Notes about Loukas Grafakos' Classical Fourier Analysis

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Abstract

This is a learning note about Chapter 1 of Grafako's *Classical Fourier Analysis*.

Contents

1 L^p Spaces and Interpolation

1.1 L^p and Weak L^p

Definition 1.1. For f a measurable function on X, the *distribution function* of f is the function d_f defined on $[0, \infty)$ as follows:

$$d_f(\alpha) = \mu\left(\{x \in X : |f(x)| > \alpha\}\right).$$

Proposition 1.2. Let (X, μ) be a σ-finite measure space. Then for f in $L^p(X, \mu)$, $0 < p < \infty$, we have

$$\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha) \mathrm{d}\alpha. \tag{1}$$

Moreover, for any increasing continuously differentiable function φ on $[0,\infty)$ with $\varphi(0)=0$ and every measurable function f on X with $\varphi(|f|)$ integrable on X, we have

$$\int_X \varphi(|f|)\mathrm{d}\mu = \int_0^\infty \varphi'(\alpha) d_f(\alpha)\mathrm{d}\alpha. \tag{2}$$

Proof.

$$\begin{aligned}
p\int_0^\infty \alpha^{p-1} d_f(\alpha)\mathrm{d}\alpha =& p\int_0^\infty \alpha^{p-1}\int_X \chi_{\{x:|f(x)|>\alpha\}}\mathrm{d}\mu(x)\mathrm{d}\alpha\\
=&\int_X\int_0^{|f(x)|} p\alpha^{p-1}\mathrm{d}\alpha\mathrm{d}\mu(x)\\
=&\int_X |f(x)|^p\mathrm{d}\mu(x)\\
=&\|f\|_{L^p}^p.
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty \varphi'(\alpha)d_f(\alpha)\mathrm{d}\alpha =& \int_0^\infty \varphi'(\alpha)\int_X \chi_{\{x:|f(x)|>\alpha\}}\mathrm{d}\mu(x)\mathrm{d}\alpha\\
=&\int_X\int_0^{|f(x)|}\varphi'(\alpha)\mathrm{d}\alpha\mathrm{d}\mu(x)\\
=&\int_X \varphi(|f|)\mathrm{d}\mu.
\end{aligned}$$

□

Definition 1.3. For $0<p<\infty$, the space *weak* $L^p(X,\mu)$ is defined as the set of all μ-measurable functions f such that

$$\begin{aligned}
\|f\|_{L^{p,\infty}} =& \inf\left\{C>0 : d_f(\alpha)\le \frac{C^p}{\alpha^p} \text{ for all } \alpha>0\right\}\\
=& \sup\left\{\gamma d_f(\gamma)^{1/p} : \gamma>0\right\}
\end{aligned}$$

is finite. The space *weak* $L^\infty(X,\mu)$ is by definition $L^\infty(X,\mu)$.

Proposition 1.4. Since

$$\alpha^p d_f(\alpha) \le \int_{\{x:|f(x)|>\alpha\}} |f(x)|^p\mathrm{d}\mu(x) \le \|f\|_{L^p}^p,$$

we get

$$\|f\|_{L^{p,\infty}} \le \|f\|_{L^p}$$

for any f in $L^{p(X,\mu)}$. Hence the embedding $L^p(X,\mu)\subset L^{p,\infty}(X,\mu)$ holds.

1.2 Convergence in Measure

Definition 1.5. Let $f, f_n, n = 1, 2, \cdots$, be measurable functions on the measure space (X, μ). The sequence f_n is said to *converge in measure* to f if for all $\varepsilon > 0$ there exists an $n_0 \in \mathbb{Z}^+$ such that

$$n > n_0 \Longrightarrow \mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) < \varepsilon. \tag{3}$$

Proposition 1.6. The preceding definition is equivalent to the following statement:

$$\text{For all } \varepsilon > 0 \quad \lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) = 0. \tag{4}$$

Proof. Clearly (4) implies (3). To see the converse, given $\varepsilon > 0$, pick $0 < \delta < \varepsilon$ and apply (3) for this δ. □

Proposition 1.7. Let $0 < p < \infty$ and f_n, f be in $L^{p,\infty}(X, \mu)$.

a. If f_n, f are in L^p and $f_n \to f$ in L^p, then $f_n \to f$ in $L^{p,\infty}$.

b. If $f_n \to f$ in $L^{p,\infty}$, then f_n converges to f in measure.

Theorem 1.8. *Let f_n and f be complex-valued measurable functions on a measure space (X, μ) and suppose that f_n converges to f in measure. Then some subsequence of f_n converges to f μ-a.e.*

Proof. For all $k = 1, 2, \cdots$, choose inductively n_k such that

$$\mu\left(\left\{x \in X : |f_{n_k}(x) - f(x)| > 2^{-k}\right\}\right) < 2^{-k} \tag{5}$$

and such that $n_1 < n_2 < \cdots < n_k < \cdots$. Define the sets

$$A_k = \left\{x \in X : |f_{n_k}(x) - f(x)| > 2^{-k}\right\}. \tag{6}$$

The left work is to prove

$$\mu\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right) = 0 \tag{7}$$

□

Definition 1.9. We say that a sequence of measurable functions $\{f_n\}$ on the measure space (X, μ) is *Cauchy in measure* if for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{Z}^+$ such that for $n, m > n_0$ we have

$$\mu\left(\{x \in X : |f_m(x) - f_n(x)| > \varepsilon\}\right) < \varepsilon.$$

Theorem 1.10. *Let f_n be a complex-valued sequence that is Cauchy in measure. Then some subsequence of f_n converges μ-a.e.*

Proof. The proof is similar to Theorem 1.8. □

1.3 A First Glimpse at Interpolation

Proposition 1.11. Let $0 < p < q \leq \infty$ and let f in $L^{p,\infty}(X,\mu) \cap L^{q,\infty}(X,\mu)$, where X is a σ-finite measure space. Then f is in $L^r(X,\mu)$ for all $p < r < q$ and

$$\|f\|_{L^r} \leq \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{\frac{1}{r}} \|f\|_{L^{p,\infty}}^{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|f\|_{L^{q,\infty}}^{\frac{\frac{1}{p}-\frac{1}{r}}{\frac{1}{p}-\frac{1}{q}}}, \tag{8}$$

with the interpretation that $\frac{1}{\infty} = 0$.

