# Geometry of Manifolds I

Based on lectures by Willian Philip Mnicozzi II

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The learning notes are an introduction to riemannian geometry. The learning resources I used are videos of the course: MIT18.965, Geometry of Manifolds, 2020 Fall, given by Willian Philip Minicozzi II. As an introduction course, the last 4 classes introduce some comparison geometry and geometric analysis which I omitted in my notes because of too much equations to type for humanity. Except this part, the notes arrange most topics the same order as the class did. Most part of the last section and some definitions come from Do Carmo's *Riemannian Geometry.*

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# 1 Tangent space, Lie brackets and flows

**Definition 1.1.** The map $u$(see Fig 1) is a *smooth map* if $y_\beta^{-1} \circ u \circ x_\alpha$ is smooth.

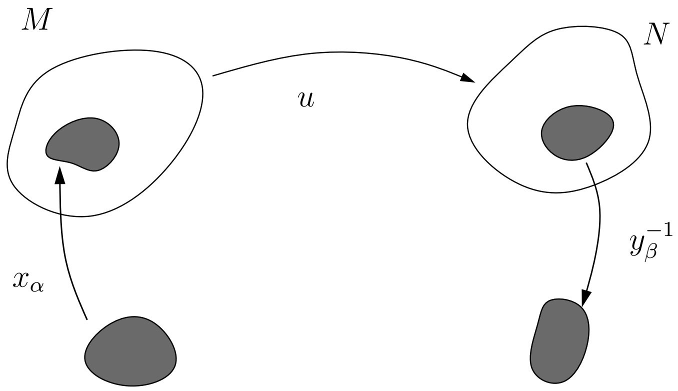


Figure 1: smooth map

Tangents,directional derivatives and vectors are equivalent.

**Definition 1.2.** Let $M$ be a manifold, $p \in M$, the *tangent space* of $M$ at $p$ is

$$T_pM = \{\text{All possible tangents to curves in } M \text{ at } p\}.$$

**Definition 1.3.** A *derivation* is a $\mathbb{R}$-linear map $X$ from functions on $M$ to functions on $M$ so that the Leibniz rule holds

$$X(fg) = fX(g) + gX(f).$$

The space of derivations and the space of vector fields on $M$ are equivalent.

**Question.** What happens to vector fields if we change coordinates?

Let $\mathbf{v} = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n b^i \frac{\partial}{\partial y^i} \in T_pM$, $\gamma$ be a curve. Set $\gamma(0) = p$, $\gamma'(0) = \mathbf{v}$.

$$\begin{aligned}
\mathbf{v}f &= \frac{\mathrm{d}}{\mathrm{d}t}\left(f\left(\gamma(t)\right)\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(f \circ y \circ y^{-1} \circ x \circ x^{-1} \circ \gamma(t)\right)\Big|_{t=0} \\
&= \sum_{i=1}^n \frac{\partial(f \circ y)}{\partial y^i} \sum_{j=1}^n \frac{\partial(y^i \circ x)}{\partial x^j} \frac{\mathrm{d}(x^j \circ \gamma(t))}{\mathrm{d}t}\Bigg|_{t=0} \\
&= \sum_{i=1}^n \frac{\partial f}{\partial y^i} \sum_{j=1}^n \frac{\partial y^i}{\partial x^j} a^j\Bigg|_{t=0}.
\end{aligned}$$

This implies $b^i = \sum_{j=1}^n \frac{\partial y^i}{\partial x^j} a^j$. From now on, we always use Einstein summation convention, i.e. $b^i = \frac{\partial y^i}{\partial x^j} a^j$.

In fact, $a^i = \mathbf{v}(x^i)$ and $b^j = \mathbf{v}(y^j)$, hence

$$b^j = \mathbf{v}(y^j) = \mathbf{v}\left(\frac{\partial y^j}{\partial x^i} x^i\right) = \frac{\partial y^j}{\partial x^i} a^i.$$

**Definition 1.4.** $X(M) \equiv$ smooth vector fields n $M$.

**Definition 1.5.** Let $\mathbf{v}, \mathbf{w} \in X(M)$, define *Lie Bracket* of $\mathbf{v}$ and $\mathbf{w}$ as

$$[\mathbf{v}, \mathbf{w}] \equiv \mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v}.$$

Consider $f \in C^\infty(M)$, then $[\mathbf{v}, \mathbf{w}]f = \mathbf{v}(\mathbf{w}(f)) - \mathbf{w}(\mathbf{v}(f))$.

**Proposition 1.6.** $[\mathbf{v}, \mathbf{w}] \in X(M)$.

*Proof.* We present two proofs, one is abstract and the other is based on coordinates.

First, we prove it in an abstract way.

$$\begin{aligned}
\mathbf{v}\left(\mathbf{w}\left(fg\right)\right) &= \mathbf{v}\left(f\mathbf{w}(g) + g\mathbf{w}(f)\right) \\
&= \mathbf{v}(f)\mathbf{w}(g) + f\mathbf{v}\left(\mathbf{w}(g)\right) + \mathbf{v}(g)\mathbf{w}(f) + g\mathbf{v}\left(\mathbf{w}(f)\right).
\end{aligned}$$

Interchange $\mathbf{v}$ and $\mathbf{w}$ we get

$$\mathbf{w}\left(\mathbf{v}(fg)\right) = \mathbf{w}(f)\mathbf{v}(g) + f\mathbf{w}\left(\mathbf{v}(g)\right) + \mathbf{w}(g)\mathbf{v}(f) + g\mathbf{w}\left(\mathbf{v}(f)\right).$$

First minus second we obtain

$$[\mathbf{v}, \mathbf{w}] = f[\mathbf{v}, \mathbf{w}](g) + g[\mathbf{v}, \mathbf{w}](f).$$

Then we prove it in another way based on coordinates. Let

$$\mathbf{v} = a^i\partial_i \text{ and } \mathbf{w} = b^j\partial_j.$$

Then

$$\begin{aligned}\mathbf{vw}f &= a^i\partial_i\left(b^j\partial_j\right)\\ &= a^i\partial_i b^j\partial_j + a^i b^j\partial_i\partial_j\end{aligned}$$

and

$$\begin{aligned}\mathbf{wv}f &= b^j\partial_j\left(a^i\partial_i\right)\\ &= b^j\partial_j a^i\partial_i + b^j a^i\partial_j\partial_i.\end{aligned}$$

Since $\partial_i\partial_j = \partial_j\partial_i$, we get

$$[\mathbf{v},\mathbf{w}] = (a^j\partial_j b^i - b^j\partial_j a^i)\partial_i.$$

□

**Proposition 1.7.** Let $X, Y, Z \in X(M)$, then we have the following called *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

**Definition 1.8.** Let $M$ be a closed(i.e., compact and no boundary) manifold. Given a smooth vector field $\mathbf{v}$, we can get a map $\varphi(x,t) : M\times\mathbb{R}\to M$ such that

$$\varphi(x,0) = x \text{ and } \frac{\mathrm{d}\varphi(x,t)}{\mathrm{d}t} = \mathbf{v}_{\varphi(x,t)}.$$

$\varphi(x,t)$ is called the *flow* of vector field $\mathbf{v}$.

**Definition 1.9.** Define $X^*(M)$ the dual space of $X(M)$.Precisely speaking, for any $\alpha \in X^*(M)$, $\alpha$ is a $C^\infty(M)$-linear map from $X(M)$ to $C^\infty(M)$. In local coordinates, we can choose $\mathrm{d}x^1, \mathrm{d}x^2, \cdots, \mathrm{d}x^n$ be the dual basis of $\partial_1, \partial_2, \cdots, \partial_n$.

# 2 Tensors and connections

**Definition 2.1.** An $(r, s)$ *tensor* is a multi-linear map over $C^\infty(M)$

$$A : \underbrace{X^*(M) \times \cdots \times X^*(M)}_{r} \times \underbrace{X(M) \times \cdots \times X(M)}_{s} \to C^\infty(M).$$

**Definition 2.2.** A Riemannian metric on a differentiable manifold $M$ is a crrespondence which associates to each point $p$ of $M$ an inner product $\langle \cdot, \cdot \rangle_p$ on the tangent space $T_p(M)$, which varies differentiably in the following sense: If $x : U \subset \mathbb{R}^n \to M$ is a system of coordinates around $p$, then $\langle \partial_i, \partial_j \rangle = g_{ij}(x_1, \cdots, x_n)$ is a differentiable function.

We can write Riemannian metric as $g = g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$. Let $A$ be a $(1, 1)$ tensor, then we can write $A = A^i_j \partial_j \otimes \mathrm{d}x^j$ with $A^i_j = A(\mathrm{d}x^i, \partial_j)$. We can do *contraction*

$$\mathrm{Tr}(A) = \sum_{i=1}^n A^i_i = \sum_{i=1}^n A(\mathrm{d}x^i, \partial_i).$$

If $y_1, \cdots, y_n$ is another choice of coordinates at the same point we still have

$$\mathrm{Tr} A = \sum_{i=1}^n A(\mathrm{d}y^i, \partial_{y^i}).$$

In fact,

$$\begin{aligned} A(\mathrm{d}y^i, \partial_{y^i}) &= A \left( \frac{\partial y^i}{\partial x^j} \mathrm{d}x^j, \frac{\partial x^k}{\partial y^i} \partial_{x^k} \right) \\ &= \frac{\partial y^i}{\partial x^j} \frac{\partial x^k}{\partial y^i} A \left( \mathrm{d}x^j, \partial_{x^k} \right) \\ &= \delta^k_j A \left( \mathrm{d}x^j, \partial_{x_k} \right) \\ &= A \left( \mathrm{d}x^j, \partial_{x^j} \right). \end{aligned}$$

**Definition 2.3.** An *affine connection* is a map

$$\nabla : X(M) \times X(M) \to X(M)$$

denoted by $\nabla_X Y$, which satisfies the following properties:

(1) $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$.

(2) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.

(3) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$.

in which $X, Y, Z \in X(M)$ and $f, g \in C^\infty(M)$.

Furthermore, it is the *Levi-Civita connection* if it satisfies the following 2 conditions:

(4) (Symmetry condition)

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

(5) (Metric compatible condition $\nabla g = 0$)

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

**Theorem 2.4.** *Levi-Civita connection exists and unique.*

*Proof.* First, we assume Levi-Civita connection exists. Then by metric compitable condition

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle X, Z\rangle &= \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Add first two identities and then subtract the third, by symmetry condition

$$\begin{aligned} &X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ =&\langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle + \langle \nabla_X Y + \nabla_Y X, Z\rangle \\ =&\langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle + 2\langle \nabla_X Y, Z\rangle - \langle [X, Y], Z\rangle. \end{aligned}$$

This means that if $\nabla_X Y$ exists, it must satisfy

$$\begin{aligned} &2\langle \nabla_X Y, Z\rangle \\ =&X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle. \end{aligned} \tag{1}$$

Since the inner product is nondegenerate, we can define the connection through the above identity. Hence we complete the unique part. For the existence, we need to check (1)–(5).

(1) It is enough to verify $\langle \nabla_{fX} Y, Z\rangle = f\langle \nabla_X Y, Z\rangle$.

$$\begin{aligned} &2\langle \nabla_{fX} Y, Z\rangle \\ =&fX\langle Y, Z\rangle + Y\langle fX, Z\rangle - Z\langle fX, Y\rangle \\ &-\langle Y, [fX, Z]\rangle - \langle fX, [Y, Z]\rangle - \langle Z, [Y, fX]\rangle \\ =&fX\langle Y, Z\rangle + fY\langle X, Z\rangle + Y(f)\langle X, Z\rangle - fZ\langle X, Y\rangle - Z(f)\langle X, Y\rangle \\ &-f\langle Y, [X, Z]\rangle + Z(f)\langle Y, X\rangle - f\langle X, [Y, Z]\rangle - f\langle Z, [Y, X]\rangle - Y(f)\langle Z, X\rangle \\ =&2f\langle \nabla_X Y, Z\rangle. \end{aligned}$$

(2) It is trivial.

(3) We need to check $fY$ is differentiated in terms 1,3,5,6 of (1).
Term 1:
$$X\left(f\langle Y, Z\rangle\right) = X(f)\langle Y, Z\rangle + fX\langle Y, Z\rangle.$$
Term 3:
$$Z\left(f\langle X, Y\rangle\right) = Z(f)\langle X, Y\rangle + fZ\langle X, Y\rangle.$$
Term 5:
$$\langle X, [fY, Z]\rangle = \langle X, f[Y, Z]\rangle - \langle X, Z(f)Y\rangle.$$
Term 6:
$$\langle Z, [fY, X]\rangle = \langle Z, f[Y, X]\rangle - \langle Z, X(f)Y\rangle.$$
Then we have
$$2\langle \nabla_X(fY), Z\rangle = 2f\langle \nabla_X Y, Z\rangle + 2X(f)\langle Y, Z\rangle.$$
Hence
$$\nabla_X(fY) = f\nabla_X Y + X(f)Y.$$

The remaining (4) and (5) are easy to verify. □

**Definition 2.5.** In some coordinate chart, we can write
$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k. \tag{2}$$
Functions $\Gamma_{ij}^k$'s are called *Christoffel symbols.*

Given two vector fields $\mathbf{v} = v^j\partial_j$ and $\mathbf{w} = w^j\partial_j$. Then
$$\begin{aligned}
\nabla_{\mathbf{w}}\mathbf{v} &= w^i\nabla_{\partial_i}\mathbf{v} \\
&= w^i\nabla_{\partial_i}\left(v^j\partial_j\right) \\
&= w^i\left(\partial_i v^j\partial_j + v^j\nabla_{\partial_i}\partial_j\right) \\
&= w^i\left(\partial_i v^j\partial_j + v^j\Gamma_{ij}^k\partial_k\right) \\
&= \left(w^i\partial_i v^k + w^i v^j\Gamma_{ij}^k\right)\partial_k.
\end{aligned}$$

Hence if we know Christoffel symbols we know how to calculate. We need formula for $\Gamma_{ij}^k$. Using (2), we have
$$2\langle \nabla_{\partial_i}\partial_j, \partial_p\rangle = 2\langle \Gamma_{ij}^k\partial_k, \partial_p\rangle.$$
Using (1), we have
$$2\langle \nabla_{\partial_i}\partial_j, \partial_p\rangle = \partial_i g_{jp} + \partial_j g_{ip} - \partial_p g_{ij}.$$

Then

$$2\Gamma_{ij}^k g_{kp} = \partial_i g_{jp} + \partial_j g_{ip} - \partial_p g_{ij}.$$

Define $g^{ij}$ the inverse metric to $g_{ij}$. Multiply the above equation by $g^{pq}$ and change indexes, we obtain

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right). \tag{3}$$

$\Gamma$'s depend on $g$ and its first derivative. If $\partial g = 0$, then $\Gamma = 0$. Christoffel symbol is NOT a tensor.

