OBSERVABILITY INEQUALITY AT TWO TIME POINTS FOR KDV EQUATIONS FROM MEASURABLE SETS

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open domain. Consider the following evolution equation

$$\partial_t u = Fu, \quad u(0,x) = u_0 \in L^2(\Omega), \tag{1.1}$$

where F is a linear operator. Sometimes we want to get the global information of the solution $u(t,x)$ of (1.1) at time t out of a small region $\omega \subset \Omega$. For example, we may establish some inequalities such as

$$\int_\Omega |u(T,x)|^2 \,\mathrm{d}x \le C(T,\Omega) \int_0^T \int_\omega |u(t,x)|^2 \,\mathrm{d}x \,\mathrm{d}t, \tag{1.2}$$

where $\omega \subset \Omega$ and $C(T,\Omega)$ is a constant depending on T and Ω. An inequality like (1.2) is called an observability inequality of (1.1).

Roughly speaking, Uncertainty Principle says that it is impossible for a nonzero function and its Fourier transform to have compact supports simultaneously. More precisely, we can illustrate this property by the following inequality:

$$\int_{\mathbb{R}^n} |f(x)|^2 \,\mathrm{d}x \le C e^{C r_1 r_2} \left(\int_{|x|\ge r_1} |f|^2 \,\mathrm{d}x + \int_{|x|\ge r_2} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \right). \tag{1.3}$$

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Here $C = C(n)$ is a constant and the inequality holds for all $r_1, r_2 > 0$ and all $f \in L^2(\mathbb{R}^n)$. This was proved by the Logvinenko-Sereda theorem in [8, 9, 10, 11] and was shown to directly follow from a Nazarov type uncertainty principle in [12].

Recently, G. Wang, M. Wang, Y. Zhang [2] found the connection between (1.3) and the observability inequality for the free Schrödinger equation

$$i\partial_t u - \Delta u = 0, \quad u(0,x) = u_0(x) \in L^2(\mathbb{R}^n). \tag{1.4}$$

In fact, it was proved in [2] that (1.3) is equivalent to the following: There exists a constant $C = C(n) > 0$ such that for all $t > 0$, all $r_1, r_2 > 0$ and all $u(t,x)$ solving (1.4),

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \, \mathrm{d}x \le Ce^{\frac{Cr_1 r_2}{t}} \left(\int_{|x| \ge r_1} |u_0(x)|^2 \, \mathrm{d}x + \int_{|x| \ge r_2} |u(t,x)|^2 \, \mathrm{d}x \right). \tag{1.5}$$

This is an observability inequality of (1.4).

Here we are interested in the linear Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} = 0, \quad u(0,x) = u_0(x) \in L^2(\mathbb{R}). \tag{1.6}$$

Our aim is to establish the similar observability inequality at two time points as (1.5) for (1.6). The newest result about this is accomplished by Z. Li and M. Wang in [6]: There exists a constant $C > 0$ so that for all $r_1, r_2, t > 0$ and all $u(t,x) \in C([0,\infty); L^2(\mathbb{R}))$ solving (1.6),

$$\int_{\mathbb{R}} |u_0(x)|^2 \, \mathrm{d}x \le Ce^{Ct^{-\frac{4}{3}}\left(r_1^4 + r_2^4\right)} \left(\int_{|x| \ge r_1} |u_0(x)|^2 \, \mathrm{d}x + \int_{|x| \ge r_2} |u(t,x)|^2 \, \mathrm{d}x \right). \tag{1.7}$$

We will prove the following theorem

Theorem 1.1. *Let A, B be two measurable sets in $\mathbb{R}$ with finite measure. Then for every $t > 0$, there exists $C = C(t, |A|, |B|) > 0$ so that when $u(t,x)$ solves the KdV equation,*

$$\int_{\mathbb{R}} |u_0|^2 \, \mathrm{d}x \le C \left(\int_{A^c} |u_0|^2 \, \mathrm{d}x + \int_{B^c} |u(t,x)|^2 \, \mathrm{d}x \right). \tag{1.8}$$

There is a major difference between the result of Z. Li and M. Wang in [6] and Theorem 1.1: The inequality (1.7) gives an exact relation between the constant and t, r_1, r_2. The inequality (1.8) fails to give such a relation, but removes the bounded condtion of two sets A and B.

