# Observability inequality at two time points for the KdV equation from measurable sets

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**Abstract**

We prove two observability inequalities at two time points for the linear KdV equation on the real line. In the two inequalities, the observable region in space is the complement of $(i)$ a measurable set with finite Lebesgue measure, and $(ii)$ of a measurable set contained in a half line with some density conditions, whose Lebesgue measure may be infinite.

*Keywords:* KdV equation, observability, unique continuation

## 1. Introduction

The classical observability inequality for the Schrödinger equation

$$i\partial_t u + \Delta u = 0, \quad u(0,x) = u_0 \in L^2(\mathbb{R}^n) \tag{1.1}$$

reads that when $u$ solves (1.1)

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \le C(n,T,E) \int_0^T \int_E |u(t,x)|^2 \,\mathrm{d}x \,\mathrm{d}t, \tag{1.2}$$

where $T > 0, E$ is a subset in $\mathbb{R}^n$ and $C(n,T,E) > 0$ is a constant depending only on $n, T, E$. The inequality (1.2) holds if $E = \{x \in \mathbb{R}^n : |x| \ge r\}$ [11] in all dimensions $n \ge 1$, and $E$ is thick [6] (a more general set class) in the dimension $n = 1$. It is well known that the inequality (1.2) holds if and only if (1.1), with controls restricted in $E$, is exactly controllable over $(0,T)$.

Recently, Wang, Wang and Zhang [13] have proved the following new observability inequality: There exists a constant $C = C(n) > 0$ such that for all $t > 0$, all $r_1, r_2 > 0$ and all $u(t,x)$ solving (1.1),

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \le C e^{\frac{Cr_1 r_2}{t}} \left( \int_{|x|\ge r_1} |u_0(x)|^2 \,\mathrm{d}x + \int_{|x|\ge r_2} |u(t,x)|^2 \,\mathrm{d}x \right). \tag{1.3}$$

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This improves the inequality (1.2) since only two time points appear on the right hand side of (1.3). By the same reason, (1.3) is called observability inequality at two time points. Moreover, it is also shown in [13, Section 5.2] that the inequality (1.3) is equivalent to the exact controllability for the impulse controlled free Schrödinger equation at two time points, with controls restricted in the set outside of a ball at each time. Further observability inequalities at two time points are proved in [5] for Schrödinger equation with potentials and in [14] for nonlinear Schrödinger equations, respectively.

The proof of (1.3) is based on the uncertainty principle of Fourier transform, which roughly says that it is impossible for a nonzero function and its Fourier transform to have compact supports simultaneously. A quantitative version is the following inequality: There exists $C = C(n) > 0$ so that for all $r_1, r_2 > 0$ and all $f \in L^2(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |f(x)|^2 \,\mathrm{d}x \le Ce^{Cr_1 r_2} \left( \int_{|x|\ge r_1} |f|^2 \,\mathrm{d}x + \int_{|x|\ge r_2} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \right), \tag{1.4}$$

where $\widehat{f}$ denotes the Fourier transform of $f$, see e.g. [10, 4, 7]. Note that [9] the solution of the Schrödinger equation (1.1) satisfies the identity

$$(2it)^{\frac{n}{2}} e^{-i|x|^2/4t} u(x,t) = \widehat{e^{i|\cdot|^2/4t} u_0}(x/2t), \ \text{ for all } t > 0. \tag{1.5}$$

Roughly speaking, this means that the solution at time $t$ is the Fourier transform of the initial data $u_0$ up to a factor with modulus length 1. With (1.5) in hand, the observability inequality (1.3) can be deduced from (1.4). By the same way, it is also proved in [13] the following observability inequality. Let $A, B \subset \mathbb{R}^n$ be measurable sets with finite Lebesgue measure, namely $|A|, |B| < \infty$. Then for all $t > 0$ and all solutions of (1.1)

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \le C(t, A, B) \left( \int_{A^c} |u_0(x)|^2 \,\mathrm{d}x + \int_{B^c} |u(t,x)|^2 \,\mathrm{d}x \right), \tag{1.6}$$

where $A^c$ denotes the complement set of $A$, $C(t, A, B) > 0$ is a constant.

Since the Korteweg-de Vries (KdV) equation is also one of the most important dispersive equations, it is interesting to see whether the above observability inequality holds for the linear KdV equation

$$u_t + u_{xxx} = 0, \quad u(0,x) = u_0(x) \in L^2(\mathbb{R}). \tag{1.7}$$

But for the KdV equation (1.7), no relation similar to (1.5) is available, thus the above argument proving (1.3) (or (1.6)) does not work now. Very recently, Li and Wang have proved in [8] the following analogue inequality of (1.3): There exists a constant $C > 0$ so that for all $r_1, r_2, t > 0$ and all $u(t,x) \in C([0,\infty); L^2(\mathbb{R}))$ solving (1.7),

$$\int_{\mathbb{R}} |u_0(x)|^2 \,\mathrm{d}x \le Ce^{Ct^{-\frac{4}{3}}(r_1^4 + r_2^4)} \left( \int_{|x|\ge r_1} |u_0(x)|^2 \,\mathrm{d}x + \int_{|x|\ge r_2} |u(t,x)|^2 \,\mathrm{d}x \right). \tag{1.8}$$

The proof relies on a quantitative analytic smooth effect of the KdV equation with compact support data and a quantitative unique continuation inequality of analytic functions.

