ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
LECTURE 16

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Proposition 1. *Let $\varphi : X \to Y$ be a morphism of varieties. Assume there exists a nonempty open subset $U \subset Y$ such that for all $p \in U$, $dim(\varphi^{-1}(p)) = n$, then we have*

$$dimX = dimY + n.$$

Proof. We prove the statement by induction over $\dim Y$. If Y is a point, then it is trivial. If $\dim Y > 0$, replacing Y by an open affine subset V(i.e. replace Y by $Y \cap \mathbb{A}^k$ for some k) and X by an open affine subset of $\varphi^{-1}(V)$, we can assume X, Y are both affine by theorem 6. In fact, $X \subset \mathbb{A}^l$ and $Y \subset \mathbb{A}^m$ for some l and some m, are closed affine subvarieties. We can write $\varphi = (F_1, \dots, F_m)$ with $F_i \in k[x_0, \dots, x_l]$. Let $g \in k[x_1, \dots, x_m]$ such that $\emptyset \neq Z(g) \cap Y \neq Y$, then we set $Y' = Z(g) \cap Y$ and $X' = \varphi^{-1}(Y')$. By definition $X' = X \cap Z(g(F_1, \dots, F_m))$ and it is not empty since its image Y' is not empty. For any point $p \in Y'$, $\varphi^{-1}(p)$ in X is also in X', hence the dimension of fibres is still equal to n. By induction any irreducible component $\tilde{X}$ of X' has the relation $\dim\tilde{X} = \dim\tilde{Y} + n$ with the corresponding $\tilde{Y}$ of Y', hence $\dim X' = \dim Y' + n$. Since $\dim Y = \dim Y' + 1$ and $\dim X = \dim X' + 1$, we get $\dim X = \dim Y + n$. □

Theorem 1 (without proof). *Let $\varphi : X \to Y$ be a surjective morphism, assume $dimX = dimY + n$, then*

(1) *for all points $p \in X$, $dim(\varphi^{-1}(p)) \geq n$;*
(2) *there is a nonempty open subset $U \subset Y$ such that for all $p \in U$, $dim\varphi^{-1}(p) = U$.*

Example 1. (1) $\dim(X \times Y) = \dim X + \dim Y$. Consider the projection map $p : X \times Y \to Y$, the inverse $p^{-1}(q) = X \times \{q\}$ has the dimension $\dim X$.
(2) Let $X \subset \mathbb{P}^n$ be a projective variety, then we have

$$\dim C(X) = \dim X + 1.$$

Consider the map $\Pi : C(X)\backslash\{0\} \to X$ that maps $(x_0, \dots, x_n)$ to $[x_0, \dots, x_n]$.

Definition 1. If $X \subset \mathbb{P}^n$ has dimension $n - k$, we say codimension $\operatorname{codim} X = k$.

Theorem 2. (1) *Let $X, Y \subset \mathbb{A}^n$ be closed subvarieties. Every irreducible component Z of $X \cap Y$ has dimension $dimZ \geq dimX + dimY - n$.*
(2) *Let $X, Y \subset \mathbb{P}^n$ be closed subvarieties, every irreducible component Z of $X \cap Y$ has dimension $dim \geq dimX + dimY - n$. In particular, if $dimX + dimY \geq n$, then $X \cap Y \neq \emptyset$.*

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Remark. The fact that $X\cap Y\neq\emptyset$ if $\dim X+\dim Y\geq n$ is special for projective space. This can be used to prove that $\mathbb{P}^1\times\mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$. If $\mathbb{P}^1\times\mathbb{P}^1\simeq\mathbb{P}^2$, then for any 1-dimension subvarieties $X,Y\subset\mathbb{P}^1\times\mathbb{P}^1$, we have $X\cap Y\neq\emptyset$. But for $X=\{p\}\times\mathbb{P}^1$ and $Y=\{q\}\times\mathbb{P}^1$ such that $p\neq q$, we have $X\cap Y=\emptyset$, which contradicts to the theorem, so $\mathbb{P}^1\times\mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Proof of Theorem 2.

(1) Trick: take the diagonal to reduce to the intersection with hyperplanes
$$\delta^{-1}(X\times Y)=\delta^{-1}((X\times Y)\cap\Delta)=X\cap Y.$$
Thus $X\cap Y\simeq(X\times Y)\cap\Delta\subset\mathbb{A}^{2n}$. In fact,
$$\Delta=Z(x_1-y_1,\dots,x_n-y_n).$$
By theorem 7, $\dim(Z\cap Z(f))\geq\dim Z-1$ where Z is a variety. By induction, we can get $\dim(X\cap Y)=\dim((X\times Y)\cap\Delta)\geq\dim X+\dim Y-n$.

