De Giorgi-Nash-Moser Theory

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Abstract

This is a learning note about the De Giorgi-Nash-Moser theory, the reference book is Qing Han and Fanghua Lin's *Elliptic Partial Differential Equations*

The main task of this note is to prove the following theorem:

Theorem 1 *Suppose $a_{ij} \in L^\infty(B_1)$ and $c \in L^q(B_1)$ for some $q > \frac{n}{2}$ satisfy the following assumptions*

$$a_{ij}(x)\xi_i\xi_j \geq \lambda |\xi|^2 \text{ for any } x \in B_1 \text{ and } |a_{ij}|_{L^\infty} + \|c\|_{L^q} \leq \Lambda$$

for some positive constants λ and Λ. Suppose that $u \in H^1(B_1)$ is a subsolution in the following sense

$$\int_{B_1} a_{ij} D_i u D_j \varphi + cu\varphi \leq \int_{B_1} f\varphi \text{ for any } \varphi \in H_0^1(B_1) \text{ and } \varphi \geq 0 \text{ in } B_1.$$

If $f \in L^q(B_1)$, then $u^+ \in L^\infty_{\mathrm{loc}}(B_1)$. Moreover, there holds for any $\theta \in (0,1)$ and any $p > 0$

$$\sup_{B_\theta} u^+ \leq C\left\{\frac{1}{(1-\theta)^{\frac{n}{p}}}\|u^+\|_{L^p(B_1)} + \|f\|_{L^q(B_1)}\right\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.

Proof. We use two approaches to prove the theorem for $\theta = 1/2$ and $p = 2$.

Method 1. Approach by De Giorgi.

For some $k \geq 0$ and $\eta \in C_0^1(B_1)$, define $v = (u-k)^+$ and set $\varphi = v\eta^2$. By Hölder inequality we have

$$\begin{aligned}\int a_{ij} D_i u D_j \varphi &= \int a_{ij} D_i u D_j v \eta^2 + \int 2 a_{ij} D_i u D_j \eta v \eta \\ &\geq \lambda \int |Dv|^2\eta^2 - 2\Lambda \int |Dv||D\eta| v\eta \\ &\geq \frac{\lambda}{2}\int |Dv|^2\eta^2 - \frac{2\Lambda^2}{\lambda}\int |D\eta|^2 v^2.\end{aligned}$$

Hence we obtain

$$\int |Dv|^2\eta^2 \leq C\left(\int v^2|D\eta|^2 + \int |c| v^2\eta^2 + k^2\int |c|\eta^2 + \int |f| v\eta^2\right).$$

Then

$$\int |D(v\eta)|^2 \le C\left(\int v^2|D\eta|^2 + \int |c|v^2\eta^2 + k^2\int |c|\eta^2 + \int |f|v\eta^2\right).$$

By the Sobolev's inequality we have

$$\left(\int_{B_1} (v\eta)^{2^*}\right)^{\frac{2}{2^*}} \le c(n)\int_{B_1} |D(v\eta)|^2$$

where $2^* = 2n/(n-2)$ for $n > 2$ and $2^* > 2$ is arbitrary if $n = 2$. Hölder's inequality implies that with δ small and $\eta \le 1$

$$\begin{aligned}\int |f|v\eta^2 \le& \left(\int |f|^q\right)^{\frac{1}{q}}\left(\int |v\eta|^{2^*}\right)^{\frac{1}{2^*}} |\{v\eta \ne 0\}|^{1-\frac{1}{2^*}-\frac{1}{q}}\\ \le& c(n)\|f\|_{L^q}\left(\int |D(v\eta)|^2\right)^{\frac{1}{2}} |\{v\eta \ne 0\}|^{\frac{1}{2}+\frac{1}{n}-\frac{1}{q}}\\ \le& \delta\int |D\,(v\eta)\,|^2 + c(n,\delta)\|f\|^2_{L^q}|\{v\eta \ne 0\}|^{1+\frac{2}{n}-\frac{2}{q}}..\end{aligned}$$

