is a noetherian topological sapce.*

Remark. Every subspace of $\mathbb{P}^n$ is noetherian. In particular, quasi-projective algebraic sets are noetherian, hence have unique decompositions into irreducible components.

Definition 4.3. A *quasi-projective variety* is an irreducible quasi-projective algebraic set.

Remark. If we use the identification $\mathbb{A}^n = U_0 \subset \mathbb{P}^n$, then $\mathbb{A}^n$ is an open set $\mathbb{A}^n = \mathbb{P}^n \backslash Z(x_0)$, i.e. $\mathbb{A}^n$ is a quasi-projective variety.

Definition 4.4. A nonempty algebraic set $X \subset \mathbb{A}^{n+1}$ is called a cone if for all $p = (a_0, \cdots, a_n) \in X$ and all $\lambda \in k$, we have $(\lambda a_0, \cdots, \lambda a_n) = \lambda p \in X$.

If $X \subset \mathbb{P}^n$ is a projective algebraic set, its affine cone is

$$C(X) := \{(a_0, \cdots, a_n) \in \mathbb{A}^{n+1} | [a_0, \cdots, a_n] \in X\} \cup \{0\} \tag{4.1}$$

Lemma 4.5. *Let $X \neq \emptyset$ be a projective algebraic set, then :*

(1) *$X = Z_p(\mathfrak{a})$, for $\mathfrak{a} \subset k[x_0, \cdots, x_n]$ a homogeneous ideal $\Rightarrow C(X) = Z_a(\mathfrak{a}) \subset \mathbb{A}^{n+1}$;*
(2) *$I_a(C(X)) = I_H(X)$.*

Theorem 4.6 (Projective Nullstellensatz)**.** *Let $\mathfrak{a} \subset k[x_0, \cdots, x_n]$ be a homogeneous ideal:*

(1) *$Z_p(\mathfrak{a}) = \emptyset \Leftrightarrow \mathfrak{a}$ contains all homogeneous polynomials of degree N for some $N \in \mathbb{N}$;*
(2) *If $Z_p(\mathfrak{a}) \neq \emptyset$, then $I_p(Z_p(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof. Let $X = Z_p(\mathfrak{a})$.

(1) $X = \emptyset \Leftrightarrow C(X) = \{0\}$. Since $C(X) = Z_a(\mathfrak{a}) \cup \{0\}$, we get

$$X = \emptyset \Leftrightarrow Z_a(\mathfrak{a}) = \emptyset \text{ or } Z_a(\mathfrak{a}) = \{0\}.$$

By affine Nullstellensatz, we get

$$\sqrt{\mathfrak{a}} = k[x_0, \cdots, x_n] \text{ or } \sqrt{\mathfrak{a}} = \langle x_0, \cdots, x_n \rangle.$$

So $\sqrt{\mathfrak{a}} \supset \langle x_0, \cdots, x_n \rangle$. Thus for any $i = 0, \cdots, n$, $\exists m_i$ s.t. $x_i^{m_i} \in \mathfrak{a}$. Let $N = m_1 + \cdots + m_n$, then any monomial of degree N in $k[x_0, \cdots, x_n]$ lies in $\mathfrak{a}$.

(2)Let $X = Z_p(\mathfrak{a} \neq \emptyset$, then

$$I_H(X) = I_a(C(X)) = I_a(Z_a(\mathfrak{a})) = \sqrt{\mathfrak{a}}. \tag{4.2}$$

$\square$

Remark. $\langle x_0, \cdots, x_n \rangle$ is called the irrelevant ideal, an ideal different from $\langle x_0, \cdots, x_n \rangle$ is called relevant.

Corollary 4.7. *There is a one-to-one correspondence between homogeneous relevant radical ideals and projective algebraic sets:*

Z_p: homogeneous relevant radical ideals in $k[x_0, \cdots, x_n] \to$ projective algebraic sets in $\mathbb{P}^n$

I_H: projective algebraic sets in $\mathbb{P}^n \to$ homogeneous relevant radical ideals in $k[x_0, \cdots, x_n]$.

Remark. We use subscripts to recognize affine spaces and projective spaces, such as $Z_p(\mathfrak{a}), Z_a(\mathfrak{a})$. Sometimes we can infer the difference from the context, so we usually write briefly as $Z(\mathfrak{a})$.

Proposition 4.8.

1. *A projective algebraic set $X \neq \emptyset \subset \mathbb{P}^n$ is irreducible if and only if $I = I_H(X)$ is a homogeneous prime ideal;*
2. *If $f \in k[x_0, \cdots, x_n]$ is a homogeneous polynomial and irreducible, then $Z_p(f)$ is irreducible.*

Proof. (1) $\Leftarrow$: Assume X reducible, then $X = X_1 \cup X_2$,$X_1, X_2 \subsetneqq X$b are closed subsets. Then we get $C(X) = C(X_1) \cup C(X_2)$, $C(X_1) \subsetneqq C(X)$,$C(X_2) \subsetneqq C(X)$ are closed, hence $C(X)$ is reducible, $I_H(X) = I(C(X))$ is not prime.

$\Rightarrow$: Assume $I_H(X)$ not prime, it means $\exists f, g \in k[x_0, \cdots, x_n]$, $fg \in I_H(X)$ and $f, g \notin I_H(X)$. Let $i, j \in \mathbb{Z} \geq 0$ be minimal such that $f^{(i)} \notin I$ and $g^{(j)} \notin I$. Subtract homogeneous components of lower degrees from f and g, we can assume f starts in degree i and g starts in degree j. Thus $f^{(i)}g^{(j)}$ is homogeneous component of minimal degree in $fg \in I$. Because I is homogeneous, we get $f^{(i)}g^{(j)} \in I$. Let $X_1 := Z(I) \cap Z(f^(i))$ and $X_2 := Z(I) \cap Z(g^{(j)})$, then $X_1, X_2 \subsetneqq X$, $X = X_1 \cup X_2$, thus X is reducible.

(2) If $I \subset k[x_0, \cdots, x_n]$ is homogeneous and prime with $Z(I) \neq \emptyset$, then follow the result from (1) we know $Z(f)$ is irreducible. $\square$

5. Functions and morphisms

Definition 5.1. Let $X \subset \mathbb{A}^n$ be an affine algebraic set, the *affine coordinate ring of X* is

$$A(X) := k[x_0, \cdots, x_n]/I(X). \tag{5.1}$$

It is a ring, also a k-algebra.

Definition 5.2. A *polynomial function on X* is a function $f : X \to k$ s.t. $f = F|_X$ for $F \in k[x_0, \cdots, x_n]$. This is the ring with pointwise addition and multiplication:

$$(f + g)(p) = f(p) + g(p), fg(p) = f(p)g(p), \forall p \in X.$$

There is a ring homomorphism:

$$\begin{gathered} k[x_0, \cdots, x_n] \to \{\text{polynomial functions on } X\} \\ F \to F|_X \end{gathered}$$

It is surjective and its kernel is $I(X)$. Thus we have the isomorphism:

$$A(X) \cong \{\text{polynomial functions on } X\}.$$

We will not distinguish them.

Remark. The zero set of a polynomial function is closed.Let X be an affine algebraic set, $f \in A(X)$, then

$$Z(f) = \{p \in X | f(p) = 0\} \tag{5.2}$$

is closed in X. $f \in A(X)$ means $f = F|_X$ for some $F \in k[x_1, \cdots, x_n]$, then

$$Z(f) = \{p \in X | F(p) = 0\} = X \cap Z(F) \tag{5.3}$$

so it is closed.

Definition 5.3. Let X be an affine variety, then $I(X)$ is prime, then $A(X)$ is integral. The *quotient field* $Q(A(X))$ is a field of rational functions on X and denoted by $K(X)$. Let $V \subset X$ be a quasi-affine variety, since $I(V) = I(X)$, we can denote its field of rational functions by $K(V) := K(X)$.

Definition 5.4. Let $p \in V$, the *local ring of* V *at* p is

$$\mathcal{O}_{V,p} := \{h \in K(V) | \exists f, g \in A(V), \text{ s.t. } h = \frac{f}{g} \text{ and } g(p) \neq 0\} \tag{5.4}$$

For simplicity in future we can write this:

$$\mathcal{O}_{V,p} = \{\frac{f}{g} \in K(V) | g(p) \neq 0\}. \tag{5.5}$$

If $U \subset V$ is an open subset, the *regular functions on* U are defined by

$$\mathcal{O}_V(U) = \underset{V,p}{\cap} \subset K(V). \tag{5.6}$$

Proposition 5.5. *We have an injective ring homomorphism:*

$$\mathcal{O}_V(U) \to \{\textit{functions from } U \textit{ to } k\}.$$

For $h \in \mathcal{O}_V(U), p \in U$, *there exists an open subset* W *and* $p \in W \subset U$, *s.t.* $h = \frac{f}{g}$ *with* $g(p) \neq 0$. *We define the homomorphism by setting* $h(p) = \frac{f(p)}{g(p)}$, *the homomorphism is*

$$h \in \mathcal{O}_V(U) \to h(p) = \frac{f(p)}{g(p)}, p \in U.$$

Proof. It is well defined: if $h = \frac{f}{g} = \frac{f'}{g'}$ with $g(p) \neq 0, g'(p) \neq 0$.Then $fg' = f'g \Rightarrow f(p)g'(p) = f'(p)g(p) \Rightarrow \frac{f(p)}{g(p)} = \frac{f'(p)}{g'(p)}$.

Injective: Let $h, h' \in \mathcal{O}_V(U)$ such that $h(p) = h'(p) \forall p \in U$. Define $l = h - h' \in \mathcal{O}_V(U)$, then $l(p) = 0, \forall p \in U$. There exists an open subset W, s.t. $l = \frac{f}{g}$ with $g(p) \neq 0 \forall p \in W$. For $p \in W$, $l(p) = \frac{f(p)}{g(p)} = 0 \Rightarrow f(p) = 0 \forall p \in W$.As zero set $Z(f)$ of f is closed, we get $f = 0 \in A(V)$, then $l = 0$ and hence $h = h'$. $\square$

Definition 5.6. We had called $\mathcal{O}_{V,p}$ a local ring of V at p. The maximal ideal at p is defined by $\mathfrak{m}(p) := \{h \in \mathcal{O}_{V,p} | h(p) = 0\}$, this is a maximal ideal in $\mathcal{O}_{V,p}$.

It is easy to verify that the local ring of a variety is a local ring.

Proposition 5.7. *For an affine variety* X, *functions which are regular functions everywhere are polynomial functions, i.e.,* $\mathcal{O}_X(X) = A(X)$.

Proof. Obviously, $A(X) \subset \mathcal{O}_X(X)$. We have to show the other inclusion. Let $h \in \mathcal{O}_X(X)$, $\forall p \in X$, $\exists F_p, G_p \in k[x_1, \cdots, x_n]$ s.t. $h = \frac{[F_p]}{[G_p]}$ and $G_p(p) \neq 0$. It is equivalent to: $\forall p \in X$, $\exists G_p \in k[x_1, \cdots, x_n]$ s.t. $h \cdot [G_p] \in A(X)$ and $[G_p(p)] \neq 0$. Let

$$\mathcal{G} := \{G \in k[x_1, \cdots, x_n] | h \cdot [G] \in A(X)\} \tag{5.7}$$

$\mathcal{G}$ is an ideal and $\mathcal{G} \supset I(X)$, so $Z(\mathcal{G}) \subset X$. But $Z(\mathcal{G}) \cap X = \emptyset$, so $Z(\mathcal{G}) = \emptyset$. By Nullstellensatz $1 \in \mathcal{G}$, so $h = h \cdot 1 \in A(X)$. □

Definition 5.8. Let $X \subset \mathbb{P}^n$ be a projective algebraic set. The *homogeneous coordinate ring of* X is defined as

$$S(X) := k[x_0, \cdots, x_n]/I_H(X) \tag{5.8}$$

If X is irreducible, then $S(X)$ is an integral domain, $Q(S(X))$ is its quotient field.

Remark. $X \subset \mathbb{P}^n$ is a quasi-projective variety, then polynomial $F \in k[x_0, \cdots, x_n]$ will not define a function $X \to k$. But we can take quotients of homogeneous polynomials of the same degree and get a well defined function.

Definition 5.9. Let $f = [F] \in S(X)$,$F \in k[x_0, \cdots, x_n]$. The *homogeneous part* $f^{(d)}$ of f is $[F^{(d)}] \in S(X)$, and $S^{(d)}(X) = \{f^{(d)} \in S(X)\}$.

Definition 5.10. Let X be a quasi-projective variety, the *field of rational functions on* X(on $V \subset X$ open subset) is

$$K(V) := K(X) := \{\frac{f}{g} \in Q(S(X)) | f, g \text{ both in } S^{(d)}(X) \text{ for some d}\}.$$

Elements of $K(X)(K(V))$ are called *rational functions on* X(on V).

Definition 5.11. Let $p \in V \subset \mathbb{P}^n$, the *local ring of* V *at* p is

$$\mathcal{O}_{V,p} := \{\frac{f}{g} \in K(V) | g(p) \neq 0\}. \tag{5.9}$$

If $U \subset V$ is open, the *ring of regular functions on* U is

$$\mathcal{O}_V(U) := \bigcap_{p \in U} \mathcal{O}_{V,p}. \tag{5.10}$$

Proposition 5.12.

(1) *(k-algebra)Constant functions $a \in k$ are regular on U. If $f, g \in \mathcal{O}_V(U)$, then $f + g$ and fg are regular on U, and if g has no zero in U, then $\frac{f}{g} \in \mathcal{O}_V(U)$.*
(2) *(Local)Let (U_i) be a open cover of U. A function $f : U \to k$ is regular if and only if $f|_{U_i}$ is regular for all i.*
(3) *Regular functions are continuous. i.e., let $h \in \mathcal{O}_V(U)$, then $h : U \to k = \mathbb{A}^1$ is continuous($k = \mathbb{A}^1$ is given Zariski topology).*

Proof. (1) By definition, $\mathcal{O}_V(U) = \bigcap_{p \in U} \mathcal{O}_{V,p}$, thus enough to show if $f, g \in \mathcal{O}_{V,p}$, then $f + g, fg \in \mathcal{O}_{V,p}$, and it is obvious. Assume g has no zero on U, then $g\frac{1}{g} \in \mathcal{O}_V(U)$, then $\frac{f}{g} \in \mathcal{O}_V(U)$.

(2) $h : U \to k$ is regular $\Leftrightarrow h \in \mathcal{O}_{V,p} \forall p \in U \Leftrightarrow h \in \mathcal{O}_{V,p} \forall p \in U_i \forall i$.

(3) $h : U \to k$ is continuous $\Leftrightarrow h|_{U_i}$ is continuous for all U_i of an open cover of U. We just replace U by a suitable U_i and show h is continuous in U_i. From the definition of regular functions, we can simply assume $h = \frac{f}{g}, f, g \in k[x_0, \cdots, x_n]$ are homogeneous of the same degree, and g has no zero on U_i. Zariski topology

on $\mathbb{A}^1$ has closed subsets $\emptyset, k$ and finite points subsets. Thus we only have to show $h^{-1}(a)$ is closed in U_i for all a in k,

$$h^{-1}(a) = \{p \in U_i | h(p) = a\} = \{p \in U_i | (f - ag)(p) = 0\}. \tag{5.11}$$

This is the zero set $Z(f - ag) \cap U$, hence the inverse of the closed sets are closed, hence h is continuous in $U_i \forall i$, hence continuous in U. □

Definition 5.13 (Polynomial Map)**.** Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be affine algebraic sets. A map

$$(F_1, \cdots, F_m) : X \to Y, p \to (F_1(p), \cdots, F_m(p)), F_1, \cdots, F_m \in k[x_1, \cdots, x_n]$$

is called a *polynomial map*. A surjective polynomial map whose inverse is also a polynomial map is an isomorphism.

Example 5.14.

(1) If X is an affine algebraic set, the polynomial map $f : X \to k$ is the polynomial function in $A(X)$.
(2) Let $X = \mathbb{A}^1$, $Y = Z(y - x^2) \subset \mathbb{A}^2$, the polynomial map
$$(t, t^2) : \mathbb{A}^2 \to Y$$
is isomorphism.

Definition 5.15. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine algebraic sets. Let

$$\varphi : X \to Y$$

be a polynomial map. The pull back of $h \in A(Y)$ is $\varphi^* h := h \circ \varphi \in A(X)$. If $h = H|_Y, H \in k[y_1, \cdots, y_m]$, $\varphi = (F_1, \cdots, F_m)$, then

$$\varphi^* h(a_1, \cdots, a_n) = h(F_1(a_1, \cdots, a_n), \cdots, F_m(a_1, \cdots, a_n).$$

i.e.,

$$\varphi^* h = H(F_1(x_1, \cdots, x_n), \cdots, F_m(x_1, \cdots, x_n))|_X \in A(X).$$

The pull back $\varphi^* : A(Y) \to A(X)$ is obviously a ring homomorphism. If $\varphi : X \to Y$ is an isomorphism, then $\varphi^* : A(Y) \to A(X)$ is an isomorphism of k-algebra.

Definition 5.16. Let X, Y be varieties, a map $\varphi : X \to Y$ is a *morphism*(regular map) if :

(1) φ is continuous;
(2) for all open subsets $U \in Y$, all regular functions $f \in \mathcal{O}_Y(U)$, we have
$$\varphi^* := f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U)).$$

Remark. Thus for each open subset $U \in Y$,

$$\varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$$

is a k-algebra homomorphism. φ is called an isomorphism if φ is bijective and φ^{-1} is also a morphism.

(1) id_X is a morphism form X itself.
(2) If $\varphi : X \to Y, \psi : Y \to Z$ are morphisms, then
$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$$
.
(3) If $\varphi : X \to Y$ is isomorphism, then $\varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism for all $U \subset Y$.

Proposition 5.17.

