Category Theory

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The learning notes are a collection of some notions and important theorems about category theory. I learned it from the note *Basic Category Theory* written by Tom Leinster. Most of the content is from this note, others are from the Stack Project and *The Rising Sea: Fundations of Algebraic Geometry*.

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1 Basic notions

Definition 1.1. A category \mathscr{A} consists of:

- a collection $ob(\mathscr{A})$ of objects;
- for each $A, B \in ob(\mathscr{A})$, a collection $\mathscr{A}(A, B)$ of maps or arrows or morphisms from A to B;
- for each $A, B, C \in ob(\mathscr{A})$, a function

$$\mathscr{A}(B,C) \times \mathscr{A}(A,B) \to \mathscr{A}(A,C)$$

 $(q,f) \mapsto q \circ f,$

called *composition*;

- for each $A \in ob(\mathscr{A})$, an element 1_A of $\mathscr{A}(A,A)$, called the *identity* on A satisfying the following axioms:
 - associativity: for each $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$ and $h \in \mathcal{A}(C, D)$, we have $(h \circ q) \circ f = h \circ (q \circ f)$;
 - identity laws: for each $f \in \mathcal{A}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Remark. We often writ $A \in \mathscr{A}$ to mean $A \in ob(\mathscr{A})$, $f : A \to B$ or $A \xrightarrow{f} B$ to mean $f \in \mathscr{A}(A, B)$, and gf to mean $g \circ f$.

Definition 1.2. Let \mathscr{A} and \mathscr{B} be categories. A functor $F: \mathscr{A} \to \mathscr{B}$ consists of:

• a function

$$ob(\mathscr{A}) \to ob(\mathscr{B}),$$

written as $A \mapsto F(A)$;

• for each $A, A' \in \mathcal{A}$, a function

$$\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A')),$$

written as $f \mapsto F(f)$,

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in \mathscr{A} ;
- $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$.

Here are two ways of constructing new categories from old.

Definition 1.3. Every category \mathscr{A} has an *opposite* or *dual* category \mathscr{A}^{op} , defined by reversing the arrows. Formally, $\operatorname{ob}(\mathscr{A}^{\text{op}}) = \operatorname{ob}(\mathscr{A})$ and $\mathscr{A}^{\text{op}}(B,A) = \mathscr{A}(A,B)$ for all objects A and B.

Definition 1.4. Given categories \mathscr{A} and \mathscr{B} , A product category $\mathscr{A} \times \mathscr{B}$ is a category in which

$$ob(\mathscr{A} \times \mathscr{B}) = ob(\mathscr{A}) \times ob\mathscr{B},$$
$$(\mathscr{A} \times \mathscr{B}) ((A, B), (A', B')) = \mathscr{A}(A, A') \times \mathscr{B}(B, B')..$$

Since we have the notion of dual category, we also have the notion of dual functor, which is formally called contravariant functor.

Definition 1.5. Let \mathscr{A} and \mathscr{B} be categories. A *contravariant functor* from \mathscr{A} to \mathscr{B} is a functor $\mathscr{A}^{\text{op}} \to \mathscr{B}$.

Definition 1.6. A functor $F: \mathscr{A} \to \mathscr{B}$ is *faithful* (respectively, *full*) if for each $A, A' \in \mathscr{A}$, the function

$$\mathscr{A}(A, A') \longrightarrow \mathscr{B}(F(A), F(A'))$$

 $f \longmapsto F(f).$

is injective (respectively, surjective).

2 Natural transformations

Definition 2.1. Let \mathscr{A} and \mathscr{B} be categories and let $\mathscr{A} \xrightarrow{F \atop G} \mathscr{B}$ be functors. A natural transformation $\alpha : F \to G$ is a family $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A \in \mathscr{A}}$ of morphisms in \mathscr{B} for every map $A \xrightarrow{f} A'$ in \mathscr{A} , the square

$$F(A) \xrightarrow{F(f)} F(A')$$

$$\alpha_{A} \downarrow \qquad \qquad \downarrow^{\alpha_{A'}}$$

$$G(A) \xrightarrow{G(f)} G(A')$$

$$(1)$$

commutes. The morphisms α_A are called the components of α . We also write

to mean that α is a natural transformation from F to G.

Given natural transformations

There is a composite natural transformation

$$\mathscr{A} = \mathbb{A}$$

$$\mathscr{A}$$

$$\mathscr{A}$$

$$\mathscr{B}$$

defined by $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ for all $A \in \mathscr{A}$. It is sometimes called *vertical composition*.

There is also an identity natural transformation

$$\mathscr{A} \underbrace{\downarrow}_{F}^{F} \mathscr{B}$$

on any functor F, defined by $(1_F)_A = 1_{F(A)}$.