Theorem 1.1 requires two sets A and B have finite measure. This limitation can be replaced by a more general condition (in which one of the sets may have infinite measure) in the case of Schrödinger equations. We will find some type of A and B with infinite measure and the inequality (1.8) still holds.

Definition 1.2. We say the set A has $|x|^{-\alpha}$ density if $\alpha > 0$ and

$$\overline{\lim_{x\to\infty}} |A \bigcap [x, x+1]| \lesssim |x|^{-\alpha}.$$

Remark 1.3. It is equivalent to say that, there exists $L > 0$ so that

$$|A \bigcap [x, x+1]| \lesssim |x|^{-\alpha}, \quad \forall |x| \geq L.$$

Theorem 1.4. *Let A and B be measurable sets with density $\alpha \in (\frac{5}{6}, 1)$, $A, B \subset (c, \infty)$ or $A, B \subset (-\infty, c)$ for some c. Then for every $t > 0$, there exists $C = C(t, A, B) > 0$ so that when $u(t, x)$ solves the KdV equation, (1.8) holds.*

2. Projection operators

Definition 2.1. Suppose that $(\mathcal{H}, \|\cdot\|)$ is a Hilbert space, $\{e_i | i \in I\}$ is an orthonormal basis of $\mathcal{H}$. Then for any linear operator T on $\mathcal{H}$ define

$$\|T\|_{\text{HS}}^2 \equiv \sum_{i\in I} \|Te_i\|^2.$$

$\|T\|_{\text{HS}}$ is called the Hilbert-Schmidt norm of T. Moreover, the bounded operator T is a Hilbert-Schmidt operator if $\|T\|_{\text{HS}}$ is finite.

Lemma 2.2. *All Hilbert-Schmidt operators are compact.*

Proposition 2.3. *Let $(\mathcal{H}, \|\cdot\|)$ be an infinite-dimensional complex Hilbert space. Let E, F be two orthogonal projections acting in $\mathcal{H}$ and $S : \mathcal{H} \to \mathcal{H}$ be an invertible operator. Define $E^\perp \equiv I - E$, $F^\perp \equiv I - F$ and $T = ESF$. If $\|S^{-1}\|\|T\| < 1$, then there exists a constant $C > 0$ sucht that for any $f \in \mathcal{H}$*

$$\|f\| \leq C \left(\|E^\perp Sf\| + \|F^\perp f\|\right), \tag{2.1}$$

or equivalently a constant $C' > 0$ such that for any $f \in \mathcal{H}$

$$\|f\|^2 \leq C' \left(\|E^\perp Sf\|^2 + \|F^\perp f\|^2\right). \tag{2.2}$$

Proof. For all $f \in \mathcal{H}$,

$$\|ESFf\| \leq \|ESF\|\|f\|.$$

This implies

$$\|ESFf\| \leq \|ESF\|\|Ff\| = \|ESF\|\|S^{-1}\|\|SFf\| \leq c_1\|SFf\|, \quad \forall f \in \mathcal{H}$$

with some $0 \leq c_1 < 1$. From this, we find that with $c_2 = \sqrt{\frac{1}{1-c_1^2}}$

$$\|SFf\| \leq c_2\|E^\perp SFf\|, \quad \forall f \in \mathcal{H}. \tag{2.3}$$

Now we use (2.3) and the property of S

$$\begin{aligned}\|f\| =&\|S^{-1}Sf\| \leq \|S^{-1}\|(\|SFf\| + \|SF^{\perp}f\|)\\ \leq& c_2\|S^{-1}\|\|E^{\perp}SFf\| + \|S^{-1}\|\|SF^{\perp}f\|\\ \leq& c_2\|S^{-1}\|\|E^{\perp}Sf\| + (1+c_2)\|S^{-1}\|\|SF^{\perp}f\|\\ \leq& C\left(\|E^{\perp}Sf\| + \|F^{\perp}f\|\right).\end{aligned}$$

This proves the theorem. □

3. Proof of Theorem 1.1

Let $S(t)$ be the solution group of linear KdV equation (1.6), namely the solution of KdV is given by

$$u(t) = S(t)u_0 = G(t,x) * u_0,$$

where G is the fundamental solution of linear KdV equation, given by

$$G(t,x) = \begin{cases} \frac{1}{(3t)^{\frac{1}{3}}}\operatorname{Ai}\left(\frac{x}{(3t)^{\frac{1}{3}}}\right), & t>0\\ \delta(x), & t=0.\end{cases}$$

