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NON-CLASSICAL CURTIS-TITS AMALGAMS WITH SIMPLY-LACED 3-SPHERICAL DIAGRAM

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Abstract. In [?] it was shown that classical Curtis-Tits amalgams with 3-spherical diagram must have a “weak system of fundamental root groups” and all amalgams possessing such a system were classified. In [?] it was then shown that all amalgams with a weak system of fundamental root groups do in fact have a non-trivial completion.

Roughly speaking a Curtis-Tits amalgam is classical if its rank-2 subamalgams are those arising from applying the Curtis-Tits theorem to the corresponding rank-2 buildings. We consider Curtis-Tits amalgams in which we replace this with a considerably weaker condition: we only require that any two rank-1 groups generate the rank-2 group that contains them. In particular, we do not require the vertex groups to form a standard pair, and we even do not require the vertex groups to be in the desired conjugacy class inside the edge groups.

1. Introduction

Local recognition results play an important role in various parts of mathematics. A key example comes from the monumental classification of finite simple groups. Local analysis of the unknown finite simple group G yields a local datum consisting of a small collection of subgroups fitting together in a particular way, called an amalgam. The Curtis-Tits theorem [8, 17, 18, 19, 20] and the Phan (-type) theorems [26, 27, 28] describe amalgams appearing in known groups of Lie type. Once the amalgam in G is identified as one of the amalgams given by these theorems, G is known.

The present paper was partly motivated by a question posed by R. Solomon and R. Lyons about this identification step, arising from their work on the classification [10, 11, 12, 13, 14, 15]: Are Curtis-Tits and Phan type amalgams uniquely determined by their subgroups? More precisely is there a way of fitting these subgroups together so that the amalgam gives rise to a different group? In many cases it is known that, indeed, depending on how one fits the subgroups together, either the resulting amalgam arises from these theorems, or it does not occur in any non-trivial group. This is due to various results of Bennet and Shpectorov [1], Gramlich [16], Dunlap [9], and R. Gramlich, M. Horn, and W. Nickel [24]. However, all of these results use, in essence, a crucial observation by Bennett and Shpectorov about tori in rank-3 groups of Lie type, which fails to hold for small fields. In the present paper we replace the condition on tori by a more effective condition on root subgroups, which holds for all fields. This condition is obtained by a careful analysis of maximal subgroups of groups of Lie type. Thus the identification step can now be made for all possible fields. A useful consequence of the identification of the group G, together with the Curtis-Tits and Phan type theorems, is that it yields a simplified version of the Steinberg presentation for G.

Note that this solves the - generally much harder - existence problem: "how can we tell if a given amalgam appears in any non-trivial group?"

The unified approach in the present paper not only extends the various results on Curtis-Tits and Phan amalgams occurring in groups of Lie type to arbitrary fields, but in fact also applies to a much larger class of Curtis-Tits and Phan type amalgams, similar to those occurring in groups of Kac-Moody type. Here, both the uniqueness and the existence problem become significantly more involved.

Groups of Kac-Moody type were introduced by J. Tits as automorphism groups of certain geometric objects called twin-buildings [30]. In the same paper J. Tits conjectured that these groups are uniquely determined by the fact that the group acts on some twin-building, together with local geometric data called Moufang foundations. As an example he sketched an approach towards classifying such foundations in the case of simply-laced diagrams. This conjecture was subsequently proved for Moufang foundations built from locally split and locally finite rank-2 residues by B. Mühlherr in [25] and refined by P. E. Caprace in [7]. All these results produce a classification of groups of Kac-Moody type using local data in the form of an amalgam, together with a global geometric assumption stipulating the existence of a twin-building on which the group acts.

Ideally, one would use the generalizations of the Curtis-Tits and Phan type theorems to describe the groups of Kac-Moody type in terms of a simplified Steinberg type presentation. However, the geometric assumption is unsatisfactory for this purpose as it is impossible to verify directly from the presentation itself.

In our unified approach we consider all possible amalgams whose local structure is any one of those appearing in the above problems. There is no condition on the field. Then, we classify those amalgams that satisfy our condition on root groups and show that in the spherical case they are unique. This explains why groups of Lie type can uniquely be recognized by their amalgam. By contrast, in the non-spherical case the amalgams are not necessarily unique and, indeed, not all such amalgams give rise to groups of Kac-Moody type. This is a consequence of the fact that we impose no global geometric condition. Nevertheless, we give a simple condition on the amalgam itself which decides whether it comes from a group of Kac-Moody type or not. As a result, we obtain a purely group theoretic definition of a large class of groups of Kac-Moody type just in terms of a finite presentation.

Finally, we note that an amalgam must satisfy the root subgroup condition to occur in a non-trivial group. A subsequent study generalizing [4, 6] shows that in fact all amalgams satisfying the root group condition do occur in non-trivial groups. Thus, in this much wider context the existence problem is also solved.

The main result of the present paper is as follows.

Theorem A. *Any weak Curtis-Tits amalgam with* 3*-spherical simply-laced diagram is classical.*

2. Curtis-Tits and Phan amalgams and their diagrams

2.1. Diagrams

For the purposes of this paper a *diagram* is a simple graph $\Gamma = (I, E)$. We shall assume that it is 3-spherical which means that it contains no circuits with ≤ 3 edges. We shall also make the harmless assumption that Γ is connected.

2.2. Weak Curtis-Tits Amalgams

Definition 2.1. An *amalgam* over a poset $(\mathcal{P}, \prec)$ is a collection $\mathscr{A} = \{\mathbf{A}_x \mid x \in \mathcal{P}\}$ of groups, together with a collection $\mathbf{a}_\bullet = \{\mathbf{a}_x^y \mid x \prec y, x, y \in \mathcal{P}\}$ of monomorphisms $\mathbf{a}_x^y \colon \mathbf{A}_x \hookrightarrow \mathbf{A}_y$, called *inclusion maps* such that whenever $x \prec y \prec z$, we have $\mathbf{a}_x^z = \mathbf{a}_y^z \circ \mathbf{a}_x^y$; we shall write $\overline{\mathbf{A}}_x = \mathbf{a}_x^y(\mathbf{A}_x) \leq \mathbf{A}_y$. A *completion* of $\mathscr{A}$ is a group A together with a collection $\alpha_\bullet = \{\alpha_x \mid x \in \mathcal{P}\}$ of homomorphisms $\alpha_x \colon \mathbf{A}_x \to A$, whose images - often denoted $A_x = \alpha_x(\mathbf{A}_x)$ - generate A, such that for any $x, y \in \mathcal{P}$ with $x \prec y$ we have $\alpha_y \circ \alpha_x^y = \alpha_x$. The amalgam $\mathscr{A}$ is *non-collapsing* if it has a non-trivial completion. As a convention, for any subgroup $\mathbf{H} \leq \mathbf{A}_J$, let $\mathrm{H} = \alpha(\mathbf{H}) \leq A$.

A completion $(\tilde{A}, \tilde{\alpha}_\bullet)$ is called *universal* if for any completion $(A, \alpha_\bullet)$ there is a unique surjective group homomorphism $\pi \colon \tilde{A} \to A$ such that $\alpha_\bullet = \pi \circ \tilde{\alpha}_\bullet$. A universal completion always exists.

Definition 2.2. Let $\Gamma = (I, E)$ be a Lie diagram. A *weak Curtis-Tits amalgam with diagram* Γ *over* $\mathbb{F}_q$ is an amalgam $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ over $\mathcal{P} = \{J \mid \emptyset \neq J \subseteq I \text{ with } |J| \leq 2\}$ ordered by inclusion such that for every $i, j \in I$, we have

(WCT1) $\mathbf{G}_i \cong \mathrm{SL}_2(q)$ or $\mathbf{G}_i \cong \mathrm{PSL}_2(q) \cong \mathrm{SO}_3(q)$.

(WCT2)

$$\mathbf{G}_{i,j} \cong \begin{cases} \mathrm{SL}_3(q) \text{ or } \mathrm{PSL}_3(q) & \text{if } \{i,j\} \in E, \\ \mathbf{G}_i \circ \mathbf{G}_j & \text{if } \{i,j\} \notin E. \end{cases}$$

(WCT3)

$$\mathbf{G}_{i,j}, = \langle \mathbf{g}_{ij}(\mathbf{G}_i), \mathbf{g}_{ji}(\mathbf{G}_j) \rangle.$$

For any subset $K \subseteq I$, we let

$$\mathscr{G}_K = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in K\}.$$

Clearly this is a weak Curtis-Tits amalgam with diagram Γ_K.

2.3. Classical Curtis-Tits amalgams

Definition 2.3. Let Γ be a diagram of type $A_1 \times A_1$ or A_2, and $q = p^e$ for some prime $p \in \mathbb{Z}$ and $e \in \mathbb{Z}_{\geq 1}$. Then a *Curtis-Tits standard pair of type* $\Gamma(q)$ is a triple $(\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2)$ of groups such that one of the following occurs:
($\Gamma = A_1 \times A_1$). Now $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ and $\mathbf{G}_1 \cong \mathbf{G}_2 \cong \mathrm{SL}_2(q)$.
($\Gamma = A_2$). Now $\mathbf{G} = \mathrm{SL}_3(q) = \mathrm{SL}(V)$ for some $\mathbb{F}_q$-vector space with basis $\{e_1, e_2, e_3\}$ and $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) is the stabilizer of the subspace $\langle e_1, e_2\rangle$ (resp. $\langle e_2, e_3\rangle$) and the vector e_3 (resp. e_1).

Explicitly we have

$$\mathbf{G}_1 = \left\{ \begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} : a, b, c, d \in \mathbb{F}_q \text{ with } ad - bc = 1 \right\},$$
$$\mathbf{G}_2 = \left\{ \begin{pmatrix} 1 & & \\ & a & b \\ & c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_q \text{ with } ad - bc = 1 \right\}.$$

Definition 2.4. Let $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ be a weak Curtis-Tits amalgam. We say that the sub-amalgam $\mathscr{G}_{i,j}$ of type $\Gamma_{i,j}$ is *classical* if $(\mathbf{G}_{i,j}, \overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ is a Curtis-Tits standard pair. A weak Curtis-Tits amalgam is called *classical* if all subamalgams $\mathscr{G}_{i,j}$ are classical.

Definition 2.5. For Curtis-Tits amalgams, the *standard identification map* will be the isomorphism $\mathbf{g}\colon \mathrm{SL}_2(q) \to \mathbf{G}_i$ sending

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the corresponding matrix of $\mathbf{G}_i$ as described above.

3. Preliminary results

3.1. Embeddings of $\mathbf{G}_i$ into $\mathbf{G}_{i,j}$

Remark 3.1. The *adjoint* representation of $\mathrm{PSL}_2(q)$ is the 3-dimensional Lie algebra $\mathfrak{sl}_2(q)$ with basis of vectors

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

on which $\mathrm{PSL}_2(q)$ acts by conjugation. The determinant $\det\colon \mathfrak{gl}_2(q) \to \mathbb{F}_q$ defines a quadratic form given by $\mathcal{Q}(eE+fF+hH) = -h^2-ef$ with associated bilinear form

$$\beta((e_1, f_1, h_1), (e_2, f_2, h_2)) = -e_1 f_2 - e_2 f_1 - 2h_1 h_2.$$

In particular, E and F are singular and form a hyperbolic line and, if q is even the radical of β is $\mathrm{Rad}\,\beta = \langle H\rangle$. This establishes an isomorphism $\mathrm{PSL}_2(q) \to \mathrm{PSO}_3(q)$.

The *dual adjoint* representation of $\mathrm{PSL}_2(q)$ is the dual of $\mathfrak{sl}_2(q)$ with the contragradient action obtained from above. Concretely, if (E^*, F^*, H^*) is the (ordered) basis dual to (E, F, H), $\varphi\colon \mathrm{SL}_2(q) \to \mathrm{GL}_3(q)$ induces the action above, then $g \in \mathrm{SL}_2(q)$ acts as ${}^t(\varphi(g))^{-1}$.

Lemma 3.2. *Let V be a 3-dimensional vector space over $\mathbb{F}_q$. Suppose ϕ : $\mathrm{SL}_2(q) \to \mathrm{PSL}_3(q) = \mathrm{PSL}(V)$ is a nontrivial map. Then one of the following holds.*

1. *V is irreducible for the action of $\mathrm{im}\,\phi$. In this case V is the adjoint module, q is odd and $\mathrm{im}\,\phi \cong \mathrm{PSL}_2(q)$ is maximal in $\mathrm{PSL}(V)$.*
2. *V is a totally decomposable module for $\mathrm{im}\,\phi$. In this case $\mathrm{im}\,\phi$ is a Levi subgroup that stabilises a decomposition $V = V_1 \oplus V_2$. Also, $\mathrm{im}\,\phi$ is included in exactly two maximal subgroups which are opposite parabolics.*
3. *V is reducible but indecomposable. In this case q is even, $\mathrm{im}\,\phi$ is included in a unique maximal subgroup M and M is a parabolic.*

Proof

□

3.2. Curtis-Tits amalgams with at least one classical edge

Proposition 3.3. *Suppose (G, γ) is a completion of a weak Curtis-Tits amalgam $\mathscr{G}$ with diagram of type A_3. If one edge of $\mathscr{G}$ is classical then $\mathscr{G}$ is classical.*

Proof Let $I = \{i, j, k\}$ so that $\{i, j\}, \{j, k\} \in E$. Suppose $\{i, j\}$ is classical. This implies that we can choose pairs of root groups $\mathbf{X}_i^{\pm} \le \mathbf{G}_i$ respectively $\mathbf{X}_j^{\pm} \le \mathbf{G}_j$ so that $[\mathbf{X}_i^+, \mathbf{X}_j^-] = 1 = [\mathbf{X}_i^-, \mathbf{X}_j^+]$.

We first prove that G_k is the derived group of a Levi subgroup. Define $\mathrm{M}^\varepsilon = \langle \mathrm{X}_j^\varepsilon, G_k\rangle$ $(\varepsilon = +, -)$. It follows that X_i^ε commutes with $\mathrm{M}^{-\varepsilon}$. Since $\mathrm{X}_i^{-\varepsilon}$ does not commute with X_j^ε, the group M^ε is a proper subgroup of $G_{j,k}$. Thus, G_k is included in two different proper subgroups of $G_{j,k}$. Note also that these two proper subgroups are not contained in the same maximal proper subgroup of $G_{j,k}$ as

$$\langle \mathrm{M}^+, \mathrm{M}^-\rangle = \langle \mathrm{X}_j^+, \mathrm{X}_j^-, \mathrm{G}_k\rangle = \langle \mathrm{G}_j, \mathrm{G}_k\rangle = \mathrm{G}_{j,k}.$$

This eliminates cases 1 and 3 of Lemma 3.2 so $G_k \cong \mathrm{SL}_2(q)$ is included in two maximal subgroups which are opposite parabolics P^+, P^-, and we may in fact assume that $\mathrm{M}^\varepsilon \le P^\varepsilon$ for $\varepsilon = +, -$. Hence, $\mathbf{G}_k$ maps into $P^- \cap P^+$, which is a Levi subgroup of $G_{j,k}$. Finally, note that γ_k is injective, and since

$\mathbf{G}_k \cong \mathrm{SL}_2(q)$ is perfect, it follows that G_k is the derived group of this Levi subgroup.

We now show that also G_j is the derived group of a Levi-subgroup. First assume that q is odd. Since $\mathbf{G}_j$ is part of the classical edge $\{i,j\}$, the image of $\gamma_j\colon \mathbf{G}_j \to G_{j,k}$ must be isomorphic to $\mathrm{SL}_2(q) \not\cong \mathrm{PSL}_2(q)$, so γ_j cannot be in the case 1 of Lemma 3.2. As q is odd, it follows, as above, from that lemma that G_j is the derived subgroup of a Levi subgroup.

Next we assume that $q > 2$ is even. Again case 1 does not arise, so either it follows as above that G_j is the derived subgroup of a Levi subgroup, or the image of $\mathbf{G}_j$ in $G_{j,k}$ is adjoint or dual adjoint.

Since $q > 2$, the torus $\mathrm{D}_i^j = N_{G_i}(G_j)$ is non-trivial, commutes with G_k, normalises G_j, hence normalises $G_{j,k}$ and so preserves the image of the amalgam $\mathscr{G}$. Consider the map given by the action on $G_{j,k} \cong \mathrm{SL}_3(q)$:

$$\kappa_i^{j,k}\colon \mathrm{D}_i^j \to C_{\mathrm{Aut}(\mathrm{SL}_3(q))}(G_k) \cong \langle \mathrm{diag}(a,b,b)\colon a,b \in \mathbb{F}_q^*\rangle \rtimes \langle\theta\rangle / \langle \mathrm{diag}(b,b,b)\colon b \in \mathbb{F}_q^*\rangle,$$

with respect to some basis $\{e_1, e_2, e_3\}$ such that G_k preserves the subspace decomposition $V = \langle e_1\rangle \oplus \langle e_2, e_3\rangle$. This map is injective, for otherwise some non-trivial element of D_i^j commutes with all of $\langle G_j, G_k\rangle = G_{j,k}$, which is not the case. Noting that $q-1$ is odd, we see that D_i^j acts as a cyclic group of non-central diagonal automorphisms $\mathrm{diag}(a,b,b)$, which normalize G_j. We shall use the following observation twice:

$$\text{The subspaces fixed by } D_i^j \text{ are exactly } \langle e_1\rangle, \langle v\rangle \text{ and } \langle e_2,e_3\rangle, \langle e_1, v\rangle \text{ where } v \in \langle e_2, e_3\rangle - \{0\} \tag{3.1}$$

Now assume that G_j has the adjoint action on V. Then it preserves the orthogonal form $\mathcal{Q}$ from Remark 3.1 with respect to a basis (E,F,H). In order for D_i^j to normalize G_j it must fix $\mathrm{Rad}\,\beta = \langle H\rangle$ and preserve the set of singular points. Namely, let $v \in V - \mathrm{Rad}\,\beta$. If v is singular, the stabilizer of a singular point $\langle v\rangle$ induces $\mathbb{F}_q^*$, whereas for v non-singular we have $\mathcal{Q}(v) = \mathcal{Q}(\lambda v) = \lambda^2\mathcal{Q}(v)$ if and only if $\lambda = 1$ as $x \mapsto x^2$ is an automorphism of $\mathbb{F}_q^*$. Thus, we have $H \in \langle e_1\rangle$ or $H \in \langle e_2, e_3\rangle$ and as

$$G_{j,k} = \langle G_j, G_k\rangle \not\leq P \text{ for any parabolic } P, \tag{3.2}$$

and G_k fixes $\langle e_1\rangle$, we must have $H \in \langle e_2, e_3\rangle$. Without loss we may assume that $H = e_2$ and $\mathcal{Q}(e_2) = 1$. Now consider the torus $D_j^i = C_{G_j}(D_i^j)$. As D_i^j preserves $\langle e_1\rangle$, also D_j^i must preserve it. It then follows that D_j^i induces $\mathbb{F}_q^*$ on $\langle e_1\rangle$. This implies that $\langle e_1\rangle$ must be singular. Finally, as we still have freedom to select e_3, we may assume that it spans the unique singular point on $\langle e_2, e_3\rangle$ and in fact so that $\beta(e_1, e_3) = 1$. We can now see that D_i^j does not preserve $\mathcal{Q}$ hence sends some singular point to a non-singular point and therefore does not normalize G_j. If G_j has the dual adjoint action on V, then ..

We claim that G_j and G_k form a standard pair in $G_{j,k}$ for any $q > 2$ (even or odd).

Now if G_j is the derived subgroup of a Levi associated to the non-incident point-line pair p, l, then in order for $d \in D_i^j$ to normalize it, we must have $d(p) = p$ and $d(l) = l$. By (3.2) $p \neq \langle e_1 \rangle$ and $l \neq \langle e_2, e_3 \rangle$. It follows that $p \leq \langle e_2, e_3 \rangle$ and $e_1 \leq l$ and so G_j and G_k form a standard pair.

Finally we consider the case $q = 2$. In this case there is only one conjugacy class of subgroups isomorphic to $\mathrm{SL}_2(2) \cong S_3$ and so all such subgroups are derived subgroups of Levi subgroups. Thus it remains to show that G_j and G_k form a standard pair. Now suppose that $L = G_k$ and $L' = G_j$ are the Levi groups associated to the non-incident point-line pairs (p, l) and (p', l'). If L and L' are not in one parabolic and do not form a standard pair it is easy to see that exactly one of the following must happen.

(1) $p \leq l'$ but $p' \not\leq l$ or, dually
(2) $p \not\leq l'$ but $p' \leq l$, or
(3) $p \not\leq l'$ and $p \not\leq l$.

In the third case, one verifies easily that p, p' and $l \cap l'$ are on a line. Moreover, there is an element x fixing p, p', $l \cap l'$ as well as l and l'. Thus, $x \in G_j' \cap G_k$ commutes with all of G_i, a contradiction to the fact that G_i and G_j form a standard pair in $G_{i,j}$. A GAP computation shows that the amalgams arising from cases 1. and 2. collapse. $\square$

Theorem 3.4. *Suppose (G, γ) is a completion of a weak CT amalgam $\mathscr{G}$ with connected simply-laced 3-spherical (i.e. containing no triangles) diagram Γ. If one edge of $\mathscr{G}$ is classical then $\mathscr{G}$ is classical.*

Proof Suppose $\{i_0, i_1\} \in E$ is any edge. Since Γ is connected and has no triangles, there exists a path $i_0, i_1, \ldots, i_{n-1}, i_n$ such that $\Gamma_{\{i_{j-1}, i_j, i_{j+1}}$ has type A_3. and $\{i_{n-1}, i_n\}$ is classical. Then, by Proposition 3.3 $\{i_0, i_1\}$ is classical. $\square$

4. Weak Curtis-Tits amalgams without classical edges

Let $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ be a weak Curtis-Tits amalgam over $\mathbb{F}_q$ ($q \geq 7$ odd) with connected 3-spherical simply-laced diagram Γ, of rank at least 3. We shall suppose that $\mathscr{G}$ has no classical edges. Our main result on such amalgams is the following.

Proposition 4.1. *Suppose $\mathscr{G}$ is a weak Curtis-Tits amalgam over $\mathbb{F}_q$ ($q \geq 7$) with connected simply-laced 3-spherical diagram without classical edges. Then, $\mathscr{G}$ collapses.*

Since Γ has at least 3 nodes, is connected and 3-spherical, for any $k \in I$, there is $K \subseteq I$ of size 3 with $k \in K$ such that the induced subamalgam $\mathscr{G}_K$ has diagram Γ_K of type A_3. Moreover, the two edges in $\mathscr{G}_K$ are non-classical. Clearly if $\mathscr{G}_K$ collapses, then so does $\mathscr{G}$.

Therefore, from now on we assume that $I = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$ and show that $\mathscr{G}$ collapses.

