Category Theory

Based on notes by Tom Leinster

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April 1, 2021 — April 12, 2021

The learning notes are a collection of some notions and important theorems about category theory. I learned it from the note *Basic Category Theory* written by Tom Leinster. Most of the content is from this note, others are from the Stack Project and *The Rising Sea: Fundations of Algebraic Geometry*.

Contents

1 Basic notions

Definition 1.1. A *category* $\mathscr{A}$ consists of:

- a collection $\mathrm{ob}(\mathscr{A})$ of objects;
- for each $A, B \in \mathrm{ob}(\mathscr{A})$, a collection $\mathscr{A}(A, B)$ of *maps* or *arrows* or *morphisms* from A to B ;
- for each $A, B, C \in \mathrm{ob}(\mathscr{A})$, a function

  $$\begin{aligned} \mathscr{A}(B, C) \times \mathscr{A}(A, B) &\to \mathscr{A}(A, C) \\ (g, f) &\mapsto g \circ f, \end{aligned}$$

  called *composition*;
- for each $A \in \mathrm{ob}(\mathscr{A})$, an element 1_A of $\mathscr{A}(A, A)$, called the *identity* on A

satisfying the following axioms:

- **associativity**: for each $f \in \mathscr{A}(A, B)$, $g \in \mathscr{A}(B, C)$ and $h \in \mathscr{A}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$;
- **identity laws**: for each $f \in \mathscr{A}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Remark. We often writ $A \in \mathscr{A}$ to mean $A \in \mathrm{ob}(\mathscr{A})$, $f : A \to B$ or $A \xrightarrow{f} B$ to mean $f \in \mathscr{A}(A, B)$, and gf to mean $g \circ f$.

Definition 1.2. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A *functor* $F : \mathscr{A} \to \mathscr{B}$ consists of:

- a function

  $$\mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{B}),$$

  written as $A \mapsto F(A)$;
- for each $A, A' \in \mathscr{A}$, a function

  $$\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A')),$$

  written as $f \mapsto F(f)$,

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$;
- $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$.

Here are two ways of constructing new categories from old.

Definition 1.3. Every category $\mathscr{A}$ has an *opposite* or *dual* category $\mathscr{A}^{\mathrm{op}}$, defined by reversing tha arrows. Formally, $\mathrm{ob}(\mathscr{A}^{\mathrm{op}}) = \mathrm{ob}(\mathscr{A})$ and $\mathscr{A}^{\mathrm{op}}(B, A) = \mathscr{A}(A, B)$ for all objects A and B.

Definition 1.4. Given categories $\mathscr{A}$ and $\mathscr{B}$, A *product category* $\mathscr{A} \times \mathscr{B}$ is a category in which

$$\begin{gathered} \mathrm{ob}(\mathscr{A} \times \mathscr{B}) = \mathrm{ob}(\mathscr{A}) \times \mathrm{ob}\mathscr{B}, \\ (\mathscr{A} \times \mathscr{B})\left((A, B), (A', B')\right) = \mathscr{A}(A, A') \times \mathscr{B}(B, B').. \end{gathered}$$

Since we have the notion of dual category, we also have the notion of dual functor, which is formally called contravariant functor.

Definition 1.5. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A *contravariant functor* from $\mathscr{A}$ to $\mathscr{B}$ is a functor $\mathscr{A}^{\mathrm{op}} \to \mathscr{B}$.

Definition 1.6. A functor $F : \mathscr{A} \to \mathscr{B}$ is *faithful* (respectively, *full*) if for each $A, A' \in \mathscr{A}$, the function

$$\begin{aligned} \mathscr{A}(A, A') &\longrightarrow \mathscr{B}(F(A), F(A')) \\ f &\longmapsto F(f). \end{aligned}$$

is injective (respectively, surjective).

2 Natural transformations

Definition 2.1. Let $\mathscr{A}$ and $\mathscr{B}$ be categories and let $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ be functors. A *natural transformation* $\alpha : F \to G$ is a family $\Big(F(A) \xrightarrow{\alpha_A} G(A)\Big)_{A \in \mathscr{A}}$ of morphisms in $\mathscr{B}$ for every map $A \xrightarrow{f} A'$ in $\mathscr{A}$, the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd} \tag{1}$$

commutes. The morphisms α_A are called the components of α. We also write

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathscr{B} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

to mean that α is a natural transformation from F to G.

Given natural transformations

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=70, "H"'{name=D}] & \mathscr{B} \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] \end{tikzcd}$$

There is a composite natural transformation

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "H"'{name=D}] & \mathscr{B} \arrow[Rightarrow, from=U, to=D, "\beta\circ\alpha"] \end{tikzcd}$$

defined by $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ for all $A \in \mathscr{A}$. It is sometimes called *vertical composition.*

There is also an identity natural transformation

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F"'{name=D}] & \mathscr{B} \arrow[Rightarrow, from=U, to=D, "1_F"] \end{tikzcd}$$

on any functor F, defined by $(1_F)_A = 1_{F(A)}$.