(1) *Let $\varphi : X \to Y$ and $(U_i)_{i\in I}$ be an open cover of X s.t. $\varphi|_{U_i} : U_i \to Y$ is a morphism. Then φ is a morphism.*
(2) *Let $Z \subset X, W \subset Y$ be varieties, let $\varphi : X \to Y$ be a morphism with $\varphi(Z) \subset W$. Then $\varphi|_Z : Z \to W$ is a morphism.*

Proof. (1) Let $W \subset Y$ be open, then we can write $\varphi^{-1}(W) = \underset{i\in I}{\cup}(\varphi|_{U_i}^{-1}(W))$, it is open so φ is continuous. Let $h \in \mathcal{O}_Y(W)$ then the pull back of regular functions h from $\mathcal{O}_Y(W)$ to $\mathcal{O}_X(U_i \cap \varphi^{-1}(W))$ is $\varphi|_{U_i}^* h = \varphi^* h|_{U_i\cap\varphi^{-1}(W)}$, since $\varphi|_{U_i}$ is a morphism we get that $U_i \cap \varphi^{-1}(W)$ is open. Then

$$\varphi^{-1}(W) = \underset{i\in I}{\cup} U_i \cap \varphi^{-1}(W) \tag{5.12}$$

and $(U_i \cap \varphi^{-1}(W))_{i\in I}$ is an open cover of $\varphi^{-1}(W)$, then we can get the conclusion that φ is a morphism by proposition 5.12.

(2) First, $\varphi|_Z$ is continuous as a restriction of a continuous map. Let $U \subset W$ be open, let $h \in \mathcal{O}_W(U)$. Replace if necessary U by a smaller open subset sucht that we can assume $h = \frac{F}{G}$. This quotient also defines a regular function H on open subset $\tilde{U} \subset Y$ s.t. $U \subset \tilde{U}$, then $\varphi^* H \in \mathcal{O}_X(\varphi^{-1}(\tilde{U}))$ is regular. Then $\varphi^* h = \varphi^* H|_{\varphi^{-1}(U)\cap Z}$ is regular on $\varphi^{-1}(U) \cap Z$. □

Now we generalize the definition of affine varieties.

Definition 5.18. An *affine variety* is a variety which is isomorphic to irreducible closed subset of some $\mathbb{A}^n$.

Theorem 5.19. *Let X, Y be subvarieties, assume $Y \subset \mathbb{A}^n$. A map $\varphi : X \to Y$ is a morphism if and only if $\exists f_1, \cdots, f_n \in \mathcal{O}_X(X)$ s.t.*

$$\varphi(p) = (f_1(p), \cdots, f_n(p)), \forall p \in X. \tag{5.13}$$

We can write $\varphi = (f_1, \cdots, f_n)$.

Proof. $\Rightarrow$: Let $\varphi : X \to Y$ be a morphism. Let $y_1, \cdots, y_n \in \mathcal{O}_Y(Y)$ be restrictions of the coordinates on $\mathbb{A}^n$ to Y, i.e., if $q = (a_1, \cdots, a_n) \in Y$, then $a_i = y_i(q)$. The pull back of y_i is

$$f_i := \varphi^* y_i = y_i \circ \varphi \in \mathcal{O}_X(X). \tag{5.14}$$

Let $p \in X$, $\varphi(p) = (b_1, \cdots, b_n)$, $b_i = y_i(\varphi(p)) = f_i(p)$, thus

$$\varphi = (f_1, \cdots, f_n)$$

where $f_i \in \mathcal{O}_X(X)$.

$\Leftarrow$ Let $\varphi := (f_1, \cdots, f_n)$, $f_i \in \mathcal{O}_X(X)$. First we show φ is continuous. Let $B \in Y$ be closed, it is equivalent to $B = Y \cap Z(G_1, \cdots, G_m)$ and $G_i \in k[x_1, \cdots, x_n]$. Since $G_i \circ \varphi = G(f_1, \cdots, f_n) \in \mathcal{O}_X(X)$, we get $\varphi^{-1}(B) = Z(G_1 \circ \varphi, \cdots, G_m \circ \varphi)$ and it is closed in X. So φ is continuous. Let $h \in \mathcal{O}_Y(U)$, write $W = \varphi^{-1}(U) \subset Y$. we need to show $h \circ \varphi \in \mathcal{O}_X(W)$. We can always make U smaller and assume $h(q) = \frac{F(q)}{G(q)}, \forall q \in U$, F and G are some polynomials and G has no zero on U. Then we have

$$h \circ \varphi = \frac{F \circ \varphi}{G \circ \varphi} = \frac{F(f_1, \cdots, f_n)}{G(f_1, \cdots, f_n)} \tag{5.15}$$

where $F(f_1, \cdots, f_n)$ and $G(f_1, \cdots, f_n)$ are regular on $\mathcal{O}_X(W)$. Since $\varphi(W) = U$ and G has no zero on U, $G(f_1, \cdots, f_n)$ also has no zero on W, i.e., $h \circ \varphi \in \mathcal{O}_X(W)$. $\square$

Remark. The regular functions on a variety X are the same as the morphisms $X \to \mathbb{A}^1$.

Corollary 5.20. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be closed subvarieties. The morphisms*

$$\varphi : X \to Y$$

are precisely the polynomial map.

Proof. We know $\varphi = (f_1, \cdots, f_m)$ and $f_i \in \mathcal{O}_X(X), \forall i$. Since $\mathcal{O}_X(X) = A(X)$, φ is a polynomial map. $\square$

Theorem 5.21. *Let X, Y be varieties, assume $Y \subset \mathbb{A}^m$ be a closed affine variety. Then there is a bijection between morphisms $X \to Y$ and k-algebra homomorphisms $A(Y) \to \mathcal{O}_X(X)$:*

$$\begin{array}{ccc} \{\textit{morphisms } X \to Y\} & \xrightarrow{\textit{bijection}} & \{\textit{homomorphisms } A(Y) \to \mathcal{O}_X(X)\} \\ \varphi & \longrightarrow & \varphi^* \end{array}$$

Proof. $\Rightarrow$: Let $\varphi : X \to Y$ be a morphism, then $\varphi^* : A(Y) \to \mathcal{O}_X(X)$ is a k-algebra homomorphism.

$\Leftarrow$: Let $\phi : A(Y) \to \mathcal{O}_X(X)$ be a k-algebraic homomorphism, let $y_1, \cdots, y_n \in A(Y)$ be the coordinate functions. We set

$$f_i = \phi(y_i) \in \mathcal{O}_X(X).$$

Let $\varphi = (f_1, \cdots, f_m) : X \to \mathbb{A}^m$. This is a morphism from X to Y. To see it is a morphism we have to show $\varphi(X) \subset Y$. Let $h \in I(Y)$, $h \circ \varphi = h(f_1, \cdots, f_m) = h(\phi(y_1), \cdots, \phi(y_m)) = \phi(h(y_1, \cdots, y_m))$. The second equality is based on the homomorphic property of ϕ, for example, if $h(x_1, x_2) = x_1^2 - x_2^3$, then $h(\phi(y_1), \phi(y_2)) = \phi(y_1)^2 - \phi(y_2)^3 = \phi(y_1^2) - \phi(y_2^3) = \phi(y_1^2 - y_2^3) = \phi(h(y_1, y_2))$. So $h(y_1, \cdots, y_m) \in A(Y)$, we choose an arbitrary element $p = (a_1, \cdots, a_m) \in Y$, then $h(y_1, \cdots, y_m)(p) = h(a_1, \cdots, a_m) = 0$ because $h \in I(Y)$. So for arbitrary $h \in I(Y)$, we get $h \circ \varphi = 0$, it implies $\varphi(X) \subset \cap_{h \in I(Y)} Z(h) = Y$. $\square$

Example 5.22. A bijective polynomial map need not to be an isomorphism. For example, let$X = \mathbb{A}^1$, $Y = Z(x_2^2 - x_1^3) \subset \mathbb{A}^2$. Then

$$\varphi = (t^2, t^3) : X \to Y$$

is a morphism and bijective and the inverse is

$$\varphi^{-1}(a, b) = \begin{cases} \frac{b}{a} & \text{if } a \neq 0 \\ 0 & \text{if } (a, b) = 0 \end{cases}$$

φ is not an isomorphism(φ^{-1} is not a morphism). To show this we see the pull back:

$$\varphi^* : A(Y) \to \mathcal{O}_X(X)$$

where $A(Y) = k[x_1, x_2]/\langle x_2^2 - x_1^3 \rangle$ and $A(X) = k[t]$. φ^* makes $x_1 \to t^2$ and $x_2 \to t^3$. Since φ^* is not surjective(there is no element maps into t), φ^* is not an isomorphism. By theorem 5.21 we know φ is not an isomorphism. So bijective morphism is not necessary to be an isomorphism.

Definition 5.23. Let $X \subset \mathbb{A}^n$ be a closed variety, $F \in k[x_1, \cdots, x_n] \backslash I(X)$. The *principal open* defined by F is $X_F := X \backslash Z(F)$.

Proposition 5.24. *X_F is an affine variety.*

Proof. Let $Z := Z(\langle I(X), F \cdot x_{n+1} - 1 \rangle) \subset \mathbb{A}^{n+1}$. We need to prove Z is a closed subvariety of $\mathbb{A}^{n+1}$ isomorphic to X_F. Let $\varphi : (x_1, \cdots, x_n, \frac{1}{F}) : X_F \to \mathbb{A}^{n+1}$, it is a bijective morphism and $\varphi(X_F) = Z$. As X_F is irreducible, Z is also irreducible. So Z is closed variety of $\mathbb{A}^{n+1}$. On the other hand, the inverse of φ is

$$\varphi^{-1} = (x_1, \cdots, x_n) : Z \to X_F$$

is a morphism, so φ is an isomorphism. □

6. Morphisms of quasi-projective varieties

Definition 6.1. Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be quasi-projective algebraic sets. A map $\varphi : X \to Y$ is called a *polynomial map* if there exists homogeneous polynomials $F_0, \cdots, F_m \in k[x_0, \cdots, x_n]$ of the same degree with no common zero on X s.t. $\varphi(p) = [F_0(p), \cdots, F_m(p)], \forall p \in X$, write $\varphi = [F_0, \cdots, F_m]$.

Definition 6.2. The intersection between a closed and an open subsets is called *locally closed set.*

Quasi-projective algebraic sets are locally closed.

Definition 6.3. The *homogenization* of $F \in k[x_0, \cdots, x_n]$ is:

$$F_a := F(1, x_1, \cdots, x_n).$$

Theorem 6.4. $\varphi_i = (\frac{x_0}{x_i}, \cdots, \hat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i}) : U_i \to \mathbb{A}^n$ *is an isomorphism.*

Proof. We can assume $i = 0$, $\varphi := \varphi_0$, $U := U_0$, then $\varphi = (\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})$. $\frac{x_i}{x_0}$ is a regular function in $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$, so φ is a morphism. We need to show that $u = \varphi^{-1}(x_1, \cdots, x_n) = [1, x_1, \cdots, x_n]$ is a morphism.

(a) $u = \varphi^{-1}$ is continuous. Let $W = Z(F_1, \cdots, F_m) \cap U$ be closed in U, $F_i \in k[x_0, \cdots, x_n]$ are homogeneous, then

$$\begin{aligned} u^{-1}(W) =& \{(a_1, \cdots, a_n) \in \mathbb{A}^n | [1, a_1, \cdots, a_n] \in W\} \\ =& \{(a_1, \cdots, a_n) \in \mathbb{A}^n | F_i(1, a_1 \cdots, a_n) = 0, \forall i = 1, \cdots, m\} \\ =& Z(F_{1a}, \cdots, F_{ma}) \end{aligned}$$

where F_{ia} is homogenization of F_i, it shows that $u^{-1}(W)$ is closed in $\mathbb{A}^n$.

(b) Let $V \subset U$ be open, $h \in \mathcal{O}_U(V)$, we need to show $u^*h \in \mathcal{O}_{\mathbb{A}^n}(u^{-1}(V))$. Making V smaller necessary, we can assume $h = \frac{F}{G}$, $F, G \in k[x_0, \cdots, x_n]$ are homogeneous polynomials of the same degree.

$$u^*h = h \circ u = \frac{F \circ u}{G \circ u} = \frac{F(1, x_1, \cdots, x_n)}{G(1, x_1, \cdots, x_n)}.$$

Thus $u^*h \in \mathcal{O}_{\mathbb{A}^n}(u^{-1}(V))$. Hence u is a morphism.

Combine the above statements we obtain that $\phi : \mathbb{A}^n \to u$ is an isomorphism. □

Remark. From theorem 6.4 we find that if we identify $\mathbb{A}^n$ with $u_0 \subset \mathbb{P}^n$, the Zariski topology on $\mathbb{A}^n$ is equivalent to the induced topology of u_0 from $\mathbb{P}^n$.

Corollary 6.5.

(1) *Every variety is isomorphic to a quasi-projective variety.*
(2) *Every variety has an open cover by affine varieties.*

Proof. (1) Let X be a variety, if X is locally closed in $\mathbb{P}^n$, then it is a quasi-projective variety, so we only need to consider the condition in $\mathbb{A}^n$. Assume X be locally closed in $\mathbb{A}^n$. $Y = \varphi_0^{-1}(X) \subset \mathbb{P}^n$ is locally closed subvariety and $\varphi_0^{-1} : X \to Y$ is an isomorphism.

(2) For varieties in $\mathbb{A}^n$, it is trivial. Let $X \subset \mathbb{P}^n$ be a quasi-projective variety, then $X = \bigcup\limits_{i=0}^{n} X \cap U_i$. $X \cap U_i$ is isomorphic to locally closed subvariety in $\mathbb{A}^n$. We can regard $X \cap U_i$ simply as $X \subset \mathbb{A}^n$, where X is locally closed. It is equivalent to prove:

For every point $p \in X$, there exists a neighborhood $U \subset X$ and U is an affine variety.

Since X is locally closed, there exist $Y, Z \subset \mathbb{A}^n$ closed in $\mathbb{A}^n$ s.t. $X = Y \backslash Z$. For any point $p \in X$, $\exists F_p \in I(Z)$ with $F_p(p) \neq 0$. Then we have $Y_{F_p} = Y \backslash Z(F_p) \subset X$ and Y_{F_p} is an affine variety. □

Theorem 6.6. *Let $X \subset \mathbb{P}^m$, $Y \subset \mathbb{P}^n$ be quasi-projective varieties. Let $\varphi : X \to Y$ be a map. The following conditions are equivalent:*

(1) *φ is a morphism;*
(2) *φ is locally given by regular functions, i.e., for all $p \in X$, there exists a neighborhood $U \subset X$, $h_0, \cdots, h_n \in \mathcal{O}_X(U)$ with no common zero on U, s.t.*
$$\varphi(q) = [h_0(q), \cdots, h_n(q)], \quad \forall q \in U.$$
We write $\varphi = [h_0, \cdots, h_n]$ on U;
(3) *φ is locally a polynomial map, i.e.:*
$\forall p \in X$, $\exists$ open neighborhood $U \subset X$, $F_0, \cdots, F_n \in k[x_0, \cdots, x_,]$ homogeneous of the same degree with no common zero s.t.
$$\varphi(q) = [F_(q), \cdots, F_n(q)] \quad \forall q \in U.$$
We write $\varphi = [F_0, \cdots, F_n]$ on U.

Proof. (1) $\Rightarrow$ (2): If $\varphi : X \to \mathbb{P}^n$ is amorphism, then $\forall p \; inX$, $\exists i$, s.t. $\varphi(p) \in U_i$. Assume $i = 0$ and then $\varphi(p) \in U_0$. Let U be an open neighborhood of p in X s.t. $\varphi(U) \subset U_0$. Then $\varphi_0 \circ \varphi : U \to \mathbb{A}^n$ is a morphism, so $\varphi_0 \circ \varphi = (h_1, \cdots, h_n)$ with $h_i \in \mathcal{O}_X(U)$. Since the inverse of φ_0 is u_0 we get

$$\varphi = u_0 \circ \varphi_0 \circ \varphi = [1, h_1, \cdots, h_n]. \tag{6.1}$$

(2) $\Rightarrow$ (3): Assume $\varphi = [h_0, \cdots, h_n]$ on $U \subset X$, where $h_i \in \mathcal{O}_X(U)$ with no common zeros on U. By making U possibly smaller we can further assume $h_i = \frac{F_i}{G_i}$, $F_i, G_i \in k[x_0, \cdots, x_m]$ are homogeneous of the same degree(F_i and G_i are of the same degree, it is not necessary that F_i and G_j are of the same degree for $i \neq j$), G_i has no zeros on U. Let $L_i = F_i \cdot G_0 \cdots \hat{G_i} \cdots G_n$, L_i are homogeneous of the same degree, we get

$$\varphi = [h_0, \cdots, h_n] = [L_0, \cdots, L_n]. \tag{6.2}$$

(3) $\Rightarrow$ (1): Let $\varphi|_U = [L_0, \cdots, L_n]$, $L_i \in k[x_0, \cdots, x_m]$ are homogeneous of the same degree with no common zero. Making U smaller, we can assume one of L_i(say

L_0) has no zero in U. Then for $i = 1, \cdots, n$, let $h_i = \frac{L_i}{L_0} \in \mathcal{O}_X(U)$. Rewrite the map as

$$\varphi = [1, h_1, \cdots, h_n] \tag{6.3}$$

$$\Rightarrow \varphi_0 \circ \varphi = (h_1, \cdots, h_n). \tag{6.4}$$

So$\varphi_0 \circ \varphi$ is amorphism, then $\varphi = u_0 \circ \varphi_0 \circ \varphi$ is a morphism. □

Definition 6.7 (Projective Transformation)**.** Let

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{6.5}$$

be a $(n+1) \times (n+1)$ matrix in k, then we can construct a map from $\mathbb{P}^n \to \mathbb{P}^n$:

$$[A] : [b_0, \cdots, b_n] \to [b_0, \cdots, b_n] \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix}^T.$$

It is called a *projective transformation*. This is a morphism and if A is inverse then it is an isomorphism.

Remark. All automorphisms of $\mathbb{P}^n$ are projective transformations and this is not easy to prove.

Definition 6.8 (Projection)**.** Let $X \subset \mathbb{P}^n$ be a variety, $W \subset \mathbb{P}^n$ be a projective subspace of $\mathbb{P}^n$ of dim$W = k$. Assume $X \cap W = \emptyset$ and there exist linear forms $H_0, \cdots, H_{n-k-1}$ such that $W = Z(H_0, \cdots, H_{n-k-1})$. The *projection from* W is

$$\Pi_W = [H_0, \cdots, H_{n-k-1}] : X \to \mathbb{P}^{n-k-1}.$$

This is a morphism(H_i have no common zero on X because $W \cap X = \emptyset$).

Remark. Π_W depends on $H_0, \cdots, H_{n-k-1}$, but if we have another relation $W = Z(L_0, \cdots, L_{n-k-1})$, then there exists a projective transformation $[A] : \mathbb{P}^{n-k-1} \to \mathbb{P}^{n-k-1}$. In particular, if $p \in \mathbb{P}^n \backslash X$, for example, $p = [0, \cdots, 0, 1]$, then $\Pi_p = [x_0, \cdots, x_{n-1}] : X \to \mathbb{P}^{n-1}$.