The main goal of this paper is to prove analogues of (1.6) for the KdV equation (1.7). Our first result is as follows.

**Theorem 1.1.** *Let $A, B$ be two measurable sets in $\mathbb{R}$ with finite measure. Then for every $t > 0$, there exists $C = C(t, |A|, |B|) > 0$ so that when $u(t, x)$ solves the KdV equation (1.7),*

$$\int_{\mathbb{R}} |u_0|^2 \,\mathrm{d}x \leq C \left( \int_{A^c} |u_0|^2 \,\mathrm{d}x + \int_{B^c} |u(t,x)|^2 \,\mathrm{d}x \right). \tag{1.9}$$

Clearly, compared to the inequality (1.8), Theorem 1.1 is an improvement in the sense that it holds for more general sets $A$ and $B$. On the other hand, we note that the explicit dependence on $t, |A|, |B|$ of the constant $C$ is lost. The main reason is that our proof, different from that of (1.8), is based on a contradiction argument. The main idea is inspired by the proof of Amerin-Berthier uncertainty principle in [1]. The proof is split into three steps:

(1) Show that (1.9) holds if

$$\|T\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} < 1, \tag{1.10}$$

where $T = \chi_B S(t) \chi_A$, $S(t) = e^{-t\partial_x^3}$ denotes the group generated by (1.7).

(2) Show that $T$ is a compact operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

(3) One can reduce (1.10) to $\|T\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \neq 1$, which shall be proved by a contradiction argument using (2).

In the proof of Theorem 1.1, the assumption that $A$ and $B$ has finite measure is used in both step (2) and step (3). With a closer look at step (2), we find that $T$ is still a compact operator if $A, B$ satisfy some density conditions, where $A, B$ may be sets with infinite measure. This will imply some new observability inequalities at two time points for the KdV equation. To state our result, we need the following definition.

**Definition 1.1.** *We say the set $A \subset \mathbb{R}$ has $|x|^{-\alpha}$ density for some $\alpha > 0$ if* [1]

$$\varlimsup_{x \to \infty} |A \bigcap [x, x+1]| \cdot |x|^{\alpha} \lesssim 1.$$

[1]Here and below, $A \lesssim B$ denote $A \leq CB$ for some not important constant $C > 0$.

**Remark 1.1.** *It is equivalent to say that, there exists $L > 0$ so that*

$$|A \bigcap [x, x+1]| \lesssim |x|^{-\alpha}, \quad \forall |x| \geq L.$$

Our second result reads as follows.

**Theorem 1.2.** *Let $A$ and $B$ be measurable sets with density $|x|^{-\alpha}$, $\alpha > \frac{5}{6}$. Assume further that either $A, B \subset (c, \infty)$ or $A, B \subset (-\infty, c)$ for some $c \in \mathbb{R}$. Then for every $t > 0$, there exists $C = C(t, A, B) > 0$ so that* (1.9) *holds when $u(t, x)$ solves the KdV equation* (1.7)*.*

In Theorem 1.2, the most interesting case is $\alpha \in (\frac{5}{6}, 1]$, since if $\alpha > 1$ then $A$ has finite Lebesgue measure, which has been treated in Theorem 1.1. Let $\alpha \in (\frac{5}{6}, 1]$ and

$$A = B = \bigcup_{k=1}^{\infty} [k, k + k^{-\alpha}].$$

Then $A, B$ satisfy the assumptions of Theorem 1.2, but have infinite Lebesgue measure. Thus, compared to Theorem 1.1, Theorem 1.2 gives some new observable sets at two time points. On the other hand, an obvious drawback of Theorem 1.2 is that $A, B$ are assumed to be contained in a half line. The assumption comes from a unique continuation property of the KdV equation, see Lemma 4.2, which will be used in the proof. We do not know whether the assumption can be removed.

Theorem 1.1 and Theorem 1.2 are quantitative version of the unique continuation property (UCP): If $u(0, \cdot) = 0$ on $A^c$ and $u(t, \cdot) = 0$ on $B^c$ with $t \neq 0$, then $u(t, x) \equiv 0$. Similar UCP results are proved by Zhang in [16, 17]. Another kind of UCP, assuming $u(0, x)$ and $u(t, x)$ have some exponential decay on $x$, is established by Escauriaza, Kenig, Ponce and Vega in [2] for KdV equations and in [3] for Schrödinger equations.

Finally, we remark that the approach used in this paper is robust. It can also be adapted to deal with other kinds of dispersive equations, e.g. higher order KdV equations and higher order Schrödinger equations.

This paper is organized as follows. In Section 2, we give a general criterion for the observability at two time points of the KdV equation. In Section 3 and Section 4, we prove Theorem 1.1 and Theorem 1.2, respectively.

## 2. A general criterion

In this section, we present a general criterion for the observability from two time points of the linear KdV equation. The main idea is inspired from the Hilbert space methods for the uncertainty principle of Fourier transform [4, Part One, Chap. 3].

