

# Modular Forms

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## Abstract

This is a learning note about Zhang Yitang's lectures on modular forms.

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## 1 Modular Group, Congruence Subgroup and Modular Forms

**Definition 1.1.** The *modular group* is the group of 2-by-2 matrices with integer entries and determinant 1:

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

The *principal congruence subgroup of level  $N$*  is

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}.$$

$\Gamma$  is a congruence subgroup if

$$\Gamma(N) \subset \Gamma \subset \mathrm{SL}_2(\mathbb{Z}) \text{ for some } N.$$

**Example 1.1.**  $\forall N \in \mathbb{N}$ ,

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N} \right\}$$

are congruence subgroups. Their relations are

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_2(N) \subset \mathrm{SL}_2(\mathbb{Z}).$$

**Definition 1.2.**  $\mathcal{H}$  is the *upper half plane* defined by

$$\mathcal{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}.$$

*Action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathcal{H}$*  is defined by

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$$

for arbitrary  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and  $\tau \in \mathcal{H}$ .

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d} = \frac{(a\tau + b)(c\bar{\tau} + d)}{|c\tau + d|^2} = \frac{ac|\tau|^2 + bd + ad\tau + bc\bar{\tau}}{|c\tau + d|^2},$$

then

$$\mathrm{Im} \gamma(\tau) = \frac{\mathrm{Im}(ad\tau + bc\bar{\tau})}{|c\tau + d|^2} = \frac{(ad - bc) \mathrm{Im} \tau}{|c\tau + d|^2} > 0.$$

Hence  $\gamma(\tau) \in \mathcal{H}$  if  $\tau \in \mathcal{H}$ .

Let  $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $\gamma' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ , then

$$\gamma\gamma' = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}.$$

It's easy to verify

$$\gamma(\gamma'(\tau)) = \gamma\gamma'(\tau).$$

Now we consider actions of  $\mathrm{SL}_2(\mathbb{Z})$  on functions  $f : \mathcal{H} \rightarrow \mathbb{C}$ . Write

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad \tau \in \mathcal{H}.$$

$$j(\gamma, \tau) := c\tau + d.$$

For  $k \in \mathbb{Z}$ , define  $[\gamma]_k :=$  the weight- $k$  operator acting on functions  $\mathcal{H} \rightarrow \mathbb{C}$  such that

$$(f[\gamma_k])(\tau) = j(\gamma, \tau)^{-k} f(\gamma(\tau)) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

**Lemma 1.3.**  $\forall \gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z}), \forall \tau \in \mathcal{H}$ , we have

- a.  $j(\gamma\gamma', \tau) = j(\gamma, \gamma'(\tau))j(\gamma', \tau);$
- b.  $[\gamma\gamma']_k = [\gamma]_k[\gamma']_k;$
- c.  $\frac{d\gamma(\tau)}{d\tau} = \frac{1}{j(\gamma, \tau)^2}.$

*Proof.*

a.

$$\begin{aligned} \gamma \begin{bmatrix} \tau \\ 1 \end{bmatrix} &= \begin{bmatrix} \gamma(\tau) \\ 1 \end{bmatrix} j(\gamma, \tau) \\ \gamma\gamma' \begin{bmatrix} \tau \\ 1 \end{bmatrix} &= \begin{bmatrix} \gamma\gamma'(\tau) \\ 1 \end{bmatrix} j(\gamma\gamma', \tau) \end{aligned}$$

Also

$$\begin{aligned} \gamma\gamma' \begin{bmatrix} \tau \\ 1 \end{bmatrix} &= \gamma \begin{bmatrix} \gamma'(\tau) \\ 1 \end{bmatrix} j(\gamma', \tau) \\ &= \begin{bmatrix} \gamma\gamma'(\tau) \\ 1 \end{bmatrix} j(\gamma, \gamma'(\tau))j(\gamma', \tau). \\ \Rightarrow j(\gamma\gamma', \tau) &= j(\gamma, \gamma'(\tau))j(\gamma', \tau). \end{aligned}$$

b.

$$\begin{aligned} &(f[\gamma\gamma']_k)(\tau) \\ &= j(\gamma\gamma', \tau)^{-k} f(\gamma\gamma'(\tau)) \\ &= j(\gamma, \gamma'(\tau))^{-k} j(\gamma', \tau)^{-k} f(\gamma\gamma'(\tau)) \\ &= j(\gamma', \tau) f([\gamma]_k)(\gamma'(\tau)) \\ &= (f[\gamma]_k[\gamma']_k)(\tau). \end{aligned}$$

c.

$$\begin{aligned}\frac{d\gamma(\tau)}{d\tau} &= \frac{a(c\tau + d) - (a\tau + b)c}{(c\tau + d)^2} \\ &= \frac{1}{(c\tau + d)^2} \\ &= \frac{1}{j(\gamma, \tau)^2}.\end{aligned}$$

□

**Definition 1.4.** Let  $\Gamma =$  congruence subgroup and  $k \in \mathbb{Z}$   $f : \mathcal{H} \rightarrow \mathbb{C}$  is a weakly modular form function of weight  $k$  with respect to  $\Gamma$  if  $f$  is meromorphic on  $\mathcal{H}$  and

$$f[\gamma]_k = f,$$

i.e.,

$$(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = f.$$

Suppose  $f$  is a weakly modular function of weight  $k$  with respect to  $\Gamma$ .

$$\Gamma \supset \Gamma(N) \Rightarrow \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix} \in \Gamma$$

$$f \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix}_k = f(\tau + N) = f(\tau)$$

$$\Rightarrow \exists \text{ minimal } h \in \mathbb{N} \text{ such that } f(\tau + h) = f(\tau)$$

$$\Rightarrow f(\tau) = g(e^{2\pi i\tau/h}) \text{ for some } g.$$

$$\tau \in \mathcal{H} \Leftrightarrow |e^{2\pi i\tau/h}| < 1$$

$$\text{Im}(\tau) \rightarrow \infty \Rightarrow e^{2\pi i\tau/h} \rightarrow 0.$$

$g(z)$  is meromorphic on  $0 < |z| < 1$ .

$$z \rightarrow 0 : g(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

$$z = e^{2\pi i\tau/h} \Rightarrow \text{Im}(\tau) \rightarrow \infty$$

$$f(\tau) = g(e^{2\pi i\tau/h}) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi in\tau/h}. \quad (1)$$

We say  $f(\tau)$  is holomorphic at  $\infty$  if  $a_n = 0$  for all  $n < 0$  in (1). In this case we write  $f(\infty) = a_0$ .

$\forall \sigma \in \text{SL}_2(\mathbb{Z})$ ,  $f[\sigma]_k$  is a weakly modular function of weight  $k$  with respect to  $\sigma^{-1}\Gamma\sigma$ . Indeed, let  $\gamma \in \Gamma$ ,

$$\begin{aligned} f[\gamma]_k(\tau) &= f(\tau) \\ j(\gamma, \tau)^{-k} f(\gamma(\tau)) &= f(\tau) \\ j(\gamma, \sigma(\tau))^{-k} f(\gamma\sigma(\tau)) &= f(\sigma(\tau)) \end{aligned}$$

$$\begin{aligned} ((f[\sigma]_k)[\sigma^{-1}\gamma\sigma]_k)(\tau) &= j(\sigma^{-1}\gamma\sigma, \tau)^{-k} (f[\sigma]_k)(\sigma^{-1}\gamma\sigma(\tau)) \\ &= j(\sigma^{-1}\gamma\sigma, \tau)^{-k} j(\sigma, \sigma^{-1}\gamma\sigma(\tau))^{-k} f(\gamma\sigma(\tau)) \\ &= j(\sigma^{-1}\gamma\sigma, \tau)^{-k} j(\sigma, \sigma^{-1}\gamma\sigma(\tau))^{-k} j(\gamma, \sigma(\tau))^k f(\sigma(\tau)) \\ &= j(\sigma, \tau)^{-k} f(\sigma(\tau)) \\ &= f[\sigma]_k. \end{aligned}$$

**Definition 1.5.**

- a.  $f : \mathcal{H} \rightarrow \mathbb{C}$  is a modular form of weight  $k$  with respect to  $\Gamma$  if  $f$  is weakly modular function and  $f\sigma_k$  is holomorphic at  $\infty \forall \sigma \in \text{SL}_2(\mathbb{Z})$ .
- b.  $f$  is a cusp form if  $f$  is a modular form and  $f\sigma_k$  vanishes at  $\infty \forall \sigma \in \text{SL}_2(\mathbb{Z})$ .

**Proposition 1.6.** Assume  $k = \text{odd}$ , Then any modular form  $f \equiv 0$ .

*Proof.*  $\gamma = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \Gamma$ , for arbitrary  $\tau \in \mathcal{H}$ ,

$$j(\gamma, \tau)^{-k} f(\gamma(\tau)) = (-1)^{-k} f\left(\frac{-\tau + 0}{0 + (-1)}\right) = (-1)^{-k} f(\tau) = -f(\tau).$$

□

## 2 Case $\Gamma = \text{SL}_2(\mathbb{Z})$ , Eisenstein Series

In this section assume  $\Gamma = \text{SL}_2(\mathbb{Z})$ .  $\Gamma$  is generated by  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

$$\left( f \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_k \right) (\tau) = f(\tau + 1)$$

since  $j\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \tau\right) = 1$ .

$$\left( f \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_k \right) = \frac{1}{\tau^k} f\left(-\frac{1}{\tau}\right)$$

since  $j\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tau\right) = \tau$ . Let  $f$  be a modular form of weight  $k$ , then

$$\begin{aligned} f(\tau + 1) &= f(\tau) \\ f\left(-\frac{1}{\tau}\right) &= \tau^k f(\tau). \end{aligned}$$

(to find a nontrivial modular form,  $k$  must be even.)

**Definition 2.1 (Eisenstein Series).** Assume  $k = \text{even} > 2$ . Define

$$G_k(\tau) = \sum_{(c,d) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(c\tau + d)^k}.$$

$G_k(\tau)$  is called Eisenstein series.

