Fourier-Stieltjes Transforms on the Line

Based on the book by Yitzhak Katznelson

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These notes are an introduction to Fourier-Stieltjes transforms on the line, which is based on Yitzhak Katznelson's book *An Introduction to Harmonic Analysis*.

Contents

1 Basic definitions

Denote by $M(\mathbb{R})$ the space of all finite Borel measures on $\mathbb{R}$. $M(\mathbb{R})$ is identified with the dual space of $C_0(\mathbb{R})$ by means of

$$\langle f, \mu \rangle = \int f \overline{\mathrm{d}\mu} \quad f \in C_0(\mathbb{R}), \mu \in M(\mathbb{R}). \tag{1}$$

The *norm* on $M(\mathbb{R})$ is defined by $\|\mu\|_{M(\mathbb{R})} := \int |\mathrm{d}\mu|$.

Definition 1.1. The *Fourier-Stieltjes transform* of a measure $\mu \in M(\mathbb{R})$ is defined by:

$$\hat{\mu}(\xi) = \int e^{-i\xi x} \mathrm{d}\mu(x) \quad \xi \in \hat{\mathbb{R}}. \tag{2}$$

It is easy to check that the transform defined above satisfies the following properties:

Proposition 1.2. *Let $\hat{\mu}(\xi)$ be the Fourier-Stieltjes transform of a measure $\mu \in M(\mathbb{R})$. Then*

a. *$\hat{\mu}(\xi)$ is bounded, i.e.,*
$$|\hat{\mu}(\xi)| \leq \|\mu\|_{M(\mathbb{R})}; \tag{3}$$

b. *$\hat{\mu}(\xi)$ is uniformly continuous.*

c. *If $\mathrm{d}\mu = f\mathrm{d}x$ for $f \in L^1(\mathbb{R})$, then*
$$\hat{\mu}(\xi) = \hat{f}(\xi). \tag{4}$$

Definition 1.3. Let $\mu \in M(\mathbb{R})$ and $f \in C_0(\mathbb{R})$, then the *convolution* is defined by

$$(\mu * f)(x) = \int f(x-y)\mathrm{d}\mu(y). \tag{5}$$

Furthermore, we can define the convolution of two measures $\mu, \nu \in M(\mathbb{R})$ by the duality

$$\langle f, \mu * \nu \rangle := \langle \overline{\mu} * f, \nu \rangle. \tag{6}$$

It is equivalent to define

$$(\mu * \nu)(E) = \int \mu(E-y)\mathrm{d}\nu(y) \tag{7}$$

for every Borel set E.

It is easy to check that $\widehat{\mu * \nu}(\xi) = \widehat{\mu}(\xi)\widehat{\nu}(\xi)$.

Remark. Consider the delta function $\delta(x) \in M(\mathbb{R})$, this implies $L^1(\mathbb{R}) \subsetneq M(\mathbb{R})$ and the failing of the Riemann-Lebesgue lemma.

Theorem 1.4 (Parseval's formula). *Let $\nu \in M(\mathbb{R})$ and let f be a continuous function in $L^1(\mathbb{R})$ such that $\hat{f} \in L^1\left(\hat{\mathbb{R}}\right)$. Then*

$$\int f(x)\mathrm{d}\mu(x) = \frac{1}{2\pi}\int \hat{f}(\xi)\hat{\mu}(-\xi). \tag{8}$$

Proof. By the theory of usual fourier transform and $\hat{f} \in L^1(\hat{\mathbb{R}})$, we have

$$f(x) = \frac{1}{2\pi}\int \hat{f}(\xi)e^{i\xi x}\mathrm{d}\xi.$$

Hence

$$\int f(x)\mathrm{d}\mu(x) = \frac{1}{2\pi}\int\int \hat{f}(\xi)e^{i\xi x}\mathrm{d}\mu(x)\mathrm{d}\xi = \frac{1}{2\pi}\int \hat{f}(\xi)\hat{\mu}(-\xi).$$

□

The condition $\hat{f} \in L^1\left(\hat{\mathbb{R}}\right)$ is used to change the order of intergration (by Fubini's theorem). Formula (8) is valid under the weaker assumption $\hat{f}(\xi)\hat{\mu}(-\xi) \in L^1(\hat{\mathbb{R}})$:

$$\begin{aligned}
\int f(x)\mathrm{d}\mu(x) &= \int \left(\lim_{\lambda\to\infty}\frac{1}{2\pi}\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)\hat{f}(\xi)e^{i\xi x}\mathrm{d}\xi\right)\mathrm{d}\mu(x) \\
&= \lim_{\lambda\to\infty}\frac{1}{2\pi}\int\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)\hat{f}(\xi)e^{i\xi x}\mathrm{d}\xi\mathrm{d}\mu(x) \\
&= \lim_{\lambda\to\infty}\frac{1}{2\pi}\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)\hat{f}(\xi)\hat{\mu}(-\xi) \\
&= \frac{1}{2\pi}\int \hat{f}(\xi)\hat{\mu}(-\xi).
\end{aligned}$$

The third identity use the assumption to change the order of integration.

