Properties of the solution map

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How to choose the solution map

Choose the orbits rather than the trajectories

Example

$$\dot{x}(t) = -x\left(t - \frac{\pi}{2}\right) \tag{1}$$

Obviously, it has a unique solution through each $(\sigma, \phi) \in \mathbb{R} \times C$. Consider two solutions:

$$x(t) = \sin t \quad \text{and} \quad x(t) = \cos t. \tag{2}$$

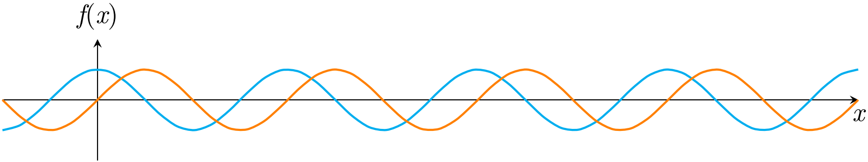

Figure 1: Two solutions intersect an infinite number of times on any interval $[\sigma, \infty)$

It is not a good way to represent these two solutions. In fact, given a $\frac{\pi}{2}$ phase shift of the solution $x(t) = \sin(t)$ we get

$$\sin(t + \frac{\pi}{2}) = \cos t.$$

It's better to consider an orbit of solutions rather than the trajectories. Further more, we need to choose a phase space that the above two solutions are in the same orbit.

Example: phase space $\mathbb{R}$ and the orbits $\bigcup_{t\geq 0} x(0,\phi)(t)$

The orbits of two solutions (2) coincide and are equal to the interval $[-1, 1]$. The difficulty is: the orbit of the solution $x = \cos t$ contain the orbit of another solution $x = 0$ and not be related in any way to a phase shift.

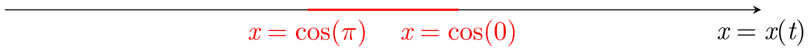

Figure 2: Orbits $\bigcup_{t\geq 0} x(0,\phi)(t)$ in phase space $\mathbb{R}$, $\phi = \cos(t+\theta), \forall\theta \in \mathbb{R}$.

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Example: phase space $C = C([-\pi/2, 0], \mathbb{R})$

The orbit of the solution $\sin t$ is the set

$$\Gamma = \left\{ \psi : \psi(\theta) = \sin(t + \theta), -\frac{\pi}{2} \leq \theta \leq 0, \text{ for } t \in [0, \infty) \right\} \tag{3}$$

of points in C. Then Γ is determined by phase shifts of a solution. Γ is a closed curve in C since $\sin t$ is periodic. This condition cannot be pictured since the dimension of Γ is infinite.

Purpose of this presentation

The purpose of the following content is to discuss some good or bad properties of the solution map $T_f(t,\sigma)$ of and RFDE(f) defined by

$$T_f(t,\sigma)\phi = x_t(\sigma,\phi,f).$$

We will assume that f is continuous and there is a unique solution of the RFDE(f) through (σ,ϕ).

Finite- or infinite-dimensional problem?

The continuation theorem:

Theorem 1 (Theorem 3.2 of Section 2.3)

Suppose Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is completely continuous, and x is a noncontinuable solution of

$$\dot{x}(t) = f(t, x_t) \tag{4}$$

on $[\sigma - r, b)$. Then, for any closed bounded set U in $\mathbb{R} \times C$, $U \in \Omega$, there exists a t_U such that $(t, x_t) \notin U$ for $t_U \leq t < b$.

Proof. Consider the first case $r = 0$ (an ordinary equation). Since $U \subset \{\sigma\} \times C \simeq C$ is a closed bounded set, the existence theorem implies there is an $\alpha > 0$ such that the equation has a solution through any $(c, y) \in U$ that exists at least on $[c, c + \alpha]$. Now suppose the assertion of the theorem is false, that is, there is a sequence $(t_k, x_{t_k} \in U, y \in \mathbb{R}^n, (b, y) \in U$ such that $t_k \to b^-, x_{t_k} \to y$ as $k \to \infty$.