$\tau \in \mathcal{H} \Rightarrow G_k(\tau)$  is absolutely convergent and holomorphic.

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) &\Rightarrow G_k\left(\frac{a\tau + b}{c\tau + d}\right) \\ &= \sum_{(c',d') \neq (0,0)} \frac{1}{\left(c'\frac{a\tau+b}{c\tau+d} + d'\right)^k} \\ &= (c\tau + d)^k \sum_{(c',d') \neq (0,0)} \frac{1}{((c'a + cd')\tau + (c'b + dd'))^k}. \end{aligned}$$

As  $(c', d')$  walks through all  $\neq (0, 0)$ , so does  $(c'a + cd', c'b + dd')$ . Hence

$$G_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k G_k(\tau).$$

When  $\text{Im}(\tau) \rightarrow \infty$ ,

$$\frac{1}{(c\tau + d)^k} \rightarrow \begin{cases} 0 & \text{if } c \neq 0, \\ d^{-k} & \text{if } c = 0. \end{cases}$$

$$\begin{aligned} \Rightarrow G_k(\infty) &= \lim_{\text{Im}(\tau) \rightarrow \infty} G_k(\tau) \\ &= \sum_{d=-\infty, d \neq 0}^{\infty} \frac{1}{d^k} = 2\zeta(k). \end{aligned}$$

Assume  $f$  is a modular form of weight  $k$ ,  $k = \text{even} > 2$ . Define

$$\begin{aligned} D &= \{q \in \mathbb{C} : |q| < 1\} \\ D' &= D - \{0\}. \end{aligned}$$

Construct the mapping

$$\begin{aligned}\mathcal{H} &\rightarrow D' \\ \tau &\mapsto e^{2\pi i\tau} = q\end{aligned}$$

and

$$\begin{aligned}g : D' &\rightarrow \mathbb{C} \\ q &\mapsto f(\log(q)/(2\pi i)).\end{aligned}$$

$g$  is well defined even though the logarithm is only determined up to  $2\pi i\mathbb{Z}$ . Then

$$f(\tau) = g(e^{2\pi i\tau}).$$

At  $\tau = \infty$ ,  $G_k(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau} = 2\zeta(k) \Rightarrow a_0 = 2\zeta(k)$ .  $a_n = ?$

**Proposition 2.2.** Let  $\tau \in \mathcal{H}$ , we have

$$\frac{1}{\tau} + \sum_{d=1}^{\infty} \left( \frac{1}{\tau - d} + \frac{1}{\tau + d} \right) = \pi \cot \pi \tau.$$

*Proof.*

$$\sin \pi \tau = \pi \tau \prod_{n=1}^{\infty} \left( 1 - \frac{\tau^2}{n^2} \right).$$

$$\log \sin \pi \tau = \log \pi + \log \tau + \sum_{n=1}^{\infty} \log \left( 1 - \frac{\tau^2}{n^2} \right).$$

Taking the derivative, we obtain

$$\pi \cot \pi \tau = \frac{1}{\tau} + \sum_{n=1}^{\infty} \frac{-2\tau/n^2}{1 - \tau^2/n^2} = \frac{1}{\tau} + \sum_{n=1}^{\infty} \left( \frac{1}{\tau - n} + \frac{1}{\tau + n} \right).$$

□

$$\begin{aligned}\frac{1}{\tau} + \sum_{d=1}^{\infty} \left( \frac{1}{\tau - d} + \frac{1}{\tau + d} \right) &= \pi i \frac{e^{i\pi\tau} + e^{-i\pi\tau}}{e^{i\pi\tau} - e^{-i\pi\tau}} \\ &= -\pi i \frac{1 + e^{2\pi i\tau}}{1 - e^{2\pi i\tau}} \\ &= -\pi i - 2\pi i \sum_{m=0}^{\infty} e^{2\pi i m \tau}.\end{aligned}$$

Differentiating  $(k-1)$  times we get

$$\begin{aligned}
(-1)^{k-1}(k-1)! \sum_{d=-\infty}^{\infty} \frac{1}{(\tau+d)^k} &= -2\pi i \sum_{m=0}^{\infty} (2\pi i m)^{k-1} e^{2\pi i m \tau}. \\
\Rightarrow \sum_{d=-\infty}^{\infty} \frac{1}{(\tau+d)^k} - \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m \tau} &= 0. \\
\Rightarrow G_k(\tau) &= \sum_{c \neq 0} \sum_{d=-\infty}^{\infty} \frac{1}{(c\tau+d)^k} + \sum_{d \neq 0} \frac{1}{d^k} \\
&= 2 \sum_{c=1}^{\infty} \sum_{d=-\infty}^{\infty} \frac{1}{(c\tau+d)^k} + 2\zeta(k) \\
&= 2 \sum_{c=1}^{\infty} \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m c \tau} + 2\zeta(k) \\
&= 2 \sum_{n=1}^{\infty} \frac{(2\pi i)^k}{(k-1)!} \left( \sum_{m|n} m^{k-1} \right) e^{2\pi i n \tau} + 2\zeta(k).
\end{aligned}$$

Then we get the following conclusion:

**Proposition 2.3.** Let  $k = \text{even} > 2$ ,  $G_k(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$ , then

$$\begin{aligned}
a_0 &= 2\zeta(k) \\
a_n &= \frac{2(2\pi i)^k}{(k-1)!} \sigma_{k-1}(n)
\end{aligned}$$

where  $n > 0$  and  $\sigma_{k-1}(n) = \sum_{m|n} m^{k-1}$ .

A group  $G$  acting on a set  $S$  gives rise to an equivalent relation  $\sim$  on  $S$ :

$$s_1 \sim s_2 \Leftrightarrow \exists g \in G \text{ such that } s_2 = gs_1.$$

The quotient space  $S/G$  = the set of the equivalent class.

**Question.**  $\text{SL}_2(\mathbb{Z})$  acts on  $\mathcal{H}$ , what is the quotient space?

**Theorem 2.4.** Let  $D = \{\tau \in \mathcal{H} : |\text{Re } \tau| \leq \frac{1}{2}, |\tau| \geq 1\}$ . Then  $D$  is a fundamental domain for  $\text{SL}_2(\mathbb{Z})$  in the sense such that

a.  $\forall \tau \in \mathcal{H}, \exists \gamma \in \text{SL}_2(\mathbb{Z})$  such that  $\gamma(\tau) \in D$ ;



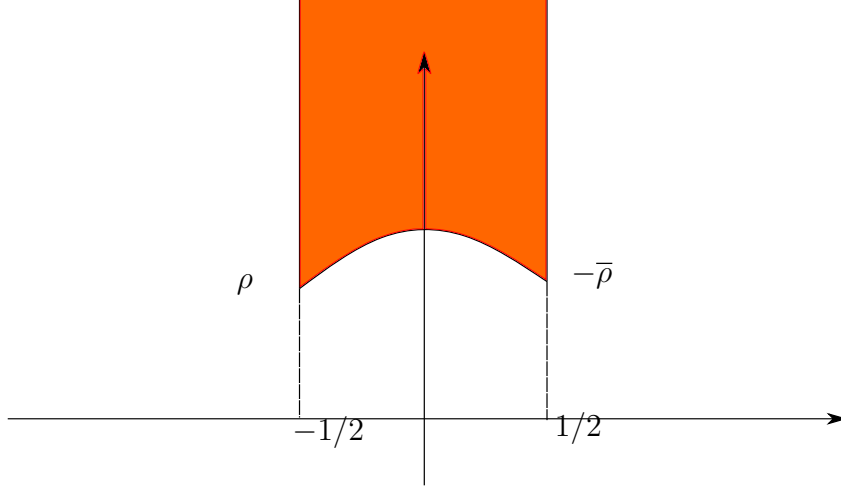


Figure 1: Domain D

- b. If  $\tau_1, \tau_2 \in D$ ,  $\tau_1 \neq \tau_2$  and  $\tau_2 = \gamma(\tau_1)$  for some  $\gamma \in \text{SL}_2(\mathbb{Z})$ , then either  $\text{Re}(\tau_1) = \pm \frac{1}{2}$ ,  $\tau_2 = \tau_1 \mp 1$  or  $|\tau_1| = 1, \tau_2 = -\frac{1}{\tau_1}$ .

*Proof.* Write  $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$ ,  $\gamma_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$ ,  $\gamma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$ . Then

$$\gamma_1^n(\tau) = \tau + n$$

$$\gamma_2(\tau) = -\frac{1}{\tau}.$$

- a.  $\forall \tau \in \mathcal{H}$ ,  $\text{Im } \gamma(\tau) = \frac{\text{Im}(\tau)}{|c\tau + d|^2}$ . There exists only finite  $(c, d)$  such that  $|c\tau + d| <$  a given volume.  $\Rightarrow (\exists \gamma \text{ such that } |c\tau + d| = \min \Rightarrow \text{Im } \gamma(\tau) = \max \text{ and } \exists n \in \mathbb{Z} \text{ such that } |\text{Re } \gamma_1^n(\gamma(\tau))| \leq \frac{1}{2})$ .

Write  $\tau_1 = \gamma_1^n(\gamma(\tau)), \tau_2 = \gamma_2(\tau_1)$ .

$$\begin{aligned} \text{Im } \tau_2 &= \frac{\text{Im } \tau_1}{|\tau_1|^2} \text{ and } \text{Im } \tau_2 \leq \text{Im } \gamma(\tau) = \text{Im } \tau_1 \\ &\Rightarrow |\tau_1| \geq 1 \Rightarrow \tau_1 \in D. \end{aligned}$$

- b. Suppose  $\tau_1, \tau_2 \in D, \tau_1 \neq \tau_2, \gamma(\tau_1) = \tau_2$  for some  $\gamma \in \text{SL}_2(\mathbb{Z})$ . We assume

$\text{Im } \tau_1 \leq \text{Im } \tau_2$ , then

$$\begin{aligned} \text{Im } \tau_1 &\leq \frac{\text{Im } \tau_1}{|c\tau_1 + d|^2} \\ &\Rightarrow |c\tau_1 + d| \leq 1. \end{aligned}$$

Since  $\text{Im } \tau_1 \geq \frac{\sqrt{3}}{2}$ , then  $c = 0, \pm 1$ .