7. Products of varieties

Theorem 7.1 (Products of Affine Varieties)**.** *If $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ are closed subvarieties, then $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ is a closed subvariety.*

Before we prove it, we need to prove a conclusion in topology.

Lemma 7.2. *Let X, Y be irreducible topological spaces. Assume we have a topology on the product $X \times Y$ s.t.:*

$$\begin{aligned} y_p : & Y \to X \times Y, \quad q \to (p, q) \text{ is continuous } \forall p \in X; \\ l_q : & X \to X \times Y, \quad p \to (p, q) \text{is continuous } \forall q \in Y. \end{aligned}$$

Then $X \times Y$ is irreducible.

Proof. Assume $X \times Y = S_1 \cup S_2$, $S_i \subsetneqq X \times Y$ are closed, $S_1 \not\subset S_2$ and $S_2 \not\subset S_1$. For $i = 1, 2$, set $T_i = \bigcap_{q \in Y} l_q^{-1}(S_i) = \{p \in X | (p, q) \in S_i \quad \forall q \in Y\}$. It is the same as $T_i = \{p \in X | \{p\} \times Y \subset S_i\}$. Since y_p is continuous and Y is irreducible, we get $y_p(Y) = \{p\} \times Y$ is irreducible. So we get $\{p\} \times Y \subset S_1$ or $\{p\} \times Y \subset S_2, \forall p \in X$(it implies $T_1 \cap T_2 = \emptyset$ and $T_i \subsetneqq X$). Hence $X = T_1 \cup T_2$. Since l_q is continuous, T_i are closed. If one of them, say $T_1 = \emptyset$, then $X = T_2$, hence $S_2 = X \times Y$, contradicts to the assumption. This implies X must be reducible, hence the proof is completed. $\square$

Proof of Theorem 7.1. Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be closed subvarieties, the product of X and Y is just

$$X \times Y = \{(p, q) \in \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m} | p \in X \text{ and } q \in Y\}.$$

Let $x_1, \cdots, x_n$ be coordinates in $\mathbb{A}^n$ and $y_1, \cdots, y_m$ be coordinates in $\mathbb{A}^m$, we can assume $X = Z(F_1, \cdots, F_k)$ and $Y = Z(G_1, \cdots, G_l)$ where $F_i \in k[x_1, \cdots, x_n], G_j \in k[y_1, \cdots, y_m]$. Then

$$X \times Y = Z(F_1, \cdots, F_k, G_1, \cdots, G_l) \subset \mathbb{A}^{n+m} \tag{7.1}$$

is a closed subset. By lemma 7.2 we only need to check $\forall q \in Y$, $l_q : X \to Y$ is continuous. Write $q = (b_1, \cdots, b_m)$, then $l_q = (x_1, \cdots, x_n, b_1, \cdots, b_m)$. It is a morphism, so it is continuous, thus we finish the proof.

Proposition 7.3 (Universal Property)**.** *Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be varieties, then*

(1) *The projections*

$$\begin{aligned} p_1 &= (x_1, \cdots, x_n) : X \times Y \to X \\ p_2 &= (y_1, \cdots, y_m) : X \times Y \to Y \end{aligned}$$

are morphisms.

(2) *Let Z be a variety. The morphism $\varphi : Z \to X \times Y$ are precisely the*

$$(f, g) : Z \to X \times Y, \quad p \to (f(p), g(p)) \quad \forall p \in Z$$

where $f : Z \to X$ and $g : Z \to Y$ are morphisms. In other words, $\varphi : Z \to X \times Y$ is a morphism if and only if both $p_1 \circ \varphi$ and $p_2 \circ \varphi$ are morphisms.

Proof. The first is obvious, we only check the second.

$\Rightarrow$: Let $\varphi : Z \to X \times Y$ be a morphism, then $f = p_1 \circ \varphi$ and $g = p_2 \circ \varphi$ are morphisms and $\varphi = (f, g)$.

$\Leftarrow$: Assume $f : Z \to X$ and $g : Z \to Y$ are both morphisms. then there exist $f_1, \cdots, f_n \in \mathcal{O}_Z(Z)$ and $g_1, \cdots, g_m \in \mathcal{O}_Z(Z)$ s.t. $f = (f_1, \cdots, f_n)$, $g = (g_1, \cdots, g_m)$. Then $(f, g) = (f_1, \cdots, f_n, g_1, \cdots, g_m)$ is a morphism. $\square$

Remark. Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be subvarieties, $X \times Y$ does not lie rationally in some projective space. Thus we need to find an embedding $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ to denote the products of quasi-projective varieties.

Definition 7.4 ([Segre Embedding)**.**] We put $N := (n+1)(m+1)-1$, let $x_0, \cdots, x_n$ be coordinates on $\mathbb{P}^n$, $y_0, \cdots, y_m$ be coordinates on $\mathbb{P}^m$. Let $z_{ij}, i = 0, \cdots, n, j = 0, \cdots, m$ be coordinates on $\mathbb{P}^N$. Define a map

$$\begin{aligned} \sigma : \mathbb{P}^n \times \mathbb{P}^m &\to \mathbb{P}^N \\ ([x_0, \cdots, x_n], [y_0, \cdots, y_m]) &\to [z_{ij}] = [x_i y_j] \end{aligned}$$

σ is called the Segre embedding.

Definition 7.5. We define the image of σ as

$$\Sigma := \sigma(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N.$$

For $i = 0, \cdots, n$, put

$$U_i := \{[x_0, \cdots, x_n] \in \mathbb{P}^n | x_i \neq 0\}.$$

For $j = 0, \cdots, m$, put

$$U_j := \{[y_0, \cdots, y_m] \in \mathbb{P}^m | y_j \neq 0\}.$$

And for $i = 0, \cdots, n, j = 0, \cdots, m$, put

$$U_{ij} := \{[z_{kl}] \in \mathbb{P}^N | z_{ij} \neq = 0\}.$$

there are isomorphisms:

$$\mathbb{A}^n \underset{\varphi_i}{\overset{u_i}{\rightleftarrows}} U_i,$$

$$\mathbb{A}^m \underset{\varphi_j}{\overset{u_j}{\rightleftarrows}} U_j,$$

$$\mathbb{A}^N \underset{\varphi_{ij}}{\overset{u_{ij}}{\rightleftarrows}} U_{ij}.$$

Since $\mathbb{P}^N = \cup_{i,j} U_{ij}$, we get $\Sigma = \cup_{i,j}(\Sigma \cap U_{ij})$, define

$$\Sigma^{ij} = \Sigma \cap U_{ij}.$$

Define the map σ^{ij}

$$\begin{aligned} \sigma^{ij} : \mathbb{A}^{n+m} & \to U_{ij} \\ (p, q) & \to \sigma(u_i(p), u_j(q)). \end{aligned}$$

By definition we know $\sigma^{ij}(\mathbb{A}^{n+m}) = \Sigma^{ij}$.

Theorem 7.6.

(1) *$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ is injective and Σ is closed in $\mathbb{P}^N$:*

$$\Sigma = Z\left(\left\{ z_{ij}z_{kl} - z_{il}z_{kj} \middle| \begin{matrix} i,k & = 0, \cdots, n \\ j,l & = 0, \cdots, m \end{matrix} \right\}\right). \tag{7.2}$$

(2) *$\sigma^{ij} : \mathbb{A}^{n+m} \to \Sigma^{ij}$ is an isomorphism.*

(3) *$\forall q \in \mathbb{P}^m$, the map*

$$\begin{aligned} \bar{i_q} : \mathbb{P}^n & \to \mathbb{P}^N \\ p & \to \sigma(p, q) \end{aligned}$$

is a morphism. Similarly, $j_p = \sigma(p, q) : \mathbb{P}^m \to \mathbb{P}^N$ is a morphism.

(4) *Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be quasi-projective varieties, then $\sigma(X \times Y) \subset \mathbb{P}^N$ is also a quasi-projective variety. What's more, if X and Y are both projective varieties, then $\sigma(X \times Y)$ is a projective variety.*

Proof. (1) If $\sigma([a_0, \cdots, a_n], [b_0, \cdots, b_m]) = \sigma([a_0{}', \cdots, a_n{}'], [b_0{}', \cdots, b_m{}'])$, then $\exists \lambda \in k\backslash\{0\}$, s.t. $\lambda a_i' b_j' = \lambda a_i b_j \ \forall i, j$. Choose i_0, j_0 s.t. $a_{i_0} b_{j_0} \neq 0$, then $\forall i = 0, \cdots, n$, $a_i b_{j_0} = \lambda a_i' b_{j_0}' \Rightarrow a_i = \left(\frac{\lambda b_{j_0}'}{b_{j_0}}\right) a_i' \Rightarrow [a_0, \cdots, a_n] = [a_0{}', \cdots, a_n{}']$. The same way can be used to prove $[b_0, \cdots, b_m] = [b_0{}', \cdots, b_m{}']$. Let W be the zero set on the right hand side of the equation (7.2), clearly we have the relation $\Sigma \subset W$. Now let $[a_{ij}] \in W$, choose i_0, j_0 s.t. $a_{i_0 j_0} \neq 0$, then we get $[a_{ij}] = [a_{i_0 j_0} a_{ij}] = [a_{i_0 j} a_{i j_0}] = [a_{i j_0} a_{i_0 j}] = \sigma([a_{0 j_0}, \cdots, a_{n j_0}], [a_{i_0 0}, \cdots, a_{i_0 m}]) \subset \Sigma$.

(2) Assume $i = j = 0$, then

$$\begin{aligned}\varphi_{00} \circ \sigma^{00}(a_1, \cdots, a_n, b_1, \cdots, b_m) \quad &= \varphi_{00}(\sigma([1, a_1, \cdots, a_n], [1, b_1, \cdots, b_m])) \\ &= (z_{ij})_{(i,j)\neq(0,0)}\end{aligned}$$

where $z_{i0} = a_i$ for $i = 1, \cdots, n$, $z_{0j} = b_j$ for $j = 1, \cdots, m$, $z_{ij} = a_i b_j$ for $i, j \geq 1$. These are all regular functions, so $\varphi_{00} \circ \sigma^{00}$ is a morphism, so σ^{00} is a morphism. Finally, σ^{00} is an isomorphism because the inverse map is

$$(\sigma^{00})^{-1} = \left(\frac{z_{10}}{z_{00}}, \cdots, \frac{z_{n0}}{z_{00}}, \frac{z_{01}}{z_{00}}, \cdots, \frac{z_{0m}}{z_{00}}\right).$$

Remark. In fact, Σ^{ij} is a quasi-projective variety. Because $\mathbb{A}^{n+m}$ is irreducible, Σ^{ij} is irreducible, hence a quasi-projective variety.

(3) Let $q = [b_0, \cdots, b_m]$, then $i_q = [x_i b_j]$, $x_i b_j$'s are homogeneous polynomials, so it is a morphism.

(4) Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be projective varieties. We can decompose the map into the following:

$$\begin{aligned}\sigma(X \times Y) \qquad &= \bigcup_{i,j} \sigma(X \times Y) \cap U_{ij} \\ &= \bigcup_{i,j} \sigma^{ij}(\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j))\end{aligned}$$

$\varphi_i(X \cap U_i)$ and $\varphi_j(Y \cap U_j)$ are closed subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively, thus $\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j)$ is closed in $\mathbb{A}^{n+m}$. Since σ^{ij} is an isomorphism, then $\sigma^{ij}(\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j))$ is closed in $\Sigma^{ij} = \Sigma \cap U_{ij}$. So $\sigma(X \times Y)$ is closed in Σ, hence closed in $\mathbb{P}^N$ because Σ itself is closed. To show its irreducible, we use the lemma 7.2. Since σ is injective we can endow $\mathbb{P}^n \times \mathbb{P}^m$ with the topological structure of $\mathbb{P}^N$, hence we can identify $\mathbb{P}^n \times \mathbb{P}^m$ with Σ provided with the topology induced from $\mathbb{P}^N$. Now we can use the lemma 7.2, we have known i_q and j_p are continuous, so $\sigma(X \times Y)$ is irreducible. For quasi-projective conditions ,we just get the conclusion by simply difference two projective varieties. □

Remark. For $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ we can now identify $X \times Y$ with $\sigma(X \times Y) \subset \mathbb{P}^N$. In particular we can identify $\mathbb{P}^n \times \mathbb{P}^m$ with Σ.

From this perspective, part (2) of the theorem just says $U_i \times U_j \subset \mathbb{P}^n \times \mathbb{P}^m$ is open and $\varphi_i \times \varphi_j : U_i \times U_j \to \mathbb{A}^{n+m}$ is an isomorphism.

Proposition 7.7 (Universal Property)**.** *Let X, Y be quasi-projective varieties, then*

(1) *The projections*

$$\begin{aligned}p_1 \quad &= (x_1, \cdots, x_n) : X \times Y \to X \\ p_2 \quad &= (y_1, \cdots, y_m) : X \times Y \to Y\end{aligned}$$

are morphisms.

(2) *Let Z be a variety. The morphism $\varphi : Z \to X \times Y$ are precisely the*

$$(f, g) : Z \to X \times Y, \quad p \to (f(p), g(p)) \quad \forall p \in Z$$

where $f : Z \to X$ and $g : Z \to Y$ are morphisms. In other words, $\varphi : Z \to X \times Y$ is a morphism if and only if both $p_1 \circ \varphi$ and $p_2 \circ \varphi$ are morphisms.

Proof. (1) It is enough to show $p_1|_{U_i \times U_j}$ is a morphism from $U_i \times U_j$ to U_i. Identify $U_i \times u_j$ with $\mathbb{A}^{n+m}$ and U_i with $\mathbb{A}^n$, then we can see that p_1 is the same as the projection defined by the proposition 7.3, so it is a morphism.

(2) $\Rightarrow$: Let $\varphi : Z \to X \times Y$ be a morphism. Then $f := p_1 \circ \varphi$ and $g := p_2 \circ \varphi$ are morphisms.

$\Leftarrow$: Let $f : Z \to X$ and $g : Z \to Y$ be morphisms. Define

$$Z^{ij} := f^{-1}(U_i) \cap g^{-1}(U_j).$$

Then (f, g) is a morphism $\Leftrightarrow (f, g)|_{Z^{ij}}$ is a morphism for $i = 1, \cdots, n, j = 1, \cdots, m$. Consider the following mapping chain

$$Z^{ij} \xrightarrow{(f,g)} (X \times Y) \cap (U_i \times U_j) \xrightarrow{\varphi_i \times \varphi_j} \varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j) \subset \mathbb{A}^{n+m}.$$

the whole chain $(\varphi_i \circ f, \varphi_j \circ g) : Z^{ij} \to \mathbb{A}^{n+m}$ is a morphism, so (f, g) is a morphism. $\square$

Corollary 7.8. *Let X_1, X_2, Y_1, Y_2 be varieties. If $f : X_1 \to Y_1$ and $X_2 \to Y_2$ are morphisms, then the map:*

$$\begin{aligned} f \times g : X_1 \times X_2 &\to Y_1 \times Y_2 \\ (p, q) &\to (f(p), g(q)) \end{aligned}$$

is a morphism. In particular, if X_1 is isomorphic to Y_1 and X_2 is isomorphic to Y_2, then $X_1 \times X_2$ is isomorphic to $Y_1 \times Y_2$

Proof. We can write $f \times g$ as $f \circ p_1$ and $g \circ p_2$, both $f \circ p_1$ and $g \circ p_2$ are morphisms, so $f \times g = (f \circ p_1, g \circ p_2)$ is a morphism. $\square$

Lemma 7.9. *The closed subset in $\mathbb{P}^n \times \mathbb{P}^m$ is the zero set of sets of polynomials of $f_k(x_0, \cdots, x_n, y_0, \cdots, y_m)$ for $k = 1, \cdots, r$ which are homogeneous in x_i and y_j, and the degree in x_i is equal to the degree in y_j. We called it behomogeneous.*

Proof. Let $W \subset \mathbb{P}^n \times \mathbb{P}^m$ be closed. $W = \sigma^{-1}(A)$, for $A \subset \mathbb{P}^N$ closed. Then A is the zero set of homogeneous polynomials in z_{ij}, write it as $A = (f_1(z_{ij}), \cdots, f_r(z_{ij}))$. Then we get $W = (f_1(x_i y_j), \cdots, f_r(x_i y_j))$. For $k = 1, \cdots, r$, $f_k(x_i y_j)$ are bihomogeneous. Conversely, assume

$$W = Z(g_1(x_0, \cdots, x_n, y_0, \cdots, y_m), \cdots, g_l(x_0, \cdots, x_n, y_0, \cdots, y_m))$$

where g_k are bihomogeneous. Then

$$\begin{aligned} (\varphi_i \times \varphi_j)(W \cap (U_i \times U_j)) = Z(&g_1(x_0, \cdots, x_i = 1, \cdots, x_n, y_0, \cdots, y_j = 1, \cdots, y_m), \\ &\cdots, g_l(x_0, \cdots, x_i = 1, \cdots, x_n, y_0, \cdots, y_j = 1, \cdots, y_m)) \end{aligned}$$

are closed in $\mathbb{A}^{n+m}$. So $W \cap (U_i \times U_j)$ are closed in $U_i \times U_j$. $U_i \times U_j$ form a finite open cover of $\mathbb{P}^n \times \mathbb{P}^m$, so W is closed. $\square$

Definition 7.10. Let X be a variety, the *diagonal* is

$$\Delta_X := \{(p, p) \in X \times X | p \in X\} \subset X \times X.$$

The *diagonal morphism* is

$$\begin{aligned} \delta_X : X &\to \Delta_X \subset X \times X \\ p &\to (p, p). \end{aligned}$$

Lemma 7.11. *Δ_X is closed in $X \times X$ and $\delta_X : X \to \Delta_X$ is an isomorphism.*

Proof. Any variety X is isomorphic to a locally closed subvariety of some projective space, so we can assume $X \subset \mathbb{P}^n$ is locally closed, then

$$\Delta_X = \Delta_{\mathbb{P}^n} \cap (X \times X).$$

Thus we know if $\Delta_{\mathbb{P}^n}$ is closed then Δ_X is closed in $X \times X$. In fact $\Delta_{\mathbb{P}^n} = Z\left(\{x_i y_j - x_j y_i | i, j = 0, \cdots, n\}\right)$ is closed.

$\delta_X : X \to \Delta_X$ is isomorphic because $p_1 : \Delta_X \to X$ is its inverse morphism. □

Remark. The fact that $\Delta_X \subset X \times X$ is closed replaces for us the Hausdorff property in general topology.

