Sobolev's Inequality

89hao

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Abstract

This is a learning note about the Sobolev's Inequality, the reference book is Evans *Partial Differential Equations* and Adams and Founier's *Sobolev Spaces*

1 Sobolev's Inequality

Definition 1 (Seminorms) *For $1 \leq p < \infty$ and for integers $j, 0 \leq j \leq m$, we introduce functionals $|\cdot|_{j,p}$ on $W^{m,p}$ as follows:*

$$|u|_{j,p} = |u|_{j,p,\Omega} = \left(\int_\Omega \sum_{|\alpha|=j} |D^\alpha u(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

Clearly $|u|_{0,p} = \|u\|_{0,p} = \|u\|_p$ is the norm on $L^p(\Omega)$ and

$$\|u\|_{m,p} = \left(\int_\Omega \sum_{j=0}^{m} |u|_{j,p}^p \right)^{\frac{1}{p}}.$$

If $j \geq 1$, we call $|\cdot|_{j,p}$ a seminorm. It has all the properties of a norm excerpt that $|u|_{j,p} = 0$ need not imply $u = 0$ in $W^{m,p}(\Omega)$.

Lemma 1 (An Averaging Lemma) *Let Ω be a domain in $\mathbb{R}^n$ where $n \geq 2$. Let k be an integer satisfying $1 \leq k \leq n$, and let $\kappa = (\kappa_1, \cdots, \kappa_k)$ be a k-tuple of integers satisfying $1 \leq \kappa_1 < \kappa_2 < \cdots \kappa_k \leq n$. Let S be the set of all $\binom{n}{k}$ such k-tuples. Given $x \in \mathbb{R}^n$, let x_κ denote the point $(x_{\kappa_1}, \cdots, x_{\kappa_k})$ in $\mathbb{R}^k$ and let $\mathrm{d}x_\kappa$ denote $\mathrm{d}x_{\kappa_1} \cdots \mathrm{d}x_{\kappa_k}$.*
For $\kappa \in S$ let E_κ be the k-dimensional plane in $\mathbb{R}^n$ spanned by the coordinate axes corresponding to the components of x_κ :

$$E_\kappa = \{x \in \mathbb{R}^n : x_i = 0 \text{ if } i \notin \kappa\}$$

and let Ω_k be the projection of Ω onto E_k :

$$\Omega_\kappa = \{x \in E_\kappa : x_\kappa = y_\kappa \text{ for some } y \in \Omega\}.$$

Let $F_\kappa(x_\kappa)$ be a function depending only on the k components of x_κ and belong to $L^\lambda(\Omega_\kappa)$, where $\lambda = \binom{n-1}{k-1}$. Then the function F defined by

$$F(x) = \prod F_\kappa(x_\kappa)$$

belongs to $L^1(\Omega)$*, and* $\|F\|_{1,\Omega} \leq \prod_{\kappa\in S} \|F_\kappa\|_{\lambda,\Omega}$*, that is,*

$$\left(\int_\Omega |F(x)|\,\mathrm{d}x\right)^\lambda \leq \prod_{\kappa\in S}\int_{\Omega_\kappa} |F_\kappa(x_\kappa)|^\lambda\,\mathrm{d}x_\kappa.$$

Proof. For each $\kappa \in S$, let $\mathbf{p}_\kappa$be the n-vector whose ith component is λ if $i \in \kappa$ and ∞ if $i \notin \kappa$. For each i, $1 \leq i \leq n$, exactly $(k/n)\binom{n}{k} = \lambda$ of the vectors $\mathbf{p}_\kappa$ have ith component equal to λ. Therefore,

$$\sum_{\kappa\in S}\frac{1}{\mathbf{p}_\kappa} = \frac{1}{\mathbf{w}},$$

where $\mathbf{w}$ is the n-vector $(1, 1, \cdots, 1)$. Let $F_\kappa(x_\kappa)$ be extended to be zero for $x_\kappa \notin \Omega_\kappa$ and consider F_κ to be defined on $\mathbb{R}^n$ but independent of x_j if $j \notin \kappa$. Then F_κ is its own suprimum over those x_j and

$$\|F_\kappa\|_{\lambda,\Omega_\kappa} = \|F_\kappa\|_{\mathbf{p}_\kappa,\mathbb{R}^n}.$$