Let $S(t)=e^{-t\partial_x^3}$ be the solution group of linear KdV equation (1.7). Then the solution of (1.7) can be written as

$$u(t)=S(t)u_0=\int_{\mathbb{R}} G(t,x-y)u_0(y)\,\mathrm{d}y, \tag{2.11}$$

where $G$ is the fundamental solution of linear KdV equation, given by

$$G(t,x)=\begin{cases} \frac{1}{(3t)^{\frac{1}{3}}}\operatorname{Ai}(\frac{x}{(3t)^{\frac{1}{3}}}), & t>0,\\ \delta(x), & t=0.\end{cases} \tag{2.12}$$

Here, $\operatorname{Ai}(x)$ is the Airy function defined via

$$\operatorname{Ai}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(xz+\frac{1}{3}z^3)}\,\mathrm{d}z.$$

According to [12, p.330], there exists a constant $C>0$ so that

$$|\operatorname{Ai}(x)|\le\begin{cases} C(1+|x|)^{-\frac{1}{4}}, & x<0,\\ Ce^{-\frac{2}{3}|x|^{\frac{3}{2}}}, & x\ge 0.\end{cases} \tag{2.13}$$

We remark that (2.12)-(2.13) will be used in Section 3-4. For the linear KdV equation (1.7), the $L^2$ norm is conservative, namely

$$\|S(t)u_0\|_{L^2(\mathbb{R})}=\|u(t,\cdot)\|_{L^2(\mathbb{R})}=\|u_0\|_{L^2(\mathbb{R})},\quad t\in\mathbb{R}. \tag{2.14}$$

Let $A,B$ be two measurable subsets of $\mathbb{R}$. Fix $t\in\mathbb{R}$. Define a linear operator $T: L^2(\mathbb{R})\mapsto L^2(\mathbb{R})$

$$Tf=\chi_B S(t)\big(\chi_A f\big),\quad f\in L^2(\mathbb{R}). \tag{2.15}$$

Here and below, the characteristic function $\chi_A(x)=1$ if $x\in A$ and $\chi_A(x)=0$ if $x\notin A$. Thanks to the conservation law (2.14), it is easy to see that

$$\|T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\le 1. \tag{2.16}$$

**Proposition 2.1.** *Assume that* $\|T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}<1$. *Then there exists* $C>0$ *so that when* $u(t,x)$ *solves* (1.7)

$$\int_{\mathbb{R}}|u_0|^2\,\mathrm{d}x\le C\left(\int_{A^c}|u_0|^2\,\mathrm{d}x+\int_{B^c}|u(t,x)|^2\,\mathrm{d}x\right).$$

*Proof.* Assume that $\|T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = c_1$ for some $0 \le c_1 < 1$. Then, by definition (2.15),

$$\|\chi_B(x)S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})} \le c_1\|u_0\|_{L^2(\mathbb{R})}, \quad \forall u_0 \in L^2(\mathbb{R}).$$

This implies that

$$\|\chi_B(x)S(t)\chi_A u_0\|_{L^2(\mathbb{R})} \le c_1\|\chi_A u_0\|_{L^2(\mathbb{R})} = c_1\|S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})}, \quad \forall u_0 \in L^2(\mathbb{R}),$$

where we used $\|\chi_A u_0\|_{L^2(\mathbb{R})} = \|S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})}$ (by (2.14)) in the last step. From this, we deduce that

$$\begin{aligned}\|S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})} &\le \|\chi_B(x)S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})} + \|\chi_{B^c}(x)S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})}\\ &\le c_1\|S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})} + \|\chi_{B^c}(x)S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})},\end{aligned}$$

which implies that, with $c_2 = 1/(1-c_1)$,

$$\|S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})} \le c_2\|S(t)(\chi_A u_0)\|_{L^2(B^c)}, \quad \forall u_0 \in L^2(\mathbb{R}). \tag{2.17}$$

Now we deduce from (2.17) that

$$\begin{aligned}\|u_0\|_{L^2(\mathbb{R})} = \|S(t)u_0\|_{L^2(\mathbb{R})} &\le \|S(t)(\chi_A u_0)\|_{L^2(\mathbb{R})} + \|S(t)(\chi_{A^c} u_0)\|_{L^2(\mathbb{R})}\\ &\le c_2\|S(t)(\chi_A u_0)\|_{L^2(B^c)} + \|S(t)(\chi_{A^c} u_0)\|_{L^2(\mathbb{R})}\\ &\le c_2\|S(t)u_0\|_{L^2(B^c)} + (1+c_2)\|S(t)(\chi_{A^c} u_0)\|_{L^2(\mathbb{R})}\\ &= c_2\|u(t,\cdot)\|_{L^2(B^c)} + (1+c_2)\|u_0\|_{L^2(A^c)}.\end{aligned}$$

This completes the proof. □

## 3. Proof of Theorem 1.1

From Proposition 2.1, the observability inequality from two time points follows from the bound $\|T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < 1$. For some technical reasons, we shall first show that

$$\|S(-t)T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < 1, \tag{3.18}$$

and then using the fact

$$\|T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \|S(-t)T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}, \tag{3.19}$$

to conclude that $\|T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < 1$. We remark that (3.19) is a direct consequence of the conservation law (2.14). The main part of this section is to prove (3.18) for some sets $A$ and $B$.

We adopt the strategy of Amerin and Berthier in [1]. Thanks to Proposition 2.1 and (3.19), Theorem 1.1 follows from the following proposition.

**Proposition 3.1.** *Let $A$, $B$ be two measurable sets in $\mathbb{R}$ with finite Lebesgue measure, namely $|A|, |B| < \infty$. Let $S(t)$ and $T$ be given by (2.11) and (2.15), respectively. Then for all $t > 0$ we have*

$$\|S(-t)T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < 1.$$

To prove Proposition 3.1, we need some lemmas.