Definition 7.12. A variety is called *separated* if $\Delta_X \subset X \times X$ is closed. By the lemma 7.11 all varieties are separated.

Corollary 7.13. *Let $\varphi, \psi : X \to Y$ be morphisms of varieties, then $W = \{p \in X | \varphi(p) = \psi(p)\}$ is closed in X. In particular, if $\varphi|_U = \psi|_U$ for an open subset of X, then $\varphi = \psi$.*

Proof. See the following chain

$$X \xrightarrow{\delta_X} \Delta_X \xrightarrow{\varphi\times\psi} Y \times Y.$$

So $W = \delta_X^{-1}((\varphi \times \psi)^{-1}(\Delta_Y))$ is closed. Because varieties are irreducible, the open set U is dense in X, let $\omega = \varphi - \psi$ and we get $l(x) = 0$ in U, hence $l = 0$ in X because of the continuity of l, hence $\varphi = \psi$. □

Definition 7.14. Let $\varphi : X \to Y$ be a morphism of varieties. The *graph* of φ is defined as

$$\Gamma_\varphi := \{(p, \varphi(p)) \,| p \in X\} \subset X \times Y. \tag{7.3}$$

Corollary 7.15. *Γ_φ is closed in $X \times Y$.*

Proof. Define the map

$$\begin{aligned} \varphi \times \mathrm{id}_Y : X \times Y &\to Y \times Y \\ (p, q) &\to (\varphi(p), q). \end{aligned}$$

Then we have $\Gamma_\varphi = (\varphi \times \mathrm{id}_Y)^{-1}(\Delta Y)$, so it is closed. In fact Γ_φ is isomorphic to X. □

Definition 7.16. A map $\varphi : X \to Y$ of topological spaces is called closed if $\varphi(Z)$ is closed in Y for all closed subsets $Z \subset X$.

Definition 7.17. A variety X is *complete* if the projection $p_2 : X \times Y \to Y$ is a closed map for all varieties Y.

Remark. Completeness replaces for us compactness in topology.

Example 7.18. $\mathbb{A}^1$ is not complete. Let $Z = Z(x_1 y_1 - 1) \subset \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$, then $p_2(Z) = \mathbb{A}^1 \backslash \{0\}$ is not closed in $\mathbb{A}^1$.

Proposition 7.19. *Let X be a complete variety, $\varphi : X \to Y$ be a morphism of varieties. Then $\varphi(X)$ is closed in Y.*

Proof. Since $\Gamma_\varphi \subset X \times Y$ is closed and $\varphi(X) = p_2(\Gamma_\varphi)$, thus if X is complete, $\varphi(X)$ is closed in Y. □

Theorem 7.20. *All projective varieties are complete.*

Proof. We finish the proof by two steps.

(1) Main step to show $p_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed. Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be closed, we can write it as

$$X = Z(f_1(x, y), \cdots, f_r(x, y))$$

where f_i is bihomogeneous, $x = (x_0, \cdots, x_n), y = (y_0, \cdots, y_m)$. We can assume all f_i have the same degree d in y. If f_j has a lower degree l, we can replace it by polynomials $y_0^{d-l} f_j, y_1^{d-l} f_j, \cdots, y_n^{d-l} f_j$. Fix a point $q \in \mathbb{P}^m$, then $q \in p_2(X) \Leftrightarrow Z(f_1(x, q), \cdots, f_r(x, q)) \neq \emptyset$. By the projective Nullstellensatz, this is equivalent to:

$$\begin{gathered}\forall s > 0,\ (*)\ \mathfrak{a} := \langle f_1(x, q), \cdots, f_r(x, q)\rangle \text{ does not contain}\\ \text{all monomials of degree } s \text{ in } x.\end{gathered}$$

It is trival for $s < d$, so it is enough to show:

$$\begin{gathered}\forall s \geq d, \text{ the set } X_s := \{q \in \mathbb{P}^m | q \text{ satisfies the condition } (*)\}\\ \text{is closed in } \mathbb{P}^m. \text{ Hence } p_2(X) = \bigcap_{s \geq d} X_s \text{ is closed in } \mathbb{P}^m.\end{gathered}$$

Denote monomials in x of degree s with $M_i(x)$, $i = 1, \cdots, \binom{n+s}{n}$. Denote monomials in x of degree $s - d$ with $N_j(x)$, $j = 1, \cdots, \binom{n+s-d}{n}$. The elements of degree s in $\mathfrak{a}$ are the linear span of $\{N_i(x) f_j(x, q) | i = 1, \cdots, \binom{n+s-d}{n}, j = 0, \cdots, r\}$. Define all $\{N_i(x) f_j(x, y)\}$ by $\{G_k(x, y), k = 1, \cdots, t\}$. The condition $(*)$ is equivalent to:

$$\{G_k(x, q)\} \text{ does not equal to the whole space of degree } s \text{ in } x.$$

We can write $G_k(x, y) = \sum_{i=1}^{\binom{n+s}{n}} A_{ik}(y) M_i(x)$. The dimension of the linear span of $\{G_k(x, q), k = 1, \cdots, t\}$ is the rank of the matrix $A := (A_{ik}(q))$. Thus the condition $(*)$ is equivalent to $\operatorname{rank}(A) < \binom{n+s}{n}$. Thus

$$\begin{aligned}&\{q \in \mathbb{P}^m | q \text{ satisfies the condition } (*)\}\\ = &\text{ zero set of all } \binom{n+s}{n} \times \binom{n+s}{n} \text{ minors of } A.\end{aligned}$$

Thus $p_2(X)$ is closed in $\mathbb{P}^m$.

(2) General case. First show $\mathbb{P}^n$ is completed. Let Y be a variety, we can assume $Y \subset \mathbb{P}^m$ is locally closed subvariety. Let $Z \subset \mathbb{P}^n \times Y$ be closed in $\mathbb{P} \times Y$, $\bar{Z}$ be the closure of Z in $\mathbb{P}^n \times \mathbb{P}^m$. Then $p_2(\bar{Z})$ is closed in $\mathbb{P}^m$, hence $p_2(Z) = p_2(\bar{Z} \cap (\mathbb{P}^n \times Y)) = p_2(\bar{Z}) \cap Y$ is closed in Y. Finally, let $X \subset \mathbb{P}^n$ be closed subvariety, $Z \subset X \times Y$ be closed, it follows that Z is also closed in $\mathbb{P}^n \times Y$, therefore by trival step $p_2(Z)$ is closed in Y. □

Corollary 7.21.

(1) *Let X be a projective variety, then $\mathcal{O}_X(X) = k$, i.e., any regular functions on the whole of X is constant;*
(2) *Let X be a projective variety, Y be an affine variety, then every morphism $\varphi : X \to Y$ maps X to a point.*

Proof.

(1) Let $f \in \mathcal{O}_X(X)$, then $f : X \to \mathbb{A}^1$ is a morphism. Since X is complete, we get $f(X)$ is closed in $\mathbb{A}^1$. Hence $f(X)$ is a point or $f(X) = \mathbb{A}^1$. Via embedding $\mathbb{A}^1$ in $\mathbb{P}^1$, f is a morphism from X to $\mathbb{P}^1$. Since $\mathbb{A}^1$ is not closed in $\mathbb{P}^1$, $f(X)$ is a point.
(2) By the definition of affine variety, we know Y is isomorphic to a closed subvariety of $\mathbb{A}^n$. Thus we can simply assume Y is a closed subvariety of $\mathbb{A}^n$, then we can write the morphism as $\varphi = (f_1, \cdots, f_n)$ with $f_i \in \mathcal{O}_X(X)$ for $i = 1, \cdots, n$. By (1) we just proved, we have $f_i = a_i$ for some $a_i \in k$. Thus $\varphi(x) = (a_1, \cdots, a_n) \in k^n$ for arbitrary $x \in X$.

□

Remark. From this corollary we know that morphisms from projective varieties to affine varieties are quite boring. In the following, we introduce morphisms from projective varieties to projective varieties, called Veronese embedding.

Definition 7.22 (Veronese Embedding)**.** Given fixed integer $d, n > 0$ and $N := \binom{n+d}{d} - 1$, we construct the map

$$\begin{array}{rcc} \nu_d : \mathbb{P}^n & \to & \mathbb{P}^N \\ [x_0, \cdots, x_n] & \to & [M_0, \cdots, M_N] \end{array} \tag{7.4}$$

where $M_i \in k[x_0, \cdots, x_n], 0 \leq i \leq N$ are all monomials of degree d. This is a morphism(note that monomials $x_0^d, x_1^d, \cdots, x_n^d$ have no common zero), so $\nu_d(\mathbb{P}^n)$ is a closed subvariety of $\mathbb{P}^N$.

Proposition 7.23. *$\nu_d : \mathbb{P}^n \to \nu_d(\mathbb{P}^N)$ is an isomorphism.*

Proof. The open subsets $U_i = \{[a_0, \cdots, a_n] \in \mathbb{P}^n | a_i \neq 0\}$ form an open cover of $\mathbb{P}^n$. For any monomial M_i of degree d in $x_0, \cdots, x_n$, we denote the corresponding coordinate on $\mathbb{P}^N$ by z_{M_i}. $\tilde{U_i} := \nu_d(\mathbb{P}^n) \backslash Z(z_{M_i})$ are open subsets of $\nu_d(\mathbb{P}^n)$ and form an open cover of $\nu_d(\mathbb{P}^n)$. For every piece U_i of $\mathbb{P}^n$, the map

$$\nu_d|_{U_i} : U_i \to \tilde{U_i}$$

is a morphism and the inverse of $\nu_d|_{U_i}$ is given by

$$\nu_d^{-1}|_{\tilde{U_i}} = [z_{x_0^d}, z_{x_0^{d-1}x_1}, \cdots, z_{x_0^{d-1}x_n}].$$

□

Example 7.24.

(1) Let $n = 1$, then we get the simplest Veronese embedding

$$\begin{aligned} \nu_d : \mathbb{P}^1 \to & \mathbb{P}^d, \\ [x_0, x_1] \to & [x_0^d, x_0^{d-1}x_1, \cdots, x_1^d]. \end{aligned}$$

$\nu_d(\mathbb{P}^1)$ is called a rational normal curve.
(2) Let $n = 2, d = 2$, we get the embedding

$$\nu_2 : \mathbb{P}^2 \to \quad \mathbb{P}^5$$

$\nu_2(\mathbb{P}^2)$ is called Veronese surface.

Remark. Let $F = \sum_{i=0}^{N} a_i M_i$ be a homogeneous polynomial of degree d in $x_0, \cdots, x_n$, $X \subset \mathbb{P}^n$ be a closed subvariety. Then we can get

$$\nu_d(X \cap Z(F)) = \nu_d(X) \cap Z(\sum_{i=0}^{N} a_i z_{M_i}). \tag{7.5}$$

Thus we can use the isomorphism between $\mathbb{P}^n$ and $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$ to reduce questions about intersections with hypersurfaces to intersections with hyperplanes.

Corollary 7.25. *Let $X \subset \mathbb{P}^n$ be a projective variety, $F \in k[x_0, \cdots, x_n]$ be a homogeneous polynomial of degree $d > 0$. Then we have the following properties*

1. *$X \backslash Z(F)$ is an affine variety;*
2. *if X is not a point, then $X \cap Z(F) \neq \emptyset$.*

Proof.

1. We identify $X \backslash Z(F)$ with $\nu_d(X) \backslash \nu_d(\tilde{F})$ where $\tilde{F} = \sum_{i=0}^{N} z_{M_i}$ is a linear polynomial in $z_0, \cdots, z_N$. We can apply projective transformation $[A] : \mathbb{P}^N \to \mathbb{P}^N$ such that $[A](Z(F)) = Z(z_0)$. Then we get $\mathbb{P}^N \backslash Z(z_0) = \mathbb{A}^N$, thus $X \backslash Z(z_0) = X \cap \mathbb{A}^N$ is an affine variety.
2. If $X \cap Z(F) = \emptyset$, then $X \cap Z(F) = X$ is both affine and projective, then X is a point, we get a contradiction.

□

Lemma 7.26. *Let φ, ψ be morphisms of varieties from X to Y. If there is a normally open subset $U \subset X$ such that $\varphi|_U = \psi|_U$, then $\varphi = \psi$*

Proof. Since X is a variety, we get that U is dense in X. Consider the morphism $l = \varphi - \psi$, it is 0 in U, from the continuity of the morphism, it is 0 in $\bar{U}$ the closure of U, which is X. □

8. Rational maps

Definition 8.1 (Rational Map)**.** A *rational map*

$$\varphi : X \dashrightarrow Y$$

is an equivalence class $\langle U, \varphi \rangle$ of pairs (U, φ) where $\emptyset \neq U \subset X$ is open and $\varphi : U \to Y$ is a morphism. Here $(U, \varphi) \sim (V, \psi) \Leftrightarrow \varphi|_{U \cap V} = \psi|_{U \cap V}$. We say $\langle U, \varphi \rangle$ is defined by (V, ψ) if $(V, \psi) \in \langle U, \varphi \rangle$.

Remark.

1. Let $\varphi : X \dashrightarrow Y$ be a rational map defined by (U, φ), then φ defines a morphism
$$\varphi : \mathrm{dom}\varphi \to Y$$
with $\mathrm{dom}\varphi = \bigcup_{(V,\psi) \sim (U,\varphi)} V$ open in X. Define $\varphi(p) := \psi(p)$ if $(V, \psi) \sim (U, \varphi)$ for $p \in V$. We call φ a rational map defined on $\mathrm{dom}\varphi$.
2. Rational maps $f : X \dashrightarrow \mathbb{A}^1$ are equivalent to rational functions $f \in K(X)$. If $f \in K(X)$, then $f \in \mathcal{O}_X(U)$ is a rational map from X to $\mathbb{A}^1$. Conversely, if f is a rational map from X to Y, then it is a morphism from an open set $U \subset X$ to Y, hence $f \in \mathcal{O}_X(U) \Leftrightarrow f \in K(X), \forall$ open subset $U \subset X$.

(3) Let X be a variety, $f_i \in K(X)$ for $i = 1, \cdots, n$. Then $(f_1, \cdots, f_n) : X \dashrightarrow \mathbb{A}^n$ is a rational map. It is a morphism from $\bigcap_{i=1}^{n} \mathrm{dom} f_i \to \mathbb{A}^n$.

(4) Let $X \subset \mathbb{P}^n$ be quasi-projective variety, $f_0, \cdots, f_m \in S^d(X)$ not all 0. Then $[f_0, \cdots, f_m] : X \dashrightarrow \mathbb{P}^m$ is a rational map. We can also construct it in a different way: let $F_0, \cdots, F_m \in k[x_0, \cdots, x_n]$ be homogeneous of the same degree and not all of them are in $I(X)$, then $[F_0, \cdots, F_m] : X \dashrightarrow \mathbb{P}^m$ is a rational map. For example, for $p = [0, \cdots, 0, 1] \in \mathbb{P}^n$, the projection from p

$$\Pi_p = [x_0, \cdots, x_{n-1}] : \mathbb{P}^n \to \mathbb{P}^{n-1}$$

is a rational map and a morphism from $\mathrm{dom}\Pi_p = \mathbb{P}^n \backslash \{p\}$.

Definition 8.2. A rational map

$$\varphi : X \dashrightarrow Y$$

is *dominant* if $\varphi(\mathrm{dom}\varphi)$ is dense in Y. (This is equivalent to say that φ is dominant if $\psi(V)$ is dense in Y for some pair $(V, \psi) \in \langle U, \varphi \rangle$.)

Remark. Let $\varphi : X \dashrightarrow Y$ be a rational map defined on $U \subset X$, $\psi : Y \dashrightarrow Z$ be a rational map defined on $V \subset Y$, then $U \cap \varphi^{-1}(V)$ is a nonempty open subset of X. Then we get a composition $\psi \circ \varphi : X \dashrightarrow Z$. Let $\varphi : X \dashrightarrow Y$ be a rational map, then $\varphi^* f := f \circ \varphi : X \dashrightarrow \mathbb{A}^1 \in K(X)$. It is easy to see $\varphi^* : K(Y) \to K(X)$ is a homomorphism.

Definition 8.3. A dominant rational map $\varphi : X \dashrightarrow Y$ is called a *birational map* if and only if there exists a dominant rational map $\varphi^{-1} : Y \dashrightarrow X$ such that $\varphi^{-1} \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \varphi^{-1} = \mathrm{id}_Y$ are both rational maps. X and Y are called *birational* or *birational equivalent* if there exists a birational map $\varphi : X \dashrightarrow Y$.

Definition 8.4. A variety is called *rational* if it is birational to $\mathbb{A}^n$ for some n.

Example 8.5.

(1) $\mathbb{P}^n$ is rational because $u_i : \mathbb{A}^n \to U_i$ is birational.

(2) $C = Z(x_2^2 - x_1^3) \subset \mathbb{A}^2$ is rational because the map $(t^2, t^3) : \mathbb{A}^1 \to C$ is birational.

(3) $\mathbb{P}^n \times \mathbb{P}^m$ is also rational because it contains an open subset $U_{ij} \simeq \mathbb{A}^{n+m}$.

Remark. If $F \in k[x_0, x_1, x_2]$ is a general homogeneous polynomial of degree $d \geq 3$, then $Z(F)$ is not rational.

Lemma 8.6. *Let X, Y be varieties and Y be affine, let $\varphi : X \to Y$ is a morphism. Then $\varphi(X)$ is dense in Y if and only if φ^* is injective.*

Proof. If φ is not dense, then there exists a closed subset $W \subsetneqq Y$ and $\varphi(X) \subset W$. We can write $W = Z(f_1, \cdots, f_r)$ for $f_1, \cdots, f_r \in A(X) \subset K(X)$. By possibly taking a bigger W we can write $W = Z(f)$ for some none zero element $f \in A(X)$. Now we find $\varphi^* f = f \circ \varphi = 0$, so φ^* is not injective. Conversely, if some $f \neq 0 \in K(X)$ satisfies $\varphi^* f = 0$, then $\varphi(X) \subset Z(f) \subsetneqq Y$ is not dense. □

Theorem 8.7. *Let X, Y be varieties, there is a bijection*

$$\left\{ \begin{array}{c} \textit{dominant rational maps} \\ \varphi : X \to Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} k - \textit{algebra monomorphisms} \\ \varphi^* : K(Y) \to K(X) \end{array} \right\}.$$

In particular, X and Y are birational if and only if $K(Y) \simeq K(X)$.