Connection is also named covariant derivative.

# 3 Parallel transports, geodesics and exponential maps

This section we concern about the following question: when is a vector field constant(parallel) through a curve?

Let $\gamma : I \to M$, $\gamma' = \mathrm{d}\gamma(t)$. Let $\mathbf{v}$ be a vector field on $M$, then $\nabla_{\gamma'}\mathbf{v}$ makes sense. We need to do this more generally. Consider $\mathbf{v}$ a vector field along $\gamma$ such that $\mathbf{v}(t) \in T_{\gamma(t)}M$. Define

$$\begin{aligned}\nabla_{\gamma'}\mathbf{v} &= \nabla_{\gamma'}\left(v^j\partial_j\right)\\ &= \nabla_{\gamma'}\left(v^j\partial_j\right)\\ &= \frac{\mathrm{d}v^j}{\mathrm{d}t}\partial_j + v^j\nabla_{\gamma'}\partial_j.\end{aligned}$$

This is well-defined and does not depend on how $I$ extend $v^j(t)$ off of $\gamma$. This definition agrees weith the case where $\mathbf{v}$ is restriction of a vector field on $M$. Moreover, it also has Leibniz rule and satisfies metric compatible conditon. We can rewrite the above equation more concisely by using (2)

$$\nabla_{\gamma'}\mathbf{v} = \left(\frac{\mathrm{d}v^k}{\mathrm{d}t} + v^j\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\Gamma^k_{ij}\right)\partial_k. \tag{4}$$

**Definition 3.1.** Let $\mathbf{v}$ be a vector field along $\gamma$. If

$$\nabla_{\gamma'}\mathbf{v} = 0,$$

we say $\mathbf{v}$ is *parallel* along $\gamma$.

Suppose $\mathbf{v}, \mathbf{w}$ are parallel through $\gamma$, then

$$\partial_t\langle\mathbf{v},\mathbf{w}\rangle = \langle\nabla_{\gamma'}\mathbf{v},\mathbf{w}\rangle + \langle\mathbf{v},\nabla_{\gamma'}\mathbf{w}\rangle = 0.$$

So $\langle\mathbf{v},\mathbf{w}\rangle$ is const, i.e., isometry.

**Definition 3.2.** Let $M$ be a Riemannian manifold, $p \in M$, $\mathbf{v}_p \in T_pM$. Suppose $\mathbf{v}(t)$ solves the ODE

$$\begin{cases}\nabla_{\gamma'}\mathbf{v} = 0,\\ \mathbf{v}(0) = \mathbf{v}_p.\end{cases}$$

We call $\mathbf{v}(t)$ the *parallel transport* through $\gamma$.

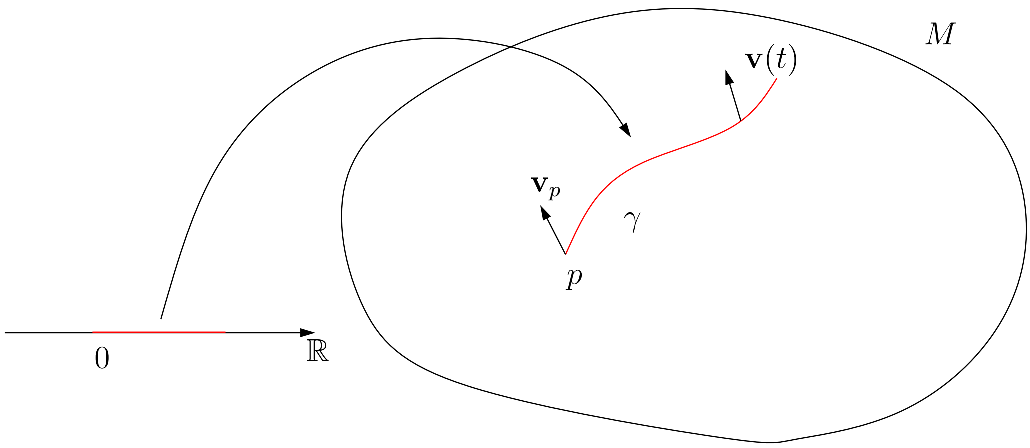


Figure 2: parallel transport

Using (4), we get

$$\frac{\mathrm{d}v^k}{\mathrm{d}t} + \Gamma_{ij}^k v^j \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} = 0, \quad k = 1, 2, \cdots, n. \tag{5}$$

$n$ first order odes in $n$ unknowns $v^j(t)$ implies there exists a unique $\mathbf{v}(t)$ which solves the ODE.

**Definition 3.3.** Let $\gamma$ be a curve in $(M, g)$, then it is a *geodesic* if

$$\nabla_{\gamma'}\gamma' = 0. \tag{6}$$

If $\gamma$ is a geodesic, then

$$\partial_t\langle\gamma', \gamma'\rangle = 2\langle\nabla_{\gamma'}\gamma', \gamma'\rangle = 0.$$

This implies $|\gamma'|$ is const on a geodesic. On $\mathbb{R}^n$, geodesics are straight lines at constant speed.

Using (5), geodesic equations in local coordinates can be wriiten as

$$\frac{\mathrm{d}^2x^k}{\mathrm{d}t^2} + \Gamma_{ij}^k \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = 0. \tag{7}$$

Given the intial value and initial tangent vector $\gamma(0)$ and $\gamma'(0)$, it can be solbed locally.

Let $p \in M$ and $\mathbf{v} \in T_pM$, we denote $\gamma(t, p, \mathbf{v})$ the geodesic starting at $p$ and direction $\mathbf{v}$. By ODE knowledge, we have

$$\gamma(t, p, \mathbf{v}) = \gamma(1, p, t\mathbf{v}).$$

**Definition 3.4.** Let $p \in M$, define the *exponential map* $\exp_p : T_pM \to M$ with

$$\exp_p(\mathbf{v}) = \gamma(1, p, \mathbf{v})$$

on a neighborhood of $0 \in T_pM$.

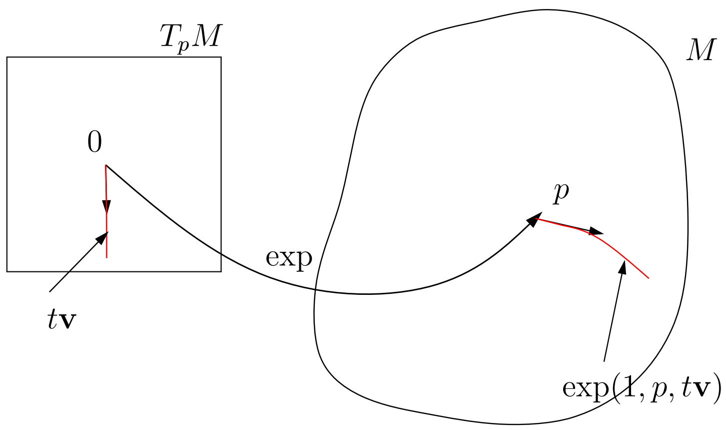


Figure 3: exponential map

Since $\exp_p : T_pM \to M$, we have

$$\mathrm{d}\exp_p : T_pM \to T_pM.$$

$$\begin{aligned}
\left(\mathrm{d}\exp_p\right)_0(\mathbf{v}) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_p(t\mathbf{v}) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(1, p, t\mathbf{v}) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t, p, \mathbf{v}) \\
&= \mathbf{v}.
\end{aligned}$$

By inverse function theorem, there exists a neighborhood where $\exp_p$ is a local diffeomorphism. If $q \in M$ is in this neighborhood of $p$, then there exists a unique geodesic from $p$ to $q$ (unique within the neighborhood).

**Theorem 3.5 (Gauss Lemma).** *Let* $\mathbf{v} \in T_pM$ *and* $\mathbf{w} \in T_{\mathbf{v}}T_pM = T_pM$*, then*

$$\langle (\mathrm{d}\exp_p)_{\mathbf{v}}(\mathbf{v}), (\mathrm{d}\exp_p)_{\mathbf{v}}(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle. \tag{8}$$

*Proof.* Let $F(t,s) = \exp_p(t(\mathbf{v} + s\mathbf{w}))$. At $(1,0)$, $F(1,0) = \exp_p(\mathbf{v})$, and

$$\begin{aligned}\partial_t = \mathbf{v} \to F_t = (\mathrm{d}\exp_p)_{\mathbf{v}}(\mathbf{v})\,(\text{at } (1,0))\\ \partial_s = \mathbf{w} \to F_s = \left(\mathrm{d}\exp_p\right)_{\mathbf{v}}(\mathbf{w})\,(\text{at } (1,0)).\end{aligned}$$

In this language, Gauss lemma is equivalent to

$$\langle \mathbf{v}, \mathbf{w}\rangle = \langle F_t(1,0, F_s(1,0)\rangle. \tag{9}$$

First we want to figure out how the right hand side of this equation depend on $t$. If $s$ is fixed, then by the definition of the exponential map we know $F(\cdot, s)$ is a geodesic, hence

$$\nabla_{F_t} F_t(t,s) = 0.$$

Do the partial derivative of $\langle F_t, F_s\rangle$ with respect to $t$

$$\begin{aligned}\partial_t\langle F_t, F_s\rangle =& \langle \nabla_{F_t}F_t, F_s\rangle + \langle F_t, \nabla_{F_t}F_s\rangle\\ =&\langle F_t, \nabla_{F_t}F_s\rangle.\end{aligned}$$

Write $F_s = \frac{\partial x^i}{\partial_s}\partial_i$ and $F_t = \frac{\partial x^j}{\partial t}\partial_j$. Then

$$\nabla_{F_s}F_t = \left(\frac{\partial^2 x^k}{\partial s \partial t} + \Gamma^k_{ij}\frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\right)\partial_k.$$

$t$ and $s$ are cleraly symmetric above, hence

$$\nabla_{F_s}F_t = \nabla_{F_t}F_s$$

or

$$[F_s, F_t] = 0.$$

Then we obtain

$$\partial_t\langle F_t, F_s\rangle = \langle F_t, \nabla_{F_s}F_t\rangle = \frac{1}{2}\partial_s\langle F_t, F_t\rangle.$$

$|F_t|^2$ depends on $(t, s)$ but constant in $t$ given a fixed $s$ since $F(\cdot, s)$ is a geodesic. At $t = 0$, $F_t = \mathbf{v} + s\mathbf{w}$. Then

$$\begin{aligned}
&|F_t|^2 = |\mathbf{v}|^2 + 2s\langle \mathbf{v}, \mathbf{w}\rangle + s^2|\mathbf{w}|^2 \\
\Rightarrow &\frac{1}{2}\partial_s |F_t|^2 = \langle \mathbf{v}, \mathbf{w}\rangle + s|\mathbf{w}|^2 \\
\Rightarrow &\partial_t \langle F_t, F_s\rangle(t, 0) = \langle \mathbf{v}, \mathbf{w}\rangle \\
\Rightarrow &\langle F_t, F_s\rangle(1, 0) = \langle \mathbf{v}, \mathbf{w}\rangle.
\end{aligned}$$

□

Now we use the exponential map to denote a curve in $M$ by

$$\gamma(t) = \exp_p\left(r(t)\mathbf{v}(t)\right), \quad |\mathbf{v}| = 1.$$

**Proposition 3.6.** See Figure 4, suppose $\gamma$ goes from $r(0) = 0$ to $r(1) = R$. Then the length of $\gamma$ is no less than $R$ and equal to if and only if $\mathbf{v}$ is constant and $r$ is monotone.

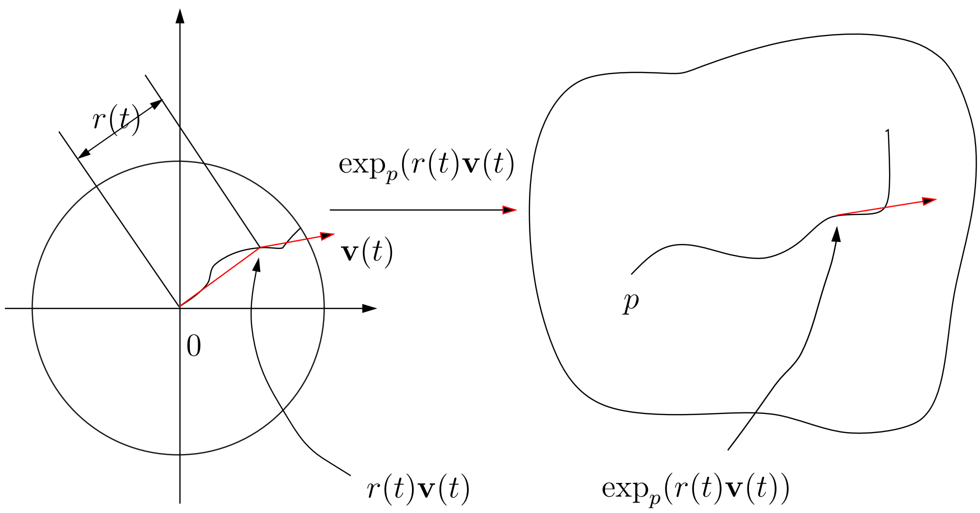


Figure 4: polar coordinate of a curve

*Proof.*

$$\gamma' = \mathrm{d}\exp_p\left(r'\mathbf{v} + r\mathbf{v}'\right) = r'\mathrm{d}\exp_p(\mathbf{v}) + r\,\mathrm{d}\exp_p(\mathbf{v}').$$

Note that $\mathbf{v} \perp \mathbf{v}'$ in $T_pM$, Gauss lemma tells $\mathrm{d}\exp_p(\mathbf{v}) \perp \mathrm{d}\exp_p(\mathbf{v}')$. Hence

$$\begin{aligned} |\gamma'|^2 &= (r')^2 \left(\mathrm{d}\exp_p(\mathbf{v})\right)^2 + r^2 \left(\mathrm{d}\exp_p(\mathbf{v}')\right)^2 \\ &\geq (r')^2 |\mathrm{d}\exp_p(\mathbf{v})|^2 \\ &= |r'|^2. \end{aligned}$$

This says

$$L(\gamma) = \int_0^1 |\gamma'| \,\mathrm{d}t \geq \int_0^1 |r'| \,\mathrm{d}t \geq r(1) - r(0) = R.$$

To make $L(\gamma) = R$, we need $\mathbf{v}' = 0$ and $r$ monotone. □

**Definition 3.7.** Let $(M, g)$ be a Riemannian manifold, $p, q \in M$, we define

$$d(p,q) \equiv \inf_{\substack{\text{piecewise curve } \gamma \\ \text{from } p \text{ to } q}} L(\gamma).$$

$d(p,q)$ is the *distance* between two points $p$ and $q$.

