Notes about
Modular Forms

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Abstract

This is a learning note about Zhang Yitang's lectures on modular forms.

Contents

1 Modular Group, Congruence Subgroup and Modular Forms

Definition 1.1. The *modular group* is the group of 2-by-2 matrices with integer entries and determinant 1:

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

The *principal congruence subgroup of level N* is

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N, b \equiv c \equiv 0 \pmod N \right\}.$$

Γ is a congruence subgroup if

$$\Gamma(N) \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z}) \text{ for some } N.$$

Example 1.1. $\forall N \in \mathbb{N}$,

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c, d & \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N, c \equiv 0 \pmod N \right\}$$

are congruence subgroups. Their relations are

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_2(N) \subset \mathrm{SL}_2(\mathbb{Z}).$$

Definition 1.2. $\mathcal{H}$ is the *upper half plane* defined by

$$\mathcal{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}.$$

Action of $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathcal{H}$ is defined by

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$$

for arbitrary $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d} = \frac{(a\tau + b)(c\overline{\tau} + d}{|c\tau + d|^2} = \frac{ac|\tau|^2 + bd + ad\tau + bc\overline{\tau}}{|c\tau + d|^2},$$

then

$$\mathrm{Im}\,\gamma(\tau) = \frac{\mathrm{Im}(ad\tau + bc\overline{\tau})}{|c\tau + d|^2} = \frac{(ad - bc)\,\mathrm{Im}\,\tau}{|c\tau + d|^2} > 0.$$

Hence $\gamma(\tau) \in \mathcal{H}$ if $\tau \in \mathcal{H}$.

Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\gamma' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, then

$$\gamma\gamma' = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}.$$

It's easy to verify

$$\gamma(\gamma'(\tau)) = \gamma\gamma'(\tau).$$

Now we consider actions of $\mathrm{SL}_2(\mathbb{Z})$ on functions $f : \mathcal{H} \to \mathbb{C}$. Write

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad \tau \in \mathcal{H}.$$

$$j(\gamma, \tau) := c\tau + d.$$

For $k \in \mathbb{Z}$, define $[\gamma]_k :=$ the weight-k operator acting on functions $\mathcal{H} \to \mathbb{C}$ such that

$$\left(f[\gamma_k]\right)(\tau) = j(\gamma, \tau)^{-k} f(\gamma(\tau)) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

Lemma 1.3. $\forall \gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z}), \forall \tau \in \mathcal{H}$, we have

a. $j(\gamma\gamma', \tau) = j(\gamma, \gamma'(\tau)) j(\gamma', \tau)$;

b. $[\gamma\gamma']_k = [\gamma]_k [\gamma']_k$;

c. $\frac{\mathrm{d}\gamma(\tau)}{\mathrm{d}\tau} = \frac{1}{j(\gamma,\tau)^2}$.

Proof.

a.

$$\gamma \begin{bmatrix} \tau \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma(\tau) \\ 1 \end{bmatrix} j(\gamma, \tau)$$
$$\gamma\gamma' \begin{bmatrix} \tau \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma\gamma'(\tau) \\ 1 \end{bmatrix} j(\gamma\gamma', \tau)$$
$$.$$

Also

$$\begin{aligned} \gamma\gamma' \begin{bmatrix} \tau \\ 1 \end{bmatrix} &= \gamma \begin{bmatrix} \gamma'(\tau) \\ 1 \end{bmatrix} j(\gamma', \tau) \\ &= \begin{bmatrix} \gamma\gamma'(\tau) \\ 1 \end{bmatrix} j(\gamma, \gamma'(\tau)) j(\gamma', \tau). \end{aligned}$$
$$\Rightarrow j(\gamma\gamma', \tau) = j(\gamma, \gamma'(\tau)) j(\gamma', \tau).$$

b.

$$\begin{aligned} &(f[\gamma\gamma']_k)(\tau) \\ =& j(\gamma\gamma', \tau)^{-k} f(\gamma\gamma'(\tau)) \\ =& j(\gamma, \gamma'(\tau))^{-k} j(\gamma', \tau)^{-k} f(\gamma\gamma'(\tau)) \\ =& j(\gamma', \tau) f([\gamma]_k)(\gamma'(\tau)) \\ =& (f[\gamma]_k[\gamma']_k)(\tau). \end{aligned}$$

c.

$$\begin{aligned}\frac{\mathrm{d}\gamma(\tau)}{\mathrm{d}\tau} =& \frac{a(c\tau+d)-(a\tau+b)c}{(c\tau+d)^2}\\ =& \frac{1}{(c\tau+d)^2}\\ =& \frac{1}{j(\gamma,\tau)^2}.\end{aligned}$$

$\square$

Definition 1.4. Let $\Gamma =$ congruence subgroup and $k \in \mathbb{Z}$ $f : \mathcal{H} \to \mathbb{C}$ is a weakly modular form function of weight k with repect to Γ if f is meromorphic on $\mathcal{H}$ and

$$f[\gamma]_k = f,$$

i.e.,

$$(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f.$$

Suppose f is a weakly modular function of weight k with respect to Γ.

$$\Gamma \supset \Gamma(N) \Rightarrow \begin{bmatrix}1 & N\\ 0 & 1\end{bmatrix} \in \Gamma$$

$$f\begin{bmatrix}1 & N\\ 0 & 1\end{bmatrix}_k = f(\tau+N) = f(\tau)$$

$$\Rightarrow \exists \text{ minimal } h \in \mathbb{N} \text{ such that } f(\tau+h) = f(\tau)$$

$$\Rightarrow f(\tau) = g(e^{2\pi i\tau/h}) \text{ for some } g.$$

$$\tau \in \mathcal{H} \Leftrightarrow |e^{2\pi i\tau/h|} < 1$$

$$\mathrm{Im}(\tau) \to \infty \Rightarrow e^{2\pi i\tau/h} \to 0.$$

$g(z)$ is meromorphic on $0 < |z| < 1$.
$z \to 0 : \; g(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.
$z = e^{2\pi i\tau/h} \Rightarrow \mathrm{Im}(\tau) \to \infty$

$$f(\tau) = g(e^{2\pi i\tau/h}) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\tau/h}. \tag{1}$$

We say $f(\tau)$ is holomorphic at ∞ if $a_n = 0$ for all $n < 0$ in (1). In this case we write $f(\infty) = a_0$.

$\forall \sigma \in \mathrm{SL}_2(\mathbb{Z})$, $f[\sigma]_k$ is a weakly modular function of weight k with respect to $\sigma^{-1}\Gamma\sigma$. Indeed, let $\gamma \in \Gamma$,

$$f[\gamma]_k(\tau) = f(\tau)$$
$$j(\gamma,\tau)^{-k} f(\gamma(\tau)) = f(\tau)$$
$$j(\gamma,\sigma(\tau))^{-k} f(\gamma\sigma(\tau)) = f(\sigma(\tau))$$

$$\begin{aligned}((f[\sigma]_k)[\sigma^{-1}\gamma\sigma]_k)(\tau) =& j(\sigma^{-1}\gamma\sigma,\tau)^{-k}(f[\sigma]_k)(\sigma^{-1}\gamma\sigma(\tau)) \\ =& j(\sigma^{-1}\gamma\sigma,\tau)^{-k} j(\sigma,\sigma^{-1}\gamma\sigma(\tau))^{-k} f\left(\gamma\sigma(\tau)\right) \\ =& j(\sigma^{-1}\gamma\sigma,\tau)^{-k} j(\sigma,\sigma^{-1}\gamma\sigma(\tau))^{-k} j(\gamma,\sigma(\tau))^{k} f(\sigma(\tau)) \\ =& j(\sigma,\tau)^{-k} f(\sigma(\tau)) \\ =& f[\sigma]_k.\end{aligned}$$

Definition 1.5.

a. $f : \mathcal{H} \to \mathbb{C}$ is a modular formof weight k with respect to Γ if f is weakly modular function and $f\sigma_k$ is holomorphic at $\infty \forall \sigma \in \mathrm{SL}_2(\mathbb{Z})$.

b. f is a cusp form if f is a modular form and $f\sigma_k$ vanishes at ∞ $\forall \sigma \in \mathrm{SL}_2(\mathbb{Z})$.

Proposition 1.6. Assume $k =$ odd, Then any modular form $f \equiv 0$.

