

F. L. Nazarov's paper Local Estimates of Exponential Polynomials and Their Applications to Inequalities of Uncertainty Principle Type

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Abstract

This is a learning note about Chapter 1 of Nazarov's paper(see [\[2\]](#)). This chapter is about the Turan lemma and its general form on measurable sets.

Contents

1	The Turan lemma: original form	2
2	Two usefull lemmas	6
3	The Turan lemma for polynomials on the unit circumference	11
4	The Turan lemma in general form	15
5	Summary: Two important techniques used	24
	Appendix A Harmonic measure	25

Definition 1. *An exponential polynomial is*

$$p(t) = \sum_{k=1}^n c_k e^{\lambda_k t} \quad (c_k \in \mathbb{C}, \lambda_k \in \mathbb{C}).$$

The main purpose of the first part of the paper is to establish the following inequality

$$\sup_{t \in I} |p(t)| \leq \left(\frac{A\mu(I)}{\mu(E)} \right)^{n-1} \sup_{t \in E} |p(t)|, \quad (1)$$

where $I \subset \mathbb{R}$ is an interval, $E \subset I$ is a measurable set of positive Lebesgue measure and A is an absolute constant.

1 The Turan lemma: original form

The following lemma was derived by Turan (see [3]).

Theorem 1. *Let z_1, \dots, z_n be complex numbers, $|z_j| \geq 1, j = 1, \dots, n$. Let*

$$b_1, \dots, b_n \in \mathbb{C}, \quad S_m \stackrel{\text{def}}{=} \sum_{k=1}^n b_k z_k^m \quad (m \in \mathbb{Z}).$$

Then

$$|S_0| \leq n \left(\frac{2e(m+n-1)}{n} \right)^{n-1} \max_{k=m+1}^{m+n} |S_k| \leq \left(\frac{4e(m+n-1)}{n} \right)^{n-1} \max_{k=m+1}^{m+n} |S_k| \quad (2)$$

for all $m \in \mathbb{Z}_+$.

Proof. To prove the lemma, we need to construct a polynomial $q(z) = 1 + \sum_{k=1}^n \gamma_k z^{m+k}$ such that

$$(1) \quad q(z_j) = 0 \text{ for each } j = 1, \dots, n \text{ and}$$

$$(2) \quad \sum_{k=1}^n |\gamma_k| \leq n \left(\frac{2e(m+n-1)}{n} \right)^{n-1}.$$

Let

$$q(z) = \prod_{k=1}^n \left(1 - \frac{z}{z_k} \right) \sigma_m(z),$$

where $\sigma_m(z)$ is the m -th partial sum of the series $\prod_{k=1}^n \left(1 - \frac{z}{z_k} \right)^{-1} = \sum_{k=0}^{\infty} \beta_k z^k$, i.e.

$$\sigma_m(z) = \sum_{k=1}^m \beta_k z^k.$$

By definition, we have

$$1 = \prod_{k=1}^n \left(1 - \frac{z}{z_k} \right) \sum_{k=0}^{\infty} \beta_k z^k.$$

This identity implies that the s -th coefficient in the expansion of the right side depends only on $\beta_{s-n}, \dots, \beta_s$. Hence the coefficients at the powers z, z^2, \dots, z^m of $q(z) = \prod_{k=1}^n \left(1 - \frac{z}{z_k}\right) \sigma_m(z)$ all vanish (since they only depend on $\sigma_m(z)$). Recalling the Taylor expansion

$$(1 - z)^{-n} = \sum_{k=0}^{\infty} \frac{(k + n - 1)!}{k!(n - 1)!} z^k,$$

hence we have (by using the condition $|z_j| \geq 1$ and assuming $|z| < 1$)

$$\left| \prod_{k=1}^n \left(1 - \frac{z}{z_k}\right)^{-1} \right| \leq (1 - |z|)^{-n} = \sum_{k=0}^{\infty} \frac{(k + n - 1)!}{k!(n - 1)!} |z|^k.$$

Thus, all coefficients of $\sigma_m(z)$ do not exceed¹

$$\frac{(m + n - 1)!}{m!(n - 1)!} \leq \left(\frac{e(m + n - 1)}{n} \right)^{n-1}.$$

Then we get the estimates

$$|\gamma_k| \leq \left(\frac{e(m + n - 1)}{n} \right)^{n-1} \sum_{s=k}^n \binom{n}{s}$$

and

$$\sum_{k=1}^n |\gamma_k| = \frac{1}{2} \sum_{k=1}^n (|\gamma_k| + |\gamma_{n+1-k}|) \leq 2^{n-1} n \left(\frac{e(m + n - 1)}{n} \right)^{n-1}.$$

Now we've constructed the desired polynomial $q(z)$.

Since

$$\begin{aligned} S_0 &= b_1 + b_2 + \dots + b_n \\ &= \sum_{j=1}^n b_j \cdot 1 \\ &= \sum_{j=1}^n \left(- \sum_{k=1}^n \gamma_k z_j^{m+k} \right) \\ &= - \sum_{k=1}^n \gamma_k S_{m+k}. \end{aligned} \tag{3}$$

¹Here needs some estimates: we need to prove

$$\binom{n}{k} \leq \left(\frac{en}{k+1} \right)^k.$$

This inequality can be proved by induction.

Hence the estimates above and (3) complete the proof. \square

Recalling the definition of an exponential polynomial

$$p(t) = \sum_{k=1}^n c_k e^{i\lambda_k t},$$

now let $t_k = t_0 + k\delta$, we have

$$p(t_k) = \sum_{j=1}^n c_j e^{i\lambda_j(t_0+k\delta)} = \sum_{j=1}^n b_j (e^{i\lambda_j\delta})^k = \sum_{j=1}^n b_j z_j^k,$$

where $z_j = e^{i\lambda_j\delta}$ and $b_j = c_j e^{i\lambda_j t_0}$. Then we can use the lemma directly and get

$$|p(t_0)| \leq \left\{ \frac{4e(m+n-1)}{n} \right\}^{n-1} \max_{k=m+1}^{m+n} |p(t_k)|. \quad (4)$$

Now the inequality (1) for the case where E is an interval can be derived in an almost immediate way (with the constant $A=4e$).

Using the same idea in

Theorem 2. *Let I be an interval, let $E \subset I$ be a measurable set of positive Lebesgue measure. Then*

$$\max_{t \in I} |p(t)| \leq 2^n \left(\frac{\mu(I)}{\mu(E)} \right)^{2n^2} \max_{t \in E} |p(t)|. \quad (5)$$

Proof. By (4), the following inequality

$$\max_{t \in I} |p(t)| \leq 2^n \max_{t \in E} |p(t)| \quad (6)$$

holds if t_0 is the first term of the arithmetic progression $t_k = t_0 + k\delta$ ($k = 0, \dots, n$) with all other terms belonging to E . The point of the proof is to find a set E_1 that is "close" to E and we can choose a δ such that all t_k 's belongs to E .

Step 1. Let $J \subset I$ is an open interval and

$$\mu(E \cap J) > \left(1 - \frac{1}{2n}\right) \mu(J).$$

Let $t_0 \in J$ be any fixed point. Such a point t_0 splits the interval J into two subintervals J_- and J_+ . At least one of them, let's say J_+ has the property

$$\mu(E \cap J_+) > \left(1 - \frac{1}{2n}\right) \mu(J).$$

Let $\varphi(t) = \chi(t)$ be the characteristic function of $J_+ \setminus E$, then by applying the lattice averaging lemma we see that the average number of points $t_k = t_0 + k\delta$ ($k \in \mathbb{N}$) belonging to $J_+ \setminus E$ as δ runs over the interval $\left(\frac{\mu(J_+)}{2n}, \frac{\mu(J_+)}{n}\right)$ is (here we write $\frac{\mu(J_+)}{2n}$ as s)

$$\begin{aligned}
\frac{\int_s^{2s} \sum_{k \in \mathbb{Z} \setminus \{0\}} \varphi(k\delta) d\delta}{\int_s^{2s} d\delta} &= \frac{1}{s} \int_1^2 \sum_{k \in \mathbb{Z} \setminus \{0\}} \varphi\left(k s \frac{\delta}{s}\right) s d\left(\frac{\delta}{s}\right) \\
&= \int_1^2 \sum_{k \in \mathbb{Z} \setminus \{0\}} \varphi(ksv) dv \\
&\leq \frac{1}{s} \int_{\mathbb{R}} \varphi(t) dt \\
&= \frac{2n}{\mu(J_+)} \mu(J_+ \setminus E) \\
&< 1.
\end{aligned} \tag{7}$$

Hence there exists a positive $\delta < \frac{\mu(J_+)}{n}$ such that none of the points t_1, \dots, t_n belongs to $J_+ \setminus E$. Since $k\delta < \frac{k\mu(J_+)}{n} \leq 1$ and t_0 is the endpoint of J_+ , all these points lie in J_+ and, consequently, in E . Since the choice of $t_0 \in J$ is arbitrary, any points in J have the property that $t_k \in E$ for each $k = 1, \dots, n$.

