ALGEBRAIC GEOMETRY
PART I

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Abstract. It is a note when I study algebraic geometry myself in YouTube from th channel ICTP Math, the speaker of the videos is Lothar Göttsche. I only writes notes for first 10 lectures because I can't view YouTube any more for some unknown reason. One day I fix the problem I will maybe finish it.

Contents

1. Affine Varieties

Definition 1 (Zariski Topology)**.** Let S be a set of arbitrary polynomials of $k[x_1, \ldots, x_n]$, k be an algebraic closed field. We define the common zero set of polynomials in S as a closed subset of k^n. The topology defined by this is called Zariski topology and denote the space as $\mathbb{A}^n$.

Remark. In all the conditions of the note, we always take k to be algebraic closed.

Proposition 1. *Let X be a set of $\mathbb{A}^n$ and S be a set of $k[x_1, \ldots, x_n]$, we define*

$$I(X) := \{f \in k[x_1, \ldots, x_n] | f(p) = 0 \text{ for all } p \in \mathbb{A}^n\}$$

and

$$Z(S) := \{p \in \mathbb{A}^n | f(p) = 0 \text{ for all } f \in S\}.$$

Then we have

$$Z(I(X)) = \bar{X}.$$

Proof. First, it is obvious that $X \subset Z(I(X))$. To show the inverse inclusion, assume a closed set Y who satisfies $X \subset Y$, then we have an ideal $\mathfrak{a}$ that satisfies $Y = Z(\mathfrak{a})$. Then $Z(\mathfrak{a}) \supset X$, hence $\mathfrak{a} \subset I(Z(\mathfrak{a})) \subset I(X)$, hence $Y = Z(\mathfrak{a}) \supset Z(I(X))$. □

Proposition 2 (Noetherian Ring)**.** *Let R be a ring,the following are equivalent:*

(1) *Every ideal $I \subset R$ is finitely generated;*

Date: June 19, 2017.

(2) *R satisfies the ascending chain condition: if $I_1 \subset I_2 \subset \dots$ is a chain of ideals, this chain becomes stationary, i.e.,*

$$\exists N, s.t. I_N = I_{N+1} = \dots$$

If R fullfills these properties, it is called noetherian.

Proof. $\Rightarrow$: Let $I_1 \subset I_2 \subset \dots$ be a chain of ideals. Let $I = \cup_{i>0} I_i$, I is an ideal. So by (1) I is finitely generated:

$$I =< f_1, f_2, \dots, f_s >$$

where f_j contained in I_{k_j}. Let $N = \max_j k_j$, we have $I \subset I_N \subset I$, so the chain is stationary.

$\Leftarrow$: Assume a ring $I \subset R$ is not finitely generated, choose an element $f_1 \in I$, $f_2 \in I \backslash \langle f_1 \rangle$, $f_3 \in I \backslash \langle f_1, f_2 \rangle$, ... Then we have a chain who is not stationary:

$$\langle f_1 \rangle \subsetneqq \langle f_1, f_2 \rangle \subsetneqq \cdots$$

□

Theorem 1 (Hilbert Base Theorem)**.** *R is a noetherian ring $\Rightarrow$ The polynomial ring $R[x_1, x_2, \dots, x_n]$ is a noetherian ring.*

Proof. Since $R[x_1, x_2, \dots, x_n] = R[x_1, x_2, \dots, x_{n-1}][x_n]$, we only need to prove $R[x]$ is a noetherian ring for a noetherian ring R. Assume $R[x]$ is not noetherian, let I be an ideal which is not finitely generated. Choose $f_1 \in I \backslash \{0\}$, $f_2 \in I \backslash \langle f_1 \rangle$, ..., $f_{i+1} \in I \backslash \langle f_1, \dots, f_i \rangle$, s.t. the degree of $f_i \in I \backslash \langle f_1, \dots, f_{i-1} \rangle$ is minimal. Let $n_i := deg(f_i)$, a_i the leading coefficient of f_i. Then we have $n_1 \le n_2 \le \dots$ and an ascending chain

$$\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \dots \tag{1.1}$$

of ideals in R. If it is stationary, then for some k we have

$$\langle a_1, \dots, a_k \rangle = \langle a_1, \dots, a_k, a_{k+1} \rangle$$

This implies $a_{k+1} \in \langle a_1, \dots, a_k \rangle$. So we can write

$$a_{k+1} = \sum_{i=1}^{k} b_i a_i b_i \in R \tag{1.2}$$

Let $g := f_{k+1} - \sum_{i=1}^{k} b_i x^{n_{k+1} - n_i} f_i$, then$g \in I \backslash \langle f_1, \dots, f_k \rangle$(otherwise $f_{k+1} = g + \sum_{i+1}^{k} b_1 x^{n_{k+1} - n_i} f_i$ would be in $\langle f_1, \dots, f_k \rangle$). The sum of the leading term in the right hand side is

$$a_{k+1} - \sum_{i=1}^{k} b_i a_i = 0.$$

It shows that $deg(g) < n_{k+1}$, which contradicts to the choose of f_i, so the chain 1.1 is not stationary, the ring R is not noetherian. □

Corollary 1. *Every affine algebraic set $X \subset \mathbb{A}^n$ is the zero set of finite algebraic polynomials.*

Proof. Every affine algebraic set is the zero set of some polynomial set S, i.e. $Z(S)$. Since $Z(S) = Z(\rangle S \langle)$, it is a zero set of an ideal, we choose the generators of the ideal, name T, then $Z(S) = Z(T)$. □

Definition 2. A topological space X is reducible if $X = X_1 \cup X_2$, where X_1,X_2 are closed subsets and $X_1 \subsetneqq X, X_2 \subsetneqq X$. X is called irreducible if it is not reducible,i.e., if $X = X_1 \cup X_2, X_i \subset$ is closed for $i = 1, 2$, then we have $X = X_1$ or $X = X_2$.

Remark. When we talk about whether a set is irreducible, it refers to its induced topology from the space where the set is on.

(1) Let X be irreducible, $\emptyset \neq U \subset X$, U is an open subset of X. Then U is dense in X. Because if it is not dense, we can write $X = (X \backslash U) \cup \overline{U}$, so X is not irreducible.
(2) U itself is also irreducible.

Definition 3. A topological space is called noetherian if every descending chain:$X \supset X_1 \supset X_2 \supset \dots$ of closed subsets becomes stationary(i.e., $X_N = X_{N+1} = \dots$ for some $N \in \mathbb{N}^+$).

Proposition 3. *Any subspace Y of noetherian topological space X is noetherian.*

Proof. Assume $Y \supset Y_1 \supset Y_2 \supset \dots$ a chain of closed subsets. Then $\forall i, Y_i = Y \cap X_i, X_i \subset X$ is closed. Let $X_i' = \cap_{1 \leq j \leq i} X_j$, $X_i' \cap Y = Y_i$. Then $X \supset X_1' \supset X_2' \supset \dots$ is a descending chain. Since X is noetherian, $\exists N$ s.t. $X_N' = X_{N+1}' = \dots$. It follows $Y_N = Y_{N+1} = \dots$. Thus $Y \supset Y_1 \supset Y_2 \supset \dots$ is stationary. □

Proposition 4. *$\mathbb{A}^n$ is noetherian topological space.*

Proof. Let $\mathbb{A}^n = X \supset X_1 \supset X_2 \supset \dots$ be a chain of closed subsets. Then we have $I(X_1) \subset I(X_2) \subset \dots$. Since $k[x_1, x_2, \dots, x_n]$ is noetherian, $\exists N, I(X_N) = I(X_{N+1}) = \dots$. Note that$X_i = Z(I(X_i))$, we get $X_N = X_{N+1} = \dots$. It shows that $\mathbb{A}^n$ is a noetherian topological space. □

Theorem 2. *Let X be a noetherian topological space.*

(1) *X is a union of finitely many irreducible closed subsets: $X = X_1 \cup \dots \cup X_r$;*
(2) *If we require $X_i \not\subset X_j$ for $i \neq j$, then this decomposition is unique.*

Proof. (1) Assume X does not have a decomposition with finitely many closed subsets. In particular, X is reducible: $X = X_1 \cup Y_1$, X_1, Y_1 are closed subsets. so one of the two sets does not have decomposition, say X_1. Repeat the argument we ge a descending chain

$$X \supsetneqq X_1 \supsetneqq X_2 \supsetneqq \dots$$

which is not stationary, it contradicts our existing condition.
(2)Let $X = X_1 \cup \dots \cup X_t = Y_1 \cup \dots \cup Y_s$. Then we have $X_i = \bigcup\limits_{j=1}^{s} (X_i \cap Y_j)$. Since X_i is irreducible, $\exists j, X_i = X_i \cap Y_j$, thus $X_i \subset Y_j$.Similarly, we can get $Y_j \subset X_k$ for some k. Then we have $X_i \subset X_k$, it implies $i = k$ and thus $X_i = Y_j$. So we get the conclusion: each X_i is equal to some Y_j and each Y_j is equal to some X_i. So $r = s$ and the Y_j's are permutations of X_i's. □

Definition 4. An affine variety is an irreducible affine algebraic set.

Proposition 5. *$X \subset \mathbb{A}^n$ is an affine algebraic set. Then we have the following equivalent relations:*

(1) *X is irreducible;*
(2) *$I(X)$ is a prime ideal.*

Proof. (1)$\Rightarrow$ (2): let X be irreducible, f, g some polynomials s.t. $fg \in I(X)$. Then we have $X \subset Z(fg) = Z(f) \cup Z(g)$, hence $X = (X \cap Z(f)) \cup (X \cap Z(g))$. Since X is irreducible, we get $X = X \cap Z(f)$ or $X = X \cap Z(g)$, so $X \subset Z(f)$ or $X \subset Z(g)$, i.e. $f \in I(X)$ or $g \in I(X)$.

(2)$\Leftarrow$ (1): Assume X is reducible, then we have $X = X_1 \cup X_2$ and $X_i \subsetneqq X$ are closed subsets. Since $Z(I(X_i)) = X_i \subsetneqq X = Z(I(X))$, we get $I(X_i) \supsetneqq I(X)$. Let $f \in I(X_1)\backslash I(X)$, $g \in I(X_2)\backslash I(X_2)$, fg vanishes on $X_1 \cup X_2 = X$, then $fg \in I(X)$, i.e., $I(X)$ is not prime. □

Example 1. $\mathbb{A}^n$ is irreducible because $I(\mathbb{A}^n) = \{0\}$ is a prime ideal.

