Algebraic Topology

Notes taken by 89hao

May 18, 2020 — May ?, 2020

Abstract

This is a learning note about Zhang Yitang's lectures on modular forms.

Contents

before 1900 Euler formula $V - E + F = 2$
Winding number

1900 H. Poincaré introduce Homology, Fundamental Group

Aimed to study "spaces"
Topological spaces and continuous mappings
Invariants
$X \in \{$ Topological spaces $\} \Rightarrow$ e.g. $G(X) \in \{$ abelian groups$\}$
If $X \to G(X)$, $Y \to G(Y)$ and $f : X \to Y$, we wish to get $G(f)$:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & G(X) \arrow[d, "G(f)"] \\ Y \arrow[r] & G(Y) \end{tikzcd}$$

and let $G(f)$ be a homomorphism of groups.

Boundary of the boundary $+C$ is 0 in Figure 2.

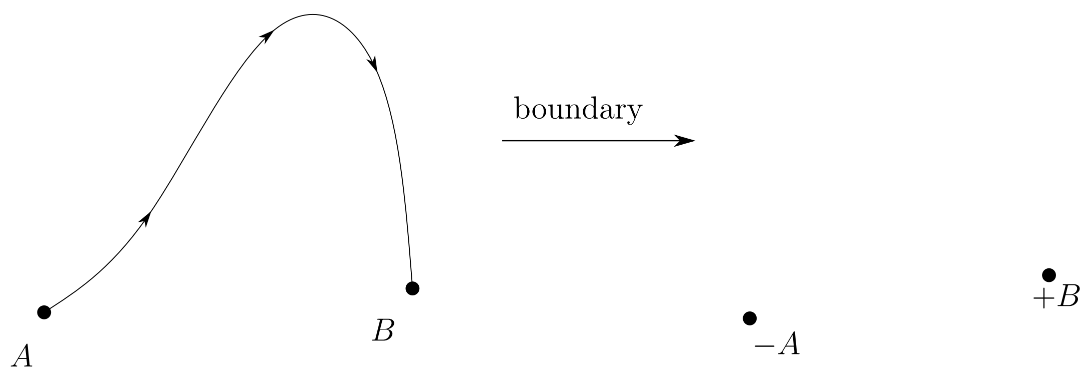

Figure 1: boundary of a segment

0 Categories, Functors, and Natural Transformations

Definition 0.1 (categories). A category $\mathcal{C}$ consists of

a. (objects) $\mathrm{Ob}(\mathcal{C})$ consists of the class of objects in $\mathcal{C}$.

b. (morphisms) $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, we have a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ s.t. $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X', Y')$ iff $X = X', Y = Y'$.

c. (composition law) $\forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, we have a map:

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
$$(f, g) \mapsto g \circ f$$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & Z \end{tikzcd}$$

which satisfy the following two axioms:

(1) (Associativity) $X \xrightarrow{f} Y, Y \xrightarrow{g} Z, Z \xrightarrow{h} W$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

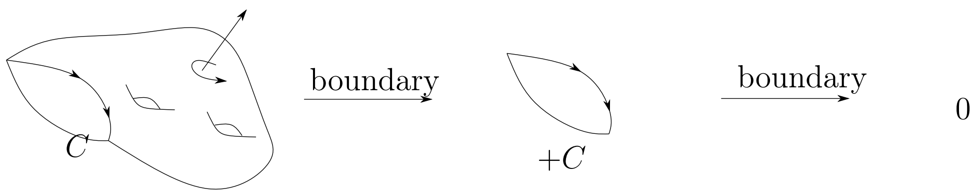

Figure 2: boundary of a surface

(2) (Identity) $\forall X \in \mathrm{Ob}(\mathcal{C})$, $\exists X \overset{1_X}{\to} X$ s.t.

$$h \circ 1_X = H, 1_X \circ k = k$$

$\forall X \overset{h}{\to} H, K \overset{k}{\to} X.$

Example 0.1. a. $\mathcal{C} =$ (set), (Ab),(Mod_R)(R is a ring), (Top),(TopGp).

b. $\mathcal{C}^{\mathrm{op}}$ (the opposite of $\mathcal{C}$):