Proof. First assume $q < \infty$. We know that

$$d_f(\alpha) \leq \min\left(\frac{\|f\|_{L^{p,\infty}}}{\alpha^p}, \frac{\|f\|_{L^{q,\infty}}^q}{\alpha^q}\right). \tag{9}$$

Set

$$B = \left(\frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p}\right)^{\frac{1}{q-p}}. \tag{10}$$

We now estimate the L^r norm of f.

$$\begin{aligned}
\|f\|_{L^r(X,\mu)}^r =& r\int_0^\infty \alpha^{r-1} d_f(\alpha)\mathrm{d}\alpha \\
\leq& r\int_0^\infty \alpha^{r-1}\min\left(\frac{\|f\|_{L^{p,\infty}}^p}{\alpha^p}, \frac{\|f\|_{L^{q,\infty}}^q}{\alpha^q}\right)\mathrm{d}\alpha \\
=& r\int_0^B \alpha^{r-1-p}\|f\|_{L^{p,\infty}}^p \mathrm{d}\alpha + r\int_B^\infty \alpha^{r-1-q}\|f\|_{L^{q,\infty}}^q \mathrm{d}\alpha \\
=& \frac{r}{r-p}\|f\|_{L^{p,\infty}}^p B^{r-p} + \frac{r}{q-r}\|f\|_{L^{q,\infty}}^q B^{r-q} \\
=& \left(\frac{r}{r-p} + \frac{r}{q-r}\right)\left(\|f\|_{L^{p,\infty}}^p\right)^{\frac{q-r}{q-p}}\left(\|f\|_{L^{q,\infty}}^q\right)^{\frac{r-p}{q-p}}.
\end{aligned}$$

The case $q = \infty$ is easier and the consequence is

$$\|f\|_{L^r}^r \leq \frac{r}{r-p}\|f\|_{L^{p,\infty}}^p \|f\|_{L^\infty}^{r-p}.$$

□

Definition 1.12. For $0 < p < \infty$, the space $L^p_{\mathrm{loc}}(\mathbb{R}^n, |\cdot|)$ or simply $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is the set of all Lebesgue-measurable functions f on $\mathbb{R}^n$ that satisfy

$$\int_K |f(x)|^p \mathrm{d}x < \infty \tag{11}$$

for any compact subset K of $\mathbb{R}^n$. Functions that satisfy (12) with $p = 1$ are called *locally integrable* functions on $\mathbb{R}^n$.

2 Convolution an Approximate Identities

2.1 Convolutin

Definition 2.1. Let f, g be in $L^1(G)$. Define the *convolution* $f \cdot g$ by

$$(f \cdot g)(x) = \int_G f(y)g(y^{-1}x)\mathrm{d}\lambda(y). \tag{12}$$

For instance, if $G = \mathbb{R}^n$ with the usual additive structure, then $y^{-1} = -y$ and the integral in (12) is written as

$$(f \cdot g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\mathrm{d}y.$$

The right-hand side of (12) is defined a.e., since the following double integral converge absolutely:

$$\begin{aligned}\int_G \int_G |f(y) \mid g(y^{-1}x)|\mathrm{d}\mathrm{d}\lambda(y)\mathrm{d}\lambda(x) &= \int_G \int_G |f(y) \mid g(y^{-1}x)|\mathrm{d}\lambda(x)\mathrm{d}\lambda(y) \\ &= \int |f(y)| \int_G |g(y^{-1}x)|\mathrm{d}\lambda(x)\mathrm{d}\lambda(y) \\ &= \int_G |f(y)| \int_G |g(x)|\mathrm{d}\lambda(x)\mathrm{d}\lambda(y) \\ &= \|f\|_{L^1(G)}\|g\|_{L^1(G)} < +\infty.\end{aligned} \tag{13}$$

The change of variables $z = x^{-1}y$ yields that (12) is in fact equal to

$$(f * g)(x) = \int_G f(xz)g(z^{-1})\mathrm{d}\lambda(z) \tag{14}$$

where the substitution of $\mathrm{d}\lambda(y)$ by $\mathrm{d}\lambda(z)$ is justified by left invariance.

Proposition 2.2. For all f, g, h in $L^1(G)$, the following properties are valid:

a. $f * (g * h) = (f * g) * h$ (associativity),

b. $f * (g + h) = f * g + f * h$ and $(f + g) * h = f * h + g * h$ (distributivity).

These imply that $L^1(G)$ is a (not necessarily commutative) Banach algebra under the convolution product.

2.2 Basic Convolution Inequalities

Theorem 2.3 (Minkowski's inequality). *Let $1 \leq p \leq \infty$, $f \in L^p(G), g \in L^1(G)$, we have that $g * f$ exists λ-a.e. and satisfies*

$$\|g * f\|_{L^p(G)} \leq \|g\|_{L^1}(G)\|f\|_{L^p(G)}. \tag{15}$$