ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE LECTURE 03

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Definition 1. Let $\mathfrak{a}$ be an ideal in a ring R. The radical of $\mathfrak{a}$ is

$$\sqrt{\mathfrak{a}} = \{r \in R | \exists n > 0, r^n \in \mathfrak{a}\}.$$

$\sqrt{\mathfrak{a}}$ is an ideal in R. $\mathfrak{a}$ is called radical ideal if $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

Remark. If $X \subset \mathbb{A}^n$ is an affine algebraic set, then $I(X)$ is a radical ideal.

Theorem 1 (Nullstallensatz)**.** *Let* $\mathfrak{a} \subset k[x_1, \ldots, x_n]$*, then* $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Definition 2. R is an integral domain, the quotient field $Q(R)$ is the set of equivalent classes of pairs $(f, g), f, g \in R, g \neq 0$, which satisfy the equivalent relation

$$(f, g) \cong (f', g') \Leftrightarrow fg' - f'g = 0.$$

We denote it by $\frac{f}{g}$.

Remark. $Q(R)$ is a field. We always identify $r \in R$ with $\frac{r}{1} \in Q(R)$, then we can say R is the subring of $Q(R).Q(k[x_1, \ldots, x_n]) := k(x_1, \ldots, x_n)$ is called field of rational functions in $x_1, x_2, \ldots, x_n$.

Now we prove the Nullstellensatz:

Proof of Nullstellensatz. Let $\mathfrak{a} = \langle f_1, \ldots, f_r \rangle, f_i \in \mathfrak{a}$. Then $I(Z(\mathfrak{a}))$ is a radical ideal containing $\mathfrak{a}$, so we get

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

Let$f \in I(Z(\mathfrak{a}))$. To show $\exists N > 0$, s.t. $f^N \in \mathfrak{a}$, we use the weak Nullstellensatz in $k[x_1, \ldots, x_n]$.

Let

$$\mathfrak{b} := \langle f_1, \ldots, f_r, f \cdot t - 1 \rangle \subset k[x_1, \ldots, x_n, t] \tag{0.1}$$

Let $(p, a) \in \mathbb{A}^{n+1}, p \in \mathbb{A}^n, a \in k$.

$$(p, a) \in Z(\mathfrak{b}) \Leftrightarrow f_1(p) = \cdots = f_r(p) = 0 \text{ and } f(p) \cdot a = 1.$$

But $f(p) = 0$, so we know $Z(\mathfrak{b}) = \emptyset$. By the weak Nullstellensatz, $1 \in \mathfrak{b}$, we can write

$$1 = g_0 \cdot (ft - 1) + \sum_{i=1}^{r} g_i \cdot f_i \tag{0.2}$$

Back to $k[x_1, \ldots, x_n]$ in $k(x_1, \ldots, x_n)$, define homomorphism:

$$\begin{aligned} \varphi : k[x_1, \ldots, x_n, t] &\to k(x_1, \ldots, x_n) \\ g(x_1, \ldots, x_n, t) &\to g(x_1, \ldots, x_n, \tfrac{1}{f}) \end{aligned}$$

Date: June 19, 2017.

Use φ to equation 0.2 we get

$$1 = \sum_{i=1}^{r} \varphi(g_i) \cdot f_i \tag{0.3}$$

where $\varphi(g_i) = \frac{G_i}{f^{n_i}}, G_i \in k[x_1, \dots, x_n]$. Let $N := \max_{1 \leq i \leq r} n_i$, multiply equation 0.3 by f^N:

$$f^N = \sum_{i=1}^{r} G_i f^{N-n_i} \cdot f_i \in \mathfrak{a} \tag{0.4}$$

□

Corollary 1. *(1) If $\mathfrak{a} \subset k[x_1, \dots, x_n]$ is a prime ideal, then $Z(\mathfrak{a})$ is irreducible;*
(2) If $f \in k[x_1, \dots, x_n]$ is irreducible, then $Z(f)$ is irreducible.

Proof. (1) Set $X = Z(\mathfrak{a})$. Prime ideals are radical, so we get $I(X) = \mathfrak{a}$ and $\mathfrak{a}$ is prime, use proposition 2 we know that X is irreducible.
(2) Since $k[x_1, \dots, x_n]$ is a UFD, we get

$$f \in k[x_1, \dots, x_n] \text{ is irreducible} \Rightarrow \langle f \rangle \text{ is a prime ideal.}$$

So $Z(f) = Z(\langle f \rangle)$ is irreducible. □

1. Projective Varieties

Definition 3. Define an equivalence relation$\sim$ in $k^{n+1} \backslash \{0\}$:

$$(a_0, \dots, a_n) \sim (b_0, \dots, b_n) \Leftrightarrow \exists \lambda \in k \backslash \{0\} \text{ s.t.} (a_0, \dots, a_n) = (\lambda b_0, \dots, \lambda b_n).$$

Then we call $k^{n+1} \backslash \{0\}$ with this relation the projective n-space and write it as $(k^{n+1} \backslash \{0\}) / \sim = \mathbb{P}^n$.

Definition 4. Let $U_i := \{[a_0, \dots, a_n] \in \mathbb{P}^n | a_i \neq 0\}$. $\varphi_i : U_i \to \mathbb{A}^n$, $[a_0, \dots, a_n] \to (\frac{a_0}{a_i}, \dots, \frac{\hat{a_i}}{a_i}, \dots, \frac{a_n}{a_i})$ is a projection, write inverse $u_i : \mathbb{A}^n \to U_i$, $(b_0, \dots, \hat{b_i}, \dots, b_n) \to [b_0, \dots, 1, \dots, b_n]$.