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathcal{C})$$
$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z) \overset{\circ_{\mathcal{C}^{\mathrm{op}}}}{\to} \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Z)$$
$$(f, g) \to g \circ_{\mathcal{C}^{\mathrm{op}}} f$$
$$X \overset{f}{\leftarrow} Y, Y \overset{g}{\leftarrow} Z \quad X \overset{f \circ g}{\leftarrow} Z.$$

Definition 0.2. $X, X' \in \mathrm{Ob}(\mathcal{C}), X \overset{f}{\mathcal{C}} X'$, f is an *isomorphism* $\Leftrightarrow \exists X' \overset{\widetilde{f}}{\to} X$ s.t.

$$\widetilde{f} \circ f = 1_X$$
$$f \circ \widetilde{f} = 1_{X'}.$$

Definition 0.3 (Functors). $\mathcal{C}, \mathcal{C}'$: categories. A covariant(contravariant) functor $F : \mathcal{C} \to \mathcal{C}'$ ($\mathcal{C} \xrightarrow{F} \mathcal{C}'$) consists of

- a rule of associating to each $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{C}')$.
- A map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ $(\mathrm{Hom}_{\mathcal{C}'}(F(Y), F(X)))$ for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$ s.t. $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f) (F(g \circ f) = F(f) \circ F(g))$ i.e.

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & Z \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & F(Y) \arrow[dr, "F(g)"] & \\ F(X) \arrow[ur, "F(f)"] \arrow[rr, "F(g\circ f)"] & & F(Z) \end{tikzcd}$$

Example 0.2.

(1) $\mathcal{C} \xrightarrow{\text{op}} \mathcal{C}^{\text{op}}$, $X^{\text{op}} := X$

(2) $\forall X \in \mathcal{O}\lfloor(\mathcal{C})$, $h_X : \mathcal{C} \to (\text{set})$,

$$h_X(Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X), \forall Y \in \mathrm{Ob}(\mathcal{C})$$

$$h_X(f) := \circ f : h_X(Y) \to h_X(Y'), \forall Y' \xrightarrow{f} Y (\to X)$$

h_X is contravariant.

Definition 0.4 (Natural Transformations). $\mathcal{C} \underset{F_2}{\overset{F_1}{\rightrightarrows}} \mathcal{C}'$ two functors of the same variance.

a. A *natural transformation* T form F_1 to F_2 (denoted as $F_1 \xrightarrow{T} F_2$) is a rule of associating to each $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $F_1(X) \xrightarrow[\mathcal{C}']{T(X)} F_2(X)$ s.t. for each $X \xrightarrow[\mathcal{C}]{f} Y$ we have :

$$\begin{tikzcd} F_1(X) \arrow[r, "T(X)"] \arrow[d, "F_1(f)"'] & F_2(X) \arrow[d, "F_2(f)"] \\ F_1(Y) \arrow[r, "T(Y)"'] & F_2(Y) \end{tikzcd}$$

b. A natural transformation $F_1 \xrightarrow{T} F_2$ is called a *natural equivalence* if $F_1(X) \xrightarrow{T(X)} F_2(X)$ is an isomorphism for each $X \in \mathrm{Ob}(\mathcal{C})$.

$F_1 \xrightarrow{T} F_2, F_2 \xrightarrow{S} F_3 \rightsquigarrow S \circ T$.

1 Singular Homolgy Groups

Definition 1.1 (Standard simplexes). $k \in \mathbb{N} \cup \{0\}$,

$$\Delta_k := \left\{ (t_0, \cdots, t_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} t_i = 1, t_i \geq 0, i = 0, \cdots, k \right\}.$$

Definition 1.2 (The i-th face inclusion). $i \leq k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \Delta_k &\xrightarrow{l_i} \Delta_{k+1} \\ (t_0, \cdots, t_k) &\mapsto (t_0, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_{k+1}). \end{aligned}$$

Definition 1.3 (Singular complexes, ude to Lefschetz-Eilenberg). X : topological space, $k \in \mathbb{N} \cup \{0\}$. A (singular) k-simplex in X is a continuous map $\sigma : \Delta_k \to X$.

