OBSERVABILITY INEQUALITY AT TWO TIME POINTS FOR KDV EQUATIONS FROM MEASURABLE SETS

Abstract.

1. Introduction

Consider the linear KdV equation

$$u_t + u_{xxx} = 0, \quad u(x,0) = u_0 \in L^2(\mathbb{R}).$$

Our result reads as follows.

Theorem 1.1. *Let A, B be two measurable sets in $\mathbb{R}$ with finite measure. Then for every $t > 0$, there exists $C = C(t, A, B) > 0$ so that when $u(t,x)$ solves the KdV equation,*

$$\int_{\mathbb{R}} |u_0|^2 \,\mathrm{d}x \le C \left(\int_{A^c} |u_0|^2 \,\mathrm{d}x + \int_{B^c} |u(t,x)|^2 \,\mathrm{d}x \right).$$

2. The proof

Let $S(t)$ be the solution group of linear KdV equation, namely the solution of KdV is given by

$$u(t) = S(t)u_0 = G(t,x) * u_0,$$

where G is the fundamental solution of linear KdV equation, given by

$$G(t,x) = \begin{cases} \frac{1}{(3t)^{\frac{1}{3}}} \operatorname{Ai}\left(\frac{x}{(3t)^{\frac{1}{3}}}\right), & t > 0 \\ \delta(x), & t = 0. \end{cases}$$

Here, $\operatorname{Ai}(x)$ is the Airy function defined via

$$\operatorname{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(xz + \frac{1}{3}z^3)} \,\mathrm{d}z.$$

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According to [Stein, p.330],

$$|\operatorname{Ai}(x)| \lesssim \begin{cases} (1+|x|)^{-\frac{1}{4}}, & x<0, \\ e^{-\frac{2}{3}|x|^{\frac{3}{2}}}, & x \geq 0. \end{cases}$$

Define an operator $T : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$

$$(Tf)(x) = \chi_B(x) S(t)(\chi_A f), \quad f \in L^2(\mathbb{R}).$$

Then we have the following

Proposition 2.1. *Let A, B be two measurable sets in $\mathbb{R}$ with $|A|, |B| < \infty$. Then the operator norm satisfies*

$$\|S(-t)T\|_{\mathcal{L}(L^2(\mathbb{R}))} < 1.$$

Before give the proof of Proposition 2.1, we first show that Theorem 1.1 follows from Proposition 2.1.

Lemma 2.2. *Proposition 2.1 implies Theorem 1.1.*

Proof. Assume that Proposition 2.1 holds, we obtain

$$\|T\|_{\mathcal{L}(L^2(\mathbb{R}))} = \|S(t)S(-t)T\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \|T\|_{\mathcal{L}(L^2(\mathbb{R}))} < 1.$$

Then for all $u_0 \in L^2(\mathbb{R})$,

$$\|\chi_B(x) S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})} \leq c_1 \|u_0\|_{L^2(\mathbb{R})}$$

with some $0 \leq c_1 < 1$. This implies that

$$\|\chi_B(x) S(t) \chi_A u_0\|^2_{L^2(\mathbb{R})} \leq c_1^2 \|\chi_A u_0\|_{L^2(\mathbb{R})} = c_1^2 \|S(t)\chi_A u_0\|^2_{L^2(\mathbb{R})}, \quad \forall u_0 \in L^2(\mathbb{R}),$$

where we used the conservation law $\|u_0\|_{L^2(\mathbb{R})} = \|S(t)u_0\|_{L^2(\mathbb{R})}$ in the last step. From this, we find that with $c_2 = \sqrt{\frac{c_1^2}{1-c_1^2} + 1}$

$$\|S(t)\chi_A u_0\|_{L^2(\mathbb{R})} \leq c_2 \|S(t)\chi_A u_0\|_{L^2(B^c)}, \quad \forall u_0 \in L^2(\mathbb{R}). \tag{2.1}$$