Usually we fix $i = 0$, view $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$ by identify the point $(a_1, \dots, a_n) \in \mathbb{A}^n$ with $[1, a_1, \dots, a_n] \in \mathbb{P}^n$. With this identification we have

$$\mathbb{P}^n = \mathbb{A}^n \cup H_\infty \tag{1.1}$$

where $H_\infty := \{[a_0, \dots, a_n] \in \mathbb{P}^n | a_0 = 0\}$ is called hyperplane at infinity.

Remark. Define projective algebraic sets are zero sets of polynomials in $k[x_0, \dots, x_n]$, but $f \in k[x_0, \dots, x_n]$ does not define a function on $\mathbb{P}^n$:

$$f(a_0, \dots, a_n) \neq f(\lambda a_0, \dots, \lambda a_n). \tag{1.2}$$

However if f is homogeneous we can still see whether $p \in \mathbb{P}^n$ is a zero point of f or not. f is homogeneous if

$$f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n). \tag{1.3}$$

Thus whether $f = 0$ is decided only on $[a_0, \dots, a_n]$.

Definition 5. Let $g \in k[x_0, \dots, x_n]$ be homogeneous, a point $p = [a_0, \dots, a_n]$ is a zero point of g if $g(a_0, \dots, a_n) = 0$. Let $S \subset k[x_0, \dots, x_n]$,

$$Z(S) := \{p \in \mathbb{P}^n | f(p) = 0 \forall f \in S\}. \tag{1.4}$$

A subset of $\mathbb{P}^n$ of the form $Z(S)$ is called a projective algebraic set.

Example 1. (1) $\emptyset = Z(1)$, $\mathbb{P}^n = Z(\emptyset)$;

(2) Any point $p = [a_0, \dots, a_n] \in \mathbb{P}^n$ is a projective algebraic set

$$\begin{aligned}\{p\} = \quad & Z(a_1x_0 - a_0x_1, a_2x_0 - a_0x_2, \dots, a_nx_0 - a_0x_n, \\ & a_2x_1 - a_1x_2, \dots, a_nx_1 - a_1x_n, \\ & \dots).\end{aligned}$$

Definition 6. A polynomial $f \in k[x_0, \dots, x_n]$ cab be written uniquely as $f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$, with $f^{(i)}$ homoegeneous of degree i. $f^{(i)}$ is called homogeneous component if f.

An ideal $\mathfrak{a} \subset k[x_0, \dots, x_n]$ is called homogeneous if for every $f \in \mathfrak{a}$ all homogeneous components $f^{(i)}$ are in $\mathfrak{a}$.

Proposition 1. *An ideal $\mathfrak{a} \subset k[x_0, \dots, x_n]$ is homogeneous $\Leftrightarrow$ It is generated by the homogeneous polynomials.*

Proof. $\Rightarrow$: Assume I homogeneous, let $(f_\alpha)_\alpha$ be a set of generators, then $(f_\alpha^{(i)})_{\alpha,i}$ is a set of homogeneous generators.

$\Leftarrow$: Let $\mathfrak{a} = \langle g_i \rangle$ and g_i be homogeneous. Let $f \in \mathfrak{a}$, then we can write

$$f = \sum_i a_i g_i. \tag{1.5}$$

Note g_i is homogeneous, thus the homogeneous part of $a_i g_i$ of degree d is just $a_i^{(d-deg(g_i))} g_i$, so

$$f^{(d)} = \sum_i a_i^{(d-deg(g_i))} g_i. \tag{1.6}$$

Since $g_i \in \mathfrak{a}$ we get $f^{(d)} \in \mathfrak{a}$. $\square$

Definition 7. Let $\mathfrak{a} \subset k[x_0, \dots, x_n]$ be a homogeneous ideal, the zero set of $\mathfrak{a}$ is written as

$$Z(\mathfrak{a}) := \{p \in \mathbb{P}^n | f(p) = 0 \text{ for all homogeneous elements } f \in \mathfrak{a}\}. \tag{1.7}$$

For a subset $X \subset \mathbb{P}^n$, the homogeneous ideal of X is

$$\begin{aligned} I(X) := \ & \text{ideal generated by } \{f \in k[x_0, \dots, x_n] | f \\ & \text{be homogeneous and } f(p) = 0 \forall p \in X\} \end{aligned} \tag{1.8}$$

By definition this is a homogeneous ideal.

Remark. If $f \in k[x_0, \dots, x_n]$ is not homogeneous, we can define

$$Z(f) := \{p \in \mathbb{P}^n | f(a_0, \dots, a_n) = 0 \text{ for all representative } (a_0, \dots, a_n) \text{ of } p\} \tag{1.9}$$

In fact, if $f = f^{(0)} + f^{(1)} \cdots + f^{(d)}$, then we have

$$Z(f) = \bigcap_{i=0}^{d} Z(f^{(i)}) \tag{1.10}$$

With this property, if $\mathfrak{a} \subset k[x_0, \dots, x_n]$ is a homogeneous ideal then formula 1.7 can be written as

$$Z(\mathfrak{a}) = \{p \in \mathbb{P}^n | f(p) = 0 \forall f \in \mathfrak{a}\} \tag{1.11}$$

and formula 1.8 can be written as

$$I(X) = \{f \in k[x_0, \dots, x_n] | f(p) = 0 \forall p \in X\} \tag{1.12}$$

2. Conclusions We Need From Previous Lectures

In lecture 02:

Proposition 2. *$X \subset \mathbb{A}^n$ is an affine algebraic set. Then we have the following equivalent relations:*

1. *X is irreducible;*
2. *$I(X)$ is a prime ideal.*

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