Corollary 1.5. *If $\hat{\mu}(\xi) = 0$ for all ξ, then $\mu = 0$.*

Proposition 1.6. *Let f be bounded and continuous on $\mathbb{R}$ and let $\{k_\lambda\}$ be a summability kernel. Then $k_\lambda * f = \int k_\lambda(x-y)f(y)\mathrm{d}y$ converges to f uniformly on compact sets on $\mathbb{R}$.*

Using this property, we obtain the gneralized Parseval's formula:

Corollary 1.7. *Let $\mu \in M(\mathbb{R})$ and let f be a bounded continuous function in $L^1(\mathbb{R})$. Then*

$$\int f(x)\mathrm{d}\mu(x) = \lim_{\lambda\to\infty} \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) \hat{f}(\xi)\hat{\mu}(-\xi). \tag{9}$$

2 Characterize Fourier-Stieltjes transforms

We have known that the Fourier-Stieltjes transform of any $\mu \in M(\mathbb{R})$ is bounded and continuous. But the converse is false.

Theorem 2.1. *Let φ be continuous on $\hat{\mathbb{R}}$, define Φ_λ by:*

$$\Phi_\lambda(x) = \frac{1}{2\pi}\int_{-\lambda}^{\lambda}\left(1 - \frac{|\xi|}{\lambda}\right)\varphi(\xi)e^{i\xi x}\mathrm{d}\xi.$$

Then φ is a Fourier-Stieltjes transform if and only if $\Phi_\lambda \in L^1(\mathbb{R})$ for all $\lambda > 0$, and $\|\Phi_\lambda\|_{L^1(\mathbb{R})}$ is bounded as $\lambda \to \infty$.

Proof. If $\varphi = \hat{\mu}$ with $\mu \in M(\mathbb{R})$, then $\Phi_\lambda = \mu * K_\lambda$ where $\widehat{K_\lambda} = \chi_{[-\lambda,\lambda]}\left(1 - \frac{|\xi|}{\lambda}\right)$ (by Proposition 1.6). It follows that for all $\lambda > 0$, $\Phi_\lambda \in L^1(\mathbb{R})$ and $\|\Phi_\lambda\|_{L^1(\mathbb{R})} \leq \|\nu\|_{M(\mathbb{R})}$.

Conversely, assuming that $\Phi_\lambda \in L^1(\mathbb{R})$ with uniformly bounded norms, we consider measures $\Phi_\lambda(x)\mathrm{d}x$ and denote by μ a weak-star limit point of $\Phi_\lambda(x)\mathrm{d}x$ ad $\lambda \to \infty$. This μ exists, because we can define

$$\langle f, \mu\rangle = \int f\overline{\mathrm{d}\mu} = \lim_{\lambda\to\infty}\int f(x)\overline{\Phi_\lambda(x)\mathrm{d}x} = \lim_{\lambda\to\infty}\frac{1}{2\pi}\int_{-\lambda}^{\lambda}\hat{f}(\xi)\overline{\left(1 - \frac{|\xi|}{\lambda}\right)\varphi(\xi)}.$$

We claim that $\varphi = \hat{\mu}$ and since both functions are continuous, this will follow if we show that

$$\int \varphi(-\xi)g(\xi)\mathrm{d}\xi = \int \hat{\mu}(-\xi)g(\xi)\mathrm{d}\xi$$

for every twice continuously differentiable g with compact support. For such g we define

$$G(x) = \frac{1}{2\pi}\int g(\xi)e^{i\xi x}\mathrm{d}\xi.$$

Then by the assumption we have $G(x) \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, hence $g = \hat{G}$. Then

$$\begin{aligned}
\int g(\xi)\varphi(-\xi)\mathrm{d}\xi &= \lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda} g(\xi)\varphi(-\xi)\left(1 - \frac{|\xi|}{\lambda}\right)\mathrm{d}\xi \\
&= \lim_{\lambda\to\infty} 2\pi\int G(x)\Phi_\lambda(x)\mathrm{d}x \\
&= 2\pi\int G(x)\mathrm{d}\mu(x) \\
&= \int g(\xi)\hat{\mu}(-\xi)\mathrm{d}\xi,
\end{aligned}$$

where the second identity use the Parseval's formula and the third identity is the definition of μ. ☐

