F. L. Nazarov's paper
Local Estimates of Exponential Polynomials and Their Applications to Inequalities of Uncertainty Pinciple Type
Part II

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Abstract

This is a learning note about Chapter 1 of Nazarov's paper(see [1]). This chapter is about the Turan lemma and its general form on measurable sets.

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Lemma 1 (the lattice averaging lemma). *Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a positive summable function, and let $\varepsilon > 0$ be fixed. Then*

$$\int_1^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \varphi(k\varepsilon v)\mathrm{d}v \leq \frac{1}{\varepsilon}\int_{\mathbb{R}} \varphi(t)\mathrm{d}t$$

and

$$\int_1^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \varphi\left(\frac{k}{\varepsilon v}\right)\mathrm{d}v \leq 4\varepsilon\int_{\mathbb{R}} \varphi(t)\mathrm{d}t.$$

Definition 1. Let $E \subset \mathbb{R}$ be a measurable set of finite measure. Consider an arbitrary function $f \in L^2(\mathbb{R})$ supported on E and fix a positive number ε. Define the random periodization g of the function f by

$$g(t) = g(\varepsilon, v|t) \stackrel{\text{def}}{=} \frac{1}{\sqrt{\varepsilon v}} \sum_{k \in \mathbb{Z}} f\left(\frac{k+t}{\varepsilon v}\right).$$

Here v is a random variable equidistributed on the interval $(1,2)$. The series in the definition of g converges in $L^1_{\text{loc}}(\mathbb{R})$ since the measure of the support of f is finite, and is a 1-periodic function.

Definition 2. We denote by $\hat{f}$ the Fourier transform of a function $f \in L^2(\mathbb{R})$ understood in the sense of the Plancherel theorem, i.e., as a limit in $L^2(\mathbb{R})$ of the functions

$$\hat{f}_n(\lambda) \stackrel{\text{def}}{=} \int_{-n}^{n} f(x) e^{-2\pi i \lambda x} \mathrm{d}x.$$

By definition, we compute the Fourier coefficients of g

$$\begin{aligned}
\hat{g}_m &= \int_0^1 g(t) e^{-2\pi i m t}\, \mathrm{d}t \\
&= \int_0^1 \frac{1}{\sqrt{\varepsilon v}} \sum_{k \in \mathbb{Z}} f\left(\frac{k+t}{\varepsilon v}\right) e^{-2\pi i m t}\, \mathrm{d}t \\
&= \frac{1}{\sqrt{\varepsilon v}} \sum_{k \in \mathbb{Z}} \int_0^1 f\left(\frac{k+t}{\varepsilon v}\right) e^{-2\pi i m t}\, \mathrm{d}t \\
&\stackrel{t=\varepsilon v \lambda}{=} \sqrt{\varepsilon v} \sum_{k \in \mathbb{Z}} \int_0^{\frac{1}{\varepsilon v}} f\left(\frac{k}{\varepsilon v} + \lambda\right) e^{-2\pi i m \varepsilon v \lambda}\, \mathrm{d}\lambda \\
&= \sqrt{\varepsilon v} \sum_{k \in \mathbb{Z}} \int_{\frac{k}{\varepsilon v}}^{\frac{k+1}{\varepsilon v}} f(\lambda) e^{-2\pi i m \varepsilon v \lambda}\, \mathrm{d}\lambda \\
&= \sqrt{\varepsilon v} \int_{-\infty}^{\infty} f(\lambda) e^{-2\pi i m \varepsilon v \lambda}\, \mathrm{d}\lambda \\
&= \sqrt{\varepsilon v} \hat{f}(m \varepsilon v).
\end{aligned}$$

Proposition 1.

(a) $\mu(\{t \in (0,1) : g(t) \neq 0\}) \leq 2\varepsilon \mu(E)$.

(b) $\mathbf{E}\|g\|^2_{L^2(0,1)} \leq 2\varepsilon |\hat{f}(0)|^2 + 2\|f\|^2_{L^2(\mathbb{R})} \leq 2(\varepsilon \mu(E) + 1) \|f\|^2_{L^2(\mathbb{R})}$.