**Lemma 3.1.** *For every $t > 0$, $T$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

*Proof.* By (2.11) and (2.15), we can rewrite the operator $T$ as an integral operator:

$$(Tf)(x) = \int_{\mathbb{R}} \chi_A(x)G(t, x-y)\chi_B(y)f(y)\,\mathrm{d}y := \int_{\mathbb{R}} K(t,x,y)f(y)\,\mathrm{d}y.$$

We claim that for all $t > 0$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{3.20}$$

Then $T$ is a Hilbert-Schmidt operator and thus a compact operator on $L^2(\mathbb{R})$, see e.g. [15, p.277].

It remains to show (3.20). In fact, for every $t > 0$, we deduce from (2.12) and (2.13) that $|G(t, x-y)| \le C(t)$, where $C(t) > 0$ is a constant depending only on $t$. Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y \le C^2(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_A(x)\chi_B(y)dx\,\mathrm{d}y = C^2(t)|A||B| < \infty.$$

This proves (3.20). □

Let $f$ be a measurable function on $\mathbb{R}$ and $A$ be a measurable set. We say that the support $\operatorname{supp} f \subset A$ if

$$f(x) = 0, \quad a.e.\, x \in A^c.$$

**Lemma 3.2.** *Suppose that $\|\chi_B S(t)(\chi_A f)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ for some $f \in L^2(\mathbb{R})$, then the support $\operatorname{supp} f \subset A$ and $\operatorname{supp} S(t)f \subset B$.*

*Proof.* We first show that

$$\operatorname{supp} f \subset A. \tag{3.21}$$

We argue by contradiction. Suppose that

$$\Big|\{x \in \mathbb{R} : |f(x)| > 0\}\backslash A\Big| > 0.$$

This implies that

$$\|f\|_{L^2(A^c)} > 0. \tag{3.22}$$

But by the assumption

$$\|f\|_{L^2(\mathbb{R})} = \|\chi_B S(t)(\chi_A f)\|_{L^2(\mathbb{R})} \leq \|\chi_A f\|_{L^2(\mathbb{R})},$$

which implies that $\|f\|_{L^2(A^c)} = 0$. This leads a contradiction with (3.22). So (3.21) holds.

Using (3.21) and the assumption, we deduce that

$$\|\chi_B S(t)f\|_{L^2(\mathbb{R})} = \|\chi_B S(t)(\chi_A f)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})} = \|S(t)f\|_{L^2(\mathbb{R})}.$$

This gives that $\operatorname{supp} S(t)f \subset B$. □

For $\lambda \in \mathbb{R}$ and $A \subset \mathbb{R}$, we denote by $A + \lambda = \{x \in \mathbb{R} | x + \lambda \in A\}$.

**Lemma 3.3.** *Let $C_0 \subset C$ be two measurable sets in $\mathbb{R}$ satisfying $0 < |C_0|, |C| < \infty$. Define a function*

$$h_C(\lambda) = |C \cup (C_0 + \lambda)|, \quad \lambda \geq 0.$$

*Then $h_C$ is a continuous function of $\lambda$, $h(0) = |C|$ and $\lim\limits_{\lambda\to\infty} h_C(\lambda) = |C| + |C_0|$.*

*Proof.* See [4, p.99]. □

We need the following lemma, which is a variant version of [1, Lemma 1].

**Lemma 3.4.** *Let $C_0 \subset C, D_0 \subset D$ be four measurable sets in $\mathbb{R}$ satisfying $0 < |C_0|, |C|$, $|D_0|, |D| < \infty$. Then for every $\varepsilon > 0$, there exists a translation $\lambda > 0$ such that*

$$|C \cup (C_0 + \lambda)| \leq |C| + \varepsilon, \quad |D \cup (D_0 + \lambda)| \leq |D| + \varepsilon \tag{3.23}$$

*and either*

$$|C| < |C \cup (C_0 + \lambda)| \tag{3.24}$$

*or*

$$|D| < |D \cup (D_0 + \lambda)|. \tag{3.25}$$

*Proof.* Define two functions

$$h_C = |C \cup (C_0 + \lambda)|, \quad h_D = |D \cup (D_0 + \lambda)|, \quad \lambda \geq 0.$$

Fix $\varepsilon > 0$. Consider the level set

$$\{h_C = |C| + \varepsilon\} = \{\lambda \geq 0 : h_C(\lambda) = |C| + \varepsilon\}.$$

Thanks to Lemma 3.3, $h_C$ is a continuous function of $\lambda$, $h(0) = |C|$ and $\lim_{\lambda\to\infty} h_C(\lambda) = |C| + |C_0|$. Thus $\{h_C = |C| + \varepsilon\}$ is a nonempty closed set. Put

$$\lambda' = \min_{\lambda \in h_C = |C|+\varepsilon} \lambda.$$

Then by the continuity of $h_C$ again, we have

$$|C| \le h_C(\lambda) \le |C| + \varepsilon, \quad \forall \lambda \in [0, \lambda']. \tag{3.26}$$

We divide the discussion into two cases.

**Case 1:** $h_D(\lambda') \le |D| + \varepsilon$. This, together with the fact $h_C(\lambda') = |C| + \varepsilon$, implies that (3.23) and (3.24) hold with $\lambda = \lambda'$.

