

# Maximum Principle

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## Abstract

This note is written by referring to Evans' PDE book[\[1\]](#)

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# 1 Second-order Elliptic Equations

Define the partial differential operator  $L$  to be

$$Lu = - \sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c(x) u. \quad (1)$$

**Definition 1** We say  $L$  is (uniformly) elliptic if there exists a constant  $\theta > 0$  such that

$$\sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad (2)$$

for a.e.  $x \in U$  and all  $\xi \in \mathbb{R}^n$ .

## 2 Maximum Principle

### 2.1 Weak Maximum Principle

In this case, we assume that  $a^{ij}, b^i, c$  are bounded, continuous and the uniform ellipticity condition (2) holds.

The maximum principle we talk about here is the classical condition.

**Theorem 1 (Weak maximum principle)** Assume  $u \in C^2(U) \cap C(\overline{U})$  and

$$c \equiv 0 \text{ in } U$$

a. If

$$Lu \leq 0 \text{ in } U, \quad (3)$$

then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

b. If

$$Lu \geq 0 \text{ in } U, \quad (4)$$

then

$$\min_{\overline{U}} u = \min_{\partial U} u.$$

*Proof.* 1. Suppose we have the strict inequality

$$Lu < 0 \text{ in } U,$$

and there is a point  $x_0 \in U$  such that

$$u(x_0) = \max_{\overline{U}} u. \quad (5)$$

At this maximum point  $x_0$ , we have

$$Du(x_0) = 0 \quad (6)$$

and

$$D^2 u(x_0) \leq 0. \quad (7)$$

Since the matrix  $A = (a^{ij}(x_0))$  is symmetric and positive definite, there exists an orthogonal matrix  $O = (o_{ij})$  so that

$$OAO^T = \text{diag}(d_1, \dots, d_n), \quad OO^T = 1, \quad (8)$$

with  $d_k > 0$  ( $k = 1, \dots, n$ ). Write  $y = x_0 + O(x - x_0)$ . Then  $x - x_0 = O^T(y - x_0)$ , and so

$$u_{x_i} = \sum_{k=1}^n u_{y_k} O_{ki}, u_{x_i x_j} = \sum_{k,l=1}^n u_{y_k y_l} O_{ki} O_{lj} \quad (i, j = 1, \dots, n).$$

Hence at the point  $x_0$ ,

$$\begin{aligned} \sum_{i,j=1}^n a^{ij} u_{x_i x_j} &= \sum_{k,l=1}^n \sum_{i,j=1}^n a^{ij} u_{y_k y_l} O_{ki} O_{lj} \\ &= \sum_{k=1}^n d_k u_{y_k y_k} \leq 0 \quad \text{by (7)}. \end{aligned} \quad (9)$$

Thus at  $x_0$

$$Lu = - \sum_{i,j=1}^n a^{ij} u_{x_i x_j} + \sum_{i=1}^n b^i u_{x_i} \geq 0$$

in light of (6) and (9). So we have a contradiction. 2. In general case that (ref3) holds, write

$$u^\epsilon(x) := u(x) + \epsilon e^{\lambda x_1} (x \in U),$$

where  $\lambda > 0$  will be selected below and  $\epsilon > 0$ . The uniform condition implies  $a^{ii}(x) \geq \theta$   $i = 1, \dots, n, x \in U$ . Therefore

$$\begin{aligned} Lu^\epsilon &= Lu + \epsilon L(e^{\lambda x_1}) \\ &\leq \epsilon e^{\lambda x_1} (-\lambda^2 a^{11} + \lambda b^1) \\ &\leq \epsilon e^{\lambda x_1} (-\lambda^2 \theta + \|\mathbf{b}\|_{L^\infty} \lambda) \\ &< 0 \quad \text{in } U, \end{aligned}$$

provided we choose  $\lambda > 0$  sufficiently large. Then according step 1 above and let  $\epsilon \rightarrow 0$  we get  $\max_{\overline{U}} u = \max_{\partial U} u$ .  $\square$