Let $\Sigma \subset \mathbb{R}$ be measurable, $0 \in \Sigma$. We consider a random lattice $\Lambda = \Lambda(\varepsilon, v) \overset{\text{def}}{=} \{s\varepsilon v : s \in \mathbb{Z}\}$ and denote $\mathfrak{M} = \{s \in \mathbb{Z} : s\varepsilon v \in \Sigma\}$.

(c) $\mathbf{E}\left(\text{card}\mathfrak{M} - 1\right) \leq \frac{\mu(\Sigma)}{\varepsilon}$.

(d) $\mathbf{E}\Sigma_{m \in \mathbb{Z}\setminus\mathfrak{M}} |\hat{g}_m|^2 \leq 2\int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2$.

Proof.

(a) The measure of the set of all points $t \in (0,1)$ for which the summand $f\left(\frac{k+t}{\varepsilon v}\right)$ in the series defining g does not vanish is equal to $\mu\left(\varepsilon v E \cap (k, k+1)\right)$. Therefore,

$$\mu\left(\{t \in (0,1) : g(t) \neq 0\}\right) \leq \sum_{k \in \mathbb{Z}} \mu\left(\{\varepsilon v E \cap (k, k+1)\}\right) = \mu\left(\varepsilon v E\right) \leq 2\varepsilon\mu\left(E\right).$$ [1]

(b)

$$\mathbf{E}\|g\|^2_{L^2(0,1)} = \mathbf{E}\sum_{k \in \mathbb{Z}} |\hat{g}_k|^2 = \mathbf{E}|\hat{g}_0|^2 + \mathbf{E}\sum_{k \in \mathbb{Z}\setminus\{0\}} |\hat{g}_k|^2.$$

But $|\hat{g}_0|^2 = \varepsilon v|\hat{f}(0)|^2 \leq 2\varepsilon|\hat{f}(0)|^2$, and

$$\begin{aligned}
\mathbf{E}\sum_{k \in \mathbb{Z}\setminus\{0\}} |\hat{g}_k|^2 = &\int_1^2 \left(\sum_{k \in \mathbb{Z}\setminus\{0\}} \varepsilon v|\hat{f}(k\varepsilon v)|^2\right) \mathrm{d}v \\
\leq &2\varepsilon\int_1^2 \left(\sum_{k \in \mathbb{Z}\setminus\{0\}} |\hat{f}(k\varepsilon v)|^2\right) \mathrm{d}v \\
\leq &2\int_{\mathbb{R}} |\hat{f}|^2 = 2\|f\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

It remains to notice that

$$|\hat{f}(0)|^2 = \left|\int_E f\right|^2 \leq \mu(E)\int_E |f|^2 = \mu(E)\|f\|^2_{L^2(\mathbb{R})}.$$

(c) Since $\text{card}\mathfrak{M} = 1 + \sum_{k \in \mathbb{Z}\setminus\{0\}} \chi_\Sigma(k\varepsilon v)$, we have

$$\mathbf{E}(\text{card}\mathfrak{M} - 1) = \int_1^2 \sum_{k \in \mathbb{Z}\setminus\{0\}} \chi_\Sigma(k\varepsilon v)\mathrm{d}v \leq \frac{1}{\varepsilon}\int_{\mathbb{R}} \chi_\Sigma = \frac{\mu(\Sigma)}{\varepsilon}.$$

[1] Remember that v is a random variable equidistributed on the interval $(1,2)$.