Remark. Denote $\mathrm{d}\mu_\lambda = \Phi_\lambda(x)\mathrm{d}x$, what we have done above is proving $\varphi(\xi) = \hat{\mu}(\xi)$. But it is not necessary that $\hat{\mu}(\xi) = \lim_{\lambda\to\infty} \hat{\mu}_n(\xi)$ pointwisely. In the case of $M(\mathbb{T})$, the weak-star convergence implies pointwise convergence of the Fourier-Stieltjes coefficients because $e^{i\xi x} \in C(\mathbb{T})$. The exponentials on $\mathbb{R}$ do not belong to $C_0(\mathbb{R})$ and it is false that weak-star convergence in $M(\mathbb{R})$ implies pointwise convergence of the Fourier-Stieltjes transforms. We give an example below to show this phenominon.

Example 2.1. Denote by $\delta_n = \delta(x-n)$ the dirac measure on $\mathbb{R}$ concentrated at $x = n$. It is easy to see that $\lim_{n\to\infty} \delta_n = 0$ in the weak-star topology of $M(\mathbb{R})$, but $\hat{\delta_n} = e^{-i\xi n}$ do not converge pointwisely.

According the argument in the above remark, we have:

Lemma 2.2. *Let $\mu_n \in M(\mathbb{R})$ and assume that $\mu_n \to \mu$ in the weeak-star topology. Assume also that $\hat{\mu}_n(\xi) \to \varphi(\xi)$ pointwise, φ being continuous on $\hat{\mathbb{R}}$. Then $\hat{\mu} = \varphi$.*

Proof. For every twice continuously differentiable g with compact support, we have

$$\begin{aligned}\int g(\xi)\varphi(-\xi)\mathrm{d}\xi &= \int g(\xi)\left(\lim_{n\to\infty} \hat{\mu}_n(-\xi)\right)\mathrm{d}\xi \\ &= \lim_{n\to\infty} \int g(\xi)\hat{\mu}_n(-\xi)\mathrm{d}\xi \\ &= 2\pi \int G(x)\mathrm{d}\mu_n(x) \\ &= 2\pi \int G(x)\mathrm{d}\mu(x) \\ &= \int g(\xi)\hat{\mu}(-\xi)\mathrm{d}\xi.\end{aligned}$$

□

Theorem 2.3. *A function φ defined and continuous on $\hat{\mathbb{R}}$, is a Fourier-Stieltjes transform if and only if there exists a constant C such that*

$$\left|\frac{1}{2\pi}\int \hat{f}(\xi)\varphi(-\xi)\mathrm{d}\xi\right| \le C\sup_x |f(x)| \tag{10}$$

for every continuous $f \in L^1(\mathbb{R})$ such that $\hat{f}$ has compact support.

Proof. If $\varphi = \hat{\mu}$, (10) follows from Parseval's formula (8) with $C = \|\mu\|_{M(\mathbb{R})}$.

Conversely, if (10) holds,

$$f \mapsto \frac{1}{2\pi}\int \hat{f}(\xi)\varphi(-\xi)\mathrm{d}\xi$$

defines a bounded linear functional on a dense subspace of $C_0(\mathbb{R})$, which by the Riesz representation theorem, has the form $f \mapsto \int f(x)\mathrm{d}\mu(x)$. Moreover, $\|\mu\| \le C$. Using (8) again we see that $\hat{mu} - \varphi$ is orthogonal to all the continuous, compactly supported functions $\hat{f}$ with $f \in L^1(\mathbb{R})$, hence $\varphi = \hat{\mu}$. □

Definition 2.4. Let $\mu \in M(\mathbb{R})$, set $E_n = E + 2\pi n$ and write $\widetilde{E} = \bigcup_{n\in\mathbb{Z}} E_n$. Define

$$\mu_{\mathbb{T}}(E) = \mu(\widetilde{E}).$$

Then $\mu_{\mathbb{T}}$ is a measure on $\mathbb{T}$ and identifies continuous functions on $\mathbb{T}$ with 2π-periodic functions on $\mathbb{R}$

$$\int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} f(x-n)\mathrm{d}x = \int_{\mathbb{T}} f(t)\mathrm{d}t. \tag{11}$$