Case 1:  $c = 0, a = d = \pm 1$

$$\begin{aligned} &\Rightarrow \tau_2 = \tau_1 \pm b \\ &\Rightarrow \begin{cases} \text{Re } \tau_1 = \pm \frac{1}{2} \\ \tau_2 = \tau_1 \mp 1 \end{cases}. \end{aligned}$$

Case 2:  $c = 1, |\tau_1 + d| \leq 1$

$$\Rightarrow \begin{cases} d = 0 \Rightarrow |\tau_1| = 1, \tau_2 = -\frac{1}{\tau_1} \\ \text{or } \tau_1 = \rho, d = 1 \\ \text{or } \tau_1 = -\bar{\rho}, d = -1 \end{cases}.$$

Case 3:  $c = -1$ , similar to Case 2.

Here  $\rho := e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ . □

The quotient space  $\mathcal{H}/\text{SL}_2(\mathbb{Z})$  is obtained by identifying the left side and right side of  $D$  and identifying the left and right parts of the bottom circle of  $D$ .

Assume  $f$  is holomorphic on  $\mathcal{H}$  and  $\infty(\text{Im } \tau \rightarrow \infty)$ .

At  $p \in \mathcal{H}$ ,  $m = \text{order of } f \text{ at } p$  means

$$\lim_{\tau \rightarrow p} \frac{f(\tau)}{(\tau - p)^m}$$

exists and not equals 0. We use  $v_p(f) = m$  to represent this meaning.

At  $\infty$ , if  $a_m \neq 0$  in  $f(\tau) = \sum_{n=m}^{\infty} a_n e^{2\pi i n \tau}$ , write  $v_{\infty}(f) = m$ .

If  $p_1, p_2$  are equivalent ( $p_2 = \gamma p_1$  for some  $\gamma \in \text{SL}_2(\mathbb{Z})$ ), then  $v_{p_1}(f) = v_{p_2}(f)$ .

**Theorem 2.5.** Suppose  $f$  is a (non-zero) modular form of weight  $k$  ( $k$  even),  $\rho = e^{\frac{2\pi i}{3}}$ . Then

$$v_{\infty}(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_{\rho}(f) + \sum_{p \in \mathcal{H}/\text{SL}_2(\mathbb{Z})}^* v_p(f) = \frac{k}{12} \quad (2)$$

where  $\sum^*$  means the sum is over  $p \in \mathcal{H}/\text{SL}_2(\mathbb{Z})$  and  $p \not\sim i, \rho$ , i.e.,  $\sum_{p \in \mathcal{H}/\text{SL}_2(\mathbb{Z})}^* := \sum_{p \in D \setminus \{i, \rho\}}$ .

$k = 2$ : right side  $= \frac{1}{6}$ , left side either  $= 0$  or  $\geq \frac{1}{3}$ . Hence modular forms of weight 2 don't exist.

$k = 4$ : right side  $= \frac{1}{3}$ ,  $f = G_4, v_\rho f = 1, v_\infty(f) = v_i(f) = 0, \sum_{p \in \mathcal{H}}^* \mathcal{H} / \text{SL}_2(\mathbb{Z}) v_p(f) = 0$ .

$k = 12$ :  $G_4^3, G_6^2, \exists$  linear combination

$$\Delta = c_1 G_4^3 + c_2 G_6^2, \Delta : \text{weight} = 12$$

such that  $\Delta(\infty) = 0$ . Since

$$v_\infty(\Delta) + \frac{1}{2}v_i(\Delta) + \frac{1}{3}v_\rho(\Delta) + \sum^* = \frac{12}{12} = 1$$

and  $v_\infty(\Delta) = 1$ , we have  $\Delta(\tau) \neq 0 \forall \tau \in \mathcal{H}$ .

**Definition 2.6.** Define

$\mathcal{M}_k :=$  space of modular forms of weight  $k$

$\mathcal{S}_k :=$  space of cusp ( $f(\infty) = 0$ ) forms of weight  $k$ .

If  $\dim \mathcal{M}_k > 0$ , then

$$\dim \mathcal{S}_k = \dim \mathcal{M}_k - 1.$$

In fact,  $\mathcal{M}_k = \mathcal{S}_k \oplus \mathbb{C}$

If  $f \in \mathcal{M}_k$ , then  $\Delta f \in \mathcal{S}_{k+12}$ , then we establish an isomorphism

$$\begin{aligned} \mathcal{M}_k &\rightarrow \mathcal{S}_{k+12} \\ f &\mapsto \Delta f. \end{aligned}$$

If we know  $\dim \mathcal{M}_k$  for all  $k \leq 12$ , then we know all the conditions.

Before the proof of Theorem 2.5, we introduce the following lemma in complex analysis:

**Lemma 2.7.**

$$\frac{1}{2\pi i} \int_C \frac{d\tau}{\tau - \rho} = -\frac{\theta}{2\pi} \quad (3)$$

where  $C$  is given by

*Proof of Theorem 2.5.* Assume  $f \neq 0$  on the boundary of  $D$ , except at  $\tau = \rho, i - \bar{\rho}$ , consider the contour  $L$  in Figure 3. Residue Theorem  $\Rightarrow$

$$\frac{1}{2\pi i} \int_L \frac{df}{f} = \sum_p^* v_p(f). \quad (4)$$

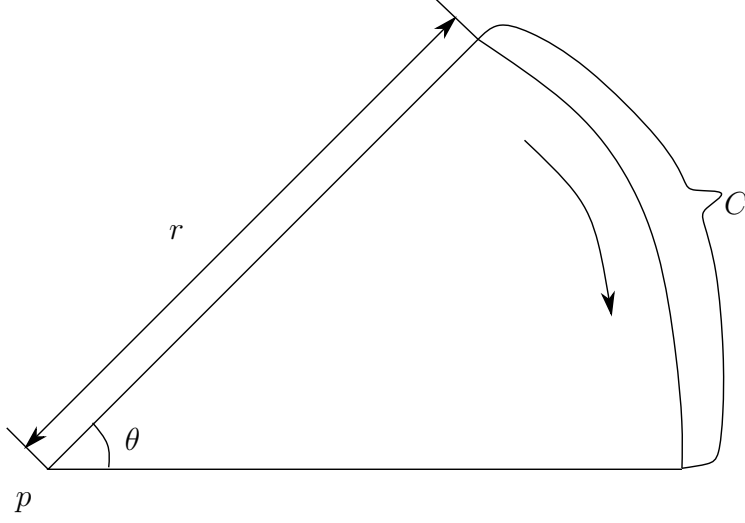


Figure 2: Argument principle

Recall  $f(\tau) = g(e^{2\pi i\tau})$  and let  $z = e^{2\pi i\tau}$ . Set  $\text{Im } A' = T$ . Then  $z = -e^{-2\pi T} \rightarrow -e^{-2\pi T}$  along a circle  $\omega$ :

$$\frac{1}{2\pi i} \int_{A'}^A \frac{df}{f} = \frac{1}{2\pi i} \int_{\omega} \frac{dg}{g} = -v_{\infty}(f). \quad (5)$$

Since

$$\frac{1}{2\pi i} \int_A^B \frac{df}{f} = \frac{1}{2\pi i} \int_A^B \frac{df(\tau+1)}{f(\tau+1)}$$

and  $\tau+1 : A' \rightarrow E'$ , we get

$$\frac{1}{2\pi i} \left( \int_A^B + \int_{E'}^{A'} \right) \frac{df(\tau)}{f(\tau)} = 0. \quad (6)$$

$\tau : B \rightarrow B'$ :

$$\frac{df}{f} \sim \frac{v_{\rho}(f)}{\tau - \rho},$$

hence

$$\frac{1}{2\pi i} \int_B^{B'} \frac{df}{f} \rightarrow -\frac{1}{6} v_{\rho}(f). \quad (7)$$

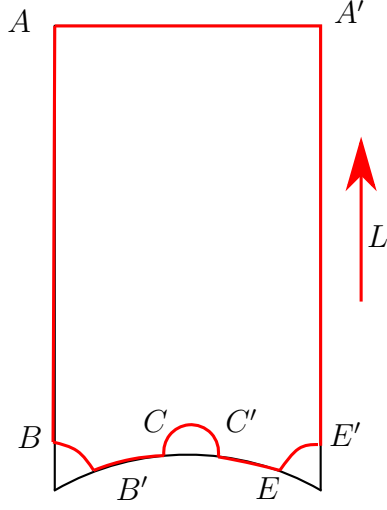


Figure 3: Countour  $L$

$$\tau : B' \rightarrow C \Rightarrow -\frac{1}{\tau} : E \rightarrow C':$$

$$\begin{aligned} \frac{df(\tau)}{f(\tau)} &= \frac{d\tau^{-k}}{\tau^{-k}} + \frac{df(-\frac{1}{\tau})}{f(-\frac{1}{\tau})} \\ &= -k \frac{d\tau}{\tau} + \frac{df(-\frac{1}{\tau})}{f(-\frac{1}{\tau})}. \end{aligned}$$

$$\frac{1}{2\pi i} \int_{B'}^C \frac{df}{f} = \frac{1}{2\pi i} \int_{B'}^C \frac{k}{\tau} d\tau + \frac{1}{2\pi i} \int_E^{C'} \frac{df(\tau)}{f(\tau)} = \frac{k}{12} - \frac{1}{2\pi i} \int_{C'}^E \frac{df(\tau)}{f(\tau)}. \quad (8)$$

$$\tau : C \rightarrow C':$$

$$\frac{df}{f} \sim \frac{v_i(f)}{\tau - i}$$

$$\frac{1}{2\pi i} \int_C^{C'} \frac{df}{f} \rightarrow -\frac{1}{2} v_i(f). \quad (9)$$

$$\tau : E \rightarrow E': \text{ Similarly}$$

$$\frac{1}{2\pi i} \int_E^{E'} \frac{df}{f} \rightarrow -\frac{1}{6} v_\rho(f). \quad (10)$$

By (5)  $\rightarrow$  (10) we obtain the conclusion.  $\square$

If  $f$  has a zero ( $\neq \rho, -\bar{\rho}, i$ ) on the boundary of  $D$ , modify  $L$  as Figure 4

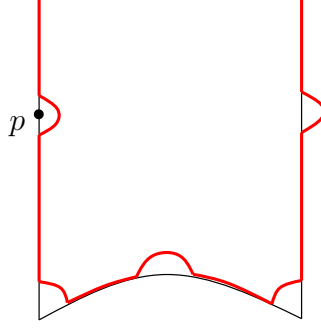


Figure 4: Modified conuntour

As vector space over  $\mathbb{C}$ , let

$\mathcal{M}_k =$  the space of modular form of weight  $k$

$\mathcal{S}_k =$  the space of cusp form of weight  $k$ .