(d)

$$\mathbf{E}\sum_{m\in\mathbb{Z}\setminus\mathfrak{M}}|\hat{g}_m|^2 = \int_1^2\left(\sum_{k\in\mathbb{Z}\setminus\{0\}}\varepsilon v|\hat{f}(k\varepsilon v)|^2\chi_{\mathbb{R}\setminus\Sigma}(k\varepsilon v)\right)\mathrm{d}v$$
$$\leq 2\varepsilon\int_1^2\left(\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(|\hat{f}(k\varepsilon v)|^2\chi_{\mathbb{R}\setminus\Sigma}\right)(k\varepsilon v)\right)\mathrm{d}v$$
$$\leq 2\int_{\mathbb{R}}|\hat{f}|^2\chi_{\mathbb{R}\setminus\Sigma} = 2\int_{\mathbb{R}\setminus\Sigma}|\hat{f}|^2.$$

□

Let E and Σ be two measurable subsets of $\mathbb{R}$. Borrrowing the terminology from Jöricke and Havin, we say that E and Σ annihilate if for every function $f\in L^2(\mathbb{R})$ the conditions $\mathrm{supp}f\subset E$, $\mathrm{spec}f\subset\Sigma$ imply that f vanishes identically. We say that E and Σ strongly annihilate if there exists a constant $C>0$ such that the inequality

$$(*)\quad \|f\|^2_{L^2(\mathbb{R})}\leq C\left(\int_{\mathbb{R}\setminus E}|f|^2+\int_{\mathbb{R}\setminus\Sigma}|\hat{f}|^2\right)$$

holds for every function $f\in L^2(\mathbb{R})$. The strong annihilation condition can be written in a form which is less symmetric but more convenient to verify: E and Σ strongly annihilate if and only if there exists a constant $C>0$ such that

$$(**)\quad \int_{\Sigma}|\hat{f}|^2\leq C'\int_{\mathbb{R}\setminus\Sigma}|\hat{f}|^2$$

holds for every $f\in L^2(\mathbb{R})$ supported on E.

There is a relationship between the best possible constants C and C' :

$$C'=\mathrm{ctg}^2\alpha,\qquad C=\frac{1}{2\sin^2\frac{\alpha}{2}}=\frac{1}{1-\cos\alpha},$$

where α is the angle between the subspaces $L^2(E)\stackrel{\mathrm{def}}{=}\{f\in L^2(\mathbb{R}):\mathrm{supp}f\subset E\}$ and $L^2\left(\hat{\Sigma}\right)\stackrel{\mathrm{def}}{=}\{f\in L^2(\mathbb{R}):\mathrm{spec}f\subset\Sigma\}$ of the Hilbert space $L^2(\mathbb{R})$. The proof of this statement is a simple exercise in geometry. Denoting by P_E and $P_{\hat{\Sigma}}$ the orthogonal projection onto $L^2(E)$ and $L^2(\hat{\Sigma})$ repectively, we have:

$$\cos\alpha=\sup\left\{|(f,g)|:f\in L^2(E),g\in L^2\left(\hat{\Sigma}\right),\|f\|^2_{L^2(\mathbb{R})}=\|g\|^2_{L^2(\mathbb{R})}=1\right\}$$
$$=\sup\{|\left(P_{\hat{\Sigma}}f,g\right)|:\cdots\}$$
$$=\sup\left\{\|P_{\hat{\Sigma}}f\|_{L^2(\mathbb{R})}:f\in L^2(E),\|f\|^2_{L^2(\mathbb{R})}=1\right\},$$

and

$$\begin{aligned}C' =& \sup\left\{\frac{\int_\Sigma |\hat f|^2}{\int_{\mathbb{R}\setminus\Sigma}|\hat f|^2} : f\in L^2(E), \|f\|^2_{L^2(\mathbb{R})}=1\right\}\\ =& \sup\left\{\frac{\|P_{\hat\Sigma}f\|^2_{L^2(\mathbb{R})}}{1-\|P_{\hat\Sigma}f\|^2_{L^2(R)}} : f\in L^2(E), \|f\|^2_{L^2(\mathbb{R})}=1\right\}\\ =& \frac{\cos^2\alpha}{1-\cos^2\alpha}\\ =& \mathrm{ctg}^2\alpha.\end{aligned}$$