4.1. **Non-classical edges**

In this subsection we shall consider a single non-classical edge amalgam $\mathscr{H} = \{\mathbf{H}_i, \mathbf{H}_{1,2} = \mathbf{H}, \mathbf{g}_{i,j} \mid \{i, j\} = \{1, 2\} = I\}$. We are interested in subgroups of $\mathbf{H}_1$ and $\mathbf{H}_2$ centralizing, normalizing or coinciding. We write $\mathbf{H} = \mathrm{SL}(V) \cong \mathrm{SL}_3(q)$.

We shall focus on transvections and (orthogonal) reflections.

We use the notation from Subsection 3.1.

A subgroup of type $\mathrm{SO}_3(q)$ is the special orthogonal group associated to a non-degenerate symmetric bilinear form β of Witt index 1 with associated quadratic form $\mathcal{Q}(\cdot) = \beta(\cdot, \cdot)$. For any β non-degenerate 1-space w we let τ_w be the orthogonal involution having eigenvalue 1 on w and eigenvalue -1 on $w^\perp$.

A subgroup of type $\mathrm{SL}_2(q)$ is the derived subgroup of the simultaneous stabilizer of a 1-space p and 2-space L such that $p \not\subseteq L$. For any 2-space U and 1-space v, let $T_{v,U} \cong \mathbb{F}_q^+$ be the group of transvections fixing U vector-wise and preserving all 2-spaces on v.

Lemma 4.2. (1) *We have*

$$N_{\mathrm{SL}_3(q)}(\mathrm{SL}_2(q)) = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} \mid (ad - bc)e = 1 \right\},$$

$$C_{\mathrm{SL}_3(q)}(\mathrm{SL}_2(q)) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & e \end{pmatrix} \mid a^2 e = 1 \right\},$$

$$Z(\mathrm{SL}_2(q)) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a = \pm 1 \right\}.$$

(2) *We have* $N_{\mathrm{SL}_3(q)}(\mathrm{SO}_3(q)) = \mathrm{SO}_3(q)$ *and* $C_{\mathrm{SL}_3(q)}(\mathrm{SO}_3(q)) = \{1\}$.

Proof (1) These are easy matrix computations.

(2) We have $\mathrm{SO}_3(q) = N_{\mathrm{SL}_3(q)}(\mathrm{SO}_3(q))$ since $\mathrm{SO}_3(q)$ is maximal in $\mathrm{SL}(V)$. Hence, $C_{\mathrm{SL}_3(q)}(\mathrm{SO}_3(q)) \leq Z(\mathrm{SO}_3(q)) = \{1\}$. $\square$

Lemma 4.3. (1) *The subspaces preserved by $T_{v,U}$ are precisely those containing v and those contained in U.*

(2) $T_{v,U} \leq \mathrm{SL}_2(q)$ *if and only if* $v \subseteq L$ *and* $p \subseteq U$.

(3) $T_{v,U} \cap \mathrm{SO}_3(q) = \{1\}$.

Now assume q is odd.

(4) *The subspaces preserved by τ_w are precisely those containing w and those contained in $w^\perp$.*
(5) *τ_w commutes with $g \in \mathrm{SL}(V)$ if and only if g preserves both w and $w^\perp$.*
(6) *τ_w commutes with $T_{v,U}$ if and only if $w \subseteq U$ and $v \perp w$.*
(7) *The commuting pairs $(\tau_w, T_{v,U})$ with $\tau_w \in \mathrm{SO}_3(q)$ and $T_{v,U} \leq \mathrm{SL}_2(q)$ are given by $v \subseteq L_1$, such that $\langle v, p_1\rangle \cap v^\perp$ contains a non-degenerate 1-space w. Then $U = \langle v, p_1\rangle$.*

Proof (1), (2), and (4) are immediate from the definitions. (3) The orbit of a singular vector under a p-element in $\mathrm{SO}_3(q)$ spans V, whereas the orbit of any vector x under $T_{v,U}$ spans a 2-space through v.

(5) This can be seen from a matrix calculation with respect to a basis in $w \cup w^\perp$. (6) This is an easy consequence of (5), but can geometrically be seen as follows. In order to centralize τ_w, the elements from $T_{v,U}$ must preserve w and hence $w \subseteq U$. Naturally, then $T_{v,U}$ must also preserve $w^\perp$, but the 2-spaces preserved by $T_{v,U}$ are exactly those containing v. Thus $v \in w^\perp$. Now one verifies that these conditions suffice for τ_w and $T_{v,U}$ to commute (e.g. using a matrix representation on a basis $\{v, w, x\}$ for some $x \in w^\perp$).

(7) Combining (2) and (6) we find that we must have $v, w \subseteq U$ and $p_1 \subseteq U$ and $v \subseteq L_1$ but $p_1 \not\subseteq L_1$. Moreover, we must have $v \perp w$. It follows that $U = \langle v, p_1\rangle$ and $w \subseteq U \cap v^\perp$.

□

4.1.1. *Type* SL-SL. In this case we assume that, for $i = 1, 2$, $\mathbf{H}_i \cong \mathrm{SL}_2(q)$ is the derived subgroup of the simultaneous stabilizer of a 1-space p_i and a 2-space L_i of V such that $p_i \not\subseteq L_i$, that is $\mathbf{H}_i$ is a Levi component of the parabolic subgroups stabilizing p_i and L_i. Note that by Lemma 3.2, these are the only two maximal subgroups containing $\mathbf{H}_i$.

Since we also assume that $\mathbf{H} = \langle \mathbf{H}_1, \mathbf{H}_2\rangle$ we must have $p_1 \neq p_2$ and $L_1 \neq L_2$. Let $p_* = L_1 \cap L_2$ and let $L_* = \langle p_1, p_2\rangle$. Also, let $r_i = L_i \cap L_*$ and $m_i = \langle p_i, p_*\rangle$ for $i = 1, 2$.

There are four possible configurations (shown in Figure 1.).

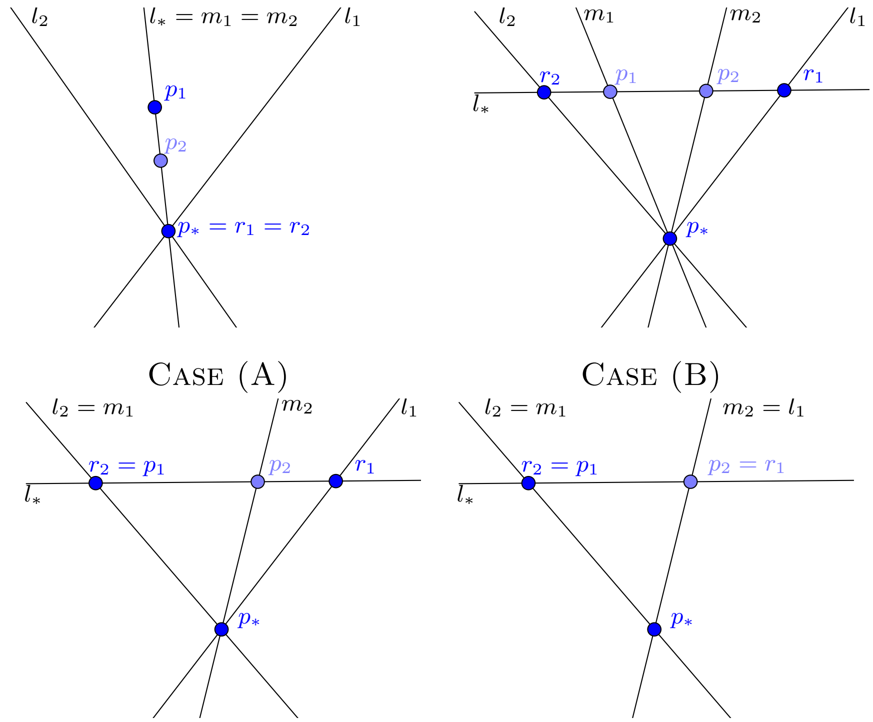
(A) $r_1 = r_2 = p_*$ and $m_1 = m_2 = l_*$,
(B) p_i, r_i, and p_* are all distinct,
(C) $p_1 = r_2 \subseteq l_2$, $p_2 \not\subseteq l_1$, and r_2 are distinct,
(D) $p_1 = r_2 \subseteq l_2$ and $p_2 = r_1 \subseteq l_1$.

We will consider the transvection groups T_{r_1,l_*}, T_{r_2,l_*}, T_{p_*,m_1}, T_{p_*,m_2} and the groups they generate depending on the cases above.

For a subspace X of V we shall write $P_X = \mathrm{Stab}_{\mathrm{SL}(V)}(X)$.

Lemma 4.4. *One of the following occurs:*

(1) *In case (A) we have $T_{r_1,l_*} = T_{r_2,l_*} = T_{p_*,m_1} = T_{p_*,m_2} \leq \mathbf{H}_1 \cap \mathbf{H}_2$.*



Case (A) Case (B)

Case (C) Case (D)

Figure 1.

(2) *In cases (B), (C), and (D) we have the following* $\langle T_{r_i,l_*}, T_{p_*,m_i}\rangle = \mathbf{H}_i$ *and* $[T_{p_*,m_1}, T_{p_*,m_2}] = 1 = [T_{r_1,l_*}, T_{r_2,l_*}]$. *Also,* $\langle T_{p_*,m_1}, T_{r_2,l_*}\rangle \leq (P_{p_1} \cap P_{l_2})$ *and* $\langle T_{p_*,m_2}, T_{r_1,l_*}\rangle \leq (P_{p_2} \cap P_{l_1})$. *Finally,*

(3) *In Case (B) we have*
$$\langle T_{p_*,m_1}, T_{r_2,l_*}\rangle \cong \mathrm{SL}_2(q) \text{ and } \langle T_{p_*,m_2}, T_{r_1,l_*}\rangle \cong \mathrm{SL}_2(q).$$

(4) *In Case (C) we have* $\langle T_{p_*,m_2}, T_{r_1,l_*}\rangle \cong \mathrm{SL}_2(q)$ *and* $\langle T_{p_*,m_1}, T_{r_2,l_*}\rangle$ *is the unipotent radical of* $P_{p_1} \cap P_{l_2}$.

(5) *In Case (D)* $\langle T_{p_*,m_1}, T_{r_2,l_*}\rangle$ *is the unipotent radical of* $P_{p_1} \cap P_{l_2}$, *and* $\langle T_{p_*,m_2}, T_{r_1,l_*}\rangle$ *is the unipotent radical of* $P_{p_2} \cap P_{l_1}$.

Proof This is an easy verification. □

Corollary 4.5. *In cases (B), (C), and (D) we have the following:*

(1) $\mathbf{H}_1 < \langle T_{r_1,L_*}, T_{r_2,L_*}, T_{p_*,M_1}\rangle \leq P_{p_1}$.
(2) $\mathbf{H}_1 < \langle T_{r_1,L_*}, T_{p_*,M_2}, T_{p_*,M_1}\rangle \leq P_{l_1}$
(3) T_{p_*,M_2} *is the unipotent radical of* $P_{l_1} \cap \mathbf{H}_2$,
(4) T_{r_2,L_*} *is the unipotent radical of* $P_{p_1} \cap \mathbf{H}_2$.

The same hold when switching the roles of 1 *and* 2.

4.1.2. *Type* SL-SO. In this subsection we shall consider the following situation. Let $H = \mathrm{SL}_3(q) = \mathrm{SL}(V)$ have subgroups $\mathbf{H}_1 \cong \mathrm{SL}_2(q)$ and $\mathbf{H}_2 \cong \mathrm{SO}_3(q)$.

Lemma 4.6. (1) *Suppose that $s_1 \in p_1^\perp \cap L_1$ is non-isotropic (which happens e.g. if p_1 is isotropic, or if l_1 is degenerate), and let $w \in s_1 - \{0\}$. Then, $\tau_w \in \mathbf{H}_1 \cap \mathbf{H}_2$.*

(2) *Let p_1 be non-isotropic so that $L_1 = p_1^\perp$ is hyperbolic. Then, $\mathbf{H}_1 \cap \mathbf{H}_2 \cong \mathrm{GO}_2^+(q) \cong D_{2(q-1)}$.*

(3) *Let p_1 be non-isotropic so that $L_1 = p_1^\perp$ is elliptic. Then, $\mathbf{H}_1 \cap \mathbf{H}_2 \cong \mathrm{GO}_2^-(q) \cong D_{2(q+1)}$.*

(4) *Suppose that $s_1 \in p_1^\perp \cap L_1$ is isotropic, let $v \in s_1$ and let $U = \langle v, p_1\rangle = s_1^\perp$. Then, $T_{v,U} \leq \mathbf{H}_1$ commutes with the group $\langle \tau_w \colon w \in U - \{s_1\}\rangle$, which has order $2q$.*

Proof (1) The reflection $\tau_w \in \mathbf{H}_2$ fixes s_1 vector-wise and preserves $p_1 \subseteq s_1^\perp$ and $s_1^\perp \cap L_1$. Hence $\tau_w \in \mathbf{H}_1$ as well.

(2) and (3) This is quite clear.

(4) By (1) we have $T_{v,U} \leq \mathbf{H}_1$. Since all $w \in s_1^\perp - s_1$ are non-degenerate, τ_w exists and by (2) τ_w commutes with $T_{v,U}$. Fixing a basis v, r for $s_1^\perp$ where $r \in p_1$, we see that if $w = r + \lambda v$, then

$$\tau_w \leftrightarrow \begin{pmatrix} 1 & 0 \\ 2\lambda & -1 \end{pmatrix}.$$

Moreover, if an element h of $\mathrm{SO}_3(q)$ fixes $s_1^\perp$ vector-wise, then it has to fix every 2-space on s_1 different from $s_1^\perp$ as well. Since each of these contains exactly one more singular 1-space, h must be scalar, hence trivial. The claim follows. □

4.1.3. *Type* SO-SO. Fix q odd. In this case we assume that, for $i = 1, 2$, $\mathbf{H}_i \cong \mathrm{SO}_3(q)$ is the special orthogonal group associated to a non-degenerate symmetric bilinear form β_i with quadratic form $\mathcal{Q}(\cdot) = \beta_i(\cdot,\cdot)$ on V. We assume that β_1 and β_2 are not proportional so that $\mathbf{H}_1 \neq \mathbf{H}_2$, and $\mathbf{H} = \langle \mathbf{H}_1, \mathbf{H}_2\rangle$.

Lemma 4.7. *Suppose that, for $i = 1, 2$, w_i is non-isotropic with respect to β_i. Then, τ_{w_1} and τ_{w_2} commute if and only if one of the following holds:*

(1) *$w_1 = w_2$ and $w_1^{\perp_1} = w_2^{\perp_2}$ or*

(2) *$w_1 \perp_i w_2$ for $i = 1, 2$.*

Proof This follows from Lemma 4.3 part (5). □

Corollary 4.8. *There exist at least one pair $(\tau_1, \tau_2) \in \mathbf{H}_1 \times \mathbf{H}_2$ with $[\tau_1, \tau_2] = 1$.*

Proof By Lemma 4.7 it suffices to count pairs (w_1, w_2) of 1-spaces such that w_i is β_i-non-isotropic and $w_1 \perp_i \beta_2$ for $i = 1, 2$.

Since $\mathbf{H}_1$ and $\mathbf{H}_2$ generate $\mathbf{H}_{1,2} = \mathrm{SL}(V_{1,2})$, β_1 and β_2 are not proportional. Hence, there exists w_1 which is β_1-non-isotropic and β_2 isotropic. Now consider $w_2 = w_1^{\perp_1} \cap w_1^{\perp_2}$. This is a 1-space since $w_1 \subseteq w_1^{\perp_2} - w_1^{\perp_1}$. Then w_2 is β_2-non-isotropic since β_2 has Witt index 1. Now τ_{w_1} and τ_{w_2} commute by Lemma 4.7. □

5. Weak Curtis-Tits amalgams of type A_3

Lemma 5.1. *In any completion G of $\mathscr{G}$ we have one of the following:*

(1) $\mathbf{G}_2 \cap \mathbf{G}_i = \{1\}$*, for both* $i = 1, 3$,

(2) *For* $i = 1, 3$, $\mathbf{G}_i \cong \mathrm{SL}_2(q)$ *is the Levi group in* $\mathbf{G}_{i,2} = \mathrm{SL}(V_{i,2})$ *associated to the pair* $(p_i, l_i = p_i^{\perp_i})$*, where* $\mathbf{G}_2$ *preserves the non-degenerate symmetric bilinear form* β_i *on* $V_{i,2}$*. In this case* $\mathbf{G}_1 \cap \mathbf{G}_2 = \mathbf{G}_2 \cap \mathbf{G}_3 = \langle \tau \rangle \cong C_2$*, where* τ *acts as* τ_{p_i} *on* $V_{i,2}$.

Proof Clearly $\mathbf{G}_2 \cap \mathbf{G}_i = \{1\}$ if $G = \{1\}$ itself, so assume otherwise. Without loss of generality assume $i = 1$. Since $[\mathbf{G}_1, \mathbf{G}_3] = 1$, we have

$$\mathbf{G}_2 \cap \mathbf{G}_3 \leq C_{\mathbf{G}_2}(\mathbf{G}_1).$$

In case $\mathbf{G}_1 \cong \mathrm{SO}_3(q)$, the claim follows from Lemma 4.2 since $C_{\mathbf{G}_2}(\mathbf{G}_1) = C_{\mathbf{G}_{1,2}}(\mathbf{G}_1) \cap \mathbf{G}_2 = \{1\}$.

So now we assume $\mathbf{G}_1 \cong \mathrm{SL}_2(q)$ is the Levi group of $\mathrm{SL}(V_{1,2})$ associated to the pair (p_1, l_1) and we assume that $1 \neq z \in \mathbf{G}_2 \cap \mathbf{G}_3$. Suppose that $p_1 = e_3$ and $l_1 = \langle e_1, e_2 \rangle$. Then, $C_{\mathbf{G}_{1,2}}(\mathbf{G}_1) = \{\mathrm{diag}(a, a, e) \colon a^2 e = 1\}$ so we have

$$z = \mathrm{diag}(a, a, e) \in C_{\mathbf{G}_2}(\mathbf{G}_1) \qquad \text{with } e = a^{-2} \text{ and } a \neq 1.$$

Now suppose $\mathbf{G}_2 = \mathrm{SO}_3(q)$ associated to some non-degenerate symmetric bilinear form β_1 of $V_{1,2}$. Note that we cannot have $a = e$ as $Z(\mathbf{G}_2) = 1$. Thus z has two distinct eigenspaces, namely l_1 of value a and p_1 of value $e = a^{-2}$. As l_1 cannot be totally isotropic, there is some non-isotropic vector v on it. But then $\beta_1(v, v) = \beta_1(z(v), z(v)) = \beta_1(av, av) = a^2 \beta_1(v, v)$ so $a = \pm 1$ and hence $e = 1$. Thus z is an involution of the form $z = \tau_w$, which means that $w = p_1$ is a non-degenerate 1-space such that $w^{\perp_1} = l_1$. Note also that in fact $z \in \mathbf{G}_1$ as well. Thus if $\mathbf{G}_2 \cap \mathbf{G}_3 \neq \{1\}$, then $\mathbf{G}_2 \cap \mathbf{G}_3 = C_{\mathbf{G}_2}(\mathbf{G}_1) = \langle z \rangle \leq \mathbf{G}_1 \cap \mathbf{G}_2$. By the case $\mathbf{G}_1 \cong \mathrm{SO}_3(q)$, we now see that $\mathbf{G}_3 \ncong \mathrm{SO}_3(q)$, so $\mathbf{G}_3 \cong \mathrm{SL}_2(q)$. But then, by symmetry $\mathbf{G}_1 \cap \mathbf{G}_2 = \mathbf{G}_2 \cap \mathbf{G}_3$.

Finally assume that $\mathbf{G}_2 \cong \mathrm{SL}_2(q)$ is also a Levi subgroup, associated to subspaces p_2 and l_2 of $V_{1,2}$. If $z \in \mathbf{G}_2$, then the only 2-spaces it could possibly stabilize must be those on p_2 or it must be l_2 itself. However, no 2-spaces on p_2 can have eigenvalue $a \neq 1$ since p_2 has eigenvalue 1 for all elements of $\mathbf{G}_2$. Thus $l_1 = l_2$. Apparently the eigenvalue e-space then must be $p_2 = p_1$. Hence $e = 1$ and $a = -1$. But this means that $\mathbf{G}_1 = \mathbf{G}_2$ which cannot generate $\mathbf{G}_{1,2}$, a contradiction. □

Corollary 5.2. *Let $\mathbf{G}_0 = \mathbf{G}_1 \cap \mathbf{G}_2 \cap \mathbf{G}_3$. In any completion G of $\mathscr{G}$ we have one of the following*

(1) $\mathbf{G}_0 = \{1\}$,
(2) *For $i = 1, 3$, $\mathbf{G}_i \cong \mathrm{SL}_2(q)$ is the Levi group in $\mathbf{G}_{i,2} = \mathrm{SL}(V_{i,2})$ associated to the pair $(p_i, l_i = p_i^{\perp_i})$, where $\mathbf{G}_2$ preserves the non-degenerate symmetric bilinear form β_i on $V_{i,2}$. In this case $\mathbf{G}_1 \cap \mathbf{G}_2 \cap \mathbf{G}_3 = \langle \tau \rangle \cong C_2$, where τ acts as τ_{p_i} on $V_{i,2}$.*

Lemma 5.3. *We have $C_{\mathbf{G}_i}(\mathbf{G}_2) = \{1\}$ for $i = 1, 3$.*

Proof Without loss of generality let $i = 1$. First of all, we have $C_{\mathbf{G}_i}(\mathbf{G}_2) \leq C_{\mathbf{G}_{1,2}}(\mathbf{G}_2)$, so if $\mathbf{G}_2 \cong \mathrm{SO}_3(q)$, then we're done by Lemma 4.2. Next, assume that $\mathbf{G}_2 \cong \mathrm{SL}_2(q)$. If $\mathbf{G}_1 \cong \mathrm{SL}_2(q)$, then the conclusion follows as in the proof of Lemma 5.1. If $\mathbf{G}_1 \cong \mathrm{SO}_3(q)$, then it follows as in the proof of Lemma 5.1 (with the roles of $\mathbf{G}_1$ and $\mathbf{G}_2$ interchanged) that if $C_{\mathbf{G}_1}(\mathbf{G}_2) \neq \{1\}$, then this is because $\mathbf{G}_2$ is the Levi group of the pair of subspaces (p_2, l_2) in $V_{1,2}$, where p_2 is β_1-non-isotropic and $p_2 \perp_1 l_2$. We then have $C_2 \cong \langle \tau_{p_2} \rangle = C_{\mathbf{G}_1}(\mathbf{G}_2) \leq \mathbf{G}_1 \cap \mathbf{G}_2$. However, by the result of Lemma 5.1, we must have $\mathbf{G}_1 \cap \mathbf{G}_2 = \{1\}$ if at least one of $\mathbf{G}_1$ and $\mathbf{G}_3$ is isomorphic to $\mathrm{SO}_3(q)$, a contradiction. □

Lemma 5.4. *If $\mathbf{G}_3 \cong \mathrm{SO}_3(q)$, then there exists no commuting pair (g_1, g_2) with $g_i \in \mathbf{G}_i - \{1\}$ for $i = 1, 2$.*

Proof Suppose that there is such a pair. Note that, by Lemma 5.1, we have $\mathbf{G}_2 \cap \mathbf{G}_3 = \{1\}$. Therefore $g_2 \notin \mathbf{G}_3$. Since $\mathbf{G}_3$ is maximal in $\mathbf{G}_{2,3}$ we have $\langle g_2, \mathbf{G}_3 \rangle = \mathbf{G}_{2,3}$. Thus, under these assumptions, g_1 commutes $\mathbf{G}_2 \leq \mathbf{G}_{2,3}$. In other words, $g_1 \in C_{\mathbf{G}_1}(\mathbf{G}_2)$. But this contradicts Lemma 5.3. □

Lemma 5.5. *If $\mathbf{G}_i \cong \mathrm{SO}_3(q)$ for some $i \in \{1, 3\}$, then $\mathscr{G}$ collapses.*

Proof Let us say $\mathbf{G}_3 \cong \mathrm{SO}_3(q)$. If $\mathbf{G}_1 \cong \mathbf{G}_2 \cong \mathrm{SO}_3(q)$, then we obtain a contradiction from Corollary 4.8 and Lemma 5.4. If $\mathbf{G}_1 \cong \mathbf{G}_2 \cong \mathrm{SL}_2(q)$, then, by Lemma 4.4, we either find a pair (g_1, g_2) of non-trivial commuting elements with $g_i \in \mathbf{G}_i - \{1\}$ for $i = 1, 2$, which contradicts Lemma 5.4, or we have $\mathbf{G}_1 \cap \mathbf{G}_2 \neq \{1\}$, which contradicts Lemma 5.1.