Note $1 + \frac{2}{n} - \frac{2}{q} > 1 - \frac{1}{q}$ if $q > n/2$. Therefore we have the following estimate:

$$\int |D(v\eta)|^2 \le C\left(\int v^2|D\eta|^2 + \int |c|v^2\eta^2 + k^2\int |c|\eta^2 + \|f\|_{L^q(B_1)}|\{v\eta \ne 0\}|^{1-\frac{1}{q}}\right).$$

Since

$$\int |c|\eta^2 \le \left(\int |c|^q\right)^{\frac{1}{q}} |\{v\eta \ne 0\}|^{1-\frac{1}{q}}$$

and by Hölder's inequality

$$\begin{aligned}\int |c|v^2\eta^2 \le& \left(\int |c|^q\right)^{\frac{1}{q}}\left(\int (v\eta)^{2^*}\right)^{\frac{2}{2^*}} |\{v\eta \ne 0\}|^{1-\frac{2}{2^*}-\frac{1}{q}}\\ \le& c(n)\int |D(v\eta)|^2\left(\int |c|^q\right)^{\frac{1}{q}} |\{v\eta \ne 0\}|^{\frac{2}{n}-\frac{1}{q}}\end{aligned}$$

if $|\{v\eta \ne 0\}|$ is small, we have

$$\begin{aligned}&\int |D(v\eta)|^2 \le\\ &C\left(\int v^2|D\eta|^2 + \int |D(v\eta)|^2|\{v\eta \ne 0\}|^{\frac{2}{n}-\frac{1}{q}} + (k^2 + \|f\|^2_{L^q(B_1)})|\{v\eta \ne 0\}|^{1-\frac{1}{q}}\right),\end{aligned}$$

and this implies

$$\int |D(v\eta)|^2 \le C\left(\int v^2|D\eta|^2 + (k^2 + \|f\|^2_{L^q(B_1)})|\{v\eta \ne 0\}|^{1-\frac{1}{q}}\right) \qquad (1)$$

if $|\{v\eta \ne 0\}|$ is small.

Applying Sobolev's inequality again we obtain

$$\int (v\eta)^2 \le \left(\int (v\eta)^{2^*}\right)^{\frac{2}{2^*}} |\{v\eta \neq 0\}|^{1-\frac{2}{2^*}} \le c(n) \int |D(v\eta)|^2 |\{v\eta \neq 0\}|^{\frac{2}{n}}.$$

Therefore we have

$$\int (v\eta)^2 \le C\left(\int v^2|D\eta|^2 |\{v\eta \neq 0\}|^{\frac{2}{n}} + (k + \|f\|_{L^q(B_1)})^2 |\{v\eta \neq 0\}|^{1+\frac{2}{n}-\frac{1}{q}}\right)$$

if $|\{v\eta \neq 0\}|$ is small. Choose $0 < \epsilon < \frac{2}{n} - \frac{1}{q}$, then we have

$$\int (v\eta)^2 \le C\left(\int v^2|D\eta|^2 |\{v\eta \neq 0\}|^{\epsilon} + (k + \|f\|_{L^q(B_1)})^2 |\{v\eta \neq 0\}|^{1+\epsilon}\right)$$

if $|\{v\eta \neq 0\}|$ is small.

Now we choose the cut-off function in the following way. For any fixed $0 < r < R \le 1$ choose $\eta \in C_0^\infty(B_R)$ such that $\eta \equiv 1$ in B_r and $0 \le \eta \le 1$ and $|D\eta| \le 2(R-r)^{-1}$ in B_1. Define the set

$$A(k,r) = \{x \in B_r | u \ge k\}.$$

If $|A(k,R)|$ is small, we obtain

$$\begin{aligned} &\int_{A(k,r)} (u-k)^2 \\ \le &C\left(\frac{1}{(R-r)^2}|A(k,R)|^{\epsilon}\int_{A(k,r)}(u-k)^2 + (k + \|f\|_{L^q(B_1)})^2|A(k,R)|^{1+\epsilon}\right) \end{aligned} \tag{2}$$

Note

$$|A(k,R)| \le \frac{1}{k}\int_{A(k,R)} u^+ \le \frac{1}{k}\|u^+\|_{L^2}.$$

Hence (2) holds if $k \ge k_0 = C\|u^+\|_{L^2}$ for some large C depending only on λ and Λ.