Now we have using (2.1) and conservation law again

$$\begin{aligned} \|u_0\|_{L^2(\mathbb{R})} = \|S(t)u_0\|_{L^2(\mathbb{R})} &\leq \|S(t)\chi_A u_0\|_{L^2(\mathbb{R})} + \|S(t)\chi_{A^c} u_0\|_{L^2(\mathbb{R})} \\ &\leq c_2 \|S(t)\chi_A u_0\|_{L^2(B^c)} + \|S(t)\chi_{A^c} u_0\|_{L^2(\mathbb{R})} \\ &\leq c_2 \|S(t)u_0\|_{L^2(B^c)} + (1+c_2)\|S(t)\chi_{A^c} u_0\|_{L^2(\mathbb{R})} \\ &= c_2 \|u(t,\cdot)\|_{L^2(B^c)} + (1+c_2)\|u_0\|_{L^2(A^c)}. \end{aligned}$$

This proves Theorem 1.1. ∎

Now we need to prove Proposition 2.1. To this end, we first note that we always have

$$\|S(-t)T\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq 1.$$

In fact, for all $f \in L^2(\mathbb{R})$

$$\|S(-t)Tf\|_{L^2(\mathbb{R})} = \|S(-t)\chi_B(x)S(t)(\chi_A f)\|_{L^2(\mathbb{R})} \leq \|S(t)(\chi_A f)\|_{L^2(\mathbb{R})} = \|\chi_A f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}.$$

Thus, it remains to show that

$$\|S(-t)T\|_{\mathcal{L}(L^2(\mathbb{R}))} \neq 1.$$

To show this, we need the following

Lemma 2.3. *For every $t \neq 0$, T is a compact operator on $L^2(\mathbb{R})$.*

Proof. We can rewrite the operator T as an integral operator:

$$(Tf)(x) = \int_{\mathbb{R}} \chi_A(x)G(t, x-y)\chi_B(y)f(y)\,\mathrm{d}y := \int_{\mathbb{R}} K(t,x,y)f(y)\,\mathrm{d}y.$$

We claim that for all $t \neq 0$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{2.2}$$

Then T is a Hilbert-Schmidt operator and thus a compact operator on $L^2(\mathbb{R})$.

It remains to show (2.2). In fact, since $|G(t,x-y)| \leq C(t)$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y \leq C^2(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_A(x)\chi_B(y)dx\,\mathrm{d}y = C^2(t)|A||B| < \infty.$$

This proves (2.2). □

Define the translation operator U_λ:

$$\mathcal{T}_\lambda f(x) = f(x-\lambda).$$

If A is a measurable set in $\mathbb{R}$ and $\lambda \in \mathbb{R}$, we shall denote the set $\lambda + A = \{\lambda + x | x \in A\}$.

Lemma 2.4. *Let C and C' be measurable sets in $\mathbb{R}^n$ with $0 < |C|, |C'| < \infty$, let A_0 and B_0 be a measurable subset of C and C' with $|A_0| > 0, |B_0| > 0$, and let $\epsilon > 0$. Then there exists a translation $\lambda \in \mathbb{R}$ such that*

$$|C| \leq |C \cup (\lambda + A_0)| < |C| + \epsilon$$

and

$$|C'| \leq |C' \cup (\lambda + B_0)| < |C'| + \epsilon.$$

Moreover, λ can be chosen such that at least one left inequality of the above is strict.

Proof. Define $h(\lambda) = |C \cup (\lambda + A_0)|$. We may express $h(\lambda)$ in terms of

$$h(\lambda) = \|\mathcal{T}_\lambda \chi_{A_0} - \chi_C\|^2_{L(\mathbb{R})} + \langle \mathcal{T}_\lambda \chi_{A_0}, \chi_C \rangle.$$

Similarly, we can define $h'(\lambda) = |C' \cup (\lambda + B_0)|$. The strong continuity of U_λ implies that h and h' are continuous functions. Hence For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|C| \leq |C \cup (\lambda + A_0)| < |C| + \epsilon$$

and

$$|C'| \leq |C' \cup (\lambda + B_0)| < |C'| + \epsilon.$$

for $0 < \lambda < \delta$.