**Case 2:** $h_D(\lambda') > |D| + \varepsilon$. Applying Lemma 3.3 to $h_D$, we find that

$$h_D(\lambda'') = |D| + \varepsilon$$

for some $\lambda'' \in (0, \lambda')$. Moreover, by (3.26) we have

$$|C| \le h_C(\lambda'') \le |C| + \varepsilon.$$

Thus, in this case, (3.23) and (3.25) hold with $\lambda = \lambda''$. $\square$

***Proof of Proposition 3.1.*** First of all, by (2.14) and (2.16), we have the trivial bound

$$\|S(-t)T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le 1.$$

Thus Proposition 3.1 holds if one can show that, for all $t > 0$

$$\|S(-t)T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \ne 1. \tag{3.27}$$

To show this, we use the contradiction argument. Fix $t > 0$. Suppose that

$$\|S(-t)T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = 1. \tag{3.28}$$

Since $\|S(-t)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = 1$ and, by Lemma 3.1, $T$ is a compact operator on $L^2(\mathbb{R})$, we find that $S(-t)T$ is a compact operator on $L^2(\mathbb{R})$. This, together with (3.28), implies that there exists a function $f \in L^2(\mathbb{R})$ such that

$$\|S(-t)Tf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})} = 1. \tag{3.29}$$

Since $\|S(-t)Tf\|_{L^2(\mathbb{R})} = \|Tf\|_{L^2(\mathbb{R})}$, by (3.29), we have $\|Tf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$. Then, by Lemma 3.2, we find

$$\mathrm{supp} f \subset A, \quad \mathrm{supp}\, S(t)f \subset B. \tag{3.30}$$

Set

$$A_0 = \text{supp} f, \quad B_0 = \text{supp}\, S(t)f.$$

By the assumption on $A, B$ and (3.30) we know that

$$0 < |A_0|, |B_0| < \infty. \tag{3.31}$$

Thanks to (3.31), we can apply Lemma 3.4 to find a sequence $\{\lambda_i\}_{i=1}^{\infty} \subset (0, \infty)$ such that

$$|A_{i-1} \cup (A_0 + \lambda_i)| \leq |A_{i-1}| + \frac{1}{2^i}, \quad |B_{i-1} \cup (B_0 + \lambda_i)| < |B_{i-1}| + \frac{1}{2^i} \tag{3.32}$$

and

$$\text{either } |A_{i-1}| < |A_{i-1} \cup (A_0 + \lambda_i)| \quad \text{or} \quad |B_{i-1}| < |B_{i-1} \cup (B_0 + \lambda_i)|, \tag{3.33}$$

where for $i = 1, 2, \cdots,$

$$A_i = A_{i-1} \cup (A_0 + \lambda_i), \quad B_i = B_{i-1} \cup (B_0 + \lambda_i). \tag{3.34}$$

It follows from (3.32)-(3.34) that

$$|\bigcup_{i=0}^{\infty} A_i| \leq |A| + 1, \quad |\bigcup_{i=0}^{\infty} B_i| \leq |B| + 1. \tag{3.35}$$

Define $f_0 = f$ and

$$f_i = \mathcal{T}_{\lambda_i} f, \quad i = 1, 2, \cdots.$$

where $\mathcal{T}_\lambda$ is the translation operator, namely

$$\mathcal{T}_\lambda f(x) = f(x - \lambda).$$

It is easy to see that $\text{supp}\, f_\lambda = A_0 + \lambda$ and $S(t)f_\lambda = \mathcal{T}_\lambda(S(t)f)$, thus $\text{supp}\, S(t)f_\lambda = B_0 + \lambda$. Then for $i = 1, 2, \cdots$

$$\text{supp} f_i = A_0 + \lambda_i \subset A_i, \quad \text{supp}\, S(t)f_i = B_0 + \lambda_i \subset B_i. \tag{3.36}$$

Let $\mathcal{A} = \bigcup_{i=0}^{\infty} A_i$ and $\mathcal{B} = \bigcup_{i=0}^{\infty} B_i$. We have the following claims.

(i) The sequence $\{f_i\}_{i=0}^{\infty}$ are linearly independent.

(ii) The operator $S(-t)\chi_{\mathcal{B}} S(t)\chi_{\mathcal{A}}$ is compact on $L^2(\mathbb{R})$.

(iii) For all $i = 0, 1, \cdots,$ $f_i$ are the eigenfunctions of $S(-t)\chi_{\mathcal{B}} S(t)\chi_{\mathcal{A}}$ corresponding to the eigenvalue 1.

To show $(i)$, we fix $m \in \mathbb{N}$. Thanks to (3.33) and (3.36), we have either $\chi_{A_m \backslash A_{m-1}} f_m \neq 0$ or $\chi_{B_m \backslash B_{m-1}} S(t) f_m \neq 0$. In both cases it can be shown that $f_m$ is not a linear combination of $f_0, f_1, \cdots, f_{m-1}$. This means that the sequence $\{f_i\}_{i=0}^m$ are linearly independent. Thus $(i)$ holds.

For $(ii)$, we first note that $|\mathcal{A}|, |\mathcal{B}| < \infty$ since (3.35). Then by Lemma 3.1, we find that $S(-t)\chi_{\mathcal{B}} S(t) \chi_{\mathcal{A}}$ is compact.