We still need to consider the following two cases: Either $\mathbf{G}_1 \cong \mathrm{SL}_2(q)$ and $\mathbf{G}_2 \cong \mathrm{SO}_3(q)$, or $\mathbf{G}_2 \cong \mathrm{SL}_2(q)$ and $\mathbf{G}_1 \cong \mathrm{SO}_3(q)$. However, by Lemma 4.6 Parts (1) and (4), we either find a pair (g_1, g_2) of non-trivial commuting elements with $g_i \in \mathbf{G}_i - \{1\}$ for $i = 1, 2$, which contradicts Lemma 5.4, or we have $\mathbf{G}_1 \cap \mathbf{G}_2 \neq \{1\}$, which contradicts Lemma 5.1. □

5.1. Weak Curtis-Tits amalgams of type LOL

Proposition 5.6. *Suppose $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ is a weak Curtis-Tits amalgam of type A_3 in which $\mathbf{G}_1 \cong \mathbf{G}_3 \cong \mathrm{SL}_2(q)$ and $\mathbf{G}_2 \cong \mathrm{SO}_3(q)$. Then, $\mathscr{G}$ collapses.*

Proof We consider the cases given by Lemma 5.1. First assume that $\mathbf{G}_i \cap \mathbf{G}_2 = \{1\}$ for $i = 1, 3$. For $i = 1, 3$, let $\mathbf{G}_2 \cong \mathrm{SO}_3(q)$ be associated to non-degenerate symmetric bilinear form β_i on $V_{i,2}$ and let $\mathbf{G}_i$ be the Levi group associated to the pair (p_i, l_i) of subspaces in $V_{i,2}$. Also, let $s_i \subseteq p_i^{\perp_i} \cap l_i$. Note that by Lemma 4.6 part (1) s_i is isotropic and in particular, p_i is non-isotropic and l_i is non-degenerate for β_i. Let $U_i = \langle s_i, p_i \rangle$. Then, by Lemma 4.6 part (4) $T_{s_i,U_i} \leq \mathbf{G}_i$ commutes with $D_{2q} \cong D_i = \langle \tau_w \colon w \subseteq U_i - \{s_i\} \rangle \leq \mathbf{G}_2$. Claim:The "orthogonal transvection group" of order q in D_i has a unique fixed 1-space p, which is isotropic, and a unique fixed 2-space $l = p^{\perp_i}$, which is degenerate. As the only proper maximal subgroups containing $\mathbf{G}_{4-i}$ are the parabolic subgroups for p_{4-i} and l_{4-i}, we have $\langle \mathbf{G}_{4-i}, D_i \rangle = \mathbf{G}_{2,4-i}$ unless possibly if p_{4-i} is isotropic or l_{4-i} is degenerate. However, as we saw above, this is not the case. But this means that T_{s_i,U_i} commutes with $\mathbf{G}_{2,4-i} \geq \mathbf{G}_{4-i}$. This contradicts Lemma 5.3.

Next, we consider case (2) of Lemma 5.1. Using the notation from above, we now know that s_i is non-isotropic and that $\mathbf{G}_1 \cap \mathbf{G}_2 = \langle \tau \rangle = \mathbf{G}_2 \cap \mathbf{G}_3$, where $\tau = \tau_{s_i}$ on $V_{i,2}$. We note that $p_i^{\perp_i} \neq l_i$. For if this is the case, then apparently p_i is non-degenerate. But then $p_i^{\perp_i}$ is either elliptic or hyperbolic and so by Lemma 4.6 $\mathbf{G}_2 \cap \mathbf{G}_i$ contains $D_{2(q-1)}$ or $D_{2(q+1)}$, which is impossible.

Now identify $V = V_{1,2} = V_{2,3}$ so that $\beta_1 = \beta_3 := \beta$. We now claim that, for some $i \in \{1, 3\}$, there is an involution $\tau_w \in \mathbf{G}_2$, where $w \subseteq V$ is a β-non-isotropic 1-space, and a group of transvections $T_{v,U} \leq \mathbf{G}_i$, where v is some 1-space on the 2-space U such that

(1) $\langle \tau_w, \mathbf{G}_{4-i} \rangle = \mathbf{G}_{2,(4-i)}$,
(2) $[T_{v,U}, \tau_w] = 1$.

It will then follow that $[T_{v,U} \leq C_{\mathbf{G}_i|(\mathbf{G}_2}$, which contradicts Lemma 5.3.

To ensure (1) we must ensure that τ_w is not in a parabolic with $\mathbf{G}_{4-i}$, which requires all of the following:

(a) $w \neq p_{4-i}$,
(b) $w \nsubseteq l_{4-i}$,
(c) $w \neq l_{4-i}^{\perp}$, and
(d) $w \nsubseteq p_{4-i}^{\perp}$.

Note here that $l_{4-i} \neq p_{4-i}^{\perp}$ (and $l_{4-i}^{\perp} \neq p_{4-i}$).

To ensure (2) we do the following. For each line U on p_i, let $v = U \cap l_i$, and let $w \subseteq v^{\perp} \cap U$. We must find U, v, w such that w is non-degenerate, since in that case τ_w commutes with $T_{v,U} \leq \mathbf{G}_i$.

We now claim that such w lie on a non-degenerate quadric $\mathcal{K}$ and that at most 3 lines U do not give a suitable w. Once this is proved we note that lines l_{4-i} and $p_{4-i}^{\perp}$ can each meet this quadric in at most 2 points, so that conditions (a)-(d) above can be satisfied once $((q+1)-3)-(2\times 2+1+1)>0$, that is $q\geq 9$.

To prove that $\mathcal{K}$ exists, first suppose that p_i is isotropic. Fix a basis $e_0\in p_i$, $e_1\in s_i$ and $e_2\in l_i-s_i$. □

Proof

(**??**) Let us assume we have a group G that admits a nonstandard diagram of type A_3, that is there are three groups G_1,G_2,G_3 so that

(a) $G_1\cong \mathrm{SL}_2(q)\cong G_3$ and $G_2\cong \mathrm{SO}_3(q)$
(b) G_1 and G_2 commute.
(c) $G_{12}=\langle G_1,G_2\rangle$ and $G_{23}=\langle G_2,G_3\rangle$ are both isomorphic to $\mathrm{SL}_3(q)$.

We shall prove that such a group does not exist. We first consider the group G_{12} in its natural action on the three dimensional space V We will also consider a quadratic form that determines the group $G_2=\mathrm{SO}_3(q)$.

First note that in its embedding in G_{12} the group G_1 can be identified as the stabilizer of a vector v_1 and a line L_1 with $v_1\notin L_1$. Note also that if $v\in V$ is a non degenerate vector then it determines uniquely a reflection $\tau_v\in G_2$ that fixes v and has $v^\perp$ as its -1 -eigenspace. Also the stabiliser in G_1 of a vector $w\in L_1$ is generated by a transvection T_w. We note that if

$$\langle u\rangle=\langle w,v_1\rangle\cap w^\perp$$

is non-degenerate, then τ_u and T_w commute.

As a consequence for each $w\in L_1$ so that $\langle w,v_1\rangle\cap w^\perp$ is non degenerate we can find a pair of elements $T\in G_1,\tau\in G_2$ that commute. It follows that T commutes with the group $\langle \tau,G_3\rangle$. We would like to find a pair T,τ so that $\langle\tau,G_3\rangle=G_{23}$, leading to a contradiction.

We first note that the number of choices of pairs T,τ depends on the choice of v_1,L_1.

The worse case scenario is when $v_1^\perp$ and L_1 are nondegenrate hyperbolic lines. In this case the point $w\in L_1$ cannot be chosen to be degenerate and it cannot be chosen so that the line $\langle w,v_1\rangle$ is degenerate. This excludes points and so we are left with $q-3$ pairs (w,u). Again in this worse case scenario, $L_1\cap v_1^\perp$ is non degenerate so two of the pairs are going to be $(v_1^\perp\cap L_1,v_1)$ and respectively $(v,L_1^\perp)$ where $L_1^\perp\in\langle v,v_1\rangle$. We also note that no three of the vectors u can be collinear.

Now without loss of generality we translate the picture in the group G_{23}, that is we have at least $q-3$ pairs (w_i,u_i) w its the property that $w_i\perp u_i$ and all the lines $\langle w,u\rangle$ pass through u_0 and that u_1 is perpendicular to all the w_i. We need to show than not all the involutions described by these pairs are included in the two parabolic groups containing G_3. We note that if G_2 is the stabiliser of a vector v_2 and a line L_2 then pair w_i,u_i is in the point stabiliser parabolic if and only if $v_2=u_i$ or $v_2\in u_i^\perp$ and in the point stabiliser parabolic if and only if $L_2 = u_i^{perp}$ or $u_i \in L_2$. Note that again in the worse case scenario we would have to exclude 6 pairs and so if $q-3>6$ we obtain a contradiction. Also, for the remaining cases note that WLOG we can assume that the excluded choices are (w_i, u_i) and that they come in two triplets so that among each of the triplets two of the u_i will have to be perpendicular to a third. in particular we have that $u_0 \perp u_i$ and $u_1 \perp u_j$. Since $u_i \perp \langle u_i.u_0\rangle \cap L_1$ we get a contradiction. A similar arguments shows that u_1 cannot be perpendicular to any of the elements. It follows that in this case only need to exclude 4 elements and solving $q=9$ as well.

5.2. **Weak Curtis-Tits amalgams of type LLL**

For $i=1,3$ let $\mathbf{G}_{i,2} = \mathrm{SL}(V_{i,2})$, where $V_{i,2}$ is a 3-dimensional vector space over $\mathbb{F}_q$. We let $\mathbf{G}_i$ be the Levi group of $\mathbf{G}_{i,2}$ associated to the pair (p_i, l_i) of non-incident 1- and 2-spaces of $V_{i,2}$. Also we let $\mathbf{G}_2$ be the Levi group of $\mathbf{G}_{i,2}$ associated to the pair (p_2^i, l_2^i) of non-incident 1- and 2-spaces of $V_{i,2}$. Note that, from Lemma 5.1 we conclude that Case (A) of Lemma 4.4 cannot occur in either $\mathbf{G}_{i,2}$. Moreover, by Proposition 3.3 we can assume that Case (D) does not occur in either $\mathbf{G}_{i,2}$.

By Corollary 4.5 there is a unique transvection group T in $P_{p_1} \cap \mathbf{G}_2$ and, likewise a unique transvection group S in $P_{l_1} \cap \mathbf{G}_2$.

Lemma 5.7. *The two parabolics containing* $\mathbf{G}_3$ *are* $\langle T, \mathbf{G}_3\rangle$ *and* $\langle S, \mathbf{G}_3\rangle$.

Proof If this is not the case, then, without loss of generality $\langle T, \mathbf{G}_3\rangle = \mathbf{G}_{2,3}$. By Lemma 4.4 there exists a transvection group $T_1 \le \mathbf{G}_1$ which commutes with T but not with S. Since T_1 commutes with $\mathbf{G}_3$, this implies it also commutes with $S \le \mathbf{G}_{2,3}$, which is a contradiction. □

Corollary 5.8. *If G is a completion of $\mathscr{G}$, then it is a completion of an amalgam whose diagram is one of those given in Figure 2., where the vertex groups are all isomorphic to $(\mathbb{F}_q,+)$, and rank-2 groups are isomorphic to one of the following:* $\mathrm{SL}_2(q)$ *(denoted by a solid edge), the Heisenberg group over $\mathbb{F}_q$ (denoted by a dashed edge) or $(\mathbb{F}_q,+)\times(\mathbb{F}_q,+)$ (denoted by a non-edge).*

Proof This follows from Lemma 5.7 and Lemma 4.4. □

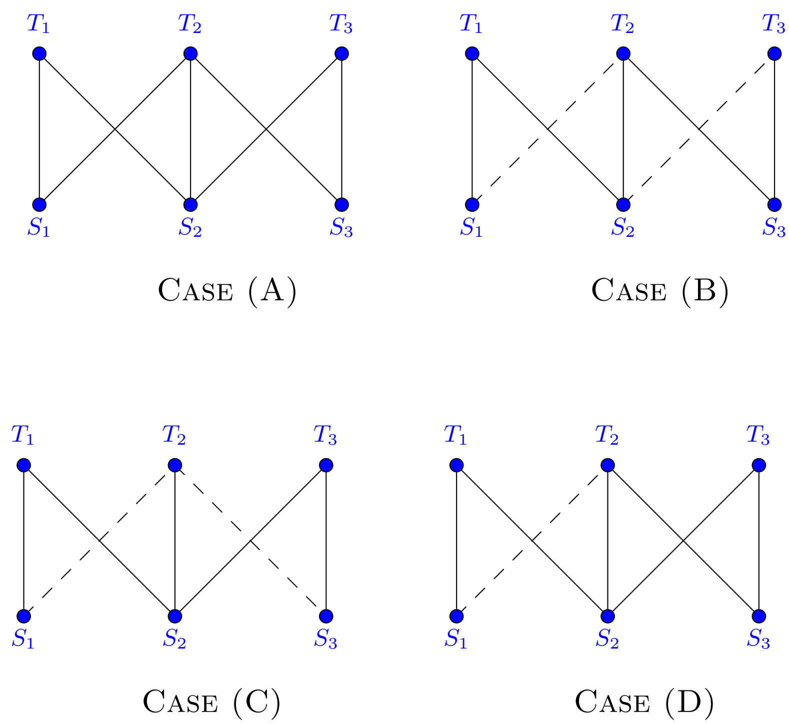


Figure 2.

6. A symplectic geometry

Let V be a 4-dimensional vector space over a field $\mathbb{K}$ with symplectic form $\beta(\cdot,\cdot)$.

We denote orthogonality between subspaces and vectors of V by $\perp$. Projective points will be denoted in lowercase, projective lines will be denoted in upper case, and projective planes will be denoted in boldface upper case. Let (e_1, f_1, e_2, f_2) be an ordered (hyperbolic) basis for V with Gram matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

6.1. **A linear subamalgam**

Let $\mathscr{Z} = \{\mathbf{Z}_i, \mathbf{Z}_{i,j}, \mathbf{z}_{i,j} \mid i,j \in \{1,2,3,4\}\}$ be the rank-2 amalgam with diagram 3

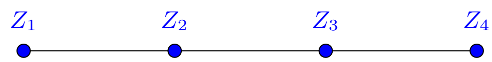


FIGURE 3. A linear diagram

Lemma 6.1. *The group* $\mathrm{Sp}(4,\mathbb{K})$ *contains a completion* Z *of* $\mathscr{Z}$. *If* $\mathrm{Char}\,\mathbb{K} \neq 2$, *we have* $Z = \mathrm{Sp}(4,\mathbb{K})$.

Proof We now set $\mathbf{Z}_i = T_{v_i}$, where $v_1 = f_1$, $v_2 = e_1 + e_2$, $v_3 = f_2$, $v_4 = e_2$. We now have that

$$\langle \mathbf{Z}_i, \mathbf{Z}_j \rangle \cong \begin{cases} \mathrm{Sp}_2(\mathbb{K}) & \text{if } v_i \not\perp v_j, \\ (\mathbb{K},+) \times (\mathbb{K},+) & \text{if } v_i \perp v_j. \end{cases}$$

Therefore we see that $\mathrm{Sp}(4,\mathbb{K})$ contains (a completion of) the amalgam, when, for $1 \le i < j \le 4$ we let $\mathbf{Z}_{i,j} = \langle \mathbf{Z}_i, \mathbf{Z}_j \rangle$ in $\mathrm{Sp}(V,\beta)$. That $\langle \mathbf{Z}_i \colon i = 1,2,3,4 \rangle = \mathrm{Sp}(4,\mathbb{K})$, when $\mathrm{Char}\,\mathbb{K} \neq 2$ follows from [2]. $\square$

For any subset $J \subseteq I = \{1,2,3,4\}$, let $\mathbf{Z}_J = \langle \mathbf{Z}_j \mid j \in J \rangle$. We briefly describe the rank-3 groups $\mathbf{Z}_{I-\{j\}}$ ($j = 1,2,3,4$). Clearly we have

$$\mathbf{Z}_{1,2,4} \cong \mathbf{Z}_{1,3,4} \cong \mathrm{SL}_2(q) \times (\mathbb{K},+)$$
$$\mathbf{Z}_{1,2,3} \cong \mathbf{Z}_{2,3,4} = P'_{\langle e_1 \rangle} = \mathrm{Stab}(e_1),$$

All isomorphisms except the last one are immediate from the amalgam and its diagram. As for the equality, we clearly have $\mathbf{Z}_{2,3,4} \subseteq P_{e_1}$ and since $\mathbf{Z}$ is perfect, in fact we have $\mathbf{Z}_{2,3,4} \subseteq P'_{\langle e_1 \rangle}$. Naturally, we have $\mathbf{Z}_{2,3,4} \le \mathrm{Stab}(e_1) \le P'_{\langle e_1 \rangle}$.

We describe a geometry Γ over $I = \{1,2,3,4\}$. Let X_i denote an object of type $i \in I$. Then,

$X_1 = p$ a 1-space in V,
$X_2 = (q, L)$ with $q \perp L$,
$X_3 = (M, \mathbf{U})$ with $M^\perp \subseteq \mathbf{U}$,
$X_4 = \mathbf{W}$ a 3-space of V.

Then incidence, denoted $\sim$, is given by:

$X_1 \sim X_2$: $\langle p, q \rangle^\perp = p^\perp \cap q^\perp = L$,
$X_1 \sim X_3$: $p \sim (M, \mathbf{U})$ iff $\langle p, M \rangle = \mathbf{U}$,
$X_1 \sim X_4$: $p \sim \mathbf{W}$ if $p \not\subseteq \mathbf{W}$,
$X_2 \sim X_3$: $(q, L) \sim (M, \mathbf{U})$ if $\mathbf{U}^\perp \subseteq L$ and $q \subseteq M$,
$X_2 \sim X_4$: $(q, L) \sim \mathbf{W}$ if $\mathbf{W} \cap L^\perp = q$ (in particular $q \subseteq W - W^\perp$).

$X_3 \sim X_4$: $(M, \mathbf{U}) \sim \mathbf{W}$ if $\mathbf{W} \cap \mathbf{U} = M^\perp$ (in particular $M^\perp \subseteq W - W^\perp$).

Lemma 6.2. (a) *The geometry* Γ *admits a type reversing involution given by*

(b) *The geometry* Γ *is linear, transversal, residually connected, and* $\mathrm{Sp}(V, \beta)$ *acts flag-transitively on* Γ.

(c)

7. Background on groups of Lie type

7.1. Automorphisms of groups of Lie type of small rank

Automorphisms of groups of Lie type are all known. In this subsection we collect some facts that we will need later on. We shall use the notation from [31]. Automorphisms of $\mathrm{SL}_n(q)$. Define automorphisms of $\mathrm{SL}_n(q)$ as follows (where $x = (x_{ij})_{i,j=1}^n \in \mathrm{SL}_n(q)$):

$$\begin{aligned} c_g &\colon x \mapsto x^g = g^{-1}xg && (g \in \mathrm{PGL}_n(q)),\\ \alpha &\colon x \mapsto x^\alpha = (x_{ij}^\alpha)_{i,j=1}^n && (\alpha \in \mathrm{Aut}(\mathbb{F}_q)),\\ \tau &\colon x \mapsto x^\tau = {}^t x^{-1} && (\text{transpose-inverse}). \end{aligned}$$

We note that for $n = 2$, τ coincides with the map $x \mapsto x^\mu$, where $\mu = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We let $\mathrm{P\Gamma L}_n(q) = \mathrm{PGL}_n(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$.
Automorphisms of $\mathrm{Sp}_{2n}(q)$. Outer automorphisms of $\mathrm{Sp}_{2n}(q)$ are of the form $\mathrm{Aut}(\mathbb{F}_q)$ as for $\mathrm{SL}_{2n}(q)$, defined with respect to a symplectic basis, or come from the group $\mathrm{GSp}_{2n}(q) \cong \mathrm{Sp}_{2n}(q).(\mathbb{F}_q^*/(\mathbb{F}_q^*)^2)$ of linear similarities of the symplectic form, where $\mathbb{F}_q^*$ acts as conjugation by

$$\delta(\lambda) = \begin{pmatrix} \lambda I_n & 0_n \\ 0_n & I_n \end{pmatrix} \qquad (\lambda \in \mathbb{F}_q^*)$$

This only provides a true outer automorphism if λ is not a square and we find that $\mathrm{PGSp}_{2n}(q) \cong \mathrm{PSp}_{2n}(q).2$ if q is odd and $\mathrm{PGSp}_{2n}(q) = \mathrm{PSp}_{2n}(q)$ if q is even. We define

$$\begin{aligned} \mathrm{\Gamma Sp}_{2n}(q) &= \mathrm{GSp}_{2n}(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)\\ \mathrm{P\Gamma Sp}_{2n}(q) &= \mathrm{PGSp}_{2n}(q) \rtimes \mathrm{Aut}(\mathbb{F}_q). \end{aligned}$$

Note that, as in $\mathrm{SL}_2(q)$, the map $\tau\colon A \to {}^tA^{-1}$ is the inner automorphism given by

$$M = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

Automorphisms of $\mathrm{SU}_n(q)$. All linear outer automorphisms of $\mathrm{SU}_n(q)$ are induced by $\mathrm{GU}_n(q)$ the group of linear isometries of the hermitian form, or are induced by $\mathrm{Aut}(\mathbb{F}_{q^2})$ as for $\mathrm{SL}_n(q^2)$ with respect to an orthonormal basis. The group $\mathrm{Aut}(\mathbb{F}_{q^2})$ has order $2e$, where $q = p^e$, p prime. We let $\Gamma\mathrm{U}_n(q) = \mathrm{GU}_n \rtimes \mathrm{Aut}(\mathbb{F}_{q^2})$ and let $\mathrm{P}\Gamma\mathrm{U}_n(q)$ denote its quotient over the center (consisting of the scalar matrices). In this case, the transpose-inverse map τ with respect to a hyperbolic basis is the composition of the inner automorphism given by

$$M = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}.$$

and the field automorphism $x \mapsto \overline{x} = x^q$ (with respect to the hyperbolic basis). The group $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$ of field automorphisms of $\mathrm{SU}_n(q)$ on a hyperbolic basis. For $\Gamma\mathrm{U}_{2n}(q)$ note that $\mathrm{Aut}(\mathbb{F}_{q^2}) = \langle\alpha\rangle$ acts with respect to an orthonormal basis $\mathcal{U} = \{u_1, \ldots, u_{2n}\}$ for the $\mathbb{F}_{q^2}$-vector space V with σ-hermitian form h preserved by the group (see [31]). . We now identify a complement $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$ of semilinear automorphisms of $\mathrm{GU}_{2n}(q)$ in $\Gamma\mathrm{U}_{2n}(q)$ with respect to a hyperbolic basis. Fix the standard hyperbolic basis $\mathcal{H} = \{e_i, f_i \colon i = 1, 2, \ldots, n\}$ so that the elements in $\mathrm{GU}(V, \mathsf{h})$ are represented by a matrix in $\mathrm{GU}_{2n}(q)$ with respect to $\mathcal{H}$. Let $\alpha \in \mathrm{Aut}(\mathbb{F}_{q^2})$ act on V via $\mathcal{U}$. Then, $\mathcal{H}^\alpha = \{e_i^\alpha, f_i^\alpha \colon i = 1, 2 \ldots, n\}$ is also a hyperbolic basis for V, so for some $A \in \mathrm{GU}_{2n}(q)$, we have $A\mathcal{H} = \mathcal{H}^\alpha$. Now the composition $\widehat{\alpha} = A^{-1} \circ \alpha$ is an α-semilinear map that fixes $\mathcal{H}$. The corresponding automorphism of $\mathrm{GU}_{2n}(q)$ acts by applying α to the matrix entries.

