ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
LECTURE 09

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Remark. Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be subvarieties, $X \times Y$ does not lie rationally in some projective space. Thus we need to find an embedding $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ to denote the products of quasi-projective varieties.

Definition 1 ([Segre Embedding).] We put $N := (n+1)\cdot(m+1)-1$, let $x_0, \dots, x_n$ be coordinates on $\mathbb{P}^n$, $y_0, \dots, y_m$ be coordinates on $\mathbb{P}^m$. Let $z_{ij}, i = 0, \dots, n, j = 0, \dots, m$ be coordinates on $\mathbb{P}^N$. Define a map

$$\begin{aligned} \sigma : \mathbb{P}^n \times \mathbb{P}^m & \to \mathbb{P}^N \\ ([x_0, \dots, x_n], [y_0, \dots, y_m]) & \to [z_{ij}] = [x_i y_j] \end{aligned}$$

σ is called the Segre embedding.

Definition 2. We define the image of σ as

$$\Sigma := \sigma(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N.$$

For $i = 0, \dots, n$, put

$$U_i := \{[x_0, \dots, x_n] \in \mathbb{P}^n | x_i \neq 0\}.$$

For $j = 0, \dots, m$, put

$$U_j := \{[y_0, \dots, y_m] \in \mathbb{P}^m | y_j \neq 0\}.$$

And for $i = 0, \dots, n, j = 0, \dots, m$, put

$$U_{ij} := \{[z_{kl}] \in \mathbb{P}^N | z_{ij} \neq= 0\}.$$

there are isomorphisms:

$$\begin{aligned} \mathbb{A}^n & \underset{\varphi_i}{\overset{u_i}{\rightleftarrows}} U_i \\ \mathbb{A}^m & \underset{\varphi_j}{\overset{u_j}{\rightleftarrows}} U_j \\ \mathbb{A}^N & \underset{\varphi_{ij}}{\overset{u_{ij}}{\rightleftarrows}} U_{ij}. \end{aligned}$$

Since $\mathbb{P}^N = \cup_{i,j} U_{ij}$, we get $\Sigma = \cup_{i,j}(\Sigma \cap U_{ij})$, define

$$\Sigma^{ij} = \Sigma \cap U_{ij}.$$

Define the map σ^{ij}

$$\begin{aligned} \sigma^{ij} : \mathbb{A}^{n+m} & \to U_{ij} \\ (p, q) & \to \sigma(u_i(p), u_j(q)). \end{aligned}$$

By definition we know $\sigma^{ij}(\mathbb{A}^{n+m}) = \Sigma^{ij}$.

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Theorem 1. (1) *$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ is injective and Σ is closed in $\mathbb{P}^N$:*

$$\Sigma = Z\left(\left\{ z_{ij}z_{kl} - z_{il}z_{kj} \Big|\begin{matrix} i,k & = 0,\dots,n \\ j,l & = 0,\dots,m \end{matrix}\right\}\right). \tag{0.1}$$

(2) *$\sigma^{ij} : \mathbb{A}^{n+m} \to \Sigma^{ij}$ is an isomorphism.*

(3) *$\forall q \in \mathbb{P}^m$, the map*

$$\begin{aligned} \bar{i_q} : \mathbb{P}^n &\to \mathbb{P}^N \\ p &\to \sigma(p,q) \end{aligned}$$

is a morphism. Similarly, $j_p = \sigma(p,q) : \mathbb{P}^m \to \mathbb{P}^N$ is a morphism.