Next we would show that there exists some $k = C\left(k_0 + \|f\|_{L^q(B_1)}\right)$ such that

$$\int_{A(k,\frac{1}{2}}(u-k)^2 = 0.$$

Let $h > k \ge k_0$ and any $0 < r < 1$. Since $A(k,r) \supset A(h,r)$, we obtain

$$\int_{A(h,r)}(u-h)^2 \le \int_{A(k,r)}(u-k)^2$$

and

$$|A(h,r)| = |B_r \cap \{u - k \ge h - k\}| \le \frac{1}{(h-k)^2}\int_{A(k,r)}(u-k)^2.$$

Therefore by (2) we have for any $h > k \geq k_0$ and $\frac{1}{2} \leq r < R \leq 1$

$$\begin{aligned}
&\int_{A(h,r)} (u-h)^2 \\
\leq & C\left(\frac{1}{(R-r)^2}\int_{A(h,R)} (u-h)^2 + (h + \|f\|_{L^q(B_1)})^2|A(h,R)|\right)|A(h,R)|^{\epsilon} \\
\leq & C\left(\frac{1}{(R-r)^2} + \frac{(h+\|f\|_{L^q(B_1)})^2}{(h-k)^2}\right)\frac{1}{(h-k)^{2\epsilon}}\left(\int_{A(k,R)} (u-k)^2\right)^{1+\epsilon}
\end{aligned}$$

or

$$\|(u-h)^+\|_{L^2(B_r)} \leq C\left(\frac{1}{R-r} + \frac{h+F}{h-k}\right)\frac{1}{(h-k)^{\epsilon}}\|(u-k)^+\|^{1+\epsilon}_{L^2(B_R)}. \tag{3}$$

Set $\varphi(k,r) = \|(u-k)^+\|_{L^2(B_r)}$, $\tau = \frac{1}{2}$ and some $k > 0$ to be determined. Define for $m = 0, 1, 2, \cdots$,

$$k_m = k_0 + k(1 - \frac{1}{2^m}) \text{ and } r_m = \tau + \frac{1}{2^m}(1-\tau).$$

By definition we have

$$k_m - k_{m-1} = \frac{k}{2^m} \text{ and } r_{m-1} - r_m = \frac{1}{2^m}(1-\tau).$$

METHOD 2. Approach by Moser.

Think about the following undergraduate level question:

Question Let $f \in C[0,1]$, then what is the value of

$$\lim_{\gamma\to\infty}\left|\int_0^1 |f(x)|^{\gamma}\,\mathrm{d}x\right|^{\frac{1}{\gamma}} =?$$

The answer is $\sup_{0\leq x\leq 1}|f(x)| = \|f\|_{L^\infty}$. This is exatly the way we transform $\|u^+\|_{L^p}$ to $\sup u^+$ in the proof of the above theorem. To make things more understanding, we assume $f = 0$. In this simplified version, We first establish the inequality

$$\left(\int_{B_r} |u^+|^{\gamma\chi}\right)^{\frac{1}{\chi}} \leq C\int_{B_R} |u^+|^{\gamma},$$

where $\chi > 1$ and $\gamma \geq 2$, $r < R$. Then we use this inequality to iterate, the iterating step makes $\chi \to \infty$, then the left side would be more and more likely to the suprimum norm of u^+ just as the question. Hence we can get the following inequality by doing this iteration:

$$\sup_{B_{\frac{1}{2}}} u^+ \leq C\|u^+\|_{L^2(B_1)}.$$

Thus the case $f = 0$ and $p = 2$ can be proved. The proof under the condition of $f \neq 0$ needs to be modified slightly. Then the general case that $p = 2$ can be proved easily by using the above special case.