Remark 7.1. The following special case will be of particular interest when considering a Curtis-Tits standard pairs of type ${}^2A_3(q)$. In this case the action of $\widehat{\alpha}$ as above on $\mathrm{SU}_4(q)$ translates via the standard identification maps (see Definition 2.5) to actions on $\mathrm{SL}_2(q)$ and $\mathrm{SL}_2(q^2)$ as follows. The action on $\mathrm{SL}_2(q^2)$ is the natural entry-wise field automorphism action. The action on $\mathrm{SL}_2(q)$ will be a product of the natural entrywise action of $\widehat{\alpha}$ and a diagonal automorphism $\mathrm{diag}(f, 1)$, where $f \in \mathbb{F}_q$ is such that $\widehat{\alpha}(\eta) = f\eta$. Note that $N_{\mathbb{F}_q/\mathbb{F}_p}(f) = -1$, so in particular $\sigma = \widehat{\alpha}^e$ translates to (left) conjugation by $\mathrm{diag}(-1, 1)$ only.

Definition 7.2. Since the norm is surjective, there exists $\zeta \in \mathbb{F}_{q^2}$ such that $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\zeta) = f^{-1}$. We then have that $\mathrm{diag}(\zeta, \zeta, \zeta^{-q}, \zeta^{-q}) \in \mathrm{GU}_4(q)$ acts trivially on $\mathrm{SL}_2(q^2)$ and acts as left conjugation by $\mathrm{diag}(f^{-1}, 1)$ on $\mathrm{SL}_2(q)$. It follows that the composition $\widetilde{\alpha}$ of $\widehat{\alpha}$ and this diagonal automorphism acts entrywise as α on both $\mathrm{SL}_2(q)$ and $\mathrm{SL}_2(q^2)$. We now define

$$\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) = \langle\widetilde{\alpha}\rangle \leq \mathrm{Aut}(\mathrm{SU}_4(q)).$$

Lemma 7.3. *(See* [29, 31]*.)*

(a) *As* $\mathrm{Sp}_2(q) = \mathrm{SL}_2(q) \cong \mathrm{SU}_2(q)$, *we have*

$$\mathrm{Aut}(\mathrm{Sp}_2(q)) = \mathrm{Aut}(\mathrm{SL}_2(q)) = \mathrm{P\Gamma L}_2(q) \cong \mathrm{P\Gamma U}_2(q) = \mathrm{Aut}(\mathrm{SU}_2(q)).$$

(b) *In higher rank we have*

$$\begin{aligned}\mathrm{Aut}(\mathrm{SL}_n(q)) &= \mathrm{P\Gamma L}_n(q) \rtimes \langle \tau \rangle\\ \mathrm{Aut}(\mathrm{Sp}_{2n}(q)) &= \mathrm{P\Gamma Sp}_{2n}(q)\\ \mathrm{Aut}(\mathrm{SU}_n(q)) &= \mathrm{P\Gamma U}_n(q)\end{aligned}$$

7.1.1. *Some normalizers and centralizers.*

Corollary 7.4. *Let* $G = \mathrm{SL}_3(q)$. *Let* $\varphi\colon \mathrm{SL}_2(q) \to G$ *given by* $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ *and let* $L = \operatorname{im} \varphi$. *Then,*

$$C_{\mathrm{Aut}(G)}(L) = \langle \mathrm{diag}(a,b,b) \colon a,b \in \mathbb{F}_q^* \rangle \rtimes \langle \theta \rangle.$$

where $\theta = \tau \circ c_\nu \colon X^\theta \mapsto {}^t(\nu^{-1} X \nu)^{-1}$ *and* $\nu = \begin{pmatrix} 1 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$.

Proof This follows easily from the fact that $\mathrm{Aut}(G) \cong \mathrm{P\Gamma L}_3(q)$. Let $\tau^i \circ \alpha \circ c_g$, where c_g denotes conjugation by $g \in \mathrm{GL}_3(q)$ and $\alpha \in \mathrm{Aut}(\mathbb{F}_q)$. Using transvection matrices from L over the fixed field $\mathbb{F}_q^\alpha$ one sees that if $i = 0$, then g must be of the form $\mathrm{diag}(a,b,b)$, and if $i = 1$, then it must be of the form $\mathrm{diag}(a,b,b)\nu$, for some $a,b \in \mathbb{F}_q^*$. Then, if $\alpha \neq \mathrm{id}$, picking transvections from L with a few entries in $\mathbb{F}_q - \mathbb{F}_q^\alpha$ one verifies that α must be the identity. $\square$

8. Classification of Curtis-Tits amalgams

8.1. Fundamental root groups in Curtis-Tits standard pairs

Lemma 8.1. *Let* q *be a power of the prime* p. *Suppose that* $(\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2)$ *is a Curtis-Tits standard pair of type* $\Gamma(q)$ *as in Subsection 2.3. For* $\{i,j\} = \{1,2\}$, *let* $\mathcal{S}_j = \mathrm{Syl}_p(\mathbf{G}_j)$.

(a) *There exist two groups* $\mathbf{X}_j^{i,\varepsilon} \in \mathcal{S}_j$ ($\varepsilon = +,-$) *such that for any* $\mathbf{X} \in \mathcal{S}_j$ *we have*

$$\langle \mathbf{G}_i, \mathbf{X} \rangle \le \mathbf{P}_i^\varepsilon \text{ if and only if } \mathbf{X} = \mathbf{X}_j^{i,\varepsilon}$$

where $\mathbf{P}_i^+$ *and* $\mathbf{P}_i^-$ *are the two parabolic subgroups of* $\mathbf{G}$ *containing* $\mathbf{G}_i$. *If* $\mathbf{X} \neq \mathbf{X}_j^{i,\varepsilon}$, *then*

$$\langle \mathbf{G}_i, \mathbf{X} \rangle = \begin{cases} (\mathbf{G}_i \times \mathbf{G}_i^x) \rtimes \langle x \rangle & \text{if } \Gamma(q) = C_2(2), \\ \mathbf{G} & \text{else.} \end{cases}$$

where in the $C_2(2)$ *case* $\mathbf{X} = \langle x \rangle$.

(b) *We can select the signs ε so that $\mathbf{X}_i^{j,\varepsilon}$ commutes with $\mathbf{X}_j^{i,-\varepsilon}$, but not with $\mathbf{X}_j^{i,\varepsilon}$ and, in fact $\langle \mathbf{X}_i^{j,\varepsilon}, \mathbf{X}_j^{i,\varepsilon}\rangle$ is contained in the unipotent radical $\mathbf{U}_{i,j}^{\varepsilon}$ of a unique Borel subgroup of $\mathbf{G}_{i,j}$, namely $\mathbf{B}_{i,j}^{\varepsilon} = \mathbf{P}_i^{\varepsilon} \cap \mathbf{P}_j^{\varepsilon}$.*

Proof We first prove part 1. by considering all cases.
$\mathbf{A}_2(q)$**,** $\mathbf{q} \geq 3$**.** View $\mathbf{G} = \mathrm{SL}_3(q) = \mathrm{SL}(V)$ for some $\mathbb{F}_q$-vector space with basis $\{e_1, e_2, e_3\}$. By symmetry we may assume that $i = 1$ and $j = 2$. Let $\mathbf{G}_1$ (resp. $\mathbf{G}_2$) stabilize $\langle e_1, e_2\rangle$ and fix e_3 (resp. stabilize $\langle e_2, e_3\rangle$) and fix e_1). A root group in $\mathbf{G}_2$ is of the form $\mathbf{X}_v = \mathrm{Stab}_{\mathbf{G}_2}(v)$ for some $v \in \langle e_2, e_3\rangle$. We let $\mathbf{X}_2^+ = \mathbf{X}_{e_2}$ and $\mathbf{X}_2^- = \mathbf{X}_{e_3}$. It clear that for $\varepsilon = +$ (resp. $\varepsilon = -$) $\langle \mathbf{G}_1, \mathbf{X}_2^{\varepsilon}\rangle = \mathbf{P}^{\varepsilon}$ is contained in (but not equal to) the parabolic subgroup stabilizing $\langle e_1, e_2\rangle$ (resp. $\langle e_3\rangle$). Now suppose that $\mathbf{X} \in \mathcal{S}_2$ is different from $\mathbf{X}_2^{\varepsilon}$ ($\varepsilon = +, -$) and $\mathbf{X} = \mathbf{X}_{\lambda e_2 + e_3}$ for some $\lambda \in \mathbb{F}_q^*$. Consider the action of a torus element $d = \mathrm{diag}(\mu, \mu^{-1}, 1) \in \mathbf{G}_1$ by conjugation on $\mathbf{G}_2$. Then $\mathbf{X}^d = \mathbf{X}_{\mu\lambda e_2 + e_3}$. Since $|\mathbb{F}_q| \geq 3$, $\mathbf{X}^d \neq \mathbf{X}$ for some d and so we have

$$\langle \mathbf{G}_i, \mathbf{X}\rangle \geq \langle \mathbf{G}_i, \mathbf{X}, \mathbf{X}^d\rangle = \langle \mathbf{G}_i, \mathbf{G}_j\rangle = \mathbf{G}. \tag{8.1}$$

$\mathbf{A}_2(2)$**.** In this case $\mathcal{S}_2 = \{\mathbf{X}_2^+, \mathbf{X} = \langle r\rangle, \mathbf{X}_2^-\}$, where r is the Coxeter element fixing e_1 and interchanging e_2 and e_3. It follows that $\mathbf{G}_1^r$ is the stabilizer of the subspace decomposition $\langle e_2\rangle \oplus \langle e_1, e_3\rangle$ and hence $\langle \mathbf{G}_1, \mathbf{X}\rangle = \mathbf{G}$.
$\mathbf{C}_2(q)$**,** $\mathbf{q} \geq 3$**, X short root.** We use the notation of Subsection 2.3. First, let $i = 2$, $j = 1$, let $\mathbf{G}_2 \cong \mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$ be the stabilizer of e_1 and e_3 and let $\mathbf{G}_1 \cong \mathrm{SL}_2(q)$ be the derived subgroup of the stabilizer of the isotropic 2-spaces $\langle e_1, e_2\rangle$ and $\langle e_3, e_4\rangle$. Root groups in $\mathbf{G}_1$ are of the form $\mathbf{X}_{u,v} = \mathrm{Stab}_{\mathbf{G}_1}(u) \cap \mathrm{Stab}_{\mathbf{G}_1}(v)$, where $u \in \langle e_1, e_2\rangle$ and $v \in \langle e_3, e_4\rangle$ are orthogonal. Let $\mathbf{X}_1^+ = \mathbf{X}_{e_1,e_4}$ and $\mathbf{X}_1^- = \mathbf{X}_{e_2,e_3}$. It is easy to verify that for $\varepsilon = +$ (resp. $\varepsilon = -$) $\langle \mathbf{G}_2, \mathbf{X}_1^{\varepsilon}\rangle = \mathbf{P}^{\varepsilon}$ is contained in the parabolic subgroup stabilizing $\langle e_1\rangle$ (resp. $\langle e_3\rangle$). Now let $\mathbf{X} = \mathbf{X}_{e_1 + \lambda e_2, e_3 - \lambda^{-1} e_4}$ for some $\lambda \in \mathbb{F}_q^*$. Consider the action of a torus element $d = \mathrm{diag}(1, \mu^{-1}, 1, \mu) \in \mathbf{G}_2$ by conjugation on $\mathbf{G}_1$. Then $\mathbf{X}^d = \mathbf{X}_{\langle e_1 + \lambda\mu e_2\rangle, e_3 - \lambda^{-1}\mu^{-1} e_4}$. Since $q \geq 3$, $\mathbf{X}^d \neq \mathbf{X}$ for some d and so, for $i = 1$, and these $\mathbf{G}_2$, $\mathbf{X}$ and d, we have (8.1) again.
$\mathbf{C}_2(q)$**,** $\mathbf{q} \geq 4$**, X long root.** Now, we let $i = 1$ and $j = 2$. Root groups in $\mathbf{G}_2$ are of the form $\mathbf{X}_u = \mathrm{Stab}_{\mathbf{G}_2}(u)$ where $u \in \langle e_2, e_4\rangle$. Let $\mathbf{X}_2^+ = \mathbf{X}_{e_2}$ and $\mathbf{X}_2^- = \mathbf{X}_{e_4}$. It is easy to verify that for $\varepsilon = +$ (resp. $\varepsilon = -$) $\langle \mathbf{G}_1, \mathbf{X}_2^{\varepsilon}\rangle = \mathbf{P}^{\varepsilon}$ is contained in the parabolic subgroup stabilizing $\langle e_1, e_2\rangle$ (resp. $\langle e_1, e_4\rangle$). Now let $\mathbf{X} = \mathbf{X}_{e_2 + \lambda e_4}$ for some $\lambda \in \mathbb{F}_q^*$.

Consider the action of a torus element $d = \mathrm{diag}(\mu, \mu^{-1}, \mu^{-1}, \mu) \in \mathbf{G}_1$ by conjugation on $\mathbf{G}_2$. Then $\mathbf{X}^d = \mathbf{X}_{\mu e_2 + \mu^{-1}\lambda e_4}$. Now if $q \geq 4$, then $\mathbf{X}^d \neq \mathbf{X}$ for some d and so so, for these $\mathbf{G}_1$, $\mathbf{X}$ and d, we have (8.1) again.
$\mathbf{C}_2(q)$**,** $\mathbf{q} = 3$**, X long root.** The proof for the case $q \geq 4$ does not yield the result since, for $q = 3$, the element d centralizes $\mathbf{G}_2$. A direct computation in GAP shows that the conclusion still holds, though. Let $x \in \mathbf{X} = \mathbf{X}_{e_2 + e_4}$

send e_2 to e_4. Then $\mathbf{G}_1$ and $\mathbf{G}_1^x$ contains two short root groups fixing e_1 and e_3. Their commutators generate a long root group fixing e_1, e_2, and e_4, while being transitive on the points $\langle e_3+\lambda e_1\rangle$. Further conjugation with an element in $\mathbf{G}_1$ interchanging the points $\langle e_1\rangle$ and $\langle e_2\rangle$ yields a long root group in $\mathbf{G}_2$ different from $\mathbf{X}$ and we obtain an equation like (8.1) again.

$\mathbf{C}_2(2)$. First note that $\mathbf{G}\cong \mathrm{Sp}_4(2)\cong O_5(2)$ is self point-line dual, so we only need to consider the case where $\mathbf{G}_2=\mathrm{Stab}_{\mathbf{G}}(e_1)\cap\mathrm{Stab}_{\mathbf{G}}(e_3)$ and $\mathbf{G}_1=\mathrm{Stab}_{\mathbf{G}}(\langle e_1,e_2\rangle)\cap\mathrm{Stab}_{\mathbf{G}}(\langle e_3,e_4\rangle)$. Now $\mathcal{S}_1=\{\mathbf{X}_1^+,\mathbf{X}_1^-,\langle x\rangle\}$, where x is the permutation matrix of $(1,2)(3,4)$. The conclusion follows easily.

$^{\mathbf{2}}\mathbf{A}_3(q)$. We use the notation of Subsection 2.3. First, let $i=2$, $j=1$, let $\mathbf{G}_2\cong\mathrm{SU}_2(q)\cong\mathrm{SL}_2(q)$ be the stabilizer of e_1 and e_3 and let $\mathbf{G}_1\cong\mathrm{SL}_2(q^2)$ be the derived subgroup of the simultaneous stabilizer in $\mathbf{G}=\mathrm{SU}_4(q)$ of the isotropic 2-spaces $\langle e_1,e_2\rangle$ and $\langle e_3,e_4\rangle$. Root groups in $\mathbf{G}_1$ are of the form $\mathbf{X}_{u,v}=\mathrm{Stab}_{\mathbf{G}_1}(u)\cap\mathrm{Stab}_{\mathbf{G}_1}(v)$, where $u\in\langle e_1,e_2\rangle$ and $v\in\langle e_3,e_4\rangle$ are orthogonal. Let $\mathbf{X}_1^+=\mathbf{X}_{e_1,e_4}$ and $\mathbf{X}_1^-=\mathbf{X}_{e_2,e_3}$. It is easy to verify that for $\varepsilon=+$ (resp. $\varepsilon=-$) $\langle\mathbf{G}_2,\mathbf{X}_1^\varepsilon\rangle=\mathbf{P}^\varepsilon$ is contained in the parabolic subgroup stabilizing $\langle e_1\rangle$ (resp. $\langle e_3\rangle$). Now let $\mathbf{X}=\mathbf{X}_{e_1+\lambda e_2,e_3-\lambda^{-\sigma}e_4}$ for some $\lambda\in\mathbb{F}_{q^2}^*$. Consider the action of a torus element $d=\mathrm{diag}(1,\mu^{-1},1,\mu)\in\mathbf{G}_2$ (with $\mu\in\mathbb{F}_q^*$) by conjugation on $\mathbf{G}_1$. Then $\mathbf{X}^d=\mathbf{X}_{e_1+\lambda\mu e_2,e_3-\lambda^{-\sigma}\mu^{-1}e_4}$. There are $q-1$ choices for μ, so if $q\ge 3$, then $\mathbf{X}^d\neq\mathbf{X}$ for some d. Hence, for $i=1$, and these $\mathbf{G}_2$, $\mathbf{X}$ and d, we have (8.1) again. The case $q=2$ is a quick GAP calculation.

Now, we let $i=1$ and $j=2$. Root groups in $\mathbf{G}_2$ are of the form $\mathbf{X}_u=\mathrm{Stab}_{\mathbf{G}_2}(u)$ where $u\in\langle e_2,e_4\rangle$ is isotropic. Let $\mathbf{X}_2^+=\mathbf{X}_{e_2}$ and $\mathbf{X}_2^-=\mathbf{X}_{e_4}$. It is easy to verify that for $\varepsilon=+$ (resp. $\varepsilon=-$) $\langle\mathbf{G}_1,\mathbf{X}_2^\varepsilon\rangle=\mathbf{P}^\varepsilon$ is contained in the parabolic subgroup stabilizing $\langle e_1,e_2\rangle$ (resp. $\langle e_1,e_4\rangle$). Now let $\mathbf{X}=\mathbf{X}_{e_2+\lambda e_4}$ for some $\lambda\in\mathbb{F}_{q^2}^*$ where $\mathrm{Tr}(\lambda)=\lambda+\lambda^\sigma=0$. Consider the action of a torus element $d=\mathrm{diag}(\mu,\mu^{-1},\mu^{-\sigma},\mu^\sigma)\in\mathbf{G}_1$ (for some $\mu\in\mathbb{F}_{q^2}^*$) by conjugation on $\mathbf{G}_2$. Then $\mathbf{X}^d=\mathbf{X}_{\mu e_2+\mu^{-\sigma}\lambda e_4}$. The q^2-1 choices for μ result in $q-1$ different conjugates. Thus, if $q-1\ge 2$, then $\mathbf{X}^d\neq\mathbf{X}$ for some d and so so, for these $\mathbf{G}_1$, $\mathbf{X}$ and d, we have (8.1) again. The case $q=2$ is a quick GAP calculation. Namely, in this case, $\mathbf{X}=\langle x\rangle$, where x is the only element of order 2 in $\mathbf{G}_2\cong S_3$ that does not belong to $\mathbf{X}_1^+\cup\mathbf{X}_1^-$; it is the Coxeter element that fixes e_1 and e_3 and interchanges e_2 and e_4. Now $\langle\mathbf{G}_1,\mathbf{G}_1^r\rangle$ contains the long root group generated by the commutators of the short root group fixing e_1 in $\mathbf{G}_1$ and $\mathbf{G}_1^r$, and likewise for e_2, e_3, and e_4. In particular, we have

$$\langle\mathbf{G}_1,\mathbf{X}\rangle\ge\langle\mathbf{G}_1,\mathbf{G}_1^r\rangle\ge\langle\mathbf{G}_1,\mathbf{G}_2\rangle=\mathbf{G}\tag{8.2}$$

We now address part 2. Note that the positive and negative fundamental root groups with respect to the torus $\mathbf{B}_{i,j}^+\cap\mathbf{B}_{i,j}^-$ satisfy the properties of $\mathbf{X}_i^{j,\varepsilon}$ and $\mathbf{X}_j^{i,\varepsilon}$ so by the uniqueness statement in 1. they must be equal. Now the claims in part 2. are the consequences of the Chevalley commutator relations. □

Remark 8.2.

Explicitly, the groups $\{\mathbf{X}_i^+, \mathbf{X}_i^-\}$ $(i = 1, 2)$, possibly up to a switch of signs, for the Curtis-Tits standard pairs are as follows.