Definition 5. Let $X \neq \emptyset$ be an irreducible topological space. The dimension of X is the largest $n \in \mathbb{Z}$ s.t. there is an ascending chain

$$\emptyset \neq X_0 \subsetneqq X_1 \subsetneqq X_2 \subsetneqq \cdots \subsetneqq X_n = X$$

of irreducible closed subsets. If X is a noetherian topological space then

$$\dim X = \text{ maximum of dimension of irreducible components of } X.$$

Remark. (1) The point $p \in \mathbb{A}^n$ has dimension 0;
(2) $\mathbb{A}^1$ has dimension 1;
(3) In the moment, we still cannot prove but true is $\dim \mathbb{A}^n = n$

It is easy to verify $\dim \mathbb{A}^n \geq n$ because we have a chain:

$$\{(0, 0, \ldots, 0)\} \subsetneqq Z(x_2, x_3, \ldots, x_n) \subsetneqq Z(x_3, \ldots, x_n) \subsetneqq \cdots \subsetneqq Z(x_n) \subsetneqq \mathbb{A}^n.$$

Theorem 3 (The Weak Form Hilbert's Nullstellensatz)**.** *Let $\mathfrak{a} \subsetneqq k[x_1, \ldots, x_n]$ be a proper ideal, then $Z(\mathfrak{a}) \neq \emptyset$*

Remark. We usually use the following form:

$$\mathfrak{a} \subset k[x_1, \ldots, x_n] \text{ and } Z(\mathfrak{a}) = \emptyset \Rightarrow 1 \in I.$$

It is true when k is algebraically closed, otherwise the theorem 3 is wrong:

$$\mathfrak{a} = \langle x^2 + 1 \rangle \in \mathbb{R}[x], Z(\mathfrak{a}) = \emptyset.$$

Definition 6. Let $\mathfrak{a}$ be an ideal in a ring R. The radical of $\mathfrak{a}$ is

$$\sqrt{\mathfrak{a}} = \{r \in R | \exists n > 0, r^n \in \mathfrak{a}\}.$$

$\sqrt{\mathfrak{a}}$ is an ideal in R. $\mathfrak{a}$ is called radical ideal if $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

Remark. If $X \subset \mathbb{A}^n$ is an affine algebraic set, then $I(X)$ is a radical ideal.

Theorem 4 (Nullstallensatz)**.** *Let $\mathfrak{a} \subset k[x_1, \ldots, x_n]$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Definition 7. R is an integral domain, the quotient field $Q(R)$ is the set of equivalent classes of pairs $(f, g), f, g \in R, g \neq 0$, which satisfy the equivalent relation

$$(f, g) \cong (f', g') \Leftrightarrow fg' - f'g = 0.$$

We denote it by $\frac{f}{g}$.

Remark. $Q(R)$ is a field. We always identify $r \in R$ with $\frac{r}{1} \in Q(R)$, then we can say R is the subring of $Q(R)$.$Q(k[x_1, \ldots, x_n]) := k(x_1, \ldots, x_n)$ is called field of rational functions in $x_1, x_2, \ldots, x_n$.

Proof of Nullstellensatz. Let $\mathfrak{a} = \langle f_1, \dots, f_r \rangle, f_i \in \mathfrak{a}$. Then $I(Z(\mathfrak{a}))$ is a radical ideal containing $\mathfrak{a}$, so we get

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

Let$f \in I(Z(\mathfrak{a}))$. To show $\exists N > 0$, s.t. $f^N \in \mathfrak{a}$, we use the weak Nullstellensatz in $k[x_1, \dots, x_n]$.

Let

$$\mathfrak{b} := \langle f_1, \dots, f_r, f \cdot t - 1 \rangle \subset k[x_1, \dots, x_n, t] \tag{1.3}$$

Let $(p, a) \in \mathbb{A}^{n+1}, p \in \mathbb{A}^n, a \in k$.

$$(p, a) \in Z(\mathfrak{b}) \Leftrightarrow f_1(p) = \cdots = f_r(p) = 0 \text{ and } f(p) \cdot a = 1.$$

But $f(p) = 0$, so we know $Z(\mathfrak{b}) = \emptyset$. By the weak Nullstellensatz, $1 \in \mathfrak{b}$, we can write

$$1 = g_0 \cdot (ft - 1) + \sum_{i=1}^{r} g_i \cdot f_i \tag{1.4}$$

Back to $k[x_1, \dots, x_n]$ in $k(x_1, \dots, x_n)$, define homomorphism:

$$\varphi : k[x_1, \dots, x_n, t] \to k(x_1, \dots, x_n)$$
$$g(x_1, \dots, x_n, t) \to g(x_1, \dots, x_n, \frac{1}{f})$$

Use φ to equation 1.4 we get

$$1 = \sum_{i=1}^{r} \varphi(g_i) \cdot f_i \tag{1.5}$$

where $\varphi(g_i) = \frac{G_i}{f^{n_i}}, G_i \in k[x_1, \dots, x_n]$. Let $N := \max\limits_{1 \le i \le r} n_i$, multiply equation 1.5 by f^N:

$$f^N = \sum_{i=1}^{r} G_i f^{N-n_i} \cdot f_i \in \mathfrak{a} \tag{1.6}$$

Corollary 2. (1) *If $\mathfrak{a} \subset k[x_1, \dots, x_n]$ is a prime ideal, then $Z(\mathfrak{a})$ is irreducible;*
(2) *If $f \in k[x_1, \dots, x_n]$ is irreducible, then $Z(f)$ is irreducible.*

Proof. (1) Set$X = Z(\mathfrak{a})$. Prime ideals are radical, so we get $I(X) = \mathfrak{a}$ and $\mathfrak{a}$ is prime, use proposition 5 we know that X is irreducible.

(2) Since $k[x_1, \dots, x_n]$ is a UFD, we get

$$f \in k[x_1, \dots, x_n] \text{ is irreducible} \Rightarrow \langle f \rangle \text{ is a prime ideal.}$$

So $Z(f) = Z(\langle f \rangle)$ is irreducible. □

2. Projective Varieties

Definition 8. Define an equivalence relation$\sim$ in $k^{n+1} \backslash \{0\}$:

$$(a_0, \dots, a_n) \sim (b_0, \dots, b_n) \Leftrightarrow \exists \lambda \in k \backslash \{0\} \text{ s.t.} (a_0, \dots, a_n) = (\lambda b_0, \dots, \lambda b_n).$$

Then we call $k^{n+1} \backslash \{0\}$ with this relation the projective n-space and write it as $(k^{n+1} \backslash \{0\}) / \sim = \mathbb{P}^n$.

Definition 9. Let $U_i := \{[a_0, \ldots, a_n] \in \mathbb{P}^n | a_i \neq 0\}$. $\varphi_i : U_i \to \mathbb{A}^n$, $[a_0, \ldots, a_n] \to (\frac{a_0}{a_i}, \ldots, \hat{\frac{a_i}{a_i}}, \ldots, \frac{a_n}{a_i})$ is a projection, write inverse $u_i : \mathbb{A}^n \to U_i$, $(b_0, \ldots, \hat{b_i}, \ldots, b_n) \to [b_0, \ldots, 1, \ldots, b_n]$.

Usually we fix $i = 0$, view $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$ by identify the point $(a_1, \ldots, a_n) \in \mathbb{A}^n$ with $[1, a_1, \ldots, a_n] \in \mathbb{P}^n$. With this identification we have

$$\mathbb{P}^n = \mathbb{A}^n \cup H_\infty \tag{2.1}$$

where $H_\infty := \{[a_0, \ldots, a_n] \in \mathbb{P}^n | a_0 = 0\}$ is called hyperplane at infinity.

Remark. Define projective algebraic sets are zero sets of polynomials in $k[x_0, \ldots, x_n]$, but $f \in k[x_0, \ldots, x_n]$ does not define a function on $\mathbb{P}^n$:

$$f(a_0, \ldots, a_n) \neq f(\lambda a_0, \ldots, \lambda a_n). \tag{2.2}$$

However if f is homogeneous we can still see whether $p \in \mathbb{P}^n$ is a zero point of f or not. f is homogeneous if

$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n). \tag{2.3}$$

Thus whether $f = 0$ is decided only on $[a_0, \ldots, a_n]$.

Definition 10. Let $g \in k[x_0, \ldots, x_n]$ be homogeneous, apoint $p = [a_0, \ldots, a_n]$ is a zero point of g if $g(a_0, \ldots, a_n) = 0$. Let $S \subset k[x_0, \ldots, x_n]$,

$$Z(S) := \{p \in \mathbb{P}^n | f(p) = 0 \forall f \in S\}. \tag{2.4}$$

A subset of $\mathbb{P}^n$ of the form $Z(S)$ is called a projective algebraic set.

Example 2. (1) $\emptyset = Z(1)$, $\mathbb{P}^n = Z(\emptyset)$;

(2) Any point $p = [a_0, \ldots, a_n] \in \mathbb{P}^n$ is a projective algebraic set

$$\begin{aligned}\{p\} = Z(a_1x_0 - a_0x_1, a_2x_0 - a_0x_2, \ldots, a_nx_0 - a_0x_n,\\ a_2x_1 - a_1x_2, \ldots, a_nx_1 - a_1x_n,\\ \ldots).\end{aligned}$$

Definition 11. A polynomial $f \in k[x_0, \ldots, x_n]$ cab be written uniquely as $f = f^{(0)} + f^{(1)} + \cdots + f^{(d)}$, with $f^{(i)}$ homoegeneous of degree i. $f^{(i)}$ is called homogeneous component if f.

An ideal $\mathfrak{a} \subset k[x_0, \ldots, x_n]$ is called homogeneous if for every $f \in \mathfrak{a}$ all homogeneous components $f^{(i)}$ are in $\mathfrak{a}$.

Proposition 6. *An ideal $\mathfrak{a} \subset k[x_0, \ldots, x_n]$ is homogeneous $\Leftrightarrow$ It is generated by the homogeneous polynomials.*

Proof. $\Rightarrow$: Assume I homogeneous, let $(f_\alpha)_\alpha$ be a set of generators, then $(f_\alpha^{(i)})_{\alpha,i}$ is a set of homogeneous generators.

$\Leftarrow$: Let $\mathfrak{a} = \langle g_i \rangle$ and g_i be homogeneous. Let $f \in \mathfrak{a}$, then we can write

$$f = \sum_i a_i g_i. \tag{2.5}$$

Note g_i is homogeneous, thus the homogeneous part of $a_i g_i$ of degree d is just $a_i^{(d-deg(g_i))} g_i$, so

$$f^{(d)} = \sum_i a_i^{(d-deg(g_i))} g_i. \tag{2.6}$$

Since $g_i \in \mathfrak{a}$ we get $f^{(d)} \in \mathfrak{a}$. ∎

Definition 12. Let $\mathfrak{a} \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal, the zero set of $\mathfrak{a}$ is written as

$$Z(\mathfrak{a}) := \{p \in \mathbb{P}^n | f(p) = 0 \text{ for all homogeneous elements } f \in \mathfrak{a}\}. \tag{2.7}$$

For a subset $X \subset \mathbb{P}^n$, the homogeneous ideal of X is

$$\begin{aligned} I(X) := \text{ } & \text{ideal generated by } \{f \in k[x_0, \ldots, x_n] | f \\ & \text{be homogeneous and } f(p) = 0 \forall p \in X\} \end{aligned} \tag{2.8}$$

By definition this is a homogeneous ideal.