For $(iii)$, by (3.36) and the definition of $\mathcal{A}$ and $\mathcal{B}$, we have

$$\mathrm{supp} f_i \subset \mathcal{A}, \quad \mathrm{supp}\, S(t) f_i \subset \mathcal{B}$$

for all $i = 0, 1, \cdots$. Hence

$$S(-t) T f_i = S(-t)\chi_{\mathcal{B}} S(t)(\chi_{\mathcal{A}} f_i) = S(-t)\chi_{\mathcal{B}} S(t) f_i = S(-t) S(t) f_i = f_i.$$

Namely, for every $i$, $f_i$ is an eigenfunction of $S(-t)\chi_{\mathcal{B}} S(t)\chi_{\mathcal{A}}$ corresponding to the eigenvalue 1. Thus $(iii)$ holds.

Finally, the claims $(i)$ and $(iii)$ imply that the operator $S(-t)\chi_{\mathcal{B}} S(t)\chi_{\mathcal{A}}$ has infinitely many linearly independent eigenfunctions corresponding to the eigenvalue 1, which contradicts to the compact property in $(ii)$. This shows that the assumption (3.28) is invalid. Thus (3.27) holds. $\square$

## 4. Proof of Theorem 1.2

We shall prove a more general theorem than Theorem 1.2 in this section. In fact, if $\alpha = \beta > \frac{5}{6}$, then $(\alpha, \beta)$ satisfy that

$$(\mathbf{H}) \qquad \begin{cases} \alpha + \beta > \frac{5}{3} \\ \alpha + 3\beta > 3 \\ 3\alpha + \beta > 3 \\ \alpha, \beta > \frac{1}{2} \end{cases}.$$

Thus, Theorem 1.2 is a direct consequence of the follow theorem.

**Theorem 4.1.** *Let $(\alpha, \beta)$ satisfy the condition $(\mathbf{H})$. Assume that $A$ and $B$ are measurable sets with density $|x|^{-\alpha}$ and $|x|^{-\beta}$, respectively. Assume further that either $A, B \subset (c, \infty)$ or $A, B \subset (-\infty, c)$ for some constant $c \in \mathbb{R}$. Then for every $t > 0$, there exists $C = C(t, A, B) > 0$ so that when $u(t, x)$ solves the KdV equation (1.7),*

$$\int_{\mathbb{R}} |u_0|^2 \,\mathrm{d}x \leq C \left( \int_{A^c} |u_0|^2 \,\mathrm{d}x + \int_{B^c} |u(t,x)|^2 \,\mathrm{d}x \right).$$

The idea is still to use Proposition 2.1, namely to show $\|T\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < 1$, where

$$Tf = \chi_B S(t)\chi_A f = \int_{\mathbb{R}} K(t,x,y)f(y)\,\mathrm{d}y, \quad f \in L^2(\mathbb{R}) \tag{4.37}$$

and

$$K(t,x,y) = \chi_B(x)G(t,x-y)\chi_A(y)f(y). \tag{4.38}$$

Here $S(t) = e^{-t\partial_x^3}$ and $G$ is given by (2.12).

**Lemma 4.1.** *Let $(\alpha,\beta)$ satisfy the condition* (**H**)*. Assume that $A$ and $B$ are measurable sets with density $|x|^{-\alpha}$ and $|x|^{-\beta}$, respectively. Then the kernel $K$, given by* (4.38)*, satisfies*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

*Proof.* Thanks to (4.38), (2.12) and (2.13), it suffices to show that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_B(x)\langle|x-y|\rangle^{-\frac{1}{2}}\chi_A(y)\,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{4.39}$$

Here and below, $\langle x\rangle = 1+|x|$. Rewrite the left hand side of (4.39) as

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_j^{j+1}\int_k^{k+1} \chi_B(x)\langle|x-y|\rangle^{-\frac{1}{2}}\chi_A(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.40}$$

But when $x \in [k,k+1]$, $y \in [j,j+1]$, then

$$\langle|x-y|\rangle \sim \langle|j-k|\rangle.$$

Here $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. Now (4.40) is bounded by

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_j^{j+1}\int_k^{k+1} \chi_B(x)\langle|j-k|\rangle^{-\frac{1}{2}}\chi_A(y)\,\mathrm{d}x\,\mathrm{d}y \\
=&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |B\cap[k,k+1]|\cdot\langle|j-k|\rangle^{-\frac{1}{2}}\cdot|A\cap[j,j+1]| \\
\lesssim& 1 + \sum_{j,k\in\mathbb{Z},|j|,|k|\ge L} |B\cap[k,k+1]|\cdot\langle|j-k|\rangle^{-\frac{1}{2}}\cdot|A\cap[j,j+1]| \\
\lesssim& \sum_{j,k\in\mathbb{Z},|j|,|k|\ge L} |j|^{-\alpha}|k|^{-\beta}\langle|j-k|\rangle^{-\frac{1}{2}}.
\end{aligned} \tag{4.41}$$

In the last line of (4.41), we have used Remark 1.1. Split the sum in (4.41) as

$$\sum_{j,k\in\mathbb{Z},|j|,|k|\ge L} |j|^{-\alpha}|k|^{-\beta}\langle|j-k|\rangle^{-\frac{1}{2}} := I + II, \tag{4.42}$$

where $I, II$ taking sum over $|j| \le |k|$ and $|j| > |k|$, respectively.