For $\Gamma = A_2$, we have

$$\mathbf{X}_1^+ = \left\{ \begin{pmatrix} 1 & b & \\ 0 & 1 & \\ & & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\}, \text{ and } \mathbf{X}_1^- = \left\{ \begin{pmatrix} 1 & 0 & \\ c & 1 & \\ & & 1 \end{pmatrix} : c \in \mathbb{F}_q \right\},$$

$$\mathbf{X}_2^+ = \left\{ \begin{pmatrix} 1 & & \\ & 1 & b \\ & 0 & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\}, \text{ and } \mathbf{X}_2^- = \left\{ \begin{pmatrix} 1 & & \\ & 1 & 0 \\ & c & 1 \end{pmatrix} : c \in \mathbb{F}_q \right\}.$$

For $\Gamma = C_2$, we have

$$\mathbf{X}_1^+ = \left\{ \begin{pmatrix} 1 & b & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & -b & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\}, \text{ and } \mathbf{X}_1^- = \left\{ \begin{pmatrix} 1 & 0 & & \\ c & 1 & & \\ & & 1 & -c \\ & & 0 & 1 \end{pmatrix} : c \in \mathbb{F}_q \right\},$$

$$\mathbf{X}_2^+ = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & b \\ & & 1 & \\ & 0 & & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\}, \text{ and } \mathbf{X}_2^- = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ & c & & 1 \end{pmatrix} : c \in \mathbb{F}_q \right\}.$$

For $\Gamma = {}^2A_3$, we have (with $\eta \in \mathbb{F}_{q^2}$ of trace 0),

$$\mathbf{X}_1^+ = \left\{ \begin{pmatrix} 1 & b & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & -b & 1 \end{pmatrix} : b \in \mathbb{F}_{q^2} \right\}, \text{ and } \mathbf{X}_1^- = \left\{ \begin{pmatrix} 1 & 0 & & \\ c & 1 & & \\ & & 1 & -c \\ & & 0 & 1 \end{pmatrix} : c \in \mathbb{F}_{q^2} \right\},$$

$$\mathbf{X}_2^+ = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & b\eta \\ & & 1 & \\ & 0 & & 1 \end{pmatrix} : b \in \mathbb{F}_q \right\}, \text{ and } \mathbf{X}_2^- = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ & c\eta^{-1} & & 1 \end{pmatrix} : c \in \mathbb{F}_q \right\}.$$

8.2. **Weak systems of fundamental groups**

In this subsection we show that a Curtis-Tits amalgam with 3-spherical diagram determines a collection of subgroups of the vertex groups, called a weak system of fundamental root groups. We then use this to determine the coefficient system of the amalgam in the sense of [3], which, in turn is applied to classify these amalgams up to isomorphism.

Definition 8.3. Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ is a CT amalgam. For each $i \in I$ let $\mathbf{X}_i^+, \mathbf{X}_i^- \leq \mathbf{G}_i$ be a pair of opposite root groups. We say that $\{\mathbf{X}_i^+, \mathbf{X}_i^- \mid i \in I\}$ is a *weak system of fundamental root groups* if, for any edge $\{i,j\} \in \mathrm{E}$ there are opposite Borel groups $\mathbf{B}_{i,j}^+$ and $\mathbf{B}_{ij}^-$ in $\mathbf{G}_{i,j}$, each of which contains exactly one of $\{\overline{\mathbf{X}}_i^+, \overline{\mathbf{X}}_i^-\}$.

We call $\mathscr{G}$ *orientable* if we can select $\mathbf{X}_i^\varepsilon$, $\mathbf{B}_{ij}^\varepsilon$ ($\varepsilon = +,-$) for all $i,j \in \mathrm{V}$ such that $\overline{\mathbf{X}}_i^\varepsilon, \overline{\mathbf{X}}_j^\varepsilon \leq \mathbf{B}_{ij}^\varepsilon$. If this is not possible, we call $\mathscr{G}$ *non-orientable*.

The relation between root groups and Borel groups is given by the following well-known fact.

Lemma 8.4. *Let q be a power of the prime p. Let $\mathbf{G}$ be a universal group of Lie type $\Gamma(q)$ and let $\mathbf{X}$ be a Sylow p-subgroup. Then, $N_{\mathbf{G}}(\mathbf{X})$ is the unique Borel group $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{X}$.*

Proposition 8.5. *Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ is a CT amalgam with connected 3-spherical diagram Γ. If $\mathscr{G}$ has a non-trivial completion (G,γ), then it has a unique weak system of fundamental root groups.*

Proof We first show that there is some weak system of fundamental root groups. For every edge $\{i,j\}$, let $\mathbf{X}_j^{i,\varepsilon}$ be the groups of Lemma 8.1. Suppose that there is some subdiagram Γ_J with $J = \{i,j,k\}$ in which j is connected to both i and k, such that $\{\mathbf{X}_j^{i,+}, \mathbf{X}_j^{i,-}\} \neq \{\mathbf{X}_j^{k,+}, \mathbf{X}_j^{k,-}\}$ as sets. Without loss of generality assume that $\Gamma_{i,j} = A_2$ (by 3-sphericity) and moreover, that $\mathbf{X}_j^{k,+} \notin \{\mathbf{X}_j^{i,+}, \mathbf{X}_j^{i,-}\}$. For any subgroup $\mathbf{H}$ of a group in $\mathscr{G}$, write $H = \gamma(\mathbf{H})$. Now note that $X_k^{j,-}$ commutes with $X_j^{k,+}$ and since Γ contains no triangles it also commutes with G_i. But then $X_k^{j,-}$ commutes with $\langle X_j^{k,+}, G_i\rangle$ which, by Lemma 8.1, equals $G_{i,j}$ (this is where we use that $\Gamma_{i,j} = A_2$), contradicting that $X_k^{j,-}$ does not commute with $X_j^{k,-} \leq G_{i,j}$. Thus, if there is a completion, then by connectedness of Γ, for each $i \in I$ we can pick a $j \in I$ so that $\{i,j\} \in E$ and set $\mathbf{X}_i^{\pm} = \mathbf{X}_i^{j,\pm}$ and drop the superscript. We claim that $\{\mathbf{X}_i^{\pm} \mid i \in I\}$ is a weak system of fundamental root groups. But this follows from part 2. of Lemma 8.1.

The uniqueness derives immediately from the fact that by Lemma 8.1, $\mathbf{g}_{i,j}(\mathbf{X}_j^{i,+})$ and $\mathbf{g}_{j,i}(\mathbf{X}_j^{i,-})$ are the only two Sylow p-subgroups in $\mathbf{g}_{j,i}(\mathbf{G}_j)$ which do not generate $\mathbf{G}_{i,j}$ with $\mathbf{g}_{i,j}(\mathbf{G}_i)$. □

An immediate consequence of the results above is the following observation.

Corollary 8.6. *Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ is a CT amalgam with connected 3-spherical diagram Γ. Then, an element of $N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ either fixes each of the pairs $(\overline{\mathbf{X}}_i^+, \overline{\mathbf{X}}_i^-)$, $(\overline{\mathbf{X}}_j^+, \overline{\mathbf{X}}_j^-)$, and $(\mathbf{B}_{i,j}^+, \mathbf{B}_{i,j}^-)$ or it reverses* *each of them. In particular,*

$$N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j) = N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\{\overline{\mathbf{X}}_i^+, \overline{\mathbf{X}}_i^-\}) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\{\overline{\mathbf{X}}_j^+, \overline{\mathbf{X}}_j^-\}).$$

8.3. **The coefficient system of a Curtis-Tits amalgam**

The automorphisms of a Curtis-Tits standard pair will be crucial in the classification of Curtis-Tits amalgams and we will need some detailed description of them.

We now fix a Curtis-Tits amalgam $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i,j \in I\}$ of type $\Gamma(q)$, where for every $i,j \in I$, $\underline{\mathbf{g}}_{i,j}$ is the standard identification map of Definition 2.5. Then, $\underline{\mathscr{G}}$ has a weak system of fundamental root groups $\mathcal{X} = \{\{\mathbf{X}_i^+, \mathbf{X}_i^-\} \colon i \in I\}$ as in Subsection 8.1.

Remark 8.7. Let $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ be a Curtis-Tits amalgam over $\mathbb{F}_q$ with given diagram Γ. Next suppose that Γ is connected 3-spherical, and that $\underline{\mathscr{G}}$ and $\mathscr{G}$ are non-collapsing. Then, by Proposition 8.5, $\underline{\mathscr{G}}$ and $\mathscr{G}$ each have a weak system of fundamental root groups. Now note that for each $i \in I$, $\mathrm{Aut}(\mathbf{G}_i)$ is 2-transitive on the set of Sylow p-subgroups. Thus, for each $i \in I$ and all $j \in I - \{i\}$, we can replace $\mathbf{g}_{i,j}$ by $\mathbf{g}_{i,j} \circ \alpha_i$, to form a new amalgam isomorphic to $\mathscr{G}$, whose weak system of fundamental root groups is exactly $\mathcal{X}$. Thus, in order to classify non-collapsing Curtis-Tits amalgams over $\mathbb{F}_q$ with diagram Γ up to isomorphism, it suffices to classify those whose weak system of fundamental root groups is exactly $\mathcal{X}$.

Definition 8.8. Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ is a Curtis-Tits amalgam over $\mathbb{F}_q$ with connected 3-spherical diagram Γ. Denote the associated weak system of fundamental root groups as $\mathcal{X} = \{\{\mathbf{X}_i^+, \mathbf{X}_i^-\} \colon i \in I\}$. The *coefficient system associated to* $\mathscr{G}$ is the collection $\mathscr{A} = \{\mathbf{A}_i, \mathbf{A}_{i,j}, \mathbf{a}_{i,j} \mid i,j \in I\}$ where, for any $i,j \in I$ we set

$$\begin{aligned}
\mathbf{A}_i &= N_{\mathrm{Aut}(\mathbf{G}_i)}(\{\mathbf{X}_i^+, \mathbf{X}_i^-\}),\\
\mathbf{A}_{i,j} &= N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\{\overline{\mathbf{X}}_i^\varepsilon \colon \varepsilon = +,-\}) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\{\overline{\mathbf{X}}_j^\varepsilon \colon \varepsilon = +,-\}),\\
\mathbf{a}_{i,j} &\colon \mathbf{A}_{i,j} \to \mathbf{A}_j \text{ is given by restriction: } \varphi \mapsto \mathbf{g}_{j,i}^{-1} \circ \rho_{i,j}(\varphi) \circ \mathbf{g}_{j,i}.
\end{aligned}$$

where $\rho_{i,j}(\varphi)$ is the restriction of φ to $\overline{\mathbf{G}}_j \le \mathbf{G}_{i,j}$.

From now on we let $\mathscr{A}$ be the coefficient system associated to $\underline{\mathscr{G}}$. The significance for the classification of Curtis-Tits amalgams with weak system of fundamental root groups is as follows:

Proposition 8.9. *Suppose that $\mathscr{G}$ and $\mathscr{G}^+$ are Curtis-Tits amalgams with diagram Γ over $\mathbb{F}_q$ with weak system of fundamental root groups $\mathcal{X}$.*

(a) *For all $i,j \in I$, we have $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}$ and $\mathbf{g}_{i,j}^+ = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}^+$ for some $\delta_{i,j}, \delta_{i,j}^+ \in \mathbf{A}_i$,*

(b) *For any isomorphism* $\phi\colon \mathscr{G} \to \mathscr{G}^+$ *and* $i,j \in I$*, we have* $\phi_i \in \mathbf{A}_i$*,* $\phi_{\{i,j\}} \in \mathbf{A}_{i,j}$*, and* $\mathbf{a}_{i,j}(\phi_{\{i,j\}}) = \delta^+_{i,j} \circ \phi_i \circ \delta^{-1}_{i,j}$*.*

Proof Part 1. follows since, for any $i,j \in I$ we have $\mathbf{g}_{i,j}^{-1} \circ \underline{\mathbf{g}}_{i,j} \in \mathrm{Aut}(\mathbf{G}_1)$ and

$$\{\mathbf{g}_{i,j}(\mathbf{X}_i^+), \mathbf{g}_{i,j}(\mathbf{X}_i^-)\} = \{\underline{\mathbf{g}}_{i,j}(\mathbf{X}_i^+), \underline{\mathbf{g}}_{i,j}(\mathbf{X}_i^-)\}.$$

Part 2. follows from Corollary 8.6 since, for any $i,j \in I$,

$$(\mathbf{G}_{i,j}, \mathbf{g}_{i,j}(\mathbf{G}_i), \mathbf{g}_{j,i}(\mathbf{G}_j)) = (\mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j}(\mathbf{G}_i), \underline{\mathbf{g}}_{j,i}(\mathbf{G}_j)) = (\mathbf{G}_{i,j}, \mathbf{g}^+_{i,j}(\mathbf{G}_i), \mathbf{g}^+_{j,i}(\mathbf{G}_j)).$$

□

We now determine the groups appearing in the coefficient system $\mathscr{A}$ associated to $\underline{\mathscr{G}}$.

Lemma 8.10. *Fix* $i \in I$ *and let* q *be such that* $\mathbf{G}_i \cong \mathrm{SL}_2(q)$*. Then,*

$$\mathbf{A}_i = \mathbf{T}_i \rtimes \mathbf{C}_i,$$

where $\mathbf{T}_i$ *is the subgroup of diagonal automorphisms in* $\mathrm{PGL}_2(q)$ *and* $\mathbf{C}_i = \langle \tau, \mathrm{Aut}(\mathbb{F}_q)\rangle$*.*

Proof This follows from the fact that via the standard embedding map $\underline{\mathbf{g}}_{i,j}$ the groups $\mathbf{X}_i^+$ and $\mathbf{X}_i^-$ of the weak system of fundamental root groups are the subgroups of unipotent upper and lower triangular matrices in $\mathrm{SL}_2(q)$. □

Lemma 8.11. *Let* $\mathscr{A}$ *be the coefficient system associated to the standard Curtis-Tits amalgam* $\underline{\mathscr{G}}$ *of type* $\Gamma(q)$ *and the weak system of fundamental root groups* $\mathcal{X}$*.*

If $\Gamma = A_1 \times A_1$*, we have* $\mathbf{G}_{i,j} = \mathbf{G}_i \times \mathbf{G}_j$*,* $\underline{\mathbf{g}}_{i,j}$ *and* $\underline{\mathbf{g}}_{j,i}$ *are identity maps, and*

$$\mathbf{A}_{i,j} = \mathbf{A}_i \times \mathbf{A}_j \cong \mathbf{T}_{i,j} \rtimes \mathbf{C}_{i,j}. \tag{8.3}$$

where $\mathbf{T}_{i,j} = \mathbf{T}_i \times \mathbf{T}_j$ *and* $\mathbf{C}_{i,j} = \mathbf{C}_i \times \mathbf{C}_j$*. Otherwise,*

$$\mathbf{A}_{i,j} = \mathbf{T}_{i,j} \rtimes \mathbf{C}_{i,j},$$

where

$$\mathbf{C}_{i,j} = \begin{cases} \mathrm{Aut}(\mathbb{F}_q) \times \langle \tau \rangle & \text{for } \Gamma = A_2, C_2 \\ \widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) \times \langle \tau \rangle & \text{for } \Gamma = {}^2A_3 \end{cases} \tag{8.4}$$

and $\mathbf{T}_{i,j}$ *denotes the image of the standard torus* T *in* $\mathrm{Aut}(\mathbf{G}_{i,j})$*. Note that*

$$\mathrm{T} = \begin{cases} \langle \mathrm{diag}(a,b,c)\colon a,b,c \in \mathbb{F}_q^*\rangle \le \mathrm{GL}_3(q) & \text{if } \Gamma = A_2 \\ \langle \mathrm{diag}(ab, a^{-1}b, a^{-1}, a)\colon a,b \in \mathbb{F}_q^*\rangle \le \mathrm{GSp}_4(q) & \text{if } \Gamma = C_2 \\ \langle \mathrm{diag}(a,b,a^{-q},b^{-q})\colon a,b \in \mathbb{F}_{q^2}^*\rangle \le \mathrm{GU}_4(q) & \text{if } \Gamma = {}^2A_3. \end{cases}$$

Remark 8.12. Remarks on Lemma 8.11

(a) We view $\mathrm{Sp}_{2n}(q)$ and $\mathrm{SU}_{2n}(q)$ as a matrix group with respect to a symplectic (resp. hyperbolic) basis for the $2n$-dimensional vector space V and $\mathrm{Aut}(\mathbb{F}_q)$ (resp. $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$) acts entrywise on the matrices.
(b) The map τ is the transpose-inverse map of Subsection 7.1.
(c) Recall that in the ${}^2A_3$ case, Remark 7.1 and Definition 7.2 describe the actions of $\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) \leq \mathbf{C}_{i,j}$ on $\mathbf{G}_i$ and $\mathbf{G}_j$ via the standard identification maps.

Proof We first consider the $A_1 \times A_1$ case of (8.3). When $\Gamma = A_1 \times A_1$, then $\mathbf{G}_{i,j} = \overline{\mathbf{G}}_i \times \overline{\mathbf{G}}_j$ and since the standard root groups $\overline{\mathbf{X}}_i^{\pm}$ generate $\overline{\mathbf{G}}_i$ ($i = 1, 2$), their simultaneous normalizer must also normalize $\overline{\mathbf{G}}_i$ and $\overline{\mathbf{G}}_j$. Thus the claim follows from Lemma 8.10.

We now deal with all remaining cases simultaneously. In the ${}^2A_3$ case we note that from Remark 7.1 and Definition 7.2 we see that $\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) \leq \mathbf{T}_{i,j} \rtimes \widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$ is simply a different complement to $\mathbf{T}_{i,j}$, so it suffices to prove the claim with $\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2})$ replaced by $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$.

Consider the descriptions of the set $\{\mathbf{X}_i^+, \mathbf{X}_i^-\}$ in all cases from Subsection 8.1. We see that since τ acts by transpose-inverse, it interchanges $\mathbf{X}_i^+$ and $\mathbf{X}_i^-$ for $i = 1, 2$ in all cases, hence it also interchanges positive and negative Borel groups (see Corollary 8.6). Thus it suffices to consider those automorphisms that normalize the positive and negative fundamental root groups. Since all field automorphisms (of $\mathrm{Aut}(\mathbb{F}_q)$ and $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$) act entrywise, they do so. Clearly so does T. Thus we have established $\supseteq$.

We now turn to the reverse inclusion. By Lemma 7.3 and the description of the automorphism groups in Subsection 7.1 any automorphism of $\mathbf{G}_{i,j}$ is a product of the form $g\alpha\tau^i$ where g is linear, α is a field automorphism (from $\widehat{\mathrm{Aut}}(\mathbb{F}_{q^2})$ in the ${}^2A_3$ case) and $i = 0, 1$. As we saw above τ and α preserve the root groups, so it suffices to describe g in case it preserves the sets of opposite root groups. A direct computation shows that g must be in T. □

Next we describe the connecting maps $\mathbf{a}_{i,j}$ of $\mathscr{A}$.

Lemma 8.13. *Let $\mathscr{A}$ be the coefficient system of the standard Curtis-Tits amalgam $\underline{\mathscr{G}}$ over $\mathbb{F}_q$ with diagram Γ and weak system of fundamental root groups $\mathcal{X}$. Fix $i, j \in I$ and let $(\mathbf{G}_{i,j}, \overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ be a Curtis-Tits standard pair in $\underline{\mathscr{G}}$ with diagram $\Gamma_{i,j}$. Denote $\mathbf{a} = (\mathbf{a}_{j,i}, \mathbf{a}_{i,j})\colon \mathbf{A}_{i,j} \to \mathbf{A}_i \times \mathbf{A}_j$. Then, we have the following:*

(a) *If $\Gamma_{i,j} = A_1 \times A_1$, then $\mathbf{a}$ is an isomorphism inducing $\mathbf{T}_{i,j} \cong \mathbf{T}_i \times \mathbf{T}_j$ and $\mathbf{C}_{i,j} \cong \mathbf{C}_i \times \mathbf{C}_j$.*
(b) *If $\Gamma_{i,j} = A_2$, or ${}^2A_3$, then $\mathbf{a}\colon \mathbf{T}_{i,j} \to \mathbf{T}_i \times \mathbf{T}_j$ is bijective.*
(c) *If $\Gamma_{i,j} = C_2$, then $\mathbf{a}\colon \mathbf{T}_{i,j} \xrightarrow{\cong} \mathbf{T}_i^2 \times \mathbf{T}_j$ has index 1 or 2 in $\mathbf{T}_i \times \mathbf{T}_j$ depending on whether q is even or odd.*

(d) *If* $\Gamma_{i,j} = A_2$ *or* C_2*, then* $\mathbf{a}\colon \mathbf{C}_{i,j} \to \mathbf{C}_i \times \mathbf{C}_j$ *is given by* $\tau^s\alpha \mapsto (\tau^s\alpha, \tau^s\alpha)$ *(for* $s \in \{0,1\}$ *and* $\alpha \in \mathrm{Aut}(\mathbb{F}_q)$*) which is a diagonal embedding.*

(e) *If* $\Gamma = {}^2A_3$*, then* $\mathbf{a}\colon \mathbf{C}_{i,j} \to \mathbf{C}_i \times \mathbf{C}_j$*, is given by* $\tau^s\widetilde{\alpha}^r \mapsto (\tau^s\alpha^r, \tau^s\alpha^r)$ *(for* $s \in \{0,1\}$*,* $r \in \mathbb{N}$*, and* $\alpha\colon x \mapsto x^p$ *for* $x \in \mathbb{F}_{q^2}$*. Here* $\widetilde{\sigma} \mapsto (\sigma, \mathrm{id})$*.*

Remark 8.14. (a) In 4. τ acts as transpose-inverse and α acts entry-wise on $\mathbf{G}_{i,j}$, $\mathbf{G}_i$ and $\mathbf{G}_j$.

Proof 1. This is immediate from Lemma 8.11.

For the remaining cases, recall that for any $\varphi \in \mathbf{A}_{i,j}$, we have $\mathbf{a}_{i,j}\colon \varphi \mapsto \underline{\mathbf{g}}_{j,i}^{-1} \circ \rho_{i,j}(\varphi) \circ \underline{\mathbf{g}}_{j,i}$, where $\rho_{i,j}(\varphi)$ is the restriction of φ to $\overline{\mathbf{G}}_j \le \mathbf{G}_{i,j}$ (Definition 8.8) and $\underline{\mathbf{g}}_{i,j}$ is the standard identification map of Definition 2.5. Note that for $\Gamma_{i,j} = A_2, C_2$ the standard identification map transforms the automorphism $\rho_{j,i}(\varphi)$ of $\overline{\mathbf{G}}_i$ essentially to the "same" automorphism φ of $\mathbf{G}_i$, whereas for $\Gamma_{i,j} = {}^2A_3$, we must take Remark 7.1 into account.