According to the above discussion, we want to establish the inequality like this:

$$\|u^+\|_{L^{\gamma\chi}(B_r)} \leq C\|u^+\|_{L^{\gamma}(B_R)}$$

for some constant C and $r < R$. This inequality estimate the $L^{\gamma\chi}$-norm by the weaker L^{γ}-norm. As a trade-off, we have to make $r < R$, via certain test function.

For some $k > 0$ and $m > 0$, set $\overline{u} = u^{+} + k$ and

$$\overline{u}_m = \begin{cases} \overline{u} & \text{if } u \leq m \\ k+m & \text{if } u \geq m \end{cases}$$

The point is that $\overline{u}_m$ is still an element of $H^1(B_1)$,but bounded below by k and from above by $(k+m)$. Then we have $D\overline{u}_m = 0$ whenever $u < 0$ or $u > m$ and $\overline{u}_m \leq \overline{u}$. Set the test function

$$\phi = \eta^2 \left(\overline{u}_m^{\beta}\overline{u} - k^{\beta+1}\right) \in H_0^1(B_1)$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. The function η is a cut-off function to be chosen later on (remember the trade-off $r < R$?). ϕ is an element of $H_0^1(B_1)$ because $\overline{u}_m$ is bounded. Direct calculation yields

$$\begin{aligned} D\phi =& \beta\eta^2\overline{u}_m^{\beta-1}D\overline{u}_m\overline{u} + D\overline{u}\eta^2\overline{u}_m^{\beta} + 2\eta D\eta\left(\overline{u}_m^{\beta}\overline{u} - k^{\beta+1}\right) \\ =& \eta^2\overline{u}_m^{\beta}\left(\beta D\overline{u}_m + D\overline{u}\right) + 2\eta D\eta\left(\overline{u}_m^{\beta}\overline{u} - k^{\beta+1}\right). \end{aligned}$$

where we used the fact that $\overline{u} = \overline{u}_m$ whenever $D\overline{u}_m \neq 0$. Then we have

$$\begin{aligned} \int a_{ij}D_iuD_j\phi =& \int a_{ij}D_i\overline{u}\left(\beta D_j\overline{u}_m + D_j\overline{u}\right)\eta^2\overline{u}_m^{\beta} + 2\int a_{ij}D_i\overline{u}\eta D_j\eta\left(\overline{u}_m^{\beta}\overline{u} - k^{\beta+1}\right) \\ \geq& \lambda\beta\int \eta^2\overline{u}_m^{\beta}\left|D\overline{u}_m\right|^2 + \lambda\int \eta^2\overline{u}_m^{\beta}\left|D\overline{u}\right|^2 - \Lambda\int \left|D\overline{u}\right|\left|D\eta\right|\overline{u}_m^{\beta}\overline{u}\eta \\ \geq& \lambda\beta\int \eta^2\overline{u}_m^{\beta}\left|D\overline{u}_m\right|^2 + \frac{\lambda}{2}\int \eta^2\overline{u}_m^{\beta}\left|D\overline{u}\right|^2 - \frac{2\Lambda^2}{\lambda}\int \left|D\eta\right|^2\overline{u}_m^{\beta}\overline{u}^2. \end{aligned}$$