In case  $\dim \mathcal{M}_k > 1$  say  $d = \dim \mathcal{M}_k$ ,  $f_1, f_2, \dots, f_d$  a basis of  $\mathcal{M}_k$ . i.e.,

$$\mathcal{M}_k = \{c_1 f_1 + c_2 f_2 + \dots + c_d f_d : c_1, \dots, c_d \in \mathbb{C}\}.$$

$$\begin{aligned} c_1 f_1 + \dots + c_d f_d \in \mathcal{S}_k &\Leftrightarrow c_1 f_1(\infty) + c_d f_d(\infty) = 0 \\ &\Rightarrow \dim \mathcal{S}_k = \dim \mathcal{M}_k - 1. \end{aligned}$$

In general  $\dim \mathcal{M}_k \leq \dim \mathcal{S}_k + 1$ .

Write

$$\begin{aligned} g_2 &= 60G_4 \\ g_3 &= 140G_6. \end{aligned}$$

Define  $\Delta = g_2^3 - 27g_3^2$ , notice that  $g_2^3$  and  $g_3^2$  are both modular forms of weight 12. Then  $\Delta \in \mathcal{M}_k$ .

$$\Delta(\infty) = 0 \Rightarrow \Delta \in \mathcal{S}_k.$$

$$\begin{aligned}
(2) & \xrightarrow{f=\Delta} v_\infty(\Delta) + \frac{1}{2}v_i(\Delta) + \frac{1}{3}v_\rho(\Delta) + \sum^* v_p(\Delta) = 1 \\
\Delta(\infty) = 0 & \implies v_{oo}(\Delta) \geq 1 \\
\text{combine with the above} & \implies v_\infty(\Delta) = 1, \quad v_i(\Delta) = v_\rho(\Delta) = v_p(\Delta) = 0 \\
& \implies \Delta(\tau) \neq 0, \forall \tau \in \mathcal{H}.
\end{aligned}$$

**Theorem 2.8.**

- a. When  $k < 0$  and  $k = 2$ ,  $\mathcal{M}_k = 0$ .
- b. The map:  $\begin{matrix} \mathcal{M}_k \rightarrow \mathcal{S}_{k+12} \\ f \mapsto \Delta f \end{matrix}$  is an isomorphism.
- c. When  $k = 0, 4, 6, 8, 10$ ,  $\dim \mathcal{M}_k = 1$ ,  $\dim S_k = 0$ , Their basis are  $1, G_4, G_6, G_8, G_{10}$  respectively.

*Proof.*

- a. Suppose  $\exists$  non-zero modular  $f \in \mathcal{M}_k$ . Left side of (2)  $\geq 0 \Rightarrow k \geq 0$ .  
For  $k = 2$ ,  $v_\infty(f), v_i(f), v_\rho(f), v_p(f)$  are non-negative integers. If one of them  $\geq 1$ , the left side of (2)  $\geq \frac{1}{3}$ . But the right side  $= \frac{1}{6}$ .
- b. It suffices to prove that  $f \mapsto \Delta f$  is surjective.  $\forall g \in S_{k+12}$ , consider  $h = \frac{g}{\Delta}$ .  

$$\begin{cases} v_\infty(g) \geq 1 \\ v_\infty(\Delta) = 1 \end{cases} \Rightarrow v_\infty(h) \geq 1 \Rightarrow h \text{ is holomorphic at } \infty.$$

$$\begin{cases} g \text{ is holomorphic on } \mathcal{H} \\ \Delta^{-1} \text{ is holomorphic and } \Delta \neq 0 \text{ on } \mathcal{H} \end{cases} \Rightarrow h \text{ is holomorphic on } \mathcal{H}.$$

$$\begin{aligned}
h\left(\frac{a\tau + b}{c\tau + d}\right) &= \frac{g\left(\frac{a\tau + b}{c\tau + d}\right)}{\Delta\left(\frac{a\tau + b}{c\tau + d}\right)} = \frac{(c\tau + d)^{k+12}g(\tau)}{(c\tau + d)^{12}\Delta(\tau)} = (c\tau + d)^k h(\tau) \\
&\Rightarrow h \in \mathcal{M}_k.
\end{aligned}$$

- c. For  $k = 0, 4, 6, 8, 10$ ,  $\exists$  non-zero  $f \in \mathcal{M}_k$

$$\left. \begin{aligned} f &\equiv 1 \text{ for } k = 0 \\ f &= G_k \text{ for } k \geq 4 \end{aligned} \right\} \Rightarrow \dim \mathcal{M}_k \geq 1.$$

By b we have  $S_k \simeq \mathcal{M}_{k-12}$ , but  $k - 12 \leq 0 \Rightarrow \mathcal{M}_{k-12} = 0 \Rightarrow S_k = 0$ .

□

**Corollary 2.9.** For  $k \geq 0$ , we have

$$\dim \mathcal{M}_k = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 2 \pmod{12}, \\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}. \end{cases}$$

*Proof.*  $0 \leq k \leq 10$  can be directly verified.

For  $k = 12$ ,  $\mathcal{S}_{12} \simeq \mathcal{M}_0 \Rightarrow \dim \mathcal{S}_{12} = 1 \Rightarrow \dim \mathcal{M}_{12} = 2$ . By  $\mathcal{S}_{k+12} \simeq \mathcal{M}_k$  we get

$$\dim \mathcal{M}_{k+12} = \dim \mathcal{S}_{k+12} + 1 = \dim \mathcal{M}_k + 1,$$

then use induction. □

**Corollary 2.10.** For  $k \geq 12$ , the set  $\left\{ G_4^\alpha G_6^\beta : 4\alpha + 6\beta = k, \alpha, \beta \geq 0 \right\}$  is a basis of  $\mathcal{M}_k$ .

*Proof.* The elements of  $\left\{ G_4^\alpha G_6^\beta : 4\alpha + 6\beta = k, \alpha, \beta \geq 0 \right\}$  are linearly independent and the number of  $G_4^\alpha G_6^\beta$  is  $\dim \mathcal{M}_k$  by Corollary 2.9. □

### 3 Complex Tori

A *Riemann surface* is an 1-dimensional connected complex manifold.

**Proposition 3.1.** If  $f : S_1 \rightarrow S_2$  is a holomorphic map of compact Riemann surfaces, then either the image of  $f$  is a point, or  $f$  is surjective.

*Proof.* Suppose  $X$  and  $Y$  are compact Riemann surfaces and  $f : X \rightarrow Y$  is holomorphic. Since  $f$  is continuous and  $X$  is compact and connected, so is the image  $f(X)$ , making  $f(X)$  closed. Unless  $f$  is constant  $f$  is open by the Open Mapping Theorem of complex analysis, applicable to Riemann surfaces since it is a local result, making  $f(X)$  open as well. So  $f(X)$  is either a single point or a connected, open, closed subset of the connected set  $Y$ , i.e., all of  $Y$ . □

Assume  $\omega_1, \omega_2 \in \mathbb{C}$ , linearly independent over  $\mathbb{R}$  (for normalization, set  $\text{Im} \frac{\omega_1}{\omega_2} > 0$ ).

Let  $\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z} = \{ \omega_1 n_1 + \omega_2 n_2 : n_1, n_2 \in \mathbb{Z} \}$ ,  $\Lambda$  is a lattice and a discrete subgroup of  $\mathbb{C}$ .



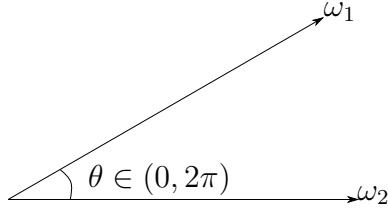


Figure 5: Relation between two numbers

**Lemma 3.2.** Let

$$\begin{aligned}\Lambda &= \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}, \\ \Lambda' &= \omega'_1 \mathbb{Z} \oplus \omega'_2 \mathbb{Z}.\end{aligned}$$

Then  $\Lambda = \Lambda' \Leftrightarrow \exists \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$  such that

$$\begin{bmatrix} \omega'_1 \\ \omega'_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}.$$

**Definition 3.3 (Complex Tori).** A *complex tori* is a quotient of  $\mathbb{C}$  by  $\Lambda$  :

$$\mathbb{C}/\Lambda = \{z + \Lambda : z \in \mathbb{C}\}.$$

In algebra:  $\mathbb{C}/\Lambda$  = an abelian group,

$$(z_1 + \Lambda) + (z_2 + \Lambda) = (z_1 + z_2) + \Lambda.$$

In topology:  $\mathbb{C}/\Lambda$  = the parallelogram on identifying the opposite side (see Figure 6).

$$z_1 + \Lambda = z_2 + \Lambda \Leftrightarrow z_1 - z_2 \in \Lambda.$$

**Proposition 3.4.** Suppose  $\varphi : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$  is holomorphic.  $\exists m, b \in \mathbb{C}$  such that

$$m\Lambda \subset \Lambda', \quad (m\Lambda = \{mz : z \in \Lambda\})$$

and

$$\varphi(z + \Lambda) = mz + b + \Lambda'.$$

$\varphi$  is invertable  $\Leftrightarrow m\Lambda = \Lambda'$ .