Proof. We only need to construct the inverse map to $\varphi \to \varphi^*$. Let $\phi : K(Y) \to K(X)$ be a k-algebra monomorphism, we want to construct a rational map $\varphi : X \to Y$ such that $\varphi^* = \phi$. Replacing Y by an open affine subset, we can now assume $Y \subset \mathbb{A}^n$ is closed. Let $y_1, \cdots, y_n \in A(Y)$ be coordinate functions, then $\phi(y_1), \cdots, \phi(y_n) \in K(X)$. We can find a nonempty open subset $U \subset X$ such that $\phi(y_i) \in \mathcal{O}_X(U)$ for all $i = 1, \cdots, n$. Then the map $x \to (\phi(y_1)(x), \cdots, \phi(y_n)(x))$ is a morphism from U to Y. In fact, we restrict ϕ to $A(Y)$, then ϕ defines an injective homomorphism from $A(Y)$ to $\mathcal{O}_X(U)$. Then by theorem 5.21, we get a morphism $\varphi : U \to Y$ and by Lemma 8.6 its image is dense in Y. Thus we find $\varphi : X \dashrightarrow Y$ which is dominant and $\varphi^* = \phi$. □

Corollary 8.8. *Let X, Y be varieties, the following statements are equivalent:*

1. *X, Y are birational;*
2. *X, Y contain open subsets isomorphic to each other;*
3. *$K(X) \simeq K(Y)$ as k-algebras.*

Proof. (1) ⇒ (2): Let $\varphi : X \dashrightarrow Y$ be a birational map with inverse $\psi : Y \dashrightarrow X$. We can check that $\psi \circ \varphi$ is the identity on $U = \mathrm{dom}\varphi \cap \varphi^{-1}(\mathrm{dom}\psi)$ and $\varphi \circ \psi$ is the identity on $V = \mathrm{dom}\psi \cap \psi^{-1}(\mathrm{dom}\varphi)$. Thus U is isomorphic to V by restrict φ on U.

(2) ⇒ (3): $K(X) \simeq K(U)$, $K(Y) \simeq K(V)$, and we know $K(U) \simeq K(V)$, thus $K(X) \simeq K(Y)$.

(3) ⇒ (1): Just the conclusion of theorem 8.7. □

9. Finite morphisms

Definition 9.1 (Finiteness)**.** Let $A \subset B$ be k-algebras. B is called *finite* over A if there exist finite many elements $b_1, \cdots, b_n \in B$ such that

$$B = b_1 A + \cdots + b_n A := \left\{ \sum b_i a_i | a_i \in A \right\}.$$

Definition 9.2 (R-module)**.** An abelian group B together with the composition $\cdot : R \cdot B \to B$ is called an *R-module* if and only if for arbitrary $r, r_1, r_2 \in R$ and arbitrary $b, b_1, b_2 \in B$, the following conditions are satisfied

1. $(r_1 \cdot r_2) \cdot b = r_1 \cdot (r_2 \cdot b)$;
2. $r_1 \cdot (b_1 + b_2) = r \cdot b_1 + r \cdot b_2$;
3. $1 \cdot b = b$.

Definition 9.3 (Finitely generated module)**.** An R-module B is called *finitely generated* if there exist $b_1, \cdots, b_n \in B$ such that

$$B = b_1 R + \cdots + b_n R.$$

Example 9.4.

1. Let R be a ring, $I \subset R$ be an ideal, then I is an R-module via multiplication in R;
2. If $I \subset R$ is an ideal and we put $A = R/I$, then A is an R-module via multiplication in quotient ring;
3. If $A \subset B$ is a subring, then B is an A-module via multiplication in B;

If A and B are k-algebras and $A \subset B$, then B is also an A-module. By definition, it is equivalent between B is a finite A-algebra and B is a finitely generated A-module. For k-algebras, it has a different definition from modules about finitely generating.

Definition 9.5. Let $A \subset B$ and A, B are k-algebra. For $b_1, \cdots, b_n \in B$, if we can denote B as

$$B = \{g(b_1, \cdots, b_n) | g \in A[x_1, \cdots, x_n]\}$$

then we call B a *finitely generated A-algebra*.

By definition, a finite A-algebra is a finitely generated A-algebra, but the converse may not true. For example, $k[x]$ is finitely generated k-algebra but not finite over k.

Proposition 9.6. *Let A, B, C be k-algebras and $A \subset B \subset C$, then we have:*

(1) *If B is finite over A and C is finite over B, then C is finite over A.If C is finite over A, then C is finite over B;*
(2) *Let $B \supset A$ be a finite A-algebra and assume B is an integral domain, then every element $x \in B$ satisfies a monic equation*
$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$
with $a_i \in A$ for $i = 0, \cdots, n-1$;
(3) *Assume b satifies a monic equation over A, then $A[b]$ is finite over A.*

Proof. (1) We can write $B = b_1A + \cdots b_mA$, $b_i \in B$ and $C = c_1B + \cdots + c_nB$, $c_i \in C$, then we get $C = \sum b_i c_j A$, hence C is finite over A. If $C = c_1A + \cdots + c_mA$, since $A \subset B$, we get $C = c_1B + \cdots + c_mB$.

(2) Assume $B = \sum\limits_{i=1}^{n} Ab_i$ for $b_1, \cdots, b_n \in B$, then for any element x in B, we can write xb_i as
$$xb_i = \sum_{j=1}^{n} d_{ij}b_j$$
with $d_{ij} \in A$. It can be rewritten as $\sum\limits_{j=1}^{n}(x\delta_{ij} - d_{ij})b_j = 0$. Thus $(b_1, \cdots, b_n)^T \in$ ker$M$ and $M = (x\delta_{ij} - d_{ij})_{i,j=1}^n$. Since B is an integral domain, we can view b_i as elements in the quotient field $Q(B)$, then we get det$M = 0$. Since det$M$ is a monic equation for x, we finish the proof.

(3) If $b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$ and $a_i \in A$ for $i = 0, \cdots, n$, then every power of b bigger than or equal to n is a linear combination of $1, b, \cdots, b^{n-1}$, i.e., $A[b] = A + Ab + \cdots + Ab^{n-1}$ is finite.

∎

Definition 9.7. Let X, Y be affine varieties. A morphism $\varphi : X \to Y$ is called *finite* if $A(X)$ is a finite $\varphi^*(A(Y))$-algebra.

Remark.

(1) (Definition of finite morphisms for general cases)By definition, we only define the finiteness of morphisms between affine varieties. In general, a morphism $\varphi : X \to Y$ of varieties is called finite if and only if Y has an open affine cover $U_1, \cdots, u_n$, $Y = U_1 \cup \cdots U_n$ such that $\varphi^{-1}(U_i) = W_i$ is affine for $i = 1, \cdots, n$ and the morphism $\varphi|_{W_i} : W_i \to U_i$ is finite.
(2) If Y is a closed subvariety of an affine variety X, the inclusion $i : Y \to X$ is a finite morphism(Because $i^* : A(X) \to A(Y)$ is surjective).
(3) Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be morphisms of affine varieties
    (a) if φ and ψ are both finite, then the composition $\psi \circ \varphi$ is finite;

(b) if $\psi \circ \varphi$ is finite, then φ is finite. In particular, if $\varphi : X \to Y$ is finite and $\varphi(X)$ is a subset of a closed subvariety W of Y, then $\varphi : X \to W$ is finite.

Theorem 9.8. *Finite morphisms are closed.*

Before we prove this theorem, we need to prove two lemmas we need to use.

Lemma 9.9. *If X is an affine variety, $I \subsetneqq A(X)$ is a proper ideal, then $Z(I) := \{p \in X | f(p) = 0, \forall f \in I\} \neq \emptyset$.*

Proof. Let

$$\pi : k[x_1, \cdots, x_n] \to A(X)$$

be a conanical map, then it is surjective. So $\pi^{-1}(I)$ is a proper ideal in $k[x_1, \cdots, x_n]$. By Nullstellensatz we know $Z(\pi^{-1}(I)) \neq \emptyset$. By definition, $Z(I) = Z(\pi^{-1}(I)) \cap X$, but $\pi^{-1}(I) \supset I(X)$, so $Z(\pi^{-1}(I)) \subset X$, hence we get $Z(I) = Z(\pi^{-1}(I)) \neq \emptyset$. □

Lemma 9.10. *Let B be a finite A-algebra and B be an integral domain, let $I \subsetneqq A$ be a proper ideal of A, then $IB \subsetneqq B$ is a proper ideal of B.*

Proof. Assume $IB = B$, since B is finite over A, we can write $B = Ab_1 + \cdots + Ab_n$, $b_1, \cdots, b_n \in B$. Then $B = IB = I(Ab_1 + \cdots + Ab_n) = Ib_1 + \cdots + Ib_n$. In particular, $b_i = \sum_{j=1}^{n} a_{ij} b_j$, $a_{ij} \in I$. Then we get $M \cdot (b_1, \cdots, b_n)^T = (0, \cdots, 0)^T$ with $M = (\delta_{ij} - a_{ij})_{i,j=1}^n$. Again view M as a matrix in $Q(B)$ we get $\det M = 0$, hence

$$0 = \det M = 1 + \sum_l c_l$$

with $c_l \in I$, it implies $1 \in I$ and hence I is not a proper ideal in A. By this contradiction we know $IB \neq B$. □

Proof of theorem 9.8. Let $\varphi : X \to Y$ be a finite morphism of affine varieties, and let W be a closed subvariety of X. We need to show $\varphi(W)$ closed in Y. Let Z be the closure of $\varphi(W)$ in Y, then we have to show $Z = \varphi(W)$. Replacing X by W and Y by Z, then our aim has changed to show a finite morphism $\varphi : X \to Y$ of varieties with dense image is surjective. As $\varphi(X)$ is dense in Y, we have that

$$\varphi^* : A(Y) \to A(X)$$

is injective, hence we can identify $A(Y)$ with the image $\varphi^*(A(Y)) \subset A(X)$. Let $Y \subset \mathbb{A}^n$, we take $x_1, \cdots, x_n$ coordinates on $\mathbb{A}^n$. For any element $p = (a_1, \cdots, a_n) \in Y$, define an ideal in $A(Y)$

$$M := \langle x_1 - a_1, \cdots, x_n - a_n \rangle.$$

Now we identify elements in M with the corresponding elements in $A(X)$, let $A(X) \cdot M$ be an ideal generated by M in $A(X)$. In addition,

$$\begin{aligned}
\varphi^{-1}(p) &= \{q \in X | \varphi(q) = p\} \\
&= \{q \in X | (x_i - a_i)(\varphi(q)) = 0, \forall i = 1, \cdots, n\} \\
&= \{q \in X | (x_i - a_i) \circ \varphi(q) = 0, \forall i = 1, \cdots, n\} \\
&= \{q \in X | \varphi^*(x_i - a_i)(q) = 0, \forall i = 1, \cdots, n\} \\
&= Z(A(X) \cdot M).
\end{aligned}$$

Thus by lemma 9.9 we only need to show $A(X)\cdot M \subsetneqq A(X)$, this is done by lemma 13.2, hence we finish the proof. $\square$

10. Noether Normalization

Theorem 10.1 (Noether Normalization)**.**

(1) *Let $Z(F) \subset \mathbb{A}^n$ be a hyperplane, then there exists a finite surjective morphism*
$$\Pi : Z(F) \to \mathbb{A}^{n-1}.$$
(2) *If $X \neq \emptyset$ is an affine variety, then there exists a finite surjective morphism*
$$\Pi : X \to \mathbb{A}^k$$
for some positive integer k.

Lemma 10.2. *Let F be a nonzero polynomial in $k[x_1, \cdots, x_n]$, then there exists a point $p = (b_1, \cdots, b_{n-1}, 1) \in \mathbb{A}^n$ s.t. $f(p) \neq 0$.*

Proof. Prove it by induction. For $n = 0$ and $n = 1$, it is obvious. Now assume $n-1$ is true , for $f \in k[x_1, \cdots, x_n]$ we can write $f = \sum\limits_i f_i x_1^i$ with $f_i \in k[x_2, \cdots, x_n]$. There exists j such that $f_j \neq 0$, by induction on $n-1$, there exists $(b_2, \cdots, b_{n-1}, 1) \in \mathbb{A}^{n-1}$ such that $f_j(b_2, \cdots, b_{n-1}, 1) \neq 0$. Then we get $g(x) := f(x, b_2, \cdots, b_{n-1}, 1)$ in $k[x]\backslash\{0\}$. Of course there exists b_1 such that $g(b_1) \neq 0$, i.e. $f(b_1, b_2, \cdots, b_{n-1}, 1) \neq 0$. $\square$

Proof of Theorem 10.1. (1) Let $F^{(d)}$ be the homogeneous part of F with the top degree, then $F^{(d)}(x_1, \cdots, x_{n-1}, 1) \neq 0$. Thus there exists $(b_1, \cdots, b_{n-1}) \in \mathbb{A}^{n-1}$ such that $F^{(d)}(b_1, \cdots, b_{n-1}, 1) \neq 0$. By change of coordinates and multiplying F by a constant, we can get $F^{(d)}(0, \cdots, 0, 1) = 1$, it is equivalent to say the coefficient of x_n^d in F is 1. Let $\Pi = (x_1, \cdots, x_{n-1}) : Z(F) \to \mathbb{A}^{n-1}$ and $[x_n] \in A(Z(F))$ be the class of the last variable x_n. Then we have
$$A(Z(F)) = \Pi^*(k[x_1, \cdots, x_{n-1}])[x_n].$$
Since $F = x_n^d + \sum\limits_{i=1}^{d-1} a_i x_n^i$ with $a_i \in k[x_1, \cdots, x_{n-1}]$, in $A(Z(F))$ we can get
$$0 = [x_n]^d + \sum_{i=1}^{d-1} \Pi^*(a_i)[x_n]^i.$$
Thus $A(Z(F))$ is finite over $\Pi^*(k[x_1, \cdots, x_{n-1}])$, i.e $\Pi : Z(F) \to \mathbb{A}^{n-1}$ is finite. Let $b = (b_1, \cdots, b_{n-1}) \in \mathbb{A}^{n-1}$, to see $\Pi^{-1}(b) \neq \emptyset$. Put $g(x) := F(b_1, \cdots, b_{n-1}, x) \in k[x]$, the coefficient of x_n of F is 1, then $g(x)$ is not constant. Hence g has a zero $b_n \in k$,
$$\Pi^{-1}(b) = \{(b_1, \cdots, b_{n-1}, b_n) | F(b_1, \cdots, b_{n-1}, b_n) = 0\} \neq \emptyset.$$
So the morphism is surjective.

(2) If $X = \mathbb{A}^n$, then it is clear. Assume $\emptyset \neq X \subsetneqq \mathbb{A}^n$ is a closed subvariety, we prove the statement by induction on n. Let $F \in I(X)\backslash\{0\}$ be irreducible. By (1) there exists a finite surjective morphism
$$\Pi : Z(F) \to \mathbb{A}^{n-1}$$
where $X \subset Z(F)$ is closed. The embedding of $i : X \to Z(F)$ is finite, so $\tilde{\Pi} = \Pi \circ i : X \to \mathbb{A}^{n-1}$ is a finite morphism. Let $Y \subset \mathbb{A}^{n-1}$ be the image of X. By induction

on n there is a finite surjective morphism $\varphi : Y \to \mathbb{A}^k$ for some k, then $\varphi \circ \Pi \circ i$ is a finite surjective morphism from $X \to \mathbb{A}^k$ for some k.

□

Lemma 10.3. *Let $\varphi : X \to Y$ be a finite surjective morphism, let Z, W be closed subvarieties of X and $Z \subsetneqq W$, then $\varphi(Z) \subsetneqq \varphi(W)$.*

Proof. We can assume $X = W$ and $Y = f(W)$, thus the lemma is equivalent to : if $Z \subsetneqq X$ is a closed subvariety, then $f(Z) \subsetneqq Y$. Let $g \in A(X) \backslash \{0\}$ such that $g|Z = 0$, since φ is finite, g satisfies a monic equation

$$g^n + \sum_{i=0}^{n-1} \varphi^*(a_i) g^i = 0$$

with $a_i \in A(Y)$. Take the one with the smallest degree n, then $\varphi^*(a_0) \neq 0$(otherwise divide by g), then we get

$$0 \neq \varphi^*(a_0) = -g(g^{n-1} + \sum_{i=1}^{n-1} \varphi^*(a_i) g^{i-1}).$$

The right hand side of the equation is in $\langle g \rangle$, thus $\varphi^*(a_0)|Z = 0 \Rightarrow a_0|_{\varphi(Z)} = 0$ $\varphi(Z) \subsetneqq Y$(if $\varphi(Z) = Y$, then $a_0 = 0 \in A(Y) \Rightarrow \varphi^*(a_0) = 0$, it makes a contradiction). □

Corollary 10.4. *Let $\varphi : X \to Y$ be a finite surjective morphism, then all the fibres of φ are finite.*

Proof. It is enough to show that every irreducible component Z of $\varphi^{-1}(y)$ is a point. Let $z \in Z$ be a point, then $\varphi(z) = y = \varphi(Z)$, by lemma 10.3 we get $\{z\} = Z$. □

11. Dimension theory

Definition 11.1 (Dimension of Varieties)**.** Let X be a variety, $\emptyset \neq X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n = X$ be a chain of irreducible closed subsets on X, we call it a *chain* in X, n is called length of the chain. The *dimension* of X is the maximal n such that there exists a chain of length n in X or ∞ if this maximum does not exist.

Lemma 11.2.

(1) *Let $Y \subset X$ be a closed subvariety, then* $\dim Y \leq \dim X$*. If $Y \subsetneqq X$ and* $\dim Y < \infty$*, then* $\dim Y < \dim X$*.*

(2) *Let $f : X \to Y$ be a surjective closd morphism, then* $\dim X \geq \dim Y$*.*

Proof. (1) Let $Y_0 \subsetneqq Y_1 \subsetneqq \cdots \subsetneqq Y_k$ be a chain in Y, it is also a chain in X, thus $\dim X \supset \dim Y$. If $Y \subsetneqq X$, then $Y_0 \subsetneqq Y_1 \subsetneqq \cdots \subsetneqq Y_k \subsetneqq X$ is a chain in X, hence if the dimension of Y is finite, we get $\dim Y < \dim X$.