Definition 2.2. For any two categories $\mathscr{A}$ and $\mathscr{B}$, there is a category whose objects are the functors between $\mathscr{A}$ and $\mathscr{B}$ and whose morphisms are the natural transformation between thenm. The composition law and identity morphism are defined and shown above. This is called the *functor category* from $\mathscr{A}$ to $\mathscr{B}$ and written as $[\mathscr{A}, \mathscr{B}]$.

Definition 2.3. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A *natural isomorphism* between functors from $\mathscr{A}$ to $\mathscr{B}$ is an isomorphism in $[\mathscr{A}, \mathscr{B}]$. In other words, let α be a natural transformation from F to G where F and G are functors from $\mathscr{A}$ to $\mathscr{B}$, then α is a natural isomorphism if and only if $\alpha_A : F(A) \to G(A)$ is an isomorphism for all $A \in \mathscr{A}$.

Since natural isomorphism is just isomorphism in a particular category $[\mathscr{A}, \mathscr{B}]$, we already have notation for this:

$$F \cong G.$$

Definition 2.4. Let F, G be two functors from $\mathscr{A}$ to $\mathscr{B}$, we say that

$$F(A) \cong G(A) \text{ naturally in } A$$

if F and G are naturally isomorphic.

Definition 2.5. An *equivalence* between categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair of functiors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ such that

$$G \circ F \cong 1_{\mathscr{A}} \text{ and } F \circ G \cong 1_{\mathscr{B}}. \tag{2}$$

We say that $\mathscr{A}$ and $\mathscr{B}$ are equivalent if there is an equivalence between them and write $\mathscr{A} \simeq \mathscr{B}$. The functors F and G are equivalences.

Remark. Consider the category of all finite sets **FinSet** (and mappings between those). That's a huge category. However in a sense it should not be so huge, since essentially there only as many finite sets as they are natural numbers. Consider annother category $\mathscr{A}$, which is only the finite sets of the form $\{1, \cdots, n\}$. Now for every $n \in \mathbb{N}$, $\mathscr{A}$ has one set-representative of that size while **FinSet** has many, but in **FinSet** all these sets of the same size are isomorphic and we should not treat isomorphic sets as being different.

Hence it doesn't make any real difference if we use **FinSet** or $\mathscr{A}$ to deal with finite sets. So they ought to be the same. And they are equivalent but not isomorphic.[1]

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[1] This remark is from here.

Definition 2.6. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor, we say F is *essentially surjective on objects* if for all $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $F(A) \cong B$.

Proposition 2.7. *A functor $F : \mathscr{A} \to \mathscr{B}$ is an equivalence if and only if it is full, faithfull and essentially surjective on objects.*

Proof. First assume two natural isomorphisms

$$\eta : G \circ F \to 1_{\mathscr{A}}, \quad \varepsilon : F \circ G \to 1_{\mathscr{B}}.$$

Let $f, f' : A \to A'$ and $F(f) = F(f') : F(A) \to F(A')$, then $G \circ F(f) = G(F(f)) = G(F(f')) = G \circ F(f') : G(F(A)) \to G\left(F(A')\right)$. Then $\eta \circ (G \circ F(f)) = \eta \circ (G \circ F(f')) \Rightarrow f = f'$. Hence F is faithuful. Let $g \in \operatorname{Mor}\left(F(A), F(A')\right)$, then $g = (F \circ G) \circ (\varepsilon(g))$. Then there exists $f = G \circ \varepsilon(g)$ s.t. $F(f) = g$, hence F is full. Given any $B \in \mathscr{B}$, let $A = G(B)$, then $F(A) = F \circ G(B) \cong B$. The converse is to construct natural isomorphisms η and ε by reversing the deduction above. ☐

Recall vertical composition introduced previously, there is also *horizontal composition*, which takes natural transformations

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "F", ""{name=U1, below}] \arrow[r, bend right=50, "G"', ""{name=D1, above}] & \mathscr{A}' \arrow[r, bend left=50, "F'", ""{name=U2, below}] \arrow[r, bend right=50, "G'"', ""{name=D2, above}] & \mathscr{A}'' \arrow[Rightarrow, from=U1, to=D1] \arrow[Rightarrow, from=U2, to=D2]
\end{tikzcd}$$

and produces a natural transformation

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "F' \circ F", ""{name=U, below}] \arrow[r, bend right=50, "G' \circ G"', ""{name=D, above}] & \mathscr{A}'. \arrow[Rightarrow, from=U, to=D, "\alpha' * \alpha"]
\end{tikzcd}$$

The component of $\alpha' * \alpha$ at $A \in \mathscr{A}$ is defined to be the diagonal of the naturality square

$$\begin{tikzcd}
F'\left(F(A)\right) \arrow[r, "F'(\alpha_A)"] \arrow[d, "\alpha'_{F(A)}"'] & F'\left(G(A)\right) \arrow[d, "\alpha'_{G(A)}"] \\
G'\left(F(A)\right) \arrow[r, "G'(\alpha_A)"'] & G'\left(G(A)\right).
\end{tikzcd}$$