**Estimate of $I$.** Let $\delta \in (0,1)$ to be determined later. Make further decomposition $I := I_1 + I_2$, where

$$I_1 = \sum_{|j|,|k| \ge L, |k|-|k|^\delta \le |j| \le |k|} |j|^{-\alpha}|k|^{-\beta}\langle |j-k|\rangle^{-\frac{1}{2}}, \tag{4.43}$$

$$I_2 = \sum_{|j|,|k| \ge L, |j| > |k|-|k|^\delta} |j|^{-\alpha}|k|^{-\beta}\langle |j-k|\rangle^{-\frac{1}{2}}. \tag{4.44}$$

For $I_1$, since $|k| - |k|^\delta \le |j| \le |k|$, we have

$$I_1 \le \sum_{|k| \ge L, |k|-|k|^\delta \le |j| \le |k|} |j|^{-\alpha}|k|^{-\beta} \lesssim \sum_{|k| \ge L} |k|^{\delta-\alpha}|k|^{-\beta} = \sum_{|k| \ge L} |k|^{\delta-\alpha-\beta}. \tag{4.45}$$

For $I_2$, since $|j| < |k| - |k|^\delta$, we have $|k-j| \ge |k| - |j| \ge |k|^\delta$, then $\langle |j-k|\rangle^{-1} \lesssim |k|^{-\delta}$. We split the discussion into two cases.

- $0 < \alpha \le 1$, we have

$$\begin{aligned} I_2 &= \sum_{|k| \ge L, |j| < |k|-|k|^\delta} |j|^{-\alpha}|k|^{-\beta}\langle |j-k|\rangle^{-\frac{1}{2}} \lesssim \sum_{|k| \ge L, |j| < |k|-|k|^\delta} |j|^{-\alpha}|k|^{-\beta-\frac{1}{2}\delta} \\ &\lesssim \begin{cases} \sum\limits_{|k| \ge L} |k|^{1-\alpha}|k|^{-\beta-\frac{1}{2}\delta} = \sum\limits_{|k| \ge L} |k|^{1-\frac{1}{2}\delta-\alpha-\beta}, & 0 < \alpha < 1, \\ \sum\limits_{|k| \ge L} \ln|k| \cdot |k|^{-\beta-\frac{1}{2}\delta}, & \alpha = 1. \end{cases} \end{aligned} \tag{4.46}$$

- $\alpha > 1$, we have

$$\begin{aligned} I_2 &= \sum_{|k| \ge L, |j| < |k|-|k|^\delta} |j|^{-\alpha}|k|^{-\beta}\langle |j-k|\rangle^{-\frac{1}{2}} \lesssim \sum_{|k| \ge L, |j| < |k|-|k|^\delta} |j|^{-\alpha}|k|^{-\beta-\frac{1}{2}\delta} \\ &\lesssim \sum_{|k| \ge L} |L|^{1-\alpha}|k|^{-\beta-\frac{1}{2}\delta} \lesssim \sum_{|k| \ge L} |k|^{-\frac{1}{2}\delta-\beta}. \end{aligned} \tag{4.47}$$

Combining (4.43)-(4.47), we obtain that

$$I \lesssim \begin{cases} \sum\limits_{|k| \ge L} |k|^{\delta-\alpha-\beta} + |k|^{1-\frac{1}{2}\delta-\alpha-\beta}, & 0 < \alpha < 1, \\ \sum\limits_{|k| \ge L} |k|^{\delta-\alpha-\beta} + \ln|k| \cdot |k|^{-\beta-\frac{1}{2}\delta}, & \alpha = 1, \\ \sum\limits_{|k| \ge L} |k|^{\delta-\alpha-\beta} + |k|^{-\frac{1}{2}\delta-\beta}, & \alpha > 1. \end{cases}$$

This implies that $I < \infty$ if there exists $\delta \in (0,1)$ so that one of the following lines holds:

$$\delta - \alpha - \beta < -1, 1 - \frac{1}{2}\delta - \alpha - \beta < -1, 0 < \alpha < 1; \tag{4.48}$$

$$\delta - \alpha - \beta < -1, -\beta - \frac{1}{2}\delta < -1, \alpha = 1; \tag{4.49}$$

$$\delta - \alpha - \beta < -1, -\frac{1}{2}\delta - \beta < -1, \alpha > 1. \tag{4.50}$$

On one hand, it is easy to see that (4.48)-(4.49) holds if

$$0 < \alpha \leq 1, \quad \alpha + \beta > 1 + \delta, \quad \alpha + \beta > 2 - \frac{1}{2}\delta.$$

By choosing $\beta = \frac{2}{3}$, we find that $I < \infty$ if

$$0 < \alpha \leq 1, \quad \alpha + \beta > \frac{5}{3}. \tag{4.51}$$

On the other hand, (4.49) holds if there exists $\delta \in (0,1)$ so that

$$\alpha > 1, \quad \alpha + \beta - 1 > \delta > 2(1 - \beta),$$

which holds if

$$\alpha > 1, \quad \alpha + \beta - 1 > 2(1 - \beta), \quad 1 > 2(1 - \beta).$$

This implies that $I < \infty$ if

$$\alpha > 1, \quad \beta > \frac{1}{2}, \quad \alpha + 3\beta > 3. \tag{4.52}$$

**Estimate of $II$.** Similarly, with the same argument, we find that $II < \infty$ if $\alpha, \beta$ satisfying either

$$0 < \beta \leq 1, \quad \alpha + \beta > \frac{5}{3} \tag{4.53}$$

or

$$\beta > 1, \quad \alpha > \frac{1}{2}, \quad \beta + 3\alpha > 3. \tag{4.54}$$

To make both the term $I$ and $II$ finite, we have four cases:

$$(4.51) + (4.53) \begin{cases} 0 < \alpha, \beta \leq 1 \\ \alpha + \beta > \frac{5}{3} \end{cases}, \quad (4.51) + (4.54) \begin{cases} \frac{1}{2} < \alpha \leq 1 \\ \beta > 1 \\ \alpha + \beta > \frac{5}{3} \\ 3\alpha + \beta > 3 \end{cases},$$

$$(4.52) + (4.53) \begin{cases} \alpha > 1 \\ \frac{1}{2} < \beta \leq 1 \\ \alpha + \beta > \frac{5}{3} \\ \alpha + 3\beta > 3 \end{cases}, \quad (4.52) + (4.54) \begin{cases} \alpha > 1 \\ \beta > 1 \end{cases}.$$