(2) Let $Y_0 \subsetneqq \cdots \subsetneqq Y_n$ be a chain in Y, we need to show that there exists a chain $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n$ in X such that $\varphi(X_i) = Y_i$ for all i. Use induction on n, it is obvious for $n = 0$. Let $Z_1, \cdots, Z_r$ be irreducible components of $f^{-1}(Y_{n-1})$, then $\bigcup_{i=1}^{r} \varphi(Z_i) = Y_{n-1}$,$f(Z_i)$ are closed, Y_{n-1} is irreducible. Thus one of the $f(Z_i)$ is equal to Y_{n-1}. Since $\varphi : Z_i \to Y_{n-1}$ is a surjective closed morphism, by induction we get a chain $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_{n-1} = Z_i$ in X with $f(X_i) = Y_i$ for $i = 0, \cdots, n-1$, then $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_{n-1} \subsetneqq X_n = X$ is a chian with $f(X_i) = Y_i$ for all i.

□

Theorem 11.3. *Let $\varphi : X \to Y$ be a finite surjective morphism of varieties, then* $\dim X = \dim Y$.

Proof. We already know $\dim X \geq \dim Y$ because φ is surjective and closed. To show $\dim Y \geq \dim X$, let $X_0 \subsetneqq \cdots, \subsetneqq X_n$ be a chain in X, for i let $Y_i = f(X_i)$, then by lemma 10.3 $Y_0 \subsetneqq Y_1 \subsetneqq \cdots \subsetneqq Y_n$ is also a chain in Y. □

Theorem 11.4.

(1) $\dim \mathbb{A}^n = n$.
(2) *Let $F \in k[x_1, \cdots, x_n] \backslash k$ be a irreducible polynomial, then* $\dim Z(F) = n-1$.
(3) *Conversely any subvariety $X \subset \mathbb{A}^n$ of dimension $n-1$ is a hypersurface, i.e. $X = Z(F)$ with F irreducible.*

Proof. We first prove $\dim Z(F) = \dim \mathbb{A}^{n-1}$ for $F \in k[x_1, \cdots, x_n] \backslash k$. By theorem 10.1 we know therre exists a surjective finite morphism from $Z(F)$ to $\mathbb{A}^{n-1}$, thus $\dim Z(F) = \dim \mathbb{A}^{n-1}$.

(1) Let $Z_i = Z(x_{i+1}, \cdots, x_n) \subset \mathbb{A}^n$, then $Z_i \simeq \mathbb{A}^i$ and thus
$$Z_0 \subsetneqq Z_1 \subsetneqq Z_2 \subsetneqq \cdots \subsetneqq Z_n = \mathbb{A}^n$$
is a chain in $\mathbb{A}^n$ of length n, it implies $\dim \mathbb{A}^n \geq n$. Now we prove the opposite inequality by induction on n. For $n = 0$, it is true, let $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_{k-1} = X \subsetneqq \mathbb{A}^n$ be a chain in $\mathbb{A}^n$. Then $X \subsetneqq \mathbb{A}^n$ is a closed subvariety, we can choose $F \in I(X)$ and F is irreducible, then $X \subset Z(F)$. Thus $k-1 \leq \dim Z(F) = \dim \mathbb{A}^{n-1} = n-1$ by induction. Since the chain we choose is arbitrary, we get $\dim \mathbb{A}^n \leq n$. Hence $\dim \mathbb{A}^n = n$.
(2) It follows from (1) immediately.
(3) Let $\emptyset \neq X \subsetneqq \mathbb{A}^n$ and $\dim X = n-1$, then there exists $F \in I(X) \backslash k$ being irreducible, thus $X \subset Z(F)$, X and $Z(F)$ are both irreducible of the same dimension, hence $X = Z(F)$.

□

Remark. In (2) of Theorem 11.4, we can drop the assumption that F on X is irreducible.

Corollary 11.5. *Every affine variety is finite dimensional.*

Proposition 11.6. *Let $X \subset \mathbb{A}^N$ be an affine variety of dimension n and $F \in k[x_1, \cdots, x_N] \backslash I(X)$. If $Z(F) \cap X \neq \emptyset$, then $\dim(Z(F) \cap X) = n-1$.($Z(F) \cap X$ may not be irreducible).*

Proof. We need to show for all irreducible components Y_i of $Z(F) \cap X$, $\dim Y_i \leq n-1$ and there exists a component Y_j with $\dim Y_j = n-1$(later we will show that all irreducible components have dimension $n-1$). By Noether normalization theorem, there is a finite surjective morphism $\Pi : X \to \mathbb{A}^n$. Identify $k[x_1, \cdots, x_n]$ with $\Pi^*(k[x_1, \cdots, x_n]) \subset A(X)$. Let $\bar{F}$ be the class of F in $A(X)$, there exists a nonzero polynomial
$$H = x_{n+1}^d + \sum_{i=0}^{d-1} a_i x_{n+1}^i$$
with $a_i \in k[x_1, \cdots, x_n]$ such that $H(x_1, \cdots, x_n, \bar{F}) = 0 \in A(X)$. Replacing H by an irreducible factor if necessary, we can assume H is irreducible. Let $\varphi :=$

$(\Pi, F) : X \to \mathbb{A}^{n+1}$. $\Pi = (x_1, \cdots, x_n) \circ \varphi$ is finite, thus φ is finite. By definition $\varphi(X) \subset Z(H)$, then $\varphi(X)$ is a closed subvariety of dimension n in $Z(H)$. Thus $\varphi(X) = Z(H)$, $\varphi : X \to Z(H)$ is a finite surjective morphism. By definition, $Z(F) \cap X = \varphi^{-1}(Z(H, x_{n+1})) = \varphi^{-1}(Z(a_0) \times \{0\})$, thus $\dim(Z(F) \cap X) = \dim Z(a_0)$ where $a_0 \in k[x_1, \cdots, x_n]$. If a_0 is constant, then $Z(F) \cap X = \emptyset$, contradict with the condition, so drop it. Now we know a_0 is a nonconstant polynomial , hence $\dim Z(a_0) = n - 1$. □

Theorem 11.7. *Let X be a variety, $\emptyset \neq U \subset X$, U is an open subset of X. Then* $\dim U = \dim X$*.*

Proof. Let $U_0 \subsetneqq U_1 \subsetneqq \cdots \subsetneqq U_n = U$ be a chain in U, let $X_i = \bar{U}$ the closure of U_i in X. By definition $U_i = U \cap X_i$, thus

$$X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n = X$$

is a chain in X, thus $\dim U \leq \dim X$.

Let $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n = X$ be a chain of largest length in X and $X_0 = \{x_0\}$ be a point, let $W \subset X$ be an open subset with $x_0 \in W$. Then we set $W_i = X_i \cap W$ for all i. Since W_{i+1} is dense in X_{i+1}, we have $W_{i+1} \supsetneqq W_i$ for all i. Thus $W_0 = \{x_0\} \subsetneqq W_1 \subsetneqq \cdots \subsetneqq W_n$ is a chain in W, we get $\dim X = \dim W$. Thus we can replace X by W and U by $W \cap U$. Now we reduce to the case X is affine.

(1) If $X = \mathbb{A}^n$, let x_0 be a point in U, X_i be affine linear subspaces containing X_{i-1} for all i. Put $U_i = X_i \cap U$, $U_0 \subsetneqq U_1 \subsetneqq \cdots \subsetneqq U_n$ is a chain in U, then $\dim U = n = \dim X$.
(2) If X is affine, there exists a finite surjective morphism $\varphi : X \to \mathbb{A}^n$. $\varphi(X \backslash U) \subsetneqq \mathbb{A}^n$ is closed, let $f \in I(\varphi(X \backslash U))$ and $V = \mathbb{A}^n \backslash Z(f)$, V is open and dense in $\mathbb{A}^n$, $\dim V = n$. Let $W = \varphi^{-1}(V) \subset X$, then $\varphi|_W : W \to V$ is surjective and closed, thus $\dim W \geq \dim V = n$, but $U \supset W$, hence $\dim U \geq \dim W \geq n$.

□

Corollary 11.8. *All varieties are finite dimensional.*

Corollary 11.9. *If X and Y are birational, then* $\dim X = \dim Y$*.*

Corollary 11.10.

(1) $\dim \mathbb{P}^n = n$.
(2) *If $F \in k[x_0, \cdots, x_n]$ is a homogeneous polynomial of positive degree, then* $\dim Z(F) = n - 1$.
(3) *If $X \subset \mathbb{P}^n$ is a closed subvariety of dimension $n-1$, then $X = Z(F)$ for some homogeneous polynomial $F \in k[x_0, \cdots, x_n]$.*

Proof. (1) It is obvious since $U_i \simeq \mathbb{A}^n$ is open dense in $\mathbb{P}^n$.
(2) By projective transformation we can set $Z(F) \not\subset H_\infty$, then $Z(F) \cap \mathbb{A}^n = Z(F(1, x_1, \cdots, x_n))$. It has dimension $n-1$ and is open in $Z(F)$, so $\dim Z(F) = n - 1$.
(3) Same as the affine condition in theorem 11.4.

□

Theorem 11.11. *Let $X \subset \mathbb{A}^n$ be an affine variety, $F \in k[x_1, \cdots, x_n] \backslash I(X)$, then every irreducible component(if there is any) of $Z(F) \cap X$ has dimension* $\dim X - 1$*.*

Proof. Let Z be a irreducible component of $Z(F) \cap X$. Take W be the union of all the other irreducible components of $Z(F) \cap X$. Take $g \in I(W) \backslash I(Z)$ and $U := X \backslash Z(g)$, then U can be viewed as an affine variety in $\mathbb{A}^{n+1}$. Since $Z(g) \supset W$, we get $U \subset Z$. Hence $U \cap Z(F) = U \cap Z$. Viewing F as a polynomial function on U(since $U = X \backslash Z(g)$ is open and dense in X, F is not zero in U, otherwise it is zero in the whole set X, contradicts with $F \notin I(X)$), then we get $\dim Z = \dim(Z \cap U) = \dim U - 1 = \dim X - 1$. The second equality $\dim(Z \cap U) = \dim U - 1$ is from proposition 11.6 by viewing it in $\mathbb{A}^{n+1}$ □

Proposition 11.12. *Let $\varphi : X \to Y$ be a morphism of varieties. Assume there exists a nonempty open subset $U \subset Y$ such that for all $p \in U$, $\dim(\varphi^{-1}(p)) = n$, then we have*

$$\dim X = \dim Y + n.$$

Proof. We prove the statement by induction over $\dim Y$. If Y is a point, then it is trivial. If $\dim Y > 0$, replacing Y by an open affine subset V(i.e. replace Y by $Y \cap \mathbb{A}^k$ for some k) and X by an open affine subset of $\varphi^{-1}(V)$, we can assume X, Y are both affine by theorem 11.7. In fact, $X \subset \mathbb{A}^l$ and $Y \subset \mathbb{A}^m$ for some l and some m, are closed affine subvarieties. We can write $\varphi = (F_1, \cdots, F_m)$ with $F_i \in k[x_0, \cdots, x_l]$. Let $g \in k[x_1, \cdots, x_m]$ such that $\emptyset \neq Z(g) \cap Y \neq Y$, then we set $Y' = Z(g) \cap Y$ and $X' = \varphi^{-1}(Y')$. By definition $X' = X \cap Z(g(F_1, \cdots, F_m))$ and it is not empty since its image Y' is not empty. For any point $p \in Y'$, $\varphi^{-1}(p)$ in X is also in X', hence the dimension of fibres is still equal to n. By induction any irreducible component $\tilde{X}$ of X' has the relation $\dim\tilde{X} = \dim\tilde{Y} + n$ with the corresponding $\tilde{Y}$ of Y', hence $\dim X' = \dim Y' + n$. Since $\dim Y = \dim Y' + 1$ and $\dim X = \dim X' + 1$, we get $\dim X = \dim Y + n$. □

Theorem 11.13 (without proof)**.** *Let $\varphi : X \to Y$ be a surjective morphism, assume* $\dim X = \dim Y + n$*, then*

1. *for all points $p \in Y$, $\dim(\varphi^{-1}(p)) \geq n$;*
2. *there is a nonempty open subset $U \subset Y$ such that for all $p \in U$, $\dim\varphi^{-1}(p) = n$.*

Example 11.14.

1. $\dim(X \times Y) = \dim X + \dim Y$. Consider the projection map $p : X \times Y \to Y$, the inverse $p^{-1}(q) = X \times \{q\}$ has the dimension $\dim X$.
2. Let $X \subset \mathbb{P}^n$ be a projective variety, then we have

$$\dim C(X) = \dim X + 1.$$

Consider the map $\Pi : C(X) \backslash \{0\} \to X$ that maps $(x_0, \cdots, x_n)$ to $[x_0, \cdots, x_n]$.

Definition 11.15. If $X \subset \mathbb{P}^n$ has dimension $n - k$, we say *codimension* $\operatorname{codim} X = k$.

Theorem 11.16.

1. *Let $X, Y \subset \mathbb{A}^n$ be closed subvarieties. Every irreducible component Z of $X \cap Y$ has dimension* $\dim Z \geq \dim X + \dim Y - n$.
2. *Let $X, Y \subset \mathbb{P}^n$ be closed subvarieties, every irreducible component Z of $X \cap Y$ has dimension* $\dim Z \geq \dim X + \dim Y - n$. *In particular, if* $\dim X + \dim Y \geq n$, *then* $X \cap Y \neq \emptyset$.

Remark. The fact that $X \cap Y \neq \emptyset$ if $\dim X + \dim Y \geq n$ is special for projective space. This can be used to prove that $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$. If $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathbb{P}^2$, then for any 1-dimension subvarieties $X, Y \subset \mathbb{P}^1 \times \mathbb{P}^1$, we have $X \cap Y \neq \emptyset$. But for $X = \{p\} \times \mathbb{P}^1$ and $Y = \{q\} \times \mathbb{P}^1$ such that $p \neq q$, we have $X \cap Y = \emptyset$, which contradicts to the theorem, so $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Proof of Theorem 11.16.

(1) *Trick: take the diagonal to reduce to the intersection with hyperplanes*

$$\delta^{-1}(X \times Y) = \delta^{-1}((X \times Y) \cap \Delta) = X \cap Y.$$

Thus $X \cap Y \simeq (X \times Y) \cap \Delta \subset \mathbb{A}^{2n}$. *In fact,*

$$\Delta = Z(x_1 - y_1, \cdots, x_n - y_n).$$

By theorem 11.11, $\dim(Z \cap Z(f)) \geq \dim Z - 1$ *where* Z *is a variety. By induction, we can get* $\dim(X \cap Y) = \dim((X \times Y) \cap \Delta) \geq \dim X + \dim Y - n$.

(2) *Reduce to (1) by using affine cones. By definition,* $C(X) \cap C(Y) = C(X \cap Y)$, $\dim C(X) = \dim X + 1$ *and same for* Y *and* $X \cap Y$. *Let* Z *be a irreducible component of* $X \cap Y$, *then* $C(Z)$ *is a irreducible component of* $C(X \cap Y)$. *By using the conclusion in (1) we get*

$$\begin{aligned} \dim Z &= \dim C(Z) - 1 \\ &\geq \dim C(X) + \dim C(Y) - (n+1) - 1 \\ &= \dim X + \dim Y - n. \end{aligned}$$

Assume $\dim X + \dim Y \geq n$, *we know* $C(X) \cap C(Y) \neq \emptyset$ *because* $0 \in C(X) \cap C(Y)$. *Every* Z *irreducible component* $C(X) \cap C(Y)$ *satisfies* $\dim Z = \dim(C(X) \cap C(Y)) \geq \dim C(X) + \dim C(Y) - (n+1) \geq 1$. *Thus* $C(X) \cap C(Y) \neq \{0\} \Rightarrow X \cap Y \neq \emptyset$. □

We know $\dim X = \dim Y$ *if* X *and* Y *are birational, and* $K(X) \simeq K(Y)$ *if* X *is birational to* Y. *Thus* $\dim X$ *must be determined by* $K(X)$. *We will see* $\dim X$ *is equal to the transcendence degree of* $K(X)$ *over* k.

Definition 11.17 (Field Extension and Finitely generated Field Extension)**.** Let K/k be a field extension. For $a_1, \cdots, a_n \in K$, denote $k(a_1, \cdots, a_n)$ as the smallest subfield of K containing k and $a_1, \cdots, a_n$. This is called *field extension* over k by $a_1, \cdots, a_n$. If there are $a_1, \cdots, a_n \in K$ such that $K = k(a_1, \cdots, a_n)$, we say K/k is *finitely generated.*

Definition 11.18 (Algebraically Independent sets)**.** Let K/k be a finitely generated field extension, elements $b_1, \cdots, b_n \in K$ are called *algebraically independent* over k if there is no polynomial $f \in k[x_1, \cdots, x_n]$ such that $f(b_1, \cdots, b_n) = 0$. In particular, if $b \in K$ is algebraically independent over k, then b is called *transcendent* over k.

Let $k(x_1, \cdots, x_n)$ *be a field of rational functions in* n *indeterminants, it is easy to see* $k(b_1, \cdots, b_n) \simeq k(x_1, \cdots, x_n)$ *if* $b_1, \cdots, b_n$ *are algebraically independent over* k.

Definition 11.19 (Transcendence Basis)**.** A maximal set of algebraically independent elements of K over k is called a *transcendence basis.*

Theorem 11.20 (without proof)**.** *Let* $K = k(a_1, \cdots, a_n)/k$ *be a finitely generated field extension, then*

(1) *there exists a transcendence basis of K/k, it can be chosen as a subset of $\{a_1, \cdots, a_n\}$;*
(2) *every transcendence basis of elements of K/k has the same number of elements, called the transcendence degree denoted by* $\mathrm{trdeg} K(X)/k$ *;*
(3) *let $b_1, \cdots, b_r$ be a transcendence basis of K/k, then $K/k(b_1, \cdots, b_r)$ is a finite algebraic extension.*

Theorem 11.21. *Every variety X is birational to a hypersurface in $\mathbb{A}^{\dim X+1}$.*

This theorem may be proved soon after.