That is

$$\left(\alpha' * \alpha\right)_A = \alpha'_{G(A)} \circ F'(\alpha_A) = G'\left(\alpha_A\right) \circ \alpha'_{F(A)}.$$

3 Adjoints

Definition 3.1. Let $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ be categories and functors. We say that F is a *left adjoint* of G, or that G is a *right adjoint* of F if there are bijections

$$\mathscr{B}(F(A), B) \to \mathscr{A}(A, G(B)) \tag{3}$$

functorial in $A \in \mathscr{A}$ and $B \in \mathscr{B}$. In other words, this means that there is a given isomorphism of functors $\mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{Set}$ from $\mathscr{B}(F(-), -)$ to $\mathscr{A}(-, G(-))$.

Remark. Here are two understandings:

a. Given (A, B) and $(A', B') \in \operatorname{ob}(\mathscr{A}^{\mathrm{op}} \times \mathscr{B})$, $f : A' \to A, g : B \to B'$, then we have

$$\begin{tikzcd}
\mathscr{B}(F(A), B) \arrow[r] \arrow[d, "\alpha_{(A,B)}"'] & \mathscr{B}(F(A'), B') \arrow[d, "\alpha_{(A',B')}"] \\
\mathscr{A}(A, G(B)) \arrow[r] & \mathscr{A}(A', G(B')).
\end{tikzcd}$$

Let $B = F(A)$, we obtain

$$\begin{tikzcd}
\mathscr{B}(F(A), F(A)) \arrow[r] \arrow[d, "\alpha_{(A,F(A))}"'] & \mathscr{B}(F(A'), B') \arrow[d, "\alpha_{(A',B')}"] \\
\mathscr{A}(A, G(F(A))) \arrow[r] & \mathscr{A}(A', G(B')).
\end{tikzcd}$$

Hence for any object A of $\mathscr{A}$ we obtain a morphism $\eta_A : A \to G(F(A))$ corresponding to $1_{F(A)}$. Similarly, for any object B of $\mathscr{B}$ we obtain a morphism $\epsilon_B : F(G(B)) \to B$ corresponding to $1_{G(B)}$. These maps are called *adjunction maps*. The adjunction maps are functorial in A and B, hence we obtain morphisms of functors

$$\eta : 1_{\mathscr{A}} \to G \circ F \quad \text{(unit)} \quad \text{and} \quad \epsilon : F \circ G \to 1_{\mathscr{B}} \quad \text{(counit)}.$$

b. Given objects $A \in \mathscr{A}$ and $B \in \mathscr{B}$, the correspondence between $F(A) \to B$ and $A \to G(B)$ is denoted by a horizontal bar, in both directions:

$$\left(F(A) \xrightarrow{g} B\right) \mapsto \left(A \xrightarrow{\bar{g}} G(B)\right),$$
$$\left(A \xrightarrow{f} G(B)\right) \mapsto \left(F(A) \xrightarrow{\bar{f}} B\right).$$

So $\bar{\bar{f}} = f$ and $\bar{\bar{g}} = g$. We call $\bar{f}$ the *transpose* of f, and similarly for g. Then "functorial in $A \in \mathscr{A}$ and $B \in \mathscr{B}$ is equivalent to

$$\overline{\left(F(A) \xrightarrow{g} B \xrightarrow{q} B'\right)} = \left(A \xrightarrow{\bar{g}} G(B) \xrightarrow{G(q)} G(B')\right) \tag{4}$$

and

$$\overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B)\right)} = \left(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\bar{f}} B\right). \tag{5}$$

The above two identities can also be written as

$$\overline{q \circ g} = G(q) \circ \bar{g} \tag{6}$$

and

$$\overline{f \circ p} = \bar{f} \circ F(p). \tag{7}$$

In fact, the bijection that satisfies the above two conditions are equivalent to the definition of adjoint functors.

Lemma 3.2. *Given an adjunction $F \dashv G$ with unit η and ϵ, the triangles*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

commute. These are called triangle identities. They are commmutative diagrams in the functor categories $[\mathscr{A}, \mathscr{B}]$ and $[\mathscr{B}, \mathscr{A}]$, respectively.

Proof. An equivalent statement is that the triangles

$$\begin{tikzcd} F(A) \arrow[r, "F(\eta_A)"] \arrow[dr, "1_{F(A)}"'] & FGF(A) \arrow[d, "\epsilon_{F(A)}"] \\ & F(A) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G(B) \arrow[r, "\eta_{G(B)}"] \arrow[dr, "1_{G(B)}"'] & GFG(B) \arrow[d, "G(\epsilon_B)"] \\ & G(B) \end{tikzcd}$$

commute for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Since $\overline{1_{GF(A)}} = \epsilon_{F(A)}$, equation (5) gives

$$\overline{\left(A \xrightarrow{\eta_A} GF(A) \xrightarrow{1} GF(A)\right)} = \left(F(A) \xrightarrow{F(\eta_A)} FGF(A) \xrightarrow{\epsilon_{F(A)}} F(A)\right).$$

But the left-hand side is $\overline{\eta_A} = \overline{\overline{1_{F(A)}}} = 1_{F(A)}$, proving the first triangle identity. The second follows by duality. □

Lemma 3.3. *Let* $\begin{tikzcd}\mathscr{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, shift left, "G"]\end{tikzcd}$ *be an adjunction, with unit* η *and counit* ϵ. *Then*

$$\overline{g} = G(g) \circ \eta_A$$

for any $g : F(A) \to B$, *and*

$$\overline{f} = \epsilon_B \circ F(f)$$

for any $f : A \to G(B)$.