Definition 2.2. For any two categories \mathscr{A} and \mathscr{B} , there is a category whose objects are the functors between \mathscr{A} and \mathscr{B} and whose morphisms are the natural transformation between thenm. The composition law and identity morphism are defined and shown above. This is called the *functor category* from \mathscr{A} to \mathscr{B} and written as $[\mathscr{A}, \mathscr{B}]$.

Definition 2.3. Let \mathscr{A} and \mathscr{B} be categories. A natural isomorphism between functors from \mathscr{A} to \mathscr{B} is an isomorphism in $[\mathscr{A},\mathscr{B}]$. In other words, let α be a natural transformation from F to G where F and G are functors from \mathscr{A} to \mathscr{B} , then α is a natural isomorphism if and only if $\alpha_A : F(A) \to G(A)$ is an isomorphism for all $A \in \mathscr{A}$.

Since natural isomorphism is just isomorphism in a particular category $[\mathscr{A}, \mathscr{B}]$, we already have notation for this:

$$F \cong G$$
.

Definition 2.4. Let F, G be two functors from \mathscr{A} to \mathscr{B} , we say that

$$F(A) \cong G(A)$$
 naturally in A

if F and G are naturally isomorphic.

Definition 2.5. An *equivalence* between categories \mathscr{A} and \mathscr{B} consists of a pair of functiors $\mathscr{A} \xleftarrow{F} \mathscr{B}$ such that

$$G \circ F \cong 1_{\mathscr{A}} \text{ and } F \circ G \cong 1_{\mathscr{B}}.$$
 (2)

We say that \mathscr{A} and \mathscr{B} are equivalent if there is an equivalence between them and write $\mathscr{A} \simeq \mathscr{B}$. The functors F and G are equivalences.

Remark. Consider the category of all finite sets **FinSet** (and mappings between those). That's a huge category. However in a sense it should not be so huge, since essentially there only as many finite sets as they are natural numbers. Consider annother category \mathscr{A} , which is only the finite sets of the form $\{1, \dots, n\}$. Now for every $n \in \mathbb{N}$, \mathscr{A} has one set-representative of that size while **FinSet** has many, but in **FinSet** all these sets of the same size are isomorphic and we should not treat isomorphic sets as being different.

Hence it doesn't make any real difference if we use **FinSet** or \mathscr{A} to deal with finite sets. So they ought to be the same. And they are equivalent but not isomorphic.¹

¹This remark is from here.

Definition 2.6. Let $F: \mathscr{A} \to \mathscr{B}$ be a functor, we say F is essentially surjective on objects if for all $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $F(A) \cong B$.

Proposition 2.7. A functor $F : \mathcal{A} \to \mathcal{B}$ is an equivalence if and only if it is full, faithfull and essentially surjective on objects.

Proof. First assume two natural isomorphisms

$$\eta: G \circ F \to 1_{\mathscr{A}}, \quad \varepsilon: F \circ G \to 1_{\mathscr{B}}.$$

Let $f, f': A \to A'$ and $F(f) = F(f'): F(A) \to F(A')$, then $G \circ F(f) = G(F(f)) = G(F(f)) = G(F(f')) = G \circ F(f'): G(F(A)) \to G(F(A'))$. Then $\eta \circ (G \circ F(f)) = \eta \circ (G \circ F(f')) \Rightarrow f = f'$. Hence F is faithuful. Let $g \in \operatorname{Mor}(F(A), F(A'))$, then $g = (F \circ G) \circ (\varepsilon(g))$. Then there exists $f = G \circ \varepsilon(g)$ s.t. F(f) = g, hence F is full. Given any $B \in \mathcal{B}$, let A = G(B), then $F(A) = F \circ G(B) \cong B$. The converse is to construct natural isomorphisms η and ε by reversing the deduction above. \square

Recall vertical composition introduced previously, there is also *horizontal com*position, which takes natural transformations

$$\mathscr{A} = \bigoplus_{G} \mathscr{A}' = \bigoplus_{G'} \mathscr{A}''$$

and produces a natural transformation

$$\mathscr{A} = \bigcap_{G' \circ G} F' \circ F$$

The component of $\alpha' * \alpha$ at $A \in \mathscr{A}$ is defined to be the diagonal of the naturality square

$$F'(F(A)) \xrightarrow{F'(\alpha_A)} F'(G(A))$$

$$\alpha'_{F(A)} \downarrow \qquad \qquad \downarrow \alpha'_{G(A)}$$

$$G'(F(A)) \xrightarrow{G'(\alpha_A)} G'(G(A)).$$

That is

$$(\alpha' * \alpha)_A = \alpha'_{G(A)} \circ F'(\alpha_A) = G'(\alpha_A) \circ \alpha'_{F(A)}.$$