Theorem 11.22. *Let X be a variety, then*

$$\dim X = \mathrm{trdeg} K(X)/k.$$

Proof. By theorem 11.21, we can assume $X = Z(F) \subset \mathbb{A}^n$ is a hypersurface, $F \in k[x_1, \cdots, x_n]$ is irreducible. We know $\dim X = n-1$. To show $\mathrm{trdeg} K(X)/k = n-1$, let $y_1, \cdots, y_n \in A(X)$ be coordinate functions. Then $K(X) = k(y_1, \cdots, y_n)$, $F(y_1, \cdots, y_n) \in A(X) = k[x_1, \cdots, x_n]/\langle F\rangle$ and $F(y_1, \cdots, y_n) = 0$ since $X = Z(F)$. Thus $y_1, \cdots, y_n$ are algebraically dependent. It follows that $\mathrm{trdeg} K(X)/k \leq n-1$. To show the equality, we assume the last variable x_n occurs in F, then we can get $y_1, \cdots, y_{n-1}$ are algebraically independent. Otherwise, there exists a nonzero element $G \in k[x_1, \cdots, x_{n-1}]$ with $G(y_1, \cdots, y_{n-1}) = 0$, then $G(y_1, \cdots, y_{n-1}) \in \langle F\rangle$. But it is impossible because F contains $x_n \Rightarrow G$ contains x_n. Thus $\mathrm{trdeg} K(X)/k = n-1$. □

Theorem 11.23 (Existence of a Primitive Element)**.** *Let k be a field of characteristic 0, L/k is a finite field extension. Then $\exists b \in L$ such that $L = k(b)$.*

Proof of Theorem 11.21. $K(X)$ is function field of X, let $a_1, \cdots, a_r$ be a transcendence basis of $K(X)/k$, then $K(X)/k(a_1, \cdots, a_r)$ is a finite algebraic extension. By theorem 11.23, there exists a primitive element $b \in K(X)$ such that $K(X) = k(a_, \cdots, a_r)(b)$ and b is algebraic over $k(a_1, \cdots, a_r)$. Since b is algebraic, there exists a polynomial $F \in k(a_1, \cdots, a_r)[x]$ such that $F(b) = 0$. Write

$$F = \sum_l \frac{G_l(a_1, \cdots, a_r)}{H_l(a_1, \cdots, a_r)} x^l$$

where $G_l(x_1, \cdots, x_r), H_l(x_1, \cdots, x_r) \in k[x_1, \cdots, x_r]$.

Now we view it as $F(x_1, \cdots, x_r, x) \in k(x_1, \cdots, x_r, x)$. Multiply F by producting H_l's and then divide it by the greatest common devisor of the new coefficients. We get $f = \tilde{h}F \in k[x_1, \cdots, x_r, x]$, it is a primitive polynomial. Let $Y = Z(f) \subset \mathbb{A}^{r+1}$, it is a irreducible hypersurface. Then $A(Y) = k[x_1, \cdots, x_r, x]/\langle f\rangle$, $K(Y) = Q(k[x_1, \cdots, x_r, x]/\langle f\rangle) \simeq Q(k[a_1, \cdots, a_r, x]/\langle f(x)\rangle) \simeq k(a_1, \cdots, a_r)[x]/\langle f\rangle \simeq k(a_1, \cdots, a_r)(b) \simeq K(X)$. Then X is birational to Y. □

This proof also implies $\dim X = \mathrm{trdeg} K(X)$.

12. Tangent Space, Singular and Nonsingular Points

First we talk about cases of hypersurfaces in $\mathbb{A}^n$.

Definition 12.1. Let $X = Z(f) \subset \mathbb{A}^n$ be a hypersurface, assume $I(X) = \langle f \rangle$. A point $p \in X$ is called a *singular point* if and only if $\frac{\partial f}{\partial x_i}(p) = 0$ for $i = 1, \cdots, n$. Otherwise, p is called a *nonsingular point.* Let

$$X_{\text{reg}} := \{p \in X | p \text{ is nonsingular.}\}.$$

X is called *smooth or nonsingular* if and only if $X = X_{\text{reg}}$.

Example 12.2. (1) $X = Z(y - x^2)$ is nonsingular.
(2) $X = Z(y^2 - x^2(x+1))$ has a singular point $(0,0)$.
(3) $X = Z(y^2 - x^3)$ has a nonsingular point $(0,0)$.

Proposition 12.3. *Let $X \subset \mathbb{A}^n$ be an irreducible hypersurface and char$k = 0$, then X_{reg} is open and dense in X.*

Proof. Let $F \in k[x_1, \cdots, x_n]$ be irreducible such that $X = Z(F)$, then $I(X) = \langle F \rangle$. Define

$$X_{\text{sing}} := \{\text{singular points of } X\}.$$

By definition $X_{\text{sing}} = Z(F, \frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}) \subset X$ is closed. Since $Z(F)$ is irreducible, the only thing we have to show is $X \neq X_{\text{sing}}$. Assume $X = X_{\text{sing}} \Rightarrow Z(\frac{\partial F}{\partial x_i}) \supset X$ $\forall i = 1, \cdots, n \Rightarrow \frac{\partial F}{\partial x_i} = 0$ $\forall i = 1, \cdots, n$. Since char$k = 0$, we get F is constant, it is impossible. $\square$

Second we talk about cases of affine algebraic sets.

Definition 12.4. Let $f \in k[x_1, \cdots, x_n]$, $p \in \mathbb{A}^n$. The *differential* of f at p is defined as

$$\mathrm{d}_p f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) x_i.$$

Let $X \subset \mathbb{A}^n$ be an affine algebraic set, the *tangent space* to X at $p \in X$ is defined as

$$T_p(X) = Z(\mathrm{d}_p f | f \in I(X)).$$

$p \in X$ is called *nonsingular* if

$$\dim T_p(X) = \dim_p X$$

where $\dim_p X$ is the maximum of dimensions of irreducible components of X passing through p.

Remark. If $I(X) = \langle f_1, \cdots, f_r \rangle$, then $T_p(X) = Z(\mathrm{d}_p f_1, \cdots, d_p f_r)$. By definition, $T_p(X) \subset Z(\mathrm{d}_p f_1, \cdots, \mathrm{d}_p f_r)$. If $h \in I(X)$, we can write $h = \sum_{i=1}^{r} h_i f_i$ with $h_i \in k[x_1, \cdots, x_n]$. Using Leibniz rule we get

$$\mathrm{d}_p h = \sum_{i=1}^{r} (\mathrm{d}_p h_i \cdot f_i(p) + h_i(p) \cdot \mathrm{d}_p f_i).$$

Since $f_i(p) = 0$, we get $\mathrm{d}_p h \in \langle \mathrm{d}_p f_1, \cdots, \mathrm{d}_p f_r \rangle$. Hence $T_p(X) = Z(\mathrm{d}_p f_1, \cdots, d_p f_r)$.

Example 12.5. If $X = Z(F) \subset \mathbb{A}^n$ and $I(X) = \langle F \rangle$, then $T_p(X) = Z(\mathrm{d}_p F)$ and $\mathrm{d}_p F = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} x_i$. Let $\frac{\partial F}{\partial x_i}(p) = 0$ $\forall i = 1, \cdots, n$ for some point $p \in X$, $\forall i = 1, \cdots, n$, then $T_p(X) = \mathbb{A}^n$. Since $\dim T_p(X) \neq \dim_p X$, p is a singular point. If $\frac{\partial F}{\partial x_i}(p) \neq 0$ for some i, then $\dim T_p(X) = n - 1$ and p is nonsingular.

Third we talk about cases of general affine varieties.

Definition 12.6 (Jacobian)**.** *Jacobian* of $f_1, \cdots, f_r \in k[x_1, \cdots, x_n]$ is a matrix defined as

$$J(f_1, \cdots, f_r) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \frac{\partial f_r}{\partial x_2} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}.$$

Definition 12.7. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be closed subvarieties. Let $p \in X$, $q \in Y$ and $\varphi = (f_1, \cdots, f_m) : X \to Y$ with $f_i \in k[x_1, \cdots, x_n]$ for $i = 1, \cdots, m$. Assume $\varphi(p) = q$. The *differential* of φ at p is

$$\mathrm{d}_p\varphi = (\mathrm{d}_p f_1, \cdots, \mathrm{d}_p f_m).$$

One can verify that $\mathrm{d}_p\varphi$ *maps* $T_p(X)$ *into* $T_q(Y)$. *We can write* $T_p(X) = \ker(J(f_1, \cdots, f_m)(p))$. $\mathrm{d}_p\varphi$ *can also be written as* $J(f_1, \cdots, f_m) \cdot x$.

Proposition 12.8.

(1) $\mathrm{d}_p\mathrm{Id} = \mathrm{Id}$.
(2) $\mathrm{d}_p(\psi \circ \varphi) = J_{\varphi(p)} \cdot \mathrm{d}_p\varphi$.

At last we talk about tengent spaces for general varieties.

Definition 12.9. Let X be a variety, $p \in X$ be a point. The *tangent space* $T_p(X)$ is

$$T_p(X) := (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$$

where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$, the symbol $*$ denotes the dual of vector space. In other words,

$$T_p(X) = \{k \text{ linear maps } \nu : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k\}$$

or

$$T_p(X) = \{k \text{ linear maps } \nu : \mathfrak{m}_p \to k \text{ with } \nu|_{\mathfrak{m}_p^2} = 0\}$$

$p \in X$ is called nonsingular if $\dim T_p(X) = \dim X$. Similarly we have definition of X_{sing} and X_{reg}. If $X = X_{\mathrm{reg}}$, X is called nonsingular or regular.

In the previous lecture, we have defined tangent spaces for affine algebraic sets and for general cases. Now we want to prove that two definitions are identical in affine cases. Recall two definitions

Definition 12.10 (Affine Cases)**.** Let $f \in k[x_1, \cdots, x_n]$ and $p \in \mathbb{A}^n$, the differential of f at p is

$$\mathrm{d}_p f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) \cdot x_i.$$

Let $X \subset \mathbb{A}^n$ be an affine algebraic set. The tangent space to X at $p \in X$ is

$$t_p(X) := Z(\mathrm{d}_p f | f \in I(X)).$$

Definition 12.11 (General Cases)**.** Let X be a variety, $p \in X$ be a point. The tangent space $T_p(X)$ is

$$T_p(X) := (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$$

where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$, the symbol $*$ denotes the dual of vector space. In other words,

$$T_p(X) = \{k \text{ linear maps } \nu : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k\}$$

or

$$T_p(X) = \{k \text{ linear maps } \nu : \mathfrak{m}_p \to k \text{ with } \nu|_{\mathfrak{m}_p^2} = 0\}.$$

For the moment, let $X \subset \mathbb{A}^n$ be an affine variety.

Definition 12.12. If $f \in A(X)$, $a = (a_1, \cdots, a_n) \in t_p(X)$, we define

$$\mathrm{d}_p f(a) := \mathrm{d}_p F(a)$$

where $[F] = f$, $f \in k[x_1, \cdots, x_n]$ and $\mathrm{d}_p F(a) = \sum_i^n \frac{\partial F}{\partial x_i}(p) \cdot a_i$.

If $h = \frac{f}{g} \in \mathrm{p}$, then $f, g \in A(X)$, $g(p) \neq 0$ and $f(p) = 0$. We define

$$\mathrm{d}_p h(a) = \frac{\mathrm{d}_p f(a)}{g(p)}.$$

Thus for $a \in t_p(X)$, we have defined a linear map

$$\partial_a : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k.$$

We define a linear map

$$\delta : t_p(X) \to T_p(X).$$

If we can prove δ is an isomorphism, then we can identify two definitions.

Theorem 12.13.
(1) *δ is an isomorphism.*
(2) *Usinng δ to identify $t_p(X)$ and $T_p(X)$, the two definitions of $\mathrm{d}_p\varphi$ for morphism $\varphi : X \to Y$ are identified.*

Proof. Let $p \in X \subset \mathbb{A}^n$, $t_i := [x_i - p_i] \in \mathfrak{m}_p$

Injectivity: For any $a \in t_p(X)$, we have $\delta(a) = \partial_a$, it is easy to check that $\partial_a(t_i) = a_i$. If $\partial_a = 0$, then $a_i = 0$ for $i = 1, \cdots, n$, then $a = 0$. Hence δ is injective.

Surjectivity: To show surjectivity, it is enough to show $t_1, \cdots, t_n$ generate $\mathfrak{m}_p/\mathfrak{m}_p^2$ as a vector space over k, If it is true , then for any $\nu : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$ let $a_i = \nu(t_i)$, we get $\nu = \delta(a)$ where $a = (a_1, \cdots, a_n)$, and it is easy to check that $a \in t_p(X)$. Now let's prove that $t_1, \cdots, t_n$ generate $\mathfrak{m}_p/\mathfrak{m}_p^2$. For $f = \frac{g}{h} \in \mathfrak{m}_p$, $f - \frac{g}{h(p)} = \frac{g\cdot(h(p)-h)}{h\cdot h(p)} \in \mathfrak{m}_p^2$, thus $f = \frac{g}{h(p)}$ in $\mathfrak{m}_p/\mathfrak{m}_p^2$. Since $\frac{g}{h(p)} \in A(X)$, we know that $\mathfrak{m}_p/\mathfrak{m}_p^2$ is generated by elements in $A(X)$.Then $\mathfrak{m}_p/\mathfrak{m}_p^2 = k[t_1, \cdots, t_n]$. For monomials of degree larger than 2 in t_i, it lies in $\mathfrak{m}_p^2$. Thus $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a vector space generated by $t_1, \cdots, t_n$. $\square$

Theorem 12.14. *Let X be a variety:*
(1) *X_{reg} is an open dense subset of X;*
(2) *for all $p \in X$, $\dim T_p X \geq \dim X$.*

Proof. Any variety X has an open cover by affine varieties. The theorem is true if it is true for each open set in the cover. Thus we can assume $X \subset \mathbb{A}^n$ is a closed subvariety. Let $I(X) = \langle f_1, \cdots, f_r \rangle$, $f_i \in k[x_1, \cdots, x_n]$. Then we get

$$\dim T_p(X) = n - \mathrm{rank}(J(f_1, \cdots, f_r)(p)).$$

this formula implies that $\dim T_p(X) \geq d$ if and only if all the $n-d+1$ minors are equal to 0. Thus for all d, $X_d := \{p \in X | \dim T_p X \geq d\}$ is closed in X. Then we get a chain

$$X_0 \supset X_1 \supset \cdots \supset X_d \supset X_{d+1} \supset .$$

Choose the largest d such that $X_d = X$ and put $X^0 := X \backslash X_{d+1}$. X^0 is open and dense in X. Then we know $\dim T_p(X) \geq d$ for all $p \in X$ and $\dim T_p(X) = d$ for all $p \in X^0$. Now we only have to show $d = \dim(X)$. Since X is birational to a hypersurface Y in $\mathbb{A}^{\dim(X)+1}$, there is a nonempty open subset $U \subset X$ that is isomorphic to an open subset of Y_{reg}. Then $\dim T_p(X) = \dim X$ for all $p \in U$. Thus for all $p \in X^0 \cap U$, $\dim X = \dim T_p(X) = d$. Thus $\dim X = d$. □

Corollary 12.15.

(1) *Let $X \subset \mathbb{A}^n$ be an affine variety, $I(X) = \langle f_1, \cdots, f_r \rangle$. Then the following is equivalent:*

$$p \in X \textit{ is nonsingular} \Leftrightarrow \operatorname{rank}(J(f_1, \cdots, f_r)(p)) \geq n - \dim X.$$

(2) *Let $X \subset \mathbb{P}^n$ be a projective variety. Assume $I(X) = \langle F_1, \cdots, F_r \rangle$, where $F_1, \cdots, F_r$ are homogeneous. Then the following is equivalent:*

$$p \in X \textit{ is nonsingular} \Leftrightarrow \operatorname{rank}(J(F_1, \cdots, F_r)(p)) \geq n - \dim X.$$

Proof. The first term is obvious. To get the second term, assume $p \in U_0 \cap X$, i.e. p can be written as $[1, a_1, \cdots, a_n]$. Then p is nonsingular if and only if $a = (a_1, \cdots, a_n)$ is nonsingular in $U_0 \cap X$. Let $f_i(x_1, \cdots, x_n) = F(1, x_1, \cdots, x_n)$ for $i = 1, \cdots, n$. Via the first term we only need to show that the rank of $J(F_1, \cdots, F_r)$ is equal to the rank of $J(f_1, \cdots, f_r)$ at p. By definition we know

$$J(F_1, \cdots, F_r)(p) = \left(\begin{array}{c|c} \frac{\partial F_1}{\partial x_0}(p) & \\ \vdots & J(f_1, \cdots, f_r)(a) \\ \frac{\partial F_r}{\partial x_0}(p) & \end{array} \right).$$

By Euler formula for homogeneous polynomial F_i of degree d_i, we have

$$\sum_{j=0}^{n} x_i \frac{\partial F_i}{\partial x_j} = d_i F_i.$$

Then we get

$$\frac{\partial F_i}{\partial x_0}(p) = -\sum_{j=1}^{n} a_i \frac{\partial f_i}{\partial x_j}(a).$$

So the first column of $J(F_1, \cdots, F_r)(p)$ is the linear combination of other columns, i.e. $J(F_1, \cdots, F_r)(p) = J(f_1, \cdots, f_r)(a)$. □

Lemma 12.16 (Nakayama)**.** *Let A be a local ring and $\mathfrak{m} \subset A$ be its maximal ideal. Let M be a finitely generated A-module:*

(1) *if $M = \mathfrak{m}M$, then $M = \{0\}$;*
(2) *write $k = A/\mathfrak{m}$, let $f_1, \cdots, f_r \in M$ such that $\bar{f}_1, \cdots, \bar{f}_r$ generate $M/\mathfrak{m}M$ as k-vector space. Then $f_1, \cdots, f_r$ generate M as an A-module.*

Proof. (1) Assume $M \neq \{0\}$, let $\{u_1, \cdots, u_r\}$ be a minimal set of generators of M as an A-module. Note $u_r \in M = \mathfrak{m}M$ i.e.

$$u_r = \sum_{i=1}^{r} m_i u_i$$

where $m_i \in \mathfrak{m}$. Then we get

$$(1-m_r)u_r = \sum_{i=1}^{r-1} m_i u_i.$$

Since $1-m_r$ is a unit(if not, then $1-m_r \in \mathfrak{m}$ since A is a local ring, then $1 = 1-m_r+m_r \in \mathfrak{m}$), we get

$$u_r = \sum_{i=1}^{r-1} m_i(1-m_r)^{-1}u_i.$$

We get a contradiction, thus $M = \{0\}$.