Remark. If $f \in k[x_0, \ldots, x_n]$ is not homogeneous, we can define

$$Z(f) := \{p \in \mathbb{P}^n | f(a_0, \ldots, a_n) = 0 \text{ for all representative } (a_0, \ldots, a_n) \text{ of } p\} \tag{2.9}$$

In fact, if $f = f^{(0)} + f^{(1)} \cdots + f^{(d)}$, then we have

$$Z(f) = \bigcap_{i=0}^{d} Z(f^{(i)}) \tag{2.10}$$

With this property, if $\mathfrak{a} \subset k[x_0, \ldots, x_n]$ is a homogeneous ideal then formula 2.7 can be written as

$$Z(\mathfrak{a}) = \{p \in \mathbb{P}^n | f(p) = 0 \forall f \in \mathfrak{a}\} \tag{2.11}$$

and formula 2.8 can be written as

$$I(X) = \{f \in k[x_0, \ldots, x_n] | f(p) = 0 \forall p \in X\} \tag{2.12}$$

Proposition 7. *Same as an affine space, in a projective space we have the following propositions:*

(1) $X \subset Y \subset \mathbb{P}^n$ *are projective algebraic sets, then* $I(X) \supset I(Y)$*;*
(2) $X \subset \mathbb{P}^n$ *is a projective algebraic set, then* $Z(I(X)) = X$*;*
(3) $\mathfrak{a} \subset k[x_0, \ldots, x_n]$ *is a homogeneous ideal, then* $I(Z(\mathfrak{a})) \supset \mathfrak{a}$*;*
(4) *If* $S \subset k[x_0, \ldots, x_n]$ *is a set of homogeneous polynomials, then* $Z(S) = Z(\langle S \rangle)$*;*
(5) *For a family* $\{S_\alpha\}$ *of sets of homogeneous polynomials,* $Z(\bigcup_\alpha S_\alpha) = \bigcap_a lpha Z(S_\alpha)$*;*
(6) *If* $T, S \subset k[x_0, \ldots, x_n]$ *are sets of homogeneous polynomials, then* $Z(ST) = Z(S) \cup Z(T)$*.*

Remark. From the proposition (5) and (6) we know that arbitrary intersections and finite unions of projective algebraic sets are projective algebraic sets, then we can define a topology through these two propositions.

Definition 13. The Zariski topology on $\mathbb{P}^n$ is the topology whose closed sets are the projective algebraic sets.

If $X \subset \mathbb{P}^n$ is a subset, we give it the induced topology, called Zariski topology on X.

Definition 14. A quasi-projective algebraic set is an open subset of a projective algebraic set. Fro example, let U and V be closed subsets, then $Y = U \backslash V \neq \emptyset$ is a quasi-projective algebraic set.

Proposition 8. *We jnow* $k[x_0, \ldots, x_n]$ *is noetherian, then follows the same proof as in affine case shows that* $\mathbb{P}^n$ *is a noetherian topological sapce.*

Remark. Every subspace of $\mathbb{P}^n$ is noetherian. In particular, quasi-projective algebraic sets are noetherian, hence have unique decompositions into irreducible components.

Definition 15. A quasi-projective variety is an irreducible quasi-projective algebraic set.

Remark. If we use the identification $\mathbb{A}^n = U_0 \subset \mathbb{P}^n$, then $\mathbb{A}^n$ is an open set $\mathbb{A}^n = \mathbb{P}^n \backslash Z(x_0)$, i.e. $\mathbb{A}^n$ is a quasi-projective variety.

Definition 16. A nonempty algebraic set $X \subset \mathbb{A}^{n+1}$ is called a cone if for all $p = (a_0, \dots, a_n) \in X$ and all $\lambda \in k$, we have $(\lambda a_0, \dots, \lambda a_n) = \lambda p \in X$.

If $X \subset \mathbb{P}^n$ is a projective algebraic set, its affine cone is

$$C(X) := \{(a_0, \dots, a_n) \in \mathbb{A}^{n+1} | [a_0, \dots, a_n] \in X\} \cup \{0\} \tag{2.13}$$

Lemma 1. *Let $X \neq \emptyset$ be a projective algebraic set, then :*

(1) *$X = Z_p(\mathfrak{a})$, for $\mathfrak{a} \subset k[x_0, \dots, x_n]$ a homogeneous ideal $\Rightarrow C(X) = Z_a(\mathfrak{a}) \subset \mathbb{A}^{n+1}$;*

(2) *$I_a(C(X)) = I_H(X)$.*

Theorem 5 (Projective Nullstellensatz)**.** *Let $\mathfrak{a} \subset k[x_0, \dots, x_n]$ be a homogeneous ideal:*

(1) *$Z_p(\mathfrak{a}) = \emptyset \Leftrightarrow \mathfrak{a}$ contains all homogeneous polynomials of degree N for some $N \in \mathbb{N}$;*

(2) *If $Z_p(\mathfrak{a}) \neq \emptyset$, then $I_p(Z_p(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Proof. Let $X = Z_p(\mathfrak{a})$.

(1) $X = \emptyset \Leftrightarrow C(X) = \{0\}$. Since $C(X) = Z_a(\mathfrak{a}) \cup \{0\}$, we get

$$X = \emptyset \Leftrightarrow Z_a(\mathfrak{a}) = \emptyset \text{ or } Z_a(\mathfrak{a}) = \{0\}.$$

By affine Nullstellensatz, we get

$$\sqrt{\mathfrak{a}} = k[x_0, \dots, x_n] \text{ or } \sqrt{\mathfrak{a}} = \langle x_0, \dots, x_n \rangle.$$

So $\sqrt{\mathfrak{a}} \supset \langle x_0, \dots, x_n \rangle$. Thus for any $i = 0, \dots, n$, $\exists m_i$ s.t. $x_i^{m_i} \in \mathfrak{a}$. Let $N = m_1 + \cdots + m_n$, then any monomial of degree N in $k[x_0, \dots, x_n]$ lies in $\mathfrak{a}$.

(2)Let $X = Z_p(\mathfrak{a} \neq \emptyset$, then

$$I_H(X) = I_a(C(X)) = I_a(Z_a(\mathfrak{a})) = \sqrt{\mathfrak{a}}. \tag{2.14}$$

□

Remark. $\langle x_0, \dots, x_n \rangle$ is called the irrelevant ideal, an ideal different from $\langle x_0, \dots, x_n \rangle$ is called relevant.

Corollary 3. *There is a one-to-one correspondence between homogeneous relevant radical ideals and projective algebraic sets:*

Z_p: homogeneous relevant radical ideals in $k[x_0, \dots, x_n] \to$ projective algebraic sets in $\mathbb{P}^n$

I_H: projective algebraic sets in $\mathbb{P}^n \to$ homogeneous relevant radical ideals in $k[x_0, \dots, x_n]$.

Remark. We use subscripts to recognize affine spaces and projective spaces, such as $Z_p(\mathfrak{a}), Z_a(\mathfrak{a})$. Sometimes we can infer the difference from the context, so we usually write briefly as $Z(\mathfrak{a})$.

Proposition 9. (1) *A projective algebraic set $X \neq \emptyset \subset \mathbb{P}^n$ is irreducible if and only if $I = I_H(X)$ is a homogeneous prime ideal;*
(2) *If $f \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial and irreducible, then $Z_p(f)$ is irreducible.*

Proof. (1) $\Leftarrow$: Assume X reducible, then $X = X_1 \cup X_2$,$X_1, X_2 \subsetneqq X$b are closed subsets. Then we get $C(X) = C(X_1) \cup C(X_2)$, $C(X_1) \subsetneqq C(X)$,$C(X_2) \subsetneqq C(X)$ are closed, hence $C(X)$ is reducible, $I_H(X) = I(C(X))$ is not prime.

$\Rightarrow$: Assume $I_H(X)$ not prime, it means $\exists f, g \in k[x_0, \ldots, x_n]$, $fg \in I_H(X)$ and $f, g \notin I_H(X)$. Let $i, j \in \mathbb{Z} \geq 0$ be minimal such that $f^{(i)} \notin I$ and $g^{(j)} \notin I$. Subtract homogeneous components of lower degrees from f and g, we can assume f starts in degree i and g starts in degree j. Thus $f^{(i)}g^{(j)}$ is homogeneous component of minimal degree in $fg \in I$. Because I is homogeneous, we get $f^{(i)}g^{(j)} \in I$. Let $X_1 := Z(I) \cap Z(f^(i))$ and $X_2 := Z(I) \cap Z(g^{(j)})$, then $X_1, X_2 \subsetneqq X$, $X = X_1 \cup X_2$, thus X is reducible.

(2) If $I \subset k[x_0, \ldots, x_n]$ is homogeneous and prime with $Z(I) \neq \emptyset$, then follow the result from (1) we know $Z(f)$ is irreducible. $\square$

3. Functions And Morphisms

Definition 17. Let $X \subset \mathbb{A}^n$ be an affine algebraic set, the affine coordinate ring of X is

$$A(X) := k[x_0, \ldots, x_n]/I(X). \tag{3.1}$$

It is a ring, also a k-algebra.

Definition 18. A polynomial function on X is a function $f : X \to k$ s.t. $f = F|_X$ for $F \in k[x_0, \ldots, x_n]$. This is the ring with pointwise addition and multiplication:

$$(f + g)(p) = f(p) + g(p), fg(p) = f(p)g(p), \forall p \in X.$$

There is a ring homomorphism:

$$\begin{gathered} k[x_0, \ldots, x_n] \to \{\text{polynomial functions on } X\} \\ F \to F|_X \end{gathered}$$

It is surjective and its kernel is $I(X)$. Thus we have the isomorphism:

$$A(X) \cong \{\text{polynomial functions on } X\}.$$

We will not distinguish them.

Remark. The zero set of a polynomial function is closed.Let X be an affine algebraic set, $f \in A(X)$, then

$$Z(f) = \{p \in X | f(p) = 0\} \tag{3.2}$$

is closed in X. $f \in A(X)$ means $f = F|_X$ for some $F \in k[x_1, \ldots, x_n]$, then

$$Z(f) = \{p \in X | F(p) = 0\} = X \cap Z(F) \tag{3.3}$$

so it is closed.