3 Adjoints

Definition 3.1. Let $\mathscr{A} \xleftarrow{F} \mathscr{B}$ be categories and functors. We say that F is a *left adjoint* of G, or that G is a *right adjoint* of F if there are bijections

$$\mathscr{B}(F(A), B) \to \mathscr{A}(A, G(B))$$
 (3)

functorial in $A \in \mathscr{A}$ and $B \in \mathscr{B}$. In other words, this means that there is a given isomorphism of functors $\mathscr{A}^{\text{op}} \times \mathscr{B} \to \mathbf{Set}$ from $\mathscr{B}(F(-), -)$ to $\mathscr{A}(-, G(-))$.

Remark. Here are two understandings:

a. Given (A, B) and $(A', B') \in ob(\mathscr{A}^{op} \times \mathscr{B}), f : A' \to A, g : B \to B'$, then we have

$$\begin{split} \mathscr{B}(F(A),B) & \longrightarrow \mathscr{B}(F(A'),B') \\ & \stackrel{\alpha_{(A,B)}}{\downarrow} & \stackrel{\alpha_{(A',B')}}{\downarrow} \\ \mathscr{A}(A,G(B)) & \longrightarrow \mathscr{A}(A',G(B')). \end{split}$$

Let B = F(A), we obtain

$$\begin{split} \mathscr{B}(F(A),F(A)) & \longrightarrow \mathscr{B}(F(A'),B') \\ & \stackrel{\alpha_{(A,F(A))}}{\downarrow} & \stackrel{\alpha_{(A',B')}}{\downarrow} \\ \mathscr{A}(A,G(F(A))) & \longrightarrow \mathscr{A}(A',G(B')). \end{split}$$

Hence for any object A of \mathscr{A} we obtain a morphism $\eta_A: A \to G(F(A))$ corresponding to $1_{F(A)}$. Similarly, for any object B of \mathscr{B} we obtain a morphism $\epsilon_B: F(G(B)) \to B$ corresponding to $1_{G(B)}$. These maps are called adjunction maps. The adjunction maps are functorial in A and B, hence we obtain morphisms of functors

$$\eta: 1_{\mathscr{A}} \to G \circ F$$
 (unit) and $\epsilon: F \circ G \to 1_{\mathscr{B}}$ (counit).

b. Given objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the correspondence between $F(A) \to B$ and $A \to G(B)$ is denoted by a horizontal bar, in both directions:

$$\left(F(A) \xrightarrow{g} B\right) \mapsto \left(A \xrightarrow{\bar{g}} G(B)\right),$$
$$\left(A \xrightarrow{f} G(B)\right) \mapsto \left(F(A) \xrightarrow{\bar{f}} B\right).$$

So $\bar{f} = f$ and $\bar{g} = g$. We call \bar{f} the *transpose* of f, and similarly for g. Then "functorial in $A \in \mathscr{A}$ and $B \in \mathscr{B}$ is equivalent to

$$\overline{\left(F(A) \xrightarrow{g} B \xrightarrow{q} B'\right)} = \left(A \xrightarrow{\bar{g}} G(B) \xrightarrow{G(q)} G(B')\right) \tag{4}$$

and

$$\overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B)\right)} = \left(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\bar{f}} B\right).$$
(5)

The above two identities can also be written as

$$\overline{q \circ g} = G(q) \circ \overline{g} \tag{6}$$

and

$$\overline{f \circ p} = \overline{f} \circ F(p). \tag{7}$$

In fact, the bijection that satisfies the above two conditions are equivalent to the definition of adjoint functors.

Lemma 3.2. Given an adjunction $F \dashv G$ with unit η and ϵ , the triangles

$$F \xrightarrow{F\eta} FGF \qquad \qquad G \xrightarrow{\eta G} GFG$$

$$\downarrow_{\epsilon F} \qquad and \qquad \downarrow_{G\epsilon}$$

$$G \xrightarrow{\eta G} GFG$$

$$\downarrow_{G\epsilon}$$

$$G \xrightarrow{\Gamma} GFG$$

commute. These are called triangle identities. They are commutative diagrams in the functor categories $[\mathscr{A},\mathscr{B}]$ and $[\mathscr{B},\mathscr{A}]$, respectively.