Proof. $\gamma = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \Gamma$, for arbitrary $\tau \in \mathcal{H}$,

$$j(\gamma,\tau)^{-k} f(\gamma(\tau)) = (-1)^{-k} f(\frac{-\tau+0}{0+(-1)}) = (-1)^{-k} f(\tau) = -f(\tau).$$

□

2 Case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, Eisenstein Series

In this section assume $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Γ is generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$\left(f \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_k \right)(\tau) = f(\tau+1)$$

since $j\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \tau \right) = 1$.

$$\left(f \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_k \right) = \frac{1}{\tau^k} f\left(-\frac{1}{\tau}\right)$$

since $j\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \tau\right) = \tau$. Let f be a modular form of weight k, then

$$\begin{aligned} f(\tau+1) =& f(\tau) \\ f\left(-\frac{1}{\tau}\right) =& \tau^k f(\tau). \end{aligned}$$

(to find a nontrivial modular form, k must be even.)

Definition 2.1 (Eisenstein Series). Assume $k = \text{even} > 2$. Define

$$G_k(\tau) = \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(c\tau+d)^k}.$$

$G_k(\tau)$ is called Eisenstein series.

$\tau \in \mathcal{H} \Rightarrow G_k(\tau)$ is absolutely convergent and holomorphic.

$$\begin{aligned} \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \Rightarrow & G_k\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= \sum_{(c',d')\neq(0,0)} \frac{1}{\left(c'\frac{a\tau+b}{c\tau+d} + d'\right)^k} \\ &= (c\tau+d)^k \sum_{(c',d')\neq(0,0)} \frac{1}{((c'a+cd')\tau + (c'b+dd'))^k}. \end{aligned}$$

As (c', d') walks through all $\neq (0,0)$, so does $(c'a+cd', c'b+dd')$. Hence

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\, G_k(\tau).$$

When $\mathrm{Im}(\tau) \to \infty$,

$$\frac{1}{(c\tau+d)^k} \to \begin{cases} 0 & \text{if } c \neq 0, \\ d^{-k} & \text{if } c = 0. \end{cases}$$

$$\begin{aligned} \Rightarrow G_k(\infty) &= \lim_{\mathrm{Im}(\tau)\to\infty} G_k(\tau) \\ &= \sum_{d=-\infty, d\neq 0}^{\infty} \frac{1}{d^k} = 2\zeta(k). \end{aligned}$$

Assume f is a modular form of weight k, $k = \text{even} > 2$. Define

$$\begin{aligned} D =& \{q \in \mathbb{C} : |q| < 1\} \\ D' =& D - \{0\}. \end{aligned}$$

Construct the mapping

$$\begin{aligned}\mathcal{H} \to& D' \\ \tau \mapsto& e^{2\pi i\tau} = q\end{aligned}$$

and

$$\begin{aligned}g : D' \to& \mathbb{C} \\ q \mapsto& f\left(\log(q)/(2\pi i)\right).\end{aligned}$$

g is well defined even though the logarithm is only determined up to $2\pi i\mathbb{Z}$. Then

$$f(\tau) = g(e^{2\pi i\tau}).$$

At $\tau = \infty$, $G_k(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau} = 2\zeta(k) \Rightarrow a_0 = 2\zeta(k)$. $a_n =?$

Proposition 2.2. Let $\tau \in \mathcal{H}$, we have

$$\frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau - d} + \frac{1}{\tau + d}\right) = \pi\cot\pi\tau.$$

Proof.

$$\sin\pi\tau = \pi\tau\prod_{n=1}^{\infty}\left(1 - \frac{\tau^2}{n^2}\right).$$

$$\log\sin\pi\tau = \log\pi + \log\tau + \sum_{n=1}^{\infty}\log\left(1 - \frac{\tau^2}{n^2}\right).$$

Taking the derivative, we obtain

$$\pi\cot\pi\tau = \frac{1}{\tau} + \sum_{n=1}^{\infty}\frac{-2\tau/n^2}{1-\tau^2/n^2} = \frac{1}{\tau} + \sum_{n=1}^{\infty}\left(\frac{1}{\tau - n} + \frac{1}{\tau + n}\right).$$

□

$$\begin{aligned}\frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau - d} + \frac{1}{\tau + d}\right) =& \pi i\frac{e^{i\pi\tau} + e^{-i\pi\tau}}{e^{i\pi\tau} - e^{-i\pi\tau}} \\ =& -\pi i\frac{1 + e^{2\pi i\tau}}{1 - e^{2\pi i\tau}} \\ =& -\pi i - 2\pi i\sum_{m=0}^{\infty} e^{2\pi i m\tau}.\end{aligned}$$

Differentiating $(k-1)$ times we get

$$(-1)^{k-1}(k-1)!\sum_{d=-\infty}^{\infty}\frac{1}{(\tau+d)^k}=-2\pi i\sum_{m=0}^{\infty}(2\pi i m)^{k-1}e^{2\pi i m\tau}.$$

$$\Rightarrow\sum_{d=-\infty}^{\infty}\frac{1}{(\tau+d)^k}-\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}m^{k-1}e^{2\pi i m\tau}=0.$$

$$\begin{aligned}\Rightarrow G_k(\tau)=&\sum_{c\neq 0}\sum_{d=-\infty}^{\infty}\frac{1}{(c\tau+d)^k}+\sum_{d\neq 0}\frac{1}{d^k}\\ =&2\sum_{c=1}^{\infty}\sum_{d=-\infty}^{\infty}\frac{1}{(c\tau+d)^k}+2\zeta(k)\\ =&2\sum_{c=1}^{\infty}\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}m^{k-1}e^{2\pi i m c\tau}+2\zeta(k)\\ =&2\sum_{n=1}^{\infty}\frac{(2\pi i)^k}{(k-1)!}\left(\sum_{m\mid n}m^{k-1}\right)e^{2\pi i n\tau}+2\zeta(k).\end{aligned}$$

Then we get the following conclusion:

Proposition 2.3. Let $k=$ even $>2, G_k(\tau)=\sum_{n=0}^{\infty}a_n e^{2\pi i n\tau}$, then

$$a_0=2\zeta(k)$$

$$a_n=\frac{2(2\pi i)^k}{(k-1)!}\sigma_{k-1}(n)$$

where $n>0$ and $\sigma_{k-1}(n)=\sum_{m\mid n}m^{k-1}$.

A group G acting on a set S gives rise to an equivalent relation $\sim$ on S:

$$s_1\sim s_2\Leftrightarrow\exists g\in G\text{ such that }s_2=gs_1.$$

The quotient space ${}^{S}\!/_{G}=$ the set of the equivalent class.

Question. $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$, what is the quotient space?

Theorem 2.4. *Let $D=\left\{\tau\in\mathcal{H}:|\operatorname{Re}\tau|\leq\frac{1}{2},|\tau|\geq 1\right\}$. Then D is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ in the sense such that*

a. $\forall\tau\in\mathcal{H}$, $\exists\gamma\in\mathrm{SL}_2(\mathbb{Z})$ *such that* $\gamma(\tau)\in D$;

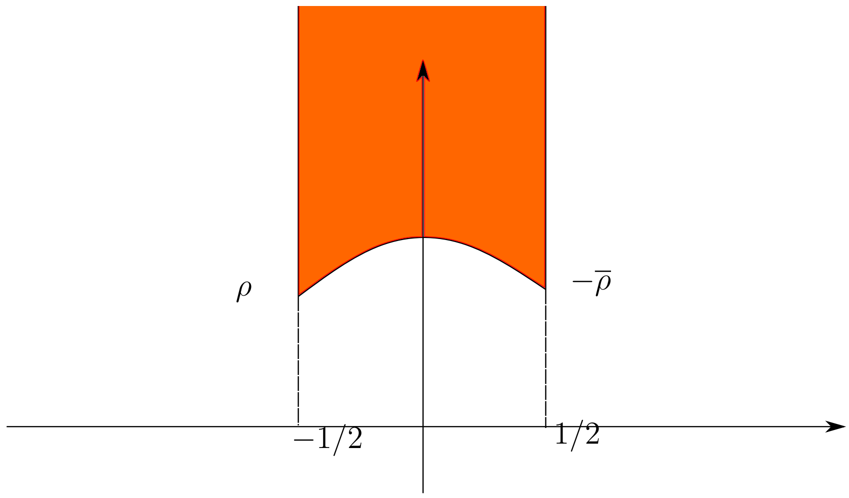

Figure 1: Domain D

b. *If* $\tau_1, \tau_2 \in D$*,* $\tau_1 \neq \tau_2$ *and* $\tau_2 = \gamma(\tau_1)$ *for some* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*, then either* $\mathrm{Re}(\tau_1) = \pm\frac{1}{2}, \tau_2 = \tau_1 \mp 1$ *or* $|\tau_1| = 1, \tau_2 = -\frac{1}{\tau_1}$*.*

Proof. Write $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\gamma_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\gamma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\gamma_1^n(\tau) = \tau + n$$
$$\gamma_2(\tau) = -\frac{1}{\tau}.$$

a. $\forall \tau \in \mathcal{H}$, $\mathrm{Im}\,\gamma(\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$. There exists only finite (c, d) such that $|c\tau + d| <$ a given volume. $\Rightarrow$ ($\exists\gamma$ such that $|c\tau + d| = \min \Rightarrow \mathrm{Im}\,\gamma(\tau) = \max$ and $\exists n \in \mathbb{Z}$ such that $|\mathrm{Re}\,\gamma_1^n(\gamma(\tau))| \leq \frac{1}{2}$).