Step 2. Let $E_1 = \bigcup \{J : J \subset I \text{ is open, } \mu(E \cap J) > (1 - \frac{1}{2n}) \mu(J)\}$. Since E_1 is the union of open sets, E_1 itself is also open, hence, the union of disjoint open intervals. Let Q be one constituent interval of E_1 , if

$$\mu(E \cap Q) > \left(1 - \frac{1}{2n}\right) \mu(Q)$$

holds, then we can find a larger open interval Q' such that $Q' \subset Q \subset E_1$, this contradicts the chosen of Q . Hence all the constituent intervals of E_1 satisfy the relation

$$\mu(E \cap Q) \leq \left(1 - \frac{1}{2n}\right) \mu(Q).$$

Thus, the set E_1 has the following two properties

$$\sup_{t \in E_1} |p(t)| \leq 2^n \sup_{t \in E} |p(t)|, \tag{8}$$

$$\mu(E_1) \geq \left(1 - \frac{1}{2n}\right)^{-1} \mu(E) \geq e^{\frac{1}{2n}} \mu(E) \text{ or } E_1 = I. \tag{9}$$

²Here we use the inequality $\frac{1}{e} \geq (1 - \frac{1}{2n})^{2n}$.

Step 3. Iterating this procedure we obtain a sequence of sets E_1, E_2, \dots such that

$$\sup_{t \in E_k} |p(t)| \leq 2^{nk} \sup_{t \in E} |p(t)|, \quad (10)$$

$$\mu(E_k) \geq e^{\frac{k}{2n}} \mu(E) \text{ or } E_k = I. \quad (11)$$

If $k > 2n \log \frac{\mu(I)}{\mu(E)}$, then the first case of (11) cannot occur. Therefore we obtain

$$E_{\lceil 2n \log \frac{\mu(I)}{\mu(E)} + 1 \rceil} = I,$$

whence

$$\sup_{t \in I} |p(t)| \leq 2^{(2n \log \frac{\mu(I)}{\mu(E)} + 1)n} \sup_{t \in E} |p(t)| \leq 2^n \left(\frac{\mu(I)}{\mu(E)} \right)^{2n^2} \sup_{t \in E} |p(t)|.$$

□

Remark. The proof of Theorem 2 is based on Theorem 1. We can regard Theorem 1 is a discrete version of Theorem 2. From the discrete version to Lebesgue measurable sets, the simplest thought is to find the discrete points which Theorem 1 can be used to. If there exists, then our problem can be solved easily. But unfortunately the arithmetic progression t_k may not exists in E for any point in I . To overcome this difficulty, we need to find an interval close to E (here the sense of "close" has exact meaning in the proof), and any point fixed t_0 in this interval satisfy the condition $t_k \in E$ for each $k = 1, \dots, n$. Finally, by iterating the procedure, the chosen set becomes strictly larger, and finally equals to I .

2 Two usefull lemmas

Lemma 1. *If $P(z)$ is an algebraic polynomial of degree n , then*

$$\mu \left(\left\{ x \in \mathbb{R} : \left| \frac{d}{dx} \log P(x) \right| > y \right\} \right) \leq \frac{8n}{y}$$

and

$$\mu \left(\left\{ z \in \mathbb{T} : \left| \frac{d}{dz} \log P(z) \right| > y \right\} \right) \leq \frac{8n}{\pi y}.$$

Proof. First we shall prove the inequality for the real line. Let z_1, \dots, z_{n_1} and $\zeta_1, \dots, \zeta_{n_2}$ ($n_1 + n_2 = n$) be complex zeros of the polynomial P enumerated in such a way that $\text{Im} z_j \leq 0$ ($j = 1, \dots, n_1$) and $\text{Im} \zeta_j > 0$ ($j = 1, \dots, n_2$). We have

$$\frac{d}{dz} \log P(z) = \sum_{j=1}^{n_1} \frac{1}{z - z_j} + \sum_{j=1}^{n_2} \frac{1}{z - \zeta_j} = \sum_1(z) + \sum_2(z).$$

The function $\sum_1(z)$ is analytic in the upper half-plane \mathbb{H} , and

$$\operatorname{Im} \sum_1(z) = \sum_{j=1}^{n_1} \frac{\operatorname{Im} z_j - \operatorname{Im} z}{|z - z_j|^2} < 0$$

for all $z \in \mathbb{H}$.

Let $h(\xi)$ be the harmonic measure of the set $\mathbb{R} \setminus [-y, y]$ with respect to the upper-half plane and a point $\xi \in \mathbb{H}$. We put $u(z) \stackrel{\text{def}}{=} h(-\sum_1(z))$, it is harmonic in \mathbb{H} .

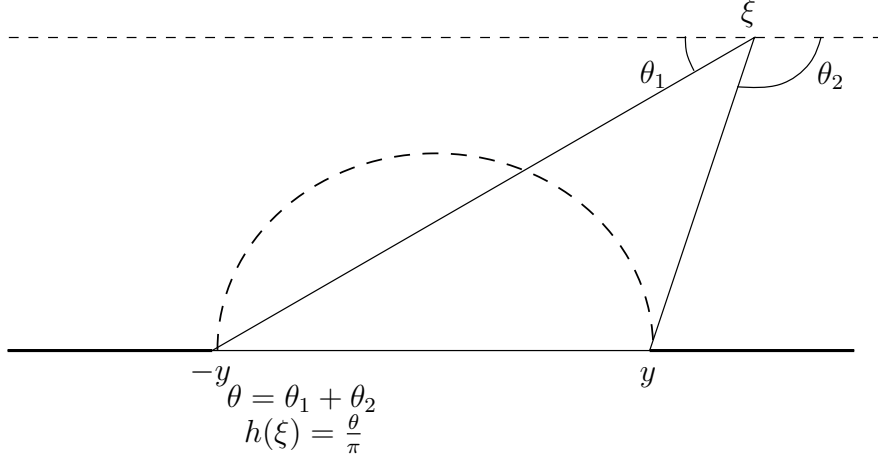


Figure 1: Harmonic function $h(\xi)$

If $t \rightarrow +\infty$, then $-\sum_1(it) \rightarrow i0^+$ and $u(it) \rightarrow 0$. If $|\sum_1(z)| \geq y$, then $u(z) \geq \frac{1}{2}$.

Hence, we have

$$\lim_{t \rightarrow +\infty} \pi t u(it) = \int_{\mathbb{R}} u(x) dx \geq \frac{1}{2} \mu \left(\left\{ x \in \mathbb{R} : \left| \sum_1(x) \right| > y \right\} \right).$$

On the other hand,

$$\begin{aligned} \lim_{t \rightarrow +\infty} \pi t u(it) &= \lim_{t \rightarrow +\infty} \pi t h \left(\frac{in_1}{t} + \mathcal{O} \left(\frac{1}{t^2} \right) \right) \\ &= \lim_{t \rightarrow +\infty} \pi t \left(2 \arctan \left(\frac{n_1}{ty} \right) / \pi \right) \\ &= \frac{2n_1}{y}. \end{aligned}$$

Hence

$$\mu \left(\left\{ x \in \mathbb{R} : \left| \sum_1(x) \right| > y \right\} \right) \leq \frac{4n_1}{y}.$$

Similarly

$$\mu \left(\left\{ x \in \mathbb{R} : \left| \sum_2(x) \right| > y \right\} \right) \leq \frac{4n_2}{y}.$$

Combining these estimates, we obtain

$$\begin{aligned} \mu \left(\left\{ x \in \mathbb{R} : \left| \sum(x) \right| > y \right\} \right) &\leq \mu \left(\left\{ x \in R : \left| \sum_1 \right| > \frac{n_1}{n} y \right\} \right) \\ &\quad + \mu \left(\left\{ x \in \mathbb{R} : \left| \sum_1 \right| > \frac{n_2}{n} y \right\} \right) \\ &\leq \frac{8n}{y}. \end{aligned}$$

Now we pass to the case of the circumference. As above, we split the zeros of $P(z)$ into two groups $z_1, \dots, z_{n_1} \in \mathbb{D}$ and $\zeta_1, \dots, \zeta_{n_2} \in \mathbb{C} \setminus \mathbb{D}$. Then

$$\frac{d}{dz} \log P(z) = \frac{1}{z} \left(\sum_{j=1}^{n_1} \frac{z}{z - z_j} + \sum_{j=1}^{n_2} \frac{z}{z - \zeta_j} \right) = \frac{1}{z} \left(\sum_1(z) + \sum_2(z) \right).$$

