# ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
# LECTURE 19

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**Lemma 1** (Nakayama)**.** *Let $A$ be a local ring and $\mathfrak{m} \subset A$ be its maximal ideal. Let $M$ be a finitely generated $A$-module:*

(1) *if $M = \mathfrak{m}M$, then $M = \{0\}$;*
(2) *write $k = A/\mathfrak{m}$, let $f_1, \ldots, f_r \in M$ such that $\bar{f}_1, \ldots, \bar{f}_r$ generate $M/\mathfrak{m}M$ as $k$-vector space. Then $f_1, \ldots, f_r$ generate $M$ as an $A$-module.*

*Proof.* (2) Let $N := \langle f_1, \ldots, f_r \rangle \subset M$. To show $N = M$ is equivalent to show $M/N = \{0\}$. Since $\bar{f}_1, \ldots, \bar{f}_r$ generate $M/\mathfrak{m}M$, we have

$$(N + \mathfrak{m}M)/\mathfrak{m}M = M/\mathfrak{m}M.$$

This equation implies

$$N + \mathfrak{m}M = M.$$

Then we get $\mathfrak{m} \cdot (M/N) = (\mathfrak{m}M + n)/N = M/N$, it implies $M/N = \{0\}$ by using the first conclusion of the lemma. $\square$

**Definition 1** (Discrete Valuation Ring)**.** Let $A$ be a local ring, $\mathfrak{m}$ be its maximal ideal. Further more, assume $A$ is also an integral domain. Then $A$ is called a discrete valuation ring(DVR) if the following conditions hold:

(1) $\mathfrak{m}$ is a principal ideal, i.e. $\mathfrak{m} = \langle t \rangle$ for some $t \in \mathfrak{m}$(such a $t$ is called a uniformizing parameter);
(2) if $t$ is a uniformizing parameter, then every element $f \in A$ can be written as $f = at^n$ for $a \in A$ a unit and $n \in \mathbb{Z}^+$.

*Remark.* If $t$ is a uniformizing parameter, then $\mathfrak{m}^n = \langle t^n \rangle$.

This remark can be proved by induction. It is obvious that $\langle t^n \rangle \subset \mathfrak{m}^n$. The opposite inclusion is true for $n = 0, 1$, assume $\langle t^{n-1} \rangle = \mathfrak{m}^{n-1}$ is true. Then every element in $\mathfrak{m}$ can be written as sum of elements of the form $abt^n = at \cdot bt^{n-1}$ with $a, b \in A$, hence $\mathfrak{m}^n \subset \langle t^n \rangle$.

**Exercise 1.** *Prove that for a curve $C$ and a nonsingular point $p \in C$, $\mathcal{O}_{C,p}$ is a discrete valuation ring.*

**Proposition 1.** (1) *Let $A$ be a ring, $I \subset A$ be an ideal, $\pi : A \to A/I$ be a projective map. Then the map*

$$\begin{array}{ccc} \{\textit{ideals of } A/I\} & \to & \{\textit{ideals of } A \textit{ containing } I\} \\ J & \to & \pi^{-1}(J) \end{array}$$

*is injective.*

(2) *If $A$ is a noetherian ring, $I \subset A$ is an ideal, then $A/I$ is also noetherian.*

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(3) *Let $X$ be a variety, $p \in X$. Then $\mathcal{O}_{X,p}$ is noetherian.*

*Proof.* (3) To show $\mathcal{O}_{X,p}$ is noetherian, as $\mathcal{O}_{X,p}$ only depends on a neighborhood of $p$, we can assume $X \subset \mathbb{A}^n$ is an affine variety. Then $A(X)$ is noetherian. The map

$$\begin{array}{ccc} \{\text{ideals in } \mathcal{O}_{X,p}\} & \to & \{\text{ideals in } A(X)\} \\ I & \to & I \cap A(X) \end{array}$$

is injective, hence $\mathcal{O}_{X,p}$ is noetherian. $\square$

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