Write $\tau_1 = \gamma_1^n(\gamma(\tau))$,$\tau_2 = \gamma_2(\tau_1)$.

$$\mathrm{Im}\,\tau_2 = \frac{\mathrm{Im}\,\tau_1}{|\tau_1|^2} \text{ and } \mathrm{Im}\,\tau_2 \leq \mathrm{Im}\,\gamma(\tau) = \mathrm{Im}\,\tau_1$$
$$\Rightarrow |\tau_1| \geq 1 \Rightarrow \tau_1 \in D.$$

b. Suppose $\tau_1, \tau_2 \in D$,$\tau_1 \neq \tau_2$,$\gamma(\tau_1) = \tau_2$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We assume

$\operatorname{Im}\tau_1 \le \operatorname{Im}\tau_2$, then

$$\begin{aligned}\operatorname{Im}\tau_1 \le & \frac{\operatorname{Im}\tau_1}{|c\tau_1+d|^2} \\ \Rightarrow & |c\tau_1+d| \le 1.\end{aligned}$$

Since $\operatorname{Im}\tau_1 \ge \frac{\sqrt{3}}{2}$, then $c = 0, \pm 1$.

Case 1: $c = 0, a = d = \pm 1$

$$\begin{aligned}\Rightarrow & \tau_2 = \tau_1 \pm b \\ \Rightarrow & \begin{cases}\operatorname{Re}\tau_1 = \pm\dfrac{1}{2} \\ \tau_2 = \tau_1 \mp 1\end{cases}.\end{aligned}$$

Case 2: $c = 1, |\tau_1 + d| \le 1$

$$\Rightarrow \begin{cases} d = 0 \Rightarrow |\tau_1| = 1, \tau_2 = -\dfrac{1}{\tau_1} \\ \text{ or } \tau_1 = \rho, d = 1 \\ \text{ or } \tau_1 = -\overline{\rho}, d = -1 \end{cases}.$$

Case 3: $c = -1$, similar to Case 2.

Here $\rho := e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. □

The quotient space $\mathcal{H}\big/\mathrm{SL}_2(\mathbb{Z})$ is obtained by identifying the left side and right side of D and identifying the left and right parts of the bottom circle of D.

Assume f is holomorphic on $\mathcal{H}$ and $\infty(\operatorname{Im}\tau \to \infty)$.

At $p \in \mathcal{H}$, $m =$ order of f at p means

$$\lim_{\tau\to p} \frac{f(\tau)}{(\tau - p)^m}$$

exists and not equals 0. We use $v_p(f) = m$ to represent this meanning.

At ∞, if $a_m \ne 0$ in $f(\tau) = \sum_{n=m}^{\infty} a_n e^{2\pi i n\tau}$, write $v_\infty(f) = m$.

If p_1, p_2 are equivalent ($p_2 = \gamma p_1$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$), then $v_{p_1}(f) = v_{p_2}(f)$.

Theorem 2.5. *Suppose f is a (non-zero) modular form of weight k (k even), $\rho = e^{\frac{2\pi i}{3}}$. Then*

$$v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum\nolimits^*_{p\in \mathcal{H}\big/\mathrm{SL}_2(\mathbb{Z})} v_p(f) = \frac{k}{12} \tag{2}$$

where $\sum^*$ means the sum is over $p \in H\big/\mathrm{SL}_2(\mathbb{Z})$ and $p \not\sim i, \rho$, i.e., $\sum^*_{p\in\mathcal{H}\big/\mathrm{SL}_2(\mathbb{Z})} := \sum_{p\in D\setminus\{i,\rho\}}$.

$k=2$: right side $=\frac{1}{6}$, left side either $=0$ or $\geq \frac{1}{3}$. Hence modular forms of weight 2 don't exist.
$k=4$: right side $=\frac{1}{3}$, $f=G_4$,$v_\rho f=1$, $v_\infty(f)=v_i(f)=0$, $\sum^*_{p\in \mathcal{H}/\mathrm{SL}_2(\mathbb{Z})} v_p(f)=0$.
$k=12$: G_4^3, G_6^2, $\exists$ linear combination

$$\Delta = c_1G_4^3+c_2G_6^2, \Delta: \text{ weight } =12$$

such that $\Delta(\infty)=0$. Since

$$v_\infty(\Delta)+\frac{1}{2}v_i(\Delta)+\frac{1}{3}v_\rho(\Delta)+\sum^* = \frac{12}{12}=1$$

and $v_\infty(\Delta)=1$, we have $\Delta(\tau)\neq 0\ \forall \tau\in\mathcal{H}$.

Definition 2.6. Define

$$\begin{aligned}\mathcal{M}_k &:= \text{space of modular forms of weight } k\\ \mathcal{S}_k &:= \text{space of cusp} \left(f(\infty)=0\right) \text{ forms of weight } k.\end{aligned}$$

If $\dim\mathcal{M}_k>0$, then

$$\dim\mathcal{S}_k=\dim\mathcal{M}_k-1.$$

In fact, $\mathcal{M}_k=\mathcal{S}_k\oplus\mathbb{C}$
If $f\in\mathcal{M}_k$, then $\Delta f\in S_{k+12}$, then we establish an isomorphism

$$\begin{aligned}\mathcal{M}_k \to& S_{k+12}\\ f\mapsto&\Delta f.\end{aligned}$$

If we know $\dim\mathcal{M}_k$ for all $k\leq 12$, then we know all the conditions.
Before tge proof of Theorem 2.5, we introduce the following lemma in complex analysis:

Lemma 2.7.

$$\frac{1}{2\pi i}\int_C\frac{\mathrm{d}\tau}{\tau-\rho}=-\frac{\theta}{2\pi} \tag{3}$$

where C is given by

Proof of Theorem 2.5. Assume $f\neq 0$ on the boundary of D, except at $\tau=\rho, i-\overline{\rho}$, consider the countour L in Figure 3. Residue Theorem $\Rightarrow$

$$\frac{1}{2\pi i}\int_L\frac{\mathrm{d}f}{f}=\sum\nolimits^*_p v_p(f). \tag{4}$$

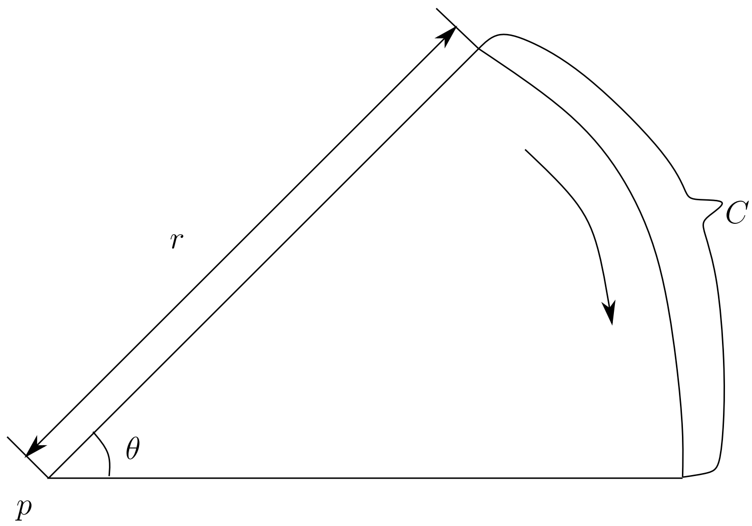

Figure 2: Argument principle

Recall $f(\tau) = g(e^{2\pi i\tau})$ and let $z = e^{2\pi i\tau}$. Set $\operatorname{Im} A' = T$. Then $z = -e^{-2\pi T} \to -e^{-2\pi T}$ along a circle ω:

$$\frac{1}{2\pi i}\int_{A'}^{A} \frac{\mathrm{d}f}{f} = \frac{1}{2\pi i}\int_{\omega} \frac{\mathrm{d}g}{g} = -v_\infty(f). \tag{5}$$