The factor $\frac{1}{z}$ can be disregarded since its absolute value is equal to 1. The estimate for $\sum_1(z)$ is essentially the same as above: having established the inequality

$$\operatorname{Re} \sum_1(z) = \sum_{j=1}^{n_1} \frac{|z|^2 - \operatorname{Re} z \bar{z}_j}{|z - z_j|^2} \geq 0$$

for $z \in \mathbb{C} \setminus \mathbb{D}$, we consider the function $u(z) \stackrel{\text{def}}{=} h(i \sum_1(z))$, which is harmonic outside the unit disk, and derive the estimate

$$u(\infty) = \frac{2 \arctan \frac{n_1}{y}}{\pi} = \int_{\mathbb{T}} u(z) d\mu(z) \geq \frac{1}{2} \mu \left(\left\{ z \in \mathbb{T} : \left| \sum_1(z) \right| > y \right\} \right),$$

which implies

$$\mu \left(\left\{ \left| \sum_1(z) \right| > y \right\} \right) \leq \frac{4}{\pi} \arctan \frac{n_1}{y} \leq \frac{4}{\pi} \frac{n_1}{y}.$$

The function $\operatorname{Re} \sum_2(z)$ may change sign. Therefore we use another inequality

$$\operatorname{Re} \sum_2(z) = \sum_j^{n_2} \frac{\operatorname{Re} z(\bar{z} - \bar{\zeta}_j)}{|z - \zeta_j|^2} = n_2 - \sum_{j=1}^{n_2} \frac{|\zeta_j|^2 - \operatorname{Re} z \bar{\zeta}_j}{|z - \zeta_j|^2} \leq n_2 \quad (z \in \mathbb{D}).$$

This time we choose the function h to be harmonic in $\mathbb{H} - in_2$. In order to obtain the estimate $\mu(|\sum_2| > y) \leq \frac{4}{\pi} \frac{n_1}{y}$, we can restrict ourselves to values $y > n_2$. Let $h(\xi)$ be harmonic measure of $(\mathbb{R} - in_2) \setminus I$ (where I is the interval cut off from the line $\mathbb{R} - in_2$ by the circle centered at 0 and of radius y) with respect to the half-plane $\mathbb{H} - in_2$ and the point $\xi \in \mathbb{H} - in_2$. One can easily check that the

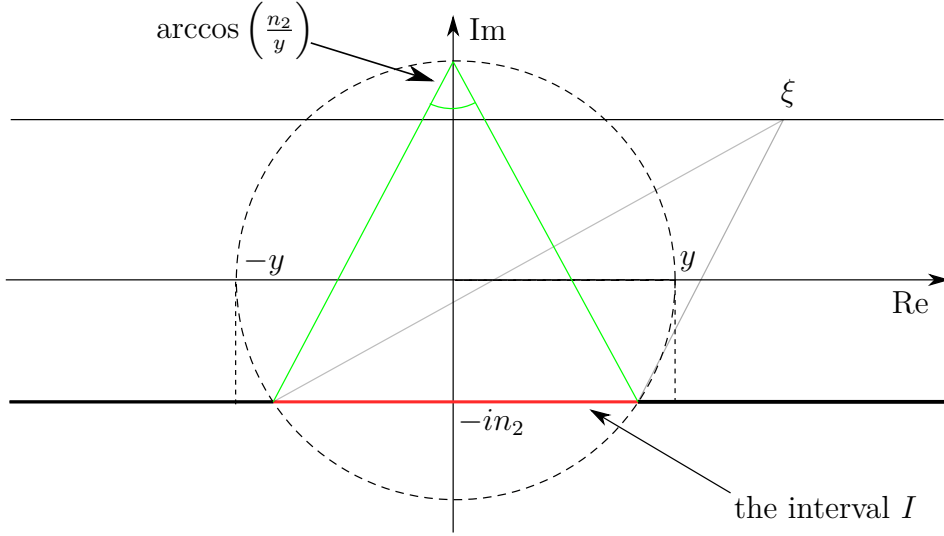


Figure 2: Domain of the harmonic function

function $u(z) \stackrel{\text{def}}{=} h(-i \sum_2(z))$ is harmonic in \mathbb{D} , $u(z) > \frac{\pi - \arccos \frac{n_2}{y}}{\pi}$ if $|\sum_2(z)| > y$, and $u(0) = h(0) = \frac{2 \arcsin \frac{n_2}{y}}{\pi}$. Therefore

$$\mu\left(\left\{\left|\sum_2\right| > y\right\}\right) \leq \frac{2 \arcsin \frac{n_2}{y}}{\pi - \arccos \frac{n_2}{y}} = \frac{2 \arcsin \frac{n_2}{y}}{\frac{\pi}{2} + \arcsin \frac{n_2}{y}}.$$

Now, to get the desired estimate it suffices to verify that $\frac{2\theta}{\pi/2+\theta} \leq \frac{4}{\pi} \sin \theta$ for each $\theta \in [0, \frac{\pi}{2}]$. The last inequality is equivalent to $\frac{\sin \theta}{\theta} + \frac{2}{\pi} \sin \theta \geq 1$. Taking into account that $\sin \theta \geq \theta - \frac{1}{6}\theta^3$ for every $\theta > 0$ and $\sin \theta \geq \frac{2}{\pi}\theta$ for every $\theta \in [0, \frac{\pi}{2}]$, we have

$$\frac{\sin \theta}{\theta} + \frac{2}{\pi} \sin \theta \geq 1 - \frac{1}{6}\theta^2 + \frac{4}{\pi^2}\theta = 1 + \theta \left(\frac{4}{\pi^2} - \frac{1}{6}\theta \right) \geq 1 + \theta \left(\frac{4}{\pi^2} - \frac{\pi}{12} \right)$$

and it remains to notice that $\pi^3 \leq 48$.

As above, the estimat of $\sum(z)$ results from the estimates of $\sum_1(z)$ and $\sum_2(z)$. Lemma 1 is proved. \square

Lemma 2 (Langer Lemma). *Let $p(z) = \sum_{k=1}^n c_k e^{i\lambda_k z}$ ($0 = \lambda_1 < \lambda_2 < \dots < \lambda_n = \lambda$) be an exponential polynomial not vanishing identically. Then the number of complex zeros of $p(z)$ in an open vertical strip $x_0 < \operatorname{Re} z < x_0 + \Delta$ of width Δ does not exceed $(n-1) + \frac{\lambda\Delta}{2\pi}$.*

Proof. Without loss of generality we assume that the coefficients c_1 and c_2 do not vanish and the boundary of the strip $x_0 < \operatorname{Re} z < x_0 + \Delta$ is free of zeros of the exponential polynomial $p(z)$. We make use of the argument principle to estimate the number of zeros of $p(z)$ in the rectangle $Q = \{z : x_0 < \operatorname{Re} z < x_0 + \Delta, |\operatorname{Im} z| \leq y\}$, as $y \rightarrow +\infty$.

On the upper edge of Q we have $p(z) = c_1 + \mathcal{O}(e^{-\lambda_2 y})$ (recall $\lambda_1 = 0$ and $\lambda_i < \lambda_{i+1}$). Therefore, the argument increment along this edge tends to 0 as $y \rightarrow +\infty$. Similarly, the representation $p(z) = c_n e^{i\lambda z} (1 + \mathcal{O}(e^{-(\lambda - \lambda_{n-1})y}))$, which is valid on the lower edge of Q , implies that the argument increment along the lower edge tends to $\lambda\Delta$ as $y \rightarrow +\infty$.

We show that the argument increment along any vertical segment

$$\{z = x + it : t \in [\alpha, \beta]\}$$

free of zeros of $p(z)$ does not exceed $\pi(n-1)$.

Here we construct a real exponential polynomial out of $p(z)$. Let

$$\xi \stackrel{\text{def}}{=} e^{i \arg p(x_0 + i\alpha)}.$$

The function

$$q(t) \stackrel{\text{def}}{=} \operatorname{Im}(\bar{\xi} p(x_0 + it)) = \sum_{k=1}^n a_k e^{-\lambda_k t} \quad (a_k = \operatorname{Im}(\bar{\xi} c_k e^{i\lambda_k x_0}) \in \mathbb{R})$$

is a real exponential polynomial. Actually, xi is used to rotate $p(x_0 + it)$ to make the imaginary part of $p(x_0 + i\alpha)$ be 0, i.e., $q(\alpha) = 0$.