Proof. An equivalent statement is that the triangles

$$F(A) \xrightarrow{F(\eta_A)} FGF(A)$$
 $G(B) \xrightarrow{\eta_{G(B)}} GFG(B)$

$$\downarrow^{\epsilon_{F(A)}} \qquad \text{and} \qquad \downarrow^{G(\epsilon_B)}$$

$$F(A) \qquad \qquad G(B) \xrightarrow{\eta_{G(B)}} GFG(B)$$

commute for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Since $\overline{1_{GF(A)}} = \epsilon_{F(A)}$, equation (5) gives

$$\overline{\left(A \xrightarrow{\eta_A} GF(A) \xrightarrow{1} GF(A)\right)} = \left(F(A) \xrightarrow{F(\eta_A)} FGF(A) \xrightarrow{\epsilon_{F(A)}} F(A)\right).$$

But the left-hand side is $\overline{\eta_A} = \overline{\overline{1_{F(A)}}} = 1_{F(A)}$, proving the first triangle identity. The second follows by duality.

Lemma 3.3. Let $\mathscr{A} \xrightarrow{F \atop \bot} \mathscr{B}$ be an adjunction, with unit η and counit ϵ . Then

$$\overline{g} = G(g) \circ \eta_A$$

for any $g: F(A) \to B$, and

$$\overline{f} = \epsilon_B \circ F(f)$$

for any $f: A \to G(B)$.

Theorem 3.4. Take categories and functors $\mathscr{A} \xrightarrow{\stackrel{F}{\underline{\perp}}} \mathscr{B}$. There is a one-to-one correspondence between:

- (a) adjunctions between F and G (with F on the left and G on the right);
- (b) pairs $\left(1_{\mathscr{A}} \xrightarrow{\eta} GF, FG \xrightarrow{\epsilon} 1_{\mathscr{B}}\right)$ of natural transformations satisfying the triangle identities.
- (a) \Rightarrow (b) direction has been proved and the proof of converse is a direct calculate.

Definition 3.5. Given categories and functors

$$\mathscr{B} \qquad \qquad \downarrow^{Q}$$

$$\mathscr{A} \xrightarrow{P} \mathscr{C},$$

the comma category $(P\Rightarrow Q)$ (often written as $(P\downarrow Q))$ is the category defined as follows:

- objects are triples (A,h,B) with $A\in\mathscr{A},\,B\in\mathscr{B},$ and $h:P(A)\to Q(B)$ in \mathscr{C} ;
- maps $(A,h,B) \to (A',h',B')$ are pairs $(f:A\to A',g:B\to B')$ of maps such that the square

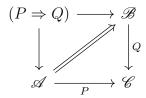
$$P(A) \xrightarrow{P(f)} P(A')$$

$$\downarrow h \qquad \qquad \downarrow h'$$

$$Q(B) \xrightarrow{Q(g)} Q(B')$$

commutes.

Remark. Given $\mathscr{A}, \mathscr{B}, \mathscr{C}, P$ and Q as above, there are canonical functors and a canonical natural transformations as shown:



in a suitable 2-categorical sense, $(P \Rightarrow Q)$ is universal with this property.

Definition 3.6. Let \mathscr{A} be a category and $A \in \mathscr{A}$. The *slice category* of \mathscr{A} over A, denoted by \mathscr{A}/A , is the category whose objects are maps into A and whose maps are commutative triangles. More precisely, an object is a pair (X,h) with $X \in \mathscr{A}$ and $h: X \to A$ in \mathscr{A} , and a map $(X,h) \to (X',h')$ in \mathscr{A}/A is a map $f: X \to X'$ in \mathscr{A} making the triangle

$$X \xrightarrow{f} X'$$

$$A$$

commute.

Slice categories are a special case of comma categories $(1_{\mathscr{A}} \Rightarrow A)$:

$$egin{array}{c} \mathbf{1} \\ \downarrow A \\ \longrightarrow & \mathscr{A} \end{array}$$

Dually, there is a coslice category $A/_{\mathscr{A}} \cong (A \Rightarrow 1_{\mathscr{A}})$, whose objects are the maps out of A.

Lemma 3.7. Take an adjunction $\mathscr{A} \xrightarrow{\frac{F}{\bot}} \mathscr{B}$ and an object $A \in \mathscr{A}$. Then the unit map $\eta_A : A \to GF(A)$ is an initial object of $(A \Rightarrow G)$.

Theorem 3.8. Take categories and functors $\mathscr{A} \xleftarrow{F}_{G} \mathscr{B}$. There is a one-to-one correspondence between:

(a) adjunctions between F and G (with F on the left and G on the right);

(b) natural transformations $\eta: 1_{\mathscr{A}} \to GF$ such that $\eta_A: A \to GF(A)$ is initial in $(A \Rightarrow G)$ for every $A \in \mathscr{A}$.

Corollary 3.9. Let $G: \mathscr{B} \to \mathscr{A}$ be a functor. Then G has a left adjoint if and only if for each $A \in \mathscr{A}$, the category $(A \Rightarrow G)$ has an intial object.

