# ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
# LECTURE 02

WANG YUNLEI

**Corollary 1.** *Every affine algebraic set $X \subset \mathbb{A}^n$ is the zero set of finite algebraic polynomials.*

*Proof.* Every affine algebraic set is the zero set of some polynomial set $S$, i.e. $Z(S)$. Since $Z(S) = Z(\langle S \rangle)$, it is a zero set of an ideal, we choose the generators of the ideal, name $T$, then $Z(S) = Z(T)$. $\square$

**Definition 1.** A topological space $X$ is reducible if $X = X_1 \cup X_2$, where $X_1$,$X_2$ are closed subsets and $X_1 \subsetneqq X, X_2 \subsetneqq X$. $X$ is called irreducible if it is not reducible,i.e., if $X = X_1 \cup X_2, X_i \subset$ is closed for $i = 1, 2$, then we have $X = X_1$ or $X = X_2$.

*Remark.* When we talk about whether a set is irreducible, it refers to its induced topology from the space where the set is on.

(1) Let $X$ be irreducible, $\emptyset \neq U \subset X$, $U$ is an open subset of $X$. Then $U$ is dense in $X$. Because if it is not dense, we can write $X = (X \backslash U) \cup \overline{U}$, so $X$ is not irreducible.
(2) $U$ itself is also irreducible.

**Definition 2.** A topological space is called noetherian if every descending chain:$X \supset X_1 \supset X_2 \supset \dots$ of closed subsets becomes stationary(i.e., $X_N = X_{N+1} = \dots$ for some $N \in \mathbb{N}^+$).

**Proposition 1.** *Any subspace $Y$ of noetherian topological space $X$ is noetherian.*

*Proof.* Assume $Y \supset Y_1 \supset Y_2 \supset \dots$ a chain of closed subsets. Then $\forall i, Y_i = Y \cap X_i, X_i \subset X$ is closed. Let $X_i' = \cap_{1 \leq j \leq i} X_j$, $X_i' \cap Y = Y_i$. Then $X \supset X_1' \supset X_2' \supset \dots$ is a descending chain. Since $X$ is noetherian, $\exists N$ s.t. $X_N' = X_{N+1}' = \dots$. It follows $Y_N = Y_{N+1} = \dots$. Thus $Y \supset Y_1 \supset Y_2 \supset \dots$ is stationary. $\square$

**Proposition 2.** *$\mathbb{A}^n$ is noetherian topological space.*

*Proof.* Let $\mathbb{A}^n = X \supset X_1 \supset X_2 \supset \dots$ be a chain of closed subsets. Then we have $I(X_1) \subset I(X_2) \subset \dots$. Since $k[x_1, x_2, \dots, x_n]$ is noetherian, $\exists N, I(X_N) = I(X_{N+1}) = \dots$. Note that$X_i = Z(I(X_i))$, we get $X_N = X_{N+1} = \dots$. It shows that $\mathbb{A}^n$ is a noetherian topological space. $\square$

**Theorem 1.** *Let $X$ be a noetherian topological space.*

(1) *$X$ is a union of finitely many irreducible closed subsets: $X = X_1 \cup \dots \cup X_r$;*
(2) *If we require $X_i \not\subset X_j$ for $i \neq j$, then this decomposition is unique.*

*Date*: June 19, 2017.

*Proof.* (1) Assume $X$ does not have a decomposition with finitely many closed subsets. In particular, $X$ is reducible: $X = X_1 \cup Y_1$, $X_1, Y_1$ are closed subsets. so one of the two sets does not have decomposition, say $X_1$. Repeat the argument we ge a descending chain

$$X \supsetneqq X_1 \supsetneqq X_2 \supsetneqq \dots$$

which is not stationary, it contradicts our existing condition.
(2)Let $X = X_1 \cup \cdots \cup X_t = Y_1 \cup \cdots \cup Y_s$. Then we have $X_i = \bigcup_{j=1}^{s} (X_i \cap Y_j)$. Since $X_i$ is irreducible, $\exists j, X_i = X_i \cap Y_j$, thus $X_i \subset Y_j$.Similarly, we can get $Y_j \subset X_k$ for some $k$. Then we have $X_i \subset X_k$, it implies $i = k$ and thus $X_i = Y_j$. So we get the conclusion: each $X_i$ is equal to some $Y_j$ and each $Y_j$ is equal to some $X_i$. So $r = s$ and the $Y_j$'s are permutations of $X_i$'s. □