(4) *Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be quasi-projective varieties, then $\sigma(X \times Y) \subset \mathbb{P}^N$ is also a quasi-projective variety. What's more, if X and Y are both projective varieties, then $\sigma(X \times Y)$ is a projective variety.*

Proof. (1) If $\sigma([a_0,\dots,a_n],[b_0,\dots,b_m]) = \sigma([a'_0,\dots,a'_n],[b'_0,\dots,b'_m])$, then $\exists \lambda \in k\backslash\{0\}$, s.t. $\lambda a'_i b'_j = \lambda a_i b_j$ $\forall i,j$. Choose i_0, j_0 s.t. $a_{i_0}b_{j_0} \neq 0$, then $\forall i = 0,\dots,n$, $a_i b_{j_0} = \lambda a'_i b'_{j_0} \Rightarrow a_i = \left(\frac{\lambda b'_{j_0}}{b_{j_0}}\right) a'_i \Rightarrow [a_0,\dots,a_n] = [a_0{}',\dots,a_n{}']$. The same way can be used to prove $[b_0,\dots,b_m] = [b_0{}',\dots,b_m{}']$. Let W be the zero set on the right hand side of the equation 0.1, clearly we have the relation $\Sigma \subset W$. Now let $[a_{ij}] \in W$, choose i_0, j_0 s.t. $a_{i_0j_0} \neq 0$, then we get $[a_{ij}] = [a_{i_0j_0}a_{ij}] = [a_{i_0j}a_{ij_0}] = [a_{ij_0}a_{i_0j}] = \sigma([a_{0j_0},\dots,a_{nj_0}],[a_{i_00},\dots,a_{i_0m}]) \subset \Sigma$.

(2) Assume $i = j = 0$, then

$$\begin{aligned} \varphi_{00} \circ \sigma^{00}(a_1,\dots,a_n,b_1,\dots,b_m) &= \varphi_{00}(\sigma([1,a_1,\dots,a_n],[1,b_1,\dots,b_m])) \\ &= (z_{ij})_{(i,j)\neq(0,0)} \end{aligned}$$

where $z_{i0} = a_i$ for $i = 1,\dots,n$, $z_{0j} = b_j$ for $j = 1,\dots,m$, $z_{ij} = a_ib_j$ for $i,j \geq 1$. These are all regular functions, so $\varphi_{00} \circ \sigma^{00}$ is a morphism, so σ^{00} is a morphism. Finally, σ^{00} is an isomorphism because the inverse map is

$$(\sigma^{00})^{-1} = \left(\frac{z_{10}}{z_{00}},\dots,\frac{z_{n0}}{z_{00}},\frac{z_{01}}{z_{00}},\dots,\frac{z_{0m}}{z_{00}}\right).$$

Remark. In fact, Σ^{ij} is a quasi-projective variety. Because $\mathbb{A}^{n+m}$ is irreducible, Σ^{ij} is irreducible, hence a quasi-projective variety.

(3) Let $q = [b_0,\dots,b_m]$, then $i_q = [x_ib_j]$, x_ib_j's are homogeneous polynomials, so it is a morphism.

(4) Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be projective varieties. We can decompose the map into the following:

$$\begin{aligned} \sigma(X \times Y) &= \bigcup_{i,j} \sigma(X \times Y) \cap U_{ij} \\ &= \bigcup_{i,j} \sigma^{ij}(\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j)) \end{aligned}$$

$\varphi_i(X \cap U_i)$ and $\varphi_j(Y \cap U_j)$ are closed subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively, thus $\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j)$ is closed in $\mathbb{A}^{n+m}$. Since σ^{ij} is an isomorphism, then $\sigma^{ij}(\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j))$ is closed in $\Sigma^{ij} = \Sigma \cap U_{ij}$. So $\sigma(X \times Y)$ is closed in Σ, hence closed in $\mathbb{P}^N$ because Σ itself is closed. To show its irreducible, we use the lemma 1. Since σ is injective we can endow $\mathbb{P}^n \times \mathbb{P}^m$ with the topological structure of $\mathbb{P}^N$, hence we can identify $\mathbb{P}^n \times \mathbb{P}^m$ with Σ provided with the topology induced from $\mathbb{P}^N$. Now we can use the lemma 1, we have known i_q and j_p are continuous, so $\sigma(X \times Y)$ is irreducible. For quasi-projective conditions ,we just get the conclusion by simply difference two projective varieties. ☐

Remark. For $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ we can now identify $X \times Y$ with $\sigma(X \times Y) \subset \mathbb{P}^N$. In particular we can identify $\mathbb{P}^n \times \mathbb{P}^m$ with Σ.