The Riemannian manifold within the above definition of distance comprise a metric space.

**Proposition 3.8.** Given $p \in M$, there exists an open set $V \subset M$, $p \in V$, numbers $\delta > 0$ and $\varepsilon_1 > 0$ and a $C^\infty$ mapping

$$\gamma : [-\delta, \delta) \times \mathcal{U} \to M, \quad \mathcal{U} = \{(q, \mathbf{v}) : q \in V, \mathbf{v} \in T_qM, |\mathbf{v}| < \epsilon\},$$

such that the curve $t \to \gamma(t, q, \mathbf{v}), t \in (-\delta, \delta)$is the unique geodesic of $M$ which at the instant $t = 0$ passing through $q$ with velocity for each $q \in V$ and for each $\mathbf{v} \in T_qM$ with $|\mathbf{v}| < \epsilon_1$.

**Corollary 3.9.** Let $p \in M$, there exists a neighborhood $V$ of $p$ in $M$, $\epsilon > 0$ and a $C^\infty$ mapping $\gamma : (-2, 2) \times \mathcal{U} \to M$, $\mathcal{U} = \{(q, \mathbf{w}) \in TM : q \in V, \mathbf{w} \in T_qM, |\mathbf{w}| < \epsilon\}$ such that $t \to \gamma(t, q, \mathbf{w}), t \in (-2, 2)$ is the unique geodesic of $M$ which at the instant $t = 0$ passing through $q$ with velocity $\mathbf{w}$ for every $q \in V$ and for every $\mathbf{w} \in T_aM$ with $|\mathbf{w}| < \epsilon$.

It is necessary to introduce some notions about neighborhoods.

**Definition 3.10.** Here are some definitions for some types of neighborhoods:

a. If $\exp_p$ is a diffeomorphism of a neighborhood $V$ of the origin in $T_pM$, $\exp_p V = U$ is called a *normal neighborhood* of $p$.

b. If $B_\epsilon(0)$ is such that $B_\epsilon(0) \subset V$, we call $\exp_p B_\epsilon(0) = B_\epsilon(p)$ the *normal ball* with center $p$ and radius $\epsilon$.

**Proposition 3.11.** Let $p \in M$, $U$ a normal neighborhood of $p$ and $B \subset U$ a normal ball of center $p$. Let $\gamma : [0,1] \to B$ be a geodesic segment with $\gamma(0) = p$. If $c[0,1] \to M$ is any piecewise differentiable curve joining $\gamma(0)$ to $\gamma(1)$ then $l(\gamma) \leq l(c)$ and if equality holds then $\gamma([0,1]) = c([0,1])$.

*Proof.* Suppose $c([0,1]) \subset B$. Write

$$c(t) = \exp_p(r(t)\mathbf{v}(t)) = f(r(t), t),$$

where $|\mathbf{v}(t)| = 1$. Except for a finite number of points,

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{\partial f}{\partial r} r'(t) + \frac{\partial f}{\partial t}.$$

From the Gauss lemma, $\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = 0$. Since $\left|\frac{\partial f}{\partial r}\right| = 1$,

$$\left|\frac{\mathrm{d}c}{\mathrm{d}t}\right|^2 = |r'(t)|^2 + \left|\frac{\partial f}{\partial t}\right|^2 \geq |r'(t)|^2$$

and so

$$\int_\epsilon^1 \left|\frac{\mathrm{d}c}{\mathrm{d}t}\right| \mathrm{d}t \geq \int_\epsilon^1 |r'(t)| \mathrm{d}t \geq \int_\epsilon^1 r'(t)\,\mathrm{d}t = r(1) - r(\epsilon).$$

Taking $\epsilon \to 0$ we obtain $l(c) \geq l(\gamma)$ because $r(1) = l(\gamma)$. If $l(c) = l(\gamma)$, then $\left|\frac{\partial f}{\partial t}\right| = 0$, i.e., $\mathbf{v}(t)$ is constant, and $|r'(t)| = r'(t) > 0$. It follows that $c$ is a monotonic reparametrization of $\gamma$, hence $c([0,1]) = \gamma([0,1])$.

If $c([0,1])$ is not contained in $B$, set $t_1 \in (0,1)$ the first point for which $c(t_1)$ belongs to the boundary of $B$. If $\rho$ is the radius of the geodesic ball $B$, we have

$$l(c) \geq l_{[0,t_1]}(c) \geq \rho > l(\gamma).$$

□

**Theorem 3.12.** *For any $p \in M$ there exists a neighborhood $W$ of $p$ and a number $\delta > 0$ such that for every $q \in W$, $\exp_q$ is a diffeomorphism on $B_\delta(0) \subset T_qM$ and $\exp_q(B_\delta(0) \supset W$, that is, $W$ is a normal neighborhood of each of its points.*

$W$ like this is called a *totally normal neighborhood* of $p \in M$.

**Corollary 3.13.** If a piecewise differentiable curve $\gamma : [a,b] \to M$ with parameter propotional to arc length, has length less or equal to the length of any other piecewise differentiable curve joining $\gamma(a)$ to $\gamma(b)$ then $\gamma$ is a geodesic. In particular, $\gamma$ is regular.

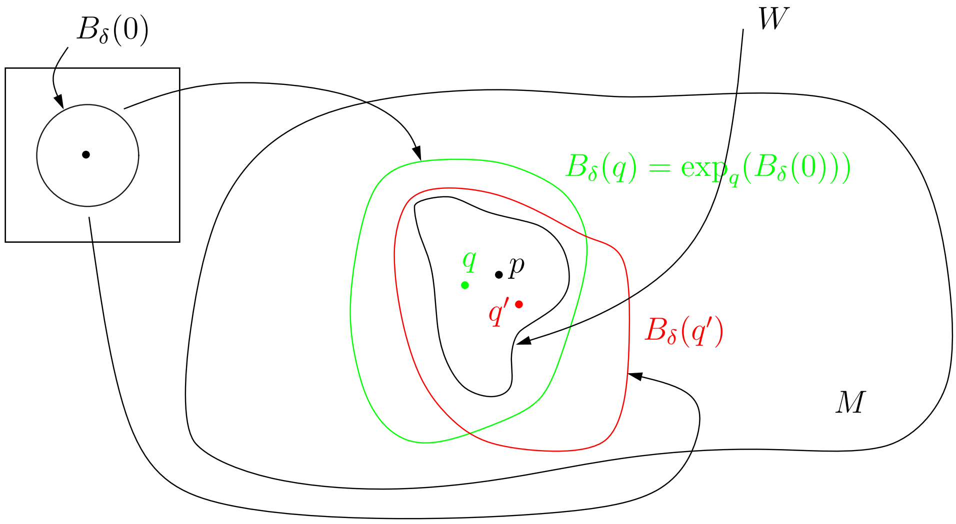


Figure 5: totally normal neighborhood

# 4 Curvature

**Definition 4.1.** The curvature $R$ of a Riemannian manifold $(M, g)$ is a $(1,3)$ tensor, or a trilinear $C^\infty(M)$ map from 3 spaces of vector fields to 1 space of vector fields, such that

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z. \tag{10}$$

**Proposition 4.2.** (0) $R(X,Y)Z$ is skew in $X, Y$,

$$R(Y,x)Z = -R(X,Y)Z.$$

(1) $\langle R(X,Y)Z, W\rangle$ is skew in $Z, W$,

$$\langle R(X,Y)W, Z\rangle = -\langle R(X,Y)Z, W\rangle.$$

(2) (Bianchi identity)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

(3) (Pair symmetry)

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle.$$

(4) (Second Bianchi identity)

$$(\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) = 0.$$

Let $B$ be a $(0,2)$ tensor, then for any $X, Y \in X(M)$, $B(X,Y)$ is in $C^\infty(M)$. We can view $B$ as a $(1,1)$ tensor by the following argument: Consider the map $X \mapsto B(X,\cdot)$, view $B(X,\cdot)$ as a vector field such that

$$\langle B(X,\cdot), Y\rangle = B(X,Y).$$

This change of viewpoint is called *raising and lowering indicies.*

Let $f \in C^\infty(M)$, then $\mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i$ is a 1-form, i.e., a $(0,1)$ tensor. We want to view it as a $(1,0)$ tensor(i.e., a vector field) $\nabla f$, by defining

$$\langle \nabla f, X\rangle = X(f) = \mathrm{d}f(X).$$

Suppose $\nabla f = a^j \partial_j$, by definition

$$\langle \nabla f, \partial_i\rangle = \partial_i f = \frac{\partial f}{\partial x_i}.$$

$$\langle \nabla f, \partial_i \rangle = \langle a^j \partial_j, \partial_i \rangle = a^j g_{ij}.$$

Combining the above two equations we obtain

$$a^j = g^{ij} \frac{\partial f}{\partial x_i}.$$

Hence

$$\nabla f = g^{ij} \frac{\partial f}{\partial x_i} \partial_j. \tag{11}$$

This is called the *gradient* of $f$.

**Definition 4.3.** *Hessian* $\mathrm{Hess}_f$ of $f$ is a $(0,2)$ tensor such that for any $X, Y \in X(M)$

$$\mathrm{Hess}_f(X,Y) = \langle \nabla_X \nabla f, Y \rangle. \tag{12}$$

*Laplacian* $\Delta f$ of $f$ is the trace of $\mathrm{Hess}_f$, i.e., for an orthonormal frame $\{e_i\}_{i=1}^n$,

$$\Delta f = \mathrm{Tr}\,(\mathrm{Hess}_f) = \sum_{i=1}^n \langle \nabla_{e_i} \nabla f, e_i \rangle. \tag{13}$$

**Lemma 4.4.** $\mathrm{Hess}_f$ is symmetric.

*Proof.* Suppose $\mathbf{v}, \mathbf{w} \in X(M)$,

$$\begin{aligned} \langle \nabla_{\mathbf{v}} \nabla f, \mathbf{w} \rangle &= \mathbf{v} \langle \nabla f, \mathbf{w} \rangle - \langle \nabla f, \nabla_{\mathbf{v}} \mathbf{w} \rangle \\ &= \mathbf{v}\left(\mathbf{w}(f)\right) - (\nabla_{\mathbf{v}} \mathbf{w})(f). \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Hess}_f(\mathbf{v},\mathbf{w}) - \mathrm{Hess}_f(\mathbf{w},\mathbf{v}) &= \mathbf{v}\left(\mathbf{w}(f)\right) - \mathbf{w}\left(\mathbf{v}(f)\right) - \left(\nabla_{\mathbf{v}}\mathbf{w}\right)(f) + \left(\nabla_{\mathbf{w}}\mathbf{v}\right)(f) \\ &= 0. \quad \text{(by symmetry of } \nabla) \end{aligned}$$

□

In local coordinates,

$$\begin{aligned} f_{ij} &\equiv \mathrm{Hess}_f\left(\partial_i, \partial_j\right) \\ &= \langle \nabla_{\partial_i} \nabla f, \partial_j \rangle \\ &= \partial_i \langle \nabla f, \partial_j \rangle - \langle \nabla f, \nabla_i \partial_j \rangle \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}. \end{aligned}$$

Then

$$\begin{aligned}\Delta f &= \sum_{i=1}^{n} g^{ij} f_{ij} \\ &= g^{ij} \frac{\partial^2 f}{x^i x^j} - g^{ij} \Gamma_{ij}^k \frac{\partial f}{\partial x^k}.\end{aligned}$$

Let $M^n$ be a submanifold of $\left(\overline{M}^{n+m}, \overline{g}\right)$, $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$. Then it induces a $\nabla$ on $M$ such that for $X, Y \in X(M)$,

$$\nabla_X Y = \left(\overline{\nabla}_X Y\right)^T.$$

**Definition 4.5.** The *second fundamental form* is defined by

$$A(X,Y) = \left(\overline{\nabla}_X Y\right)^{\perp} = \overline{\nabla}_X Y - \nabla_X Y. \tag{14}$$

**Lemma 4.6.** *The second fundamental form $A$ is a well-defined $(0,2)$ tensor in $X, Y$ and symmetric.*

*Proof.* The "hard" part is $C^{\infty}(M)$ linear in $Y$.

$$\begin{aligned}\left(\overline{\nabla}_X Y\right)^{\perp} &= \left(f\overline{\nabla}_X Y + X(f)Y\right)^{\perp} \\ &= f\left(\overline{\nabla}_X Y\right)^{\perp}.\end{aligned}$$

$A$ is symmetric because

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y].$$

□

**Theorem 4.7.** *Let $X, Y, Z \in T_pM \subset T_p\overline{M}$, then*

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + \langle A(Y,Z), A(X,W)\rangle - \langle A(Y,W), A(X,Z)\rangle. \tag{15}$$

*Proof.*

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= \langle \overline{R}(X,Y)Z, W\rangle \\ &= \langle \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_X \overline{\nabla}_Y Z + \overline{\nabla}_{[X,Y]} Z, W\rangle \\ &= \langle \overline{\nabla}_Y^T \overline{\nabla}_X Z - \overline{\nabla}_X^T \overline{\nabla}_Y Z, W\rangle + \langle \overline{\nabla}_{[X,Y]}^T Z, W\rangle \\ &= \langle \overline{\nabla}_Y^T \left(\overline{\nabla}_X^T Z + \overline{\nabla}_X^{\perp} Z\right), W\rangle - \langle \overline{\nabla}_X^T \left(\overline{\nabla}_Y^T Z + \overline{\nabla}_Y^{\perp} Z\right), W\rangle \\ &+ \langle \nabla_{[X,Y]} Z, W\rangle \\ &= R(X,Y,Z,W) + \langle \overline{\nabla}_Y^T A(X,Z), W\rangle - \langle \overline{\nabla}_X^T A(Y,Z), W\rangle \\ &= R(X,Y,Z,W) - \langle A(X,Z), \overline{\nabla}_Y^{\perp} W\rangle + \langle A(Y,Z), \overline{\nabla}_X^{\perp} W\rangle \\ &= R(X,Y,Z,W) + \langle A(Y,Z), A(X,W)\rangle - \langle A(Y,W), A(X,Z)\rangle.\end{aligned}$$

□

**Corollary 4.8.** Let $e_1, e_2$ be orthonormal in $T_pM$, the *sectional curvature* of $e_1-e_2$ plane in $M$ defined by

$$K_{12} \equiv R\left(e_1, e_2, e_1, e_2\right)$$

and

$$\overline{K}_{12} \equiv \overline{R}\left(e_1, e_2, e_1, e_2\right)$$

have the relation

$$\overline{K}_{12} = K_{12} + \langle A_{12}, A_{12}\rangle - \langle A_{22}, A_{11}\rangle. \tag{16}$$

**Example 4.1.** Let $\overline{M} = \mathbb{R}^n$, then $\overline{R} = 0$, this implies $\overline{K}_{12} = 0$, hence

$$K_{12} = \langle A_{22}, A_{11}\rangle - |A_{12}|^2.$$

In particular, let $n = 3$, $M$ be a two dimensional surface. Let $\mathbf{v}, \mathbf{w} \in T_pM$, two curves $v$ and $w$ with $v'(0) = \mathbf{v}$ and $w'(0) = \mathbf{w}$,

$$A\left(\mathbf{v}, \mathbf{w}\right) = (\overline{\nabla}_{\mathbf{v}}\mathbf{w})^{\perp} = \left(\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t}\left(v(t)\right)\Big|_{t=0}\right)^{\perp}.$$

$$\begin{aligned}
&\left(\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t}\left(v(t)\right)\Big|_{t=0}\right)^{\perp} \cdot \mathbf{N} \\
=&\left(\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t}\left(v(t)\right)\Big|_{t=0}\right) \cdot \mathbf{N} \\
=&-\mathbf{w} \cdot \mathrm{d}\mathbf{N}_p\left(\mathbf{v}\right).
\end{aligned}$$

This is exactly the second fundamental form in classical differential geometry. Hence $K_{12}$ is the Gaussian curvature in this classical condition.

