ALGEBRAIC GEOMETRY - LOTHAR GÖTTSCHE
LECTURE 13

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Definition 1 (Finiteness)**.** Let $A \subset B$ be k-algebras. B is called finite over A if there exist finite many elements $b_1, \dots, b_n \in B$ such that

$$B = b_1 A + \cdots + b_n A := \{\sum b_i a_i | a_i \in A\}.$$

Definition 2 (R-module)**.** Let R. An abelian group B together with the composition $\cdot : R \cdot B \to B$ is called an R-module if and only if for arbitrary $r, r_1, r_2 \in R$ and arbitrary $b, b_1, b_2 \in B$, the following conditions are satisfied

(1) $(r_1 \cdot r_2) \cdot b = r_1 \cdot (r_2 \cdot b)$;
(2) $r_1 \cdot (b_1 + b_2) = r \cdot b_1 + r \cdot b_2$;
(3) $1 \cdot b = b$.

Definition 3 (Finitely generated module)**.** An R-module B is called finitely generated if there exist $b_1, \dots, b_n \in B$ such that

$$B = b_1 R + \cdot b_n R.$$

Example 1. (1) Let R be a ring, $I \subset R$ be an ideal, then I is an R-module via multiplication in R;
(2) If $I \subset R$ is an ideal and we put $A = R/I$, then A is an R-module via multiplication in quotient ring;
(3) If $A \subset B$ is a subring, then B is an A-module via multiplication in B;

If A and B are k-algebras and $A \subset B$, then B is also an A-module. By definition, it is equivalent between B is a finite A-algebra and B is a finitely generated A-module. For k-algebras, it has a different definition from modules about finitely generating.

Definition 4. Let $A \subset B$ and A, B are k-algebra. For $b_1, \dots, b_n \in B$, if we can denote B as

$$B = \{g(b_1, \dots, b_n) | g \in A[x_1, \dots, x_n]\}$$

then we call B a finitely generated A-algebra.

By definition, a finite A-algebra is a finitely generated A-algebra, but the converse is not true. For example $k[x]$ is finitely generated k-algebra but not finite.

Proposition 1. *Let A, B, C be k-algebras and $A \subset B \subset C$, then we have*

(1) *if B is finite over A and C is finite over B, then C is finite over A.If C is finite over A, then C is finite over B;*

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(2) *let $B \supset A$ be a finite A-algebra and assume B is an integral domain, then every element $x \in B$ satisfies a monic equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

with $a_i \in A$ for $i = 0, \ldots, n-1$;

(3) *assume b satifies a monic equation over A, then $A[b]$ is finite over A.*

Proof. (1) We can write $B = b_1A + \cdots_m A$, $b_i \in B$ and $C = c_1B + \cdots + c_nB$, $c_i \in C$, then we get $C = \sum b_ic_jA$, hence C is finite over A. If $C = c_1A + \cdots + c_mA$, since $A \subset B$, we get $C = c_1B + \cdots + c_mB$.

(2) Assume $B = \sum_{i=1}^{n} Ab_i$ for $b_1, \ldots, b_n \in B$, then for any element x in B, we can write xb_i as

$$xb_i = \sum_{j=1}^{n} d_{ij}b_j$$

with $d_{ij} \in A$. It can be rewritten as $\sum_{j=1}^{n}(x\delta_{ij} - d_{ij})b_j = 0$. Thus $(b_1, \ldots, b_n)^T \in$ ker$M$ and $M = (x\delta_{ij} - d_{ij})_{i,j=1}^n$. Since B is an integral domain, we can view b_i as elements in the quotient field $Q(B)$, then we get det$M = 0$. Since det$M$ is a monic equation for x, we finish the proof.

(3) If $b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$ and $a_i \in A$ for $i = 0, \ldots, n$, then every power of b bigger than or equal to n is a linear combination of $1, b, \ldots, b^{n-1}$, i.e., $A[b] = A + Ab + \cdots + Ab^{n-1}$ is finite.

□

Definition 5. Let X, Y be affine varieties. A morphism $\varphi : X \to Y$ is called finite if $A(X)$ is a finite $\varphi^*(A(Y))$-algebra.

Remark. (1) (Definition of finite morphisms for general cases)By definition, we only define the finiteness of morphisms between affine varieties. In general, a morphism $\varphi : X \to Y$ of varieties is called finite if and only if Y has an open affine cover $U_1, \ldots, u_n$, $Y = U_1 \cup \ldots U_n$ such that $\varphi^{-1}(U_i) = W_i$ is affine for $i = 1, \ldots, n$ and the morphism $\varphi|_{W_i} : W_i \to U_i$ is finite.

