ALGEBRAIC GEOMETRY
AFFINE VARIETIES

WANG YUNLEI

Exercise 1.

Proof.
(a) $A(Y) = k[x,y]/(y - x^2) \simeq k[x, x^2] = k[x]$.
(b) $A(Z) = k[x,y]/(xy-1) \simeq k[x, \frac{1}{x}]$. If $A(Z)$ is isomorphic to some polynomial ring, then x is mapped to an element in k because x is invertible. Hence the map is not surjective, which contradicts to the isomorphism.
(c) The general form of a quadratic polynomial is $f = x^2+axy+by^2+cx+dy+e$. We can use the invertible linear transformation to simplify this into $f = x^2 + ay^2 + bx + cy + d$. If $a \neq 0$, we can use the translation and linear transformation to get $f = x^2 - y^2 - d$ and $d \neq 0$, it can be transformed into $f = xy - d$ with $d \neq 0$ at last. If $a = 0$, we can transform this into $x^2 - y$. □

Exercise 2.

Proof. $\dim Y = \dim A(Y) = 1$, since $A(Y) = k[x,y,z]/(y-x^2, z-x^3) \simeq k[x, x^2, x^3] = k[x]$ has dimension 1. □

Exercise 3.

Proof.

$$\begin{aligned} Y &= Z(x^2 - yz, xz - x) \\ &= Z(x^2 - yz) \cap (Z(x) \cup Z(z-1)) \\ &= Z(x^2 - yz, x) \cup Z(x^2 - yz, z - 1) \\ &= Z(yz, x) \cup Z(x^2 - y, z-1) \\ &= Z(y,x) \cup Z(z,x) \cup (x^2 - y, z - 1) \end{aligned} \tag{0.1}$$

□

Exercise 4.

Proof. $Z(xy - 1)$ is not closed in $\mathbb{A}^1 \times \mathbb{A}^1$. □

E-mail address: wcghdpwyl@126.com

Date: June 19, 2017.