**Definition 4.9.** A submanifold is *minimal* if $\mathrm{Tr}(A) = 0$.

To a surface, if it is minimal, then $A_{11} = -A_{22} \Rightarrow K_{12} = -|A_{11}|^2 - |A_{12}|^2 \leq 0$.

**Example 4.2.** Let $S^n \hookrightarrow \mathbb{R}^{n+1}$. We notice that for a position vector $\mathbf{x} \in \mathbb{R}^{n+1}$, we have

$$\overline{\nabla}_{\mathbf{v}}\mathbf{x} = \mathbf{x}. \tag{17}$$

For $|\mathbf{x}| = 1$, at position $\mathbf{x}$, we have

$$A(\mathbf{v}, \mathbf{w}) = \langle A(\mathbf{v}, \mathbf{w}), \mathbf{x}\rangle\mathbf{x}.$$

$$\begin{aligned}&\langle A(\mathbf{v},\mathbf{w},\mathbf{x}\rangle\\=&\langle\overline{\nabla}^{\perp}_{\mathbf{v}}\mathbf{w},\mathbf{x}\rangle\\=&\langle\overline{\nabla}_{\mathbf{v}}\mathbf{w},\mathbf{x}\rangle\\=&-\langle\mathbf{w},\overline{\nabla}_{\mathbf{v}}\mathbf{x}\rangle\\=&-\langle\mathbf{w},\mathbf{v}\rangle.\end{aligned}$$

Hence

$$A(\mathbf{v},\mathbf{w})=-\langle\mathbf{v},\mathbf{w}\rangle\mathbf{x}.$$

Use this in (15),

$$R(X,Y,Z,W)=\langle Y,W\rangle\langle X,Z\rangle-\langle Y,Z\rangle\langle X,W\rangle.$$

Let $X=Z=e_1, Y=W=e_2$, then $K_{12}=1-0=1$. Hence $S^n$ has constant sectional curvature 1.

In fact, for any Riemannian manifold $(M,g)$, when

$$R(X,Y,Z,W)=k\left(\langle Y,W\rangle\langle X,Z\rangle-\langle Y,Z\rangle\langle X,W\rangle\right),$$

then $M$ has constant sectional curvature $k$.

From now on, $\{e_i\}_{i=1}^n$ is always an orthonormal basis of $T_pM$ at some point $p\in M$.

**Definition 4.10.** The *Ricci tensor* is

$$\mathrm{Ric}(\mathbf{v},\mathbf{w})=\sum_{i=1}^n R(\mathbf{v},e_i,\mathbf{w},e_i). \tag{18}$$

The *Ricci curvature* of point $p$ in the direction $\mathbf{x}$ is

$$\mathrm{Ric}_p(\mathbf{x})=\sum_{i=1}^n\langle R(\mathbf{x},e_i)\mathbf{x},e_i\rangle. \tag{19}$$

Ricci tensor of $S^n$ is

$$\begin{aligned}\mathrm{Ric}(X,Z)=&\sum_{i=1}^n R(X,e_i,Z,e_i)\\=&\sum_{i=1}^n\left(\langle e_i,e_i\rangle\langle X,Z\rangle-\langle e_i,X\rangle\langle e_i,Z\rangle\right)\\=&n\langle X,Z\rangle-\langle X,Z\rangle\\=&(n-1)\langle X,Z\rangle.\end{aligned}$$

Hence for $M = S^n$, we have

$$\mathrm{Ric} = (n-1)g.$$

Any manifold where $\mathrm{Ric} = \lambda g, \lambda \in \mathbb{R}$ is called *Einstein*.

**Definition 4.11.** The *scalar curvature* $K$ is defined as

$$K(p) = \sum_{i=1}^{n} \mathrm{Ric}\left(e_i, e_i\right). \tag{20}$$

The scalar curvature of $S^n$ is

$$\begin{aligned} K(p) &= \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i) \\ &= (n-1) \sum_{i=1}^{n} \langle e_i, e_i \rangle \\ &= n(n-1). \end{aligned}$$

**Definition 4.12.** The *mean curvature vector* is defined as

$$H = -\mathrm{Tr}(A) = -\sum_{i=1}^{n} \left(\overline{\nabla}_{e_i} e_i\right)^{\perp}. \tag{21}$$

By definition we know that a submanifold is minimal if and only if $H = 0$.

**Theorem 4.13.** *Let $M^n \hookrightarrow \mathbb{R}^N$ be a minimal submanifold, then coordinate functions are harmonic, i.e.,*

$$\Delta x_i = 0, \quad i = 1, 2, \cdots, N.$$

*Proof.*

$$\begin{aligned} \Delta x_i &= \sum_{j=1}^{n} \langle \overline{\nabla}^T_{e_j} \overline{\nabla}^T x_i, e_j \rangle \\ &= \sum_{j=1}^{n} \langle \overline{\nabla}_{e_j} \left( \overline{\nabla} x_i - \overline{\nabla}^{\perp} x_i \right), e_j \rangle \\ &= -\sum_{j=1}^{n} \langle \overline{\nabla}_{e_j} \overline{\nabla}^{\perp} x_i, e_j \rangle \\ &= \sum_{j=1}^{n} \langle \overline{\nabla}^{\perp} x_i, \overline{\nabla}^{\perp}_{e_j} e_j \rangle \\ &= \langle \overline{\nabla}^{\perp} x_i, -H \rangle \\ &= 0.. \end{aligned}$$

□

**Example 4.3.** Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be oriented with unit normal vector field $\mathbf{n}$. Then

$$\mathbf{n} : M^n \to S^n.$$

It is called the Gauss map. For example, if $M^n = S^n$, then the Gauss map is identity. If $M^n = \mathbb{R}^n$, or any hyperplane, then the Gauss map is constant(in fact the statement can be strengthen to if and only if).

Let $\mathbf{v}, \mathbf{w} \in X(M)$,

$$\langle \overline{\nabla}_{\mathbf{v}} \mathbf{n}, \mathbf{w} \rangle = -\langle \mathbf{n}, \overline{\nabla}_{\mathbf{v}} \mathbf{w} \rangle = -\langle \mathbf{n}, A(\mathbf{v}, \mathbf{w}) \rangle.$$

Hence $A$ measures change of $\mathbf{n}$ intangent directions.

$$\langle \overline{\nabla}_{\mathbf{v}} \mathbf{n}, \mathbf{n} \rangle = \frac{1}{2} \mathbf{v} \langle \mathbf{n}, \mathbf{n} \rangle = 0.$$

Since $\mathbf{n} : M^n \to S^n$,

$$\mathrm{d}\mathbf{n}_p : T_pM \to T_{\mathbf{n}(p)} S^n = T_pM.$$

Let $\mathbf{v} \in T_pM$, choose a curve $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = \mathbf{v}$. Then we can get a curve $\mathbf{n}(\gamma(t))$ in $S^n$ and $\mathbf{n}(\gamma(0)) = \mathbf{n}(p)$. Define

$$\mathrm{d}\mathbf{n}_p(\mathbf{v}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{n}(\gamma(t)).$$

What is $\langle \mathrm{d}\mathbf{n}_p(\mathbf{v}), \mathbf{w} \rangle$? For any $\mathbf{v}, \mathbf{w} \in T_pM$, we calculate

$$\begin{aligned}
\langle \mathrm{d}\mathbf{n}_p(\mathbf{v}), \mathbf{w} \rangle &= \langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{n}(\gamma(t)), \mathbf{w} \rangle \\
&= \langle \overline{\nabla}_{\mathbf{v}} \mathbf{n}, \mathbf{w} \rangle \\
&= -\langle A(\mathbf{v}, \mathbf{w}), \mathbf{n} \rangle.
\end{aligned}$$

Hence the change in $\mathbf{n}$ is measured by $A(\cdot, \cdot)$. If $\mathbf{n} = \text{const}$, then $A(\cdot, \cdot) = 0$.

**Definition 4.14.** Let $\mathbf{v} \in T_pM$, $\eta \perp T_pM$, define

$$\nabla^{\perp}_{\mathbf{v}} \eta \equiv \left( \overline{\nabla}_{\mathbf{v}} \eta \right)^{\perp} \tag{22}$$

be a *connection on normal bundle.* It is easy to check that $\nabla^{\perp}_{\mathbf{v}} \eta$ is $C^{\infty}(M)$ linear in $\mathbf{v}$ and satisfies Leibniz rule in $\eta$. It is also metric compatible.

Let $\mathbf{v}, X, Y$ be tangent, we define

$$\left( \nabla_{\mathbf{v}} A \right)(X, Y) \equiv \nabla^{\perp}_{\mathbf{v}} \left( A(X, Y) \right) - A\left( \nabla_{\mathbf{v}} X, Y \right) - A\left( X, \nabla_{\mathbf{v}} Y \right). \tag{23}$$

This definition also satisfies Leibniz rule.

**Theorem 4.15 (Codazzi Equation).** *Let $V, W, X$ be tengent vector fields, then*

$$\left(\overline{R}(V,W)X\right)^{\perp} = (\nabla_W A)(V,X) - (\nabla_V A)(W,X). \tag{24}$$

*Proof.*

$$\begin{aligned}\left(\overline{R}(V,W)X\right)^{\perp} &= \overline{\nabla}^{\perp}_W \overline{\nabla}_V X - \overline{\nabla}^{\perp}_V \overline{\nabla}_W X + \overline{\nabla}^{\perp}_{[V,W]} X \\ &= A(W, \nabla_V X) + \nabla^{\perp}_W (A(V,X)) - A(V, \nabla_W X) \\ &\quad - \nabla^{\perp}_V (A(W,X)) + A([V,W],X) \\ &= (\nabla_W A)(V,X) + A(\nabla_W V, X) - (\nabla_V A)(W,X) \\ &\quad - A(\nabla_V W, X) + A([V,W],X) \\ &= (\nabla_W A)(V,X) - (\nabla_V A)(W,X).\end{aligned}$$

□

**Corollary 4.16.** Let $\overline{M} = \mathbb{R}^{n+1}$, then $(\nabla_{.} A)(\cdot,\cdot)$ is fully symmetric in all three entries.

*Proof.* Notice $\overline{R} = 0$. □

**Remark.** 3-tensor that is fully symetric is called a Codazzi tensor. There are very few such tensors.

**Example 4.4.** Let $M \hookrightarrow S^{n+1}$, then

$$\overline{R}(X,Y,Z,W) = \langle Y,W\rangle\langle X,Z\rangle - \langle Y,Z\rangle\langle X,W\rangle.$$

If 3 are tengent and one is $\perp$, then this is 0. This implies that $\nabla_{.} A(\cdot,\cdot)$ is fully symmetric for submanifolds of $S^{n+1}$.

Back to hypersurfaces, let $\mathbf{n}$ be the unit normal vector, $\langle A(X,Y),\mathbf{n}\rangle$ is symmetric at each point in $X, Y \in X(M)$. View it as a bilinear form, then it has eigenvalues. If at point $p$ there exists only one eigenvalue, we say the point $p$ is *umbillic*. If every point of the submanifold is umbillic, we call it umbillic submanifold. For this cse, we have

$$\langle A(X,Y),\mathbf{n}\rangle_p = f(p)\langle X,Y\rangle, \quad p \in M.$$

**Theorem 4.17.** *Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be umbillic and $n \geq 2$, then $M$ is a sphere or plane(or part of them).*

*Proof.* Write
$$\langle A, \mathbf{n}\rangle = fg.$$

Step 1, show $f$ is constant.

$$\nabla_X(fg) = X(f)g + f\nabla_X g = X(f)g \tag{25}$$

Since $(\nabla.A)(\cdot,\cdot)$ is fully symmetric by Codazzi equation, the term $(\nabla_X(fg))(Y, Z) = \langle(\nabla_X A)(Y, Z), \mathbf{n}\rangle$ is also fully symmetric. Use (25) we know $X(f)g(Y, Z)$ is fully symmetric.

Choose orthonormal frame $\{e_i\}_{i=1}^n$, Let $e_1 = \frac{\nabla f}{|\nabla f|}$. Then $e_i(f)\delta_{jk}$ is symmetric to $i, j, k$, hence $e_i(f) = e_i(f)\delta_{jj} = e_j(f)\delta_{ij} = 0$ for $i \neq j$. Hence $\nabla f = 0$, i.e., $f$ is a constant.

In the first step we use the conditon $n \geq 2$ since for $n = 1$ $M$ has no intrinsic metric structure.