From the mixed-norm Hölder inequality

$$\|F\|_{1,\Omega} \leq \|F\|_{\mathbf{w},\mathbb{R}^n} \leq \prod_{\kappa\in S}\|F_\kappa\|_{\mathbf{p}_\kappa,\mathbb{R}^n} = \prod_{\kappa\in S}\|F_\kappa\|_{\lambda,\Omega_\kappa}$$

as required. □

The Sobolev imbedding theorem tells us that for all $\phi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\|\phi\|_q \leq K\|\phi\|_{m,p}$$

for some q and a constant K. We now want to replace $\|\cdot\|_{m,p}$ with $|\cdot|_{m,p}$. That is, we want to esablish

$$\int_{\mathbb{R}^n} |\phi(x)|^q\,\mathrm{d}x \leq K^q\left(\sum_{|\alpha|=m}\int_{\mathbb{R}^n} |D^\alpha\phi(x)|^p\,\mathrm{d}x\right)^{\frac{q}{p}}.$$

It must also holds for $\phi_t(x) = \phi(tx), 0 < t < \infty$. Since $\|\phi_t\|_q = t^{-\frac{n}{q}}\|\phi\|_q$ and $\|D^\alpha\phi_t\|_p = t^{m-(n/p}\|D^\alpha\phi\|_p$. We must have

$$\int_{\mathbb{R}^n} |\phi(x)|^q\,\mathrm{d}x \leq K^q t^{n+mq-(nq/p)}\left(\sum_{|\alpha|=m} |D^\alpha\phi(x)|^p\,\mathrm{d}x\right)^{\frac{q}{p}}.$$

It is posiible for all $t > 0$ if and only if the exponent of t is zero, that is, $q = p^* = np/(n - mp)$.

Theorem 1 (Sobolev's Inequality) *When* $mp < n$*, there exists a finite constant* K *such that*

$$\|\phi\|_{q,\mathbb{R}^n} \leq K\,|\phi|_{m,p,\mathbb{R}^n}$$

holds for every ϕ *in* $C_0^\infty(\mathbb{R}^n)$ *if and only if* $q = p^* = np/(n - mp)$*.*

Proof. The "only if" part was demonstrated above. For the "if" part note first that it is sufficient to establish the inequality for $m = 1$ as its validity for higher m (with $mp < n$) can be confirmed by induction on m.
Now we prove the case $m = 1$, $p = 1$, that is

$$\int_{\mathbb{R}^n} |\phi(x)|^{n/(n-1)} \, \mathrm{d}x \leq K \left(\sum_{j=1}^{n} \int_{\mathbb{R}^n} |D_j \phi(x)| \, \mathrm{d}x \right)^{n/(n-1)} .$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$ and $1 \leq j \leq n$ let $\hat{x}_j = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$.
Let

$$u_j(\hat{x}_j) = \left(\int_{-\infty}^{\infty} |D\phi(x)| \, \mathrm{d}x_j \right)^{\frac{1}{n-1}} .$$

which is evidently independent of x_j and satisfies

$$\left(\|u_j\|_{n-1,\mathbb{R}^{n-1}} \right)^{n-1} \leq |\phi|_{1,1,\mathbb{R}^n} .$$

Since

$$\phi(x) = \int_{-\infty}^{x_1} D_1 \phi(t, \hat{x}_1) \mathrm{d}t$$

we have

$$|\phi(x)| \leq \int_{-\infty}^{\infty} |D_1 \phi(t, \hat{x}_1)| \, \mathrm{d}t \leq \left(u_1 \left(\hat{x}_1 \right) \right)^{n-1} .$$

Similarly, $|\phi(x)| \leq \left(u_j \left(\hat{x}_j \right) \right)^{n-1}$. Applying the Lemma above with $k = n-1 = \lambda$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |\phi(x)|^{n/(n-1)} \, \mathrm{d}x &\leq \int_{\mathbb{R}^n} \prod_{j=1}^{n} u_j(\hat{x}_j) \mathrm{d}x \\
&\leq \left(\prod_{j=1}^{n} \int_{\mathbb{R}^{n-1}} |u_j(\hat{x}_j)|^{n-1} \, \mathrm{d}\hat{x}_j \right)^{\frac{1}{n-1}} \\
&\leq |\phi|_{1,1,\mathbb{R}^n}^{n/(n-1)} .
\end{aligned}$$

For $1 < p < n$ and $p^* = np/\left(n - p\right)$ we can apply the above inequality to $|\phi(x)|^s$ where $s = (n-1)p^*/n$ and obtain, using Hölder's inequality,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\phi(x)|^{p^*} \, \mathrm{d}x \leq & K \left(\sum_{j=1}^{n} s \, |\phi(x)|^{s-1} \, |D_j \phi(x)| \, \mathrm{d}x \right)^{n/(n-1)} \\
\leq & K_1 \left(\sum_{j=1}^{n} \|\phi\|_{(s-1)p'}^{s-1} \|D_j \phi\|_p \right)^{n/(n-1)} .
\end{aligned}$$

