ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
LECTURE 07

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Theorem 1. *Let X, Y be varieties, assume $Y \subset \mathbb{A}^m$ be a closed affine variety. Then there is a bijection between morphisms $X \to Y$ and k-algebra homomorphisms $A(Y) \to \mathcal{O}_X(X)$:*

$$\begin{array}{ccc} \{\textit{morphisms } X \to Y\} & \xrightarrow{\textit{bijection}} & \{\textit{homomorphisms } A(Y) \to \mathcal{O}_X(X)\} \\ \varphi & \longrightarrow & \varphi^* \end{array}$$

Proof. $\Rightarrow$: Let $\varphi : X \to Y$ be a morphism, then $\varphi^* : A(Y) \to \mathcal{O}_X(X)$ is a k-algebra homomorphism.

$\Leftarrow$: Let $\phi : A(Y) \to \mathcal{O}_X(X)$ be a k-algebraic homomorphism, let $y_1, \ldots, y_n \in A(Y)$ be the coordinate functions. We set

$$f_i = \phi(y_i) \in \mathcal{O}_X(X).$$

Let $\varphi = (f_1, \ldots, f_m) : X \to \mathbb{A}^m$. This is a morphism from X to Y. To see it is a morphism we have to show $\varphi(X) \subset Y$. Let $h \in I(Y)$, $h \circ \varphi = h(f_1, \ldots, f_m) = h(\phi(y_1), \ldots, \phi(y_m)) = \phi(h(y_1, \ldots, y_m))$. The second equality is based on the homomorphic property of ϕ, for example, if $h(x_1, x_2) = x_1^2 - x_2^3$, then $h(\phi(y_1), \phi(y_2)) = \phi(y_1)^2 - \phi(y_2)^3 = \phi(y_1^2) - \phi(y_2^3) = \phi(y_1^2 - y_2^3) = \phi(h(y_1, y_2))$. So $h(y_1, \ldots, y_m) \in A(Y)$, we choose an arbitrary element $p = (a_1, \ldots, a_m) \in Y$, then $h(y_1, \ldots, y_m)(p) = h(a_1, \ldots, a_m) = 0$ because $h \in I(Y)$. So for arbitrary $h \in I(Y)$, we get $h \circ \varphi = 0$, it implies $\varphi(X) \subset \cap_{h \in I(Y)} Z(h) = Y$. $\square$

Example 1. A bijective polynomial map need not to be an isomorphism. For example, let$X = \mathbb{A}^1$, $Y = Z(x_2^2 - x_1^3) \subset \mathbb{A}^2$. Then

$$\varphi = (t^2, t^3) : X \to Y$$

is a morphism and bijective and the inverse is

$$\varphi^{-1}(a, b) = \begin{cases} \frac{b}{a} & \text{if } a \neq 0 \\ 0 & \text{if } (a, b) = 0 \end{cases}$$

φ is not an isomorphism(φ^{-1} is not a morphism). To show this we see the pull back:

$$\varphi^* : A(Y) \to \mathcal{O}_X(X)$$

where $A(Y) = k[x_1, x_2]/\langle x_2^2 - x_1^3 \rangle$ and $A(X) = k[t]$. φ^* makes $x_1 \to t^2$ and $x_2 \to t^3$. Since φ^* is not surjective(there is no element maps into t), φ^* is not an isomorphism. By theorem 1 we know φ is not an isomorphism. So bijective morphism is not necessary to be an isomorphism.

Definition 1. Let $X \subset \mathbb{A}^n$ be a closed variety, $F \in k[x_1, \ldots, x_n] \backslash I(X)$. The principal open defined by F is $X_F := X \backslash Z(F)$.

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Proposition 1. *X_F is an affine variety.*

Proof. Let $Z := Z(\langle I(X), F \cdot x_{n+1} - 1\rangle) \subset \mathbb{A}^{n+1}$. We need to prove Z is a closed subvariety of $\mathbb{A}^{n+1}$ isomorphic to X_F. Let $\varphi : (x_1, \dots, x_n, \frac{1}{F}) : X_F \to \mathbb{A}^{n+1}$, it is a bijective morphism and $\varphi(X_F) = Z$. As X_F is irreducible, Z is also irreducible. So Z is closed variety of $\mathbb{A}^{n+1}$. On the other hand, the inverse of φ is

$$\varphi^{-1} = (x_1, \dots, x_n) : Z \to X_F$$

is a morphism, so φ is an isomorphism. □

Definition 2. Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be quasi-projective algebraic sets. A map $\varphi : X \to Y$ is called a polynomial map if there exists homogeneous polynomials $F_0, \dots, F_m \in k[x_0, \dots, x_n]$ of the same degree with no common zero on X s.t. $\varphi(p) = [F_0(p), \dots, F_m(p)]$, $\forall p \in X$, write $\varphi = [F_0, \dots, F_m]$.

Definition 3. The homogenization of $F \in k[x_0, \dots, x_n]$ is:

$$F_a := F(1, x_1, \dots, x_n).$$

Theorem 2. $\varphi_i = (\frac{x_0}{x_i}, \dots, \frac{\hat{x_i}}{x_i}, \dots, \frac{x_n}{x_i}) : U_i \to \mathbb{A}^n$ *is an isomorphism.*

Proof. We can assume $i = 0$, $\varphi := \varphi_0$, $U := U_0$, then $\varphi = (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$. $\frac{x_i}{x_0}$ is a regular function in $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$, so φ is a morphism. We need to show that $u = \varphi^{-1}(x_1, \dots, x_n) = [1, x_1, \dots, x_n]$ is a morphism.

(a) $u = \varphi^{-1}$ is continuous. Let $W = Z(F_1, \dots, F_m) \cap U$ be closed in U, $F_i \in k[x_0, \dots, x_n]$ are homogeneous, then

$$\begin{aligned} u^{-1}(W) = &\quad \{(a_1, \dots, a_n) \in \mathbb{A}^n | [1, a_1, \dots, a_n] \in W\} \\ = &\quad \{(a_1, \dots, a_n) \in \mathbb{A}^n | F_i(1, a_1 \dots, a_n) = 0, \forall i = 1, \dots, m\} \\ = &\quad Z(F_{1a}, \dots, F_{ma}) \end{aligned}$$

where F_{ia} is homogenization of F_i, it shows that $u^{-1}(W)$ is closed in $\mathbb{A}^n$.

(b) Let $V \subset U$ be open, $h \in \mathcal{O}_U(V)$, we need to show $u^*h \in \mathcal{O}_{\mathbb{A}^n}(u^{-1}(V))$. Making V smaller necessary, we can assume $h = \frac{F}{G}$, $F, G \in k[x_0, \dots, x_n]$ are homogeneous polynomials of the same degree.

$$u^*h = h \circ u = \frac{F \circ u}{G \circ u} = \frac{F(1, x_1, \dots, x_n)}{G(1, x_1, \dots, x_n)}.$$

Thus $u^*h \in \mathcal{O}_{\mathbb{A}^n}(u^{-1}(V))$, $phi : \mathbb{A}^n \to u$ is an isomorphism. □

Remark. From theorem 2 we find that if we identify $\mathbb{A}^n$ with $u_0 \subset \mathbb{P}^n$, the Zariski topology on $\mathbb{A}^n$ is equivalent to the induced topology of u_0 from $\mathbb{P}^n$.

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