Since we have assumed that there are no zeros of $p(x + it)$ among $t \in [\alpha, \beta]$, $p(x + it)$ cannot pass through 0. If $q \equiv 0$, along with $p(x + i\alpha) \in \mathbb{R}$ and $\bar{\xi} p(x + i\alpha) > 0$ by definition, then all values $p(x_0 + it)$ for $t \in [\alpha, \beta]$ lie on the ray $\{\xi y : y > 0\}$. Therefore $\Delta_{[\alpha, \beta]} \arg p(x_0 + it) = 0$. Otherwise, real zeros of $q(t)$ split the segment $[\alpha, \beta]$ into at most $n-1$ intervals I_j (it is well known that a real exponential polynomial of order n has at most $n-1$ zeros). Within each of intervals I_j ($q(\alpha) = 0$, hence there are $n-1$ intervals not n intervals), the values $p(x_0 +$

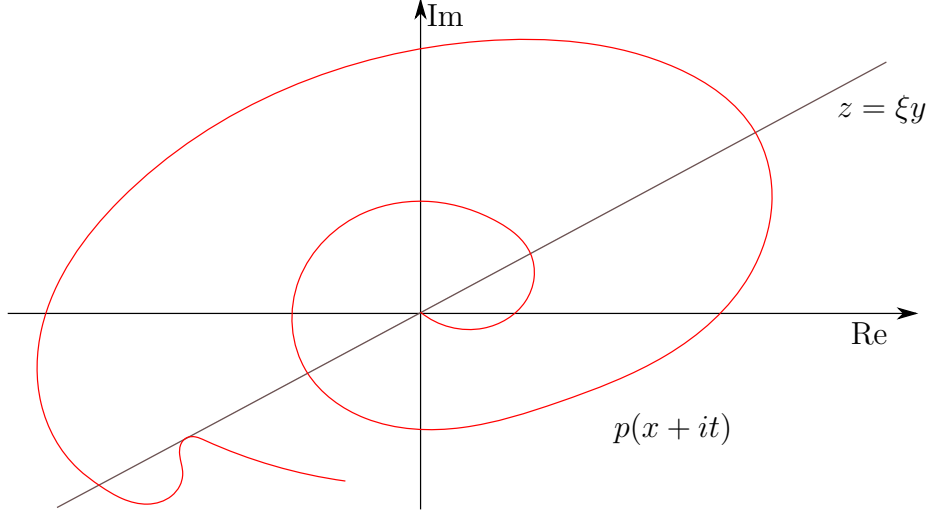


Figure 3: Argument of $p(x + it)$

it) lie in one of the two half-planes generated by the line $\{\xi y : y \in \mathbb{R}\}$, whence $|\Delta_{I_j} \arg p(x_0 + it)| \leq \pi$.

Adding these inequalities, we obtain argument increment along each of the lateral edges of Q does not exceed $\pi(n - 1)$. So the total argument increment of $p(z)$ along the boundary of Q traced counter clockwise can be estimated from above by a quantity tending to $2\pi \left(\frac{\Delta\lambda}{2\pi} + (n - 1) \right)$ as $y \rightarrow +\infty$, whence Lemma 2 follows. \square

3 The Turan lemma for polynomials on the unit circumference

Here we shall prove inequality (1) for the case of a 1-periodic exponential polynomial $p(t) = \sum_{k=1}^n c_k e^{2\pi i m_k t}$, where $c_k \in \mathbb{C}$, $m_1 < \dots < m_n \in \mathbb{Z}$, and for the segment $I = [0, 1]$.

Theorem 3. *Let $p(z) = \sum_{k=1}^n c_k z^{m_k}$ ($c_k \in \mathbb{C}$, $m_1 < \dots < m_n \in \mathbb{Z}$) be a trigonometric polynomial on the unit circumference T , and let E be a measurable subset of \mathbb{T} . Then*

$$\|p\|_W \stackrel{\text{def}}{=} \sum_{k=1}^n |c_k| \leq \left(\frac{16e}{\pi} \frac{1}{\mu(E)} \right)^{n-1} \sup_{z \in E} |p(z)| \leq \left(\frac{14}{\mu(E)} \right)^{n-1} \sup_{z \in E} |p(z)|. \quad (12)$$

Proof.

Step 1. We shall construct by induction a sequence of polynomials p_n, \dots, p_1 such that

- (1) $p_n = p$;
- (2) $\text{ord } p_k = k$ ($k = 1, \dots, n$) ;
- (3) $\|p_{k-1}\|_W \geq \frac{\pi}{16} \|p_k\|_W$;
- (4) the ratio $\varphi_k \stackrel{\text{def}}{=} \left| \frac{p_{k-1}}{p_k} \right|$ admits the weak type estimate

$$\mu(\{z \in \mathbb{T} : \varphi_k(z) > t\}) \leq \frac{1}{t}$$

for all $t > 0$.

The construction is as follows. Let $p_n = p$. The polynomial $p_k(z) = \sum_{s=1}^k d_s z^{r_s}$ ($r_1 < r_2 < \dots < r_k \in \mathbb{Z}$ being chosen, we introduce two polynomials

$$\underline{q} \stackrel{\text{def}}{=} \frac{d}{dz} (z^{-r_1} p_k(z))$$

and

$$\bar{q} \stackrel{\text{def}}{=} \frac{d}{dz} (z^{-r_k} p_k(z)) .$$

Obviously, $\text{ord } \underline{q} = \text{ord } \bar{q} = k - 1$. We have

$$\|\underline{q}\|_W = \sum_{s=1}^k |d_s| (r_s - r_1), \quad \|\bar{q}\|_W = \sum_{s=1}^k |d_s| (r_k - r_s),$$

whence

$$\|\underline{q}\|_W + \|\bar{q}\|_W = (r_k - r_1) \sum_{s=1}^k |d_s| = r \|p_k\|_W,$$

where $r \stackrel{\text{def}}{=} r_k - r_1$. Hence at least one of the norms larger than or equal to $\frac{r}{2} \|p_k\|_W$. We assume $\|\bar{q}\|_W \geq \frac{r}{2} \|p_k\|_W$ (the other case is similar). Put $p_{k-1}(z) = \frac{\pi}{8r} \bar{q}(z)$, then conditions (2) and (3) are satisfied. It remains to check condition (4). Since $r_1 < r_2 < \dots < r_k \in \mathbb{Z}$, let $g(\frac{1}{z}) = z^{-r_k} p_k(z)$, then $g(z)$ is an algebraic polynomial of degree r . Then

$$\bar{q}(z) = \frac{d}{dz} (z^{-r_k} p_k(z)) = \frac{d}{dz} \left(g \left(\frac{1}{z} \right) \right) = -\frac{1}{z^2} g' \left(\frac{1}{z} \right).$$

Since $g\left(\frac{1}{z}\right)$ is an algebraic polynomial of degree r , we can use Lemma 1 and get ³

$$\mu(\{z \in \mathbb{T} : \varphi_k(z) > t\}) = \mu\left(\{z \in \mathbb{T} : \left|\frac{g'(1/z)}{g(1/z)}\right| > \frac{8r}{\pi}t\}\right) \leq \frac{1}{t}$$

since

$$\left|\frac{p_{k-1}}{p_k} = \frac{\pi}{8r} \frac{\bar{q}(z)}{p_k}\right| = \left|\frac{\pi}{8r} \frac{g'(1/z)(-1/z^2)}{g(1/z)z^{rk}}\right| = \frac{\pi}{8r} \left|\frac{g'(1/z)}{g(1/z)}\right|.$$

The above inequality also explains how the weird coefficient $\frac{\pi}{16}$ of condition (3) chooses.

Step 2. Before proving the theorem, we first illustrate what the step 2 does. By step 1, we have constructed a sequence of polynomials and they have the relation

$$\|p_{k-1}\|_W \geq \frac{\pi}{16} \|p_k\|_W.$$

Hence we can get

$$\left(\frac{\pi}{16}\right)^{n-1} \|p\|_W \leq \|p_1\|_W.$$

Since $\text{ord } p_1 = 1$, the norm of p_1 is equivalent to any $|p_1(z)|$. We want to get the inequality (12), that means we may need to establish the inequality between $|p_1(z)|$ and $|p(z)|$ for $z \in E$. More precisely, we want to find some point $z_0 \in E$ such that

$$\left|\frac{p_1(z_0)}{p(z_0)}\right| < \text{some large number.} \quad (13)$$

The constant can be chosen large enough so that the measure of points which don't satisfy condition (13) is less than $\mu(E)$, hence cannot cover all points of E , i.e., the point $z_0 \in E$ satisfies the condition exists.

