Maximum Principle

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December 31, 2019

Abstract

This note is written by referring to Evans' PDE book[1]

Contents

1 Second-order Elliptic Equations

Define the partial differential operator L to be

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u. \tag{1}$$

Definition 1 *We say L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta\,|\xi|^2 \tag{2}$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

2 Maximum Principle

2.1 Weak Maximum Principle

In this case, we assume that a^{ij}, b^i, c are bounded, continuous and the uniform ellipticity condition (2) holds.

The maximum principle we talk about here is the classical condition.

Theorem 1 (Weak maximum principle) *Assume $u \in C^2(U) \cap C(\overline{U})$ and*

$$c \equiv 0 \text{ in } U$$

a. *If*

$$Lu \leq 0 \text{ in } U, \tag{3}$$

then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

b. *If*

$$Lu \geq 0 \text{ in } U, \tag{4}$$

then

$$\min_{\overline{U}} u = \min_{\partial U} u.$$

Proof. 1. Suppose we have the strict inequality

$$Lu < 0 \text{ in } U,$$

and there is a point $x_0 \in U$ such that

$$u(x_0) = \max_{\overline{U}} u. \tag{5}$$

At this maximum point x_0, we have

$$Du(x_0) = 0 \tag{6}$$

and

$$D^2u(x_0) \leq 0. \tag{7}$$

Since the matrix $A = \left(a^{ij}(x_0)\right)$ is symmetric and positive definite, there exists an orthognal matrix $O = (o_{ij})$ so that

$$OAO^T = \operatorname{diag}(d_1, \cdots, d_n), \quad OO^T = 1, \tag{8}$$

with $d_k > 0$ $(k = 1, \cdots, n)$. Write $y = x_0 + O(x - x_0)$. Then $x - x_0 = O^T(y - x_0)$, and so

$$u_{x_i} = \sum_{k=1}^{n} u_{y_k} O_{ki}, u_{x_i x_j} = \sum_{k,l=1}^{n} u_{y_k y_l} O_{ki} O_{lj} \quad (i, j = 1, \cdots, n)\,.$$

Hence at the point x_0,

$$\begin{aligned}\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} &= \sum_{k,l=1}^{n} \sum_{i,j=1}^{n} a^{ij} u_{y_k y_l} o_{ki} o_{lj} \\ &= \sum_{k=1}^{n} d_k u_{y_k y_k} \leq 0 \quad \text{by (7)}.\end{aligned} \tag{9}$$

Thus at x_0

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} \geq 0$$

in light of (6) and (9). So we have a contradiction. 2. In general case that (ref3) holds, write

$$u^\epsilon(x) := u(x) + \epsilon e^{\lambda x_1} (x \in U),$$

where $\lambda > 0$ will be selected below and $\epsilon > 0$. The uniform condition implies $a^{ii}(x) \geq \theta$ $i = 1, \cdots, n, x \in U$. Therefore

$$\begin{aligned}Lu^\epsilon =& Lu + \epsilon L\left(e^{\lambda x_1}\right) \\ \leq& \epsilon e^{\lambda x_1}\left(-\lambda^2 a^{11} + \lambda b^1\right) \\ \leq& \epsilon e^{\lambda x_1}\left(-\lambda^2 \theta + \|\mathbf{b}\|_{L^\infty} \lambda\right) \\ <& 0 \quad \text{ in } U, .\end{aligned}$$

provided we choose $\lambda > 0$ sufficiently large. Then according step 1 above and let $\epsilon \to 0$ we get $\max_{\overline{U}} u = \max_{\partial U} u$. $\square$

Theorem 2 (Weak maximum principle for $c \geq 0$) *Assume $u \in C^2(U) \cap C(\overline{U})$ and*

$$c \geq 0 \text{ in } U.$$

a. *If*

$$Lu \leq 0 \text{ in } U,$$

then

$$\max_{\overline{U}} \leq \max_{\partial U} u^+. \tag{10}$$

b. *Likewise, if*

$$Lu \geq 0 \text{ in } U,$$

then

$$\min_{\overline{U}} \geq -\max_{\partial U} u^-. \tag{11}$$

Proof. 1. Let u ve a subsolution and set $V := \{x \in U | u(x) > 0\}$. Then

$$Ku := Lu - cu \leq -cu \leq 0 \quad \text{ in } V.$$

The operator K has no zeroth=order term and consequently Theorem 1 implies $\max_{\overline{V}} u = \max_{\partial V} u = \max_{\partial U} u^+$. This gives (10) in the case that $V \neq$. Otherwise $u \leq 0$ everywhere in U,and (10) likewise follows.

2. Assertion b follows from a applied to $-u$, once we observe that $(-u)^+ = u^-$. $\square$

2.2 Strong Maximum Principle

Lemma 1 (Hopf's Lemma) *Assume $u \in C^2(U) \cap C^1(\overline{U})$ and*

$$c \equiv 0 \textit{ in } U.$$

Suppose further

$$Lu \leq 0 \textit{ in } U$$

and there exists a point $x^0 \in \partial U$ such that

$$u(x^0) > u(x) \textit{ for all } x \in U. \tag{12}$$

Assume finially that U satisfies the interior ball condition at x^0 ; that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$.

a. *Then*

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

where ν is the outer unit normal to B at x^0.