(2) Reduce to (1) by using affine cones. By definition, $C(X)\cap C(Y)=C(X\cap Y)$, $\dim C(X)=\dim X+1$ and same for Y and $X\cap Y$. Let Z be a irreducible component of $X\cap Y$, then $C(Z)$ is a irreducible component of $C(X\cap Y)$. By using the conclusion in (1) we get
$$\begin{array}{rcc}\dim Z&=&\dim C(Z)-1\\&\geq&\dim C(X)+\dim C(Y)-(n+1)-1\\&=&\dim X+\dim Y-n\end{array}.$$

Assume $\dim X+\dim Y\geq n$, we know $C(X)\cap C(Y)\neq\emptyset$ because $0\in C(X)\cap C(Y)$. Every Z irreducible component $C(X)\cap C(Y)$ satisfies $\dim Z=\dim(C(X)\cap C(Y))\geq\dim C(X)+\dim C(Y)-(n+1)\geq 1$. Thus $C(X)\cap C(Y)\neq\{0\}\Rightarrow X\cap Y\neq\emptyset$. □

We know $\dim X=\dim Y$ if X and Y are birational, and $K(X)\simeq K(Y)$ if X is birational to Y. Thus $\dim X$ must be determined by $K(X)$. We will see $\dim X$ is equal to the transcendence degree of $K(X)$ over k.

Definition 2 (Field Extension and Finitely generated Field Extension)**.** Let K/k be a field extension. For $a_1,\dots,a_n\in K$, denote $k(a_1,\dots,a_n)$ as the smallest subfield of K containing k and $a_1,\dots,a_n$. This is called field extension over k by $a_1,\dots,a_n$. If there are $a_1,\dots,a_n\in K$ such that $K=k(a_1,\dots,a_n)$, we say K/k is finitely generated.

Definition 3 (Algebraically Independent sets)**.** Let K/k be a finitely generated field extension, elements $b_1,\dots,b_n\in K$ are called algebraically independent over k if there is no polynomial $f\in k[x_1,\dots,x_n]$ such that $f(b_1,\dots,b_n)=0$. In particular, if $b\in K$ is algebraically independent over k, then b is called transcendent over k.

Let $k(x_1,\dots,x_n)$ be a field of rational functions in n indeterminants, it is easy to see $k(b_1,\dots,b_n)\simeq k(x_1,\dots,x_n)$ if $b_1,\dots,b_n$ are algebraically independent over k.

Definition 4 (Transcendence Basis)**.** A maximal set of algebraically independent elements of K over k is called a transcendence basis.

Theorem 3 (without proof)**.** *Let $K=k(a_1,\dots,a_n)/k$ be a finitely generated field extension, then*

(1) *there exists a transcendence basis of K/k, it can be chosen as a subset of $\{a_1,\dots,a_n\}$;*

(2) *every transcendence basis of elements of K/k has the same number of elements, called the transcendence degree;*
(3) *let $b_1, \ldots, b_r$ be a transcendence basis of K/k, then $K/k(b_1, \ldots, b_r)$ is a finite algebraic extension.*

Theorem 4. *Every variety X is birational to a hypersurface in $\mathbb{A}^{dimX+1}$.*

This theorem may be proved next time.

Theorem 5. *Let X be a variety, then*

$$dimX = trdegK(X)/k.$$

Proof. By theorem 4, we can assume $X = Z(F) \subset \mathbb{A}^n$ is a hypersurface, $F \in k[x_1, \ldots, x_n]$ is irreducible. We know $\dim X = n-1$. To show $\text{trdeg}K(X)/k = n-1$, let $y_1, \ldots, y_n \in A(X)$ be coordinate functions. Then $K(X) = k(y_1, \ldots, y_n)$, $F(y_1, \ldots, y_n) \in A(X) = k[x_1, \ldots, x_n]/\langle F \rangle$ and $F(y_1, \ldots, y_n) = 0$ since $X = Z(F)$. Thus $y_1, \ldots, y_n$ are algebraically dependent. It follows that $\text{trdeg}K(X)/k \leq n-1$. To show the equality, we assume the last variable x_n occurs in F, then we can get $y_1, \ldots, y_{n-1}$ are algebraically independent. Otherwise, there exists a nonzero element $G \in k[x_1, \ldots, x_{n-1}]$ with $G(y_1, \ldots, y_{n-1}) = 0$, then $G(y_1, \ldots, y_{n-1}) \in \langle F \rangle$. But it is impossible because F contains $x_n \Rightarrow G$ contains x_n. Thus $\text{trdeg}K(X)/k = n-1$. □

1. Conclusions We Need From Previous Lectures

In lecture 15:

Theorem 6. *Let X be a variety, $\emptyset \neq U \subset X$, U is an open subset of X. Then $dimU = dimX$.*

Theorem 7. *Let $X \subset \mathbb{A}^n$ be an affine variety, $F \in k[x_1, \ldots, x_n] \backslash I(X)$, then every irreducible component(if there is any) of $Z(F) \cap X$ has dimension $dimX - 1$.*

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