ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
LECTURE 04

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Proposition 1. *Same as an affine space, in a projective space we have the following propositions:*

1. $X \subset Y \subset \mathbb{P}^n$ *are projective algebraic sets, then* $I(X) \supset I(Y)$*;*
2. $X \subset \mathbb{P}^n$ *is a projective algebraic set, then* $Z(I(X)) = X$*;*
3. $\mathfrak{a} \subset k[x_0, \ldots, x_n]$ *is a homogeneous ideal, then* $I(Z(\mathfrak{a})) \supset \mathfrak{a}$*;*
4. *If* $S \subset k[x_0, \ldots, x_n]$ *is a set of homogeneous polynomials, then* $Z(S) = Z(\langle S \rangle)$*;*
5. *For a family* $\{S_\alpha\}$ *of sets of homogeneous polynomials,* $Z(\bigcup\limits_{\alpha} S_\alpha) = \bigcap\limits_{a} lphaZ(S_\alpha)$*;*
6. *If* $T, S \subset k[x_0, \ldots, x_n]$ *are sets of homogeneous polynomials, then* $Z(ST) = Z(S) \cup Z(T)$*.*

Remark. From the proposition (5) and (6) we know that arbitrary intersections and finite unions of projective algebraic sets are projective algebraic sets, then we can define a topology through these two propositions.

Definition 1. The Zariski topology on $\mathbb{P}^n$ is the topology whose closed sets are the projective algebraic sets.

If $X \subset \mathbb{P}^n$ is a subset, we give it the induced topology, called Zariski topology on X.

Definition 2. A quasi-projective algebraic set is an open subset of a projective algebraic set. Fro example, let U and V be closed subsets, then $Y = U \backslash V \neq \emptyset$ is a quasi-projective algebraic set.

Proposition 2. *We jnow* $k[x_0, \ldots, x_n]$ *is noetherian, then follows the same proof as in affine case shows that* $\mathbb{P}^n$ *is a noetherian topological sapce.*

Remark. Every subspace of $\mathbb{P}^n$ is noetherian. In particular, quasi-projective algebraic sets are noetherian, hence have unique decompositions into irreducible components.

Definition 3. A quasi-projective variety is an irreducible quasi-projective algebraic set.

Remark. If we use the identification $\mathbb{A}^n = U_0 \subset \mathbb{P}^n$, then $\mathbb{A}^n$ is an open set $\mathbb{A}^n = \mathbb{P}^n \backslash Z(x_0)$, i.e. $\mathbb{A}^n$ is a quasi-projective variety.

Definition 4. A nonempty algebraic set $X \subset \mathbb{A}^{n+1}$ is called a cone if for all $p = (a_0, \ldots, a_n) \in X$ and all $\lambda \in k$, we have $(\lambda a_0, \ldots, \lambda a_n) = \lambda p \in X$.

If $X \subset \mathbb{P}^n$ is a projective algebraic set, its affine cone is

$$C(X) := \{(a_0, \ldots, a_n) \in \mathbb{A}^{n+1} | [a_0, \ldots, a_n] \in X\} \cup \{0\} \tag{0.1}$$

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Lemma 1. *Let $X \neq \emptyset$ be a projective algebraic set, then :*

1. $X = Z_p(\mathfrak{a})$*, for* $\mathfrak{a} \subset k[x_0, \dots, x_n]$ *a homogeneous ideal* $\Rightarrow C(X) = Z_a(\mathfrak{a}) \subset \mathbb{A}^{n+1}$*;*
2. $I_a(C(X)) = I_H(X)$*.*

Theorem 1 (Projective Nullstellensatz)**.** *Let $\mathfrak{a} \subset k[x_0, \dots, x_n]$ be a homogeneous ideal:*

1. $Z_p(\mathfrak{a}) = \emptyset \Leftrightarrow \mathfrak{a}$ *contains all homogeneous polynomials of degree N for some $N \in \mathbb{N}$;*
2. *If $Z_p(\mathfrak{a}) \neq \emptyset$, then $I_p(Z_p(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof. Let $X = Z_p(\mathfrak{a})$.