Since

$$\frac{1}{2\pi i}\int_{A}^{B} \frac{\mathrm{d}f}{f} = \frac{1}{2\pi i}\int_{A}^{B} \frac{\mathrm{d}f(\tau+1)}{f(\tau+1)}$$

and $\tau + 1 : A' \to E'$, we get

$$\frac{1}{2\pi i}\left(\int_{A}^{B} + \int_{E'}^{A'}\right)\frac{\mathrm{d}f(\tau)}{f(\tau)} = 0. \tag{6}$$

$\tau : B \to B'$:

$$\frac{\mathrm{d}f}{f} \sim \frac{v_\rho(f)}{\tau - \rho},$$

hence

$$\frac{1}{2\pi i}\int_{B}^{B'} \frac{\mathrm{d}f}{f} \to -\frac{1}{6}v_\rho(f). \tag{7}$$

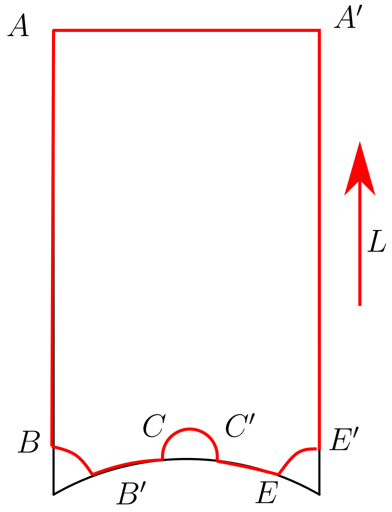

Figure 3: Countour L

$\tau : B' \to C \Rightarrow -\frac{1}{\tau} : E \to C'$:

$$\begin{aligned}\frac{\mathrm{d}f(\tau)}{f(\tau)} &= \frac{\mathrm{d}\tau^{-k}}{\tau^{-k}} + \frac{\mathrm{d}f(-\frac{1}{\tau})}{f(-\frac{1}{\tau})} \\ &= -k\frac{\mathrm{d}\tau}{\tau} + \frac{\mathrm{d}f(-\frac{1}{\tau})}{f(-\frac{1}{\tau})}.\end{aligned}$$

$$\frac{1}{2\pi i}\int_{B'}^{C}\frac{\mathrm{d}f}{f} = \frac{1}{2\pi i}\int_{B'}^{C}\frac{k}{\tau}\mathrm{d}\tau + \frac{1}{2\pi i}\int_{E}^{C'}\frac{\mathrm{d}f(\tau)}{f(\tau)} = \frac{k}{12} - \frac{1}{2\pi i}\int_{C'}^{E}\frac{\mathrm{d}f(\tau)}{f(\tau)}. \tag{8}$$

$\tau : C \to C'$:

$$\frac{\mathrm{d}f}{f} \sim \frac{v_i(f)}{\tau - i}$$

$$\frac{1}{2\pi i}\int_{C}^{C'}\frac{\mathrm{d}f}{f} \to -\frac{1}{2}v_i(f). \tag{9}$$

$\tau : E \to E'$: Similarly

$$\frac{1}{2\pi i}\int_{E}^{E'}\frac{\mathrm{d}f}{f} \to -\frac{1}{6}v_\rho(f). \tag{10}$$

By (5)$\to$ (10) we obtain the conclusion. $\square$

If f has a zero ($\neq \rho, -\overline{\rho}, i$) on the boundary of D, modify L as Figure 4

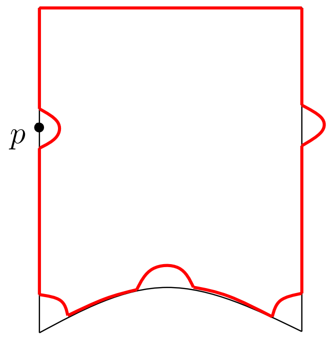

Figure 4: Modified conuntour

As vector space over $\mathbb{C}$, let

$$\mathcal{M}_k = \text{ the space of modular form of weight } k$$

$$\mathcal{S}_k = \text{ the space of cusp form of weight } k.$$

In case $\dim\mathcal{M}_k > 1$ say $d = \dim\mathcal{M}_k$, $f_1, f_2, \cdots, f_d$ a basis of $\mathcal{M}_k$. i.e.,

$$\mathcal{M}_k = \{c_1 f_1 + c_2 f_2 + \cdots + c_d f_d : c_1, \cdots, c_d \in \mathbb{C}\}.$$

$$\begin{aligned} c_1 f_1 + \cdots c_d f_d \in S_k \Leftrightarrow & c_1 f_1(\infty) + c_d f_d(\infty) = 0 \\ \Rightarrow & \dim\mathcal{S}_k = \dim\mathcal{M}_k - 1. \end{aligned}$$

In general $\dim\mathcal{M}_k \leq \dim\mathcal{S}_k + 1$.

Write

$$\begin{aligned} g_2 =& 60G_4 \\ g_3 =& 140G_6. \end{aligned}$$

Define $\Delta = g_2^3 - 27g_3^2$, notice that g_2^3 and g_3^2 are both modular forms of weight 12. Then $\Delta \in \mathcal{M}_k$.

$$\Delta(\infty) = 0 \Rightarrow \Delta \in \mathcal{S}_k.$$

$$
\begin{aligned}
&(2) \xRightarrow{f=\Delta} v_\infty(\Delta) + \frac{1}{2}v_i(\Delta) + \frac{1}{3}v_\rho(\Delta) + \sum\nolimits^* v_p(\Delta) = 1 \\
\Delta(\infty) = 0 &\Longrightarrow v_{oo}(\Delta) \geq 1 \\
&\xRightarrow{\text{combine with the above}} v_\infty(\Delta) = 1, \quad v_i(\Delta) = v_\rho(\Delta) = v_p(\Delta) = 0 \\
&\Longrightarrow \Delta(\tau) \neq 0, \forall \tau \in \mathcal{H}.
\end{aligned}
$$

Theorem 2.8.

a. *When $k < 0$ and $k = 2$, $\mathcal{M}_k = 0$.*

b. *The map:* $\begin{matrix} \mathcal{M}_k \to & \mathcal{S}_{k+12} \\ f \mapsto & \Delta f \end{matrix}$ *is an isomorphism.*

c. *When $k = 0, 4, 6, 8, 10$,*$\dim \mathcal{M}_k = 1$, $\dim S_k = 0$, *Their basis are* $1, G_4, G_6, G_8, G_{10}$ *respectively.*

Proof.

a. Suppose $\exists$ non-zero modular $f \in \mathcal{M}_k$. Left side of (2) $\geq 0 \Rightarrow k \geq 0$.

   For $k = 2$, $v_\infty(f), v_i(f), v_\rho(f), v_p(f)$ are non-negative integers. If one of them ≥ 1, the left side of (2) $\geq \frac{1}{3}$. But the right side $= \frac{1}{6}$.

b. It suffices to prove that $f \mapsto \Delta f$ is surjective. $\forall g \in S_{k+12}$, consider $h = \frac{g}{\Delta}$.

$$
\begin{cases} v_\infty(g) \geq & 1 \\ v_\infty(\Delta) = 1 \end{cases} \Rightarrow v_\infty(h) \geq 1 \Rightarrow h \text{ is holomorphic at } \infty.
$$

$$
\begin{cases} g \text{ is holomorphic on } \mathcal{H} \\ \Delta^{-1} \text{ is holomorphic and } \Delta \neq 0 \text{ on } \mathcal{H} \end{cases} \Rightarrow h \text{ is holomorphic on } \mathcal{H}.
$$

$$
h\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{g\left(\frac{a\tau+b}{c\tau+d}\right)}{\Delta\left(\frac{a\tau+b}{c\tau+d}\right)} = \frac{(c\tau+d)^{k+12}g(\tau)}{(c\tau+d)^{12}\Delta(\tau)} = (c\tau+d)^k h(\tau)
$$

$$
\Rightarrow h \in \mathcal{M}_k.
$$

c. For $k = 0, 4, 6, 8, 10$, $\exists$ non-zero $f \in \mathcal{M}_k$

$$
\left.\begin{matrix} f \equiv 1 \text{ for } k = 0 \\ f = G_k \text{ for } k \geq 4 \end{matrix}\right\} \Rightarrow \dim \mathcal{M}_k \geq 1.
$$

   By b we have $S_k \simeq \mathcal{M}_{k-12}$, but $k - 12 \leq 0 \Rightarrow \mathcal{M}_{k-12} = 0 \Rightarrow S_k = 0$.