Definition 19. Let X be an affine variety, then $I(X)$ is prime, then $A(X)$ is integral. The quotient field $Q(A(X))$ is a field of rational functions on X and denoted by $K(X)$. Let $V \subset X$ be a quasi-affine variety, since $I(V) = I(X)$, we can denote its field of rational functions by $K(V) := K(X)$.

Definition 20. Let $p \in V$, the local ring of V at p is

$$\mathcal{O}_{V,p} := \{h \in K(V) | \exists f, g \in A(V), \text{ s.t. } h = \frac{f}{g} \text{ and } g(p) \neq 0\} \tag{3.4}$$

For simplicity in future we can write this:

$$\mathcal{O}_{V,p} = \{\frac{f}{g} \in K(V) | g(p) \neq 0\}. \tag{3.5}$$

If $U \subset V$ is an open subset, the regular functions on U are defined by

$$\mathcal{O}_V(U) = \underset{V,p}{\cap} \subset K(V). \tag{3.6}$$

Proposition 10. *We have an injective ring homomorphism:*

$$\mathcal{O}_V(U) \to \{\textit{functions from } U \textit{ to } k\}.$$

For $h \in \mathcal{O}_V(U), p \in U$, there exists an open subset W and $p \in W \subset U$, s.t. $h = \frac{f}{g}$ with $g(p) \neq 0$. We define the homomorphism by setting $h(p) = \frac{f(p)}{g(p)}$, the homomorphism is

$$h \in \mathcal{O}_V(U) \to h(p) = \tfrac{f(p)}{g(p)}, p \in U.$$

Proof. It is well defined: if $h = \frac{f}{g} = \frac{f'}{g'}$ with $g(p) \neq 0, g'(p) \neq 0$.Then $fg' = f'g \Rightarrow f(p)g'(p) = f'(p)g(p) \Rightarrow \frac{f(p)}{g(p)} = \frac{f'(p)}{g'(p)}$.

Injective: Let $h, h' \in \mathcal{O}_V(U)$ such that $h(p) = h'(p) \forall p \in U$. Define $l = h - h' \in \mathcal{O}_V(U)$, then $l(p) = 0, \forall p \in U$. There exists an open subset W, s.t. $l = \frac{f}{g}$ with $g(p) \neq 0 \forall p \in W$. For $p \in W$, $l(p) = \frac{f(p)}{g(p)} = 0 \Rightarrow f(p) = 0 \forall p \in W$.As zero set $Z(f)$ of f is closed, we get $f = 0 \in A(V)$, then $l = 0$ and hence $h = h'$. ☐

Remark. We had called $\mathcal{O}_{V,p}$ a local ring of V at p. The maximal ideal at p is $\mathfrak{m}(p) := \{h \in \mathcal{O}_{V,p} | h(p) = 0\}$, this is a maximal ideal in $\mathcal{O}_{V,p}$. It is easy to verify that the local ring of a variety is alocal ring.

Proposition 11. *For an affine variety X, functions which are regular functions everywhere are polynomial functions, i.e., $\mathcal{O}_X(X) = A(X)$.*

Proof. Obviously, $A(X) \subset \mathcal{O}_X(X)$. We have to show the other inclusion. Let $h \in \mathcal{O}_X(X)$, $\forall p \in X$, $\exists F_p, G_p \in k[x_1, \ldots, x_n]$ s.t. $h = \frac{[F_p]}{[G_p]}$ and $G_p(p) \neq 0$. It is equivalent to: $\forall p \in X$, $\exists G_p \in k[x_1, \ldots, x_n]$ s.t. $h \cdot [G_p] \in A(X)$ and $[G_p(p)] \neq 0$. Let

$$\mathcal{G} := \{G \in k[x_1, \ldots, x_n] | h \cdot [G_p] \in A(X)\} \tag{3.7}$$

$\mathcal{G}$ is an ideal and $\mathcal{G} \supset I(X)$, so $Z(\mathcal{G}) \subset X$. But $Z(\mathcal{G}) \cap X = \emptyset$, so $Z(\mathcal{G}) = \emptyset$. By Nullstellensatz $1 \in \mathcal{G}$, so $h = h \cdot 1 \in A(X)$. ☐

Definition 21. Let $X \subset \mathbb{P}^n$ be a projective algebraic set. The homogeneous coordinate ring of X is defined as

$$S(X) := k[x_0, \ldots, x_n]/I_H(X) \tag{3.8}$$

If X is irreducible, then $S(X)$ is an integral domain, $Q(S(X))$ is its quotient field.

Remark. $X \subset \mathbb{P}^n$ is a quasi-projective variety, then polynomial $F \in k[x_0, \ldots, x_n]$ will not define a function $X \to k$. But we can take quotients of homogeneous polynomials of the same degree and get a well defined function.

Definition 22. Let $f = [F] \in S(X), F \in k[x_0, \ldots, x_n]$. The homogeneous part $f^{(d)}$ of f is $[F^{(d)}] \in S(X)$, and $S^{(d)}(X) = \{f^{(d)} \in S(X)\}$.

Definition 23. X is a quasi-projective variety, the field of rational functions on X(on $V \subset X$ open subset) is $K(V) := K(X) := \{\frac{f}{g} \in Q(S(X)) | f, g$ both in $S^{(d)}(X)$ for some d$\}$. Elements of $K(X)(K(V))$ are called rational functions on X(on V).

Definition 24. Let $p \in V \subset \mathbb{P}^n$, the local ring of V at p is

$$\mathcal{O}_{V,p} := \{\frac{f}{g} \in K(V) | g(p) \neq 0\}. \tag{3.9}$$

If $U \subset V$ is open, the ring of regular functions on U is

$$\mathcal{O}_V(U) := \bigcap_{p \in U} \mathcal{O}_{V,p}. \tag{3.10}$$

Proposition 12. (1) *(k-algebra)Constant functions $a \in k$ are regular on U. If $f, g \in \mathcal{O}_V(U)$, then $f + g$ and fg are regular on U, and if g has no zero in U, then $\frac{f}{g} \in \mathcal{O}_V(U)$.*
(2) *(Local)Let (U_i) be a open cover of U. A function $f : U \to k$ is regular if and only if $f|_{U_i}$ is regular for all i.*
(3) *Regular functions are continuous. i.e., let $h \in \mathcal{O}_V(U)$, then $h : U \to k = \mathbb{A}^1$ is continuous($k = \mathbb{A}^1$ is given Zariski topology).*

Proof. (1) By definition, $\mathcal{O}_V(U) = \bigcap_{p \in U} \mathcal{O}_{V,p}$, thus enough to show if $f, g \in \mathcal{O}_{V,p}$, then $f + g, fg \in \mathcal{O}_{V,p}$, and it is obvious. Assume g has no zero on U, then $g\frac{1}{g} \in \mathcal{O}_V(U)$, then $\frac{f}{g} \in \mathcal{O}_V(U)$.

(2) $h : U \to k$ is regular $\Leftrightarrow h \in \mathcal{O}_{V,p} \forall p \in U \Leftrightarrow h \in \mathcal{O}_{V,p} \forall p \in U_i \forall i$.

(3) $h : U \to k$ is continuous $\Leftrightarrow h|_{U_i}$ is continuous for all U_i of an open cover of U. We just replace U by a suitable U_i and show h is continuous in U_i. From the definition of regular functions, we can simply assume $h = \frac{f}{g}, f, g \in k[x_0, \ldots, x_n]$ are homogeneous of the same degree, and g has no zero on U_i. Zariski topology on $\mathbb{A}^1$ has closed subsets $\emptyset, k$ and finite points subsets. Thus we only have to show $h^{-1}(a)$ is closed in U_i for all a in k,

$$h^{-1}(a) = \{p \in U_i | h(p) = a\} = \{p \in U_i | (f - ag)(p) = 0\}. \tag{3.11}$$

This is the zero set $Z(f - ag) \cap U$, hence the inverse of the closed sets are closed, hence h is continuous in $U_i \forall i$, hence continuous in U. □

Definition 25 (Polynomial Map)**.** Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be affine algebraic sets. A map

$$(F_1, \ldots, F_m) : X \to Y, p \to (F_1(p), \ldots, F_m(p)), F_1, \ldots, F_m \in k[x_1, \ldots, x_n]$$

is called a polynomial map. A surjective polynomial map whose inverse is also a polynomial map is an isomorphism.

Example 3. (1) If X is an affine algebraic set, the polynomial map $f : X \to k$ is the polynomial function in $A(X)$.
(2) Let $X = \mathbb{A}^1$, $Y = Z(y - x^2) \subset \mathbb{A}^2$, the polynomial map

$$(t, t^2) : \mathbb{A}^2 \to Y$$

is isomorphism.

Definition 26. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine algebraic sets. Let

$$\varphi : X \to Y$$

be a polynomial map. The pull back of $h \in A(Y)$ is $\varphi^* h := h \circ \varphi \in A(X)$. If $h = H|_Y, H \in k[y_1, \dots, y_m]$, $\varphi = (F_1, \dots, F_m)$, then

$$\varphi^* h(a_1, \dots, a_n) = h(F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)).$$

i.e.,

$$\varphi^* h = H(F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n))|_X \in A(X).$$

The pull back $\varphi^* : A(Y) \to A(X)$ is obviously a ring homomorphism. If $\varphi : X \to Y$ is an isomorphism, then $\varphi^* : A(Y) \to A(X)$ is an isomorphism of k-algebra.

Definition 27. Let X, Y be varieties, a map $\varphi : X \to Y$ is a morphism(regular map) if :

(1) φ is continuous;
(2) for all open subsets $U \in Y$, all regular functions $f \in \mathcal{O}_Y(U)$, we have

$$\varphi^* := f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(U)).$$

Remark. Thus for each open subset $U \in Y$,

$$\varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$$

is a k-algebra homomorphism. φ is called an isomorphism if φ is bijective and φ^{-1} is also a morphism.

(1) id_X is a morphism form X itself.
(2) If $\varphi : X \to Y, \psi : Y \to Z$ are morphisms, then

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$$

.

(3) If $\varphi : X \to Y$ is isomorphism, then $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism for all $U \subset Y$.