Since

$$\begin{aligned}
(s-1)p' &= \left(\frac{(n-1)p^*}{n} - 1 \right) \frac{p}{p-1} = \left(\frac{n-1}{n} \cdot \frac{np}{n-p} - 1 \right) \frac{p}{p-1} \\
&= \left(\frac{(n-1)p}{n-p} - 1 \right) \frac{p}{p-1} = \frac{np - p - n + p}{n-p} \cdot \frac{p}{p-1} \\
&= \frac{n(p-1)}{n-p} \cdot \frac{p}{p-1} = \frac{np}{n-p} = p^*
\end{aligned}$$

and

$$p^* - \frac{(s-1)n}{n-1} = \frac{n}{n-1},$$

it follows by cancellation that

$$\|\phi\|_{p^*} \leq K_2 \, |\phi|_{1,p} \, .$$

For the case $m > 1$, it can be proved by induction on m. $\square$

In the above proof, we used the given lemma to get the inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} \phi_j(\hat{x}_j) \mathrm{d}x \leq \int \left(\prod_{j=1}^{n} \int_{\mathbb{R}^{n-1}} |\phi_j(\hat{x}_j)|^{n-1} \, \mathrm{d}\hat{x}_j \right)^{\frac{1}{n-1}} .$$

In fact, this inequality can also be proved as follows:

First we have

$$|\phi(x)|^{\frac{n}{n-1}} \leq \prod_{j=1}^{n} \phi_j(\hat{x}_j).$$

To make the terms short and clear, we denote

$$\int_{\mathbb{R}} |D\phi| \, \mathrm{d}y_i = \int_{\mathbb{R}} |D\phi(x_1, \cdots, y_i, \cdots, x_n| \, \mathrm{d}y_i.$$

Integrate this inequality with respect to x_1:

$$\begin{aligned}
\int_{\mathbb{R}} |\phi(x)|^{\frac{n}{n-1}} \, \mathrm{d}x_1 &\leq \int_{\mathbb{R}} \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |D\phi| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\
&= \left(\int_{\mathbb{R}} |D\phi| \, \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} |D\phi| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\
&\leq \left(\int_{\mathbb{R}} |D\phi| \, \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{\mathbb{R}^2} |D\phi| \, \mathrm{d}x_1 \mathrm{d}y_i \right)^{\frac{1}{n-1}},
\end{aligned}$$

the last inequality resulting from the general Hölder inequality.

Now integrate the above inequality with respect to x_2:

$$\int_{\mathbb{R}^2} |\phi|^{\frac{n}{n-1}} \, \mathrm{d}x_1 \mathrm{d}x_2 \leq \left(\int_{\mathbb{R}^2} |D\phi| \, \mathrm{d}x_1 \mathrm{d}y_2 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \prod_{i=1, i\neq 2}^{n} I_i^{\frac{1}{n-1}} \mathrm{d}x_2,$$

for

$$I_1 := \int_{\mathbb{R}} |D\phi| \, \mathrm{d}y_1, \quad I_i := \int_{\mathbb{R}^2} |D\phi| \, \mathrm{d}x_1 \mathrm{d}y_i \quad (i = 3, \cdots, n) \, .$$

Applying once more the extended Hölder inequality, we find

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\phi|^{\frac{n}{n-1}} \, \mathrm{d}x_1 \mathrm{d}x_2 \\
&\leq \left(\int_{\mathbb{R}^2} |D\phi| \, \mathrm{d}x_1 \mathrm{d}y_2 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}^2} |D\phi| \, \mathrm{d}y_1 \mathrm{d}x_2 \right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{\mathbb{R}^3} |D\phi| \, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_i \right)^{\frac{1}{n-1}} .
\end{aligned}$$

We continue by integrating with respect to $x_3, \cdots, x_n$, eventually to find

$$\int_{\mathbb{R}^n} |\phi|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |D\phi| \, \mathrm{d}x_1 \cdots \mathrm{d}y_i \cdots \mathrm{d}x_n \right)^{\frac{1}{n-1}} = \left(\int_{\mathbb{R}^n} |D\phi| \, \mathrm{d}x \right)^{\frac{n}{n-1}} .$$