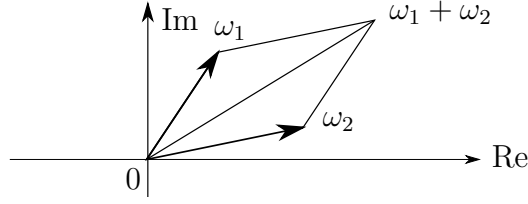


Figure 6: The parallelogram  $\mathbb{C}/\Lambda$

*Proof.* Let

$$\begin{aligned} p : \mathbb{C} &\rightarrow \mathbb{C}/\Lambda \\ z &\mapsto z + \Lambda \end{aligned}$$

and

$$\begin{aligned} p' : \mathbb{C} &\rightarrow \mathbb{C}/\Lambda' \\ z &\mapsto z + \Lambda' \end{aligned}$$

By universal cover lifting:  $\exists \tilde{\varphi} : \mathbb{C} \rightarrow \mathbb{C}$  holomorphic such that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{\varphi}} & \mathbb{C} \\ p \downarrow & & \downarrow p' \\ \mathbb{C}/\Lambda & \xrightarrow{\varphi} & \mathbb{C}/\Lambda' \end{array}$$

is commutative:  $p' \circ \tilde{\varphi} = \varphi \circ p$ .  $\forall \lambda \in \Lambda$ ,

$$\begin{aligned} & p'(\tilde{\varphi}(z + \lambda) - \tilde{\varphi}(z)) \\ &= \varphi(p(z + \lambda)) - \varphi(p(z)) \\ &= \varphi(p(z)) - \varphi(p(z)) \\ &= 0 + \Lambda' \\ &\Rightarrow \tilde{\varphi}(z + \lambda) - \tilde{\varphi}(z) \in \Lambda'. \end{aligned}$$

Left side = holomorphic function in  $z$  taking values in discrete  $\Lambda' \Rightarrow$  it is constant

$$\Rightarrow \tilde{\varphi}'(z + \lambda) - \tilde{\varphi}'(z) = 0 \text{ (here the prime symbol means taking derivative)}$$

$$\tilde{\varphi}' = \text{entire function and period} = \lambda \in \Lambda$$

$$\Rightarrow \tilde{\varphi}' \text{ is bounded}$$

$$\Rightarrow \tilde{\varphi}' = \text{constant by Liouville's Theorem}$$

$$\Rightarrow \tilde{\varphi} = mz + b \text{ for some } m, b \in \mathbb{C}$$

$$\Rightarrow \varphi(z + \Lambda) = mz + b + \Lambda'.$$

To prove  $m\Lambda \subset \Lambda'$ ,  $\forall z \in \Lambda$ ,

$$\left. \begin{aligned} \varphi(z + \Lambda) &= mz + b + \Lambda' \\ \varphi(0 + \Lambda) &= b + \Lambda' \end{aligned} \right\} \Rightarrow mz \in \Lambda' \text{ since } \varphi(z + \Lambda) = \varphi(0 + \Lambda).$$

$\varphi$  is invertable  $\Leftrightarrow$  directly verified.  $\square$

**Corollary 3.5.** Suppose  $\varphi : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$  is holomorphic:

$$\varphi(z + \Lambda) = mz + b + \Lambda', \quad m\Lambda \subset \Lambda'.$$

Then  $\varphi$  is a group homomorphism  $\Leftrightarrow b \in \Lambda'$ .

**Definition 3.6.** A nonzero holomorphic homomorphism:

$$\mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$$

is called an *isogeny*.

Isogeny is surjective, its kernel is finite (it is discrete, otherwise the map is zero).

A *curve*  $C$  (in  $\mathbb{R}^2$ ) means  $\exists$  polynomial  $F(x, y)$  such that

$$(x, y) \in C \Leftrightarrow F(x, y) = 0.$$

Assume  $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ , then  $\mathbb{C}/\Lambda$  = a complex curve, why?

Define (Weierstrass- $p$  function)

$$p = p_\Lambda : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$$

by

$$p(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \left( \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right).$$

At each  $z \in \Lambda$ ,  $p(z)$  has a double pole, otherwise  $p(z)$  is holomorphic.

$\forall \lambda \in \Lambda$ ,  $\lambda \neq 0$ ,

$$p(z - \lambda) = \frac{1}{(z - \lambda)^2} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0 \\ \omega \neq -\lambda}} \left( \frac{1}{(z - \lambda - \omega)^2} - \frac{1}{\omega^2} \right) + \left( \frac{1}{z^2} - \frac{1}{\lambda^2} \right).$$

By the virtue of

$$\lim_{z \rightarrow \infty} \left( \sum_{\substack{\omega \in \Lambda \\ |\omega| < z \\ \omega \neq 0, -\lambda}} \frac{1}{\omega^2} - \sum_{\substack{\omega \in \Lambda \\ |\omega| < z \\ \omega \neq 0, -\lambda}} \frac{1}{(\omega + \lambda)^2} \right) = 0$$

$\Rightarrow p(z)$  is  $\Lambda$ -periodical,

i.e.,  $p(z - \lambda) = p(z), \forall \lambda \in \Lambda$ .

$$p'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^3}, p'(z - \lambda) = p'(z), \forall \lambda \in \Lambda.$$

Identify  $\mathbb{C}/\Lambda$  with the parallelogram (see Figure 6), consider

$$\begin{aligned} \left\{ \frac{\mathbb{C}}{\Lambda} \right\} &\rightarrow \mathbb{C}^2 \\ z &\mapsto (p(z), p'(z)). \end{aligned}$$

For  $k = \text{even} > 2$ ,

$$G_k(\Lambda) = \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \frac{1}{\omega^k} = \sum_{(c,d) \neq (0,0)} \frac{1}{(c\omega_1 + d\omega_2)^k}.$$

Laurent expansion of  $p(z)$  and  $p'(z)$  at  $z = 0, \omega \neq 0, \omega \in \Lambda, |z| < |\omega| \Rightarrow \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2} \left( \frac{1}{(1-\frac{z}{\omega})^2} - 1 \right) = \dots \Rightarrow$  when  $z \rightarrow 0$ :

$$p(z) = \frac{1}{z^2} + 3G_4(\Lambda)z^2 + 5G_6(\Lambda)z^4 + \mathcal{O}(|z|^6) \quad (11)$$

$$p'(z) = -\frac{2}{z^3} + 6G_4(\Lambda)z + 20G_6(\Lambda)z^3 + \mathcal{O}(|z|^5). \quad (12)$$

Write

$$\begin{aligned} g_2(\Lambda) &= 60G_4(\Lambda) \\ g_3(\Lambda) &= 140G_6(\Lambda). \end{aligned}$$

(11),(12)  $\Rightarrow$  the function

$$F(z) = p'^2(z) - [4p^3(z) - g_2(\Lambda)p(z) - g_3(\Lambda)]$$

is holomorphic and  $F(0) = 0 \Rightarrow F(z) \equiv 0$  (bounded entire function)  $\Rightarrow$  the point  $(p(z), p'(z))$  lies in the curve

$$E : y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda). \quad (13)$$

**Proposition 3.7.** The map  $\varphi : \frac{\mathbb{C}}{\Lambda} \setminus \{0\} \rightarrow E$   
 $z \mapsto (p(z), p'(z))$  is bijective.

*Proof.* (i) We prove  $\forall s \in \mathbb{C}$ , the function  $p(z) - s$  has exactly two roots on  $\mathbb{C}/\Lambda$ . First assume  $\varphi(z) - s \neq 0$  on boundary of  $\mathbb{C}/\Lambda$ .

$$\# \text{ of roots} = \frac{1}{2\pi i} \int_L \frac{p'(z)}{p(z) - s} ds$$

where  $L$  is the countour in Figure 7.

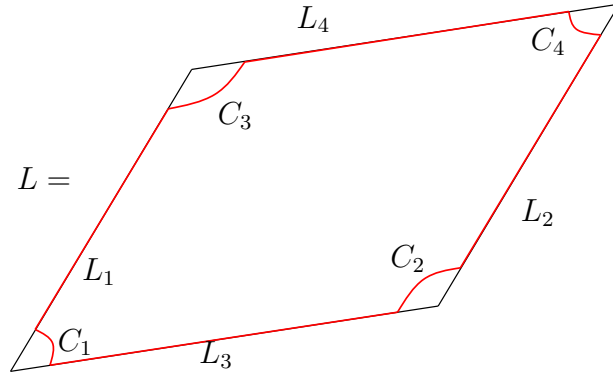


Figure 7: Countour L on boundary of  $\mathbb{C}/\Lambda$

$$\begin{aligned} p(z + \omega_2) = p(z) &\Rightarrow \int_{L_1} + \int_{L_2} = 0, \\ p(z + \omega_1) = p(z) &\Rightarrow \int_{L_3} + \int_{L_4} = 0. \end{aligned}$$

$$z \rightarrow 0, \quad \begin{aligned} p(z) - s &\sim \frac{1}{z^2} \\ p'(z) &\sim -\frac{2}{z^3} \end{aligned} \Rightarrow \frac{p'(z)}{p(z) - s} \sim -\frac{2}{z}$$

$$\Rightarrow \frac{1}{2\pi i} \int_{C_1} \rightarrow -2 \frac{-\theta}{2\pi} = \frac{\theta}{\pi}.$$

Similarly,

$$\frac{1}{2\pi i} \int_{C_3} \rightarrow \frac{\pi - \theta}{\pi}.$$

$$\Rightarrow \frac{1}{2\pi i} \left( \int_{C_1} + \int_{C_3} \right) \rightarrow 1.$$

Similarly,

$$\frac{1}{2\pi i} \left( \int_{C_2} + \int_{C_4} \right) \rightarrow 1.$$

Hence

$$\frac{1}{2\pi i} \int_L \frac{p'(z)}{p(z) - s} ds = 2.$$

**Remark.** If

$$\begin{aligned} p(z) - s &= 0 \\ z &\neq -z + \Lambda \in \mathbb{C}/\Lambda \end{aligned}$$

then  $p(z) - s = 0$  has two distinct roots (if  $z$  is the root of  $p(z) - s = 0$ , then  $-z + \lambda, \lambda \in \Lambda$  is also the root).