Hence we obtain by noting $\overline{u} \geq k$

$$\begin{aligned} &\beta\int \eta^2\overline{u}_m^{\beta}\left|D\overline{u}_m\right|^2 + \int \eta^2\overline{u}_m^{\beta}\left|D\overline{u}\right|^2 \\ \leq& C\left\{\int \left|D\eta\right|^2\overline{u}_m^{\beta}\overline{u}^2 + \int \left(|c|\,\eta^2\overline{u}_m^{\beta}\overline{u}^2 + |f|\,\eta^2\overline{u}_m^{\beta}\overline{u}\right)\right\} \\ \leq& C\left\{\int \left|D\eta\right|^2\overline{u}_m^{\beta}\overline{u}^2 + \int c_0\eta^2\overline{u}_m^{\beta}\overline{u}^2\right\} \end{aligned}$$

where c_0 is defined as

$$c_0 = |c| + \frac{|f|}{k}.$$

Choose $k = \|f\|_{L^q}$ if f is not identically zero. Otherwise choose arbitrary $k > 0$ and eventually let $k \to 0^{+}$. By assumption we have

$$\|c_0\|_{L^q} \leq \Lambda + 1.$$

Set $w = \overline{u}_m^{\frac{\beta}{2}}\overline{u}$. Then

$$|Dw| = \overline{u}_m^{\frac{\beta}{2}}\left(\frac{\beta}{2}\cdot D\overline{u}_m + D\overline{u}\right),$$

therefore

$$
\begin{aligned}
|Dw|^2 =& \overline{u}_m^\beta \left| \frac{\beta}{2} D\overline{u}_m + D\overline{u} \right|^2 \\
=& \overline{u}_m^\beta \left(\frac{\beta^2}{4} |D\overline{u}_m|^2 + \beta D\overline{u}_m D\overline{u} + |D\overline{u}|^2 \right) \\
=& \overline{u}_m^\beta \left(\beta \left(\frac{\beta}{4} + 1 \right) |D\overline{u}_m|^2 + |D\overline{u}|^2 \right) \\
\leq& \overline{u}_m^\beta (\beta + 1) \left(\beta |D\overline{u}_m|^2 + |D\overline{u}|^2 \right).
\end{aligned}
$$

Therefore we have

$$
\int |Dw|^2 \eta^2 \leq C(1+\beta) \left(\int w^2 |D\eta|^2 + \int c_0 w^2 \eta^2 \right)
$$

and so

$$
\begin{aligned}
\int |D(w\eta)|^2 \leq& 2 \int \left(|D\eta|^2 w^2 + |Dw|^2 \eta^2 \right) \\
\leq& C(1+\beta) \left(\int w^2 |D\eta|^2 + \int c_0 w^2 \eta^2 \right).
\end{aligned}
$$

Hölder inequality implies

$$
\int c_0 w^2 \eta^2 \leq \left(\int c_0^q \right)^{\frac{1}{q}} \left(\int (\eta w)^{\frac{2q}{q-1}} \right)^{1-\frac{1}{q}} \leq (\Lambda + 1) \left(\int (\eta w)^{\frac{2q}{q-1}} \right)^{1-\frac{1}{q}}.
$$

What we do in this inequality is to split c_0 and $w^2\eta^2$.If $c = f = 0$, then the operation here would not be needed and the proof can be simpler. By interpolation inequality and Sobolev's inequality with $2^* = \frac{2n}{n-2} > \frac{2q}{q-1} > 2$ if $q > \frac{n}{2}$, we have

$$
\begin{aligned}
\|\eta w\|_{L^{\frac{2q}{q-1}}} \leq& \varepsilon \|\eta w\|_{L^{2^*}} + C(n,q) \varepsilon^{-\frac{n}{2q-n}} \|\eta w\|_{L^2} \\
\leq& \varepsilon \|D(\eta w)\|_{L^2} + C(n,q) \varepsilon^{-\frac{n}{2q-n}} \|\eta w\|_{L^2}
\end{aligned}
$$

for any small $\varepsilon > 0$. Therefore we obtain

$$
\int |D(w\eta)|^2 \leq C \left((1+\beta) \int w^2 |D\eta|^2 + (1+\beta)^{\frac{2q}{2q-n}} \int w^2 \eta^2 \right)
$$

and in particular

$$
\int |D(w\eta)|^2 \leq C(1+\beta)^\alpha \int \left(|D\eta|^2 + \eta^2 \right) w^2
$$

where α is a positive number depending only on n and q. From the Sobolev's inequality, with $\chi = n/(n-2) > 1$ for $n > 2$ and any fixed $\chi > 2$ for $n = 2$, we get

$$
\left(\int |\eta w|^{2\chi} \right)^{\frac{1}{\chi}} \leq C(1+\beta)^\alpha \int \left(|D\eta|^2 + \eta^2 \right) w^2.
$$

