ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
LECTURE 12

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Lemma 1. *Let X, Y be varieties and Y be affine, let $\varphi : X \to Y$ is a morphism. Then $\varphi(X)$ is dense in Y if and only if φ^* is injective.*

Proof. If φ is not dense, then there exists a closed subset $W \subsetneqq Y$ and $\varphi(X) \subset W$. We can write $W = Z(f_1, \dots, f_r)$ for $f_1, \dots, f_r \in A(X) \subset K(X)$. By possibly taking a bigger W we can write $W = Z(f)$ for some none zero element $f \in A(X)$. Now we find $\varphi^* f = f \circ \varphi = 0$, so φ^* is not injective. Conversely, if some $f \neq 0 \in K(X)$ satisfies $\varphi^* f = 0$, then $\varphi(X) \subset Z(f) \subsetneqq Y$ is not dense. □

Theorem 1. *Let X, Y be varieties, there is a bijection*

$$\left\{ \begin{array}{c} \textit{dominant rational maps} \\ \varphi : X \to Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} k-\textit{algebra homomorphisms} \\ \varphi^* : K(Y) \to K(X) \end{array} \right\}.$$

In particular, X and Y are birational if and only if $K(Y) \simeq K(X)$.

Proof. We only need to construct the inverse map to $\varphi \to \varphi^*$. Let $\phi : K(Y) \to K(X)$ be a k-algebra homomorphism, we want to construct a rational map $\varphi : X \to Y$ such that $\varphi^* = \phi$. Replacing Y by an open affine subset, we can now assume $Y \subset \mathbb{A}^n$ is closed. Let $y_1, \dots, y_n \in A(Y)$ be coordinate functions, then $\phi(y_1), \dots, \phi(y_n) \in K(X)$. We can find a nonempty open subset $U \subset X$ such that $\phi(y_i) \in \mathcal{O}_X(U)$ for all $i = 1, \dots, n$. Then the map $x \to (\phi(y_1)(x), \dots, \phi(y_n)(x))$ is a morphism from U to Y. In fact, we restrict ϕ to $A(Y)$, then ϕ defines an injective homomorphism from $A(Y)$ to $\mathcal{O}_X(U)$. Then by theorem 2, we get a morphism $\varphi : U \to Y$ and by lemma its image is dense in Y. Thus we find $\varphi : X \dashrightarrow Y$ which is dominant and $\varphi^* = \phi$. □

Corollary 1. *Let X, Y be varieties, the following statements are equivalent*

(1) *X, Y are birational;*
(2) *X, Y contain open subsets isomorphic to each other;*
(3) *$K(X) \simeq K(Y)$ as k-algebras.*

Proof. (1) $\Rightarrow$ (2): Let $\varphi : X \dashrightarrow Y$ be a birational map with inverse $\psi : Y \dashrightarrow X$. We can check that $\psi \circ \varphi$ is the identity on $U = \mathrm{dom}\varphi \cap \varphi^{-1}(\mathrm{dom}\psi)$ and $\varphi \circ \psi$ is the identity on $V = \mathrm{dom}\psi \cap \psi^{-1}(\mathrm{dom}\varphi)$. Thus U is isomorphic to V by restrict φ on U.

(2) $\Rightarrow$ (3): $K(X) \simeq K(U)$, $K(Y) \simeq K(V)$, and we know $K(U) \simeq K(V)$, thus $K(X) \simeq K(Y)$.

(3) $\Rightarrow$ (1): Just the conclusion of theorem 1. □

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1. Conclusions We Need From Previous Lectures

Theorem 2. *Let X, Y be varieties, assume $Y \subset \mathbb{A}^m$ be a closed affine variety. Then there is a bijection between morphisms $X \to Y$ and k-algebra homomorphisms $A(Y) \to \mathcal{O}_X(X)$:*

$$\begin{array}{ccc} \{\textit{morphisms } X \to Y\} & \xrightarrow{\textit{bijection}} & \{\textit{homomorphisms } A(Y) \to \mathcal{O}_X(X)\} \\ \varphi & \longrightarrow & \varphi^* \end{array}$$

Definition 1. Take coordinates x_1, x_2 on $\mathbb{A}^2$ and coordinates y_0, y_1 on $\mathbb{P}^1$, the blowup $\hat{\mathbb{A}^2}$ of $\mathbb{A}^2$ at 0 is $\hat{\mathbb{A}^2} = Z(x_1y_1 - x_2y_0) \subset \mathbb{A}^2 \times \mathbb{P}^1$.

Remark. $\hat{\mathbb{A}^2}$ is closed in $\mathbb{A}^2 \times \mathbb{P}^1$. Closed subsets of $\mathbb{A}^n \times \mathbb{P}^m$ are zero sets $Z(F_1, \ldots, F_r)$ where $F_i \in k[x_1, \ldots, x_n, y_0, \ldots, y_m]$ are homogeneous in y_i.