(1) $X = \emptyset \Leftrightarrow C(X) = \{0\}$. Since $C(X) = Z_a(\mathfrak{a}) \cup \{0\}$, we get

$$X = \emptyset \Leftrightarrow Z_a(\mathfrak{a}) = \emptyset \text{ or } Z_a(\mathfrak{a}) = \{0\}.$$

By affine Nullstellensatz, we get

$$\sqrt{\mathfrak{a}} = k[x_0, \dots, x_n] \text{ or } \sqrt{\mathfrak{a}} = \langle x_0, \dots, x_n \rangle.$$

So $\sqrt{\mathfrak{a}} \supset \langle x_0, \dots, x_n \rangle$. Thus for any $i = 0, \dots, n$, $\exists m_i$ s.t. $x_i^{m_i} \in \mathfrak{a}$. Let $N = m_1 + \cdots + m_n$, then any monomial of degree N in $k[x_0, \dots, x_n]$ lies in $\mathfrak{a}$.

(2)Let $X = Z_p(\mathfrak{a} \neq \emptyset$, then

$$I_H(X) = I_a(C(X)) = I_a(Z_a(\mathfrak{a})) = \sqrt{\mathfrak{a}}. \tag{0.2}$$

□

Remark. $\langle x_0, \dots, x_n \rangle$ is called the irrelevant ideal, an ideal different from $\langle x_0, \dots, x_n \rangle$ is called relevant.

Corollary 1. *There is a one-to-one correspondence between homogeneous relevant radical ideals and projective algebraic sets:*

Z_p: homogeneous relevant radical ideals in $k[x_0, \dots, x_n] \to$ projective algebraic sets in $\mathbb{P}^n$

I_H: projective algebraic sets in $\mathbb{P}^n \to$ homogeneous relevant radical ideals in $k[x_0, \dots, x_n]$.

Remark. We use subscripts to recognize affine spaces and projective spaces, such as $Z_p(\mathfrak{a}), Z_a(\mathfrak{a})$. Sometimes we can infer the difference from the context, so we usually write briefly as $Z(\mathfrak{a})$.

Proposition 3.

1. *A projective algebraic set $X \neq \emptyset \subset \mathbb{P}^n$ is irreducible if and only if $I = I_H(X)$ is a homogeneous prime ideal;*
2. *If $f \in k[x_0, \dots, x_n]$ is a homogeneous polynomial and irreducible, then $Z_p(f)$ is irreducible.*

Proof. (1) $\Leftarrow$: Assume X reducible, then $X = X_1 \cup X_2$,$X_1, X_2 \subsetneqq X$b are closed subsets. Then we get $C(X) = C(X_1) \cup C(X_2)$, $C(X_1) \subsetneqq C(X)$,$C(X_2) \subsetneqq C(X)$ are closed, hence $C(X)$ is reducible, $I_H(X) = I(C(X))$ is not prime.

$\Rightarrow$: Assume $I_H(X)$ not prime, it means $\exists f, g \in k[x_0, \dots, x_n]$, $fg \in I_H(X)$ and $f, g \notin I_H(X)$. Let $i, j \in \mathbb{Z} \geq 0$ be minimal such that $f^{(i)} \notin I$ and $g^{(j)} \notin I$. Subtract homogeneous components of lower degrees from f and g, we can assume f starts in degree i and g starts in degree j. Thus $f^{(i)} g^{(j)}$ is homogeneous component of minimal degree in $fg \in I$. Because I is homogeneous, we get $f^{(i)} g^{(j)} \in I$. Let

$X_1 := Z(I) \cap Z(f^(i))$ and $X_2 := Z(I) \cap Z(g^{(j)})$, then $X_1, X_2 \subsetneqq X$, $X = X_1 \cup X_2$, thus X is reducible.

(2) If $I \subset k[x_0, \ldots, x_n]$ is homogeneous and prime with $Z(I) \neq \emptyset$, then follow the result from (1) we know $Z(f)$ is irreducible. $\square$

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