If  $z = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2}$ , then

$$\begin{aligned} -z + \Lambda &= z \\ \Rightarrow p'(z) &= p'(-z) = -p'(z) \\ \Rightarrow p'(z) &= 0 \\ \Rightarrow z &\text{ is a double root.} \end{aligned}$$

If  $p(z) = 0$  on the boundary, then modify  $L$ .

(ii)  $\varphi$  is surjective.

$\forall (x, y) \in E, (i) \Rightarrow \exists z$  such that  $p(z) = x$ . Let  $y' = p'(z)$ , then

$$(x, y) \in E, (x, y') \in E \Rightarrow y'^2 = y^2 \Rightarrow y' = \pm y.$$

If  $y' = -y \neq 0$ , then:

$\exists \lambda \in \Lambda$  such that  $-z + \lambda \neq z \Rightarrow$  and :

$$\begin{cases} p(-z + \lambda) = x, \\ p'(-z + \lambda) = -y' = y \end{cases} \quad (\text{recall } p'(-z + \lambda) = -p'(z)).$$

(iii)  $\varphi$  is injective.

Suppose  $\varphi(z_1) = \varphi(z_2) = (x, y)$

$$\Rightarrow p(z_1) = p(z_2) = x \Rightarrow z_2 = z_1 \text{ or } z_2 = -z_1 + \lambda.$$

In case  $z_2 = -z_1 + \lambda$ ,

$$\begin{aligned} \Rightarrow y &= p'(z_2) = p'(-z_1 + \lambda) = -p'(z_1) = -y \\ \Rightarrow p'(z_1) &= 0 \end{aligned}$$

$\Rightarrow p'(z_1) = 0$ ,  $z_1$  is a double root of  $p(z) - x = 0 \Rightarrow z_2 = z_1$ . □

Recall

$$G_k(\Lambda) = \sum_{(c,d) \neq (0,0)} \frac{1}{(c\omega_1 + d\omega_2)^k}$$

and

$$\begin{aligned} G_k(\tau) &= \sum_{(c,d) \neq (0,0)} \frac{1}{c\tau + d} \\ \Rightarrow G_k(\Lambda) &= \frac{1}{\omega_2^k} G_k\left(\frac{\omega_1}{\omega_2}\right). \end{aligned}$$

$$\begin{aligned} \delta(\tau) &= g_2^3(\tau) - 27g_3^2(\tau) \\ \Rightarrow g_2^3(\Lambda) - 27g_3^2(\Lambda) &= \delta\left(\frac{\omega_1}{\omega_2}\right) \frac{1}{\omega_2^{12}} \neq 0. \text{ ( Recall } \delta(\infty) = 0, \delta(\tau) \neq 0 \forall \tau \in \mathcal{H} \text{ )} \end{aligned}$$

**Definition 3.8.** Suppose  $C : F(x, y)$  is a curve. If  $\forall (x_0, y_0) \in C$  we have

$$\frac{\partial F}{\partial x}|_{(x_0, y_0)} \neq 0 \text{ or } \frac{\partial F}{\partial y}|_{(x_0, y_0)} \neq 0,$$

then  $C$  is a non-singular curve.

Let  $E : y^2 - (4x^3 - g_2(\Lambda)x - g_3(\Lambda)) = 0$ , then

$$\begin{aligned} E \text{ is non-singular} &\Leftrightarrow 4x^3 - g_2(\Lambda)x - g_3(\Lambda) = 0 \text{ has no multiple roots} \\ &\Leftrightarrow g_2^3(\tau) - 27g_3^2(\tau) \neq 0. \end{aligned}$$

Given  $E : y = 4x^3 - C_2x - C_3$ ,  $\Delta = C_2^3 - C_3^2$ . If  $\exists \Lambda$  such that

$$\begin{cases} C_2 = g_2(\Lambda) \\ C_3 = g_3(\Lambda) \end{cases}$$

then  $\Delta \neq 0$ ,  $E$  is a non-singular curve. Let

$$j(\tau) := \frac{1728g_2^3(\tau)}{\Delta(\tau)}. \tag{14}$$

It is easy to verify that

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau), \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

$j(\tau)$  is holomorphic on  $\mathcal{H}$  ( $\Delta(\tau) \neq 0$  on  $\mathcal{H}$ ) and  $j(\tau)$  has a simple pole at  $\infty$ .

**Lemma 3.9.** The map  $\begin{matrix} \mathcal{H} & \rightarrow & \mathbb{C} \\ \tau & \mapsto & j(\tau) \end{matrix}$  is surjective.

*Proof.*  $\forall s \in \mathbb{C}$ , let  $f = f_s = 1728g_2^3 - s\Delta$ , i.e.,

$$f(\tau) = 1728g_2^3(\tau) - s\Delta(\tau).$$

$f$  is of weight 12 modular form.

$$\begin{aligned} f(\infty) &\neq 0 \\ \Rightarrow \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum^* v_p(f) &= 1 \\ \Rightarrow \text{one of } v_i(f), v_\rho(f), v_p(f) &> 0. \end{aligned}$$

□

**Proposition 3.10.** If  $a_2^3 - 27a_3^2 \neq 0$  then  $\exists \Lambda$  such that

$$\begin{cases} g_2(\Lambda) = a_2 \\ g_3(\Lambda) = a_3. \end{cases}$$

*Proof.* Lemma 3.9  $\Rightarrow \exists \tau \in \mathcal{H}$  such that

$$\begin{aligned} j(\tau) &= \frac{1728g_2^3(\tau)}{\Delta(\tau)} \\ \Rightarrow \frac{g_2(\tau)^3}{g_2^3(\tau) - 27g_3^2(\tau)} &= \frac{a_2^3}{a_2^3 - 27a_3^2} \\ \Rightarrow \frac{a_2^3}{g_2^3(\tau)} &= \frac{a_3^2}{g_3^2(\tau)}. \end{aligned}$$

Choose  $\omega_2 \neq 0$  such that  $\frac{a_2}{g_2(\tau)} = \omega_2^4$ , let  $\omega_1 = \tau\omega_2 \Rightarrow \frac{a_2}{g_2(\Lambda)} = \frac{a_2}{g_2(\tau)\omega_2^4} = 1 \Rightarrow \frac{a_3^2}{g_3^2(\Lambda)} = \frac{a_3^2}{g_3^2(\tau)\omega_2^6} = 1 \Rightarrow \frac{a_3}{g_3(\Lambda)} = \pm 1$ . Replace  $\omega_2$  by  $i\omega_2$  if necessary  $\Rightarrow \frac{a_3}{g_3(\Lambda)} = 1$ . □



**Remark.**  $\exists$  a surjection between  $\{ \text{complex tori} \}$  and  $\{ \text{curves } E : y^2 = 4x^3 - a_2x - a_3, a_2^3 - 27a_3^2 \neq 0 \}$ . Write

$$\begin{aligned}\Lambda &= \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \quad \tau = \frac{\omega_1}{\omega_2} \in \mathcal{H}, \\ \Lambda' &= \omega'_1\mathbb{Z} \oplus \omega'_2\mathbb{Z} \quad \tau' = \frac{\omega'_1}{\omega'_2} \in \mathcal{H}.\end{aligned}$$

Recall that

$\varphi : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$  is holomorphically group-homomorphism  $\Leftrightarrow \exists m \in \mathbb{C}$  such that  $\varphi(z + \Lambda) = mz + \Lambda', m\Lambda \subset \Lambda'$ .

$\varphi$  is isomorphic  $\Leftrightarrow m\Lambda = \Lambda'$ .

$\mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda' \Leftrightarrow m \in \mathbb{C}$  such that  $m\Lambda = \Lambda' \Leftrightarrow \begin{bmatrix} m\omega_1 \\ m\omega_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega'_1 \\ \omega'_2 \end{bmatrix}$  for some

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z})$$

$$\begin{aligned}\frac{m\omega_1}{m\omega_2} &= \frac{a\omega'_1 + b\omega'_2}{c\omega'_1 + d\omega'_2} \\ \Leftrightarrow \tau &= \frac{a\tau' + b}{c\tau' + d} \\ \Leftrightarrow \tau, \tau' &\text{ are } \text{SL}_2(\mathbb{Z})\text{-equivalent}\end{aligned}$$

$\Rightarrow \exists$  a bijection between  $\{ \text{isomorphism class of } \mathbb{C}/\Lambda \}$  and  $\{ \text{SL}_2(\mathbb{Z})\text{-equivalence class of } \mathcal{H} \}$ .

$$\mathbb{C}/\Lambda \rightarrow \tau.$$

## 4 The Congruence Subgroup Case: Basic Results

Let

$$\begin{aligned}\Gamma &:= \text{a congruence subgroup,} \\ s, s' &\in \mathbb{Q} \cup \{\infty\}, \infty := \lim_{\text{Im } \tau \rightarrow \infty} \tau, \tau \in \mathcal{H},\end{aligned}$$

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$$

$$\gamma(s) := \frac{as + b}{cs + d}.$$

Then

$$\gamma(\infty) = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases} \quad \text{and} \quad \gamma(s) = \infty \text{ if } cs + d = 0, s \in \mathcal{H}.$$

If  $s' = \gamma(s), \gamma \in \Gamma, s' \neq s$  are  $\Gamma$ -equivalent, denoted by  $s' \sim s$ .

**Definition 4.1.** A *cusps* of  $\Gamma$  is a  $\Gamma$ -equivalence class of points in  $\mathbb{Q} \cup \{\infty\}$ .

**Exercise 4.1.**

- a. Let  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ , only one cusp  $= \{\infty\}$ .
- b. Let  $p$  be prime,  $\Gamma_0(p) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{p} \right\}$ . How many cusps?

**Solution.** The first is obvious. Consider  $\frac{m}{n} \in \mathbb{Q}, (m, n) = 1, n > 0$ .

$$\begin{aligned} \frac{m}{n} &\sim \infty \\ \Leftrightarrow c \frac{m}{n} + d &= 0 \Leftrightarrow n \equiv 0 \pmod{p}. \end{aligned}$$

Hence one cusp  $= \left\{ \frac{m}{n}, n \equiv 0 \pmod{p} \right\} \cup \{\infty\}$  and another cusp  $= \left\{ \frac{m}{n} : (m, n) = (n, p) = 1 \right\}$ .