b. *If*

$$c \geq 0 \textit{ in } U$$

the same conclusion holds provided

$$u(x^0) \geq 0.$$

Proof. 1. Assume $c \geq 0$ and $B = B^0(0, r)$ for some radius $r > 0$. Define

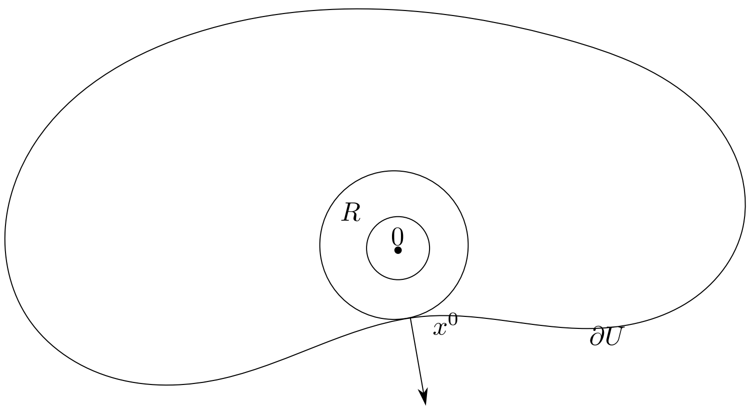

Figure 1: Interior ball condition

$$v(x) := e^{-\lambda |x|^2} - e^{-\lambda r^2} \quad (x \in B\,(0, r))$$

for $\lambda > 0$ as selected below. Then using the uniform condition, we compute

$$
\begin{aligned}
Lv = & -\sum_{i,j=1}^{n} a^{ij} v_{x_i x_j} + \sum_{i=1}^{n} b^i v_{x_i} = cv \\
= & e^{-\lambda|x|^2} \sum_{i,j=1}^{n} a^{ij} \left(-4\lambda^2 x_i x_j + 2\lambda \delta_{ij}\right) \\
& - e^{-\lambda|x|^2} \sum_{i=1}^{n} b^i 2\lambda x_i + c\left(e^{-\lambda|x|^2} - e^{-\lambda r^2}\right) \\
\leq & e^{-\lambda|x|^2} \left(-4\theta\lambda^2 |x|^2 + 2\lambda \operatorname{tr} A + 2\lambda |b| |x| + c\right),
\end{aligned}
$$

for $A = (a_{ij})$, $b = \left(b^1, \cdots, b^n\right)$. Consider next the open annular region $R := B^0(0, r) - B\left(0, \frac{r}{2}\right)$. We have

$$
Lv \leq 0 \tag{13}
$$

in R, provided λ is large enough. 2. In view of $u(x^0) > u(x)$ for all $x \in U$, there exists a constant $\epsilon > 0$ so small that

$$
u(x^0) \geq u(x) + \epsilon v(x) \quad \left(x \in \partial B(0, \frac{r}{2}\right). \tag{14}
$$

In addition note

$$
u(x^0) \geq u(x) + \epsilon v(x) \quad (x \in \partial B(0, r)), \tag{15}
$$

since $v \equiv 0$ on $\partial B(0, r)$.

3. From (13) we see

$$
L\left(u + \epsilon v - u(x^0)\right) \leq -cu(x^0) \leq 0 \quad \text{in } R,
$$

and from (14), (15) we observe

$$
u + \epsilon v - u(x^0) \leq 0 \quad \text{in } \partial R,
$$

In view of the weak maximum principle, Theorem 2, we get $u + \epsilon v - u(x^0) \leq 0$ in R. But $u(x^0) + \epsilon v(x^0) - u(x^0) = 0$, and so

$$
\frac{\partial u}{\partial \nu}(x^0) + \epsilon \frac{\partial v}{\partial \nu}(x^0) \geq 0.
$$

Consequently

$$
\frac{\partial u}{\partial \nu}\left(x^0\right) \geq -\epsilon \frac{\partial v}{\partial \nu}\left(x^0\right) = -\frac{\epsilon}{r} Dv(x^0) \cdot x^0 = 2\lambda\epsilon r e^{-\lambda r^2} > 0,
$$

as required. $\square$

Theorem 3 (Strong maximum principle) *Assume $u \in C^2(U) \cap C\left(\overline{U}\right)$ and*

$$
c \equiv 0 \quad \textit{in } U.
$$

Suppose also U is connected, open and bounded.

a. *If*

$$
Lu \leq 0 \quad \textit{in } U
$$

and u attains its minimum over $\overline{U}$ at an interior point, thenm u is constant within U.

b. *Similarly, if*

$$Lu \geq 0 \quad \text{in } U$$

and u attains its minimum over $\overline{U}$ at an interior point, then u is constant within U.

Proof. Write $M := \max_{\overline{U}} u$ and $C := \{x \in U | u(x) = M\}$. Then if $u \not\equiv M$, set

$$V := \{x \in U | u(x) < M\}.$$

Choose a point $y \in V$ satisfying $\operatorname{dist}(y, C) < \operatorname{dist}(y, \partial U)$, and let B denote the largest ball with center y whose interior lies in V. Then there exists some point $x^0 \in C$, with $x^0 \in \partial B$. Clearly V satisfies the interior ball condition at x^0, whence Hopf's Lemma (a) implies

$$\frac{\partial u}{\partial \nu}\left(x^0\right) > 0.$$

But this is a contradiction: since u attains its maximum at $x^0 \in U$, we have $Du(x^0) = 0$. □

Similarly, we have the $c \geq 0$ version of the strong maximum principle, and the proof is like the above.

Theorem 4 (Strong maximum principle with c ≥ 0) *Assume $u \in C^2(U) \cap C\left(\overline{U}\right)$ and*

$$c \geq 0 \quad \text{in } U.$$

Suppose also U is connect.

a. *If*

$$Lu \leq 0 \quad \text{in } U$$

and u attains a nonnegative maximum over $\overline{U}$ at an interior point, then u is constant with U.

b. *Similarly, if*

$$Lu \geq 0 \quad \text{in } U$$

and u attains a nonpositive minimum over $\overline{U}$ at an interior point, then u is constant within U.

References

[1] Lawrence C Evans. Partial differential equations. *Providence, RI*, 1998.