Theorem 3.4. *Take categories and functors* $\begin{tikzcd}\mathscr{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, shift left, "G"]\end{tikzcd}$. *There is a one-to-one correspondence between:*

(a) *adjunctions between* F *and* G *(with* F *on the left and* G *on the right);*

(b) *pairs* $\left(1_{\mathscr{A}} \xrightarrow{\eta} GF, FG \xrightarrow{\epsilon} 1_{\mathscr{B}}\right)$ *of natural transformations satisfying the triangle identities.*

(a) ⇒ (b) direction has been proved and the proof of converse is a direct calculate.

Definition 3.5. Given categories and functors

$$\begin{tikzcd} & \mathscr{B} \arrow[d, "Q"] \\ \mathscr{A} \arrow[r, "P"'] & \mathscr{C}, \end{tikzcd}$$

the *comma category* $(P \Rightarrow Q)$ (often written as $(P \downarrow Q)$) is the category defined as follows:

- objects are triples (A, h, B) with $A \in \mathscr{A}$, $B \in \mathscr{B}$, and $h : P(A) \to Q(B)$ in $\mathscr{C}$;
- maps $(A, h, B) \to (A', h', B')$ are pairs $(f : A \to A', g : B \to B')$ of maps such that the square

$$\begin{tikzcd} P(A) \arrow[r, "P(f)"] \arrow[d, "h"'] & P(A') \arrow[d, "h'"] \\ Q(B) \arrow[r, "Q(g)"'] & Q(B') \end{tikzcd}$$

commutes.

Remark. Given $\mathscr{A}, \mathscr{B}, \mathscr{C}$, P and Q as above, there are canonical functors and a canonical natural transformations as shown:

$$\begin{tikzcd}
(P \Rightarrow Q) \arrow[r] \arrow[d] & \mathscr{B} \arrow[d, "Q"] \\
\mathscr{A} \arrow[r, "P"'] \arrow[ur, Rightarrow] & \mathscr{C}
\end{tikzcd}$$

in a suitable 2-categorical sense, $(P \Rightarrow Q)$ is universal with this property.

Definition 3.6. Let $\mathscr{A}$ be a category and $A \in \mathscr{A}$. The *slice category* of $\mathscr{A}$ over A, denoted by ${}^{\mathscr{A}}\!/_{A}$, is the category whose objects are maps into A and whose maps are commutative triangles. More precisely, an object is a pair (X, h) with $X \in \mathscr{A}$ and $h : X \to A$ in $\mathscr{A}$, and a map $(X, h) \to (X', h')$ in ${}^{\mathscr{A}}\!/_{A}$ is a map $f : X \to X'$ in $\mathscr{A}$ making the triangle

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"'] & & X' \arrow[dl, "h'"] \\
& A &
\end{tikzcd}$$

commute.

Slice categories are a special case of comma categories $(1_{\mathscr{A}} \Rightarrow A)$:

$$\begin{tikzcd}
& \mathbf{1} \arrow[d, "A"] \\
\mathscr{A} \arrow[r, "1_{\mathscr{A}}"'] & \mathscr{A}.
\end{tikzcd}$$

Dually, there is a *coslice category* ${}^{A}\!/_{\mathscr{A}} \cong (A \Rightarrow 1_{\mathscr{A}})$, whose objects are the maps out of A.

Lemma 3.7. *Take an adjunction* $\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] \arrow[r, shift right, "G"'] & \mathscr{B} \end{tikzcd}$ *and an object* $A \in \mathscr{A}$*. Then the unit map* $\eta_A : A \to GF(A)$ *is an initial object of* $(A \Rightarrow G)$*.*

Theorem 3.8. *Take categories and functors* $\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "F"] & \mathscr{B} \arrow[l, shift left, "G"] \end{tikzcd}$. *There is a one-to-one correspondence between:*

(a) *adjunctions between F and G (with F on the left and G on the right);*

(b) *natural transformations* $\eta : 1_{\mathscr{A}} \to GF$ *such that* $\eta_A : A \to GF(A)$ *is initial in* $(A \Rightarrow G)$ *for every* $A \in \mathscr{A}$.