**Proposition 4.2.** Let  $\Gamma$  be a congruent subgroup,  $\mathbb{C}/\Lambda, \Gamma \backslash \mathcal{H}$  be the quotient space of  $\Gamma$  acting on  $\mathcal{H}$ . Let

$$X(\Gamma) = \Gamma \backslash \mathcal{H} \cup [\text{cusps of } \Gamma].$$

Then  $X(\Gamma)$  has a natural structure as a compact Riemann surface.

**Definition 4.3 (Elliptic points w.r.t  $\Gamma$ ).**  $\tau \in \mathcal{H}$  is an *elliptic point* of  $\Gamma$  if  $\exists \gamma \in \Gamma, \gamma \neq \pm I$  such that  $\gamma(\tau) = \tau$ .

$$\gamma(\tau) = \tau \Leftrightarrow \sigma \gamma \sigma^{-1}(\sigma(\tau)) = \sigma(\tau).$$

$\gamma$  is an elliptic point of  $\Gamma \Rightarrow \tau$  is an elliptic point of  $\mathrm{SL}_2(\mathbb{Z})$ .

**Lemma 4.4.** Let  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ .

- a. If  $\gamma^2 = -I$ , then  $\gamma$  is conjugate to  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{\pm 1}$ , i.e.,

$$\sigma \gamma \sigma^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{\pm 1}.$$

- b. If  $\gamma^2 + \gamma + I = 0$ , then  $\gamma$  is conjugate to  $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^{\pm 1}$ .

c. If  $\gamma^2 - \gamma + I = 0$ , then  $\gamma$  is conjugate to  $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{\pm 1}$ .

**Lemma 4.5.** Any elliptic point of  $\text{SL}_2(\mathbb{Z})$  is equivalent to  $i$  or  $\rho = e^{\frac{2\pi i}{3}}$ .

*Proof.* For  $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \pm I$ ,

$$\begin{aligned} \gamma(\tau) = \tau &\Leftrightarrow a\tau + b = c\tau^2 + d\tau \\ &\Leftrightarrow c\tau^2 + (d - a)\tau - b = 0. \end{aligned}$$

If  $c = 0$ , then  $a = d \Rightarrow \gamma = \pm I$ . Assume  $c \neq 0$ ,  $\tau \notin \mathbb{R} \Rightarrow (d - a)^2 + 4bc < 0$

$$\begin{aligned} &\Rightarrow (d + a)^2 + 4(bc - ad) < 0 \\ &\Rightarrow (d + a)^2 - 4 < 0 \\ &\Rightarrow |d + a| < 2 \\ &\Rightarrow \left| \begin{matrix} a - x & b \\ c & d - x \end{matrix} \right| = x^2 + 1 \text{ or } x^2 \pm x + 1 \\ &\Rightarrow \gamma^2 + I = 0 \text{ or } \gamma^2 \pm \gamma + I = 0. \end{aligned}$$

By Lemma 4.4:

$$\begin{aligned} \gamma^2 + I = 0 &\Rightarrow \gamma = \sigma \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{\pm 1} \sigma^{-1} \\ \gamma(\tau) = \tau &\Rightarrow \tau \sim i. \end{aligned}$$

Similarly  $\gamma^2 \pm \gamma + I = 0 \Rightarrow \tau \sim \rho$ . □

Let

$$\begin{aligned} \mathcal{M}_k(\Gamma) &= \text{space of weight } k \text{ modular forms w.r.t. } \Gamma, \\ \mathcal{S}_k(\Gamma) &= \text{space of weight } k \text{ cusp forms w.r.t. } \Gamma, \\ g &= \text{genus of } X(\Gamma). \end{aligned}$$

**Theorem 4.6.** Suppose  $k$  is even,

$$\begin{aligned} \varepsilon_2 &= \text{number of elliptic points of } \Gamma \text{ which are } \text{SL}_2(\mathbb{Z}) \sim i, \\ \varepsilon_3 &= \text{number of elliptic points of } \Gamma \text{ which are } \text{SL}_2(\mathbb{Z}) \sim \rho, \\ \varepsilon_\infty &= \text{number of cusps of } \Gamma. \end{aligned}$$

Then

$$\dim \mathcal{M}_k(\Gamma) = \begin{cases} (k-1)(g-1) + \left[\frac{k}{4}\right] \varepsilon_2 + \left[\frac{k}{3}\right] \varepsilon_3 + \frac{k}{2} \varepsilon_\infty & k \geq 2, \\ 1 & k = 0, \\ 0 & k < 0. \end{cases}$$

$$\dim \mathcal{S}_k(\Gamma) = \begin{cases} (k-1)(g-1) + \left[\frac{k}{4}\right] \varepsilon_2 + \left[\frac{k}{3}\right] \varepsilon_3 + \left(\frac{k}{2} - 1\right) \varepsilon_\infty & k \geq 4, \\ g & k = 2, \\ 0 & k \leq 0. \end{cases}$$

$X(\Gamma)$   $\stackrel{\text{called}}{=}$  a modular curve.

**Modularity Theorem (Version  $X_{\mathbb{C}}$ ).** Suppose  $\mathbb{C}/\Lambda$  is a complex elliptic curve with  $j(\Lambda) \in \mathbb{Q}$ . Then for some  $N \in \mathbb{N}$ , there exists a surjective holomorphic function  $X(\Gamma_0(N)) \rightarrow \mathbb{C}/\Lambda$ .

## 5 Hecke Operators

Let  $\Gamma_1$  and  $\Gamma_2$  be congruence subgroups,

$$\mathrm{GL}_2^+(\mathbb{Q}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Q}, ad - bc > 0 \right\}.$$

Let  $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ , write

$$\Gamma_1 \alpha \Gamma_2 := \{ \gamma_1 \alpha \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2 \}.$$

$\Gamma_1 \alpha \Gamma_2$  is called a double coset in  $\mathrm{GL}_2^+(\mathbb{Q})$ .

**Remark.** Let  $G$  = group,  $S$  = set, then  $G$  acts on  $S$  is denoted by  $G \backslash S = \{ \text{orbits of } G \text{ on } S \} = \{Gs : s \in S\}$ . Indeed, if  $S$  = group  $G \triangleleft S \Rightarrow G \backslash S = \{Gs : s \in S\} = S/G$ .

**Fact:**  $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2 = \{ \Gamma_1 \alpha \gamma_2 : \gamma_2 \in \Gamma_2 \}$  is finite.

$\Gamma$  = congruence subgroup  $\Rightarrow [\mathrm{SL}_2(\mathbb{Z})_2(\mathbb{Z}) : \Gamma] < \infty$ .

**Lemma 5.1.** Let  $\Gamma$  be a congruence subgroup and  $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ , then  $\alpha^{-1} \Gamma \alpha \cap \mathrm{GL}_2^+(\mathbb{Q})$  is a congruence subgroup.

*Proof.*  $\exists \tilde{N} \in \mathbb{N}$  such that  $\Gamma(\tilde{N}) \subset \Gamma$  and

$$\tilde{N} \alpha \in M_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \right\}, \tilde{N} \alpha^{-1} \in M_2(\mathbb{Z}).$$

Let  $N = \tilde{N}^3$ ,

$$\begin{aligned}\alpha\Gamma(N)\alpha^{-1} &\subset \alpha(I + NM_2(\mathbb{Z}))\alpha^{-1} \\ &= I + \tilde{N}\tilde{N}\alpha M_2(\mathbb{Z})\tilde{N}\alpha^{-1} \\ &\subset I + \tilde{N}M_2(\mathbb{Z})\end{aligned}$$

$$\begin{aligned}\Rightarrow \alpha\Gamma(N)\alpha^{-1} &\subset \mathrm{SL}_2(\mathbb{Z}) \cap (I + \tilde{N}M_2(\mathbb{Z})) = \Gamma(\tilde{N}) \\ \Leftrightarrow \alpha\Gamma(N)\alpha^{-1} &\subset \Gamma(\tilde{N}) \\ \Rightarrow \Gamma(N) &\subset \alpha^{-1}\Gamma(\tilde{N})\alpha \subset \alpha^{-1}\Gamma\alpha \\ \Rightarrow \Gamma(N) &\subset \alpha^{-1}\Gamma\alpha \cap \mathrm{SL}_2(\mathbb{Z}) \\ \Rightarrow \alpha^{-1}\Gamma\alpha \cap \mathrm{SL}_2(\mathbb{Z}) &\text{ is a congruence sub group.}\end{aligned}$$

□

**Lemma 5.2.** Write  $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$  ( $\Gamma_3 \subset \Gamma_2$ ),

$$\Gamma_1 \backslash \Gamma_1\alpha\Gamma_2 = \{\Gamma_1\alpha\gamma_2 : \gamma_2 \in \Gamma_2\}.$$

The map  $\varphi : \begin{array}{ccc} \Gamma_2 & \rightarrow & \Gamma_1 \backslash \Gamma_1\alpha\Gamma_2 \\ \gamma_2 & \mapsto & \Gamma_1\alpha\gamma_2 \end{array}$  induces a bijection

$$\Gamma_3 \backslash \Gamma_2 \rightarrow \Gamma_1 \backslash \Gamma_1\alpha\Gamma_2.$$

*Proof.* It is obvious that  $\varphi$  is surjective. Suppose  $\varphi(\gamma_2) = \varphi(\gamma'_2)$ ,

$$\begin{aligned}\Gamma_1\alpha\gamma_2 &= \Gamma_1\alpha\gamma'_2 \\ \Leftrightarrow \Gamma_1\alpha\gamma_2\gamma'^{-1}_2 &= \Gamma_1\alpha \\ \Leftrightarrow \alpha(\gamma_2\gamma'^{-1}_2)\alpha^{-1} &\in \Gamma_1 \\ \Leftrightarrow \gamma_2\gamma'^{-1}_2 &\in \alpha^{-1}\Gamma_1\alpha \\ \Leftrightarrow \gamma_2\gamma'^{-1}_2 &\in \Gamma_3.\end{aligned}$$

