Chapter 1

hello

1.1 Classical Field Theory

Analogy between classical field theory and the system of finite particles

1.1.1 The System of finite particles

$$S = \int \mathrm{d}t \sum_a L\left(q_a, \dot{q}_a\right).$$

$$\frac{\delta S}{\delta q_a\left(t\right)} = 0.$$

$$\frac{\delta L}{\delta q_a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_a} = 0.$$

The generalized momentum is

$$p^a = \frac{\partial L}{\partial \dot{q}_a}.$$

Then the Hamiltonian is

$$H = p^a \dot{q}_a - L.$$

We use the Einstein summation convention here and later on.

1.1.2 Classical Field

Here the generalized coordinate is changed to $\varphi\left(x\right)$, t and $\vec{x}$ are parameters,

$$\varphi\left(x\right) = \varphi\left(t, \vec{x}\right).$$

$$\int \mathrm{d}t \int \mathrm{d}^3 x \mathcal{L}\left(\varphi\left(x\right), \partial_\mu \varphi\left(x\right)\right).$$

$$\frac{\delta S}{\delta \varphi\left(x\right)} = 0.$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi\right)} = 0.$$

The generalized momentum is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}.$$

The Hamiltonian is

$$H = \int \mathrm{d}^3 x \mathcal{H} = \int \mathrm{d}^3 x \left[\pi(x)\dot{\varphi}(x) - \mathcal{L}(x)\right].$$

1.1.3 Klein-Gordon Theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2.$$

$$\pi = \dot{\varphi}.$$

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2.$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = -m^2\varphi^2 - \partial_\mu\partial^\mu\varphi = 0.$$

i.e.,

$$(\partial^2 + m^2)\varphi = 0.$$

This is the Klein-Gordon equation.

Noether's Theorem

- continuous symmetry $\to$ conserved current j^μ $\partial_\mu j^\mu = 0$
- conserved current $\to$ conserved charge $Q = \int_{\mathbb{R}^3} \mathrm{d}^3x j^0$

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= \int_{\mathbb{R}^3} \mathrm{d}^3x \frac{\partial j^0}{\partial t} \\ &= -\int_{\mathbb{R}^3} \mathrm{d}^3x \nabla\cdot\mathbf{j} \\ &= -\int_{\partial\mathbb{R}^3} \mathbf{j}\cdot\mathrm{d}\mathbf{s} \\ &= (\text{If } \mathbf{j}\to 0 \text{ quickly enough as } |\mathbf{x}|\to\infty)\ .\end{aligned}$$

Let $\delta\varphi = Y(\varphi)$ is a symmetry if

$$\begin{aligned}\delta\mathcal{L} =& \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_\mu(\delta\varphi) \\ =& \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right).\end{aligned}$$

The first term is 0 if the equation of motion is satisfied.

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi}Y(\varphi)\right) = \partial_\mu F^\mu(\varphi).$$

Then we have

$$\partial_\mu j^\mu = 0$$

where $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}$.

Translations

$$x^\nu \to x^\nu - \epsilon^\nu.$$

$$\varphi(x) \to \varphi(x) + \epsilon^\nu \partial_\nu \varphi(x).$$

$$\mathcal{L}(x) \to \mathcal{L}(x) + \epsilon^\nu \partial_\nu \mathcal{L}.$$

$$\mathbf{j}^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu{}_\nu \mathcal{L} \equiv T^\mu{}_\nu .$$

This is called stress-energy tensor. The 4-momentum is

$$P^\mu = \int \mathrm{d}^3 x T^{0\mu}.$$

$U(1)$ **symmetry**

Consider complex scalar field $\Phi(x)$

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - V(\left|\Phi^2\right|).$$

$$\begin{aligned} \Phi &\to e^{i\theta} \Phi \\ \Phi^* &\to e^{-i\theta} \Phi^*. \end{aligned}$$

$$\delta \Phi = i\theta \Phi$$

$$\delta \Phi^* = -i\theta \Phi^*.$$

$$\delta \mathcal{L} = 0.$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} \delta \Phi^*.$$

Drop the θ term and rewrite it as

$$j^\mu = \Phi \partial^\mu \Phi^* - \Phi^* \partial^\mu \Phi.$$

The charge is the electric charge or the particle number.

1.1.4 Solution of the Klein-Gordon Equation

$$\varphi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{\varphi}(t, \mathbf{k}).$$

$$\dot{\varphi}^2 - (\nabla \varphi)^2 + m^2 \varphi^2 = 0.$$

$$\dot{\tilde{\varphi}}^2 + (\mathbf{k}^2 + m^2) \tilde{\varphi} = 0.$$

$$\tilde{\varphi}(t, \mathbf{k}) = A(\mathbf{k}) e^{-iE_{\mathbf{k}} t} + B(\mathbf{k}) e^{iE_{\mathbf{k}} t}$$

where $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$.