2. Let $\Gamma_{i,j} = A_2$. Every element of $\mathbf{T}_{i,j}$ ($\mathbf{T}_i$, and $\mathbf{T}_j$ respectively) is given by a unique matrix of the form $\mathrm{diag}(a,1,c)$ ($\mathrm{diag}(a,1)$, and $\mathrm{diag}(1,c)$), and we have

$$(\mathbf{a}_{j,i}, \mathbf{a}_{i,j})\colon\ \mathrm{diag}(a,1,c) \mapsto (\mathrm{diag}(a,1), \mathrm{diag}(1,c)) \qquad (a, c \in \mathbb{F}_q^*),$$

which is clearly bijective. In the ${}^2A_3$ case, every element of $\mathbf{T}_{i,j}$ ($\mathbf{T}_i$, and $\mathbf{T}_j$ respectively) is given by a unique matrix of the form $\mathrm{diag}(ab^{-1}, 1, a^{-q}b^{-1}, b^{-(q+1)})$ ($\mathrm{diag}(1, b^{-(q+1)})$, and $\mathrm{diag}(ab^{-1}, 1)$), and we have

$$\mathbf{a}\colon\ \mathrm{diag}(ab^{-1}, 1, a^{-q}b^{-1}, b^{-(q+1)}) \mapsto (\mathrm{diag}(1, b^{-(q+1)}), \mathrm{diag}(ab^{-1}, 1)).$$

This map is onto since $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}\colon b \mapsto b^{q+1}$ is onto. Its kernel is trivial, as it is given by pairs $(a,b) \in \mathbb{F}_{q^2}$, with $a = b$ and $b^{q+1} = 1$ so that also $a^{-q}b^{-1} = 1$.

3. In the C_2-case, every element of $\mathbf{T}_{i,j}$ is given by a unique diagonal matrix $\mathrm{diag}(a^2b, b, 1, a^2)$ ($a, b \in \mathbb{F}_q$). Every element of $\mathbf{T}_i$ (resp. $\mathbf{T}_j$) is given by a unique $\mathrm{diag}(c,1)$ (resp. $\mathrm{diag}(d,1)$). Now we have

$$\mathbf{a}\colon\ \mathrm{diag}(a^2b, b, 1, a^2) \mapsto (\mathrm{diag}(a^2,1), \mathrm{diag}(ba^{-2},1)).$$

It follows that $\mathbf{a}$ is injective and has image $\mathbf{T}_i^2 \times \mathbf{T}_j$. The rest of the claim follows.

4. The field automorphism $\alpha \in \mathrm{Aut}(\mathbb{F}_q)$ acts entrywise on the matrices in $\mathbf{G}_{i,j} = \mathrm{SL}_3(q)$, or $\mathrm{Sp}_4(q)$, and $\mathbf{G}_i = \mathbf{G}_j = \mathrm{SL}_2(q)$. In the case $\mathbf{G} = \mathrm{Sp}_4(q)$, we saw in Subsection 7.1 that τ is inner and coincides with conjugation by M. This clearly restricts to conjugation by μ on both $\overline{\mathbf{G}}_2 = \mathrm{Sp}_2(q)$ and $\overline{\mathbf{G}}_1 = \mathrm{SL}_2(q)$, which is again τ. Clearly these actions correspond to each other via the standard identification maps $\underline{\mathbf{g}}_{i,j}$ and $\underline{\mathbf{g}}_{j,i}$.

5. The action of $\widetilde{\mathrm{Aut}}(\mathbb{F}_{q^2}) \le \mathbf{C}_{i,j}$ on $\mathbf{G}_i$ and $\mathbf{G}_j$ via $\mathbf{a}$ was explained in Remark 7.1 and Definition 7.2. In case $\mathbf{G}_{i,j} = \mathrm{SU}_4(q)$, τ is given by conjugation

by M composed with the field automorphism $\widehat{\sigma}$, where $\sigma\colon x \mapsto x^q$ for $x \in \mathbb{F}_{q^2}$. The same holds for $\overline{\mathbf{G}}_j = \mathrm{SU}_2(q)$ and τ restricts to $\overline{\mathbf{G}}_i = \mathrm{SL}_2(q^2)$ as transpose-inverse. In view of Remark 7.1 we see that via the standard identification map each restricts to transpose inverse on $\mathbf{G}_i$ and $\mathbf{G}_j$. □

8.4. **A standard form for Curtis-Tits amalgams**

Suppose that $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i,j \in I\}$ is a Curtis-Tits amalgam over $\mathbb{F}_q$ with 3-spherical diagram Γ. Without loss of generality we will assume that all inclusion maps $\underline{\mathbf{g}}_{i,j}$ are the standard identification maps of Definition 2.5.

By Proposition 8.5 it possesses a weak system of fundamental root groups

$$\mathcal{X} = \{\{\mathbf{X}_i^+, \mathbf{X}_i^-\}\colon i \in I\},$$

which via the standard embeddings $\underline{\mathbf{g}}_{i,j}$ can be identified with those given in Subsection 8.1 (note that orienting $\mathcal{X}$ may involve changing some signs). Let $\mathscr{A} = \{\mathbf{A}_i, \mathbf{A}_{i,j}, \mathbf{a}_{i,j} \mid i,j \in I\}$ be the coefficient system associated to $\underline{\mathscr{G}}$ and $\mathcal{X}$.

We wish to classify all Curtis-Tits amalgams $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i,j \in I\}$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$ with weak system of fundamental root groups $\mathcal{X}$ up to isomorphism of Curtis-Tits amalgams. By Proposition 8.9 we may restrict to those amalgams whose connecting maps are of the form $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}$ for $\delta_{i,j} \in \mathbf{A}_i$ for all $i \in I$.

Definition 8.15. The *trivial support* of $\mathscr{G}$ (with respect to $\underline{\mathscr{G}}$) is the set $\{(i,j) \in I \times I \mid \mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j}\}$ (that is, $\delta_{i,j} = \mathrm{id}_{\mathbf{G}_i}$ in the notation of Proposition 8.9). The word "trivial" derives from the assumption that the $\underline{\mathbf{g}}_{i,j}$'s are the standard identification maps of Definition 2.5.

Fix some spanning tree $\Sigma \subseteq \Gamma$ and suppose that $\mathrm{E} - \mathrm{E}\,\Sigma = \{\{i_s, j_s\}\colon s = 1,2,\ldots,r\}$ so that $H_1(\Gamma,\mathbb{Z}) \cong \mathbb{Z}^r$.

Proposition 8.16. *There is a Curtis-Tits amalgam $\mathscr{G}(\Sigma)$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$ and the same $\mathcal{X}$, which is isomorphic to $\mathscr{G}$ and has the following properties:*

(a) *$\mathscr{G}$ has trivial support $S = \{(i,j) \in I \times I \mid \{i,j\} \in \mathrm{E}\,\Sigma\} \cup \{(i_s,j_s)\colon s = 1,2\ldots,r\}$.*
(b) *for each $s = 1,2,\ldots,r$, we have $\mathbf{g}_{j_s,i_s} = \underline{\mathbf{g}}_{j_s,i_s} \circ \gamma_{j_s,i_s}$, where $\gamma_{j_s,i_s} \in \mathbf{C}_{j_s}$.*

Lemma 8.17. *There is a Curtis-Tits amalgam $\mathscr{G}^+$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$ and the same $\mathcal{X}$, which is isomorphic to $\mathscr{G}$ and has the following properties: For any $u,v \in I$, if $\mathbf{g}_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v} \circ d_{u,v}$, for some $\gamma_{u,v} \in \mathbf{C}_u$ and $d_{u,v} \in \mathbf{T}_u$, then $\mathbf{g}^+_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$.*

Proof Note that we have $|I| \geq 2$ and that Γ is connected. Fix $u \in I$. Since Γ is 3-spherical, there is at most one $w \in I$ such that $(\mathbf{G}_{u,w}, \overline{\mathbf{G}}_u, \overline{\mathbf{G}}_w)$ is a Curtis-Tits standard pair of type B_2 or C_2. If there is no such w, let w be an arbitrary vertex such that $\{u,v\} \in \mathrm{E}\,\Gamma$. We define $\mathscr{G}^+$ by setting $\mathbf{g}^+_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$ for all $v \neq u$.

Next we define $\phi\colon \mathscr{G} \to \mathscr{G}^+$ setting $\phi_u = d_{u,w}$ and $\phi_v = \mathrm{id}_{\mathbf{G}_v}$ for all $v \neq u$. Now note that setting $\phi_{u,w} = \mathrm{id}_{\mathbf{G}_{u,v}}$, $\{\phi_{u,w}, \phi_u, \phi_w\}$ is an isomorphism of the subamalgams of $\mathscr{G}_{\{u,w\}}$ and $\mathscr{G}^+_{\{u,w\}}$. As for $\phi_{u,v}$ for $v \neq w$, note that in order for $\{\phi_{u,v}, \phi_u, \phi_v\}$ to be an isomorphism of the subamalgams of $\mathscr{G}_{\{u,v\}}$ and $\mathscr{G}^+_{\{u,v\}}$, we must have

$$\mathbf{g}^+_{u,v}\phi_u = \phi_{u,v} \circ \mathbf{g}_{u,v}$$
$$\mathbf{g}^+_{v,u}\phi_v = \phi_{u,v} \circ \mathbf{g}_{v,u}$$

which translates as

$$\underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v} \circ d_{u,w} = \phi_{u,v} \circ \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v} \circ d_{u,v}$$
$$\underline{\mathbf{g}}_{v,u} \circ \delta_{v,u} = \phi_{u,v} \circ \underline{\mathbf{g}}_{v,u} \circ \delta_{v,u}$$

or in other words

$$\gamma_{u,v} \circ d_{u,w} \circ d^{-1}_{u,v} \circ \gamma^{-1}_{u,v} = \mathbf{a}_{v,u}(\phi_{u,v})$$
$$\mathrm{id}_{\mathbf{G}_v} = \mathbf{a}_{u,v}(\phi_{u,v}).$$

Note that $\gamma_{u,v} \circ d_{u,w} \circ \circ d^{-1}_{u,v} \circ \gamma^{-1}_{u,v} \in \mathbf{T}_u \lhd \mathbf{A}_u$. Now by Lemma 8.13 as $(\mathbf{G}_{u,v}, \overline{\mathbf{G}}_u, \overline{\mathbf{G}}_v)$ is not of type B_2 or C_2 the map $(\mathbf{a}_{j,i}, \mathbf{a}_{i,j})\colon \mathbf{T}_{i,j} \to \mathbf{T}_i \times \mathbf{T}_j$ is onto. In particular, the required $\phi_{u,v} \in \mathbf{T}_{u,v}$ can be found. This completes the proof. $\square$

By Lemma 8.17 in order to prove Proposition 8.16 we may now assume that $\mathbf{g}_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$ for some $\gamma_{u,v} \in \mathbf{C}_u$ for all $u, v \in I$.

Let $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_j, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \gamma_{i,j} \mid i,j \in I\}$ be a Curtis-Tits amalgam over $\mathbb{F}_q$ with $|I| = 2$ and $\gamma_{i,j} \in \mathbf{C}_i$ and $\gamma_{j,i} \in \mathbf{C}_j$. We will describe all possible amalgams $\mathscr{G}^+ = \{\mathbf{G}_i, \mathbf{G}_j, \mathbf{G}_{i,j}, \mathbf{g}^+_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \gamma^+_{i,j} \mid i,j \in I\}$ with $\gamma^+_{i,j} \in \mathbf{C}_i$ and $\gamma^+_{j,i} \in \mathbf{C}_j$, isomorphic to $\mathscr{G}$ via an isomorphism ϕ with $\phi_i \in \mathbf{C}_i$, $\phi_j \in \mathbf{C}_j$ and $\phi_{i,j} \in \mathbf{C}_{i,j}$.

$$\begin{tikzcd}
 & \mathbf{G}_{i,j} \arrow[ddd, dotted, "\phi_{i,j}"'] & \\
\mathbf{G}_i \arrow[ur, "\mathbf{g}_{i,j}"] \arrow[d, "\phi_i"'] & & \mathbf{G}_j \arrow[ul, "\mathbf{g}_{j,i}"'] \arrow[d, "\phi_j"] \\
\mathbf{G}_i \arrow[dr, "\mathbf{g}^+_{i,j}"'] & & \mathbf{G}_j \arrow[dl, "\mathbf{g}^+_{j,i}"] \\
 & \mathbf{G}_{i,j} &
\end{tikzcd}$$

FIGURE 4. The commuting hexagon of Corollary 8.18.

Corollary 8.18. *With the notation introduced above, fix the maps $\gamma_{i,j}, \gamma^+_{i,j}, \phi_i \in \mathbf{C}_i$ as well as $\gamma_{j,i} \in \mathbf{C}_j$. Then for any one of $\gamma^+_{j,i}, \phi_j \in \mathbf{C}_j$, there exists a choice $\gamma \in \mathbf{C}_i$ for the remaining map in $\mathbf{C}_j$ so that there exists $\phi_{i,j}$ making the diagram in Figure 4 commute. Moreover, if $\Gamma_{i,j}$ is one of A_2, B_2, C_2, ${}^2A_3$, then γ is unique, whereas if $\Gamma_{i,j} = {}^2D_3$, then there are exactly two choices for γ.*

Proof The first claim follows immediately from the fact that the restriction maps $\mathbf{a}_{j,i} \colon \mathbf{C}_{i,j} \to \mathbf{C}_i$ and $\mathbf{a}_{i,j} \colon \mathbf{C}_{i,j} \to \mathbf{C}_j$ in part 4. and 5. of Lemma 8.13 are both surjective. The second claim follows from the fact that $\mathbf{a}_{j,i} \colon \mathbf{C}_{i,j} \to \mathbf{C}_i$ is injective except if $\Gamma_{i,j} = {}^2D_3$ in which case it has a kernel of order 2. $\square$

Proof (of Proposition 8.16) By Lemma 8.17 we may assume that $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \gamma_{i,j}$ for some $\gamma_{i,j} \in \mathbf{C}_i$ for all $i, j \in I$.

For any (possibly empty) subset $T \subseteq \mathrm{V}$ let $S(T)$ be the set of pairs $(i, j) \in S$ such that $i \in T$. Clearly the trivial support of $\mathscr{G}$ contains $S(\emptyset)$.

We now show that if T is the vertex set of a (possibly empty) proper subtree of Σ, and u is a vertex such that $T \cup \{u\}$ is also the vertex set of a subtree of Σ, then for any Curtis-Tits amalgam $\mathscr{G}$ whose trivial support contains $S(T)$, there is a Curtis-Tits amalgam $\mathscr{G}^+$ isomorphic to $\mathscr{G}$, whose trivial support contains $S(T \cup \{u\})$.

Once this is proved, Claim 1. follows since we can start with $T = \emptyset$ and end with a Curtis-Tits amalgam, still isomorphic to $\mathscr{G}$, whose trivial support contains S.

Now let T and u be as above. We first deal with the case where $T \neq \emptyset$. Let t be the unique neighbor of u in the subtree of Σ with vertex set $T \cup \{u\}$. We shall define an amalgam $\mathscr{G}^+ = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}^+_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \gamma^+_{i,j} \mid i, j \in I\}$ and an isomorphism $\phi \colon \mathscr{G} \to \mathscr{G}^+$, where $\gamma^+_{i,j}, \phi_i \in \mathbf{C}_i$ and $\phi_{\{i,j\}} \in \mathbf{C}_{i,j}$ for all $i, j \in I$. First note that it suffices to define $\mathbf{g}^+_{i,j}$, ϕ_i and $\phi_{\{i,j\}}$ for $\{i, j\} \in \mathrm{E}$: given this data, by the $A_1 \times A_1$ case in Lemma 8.11 and Lemma 8.18, for any non-edge $\{k,l\}$ there is a unique $\phi_{\{k,l\}} \in \mathbf{C}_{k,l}$ such that $(\phi_{k,l}, \phi_k, \phi_l)$ is an isomorphism between $\mathscr{G}_{\{k,l\}}$ and $\mathscr{G}^+_{\{k,l\}}$.

Before defining inclusion maps on edges, note that since Γ is 3-spherical, no two neighbors of u in Γ are connected by an edge. Therefore we can unambiguously set

$$\mathbf{g}^+_{i,j} = \mathbf{g}_{i,j} \text{ for } u \notin \{i,j\} \in \mathrm{E}\,\Gamma.$$

Note that both maps $\mathbf{g}^+_{t,u}$ and $\mathbf{g}^+_{u,t}$ are forced upon us, but at this point for any other neighbor v of u, only one of $\mathbf{g}^+_{u,v}$ and $\mathbf{g}^+_{v,u}$ is forced upon us. We set

$$\begin{aligned}
\mathbf{g}^+_{t,u} &= \mathbf{g}_{t,u}, \text{ and} \\
\mathbf{g}^+_{v,u} &= \mathbf{g}_{v,u} \text{ for } v \in I \text{ with } (u,v) \notin S \text{ and } (v,u) \in S.
\end{aligned}$$

To extend the trivial support as required, we set

$$\mathbf{g}^+_{u,v} = \underline{\mathbf{g}}_{u,v} \text{ for } v \in I \text{ with } (u,v) \in S.$$

We can already specify part of ϕ: Set

$$\begin{aligned}
\phi_i &= \mathrm{id}_{\mathbf{G}_i} \text{ for } i \in I - \{u\}, \\
\phi_{\{i,j\}} &= \mathrm{id}_{\mathbf{G}_{i,j}} \text{ for } u \notin \{i,j\} \in \mathrm{E}\,\Gamma.
\end{aligned}$$

Thus, what is left to specify is the following: ϕ_u and $\phi_{\{u,t\}}$ and, for all neighbors $v \neq t$ of u we must specify $\phi_{\{u,v\}}$ as well as

$$\begin{aligned}
&\mathbf{g}^+_{u,v} \text{ if } (u,v) \notin S, \\
&\mathbf{g}^+_{v,u} \text{ if } (u,v) \in S.
\end{aligned}$$

Figures 5 and 6 describe the amalgam $\mathscr{G}$ (top half) and $\mathscr{G}^+$ (bottom half) at the vertex u, where $t \in \mathrm{V}\,\Sigma$, $v \in \mathrm{V}\,\Gamma$, and $\{u,t\}, \{u,v\} \in \mathrm{E}\,\Gamma$. Inclusion maps from $\mathscr{G}^+$ forced upon us are indicated in bold, the dotted arrows are those we must define so as to make the diagram commute.

$$\begin{tikzcd}
 & \mathbf{G}_{t,u} \arrow[ddd, dotted, "\phi_{u,t}"'] & & \mathbf{G}_{u,v} \arrow[ddd, dotted, "\phi_{u,v}"] & \\
\mathbf{G}_t \arrow[ur, "\underline{\mathbf{g}}_{t,u}"] \arrow[d, "\phi_t := \mathrm{id}_{\mathbf{G}_t}"'] & & \mathbf{G}_u \arrow[ul, "\mathbf{g}_{u,t}"'] \arrow[ur, "\mathbf{g}_{u,v}"] \arrow[d, dotted, "\phi_u"] & & \mathbf{G}_v \arrow[ul, "\mathbf{g}_{v,u}"'] \arrow[d, "\phi_v := \mathrm{id}_{\mathbf{G}_v}"] \\
\mathbf{G}_t^+ \arrow[dr, thick, "\underline{\mathbf{g}}_{t,u}"'] & & \mathbf{G}_u^+ \arrow[dl, thick, "\underline{\mathbf{g}}_{u,t}"] \arrow[dr, thick, "\underline{\mathbf{g}}_{u,v}"'] & & \mathbf{G}_v^+ \arrow[dl, dotted, "\mathbf{g}^+_{v,u}"] \\
 & \mathbf{G}^+_{t,u} & & \mathbf{G}^+_{u,v} &
\end{tikzcd}$$

FIGURE 5. The case $(u,v) \in S$ and $(v,u) \notin S$.

$$\begin{tikzcd}
 & \mathbf{G}_{t,u} \arrow[ddd, dotted, "\phi_{u,t}"'] & & \mathbf{G}_{u,v} \arrow[ddd, dotted, "\phi_{u,v}"] & \\
\mathbf{G}_t \arrow[ur, "\underline{\mathbf{g}}_{t,u}"] \arrow[d, "\phi_t := \mathrm{id}_{\mathbf{G}_t}"'] & & \mathbf{G}_u \arrow[ul, "\mathbf{g}_{u,t}"'] \arrow[ur, "\mathbf{g}_{u,v}"] \arrow[d, dotted, "\phi_u"] & & \mathbf{G}_v \arrow[ul, "\mathbf{g}_{v,u}"'] \arrow[d, "\phi_v := \mathrm{id}_{\mathbf{G}_v}"] \\
\mathbf{G}_t^+ \arrow[dr, thick, "\underline{\mathbf{g}}_{t,u}"'] & & \mathbf{G}_u^+ \arrow[dl, thick, "\underline{\mathbf{g}}_{u,t}"] \arrow[dr, dotted, "\mathbf{g}^+_{u,v}"'] & & \mathbf{G}_v^+ \arrow[dl, thick, "\mathbf{g}_{v,u}"] \\
 & \mathbf{G}^+_{t,u} & & \mathbf{G}^+_{u,v} &
\end{tikzcd}$$

FIGURE 6. The case $(u,v) \notin S$ and $(v,u) \in S$.

In these figures all non-dotted maps are of the form $\underline{\mathbf{g}}_{i,j} \circ \gamma_{i,j}$ for some $\gamma_{i,j} \in \mathbf{C}_i$ hence we can find the desired maps using Corollary 8.18.

In case $T = \emptyset$, the situation is as described in Figures 5 and 6 after removing the $\{u,t\}$-hexagon and any conditions it may impose on ϕ_u, and letting v run over all neighbors of u. That is, we must now define ϕ_u, and for any neighbor v of u, we must find $\phi_{u,v}$ as well as

$$\begin{aligned} &\mathbf{g}^+_{u,v} \text{ if } (u,v) \notin S, \\ &\mathbf{g}^+_{v,u} \text{ if } (u,v) \in S. \end{aligned}$$

To do so we let $\phi_u = \mathrm{id}_{\mathbf{G}_u} \in \mathbf{C}_u$. Finally, for each neighbor v of u we simply let $\mathbf{g}^+_{u,v} = \mathbf{g}_{u,v}$ (so that $\phi_{u,v} = \mathrm{id}_{\mathbf{G}_{i,j}} \in \mathbf{C}_{i,j}$) if $(u,v) \notin S$, and we obtain $\mathbf{g}^+_{v,u}$ and $\phi_{u,v} \in \mathbf{C}_{u,v}$ using Corollary 8.18 if $(u,v) \in S$. $\square$

8.5. Classification of Curtis-Tits amalgams with 3-spherical diagram

In the case where $\mathscr{G}$ is a Curtis-Tits amalgam over $\mathbb{F}_q$ whose diagram is a 3-spherical tree, Proposition 8.16 says that $\mathscr{G} \cong \underline{\mathscr{G}}$.