**Definition 3.** An affine variety is an irreducible affine algebraic set.

**Proposition 3.** *$X \subset \mathbb{A}^n$ is an affine algebraic set. Then we have the following equivalent relations:*

(1) *$X$ is irreducible;*
(2) *$I(X)$ is a prime ideal.*

*Proof.* (1)$\Rightarrow$ (2): let $X$ be irreducible, $f, g$ some polynomials s.t. $fg \in I(X)$. Then we have $X \subset Z(fg) = Z(f) \cup Z(g)$, hence $X = (X \cap Z(f)) \cup (X \cap Z(g))$. Since $X$ is irreducible, we get $X = X \cap Z(f)$ or $X = X \cap Z(g)$, so $X \subset Z(f)$ or $X \subset Z(g)$, i.e. $f \in I(X)$ or $g \in I(X)$.

(2)$\Leftarrow$ (1): Assume $X$ is reducible, then we have $X = X_1 \cup X_2$ and $X_i \subsetneqq X$ are closed subsets. Since $Z(I(X_i)) = X_i \subsetneqq X = Z(I(X))$, we get $I(X_i) \supsetneqq I(X)$. Let $f \in I(X_1) \backslash I(X)$, $g \in I(X_2) \backslash I(X_2)$, $fg$ vanishes on $X_1 \cup X_2 = X$, then $fg \in I(X)$, i.e., $I(X)$ is not prime. □

**Example 1.** $\mathbb{A}^n$ is irreducible because $I(\mathbb{A}^n) = \{0\}$ is a prime ideal.

**Definition 4.** Let $X \neq \emptyset$ be an irreducible topological space. The dimension of $X$ is the largest $n \in \mathbb{Z}$ s.t. there is an ascending chain

$$\emptyset \neq X_0 \subsetneqq X_1 \subsetneqq X_2 \subsetneqq \cdots \subsetneqq X_n = X$$

of irreducible closed subsets. If $X$ is a noetherian topological space then

$$\dim X = \text{ maximum of dimension of irreducible components of } X.$$

*Remark.* (1) The point $p \in \mathbb{A}^n$ has dimension 0;
(2) $\mathbb{A}^1$ has dimension 1;
(3) In the moment, we still cannot prove but true is $\dim \mathbb{A}^n = n$

It is easy to verify $\dim \mathbb{A}^n \geq n$ because we have a chain:

$$\{(0, 0, \dots, 0)\} \subsetneqq Z(x_2, x_3, \dots, x_n) \subsetneqq Z(x_3, \dots, x_n) \subsetneqq \cdots \subsetneqq Z(x_n) \subsetneqq \mathbb{A}^n.$$

**Theorem 2** (The Weak Form Hilbert's Nullstellensatz)**.** *Let $\mathfrak{a} \subsetneqq k[x_1, \dots, x_n]$ be a proper ideal, then $Z(\mathfrak{a}) \neq \emptyset$*

*Remark.* We usually use the following form:

$$\mathfrak{a} \subset k[x_1, \dots, x_n] \text{ and } Z(\mathfrak{a}) = \emptyset \Rightarrow 1 \in I.$$

It is true when $k$ is algebraically closed, otherwise the theorem 2 is wrong:

$$\mathfrak{a} = \langle x^2 + 1 \rangle \in \mathbb{R}[x], Z(\mathfrak{a}) = \emptyset.$$

*Email address*: wcghdpwyl@126.com