Corollary 3.9. *Let* $G : \mathscr{B} \to \mathscr{A}$ *be a functor. Then* G *has a left adjoint if and only if for each* $A \in \mathscr{A}$, *the category* $(A \Rightarrow G)$ *has an intial object.*

4 Yoneda lemma

Definition 4.1. Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. We define a functor

$$H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$$

as follows:

- for objects $B \in \mathscr{A}$, put $H^A(B) = \mathscr{A}(A, B)$;
- for maps $B \xrightarrow{g} B'$ in $\mathscr{A}$, define

  $$H^A(g) = \mathscr{A}(A, g) = g_* = g \circ - : \mathscr{A}(A, B) \to \mathscr{A}(A, B')$$

  by

  $$p \mapsto g \circ p$$

  for all $p : A \to B$.

Definition 4.2. Let $\mathscr{A}$ be a locally small category. A functor $X : \mathscr{A} \to \mathbf{Set}$ is *representable* if $X \cong H^A$ for some $A \in \mathscr{A}$. A *representation* of X is a choice of an object $A \in \mathscr{A}$ and an isomorphism between H^A and X.

Any map $A' \xrightarrow{f} A$ induces a natural transformation

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "H^A"{name=U}] \arrow[r, bend right=50, "H^{A'}"'{name=D}] & \mathbf{Set} \arrow[Rightarrow, from=U, to=D, "H^f"]
\end{tikzcd}$$

(also called $\mathscr{A}(f, -)$, f^* or $- \circ f$), whose B-component (for $B \in \mathscr{A}$) is the function

$$H^A(B) = \mathscr{A}(A, B) \to H^{A'}(B) = \mathscr{A}(A', B)$$
$$p \mapsto p \circ f.$$

Each H^A is covariant, but they come together to form a contravariant functor, as in the following definition.

Definition 4.3. Let $\mathscr{A}$ be a locally small category. The functor

$$H^\bullet : \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathbf{Set}]$$

is defined on objects A by $H^\bullet(A) = H^A$ and on maps f by $H^\bullet(f) = H^f$.

Definition 4.4. Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. We define a functor

$$H_A = \mathscr{A}(-, A) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$$

as follows:

- for objects $B \in \mathscr{A}$, put $H_A(B) = \mathscr{A}(B, A)$;
- for maps $B' \xrightarrow{g} B$ in $\mathscr{A}$, define

  $$H_A(g) = \mathscr{A}(g, A) = g^* = - \circ g : \mathscr{A}(B, A) \to \mathscr{A}(B', A)$$

  by

  $$p \mapsto p \circ g$$

  for all $p : B \to A$

Definition 4.5. Let $\mathscr{A}$ be a locally small category. A functor $X : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ is *representable* if $X \cong H_A$ for some $A \in \mathscr{A}$. A *representation* of X is a choice of an object $A \in \mathscr{A}$ and an isomorphism between H_A and X.

Any map $A \xrightarrow{f} A'$ in $\mathscr{A}$ induces a natural transformation

$$\begin{tikzcd}
\mathscr{A}^{\mathrm{op}} \arrow[r, bend left=50, "H_A", ""{name=U, below}] \arrow[r, bend right=50, "H_{A'}"', ""{name=D}] & \mathscr{A}' \arrow[Rightarrow, from=U, to=D, "H_f"]
\end{tikzcd}$$

(also called $\mathscr{A}(-, f)$, f_* or $f \circ p$), whose B-component (for $B \in \mathscr{A}$) is the function

$$\begin{gathered} H_A(B) = \mathscr{A}(B, A) \to H_{A'}(B) = \mathscr{A}(B, A') \\ p \mapsto f \circ p. \end{gathered}$$

Definition 4.6. Let $\mathscr{A}$ be a locally small category. The *Yoneda embedding* of $\mathscr{A}$ is the functor

$$H_\bullet : \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$$

defined on objects A by $H_\bullet(A) = H_A$ and on maps f by $H_\bullet(f) = H_f$.

Proposition 4.7. *Let $\mathscr{A}$ be a locally small category, and let $A, A' \in \mathscr{A}$ with $H_A \cong H_{A'}$. Then we have $A \cong A'$.*

Proof. By definition, we have a natural isomorphism α such that for any $B, B' \in \mathscr{A}$ and any map $f : B' \to B$, the square

$$\begin{tikzcd}
\mathscr{A}(B,A) \arrow[r, "H_A(f)"] \arrow[d, "\alpha_B"'] & \mathscr{A}(B',A) \arrow[d, "\alpha_{B'}"] \\
\mathscr{A}(B,A') \arrow[r, "H_{A'}(f)"'] & \mathscr{A}(B',A')
\end{tikzcd}$$

commutes. Let $B = A$ and $B' = A'$, then for any $g \in \mathscr{A}(A,A)$ and $f : A' \to A$, we have

$$\alpha_A(g) \circ f = \alpha_{A'}(g \circ f)$$

for any $g : A \to A$. Then $\alpha_A(g) \circ f \in \mathscr{A}(A', A')$. Let $f = \alpha_{A'}^{-1}(1_{A'})$ and $g = 1_A$, we obtain

$$\alpha_A(1_A) \circ \alpha_{A'}^{-1}(1_{A'}) = 1_{A'}.$$

Similarly, we have

$$\alpha_{A'}(1_{A'}) \circ \alpha_A^{-1}(1_A) = 1_A.$$

□

Definition 4.8. Let $\mathscr{A}$ be a locally small category. The functor

$$\mathrm{Hom}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathbf{Set}$$

is defined by

$$\begin{tikzcd}
(A,B) \arrow[dd, shift left, "g"] & \mapsto & \mathscr{A}(A,B) \arrow[dd, "g\circ - \circ f"] \\
 & \mapsto & \\
(A',B') \arrow[uu, shift left, "f"] & \mapsto & \mathscr{A}(A',B').
\end{tikzcd}$$