Proposition 13. (1) *Let $\varphi : X \to Y$ and $(U_i)_{i\in I}$ be an open cover of X s.t. $\varphi|_{U_i} : U_i \to Y$ is a morphism. Then φ is a morphism.*
(2) *Let $Z \subset X, W \subset Y$ be varieties, let $\varphi : X \to Y$ be a morphism with $\varphi(Z) \subset W$. Then $\varphi|_Z : Z \to W$ is a morphism.*

Proof. (1) Let $W \subset Y$ be open, then we can write $\varphi^{-1}(W) = \underset{i\in I}{\cup} (\varphi|_{U_i}^{-1}(W))$, it is open so φ is continuous. Let $h \in \mathcal{O}_Y(W)$ then the pull back of regular functions h from $\mathcal{O}_Y(W)$ to $\mathcal{O}_X(U_i \cap \varphi^{-1}(W))$ is $\varphi|_{U_i}^* h = \varphi^* h|_{U_i \cap \varphi^{-1}(W)}$, since $\varphi|_{U_i}$ is a morphism we get that $U_i \cap \varphi^{-1}(W)$ is open. Then

$$\varphi^{-1}(W) = \underset{i\in I}{\cup} U_i \cap \varphi^{-1}(W) \tag{3.12}$$

and $(U_i \cap \varphi^{-1}(W))_{i\in I}$ is an open cover of $\varphi^{-1}(W)$, then we can get the conclusion that φ is a morphism by proposition 12.

(2) First, $\varphi|_Z$ is continuous as a restriction of a continuous map. Let $U \subset W$ be open, let $h \in \mathcal{O}_W(U)$. Replace if necessary U by a smaller open subset sucht that we can assume $h = \frac{F}{G}$. This quotient also defines a regular function H on open subset $\tilde{U} \subset Y$ s.t. $U \subset \tilde{U}$, then $\varphi^* H \in \mathcal{O}_X(\varphi^{-1}(\tilde{U}))$ is regular. Then $\varphi^* h = \varphi^* H|_{\varphi^{-1}(U)\cap Z}$ is regular on $\varphi^{-1}(U) \cap Z$. □

Definition 28. An affine variety is a variety which is isomorphis to irreducible closed subset of some $\mathbb{A}^n$.

Theorem 6. *Let X, Y be subvarieties, assume $Y \subset \mathbb{A}^n$. A map $\varphi : X \to Y$ is a morphism if and only if $\exists f_1, \ldots, f_n \in \mathcal{O}_X(X)$ s.t.*

$$\varphi(p) = (f_1(p), \ldots, f_n(p)), \forall p \in X. \tag{3.13}$$

We can write $\varphi = (f_1, \ldots, f_n)$.

Proof. $\Rightarrow$: Let $\varphi : X \to Y$ be a morphism. Let $y_1, \ldots, y_n \in \mathcal{O}_Y(Y)$ be restrictions of the coordinates on $\mathbb{A}^n$ to Y, i.e., if $q = (a_1, \ldots, a_n) \in Y$, then $a_i = y_i(q)$. The pull back of y_i is

$$f_i := \varphi^* y_i = y_i \circ \varphi \in \mathcal{O}_X(X). \tag{3.14}$$

Let $p \in X$, $\varphi(p) = (b_1, \ldots, b_n)$, $b_i = y_i(\varphi(p)) = f_i(p)$, thus

$$\varphi = (f_1, \ldots, f_n)$$

where $f_i \in \mathcal{O}_X(X)$.

$\Leftarrow$ Let $\varphi := (f_1, \ldots, f_n), f_i \in \mathcal{O}_X(X)$. First we show φ is continuous. Let $B \in Y$ be closed, it is equivalent to $B = Y \cap Z(G_1, \ldots, G_m)$ and $G_i \in k[x_1, \ldots, x_n]$. Since $G_i \circ \varphi = G(f_1, \ldots, f_n) \in \mathcal{O}_X(X)$, we get $\varphi^{-1}(B) = Z(G_1 \circ \varphi, \ldots, G_m \circ \varphi)$ and it is closed in X. So φ is continuous. Let $h \in \mathcal{O}_Y(U)$, write $W = \varphi^{-1}(U) \subset Y$. we need to show $h \circ \varphi \in \mathcal{O}_X(W)$. We can always make U smaller and assume $h(q) = \frac{F(q)}{G(q)}, \forall q \in U$, F and G are some polynomials and G has no zero on U. Then we have

$$h \circ \varphi = \frac{F \circ \varphi}{G \circ \varphi} = \frac{F(f_1, \ldots, f_n)}{G(f_1, \ldots, f_n)} \tag{3.15}$$

where $F(f_1, \ldots, f_n)$ and $G(f_1, \ldots, f_n)$ are regular on $\mathcal{O}_X(W)$. Since $\varphi(W) = U$ and G has no zero on U, $G(f_1, \ldots, f_n)$ also has no zero on W, i.e., $h \circ \varphi \in \mathcal{O}_X(W)$. □

Remark. The regular functions on a variety X are the same as the morphisms $X \to \mathbb{A}^1$.

Corollary 4. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be closed subvarieties. The morphisms*

$$\varphi : X \to Y$$

are precisely the polynomial map.

Proof. From theorem 6 we know $\varphi = (f_1, \ldots, f_m)$ and $f_i \in \mathcal{O}_X(X) \forall i$. From theorem 11 we know $f_i \in A(X)$, so φ is a polynomial map. □

Theorem 7. *Let X, Y be varieties, assume $Y \subset \mathbb{A}^m$ be a closed affine variety. Then there is a bijection between morphisms $X \to Y$ and k-algebra homomorphisms $A(Y) \to \mathcal{O}_X(X)$:*

$$\{\textit{morphisms } X \to Y\} \xrightarrow{\textit{bijection}} \{\textit{homomorphisms } A(Y) \to \mathcal{O}_X(X)\}$$
$$\varphi \longrightarrow \varphi^*$$

Proof. $\Rightarrow$: Let $\varphi : X \to Y$ be a morphism, then $\varphi^* : A(Y) \to \mathcal{O}_X(X)$ is a k-algebra homomorphism by definition 27.

$\Leftarrow$: Let $\phi : A(Y) \to \mathcal{O}_X(X)$ be a k-algebraic homomorphism, let $y_1, \ldots, y_n \in A(Y)$ be the coordinate functions. We set

$$f_i = \phi(y_i) \in \mathcal{O}_X(X).$$

Let $\varphi = (f_1, \dots, f_m) : X \to \mathbb{A}^m$. This is a morphism from X to Y. To see it is a morphism we have to show $\varphi(X) \subset Y$. Let $h \in I(Y)$, $h \circ \varphi = h(f_1, \dots, f_m) = h(\phi(y_1), \dots, \phi(y_m)) = \phi(h(y_1, \dots, y_m))$. The second equality is based on the homomorphic property of ϕ, for example, if $h(x_1, x_2) = x_1^2 - x_2^3$, then $h(\phi(y_1), \phi(y_2)) = \phi(y_1)^2 - \phi(y_2)^3 = \phi(y_1^2) - \phi(y_2^3) = \phi(y_1^2 - y_2^3) = \phi(h(y_1, y_2))$. So $h(y_1, \dots, y_m) \in A(Y)$, we choose an arbitrary element $p = (a_1, \dots, a_m) \in Y$, then $h(y_1, \dots, y_m)(p) = h(a_1, \dots, a_m) = 0$ because $h \in I(Y)$. So for arbitrary $h \in I(Y)$, we get $h \circ \varphi = 0$, it implies $\varphi(X) \subset \cap_{h \in I(Y)} Z(h) = Y$. □

Example 4. A bijective polynomial map need not to be an isomorphism. For example, let$X = \mathbb{A}^1$, $Y = Z(x_2^2 - x_1^3) \subset \mathbb{A}^2$. Then

$$\varphi = (t^2, t^3) : X \to Y$$

is a morphism and bijective and the inverse is

$$\varphi^{-1}(a, b) = \begin{cases} \frac{b}{a} & \text{if } a \neq 0 \\ 0 & \text{if } (a, b) = 0 \end{cases}$$

φ is not an isomorphism(φ^{-1} is not a morphism). To show this we see the pull back:

$$\varphi^* : A(Y) \to \mathcal{O}_X(X)$$

where $A(Y) = k[x_1, x_2]/\langle x_2^2 - x_1^3 \rangle$ and $A(X) = k[t]$. φ^* makes $x_1 \to t^2$ and $x_2 \to t^3$. Since φ^* is not surjective(there is no element maps into t), φ^* is not an isomorphism. By theorem 7 we know φ is not an isomorphism. So bijective morphism is not necessary to be an isomorphism.

Definition 29. Let $X \subset \mathbb{A}^n$ be a closed variety, $F \in k[x_1, \dots, x_n] \backslash I(X)$. The principal open defined by F is $X_F := X \backslash Z(F)$.

Proposition 14. *X_F is an affine variety.*

Proof. Let $Z := Z(\langle I(X), F \cdot x_{n+1} - 1 \rangle) \subset \mathbb{A}^{n+1}$. We need to prove Z is a closed subvariety of $\mathbb{A}^{n+1}$ isomorphic to X_F. Let $\varphi : (x_1, \dots, x_n, \frac{1}{F}) : X_F \to \mathbb{A}^{n+1}$, it is a bijective morphism and $\varphi(X_F) = Z$. As X_F is irreducible, Z is also irreducible. So Z is closed variety of $\mathbb{A}^{n+1}$. On the other hand, the inverse of φ is

$$\varphi^{-1} = (x_1, \dots, x_n) : Z \to X_F$$

is a morphism, so φ is an isomorphism. □

4. Morphisms of Quasi-projective varieties

Definition 30. Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be quasi-projective algebraic sets. A map $\varphi : X \to Y$ is called a polynomial map if there exists homogeneous polynomials $F_0, \dots, F_m \in k[x_0, \dots, x_n]$ of the same degree with no common zero on X s.t. $\varphi(p) = [F_0(p), \dots, F_m(p)]$, $\forall p \in X$, write $\varphi = [F_0, \dots, F_m]$.

Definition 31. The homogenization of $F \in k[x_0, \dots, x_n]$ is:

$$F_a := F(1, x_1, \dots, x_n).$$

Theorem 8. $\varphi_i = (\frac{x_0}{x_i}, \dots, \hat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}) : U_i \to \mathbb{A}^n$ *is an isomorphism.*

Proof. We can assume $i = 0$, $\varphi := \varphi_0$, $U := U_0$, then $\varphi = (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$. $\frac{x_i}{x_0}$ is a regular function in $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)$, so φ is a morphism. We need to show that $u = \varphi^{-1}(x_1, \dots, x_n) = [1, x_1, \dots, x_n]$ is a morphism.

(a) $u = \varphi^{-1}$ is continuous. Let $W = Z(F_1, \dots, F_m) \cap U$ be closed in U, $F_i \in k[x_0, \dots, x_n]$ are homogeneous, then

$$
\begin{aligned}
u^{-1}(W) =& \{(a_1, \dots, a_n) \in \mathbb{A}^n | [1, a_1, \dots, a_n] \in W\} \\
=& \{(a_1, \dots, a_n) \in \mathbb{A}^n | F_i(1, a_1 \dots, a_n) = 0, \forall i = 1, \dots, m\} \\
=& Z(F_{1a}, \dots, F_{ma})
\end{aligned}
$$

where F_{ia} is homogenization of F_i, it shows that $u^{-1}(W)$ is closed in $\mathbb{A}^n$.