Using the fact that f is bounded in a neighborhood of (b, y), the function x is uniformly continuous on $[\sigma, b)$ and $x(t) \to y$ as $t \to b^-$. There is obviously an extension of x to the interval $[\sigma, b+\alpha]$. Since $b+\alpha > b$, this is a contradiction. Consider the second case $r > 0$. Suppose the conclusion of the theorem is not true. Then there is a sequence of real numbers $t_k \to b^-$ such that $(t_k, x_{t_k}) \in U$ for all k. Since $t > 0$, this implies that $x(t), \sigma - r \leq t < b$ is bounded. Consiquently, there is a constant M such that $|f(\tau, \phi)| \leq M$ for (τ, ϕ) in the closure of $\{(t, x_t) : \sigma \leq t < b\}$. The integral equation for the solution of Equation (4) imply

$$|x(t+\tau) - x(t)| = |\int_t^{t+\tau} f(s, x_s)\mathrm{d}x| \leq M\tau$$

for all $t, t+\tau < b$. Thus, x is uniformly continuous on $[\sigma - r, b)$. This implies $\{(t, x_t) : \sigma \leq t < b\}$ belongs to a compact set in Ω. This contradicts Theorem 3.1 of Section 2.3 and proves the theorem. □

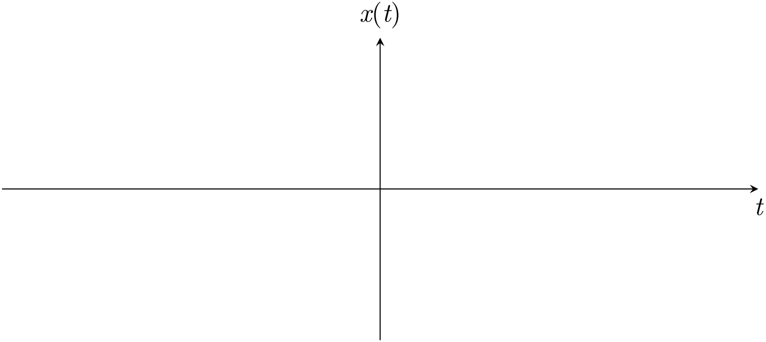
Property 1

The continuation theorem is not valid if f is not a completely continuous map.

Proof. Let $\Delta(t) = t^2$ and

$$a_1 < b_1 < a_2 < b_2 < a_3 < b_3 \cdots, a_k \to 0, b_k \to 0 \text{ as } k \to \infty.$$

For example, choose $b_k = -2^{-k}$. Define $\psi(t)$ as the following:



Let

$$h(t-\Delta(t),\psi(t-\Delta(t)))=\psi'(t).$$

Now consider the equation

$$\dot{x}(t)=h(t-\Delta(t),x(t-\Delta(t))),\quad t<0 \text{ and } \Delta(t)=t^2.$$

$\square$

Property 2

$T(t,\sigma)$ is locally bounded for $t \geq \sigma$.

Proof. Since $T(t,\sigma)\phi$ is assumed to be continuous in (t,σ,ϕ), it follows that for any $t \geq \sigma, \phi \in C$ for which $(\sigma,\phi) \in \Omega$ and $T(t,\sigma)\phi$ is defined, there is a neighborhood $V(t,\sigma,\phi)$ of ϕ in C such that $T(t,\sigma)V(t,\sigma,\phi)$ is bounded. $\square$

Property 3

$T(t,\sigma)$ may not be a bounded map.

Proof. Let $r=\frac{1}{4}$, $C=C([-r,0],\mathbb{R})$, consider the equation

$$\dot{x}(t)=f(t,x_t):=x^2(t)-\int_{\min(t-r,0)}^{0}|x(s)|\mathrm{d}s. \tag{5}$$

Let $B=\{\phi\in C:|\phi|\leq 1\}$ and $x(b)$ be the solution. For $b\neq 0, x(b)(0)\leq 1$, then

$$\dot{x}(b)(t)<x^2(t)$$

for all t. Let $\dot{y}=y^2(t), y(0)=1$, then $x(t)<y(t)$ for all $0<t<1$,

$$x(b)(t)<y(t)=\frac{1}{1-t}.$$

Hence $x(b)(r)<(1-r)^{-1}$ for all $b\in B$. For $t\geq r$, $\dot{x}(b)(t)=x^2(b)(t)$ and the fact that $x(b)(r)<(1-r)^{-1}$ implies $x(b)(t)$ exists for $-r\leq t\leq 1$.