Now we estimate the measure of the set of all points $z \in \mathbb{T}$ for which $|p_1(z)|$ is essentially greater than $|p_n(z)| = |p(z)|$ (the meaning of "essentially greater" would be clear later). We have

$$\left|\frac{p_1(z)}{p_n(z)}\right| = \prod_{k=2}^n \varphi_k(z) \leq \exp\left(\sum_{k=2}^n \psi_k(z)\right),$$

where $\psi_k(z) \stackrel{\text{def}}{=} \log_+ \varphi_k(z)$ ($\log_+ x$ means $\log_+ x = 0$ if $\log x < 0$). The weak type estimate of φ_k gives the inequality

$$\mu(\psi_k > t) \leq e^{-t}$$

³In Lemma 1, the term $\left|\frac{P'(z)}{P(z)}\right|$ can be changed into $\left|\frac{P'(1/z)}{P(1/z)}\right|$ since the substitution $z \mapsto 1/z$ preserves Lebesgue measure on the unit circumference.

for all $t > 0$. Let $\alpha > 0$, we decompose $\psi_k(z)$ into the sum of $\eta_k(z) \stackrel{\text{def}}{=} \min(\psi_k(z), \alpha)$ and $\omega_k(z) \stackrel{\text{def}}{=} \psi_k(z) - \eta_k(z)$. Then $\sum_{k=2}^n \eta_k(z) \leq \alpha(n-1)$ for all $z \in \mathbb{T}$. Since for a nonnegative measurable function in measure space (X, \mathcal{M}, μ) we have

$$\int f(x) d\mu(x) = \int_0^\infty \mu(f(x) > t) dt,$$

we obtain

$$\int_{\mathbb{T}} \omega_k(z) d\mu(z) = \int_\alpha^\infty \mu(\psi_k > t) dt \leq \int_\alpha^\infty e^{-t} dt = e^{-\alpha}.$$

Therefore,

$$\int_{\mathbb{T}} \left(\sum_{k=2}^n \omega_k(z) \right) d\mu(z) \leq e^{-\alpha}(n-1). \quad (14)$$

Since

$$\sum_{k=2}^n \omega_k(z) = \sum_{k=2}^n \psi_k(z) - \sum_{k=2}^n \eta_k(z)$$

and $\sum_{k=2}^n \eta_k(z) \leq \alpha(n-1)$, we have

$$\mu \left(\left\{ z \in \mathbb{T} : \sum_{k=2}^n \psi_k(z) > (\alpha+1)(n-1) \right\} \right) \leq \mu \left(\left\{ z \in \mathbb{T} : \sum_{k=2}^n \omega_k(z) > n-1 \right\} \right).$$

Let $F \stackrel{\text{def}}{=} \{z \in \mathbb{T} : \sum_{k=2}^n \omega_k(z) > n-1\}$, then we have

$$\mu(F) < \frac{1}{n-1} \int_F \sum_{k=2}^n \omega_k(z) d\mu(z) \leq e^{-\alpha}$$

by using (14). Hence

$$\mu \left(\left\{ z \in \mathbb{T} : \sum_{k=2}^n \psi_k(z) > (\alpha+1)(n-1) \right\} \right) < e^{-\alpha}. \quad (15)$$

Let $\alpha = \log \frac{1}{\mu(E)}$, then $e^{-\alpha} = \mu(E)$. Substitute this into (refexists) then this inequality implies that there exists a point $z_0 \in E$ for which $\sum_{k=2}^n \psi_k(z_0) \leq$

$(\alpha + 1)(n - 1)$. Now we have

$$\begin{aligned}
\left(\frac{\pi}{16}\right)^{n-1} \|p\|_W &\leq \|p_1\|_W^{\text{(ord } p_1 = 1!)} \|p_1(z_0)\| \\
&\leq \exp\left(\left(1 + \log \frac{1}{\mu(E)}\right)(n-1)\right) |p(z_0)| \\
&= \left(\frac{e}{\mu(E)}\right)^{n-1} |p(z_0)| \\
&\leq \left(\frac{e}{\mu(E)}\right)^{n-1} \sup_{z \in E} |p(z)|,
\end{aligned}$$

and the theorem is proved. \square

Remark. We first construct the polynomial sequence $p_n = p, p_{n-1}, \dots, p_2, p_1$, and they satisfy $\|p_{k-1}\|_W \geq \frac{\pi}{16} \|p_k\|_W$, $\text{ord } p_k = k$ and so on. Then we can get

$$\left(\frac{\pi}{16}\right)^{n-1} \|p\|_W \leq \|p_1\|_W.$$

This means we transform the question into the proof of the certain inequality between $\|p_1\|_W = |p_1(z)| \forall z \in \mathbb{T}$ and $p = p_n$. Then we need to find a point $z_0 \in \mathbb{T}$ such that $|p_1(z_0)| \leq \exp\left(\left(1 + \log \frac{1}{\mu(E)}\right)(n-1)\right) |p(z_0)|$, this step needs to estimate the amount or measure of the points that have large function values. If the measure of these points are smaller than $\mu(E)$, then we can get a point $z_0 \in E$ that satisfies the condition.

4 The Turan lemma in general form

Theorem 4. Let $p(t) = \sum_{k=1}^n c_k e^{i\lambda_k t}$ where $c_k \in \mathbb{C}$ and $\lambda_1 < \dots < \lambda_n \in \mathbb{R}$. If E is a measurable subset of the segment $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, then

$$\sup_{t \in I} |p(t)| \leq \left(\frac{316}{\mu(E)}\right)^{n-1} \sup_{t \in E} |p(t)|.$$

Before proving Theorem 4, we first introduce a weak type estimate:

Lemma 3. Let $g(t) = \sum_{k=1}^n c_k e^{i\lambda_k t}$, ($c_k \in \mathbb{C}$, $0 = \lambda_1 < \lambda_2 < \dots < \lambda_n = \lambda$). If $\lambda \geq n - 1$, then

$$\mu\left(\left\{t \in \left[-\frac{1}{2}, \frac{1}{2}\right] : \left|\frac{d}{dt} \log g(t)\right| > y\right\}\right) \leq \frac{29\lambda}{y}$$

for all $y > 0$.

Proof. Let z_j be the complex zeros of $g(z)$ enumerated in the order of increase of $|\operatorname{Re} z_j|$. The Langer lemma yields $|\operatorname{Re} z_j| \geq \pi \frac{j-(n-1)}{(n-1)} \geq \frac{\pi}{\lambda} (j - (n-1))$ (otherwise, there are j zeros in the strip of width $\Delta < 2\pi \frac{j-(n-1)}{(n-1)}$, but by Langer lemma the number of zero points in the strip is less than $\frac{\lambda\Delta}{2\pi} + n - 1 < \frac{\lambda(j-(n-1))}{(n-1)} + (n-1) = (\frac{j}{n-1} - 1)\lambda + (n-1) \leq (\frac{j}{n-1} - 1)(n-1) + (n-1) = j$). We write the Hadamard factorization

$$g(z) = ce^{az} \prod_{j \leq 2\lambda} (z - z_j) \prod_{j > 2\lambda} \left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j}} \stackrel{\text{def}}{=} ce^{az} Q(z) R(z).$$

The estimate for $\left|\frac{d}{dz} \log R(z)\right|$

Notice that $|\operatorname{Re} z_j| \geq \pi$ if $j > 2\lambda$. Let $|\operatorname{Re} z| < \frac{\pi}{2}$, then

$$\begin{aligned} \left|\frac{d}{dz} \log R(z)\right| &\leq |z| \sum_{j > 2\lambda} \frac{1}{|\operatorname{Re} z_j| (|\operatorname{Re} z_j| - \pi/2)} \\ &\leq 2|z| \sum_{j > 2\lambda} \frac{1}{|\operatorname{Re} z_j|^2} \\ &\leq 2|z| \sum_{j > 2\lambda} \frac{\lambda^2}{\pi^2 (j - (n-1))^2} \\ &\leq 2\frac{\lambda^2}{\pi^2} |z| \sum_{j > 2\lambda} \int_{j-(n-1)-1/2}^{j-(n-1)+1/2} \frac{dt}{t^2}. \end{aligned}$$

But if $j > 2\lambda > 2(n-1)$, then $j \geq 2n-1$, and $j - (n-1) - 1/2 \geq j/2 \geq \lambda$. Therefore

$$\sum_{j > 2\lambda} \int_{j-(n-1)-1/2}^{j-(n-1)+1/2} \frac{dt}{t^2} \leq \int_{\lambda}^{\infty} \frac{dt}{t^2} = \frac{1}{\lambda}$$

and $\left|\frac{d}{dz} \log R(z)\right| \leq \frac{2|z|\lambda}{\pi^2}$ if $|\operatorname{Re} z| < \pi/2$. In particular, $\left|\frac{d}{dz} \log R(z)\right| \leq \frac{\lambda}{\pi^2}$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

The estimate for $|a|$

It can be estimated by considering the argument increment of $g(z)$ along segment $\left[-i\omega \frac{\bar{a}}{|a|}, i\omega \frac{\bar{a}}{|a|}\right]$, similar to the proof of Case 1 Theorem 4 below. Here we use another approach. Consider an exponential polynomial $\tilde{g}(t) \stackrel{\text{def}}{=} e^{\lambda t} g\left(-\frac{\bar{a}}{|a|}t\right)$ on the interval $t \in \left[-\frac{3}{2}, \frac{3}{2}\right]$, then

$$\tilde{g}(t) = e^{\lambda t} \sum_{k=1}^n c_k e^{-i\lambda_k \frac{\bar{a}}{|a|}t} = \sum_{k=1}^n c_k e^{(\lambda - i\lambda_k \frac{\bar{a}}{|a|})t}.$$

Its remarkable property is that the real parts of exponent in its terms are nonnegative ($\operatorname{Re} \left(\lambda - i\lambda_k \frac{\bar{a}}{|a|} \right) \geq 0$), then it satisfies the condition $|z_j| \geq 1$ of Theorem 1). The reasoning of the first half of Section 1 ensures the estimate

$$\sup_{t \in [-\frac{3}{2}, -\frac{1}{2}]} |\tilde{g}(t)| \leq \sup_{t \in [-\frac{3}{2}, \frac{3}{2}]} |\tilde{g}(t)| \leq (12e)^{n-1} \sup_{t \in [\frac{1}{2}, \frac{3}{2}]} |\tilde{g}(t)| \leq (12e)^\lambda \sup_{t \in [\frac{1}{2}, \frac{3}{2}]} |\tilde{g}(t)|.$$