Here, $\operatorname{Ai}(x)$ is the Airy function defined via

$$\operatorname{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(xz+\frac{1}{3}z^3)}\,\mathrm{d}z.$$

According to [Stein, p.330],

$$|\operatorname{Ai}(x)| \lesssim \begin{cases}(1+|x|)^{-\frac{1}{4}}, & x<0,\\ e^{-\frac{2}{3}|x|^{\frac{3}{2}}}, & x\geq 0.\end{cases}$$

Let $Eu = \chi_B u$, $Fu = \chi_A u$,$S = S(t)$,$\mathcal{H} = L^2(\mathbb{R})$, then

$$Tf = ESFf, \quad f \in L^2(\mathbb{R}).$$

If E, S, F satisfy the condition of Proposition 2.3, then Theorem 1.1 can be verified directly. Since $\|Sf\| = \|S(t)f\| = \|f\|$ and $S^{-1}(t) = S(-t)$, we only need to prove the following

Proposition 3.1. *Let A, B be two measurable sets in $\mathbb{R}^n$ with $|A|, |B| < \infty$. Then we have*

$$\|T\| < 1.$$

It is equivalent to prove

$$\|S^{-1}T\| < 1.$$

Before proving Proposition 3.1, we first note that we always have

$$\|S^{-1}T\| \leq 1.$$

In fact, for all $f \in L^2(\mathbb{R})$

$$\|S(-t)Tf\| = \|S(-t)\chi_B S(t)(\chi_A f)\| \leq \|S(t)(\chi_A f)\| = \|\chi_A f\| \leq \|f\|.$$

Thus, it remains to show that

$$\|S(-t)T\| \neq 1.$$

To show this, we need the following

Lemma 3.2. *For every $t \neq 0$, T is a compact operator on $L^2(\mathbb{R}^n)$.*

Proof. We can rewrite the operator T as an integral operator:

$$(Tf)(x) = \int_{\mathbb{R}} \chi_A(x) G(t, x-y) \chi_B(y) f(y)\,\mathrm{d}y := \int_{\mathbb{R}} K(t,x,y) f(y)\,\mathrm{d}y.$$

We claim that for all $t \neq 0$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} K^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{3.1}$$

Then T is a Hilbert-Schmidt operator and thus a compact operator on $L^2(\mathbb{R})$ by Lemma 2.2.

It remains to show (3.1). In fact, since $|G(t, x-y)| \leq C(t)$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} K^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y \leq C^2(t) \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(x)\chi_B(y) dx\,\mathrm{d}y = C^2(t)|A||B| < \infty.$$

This proves (3.1). $\square$

Define the translation operator $\mathcal{T}_\lambda$ in $L^2(R)$

$$\mathcal{T}_\lambda f(x) = f(x-\lambda).$$

If A is a measurable set in $\mathbb{R}$ and $\lambda \in \mathbb{R}$, we shall denote the set $A+\lambda = \{x \in \mathbb{R} | x+\lambda \in A\}$. The following lemma is Lemma 1 in [5].

Lemma 3.3. *Let C be a measurable set in $\mathbb{R}^n$ with $0 < |C| < \infty$, let C_0 be a measurable subset of C with $|C_0| > 0$, and let $\epsilon > 0$. Then there exists a translation $\lambda \in \mathbb{R}^n$ such that*

$$|C| < |C \cup (C_0 + \lambda)| < |C| + \epsilon.$$

Proof of Theorem 1.1. Suppose by way of contradiction that

$$\|S(-t)T\| = 1. \tag{3.2}$$

By Lemma 3.3, $S(-t)T$ is a compact operator. Note that, by compactness, the L^2 operator norm of $S(-t)T$ is 1 if and only if there exists a function $f \in L^2(\mathbb{R})$ such that $\|S(-t)Tf\| = \|f\|$. This implies $\mathrm{supp} f \subset B$ and $\mathrm{supp} S(t)f \subset A$.