Theorem 2.5. *A function φ defined and continuous on $\hat{\mathbb{R}}$, is a Fourier-Stieltjes transform if and only if there exists a constant $C > 0$ such that for all $\lambda > 0$, $\{\varphi(\lambda n)\}_{n=-\infty}^{\infty}$ are the Fourier-Stieltjes coefficients of a measure of norm $\le C$ on $\mathbb{T}$.*

Proof. If $\varphi = \hat{\mu}$ with $\mu \in M(\mathbb{R})$, we have $\varphi(n) = \hat{\mu}(n) = \hat{\mu}_{\mathbb{T}}(n)$ and $\|\mu_{\mathbb{T}}\| \le \|\mu\|$. Writing $\mathrm{d}\mu(x/\lambda)$ for the measure satisfying

$$\int f(x)\mathrm{d}\mu\left(\frac{x}{\lambda}\right) = \int f(\lambda x)\mathrm{d}\mu(x)$$

we have $\|\mu(x/\lambda)\|_{M(\mathbb{R})} = \|\mu\|_{M(\mathbb{R})}$ and $\widehat{\mu\,(x/\lambda)}(\xi) = \hat{\mu}(\xi\lambda)$. This implies $\varphi(\lambda n) = \widehat{\mu(x/\lambda)}_{\mathbb{T}}(n)$ and the "only if" part is established.

Conversely we use Theorem 2.3. Let f be continuous and integrable on $\mathbb{R}$ and assume that $\hat{f}$ is infinitely differentiable and compactly supported. We need to estimate the integral $\frac{1}{2\pi}\int \hat{f}(\xi)\varphi(-\xi)\mathrm{d}\xi$. Since the integrand is continuous and compactly supported, we can approximate the integral by its Riemann sums. Thus for aritrary $\varepsilon > 0$, if λ is small enough:

$$\left|\frac{1}{2\pi}\int \hat{f}(\xi)\varphi(-\xi)\mathrm{d}\xi\right| < \left|\frac{\lambda}{2\pi}\sum \hat{f}(\lambda n)\varphi(-\lambda n)\right| + \varepsilon. \tag{12}$$

Now, $(\lambda/2\pi)\hat{f}(\lambda n)$ are the Fourier coefficients of the function $\psi_\lambda(t) = \sum_{m\in\mathbb{Z}} f\left((t+2\pi m)/\lambda\right)$ on $\mathbb{T}$, and since the infinite differentiability of $\hat{f}$ implies a very fast decrease of $f(x)$ as $|x| \to \infty$, we see that if λ is sufficiently small

$$\sup|\psi_\lambda(t)| \le \sup|f(x)| + \varepsilon. \tag{13}$$

Assuming that $\varphi(\lambda n) = \hat{\mu}_\lambda(n)$, $\mu_\lambda \in M(\mathbb{T})$ and $\|\mu_\lambda\|_{M(\mathbb{T})} \leq C$, we obtain from Parseval's formula

$$\left|\frac{\lambda}{2\pi}\sum \hat{f}(\lambda n)\varphi(-\lambda n)\right| = \left|\sum \hat{\psi}_\lambda(n)\hat{\mu}_\lambda(-n)\right| \leq C\sup|\psi_\lambda(t)|.$$

By (12) and (13)

$$\left|\frac{1}{2\pi}\int \hat{f}(\xi)\varphi(-\xi)\mathrm{d}\xi\right| \leq C\sup|f(x)| + (C+1)\varepsilon$$

and since $\varepsilon > 0$ is arbitrary, (10) is satisfied.

$\square$

3 Fourier-Stieltjes transform of a positive measure

Theorem 3.1. *Let φ be a bounded and continuous function on $\hat{\mathbb{R}}$. Then φ is the Fourier-Stieltjes transform of a positive measure on $\mathbb{R}$ if and only if*

$$\int \hat{f}(\xi)\varphi(-\xi) \geq 0 \tag{14}$$

for every nonnegative function f which is infinitely differentiable and compactly supported.

Proof. The "only if" part is obvious by Parseval's formula. To complete the proof we only need to show that (14) implies (10) with $C = \varphi(0)$ for every real-valued, compactly suppported infinitely differentiable f.