Step 2, Let $f = c$, we need two split it into two cases

(1) $c = 0$. Then

$$\begin{aligned}
&\langle A(\cdot,\cdot), \mathbf{n}\rangle = 0 \\
\Rightarrow&\langle \overline{\nabla}_X^{\perp} Y, \mathbf{n}\rangle = 0 \\
\Rightarrow&\langle \overline{\nabla}_X Y, \mathbf{n}\rangle = 0 \\
\Rightarrow&\langle Y, \overline{\nabla}_X \mathbf{n}\rangle \\
\Rightarrow&\mathbf{n} = \text{const.}
\end{aligned}$$

This means $M$ is a plane or part of the plane.

(2) $c \neq 0$. Let $\eta(p) \in T_pM^{\perp}$ for any $p \in M$ and $|\eta| = 1$. Assume $X \in T_pM$,

$$\begin{aligned}
&\langle \overline{\nabla}_X \eta, \eta\rangle = 0 \\
\Rightarrow&\langle \overline{\nabla}_X \eta, Y\rangle = -\langle \eta, \overline{\nabla}_X Y\rangle \\
\Rightarrow&\langle \overline{\nabla}_X \eta, Y\rangle = -\langle \eta, \overline{\nabla}_X^{\perp} Y\rangle \\
\Rightarrow&\langle \overline{\nabla}_X \eta, Y\rangle = -\langle \eta, A(X, Y)\rangle \\
\Rightarrow&\langle \overline{\nabla}_X \eta, Y\rangle = -cg(X, Y) \\
\Rightarrow&\overline{\nabla}_X \eta = -cX.
\end{aligned}$$

Define a map $F : M \to \mathbb{R}^{n+1}$ such that

$$F(p) \equiv p + \frac{1}{c}\eta(p).$$

We want to show $F = \text{const}$ (this constant vector is the centre of the sphere). Lets do the covariant derivative of $F$,

$$\begin{aligned}\overline{\nabla}_X F =& \overline{\nabla}_X p + \frac{1}{c}\overline{\nabla}_X \eta \\ =& X + \frac{1}{c}(-cX) \\ =& 0.\end{aligned}$$

$\square$

# 5 Jacobi Fields

Suppose $\gamma(s,t)$ maps to $M$. Assume for each $s$, $t \to \gamma(s,t)$ is a geodesic. Then we may get two vector fields along the curve,

$$\begin{aligned}\gamma_s =& \mathrm{d}\gamma\left(\frac{\partial}{\partial s}\right)\\ \gamma_t =& \mathrm{d}\gamma\left(\frac{\partial V}{\partial t}\right).\end{aligned}$$

Since for a fixed $s$, $\gamma(s,\cdot)$ is a geodesic, we obtain

$$\nabla_{\gamma_t}\gamma_t = 0.$$

This is true for any $s$, hence

$$\nabla_{\gamma_s}\nabla_{\gamma_t}\gamma_t = 0.$$

How about reorder $\nabla_{\gamma_s}\nabla_{\gamma_t}$ to $\nabla_{\gamma_t}\nabla_{\gamma_s}$? Remember that $[\gamma_t,\gamma_s]=0$ we have

$$R(\gamma_t,\gamma_s)\gamma_t = \nabla_{\gamma_s}\nabla_{\gamma_t}\gamma_t - \nabla_{\gamma_t}\nabla_{\gamma_s}\gamma_t.$$

Hence

$$\nabla_{\gamma_t}\nabla_{\gamma_s}\gamma_s = -R\left(\gamma_t,\gamma_s\right)\gamma_t.$$

Replace $\gamma_s$ with $J$ and $\gamma(s,t)$ with $\gamma(t)$(still a geodesic), we get

$$\nabla_{\gamma_t}\nabla_{\gamma_t}J = -R(\gamma_t,J)\gamma_t. \tag{26}$$

This equation is called the *Jacobi equation.* The solution of (26) is called the *Jacobi filed* along the curve $\gamma(t)$.

Note that it is a second order linear ODE along $\gamma$, hence specify $J$ and $J'$ at $0$ we get unique solution. Choose parallel frame $e_i$ along $\gamma$ with $e_1 = \frac{\gamma_t}{|\gamma_t|}$, write $J = J^i e_i$. Then

$$\nabla_{\gamma_t}J = \frac{\mathrm{d}J^i}{\mathrm{d}t}e_i.$$

Do the covariant derivative again

$$\begin{aligned}\nabla_{\gamma_t}\nabla_{\gamma_t}J &= \frac{\mathrm{d}^2J^i}{\mathrm{d}t^2}e_i\\ &= -R\left(\gamma_t,J\right)\gamma_t\\ &= -|\gamma_t|^2 R\left(e_1,J^ie_i\right)e_1\\ &= -|\gamma_t|^2 J^i R\left(e_1,e_i\right)e_1.\end{aligned}$$

Since $R(e_1, e_i)e_1 = R_{1i1}{}^k e_k = -R^k{}_{11i} e_k$, we have

$$\frac{\mathrm{d}^2 J^k}{\mathrm{d}t^2} = -|\gamma_t|^2 J^i R_{1i1}{}^k. \tag{27}$$

We consider the simplist case that $J^2 = J^3 = \cdots J^n = 0$, then

$$\frac{\mathrm{d}^2 J^1}{\mathrm{d}t^2} = 0 \Rightarrow J^1 = a + bt.$$

**Example 5.1.** Let $M = \mathbb{R}^n$, then the solution is linear for every $J^i$. Let $M = S^n$, then

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} \Rightarrow R_{1i1j} = \delta_{ij} - \delta_{i1}\delta_{j1}.$$

Assume $|\gamma_t| = 1$, then for $k > 1$, we obtain

$$\frac{\mathrm{d}^2 J^k}{\mathrm{d}t^2} = -J^k.$$

Hence for $k > 1$,

$$J^k(t) = a\cos t + b\sin t.$$

Since $J^k(0) = 0$, we have

$$J^k(t) = b^k \sin t.$$

We can see they vanish again at $t = \pi$.

**Proposition 5.1.** Let $\gamma : [0, a] \to M$ be a geodesic and $J$ be a Jacobi field along $\gamma$ with $J(0) = 0$. Put $\nabla_{\gamma_t} J(0) = \mathbf{w}$ and $\gamma'(0) = \mathbf{v}$. Consider $\mathbf{w}$ as an element of $T_{a\mathbf{v}}\left(T_{\gamma(0)}M\right)$ and construct a curve $v(s)$ in $T_{\gamma(0)}M$ with $v(0) = a\mathbf{v}, v'(0) = \mathbf{w}$. Put $f(s,t) = \exp_p\left(\frac{t}{a}v(s)\right)$, $p = \gamma(0)$, and define a Jacobi field $\overline{J}$ by $\overline{J}(t) = \frac{\partial f}{\partial s}(0,t)$. Then $\overline{J} = J$ on $[0, a]$.

*Proof.*

$$\frac{\partial f}{\partial s} = \frac{\partial \exp_p\left(\frac{t}{a}v(s)\right)}{\partial s} = (\mathrm{d}\exp_p)_{\frac{t}{a}v(s)} \frac{t}{a} v'(s).$$

Let $s = 0$ we obtain

$$\overline{J}(t) = t\left(\mathrm{d}\exp_p\right)_{t\mathbf{v}} \mathbf{w}.$$

Then

$$\begin{aligned}\nabla_{\gamma_t} \frac{\partial f}{\partial s} &= \nabla_{\gamma_t}\left(t\left(\mathrm{d}\exp_p\right)_{t\mathbf{v}} \mathbf{w}\right) \\ &= \left(\mathrm{d}\exp_p\right)_{t\mathbf{v}} \mathbf{w} + t\nabla_{\gamma_t}\left(\left(\mathrm{d}\exp_p\right)_{t\mathbf{v}} \mathbf{w}\right)\end{aligned}$$

.

Let $t = 0$, we obtain $\nabla_{\gamma_t}\overline{J}(0) = \mathbf{w}$. Since $J(0) = \overline{J}(0) = 0$ and $\nabla_{\gamma_t} J(0) = \nabla_{\gamma_t}\overline{J}(0) = \mathbf{w}$, we conclude form the uniqueness theorem that $J = \overline{J}$ on $[0, a]$. $\square$

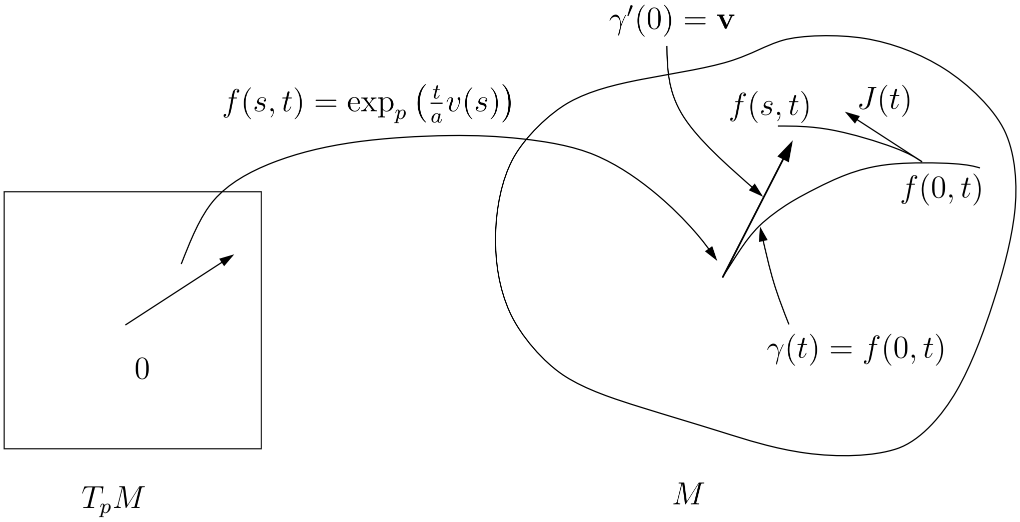


Figure 6: Jacobi field

**Corollary 5.2.** Let $\gamma : [0,a] \to M$ be a geodesic. Then a Jacobi field $J$ along $\gamma$ with $J(0) = 0$ is given by

$$J(t) = \left(\mathrm{d}\exp_p\right)_{t\gamma'(0)} (tJ'(0)), \quad t \in [0,a]. \tag{28}$$

**Definition 5.3.** Let $\gamma : [0,a] \to M$ be a geodesic, $\gamma(t_0)$ is said to be conjugate to $\gamma(0)$ along $\gamma$,$t_0 \in [0,a]$, if there exists a Jacobi field $J$ along $\gamma$ not identically 0 with $J(0) = 0 = J(t_0)$. THe maximum number of such linearly independent fields is caleed multiplicity of the conjugate point $\gamma(t_0)$.

**Example 5.2.**
- $M = \mathbb{R}^n$, no conjugate points.
- $M = S^n$, antipodal points of conjugate multiplicity of $n-1$.

**Proposition 5.4.** Let $\gamma(0)$ and $\gamma(l)$ be conjuegate. Then $l\left(\gamma'(0)\right)$ is a critical point for $\exp_{\gamma(0)}(\cdot)$. In fact, the multiplicity is equal to $\dim Ker[\mathrm{d}\exp_{\gamma(0)}]_{l\gamma'(0)}$.

*Proof.* Let

$$J(t) = \left(\mathrm{d}\exp_{\gamma(0)}\right)_{t\gamma'(0)} (tJ'(0)).$$

$J(l) = 0$ implies

$$\left(\mathrm{d}\exp_{\gamma(0)}\right)_{l\gamma'(0)} (lJ'(0)) = 0.$$

Since $J'(0) \neq 0$, the point is critical. □

# 6 Hopf-Rinow Theorem

From now on, except when explicitly mentioned otherwise, all manifolds will be supposed connected.

**Definition 6.1.** A Riemanian manifold $M$ is *geodesically complete* if for all $p \in M$, the exponential map $\exp_p$ is defined for all $\mathbf{v} \in T_pM$, i.e., if any geodesic $\gamma(t)$ starting from $p$ is defined for all values of the parameter $t \in \mathbb{R}$.

**Theorem 6.2 (Hopf-Rinow Theorem).** *Let $M$ be a Riemannian manifold and let $p \in M$. The following assertions are equivalent:*

a. $\exp_p$ *is defined on all of $T_pM$.*

b. *The closed and bounded sets of $M$ are compact.*

c. *$M$ is complete as a metric space.*

d. *$M$ is geodesically complete.*

e. *There exists a sequence of compact subsets $K_n \subset M$, $K_n \subset K_{n+1}$ and $\bigcup_n K_n = M$, such taht if $q_n \notin K_n$ then $d(p, q_n) \to \infty$.*

*In addition, any of the statements above implies that*

f. *For any $q \in M$ there exists a geodesic $\gamma$ joining $p$ to $q$ with $l(\gamma) = d(p, q)$.*

*Proof.* The critical point is to prove $(a) \Rightarrow (f)$.

Let $B_\delta(p) =$ be a normal ball and $q \notin B_\delta(p)$. There exists a point $x_0 \in \partial B_\delta(\delta)$ which is closest to $q$. Let $|\mathbf{v}| = 1$ and $\exp_p(\delta\mathbf{v}) = x_0$. Then $r = d(p, q) > \delta$. Define $\gamma : [0, r] \to M$ by

$$\gamma(s) = \exp_p(s\mathbf{v}).$$

Then $\gamma(0) = p$ and $\gamma(\delta) = p$. We need to show $\gamma(r) = q$. To accomplish this, we define

$$A = \{s | d(q, \gamma(s)) + s = r\}.$$

Then $\gamma(r) = q$ is equivalent to $r \in A$. Since $d(q, \gamma(s)) + s$ is continuous function of $s$, $A$ is closed. Since $0 \in A$, $A$ is nonempty.

Since $[0, r]$ is connected, we just have to show $A$ is open.