Define $f_\lambda \equiv \mathcal{T}_\lambda f$. Then $\mathrm{supp} f_\lambda = A + \lambda$. Since

$$\begin{aligned} S(t)f_\lambda =& S(t)\mathcal{T}_\lambda f \\ =& \int_{\mathbb{R}} G(t, x-y) f(y-\lambda)\, \mathrm{d}y \\ =& \int_{\mathbb{R}} G(t, x-\lambda-y) f(y)\, \mathrm{d}y \\ =& \mathcal{T}_\lambda(S(t)f), \end{aligned}$$

we have $\mathrm{supp} S(t) f_\lambda = B + \lambda$.

Now we define a sequence $\{f_i\}_{i=1}^\infty$ recursively. The method we constructed the sequence below derives from [5]. By the strong continuity of $\mathcal{T}_\lambda$, there exists a number $\delta > 0$ such that for any translation $\lambda_i < \delta$ such that

$$|A_{i-1}| \le |A_{i-1} \cup (A_0 + \lambda_i)| < |A_{i-1}| + \frac{1}{2^i},$$

and

$$|B_{i-1}| \le |B_{i-1} \cup (B_0 + \lambda_i)| < |B_{i-1}| + \frac{1}{2^i},$$

and we set $A_i = A_{i-1} \cup (A_0 + \lambda_i), B_i = B_{i-1} \cup (B_0 + \lambda_i)$. By Lemma 3.3 we can choose λ_i such that at least one left inequality of the above two is strict less-than sign. Using the above inequality recursively, we obtain

$$|\bigcup_{i=0}^\infty A_i| < |A| + 1, \quad |\bigcup_{i=0}^\infty B_i| < |B| + 1.$$

Define $f_i = \mathcal{T}_{\lambda_i} f$ and $f_0 = f$, then $\mathrm{supp} f_i \subset A_i \subset \bigcup_{i=0}^\infty A_i$ and $\mathrm{supp} S(t) f_i \subset B_i \subset \bigcup_{i=0}^\infty B_i$. We shall prove that the sequence $\{f_i\}_{i=0}^\infty$ are linearly independent. Denote the projection operator $E_U f = \chi_U f$. Since $A_m = A_0 \cup (A_0 + \lambda_1) \cup \cdots \cup (A_0 + \lambda_m)$ and $B_m = B_0 \cup (B_0 + \lambda_1) \cup \cdots \cup (B_0 + \lambda_m)$, we have $E_{A_m} f_i = f_i$ and $E_{B_m} S(t) f_i = S(t) f_i$ for all $i = 0, 1, \cdots, m$. By the choice of λ_i, we have either $E_{A_m \setminus A_{m-1}} f_m \neq 0$ or $E_{B_m \setminus B_{m-1}} S(t) f \neq 0$, both can show that f_m is not a linear combination of $f_0, f_1, \cdots, f_{m-1}$. This means that the sequence $\{f_i\}_{i=0}^\infty$ are linearly independent. Let $A' = \bigcup_{i=0}^\infty A_i$ and $B' = \bigcup_{i=0}^\infty B_i$, then the eigensppace of the new operator $S(-t)Tf = S(-t)\chi_{B'} S(t)(\chi_{A'} f)$ has infinitely many eigenfunctions of eigenvalue 1, which contradicts to the compact property of $S(-t)T$. Hence the assumption (3.2) is invalid. □

4. Proof of Theorem 1.4

In fact, the density condtion of A and B in Theorem 1.4 can be extended to the following more general condition.

Lemma 4.1. *Assume that A has density $|x|^{-\alpha}$ and B has density $|x|^{-\beta}$ with*

$$\begin{cases} \alpha+\beta > \frac{5}{3} \\ \alpha+3\beta > 3 \\ 3\alpha+\beta > 3 \\ \alpha, \beta > \frac{1}{2} \end{cases}.$$

We have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K^2(t,x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Proof. It suffices to show that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_B(x)\langle|x-y|\rangle^{-\frac{1}{2}}\chi_A(y)\,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{4.1}$$

Rewrite the LHS of (4.1) as

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_j^{j+1}\int_k^{k+1} \chi_B(x)\langle|x-y|\rangle^{-\frac{1}{2}}\chi_A(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.2}$$