If we show that for any $\varepsilon>0$ there is a $b\in B$ such that

$$x(b)(r) \geq (1-r)^{-1} - \varepsilon,$$

then the set $x(B)(1)$ is not bounded.

Let $\psi = y - x$, we need to find x such that $\psi \leq \varepsilon$ for $0 < t < r$. Let $C = (1-r)^{-1}$,$\lambda = \int_{-r}^{0} |b(s)|\mathrm{d}s$ then

$$\begin{aligned}
\dot{\psi}(t) =& \dot{y}(t) - \dot{x}(t) \\
=& y^2(t) - x^2(t) + \int_{\min(t-r,0)}^{0} |b(s)|\mathrm{d}s \\
\leq& (y(t) + x(t))\,\psi(t) + \int_{-r}^{0} |b(s)|\mathrm{d}s \\
\leq& 2C\psi(t) + \lambda \\
\leq& 2C\left(\psi(t) + \frac{\lambda}{2C}\right).
\end{aligned}$$

Since $\psi(0) = 0$,

$$\psi(t) + \frac{\lambda}{2C} \leq \frac{\lambda}{2C} e^{2Ct}.$$

To obtain $\psi \leq \varepsilon$, it is enough to get

$$\begin{aligned} &(e^{2Ct} - 1)\frac{\lambda}{2C} \leq \varepsilon \\ \Leftarrow &\lambda \leq \frac{2C}{e^{2Ct} - 1}\varepsilon \\ \Leftarrow &\lambda \leq 2C\varepsilon \text{ since } e^{\frac{2r}{1-r}} - 1 < 1. \end{aligned}$$

$\square$

Property 4

Bang-bang controls are not always possible for RFDE.

Proof. Suppose

$$\phi = 0$$

and consider

$$\dot{x}(t) = x(t-1) + u(t), \quad |u| \leq 1. \tag{6}$$

Then

$$x(0,u)(t) = \int_0^t u(s)\mathrm{d}s$$

for $0 \leq t \leq 1$ and $\mathcal{A}(1,0)$ contains zero since the control $u(t) = 0, 0 \leq t \leq 1$, gives $x_1(0,u) = 0$. On the other hand, there is no way to reach zero with a bang-bang control. □

Equivalence class of solutions

Property 1

The map $T(t,\sigma)$ may not be one-to-one.

Proof. Consider the equation

$$\dot{x}(t) = -x(t-r)[1-x^2(t)]. \tag{7}$$

Equation (7) has the solution $x(t)=1$ for all t in $(-\infty,\infty)$. If $r=1, \sigma=0$, and $\phi \in C$, then there is a unique solution $x(0,\phi)$ of Equation (7) through $(0,\phi)$ that depends continuously on ϕ. If $-1 \le \phi(0) \le 1$, these solutions are actually defined on $[-1,\infty)$. On the other hand, if $\phi \in C, \phi(0)=1$, then $x(0,\phi)(t)=1$ for all $t \ge 0$. Therefore, for all such initial values, $x_t(0,\phi), t \ge 1$, is the constant function 1. A translation of a subspace of C of codimension one is mapped into a point by $T(t,0)$ for all $t \ge 1$. □