(b) Let $V \subset U$ be open, $h \in \mathcal{O}_U(V)$, we need to show $u^*h \in \mathcal{O}_{\mathbb{A}^n}(u^{-1}(V))$. Making V smaller necessary, we can assume $h = \frac{F}{G}$, $F, G \in k[x_0, \dots, x_n]$ are homogeneous polynomials of the same degree.

$$
u^*h = h \circ u = \frac{F \circ u}{G \circ u} = \frac{F(1, x_1, \dots, x_n)}{G(1, x_1, \dots, x_n)}.
$$

Thus $u^*h \in \mathcal{O}_{\mathbb{A}^n}(u^{-1}(V))$, $phi : \mathbb{A}^n \to u$ is an isomorphism. □

Remark. From theorem 8 we find that if we identify $\mathbb{A}^n$ with $u_0 \subset \mathbb{P}^n$, the Zariski topology on $\mathbb{A}^n$ is equivalent to the induced topology of u_0 from $\mathbb{P}^n$.

Corollary 5. (1) *Every variety is isomorphic to a quasi-projective variety.*
(2) *Every variety has an open cover by affine varieties.*

Proof. (1) Let X be a variety, if X is locally closed in $\mathbb{P}^n$, then it is a quasi-projective variety, so we only need to consider the condition in $\mathbb{A}^n$. Assume X be locally closed in $\mathbb{A}^n$. $Y = \varphi_0^{-1}(X) \subset \mathbb{P}^n$ is locally closed subvariety and $\varphi_0^{-1} : X \to Y$ is an isomorphism.

(2) For varieties in $\mathbb{A}^n$, it is trivial. Let $X \subset \mathbb{P}^n$ be a quasi-projective variety, then $X = \bigcup\limits_{i=0}^{n} X \cap U_i$. $X \cap U_i$ is isomorphic to locally closed subvariety in $\mathbb{A}^n$. We can regard $X \cap U_i$ simply as $X \subset \mathbb{A}^n$, where X is locally closed. It is equivalent to prove:

For every point $p \in X$, there exists a neighborhood $U \subset X$ and U is an affine variety.

Since X is locally closed, there exist $Y, Z \subset \mathbb{A}^n$ closed in $\mathbb{A}^n$ s.t. $X = Y \backslash Z$. For any point $p \in X$, $\exists F_p \in I(Z)$ with $F_p(p) \neq 0$. Then we have $Y_{F_p} = Y \backslash Z(F_p) \subset X$. According to proposition 14, Y_{F_p} is an affine variety. □

Theorem 9. *Let $X \subset \mathbb{P}^m$, $Y \subset \mathbb{P}^n$ be quasi-projective varieties. Let $\varphi : X \to Y$ be a map. The following conditions are equivalent:*

(1) *φ is a morphism;*
(2) *φ is locally given by regular functions, i.e., for all $p \in X$, there exists a neighborhood $U \subset X$, $h_0, \dots, h_n \in \mathcal{O}_X(U)$ with no common zero on U, s.t.*

$$
\varphi(q) = [h_0(q), \dots, h_n(q)], \quad \forall q \in U.
$$

We write $\varphi = [h_0, \dots, h_n]$ on U;
(3) *φ is locally a polynomial map, i.e.:*

$\forall p \in X$, $\exists$ open neighborhood $U \subset X$, $F_0, \dots, F_n \in k[x_0, \dots, x_,]$ homogeneous of the same degree with no common zero s.t.

$$\varphi(q) = [F_(q), \dots, F_n(q)] \quad \forall q \in U.$$

We write $\varphi = [F_0, \dots, F_n]$ on U.

Proof. (1) $\Rightarrow$ (2): If $\varphi : X \to \mathbb{P}^n$ is amorphism, then $\forall p \ inX$, $\exists i$, s.t. $\varphi(p) \in U_i$. Assume $i = 0$ and then $\varphi(p) \in U_0$. Let U be an open neighborhood of$p$ in X s.t. $\varphi(U) \subset U_0$. Then $\varphi_0 \circ \varphi : U \to \mathbb{A}^n$ is a morphism, so $\varphi_0 \circ \varphi = (h_1, \dots, h_n)$ with $h_i \in \mathcal{O}_X(U)$. Since the inverse of φ_0 is u_0 we get

$$\varphi = u_0 \circ \varphi_0 \circ \varphi = [1, h_1, \dots, h_n]. \tag{4.1}$$

(2) $\Rightarrow$ (3): Assume $\varphi = [h_0, \dots, h_n]$ on $U \subset X$, where $h_i \in \mathcal{O}_X(U)$ with no common zeros on U. By making U possibly smaller we can further assume $h_i = \frac{F_i}{G_i}$, $F_i, G_i \in k[x_0, \dots, x_m]$ are homogeneous of the same degree(F_i and G_i are of the same degree, it is not necessary that F_i and G_j are of the same degree for $i \neq j$), G_i has no zeros on U. Let $L_i = F_i \cdot G_0 \cdot \hat{G}_i \cdot G_n$, L_i are homogeneous of the same degree, we get

$$\varphi = [h_0, \dots, h_n] = [L_0, \dots, L_n]. \tag{4.2}$$

(3) $\Rightarrow$ (1): Let $\varphi|_U = [L_0, \dots, L_n]$, $L_i \in k[x_0, \dots, x_m]$ are homogeneous of the same degree with no common zero. Making U smaller, we can assume one of L_i(say L_0) has no zero in U. Then for $i = 1, \dots, n$, let $h_i = \frac{L_i}{L_0} \in \mathcal{O}_X(U)$. Rewrite the map as

$$\varphi = [1, h_1, \dots, h_n] \tag{4.3}$$

$$\Rightarrow \varphi_0 \circ \varphi = (h_1, \dots, h_n). \tag{4.4}$$

So$\varphi_0 \circ \varphi$ is amorphism, then $\varphi = u_0 \circ \varphi_0 \circ \varphi$ is a morphism. □

Definition 32 (Projective Transformation)**.** Let

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{4.5}$$

be a $(n+1) \times (n+1)$ matrix in k, then we can construct a map from $\mathbb{P}^n \to \mathbb{P}^n$:

$$[A] : [b_0, \dots, b_n] \to [b_0, \dots, b_n] \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix}^T.$$

It is called a projective transformation. This is a morphism and if A is inverse then it is an isomorphism.

Remark. All automorphisms of $\mathbb{P}^n$ are projective transformations. It is not so easy to prove.

Definition 33 (Projection)**.** Let $X \subset \mathbb{P}^n$ be a variety, $W \subset \mathbb{P}^n$ be a projective subspace of $\mathbb{P}^n$ of $\dim W = k$. Assume $X \cap W = \emptyset$ and there exist linear forms $H_0, \ldots, H_{n-k-1}$ such that $W = Z(H_0, \ldots, H_{n-k-1})$. The projection from W is

$$\Pi_W = [H_0, \ldots, H_{n-k-1}] : X \to \mathbb{P}^{n-k-1}.$$

This is a morphism(H_i have no common zero on X because $W \cap X = \emptyset$).

Remark. Π_W depends on $H_0, \ldots, H_{n-k-1}$, but if we have another relation $W = Z(L_0, \ldots, L_{n-k-1})$, then there exists a projection transformation $[A] : \mathbb{P}^{n-k-1} \to \mathbb{P}^{n-k-1}$. In particular, if $p \in \mathbb{P}^n \backslash X$, for example, $p = [0, \ldots, 0, 1]$, then $\Pi_p = [x_0, \ldots, x_{n_1}] : X \to \mathbb{P}^{n-1}$.

5. Products of Varieties

Theorem 10 (Products of Affine Varieties)**.** *If $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ are closed subvarieties, then $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ is a closed subvariety.*

Before we prove it, we need to prove a conclusion in topology.

Lemma 2. *Let X, Y be irreducible topological spaces. Assume we have a topology on the product $X \times Y$ s.t.:*

$$\begin{aligned} y_p :& Y \to X \times Y, \quad q \to (p, q) \text{ is continuous } \forall p \in X; \\ l_q :& X \to X \times Y, \quad p \to (p, q) \text{is continuous } \forall q \in Y. \end{aligned}$$

Then $X \times Y$ is irreducible.

Proof. Assume $X \times Y = S_1 \cup S_2$, $S_i \subsetneqq X \times Y$ are closed. For $i = 1, 2$, set $T_i = \bigcap_{q \in Y} l_q^{-1}(S_i) = \{p \in X | (p, q) \in S_i \quad \forall q \in Y\}$. It is the same as $T_i = \{p \in X | \{p\} \times Y \subset S_i\}$. Since y_p is continuous and Y is irreducible, we get $y_p(Y) = \{p\} \times Y$ is irreducible. So we get $\{p\} \times Y \subset S_1$ or $\{p\} \times Y \subset S_2 \quad \forall p \in X$(it implies $T_1 \cap T_2 = \emptyset$ and $T_i \subsetneqq X$). Hence $X = T_1 \cup T_2$. Since l_q is continuous, T_i are closed, then X is reducible. □

Proof of Theorem 10. Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be closed subvarieties, the product of X and Y is just

$$X \times Y = \{(p, q) \in \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m} | p \in X \text{ and } q \in Y\}.$$

Let $x_1, \ldots, x_n$ be coordinates in $\mathbb{A}^n$ and $y_1, \ldots, y_m$ be coordinates in $\mathbb{A}^m$, we can assume $X = Z(F_1, \ldots, F_k)$ and $Y = Z(G_1, \ldots, G_l)$ where $F_i \in k[x_1, \ldots, x_n], G_j \in k[y_1, \ldots, y_m]$. Then

$$X \times Y = Z(F_1, \ldots, F_k, G_1, \ldots, G_l) \subset \mathbb{A}^{n+m} \tag{5.1}$$

is a closed subset. By lemma 2 we only need to check $\forall q \in Y$, $l_q : X \to Y$ is continuous. Write $q = (b_1, \ldots, b_m)$, then $l_q = (x_1, \ldots, x_n, b_1, \ldots, b_m)$. It is a morphism, so it is continuous, thus we finish the whole proof.

Proposition 15 (Universal Property)**.** *Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be varieties, then*

(1) *The projections*

$$\begin{aligned} p_1 = (x_1, \ldots, x_n) : X \times Y \to X \\ p_2 = (y_1, \ldots, y_m) : X \times Y \to Y \end{aligned}$$

are morphisms.