But when $x\in[k,k+1]$, $y\in[j,j+1]$, then

$$\langle|x-y|\rangle \sim \langle|j-k|\rangle.$$

Now (4.2) is bounded by

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_j^{j+1}\int_k^{k+1} \chi_B(x)\langle|j-k|\rangle^{-\frac{1}{2}}\chi_A(y)\,\mathrm{d}x\,\mathrm{d}y \\
=&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |B\cap[k,k+1]|\cdot\langle|j-k|\rangle^{-\frac{1}{2}}\cdot|A\cap[j,j+1]| \\
\lesssim& 1+\sum_{j,k\in\mathbb{Z},|j|,|k|\geq L} |B\cap[k,k+1]|\cdot\langle|j-k|\rangle^{-\frac{1}{2}}\cdot|A\cap[j,j+1]| \\
\lesssim& \sum_{j,k\in\mathbb{Z},|j|,|k|\geq L} |j|^{-\alpha}|k|^{-\beta}\langle|j-k|\rangle^{-\frac{1}{2}}. \qquad (4.3)
\end{aligned}$$

We first assume that $|j| \leq |k|$. Arbitrarily given $\delta \in (0,1)$. To make the term

$$\sum_{j,k\in\mathbb{Z},L\leq|j|\leq|k|} |j|^{-\alpha}|k|^{-\beta}\langle|j-k|\rangle^{-\frac{1}{2}}$$

finite, we consider this into two cases.

(1) $|k|-|k|^{\delta} \leq |j| \leq |k|$. In this case, we have

$$\sum_{|k|\geq L,|k|-|k|^{\delta}\leq j\leq|k|} |j|^{-\alpha}|k|^{-\beta}\langle|j-k|\rangle^{-\frac{1}{2}} \leq \sum_{|k|\geq L,|k|-|k|^{\delta}\leq|j|\leq|k|} |j|^{-\alpha}|k|^{-\beta}$$
$$\leq \sum_{|k|\geq L} |k|^{\delta-\alpha}|k|^{-\beta} = \sum_{|k|\geq L} |k|^{\delta-\alpha-\beta}.$$

(2) $|j| < |k|-|k|^{\delta}$. In this case, we have

$$|k-j| \geq |k|-|j| \geq |k|^{\delta}.$$

This gives that $\langle|j-k|\rangle^{-1} \lesssim |k|^{-\delta}$. Then if

- $\alpha < 1$, we have

$$\sum_{|k|\geq L,|j|<|k|-|k|^{\delta}} |j|^{-\alpha}|k|^{-\beta}\langle|j-k|\rangle^{-\frac{1}{2}} \lesssim \sum_{|k|\geq L,|j|<|k|-|k|^{\delta}} |j|^{-\alpha}|k|^{-\beta-\frac{1}{2}\delta}$$
$$\lesssim \sum_{|k|\geq L} |k|^{1-\alpha}|k|^{-\beta-\frac{1}{2}\delta} = \sum_{|k|\geq L} |k|^{1-\frac{1}{2}\delta-\alpha-\beta}.$$

Then we have

$$\begin{cases} \delta-\alpha-\beta < -1 \\ 1-\frac{1}{2}\delta-\alpha-\beta < -1 \end{cases} \Rightarrow \begin{cases} \alpha+\beta > \delta+1 \\ \alpha+\beta > 2-\frac{1}{2}\delta \end{cases}.$$

Choose the best $\delta = \frac{2}{3}$, we obtain $\alpha+\beta > \frac{5}{3}$.

- $\alpha = 1$, we have

$$\sum_{|k|\geq L,|j|<|k|-|k|^{\delta}} |j|^{-\alpha}|k|^{-\beta}\langle|j-k|\rangle^{-\frac{1}{2}} \lesssim \sum_{|k|\geq L,|j|<|k|-|k|^{\delta}} |j|^{-\alpha}|k|^{-\beta-\frac{1}{2}\delta}$$
$$\lesssim \sum_{|k|\geq L} \ln|k|\cdot|k|^{-\beta-\frac{1}{2}\delta}.$$

Then we have

$$\begin{cases} \delta-\alpha-\beta < -1 \\ -\beta-\frac{1}{2}\delta < -1 \end{cases} \Rightarrow \begin{cases} \beta > \delta \\ \beta > 1-\frac{1}{2}\delta \end{cases}.$$

Choose the best $\delta = \frac{2}{3}$, we obtain $\beta > \frac{2}{3}$.