In other words, $\mathrm{Hom}_{\mathscr{A}}(A,B) = \mathscr{A}(A,B)$ and $\left(\mathrm{Hom}_{\mathscr{A}}(f,g)\right)(p) = g \circ p \circ f$, whenever $A' \xrightarrow{f} A \xrightarrow{p} B \xrightarrow{g} B'$.

Remark. (a) Given sets A and B, we have product $A \times B$ and the set B^A (or $\mathbf{Set}(A,B)$) of functions from A to B. Fix a set B, taking the product with B defines a functor

$$\begin{aligned} - \times B : \mathbf{Set} &\longrightarrow \mathbf{Set} \\ A &\longmapsto A \times B. \end{aligned}$$

There is also a functor

$$\begin{aligned}(-)^B : \mathbf{Set} &\longrightarrow \mathbf{Set}\\ C &\longmapsto C^B.\end{aligned}$$

Moreover, there is an adjunction between them, i.e.,

$$\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, C^B)$$

for any sets A and C.

(b) Similarly, for any category $\mathscr{B}$, there is an adjunction $(- \times \mathscr{B}) \dashv [\mathscr{B}, -]$ of functors $\mathbf{CAT} \to \mathbf{CAT}$, that is, there is a canonical bijection

$$\mathbf{CAT}\,(\mathscr{A} \times \mathscr{B}, \mathscr{C}) \cong \mathbf{CAT}\,(\mathscr{A}, [\mathscr{B}, \mathscr{C}])$$

for $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbf{CAT}$. Under this bijection, the functors

$$\mathrm{Hom}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathbf{Set} \quad \text{and} \quad H^{\bullet} : \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathbf{Set}]$$

correspond to one another and carries the same information.

(c) Now we can use this bijection to explain naturality in the definition of adjunction. Take categories $\begin{tikzcd}\mathscr{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, shift left, "G"]\end{tikzcd}$, then we have

$$\begin{tikzcd}
\mathscr{A}^{\mathrm{op}} \times \mathscr{B} \arrow[r, "1 \times G"] \arrow[d, "F^{\mathrm{op}} \times 1"'] & \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \arrow[d, "\mathrm{Hom}_{\mathscr{A}}"] \\
\mathscr{B}^{\mathrm{op}} \times \mathscr{B} \arrow[r, "\mathrm{Hom}_{\mathscr{B}}"'] & \mathbf{Set}.
\end{tikzcd}$$

Definition 4.9. Let A be an object of a category. A *generalized element* of A is a map with codomain A. A map $S \to A$ is a generalized element of A of *shape* S.

Definition 4.10. Let $\mathscr{A}$ be a category. A *presheaf* on $\mathscr{A}$ is a functor $\mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$.

For each $A \in \mathscr{A}$ we have a representable presheaf H_A, and we are asking how the rest of the presheaf category $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ looks from the viewpoint of H_A. In other words, if X is another presheaf, what are the maps $H_A \to X$?

Theorem 4.11 (Yoneda). *Let $\mathscr{A}$ be a locally small category. Then*

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]\,(H_A, X) \cong X(A) \tag{8}$$

naturally in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$.

5 Limits

Definition 5.1. Let $\mathscr{A}$ be a category and $X, Y \in \mathscr{A}$. A *product* of X and Y consists of an object P and maps

$$\begin{tikzcd}
& P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X & & Y
\end{tikzcd}$$

with the property that for all objects and maps

$$\begin{tikzcd}
& A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
X & & Y
\end{tikzcd} \tag{9}$$

in $\mathscr{A}$, there exists a unique map $\bar{f} : A \to P$ such that

$$\begin{tikzcd}
& A \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] \arrow[d, dotted, "\bar{f}"] & \\
& P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X & & Y
\end{tikzcd} \tag{10}$$

commutes. The maps p_1 and p_2 are called *projections*.

Definition 5.2. Let $\mathscr{A}$ be a category, I a set, and $(X_i)_{i \in I}$ a family of objects of $\mathscr{A}$. A *product* of $(X_i)_{i \in I}$ consists of an object P and a family of maps

$$\left(P \xrightarrow{p_i} X_i \right)_{i \in I}$$

with the property that for all objects A and families of maps

$$\left(A \xrightarrow{f_i} X_i \right)_{i \in I} \tag{11}$$

there exists a unique map $\bar{f} : A \to P$ such that $p_i \circ \bar{f} = f_i$ for all $i \in I$.