(2) *Let Z be a variety. The morphism $\varphi : Z \to X \times Y$ are precisely the*

$$(f, g) : Z \to X \times Y, \quad p \to (f(p), g(p)) \quad \forall p \in Z$$

where $f : Z \to X$ and $g : Z \to Y$ are morphisms. In other words, $\varphi : Z \to X \times Y$ is a morphism if and only if both $p_1 \circ \varphi$ and $p_2 \circ \varphi$ are morphisms.

Proof. The first is obvious, we only check the second.

$\Rightarrow$: Let $\varphi : Z \to X \times Y$ be a morphism, then $f = p_1 \circ \varphi$ and $g = p_2 \circ \varphi$ are morphisms and $\varphi = (f, g)$.

$\Leftarrow$: Assume $f : Z \to X$ and $g : Z \to Y$ are both morphisms. then there exist $f_1, \dots, f_n \in \mathcal{O}_Z(Z)$ and $g_1, \dots, g_m \in \mathcal{O}_Z(Z)$ s.t. $f = (f_1, \dots, f_n)$, $g = (g_1, \dots, g_m)$. Then $(f, g) = (f_1, \dots, f_n, g_1, \dots, g_m)$ is a morphism. □

Remark. Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be subvarieties, $X \times Y$ does not lie rationally in some projective space. Thus we need to find an embedding $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ to denote the products of quasi-projective varieties.

Definition 34 ([Segre Embedding)**.**] We put $N := (n + 1) \cdot (m + 1) - 1$, let $x_0, \dots, x_n$ be coordinates on $\mathbb{P}^n$, $y_0, \dots, y_m$ be coordinates on $\mathbb{P}^m$. Let $z_{ij}, i = 0, \dots, n, j = 0, \dots, m$ be coordinates on $\mathbb{P}^N$. Define a map

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$$
$$([x_0, \dots, x_n], [y_0, \dots, y_m]) \to [z_{ij}] = [x_i y_j]$$

σ is called the Segre embedding.

Definition 35. We define the image of σ as

$$\Sigma := \sigma(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^N.$$

For $i = 0, \dots, n$, put

$$U_i := \{[x_0, \dots, x_n] \in \mathbb{P}^n | x_i \neq 0\}.$$

For $j = 0, \dots, m$, put

$$U_j := \{[y_0, \dots, y_m] \in \mathbb{P}^m | y_j \neq 0\}.$$

And for $i = 0, \dots, n, j = 0, \dots, m$, put

$$U_{ij} := \{[z_{kl}] \in \mathbb{P}^N | z_{ij} \neq= 0\}.$$

there are isomorphisms:

$$\mathbb{A}^n \underset{\varphi_i}{\overset{u_i}{\rightleftarrows}} U_i$$
$$\mathbb{A}^m \underset{\varphi_j}{\overset{u_j}{\rightleftarrows}} U_j$$
$$\mathbb{A}^N \underset{\varphi_{ij}}{\overset{u_{ij}}{\rightleftarrows}} U_{ij}.$$

Since $\mathbb{P}^N = \cup_{i,j} U_{ij}$, we get $\Sigma = \cup_{i,j} (\Sigma \cap U_{ij})$, define

$$\Sigma^{ij} = \Sigma \cap U_{ij}.$$

Define the map σ^{ij}

$$\sigma^{ij} : \mathbb{A}^{n+m} \to U_{ij}$$
$$(p, q) \to \sigma(u_i(p), u_j(q)).$$

By definition we know $\sigma^{ij}(\mathbb{A}^{n+m}) = \Sigma^{ij}$.

Theorem 11. (1) *$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ is injective and Σ is closed in $\mathbb{P}^N$:*

$$\Sigma = Z\left(\left\{ z_{ij}z_{kl} - z_{il}z_{kj} \Big|\begin{matrix} i,k & = 0,\dots,n \\ j,l & = 0,\dots,m \end{matrix}\right\}\right). \tag{5.2}$$

(2) *$\sigma^{ij} : \mathbb{A}^{n+m} \to \Sigma^{ij}$ is an isomorphism.*

(3) *$\forall q \in \mathbb{P}^m$, the map*

$$\begin{aligned} \bar{i_q} : \mathbb{P}^n &\to \mathbb{P}^N \\ p &\to \sigma(p,q) \end{aligned}$$

is a morphism. Similarly, $j_p = \sigma(p,q) : \mathbb{P}^m \to \mathbb{P}^N$ is a morphism.

(4) *Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be quasi-projective varieties, then $\sigma(X \times Y) \subset \mathbb{P}^N$ is also a quasi-projective variety. What's more, if X and Y are both projective varieties, then $\sigma(X \times Y)$ is a projective variety.*

Proof. (1) If $\sigma\left([a_0,\dots,a_n],[b_0,\dots,b_m]\right) = \sigma\left([a_0',\dots,a_n'],[b_0',\dots,b_m']\right)$, then $\exists \lambda \in k\backslash\{0\}$, s.t. $\lambda a_i' b_j' = \lambda a_i b_j$ $\forall i,j$. Choose i_0, j_0 s.t. $a_{i_0}b_{j_0} \neq 0$, then $\forall i = 0,\dots,n$, $a_i b_{j_0} = \lambda a_i' b_{j_0}' \Rightarrow a_i = \left(\frac{\lambda b_{j_0}'}{b_{j_0}}\right) a_i' \Rightarrow [a_0,\dots,a_n] = [a_0',\dots,a_n']$. The same way can be used to prove $[b_0,\dots,b_m] = [b_0',\dots,b_m']$. Let W be the zero set on the right hand side of the equation 5.2, clearly we have the relation $\Sigma \subset W$. Now let $[a_{ij}] \in W$, choose i_0, j_0 s.t. $a_{i_0 j_0} \neq 0$, then we get $[a_{ij}] = [a_{i_0j_0}a_{ij}] = [a_{i_0 j}a_{ij_0}] = [a_{ij_0}a_{i_0j}] = \sigma([a_{0j_0},\dots,a_{nj_0}],[a_{i_00},\dots,a_{i_0m}]) \subset \Sigma$.

(2) Assume $i = j = 0$, then

$$\begin{aligned} \varphi_{00} \circ \sigma^{00}(a_1,\dots,a_n,b_1,\dots,b_m) &= \varphi_{00}(\sigma([1,a_1,\dots,a_n],[1,b_1,\dots,b_m])) \\ &= (z_{ij})_{(i,j)\neq(0,0)} \end{aligned}$$

where $z_{i0} = a_i$ for $i = 1,\dots,n$, $z_{0j} = b_j$ for $j = 1,\dots,m$, $z_{ij} = a_i b_j$ for $i,j \geq 1$. These are all regular functions, so $\varphi_{00} \circ \sigma^{00}$ is a morphism, so σ^{00} is a morphism. Finally, σ^{00} is an isomorphism because the inverse map is

$$(\sigma^{00})^{-1} = \left(\frac{z_{10}}{z_{00}},\dots,\frac{z_{n0}}{z_{00}},\frac{z_{01}}{z_{00}},\dots,\frac{z_{0m}}{z_{00}}\right).$$

Remark. In fact, Σ^{ij} is a quasi-projective variety. Because $\mathbb{A}^{n+m}$ is irreducible, Σ^{ij} is irreducible, hence a quasi-projective variety.

(3) Let $q = [b_0,\dots,b_m]$, then $i_q = [x_i b_j]$, $x_i b_j$'s are homogeneous polynomials, so by theorem 9 we know it is a morphism.

(4) Let $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$ be projective varieties. We can decompose the map into the following:

$$\begin{aligned} \sigma(X \times Y) &= \bigcup_{i,j} \sigma(X \times Y) \cap U_{ij} \\ &= \bigcup_{i,j} \sigma^{ij}(\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j)) \end{aligned}$$

$\varphi_i(X \cap U_i)$ and $\varphi_j(Y \cap U_j)$ are closed subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively. By the theorem 10 $\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j)$ is closed in $\mathbb{A}^{n+m}$. Since σ^{ij} is an isomorphism, then $\sigma^{ij}(\varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j))$ is closed in $\Sigma^{ij} = \Sigma \cap U_{ij}$. So $\sigma(X \times Y)$ is closed in Σ, hence closed in $\mathbb{P}^N$ because Σ itself is closed. To show its irreducible, we use the lemma 2. Since σ is injective we can endow $\mathbb{P}^n \times \mathbb{P}^m$ with the topological structure of $\mathbb{P}^N$, hence we can identify $\mathbb{P}^n \times \mathbb{P}^m$ with Σ provided with the topology induced from $\mathbb{P}^N$. Now we can use the lemma 2, we have known i_q and j_p are continuous, so $\sigma(X \times Y)$ is irreducible. For quasi-projective conditions ,we just get the conclusion by simply difference two projective varieties. $\square$

Remark. For $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ we can now identify $X \times Y$ with $\sigma(X \times Y) \subset \mathbb{P}^N$. In particular we can identify $\mathbb{P}^n \times \mathbb{P}^m$ with Σ.

From this perspective, part (2) of the theorem just says $U_i \times U_j \subset \mathbb{P}^n \times \mathbb{P}^m$ is open and $\varphi_i \times \varphi_j : U_i \times U_j \to \mathbb{A}^{n+m}$ is an isomorphism.

Proposition 16 (Universal Property). *Let X, Y be quasi-projective varieties, then*

(1) *The projections*

$$p_1 = (x_1, \dots, x_n) : X \times Y \to X$$
$$p_2 = (y_1, \dots, y_m) : X \times Y \to Y$$

are morphisms.

(2) *Let Z be a variety. The morphism $\varphi : Z \to X \times Y$ are precisely the*

$$(f, g) : Z \to X \times Y, \quad p \to (f(p), g(p)) \quad \forall p \in Z$$

where $f : Z \to X$ and $g : Z \to Y$ are morphisms. In other words, $\varphi : Z \to X \times Y$ is a morphism if and only if both $p_1 \circ \varphi$ and $p_2 \circ \varphi$ are morphisms.

Proof. (1) It is enough to show $p_1|_{U_i \times U_j}$ is a morphism from $U_i \times U_j$ to U_i. Identify $U_i \times u_j$ with $\mathbb{A}^{n+m}$ and U_i with $\mathbb{A}^n$, then we can see that p_1 is the same as the projection defined by the proposition 15, so it is a morphism.

(2) $\Rightarrow$: Let $\varphi : Z \to X \times Y$ be a morphism. Then $f := p_1 \circ \varphi$ and $g := p_2 \circ \varphi$ are morphisms.