In fact, the above two cases can be written together as below

$$\begin{cases} 0 \le \alpha \le 1 \\ \alpha + \beta > \frac{5}{3} \end{cases} . \tag{4.4}$$

- $\alpha > 1$, we have

$$\sum_{|k|\ge L, |j|<|k|-|k|^\delta} |j|^{-\alpha}|k|^{-\beta}\langle |j-k|\rangle^{-\frac{1}{2}} \lesssim \sum_{|k|\ge L, |j|<|k|-|k|^\delta} |j|^{-\alpha}|k|^{-\beta-\frac{1}{2}\delta}$$
$$\lesssim \sum_{|k|\ge L} |L|^{1-\alpha}|k|^{-\beta-\frac{1}{2}\delta} \lesssim \sum_{|k|\ge L} |k|^{-\frac{1}{2}\delta-\beta}.$$

Then we have

$$\begin{cases} \delta - \alpha - \beta < -1 \\ -\frac{1}{2}\delta - \beta < -1 \end{cases} \Leftrightarrow 2(1-\beta) < \delta < \alpha + \beta - 1.$$

Since $0 < \delta < 1$, we must ensure

$$\begin{cases} 2(1-\beta) < \alpha + \beta - 1 \\ 2(1-\beta) < 1 \\ \alpha + \beta - 1 > 0 \end{cases} \Leftrightarrow \begin{cases} \beta > \frac{1}{2} \\ \alpha + \beta > 1 \\ \alpha + 3\beta > 3 \end{cases} .$$

Hence

$$\begin{cases} \alpha > 1 \\ \beta > \frac{1}{2} \\ \alpha + \beta > 1 \\ \alpha + 3\beta > 3 \end{cases} . \tag{4.5}$$

For $|j| > |k|$, to make the term

$$\sum_{j,k\in\mathbb{Z}, L\le |k|<|j|} |j|^{-\alpha}|k|^{-\beta}\langle |j-k|\rangle^{-\frac{1}{2}}$$

finite, we can get the condition similar to (4.4), (4.5):

$$\begin{cases} 0 \le \beta \le 1 \\ \alpha + \beta > \frac{5}{3} \end{cases} \tag{4.6}$$

or

$$\begin{cases} \alpha > \frac{1}{2} \\ \beta > 1 \\ \alpha + \beta > 1 \\ 3\alpha + \beta > 3 \end{cases} . \tag{4.7}$$

To make the both terms finite, we have four cases:

- (4.4) and (4.6):

$$\begin{cases} 0 \leq \alpha, \beta \leq 1 \\ \alpha + \beta > \frac{5}{3} \end{cases} . \tag{4.8}$$

- (4.4) and (4.7):

$$\begin{cases} \frac{1}{2} < \alpha \leq 1 \\ \beta > 1 \\ \alpha + \beta > \frac{5}{3} \\ 3\alpha + \beta > 3 \end{cases} . \tag{4.9}$$

- (4.5) and (4.6)

$$\begin{cases} \alpha > 1 \\ \frac{1}{2} < \beta \leq 1 \\ \alpha + \beta > \frac{5}{3} \\ \alpha + 3\beta > 3 \end{cases} . \tag{4.10}$$

- (4.5) and (4.7)

$$\begin{cases} \alpha > 1 \\ \beta > 1 \end{cases} . \tag{4.11}$$

Figure 1 shows the allowed region of four conditions. Combine all these four conditions we complete the proof. □

To obtain Theorem 1.4, we need the following important theorem in[3, p. 60].

Theorem 4.2. *Let $u(t, x)$ be a mild solution of (1.6). If there exists $t_1 < t_2$ such that*

