Homological Algebra

Based on lectures by Nicolas Addington
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1 Lecture 1 Motivation and goals

Example 1.1. Let k be a field and $R = k[x, y, z]$, $I = (y - x^2, z)$, $J = (x)$, then $R/I \otimes R/J = R/I + J = R\big/(x, y, z)$. Now we are pretending that we know what Tor means, we have

$$\text{codim}I \cap \text{codim}J = 3 = \text{codim}I + \text{codim}J,$$

and

$$\text{Tor}_1^R \left(R/I, R/J\right) = 0.$$

For $J' = (y)$, we have $R/I \otimes R/J' = R/I + J' = R\big/(x^2, y, z) \neq R\big/(x, y, z)$.

$$\text{Tor}_1^R(R/I, R/J') = 0.$$

For $K = (x, y)$, not transverse! $R/I \otimes R/K = R/I + K = R\big/(x, y, z)$. We have

$$\text{codim}I \cap \text{codim}K = 3 < 4 = \text{codim}I + \text{codim}K,$$

$$\text{Tor}_1^R \left(R/I, R/K\right) = R\big/(x, y, z),$$

and

$$\text{Tor}_2 = 0.$$

$R/K \otimes R/K = R/K$,

$$\text{Tor}_1 \left(R/K, R/K\right) = \left(R/K\right)^2,$$

$$\text{Tor}_2 = R/K,$$

and

$$\text{Tor}_3 = 0.$$

In general, if $I, J \subset \mathbb{C}[x_1, \cdots, x_n]$, cut-out $X, Y \subset \mathbb{C}^n$ (smooth), we know $\text{codim}(X \cap Y) \leq \text{codim}X + \text{codim}Y$, difference tells the last non-zero $\text{Tor}_i \left(R/I, R/J\right)$.

Example 1.2. Let R be a Noetherian commutative ring and M a finitely genrated module.

$$\text{projdim}(M) = \text{ length of shortest projective resolution},$$

$$\text{globdim}(R) = \max\left(\text{projdim}\left(\text{ finitely generated modules}\right)\right)$$

Serre: The global dimension is finite if and only if R is regular. If $k = \overline{k}$ and $R = k[x_1, \cdots, x_n\big/I$, then regular $\Leftrightarrow$ smooth. But $\mathbb{Z}$ is also regular and we will see it's one dimensional, $\mathbb{Z}[\sqrt{-5}]$ is regular, $\mathbb{Z}[\sqrt{-3}]$ is not, $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is regular.