$\square$

Corollary 2.9. For $k \geq 0$, we have

$$\dim \mathcal{M}_k = \begin{cases} \left[\dfrac{k}{12}\right] & \text{if } k \equiv 2 \pmod{12}, \\ \left[\dfrac{k}{12}\right] + 1 & \text{if } k \not\equiv 2 \pmod{12}. \end{cases}$$

Proof. $0 \leq k \leq 10$ can be directly verified.

For $k = 12$, $\mathcal{S}_{12} \simeq \mathcal{M}_0 \Rightarrow \dim \mathcal{S}_1 2 = 1 \Rightarrow \dim \mathcal{M}_{12} = 2$. By $\mathcal{S}_{k+12} \simeq \mathcal{M}_k$ we get

$$\dim \mathcal{M}_{k+12} = \dim \mathcal{S}_{k+12} + 1 = \dim \mathcal{M}_k + 1,$$

then use induction. $\square$

Corollary 2.10. For $k \geq 12$, the set $\left\{G_4^\alpha G_6^\beta : 4\alpha + 6\beta = k, \alpha, \beta \geq 0\right\}$ is a basis of $\mathcal{M}_k$.

Proof. The elements of $i \left\{G_4^\alpha G_6^\beta : 4\alpha + 6\beta = k, \alpha, \beta \geq 0\right\}$ are linearly independent and the number of $G_4^\alpha G_6^\beta$ is $\dim \mathcal{M}_k$ by Corollary 2.9. $\square$

3 Complex Tori

A *Riemann surface* is an 1-dimensional connected complex manifold.

Proposition 3.1. If $f : S_1 \to S_2$ is a holomorphic map of compact Riemann surfaces, then either the image of f is a point, or f is surjective.

Proof. Suppose X and Y are compact Riemann surfaces and $f : X \to Y$ is holomorphic. Since f is continuous and X is compact and connected, so is the image $f(X)$, making $f(X)$ closed. Unless f is constant f is open by the Open Mapping Theorem of complex analysis, applicable to Riemann surfaces since it is a local result, making $f(X)$ open as well. So $f(X)$ is either a single point or a connected, open, closed subset of the connected set Y, i.e., all of Y. $\square$

Assume $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$ (for normalization, set $\operatorname{Im} \frac{\omega_1}{\omega_2} > 0$).

Let $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} = \{\omega_1 n_1 + \omega_2 n_2 : n_1, n_2 \in \mathbb{Z}\}$, Λ is a lattice and a discrete subgroup fo $\mathbb{C}$.

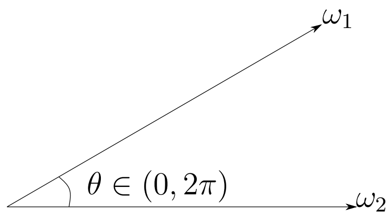

Figure 5: Relation between two numbers

Lemma 3.2. Let

$$\begin{aligned}\Lambda =& \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z},\\ \Lambda' =& \omega_1'\mathbb{Z} \oplus \omega_2'\mathbb{Z}.\end{aligned}$$

Then $\Lambda = \lambda' \Leftrightarrow \exists \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}.$$

Definition 3.3 (Complex Tori). A *complex tori* is a quotien of $\mathbb{C}$ by Λ :

$$\mathbb{C}/\Lambda = \{z + \Lambda : z \in \mathbb{C}\}.$$

In algebra: $\mathbb{C}/\Lambda$ = an abelian group,

$$(z_1 + \Lambda) + (z_2 + \Lambda) = (z_1 + z_2) + \Lambda.$$

In topology: $\mathbb{C}/\Lambda$ = the parallelogram on identifying the opposite side (see Figure 6).

$$z_1 + \Lambda = z_2 + \Lambda \Leftrightarrow z_1 - z_2 \in \Lambda.$$

Proposition 3.4. Suppose $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is holomorphic. $\exists m, b \in \mathbb{C}$ such that

$$m\Lambda \subset \Lambda', \quad (m\Lambda = \{mz : z \in \Lambda\})$$

and

$$\varphi(z + \Lambda) = mz + b + \Lambda'.$$

φ is invertable $\Leftrightarrow m\Lambda = \Lambda'$.

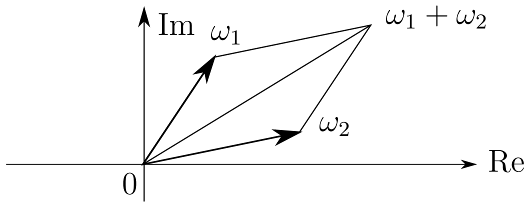

Figure 6: The parallelogram $\mathbb{C}/_{\Lambda}$

Proof. Let

$$\begin{array}{rcl} p : \mathbb{C} & \to & \mathbb{C}/_{\Lambda} \\ z & \mapsto & z + \Lambda \end{array}$$

and

$$\begin{array}{rcl} p' : \mathbb{C} & \to & \mathbb{C}/_{\Lambda'} \\ z & \mapsto & z + \Lambda'. \end{array}$$

By universal cover lifting: $\exists \widetilde{\varphi} : \mathbb{C} \to \mathbb{C}$ holomorphic such that the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "p"'] & \mathbb{C} \arrow[d, "p'"] \\ \mathbb{C}/_{\Lambda} \arrow[r, "\varphi"] & \mathbb{C}/_{\Lambda'} \end{tikzcd}$$

is commutative: $p' \circ \widetilde{\varphi} = \varphi \circ p$. $\forall \lambda \in \Lambda$,

$$\begin{aligned} & p'\left(\widetilde{\varphi}(z+\lambda) - \widetilde{\varphi}(z)\right) \\ =& \varphi\left(p(z+\lambda)\right) - \varphi\left(p(z)\right) \\ =& \varphi\left(p(z)\right) - \varphi\left(p(z)\right) \\ =& 0 + \Lambda' \\ \Rightarrow & \widetilde{\varphi}\left(z+\lambda\right) - \tilde{\varphi}(z) \in \Lambda'. \end{aligned}$$

Left side = holomorphic function in z taking values in discrete $\Lambda' \Rightarrow$ it is constant

$$\begin{aligned} \Rightarrow & \widetilde{\varphi}'(z+\lambda) - \tilde{\varphi}'(z) = 0 \text{ (here the prime symbol means taking derivative)} \\ & \widetilde{\varphi}' = \text{ entire function and period } = \lambda \in \Lambda \\ \Rightarrow & \widetilde{\varphi}' \text{ is bounded} \\ \Rightarrow & \widetilde{\varphi}' = \text{ constant by Liouville's Theorem} \\ \Rightarrow & \widetilde{\varphi} = mz + b \text{ for some } m, b \in \mathbb{C} \\ \Rightarrow & \varphi(z+\Lambda) = mz + b + \Lambda'. \end{aligned}$$

To prove $m\Lambda \subset \Lambda'$, $\forall z \in \Lambda$,

$$\left.\begin{array}{rr} \varphi(z+\Lambda) = & mz+b+\Lambda' \\ \varphi(0+\Lambda) = & b+\Lambda' \end{array}\right\} \Rightarrow mz \in \Lambda' \text{ since } \varphi(z+\Lambda) = \varphi(0+\Lambda).$$

φ is invertable $\Leftrightarrow$ directly verified. □

Corollary 3.5. Suppose $\varphi : \mathbb{C}/_{\Lambda} \to \mathbb{C}/_{\Lambda'}$ is holomorphic:

$$\varphi(z+\Lambda) = mz+b+\Lambda', \quad m\Lambda \subset \Lambda'.$$

Then φ is a group homomorphism $\Leftrightarrow b \in \Lambda'$.

Definition 3.6. A nonzero holomorphic homomorphism:

$$\mathbb{C}/_{\Lambda} \to \mathbb{C}/_{\Lambda'}$$

is called an *isogeny* .

Isogeny is surjective, its kernel is finite (it is discrete, otherwise the map is zero).

A *curve* C (in $\mathbb{R}^2$) means $\exists$ polynomial $F(x,y)$ such that

$$(x,y) \in C \Leftrightarrow F(x,y) = 0.$$

Assume $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$, then $\mathbb{C}/_{\Lambda}$ = a complex curve, why?

Define (Weierstrass-p function)

$$p = p_\Lambda : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$$

by

$$p(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right).$$

At each $z \in \Lambda$, $p(z)$ has a double pole, otherwise $p(z)$ is holomorphic.