The computation of the constant C is slightly more complicated. Denote by β and γ the angles between f and the subspaces $L^2(E)$ and $L^2\left(\hat\Sigma\right)$, respectively. It is clear that $0<\beta,\gamma<\frac{\pi}{2}, \beta+\gamma\geq\alpha$. Since

$$\begin{aligned}\int_{\mathbb{R}\setminus E}|f|^2+\int_{\mathbb{R}\setminus\Sigma}|\hat f|^2 =& \|f-P_Ef\|^2_{L^2(\mathbb{R})}+\|f-P_{\hat\Sigma}f\|^2_{L^2(\mathbb{R})}\\ =& \|f\|^2-2(P_Ef,f)+\|P_Ef\|^2+\|f\|^2-2(P_{\hat\Sigma}f,f)+\|P_{\hat\Sigma}f\|^2\\ =& \left(\sin^2\beta+\sin^2\gamma\right)\|f\|^2_{L^2(\mathbb{R})}\geq 2\sin^2\frac{\alpha}{2}\|f\|^2_{L^2(\mathbb{R})},\end{aligned}$$

we have $C\leq\frac{1}{2\sin^2\frac{\alpha}{2}}$. To verify the reverse inequality, it suffices to exhibit a function f for which the angles β and γ are close to $\frac{\alpha}{2}$. This can be done as follows. One can choose $g\in L^2(E)$ and $h\in L^2(\hat\Sigma)$ so that $\|g\|_{L^2(\mathbb{R})}=\|h\|_{L^2(\mathbb{R})}=1$ and $\mathrm{Re}(h,g)\approx\cos\alpha$, and then put $f\stackrel{\mathrm{def}}{=}\frac{1}{2}(g+h)$.

It should be noted that, proceeding in the same way, one can describe the image of the unit ball of $L^2(\mathbb{R})$ under the mapping

$$L^2(\mathbb{R})\ni f\to\left(\int_{\mathbb{R}\setminus E}|f|^2,\int_{\mathbb{R}\setminus\Sigma}|\hat f|^2\right)\in\mathbb{R}^2_+$$

provided that each of the subspaces $L^2(E)$ and $L^2(\hat\Sigma)$ contains a vector making an angle arbitrarily close to $\frac{\pi}{2}$ with the other subspace (this condition is certainly satisfied if both E and Σ have zero density at infinity, i.e., if $\lim_{A\to+\infty}\frac{\mu(E\cap[-A,A]}{A}=\lim_{A\to+\infty}\frac{\mu(\Sigma\cap[-A,A]}{A}=0$; the corresponding vectors can be chosen among those of the form $fe^{i\lambda t}$ and $\tau_\lambda g$, where $f\in L^2(E)$, $g\in L^2(\hat\Sigma)$ and λ is a suitable number from a sufficiently large interval centered at 0). This image turns out to be the square $[0,1]^2$ with the upper-right angle cut off along the curve $\arccos\sqrt{x}+\arccos\sqrt{y}=\alpha$.

Excluding α from the formulas for C and C', we get

$$C=C'+1+\sqrt{C'(C'+1)}\leq 2C'+\frac{3}{2}.$$

Now we state the main theorem of this section.

Theorem 1. *For every two sets E and Σ of finite measure and every function $f \in L^2(\mathbb{R})$, the following inequality holds:*

$$\|f\|^2_{L^2(\mathbb{R})} \leq 130 e^{66\mu(E)\mu(\Sigma)} \left(\int_{\mathbb{R}\setminus E} |f|^2 + \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2 \right).$$

Proof. As it was shown above, it suffices to prove that

$$\int_{\Sigma} |\hat{f}|^2 \leq 64 e^{\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2$$

for every function $f \in L^2(E)$. We set $\varepsilon = \frac{1}{4\mu(E)}$ and introduce the random periodization g of the function f. By (a),

$$\mu\left(\{t \in (0,1) : g(t) = 0\}\right) \stackrel{\text{def}}{=} \mu(F) \geq 1 - 2\varepsilon\mu(E) = \frac{1}{2}.$$

We decompose g into a sum $p + q$, where

$$p(t) \stackrel{\text{def}}{=} \sum_{m : m\varepsilon v \in \Sigma\cup\{0\}} \hat{g}_m e^{2\pi i m t} \stackrel{\text{def}}{=} \sum_{m\in\mathfrak{M}} \hat{g}_m e^{2\pi i m t}$$

and

$$q(t) \stackrel{\text{def}}{=} \sum_{m\in\mathbb{Z}\setminus\mathfrak{M}} \hat{g}_m e^{2\pi i m t}.$$