As usual, we denote the Fejér kernel

$$K_\lambda(x) = \lambda K(\lambda x) = \frac{\lambda}{2\pi}\left(\frac{\sin \lambda x/2}{\lambda x/2}\right)^2.$$

Note that $\frac{1}{2\pi}\left(\frac{\sin \lambda x/2}{\lambda x/2}\right)^2 \to \frac{1}{2\pi}$ and nonnegative as $\lambda \to 0$, uniformly on compact subsets of $\mathbb{R}$. The Fourier transform of $\lambda^{-1}K_\lambda(x)$ is $\lambda^{-1}\max(1 - |\xi|/\lambda, 0)$ and, as $\varphi(\xi)$ is continuous at $\xi = 0$,

$$\lim_{\lambda\to 0}\int \frac{1}{\lambda}\hat{K}_\lambda(\xi)\varphi(-\xi)\mathrm{d}\xi = \varphi(0). \tag{15}$$

If f is real-valued and compactly supported and $\varepsilon > 0$, then, for sufficiently small λ and all x,

$$2\pi(\varepsilon + \sup|f|)K(\lambda x) - f(x) \geq 0. \tag{16}$$

Hence by (14),(15) and (16) (replace f in (14) by the left hand side of (16)), if $\hat{f} \in L^1(\hat{\mathbb{R}})$,

$$\frac{1}{2\pi}\int \hat{f}(\xi)\varphi(-\xi)\mathrm{d}\xi \leq \varphi(0)\left(2\varepsilon + \sup|f|\right). \tag{17}$$

Rewritting (17) for $-f$ and letting $\varepsilon \to 0$ we obtain

$$\left|\frac{1}{2\pi}\int \hat{f}(\xi)\varphi(-\xi)\mathrm{d}\xi\right| \leq \varphi(0)\sup|f|. \tag{18}$$

□

The analog to Theorem 2.5 is:

Theorem 3.2. *A function φ defined and continuous on $\hat{\mathbb{R}}$, is the Fourier-Stieltjes transform of a positive measure if and only if for all $\lambda > 0$, $\{\varphi(\lambda n)\}_{n=-\infty}^{\infty}$ are the Fourier-Stieltjes coefficients of a positive measure on $\mathbb{T}$.*

4 Positive definite functions

Definition 4.1. A function φ defined on $\hat{\mathbb{R}}$ is said to be *positive definite* if, for every choice of $\xi_1, \cdots, \xi_N \in \hat{\mathbb{R}}$ and complex numbers $z_1, \cdots, z_N$, we have

$$\sum_{j,k=1}^{N} \varphi(\xi_j - \xi_k) z_j \overline{z_k} \geq 0. \tag{19}$$

Let $N = 2, z_1 = 1, z_2 = z$, then (19) reads

$$\varphi(0)(1 + |z|^2) + \varphi(\xi)z + \varphi(-\xi)\overline{z} \geq 0.$$

Set $z = 1$, we get $\varphi(\xi) + \varphi(-\xi)$ real. Set $z = 1$, we get $i(\varphi(\xi) - \varphi(-\xi))$ real, hence

$$\varphi(-\xi) = \overline{\varphi(\xi)}. \tag{20}$$

If we take z such that $z\varphi(\xi) = -|\varphi(\xi)|$, we obtain

$$|\varphi(\xi)| \leq \varphi(0). \tag{21}$$

Theorem 4.2 (Bochner). *A function φ defined on $\hat{\mathbb{R}}$, is a Fourier-Stieltjes transform of a positive measure if and only if it is positive definite and cntinuous.*

Proof. Assume first $\varphi = \hat{\mu}$ with $\mu \geq 0$. Let $\xi_1, \cdots, \xi_N \in \hat{\mathbb{R}}$ and $z_1, \cdots, z_N$ be complex numbers. Then

$$\begin{aligned} \sum_{j,k} \varphi(\xi_i - \xi_j) z_j \overline{z_k} &= \int \sum e^{-i\xi_j x} z_j e^{i\xi_k x} \overline{z_k} \mathrm{d}\mu(x) \\ &= \int \left| \sum_{1}^{N} z_j e^{-i\xi_j x} \right|^2 \mathrm{d}\mu(x) \geq 0. \end{aligned}$$

So the Fourier-Stieltjes transform of a positive measure is positive definite.

Conversely, we assume that φ is positive definite, it follows that for all $\lambda > 0$, $\{\varphi(\lambda n)\}$ is a positive definite sequence (cf. I.7.6). By Herglotz' theorem I.7.6, $\varphi(\lambda n) = \hat{\mu}_\lambda(n)$ for some positive measure μ_λ on $\mathbb{T}$, and by Theorem 3.2, $\varphi = \hat{\mu}$ for some positive $\mu \in M(\mathbb{R})$. □