$\forall \lambda \in \Lambda$, $\lambda \neq 0$,

$$p(z-\lambda) = \frac{1}{(z-\lambda)^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0 \\ \omega\neq-\lambda}} \left(\frac{1}{(z-\lambda-\omega)^2} - \frac{1}{\omega^2} \right) + \left(\frac{1}{z^2} - \frac{1}{\lambda^2} \right).$$

By the virtue of

$$\lim_{z\to\infty} \left(\sum_{\substack{\omega\in\Lambda \\ |\omega|<z \\ \omega\neq 0,-\lambda}} \frac{1}{\omega^2} - \sum_{\substack{\omega\in\Lambda \\ |\omega|<z \\ \omega\neq 0,-\lambda}} \frac{1}{(\omega+\lambda)^2} \right) = 0$$

$$\Rightarrow p(z) \text{ is } \Lambda\text{-periodical},$$

i.e., $p(z-\lambda) = p(z), \forall \lambda \in \Lambda$.

$$p'(z) = -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}, p'(z-\lambda) = p'(z), \forall \lambda \in \Lambda.$$

Identify $\mathbb{C}/_{\Lambda}$ with the parallelogram (see Figure 6), consider

$$\begin{Bmatrix} \mathbb{C}/_{\Lambda} \\ z \neq 0 \end{Bmatrix} \to \mathbb{C}^2$$
$$z \mapsto (p(z), p'(z)).$$

For $k =$ even > 2,

$$G_k(\Lambda) = \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{\omega^k} = \sum_{(c,d)\neq(0,0)} \frac{1}{(c\omega_1 + d\omega_2)^k}.$$

Laurent expansion of $p(z)$ and $p'(z)$ at $z = 0, \omega \neq 0, \omega \in \Lambda, |z| < |\omega| \Rightarrow \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\left(\frac{1}{(1-\frac{z}{\omega})^2} - 1\right) = \cdots \Rightarrow$ when $z \to 0$:

$$p(z) = \frac{1}{z^2} + 3G_4(\Lambda)z^2 + 5G_6(\Lambda)z^4 + \mathcal{O}(|z|^6) \tag{11}$$

$$p'(z) = -\frac{2}{z^3} + 6G_4(\Lambda)z + 20G_6(\Lambda)z^3 + \mathcal{O}(|z|^5). \tag{12}$$

Write

$$\begin{aligned} g_2(\Lambda) =& 60G_4(\Lambda) \\ g_3(\Lambda) =& 140G_6(\Lambda). \end{aligned}$$

(11),(12) $\Rightarrow$ the function

$$F(z) = p'^2(z) - \left[4p^3(z) - g_2(\Lambda)p(z) - g_3(\Lambda)\right]$$

is holomorphic and $F(0) = 0 \Rightarrow F(z) \equiv 0$ (bounded entire function) $\Rightarrow$ the point $(p(z), p'(z))$ lies in the curve

$$E : y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda). \tag{13}$$

Proposition 3.7. The map $\begin{array}{cccc} \varphi : & \mathbb{C}/_{\Lambda} \setminus \{0\} & \to & E \\ & z & \mapsto & (p(z), p'(z)) \end{array}$ is bijective.

Proof. (i) We prove $\forall s \in \mathbb{C}$, the function $p(z) - s$ has exactly two roots on ${}^{\mathbb{C}}\!/_{\Lambda}$. First assume $\varphi(z) - s \neq 0$ on boundary of ${}^{\mathbb{C}}\!/_{\Lambda}$.

$$\# \text{ of roots } = \frac{1}{2\pi i} \int_L \frac{p'(z)}{p(z) - s} \mathrm{d}s$$

where L is the countour in Figure 7.

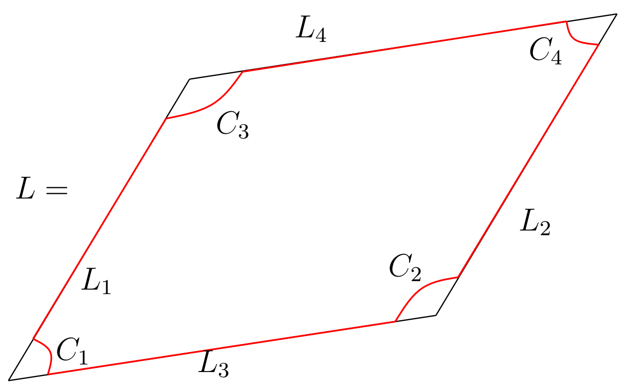

Figure 7: Countour L on boundary of ${}^{\mathbb{C}}\!/_{\Lambda}$

$$p(z + \omega_2) = p(z) \Rightarrow \int_{L_1} + \int_{L_2} = 0,$$
$$p(z + \omega_1) = p(z) \Rightarrow \int_{L_3} + \int_{L_4} = 0.$$

$$z \to o, \begin{array}{ccc} p(z) - s & \sim & \frac{1}{z^2} \\ p'(z) & \sim & -\frac{2}{z^3} \end{array} \Rightarrow \tfrac{p'(z)}{p(z)-s} \sim -\tfrac{2}{z}$$

$$\Rightarrow \frac{1}{2\pi i} \int_{C_1} \to -2\frac{-\theta}{2\pi} = \frac{\theta}{\pi}.$$

Similarly,

$$\frac{1}{2\pi i} \int_{C_3} \to \frac{\pi - \theta}{\pi}.$$

$$\Rightarrow \frac{1}{2\pi i}\left(\int_{C_1}+\int_{C_3}\right)\to 1.$$

Similarly,

$$\frac{1}{2\pi i}\left(\int_{C_2}+\int_{C_4}\right)\to 1.$$

Hence

$$\frac{1}{2\pi i}\int_L \frac{p'(z)}{p(z)-s}\mathrm{d}s = 2.$$

Remark. If

$$\begin{aligned} &p(z)-s=0\\ &z\neq -z+\Lambda\in \mathbb{C}/\Lambda \end{aligned}$$

then $p(z)-s=0$ has two distinct roots(if z is the root of $p(z)-s=0$, then $-z+\lambda, \lambda\in\Lambda$ is also the root).

If $z=\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_1+\omega_2}{2}$, then

$$\begin{aligned} &-z+\Lambda = z\\ \Rightarrow &p'(z)=p'(-z)=-p'(z)\\ \Rightarrow &p'(z)=0\\ \Rightarrow &z \text{ is a double root.} \end{aligned}$$

If $p(z)=0$ ont the boundary, then modify L.

(ii) φ is surjective.
$\forall (x,y)\in E$, $(i)\Rightarrow \exists z$ such that $p(z)=x$. Let $y'=p'(z)$, then

$$(x,y)\in E,(x,y')\in E\Rightarrow y'^2=y^2\Rightarrow y'=\pm y.$$

If $y'=-y\neq 0$, then:
$\exists \lambda\in\Lambda$ such that $-z+\lambda\neq z\Rightarrow$ and :

$$\begin{cases} p(-z+\lambda)=x,\\ p'(-z+\lambda)=-y'=y \quad (\text{ recall } p'(-z+\lambda)=-p'(z)). \end{cases}$$

(iii) φ is injective.
Suppose $\varphi(z_1)=\varphi(z_2)=(x,y)$

$$\Rightarrow p(z_1)=p(z_2)=x\Rightarrow z_2=z_1 \text{ or } z_2=-z_1+\lambda.$$

In case $z_2 = -z_1 + \lambda$,

$$\begin{aligned}
&\Rightarrow y = p'(z_2) = p'(-z_1 + \lambda) = -p'(z_1) = -y \\
&\Rightarrow p'(z_1) = 0
\end{aligned}$$

$\Rightarrow p'(z_1) = 0$, z_1 is a double root of $p(z) - x = 0 \Rightarrow z_2 = z_1$. $\square$

Recall

$$G_k(\Lambda) = \sum_{(c,d)\neq(0,0)} \frac{1}{(c\omega_1 + d\omega_2)^k}$$

and

$$G_k(\tau) = \sum_{(c,d)\neq(0,0)} \frac{1}{c\tau + d)^k}$$

$$\Rightarrow G_k(\Lambda) = \frac{1}{\omega_2^k} G_k\left(\frac{\omega_1}{\omega_2}\right).$$

$$\begin{aligned}
\delta(\tau) &= g_2^3(\tau) - 27g_3^2(\tau) \\
\Rightarrow g_2^3(\Lambda) - 27g_3^2(\Lambda) &= \delta\left(\frac{\omega_1}{\omega_2}\right)\frac{1}{\omega_2^{12}} \neq 0. \text{(Recall } \delta(\infty) = 0, \delta(\tau) \neq 0 \forall \tau \in \mathcal{H})
\end{aligned}$$

Definition 3.8. Suppose $C : F(x, y)$ is a curve. If $\forall (x_0, y_0) \in C$ we have

$$\frac{\partial F}{\partial x}|_{(x_0,y_0)} \neq 0 \text{ or } \frac{\partial F}{\partial y}|_{(x_0,y_0)} \neq 0,$$

then C is a non-singular curve.