4 Yoneda lemma

Definition 4.1. Let \mathscr{A} be a locally small category and $A \in \mathscr{A}$. We define a functor

$$H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$$

as follows:

- for objects $B \in \mathcal{A}$, put $H^A(B) = \mathcal{A}(A, B)$;
- for maps $B \xrightarrow{g} B'$ in \mathscr{A} , define

$$H^A(g) = \mathscr{A}(A,g) = g_* = g \circ - : \mathscr{A}(A,B) \to \mathscr{A}(A,B')$$

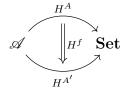
by

$$p \mapsto g \circ p$$

for all $p: A \to B$.

Definition 4.2. Let \mathscr{A} be a locally small category. A functor $X: \mathscr{A} \to \mathbf{Set}$ is representable if $X \cong H^A$ for some $A \in \mathscr{A}$. A representation of X is a choice of an object $A \in \mathscr{A}$ and an isomorphism between H^A and X.

Any map $A' \xrightarrow{f} A$ induces a natural transformation



(also called $\mathscr{A}(f,-),\,f^*$ or $-\circ f),$ whose B-component (for $B\in\mathscr{A}$) is the function

$$H^A(B) = \mathscr{A}(A,B) \to H^{A'}(B) = \mathscr{A}(A',B)$$

$$p \mapsto p \circ f.$$

Each ${\cal H}^A$ is covariant, but they come together to form a contravariant functor, as in the following definition.

Definition 4.3. Let \mathscr{A} be a locally small category. The functor

$$H^{\bullet}: \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathbf{Set}]$$

is defined on objects A by $H^{\bullet}(A) = H^{A}$ and on maps f by $H^{\bullet}(f) = H^{f}$.

Definition 4.4. Let \mathscr{A} be a locally small category and $A \in \mathscr{A}$. We define a functor

$$H_A = \mathscr{A}(-,A) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$$

as follows:

- for objects $B \in \mathcal{A}$, put $H_A(B) = \mathcal{A}(B, A)$;
- for maps $B' \xrightarrow{g} B$ in \mathscr{A} , define

$$H_A(g) = \mathscr{A}(g, A) = g^* = -\circ g : \mathscr{A}(B, A) \to \mathscr{A}(B', A)$$

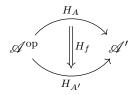
by

$$p \mapsto p \circ g$$

for all $p: B \to A$

Definition 4.5. Let \mathscr{A} be a locally small category. A functor $X : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ is representable if $X \cong H_A$ for some $A \in \mathscr{A}$. A representation of X is a choice of an object $A \in \mathscr{A}$ and an isomorphism between H_A and X.

Any map $A \xrightarrow{f} A'$ in $\mathscr A$ induces a natural transformation



(also called $\mathscr{A}(-,f)$, f_* or $f \circ p$), whose B-component (for $B \in \mathscr{A}$) is the function

$$H_A(B) = \mathscr{A}(B, A) \to H_{A'}(B) = \mathscr{A}(B, A')$$

 $p \mapsto f \circ p.$

Definition 4.6. Let $\mathscr A$ be a locally small category. The *Yoneda embedding* of $\mathscr A$ is the functor

$$H_{\bullet}: \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$$

defined on objects A by $H_{\bullet}(A) = H_A$ and on maps f by $H_{\bullet}(f) = H_f$.

Proposition 4.7. Let \mathscr{A} be a locally small category, and let $A, A' \in \mathscr{A}$ with $H_A \cong H_{A'}$. Then we have $A \cong A'$.

Proof. By definition, we have a natural isomorphism α such that for any $B, B' \in \mathcal{A}$ and any map $f: B' \to B$, the square

$$\mathscr{A}(B,A) \xrightarrow{H_A(f)} \mathscr{A}(B',A)$$

$$\downarrow^{\alpha_{B'}} \qquad \qquad \downarrow^{\alpha_{B'}}$$

$$\mathscr{A}(B,A') \xrightarrow{H_{A'}(f)} \mathscr{A}(B',A')$$

commutes. Let B = A and B' = A', then for any $g \in \mathscr{A}(A,A)$ and $f : A' \to A$, we have

$$\alpha_A(g) \circ f = \alpha_{A'} (g \circ f)$$

for any $g: A \to A$. Then $\alpha_A(g) \circ f \in \mathcal{A}(A', A')$. Let $f = \alpha_{A'}^{-1}(1_{A'})$ and $g = 1_A$, we obtain