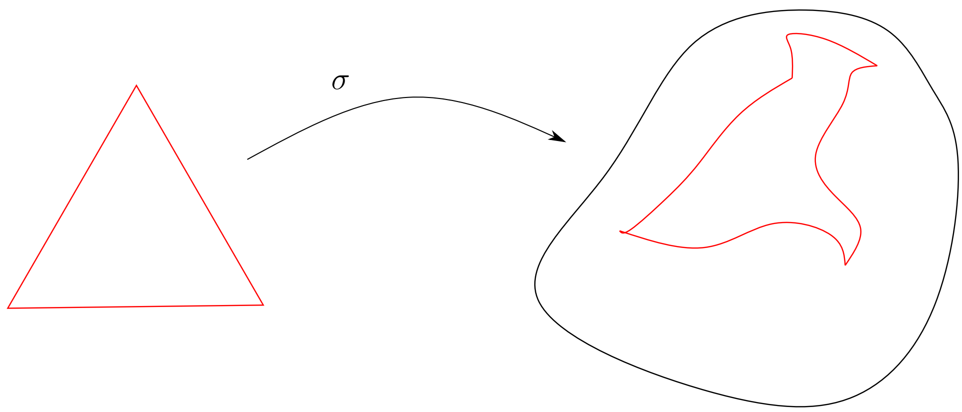

Figure 3: singular complexes

Definition 1.4 (Faces of a singular simplex). $\sigma : \Delta_k \to X$ continuous, $\sigma_i := \sigma \circ l_i, i = 0, \cdots, k$.

Definition 1.5 (Singular chain groups). $k \in \mathbb{Z}$,

$$\begin{aligned} S_k(X) :=& \text{ the free abelian group generated by all singular } k\text{-simplexes in } X \\ &= \oplus_{\sigma:\Delta_k \to X} \mathbb{Z}\sigma, k \geq 0 \\ &= \{0\}, k < 0. \end{aligned}$$

$X \xrightarrow[\text{continuous}]{f} Y$, we can define $S_k(X) \xrightarrow{S_k(f)=f_\#} S_k(Y)$:

$$\sigma : \Delta_k \to X \mapsto \begin{tikzcd} \Delta_k \arrow[rr, "f\circ\sigma"] \arrow[dr, "\sigma"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd} .$$

$S_k : (\text{Top}) \to (\text{Ab})$ is a covariant functor.

Definition 1.6 (Boundary operation). $S_k(X) \xrightarrow{\partial_k} S_{k-1}(X)$

$$\partial_k := \sum_{i=0}^{k} (-1)^i \sigma_i$$

Exercise 1.1. The following two diagrams are commutative:

$$\begin{tikzcd} S_k(X) \arrow[r, "f_\#"] \arrow[d, "\partial_k"'] & S_k(Y) \arrow[d, "\partial_\#"] \\ S_{k-1}(X) \arrow[r, "f_\#"'] & S_{k-1}(Y) \end{tikzcd}$$

$$\begin{tikzcd} \Delta_k \arrow[r, "l_j"] \arrow[d, "l_{i-1}"'] & \Delta_{k-1} \arrow[d, "l_i"] \\ \Delta_{k-1} \arrow[r, "l_j"'] & \Delta_k \end{tikzcd}$$

if $1 \leq j+1 \leq i \leq k, k \geq 2$.

Definition 1.7 (Singular chain complexes). $\sigma : \Delta_k \to X$: a singular k-simplex in X,

$$\begin{aligned} \partial_{k-1}\left(\partial_k \sigma\right) =& \sum_{j=0}^{k-1} \sum_{i=0}^{k} (-1)^{i+j} (\sigma_i)_j \\ =& \sum_{k-1 \geq j \geq i \geq 0} (-1)^{i+j} \sigma \circ l_i \circ l_j + \sum_{1 \leq j+1 \leq i \leq k} (-1)^{i+j} \sigma \circ l_i \circ l_j \\ =& \sum_{k-1 \geq j \geq i \geq 0} (-1)^{i+j} \sigma \circ l_i \circ l_j + \sum_{1 \leq j+1 \leq i \leq k} \sigma \circ l_j \circ l_{i-1} \\ =& 0. \end{aligned}$$

Then, we have the chain complex:

$$\cdots \xrightarrow{\partial_{k+2}} S_{k+1}(X) \xrightarrow{\partial_{k+1}} S_k(X) \xrightarrow{\partial_k} S_{k-1} \xrightarrow{\partial_{k-1}} S_{k-2} \cdots .$$