Let $E : y^2 - (4x^3 - g_2(\Lambda)x - g_3(\Lambda)) = 0$, then

$$\begin{aligned}
E \text{ is non-singular} &\Leftrightarrow 4x^3 - g_2(\Lambda)x - g_3(\Lambda) = 0 \text{ has no multiple roots} \\
&\Leftrightarrow g_2^3(\tau) - 27g_3^2(\tau) \neq 0.
\end{aligned}$$

Given $E : y = 4x^3 - C_2x - C_3$, $\Delta = C_2^3 - C_3^2$. If $\exists \Lambda$ such that

$$\begin{cases} C_2 = g_2(\Lambda) \\ C_3 = g_3(\Lambda) \end{cases}$$

then $\Delta \neq 0$, E is a non-singular curve. Let

$$j(\tau) := \frac{1728 g_2^3(\tau)}{\Delta(\tau)}. \tag{14}$$

It is easy to verify that

$$j\left(\frac{a\tau+b}{c\tau+d}\right)=j(\tau),\forall\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\mathrm{SL}_2(\mathbb{Z}),$$

$j(\tau)$ is holomorphic on $\mathcal{H}$ ($\Delta(\tau)\neq 0$ on $\mathcal{H}$) and $j(\tau)$ has a simple pole at ∞.

Lemma 3.9. The map $\begin{matrix}\mathcal{H} & \to & \mathbb{C}\\ \tau & \mapsto & j(\tau)\end{matrix}$ is surjective.

Proof. $\forall s\in\mathbb{C}$, let $f=f_s=1728g_2^3-s\Delta$, i.e.,

$$f(\tau)=1728g_2^3(\tau)-s\Delta(\tau).$$

f is of weight 12 modular form.

$$\begin{aligned}&f(\infty)\neq 0\\ \Rightarrow&\frac{1}{2}v_i(f)+\frac{1}{3}v_\rho(f)+\sum{}^{*}v_p(f)=1\\ \Rightarrow&\text{ one of } v_i(f),v_\rho(f),v_p(f)>0.\end{aligned}$$

□

Proposition 3.10. If $a_2^3-27a_3^2\neq 0$ then $\exists\Lambda$ such that

$$\begin{cases}g_2(\Lambda)=a_2\\ g_3(\Lambda)=a_3.\end{cases}$$

Proof. Lemma 3.9 $\Rightarrow\exists\tau\in\mathcal{H}$ such that

$$\begin{aligned}j(\tau)&=\frac{1728g_2^3(\tau)}{\Delta(\tau)}\\ \Rightarrow&\frac{g_2(\tau)^3}{g_2^3(\tau)-27g_3^2(\tau)}=\frac{a_2^3}{a_2^3-27a_3^2}\\ \Rightarrow&\frac{a_2^3}{g_2^3(\tau)}=\frac{a_3^2}{g_3^2(\tau)}.\end{aligned}$$

Choose $\omega_2\neq 0$ such that $\frac{a_2}{g_2(\tau)}=\omega_2^4$, let $\omega_1=\tau\omega_2\Rightarrow\frac{a_2}{g_2(\Lambda)}=\frac{a_2}{g_2(\tau)\omega_2^4}=1\Rightarrow\frac{a_3^2}{g_3^2(\Lambda)}=\frac{a_2^3}{g_2^3(\tau)\omega_2^6}=1\Rightarrow\frac{a_3}{g_3(\Lambda)}=\pm 1$. Replace ω_2 by $i\omega_2$ if necessary $\Rightarrow\frac{a_3}{g_2(\Lambda)}=1$. □

Remark. $\exists$ a surjection between $\{$ complex tori $\}$ and $\{$ curves $E: y^2 = 4x^3 - a_2x - a_3, a_2^3 - 27a_3^2 \neq 0\}$. Write

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \quad \tau = \frac{\omega_1}{\omega_2} \in \mathcal{H},$$
$$\Lambda' = \omega_1'\mathbb{Z} \oplus \omega_2'\mathbb{Z} \quad \tau' = \frac{\omega_1'}{\omega_2'} \in \mathcal{H}.$$

Recall that
$\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is holomorphically group-homomorphism $\Leftrightarrow \exists\, m \in \mathbb{C}$ such that $\varphi(z + \Lambda) = mz + \Lambda', m\Lambda \subset \Lambda'$.
φ is isomorphic $\Leftrightarrow m\Lambda = \Lambda'$.
$\mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda' \Leftrightarrow m \in \mathbb{C}$ such that $m\Lambda = \Lambda' \Leftrightarrow \begin{bmatrix} m\omega_1 \\ m\omega_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix}$ for some $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$\begin{aligned} &\frac{m\omega_1}{m\omega_2} = \frac{a\omega_1' + b\omega_2'}{c\omega_1' + d\omega_2'} \\ \Leftrightarrow &\tau = \frac{a\tau' + b}{c\tau' + d} \\ \Leftrightarrow &\tau, \tau' \text{ are } \mathrm{SL}_2(\mathbb{Z})\text{-equivalent} \end{aligned}$$
.

$\Rightarrow \exists$ a bijection between $\{$ isomorphism class of $\mathbb{C}/\Lambda\}$ and $\{\mathrm{SL}_2(\mathbb{Z})$-equivalence class of $\mathcal{H}\}$.

$$\mathbb{C}/\Lambda \to \tau.$$

4 The Congruence Subgroup Case: Basic Results

Let

$$\begin{aligned} \Gamma &:= \text{ a congruence subgroup}, \\ s, s' &\in \mathbb{Q} \cup \{\infty\}, \infty := \lim_{\operatorname{Im}\tau \to \infty} \tau, \tau \in \mathcal{H}, \\ \gamma &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \\ \gamma(s) &:= \frac{as + b}{cs + d}. \end{aligned}$$

Then

$$\gamma(\infty) = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases} \quad \text{and} \quad \gamma(s) = \infty \text{ if } cs + d = 0, s \in \mathcal{H}.$$

If $s' = \gamma(s), \gamma \in \Gamma$, $s' \neq s$ are Γ-equivalent, denoted by $s' \sim s$.

Definition 4.1. A *cusp* of Γ is a Γ-equivalence class of points in $\mathbb{Q} \cup \{\infty\}$.

Exercise 4.1.

a. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, only one cusp $= \{\infty\}$.

b. Let p be prime, $\Gamma_0(p) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{p} \right\}$. How many cusps?

Solution. The first is obvious. Consider $\frac{m}{n} \in \mathbb{Q}$, $(m,n) = 1, n > 0$.

$$\begin{aligned} &\frac{m}{n} \sim \infty \\ \Leftrightarrow &c\frac{m}{n} + d = 0 \Leftrightarrow n \equiv 0 \pmod{p}. \end{aligned}$$

Hence one cusp $= \left\{\frac{m}{n}, n \equiv 0 \pmod{p}\right\} \cup \{\infty\}$ and another cusp $= \left\{\frac{m}{n} : (m,n) = (n,p) = 1\right\}$.

Proposition 4.2. Let Γ be a congruent subgroup, $\mathbb{C}/\Lambda$, $\Gamma\backslash\mathcal{H}$ be the quotient space of Γ acting on $\mathcal{H}$. Let

$$X(\Gamma) = \Gamma\backslash\mathcal{H} \cup [\text{ cusps of } \Gamma].$$

Then $X(\Gamma)$ has a natural structure as a compact Riemann surface.

Definition 4.3 (Elliptic points w.r.t Γ). $\tau \in \mathcal{H}$ is an *elliptic point* of Γ if $\exists \gamma \in \Gamma$, $\gamma \neq \pm I$ such that $\gamma(\tau) = \tau$.

$$\gamma(\tau) = \tau \Leftrightarrow \sigma\gamma\sigma^{-1}(\sigma(\tau)) = \sigma(\tau).$$

γ is an elliptic point of $\Gamma \Rightarrow \tau$ is an elliptic point of $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 4.4. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

a. If $\gamma^2 = -I$, then γ is conjugate to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{\pm 1}$, i.e.,

$$\sigma\gamma\sigma^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{\pm 1}.$$

b. If $\gamma^2 + \gamma + I = 0$, then γ is conjugate to $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^{\pm 1}$.

c. If $\gamma^2 - \gamma + I = 0$, then γ is conjugate to $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\pm 1}$.