(2) Let $N := \langle f_1, \cdots, f_r \rangle \subset M$. To show $N = M$ is equivalent to show $M/N = \{0\}$. Since $\bar{f}_1, \cdots, \bar{f}_r$ generate $M/\mathfrak{m}M$, we have

$$(N+\mathfrak{m}M)/\mathfrak{m}M = M/\mathfrak{m}M.$$

This equation implies

$$N + \mathfrak{m}M = M.$$

Then we get $\mathfrak{m} \cdot (M/N) = (\mathfrak{m}M+N)/N = M/N$, it implies $M/N = \{0\}$ by using the first conclusion of the lemma. □

Definition 12.17 (Discrete Valuation Ring)**.** Let A be a local ring, $\mathfrak{m}$ be its maximal ideal. Further more, assume A is also an integral domain. Then A is called a *discrete valuation ring*(DVR) if the following conditions hold:

1. (1) $\mathfrak{m}$ is a principal ideal, i.e. $\mathfrak{m} = \langle t \rangle$ for some $t \in \mathfrak{m}$(such a t is called a uniformizing parameter);
2. (2) if t is a uniformizing parameter, then every element $f \in A$ can be written as $f = at^n$ for $a \in A$ a unit and $n \in \mathbb{Z}^+$.

Remark. If t is a uniformizing parameter, then $\mathfrak{m}^n = \langle t^n \rangle$.

This remark can be proved by induction. It is obvious that $\langle t^n \rangle \subset \mathfrak{m}^n$. The opposite inclusion is true for $n = 0, 1$, assume $\langle t^{n-1} \rangle = \mathfrak{m}^{n-1}$ is true. Then every element in $\mathfrak{m}$ can be written as sum of elements of the form $abt^n = at \cdot bt^{n-1}$ with $a, b \in A$, hence $\mathfrak{m}^n \subset \langle t^n \rangle$.

Exercise 12.18. *Prove that for a curve C and a nonsingular point $p \in C$, $\mathcal{O}_{C,p}$ is a discrete valuation ring.*

Proposition 12.19.

1. (1) *Let A be a ring, $I \subset A$ be an ideal, $\pi : A \to A/I$ be a projective map. Then the map*
$$\{\textit{ideals of } A/I\} \to \{\textit{ideals of } A \textit{ containing } I\}$$
$$J \to \pi^{-1}(J)$$
is injective.
2. (2) *If A is a noetherian ring, $I \subset A$ is an ideal, then A/I is also noetherian.*
3. (3) *Let X be a variety, $p \in X$. Then $\mathcal{O}_{X,p}$ is noetherian.*

Proof. (1) It is trivial.

(2) If

$$J_1 \subset J_2 \subset \cdots$$

is an ascending chain of ideals in A/J, then

$$\pi^{-1}(J_1) \subset \pi^{-1}(J_2) \subset \cdots$$

is an asending chain of ideals in A, and it is stationary by (1).

(3) To show $\mathcal{O}_{X,p}$ is noetherian, as $\mathcal{O}_{X,p}$ only depends on a neighborhood of p, we can assume that $X \subset \mathbb{A}^n$ is an affine variety. Then $A(X)$ is noetherian by (2). The map

$$\begin{aligned} \{\text{ideals in } \mathcal{O}_{X,p}\} \to & \qquad \{\text{ideals in } A(X)\} \\ I \to & \qquad I \cap A(X) \end{aligned}$$

is injective, hence $\mathcal{O}_{X,p}$ is noetherian. □

Theorem 12.20. *Let p be a nonsingular point on a curve C. Then $\mathcal{O}_{C,p}$ is a DVR.*

Proof. $\mathcal{O}_{C,p}$ is a subring of a field $K(C)$, thn $\mathcal{O}_{C,p}$ is an integral domain. We have seen $\mathcal{O}_{C,p}$ is noetherian, let $\mathfrak{m} \subset \mathcal{O}_{C,p}$ be a maximal ideal, then $\mathfrak{m}$ is finitely generated ieal because $\mathcal{O}_{C,p}$ is noetherian. We know that $1 = \dim(T_pC) = \dim\left(\mathfrak{m}/\mathfrak{m}^2\right)^*$, let $t \in \mathfrak{m}$ be an element that its class $\bar{t}$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama Lemma we obtain $\langle t \rangle = \mathfrak{m}$ as $\mathcal{O}_{C,p}$-module, i.e., t is an uniformizing parameter. Define

$$M := \bigcap_{n \geq 0} \mathfrak{m}^n,$$

we want to show $M = \{0\}$. M is an ideal in $\mathcal{O}_{C,p}$, hence it is finitely generated because $\mathcal{O}_{C,p}$ is noetherian. And $\mathfrak{m}M = \bigcap_{n>0} \mathfrak{m}^n = M$, by Nakayama Lemma $M = 0$. This means every element $f \in \mathcal{O}_{C,p}$ lies in $\mathfrak{m}^n \backslash \mathfrak{m}^{n+1}$ for some $n \geq 0$. Hence $f = at^n$ for $a \in \mathcal{O}_{C,p}$ unit. □

Definition 12.21. Let p be a nonsingular point on a curve C. Define

$$\begin{aligned} \nu_p : \mathcal{O}_{C,p} \backslash \{0\} &\longrightarrow \mathbb{Z}_{\geq 0} \\ f &\longmapsto \nu_p(f) = n, f = at^n, \end{aligned}$$

where a is a unit and t is a uniformazing parameter. $\nu_p(f)$ is the *order* of f at p.

Remark. ν_p has the following properties:

(1) $\nu_p(fg) = \nu_p(f) + \nu_p(g)$;
(2) $\nu_p(f+g) \geq \min\left(\nu_p(f), \nu_p(g)\right)$ with equality if $\nu_p(f) \neq \nu_p(g)$;
(3) f is a unit $\Leftrightarrow \nu_p(f) = 0$.

Definition 12.22. Note $K(C)$ is also the quotient field of $\mathcal{O}_{C,p}$, we can extend $\nu_p : \mathcal{O}_{C,p} \backslash \{0\} \to \mathbb{Z}_{\geq 0}$ to $\nu_p : K(C) \backslash \{0\} \to \mathbb{Z}$ by $\nu\left(\frac{f}{g}\right) := \nu_p(f) - \nu_p(g)$.

Let $h \in K(C) \backslash \{0\}$, $n = \nu_p(h)$, then we say h has a zero at p of order n if $n \geq 0$, h has a pole at p of order $-n$ if $n < 0$.

Example 12.23. Let $C = \mathbb{A}^1, p = 0, \mathfrak{m} = \langle x \rangle$. If f is a rational function in x, $f = \frac{g(x)}{h(x)}, g, h \in k[x]$, $g = x^l g'$ with $g'(0) \neq 0$, $h = x^m h'$ with $h'(0) \neq 0$, then $\nu_p(f) = l - m$. Write $f = x^{l-m}\frac{g'(x)}{h'(x)}$, $\frac{g'(x)}{h'(x)}$ is a unit in $\mathcal{O}_{\mathbb{A}^1,0}$.

Proposition 12.24. *Let p be a nonsingular point on a curve C :*

(1) *Let* $f \in K(C)\setminus\{0\}$*. Then we can write*

$$f = a \cdot t^{\nu_p(f)}$$

for t *a uniformizing parameter,* a *a unit in* $\mathcal{O}_{C,p}$ *;*

(2) $\mathcal{O}_{C,p} = \{f \in K(C)\setminus\{0\} \,|\nu_p(f) \geq 0\} \cup \{0\}$.

Proof. (1) We know by definition $\nu_p(f) = n \Leftrightarrow f = \frac{g}{h}, g, h \in \mathcal{O}_{C,p}$, $\nu_p(g) - \nu_p(h) = n$. It follows $g = at^{m+n}$, $h = bt^m$ with a, b units. Then $f = \frac{g}{h} = \frac{a}{b}t^n$ and $\frac{a}{b}$ is a unit in $\mathcal{O}_{C,p}$.

(2) We know that if $f \in \mathcal{O}_{C,p}\setminus\{0\}$, then $\nu_p(f) \geq 0$. Conversely let $\nu_p(f) \geq 0$, Then by (1) $f = a \cdot t^{\nu_p(f)}$, a aunit in $\mathcal{O}_{C,p}$. □

Theorem 12.25. *Let* C *be a nonsigular curve,* $p \in C$*,* $\varphi_0 : C\setminus\{p\} \to Y$ *be a morphism to a projective variety* Y*. Then* φ *can be extended to* $\varphi : C \to Y$*.*

Proof. Assume we can extend $\varphi_0 : C\setminus\{p\} \to Y$ to $\varphi : C \to \mathbb{P}^n$ (i.e., may be outside of Y). Then $\varphi^{-1}(Y)$ is closed in C and it contains $C\setminus\{p\}$. Thus $\varphi^{-1}(Y) = C$, hence $\varphi : C \to Y$. We can replace C by a neighborhood of p in C. Making this neighborhood sufficiently small we can assume

$$\varphi_0 = [f_0, \cdots, f_n], \quad f_i \in \mathcal{O}_C\left(C\setminus\{p\}\right) i = 0, \cdots, n$$

without common zeros. Let t be a uniformizing parameter at p, we can write $f_i = a_i t^{m_i}, m_i \in \mathbb{Z}$ and a_i units in $\mathcal{O}_{C,p}$. We can assume $a_i, i = 0, \cdots, n$ have no zero on C, t has no zero on $C\setminus\{p\}$. Let $m_j = \min_{0\leq i\leq n} m_i$. For all i, let $g_i = a_i \cdot t^{m_i - m_j} = \frac{f_i}{t^{m_j}}$. Then for all i, $g_i \in \mathcal{O}_{C,p}$, make the neighborhood small enough and replace it by C, we can ensure $g_i \in \mathcal{O}_C(C)$, $\forall i = 0, \cdots, n$. Then

$$\varphi : [g_0, \cdots, g_n] : C \to \mathbb{P}^n,$$

$g_0, \cdots, g_n$ have no common zeros because $g_j = a_j$ for smome j has no zeros in C.

Furthermore on $C\setminus\{p\}$, $\varphi = [g_0, \cdots, g_n] = [t^{m_j}g_0, \cdots, t^{m_j}g_n] = \varphi_0$. □

Corollary 12.26. *Let* C *be a nonsingular curve,* $U \subset C$ *be open nonempty. Then* $\varphi_0 : U \to Y$ *can be extended to* $\varphi : C \to Y$*.(Notice that* $C\backslash U$ *is a finite set of points)*

Corollary 12.27. *Let* C, D *be projective nonsingular curves. If* C *and* D *are birational, then they are isomorphic.*

Remark. (1) Let $C = \mathbb{A}^1$ and $D = \mathbb{P}^1$, they are birational but not isomorphic.

(2) $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birational but not isomorphic.

13. Divisors and Riemann-Roch Theorem

In the following, we always assume that curve C *is nonsingular and projective.*

Definition 13.1. A *divisor* on C is a formal sum

$$\sum_{p\in C} a_p \cdot p, \quad a_p \in \mathbb{Z} \text{ all but finitely many } a_p \text{ are zero.}$$

Divisors on C form an abelian group $\mathrm{Div}(C)$: For $D = \sum_{p\in C} a_p \cdot p, E = \sum_{p\in C}$,

$$D + E := \sum_{p\in C}(a_p + b_p)p,$$

$$0 := \sum_{p\in C} 0 \cdot p.$$

If $D = \sum_{p\in C} a_p p$, write $\nu_p(D) := a_p$. D is called *effective* if $\nu_p(D) \geq 0, \forall p \in C$, write it as $D \geq 0$.

The *degree* of $D = \sum_{p\in C} a_p p$ is

$$\deg(D) := \sum_{p\in C} a_p \in \mathbb{Z}.$$

If $h \in K(C)\setminus\{0\}$, $\forall p \in C$, we have defined $\nu_p(h) \in \mathbb{Z}$. The *divisor of h* is

$$\operatorname{div}(h) := \sum_{p\in C} \nu_p(h) \cdot p.$$

A divisor of this form is called *principal devisor*.

Theorem 13.2 (without proof)**.** *The degree of a principal divisor is* 0.

Definition 13.3. Divisors D, E on C are called *linearly equivalent* if their difference is a principal divisor.

Write $D \sim E$ if they are linearly equivalent, $[D]$ denotes the corresponding equivalence class. The *Picard group* of C denoted by $\operatorname{Pic}(C)$ is $\operatorname{Div}(C)/\sim$.

Write $\operatorname{Div}^0(C) :=$ *divisors on C of degree* 0, $\operatorname{Pic}^0(C) = \operatorname{Div}^0(C)/\sim$.

Proposition 13.4. *If $C \simeq \mathbb{P}^1$, then $\operatorname{Pic}^0(C) = \{0\}$.(In fact, it is if and only if relation)*

Proof. It is easy to see that $\operatorname{Pic}^0(C)$ is equivalent to: $\forall P, Q \in C$, $P \sim Q$. Let $P = [a : b], Q = [c : d]$, define $f := \frac{ax_1 - bx_0}{cx_1 - dx_0}$, then $\operatorname{div}(f) = P - Q$. □

Theorem 13.5 (without proof)**.** $\operatorname{Pic}^0(C)$ *can be given the structure of a nonsingular variety of dimension $g(C)$. $g(C)$ is the genus of C defined later.*

Definition 13.6. Let D be a divisor on C, the *space of sections of D* is

$$L(D) := \{f \in K(C)\setminus\{0\} \,|\operatorname{div}(f) + D \geq 0\} \cup \{0\}.$$

Let $D = \sum_{p\in C} a_p \cdot p$, then $\nu_p(f) \geq -a_p$ for $f \in L(D)$.
$L(D)$ is a finite dimensional sub k-vectorspace $K(C)$. Define

$$l(D) := \dim(L(D)).$$

If $D \sim E$, then $L(D) \simeq L(E)$ as k-vectorspace. $E = D + \operatorname{div}(h)$, then

$$\begin{aligned} L(D) \to & E, \\ f \mapsto & f \cdot h. \end{aligned}$$

If $D = 0$, then $L(D) = \{f \in K(C) | f \text{ regular on } C\} = k$.
If $\deg(D) < 0$, then $L(D) = \{0\}$ by Theorem 13.2.

Definition 13.7. A *differential form* K_C on C is an expression

$$\omega = \sum_i f_i \mathrm{d}g_i, \quad f_i, g_i \in K(C)$$

with the following relations:

(1) $\mathrm{d}(g + f) = \mathrm{d}g + \mathrm{d}f$;

(2) $\mathrm{d}(fg) = f\mathrm{d}g + \mathrm{d}f$;
(3) $\mathrm{d}a = 0$ for $a \in k$.

If $p \in C$, ω is a differential form, we can write

$$\omega = f \cdot \mathrm{d}t, \quad t \text{ uniformizing parameter at } p, f \in K(C).$$

Put $\nu_p(\omega) := \nu_p(f)$ and

$$\mathrm{div}(\omega) := \sum_{p \in C} \nu_p(\omega) \cdot p.$$

Theorem 13.8. *For all differential forms ω_1, ω_2 on C, we have*

$$\mathrm{div}(\omega_1) \sim \mathrm{div}(\omega_2).$$

We call K_C in $\mathrm{Pic}(C)$ the divisor class of a differential form(canonical divisor in C).

Definition 13.9. The *genus* of C is

$$g(C) := l(K_C).$$

Example 13.10. If $C = \mathbb{P}^1$, then $g(C) = 0$. Indeed, View $\mathbb{P}^1 = U_0 \cup \infty = U_0 \cup U_1, U_0 \simeq U_1 \simeq \mathbb{A}^1$. Then U_1 is an open neighborhood of ∞. Consider the poit $z = [t_0, t_1] \in \mathbb{P}^1$. Then for $z \in U_0$, we can write $z = \frac{t_1}{t_0}$, $\mathrm{d}z$. Choose any point $p \in U_1$, then $z - p$ is a uniformizing parameter and then $\mathrm{d}(z-p) = \mathrm{d}z$. Hence $\nu(p) = 0$ for $p \in U_0$. For $p = \infty \in U_1$, we denote $p = \frac{1}{z} = \frac{t_0}{t_1}$. A uniformizing parameter at ∞ is $t := \frac{1}{z}$. Hence $\mathrm{d}z = \mathrm{d}\left(\frac{1}{t}\right)$ at ∞. Hence $\mathrm{div}(\mathrm{d}z) = -2 \cdot [\infty]$. Then $\deg(\mathrm{div}(\mathrm{d}z)) \Rightarrow l(\mathrm{div}(\mathrm{d}z)) = 0$.

Theorem 13.11 (Riemann-Roch Theorem)**.** *Let D be a divisor on C, then*

$$l(D) - l(K_C - D) = \deg(D) + 1 - g(C).$$

Corollary 13.12.

$$\deg(K_C) = 2g - 2.$$

Proof. Apply Riemann-Roch theorem to K_C :

$$l(K_C) - l(0) = \deg(K_C) + 1 - g(C).$$

Then by $g(C) = l(K_C), l(0) = 1$ we obtain the corollary. □

Theorem 13.13. *Let C be a nonsingular curve of degree d in $\mathbb{P}^2$. Then*

$$g(C) = \frac{1}{2}(d-1)(d-2).$$

Proof. Prove it by calculating the term in Riemann-Roch. □

Define

$$|D| := \{E \text{ effective}, E \sim D\} = \{D + \mathrm{div}(f) | f \in L(D)\}.$$

Let D be a divisor on C. We can define a morphism $\varphi_{|D|} : C \to \mathbb{P}^{l(D)-1}$ by $\varphi_{|D|} := \left[h_0, \cdots, h_{l(D)-1}\right]$ where $h_0, \cdots, h_{l(D)-1}$ are a basis of $L(D)$.

The elements $E \in |D|$ are precisely the inverse images of the intersections of $\varphi_{|D|}$ with hyperplanes in $\mathbb{P}^{l(D)-1}$

Theorem 13.14. *Let D be a divisor on a curve C of genus g. If $\deg(D) \geq 2g+1$, then*

$$\varphi_{|D|} : C \to \mathbb{P}^{\deg(D)-g}$$

is an embedding.

Further topics: If X is a nonsingular projective variety of bigger dimension d, a divisor on X is formal sum

$$D := \sum_z a_z \cdot z, a_z \in \mathbb{Z},$$

where z are irreducible subsets of dimension $d-1$. For $f \in K(X)$, $\operatorname{div}(f)$ is the corresponding principal divisor. Again we can define

$$L(D) := \{f \in K(X) | \operatorname{div}(f) + D \geq 0\}.$$

To divisor D associate morphism

$$\varphi_{|D|} : X \to \mathbb{P}^{l(D)-1}.$$

Cohomology: D a divisor, sheaf $\mathcal{O}(D)$, there is a so called sheaf cohomology $H^i(X, \mathcal{F})$. Then $H^0(X, \mathcal{O}(D)) = L(D)$. We also have Riemann-Roch Theorem under this general case

Theorem 13.15.

$$\sum_{i=0}^{\dim X} \dim H^i\left(X, \mathcal{O}(D)\right) = \textit{some topological quantities.}$$

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