(2) If Y is a closed subvariety of an affine variety X, the inclusion $i : Y \to X$ is a finite morphism(Because $i^* : A(X) \to A(Y)$ is surjective).

(3) Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be morphisms of affine varieties
   (a) if φ and ψ are both finite, then the composition $\psi \circ \varphi$ is finite;
   (b) if $\psi \circ \varphi$ is finite, then φ is finite. In particular, if $\varphi : X \to Y$ is finite and $\varphi(X)$ is a subset of aclosed subvariety W of Y, then $\varphi : X \to W$ is finite.

Theorem 1. *Finite morphisms are closed.*

Before we prove this theorem, we need to prove two lemmas we need to use.

Lemma 1. *If X is an affine variety, $I \subsetneqq A(X)$ is a proper ideal, then $Z(I) := \{p \in X | f(p) = 0, \forall f \in I\} \neq \emptyset$.*

Proof. Let

$$\pi : k[x_1, \ldots, x_n] \to A(X)$$

be a conanical map, then it is surjective. So $\pi^{-1}(I)$ is a proper ideal in $k[x_1,\ldots,x_n]$. By Nullstellensatz we know $Z(\pi^{-1}(I))\neq\emptyset$. By definition, $Z(I)=Z(\pi^{-1}(I))\cap X$, but $\pi^{-1}(I)\supset I(X)$, so $Z(\pi^{-1}(I))\subset X$, hence we get $Z(I)=Z(\pi^{-1}(X))\neq\emptyset$. □

Lemma 2. *Let B be a finite A-algebra and B be an integral domain, let $I\subsetneqq A$ be a proper ideal of A, then $IB\subsetneqq B$ is a proper ideal of B.*

Proof. Assume $IB=B$, since B is finite over A, we can write $B=Ab_1+\cdots+Ab_n$, $b_1,\ldots,b_n\in B$. Then $B=IB=I(Ab_1+Ab_n)=Ib_1+Ib_n$. In particular, $b_i=\sum\limits_{j=1}^{n}a_{ij}b_j$, $a_{ij}\in I$. Then we get $M\cdot(b_1,\ldots,b_n)^T=(0,\ldots,0)^T$ with $M=(\delta_{ij}-a_{ij})_{i,j=1}^{n}$. Again view M as a matrix in $Q(B)$ we get $\det M=0$, hence

$$0=\det M=1+\sum_l c_l$$

with $c_l\in I$, it implies $1\in I$ and hence I is not a proper ideal in A. By this contradiction we know $IB\neq B$. □

Proof of theorem 1. Let $\varphi: X\to Y$ be a finite morphism of affine varieties, and let W be a closed subvariety of X. We need to show $\varphi(W)$ closed in Y. Let Z be the closure of $\varphi(W)$ in Y, then we have to show $Z=\varphi(W)$. Replacing X by W and Y by Z, then our aim has changed to show a finite morphism $\varphi: X\to Y$ of varieties with dense image is surjective. As $\varphi(X)$ is dense in Y, we have that

$$\varphi^*: A(Y)\to A(X)$$

is injective, hence we can identify $A(Y)$ with the image $\varphi^*(A(Y))\subset A(X)$. Let $Y\subset\mathbb{A}^n$, we take $x_1,\ldots,x_n$ coordinates on $\mathbb{A}^n$. For any element $p=(a_1,\ldots,a_n)\in Y$, define an ideal in $A(Y)$

$$M:=\langle x_1-a_1,\ldots,x_n-a_n\rangle.$$

Now we identify elements in M with the corresponding elements in $A(X)$, let $A(X)\cdot M$ be an ideal generated by M in $A(X)$. In addition,

$$\begin{array}{rcl}\varphi^{-1}(p) & = & \{q\in X|\varphi(q)=p\}\\ & = & \{q\in X|(x_i-a_i)(f(q))=0\forall i=1,\ldots,n\}\\ & = & \{q\in X|(x_i-a_i)\circ\varphi(q)=0,\forall i=1,\ldots,n\}\\ & = & \{q\in X|\varphi^*(x_i-a_i)(q)=0,\forall i=1,\ldots,n\}\\ & = & Z(A(X)\cdot M).\end{array}$$

Thus by lemma 1 we only need to show $A(X)\cdot M\subsetneqq A(X)$, this is done by lemma 2, hence we finish the proof. □

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