**Claim**: If $s \in A$, then $[0, s] \subset A$.

By triangular inequality we obtain for any $t < s$

$$d(q, \gamma(t)) + t \geq r.$$

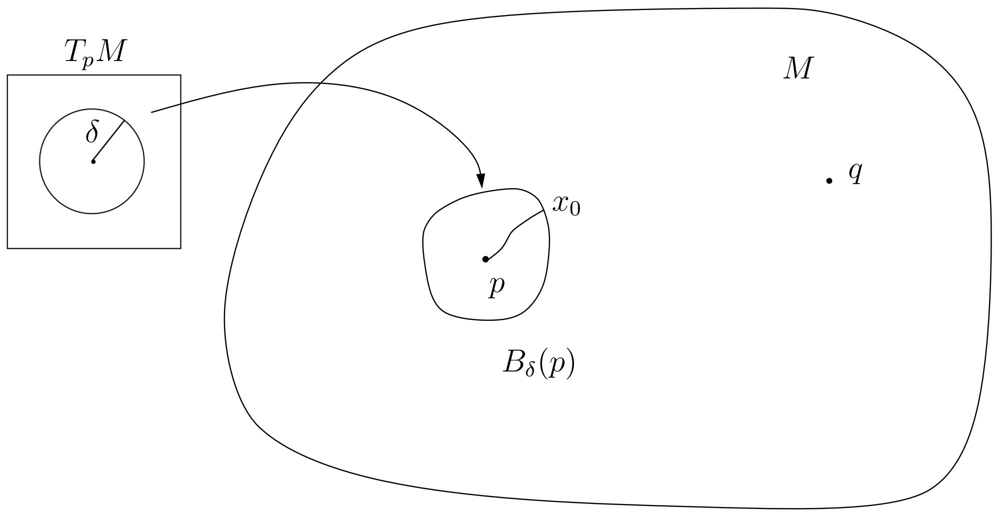


Figure 7: the closest point $x_0$ in the boundary

We need the opposite inequality for $t < s$. By triangular inequality again

$$\begin{aligned} d(q,\gamma(t)) &\le d(q,\gamma(s)) + d(\gamma(s),\gamma(t)) \\ &\le d(q,\gamma(s)) + s - t \\ &= r - s + s - t = r - t. \end{aligned}$$

Hence $t \in A$. The claim is proved.

**Claim**: $\delta \in A$.

The inequality $d(q,\gamma(\delta)) + \delta \ge r$ is trivial. We need to show the inverse inequality. Let $\sigma$ be a curve from $p$ to $q$, it must hit $\partial B_\delta(p)$. Let $w \in \partial B_\delta(p) \cap \sigma$. Then

$$\begin{aligned} L(\sigma) &\ge d(p,w) + d(w,q) \\ &= \delta + d(w,q) \\ &\ge t + d(x_0,q). \end{aligned}$$

Since $\sigma$ is arbitrary, we obtain

$$r = d(p,q) \ge \delta + d(\gamma(\delta),q).$$

Hence $\delta \in A$.

**Claim**: $A$ is open.

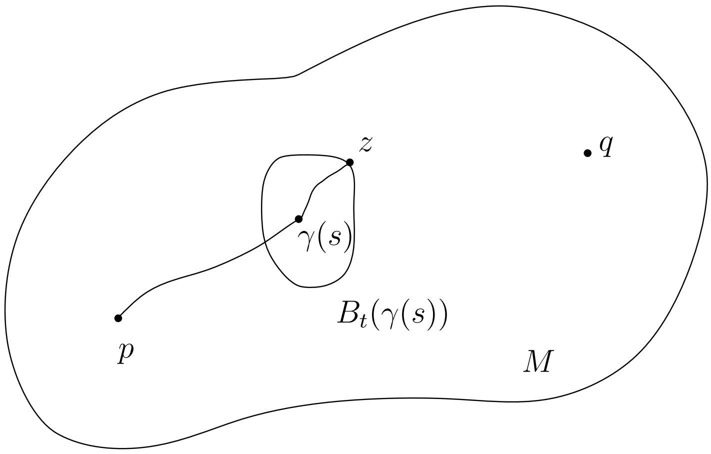


Figure 8: A normal ball $B_t(\gamma(s))$ such that $q \notin \overline{B_t}(\gamma(s))$

Assume $s \in A$. Choose a normal ball $B_t(\gamma(s))$ such that $q \notin \overline{B_t}(\gamma(s))$. Suppose $z$ is the closest point to $q$ in the boundary of the normal ball. see Figure 8.

By the claim before, we know

$$d(q, z) + t = d(\gamma(s), q) = r - s.$$

Then

$$d(q, z) = r - (s + t) \Rightarrow d(p, z) \geq s + t.$$

But $d(p, z) \leq s + t$, hence

$$d(p, z) = s + t.$$

But doing $\gamma$ from $p$ to $\gamma(s)$ and then a ray from $\gamma(s)$ to $z$ has length $s + t$, this possibly broken curve is minimizing $\Rightarrow$ It's a smooth geodesic(by Corollary 3.13) $\Rightarrow$ It's just $\gamma$ itself $\Rightarrow \gamma(s + t) = z$ and we are done.

So it's open $\Rightarrow [0, r] = A$ and $(a) \Rightarrow (f)$.

$(a) \Rightarrow (b)$. Let $K$ be a closed and bounded set, then $K \subset B_R(p)$ for some $R > 0$. Hence $K \subset \exp_p(\overline{B}_R(0))$. Hence $K$ is a subset of continuous image of a compact set. But closed subsets of compact sets are compact, this implies $K$ is compact.

$(b) \Rightarrow (c)$. Assume $\{p_n\}$ is Cauchy, then $\{p_n\}$ is bounded. By $(b)$ it is contained in a compact set. Hence there exists a subsequence $\{p_{n'}\}$ that converges. Then the whole sequence converges.

$(c) \Rightarrow (d)$. Suppose that $M$ is not geodesically complete, then there exists some normalised geodesic $\gamma$ defined for $s < s_0$ but not for $s_0$. Let $\{s_n\}$ be a convergent sequence converging to $s_0$ with $s_n < s_0$. Given $\epsilon > 0$, there exists an index $n_0$ such that if $n, m > n_0$, then $|s_n - s_m| < \epsilon$. It follows that

$$d(\gamma(s_n), \gamma(s_m)) \leq |s_n - s_m| < \epsilon,$$

and hence the sequence $\{\gamma(s_n)\}$ is a Cauchy sequence in $M$. Since $M$ is complete in the metric $d$, $\{\gamma(s_n)\} \to p_0 \in M$.

Let $(W, \delta)$ be a totally normal neighborhood of $p_0$. Choose $n_1$ such that if $n, m > n_1$, then $|s_m - s_n| < \delta$ and $\gamma(s_n), \gamma(s_m) \in W$. Then there exists a unique geodesic $g$ whose length is less than $\delta$ joining $\gamma(s_n)$ to $\gamma(s_m)$. It is clear that $g$ coincides with $\gamma$, whenever $\gamma$ is defined. Since $\exp_{\gamma(s_n)}$ is a diffeomorphism on $B_\delta(0)$ and $\exp_{\gamma(s_n)}(B_\delta(0)) \supset W$, $g$ extends $\gamma$ beyond $s_0$.

$(d) \Rightarrow (a)$ is obvious and $(b) \Leftrightarrow (e)$ is general topology. □

**Definition 6.3.** $F : M \to N$ is a *covering map* if it's a local diffeomorphism and for each point $q \in N$, there is an opens set $U$ of $q$ such that

$$f^{-1}(U) = \bigcup_\alpha V_\alpha \subset M$$

and $U \simeq V_\alpha$for each $\alpha$.

The biggest cover of $N$ is called the *universal cover*. Let $\widetilde{M}$ be a universal cover of $M$, then $\pi_1(\widetilde{M}) = \{0\}$ and the degree of covering $= |\pi_1(M)|$.

**Theorem 6.4 (Hadamard).** *Let $M$ be complete, sectional curvature $\leq 0$ and $\pi_1(M) = \{0\}$. Then $\exp_p$ is global diffeomorphism. In particular, $M$ is diffeomorphic to $\mathbb{R}^n$.*

**Corollary 6.5.** Let $M$ be complete, sectional curvature $\leq 0$. Then the universal cover $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$. In particular, if $\pi_1(M) = \{0\}$, then $M$ is diffeomorphic to $\mathbb{R}^n$.

**Lemma 6.6.** Let $M$ be complete with $K(p, \sigma) \leq 0$ for all $p \in M$ and all two dimensional vector subspace $\sigma \subset T_pM$. Then for all $p \in M$, then conjugate locus $C(p) = \emptyset$. In particular, the map $\exp_p : T_pM \to M$ is a local diffeomorphism.

*Proof.* Let $J$ be a non-trivial Jacobi field along $\gamma : [0, +\infty) \to M$ where $\gamma(0) = p$ and $J(0) = 0$. Then from the hypothesis on the curvature and from the Jacobi equation

$$\begin{aligned}\langle J, J\rangle'' =& 2\langle J', J'\rangle + 2\langle J'', J\rangle \\ =& 2\langle J', J'\rangle - 2\langle R(\gamma', J)\gamma', J\rangle \\ =& 2|J'|^2 - 2K(\gamma', J)|\gamma' \wedge J|^2 \geq 0.\end{aligned}$$

Therefore $\langle J, J\rangle'(t_2) \geq \langle J, J\rangle'(t_1)$ whenever $t_2 > t_1$. Since $J'(0) \neq 0$ and $\langle J, J\rangle'(0) = 0$, it follows that for $t$ sufficiently small positive number

$$\langle J, J\rangle(t) > \langle J, J\rangle(0) = 0.$$

It follows that for all $t > 0$, $\langle J, J\rangle(t) > 0$ and $\gamma(t)$ is not conjugate to $\gamma(0)$ along $\gamma$. $\square$

**Lemma 6.7.** Let $F : M \to N$ be local isomertry between complete manifolds. Then $F$ is a covering map.

The proof of Lemma use the path lifting property.
*Proof of the Hadamard theorem.* Since $M$ is complete, $\exp_p$ is defined for all $p \in M$ and is surjective. By Lemma 6.6, $\exp_p$ is a local diffeomorphism. This allows us to introduce a Riemannian metric on $T_pM$ in such a way that $\exp_p$ is a local isometry. Such a metric is complete, since the geodesic of $T_pM$ passing through the origin are straight lines. From Lemma 6, $\exp_p$ is a covering map. Since $M$ is simply connected, $\exp_p$ is a diffeomorphism. $\square$

# 7 Calculus of variations

Philosophy: we see space of curves as an $\infty$-dimensional manifold.

**Definition 7.1.** Let $\gamma : [0,l] \to M$ be a piecewise differentiable curve in a manifold $M$. A *variation* of $\gamma$ is a continuous map

$$f : (-\epsilon, \epsilon) \times [0,l] \to M$$

such that:

(a) $f(0,t) = \gamma(t)$.

(b) there exists a subdivision of $[0,l]$ by points $0 = t_0 < t_1 < \cdots < t_{k+1} = l$ such that the restriction of $f$ to each $(-\epsilon, \epsilon) \times [t_i, t_{i+1}]$ is differentiable.

A variation is said to be is *proper* if the endpoints are fixed, i.e.,

$$f(s,0) = \gamma(0), \quad f(s,l) = \gamma(l)$$

for all $s \in (-\epsilon, \epsilon)$. If $f$ is differentiable, the variation is ssaid to be differentiable.

**Definition 7.2.** The *tangent vector to a curve* in the space of curves is a vector field along the curve.

Consider a variation $f(s,t)$ of a curve $\gamma : [0,l] \to M$, $f(0,t) = \gamma(t)$ and assume the variation is proper. Let

$$\mathbf{v}(t) = \mathrm{d}f \left( \frac{\partial}{\partial s} \right) \bigg|_{(0,t)} = f_s.$$

Since the variation is proper, we have

$$\mathbf{v}(0) = \mathbf{v}(l) = 0.$$

We'd like to allow $\gamma$ to be piecewise smooth, then $f_s$ is piecewise differentiable.

Fix $0 = t_0 < t_1 < \cdots < t_k < l = t_{k+1}$.

Let's look at some functions and find their critical points. There are two favoriate functions:

- $L(\gamma) = \int_0^l |\gamma'(t)| \, \mathrm{d}t$.
- $E(\gamma) = \int_0^l |\gamma'(t)|^2 \, \mathrm{d}t$.

Use Cauchy-Schwartz inequality we obtain

$$\left(\int_0^l |u|^2 \, \mathrm{d}t\right) \leq l \int_0^l |u|^2 \, \mathrm{d}t.$$

Hence $L^2(\gamma) \leq lE(\gamma)$. The equality is true for $\gamma$ has constant speed, $|\gamma'| = \frac{L}{l}$.

**Proposition 7.3 (Formula for the first variation of the energy of a curve).** Let $\gamma : [0, l] \to M$ be a piecewise differentiable curve and let $f : (-\epsilon, \epsilon) \times [0, l] \to M$ be a variation of $\gamma$. Then

$$\begin{aligned}\frac{1}{2}E'(0) = &- \int_0^l \langle \mathbf{v}(t), \nabla_{\gamma'(t)}\gamma'(t)\rangle \, \mathrm{d}t \\ &- \sum_{i=1}^{k} \langle \mathbf{v}(t_i), \gamma'(t_i^+) - \gamma'(t_i^-)\rangle \\ &- \langle \mathbf{v}(0), \gamma'(0)\rangle + \langle \mathbf{v}(l), \gamma'(l)\rangle. \end{aligned} \tag{29}$$

*Proof.* By definition,

$$E(s) = \int_0^l \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle \, \mathrm{d}t = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle \, \mathrm{d}t.$$

Differentiating under the integral sign and using the symmetry of the Riemannian connection, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s} \int_{t_{i+1}}^{t_{i+1}} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle \, \mathrm{d}t = \int_{t_i}^{t_{i+1}} 2\langle \nabla_{\partial/\partial s}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle \, \mathrm{d}t \\ &= 2\int_{t_i}^{t_{i+1}} \langle \nabla_{\gamma'}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \, \mathrm{d}t \\ &= 2\int_{t_i}^{t_{i+1}} \frac{\partial}{\partial t}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \, \mathrm{d}t - 2\int_{t_i}^{t_{i+1}} \langle \frac{\partial f}{\partial s}, \nabla_{\gamma'}\frac{\partial f}{\partial t}\rangle \, \mathrm{d}t \\ &= 2\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \Big|_{t_i}^{t_{i+1}} - 2\int_{t_i}^{t_{i+1}} \langle \frac{\partial f}{\partial s}, \nabla_{\gamma'}\frac{\partial f}{\partial t}\rangle \, \mathrm{d}t.\end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}s} = \sum_{i=0}^{k} \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \Big|_{t_i}^{t_{i+1}} - \int_0^a \langle \frac{\partial f}{\partial s}, \nabla_{\gamma'}\frac{\partial f}{\partial t}\rangle \, \mathrm{d}t. \tag{30}$$