Definition 5.3. A *fork* in a category consists of objects and maps

$$A \xrightarrow{f} X \underset{t}{\overset{s}{\rightrightarrows}} Y \tag{12}$$

such that $sf = tf$. Let $\mathscr{A}$ be a category and let $X \underset{t}{\overset{s}{\rightrightarrows}} Y$ be objects and maps in $\mathscr{A}$. An *equalizer* of s and t is an object E together with a map $E \xrightarrow{i} X$ such that

$$E \xrightarrow{i} X \underset{t}{\overset{s}{\rightrightarrows}} Y$$

is a fork, and with the property that for any fork (12), there exists a unique map $\bar{f} : A \to E$ such that

$$\begin{tikzcd} A \arrow[d, dotted, "\bar{f}"'] \arrow[dr, "f"] & \\ E \arrow[r, "i"'] & X \end{tikzcd} \tag{13}$$

commutes.

Definition 5.4. Let $\mathscr{A}$ be a category, and take objects and maps

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd} \tag{14}$$

in $\mathscr{A}$. A *pullback* of this diagram is an object $P \in \mathscr{A}$ together with maps $p_1 : P \to X$ and $p_2 : P \to Y$ such that

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd} \tag{15}$$

commutes, and with the property that for any commutative squar

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd} \tag{16}$$

in $\mathscr{A}$, there is a unique map $\bar{f} : A \to P$ such that

$$\begin{tikzcd}
A \arrow[rrd, bend left, "f_2"] \arrow[rdd, bend right, "f_1"'] \arrow[rd, dotted, "\bar{f}"] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\
& X \arrow[r, "s"'] & Z
\end{tikzcd} \tag{17}$$

commutes.

Definition 5.5. Let $\mathscr{A}$ be a category and $\mathbf{I}$ a small category. A functor $\mathbf{I} \to \mathscr{A}$ is called a *diagram* in $\mathscr{A}$ of *shape* $\mathbf{I}$.

Definition 5.6. Let $\mathscr{A}$ be a category, $\mathbf{I}$ a small category, and $D : \mathbf{I} \to \mathscr{A}$ a diagram in $\mathscr{A}$.

(a) A *cone* on D is an object $A \in \mathscr{A}$ (the vertex of the cone) together with a family

$$\left(A \xrightarrow{f_I} D(I)\right)_{I \in \mathbf{I}} \tag{18}$$

of maps in $\mathscr{A}$ such that for all maps $I \xrightarrow{u} J$ in $\mathbf{I}$, the triangle

$$\begin{tikzcd}
& D(I) \arrow[dd, "Du"] \\
A \arrow[ru, "f_I"] \arrow[rd, "f_J"] & \\
& D(J)
\end{tikzcd}$$

commutes.

(b) A *limit* of D is a cone $\left(L \xrightarrow{p_I} D(I)\right)_{I \in \mathbf{I}}$ with the property that for any cone (18) on D, there exists a unique map $\bar{f} : A \to L$ such that $p_I \circ \bar{f} = f_I$ for all $I \in \mathbf{I}$. The maps p_I are called the *projections* of the limit. We write $L = \lim\limits_{\leftarrow I} D$.

Example 5.1. Let categories $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{P}$ be

$$\mathbf{T} = \bullet \quad \bullet, \quad \mathbf{E} = \begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}, \quad \mathbf{P} = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \tag{19}$$

Let $\mathscr{A}$ be any category.

(a) A diagram D of shape $\mathbf{T}$ in $\mathscr{A}$ is a pair (X, Y) of objects of $\mathscr{A}$. A cone on D is an object A together with maps $f_1 : A \to X$ and $f_2 : A \to Y$, and a limit of D is a product of X and Y.

More generally, let I be a set and write $\mathbf{I}$ for the discrete category on I. A functor $D : \mathbf{I} \to \mathscr{A}$ is an I-indexed family $(X_i)_{i\in I}$ of objects of $\mathscr{A}$, and a limit of D is exactly a product of the family $(X_i)_{i\in I}$.

(b) A diagram D of shape $\mathbf{E}$ in $\mathscr{A}$ is a parallel pair $\begin{tikzcd} X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \end{tikzcd}$ of maps in $\mathscr{A}$. A cone on D consists of objects and maps

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X \arrow[rr, shift left, "s"] \arrow[rr, shift right, "t"'] & & Y
\end{tikzcd}$$

such that $s \circ f = g$ and $t \circ f = g$. But since g is determined by f, it is equivalent to say that a cone on D consists of an object A and a map $f : A \to X$ such that

$$\begin{tikzcd}
A \arrow[r, "f"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y
\end{tikzcd}$$

is a fork. A limit of D is a universal fork on s and t, that is, an equalizer of s and t.

(c) A diagram D of shape $\mathbf{P}$ in $\mathscr{A}$ consists of objects and maps

$$\begin{tikzcd}
 & Y \arrow[d, "t"] \\
X \arrow[r, "s"] & Z
\end{tikzcd}$$

in $\mathscr{A}$. Performing a simplification similar to that in (b), we see that a cone on D is a commutative square (16).