**Theorem 2 (Weak maximum principle for  $c \geq 0$ )** Assume  $u \in C^2(U) \cap C(\overline{U})$  and

$$c \geq 0 \text{ in } U.$$

a. If

$$Lu \leq 0 \text{ in } U,$$

then

$$\max_{\overline{U}} u \leq \max_{\partial U} u^+. \quad (10)$$

b. Likewise, if

$$Lu \geq 0 \text{ in } U,$$

then

$$\min_{\overline{U}} u \geq -\max_{\partial U} u^-. \quad (11)$$

*Proof.* 1. Let  $u$  be a subsolution and set  $V := \{x \in U | u(x) > 0\}$ . Then

$$Ku := Lu - cu \leq -cu \leq 0 \quad \text{in } V.$$

The operator  $K$  has no zeroth-order term and consequently Theorem 1 implies  $\max_{\overline{V}} u = \max_{\partial V} u = \max_{\partial U} u^+$ . This gives (10) in the case that  $V \neq \emptyset$ . Otherwise  $u \leq 0$  everywhere in  $U$ , and (10) likewise follows.

2. Assertion b follows from a applied to  $-u$ , once we observe that  $(-u)^+ = u^-$ .  $\square$

## 2.2 Strong Maximum Principle

**Lemma 1 (Hopf's Lemma)** Assume  $u \in C^2(U) \cap C^1(\overline{U})$  and

$$c \equiv 0 \text{ in } U.$$

Suppose further

$$Lu \leq 0 \text{ in } U$$

and there exists a point  $x^0 \in \partial U$  such that

$$u(x^0) > u(x) \text{ for all } x \in U. \quad (12)$$

Assume finally that  $U$  satisfies the interior ball condition at  $x^0$ ; that is, there exists an open ball  $B \subset U$  with  $x^0 \in \partial B$ .

a. Then

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

where  $\nu$  is the outer unit normal to  $B$  at  $x^0$ .

b. If

$$c \geq 0 \text{ in } U$$

the same conclusion holds provided

$$u(x^0) \geq 0.$$

*Proof.* 1. Assume  $c \geq 0$  and  $B = B^0(0, r)$  for some radius  $r > 0$ . Define

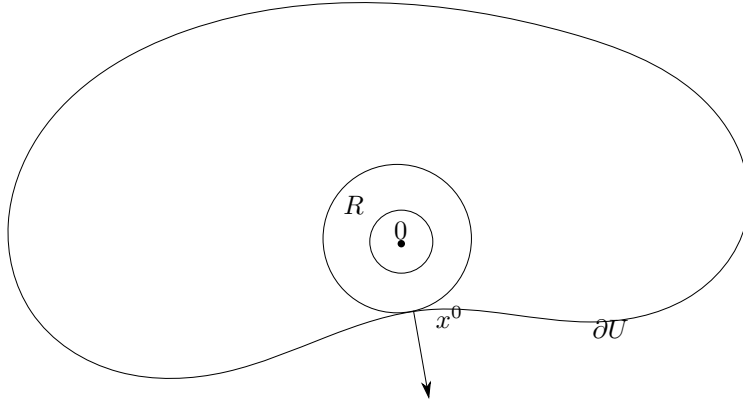


Figure 1: Interior ball condition

$$v(x) := e^{-\lambda|x|^2} - e^{-\lambda r^2} \quad (x \in B(0, r))$$

for  $\lambda > 0$  as selected below. Then using the uniform condition, we compute

$$\begin{aligned}
Lv &= - \sum_{i,j=1}^n a^{ij} v_{x_i x_j} + \sum_{i=1}^n b^i v_{x_i} = cv \\
&= e^{-\lambda|x|^2} \sum_{i,j=1}^n a^{ij} (-4\lambda^2 x_i x_j + 2\lambda \delta_{ij}) \\
&\quad - e^{-\lambda|x|^2} \sum_{i=1}^n b^i 2\lambda x_i + c \left( e^{-\lambda|x|^2} - e^{-\lambda r^2} \right) \\
&\leq e^{-\lambda|x|^2} \left( -4\theta\lambda^2 |x|^2 + 2\lambda \text{tr} A + 2\lambda |b| |x| + c \right),
\end{aligned}$$