$$\alpha_A(1_A) \circ \alpha_{A'}^{-1}(1_{A'}) = 1_{A'}.$$

Similarly, we have

$$\alpha_{A'}(1_{A'}) \circ \alpha_A^{-1}(1_A) = 1_A.$$

Definition 4.8. Let \mathscr{A} be a locally small category. The functor

$$\operatorname{Hom}_{\mathscr{A}}: \mathscr{A}^{\operatorname{op}} \times \mathscr{A} \to \mathbf{Set}$$

is defined by

$$(A,B) \qquad \mapsto \qquad \mathscr{A}(A,B)$$

$$\uparrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

In other words, $\operatorname{Hom}_{\mathscr{A}}(A,B) = \mathscr{A}(A,B)$ and $(\operatorname{Hom}_{\mathscr{A}}(f,g))(p) = g \circ p \circ f$, whenever $A' \xrightarrow{f} A \xrightarrow{p} B \xrightarrow{g} B'$.

Remark. (a) Given sets A and B, we have product $A \times B$ and the set B^A (or $\mathbf{Set}(A,B)$) of functions from A to B. Fix a set B, taking the product with B defines a functor

$$- \times B : \mathbf{Set} \longrightarrow \mathbf{Set}$$

$$A \longmapsto A \times B.$$

There is also a functor

$$(-)^B : \mathbf{Set} \longrightarrow \mathbf{Set}$$
 $C \longmapsto C^B.$

Moreover, there is an adjunction between them, i.e.,

$$\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, C^B)$$

for any sets A and C.

(b) Similarly, for any category \mathscr{B} , there is an adjunction $(-\times\mathscr{B})\dashv [\mathscr{B},-]$ of functors $\mathbf{CAT}\to\mathbf{CAT}$, that is, there is a canonical bijection

$$\mathbf{CAT}(\mathscr{A} \times \mathscr{B}, \mathscr{C}) \cong \mathbf{CAT}(\mathscr{A}, [\mathscr{B}, \mathscr{C}])$$

for $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbf{CAT}$. Under this bijection, the functors

$$\operatorname{Hom}_{\mathscr{A}}: \mathscr{A}^{\operatorname{op}} \times \mathscr{A} \to \mathbf{Set} \quad \text{ and } \quad H^{\bullet}: \mathscr{A}^{\operatorname{op}} \to [\mathscr{A}, \mathbf{Set}]$$

correspond to one another and carries the same information.

(c) Now we can use this bijection to explain naturality in the definition of adjunction. Take categories $\mathscr{A} \xrightarrow{F} \mathscr{B}$, then we have

$$\begin{array}{ccc} \mathscr{A}^{\mathrm{op}} \times \mathscr{B} & \xrightarrow{1 \times G} \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \\ F^{\mathrm{op}} \times 1 & & & \downarrow \operatorname{Hom}_{\mathscr{A}} \\ \mathscr{B}^{\mathrm{op}} \times \mathscr{B} & \xrightarrow{\operatorname{Hom}_{\mathscr{B}}} & \mathbf{Set}. \end{array}$$

Definition 4.9. Let A be an object of a category. A generalized element of A is a map with codomain A. A map $S \to A$ is a generalized element of A of shape S.

Definition 4.10. Let \mathscr{A} be a category. A *presheaf* on \mathscr{A} is a functor $\mathscr{A}^{op} \to \mathbf{Set}$.

For each $A \in \mathscr{A}$ we have a representable presheaf H_A , and we are asking how the rest of the presheaf category $[\mathscr{A}^{op}, \mathbf{Set}]$ looks from the viewpoint of H_A . In other words, if X is another presheaf, what are the maps $H_A \to X$?

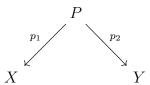
Theorem 4.11 (Yoneda). Let \mathscr{A} be a locally small category. Then

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] (H_A, X) \cong X(A) \tag{8}$$

naturally in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$.

5 Limits

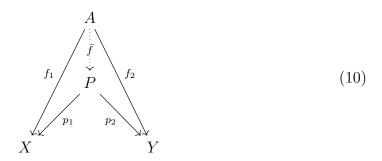
Definition 5.1. Let \mathscr{A} be a category and $X,Y\in\mathscr{A}$. A product of X and Y consists of an object P and maps



with the property that for all objects and maps



in \mathscr{A} , there exists a unique map $\bar{f}:A\to P$ such that



commutes. The maps p_1 and p_2 are called *projections*.

Definition 5.2. Let \mathscr{A} be a category, I a set, and $(X_i)_{i\in I}$ a family of objects of \mathscr{A} . A product of $(X_i)_{i\in I}$ consists of an object P and a family of maps

$$\left(P \xrightarrow{p_i} X_i\right)_{i \in I}$$

with the property that for all objects A and families of maps

$$\left(A \xrightarrow{f_i} X_i\right)_{i \in I} \tag{11}$$

there exists a unique map $\bar{f}: A \to P$ such that $p_i \circ \bar{f} = f_i$ for all $i \in I$.