Lemma 4.5. Any elliptic point of $\mathrm{SL}_2(\mathbb{Z})$ is equivalent to i or $\rho = e^{\frac{2\pi i}{3}}$.

Proof. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \pm I$,

$$\begin{aligned} \gamma(\tau) = \tau \Leftrightarrow & a\tau + b = c\tau^2 + d\tau \\ \Leftrightarrow & c\tau^2 + (d-a)\tau - b = 0. \end{aligned}$$

If $c = 0$, then $a = d \Rightarrow \gamma = \pm I$. Assume $c \neq 0$, $\tau \notin \mathbb{R} \Rightarrow (d-a)^2 + 4bc < 0$

$$\begin{aligned} \Rightarrow & (d+a)^2 + 4(bc - ad) < 0 \\ \Rightarrow & (d+a)^2 - 4 < 0 \\ \Rightarrow & |d+a| < 2 \\ \Rightarrow & \begin{vmatrix} a - x & b \\ c & d - x \end{vmatrix} = x^2 + 1 \text{ or } x^2 \pm x + 1 \\ \Rightarrow & \gamma^2 + I = 0 \text{ or } \gamma^2 \pm \gamma + I = 0. \end{aligned}$$

By Lemma 4.4:

$$\begin{aligned} \gamma^2 + I = 0 \Rightarrow & \gamma = \sigma \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{\pm 1} \sigma^{-1} \\ \gamma(\tau) = \tau \Rightarrow & \tau \sim i. \end{aligned}$$

Similarly $\gamma^2 \pm \gamma + I = 0 \Rightarrow \tau \sim \rho$. ∎

Let

$$\begin{aligned} \mathcal{M}_k(\Gamma) &= \text{space of weight } k \text{ modular forms w.r.t. } \Gamma, \\ \mathcal{S}_k(\Gamma) &= \text{space of weight } k \text{ cusp forms w.r.t. } \Gamma, \\ g &= \text{genus of } X(\Gamma). \end{aligned}$$

Theorem 4.6. *Suppose k is even,*

$$\begin{aligned} \varepsilon_2 &= \textit{number of elliptic points of } \Gamma \textit{ which are } \mathrm{SL}_2(\mathbb{Z}) \sim i, \\ \varepsilon 3 &= \textit{number of elliptic points of } \Gamma \textit{ which are } \mathrm{SL}_2(\mathbb{Z}) \sim \rho, \end{aligned}$$

$\varepsilon_\infty =$ *number of cusps of* Γ.

Then

$$\dim \mathcal{M}_k(\Gamma) = \begin{cases} (k-1)(g-1) + \left[\frac{k}{4}\right]\varepsilon_2 + \left[\frac{k}{3}\right]\varepsilon_3 + \frac{k}{2}\varepsilon_\infty & k >= 2, \\ 1 & k = 0, \\ 0 & k < 0. \end{cases}$$

$$\dim \mathcal{S}_k(\Gamma) = \begin{cases} (k-1)(g-1) + \left[\frac{k}{4}\right]\varepsilon_2 + \left[\frac{k}{3}\right]\varepsilon_3 + \left(\frac{k}{2} - 1\right)\varepsilon_\infty & k >= 4, \\ g & k = 2, \\ 0 & k <= 0. \end{cases}$$

$X(\Gamma) \overset{\text{called}}{=}$ a *modular curve*.

Modularity Theorem(Version $X_{\mathbb{C}}$). Suppose ${}^{\mathbb{C}}/_{\Lambda}$ is a complex elliptic curve with $j(\Lambda) \in \mathbb{Q}$. Then for some $N \in \mathbb{N}$, there exists a surjective holomorphic function $X(\Gamma_0(N)) \to {}^{\mathbb{C}}/_{\Lambda}$.

5 Hecke Operators

Let Γ_1 and Γ_2 be congruence subgroups,

$$\mathrm{GL}_2^+(\mathbb{Q}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Q}, ad - bc > 0 \right\}.$$

Let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, write

$$\Gamma_1 \alpha \Gamma_2 := \{\gamma_1 \alpha \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}.$$

$\Gamma_1 \alpha \Gamma_2$ is called a double coset in $\mathrm{GL}_2^+(\mathbb{Q})$.

Remark. Let G = group, S = set, then G acts on S is denoted by ${}_{G}\backslash^{S} =$ $\{$ orbits of G on $S\} = \{Gs : s \in S\}$. Indeed, if S = group $G \lhd S \Rightarrow {}_{G}\backslash^{S} =$ $\{Gs : s \in S\} = {}^{S}/_{G}$.

Fact: ${}_{\Gamma_1}\backslash^{\Gamma_1 \alpha \Gamma_2} = \{\Gamma_1 \alpha_2 \gamma_2 : \gamma_2 \in \Gamma_2\}$ is finite.
Γ = congruence subgroup $\Rightarrow [\mathrm{SL}_2(\mathbb{Z})_2(\mathbb{Z}) : \Gamma] < \infty$.

Lemma 5.1. Let Γ be a congruence subgroup and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, then $\alpha^{-1}\Gamma\alpha \cap$ $\mathrm{GL}_2^+(\mathbb{Q})$ is a congruence subgroup.

Proof. $\exists \widetilde{N} \in \mathbb{N}$ such that $\Gamma(\widetilde{N}) \subset \Gamma$ and

$$\widetilde{N}\alpha \in M_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \right\}, \widetilde{N}\alpha^{-1} \in M_2(\mathbb{Z}).$$

Let $N = \widetilde{N}^3$,

$$\begin{aligned}\alpha\Gamma(N)\alpha^{-1} \subset& \alpha(I + NM_2(\mathbb{Z}))\alpha^{-1}\\ =& I + \widetilde{N}\widetilde{N}\alpha M_2(\mathbb{Z})\widetilde{N}\alpha^{-1}\\ \subset& I + \widetilde{N}M_2(\mathbb{Z})\end{aligned}$$

$$\begin{aligned}&\Rightarrow \alpha\Gamma(N)\alpha^{-1} \subset \mathrm{SL}_2(\mathbb{Z}) \cap \left(I + \widetilde{N}M_2(\mathbb{Z})\right) = \Gamma(\widetilde{N})\\ &\Leftrightarrow \alpha\Gamma(N)\alpha^{-1} \subset \Gamma(\widetilde{N})\\ &\Rightarrow \Gamma(N) \subset \alpha^{-1}\Gamma(\widetilde{N})\alpha \subset \alpha^{-1}\Gamma\alpha\\ &\Rightarrow \Gamma(N) \subset \alpha^{-1}\Gamma\alpha \cap \mathrm{SL}_2(\mathbb{Z})\\ &\Rightarrow \alpha^{-1}\Gamma\alpha \cap \mathrm{SL}_2(\mathbb{Z}) \text{ is a congruence sub group.}\end{aligned}$$

□

Lemma 5.2. Write $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2\,(\Gamma_3 \subset \Gamma_2)$,

$$\Gamma_1\backslash\Gamma_1\alpha\Gamma_2 = \{\Gamma_1\alpha\gamma_2 : \gamma \in \Gamma_2\}.$$

The map $\begin{array}{rccc}\varphi: & \Gamma_2 & \to & \Gamma_1\backslash\Gamma_1\alpha\Gamma_2\\ & \gamma_2 & \mapsto & \Gamma_1\alpha\gamma_2\end{array}$ induces a bijection

$$\Gamma_3\backslash\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2.$$

Proof. It is obvious that φ is surjective. Suppose $\varphi(\gamma_2) = \varphi(\gamma_2')$,

$$\begin{aligned}&\Gamma_1\alpha\gamma_2 = \Gamma_1\alpha\gamma_2'\\ &\Leftrightarrow \Gamma_1\alpha\gamma_2\gamma_2'^{-1} = \Gamma_1\alpha\\ &\Leftrightarrow \alpha(\gamma_2\gamma_2'^{-1})\alpha^{-1} \in \Gamma_1\\ &\Leftrightarrow \gamma_2\gamma_2'^{-1} \in \alpha^{-1}\Gamma_1\alpha\\ &\Leftrightarrow \gamma_2\gamma_2'^{-1} \in \Gamma_3.\end{aligned}$$

□

Zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \mathrm{Re}\, s > 1.$$

Let f = cusp form at ∞ : $f(\tau) = \sum_{n=1}^{\infty} a_n(f)e^{2\pi n\tau}$. Define

$$L(f, s) = \sum_{n=1}^{\infty} a_n(f)n^{-s}$$

for some f. Then

$$L(f, s) = \prod_p \left(1 - \frac{a_p(f)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}.$$