We have

$$\mathbf{E}\|q\|^2_{L^2(0,1)} = \mathbf{E} \sum_{m\in\mathbb{Z}\setminus\mathfrak{M}} |\hat{g}_m|^2 \stackrel{\text{(d)}}{\leq} 2 \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2,$$

whence

$$\mathbf{P}\left(\left\{ \|q\|^2_{L^2(0,1)} > 4 \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2 \right\} \right) < \frac{1}{2}.$$

Next,

$$\mathbf{E}(\operatorname{ord} p - 1) = \mathbf{E}(\text{card} - 1) \stackrel{\text{(c)}}{\leq} \frac{\mu(\Sigma)}{\varepsilon} = 4\mu(E)\mu(\Sigma).$$

Consequently,

$$\mathbf{P}\left(\operatorname{ord} p > 1 + 8\mu(E)\mu(\Sigma)\right) < \frac{1}{2}.$$

We see that, with positive probability, the following 4 events take place simultaneously:

(a) $\mu(F) \geq \frac{1}{2}$;

(b) $\|q\|^2_{L^2(0,1)} \leq 4 \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2$;

(c) $\operatorname{ord} p \leq 1 + 8\mu(E)\mu(\Sigma)$;

(d) $\varepsilon|\hat{f}(0)|^2 = \frac{1}{4\mu(E)}|\hat{f}(0)|^2 \leq |\hat{p}_0|^2 = |\hat{g}_0|^2$.

Indeed, (a) and (d) always hold, while each of (b) and (c) does not hold with probability less than $\frac{1}{2}$. Since $g|_F \equiv 0$, we have $p|_F = q|_F$ and $\int_F |p|^2 = \int_F |q|^2$. Hence

$$\mu\left(\left\{t \in F : |p(t)|^2 \geq 16 \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2\right\}\right) \leq \frac{1}{4},[2]$$

and, since $\mu(F) \geq \frac{1}{2}$, we get

$$\mu\left(\left\{t \in (0,1) : |p(t)| \leq 4 \left(\int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2\right)^{1/2}\right\}\right) \geq \frac{1}{4}.$$

Now a special case of the Turan lemma (Theorem 3 in Part I) implies

$$\begin{aligned}\frac{1}{4\mu(E)}|\hat{f}(0)|^2 \leq& |\hat{p}_0|^2 \leq \left(\sum_k |\hat{p}_k|\right)^2 \leq \left(\left(\frac{14}{1/4}\right)^{\operatorname{ord} p-1} 4\left(\int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2\right)^{1/2}\right)^2 \\ \leq& 16 \times 56^{16\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2,\end{aligned}$$

whence

$$|\hat{f}(0)|^2 \leq 64\mu(E)e^{16\log 56\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2.$$

If we take the function $f_1(x) \stackrel{\text{def}}{=} f(x)e^{-2\pi ixy}$ instead of $f(x)$ and the set $\Sigma - y$ instead of Σ, we arrive at the same estimate for $|\hat{f}(y)|$. Integrating this estimate over Σ, we get the inequality

$$\int_\Sigma |\hat{f}|^2 \leq 64\mu(E)\mu(\Sigma)e^{16\log 56\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2 \leq 64e^{66\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2,$$

which proves the theorem. $\square$

[2]Indeed, if $\mu\left(\left\{t \in F : |p(t)|^2 \geq 16 \int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2\right\}\right) > \frac{1}{4}$, we would obtain $\|q\|^2_{L^2(0,1)} > 4\int_{\mathbb{R}\setminus\Sigma} |\hat{f}|^2$, this contradicts the event (b).

References

[1] FL Nazarov. “Local estimates of exponential polynomials and their applications to inequalities of uncertainty priciple type”. In: *St Petersburg Mathematical Journal* 5.4 (1994), pp. 663–718.