(d) Let $\mathbf{I} = (\mathbb{N}, \leq)^{\mathrm{op}}$. A diagram $D : \mathbf{I} \to \mathscr{A}$ consists of objects and maps

$$\cdots \xrightarrow{s_3} X_2 \xrightarrow{s_2} X_1 \xrightarrow{s_1} X_0.$$

For example, suppose that we have a set X_0 and a chain of subsets

$$\cdots \subset X_2 \subset X_1 \subset X_0.$$

The inclusion maps form a diagram in **Set** of the type above, and its limit is $\bigcap_{i\in\mathbb{N}} X_i$.

Definition 5.7. (a) Let $\mathbf{I}$ be a small category. A category $\mathscr{A}$ *has limits of shape* $\mathbf{I}$ if for every diagram D of shape $\mathbf{I}$ in $\mathscr{A}$, a limit of D exists.

(b) A category *has all limits* (or properly, *has small limits*) if it has limits of shape $\mathbf{I}$ for all small categories.

Definition 5.8. A category is *finite* if it contains only finitely many maps (in which case it also contains only finitely many objects). A *finite limit* is a limit of shape $\mathbf{I}$ for some finite category $\mathbf{I}$.

Proposition 5.9. *Let $\mathscr{A}$ be a category.*

(a) *If $\mathscr{A}$ has all products and equalizers then $\mathscr{A}$ has all limits.*

(b) *If $\mathscr{A}$ has binary products, a terminal object and equalizers then $\mathscr{A}$ has finite limits.*

Definition 5.10. Let $\mathscr{A}$ be a category. A map $X \xrightarrow{f} Y$ in $\mathscr{A}$ is *monic* (or a *monomorphism*) if for all objects A and maps $A \underset{x'}{\overset{x}{\rightrightarrows}} X$,

$$f \circ x = f \circ x' \quad \implies \quad x = x'.$$

Definition 5.11. Let $\mathscr{A}$ be a category and $\mathbf{I}$ a small category. Let $D : \mathbf{I} \to \mathscr{A}$ be a diagram in $\mathscr{A}$, and write D^{op} for the corresponding functor $\mathbf{I}^{\mathrm{op}} \to \mathscr{A}^{\mathrm{op}}$. A *cocone* on D is a cone on D^{op}, and a *colimit* of D is a limit of D^{op}.

Explicitly, a cocone on D is an object $A \in \mathscr{A}$ together with a family

$$\left(D(I) \xrightarrow{f_I} A\right)_{I \in \mathbf{I}} \tag{20}$$

of maps in $\mathscr{A}$ such that for all maps $I \xrightarrow{u} J$ in $\mathbf{I}$, the diagram

$$\begin{tikzcd}
D(I) \arrow[dd, "Du"'] \arrow[dr, "f_I"] & \\
& A \\
D(J) \arrow[ur, "f_J"'] &
\end{tikzcd}$$

commutes. A colimit of D is a cocone

$$\left(D(I) \xrightarrow{p_I} C\right)_{I \in \mathbf{I}}$$

with the property that for any cocone (20) on D, there is a unique map $\bar{f} : C \to A$ such that $\bar{f} \circ p_I = f_I$ for all $I \in \mathbf{I}$.

Definition 5.12. A *sum* or *coproduct* is a colimit over a discrete category.

Let $(X_i)_{i\in I}$ be a family of objects of a category. Their sum (if it exists) is written as $\sum_{i\in I} X_i$ or $\coprod_{i\in I} X_i$.

Definition 5.13. A *coeualizer* is a colimit of shape $\mathbf{E}$.

In other words, given a diagram $\begin{tikzcd} X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \end{tikzcd}$, a coequalizer of s and t is a map $Y \xrightarrow{p} C$ satisfying $p \circ s = p \circ t$ and universal with this property.

Definition 5.14. A *pushout* is a colimit of shape

$$\mathbf{P}^{\mathrm{op}} = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} .$$

In other words, the pushout of a diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \\ Z & \end{tikzcd} \tag{21}$$

is (if it exists) a commutative square

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d] \\ Z \arrow[r] & \cdot \end{tikzcd}$$

that is universal as such. In other words still, a pushout in a category $\mathscr{A}$ is a pullback in $\mathscr{A}^{\mathrm{op}}$.

Example 5.2. A diagram $D : (\mathbb{N}, \leq) \to \mathscr{A}$ consists of objects and maps

$$X_0 \xrightarrow{s_1} X_1 \xrightarrow{s_2} X_2 \xrightarrow{s_3} \cdots$$

in $\mathscr{A}$. Colimits of such diagrams are traditionally called *direct limits*.

Definition 5.15. Let $\mathscr{A}$ be a category. A map $X \xrightarrow{f} Y$ in $\mathscr{A}$ is *epic* (or an *epimorphism*) if for all objects Z and maps $\begin{tikzcd} Y \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & Z \end{tikzcd}$,

$$g \circ f = g' \circ f \quad \Longrightarrow \quad g = g'.$$