Definition 2. Let $\Pi = p_1 : \hat{\mathbb{A}^2} \to \mathbb{A}^2$, $E := \Pi^{-1}(0)$ is called the exceptional divisor.

We can see that Π is a birational morphism because $\Pi|_{\hat{\mathbb{A}^2}\setminus E} : \mathbb{A}^2\setminus E \to \mathbb{A}^2\setminus\{0\}$ is an isomorphism. In fact $E = \{0\} \times \mathbb{P}^1$.

Remark. (1) Let $((x_1, x_2), [y_0, y_1]) \in \hat{\mathbb{A}^2}\setminus E$, assume $x_1 \neq 0$, then we get $x_2 = \frac{y_1}{y_0}x_1$. It is equivalent to $(x_1, x_2) \in [y_0, y_1]$. So $\hat{\mathbb{A}^2}$ is the graph of the canonical morphism

$$\begin{array}{ccc} \mathbb{A}^2\setminus\{0\} & \to & \mathbb{P}^1 \\ (x_1, x_2) & \to & [x_1, x_2]. \end{array}$$

(2) $U_{y_0} := \hat{\mathbb{A}^2}\setminus Z(y_0) = \{((x_1, x_2), [1, u]) \in \mathbb{A}^2 \times \mathbb{P}^1 | x_2 = ux_1\}$ is isomorphic to $\mathbb{A}^2$ by the morphism $((x_1, x_2), [1, u]) \to (x_1, u)$ with inverse $(x_1, u) \to ((x_1, ux_1), [1, u])$. In the same way $U_{y_1} := \hat{\mathbb{A}^2}\setminus Z(y_1) = \{((x_1, x_2), [t, 1]) \in \mathbb{A}^2 \times \mathbb{P}^1 | x_1 = tx_2\}$ is isomorphic to $\mathbb{A}^2$. Then we get $\hat{\mathbb{A}^2} = U_{y_0} \cup U_{y_1}$, the union of irreducible open cover, so $\hat{\mathbb{A}^2}$ itself is also irreducible.

In U_{y_0} the exceptional divisor E is $Z(x_1)$, and in U_{y_1} the exceptional divisor E is $Z(x_2)$.

Example 1. What does the blowup do to a curve $C \subset \mathbb{A}^2$ through 0? Let $C \subset \mathbb{A}^2$ be a curve, the strict transform of C is

$$\hat{C} = \text{ closure of } \Pi^{-1}(C\setminus\{0\})$$

in $\hat{\mathbb{A}^2}$. For example, let $C = Z(F)$, where $F = x_2^2 - x_1^2(x_1 + 1)$. In the chart $y_1 \neq 0$, i.e. U_{y_0}, we have

$$\Pi^{-1}(C) = Z(x_1^2(u^2 - (x_1 + 1))) = Z(x_1^2) \cup Z(u^2 - (x + 1))$$

(Note that we talk about this in U_{y_0}, so the exact equation is $\Pi^{-1}(C) = Z(x_1^2(u^2 - (x_1 + 1))) \cap U_{y_0}$. But we ignore it for simplicity). Since $Z(x_1^2)$ is exactly the exceptional divisor E, we get $\hat{C} = Z(u^2 - (x + 1))$, and it is isomorphic to $\mathbb{A}^1$ by mapping $((x_1, x_2), [1, u]) \to u$. The inverse of the map is $u \to ((u^2-1, u^3-u), [1, u])$. We can see C and $\hat{C}$ in the real axes condition Now we can find that there is no singular point after blowing up.

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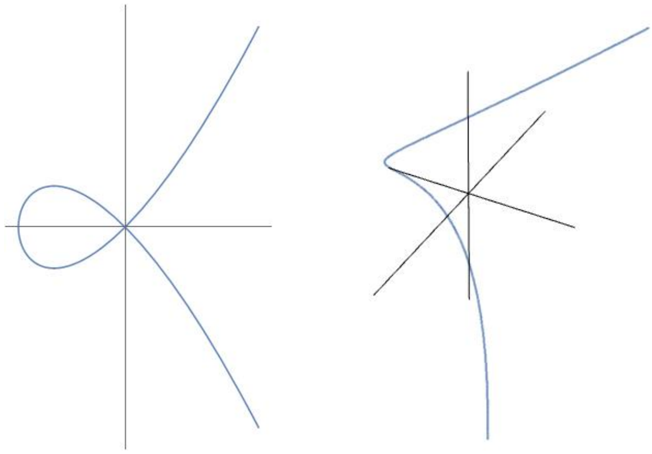

FIGURE 1. $\hat{C}$ in real space