Choose the cut-off function η as follows. For any $0 < r < R \le 1$ set $\eta \in C_0^1(B_R)$ with the property

$$\eta \equiv 1 \text{ in } B_r \text{and } |D\eta| \le \frac{2}{R-r}.$$

Then we obtain

$$\left(\int_{B_r} w^{2\chi}\right)^{\frac{1}{\chi}} \le \frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} w^2.$$

Since by definition of $w = \overline{u}_m^\beta \overline{u}$, we have

$$\left(\int_{B_r} \overline{u}^{2\chi}\overline{u}_m^{\beta\chi}\right)^{\frac{1}{\chi}} \le C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} \overline{u}^2\overline{u}_m^\beta.$$

Set $\gamma = \beta + 2 \ge 2$. Then we obtain

$$\left(\int_{B_r} \overline{u}_m^{\gamma\chi}\right)^{\frac{1}{\chi}} \le C\frac{(\gamma-1)^\alpha}{(R-r)^2}\int_{B_R} \overline{u}^\gamma$$

provided the integral in the right-hand side is bounded. By letting $m \to \infty$ we obtain

$$\|\overline{u}\|_{L^{\gamma\chi}(B_r)} \le \left(C\frac{(\gamma-1)^\alpha}{(R-r)^2}\right)^{\frac{1}{\gamma}} \|\overline{u}\|_{L^\gamma(B_R)}$$

provided $\|\overline{u}\|_{L^\gamma(B_R)} < \infty$, where $C = C(n, q, \lambda, \Lambda)$ is a positive constant independent of γ.

Then we do the iteration, taking successively the values $\gamma = 2, 2\chi, 2\chi^2, \cdots$. Define, for all $i = 1, 2, \cdots$,

$$\gamma_i = 2\chi^i \text{ and } r_i = 2 + \frac{1}{2^{i-1}}.$$

For any $i \ge 0$, $\gamma_{i+1} = \chi\gamma_i$, $r_i - r_{i+1} = \frac{1}{2^{i+2}}$, we have

$$\|\overline{u}\|_{L^{\gamma_{i+1}}(B_{r_{i+1}})} \le C\left(n, q, \lambda, \Lambda\right)^{\frac{1}{\gamma_i}} \|\overline{u}\|_{L^{\gamma_i}\left(B_{r_i}\right)},$$

that is,

$$\|\overline{u}\|_{L^{\gamma_{i+1}}(B_{r_{i+1}})} \le C^{\frac{i}{\chi^i}} \|\overline{u}\|_{L^{\gamma_i}(B_{r_i})}.$$

Hence by iteration we obtain

$$\|\overline{u}\|_{L^{\gamma_{i+1}}\left(B_{r_{i+1}}\right)} \le C^{\sum_{j=1}^i \frac{j}{\chi^j}} \|\overline{u}\|_{L^2(B_1)}$$

in particular

$$\|\overline{u}\|_{L^{\gamma_{i+1}}\left(B_{\frac{1}{2}}\right)} \le C^{\sum_{j=1}^i \frac{j}{\chi^j}} \|\overline{u}\|_{L^2(B_1)}$$

Letting $i \to \infty$ we get

$$\sup_{B_{\frac{1}{2}}} \overline{u} \le C\|\overline{u}\|_{L^2(B_1)},$$

hence

$$\sup_{B_{\frac{1}{2}}} u^+ \le C\left(\|u^+\|_{L^2(B_1)} + k\right).$$

Since $k = \|f\|_{L^q}$, we finish the proof for $p = 2$.

$\square$