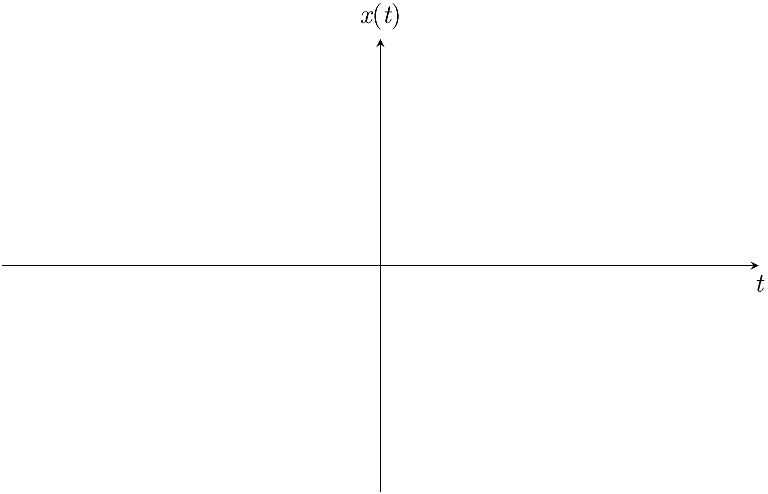
Equivalence class of solutions

Definition 2

Suppose $\Omega = \mathbb{R} \times C$ and all solutions $x(\sigma, \phi)$ of the RFDE(f) are defined on $[\sigma - r, \infty)$. We say $(\sigma, \phi) \in \mathbb{R} \times C$ is *equivalent* to $(\sigma, \psi) \in \mathbb{R} \times C$, if there is a $\tau \geq \sigma$ such that $x_\tau(\sigma, \phi) = x_\tau(\sigma, \psi)$.

Be careful of the difference between equivalence relation defined here and orbits defined before.

Then the space can be decomposed into equivalence classes $\{V_\alpha\}$ for each fixed σ.


$x(t)$
t

Choose the representation element

For each equivalence class V_α, choose a representation element $\phi^{\sigma,\alpha}$ and let

$$W(\sigma) = \bigcup_{\alpha} \phi^{\sigma,\alpha}. \tag{8}$$

It is important to choose an appropriate $\phi^{\sigma,\alpha} \in V_\alpha$.

Example: Equation (7)

A good choice for $W(0)$ in Equation (7) would be

$$C \setminus \{(C_1 \setminus \{1\}) \cup (C_{-1} \setminus \{-1\})\}$$

where $C_a = \{f \in C : \phi(0) = a\}$.

Determined in finite time

Definition 3

We say that an *equivalence class* V_α is *determined in a finite time* if there exists $\tau > 0$ such that for any $\phi, \psi \in V_\alpha$, $x_{\sigma+t}(\sigma, \phi) = x_{\sigma+t}(\sigma, \psi)$ for $t \geq \tau$.

Given two fixed $\phi, \psi \in V_\alpha$, there must exists $\tau > 0$ such that $x_{\sigma+t}(\sigma, \phi) = x_{\sigma+t}(\sigma, \psi)$ for $t \geq \tau$. The choice of τ here may be relevant to ϕ and ψ. *Determined in a finite time* means the choice of τ can be chosen as the same number, i.e., irrelevant to the choice of ϕ and ψ.

Property 2

The equivalence classes may not be determined in finite time.

To prove the Property 2, we consider the equation

$$\dot{x}(t) = \beta[|x_t| - x(t)], \quad \beta > 0. \tag{9}$$

We first establish some lemmas.

Lemma 4

Suppose $\phi(0) \geq 0$, then the solution $x(t)$ of Equation (9) is a constant for $t \geq 1$. Further more, for any positive constant function, the corresponding equivalence class contains more than one element and equivalence classs corresponding to the constant function zero contains only zero.

Proof. If $\phi(0) \geq 0, \phi \neq 0$, combined with $\dot{x}(t) \geq 0$ by Equation (9), then $|x_t| = x(t)$ for $t \geq 1$ and implies $x(t)$ is a constant $\geq \phi(0)$ for $t \geq 1$. If $\phi(0) = 0$ and $\phi \neq 0$, then $\dot{x}(0) > 0$ and $x(t) > 0$ for $t \geq 1$. Therefore, for any positive constant function, the corresponding equivalence class contains more than one element. If $x(t) = 0, t \geq a > 0$, then $x(t)$ must be zero at $[a-1, a]$ by preceding argument, hence the equivalence class corresponding to the constant function zero contains only zero. □

Lemma 5

Suppose $\phi(0) < 0$ and $x(\phi, \beta)(t)$ has a zero $z(\phi, \beta)$. Then it must be simple.

Proof. Given $\phi(0) < 0$, it is clear that $x(\phi, \beta)(t)$ approaches a constant as $t \to \infty$. If $x(\phi, \beta)(t)$ has a zero $z = z(\phi, \beta)$, then $x(t) \neq 0$ as a function in $C([z-1, z], \mathbb{R})$, hence $\dot{x}(z) = \beta|x_t| > 0$, i.e., z is simple. This lemma can also be proved by using the last part of Lemma 4. □

Lemma 6

For any $\beta > 0$, there is a $\phi \in C, \phi(0) < 0$ such that $z(\phi, \beta)$ exists.