The function $Q \left(-\frac{\bar{a}}{|a|} t \right)$ is an algebraic polynomial of degree at most 2λ , consequently, it is a limit of exponential polynomials of order at most $2\lambda + 1$ with purely imaginary exponents⁴. Applying the Turan lemma again, we obtain the inequalities

$$\begin{aligned} \sup_{t \in [-\frac{3}{2}, -\frac{1}{2}]} \left| Q \left(-\frac{\bar{a}}{|a|} t \right) \right| &\geq (12e)^{-2\lambda} \sup_{t \in [-\frac{3}{2}, \frac{3}{2}]} \left| Q \left(-\frac{\bar{a}}{|a|} t \right) \right| \\ &\geq (12e)^{-2\lambda} \sup_{t \in [\frac{1}{2}, \frac{3}{2}]} \left| Q \left(-\frac{\bar{a}}{|a|} t \right) \right| \end{aligned}$$

and

$$\begin{aligned} &\inf_{t \in [-\frac{3}{2}, -\frac{1}{2}]} \left| R \left(-\frac{\bar{a}}{|a|} t \right) \right| \\ &\geq \exp \left(- \int_{-3/2}^{3/2} \left| \frac{d}{dt} \log R \left(-\frac{\bar{a}}{|a|} t \right) \right| dt \right) \sup_{t \in [-\frac{3}{2}, \frac{3}{2}]} \left| R \left(-\frac{\bar{a}}{|a|} t \right) \right| \\ &\geq e^{-\lambda/2} \sup_{t \in [\frac{1}{2}, \frac{3}{2}]} \left| R \left(-\frac{\bar{a}}{|a|} t \right) \right|. \end{aligned}$$

If $|a| > \lambda$, then $\inf_{t \in [-\frac{3}{2}, -\frac{1}{2}]} |ce^{(\lambda-|a|)t}| \geq e^{|a|-\lambda} \sup_{t \in [\frac{1}{2}, \frac{3}{2}]} |ce^{(\lambda-|a|)t}|$. Since

$$\tilde{g}(t) = ce^{(\lambda-|a|)t} Q \left(-\frac{\bar{a}}{|a|} t \right) R \left(-\frac{\bar{a}}{|a|} t \right),$$

we have

$$\sup_{t \in [-\frac{3}{2}, -\frac{1}{2}]} |\tilde{g}(t)| \geq \sup_{t \in [-\frac{3}{2}, -\frac{1}{2}]} \left| Q \left(-\frac{\bar{a}}{|a|} t \right) \right| \inf_{t \in [-\frac{3}{2}, -\frac{1}{2}]} \left| R \left(-\frac{\bar{a}}{|a|} t \right) \right| \inf_{t \in [-\frac{3}{2}, -\frac{1}{2}]} |ce^{(\lambda-|a|)t}|.$$

⁴Consider $g_\lambda(x) = \frac{e^{\lambda x}}{\lambda} - 1$, it is easy to check that

$$\sup_{x \in [0,1]} |g_\lambda - x| \leq \frac{1}{2}\lambda.$$

These inequalities may hold simultaneously only if $|a| \leq \lambda(3 \log(12e) + 1/2 + 1) \leq \frac{25}{2}\lambda$.

The polynomial $Q(z)$

By Lemma 1, the polynomial $Q(z)$ satisfies the weak type estimate

$$\mu \left(\left\{ t \in \left[-\frac{1}{2}, \frac{1}{2} \right] : \left| \frac{d}{dt} \log Q(t) \right| > y \right\} \right) \leq \frac{16\lambda}{y}$$

on the segment $\left[-\frac{1}{2}, \frac{1}{2} \right]$.

Combine all the above estimates and make use of the inequality $\left| \frac{d}{dt} \log g(t) \right| \leq |a| + \left| \frac{d}{dt} \log R(t) \right| + \left| \frac{d}{dt} \log Q(t) \right|$, we obtain

$$\begin{aligned} & \mu \left(\left\{ t \in \left[-\frac{1}{2}, \frac{1}{2} \right] : \left| \frac{d}{dt} \log g(t) \right| > y \right\} \right) \\ & \leq \mu \left(\left\{ t \in \left[-\frac{1}{2}, \frac{1}{2} \right] : \left| \frac{d}{dt} \log Q(t) \right| > y - 13\lambda \right\} \right) \\ & \leq \frac{16\lambda}{y - 13\lambda} \leq \frac{29\lambda}{y} \end{aligned}$$

for $y \geq 29\lambda$. But if $y < 29\lambda$, then the corresponding estimate becomes trivial because $\frac{29\lambda}{y} \geq 1 = \mu \left(\left[-\frac{1}{2}, \frac{1}{2} \right] \right)$. Lemma 2 is proved. \square

Now we go back to the proof of Theorem 4.

Proof of Theorem 4. Let $\lambda \stackrel{\text{def}}{=} \lambda_n - \lambda_1$, we prove the theorem separately in two cases.

Case $\lambda \leq n - 1$. If $n = 1$, the statement is obvious. Let $n > 1$, without loss of generality, we assume that $0 = \lambda_1 < \dots < \lambda_n = \lambda \leq n - 1$. By virtue of the Langer lemma, complex zeros of the exponential polynomial $p(z)$ are well separated, i.e., each vertical strip of width Δ contains at most $\frac{\Delta\lambda}{2\pi} + (n - 1) \leq \left(1 + \frac{\Delta}{2\pi}\right)(n - 1)$ zeros.

Lets enumerate z_j in the order of increase of $|\operatorname{Re} z_j|$. For every $j \in \mathbb{N}$, the inequality $|\operatorname{Re} z_j| \geq \pi \frac{j-(n-1)}{(n-1)}$ holds (otherwise the zeros z_1, \dots, z_j would lie in a vertical strip of width $\Delta < 2\pi \frac{j-(n-1)}{(n-1)}$, but this strip can contain at most $\left(1 + \frac{\Delta}{2\pi}\right)(n - 1) < j$ zeros). Now we write the Hadamard factorization of $p(z)$:

$$p(z) = ce^{az} \prod_{j=1}^{2(n-1)} (z - z_j) \prod_{j>2(n-1)} \left(1 - \frac{z}{z_j}\right) e^{z/z_j} \stackrel{\text{def}}{=} ce^{az} Q(z) R(z).$$

We shall examine the behavior of each of the above three factors separately.

The canonical product $R(z) = \prod_{j>2(n-1)} \left(1 - \frac{z}{z_j}\right) e^{z/z_j}$

First of all, we notice that $|\operatorname{Re} z_j| \geq \pi$ if $j > 2(n-1)$ (since $|\operatorname{Re} z_j| \geq \pi \frac{j-(n-1)}{(n-1)} > \pi$). We have (since $|\operatorname{Re} z| \leq 1/2 < \pi$):

$$\begin{aligned} \left| \frac{d}{dz} \log R(z) \right| &= \left| \sum_{j>2(n-1)} \left(\frac{1}{z_j} + \frac{1}{z - z_j} \right) \right| \leq |z| \sum_{j>2(n-1)} \frac{1}{|z_j| |z - z_j|} \\ &\leq |z| \sum_{j>2(n-1)} \frac{1}{|\operatorname{Re} z_j| (|\operatorname{Re} z_j| - |\operatorname{Re} z|)}. \end{aligned}$$

whence it follows, since $z \in [-\frac{1}{2}, \frac{1}{2}]$, that

$$\begin{aligned} \left| \frac{d}{dz} \log R(z) \right| &\leq |z| \sum_{j>2(n-1)} \frac{1}{|\operatorname{Re} z_j| (|\operatorname{Re} z_j| - \frac{1}{2})} \\ &\leq 2|z| \sum_{j=1}^{\infty} \frac{1}{|\operatorname{Re} z_{2(n-1)+j}|^2} \\ &\leq 2|z| \sum_{j=1}^{\infty} \frac{1}{(\pi + \frac{\pi j}{n-1})^2} \\ &\leq \frac{2(n-1)}{\pi} |z| \int_{\pi}^{\infty} \frac{dt}{t^2} \\ &= \frac{2|z|}{\pi^2} (n-1). \end{aligned}$$

Now,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{d}{dz} \log R(z) \right| dz \leq \frac{2(n-1)}{\pi^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |z| dz = \frac{n-1}{2\pi^2},$$

and, therefore

$$\max_{z \in [-\frac{1}{2}, \frac{1}{2}]} |R(z)| \leq \exp \left(\frac{n-1}{2\pi^2} \right) \min_{z \in [-\frac{1}{2}, \frac{1}{2}]} |R(z)|.$$