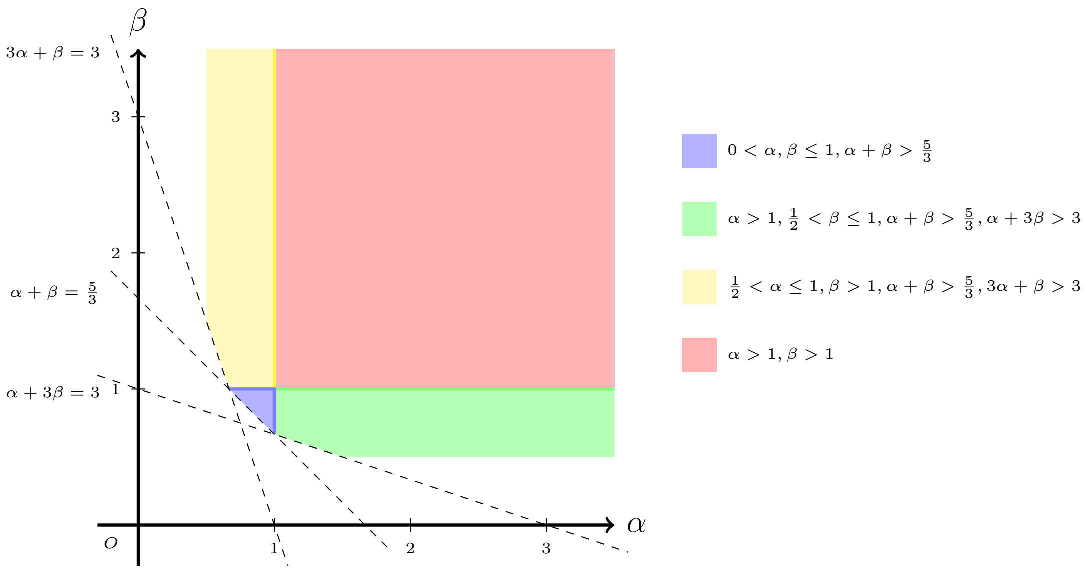


Figure 1: The range of $(\alpha, \beta)$ ensures $\int_{\mathbb{R}^2} K^2 \, \mathrm{d}x\mathrm{d}y < \infty$.

This completes the proof since the union of the four cases is equivalent to that $(\alpha, \beta)$ satisfies the condition (**H**), see Figure 1. □

To obtain Theorem 1.2, we need the following unique continuation result.

**Lemma 4.2.** *Let $u(t,x) \in C(\mathbb{R}, L^2(\mathbb{R}))$ be a solution of $\partial_t u + \partial_x^3 u = 0$. If there exist two time points $t_1 \neq t_2$ such that*

$$\operatorname{supp} u(t_j, \cdot) \subset (-\infty, c), \quad j = 1, 2 \tag{4.55}$$

*or*

$$\operatorname{supp} u(t_j, \cdot) \subset (c, \infty), \quad j = 1, 2 \tag{4.56}$$

*for some $c \in \mathbb{R}$, then*

$$u(t,x) \equiv 0, \quad -\infty < t, x < \infty.$$

*Proof.* See Theorem 3.1 in [16, p. 60]. □

***Proof of Theorem 1.2.*** Fix $t > 0$. Similar to the proof of Theorem 1.2, it suffices to show that

$$\|T\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \neq 1.$$

We argue by contradiction. Suppose that $\|T\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = 1$. Thanks to Lemma 4.1, we know $T$ is a compact operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Thus, there exists $f \in L^2(\mathbb{R})$ so that

$$\|Tf\|_{L^2(\mathbb{R})} = \|\chi_B S(t) \chi_A f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})} = 1. \tag{4.57}$$

With (4.57) in hand, it follows from Lemma 3.2 that $\operatorname{supp} f \subset A$ and $\operatorname{supp} S(t)f \subset B$. Let $u(\tau,\cdot) = S(\tau)f$. Then $u(\tau,\cdot)$ is a solution of the equation

$$\partial_\tau u + \partial_x^3 u = 0.$$

By the assumption on $A, B$, we have either $\operatorname{supp} u(\tau,\cdot) \subset (-\infty, c)$ or $(c,\infty)$ for $\tau = 0, t$, according to Lemma 4.2, we conclude that $u(\tau,\cdot) \equiv 0$. In particular, this implies that $f = 0$. But this is impossible since $\|f\|_{L^2(\mathbb{R})} = 1$ by (4.57). So we complete the proof. $\square$

## Acknowledgements

This work is partially supported by the National Natural Science Foundation of China under grant No.11701535, and the Fundamental Research Funds for the Central Universities, China University of Geosciences(Wuhan) under grant No.CUGSX01.

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