Theorem 8.19. *Suppose that $\mathscr{G}$ is a Curtis-Tits amalgam with a diagram that is a 3-spherical tree. Then, $\mathscr{G}$ is unique up to isomorphism. In particular any Curtis-Tits amalgam with spherical diagram is unique.*

Lemma 8.20. *Given a Curtis-Tits amalgam over $\mathbb{F}_q$ with connected 3-spherical diagram Γ there is a spanning tree Σ such that the set of edges in $\mathrm{E}\,\Gamma - \mathrm{E}\,\Sigma = \{\{i_s, j_s\} \colon s = 1, 2, \ldots, r\}$ has the property that*

(a) $(\mathbf{G}_{\{i_s,j_s\}}, \mathbf{g}_{i_s,j_s}(\mathbf{G}_{i_s}), \mathbf{g}_{i_s,j_s}(\mathbf{G}_{j_s}))$ *has type $A_2(q^{e_s})$, where e_s is some power of 2.*

(b) *There is a loop Λ_s containing $\{i_s, j_s\}$ such that any vertex group of Λ_s is isomorphic to $\mathrm{SL}_2(q^{e_s 2^l})$ for some $l \geq 0$.*

Proof Induction on the rank r of $H^1(\Gamma, \mathbb{Z})$. If $r = 0$, then there is no loop at all and we are done.

Consider the collection of all edges $\{i, j\}$ of Γ such that $\Gamma_{\{i,j\}}$ has type A_2 and $H^1(\Gamma - \{i, j\}, \mathbb{Z})$ has rank $r - 1$, and choose one such that $\mathbf{G}_i \cong \mathrm{SL}_2(q^{e_1})$ where e_1 is minimal among all these edges. Next replace Γ by $\Gamma - \{i, j\}$ and use induction. Suppose $\{i_s, j_s\} \mid s = 1, 2, \ldots, r\}$ is the resulting selection of edges so that $\Sigma = \Gamma - \{i_s, j_s\} \mid s = 1, 2, \ldots, r\}$ is a spanning tree and condition (a) is satisfied. Note that by choice of these edges, also condition (b) is satisfied by at least one of the loops of $\Gamma - \{\{i_t, j_t\} \colon t = 1, 2, \ldots, s - 1\}$ that contains $\{i_s, j_s\}$. Note that this uses the fact that by 3-sphericity every vertex belongs to at least one subdiagram of type A_2. $\square$

Definition 8.21. Fix a connected 3-spherical diagram Γ and a prime power q. Let Σ be a spanning tree and let the set of edges $\mathrm{E}\,\Gamma - \mathrm{E}\,\Sigma = \{\{i_s, j_s\} \colon s = 1, 2, \ldots, r\}$ together with the integers $\{e_s \colon s = 1, 2, \ldots, r\}$ satisfy the conclusions of Lemma 8.20. Let $\mathsf{CT}(\Gamma, q)$ be the collection of isomorphism classes of Curtis-Tits amalgams of type $\Gamma(q)$ and let $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i, j \in I\}$ be the standard Curtis-Tits amalgam over $\mathbb{F}_q$ with diagram Γ as in Subsection 8.4.

Consider the following map:

$$\kappa \colon \prod_{s=1}^{r} \mathrm{Aut}(\mathbb{F}_{q^{e_s}}) \times \langle \tau \rangle \to \mathsf{CT}(\Gamma).$$

where $\kappa((\alpha_s)_{s=1}^r)$ is the isomorphism class of the amalgam $\mathscr{G}^+ = \mathscr{G}((\alpha_s)_{s=1}^r)$ given by setting $\mathbf{g}^+_{j_s,i_s} = \underline{\mathbf{g}}_{j_s,i_s} \circ \alpha_s$ for all $s = 1, 2, \ldots, r$.

We now have

Corollary 8.22. *The map κ is onto.*

Proof Note that, for each $s = 1, 2, \ldots, r$, the Curtis-Tits standard pair $(\mathbf{G}_{\{i_s,j_s\}},\ \mathbf{g}_{i_s,j_s}(\mathbf{G}_{i_s}),\ \mathbf{g}_{i_s,j_s}(\mathbf{G}_{j_s}))$ has type $A_2(q^{e_s})$ and so $\mathbf{C}_{j_s} = \mathrm{Aut}(\mathbb{F}_{q^{e_s}})$. Thus the claim is an immediate consequence of Proposition 8.16. $\square$

We note that if we select Σ differently, the map κ will still be onto. However, the "minimal" choice made in Lemma 8.20 ensures that κ is injective as well, as we will see.

Lemma 8.23. *Suppose $\Gamma(q)$ is a 3-spherical diagram Γ that is a simple loop. Then, κ is injective.*

Proof Suppose there is an isomorphism $\kappa(\alpha) = \mathscr{G} \xrightarrow{\phi} \mathscr{G}^+ = \kappa(\beta)$, for some $\alpha, \beta \in \mathrm{Aut}(\mathbb{F}_q) \times \langle\tau\rangle$. Write $I = \{0, 1, \ldots, n-1\}$ so that $\{i, i+1\} \in \mathrm{E}\,\Gamma$ for all $i \in I$ (subscripts modulo n). Without loss of generality assume that $(i_1, j_1) = (1, 0)$ so that by Proposition 8.16 we may assume that $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} = \mathbf{g}^+_{i,j}$ for all $(i,j) \neq (1,0)$. This means that $\mathbf{a}\colon \mathbf{C}_{i,i+1} \to \mathbf{C}_i \times \mathbf{C}_{i+1}$ sends $\phi_{i,i+1}$ to (ϕ_i, ϕ_{i+1}) for any edge $\{i, i+1\} \neq \{0, 1\}$. Now note that by minimality of q, $\mathbf{C}_i$ (and $\mathbf{C}_{i,i+1}$) has a quotient $\overline{\mathbf{C}}_i$ (and $\overline{\mathbf{C}}_{i,i+1}$) isomorphic to $\langle\mathrm{Aut}(\mathbb{F}_q)\rangle \times \langle\tau\rangle$ for every $i \in I$, by considering the action of $\mathbf{C}_i$ on the subgroup of $\mathbf{G}_i$ isomorphic to $\mathrm{SL}_2(q)$. By Part 4 and 5 of Lemma 8.13 the maps $\mathbf{a}^{-1}_{i+1,i}$ and $\mathbf{a}_{i,i+1}$ induce isomorphisms $\overline{\mathbf{C}}_i \to \overline{\mathbf{C}}_{i,i+1}$ and $\overline{\mathbf{C}}_{i,i+1} \to \overline{\mathbf{C}}_{i+1}$, which compose to an isomorphism

$$\phi_i \mapsto \underline{\mathbf{g}}^{-1}_{i+1,i} \circ \underline{\mathbf{g}}_{i,i+1} \circ \phi_i \circ \underline{\mathbf{g}}^{-1}_{i,i+1} \circ \underline{\mathbf{g}}_{i+1,i},$$

sending the image of τ and α in $\overline{\mathbf{C}}_i$ to the image of τ (and α respectively) in $\overline{\mathbf{C}}_{i+1}$, where $\alpha\colon x \mapsto x^p$ for x in the appropriate extension of $\mathbb{F}_q$ defining $\mathbf{G}_{i,i+1}$. Concatenating these maps along the path $\{1, 2, \ldots, n-1, 0\}$ and considering the edge $\{0, 1\}$ we see that the images of $\beta^{-1}\phi_1\alpha$ and ϕ_1 in $\overline{\mathbf{C}}_1 = \mathbf{C}_1$ coincide. Since $\mathbf{C}_1$ is abelian this means that $\beta = \alpha$. $\square$

Theorem 8.24. *Let Γ be a connected 3-spherical diagram with spanning tree Σ and set of edges $\mathrm{E}\,\Gamma - \mathrm{E}\,\Sigma = \{\{i_s, j_s\}\colon s = 1, 2, \ldots, r\}$ together with the integers $\{e_s\colon s = 1, 2, \ldots, r\}$ satisfying the conclusions of Lemma 8.20. Then κ is a bijection between the elements of $\prod_{s=1}^r \mathrm{Aut}(\mathbb{F}_{q^{e_s}}) \times \langle\tau\rangle$ and the type preserving isomorphism classes of Curtis-Tits amalgams with diagram Γ over $\mathbb{F}_q$.*

Proof Again, it suffices to show that κ is injective. This in turn follows from Lemma 8.23, for if two amalgams are isomorphic (via a type preserving isomorphism), then the amalgams induced on subgraphs of Γ must be isomorphic and Lemma 8.23 shows that κ is injective on the subamalgams supported by the loops Λ_s $(s = 1, 2, \ldots, r)$. $\square$

9. Classification of Phan amalgams

9.1. Introduction

The classification problem is formulated as follows: Determine, up to isomorphism of amalgams, all Phan amalgams $\mathscr{G}$ with given diagram Γ possessing a non-trivial (universal) completion.

9.2. Classification of Phan amalgams with 3-spherical diagram

9.2.1. *Tori in Phan standard pairs.* Let $\mathscr{G} = \{\mathbf{G}_{i,j}, \mathbf{G}_i, \mathbf{g}_{i,j} \mid i, j \in I\}$ be a Phan amalgam over $\mathbb{F}_q$ with 3-spherical diagram $\Gamma = (I, E)$. This means that the subdiagram of Γ induced on any set of three vertices is spherical. This is equivalent to Γ not containing triangles of any kind and such that no vertex is on more than one C_2-edge.

Definition 9.1. For any $i, j \in I$ with $\{i, j\} \in \mathrm{E}\,\Gamma$, let

$$\mathbf{D}_i^j = N_{\mathbf{G}_{i,j}}(\mathbf{g}_{j,i}(\mathbf{G}_j)) \cap \mathbf{g}_{i,j}(\mathbf{G}_i)$$

Lemma 9.2. *Suppose that $(\mathbf{G}, \mathbf{G}_1, \mathbf{G}_2)$ is a Phan standard pair of type $\Gamma(q)$ as in Subsection* **??***.*

(a) *If $\Gamma(q) = A_2(q)$, then $\langle \mathbf{D}_1^2, \mathbf{D}_2^1\rangle$ is the standard torus stabilizing the orthonormal basis $\{e_1, e_2, e_3\}$. Here $\mathbf{D}_1^2$ (resp. $\mathbf{D}_2^1$) is the stabilizer in this torus of e_1 (resp. e_3).*

(b) *If $\Gamma(q) = C_2(q)$, then $\langle \mathbf{D}_1^2, \mathbf{D}_2^1\rangle$ is the standard torus stabilizing the basis $\{e_1, e_2, e_3 = f_1, e_4 = f_2\}$ which is hyperbolic for the symplectic form of $\mathrm{Sp}_4(q^2)$ and orthonormal for the unitary form of $\mathrm{SU}_4(q)$; Here $\mathbf{D}_2^1$ (resp. $\mathbf{D}_1^2$) is the stabilizer of $\langle e_2\rangle$ and $\langle f_2\rangle$ (resp. the pointwise stabilizer of both $\langle e_1, f_1\rangle$ and $\langle e_2, f_2\rangle$). Thus,*

$$\mathbf{D}_2^1 = \langle \mathrm{diag}(1, a, 1, a^\sigma) \colon a \in \mathbb{F}_{q^2} \text{ with } aa^\sigma = a^{q+1} = 1\rangle,$$
$$\mathbf{D}_1^2 = \langle \mathrm{diag}(a, a^\sigma, a^\sigma, a) \colon a \in \mathbb{F}_{q^2} \text{ with } aa^\sigma = a^{q+1} = 1\rangle.$$

(c) *In either case, for $\{i, j\} = \{1, 2\}$, $\mathbf{D}_i^j = C_{\mathbf{G}_{i,j}}(\mathbf{D}_j^i) \cap \mathbf{G}_i$ and $\mathbf{D}_j^i$ is the unique torus of $\mathbf{G}_j$ normalized by $\mathbf{D}_i^j$.*

Proof Parts 1. and 2. as well as the first claim of Part 3. are straightforward matrix calculations. As for the last claim note that in both cases, $\mathbf{D}_i^j$ acts diagonally on $\mathbf{G}_j$ viewed as $\mathrm{SU}_2(q)$ in it natural representation V via the standard identification map; in fact (in the case C_2, $\mathbf{D}_2^1$ acts even innerly on $\mathbf{G}_1$). If $\mathbf{D}_i^j$ normalizes a torus $\mathbf{D}'$ in $\mathbf{G}_j$ then it will have to stabilize its eigenspaces. Since $q + 1 \geq 3$, the eigenspaces of $\mathbf{D}_i^j$ in its action on V have dimension 1, so $\mathbf{D}_i^j$ and $\mathbf{D}'$ must share these eigenspaces. This means that $\mathbf{D}' = \mathbf{D}_j^i$. $\square$

9.2.2. *Property (D) for Phan amalgams.* We state Property (D) for 3-spherical Phan amalgams, extending the definition from [5] which was given for Curtis-Tits amalgams under the assumption that $q \geq 4$.

Definition 9.3. (property (D)) We say that $\mathscr{G}$ *has property (D)* if there is a *system of tori* $\mathcal{D} = \{\mathbf{D}_i \colon i \in I\}$ such that for all edges $\{i,j\} \in \mathrm{E}\,\Gamma$ we have $\mathbf{g}_{i,j}(\mathbf{D}_i) = \mathbf{D}_i^j$.

Lemma 9.4. *Suppose that $\mathscr{G}$ has a completion (G,γ) so that γ_i is non-trivial for all $i \in I$. Then, for any $i,j,k \in I$ such that $\{i,j\},\{j,k\} \in \mathrm{E}\,\Gamma$, there is a torus $\mathbf{D}_j \leq \mathbf{G}_j$ such that $\mathbf{g}_{j,i}(\mathbf{D}_j) = \mathbf{D}_j^i$ and $\mathbf{g}_{j,k}(\mathbf{D}_j) = \mathbf{D}_j^k$. In particular, $\mathscr{G}$ has property (D).*

Proof First note that in case $q = 2$, the conclusion of the lemma is trivially true as, for all $i \in I$, $\mathbf{G}_i \cong S_3$ has a unique Phan torus.

We now consider the general case. For $\Gamma(q) = A_3(q)$, this was proved by Bennett and Shpectorov in [1] (see also [5]). For completeness we recall the argument, which applies in this more general case as well. We shall prove that

$$\gamma(\mathbf{D}_j^i) = \gamma(\mathbf{D}_j^k)$$

and then let $\mathbf{D}_j \leq \mathbf{G}_j$ be such that $\gamma(\mathbf{D}_j) = \gamma(\mathbf{D}_j^i) = \gamma(\mathbf{D}_j^k)$. Note that since γ_j is non-trivial, it now follows that $\mathbf{g}_{j,i}(\mathbf{D}_j) = \mathbf{D}_j^i$ and $\mathbf{g}_{j,k}(\mathbf{D}_j) = \mathbf{D}_j^k$.

Recall that for any subgroup $\mathbf{H}$ of a group in $\mathscr{G}$ we'll write $H = \gamma(\mathbf{H})$. We show that D_j^i is normalized by D_k^j and use Lemma 9.2 to conclude that $D_j^i = D_j^k$. To that end we let $h \in D_k^j$ and prove that $hD_j^ih^{-1} = D_j^i$. To achieve this we show that $hD_j^ih^{-1}$ is normalized by D_i^j and again use Lemma 9.2. So now let $g \in D_i^j$ and note that since Γ is 3-spherical, $\{i,k\} \notin \mathrm{E}\,\Gamma$ so that g and h commute. In addition note that by Lemma 9.2, $gD_j^ig^{-1} = D_j^i$. Therefore we have

$$ghD_j^ih^{-1}g^{-1} = hgD_j^ig^{-1}h^{-1} = hD_j^ih^{-1},$$

as required. □

9.2.3. *The coefficient system of a Phan amalgam.*

Definition 9.5. We now fix a standard Phan amalgam $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i,j \in I\}$ over $\mathbb{F}_q$ with diagram $\Gamma(q)$, where for every $i,j \in I$, $\underline{\mathbf{g}}_{i,j}$ is the standard identification map of Definition **??**. Then, $\underline{\mathscr{G}}$ has property (D) with system of tori $\mathcal{D} = \{\mathbf{D}_i \colon i \in I\}$ as in Lemma 9.2.

If $\mathscr{G}$ is any other non-collapsing Phan amalgam over $\mathbb{F}_q$ with diagram Γ, then since all tori of $\mathbf{G}_i$ are conjugate under $\mathrm{Aut}(\mathbf{G}_i)$, by adjusting the inclusion maps $\mathbf{g}_{i,j}$ we can replace $\mathscr{G}$ by an isomorphic amalgam whose system of tori is exactly $\mathcal{D}$.

From now on we assume that $\underline{\mathscr{G}}$, $\mathcal{D} = \{\mathbf{D}_i : i \in I\}$ and $\mathscr{G}$ are as in Definition 9.5

Definition 9.6. Suppose that $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ is a Phan amalgam with connected 3-spherical diagram Γ having property (D). Let $\mathcal{D} = \{\mathbf{D}_i : i \in I\}$ be the associated system of tori. The *coefficient system associated to* $\mathscr{G}$ is the collection $\mathscr{A} = \{\mathbf{A}_i, \mathbf{A}_{i,j}, \mathbf{a}_{i,j} \mid i, j \in I\}$ where, for any $i, j \in I$ we set

$$\begin{aligned}
\mathbf{A}_i &= N_{\mathrm{Aut}(\mathbf{G}_i)}(\mathbf{D}_i),\\
\mathbf{A}_{i,j} &= N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{g}_{i,j}(\mathbf{G}_i)) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{g}_{j,i}(\mathbf{G}_j)),\\
\mathbf{a}_{i,j}&\colon \mathbf{A}_{i,j} \to \mathbf{A}_j \text{ is given by restriction: } \varphi \mapsto \mathbf{g}_{j,i}^{-1} \circ \rho_{i,j}(\varphi) \circ \mathbf{g}_{j,i}.
\end{aligned}$$

where $\rho_{i,j}(\varphi)$ is the restriction of φ to $\overline{\mathbf{G}}_j \leq \mathbf{G}_{i,j}$.

From now on we let $\mathscr{A}$ be the coefficient system associated to $\underline{\mathscr{G}}$ with respect to the system of tori $\mathcal{D}$. The fact that the $\mathbf{a}_{i,j}$ are well-defined follows from the following simple observation.

Lemma 9.7. *For any $i, j \in I$ with $\{i, j\} \in \mathrm{E}\,\Gamma$, we have*

$$\mathbf{A}_{i,j} \leq N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{g}_{i,j}(\mathbf{D}_i)) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{g}_{j,i}(\mathbf{D}_j)).$$

Proof The inclusion $\leq$ is immediate from the definitions. $\square$

The significance for the classification of Phan amalgams with the same system of tori is as follows:

Proposition 9.8. *Suppose that $\mathscr{G}$ and $\mathscr{G}^+$ are Phan amalgams of type $\underline{\mathscr{G}}$ with the same system of tori $\mathcal{D} = \{\mathbf{D}_i : i \in I\}$.*

(a) *For all $i, j \in I$, we have $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}$ and $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}^+$ for some $\delta_{i,j}, \delta_{i,j}^+ \in \mathbf{A}_i$,*

(b) *For any isomorphism $\phi\colon \mathscr{G} \to \mathscr{G}^+$ and $i, j \in I$, we have $\phi_i \in \mathbf{A}_i$, $\phi_{\{i,j\}} \in \mathbf{A}_{i,j}$, and $\mathbf{a}_{i,j}(\phi_{\{i,j\}}) = \delta_{i,j}^+ \circ \phi_i \circ \delta_{i,j}^{-1}$.*

Proof Part 1. follows since, for any $i, j \in I$ we have $\mathbf{g}_{i,j}^{-1} \circ \underline{\mathbf{g}}_{i,j} \in \mathrm{Aut}(\mathbf{G}_i)$ and

$$\mathbf{g}_{i,j}(\mathbf{D}_i) = \underline{\mathbf{g}}_{i,j}(\mathbf{D}_i).$$

Part 2. follows from Lemma 9.7 since, for any $i, j \in I$,

$$(\mathbf{G}_{i,j}, \mathbf{g}_{i,j}(\mathbf{G}_i), \mathbf{g}_{j,i}(\mathbf{G}_j)) = (\mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j}(\mathbf{G}_i), \underline{\mathbf{g}}_{j,i}(\mathbf{G}_j)) = (\mathbf{G}_{i,j}, \mathbf{g}_{i,j}^+(\mathbf{G}_i), \mathbf{g}_{j,i}^+(\mathbf{G}_j)).$$

$\square$

We now determine the groups appearing in a coefficient system by looking at standard pairs.

Lemma 9.9. *Fix $i \in I$ and let q be such that $\mathbf{G}_i \cong \mathrm{SU}_2(q)$. Then,*

$$\mathbf{A}_i = \mathbf{T}_i \rtimes \mathbf{C}_i,$$

where $\mathbf{T}_i$ is the subgroup of diagonal automorphisms in $\mathrm{PGU}_2(q)$ and $\mathbf{C}_i = \mathrm{Aut}(\mathbb{F}_{q^2})$.

Proof This follows from the fact that via the standard embedding map $\underline{\mathbf{g}}_{i,j}$ the groups $\mathbf{D}_i$ of the system of tori are the subgroups of standard diagonal matrices in $\mathrm{SU}_2(q)$.

To see this note that $\mathbf{G}_i \cong \mathrm{SU}_2(q)$ and that $\mathrm{Aut}(\mathbf{G}_i) \cong \mathrm{PGU}_2(q) \rtimes \mathrm{Aut}(\mathbb{F}_{q^2})$. Also, $\mathbf{D}_i = \langle d \rangle$ for some $d = \mathrm{diag}(\zeta, \zeta^q)$ and ζ a primitive $q+1$-th root of 1 in $\mathbb{F}_{q^2}$. A quick calculation now shows that τ and σ are the same in their action, which is inner and one verifies that $N_{\mathrm{GU}_2(q)}(\mathbf{D}_i) = \langle \tau, \mathrm{diag}(a,b) \colon a, b \in \mathbb{F}_q^2 \rangle$. $\square$

Lemma 9.10. *Let $\mathscr{A}$ be the coefficient system associated to the standard Phan amalgam $\underline{\mathscr{G}}$ of type $\Gamma(q)$ and the system of tori $\mathcal{D}$.*

If $\Gamma = A_1 \times A_1$, we have $\mathbf{G}_{i,j} = \mathbf{G}_i \times \mathbf{G}_j$, $\underline{\mathbf{g}}_{i,j}$ and $\underline{\mathbf{g}}_{j,i}$ are identity maps, and

$$\mathbf{A}_{i,j} = \mathbf{A}_i \times \mathbf{A}_j \cong \mathbf{T}_{i,j} \rtimes \mathbf{C}_{i,j}. \tag{9.1}$$

where $\mathbf{T}_{i,j} = \mathbf{T}_i \times \mathbf{T}_j$ and $\mathbf{C}_{i,j} = \mathbf{C}_i \times \mathbf{C}_j$. Otherwise,

$$\mathbf{A}_{i,j} = \mathbf{T}_{i,j} \rtimes \mathbf{C}_{i,j},$$

where $\mathbf{C}_{i,j} = \mathrm{Aut}(\mathbb{F}_{q^2})$ and $\mathbf{T}_{i,j}$ denotes the image of the standard torus T in $\mathrm{Aut}(\mathbf{G}_{i,j})$. Note that T is as follows

$$\langle \mathrm{diag}(a,b,c) \colon a,b,c \in \mathbb{F}_{q^2} \text{ with } aa^\sigma = bb^\sigma = cc^\sigma = 1 \rangle \text{ if } \Gamma = A_2,$$
$$\langle \mathrm{diag}(c^\sigma b, ab, c, a^\sigma) \colon a,b,c \in \mathbb{F}_{q^2} \text{ with } aa^\sigma = bb^\sigma = cc^\sigma = 1 \rangle \text{ if } \Gamma = C_2.$$

Remark 9.11.
(a) In case $\Gamma = C_2$, $\mathbf{G} \cong \mathrm{Sp}_4(q)$ is realized as $\mathrm{Sp}_4(q^2) \cap \mathrm{SU}_4(q)$ with respect to a basis that is hyperbolic for the symplectic form and orthonormal for the unitary form, and $\mathrm{Aut}(\mathbb{F}_{q^2})$ acts entry-wise on these matrices. Moreover, τ acts as transpose-inverse on these matrices.
(b) In all cases τ coincides with σ.