□

Zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \text{ Re } s > 1.$$

Let  $f = \text{cusp form at } \infty : f(\tau) = \sum_{n=1}^{\infty} a_n(f)e^{2\pi n\tau}$ . Define

$$L(f, s) = \sum_{n=1}^{\infty} a_n(f)n^{-s}$$

for some  $f$ . Then

$$L(f, s) = \prod_p \left( 1 - \frac{a_p(f)}{p^s} + \frac{1}{p^{2s}} \right)^{-1}.$$

## 5.1 How to draw the fundamental domain for some types of congruence subgroup $\Gamma$ (by a student in class)

### 1. Principle

$\Gamma =$  congruence subgroup,  $\exists \min h > 0$  s.t.  $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \in \Gamma$ . Recall

$$\text{Im } \gamma(\tau) = \frac{\text{Im}(\tau)}{|c\tau + d|^2}$$

$$\begin{aligned} D &:= \{ \tau \in \mathcal{H} : 0 \leq \text{Re}(\tau) \leq h, \text{Im}(\tau) \text{ max on } \Gamma\tau \} \\ &= \left\{ \tau \in \mathcal{H} : 0 \leq \text{Re}(\tau) \leq h, \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, |c\tau + d| \geq 1 \right\}. \end{aligned}$$

**Proposition 5.3.**  $D$  is a fundamental domain for  $\Gamma \backslash \mathcal{H}$ .

*Proof.*

- a.  $\forall \tau \in \mathcal{H}, \exists \gamma \in \Gamma$  s.t.  $\gamma(\tau) \in D$
- b. If  $\tau \in D$ ,  $\Gamma\tau \cap D = \{\tau\}$ .

□

### 2. An example

$\Gamma = \Gamma_0(13)$

$$\begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}.$$

$\gamma \in \text{SL}_2(\mathbb{Z})$  maintain

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Circles in this metric are geodesics, hence  $\gamma$  maps circles to circles or vertical lines (See Figure 8).

Then we can draw the fundamental domain  $D$  (See Figure 9).

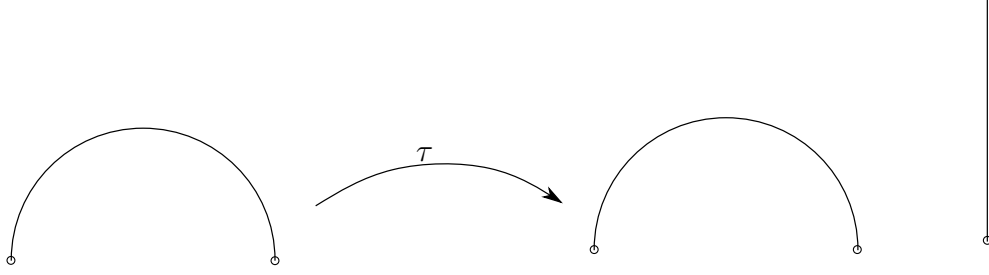


Figure 8: transform of circles

$$\textcircled{1} \quad \gamma = \begin{bmatrix} 1 & 0 \\ -13 & 1 \end{bmatrix} : \begin{matrix} 0 \\ \frac{2}{13} \end{matrix} \rightarrow \begin{matrix} 0 \\ -\frac{2}{13} \end{matrix} \rightarrow \begin{matrix} 1 \\ \frac{11}{13} \end{matrix}$$

$$\textcircled{2} \quad \gamma = \begin{bmatrix} -6 & 1 \\ -13 & 2 \end{bmatrix} : \begin{matrix} \frac{1}{13} \\ \frac{3}{13} \end{matrix} \rightarrow \begin{matrix} \frac{7}{13} \\ \frac{5}{13} \end{matrix}$$

...

### 3. What can we see from the fundamental domain

$$d := [\mathrm{SL}_2(\mathbb{Z}) : \{\pm I\} \Gamma].$$

Let  $D_0 = \Gamma \backslash \mathcal{H}$ , then  $d = \frac{|D|}{|D_0|}$ .

$$\begin{aligned} S &= \iint_S \frac{dx dy}{y^2} \\ &= \int_{r \cos \alpha}^{r \cos \beta} \int_{\sqrt{r^2 - x^2}}^{\infty} \frac{dy}{y^2} dx \\ &= \int_{r \cos \alpha}^{r \cos \beta} \frac{1}{\sqrt{r^2 - x^2}} dx \\ &= \beta - \alpha. \end{aligned}$$

We can also calculate it by Gauss-Bonnet Theorem.

$$d = (\Gamma_0(13)) = 14.$$

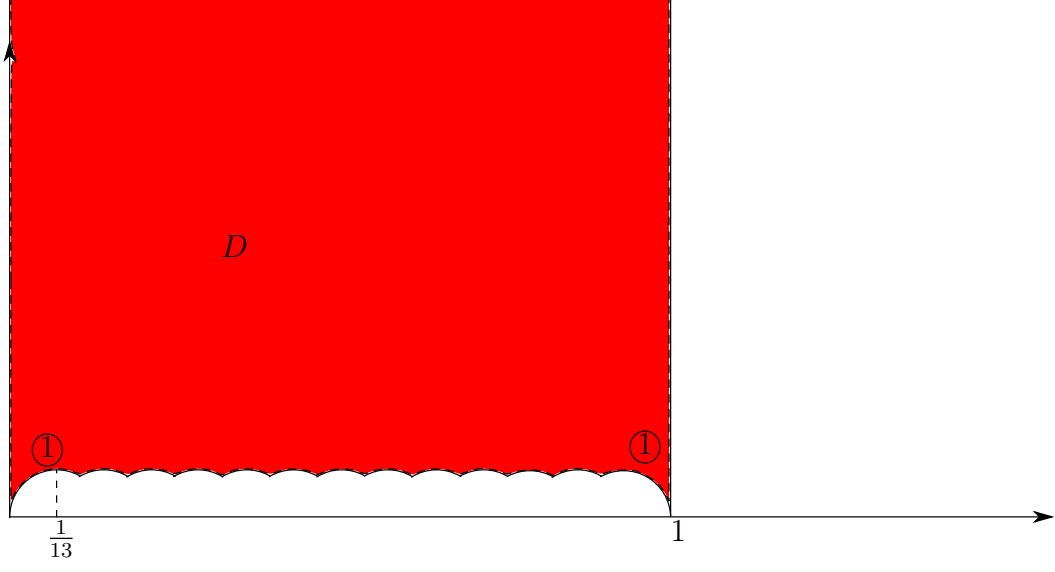


Figure 9: fundamental domain of  $\Gamma_0(13)$

We can easily calculate  $\varepsilon_2 = 2, \varepsilon_3 = 2, \varepsilon_\infty = 2$ . Since

$$2 - 2g = V - E + F,$$

and  $V = 8, E = 7, F = 1$ , we get  $g = 0$ .

**Proposition 5.4 (Genus formula).**

$$g = 1 + \frac{d}{12} - \frac{\varepsilon_\infty}{2} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_2}{4}$$

*Proof.* Cusp points:  $n_1, n_2, \dots, n_t$ ,

$$|D| = \pi(n_1 + \dots + n_t) + \frac{\pi}{3}(\varepsilon_3 + 3(V - (\varepsilon_\infty - 1) - \varepsilon_2 - \varepsilon_3))$$

On the other hand,

$$\begin{aligned} |D| &= d \frac{\pi}{3}. \\ \Rightarrow d &= 3V - 3\varepsilon_2 - 2\varepsilon_3 + 3\delta \\ V &= \frac{d}{3} + \varepsilon_2 + \frac{2}{3}\varepsilon_3 - \delta \end{aligned}$$



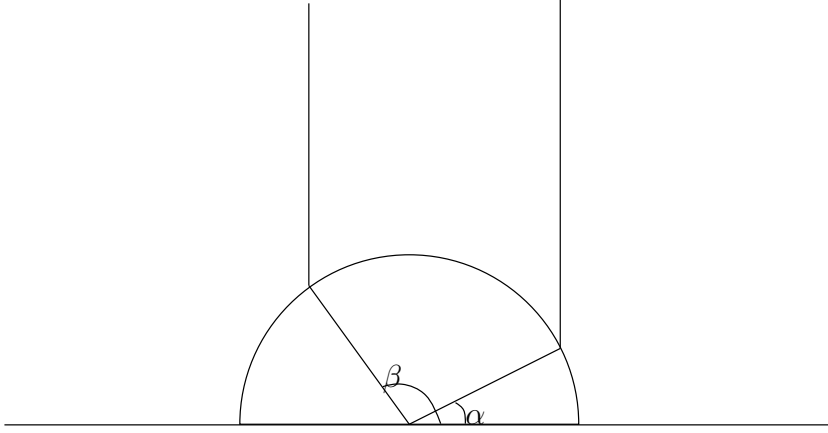


Figure 10: area of the domain

where  $\delta = (n_1 + \cdots + n_t) - (\varepsilon_\infty - 1)$ .

$$\begin{aligned}
 2E &= 3(V - (\varepsilon_\infty - 1) - \varepsilon_2 - \varepsilon_3) + (n_1 + \cdots + n_t) + \varepsilon_2 + \varepsilon_3 \\
 \Rightarrow E &= \frac{3V}{2} - (\varepsilon_\infty - 1) - \varepsilon_2 - \varepsilon_3 + \frac{\delta}{2} \\
 &= \frac{d}{2} - (\varepsilon_\infty - 1) + \frac{\varepsilon_2}{2} - \delta.
 \end{aligned}$$

$$F = 1.$$

□

**Proposition 5.5.** Let  $0 \neq f \in \mathcal{M}_k(\Gamma)$ , then

$$\left( \sum_{p \in \text{cusp}} v_p(f) + \frac{1}{2} \left( \sum_{p \in \#2} v_p(f) \right) + \frac{1}{3} \left( \sum_{p \in \#3} v_p(f) \right) \right) + \sum^* v_p(f) = \frac{dk}{12}$$