The factor ce^{az}

The simplest way to estimate $|\operatorname{Re} a|$ is to consider the argument increment of $p(z)$ along a segment $[-i\omega, i\omega]$ ($\omega > 0$). It follows from the proof of the Langer lemma that $|\Delta_{[-i\omega, i\omega]} \arg p| \leq \pi(n-1)$. The argument increment brought in by each of the zeros of $Q(z)$ does not exceed π . So, we have

$$\begin{aligned} |\Delta_{[-i\omega, i\omega]} \arg Q| &\leq 2\pi(n-1), \\ |\Delta_{[-i\omega, i\omega]} \arg R| &\leq \int_{-\omega}^{\omega} \left| \frac{d}{dz} \log R(it) \right| dt \leq \frac{n-1}{\pi^2} \int_{-\omega}^{\omega} |t| dt = \frac{n-1}{\pi^2} \omega^2, \end{aligned}$$

and

$$\Delta_{[-i\omega, i\omega]} \arg(cc^{az}) = 2\omega \operatorname{Re} a.$$

The identity

$$\Delta_{[-i\omega, i\omega]} \arg = 2\omega \operatorname{Re} a + \Delta_{[-i\omega, i\omega]} \arg Q + \Delta_{[-i\omega, i\omega]} \arg R$$

implies

$$|\operatorname{Re} a| \leq \min_{\omega > 0} \left(\frac{3\pi}{2\omega} + \frac{\omega}{2\pi^2} \right) (n-1) = \sqrt{\frac{3}{\pi}} (n-1).$$

It remains to examine

The behavior of the polynomial $Q(z)$.

Let $0 < h < \frac{1}{8}$. We shall carry out the Cartan lemma construction. Let n_1 be the maximal integer for which there exists a disk D_1 of radius $\frac{n_1}{n-1}h$ containing at least n_1 zeros of the polynomial Q . It is clear that D_1 contains exactly n_1 zeros of Q because otherwise n_1 could be enlarged (the strip of width $\frac{h}{n-1}$ contains at most 1 point according to Langer lemma). Let n_2 be the maximal integer for which there exists a disk D_2 of radius $\frac{n_2}{n-1}h$ containing at least n_2 zeros of Q among those not lying in D_1 , and so on, till all the zeros of Q are covered. Putting $D'_k = 2D_k$ (i.e., the disk centered at the same point and of double radius), we obtain the corresponding sequence of integers $n_1 \geq \dots \geq n_s$ with the sum $n_1 + \dots + n_s = 2(n-1)$ and the corresponding sequencedisks D'_1, \dots, D'_s with the sum of radii equal to $4h$. We fix a point

$$z \in \left[-\frac{1}{2}, \frac{1}{2} \right] \setminus \bigcup_{k=1}^s D'_k$$

and enumerate the zeros of Q in the order of increase of $|z - z_j|$. Following Cartan, we shall show that $|z - z_j| \geq \frac{j}{n-1}h$. Indeed, if this is not the case, then the disk D centered at z and of radius $\frac{j}{n-1}h$ contains at least j zeros of Q . Choose an $m \in \{1, \dots, s\}$ such that $n_1 \geq \dots \geq n_m \geq j > n_{m+1} \geq \dots \geq n_s$. For every $z \notin \bigcup_{k=1}^s D'_k$ and $k \leq m$, we have $z_j \notin D'_{n_k}$, hence the distance between z and the center of D_k is at least

$$\frac{2n_k}{n-1}h \geq \frac{n_k}{n-1}h + \frac{j}{n-1}h.$$

Hence D does not intersect any of the disks D_1, \dots, D_m . But if this were true, the disk D (or a disk with larger number of zeros) would have been taken instead of D_{m+1} at the $m+1$ -th step. This contradiction proves the claim.

Besides, the Langer lemma implies the inequality $|z - z_j| \geq \pi \frac{j-(n-1)}{(n-1)}$ (otherwise the zeros z_1, \dots, z_j would lie in a disk of radius strictly less than $\pi \frac{j-(n-1)}{(n-1)}$, and,

consequently, in the strip of width $\Delta < 2\pi^{\frac{j-(n-1)}{(n-1)}}$. Thus, we have

$$\begin{aligned}
\frac{|Q(z)|}{\max \{|Q(t)| : t \in [-\frac{1}{2}, \frac{1}{2}]\}} &\geq \prod_{j=1}^{2(n-1)} \frac{|z - z_j|}{\max \{|t - z_j| : t \in [-\frac{1}{2}, \frac{1}{2}]\}} \\
&\geq \prod_{j=1}^{2(n-1)} \frac{|z - z_j|}{1 + |z - z_j|} \\
&= \prod_{j=1}^{n-1} \frac{|z - z_j|}{1 + |z - z_j|} \times \prod_{j=1}^{n-1} \frac{|z - z_{n-1+j}|}{1 + |z - z_{n-1+j}|} \\
&\geq \prod_{j=1}^{n-1} \frac{\frac{j}{n-1}h}{1 + \frac{j}{n-1}h} \times \prod_{j=1}^{n-1} \frac{\frac{\pi j}{n-1}}{1 + \frac{\pi j}{n-1}} \\
&\geq \prod_{j=1}^{n-1} \frac{\frac{j}{n-1}h}{1 + \frac{j}{n-1}\frac{1}{8}} \times \prod_{j=1}^{n-1} \frac{\frac{\pi j}{n-1}}{1 + \frac{\pi j}{n-1}} \\
&\geq (8h)^{n-1} \times \prod_{j=1}^{n-1} \frac{\frac{j}{n-1}\frac{1}{8}}{1 + \frac{j}{n-1}\frac{1}{8}} \times \prod_{j=1}^{n-1} \frac{\frac{3j}{n-1}}{1 + \frac{3j}{n-1}}.
\end{aligned}$$

But for each $\theta > 0$ we have

$$\prod_{j=1}^{n-1} \frac{\frac{j}{n-1}\theta}{1 + \frac{j}{n-1}\theta} \geq \exp \left((n-1) \int_0^1 \log \frac{\theta t}{1 + \theta t} dt \right) = \left(\frac{\theta}{(1 + \theta)^{1 + \frac{1}{\theta}}} \right)^{n-1},$$

whence it follows that

$$\frac{|Q(z)|}{\max \{|Q(t)| : t \in [-\frac{1}{2}, \frac{1}{2}]\}} \geq (8h)^{n-1} \left(8 \times \left(\frac{9}{8} \right)^9 \times \frac{4\sqrt[3]{4}}{3} \right)^{-(n-1)} \geq \left(\frac{8h}{32\sqrt[3]{4}} \right)^{n-1}.$$

Observe that the measure of the exceptional set $[-\frac{1}{2}, \frac{1}{2}] \cap (\bigcup_{k=1}^s D'_k)$ is at most

$8h$, we can set $z \in E$ if $h = \mu(E)/8$. Combining all these estimates, we find

$$\begin{aligned}
\sup_{t \in I} |p(t)| &\leq \sup_{t \in I} |ce^{at}| \times \sup_{t \in I} |R(t)| \times \sup_{t \in I} |Q(t)| \\
&\leq \left| c \exp \left(\sqrt{\frac{3}{\pi}} (n-1) \right) \right| \times \exp \left(\frac{n-1}{2\pi^2} \right) \min_{t \in I} |R(t)| \times \sup_{t \in I} |Q(t)| \\
&\leq |c| \times \exp \left(\left(\sqrt{\frac{3}{\pi}} + \frac{1}{2\pi^2} \right) (n-1) \right) \min_{t \in I} |R(t)| \times \sup_{t \in I} |Q(t)| \\
&\leq |c| 3^{n-1} \min_{t \in I} |R(t)| \times \left(\frac{32\sqrt[3]{4}}{8h} \right)^{n-1} |Q(z)| \\
&\leq \left(\frac{154}{\mu(E)} \right)^{n-1} \sup_{t \in E} |p(t)|.
\end{aligned}$$

Case $\lambda > n - 1$. We shall reduce this case to Case 1 in the same way as in Section 3. This is why we need Lemma 3. We can finish the proof by constructing a sequence of exponential polynomials $p_n, p_{n-1}, \dots, p_s (s \geq 1)$ such that

- (1) $p_n = p$;
- (2) $\text{ord } p_k = k$ ($k = s, \dots, n$) ;
- (3) $\|p_{k-1}\|_\infty \geq \frac{1}{58} \|p_k\|_\infty$ ($k = s+1, \dots, n$) ;
- (4) the ratio $\varphi_k \stackrel{\text{def}}{=} \left| \frac{p_{k-1}}{p_k} \right|$ satisfies the weak type estimate

$$\mu \left(\left\{ x \in \left[-\frac{1}{2}, \frac{1}{2} : \varphi_k(x) > t \right] \right\} \right) \leq \frac{1}{t}$$

for $t > 0$;

- (5) the difference between the greatest and the smallest exponent of p_s does not exceed $s - 1$ (i.e., p_s meets the condition of Case 1 investigated above).