Definition 5.3. A fork in a category consists of objects and maps

$$A \xrightarrow{f} X \xrightarrow{s} Y \tag{12}$$

such that sf = tf. Let \mathscr{A} be a category and let $X \xrightarrow{s} Y$ be objects and maps in \mathscr{A} . An equalizer of s and t is an object E together with a map $E \xrightarrow{i} X$ such that

$$E \xrightarrow{i} X \xrightarrow{s} Y$$

is a fork, and with the property that for any fork (12), there exists a unique map $\bar{f}:A\to E$ such that

$$\begin{array}{ccc}
A \\
& \downarrow \\
E & \longrightarrow X
\end{array} \tag{13}$$

commutes.

Definition 5.4. Let \mathscr{A} be a category, and take objects and maps

$$\begin{array}{c}
Y \\
\downarrow t \\
X \xrightarrow{s} Z
\end{array} \tag{14}$$

in \mathscr{A} . A *pullback* of this diagram is an object $P \in \mathscr{A}$ together with maps $p_1: P \to X$ and $p_2: P \to Y$ such that

$$P \xrightarrow{p_2} Y$$

$$\downarrow t$$

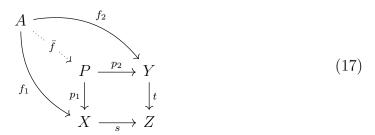
$$X \xrightarrow{s} Z$$

$$(15)$$

commutes, and with the property that for any commutative squar

$$\begin{array}{ccc}
A & \xrightarrow{f_2} & Y \\
f_1 \downarrow & & \downarrow t \\
X & \xrightarrow{s} & Z
\end{array}$$
(16)

in \mathscr{A} , there is a unique map $\bar{f}:A\to P$ such that



commutes.

Definition 5.5. Let \mathscr{A} be a category and \mathbf{I} a small category. A functor $\mathbf{I} \to \mathscr{A}$ is called a *diagram* in \mathscr{A} of *shape* \mathbf{I} .

Definition 5.6. Let \mathscr{A} be a category, **I** a small category, and $D: \mathbf{I} \to \mathscr{A}$ a diagram in \mathscr{A} .

(a) A cone on D is an object $A \in \mathcal{A}$ (the vertex of the cone) together with a family

$$\left(A \xrightarrow{f_I} D(I)\right)_{I \in \mathbf{I}} \tag{18}$$

of maps in \mathscr{A} such that for all maps $I \xrightarrow{u} J$ in **I**, the triangle

$$A \xrightarrow{f_I} D(I)$$

$$D_{Uu}$$

$$D(J)$$

commutes.

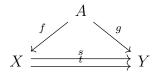
(b) A limit of D is a cone $\left(L \xrightarrow{p_I} D(I)\right)_{I \in \mathbf{I}}$ with the property that for any cone (18) on D, there exists a unique map $\bar{f}: A \to L$ such that $p_I \circ \bar{f} = f_I$ for all $I \in \mathbf{I}$. The maps p_I are called the *projections* of the limit. We write $L = \lim_{t \to I} D$.

Example 5.1. Let categories T, E and P be

$$T = \bullet \quad \bullet, \quad E = \quad \bullet \implies \bullet \quad , \quad P = \qquad \qquad \downarrow$$
 (19)

Let \mathscr{A} be any category.

- (a) A diagram D of shape \mathbf{T} in \mathscr{A} is a pair (X,Y) of objects of \mathscr{A} . A cone on D is an object A together with maps $f_1:A\to X$ and $f_2:A\to Y$, and a limit of D is a product of X and Y.
 - More generally, let I be a set and write \mathbf{I} for the discrete category on I. A functor $D: \mathbf{I} \to \mathscr{A}$ is an I-indexed family $(X_i)_{i \in I}$ of objects of \mathscr{A} , and a limit of D is exactly a product of the family $(X_i)_{i \in I}$.
- (b) A diagram D of shape \mathbf{E} in \mathscr{A} is a parallel pair $X \xrightarrow{s} Y$ of maps in \mathscr{A} . A cone on D consists of objects and maps



such that $s \circ f = g$ and $t \circ f = g$. But since g is determined by f, it is equivalent to say that a cone on D consists of an object A and a map $f: A \to X$ such that

$$A \xrightarrow{f} X \xrightarrow{s} Y$$

is a fork. A limit of D is a universal fork on s and t, that is, an equalizer of s and t.

(c) A diagram D of shape \mathbf{P} in \mathscr{A} consists of objects and maps

$$\begin{array}{c} Y \\ \downarrow^t \\ X \xrightarrow{s} Z \end{array}$$

in \mathscr{A} . Performing a simplification similar to that in (b), we see that a cone on D is a commutative square (16).