Klein-Gordon field is a real field hence $\varphi = \varphi^*$,

$$\varphi^*(t, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \tilde{\varphi}^*(t, \mathbf{k})$$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{\varphi}^*(t, -\mathbf{k}).$$

$$\tilde{\varphi}(t, kve) = \tilde{\varphi}^*(t, -\mathbf{k}).$$

i.e.,

$$B(\mathbf{k}) = A^*(t, -\mathbf{k}).$$

$$\varphi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[A(\mathbf{k}) e^{-iE_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}} + A^*(-\mathbf{k}) e^{iE_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}}\right].$$

Rewrite it as

$$\varphi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[A(\mathbf{k}) e^{-ik\cdot x} + A^*(\mathbf{k}) e^{ik\cdot x}\right].$$

Lorentz invariant term:

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_{\mathbf{k}}} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\Theta(k^0)$$

where $\Theta\,(x)$ is Heaviside function and invariant under orthochronous Lorentz transformations. Rewrite the general solution as following:

$$\varphi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_{\mathbf{k}}} \left[a(\mathbf{k}) e^{-ik\cdot x} + a^*(\mathbf{k}) e^{ik\cdot x}\right].$$

1.2 Canonical Quantization of the Klein-Gordon Field

1.2.1 Quantization

In quantum mechanics

$$
\begin{aligned}
[q_a, p_b] &= i\delta_{ab}. \\
[q_a, q_b] &= 0. \\
[p_a, p_b] &= 0.
\end{aligned}
$$

Similarly, quantize the Klein-Gordon field as following

$$
\begin{aligned}
[\varphi(\mathbf{x}, \pi(\mathbf{y})] &= i\delta(\mathbf{x}-\mathbf{y}) \\
[\varphi(\mathbf{x}, \varphi(\mathbf{y}] &= 0 \\
[\pi(\mathbf{x}), \pi(\mathbf{y})] &= 0.
\end{aligned}
$$

In classical field theory, the coefficients $a(\mathbf{k})$ and $a^*(\mathbf{k})$ are numbers, after quantization, they are changed into operators

$$
\begin{aligned}
a(\mathbf{k}) \rightarrow & a_{\mathbf{k}} \\
a^*(\mathbf{k}) \rightarrow & a_{\mathbf{k}}^\dagger.
\end{aligned}
$$

$$
\varphi(\mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_{\mathbf{k}}} \left[a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right].
$$

$$
\begin{aligned}
\left[a_{\mathbf{k}}, a_{\mathbf{p}}^\dagger\right] &= (2\pi)^3 2E_{\mathbf{k}} \delta^{(3)}(\mathbf{k}-\mathbf{p}) \\
[a_{\mathbf{k}}, a_{\mathbf{p}}] &= 0 \\
\left[a_{\mathbf{k}}^\dagger, a_{\mathbf{p}}^\dagger\right] &= 0.
\end{aligned}
$$

The Hamiltonian is

$$
H = \frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E_{\mathbf{k}}} E_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger\right)
$$

$$
H = \frac{1}{4} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger\right).
$$

1.2.2 States

Vacuum state $|0\rangle$

$$
\begin{aligned}
a_{\mathbf{k}}|0\rangle &= 0 \\
\langle 0|0\rangle &= 1.
\end{aligned}
$$

$$
\begin{aligned}
H|0\rangle = & E_0|0\rangle \\
= & \frac{1}{4} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} a_{\mathbf{k}} a_{\mathbf{k}}^\dagger |0\rangle \\
= & \frac{1}{2} \int \mathrm{d}^3 k E_{\mathbf{k}} \delta^{(3)}(\mathbf{k}-\mathbf{k})|0\rangle \\
= & \infty|0\rangle.
\end{aligned}
$$

The vacuum energy is infinite.

1.2.3 IR-regulate

IR-regulate by putting theory in a box of size L.

$$\begin{aligned}(2\pi)^3\delta^{(3)}(\mathbf{0}) &= \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\Big|_{\mathbf{p}=0} \\ &= \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}^3x \\ &= \lim_{L\to\infty} V.\end{aligned}$$

$$\rho_0 = \frac{E_0}{V} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2}E_{\mathbf{k}}.$$

Total energy diverges if V diverges unless $\rho_0 = 0$. This is a UV divergence.

Normal Hamiltonian is

$$:H:= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}} E_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}.$$

1.2.4 One Particle States

Let

$$|\mathbf{k}\rangle = a^\dagger_{\mathbf{k}}|0\rangle.$$

$|\mathbf{k}\rangle$ has definite momentum and energy, sometimes also be denoted by $|k\rangle$.

$$\begin{aligned}\langle\mathbf{p}|\mathbf{k}\rangle =&\langle 0|a_{\mathbf{p}}a^\dagger_{\mathbf{k}}|0\rangle \\ =&(2\pi)^3 2E_{\mathbf{k}}\delta^{(3)}(\mathbf{p}-\mathbf{k}).\end{aligned}$$

This is Lorentz invariant.