From this perspective, part (2) of the theorem just says $U_i \times U_j \subset \mathbb{P}^n \times \mathbb{P}^m$ is open and $\varphi_i \times \varphi_j : U_i \times U_j \to \mathbb{A}^{n+m}$ is an isomorphism.

Proposition 1 (Universal Property)**.** *Let X, Y be quasi-projective varieties, then*

(1) *The projections*

$$\begin{aligned} p_1 &= (x_1, \ldots, x_n) : X \times Y \to X \\ p_2 &= (y_1, \ldots, y_m) : X \times Y \to Y \end{aligned}$$

are morphisms.

(2) *Let Z be a variety. The morphism $\varphi : Z \to X \times Y$ are precisely the*

$$(f, g) : Z \to X \times Y, \quad p \to (f(p), g(p)) \quad \forall p \in Z$$

where $f : Z \to X$ and $g : Z \to Y$ are morphisms. In other words, $\varphi : Z \to X \times Y$ is a morphism if and only if both $p_1 \circ \varphi$ and $p_2 \circ \varphi$ are morphisms.

Proof. (1) It is enough to show $p_1|_{U_i \times U_j}$ is a morphism from $U_i \times U_j$ to U_i. Identify $U_i \times u_j$ with $\mathbb{A}^{n+m}$ and U_i with $\mathbb{A}^n$, then we can see that p_1 is the same as the projection defined by the proposition **??**, so it is a morphism.

(2) $\Rightarrow$: Let $\varphi : Z \to X \times Y$ be a morphism. Then $f := p_1 \circ \varphi$ and $g := p_2 \circ \varphi$ are morphisms.

$\Leftarrow$: Let $f : Z \to X$ and $g : Z \to Y$ be morphisms. Define

$$Z^{ij} := f^{-1}(U_i) \cap g^{-1}(U_j).$$

Then (f, g) is a morphism $\Leftrightarrow$ $(f, g)|_{Z^{ij}}$ is a morphism for $i = 1, \ldots, n, j = 1, \ldots, m$. Consider the following mapping chain

$$Z^{ij} \xrightarrow{(f,g)} (X \times Y) \cap (U_i \times U_j) \xrightarrow{\varphi_i \times \varphi_j} \varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j) \subset \mathbb{A}^{n+m}.$$

the whole chain $(\varphi_i \circ f, \varphi_j \circ g) : Z^{ij} \to \mathbb{A}^{n+m}$ is a morphism, so (f, g) is a morphism. □

Corollary 1. *Let X_1, X_2, Y_1, Y_2 be varieties. If $f : X_1 \to Y_1$ and $X_2 \to Y_2$ are morphisms, then the map:*

$$\begin{aligned} f \times g : X_1 \times X_2 &\to Y_1 \times Y_2 \\ (p, q) &\to (f(p), g(q)) \end{aligned}$$

is a morphism. In particular, if X_1 is isomorphic to Y_1 and X_2 is isomorphic to Y_2, then $X_1 \times X_2$ is isomorphic to $Y_1 \times Y_2$

Proof. We can write $f \times g$ as $f \circ p_1$ and $g \circ p_2$, both $f \circ p_1$ and $g \circ p_2$ are morphisms, so $f \times g = (f \circ p_1, g \circ p_2)$ is a morphism. □

1. Conclusions We Need From Previous Lectures

In Lecture 09:

Lemma 1. *Let X, Y be irreducible topological spaces. Assume we have a topology on the product $X \times Y$ s.t.:*

$$\begin{aligned} y_p : \quad & Y \to X \times Y, \quad q \to (p, q) \text{ is continuous } \forall p \in X; \\ l_q : \quad & X \to X \times Y, \quad p \to (p, q) \text{is continuous } \forall q \in Y. \end{aligned}$$

Then $X \times Y$ is irreducible.

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