$\Leftarrow$: Let $f : Z \to X$ and $g : Z \to Y$ be morphisms. Define

$$Z^{ij} := f^{-1}(U_i) \cap g^{-1}(U_j).$$

Then (f, g) is a morphism $\Leftrightarrow$ $(f, g)|_{Z^{ij}}$ is a morphism for $i = 1, \dots, n, j = 1, \dots, m$. Consider the following mapping chain

$$Z^{ij} \xrightarrow{(f,g)} (X \times Y) \cap (U_i \times U_j) \xrightarrow{\varphi_i \times \varphi_j} \varphi_i(X \cap U_i) \times \varphi_j(Y \cap U_j) \subset \mathbb{A}^{n+m}.$$

the whole chain $(\varphi_i \circ f, \varphi_j \circ g) : Z^{ij} \to \mathbb{A}^{n+m}$ is a morphism, so (f, g) is a morphism. $\square$

Corollary 6. *Let X_1, X_2, Y_1, Y_2 be varieties. If $f : X_1 \to Y_1$ and $X_2 \to Y_2$ are morphisms, then the map:*

$$f \times g : X_1 \times X_2 \to Y_1 \times Y_2$$
$$(p, q) \to (f(p), g(q))$$

is a morphism. In particular, if X_1 is isomorphic to Y_1 and X_2 is isomorphic to Y_2, then $X_1 \times X_2$ is isomorphic to $Y_1 \times Y_2$

Proof. We can write $f \times g$ as $f \circ p_1$ and $g \circ p_2$, both $f \circ p_1$ and $g \circ p_2$ are morphisms, so $f \times g = (f \circ p_1, g \circ p_2)$ is a morphism. $\square$

Lemma 3. *The closed subset in $\mathbb{P}^n \times \mathbb{P}^m$ is the zero set of sets of polynomials of $f_k(x_0, \dots, x_n, y_0, \dots, y_m)$ for $k = 1, \dots, r$ which are homogeneous in x_i and y_j, and the degree in x_i is equal to the degree in y_j, we called it behomogeneous.*

Proof. Let $W \subset \mathbb{P}^n \times \mathbb{P}^m$ be closed. $W = \sigma^{-1}(A)$, for $A \subset \mathbb{P}^N$ closed. Then A is the zero set of homogeneous polynomials in z_{ij}, write it as $A = (f_1(z_{ij}), \dots, f_r(z_{ij})$. Then we get $W = (f_1(x_i y_j), \dots, f_r(x_i y_j))$. For $k = 1, \dots, r$, $f_k(x_i y_j)$ are bihomogeneous. Conversely, assume

$$W = Z(g_1(x_0, \dots, x_n, y_0, \dots, y_m), \dots, g_l(x_0, \dots, x_n, y_0, \dots, y_m))$$

where g_k are bihomogeneous. Then

$(\varphi_i \times \varphi_j)(W \cap (U_i \times U_j)) = Z(g_1(x_0, \dots, x_i = 1, \dots, x_n, y_0, \dots, y_j = 1, \dots, y_m), \dots, g_l(x_0, \dots, x_i = 1, \dots, x_n, y_0$

are closed in $\mathbb{A}^{n+m}$. So $W \cap (U_i \times U_j)$ are closed in $U_i \times U_j$. $U_i \times U_j$ form a finite open cover of $\mathbb{P}^n \times \mathbb{P}^m$, so W is closed. $\square$

Definition 36. Let X be a variety, the diagonal is

$$\Delta_X := \{(p,p) \in X \times X | p \in X\} \subset X \times X.$$

The diagonal morphism is

$$\begin{aligned} \delta_X : X &\to \Delta_X \subset X \times X \\ p &\to (p,p). \end{aligned}$$

Lemma 4. *Δ_X is closed in $X \times X$ and $\delta_X : X \to \Delta_X$ is an isomorphism.*

Proof. Any variety X is isomorphic to a locally closed subvariety of some projective space, so we can assume $X \subset \mathbb{P}^n$ is locally closed, then

$$\Delta_X = \Delta_{\mathbb{P}^n} \cap (X \times X).$$

Thus we know if $\Delta_{\mathbb{P}^n}$ is closed then Δ_X is closed in $X \times X$. In fact $\Delta_{\mathbb{P}^n} = Z(\{x_i y_j - x_j y_i | i, j = 0, \dots, n\})$ is closed.

$\delta_X : X \to \Delta_X$ is isomorphic because $p_1 : \Delta_X \to X$ is its inverse morphism. $\square$

Remark. The fact that $\Delta_X \subset X \times X$ is closed replaces for us the Hausdorff property in topology.

Definition 37. A variety is called separated if $\Delta_X \subset X \times X$ is closed. By the lemma 4 all varieties are separated.

Corollary 7. *Let $\varphi, \psi : X \to Y$ be morphisms of varieties, then $W = \{p \in X | \varphi(p) = \psi(p)\}$ is closed in X. In particular, if $\varphi|_U = \psi|_U$ for an open subset of X, then $\varphi = \psi$.*

Proof. See the following chain

$$X \xrightarrow{\delta_X} \Delta_X \xrightarrow{\varphi \times \psi} Y \times Y.$$

So $W = \delta^{-1}((\varphi \times \psi)^{-1}(\Delta_Y))$ is closed. Because varieties are irreducible, the open set U is dense in X, let $\omega = \varphi - \psi$ and we get $l(x) = 0$ in U, hence $l = 0$ in X because of the continuity of l, hence $\varphi = \psi$. $\square$

Definition 38. Let $\varphi : X \to Y$ be a morphism of varieties. The graph of φ is defined as

$$\Gamma_\varphi := \{(p, \varphi(p)) | p \in X\} \subset X \times Y. \tag{5.3}$$

Corollary 8. *Γ_φ is closed in $X \times Y$.*

Proof. Define the map

$$\begin{aligned} \varphi \times \mathrm{id}_Y : X \times Y &\to Y \times Y \\ (p, q) &\to (\varphi(p), q). \end{aligned}$$

Then we have $\Gamma_\varphi = (\varphi \times \mathrm{id}_Y)^{-1}(\Delta Y)$, so it is closed. In fact Γ_φ is isomorphic to X. $\square$

Definition 39. A map $\varphi : X \to Y$ of topological spaces is called closed if $\varphi(Z)$ is closed in Y for all closed subsets $Z \subset X$.

Definition 40. A variety complete if the projection $p_2 : X \times Y \to Y$ is a closed map for all varieties Y.

Remark. Completeness replaces for us compactness in topology.

Example 5. $\mathbb{A}^1$ is not complete. Let $Z = Z(x_1 y_1 - 1) \subset \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$, then $p_2(Z) = \mathbb{A}^1 \backslash \{0\}$ is not closed in $\mathbb{A}^1$.

Proposition 17. *Let X be a complete variety, $\varphi : X \to Y$ be a morphism of varieties. Then $\varphi(X)$ is closed in Y.*

Proof. Since $\Gamma_\varphi \subset X \times Y$ is closed and $\varphi(X) = p_2(\Gamma_\varphi)$, thus if X is complete, $\varphi(X)$ is closed in Y. $\square$

Theorem 12. *All projective varieties are complete.*

Proof. We finish the proof by two steps.

(1) Main step to show $p_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed. Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be closed, we can write it as

$$X = Z(f_1(x, y), \ldots, f_r(x, y))$$

where f_i isbihomogeneous, $x = (x_0, \ldots, x_n), y = (y_0, \ldots, y_m)$. We can assume all f_i have the same degree d in y. If f_j has a lower degree l, we can replace it by polynomials $y_0^{d-l} f_j, y_1^{d-l} f_j, \ldots, y_n^{d-l} f_j$. Fix a point $q \in \mathbb{P}^m$, then $q \in p_2(X) \Leftrightarrow Z(f_1(x, q), \ldots, f_r(x, q)) \neq \emptyset$. By the projective Nullstellensatz, this is equivalent to:

$$\begin{gathered} \forall s > 0,\ (*)\ \mathfrak{a} := \langle f_1(x, q), \ldots, f_r(x, q) \rangle \text{ does not contain} \\ \text{all monomials of degree } s \text{ in } x. \end{gathered}$$

It is trival for $s < d$, so it is enough to show:

$$\begin{gathered} \forall s \geq d, \text{ the set } X_s := \{q \in \mathbb{P}^m | q \text{ satisfies the condition } (*)\} \\ \text{is closed in } \mathbb{P}^m. \text{ Hence } p_2(X) = \bigcap_{s \geq d} X_s \text{ is closed in } \mathbb{P}^m. \end{gathered}$$

Denote monomials in x of degree s with $M_i(x)$, $i = 1, \ldots, \binom{n+s}{n}$. Denote monomials in x of degree $s - d$ with $N_j(x)$, $j = 1, \ldots, \binom{n+s-d}{n}$. The elements of degree s in $\mathfrak{a}$ are the linear span of $\{N_i(x) f_j(x, q) | i = 1, \ldots, \binom{n+s-d}{n}, j = 0, \ldots, r\}$. Define all $\{N_i(x) f_j(x, y)\}$ by $\{G_k(x, y), k = 1, \ldots, t\}$. The condition $(*)$ is equivalent to:

$$\{G_k(x, q)\} \text{ does not equal to the whole space of degree } s \text{ in } x.$$

We can write $G_k(x,y) = \sum_{i=1}^{\binom{n+s}{n}} A_{ik}(y)M_i(x)$. The dimension of the linear span of $\{G_k(x,q), k = 1,\ldots,t\}$ is the rank of the matrix $A := (A_{ik}(q))$. Thus the condition $(*)$ is equivalent to $\mathrm{rank}(A) < \binom{n+s}{n}$. Thus

$$\{q \in \mathbb{P}^m | q \text{ satisfies the condition } (*)\} = \text{ zero set of all } \binom{n+s}{n} \times \binom{n+s}{n} \text{ minors of } A.$$

Thus $p_2(X)$ is closed in $\mathbb{P}^m$.
(2) General case. First show $\mathbb{P}^n$ is completed. Let Y be a variety, we can assume $Y \subset \mathbb{P}^m$ is locally closed subvariety. Let $Z \subset \mathbb{P}^n \times Y$ be closed in $\mathbb{P} \times Y$, $\bar{Z}$ be the closure of Z in $\mathbb{P}^n \times \mathbb{P}^m$. Then $p_2(\bar{Z})$ is closed in $\mathbb{P}^m$, hence $p_2(Z) = p_2(\bar{Z} \cap (\mathbb{P}^n \times Y)) = p_2(\bar{Z}) \cap Y$ is closed in Y. Finally, let $X \subset \mathbb{P}^n$ be closed subvariety, $Z \subset X \times Y$ be closed, it follows that Z is also closed in $\mathbb{P}^n \times Y$, therefore by trival step $p_2(Z)$ is closed in Y. □

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