Putting $s = 0$ in (30), this yields (29). ☐

**Proposition 7.4 (Formula for the second variation).** Let $\gamma$ be a geodesic and let $f : (-\epsilon, \epsilon) \times [0, a] \to M$ be a proper variation of $\gamma$. Then

$$\frac{1}{2}E''(0) = -\int_0^l \langle \mathbf{v}(t), \nabla_{\gamma'(t)}\nabla_{\gamma'(t)}\mathbf{v}(t) + R\left(\gamma'(t), \mathbf{v}(t)\right)\gamma'(t)\rangle \,\mathrm{d}t$$
$$-\sum_{i=1}^{k}\langle \mathbf{v}(t_i), \nabla_{\gamma'(t_i^+)}\mathbf{v}(t_i^+) - \nabla_{\gamma'(t_i^-}\mathbf{v}(t_i^-)\rangle. \tag{31}$$

*Proof.* Taking the deriative of (30), we obtain

$$\frac{1}{2}\frac{\mathrm{d}^2E}{\mathrm{d}s^2} = \sum_{i=0}^{k}\langle \nabla_{\partial/\partial s}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle\Big|_{t_i}^{t_{i+1}} + \sum_{i=0}^{k}\langle \frac{\partial f}{\partial s}, \nabla_{\partial/\partial_s}\frac{\partial f}{\partial t}\rangle\Big|_{t_i}^{t_{i+1}}$$
$$-\int_0^a \langle \nabla_{\partial/\partial s}\frac{\partial f}{\partial s}, \nabla_{\gamma'}\frac{\partial f}{\partial t}\rangle \,\mathrm{d}t - \int_0^a \langle \frac{\partial f}{\partial s}, \nabla_{\partial/\partial s}\nabla_{\gamma'}\frac{\partial f}{\partial t}\rangle \,\mathrm{d}t.$$

Putting $s = 0$ in the expression above, we obtain that the first and the third terms are zero, since $f$ is proper and $\gamma$ is geodesic. Since

$$\nabla_{\partial/\partial s}\nabla_{\gamma'}\frac{\partial f}{\partial t} = \nabla_{\gamma'}\nabla_{\partial/\partial s}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t},$$

we have at $s = 0$,

$$\nabla_{\partial/\partial s}\nabla_{\gamma'}\frac{\partial f}{\partial t} = \nabla_{\gamma'}\nabla_{\gamma'}\mathbf{v} + R\left(\gamma', \mathbf{v}\right)\gamma'.$$

Further use of the fact that the variation is proper yields

$$\sum_{i=0}^{k}\langle \frac{\partial f}{\partial s}, \nabla_{\partial/\partial s}\frac{\partial f}{\partial t}\rangle\Big|_{t_i}^{t_{i+1}} = -\sum_{i=1}^{k}\langle \mathbf{v}(t_i), \nabla_{\gamma'}(t_i^+) - \nabla_{\gamma'}\mathbf{v}(t_i^-)\rangle. \tag{32}$$

Putting all these together, we obtain (31).

□

**Lemma 7.5.** If $\gamma$ minimises $E$, then it is a geodesic. In fact, $\gamma$ is a critical point for $E$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(s) = 0.$$

**Theorem 7.6 (Bonnet-Myers Theorem).** *Let $M$ be a complete Riemannian manifold. Suppose*

$$\mathrm{Ric}_p(\mathbf{v}) \geq \frac{n-1}{r^2} > 0$$

*for all $p \in M$ and for all $\mathbf{v} \in T_pM$. Then $\mathrm{diam}(M) \leq \pi r$ and $M$ is compact.*

*Proof.* To show the diameter bound, we need to show any length-minimise geodesic $\gamma$ has $L(\gamma) \leq \pi r$. Since $\gamma$ is a geodesic that minimises the length, it also minimises the energy $E(\gamma)$. Hence $E''(0) \geq 0$ for a minimiser. Let $\{e_i\}$ be a parallel frame with $|\gamma'| = 1$ and $e_1 = \gamma'$.

Set $\mathbf{v}_j = \sin\left(\frac{\pi t}{l}\right) e_j$ for $j = 2, \cdots, n$. Then

$$\nabla_{\gamma'}\mathbf{v}_j = \frac{\pi}{l}\cos\left(\frac{\pi t}{l}\right) e_j \text{ and } \nabla_{\gamma'}\nabla_{\gamma'}\mathbf{v}_j = -\left(\frac{\pi}{l}\right)^2 \mathbf{v}_j.$$

$$R\left(\gamma', \mathbf{v}_j\right)\gamma' = \left(R(e_1, e_j)e_1\right)\sin\left(\frac{\pi t}{l}\right).$$

Then

$$\frac{1}{2}E''(0) = -\int_0^l \left[-\left(\frac{\pi}{l}\right)^2 \sin^2\left(\frac{\pi t}{l}\right) + R_{1j1j}\sin^2\left(\frac{\pi t}{l}\right)\right] \geq 0.$$

Sum over $j = 2, \cdots, n$,

$$\sum_{j=2}^{n} R_{1j1j} = \mathrm{Ric}(e_1, e_1) \geq \frac{n-1}{r^2}.$$

Hence

$$\frac{n-1}{r^2}\int_0^l \sin^2\frac{\pi t}{l}\,\mathrm{d}t \leq (n-1)\left(\frac{\pi}{l}\right)^2\int_0^l \sin^2\frac{\pi t}{l}\,\mathrm{d}t$$
$$\Rightarrow l \leq \pi r.$$

□

Consider

$$\frac{1}{2}E''(0) = -\int_0^l \langle \nabla_\gamma\nabla_\gamma\mathbf{v} + R(\gamma', \mathbf{v})\gamma', \mathbf{v}\rangle. \tag{33}$$

The operator

$$J(\mathbf{v}) = \nabla_{\gamma'}\nabla_{\gamma'}\mathbf{v} + R(\gamma', \mathbf{v})\gamma'$$

is called the *Jacobi operator*. Then the equation (33) can be written as

$$\frac{1}{2}E''(0) = -\int_0^l \langle J(\mathbf{v}), \mathbf{v}\rangle. \tag{34}$$

Define the $L^2$ inner product

$$\langle\langle \mathbf{v}, \mathbf{w}\rangle\rangle = \int_0^l \langle \mathbf{v}, \mathbf{w}\rangle.$$

**Proposition 7.7.** $J$ is symmetric w.r.t $\langle\langle\cdot,\cdot\rangle\rangle$, i.e.

$$\langle\langle J(\mathbf{v}),\mathbf{w}\rangle\rangle = \langle\langle \mathbf{v}, J(\mathbf{w})\rangle\rangle.$$

*Proof.* It is because

$$\int_0^l \partial_t \left(\langle \nabla_{\gamma'}\mathbf{v},\mathbf{w}\rangle - \langle \mathbf{v},\nabla_{\gamma'}\mathbf{w}\rangle\right)\,\mathrm{d}t = 0.$$

□

Write

$$\begin{aligned}\langle\langle J(\mathbf{v}),\mathbf{v}\rangle\rangle = &-\int_0^l \langle \nabla_{\gamma'}\nabla_{\gamma'}\mathbf{v},\mathbf{v}\rangle\,\mathrm{d}t\\ &-\int_0^l \langle R\left(\gamma',\mathbf{v}\right)\gamma',\mathbf{v}\rangle\,\mathrm{d}t.\end{aligned}$$

Notice

$$0=\int_0^l \partial_t\langle \nabla_{\gamma'\mathbf{v}},\mathbf{v}\rangle\,\mathrm{d}t = \int_0^l \langle \nabla_{\gamma'}\nabla_{\gamma'}\mathbf{v},\mathbf{v}\rangle\,\mathrm{d}t + \int_0^l \langle \nabla_{\gamma'}\mathbf{v},\nabla_{\gamma'}\mathbf{v}\rangle\,\mathrm{d}t,$$

substitute it into previous equation we obtain

$$\langle\langle J(\mathbf{v}),\mathbf{v}\rangle\rangle = \int_0^l \left(|\nabla_{\gamma'}\mathbf{v}|^2 - \langle R(\gamma'\mathbf{v})\gamma',\mathbf{v}\rangle\right)\,\mathrm{d}t.$$

The first term is obviously positive definite, the sign of the second term is unknown but it's a bounded operator.

# 8 The Morse Index Theorem

**Definition 8.1.** Let $\gamma$ be a geodesic with $|\gamma'| = 1$ on $[0, l]$. Define the *index form*

$$I(\mathbf{v}, \mathbf{w}) = \int_0^l \left( \langle \mathbf{v}', \mathbf{w}' \rangle - \langle R(\gamma', \mathbf{v})\gamma', \mathbf{w} \rangle \right) \mathrm{d}t, \tag{35}$$

where $\mathbf{v}, \mathbf{w}$ are continuous and piecewise differentiable. Here $\mathbf{v}' = \nabla_{\gamma'}\mathbf{v}$ and $\mathbf{w}' = \nabla_{\gamma'}\mathbf{w}$.

**Lemma 8.2.**

$$I(\mathbf{v}, \mathbf{w}) = -\int_0^l \langle J(\mathbf{v}), \mathbf{w} \rangle \,\mathrm{d}t + \langle \mathbf{v}', \mathbf{w} \rangle(l) - \langle \mathbf{v}', \mathbf{w} \rangle\,(0) - \sum_{i=1}^k \langle \mathbf{v}'(t_j^+) - \mathbf{v}'(t_j^-), \mathbf{w}(t_j) \rangle. \tag{36}$$

**Lemma 8.3 (The Index Lemma).** Let $\gamma$ be a geodesic on $[0, l]$ without conjugate points. Let $\mathbf{v}$ be a vector field with $\mathbf{v}(0) = 0$. Let $\mathbf{w}$ be a Jacobi field with $\mathbf{w}(0) = 0$ and $\mathbf{w}(l) = \mathbf{v}(l)$. Then

$$I(\mathbf{v}, \mathbf{v}) \geq I(\mathbf{w}, \mathbf{w}).$$

In particular, the equality holds if and only if $\mathbf{v} = \mathbf{w}$.

**Remark.** Before we prove the lemma, we consider a baby version: Let $\phi$ be a function on $[0, l]$, define

$$I(u, v) = \int_0^l (u'v' - \phi uv)\,\mathrm{d}t.$$

If $u'' = -\phi u$, and $f$ is any function such that $u = 0$ at 0 and $l$, then

$$\int_0^l \left( |(fu)'|^2 - \phi (fu)^2 \right) \mathrm{d}t \geq 0$$

and the equality holds if and only if $f = \text{const}$ ($\int |f'|^2 u = 0$).

Observing that $(f^2uu')$ vanishes at 0 and $l$, we obtain

$$\int (f^2uu')' = 0.$$

Then

$$\begin{aligned} 0 &= \int \left[ 2ff'uu' + f^2(u')^2 + f^2uu'' \right] \\ &= \int \left[ 2ff'uu' + f^2(u')^2 - \phi f^2u^2 \right]. \end{aligned}$$

The first two terms show up in $|(fu)'|^2 = 2ff'uu' + (f'u)^2 + (fu')^2$. Then

$$0 = \int \left[|(fu)'|^2 - (f'u)^2 - \phi f^2 u^2\right] \Rightarrow 0 \leq \int (f'u)^2 = I(fu, fu).$$

*Proof of the Index Lemma.* Let $v_1, v_2, \cdots, v_n$ be a basis of Jacobi fields with $v_i(0) = 0$ and $v_i'(0) = e_i$, and $\{e_i\}$ is the unit basis at $\gamma(0)$. They do not vanish again on $[0, l]$ since the geodesic $\gamma$ has not conjugate points on $[0, l]$. In fact, they have full rank $n$ at each point of $\gamma$ on $(0, l]$. Any vector field with $\mathbf{v}(0) = 0$ can be written as

$$\mathbf{v}(t) = \sum_{i=1}^{n} f_i(t) v_i(t).$$

Then we can write

$$\mathbf{w}(t) = \sum_{i=1}^{n} f_i(l) v_i(t).$$

Differentiating the term $f_i f_j \langle v_i', v_j \rangle$, we obtain

$$(f_i f_j \langle v_i', v_j \rangle)' = \underbrace{f_i' f_j \langle v_i', v_j \rangle}_{(1)} + \underbrace{f_i f_j' \langle v_i', v_j \rangle}_{(2)} + \underbrace{f_i f_j \langle v_i', v_j' \rangle}_{(3)} + \underbrace{f_i f_j \langle v_i'', v_j \rangle}_{(4)}.$$

Write $I(\mathbf{v}, \mathbf{v})$ explicitly, we obtain

$$\begin{aligned} I(\mathbf{v}, \mathbf{v}) &= I\left(\sum_{i=1}^{n} f_i v_i, \sum_{j=1}^{n} f_j v_j\right) \\ &= \sum_{ij} I\left(f_i v_i, f_j v_j\right). \end{aligned}$$

$$\begin{aligned} I(f_i v_i, f_j v_j) &= \int \left[\langle f_i' v_i + f_i v_i', f_j' v_j + f_j v_j' \rangle - \langle R(\gamma', f_i v_i)\gamma', f_j v_j \rangle\right] \\ &= \int \Bigg[\underbrace{f_i' f_j' \langle v_i, v_j \rangle}_{(A)} + \underbrace{f_i' f_j \langle v_i, v_j' \rangle}_{(B)} + \underbrace{f_i f_j' \langle v_i', v_j \rangle}_{(2)} \\ &\quad + \underbrace{f_i f_j \langle v_i', v_j' \rangle}_{(3)} + \underbrace{f_i f_j \langle R(\gamma', v_i)\gamma', v_j \rangle}_{(C)}\Bigg]. \end{aligned}$$

Since

$$\begin{aligned}\big(\langle v_i', v_j\rangle - \langle v_i, v_j'\rangle\big)' &= \langle v_i'', v_j\rangle - \langle v_i, v_j''\rangle \\ &= -\langle R(\gamma', v_i)\gamma', v_j\rangle + \langle R(\gamma', v_j)\gamma', v_i\rangle \\ &= 0,\end{aligned}$$

then we obtain $(B) = (1)$. Then

$$I(\mathbf{v},\mathbf{v}) = \sum_{i,j}\int (1)+(2)+(3)+(A)-(C).$$

$$\begin{aligned}\sum_{i,j}\int (f_i f_j\langle v_i', v_j\rangle)' &= \sum_{i,j}\int (1)+(2)+(3)+(4) \\ &= \sum_{i,j} f_i f_j \langle v_i', v_j\rangle(l) \\ &= I(\mathbf{w},\mathbf{w}) \quad \text{(by Lemma 8.2)}.\end{aligned}$$

Observing that $(4) = -(C)$, we obtain

$$\begin{aligned}I(\mathbf{v},\mathbf{v}) &= \sum_{i,j}\int (1)+(2)+(3)+(4)+(A) \\ &= I(\mathbf{w},\mathbf{w}) + \sum_{i,j}\int f_i' f_j' \langle v_i, v_j\rangle \\ &\geq I(\mathbf{w},\mathbf{w}).\end{aligned}$$

□

Let $\gamma : [0,l] \to M$ be a geodesic. Denote by $\mathcal{V}$ the vector space formed by vector fields $\mathbf{v}$ along $\gamma$, which are piecewise differentiable and vanish at the endpoints of $\gamma$, that is, $\mathbf{v}(0) = \mathbf{v}(l) = 0$.

