ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
LECTURE 17

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Theorem 1 (Existence of a Primitive Element)**.** *Let k be a field of characteristic 0, L/k is a finite field extension. Then $\exists b \in L$ such that $L = k(b)$.*

Proof of Theorem 2. $K(X)$ is function field of X, let $a_1, \ldots, a_r$ be a transcendence basis of $K(X)/k$, then $K(X)/k(a_1, \ldots, a_r)$ is a finite algebraic extension. By theorem 1, there exists a primitive element $b \in K(X)$ such that $K(X) = k(a_, \ldots, a_r)(b)$ and b is algebraic over $k(a_1, \ldots, a_r)$. Since b is algebraic, there exists a polynomial $F \in k(a_1, \ldots, a_r)[x]$ such that $F(b) = 0$. Write

$$F = \sum_l \frac{G_l(a_1, \ldots, a_r)}{H_l(a_1, \ldots, a_r)} x^l$$

where $G_l(x_1, \ldots, x_r), H_l(x_1, \ldots, x_r) \in k[x_1, \ldots, x_r]$.

Now we view it as $F(x_1, \ldots, x_r, x) \in k(x_1, \ldots, x_r, x)$. Multiply F by producting H_l's and then divide it by the greatest common devisor of the new coefficients. We get $f = \tilde{h}F \in k[x_1, \ldots, x_r, x]$, it is a primitive polynomial. Let $Y = Z(f) \subset \mathbb{A}^{r+1}$, it is a irreducible hypersurface. Then $A(Y) = k[x_1, \ldots, x_r, x]/\langle f \rangle$, $K(Y) = Q(k[x_1, \ldots, x_r, x]/\langle f \rangle) \simeq Q(k[a_1, \ldots, a_r, x]/\langle f(x) \rangle) \simeq k(a_1, \ldots, a_r)[x]/\langle f \rangle$ $\simeq k(a_1, \ldots, a_r)(b) \simeq K(X)$. Then X is birational to Y. $\square$

This proof also implies $\dim X = \operatorname{trdeg} K(X)$.

1. Tangent Space, Singular And Nonsingular Points

First we talk about cases of hypersurfaces in $\mathbb{A}^n$.

Definition 1. Let $X = Z(f) \subset \mathbb{A}^n$ be a hypersurface, assume $I(X) = \langle f \rangle$. A point $p \in X$ is called a singular point if and only if $\frac{\partial f}{\partial x_i}(p) = 0$ for $i = 1, \ldots, n$. Otherwise, p is called a nonsingular point. Let

$$X_{\text{reg}} := \{p \in X | p \text{ is nonsingular.}\}.$$

X is called smooth or nonsingular if and only if $X = X_{\text{reg}}$.

Example 1.
(1) $X = Z(y - x^2)$ is nonsingular.
(2) $X = Z(y^2 - x^2(x+1))$ has a singular point $(0,0)$.
(3) $X = Z(y^2 - x^3)$ has a nonsingular point $(0,0)$.

Proposition 1. *Let $X \subset \mathbb{A}^n$ be an irreducible hypersurface and $\operatorname{char} k = 0$, then X_{reg} is open and dense in X.*

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Proof. Let $F \in k[x_1, \dots, x_n]$ be irreducible such that $X = Z(F)$, then $I(X) = \langle F \rangle$. Define

$$X_{\text{sing}} := \{\text{singular points of } X\}.$$

By definition $X_{\text{sing}} = Z(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}) \subset X$ is closed. Since $Z(F)$ is irreducible, the only thing we have to show is $X \neq X_{\text{sing}}$. Assume $X = X_{\text{sing}} \Rightarrow Z(\frac{\partial F}{\partial x_i}) \supset X$ $\forall i = 1, \dots, n \Rightarrow \frac{\partial F}{\partial x_i} = 0$ $\forall i = 1, \dots, n$. Since $\text{char} k = 0$, we get F is constant, it is impossible. $\square$

Second we talk about cases of affine algebraic sets.