Choose σ such that $0 < 2\sigma < |A_0|$ and a ball $S_r = [-r, r]$ such that $|C \cap S_r^c| < \sigma$. Let $\lambda \in \mathbb{R}$ be such that $|\lambda| > 2r$. Since $A_0 \subset C$, we obtain $|\lambda C_0 \cap S_r| < \sigma$. Thus

$$\begin{aligned} h(\lambda) =& |C \cup \lambda A_0| \\ \geq& |C \cap S_r| + |\lambda C_0 \cap S_r'| \\ \geq& |C| - \sigma + |\lambda A_0| - \sigma \\ =& |C| + |A_0| - 2\sigma \\ >& |C| = h(0). \end{aligned}$$

This shows that h is not a constant, and h' is not a constant the same way. Hence the last claim of the Lemma 2.4 is true. □

Go back to the proof of Proposition 2.1. Suppose by way of contradiction that

$$\|S(-t)T\|_{\mathcal{L}(L^2(\mathbb{R}))} = 1,$$

then by Lemma 2.3 there exists a function $f \in L^2(\mathbb{R})$ such that $\|S(-t)Tf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$. It is supported on B and $S(t)f$ is supported on A.

Define $f_\lambda = \mathcal{T}_\lambda f$. Then $\mathrm{supp} f_\lambda = \lambda A$. Since

$$\begin{aligned} S(t) f_\lambda =& S(t)\mathcal{T}_\lambda f \\ =& \int_{\mathbb{R}} G(t, x-y) f(y-\lambda) \, \mathrm{d}y \\ =& \int_{\mathbb{R}} G(t, x-\lambda-y) f(y) \, \mathrm{d}y \\ =& \mathcal{T}_\lambda (S(t)f), \end{aligned}$$

we have $\mathrm{supp} S(t) f_\lambda = \lambda + B$.

Now we define a sequence $\{f_i\}_{i=1}^{\infty}$ recursively. By Lemma 2.4 with $\epsilon = \frac{1}{2^i}$, $C = A_{i-1}, A_0 = A$ and $C' = B_{i-1}, B_0 = B$, we choose a translation λ_i such that

$$|A_{i-1}| \leq |A_{i-1} \cup (\lambda_i + A_0)| < |A_{i-1}| + \frac{1}{2^i},$$

and

$$|B_{i-1}| \leq |B_{i-1} \cup (\lambda_i + B_0)| < |B_{i-1}| + \frac{1}{2^i},$$

and we set $A_i = A_{i-1} \cup (\lambda_i + A_0), B_i = B_{i-1} \cup (\lambda_i + B_0)$. By the last sentence of Lemma 2.4, λ_i can be chosen such that at least one left inequality of the above two is strict. Using the above inequality recursively, we obtain

$$|\bigcup_{i=0}^{\infty} A_i| < |A| + 1, \quad |\bigcup_{i=0}^{\infty} B_i| < |B| + 1.$$

Define $f_i = \mathcal{T}_{\lambda_i} f$ and $f_0 = f$, then supp$f_i \subset A_i \subset \bigcup_{i=0}^{\infty} A_i$ and supp$S(t)f_i \subset B_i \subset \bigcup_{i=0}^{\infty} B_i$. We shall prove that the sequence $\{f_i\}_{i=0}^{\infty}$ are linearly independent. Denote the projection operator $E_U f = \chi_U f$. Since $A_m = A_0 \cup (\lambda_1 + A_0) \cup \cdots \cup (\lambda_m + A_0)$ and $B_m = B_0 \cup (\lambda_1 + B_0) \cup \cdots \cup (\lambda_m + B_0)$, we have $E_{A_m} f_i = f_i$ and $E_{B_m} S(t) f_i = S(t) f_i$ for all $i = 0, 1, \cdots, m$. By the choice of λ_i, we have either $E_{A_m \setminus A_{m-1}} f_m \neq 0$ or $E_{B_m \setminus B_{m-1}} S(t) f \neq 0$, both can show that f_m is not a linear combination of $f_0, f_1, \cdots, f_{m-1}$. This means that the sequence $\{f_i\}_{i=0}^{\infty}$ are linearly independent. Let $A' = \bigcup_{i=0}^{\infty} A_i$ and $B' = \bigcup_{i=0}^{\infty} B_i$, then the eigensppace of the new operator $S(-t)T = S(-t)\chi_{B'} S(t)(\chi_{A'} f)$ has infinitely many eigenfunctions of eigenvalue 1, which contradicts to the compact proposition of $S(-t)T$ by Lemma 2.3. Thus we complete the proof of Proposition 2.1.

3. The case one of the sets has infinite measure

drawing

Figure 1. drawing

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