$\varphi(\mathbf{x})|0\rangle$ is an one-particle state localized at $\mathbf{x}$.

$$N = \int \frac{\mathrm{d}^3k}{(2\pi^3)2E_{\mathbf{k}}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$$

$$N a^\dagger_{\mathbf{p}}|0\rangle = a^\dagger_p|0\rangle$$
$$N\varphi(\mathbf{x})|0\rangle = \varphi(\mathbf{x})|0\rangle$$
$$\langle\mathbf{k}|\varphi(\mathbf{x})|0\rangle = e^{-i\mathbf{k}\cdot\mathbf{x}}.$$

This formula is similar to $\langle\mathbf{k}|\mathbf{x}\rangle = e^{-i\mathbf{k}\cdot\mathbf{x}}$ in quantum mechanics.

1.2.5 Multiparticle States

$$|\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n\rangle = a^\dagger_{\mathbf{k}_1}a^\dagger_{\mathbf{k}_2}\cdots a_{\mathbf{k}_n}|0\rangle$$

The operators in the right hand are commutative $\to$ bosons.

$$[:H:, N] = 0.$$

The state space is Fock space $\mathcal{F} = \oplus_{n=0}^{\infty}\mathcal{H}_n$.

1.2.6 Heisenberg Picture

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt}.$$
$$a_{\mathbf{p}}(t) = e^{iHt}a_{\mathbf{p}}e^{-iHt}.$$

Using $e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \cdots$ we get

$$[H, a_{\mathbf{p}}] = -E_{\mathbf{p}}a_{\mathbf{p}}$$

and

$$a_{\mathbf{p}}(t) = e^{-iE_{\mathbf{p}}t}a_{\mathbf{p}}$$
$$a_{\mathbf{p}}^{\dagger}(t) = e^{iE_{\mathbf{p}}t}a_{\mathbf{p}}^{\dagger}.$$
$$\varphi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}}\left[a_{\mathbf{k}}e^{-ik\cdot x} + a_{\mathbf{k}}^{\dagger}e^{ik\cdot x}\right].$$

$$\begin{aligned}[:H:, \varphi] =& \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\left[E_{\mathbf{k}}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}, a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger}e^{ip*x}\right] \\ =& \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}}\left(-E_{\mathbf{k}}a_{\mathbf{k}}e^{-ik\cdot x} + E_{\mathbf{k}}a_k^{\dagger}e^{ik\cdot x}\right) \\ =& - i\partial_t\varphi(t, \mathbf{x}).\end{aligned}$$

The interaction field can be written as

$$\Phi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}\left[b_{\mathbf{p}}(t)e^{-ip\cdot x} + b_{\mathbf{p}}^{\dagger}(t)e^{ip\cdot x}\right].$$

At any fixed time $b_{\mathbf{p}}^{\dagger}(t)$ and $b_{\mathbf{p}}(t)$ satisfy the same algebra as free theory.

Propagator

$$\begin{aligned}D(x-y) =& \langle 0|\varphi(x)\varphi(y)|0\rangle \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}\frac{\mathrm{d}^3k}{2E_{\mathbf{k}}}e^{-ip\cdot x + ip\cdot y}\langle 0|a_{\mathbf{p}}a_{\mathbf{k}}^{\dagger}|0\rangle \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}.\end{aligned}$$

Space like: $x^0 = y^0, \mathbf{x} - \mathbf{y} = \mathbf{r} \neq 0$

$$\begin{aligned}D(x-y) =& \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot r\mathbf{p}} \\ \sim& e^{-mr} \neq 0.\end{aligned}$$

If $\Delta(x, y) = [\varphi(x), \varphi(y)] = 0$, then the measurement at x cannot affect y.

$$[\varphi(x), \varphi(y)] = D(x-y) - D(y-x) = \langle 0|\left[\varphi(x), \varphi(y)\right]|0\rangle.$$

These two amplitude eliminate with each other when $(x-y)^2 < 0$.

For time like separation, assume $x^0 > y^0$

$$\begin{aligned}
\Delta(x,y) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \\
&= \int \frac{\mathrm{d}^3 p}{2E_{\mathbf{p}}} \left(\frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \Big|_{p^0=E_{\mathbf{p}}} + \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot x-y} \Big|_{p^0=-E_{\mathbf{p}}} \right) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int_{C_R} \frac{\mathrm{d}p^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-ip\cdot(x-y)}.
\end{aligned}$$

$$\Delta_R(x-y) = \Theta(x^0-y^0)\langle 0| \left[\varphi(x), \varphi(y)\right] |0\rangle = \int_{C_R'} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip\cdot(x-y)}.$$