Proof The $A_1 \times A_1$ case is self evident. Now consider the case $\Gamma = A_2$. As in the proof of Lemma 9.9, $\mathrm{Aut}(\mathbb{F}_{q^2}) \leq N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{G}_i) \cap N_{\mathrm{Aut}(\mathbf{G}_{i,j})}(\mathbf{G}_j)$, $A^\tau = {}^tA^{-1} = A^\sigma$ and $\mathrm{Aut}(\mathbf{G}_{i,j}) \cong \mathrm{PGU}_3(q) \rtimes \mathrm{Aut}(\mathbb{F}_{q^2})$, so it suffices to consider linear automorphisms. As before this is an uncomplicated calculation.

Now consider the case $\Gamma = C_2$. Writing $\Gamma\mathrm{L}(V) \cong \mathrm{GL}_4(q^2) \rtimes \mathrm{Aut}(\mathbb{F}_{q^2})$ with respect to the basis $\mathcal{E} = \{e_1, e_2, e_3 = f_1, e_4 = f_2\}$, which is hyperbolic for the symplectic form of $\mathrm{Sp}_4(q^2)$ and orthonormal for the unitary form of $\mathrm{SU}_4(q)$, we have $\mathbf{G}_{i,j} = \mathrm{Sp}_4(q^2) \cap \mathrm{SU}_4(q)$.

There is an isomorphism $\Phi\colon \mathbf{G}_{i,j} \to \mathrm{Sp}_4(q)$ as in [22]. Abstractly, we have $\mathrm{Aut}(\mathrm{Sp}_4(q)) = \mathrm{GSp}_4(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$ (with respect to a suitable basis E for V). Since the embedding of $\mathrm{Sp}_4(q)$ into $\mathrm{Sp}_4(q^2)$ is non-standard, we are reconstructing the automorphism group here.

We first note that changing bases just replaces $\mathrm{Aut}(\mathbb{F}_{q^2})$ with a different complement to the linear automorphism group. As for linear automorphisms we claim that

$$\mathrm{GSp}_4(q^2) \cap \mathrm{GU}_4(q) = \mathrm{GSp}_4(q)$$

(viewing the latter as a matrix group w.r.t. E). Clearly, up to a center, we have $\mathbf{G}_{i,j} \leq \mathrm{GSp}_4(q^2) \cap \mathrm{GU}_4(q) \leq \mathrm{GSp}_4(q)$ and we note that $\mathrm{GSp}_4(q)/\mathrm{Sp}_4(q) \cong (\mathbb{F}_q^*)^2/(\mathbb{F}_q^*)$. Thus for q even, the claim follows. For q odd, let $\mathbb{F}_{q^2}^* = \langle\zeta\rangle$ and define $\beta = \mathrm{diag}(\zeta^{q-1}, \zeta^{q-1}, 1, 1)$. Then $\beta \in \mathrm{GSp}_4(q^2) \cap \mathrm{GU}_4(q)$ acts on $\mathbf{G}_{i,j}$ as $\mathrm{diag}(\zeta^q, \zeta^q, \zeta, \zeta)$, which scales the symplectic form of $\mathrm{Sp}_4(q^2)$ by ζ^{q+1}. By [22] the form of $\mathrm{Sp}_4(q)$ is proportional and since ζ^{q+1} is a non-square in $\mathbb{F}_q$, β is a linear outer automorphism of $\mathrm{Sp}_4(q)$. Thus, $\mathrm{GSp}_4(q) = \langle \mathrm{Sp}_4(q), \beta\rangle$ and the claim follows.

We now determine $\mathbf{A}_{i,j}$. First we note that β, as well as the group $\mathrm{Aut}(\mathbb{F}_{q^2})$ with respect to the basis $\mathcal{E}$, clearly normalize $\mathbf{G}_i$ and $\mathbf{G}_j$ hence by Lemma 9.7, $\mathrm{Aut}(\mathbb{F}_{q^2}) \leq \mathbf{A}_{i,j}$. So it suffices to determine inner automorphisms of $\mathrm{Sp}_4(q)$ normalizing $\mathbf{D}_i^j$ and $\mathbf{D}_j^i$.

Any inner automorphism in $\mathrm{Sp}_4(q)$ is induced by an inner automorphism of $\mathrm{Sp}_4(q^2)$. So now the claim reduces to a matrix calculation in the group $\mathrm{Sp}_4(q^2)$. □

Next we describe the restriction maps $\mathbf{a}_{i,j}$ for Phan amalgams made up of a single standard pair with trivial inclusion maps.

Lemma 9.12. *Let $\mathscr{A}$ be the coefficient system of the standard Phan amalgam $\underline{\mathscr{G}}$ over $\mathbb{F}_q$ with diagram Γ and system of tori $\mathcal{D}$. Fix $i, j \in I$ and let $(\mathbf{G}_{i,j}, \overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j)$ be a Phan standard pair in $\underline{\mathscr{G}}$ with diagram $\Gamma_{i,j}$. Denote $\mathbf{a} = (\mathbf{a}_{j,i}, \mathbf{a}_{i,j})\colon \mathbf{A}_{i,j} \to \mathbf{A}_i \times \mathbf{A}_j$. Then, we have the following:*

(a) *If $\Gamma_{i,j} = A_1 \times A_1$, then $\mathbf{a}$ is an isomorphism inducing $\mathbf{T}_{i,j} \cong \mathbf{T}_i \times \mathbf{T}_j$ and $\mathbf{C}_{i,j} \cong \mathbf{C}_i \times \mathbf{C}_j$.*
(b) *If $\Gamma_{i,j} = A_2$ or C_2, then $\mathbf{a}$ induces an isomorphism $\mathbf{T}_{i,j} \to \mathbf{T}_i \times \mathbf{T}_j$.*
(c) *If $\Gamma(q) = A_2(q)$ or $\Gamma(q) = C_2(q)$, then $\mathbf{a}\colon \mathbf{C}_{i,j} \to \mathbf{C}_i \times \mathbf{C}_j$ is given by $\alpha \mapsto (\alpha, \alpha)$ (for $\alpha \in \mathrm{Aut}(\mathbb{F}_{q^2})$) which is a diagonal embedding.*

Proof 1. This is immediate from Lemma 9.10.

For the remaining cases, recall that for any $\varphi \in \mathbf{A}_{i,j}$, we have $\mathbf{a}_{i,j}\colon \varphi \mapsto \underline{\mathbf{g}}_{j,i}^{-1} \circ \rho_{i,j}(\varphi) \circ \underline{\mathbf{g}}_{j,i}$, where $\rho_{i,j}(\varphi)$ is the restriction of φ to $\overline{\mathbf{G}}_j \leq \mathbf{G}_{i,j}$ (Definition 9.6) and $\underline{\mathbf{g}}_{i,j}$ is the standard identification map of Definition **??**. Note that the standard identification map transforms the automorphism $\rho_{j,i}(\varphi)$ of $\overline{\mathbf{G}}_i$ essentially to the "same" automorphism φ of $\mathbf{G}_i$.

First let $\Gamma(q) = A_2(q)$. The map $\mathbf{a}$ is well-defined. On T, it is induced by the homomorphism

$$\mathrm{diag}(ac, c, ec) \mapsto (\mathrm{diag}(1, e), \mathrm{diag}(a, 1)),$$

where $a, c, e \in \mathbb{F}_{q^2}$ are such that $aa^\sigma = cc^\sigma = ee^\sigma = 1$. Note that the kernel is $Z(\mathrm{T})$ so that $\mathbf{a}$ is injective. The map is obviously surjective, so we are done. Thus if we factor $\mathbf{a}$ by $\mathbf{T}_{i,j}$ and $\mathbf{T}_i \times \mathbf{T}_j$, we get

$$\mathbf{C}_{i,j} \hookrightarrow \mathbf{C}_i \times \mathbf{C}_j \tag{9.2}$$

which is a diagonal embedding given by $\alpha^r \mapsto (\alpha^r, \alpha^r)$, where $r \in \mathbb{N}$ and $\alpha\colon x \mapsto x^p$ for $x \in \mathbb{F}_{q^2}$.

Next let $\Gamma(q) = C_2(q)$. We can rewrite the elements of T as a diagonal matrix $\mathrm{diag}(xyz, xz, zy^{-1}, z)$, by taking $z = a^\sigma$, $y = (ac)^{-1}$, $x = a^2 b$. On T the map $\mathbf{a}$ is induced by the homomorphism

$$\mathrm{diag}(xyz, xz, y^{-1}z, z) \mapsto (\mathrm{diag}(y, 1), \mathrm{diag}(x, 1))$$

with kernel $\{\mathrm{diag}(z, z, z, z)\colon z \in \mathbb{F}_{q^2} \text{ with } zz^\sigma = 1\} = Z(\mathrm{GU}_4(q))$. Clearly $\mathbf{a}\colon \mathbf{T}_{i,j} \to \mathbf{T}_i \times \mathbf{T}_j$ is an isomorphism. Taking the quotient over these groups, $\mathbf{a}$ induces a diagonal embedding as in (9.2), where we now interpret it in the $C_2(q)$ setting. □

9.2.4. *A standard form for Phan amalgams.* Suppose that $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i, j \in I\}$ is a Phan amalgam over $\mathbb{F}_q$ with 3-spherical diagram Γ. Without loss of generality we will assume that all inclusion maps $\underline{\mathbf{g}}_{i,j}$ are the standard identification maps of Definition **??**. By Lemma 9.4, $\underline{\mathscr{G}}$ has Property (D) and possesses a system $\mathcal{D} = \{\mathbf{D}_i\colon i \in I\}$ of tori, which, as noted in Definition 9.5 via the standard embeddings $\underline{\mathbf{g}}_{i,j}$ can be identified with those given in Lemma 9.2.

We wish to classify all Phan amalgams $\mathscr{G} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \mathbf{g}_{i,j} \mid i, j \in I\}$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$. As noted in Definition 9.5 we may assume that all such amalgams share $\mathcal{D}$. Let $\mathscr{A} = \{\mathbf{A}_i, \mathbf{A}_{i,j}, \mathbf{a}_{i,j} \mid i, j \in I\}$ be the coefficient system of $\underline{\mathscr{G}}$ associated to $\mathcal{D}$. By Proposition 9.8, we may restrict to those amalgams whose connecting maps are of the form $\mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j} \circ \delta_{i,j}$ for $\delta_{i,j} \in \mathbf{A}_i$ for all $i \in I$.

Definition 9.13. The *trivial support* of $\mathscr{G}$ (with respect to $\underline{\mathscr{G}}$) is the set $\{(i,j) \in I \times I \mid \mathbf{g}_{i,j} = \underline{\mathbf{g}}_{i,j}\}$ (that is, $\delta_{i,j} = \mathrm{id}_{\mathbf{G}_i}$ in the notation of Proposition 9.8). The word "trivial" derives from the assumption that the $\underline{\mathbf{g}}_{i,j}$'s are the standard identification maps of Definition **??**.

Fix some spanning tree $\Sigma \subseteq \Gamma$ and suppose that $\mathrm{E} - \mathrm{E}\,\Sigma = \{\{i_s, j_s\}\colon s = 1, 2, \ldots, r\}$ so that $H_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^r$. We now have

Proposition 9.14. *There is a Phan amalgam $\mathscr{G}(\Sigma)$ with the same diagram as $\underline{\mathscr{G}}$ and the same $\mathcal{D}$, which is isomorphic to $\mathscr{G}$ and has the following properties:*

(a) *$\mathscr{G}$ has trivial support $S = \{(i,j) \in I \times I \mid \{i,j\} \in \mathrm{E}\,\Sigma\} \cup \{(i_s, j_s) \colon s = 1, 2\dots, r\}$.*

(b) *for each $s = 1, 2, \dots, r$, we have $\mathbf{g}_{j_s,i_s} = \underline{\mathbf{g}}_{j_s,i_s} \circ \gamma_{j_s,i_s}$, where $\gamma_{j_s,i_s} \in \mathbf{C}_{j_s}$.*

Lemma 9.15. *There is a Phan amalgam $\mathscr{G}^+$ over $\mathbb{F}_q$ with the same diagram as $\underline{\mathscr{G}}$ and the same $\mathcal{D}$, which is isomorphic to $\mathscr{G}$ and has the following properties: For any $u, v \in I$, if $\mathbf{g}_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v} \circ d_{u,v}$, for some $\gamma_{u,v} \in \mathbf{C}_u$ and $d_{u,v} \in \mathbf{T}_u$, then $\mathbf{g}^+_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$.*

Proof The proof follows the same steps as that of Lemma 8.17 using Part 2 of Lemma 9.12 instead of Lemma 8.13 Part 2. $\square$

By Lemma 9.15 in order to prove Proposition 9.14 we may now assume that $\mathbf{g}_{u,v} = \underline{\mathbf{g}}_{u,v} \circ \gamma_{u,v}$ for some $\gamma_{u,v} \in \mathbf{C}_u$ for all $u, v \in I$.

We now prove a Corollary for Phan amalgams analogous to, but stronger than Corollary 8.18. To this end consider the situation of Figure 4 interpreted in the Phan setting.

Corollary 9.16. *With the notation introduced in Figure 4, fix the maps $\gamma_{i,j}$, $\gamma^+_{i,j}$, $\phi_i \in \mathbf{C}_i$ as well as $\gamma_{j,i} \in \mathbf{C}_j$. Then for any one of $\gamma^+_{j,i}, \phi_j \in \mathbf{C}_j$, there exists a unique choice $\gamma \in \mathbf{C}_i$ for the remaining map in $\mathbf{C}_j$ so that there exists $\phi_{i,j}$ making the diagram in Figure 4 commute.*

Proof This follows immediately from the fact that the maps $\mathbf{a}_{j,i} \colon \mathbf{C}_{i,j} \to \mathbf{C}_i$ and $\mathbf{a}_{i,j} \colon \mathbf{C}_{i,j} \to \mathbf{C}_j$ in part 3. of Lemma 9.12 are isomorphisms. $\square$

Proof (of Proposition 9.14) The proof follows the same steps as that of Proposition 8.16, replacing Lemma 8.17 and Corollary 8.18 by Lemma 9.15 and Corollary 9.16. $\square$

9.2.5. *Classification of Phan amalgams with* 3*-spherical diagram.* In the case where $\mathscr{G}$ is a Phan amalgam over $\mathbb{F}_q$ whose diagram is a 3-spherical tree, Proposition 9.14 says that $\mathscr{G} \cong \underline{\mathscr{G}}$.

Theorem 9.17. *Suppose that $\mathscr{G}$ is a Curtis-Tits amalgam with a diagram that is a* 3*-spherical tree. Then, $\mathscr{G}$ is unique up to isomorphism. In particular any Phan amalgam with spherical diagram is unique.*

Definition 9.18. Fix a connected 3-spherical diagram Γ and a prime power q. Let Σ be a spanning tree and let the set of edges $\mathrm{E}\,\Gamma - \mathrm{E}\,\Sigma = \{\{i_s, j_s\} \colon s = 1, 2, \dots, r\}$ together with the integers $\{e_s \colon s = 1, 2, \dots, r\}$ satisfy the conclusions of Lemma 8.20. Note that since in the Phan case we do not have subdiagrams of type ${}^2A_3(q)$, we have $e_s = 1$ for all $s = \{1, 2, \dots, r\}$.

Let $\mathsf{Ph}(\Gamma, q)$ be the collection of isomorphism classes of Phan amalgams of type $\Gamma(q)$ and let $\underline{\mathscr{G}} = \{\mathbf{G}_i, \mathbf{G}_{i,j}, \underline{\mathbf{g}}_{i,j} \mid i, j \in I\}$ be a Phan amalgam over $\mathbb{F}_q$ with diagram Γ.

Consider the following map:

$$\kappa\colon \prod_{s=1}^{r} \mathrm{Aut}(\mathbb{F}_{q^2}) \to \mathsf{Ph}(\Gamma).$$

where $\kappa((\alpha_s)_{s=1}^r)$ is the isomorphism class of the amalgam $\mathscr{G}^+ = \mathscr{G}((\alpha_s)_{s=1}^r)$ given by setting $\mathbf{g}^+_{j_s,i_s} = \mathbf{g}_{j_s,i_s} \circ \alpha_s$ for all $s = 1, 2, \ldots, r$.

As for Curtis-Tits amalgams, one shows the following.

Corollary 9.19. *The map κ is onto.*

Lemma 9.20. *Suppose $\Gamma(q)$ is a 3-spherical diagram Γ that is a simple loop. Then, κ is injective.*

Proof The proof is identical to that of Lemma 8.23 replacing Proposition 8.16 by Proposition 9.14 and Lemma 8.13 by Lemma 9.12, and noting that in the Phan case, we can consider the group $\mathbf{C}_i$ and $\mathbf{C}_{i,j}$ themselves rather than some suitably chosen quotient. □

Theorem 9.21. *Let Γ be a connected 3-spherical diagram with spanning tree Σ and set of edges $\mathrm{E}\,\Gamma - \mathrm{E}\,\Sigma = \{\{i_s, j_s\}\colon s = 1, 2, \ldots, r\}$. Then κ is a bijection between the elements of $\prod_{s=1}^r \mathrm{Aut}(\mathbb{F}_{q^2})$ and the isomorphism classes of Curtis-Tits amalgams with diagram Γ over $\mathbb{F}_q$.*

Proof This follows from Lemma 9.20 just as Theorem 8.24 follows from Lemma 8.23. □

References

[1] C. D. Bennett and S. Shpectorov. A new proof of a theorem of Phan. *J. Group Theory*, 7(3):287–310, 2004.

[2] R. J. Blok. The generating rank of the symplectic grassmannians: Hyperbolic and isotropic geometry. *Europ. J. Comb.*, 28(5):1368–1394, 2007.

[3] R. J. Blok and C. G. Hoffman. 1-cohomology of simplicial amalgams of groups. *J. Algebraic Combin.*, 37(2):381–400, 2013.

[4] R. J. Blok and C. G. Hoffman. Curtis–Tits groups generalizing Kac–Moody groups of type $\tilde{A}_{n-1}$. *J. Algebra*, 399:978–1012, 2014.

[5] R. J. Blok and C. Hoffman. A classfication of Curtis-Tits amalgams. In N. Sastry, editor, *Groups of Exceptional Type, Coxeter Groups and Related Geometries*, volume 149 of *Springer Proceedings in Mathematics & Statistics*, pages 1–26. Springer, January 2014.

[6] R. J. Blok and C. G. Hoffman. Curtis-Tits groups of simply-laced type. To appear in J. Comb. Th. Ser. A.

[7] P. E. Caprace. On 2-spherical Kac-Moody groups and their central extensions. *Forum Math.*, 19(5):763–781, 2007.

[8] C. W. Curtis. Central extensions of groups of Lie type. *J. Reine Angew. Math.*, 220:174–185, 1965.
[9] J. Dunlap. *Uniqueness of Curtis-Phan-Tits amalgams.* PhD thesis, Bowling Green State University, 2005.
[10] D. Gorenstein. *The classification of finite simple groups. Vol. 1.* The University Series in Mathematics. Plenum Press, New York, 1983. Groups of noncharacteristic 2 type.
[11] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 2. Part I. Chapter G*, volume 40 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1996. General group theory.
[12] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*, volume 40 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1998. Almost simple K-groups.
[13] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 4. Part II. Chapters 1–4*, volume 40 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1999. Uniqueness theorems, With errata: ıt The classification of the finite simple groups. Number 3. Part I. Chapter A [Amer. Math. Soc., Providence, RI, 1998; MR1490581 (98j:20011)].
[14] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 5. Part III. Chapters 1–6*, volume 40 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2002. The generic case, stages 1–3a.
[15] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 6. Part IV*, volume 40 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2005. The special odd case.
[16] R. Gramlich. Weak Phan systems of type C_n. *J. Algebra*, 280(1):1–19, 2004.
[17] F. G. Timmesfeld. Presentations for certain Chevalley groups. *Geom. Dedicata*, 73(1):85–117, 1998.
[18] F. G. Timmesfeld. On the Steinberg-presentation for Lie-type groups. *Forum Math.*, 15(5):645–663, 2003.
[19] F. G. Timmesfeld. The Curtis-Tits-presentation. *Adv. Math.*, 189(1):38–67, 2004.
[20] F. G. Timmesfeld. Steinberg-type presentation for Lie-type groups. *J. Algebra*, 300(2):806–819, 2006.
[21] R. Gramlich. Weak Phan systems of type C_n. *J. Algebra*, 280(1):1–19, 2004.
[22] R. Gramlich, C. Hoffman, and S. Shpectorov. A Phan-type theorem for $\mathrm{Sp}(2n, q)$. *J. Algebra*, 264(2):358–384, 2003.
[23] R. Gramlich, M. Horn, and W. Nickel. The complete Phan-type theorem for $\mathrm{Sp}(2n, q)$. *J. Group Theory*, 9(5):603–626, 2006.
[24] R. Gramlich, M. Horn, and W. Nickel. The complete Phan-type theorem for $\mathrm{Sp}(2n, q)$. *J. Group Theory*, 9(5):603–626, 2006.
[25] B. Mühlherr. Locally split and locally finite twin buildings of 2-spherical type. *J. Reine Angew. Math.*, 511:119–143, 1999.
[26] K.-W. Phan. A characterization of the unitary groups PSU(4, q^2), q odd. *J. Algebra*, 17:132–148, 1971.
[27] K. W. Phan. On groups generated by three-dimensional special unitary groups. I. *J. Austral. Math. Soc. Ser. A*, 23(1):67–77, 1977.
[28] K.-W. Phan. On groups generated by three-dimensional special unitary groups. II. *J. Austral. Math. Soc. Ser. A*, 23(2):129–146, 1977.

[29] O. Schreier and B. Van der Waerden. Die automorphismen der projektiven gruppen. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 6(1):303–322, December 1928.

[30] J. Tits. Twin buildings and groups of Kac-Moody type. In *Groups, combinatorics & geometry (Durham, 1990)*, volume 165 of *London Math. Soc. Lecture Note Ser.*, pages 249–286. Cambridge Univ. Press, Cambridge, 1992.

[31] R. A. Wilson. *The finite simple groups*, volume 251 of *Graduate Texts in Mathematics*. Springer-Verlag London Ltd., London, 2009.

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