Definition 2. Let $f \in k[x_1, \dots, x_n]$, $p \in \mathbb{A}^n$. The differential of f at p is defined as

$$\mathrm{d}_p f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) x_i.$$

Let $X \subset \mathbb{A}^n$ be an affine algebraic set, the tangent space to X at $p \in X$ is defined as

$$T_p(X) = Z(\mathrm{d}_p f | f \in I(X)).$$

$p \in X$ is called nonsingular if

$$\dim T_p(X) = \dim_p X$$

where $\dim_p X$ is the maximum of dimensions of irreducible components of X passing through p.

Remark. If $I(X) = \langle f_1, \dots, f_r \rangle$, then $T_p(X) = Z(\mathrm{d}_p f_1, \dots, d_p f_r)$. By definition, $T_p(X) \subset Z(\mathrm{d}_p f_1, \dots, \mathrm{d}_p f_r)$. If $h \in I(X)$, we can write $h = \sum_{i=1}^{r} h_i f_i$ with $h_i \in k[x_1, \dots, x_n]$. Using Leibniz rule we get

$$\mathrm{d}_p h = \sum_{i=1}^{r} (\mathrm{d}_p h_i \cdot f_i(p) + h_i(p) \cdot \mathrm{d}_p f_i).$$

Since $f_i(p) = 0$, we get $\mathrm{d}_p h \in \langle \mathrm{d}_p f_1, \dots, \mathrm{d}_p f_r \rangle$. Hence $T_p(X) = Z(\mathrm{d}_p f_1, \dots, d_p f_r)$.

Example 2. If $X = Z(F) \subset \mathbb{A}^n$ and $I(X) = \langle F \rangle$, then $T_p(X) = Z(\mathrm{d}_p F)$ and $\mathrm{d}_p F = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} x_i$. Let $\frac{\partial F}{\partial x_i}(p) = 0$ $\forall i = 1, \dots, n$ for some point $p \in X$, $\forall i = 1, \dots, n$, then $T_p(X) = \mathbb{A}^n$. Since $\dim T_p(X) \neq \dim_p X$, p is a singular point. If $\frac{\partial F}{\partial x_i}(p) \neq 0$ for some i, then $\dim T_p(X) = n - 1$ and p is nonsingular.

Third we talk about cases of general affine varieties.

Definition 3 (Jacobian)**.** Jacobian of $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ is a matrix defined as

$$J(f_1, \dots, f_r) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \frac{\partial f_r}{\partial x_2} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}.$$

Definition 4. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be closed subvarieties. Let $p \in X$, $q \in Y$ and $\varphi = (f_1, \dots, f_m) : X \to Y$ with $f_i \in k[x_1, \dots, x_n]$ for $i = 1, \dots, m$. Assume $\varphi(p) = q$. The differential of φ at p is

$$\mathrm{d}_p\varphi = (\mathrm{d}_p f_1, \dots, \mathrm{d}_p f_m).$$

One can verify that $\mathrm{d}_p\varphi$ maps $T_p(X)$ into $T_q(Y)$. We can write $T_p(X) = \ker(J(f_1, \dots, f_m)(p))$. $\mathrm{d}_p\varphi$ can also be written as $J(f_1, \dots, f_m) \cdot x$.

Proposition 2. (1) $\mathrm{d}_p\mathrm{Id} = \mathrm{Id}$.
(2) $\mathrm{d}_p(\psi \circ \varphi) = J_{\varphi(p)} \circ \mathrm{d}_p\varphi$.

At last we talk about tengent spaces for general varieties.

Definition 5. Let X be a variety, $p \in X$ be a point. The tangent space $T_p(X)$ is

$$T_p(X) := (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$$

where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$, the symbol $*$ denotes the dual of vector space. In other words,

$$T_p(X) = \{k \text{ linear maps } \nu : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k\}$$

or

$$T_p(X) = \{k \text{ linear maps } \nu : \mathfrak{m}_p \to k \text{ with } \nu|_{\mathfrak{m}_p^2} = 0\}$$

$p \in X$ is called nonsingular if $\dim T_p(X) = \dim X$. Similarly we have definition of X_{sing} and X_{reg}. If $X = X_{\mathrm{reg}}$, X is called nonsingular or regular.

2. Conclusions We Need From Previous Lectures

In lecture 16:

Theorem 2. *Every variety X is birational to a hypersurface in $\mathbb{A}^{dim X+1}$.*

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