Proof. Let $\phi(0) = -1, \phi(\theta) = -\gamma, \gamma > 1, -1 \leq \theta \leq -\frac{1}{2}$ and let $\phi(\theta)$ be a monotone increasing function for $-\frac{1}{2} \leq \theta \leq 0$. As long as $x(t) \leq 0$ and $0 \leq t \leq \frac{1}{2}$, we have $|x_t| = \gamma$ and

$$\dot{x}(t) = \beta[\gamma - x(t)] \geq \beta\gamma.$$

Therefore, $x(t) \geq \beta\gamma t - 1$ if $x(t) \leq 0$ and $0 \leq t \leq \frac{1}{2}$. For $\beta\gamma/2 > 1$, $x(\frac{1}{2}) \geq \frac{\beta\gamma}{2} - 1 > 0$, hence x must have a zero $z(\phi, \beta) < \frac{1}{2}$. □

Proof of Property 2. Define

$$C_{-1} = \{\phi \in C : \phi(0) = -1\}$$
$$C_{-1^0} = \{\phi \in C_{-1} : z(\phi,\beta) \text{ exists}\}$$
$$C_{-1^n} = \{\phi \in C_{-1} : z(\phi,\beta) \text{ does not exist}\}.$$

Since $z(\phi,\beta)$ is continuous, the set C_{-1^0} is open and C_{-1^n} is closed. If C_{-1^n} is not empty, set $\phi \in C_{-1^n}$ and the corresponding solution $x(t) \to 0\,(t \to \infty)$ by Lemma 5. Then there is $\phi_j \in C_{-1^0}$, $\phi_j \to \phi \in C_{-1^n}$ as $j \to \infty$ and $z(\phi_j,\beta) \to \infty$. Now we claim that C_{-1^n} is not empty. To prove it, choose $\beta_0 > 0$ less than or equal to the value β for which the equation

$$\lambda + \beta = -\beta e^{-\lambda}$$

has a real root λ_0 of multiplicity two. For this β_0, the equation $\lambda + \beta = -\beta e^{-\lambda}$ has two real negative roots. If $-\lambda_0$ is one of these roots, then $x(t) = -e^{-\lambda_0 t}$

is a solution of Equation (9) with initial value $\phi_0(\theta) = -e^{-\lambda_0\theta}$,$-1 \leq \theta \leq 0$, $\phi_0 \in C_{-1}$. Therefore C_{-1^n} is not empty. It follows that

$$\delta(\beta_0) := \sup\{z(\phi, \beta_0) : \phi \in C_{-1^0}\} = \infty.$$

Since the original equation is positive homogeneous of degree 1 in x, it follows that, for any positive constants a and t_0, there exists $\phi \in C$, such that $x(\phi, \beta_0)(t) = a$, $t \geq t_0$, and $x(\phi, \beta_0)(t) < a$ for $0 \leq t < t_0$. $\square$

Small solutions for linear equations

Small solution

Definition 7

A *small solution* x is a solution such that

$$\lim_{t\to\infty} e^{kt}x(t) = 0 \text{ for all } k \in \mathbb{R}. \tag{10}$$

In this section we study the existence of small solutions of linear autonomous RFDE(L)

$$\begin{cases} \dot{x}(t) = \int_{-r}^{0} \mathrm{d}[\eta(\theta)]x(t+\theta) \\ x_0 = \phi. \end{cases} \tag{11}$$

Nontrivial small solutions

Definition 8

If there are initial conditions $\phi \neq 0$ such that $x(\cdot, \phi)$ to System (11) is a small solution, then such solutions are called *nontrivial* small solutions.

Example

Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-1) \\ \dot{x}_2(t) &= x_1(t). \end{aligned} \tag{12}$$

Any initial condition $\phi = (\phi_1, \phi_2)^T$ with $\phi_1(0) = 0$ and $\phi_2 = 0$ yields a small solution $x_1(t) = x_2(t) = 0, t \geq 0$.