

Maximum Principle

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Abstract

This note is written by referring to Evans' PDE book[\[1\]](#)

Contents

1	Second-order Elliptic Equations	2
2	Maximum Principle	2
2.1	Weak Maximum Principle	2
2.2	Strong Maximum Principle	4

1 Second-order Elliptic Equations

Define the partial differential operator L to be

$$Lu = - \sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c(x) u. \quad (1)$$

Definition 1 We say L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad (2)$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

2 Maximum Principle

2.1 Weak Maximum Principle

In this case, we assume that a^{ij}, b^i, c are bounded, continuous and the uniform ellipticity condition (2) holds.

The maximum principle we talk about here is the classical condition.

Theorem 1 (Weak maximum principle) Assume $u \in C^2(U) \cap C(\overline{U})$ and

$$c \equiv 0 \text{ in } U$$

a. If

$$Lu \leq 0 \text{ in } U, \quad (3)$$

then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

b. If

$$Lu \geq 0 \text{ in } U, \quad (4)$$

then

$$\min_{\overline{U}} u = \min_{\partial U} u.$$

Proof. 1. Suppose we have the strict inequality

$$Lu < 0 \text{ in } U,$$

and there is a point $x_0 \in U$ such that

$$u(x_0) = \max_{\overline{U}} u. \quad (5)$$

At this maximum point x_0 , we have

$$Du(x_0) = 0 \quad (6)$$

and

$$D^2 u(x_0) \leq 0. \quad (7)$$

Since the matrix $A = (a^{ij}(x_0))$ is symmetric and positive definite, there exists an orthogonal matrix $O = (o_{ij})$ so that

$$OAO^T = \text{diag}(d_1, \dots, d_n), \quad OO^T = 1, \quad (8)$$

with $d_k > 0$ ($k = 1, \dots, n$). Write $y = x_0 + O(x - x_0)$. Then $x - x_0 = O^T(y - x_0)$, and so

$$u_{x_i} = \sum_{k=1}^n u_{y_k} O_{ki}, u_{x_i x_j} = \sum_{k,l=1}^n u_{y_k y_l} O_{ki} O_{lj} \quad (i, j = 1, \dots, n).$$

Hence at the point x_0 ,

$$\begin{aligned} \sum_{i,j=1}^n a^{ij} u_{x_i x_j} &= \sum_{k,l=1}^n \sum_{i,j=1}^n a^{ij} u_{y_k y_l} O_{ki} O_{lj} \\ &= \sum_{k=1}^n d_k u_{y_k y_k} \leq 0 \quad \text{by (7)}. \end{aligned} \quad (9)$$

Thus at x_0

$$Lu = - \sum_{i,j=1}^n a^{ij} u_{x_i x_j} + \sum_{i=1}^n b^i u_{x_i} \geq 0$$

in light of (6) and (9). So we have a contradiction. 2. In general case that (3) holds, write

$$u^\epsilon(x) := u(x) + \epsilon e^{\lambda x_1} (x \in U),$$

where $\lambda > 0$ will be selected below and $\epsilon > 0$. The uniform condition implies $a^{ii}(x) \geq \theta$ $i = 1, \dots, n, x \in U$. Therefore

$$\begin{aligned} Lu^\epsilon &= Lu + \epsilon L(e^{\lambda x_1}) \\ &\leq \epsilon e^{\lambda x_1} (-\lambda^2 a^{11} + \lambda b^1) \\ &\leq \epsilon e^{\lambda x_1} (-\lambda^2 \theta + \|\mathbf{b}\|_{L^\infty} \lambda) \\ &< 0 \quad \text{in } U, \end{aligned}$$

provided we choose $\lambda > 0$ sufficiently large. Then according step 1 above and let $\epsilon \rightarrow 0$ we get $\max_{\overline{U}} u = \max_{\partial U} u$. \square

Theorem 2 (Weak maximum principle for $c \geq 0$) Assume $u \in C^2(U) \cap C(\overline{U})$ and

$$c \geq 0 \text{ in } U.$$

a. If

$$Lu \leq 0 \text{ in } U,$$

then

$$\max_{\overline{U}} u \leq \max_{\partial U} u^+. \quad (10)$$

b. Likewise, if

$$Lu \geq 0 \text{ in } U,$$

then

$$\min_{\overline{U}} u \geq -\max_{\partial U} u^-. \quad (11)$$

Proof. 1. Let u be a subsolution and set $V := \{x \in U | u(x) > 0\}$. Then

$$Ku := Lu - cu \leq -cu \leq 0 \quad \text{in } V.$$

The operator K has no zeroth-order term and consequently Theorem 1 implies $\max_{\overline{V}} u = \max_{\partial V} u = \max_{\partial U} u^+$. This gives (10) in the case that $V \neq \emptyset$. Otherwise $u \leq 0$ everywhere in U , and (10) likewise follows.