(d) Let $\mathbf{I} = (\mathbb{N}, \leq)^{\mathrm{op}}$. A diagram $D: \mathbf{I} \to \mathscr{A}$ consists of objects and maps

$$\cdots \xrightarrow{s_3} X_2 \xrightarrow{s_2} X_1 \xrightarrow{s_1} X_0.$$

For example, suppose that we have a set X_0 and a chain of subsets

$$\cdots \subset X_2 \subset X_1 \subset X_0.$$

The inclusion maps form a diagram in **Set** of the type above, and its limit is $\bigcap_{i\in\mathbb{N}} X_i$.

Definition 5.7. (a) Let **I** be a small category. A category \mathscr{A} has limits of shape **I** if for every diagram D of shape **I** in \mathscr{A} , a limit of D exists.

(b) A category has all limits (or properly, has small limits) if it has limits of shape I for all small categories.

Definition 5.8. A category is *finite* if it contains only finitely many maps (in which case it also contains only finitely many objects). A *finite limit* is a limit of shape \mathbf{I} for some finite category \mathbf{I} .

Proposition 5.9. Let \mathscr{A} be a category.

- (a) If \mathscr{A} has all products and equalizers then \mathscr{A} has all limits.
- (b) If $\mathscr A$ has binary products, a terminal object and equalizers then $\mathscr A$ has finite limits.

Definition 5.10. Let \mathscr{A} be a category. A map $X \xrightarrow{f} Y$ in \mathscr{A} is *monic* (or a *monomorphism*) if for all objects A and maps $A \xrightarrow{x'} X$,

$$f \circ x = f \circ x' \implies x = x'.$$

Definition 5.11. Let \mathscr{A} be a category and \mathbf{I} a small category. Let $D: \mathbf{I} \to \mathscr{A}$ be a diagram in \mathscr{A} , and write D^{op} for the corresponding functor $\mathbf{I}^{\mathrm{op}} \to \mathscr{A}^{\mathrm{op}}$. A cocone on D is a cone on D^{op} , and a colimit of D is a limit of D^{op} .

Explicitly, a cocone on D is an object $A \in \mathcal{A}$ together with a family

$$\left(D(I) \xrightarrow{f_I} A\right)_{I \in \mathbf{I}} \tag{20}$$

of maps in \mathscr{A} such that for all maps $I \xrightarrow{u} J$ in **I**, the diagram

commutes. A colimit of D is a cocone

$$\left(D(I) \xrightarrow{p_I} C\right)_{I \in \mathbf{I}}$$

with the property that for any cocone (20) on D, there is a unique map $\bar{f}: C \to A$ such that $\bar{f} \circ p_I = f_I$ for all $I \in \mathbf{I}$.

Definition 5.12. A sum or coproduct is a colimit over a discrete category.

Let $(X_i)_{i\in I}$ be a family of objects of a category. Their sum (if it exists) is written as $\sum_{i\in I} X_i$ or $\coprod_{i\in I} X_i$.

Definition 5.13. A coeualizer is a colimit of shape **E**.

In other words, given a diagram $X \xrightarrow{s} Y$, a coequalizer of s and t is a map $Y \xrightarrow{p} C$ satisfying $p \circ s = p \circ t$ and universal with this property.

Definition 5.14. A pushout is a colimit of shape

$$\mathbf{P}^{\mathrm{op}} = igcup_{ullet} egin{matrix} ullet & \longrightarrow ullet \ \mathbf{P}^{\mathrm{op}} & \downarrow \ ullet \ ullet \ \end{pmatrix}$$
 .

In other words, the pushout of a diagram

$$\begin{array}{ccc}
X & \xrightarrow{s} & Y \\
\downarrow & & \\
Z & & \end{array}$$
(21)

is (if it exists) a commutative square

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \cdot \end{array}$$

that is universal as such. In other words still, a pushout in a category \mathscr{A} is a pullback in $\mathscr{A}^{\mathrm{op}}$.

Example 5.2. A diagram $D: (\mathbb{N}, \leq) \to \mathscr{A}$ consists of objects and maps

$$X_0 \xrightarrow{s_1} X_1 \xrightarrow{s_2} X_2 \xrightarrow{s_3} \cdots$$

in \mathscr{A} . Colimits of such diagrams are traditionally called *direct limits*.

Definition 5.15. Let \mathscr{A} be a category. A map $X \xrightarrow{f} Y$ in \mathscr{A} is *epic* (or an *epimorphism*) if for all objects Z and maps $Y \xrightarrow{g} Z$,

$$g \circ f = g' \circ f \implies g = g'.$$