for  $A = (a_{ij})$ ,  $b = (b^1, \dots, b^n)$ . Consider next the open annular region  $R := B^0(0, r) - B(0, \frac{r}{2})$ . We have

$$Lv \leq 0 \quad (13)$$

in  $R$ , provided  $\lambda$  is large enough. 2. In view of  $u(x^0) > u(x)$  for all  $x \in U$ , there exists a constant  $\epsilon > 0$  so small that

$$u(x^0) \geq u(x) + \epsilon v(x) \quad \left( x \in \partial B(0, \frac{r}{2}) \right). \quad (14)$$

In addition note

$$u(x^0) \geq u(x) + \epsilon v(x) \quad (x \in \partial B(0, r)), \quad (15)$$

since  $v \equiv 0$  on  $\partial B(0, r)$ .

3. From (13) we see

$$L(u + \epsilon v - u(x^0)) \leq -cu(x^0) \leq 0 \quad \text{in } R,$$

and from (14), (15) we observe

$$u + \epsilon v - u(x^0) \leq 0 \quad \text{in } \partial R,$$

In view of the weak maximum principle, Theorem 2, we get  $u + \epsilon v - u(x^0) \leq 0$  in  $R$ . But  $u(x^0) + \epsilon v(x^0) - u(x^0) = 0$ , and so

$$\frac{\partial u}{\partial \nu}(x^0) + \epsilon \frac{\partial v}{\partial \nu}(x^0) \geq 0.$$

Consequently

$$\frac{\partial u}{\partial \nu}(x^0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\epsilon}{r} Dv(x^0) \cdot x^0 = 2\lambda \epsilon r e^{-\lambda r^2} > 0,$$

as required.  $\square$

**Theorem 3 (Strong maximum principle)** Assume  $u \in C^2(U) \cap C(\bar{U})$  and

$$c \equiv 0 \quad \text{in } U.$$

Suppose also  $U$  is connected, open and bounded.

a. If

$$Lu \leq 0 \quad \text{in } U$$

and  $u$  attains its minimum over  $\bar{U}$  at an interior point, then  $u$  is constant within  $U$ .

b. Similarly, if

$$Lu \geq 0 \quad \text{in } U$$

and  $u$  attains its minimum over  $\overline{U}$  at an interior point, then  $u$  is constant within  $U$ .

*Proof.* Write  $M := \max_{\overline{U}} u$  and  $C := \{x \in U \mid u(x) = M\}$ . Then if  $u \not\equiv M$ , set

$$V := \{x \in U \mid u(x) < M\}.$$

Choose a point  $y \in V$  satisfying  $\text{dist}(y, C) < \text{dist}(y, \partial U)$ , and let  $B$  denote the largest ball with center  $y$  whose interior lies in  $V$ . Then there exists some point  $x^0 \in C$ , with  $x^0 \in \partial B$ . Clearly  $V$  satisfies the interior ball condition at  $x^0$ , whence Hopf's Lemma (a) implies

$$\frac{\partial u}{\partial \nu}(x^0) > 0.$$

But this is a contradiction: since  $u$  attains its maximum at  $x^0 \in U$ , we have  $Du(x^0) = 0$ .  $\square$

Similarly, we have the  $c \geq 0$  version of the strong maximum principle, and the proof is like the above.

**Theorem 4 (Strong maximum principle with  $c \geq 0$ )** Assume  $u \in C^2(U) \cap C(\overline{U})$  and

$$c \geq 0 \quad \text{in } U.$$

Suppose also  $U$  is connect.

a. If

$$Lu \leq 0 \quad \text{in } U$$

and  $u$  attains a nonnegative maximum over  $\overline{U}$  at an interior point, then  $u$  is constant with  $U$ .

b. Similarly, if

$$Lu \geq 0 \quad \text{in } U$$

and  $u$  attains a nonpositive minimum over  $\overline{U}$  at an interior point, then  $u$  is constant within  $U$ .

## References

- [1] Lawrence C Evans. Partial differential equations. *Providence, RI*, 1998.