Given a symmetric bilinear form $B$ over a vector space $\mathcal{V}$, we define the *index* of $B$ as the maximal dimension of all subspaces of $\mathcal{V}$ on which the quadratic form associated to $B$ is negative definite. The *nullity* of $B$ is defined to be the dimension of the subspace of $\mathcal{V}$ formed by the elements $\mathbf{v} \in \mathcal{V}$ such that $B(\mathbf{v},\mathbf{w}) = 0$ for all $\mathbf{w} \in \mathcal{V}$.

Let

$$I_l(\mathbf{v},\mathbf{w}) = \int_0^l [\langle \mathbf{v}', \mathbf{w}'\rangle - \langle R(\gamma', \mathbf{v})\gamma', \mathbf{w}\rangle]\, \mathrm{d}t, \quad \mathbf{v},\mathbf{w} \in \mathcal{V}.$$

**Theorem 8.4 (The Morse Index Theorem).** *The index of the form $I_l$ is finite and equals the number of points $\gamma(t), 0 < t < l$, conjugate to $\gamma(0)$, each counted with its multiplicity.*

**Proposition 8.5.**

$$Ker(I) = \text{Jacobi fields that vanish at } 0, l.$$

*Proof.* By Lemma 8.2, we have

$$I(\mathbf{v}, \mathbf{w}) = -\int_0^l \langle \mathbf{v}'' + R(\gamma', \mathbf{v})\gamma', \mathbf{w}\rangle \, \mathrm{d}t - \sum_{j=1}^{k} \langle \mathbf{v}'(t_j^+) - \mathbf{v}'\left(t_j^-\right), \mathbf{w}(t_j)\rangle.$$

If $\mathbf{v}$ is a Jacobi field, then by the above equation, $\mathbf{v}$ is in the null spce of $I_l$.
COnversely, suppose that $I_l(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in \mathcal{V}$. Let

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$$

be a subdivision of $[0, l]$ such that the restriction $\mathbf{v}|_{[t_{j-1}, t_j]}$ is differentiable, $j = 1, \cdots, k+1$. Let $f : [0, l] \to \mathbb{R}$ be a differentiable function with $f(t) > 0$ for $t \neq t_j$ and $f(t) = 0$ for $t = t_j$. Define $\mathbf{w}$ by

$$\mathbf{w}(t) = f(t)\left(\mathbf{v}'' + R\left(\gamma', \mathbf{v}\right)\gamma'\right).$$

Then

$$0 = I_l(\mathbf{v}, \mathbf{w}) = -\int_0^l f(t)\|\mathbf{v}'' + R(\gamma', \mathbf{v})\gamma'\|^2 \, \mathrm{d}t.$$

It follows that the integrand is zero and therefore the restriction $\mathbf{v}|_{[t_{j-1}, t_j]}$ is a Jacobi field. Choose $\mathbf{t} \in \mathcal{V}$ in such a way that

$$\mathbf{t}(t_j) = \mathbf{v}'(t_j^+) - \mathbf{v}'\left(t_j^-\right), \quad j = 1, \cdots, k.$$

Since

$$0 = I_l(\mathbf{v}, \mathbf{t}) = -\sum_{j=1}^{k} \|\mathbf{v}'\left(t_j^+\right) - \mathbf{v}\left(t_j^-\right)\|^2,$$

we conclude that $\mathbf{v}$ is of class $C^1$ at each $t_j$. By the uniqueness of the solution to an ODE, $\mathbf{v}$ is $C^\infty$. Therefore $\mathbf{v}$ is a Jacobi field. □

Set

$$i(t) := \text{index of } I \text{ on } [0, t],$$

and

$$d(t) := \text{nullity of } I \text{ on } [0, t].$$

Here $t \leq l$.

**Corollary 8.6.** $I_l$ is degenerate if and only if $\gamma(0)$ and $\gamma(a)$ are conjugate along $\gamma$. Further more, the nullity of $I_l$ is equal to the multiplicity of $\gamma(l)$ as a conjugate point.

Fix a subdivision

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$$

of $[0, l]$ such that $\gamma|_{[t_{j-1}, t_j]}, j = 1, \cdots, k+1$, is contained in a totally normal neighborhood. We denote by $\mathcal{V}^-$ the subspace of $\mathcal{V}$ formed from the fields $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{v}|_{(t_{i-1}, t_i)}, i = 1, \cdots, k+1$, is a Jacobi field. $\mathcal{V}^-$ has finite dimension. We denote by $\mathcal{V}^+$ be the subspace of $\mathcal{V}$ consisting of vector fields $\mathbf{w}$ such that $\mathbf{w}(t_1) = \mathbf{w}(t_2) = \cdots \mathbf{w}(t_k) = 0$.

**Proposition 8.7.** $\mathcal{V} = \mathcal{V}^- \oplus \mathcal{V}^+$ and the subspaces $\mathcal{V}^+$ and $\mathcal{V}^-$ are orthogonal with respect to $I_l$. In addition, $I_l$ restricted to $\mathcal{V}^+$ is positive definite.

*Proof.* Given $\mathbf{v} \in \mathcal{V}$, let $\mathbf{w}$ be a vector field in $\mathcal{V}^-$ given by $\mathbf{w}(t_j) = \mathbf{v}(t_j)$. Since $\gamma|_{[t_{j-1}, t_j]}$ does not have any conjugate points, such a $\mathbf{w}$ exists and unique. Hence $\mathbf{v} - \mathbf{w} \in \mathcal{V}^+$ and, therefore

$$\mathcal{V} = \mathcal{V}^- \oplus \mathcal{V}^+.$$

In addition, if $X \in \mathcal{V}^-$ and $Y \in \mathcal{V}^+$, we have

$$I_l(X, Y) = -\sum_{j=1}^{k} \langle 0, \nabla_{\gamma'} X\left(t_j^+\right) - \nabla_{\gamma'} X\left(t_j^-\right)\rangle = 0.$$

This proves the first part the proposition.

Since $\gamma|_{[t_{j-1}, t_j]}$ are minimizing geodesics, they have less energy than any other paths between their endpoints. Therefore, if $\mathbf{v} \in \mathcal{V}^+$, then $I_l(\mathbf{v}, \mathbf{v}) \geq 0$.

It remains to show that $I_l(\mathbf{v}, \mathbf{v}) > 0$ if $\mathbf{v} \in \mathcal{V}^+ - \{0\}$. This can be obtained by the index lemma. ☐

**Lemma 8.8.**

$$i(t) + d(t) \leq nk.$$

*Proof.* If not, let $W$ be $(nk+1)$-dimensional subspace of $\mathcal{V}$ such that $I_t(\mathbf{v}, \mathbf{v}) \leq 0$ for any $\mathbf{v} \in W \setminus \{0\}$. Define a mapping $\psi : W \to \mathbb{R}^{nk}$ such that

$$\psi(\mathbf{v}) = \left(\mathbf{v}(t_1), \cdots, \mathbf{v}(t_k)\right).$$

It is a linear map. Since

$$\dim \operatorname{Im} \dim Ker = nK + 1$$

and $\dim \operatorname{Im} \leq nk$, we obtain

$$\dim Ker \geq 1.$$

Hence there exists $\mathbf{v} \neq 0$ and $\mathbf{v} \in Ker\psi \Rightarrow \mathbf{v} \in W \cap \mathcal{V}^+$, this makes a contradiction. □

Observe that $i(t)$ does not depend on the choice of normal subdivision of $[0, l]$, we can therefore choose such subdivisions in a way that $t \in (t_{j-1}, t_j)$. The index of $I_t$ is the index of the restriction of $I_t$ to the subspace $\mathcal{V}^-(0, t)$. Such a restriction will be again denoted by $I_t$. Since each element of $\mathcal{V}^-(0, t)$ is determined by its value at the points $\gamma(t_1), \cdots, \gamma(t_{j-1})$, we have

$$\mathcal{V}^-(0, t) \simeq T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_{j-1})}M =: S_j.$$

**Lemma 8.9.** If $\epsilon > 0$ is sufficiently small, $i(t - \epsilon) = i(t)$.

*Proof.* Since $i(t)$ is non-decreasing, $i(t - \epsilon) \leq i(t)$ for all $\epsilon$. On the other hand, if $I_t$ is negative definite on a subspace $\overline{S} \subset S_j$ with $\dim \overline{S} = i(t)$, then, by continuity of $I_t$, there exists $\epsilon > 0$ such that $I_{t-\epsilon}$ is still negative definite on $\overline{S}$, hence $i(t - \epsilon) \geq i(t)$. □

**Lemma 8.10.** If $\epsilon > 0$ is small enough, then

$$i(t + \epsilon) = i(t) + d(t).$$

*Proof.* We first show that $i(t + \epsilon) \leq i(t) + d(t)$. Indeed, since $\dim S_j = n(j - 1)$, $I_t$ is positive definite on a subspace of dimension $n(j - 1) - i(t) - d(t)$. By continuity, $I_{t+\epsilon}$ is still positive definite on this subspace, for $\epsilon > 0$ sufficiently small. Hence

$$i(t + \epsilon) \leq n(j - 1) - (n(j - 1) - i(t) - d(t)) = i(t) + d(t).$$

Now we show the converse inequality. Let $\mathbf{v} \in S_j$ with $\mathbf{v}(t_{j-1}) \neq 0$ and denote by $\mathbf{v}_{t_0}$ the "broken" Jacobi field which coincides with $\mathbf{v}(t_i)$ at $t_i, i = 1, \cdots, j - 1$, and which vanishes at the point $t_0 \in (t_{j-1}, t_j)$. We claim that

$$I_{t_0}(\mathbf{v}_{t_0}, \mathbf{v}_{t_0}) > I_{t_0+\epsilon}(\mathbf{v}_{t_0+\epsilon}, \mathbf{v}_{t_0+\epsilon}).$$

In fact, if we denote by $W_{t_0}$ (see Fig 9) the vector field defined along $\gamma([0, t_0 + \epsilon)$ by

$$W_{t_0}(t) = \begin{cases} V_{t_0}(t), & t \in [0, t_0], \\ 0, & t \in [t_0, t_0 + \epsilon], \end{cases}$$

we have, from the Index Lemma,

$$I_{t_0}(\mathbf{v}_{t_0}, \mathbf{v}_{t_0}) = I_{t_0+\epsilon}(\mathbf{w}_{t_0}, \mathbf{w}_{t_0}) > I_{t_0+\epsilon}(\mathbf{v}_{t_0+\epsilon}, \mathbf{v}_{t_0+\epsilon}).$$

Therefore, if $\mathbf{v} \in S_j$ and $I_t(\mathbf{v}, \mathbf{v}) \leq 0$, then $I_{t+\epsilon}(\mathbf{v}, \mathbf{v}) < 0$. Hence, if $I_t$ is negative definite on a subspace $\overline{S} \subset S_j$, $I_{t+\epsilon}$ will still be negative definite on the direct sum of $\overline{S}$ with the null space of $I_t$. Therefore,

$$i(t + \epsilon) \geq i(t) + d(t).$$

□

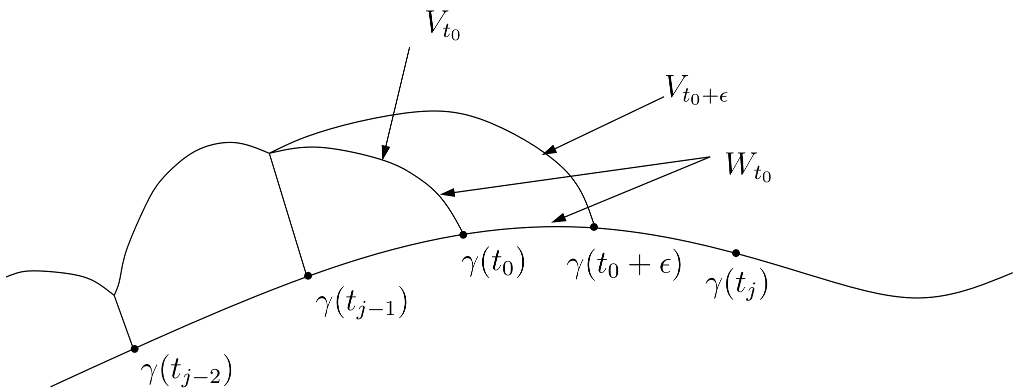


Figure 9: “broken” Jacobi field

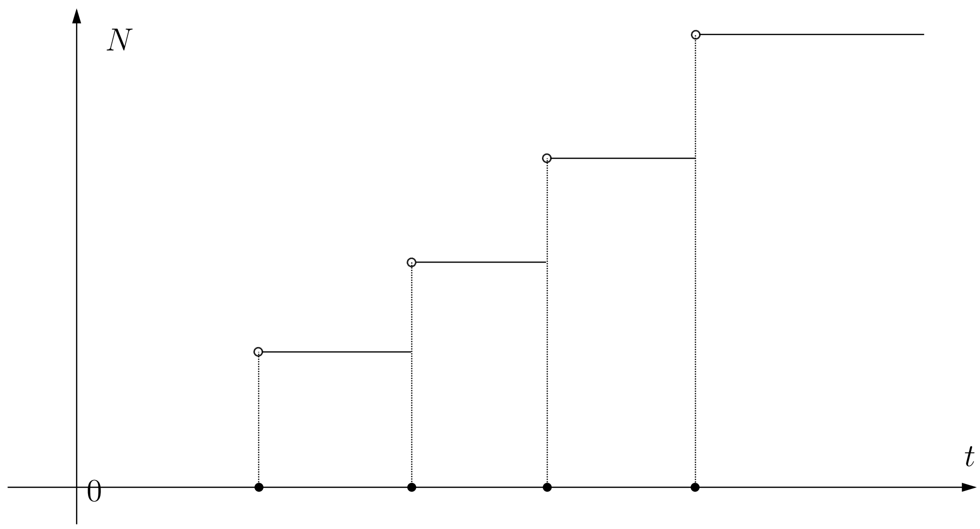


Figure 10: the function $i(t)$