The construction is almost the same as in Section 3. The difference is that, firstly, we make use of the identity $\underline{q}(t) - \bar{q}(t) = i(\rho_k - \rho_1) p_k(t)$, where

$$p_k(t) \stackrel{\text{def}}{=} \sum_{m=1}^k d_m e^{i\rho_m t} \quad (\rho_1 < \dots < \rho_k \in \mathbb{R}),$$

$$\underline{q}(t) \stackrel{\text{def}}{=} e^{i\rho_1 t} \frac{d}{dt} (e^{-i\rho_1 t} p_k(t)),$$

$$\bar{q}(t) \stackrel{\text{def}}{=} e^{i\rho_k t} \frac{d}{dt} (e^{-i\rho_k t} p_k(t))$$

to estimate the sum of norms $\|\underline{q}\|_\infty + \|\bar{q}\|_\infty$ from below, and, secondly, we stop the sequence at the polynomial p_s satisfying the condition of Case 1, i.e., at that very moment when we cannot apply Lemma 3 to estimate φ_s once more.

Since $\|p_{k-1}\|_\infty \geq \left(\frac{1}{58}\right) \|p_k\|_\infty$, we obtain

$$\left(\frac{1}{58}\right)^{n-s} \|p\|_\infty \leq \|p_s\|_\infty. \quad (16)$$

By the construction procedure, p_s satisfies the condition of Case 1, hance for a measurable set F we have

$$\|p_s\|_\infty \leq \left(\frac{154}{\mu(F)}\right)^{s-1} \sup_{t \in F} |p_s(t)|. \quad (17)$$

Now we use the same reasoning as in Section 3 to establish $\left|\frac{p_s(t)}{p_n(t)}\right| \leq \left(\frac{2e}{\mu(E)}\right)^{n-s}$ outside an exceptional set E' of measure $\mu(E') \leq \mu(E)/2$. We have

$$\left|\frac{p_s(x)}{p_n(x)}\right| = \prod_{k=s+1}^n \varphi_k(z) \leq \exp\left(\sum_{k=s+1}^n \psi_k(x)\right),$$

where $\psi_k(x) \stackrel{\text{def}}{=} \log_+ \varphi_k(x)$. The weak type estimate of φ_k gives the inequality $\mu(\psi_k > t) \leq e^{-t}$ for all $t > 0$. Let $\alpha > 0$, we decompose $\psi_k(x)$ into the sum of $\eta_k(x) \stackrel{\text{def}}{=} \min(\psi_k(x), \alpha)$ and $\omega_k(x) \stackrel{\text{def}}{=} \psi_k(x) - \eta_k(x)$. Then $\sum_{k=s+1}^n \eta_k(x) \leq \alpha(n-s)$ for all $x \in [-\frac{1}{2}, \frac{1}{2}]$. We also have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \omega_k(x) dx = \int_\alpha^\infty \mu(\psi_k > t) dt \leq \int_\alpha^\infty e^{-t} dt = e^{-\alpha}.$$

Therefore,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k=s+1}^n \omega_k(z)\right) d\mu(z) \leq e^{-\alpha}(n-s).$$

Since

$$\sum_{k=s+1}^n \omega_k(x) = \sum_{k=s+1}^n \psi_k(x) - \sum_{k=s+1}^n \eta_k(x)$$

and $\sum_{k=s+1}^n \eta_k(x) \leq \alpha(n-s)$, we have

$$\begin{aligned} & \mu \left(\left\{ x \in \left[-\frac{1}{2}, \frac{1}{2} \right] : \sum_{k=s+1}^n \psi_k(x) > (\alpha+1)(n-s) \right\} \right) \\ & \leq \mu \left(\left\{ x \in \left[-\frac{1}{2}, \frac{1}{2} \right] : \sum_{k=s+1}^n \omega_k(x) > n-s \right\} \right) < e^{-\alpha}. \end{aligned}$$

Let $\alpha = \log \left(\frac{2}{\mu(E)} \right)$, then we have

$$\mu \left(\left\{ x \in \left[-\frac{1}{2}, \frac{1}{2} \right] : \sum_{k=s+1}^n \psi_k(x) > (\alpha+1)(n-s) \right\} \right) < \frac{\mu(E)}{2}.$$

Thus the measure of the set $E' = \left\{ x \in \left[-\frac{1}{2}, \frac{1}{2} \right] : \left| \frac{p_s(x)}{p_n(x)} \right| > \left(\frac{2e}{\mu(E)} \right)^{n-s} \right\}$ satisfies

$$\mu(E') < \frac{\mu(E)}{2}$$

and hence

$$\mu(E \setminus E') \geq \frac{\mu(E)}{2}. \quad (18)$$

By definition of the set E' , we know $\left| \frac{p_s(x)}{p_n(x)} \right| \leq \left(\frac{2e}{\mu(E)} \right)^{n-s}$ for each $x \in E \setminus E'$. By using (17) (let $F = E \setminus E'$), (16) and (18) we obtain

$$\begin{aligned} \left(\frac{1}{58} \right)^{n-s} \|p\|_{\infty} & \leq \|p_s\|_{\infty} \leq \left(\frac{154}{\mu(E \setminus E')} \right)^{s-1} \sup_{t \in E \setminus E'} |p_s(t)| \\ & \leq \left(\frac{308}{\mu(E)} \right)^{s-1} \left(\frac{2e}{\mu(E)} \right)^{n-s} \sup_{t \in E} |p(t)|. \end{aligned}$$

Now Theorem (4) easily follows if we take into account the inequality $116e < 316$.

□

5 Summary: Two important techniques used

- Construct a sequence of polynomials like p_k, p_{k-1}, \dots, p_1 to decrease the order of p_k . In this note, the order is the order of exponential polynomials, it may have different meaning when we solve other problems.
- Weak type estimates allow us to get an upper bound of a measure of a set A that satisfies some property P , then compare it to the measure of a given set B . If the latter is strictly larger than the former, then there must be some point in B which does not meet the property P .

A Harmonic measure

Let \mathbb{H} be the upper half-plane. Suppose $a < b$ are real. Then the function

$$\theta = \theta(z) = \arg \left(\frac{z - b}{z - a} \right) = \operatorname{Im} \log \left(\frac{z - b}{z - a} \right)$$

is harmonic on \mathbb{H} , and $\theta = \pi$ on (a, b) and $\theta = 0$ on $\mathbb{R} \setminus [a, b]$.

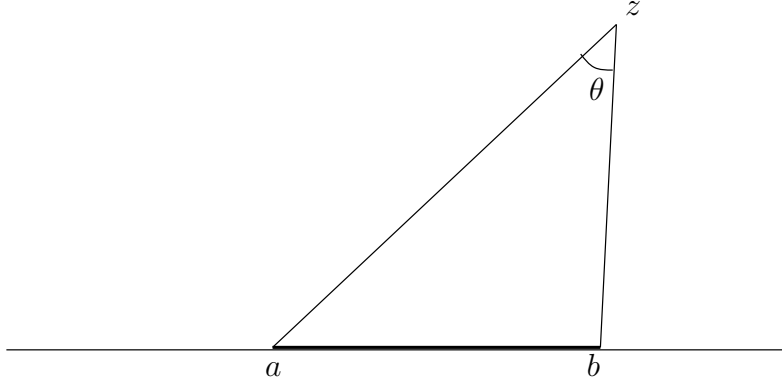


Figure 4: Harmonic function $\theta(z)$

Viewed geometrically, $\theta(z) = \operatorname{Re} \varphi(z)$ where $\varphi(z)$ is any conformal mapping from \mathbb{H} to the strip $\{0 < \operatorname{Re} z < \pi\}$ which maps (a, b) onto $\{z : \operatorname{Re} z = \pi\}$ and $\mathbb{R} \setminus [a, b]$ into $\{z : \operatorname{Re} z = 0\}$.

Definition 2. Let $E \subset \mathbb{R}$ be a finite union of open intervals and write $E = \bigcup_{j=1}^n (a_j, b_j)$ with $b_{j-1} < a_j < b_j$. Set

$$\theta_j = \theta_j(z) = \arg \left(\frac{z - b_j}{z - a_j} \right).$$

Then the harmonic measure of E at $z \in \mathbb{H}$ is

$$\omega(z, E, \mathbb{H}) \stackrel{\text{def}}{=} \sum_{j=1}^n \frac{\theta_j}{\pi}. \tag{19}$$

It satisfies the following properties:

- a. $0 < \omega(z, E, \mathbb{H}) < 1$ for $z \in \mathbb{H}$,
- b. $\omega(z, E, \mathbb{H}) \rightarrow 1$ as $z \rightarrow E$, and
- c. $\omega(z, E, \mathbb{H}) \rightarrow 0$ as $z \rightarrow \mathbb{R} \setminus \overline{E}$.

The function $\omega(z, E, \mathbb{H})$ is the unique harmonic function on \mathbb{H} that satisfies a,b and c. The uniqueness of $\omega(z, E, \mathbb{H})$ is a consequence of Lindelöf's maximum principle (see [1, p. 2]).

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