2. Assertion b follows from a applied to $-u$, once we observe that $(-u)^+ = u^-$. \square

2.2 Strong Maximum Principle

Lemma 1 (Hopf's Lemma) Assume $u \in C^2(U) \cap C^1(\overline{U})$ and

$$c \equiv 0 \text{ in } U.$$

Suppose further

$$Lu \leq 0 \text{ in } U$$

and there exists a point $x^0 \in \partial U$ such that

$$u(x^0) > u(x) \text{ for all } x \in U. \quad (12)$$

Assume finally that U satisfies the interior ball condition at x^0 ; that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$.

a. Then

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

where ν is the outer unit normal to B at x^0 .

b. If

$$c \geq 0 \text{ in } U$$

the same conclusion holds provided

$$u(x^0) \geq 0.$$

Proof. 1. Assume $c \geq 0$ and $B = B^0(0, r)$ for some radius $r > 0$. Define

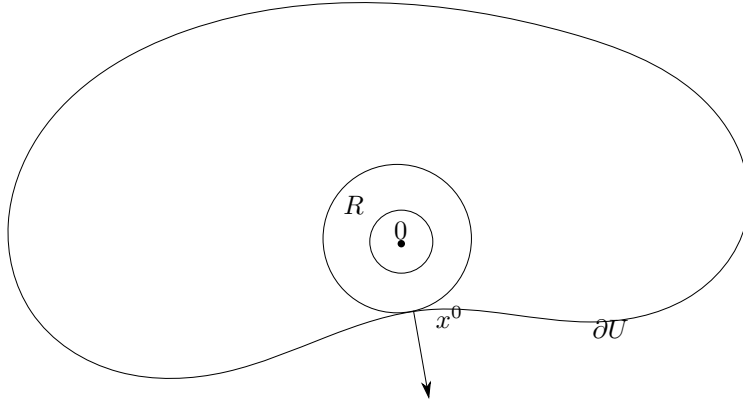


Figure 1: Interior ball condition

$$v(x) := e^{-\lambda|x|^2} - e^{-\lambda r^2} \quad (x \in B(0, r))$$

for $\lambda > 0$ as selected below. Then using the uniform condition, we compute

$$\begin{aligned}
Lv &= - \sum_{i,j=1}^n a^{ij} v_{x_i x_j} + \sum_{i=1}^n b^i v_{x_i} = cv \\
&= e^{-\lambda|x|^2} \sum_{i,j=1}^n a^{ij} (-4\lambda^2 x_i x_j + 2\lambda \delta_{ij}) \\
&\quad - e^{-\lambda|x|^2} \sum_{i=1}^n b^i 2\lambda x_i + c \left(e^{-\lambda|x|^2} - e^{-\lambda r^2} \right) \\
&\leq e^{-\lambda|x|^2} \left(-4\theta\lambda^2 |x|^2 + 2\lambda \text{tr} A + 2\lambda |b| |x| + c \right),
\end{aligned}$$

for $A = (a_{ij})$, $b = (b^1, \dots, b^n)$. Consider next the open annular region $R := B^0(0, r) - B(0, \frac{r}{2})$. We have

$$Lv \leq 0 \quad (13)$$

in R , provided λ is large enough. 2. In view of $u(x^0) > u(x)$ for all $x \in U$, there exists a constant $\epsilon > 0$ so small that

$$u(x^0) \geq u(x) + \epsilon v(x) \quad \left(x \in \partial B(0, \frac{r}{2}) \right). \quad (14)$$

In addition note

$$u(x^0) \geq u(x) + \epsilon v(x) \quad (x \in \partial B(0, r)), \quad (15)$$

since $v \equiv 0$ on $\partial B(0, r)$.

3. From (13) we see

$$L(u + \epsilon v - u(x^0)) \leq -cu(x^0) \leq 0 \quad \text{in } R,$$

and from (14), (15) we observe

$$u + \epsilon v - u(x^0) \leq 0 \quad \text{in } \partial R,$$

In view of the weak maximum principle, Theorem 2, we get $u + \epsilon v - u(x^0) \leq 0$ in R . But $u(x^0) + \epsilon v(x^0) - u(x^0) = 0$, and so

$$\frac{\partial u}{\partial \nu}(x^0) + \epsilon \frac{\partial v}{\partial \nu}(x^0) \geq 0.$$

Consequently

$$\frac{\partial u}{\partial \nu}(x^0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\epsilon}{r} Dv(x^0) \cdot x^0 = 2\lambda \epsilon r e^{-\lambda r^2} > 0,$$

as required. \square

Theorem 3 (Strong maximum principle) Assume $u \in C^2(U) \cap C(\bar{U})$ and

$$c \equiv 0 \quad \text{in } U.$$

Suppose also U is connected, open and bounded.

a. If

$$Lu \leq 0 \quad \text{in } U$$

and u attains its maximum over \bar{U} at an interior point, then u is constant within U .

b. Similarly, if

$$Lu \geq 0 \quad \text{in } U$$

and u attains its minimum over \overline{U} at an interior point, then u is constant within U .

Proof. Write $M := \max_{\overline{U}} u$ and $C := \{x \in U \mid u(x) = M\}$. Then if $u \not\equiv M$, set

$$V := \{x \in U \mid u(x) < M\}.$$

Choose a point $y \in V$ satisfying $\text{dist}(y, C) < \text{dist}(y, \partial U)$, and let B denote the largest ball with center y whose interior lies in V . Then there exists some point $x^0 \in C$, with $x^0 \in \partial B$. Clearly V satisfies the interior ball condition at x^0 , whence Hopf's Lemma (a) implies

$$\frac{\partial u}{\partial \nu}(x^0) > 0.$$

But this is a contradiction: since u attains its maximum at $x^0 \in U$, we have $Du(x^0) = 0$. \square

Similarly, we have the $c \geq 0$ version of the strong maximum principle, and the proof is like the above.

Theorem 4 (Strong maximum principle with $c \geq 0$) Assume $u \in C^2(U) \cap C(\overline{U})$ and

$$c \geq 0 \quad \text{in } U.$$

Suppose also U is connect.

a. If

$$Lu \leq 0 \quad \text{in } U$$

and u attains a nonnegative maximum over \overline{U} at an interior point, then u is constant with U .

b. Similarly, if

$$Lu \geq 0 \quad \text{in } U$$

and u attains a nonpositive minimum over \overline{U} at an interior point, then u is constant within U .

References

- [1] Lawrence C Evans. Partial differential equations. *Providence, RI*, 1998.