$$\mathrm{supp} u(\cdot, t_j) \subset (-\infty, c), \quad j = 1, 2 \tag{4.12}$$

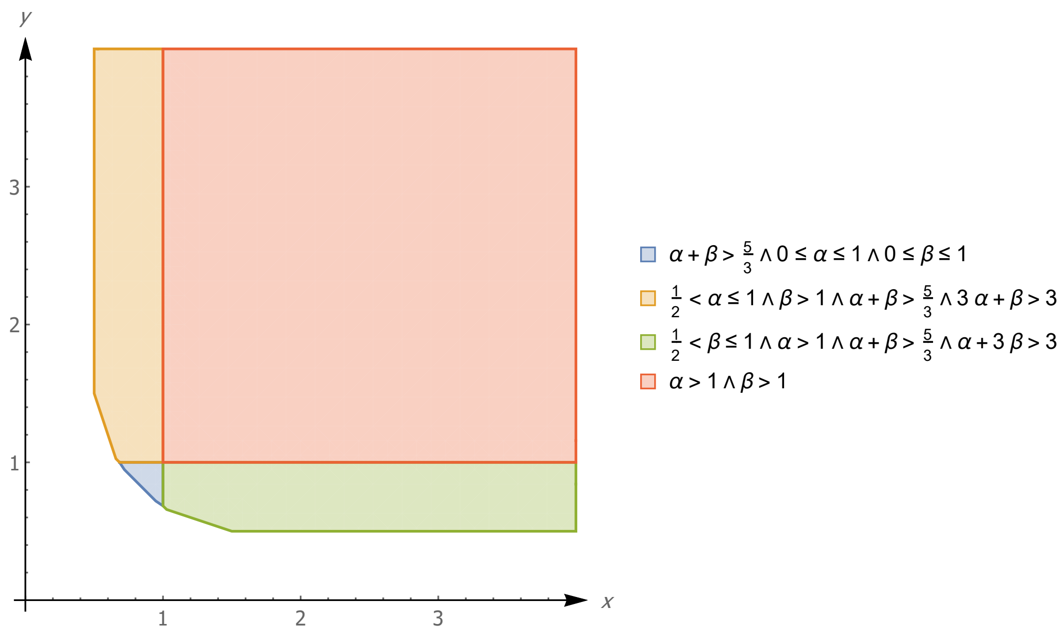

FIGURE 1. Region that ensures the integral finite

or

$$\text{supp}u(\cdot, t_j) \subset (c, \infty), \quad j = 1, 2 \tag{4.13}$$

for some $c \in \mathbb{R}$, *then*

$$u(x, t) \equiv 0, \quad -\infty < x, t < \infty.$$

Proof of Theorem 1.4. Similar to the proof of Theorem 1.4, we need to show $\|S(-t)T\| \neq 1$. It is equivalent to show that there does not exist nonzero function $f \in L^2(\mathbb{R})$ such that $\|S(-t)Tf\| = \|f\|$. If not, then supp$f \subset B$ and supp$S(t)f \subset A$. But Theorem 4.2 implies $f = 0$, which makes a contradiction. □

5. Generalization

Definition 5.1. Let S be a linear operator mapping the Hilbert space $(\mathcal{H}(R^n), \|\cdot\|)$ into itself and $E \subset \mathbb{R}^n$ be a measurable set. We call the operator S antilocal with respect to O if for any $f \in \mathcal{H}$

$$\chi_O f = \chi_O S f = 0 \Rightarrow f = 0.$$

If the above is true for any measurable set $|O| > 0$, then we call the operator S completely antilocal.

We can abstract and generalize the proof of Theorem 1.4 to the following theorem directly.

Theorem 5.2. *Let $(\mathcal{H}(\mathbb{R}^n), \|\cdot\|)$ be an infinite-dimensional complex Hilbert space. Let A and B be two measurable sets. Define $Ef = \chi_B f$ and $Ff = \chi_A f$. $S : \mathcal{H} \to \mathcal{H}$ is an invertible linear operator which satisfies $\|S^{-1}\|\|T\| < 1$. If S is antilocal with respect to some measurable set O and $A, B \subset O$, then the inequalities (2.1) and (2.2) in Theorem 4.2 still holds.*

Example 5.3. The Hilbert transform

$$Hf(x) \equiv \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x-t)}{t}\,\mathrm{d}t$$

is completely antilocal[7, p. 485] and an isometry in $L^2(\mathbb{R})$. Then we can use Theorem 5.2 to obtain the following: Let A and B be two measurable sets with finite measure, then there exists a constant $C = C(n, A, B)$

$$\int_{\mathbb{R}} |f|^2\,\mathrm{d}x \le C\left(\int_{A^c} |f|^2\,\mathrm{d}x + \int_{B^c} |Hf|^2\,\mathrm{d}x\right).$$

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