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MONOTONE APPROXIMATION BY SPLINES*

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Abstract. We prove Jackson type estimates for the approximation of monotone nondecreasing functions by monotone nondecreasing splines with equally spaced knots. Our results are of the same order as the Jackson type estimates for unconstrained approximation by splines with equally spaced knots.

1. Introduction. We are interested in how well we can approximate a monotone nondecreasing function by monotone nondecreasing splines. For $r, n \geqq 1$, let $\mathcal{S}(r, n)$ denote the space of splines of order r (degree $r-1$) with knots $\{i/n\}_0^n$, i.e., $S \in \mathcal{S}(r, n)$ if and only if $S^{(r-2)}$ is continuous on $[0, 1]$ and on each interval $[i/n, (i+1)/n]$, $i = 0, 1, \cdots, n-1$, S is a polynomial of degree $\leqq r-1$. If f is a monotone, nondecreasing function on $[0, 1]$ ($f\uparrow$), then we define the error of monotone approximation by splines to be

$$E_n^*(f, r) \equiv \inf_{S \in \mathcal{S}^*(r, n)} \|f - S\|,$$

where $\|\cdot\|$ is the supremum norm on $[0, 1]$ and $\mathcal{S}^*(r, n)$ is the set of those splines S in $\mathcal{S}(r, n)$ with $S\uparrow$. The question then is how fast does $E_n^*(f, r) \to 0$, $n \to \infty$, in relation to the smoothness of f? Our main result is the following theorem.

THEOREM 1. *There is a constant $C > 0$, depending only on r, such that whenever $f\uparrow$ and $f^{(k)}$ is continuous, $0 \leqq k \leqq r-1$, then*

$$E_n^*(f, r) \leqq Cn^{-k}\omega(f^{(k)}, n^{-1}), \qquad n = 1, 2, \cdots. \tag{1.1}$$

This is a Jackson type theorem for monotone approximation by splines. It is exactly the same as the Jackson type theorem for unrestricted spline approximation.

Theorem 1 shows that monotone approximation by splines is as efficient as the unrestricted approximation by splines, at least in the sense of Jackson type estimates of the form (1.1). There is some deficiency in (1.1) however, in that it is preferable to give the Jackson type estimates in terms of the rth order modulus of smoothness, $\omega_r(f, t)$, rather than just the first order as in (1.1). The rth order modulus of smoothness is needed to completely characterize the degree of approximation by splines in the unrestricted case in terms of both direct and inverse theorems (see K. Scherer [9]). We have more to say on this in the remarks section (§ 6).

Theorem 1 is already known in the case of approximation by step functions or piecewise linear functions ($r = 1, 2$) and also in the general case $r \geqq 1$ if $k = 0$ or 1. Here, the variation diminishing splines of Schoenberg are monotone when f is and they also provide the estimate (1.1) in case $k = 0, 1$ (see M. Marsden [7] and De

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Vore [3]). The variation diminishing splines can not give the result (1.1) for $k \geqq 2$ because they are positive operators and hence saturated. This limitation exists not only for variation diminishing splines but for any linear method of approximation since any such method would have to preserve positivity (of the derivative of f) and hence be restricted in its degree of approximation by the saturation phenomena for positive linear operators. Thus, we must go to nonlinear techniques to prove the general case of (1.1) and this makes the situation more difficult. For example, there is no easy proof of (1.1) even for quadratic or cubic splines.

This situation is paralleled in monotone approximation by polynomials. For this problem, G. G. Lorentz and K. Zeller [6] and G. G. Lorentz [5] have given Jackson type theorems of the same form as (1.1), for $k = 0, 1$ (here n is the degree of the approximating polynomials), but these results have not been extended to $k \geqq 2$. It is possible to use the techniques and results of this paper to prove the higher order Jackson theorems for monotone approximation by polynomials. This is given in the next article of this journal [10].

Our proof of (1.1) is somewhat complicated by the presence of many constants whose actual values are usually not important but they are sometimes used in the definition of other constants. We will use the following conventions in labeling constants. Constants that appear in inequalities for general splines, e.g. B-splines, will be denoted by α_1, α_2, etc. Constants that appear in the approximation of f or its derivatives by splines will be denoted by C_1, C_2, etc. Constants that appear in upper estimates for f' will be denoted by A_1, A_2, etc., while constants appearing in lower estimates for f' will be denoted by $B_1, B_2, \cdots$.

2. *B*-Splines. We will on several occasions have to make local corrections of splines. This is best done by using splines with small support, such as the B-splines. Let $t_j = j/n$, for $j = 0, 1, \cdots, n$, $t_j = 0$ for $j < 0$, and $t_j = 1$, for $j > n$. If $M(x; t) = r(t-x)_+^{r-1}$ then the B-splines of order r are given by

$$N_{i,n,r}(x) = M(x; t_i, \cdots, t_{i+r}), \qquad -r+1 \leqq i \leqq n-1,$$

where the notation means that for fixed x we take the rth divided difference of M with respect to the variable t at the points $t_i, \cdots, t_{i+r}$. We mention now some properties of B-splines which can be found either in [1] or [2].

The B-splines have minimal support. For each i, $N_{i,n,r}$ vanishes outside of (t_i, t_{i+r}) and is strictly positive on (t_i, t_{i+r}). With our normalization we have

$$\int_{-\infty}^{\infty} N_{i,n,r}(x)\, dx = 1.$$

Actually, we will be more interested in the B-splines of order $r-1$ (degree $r-2$) since these will be used in the approximation of f'. Accordingly, let us introduce the notation that $N_i = N_{i,n,r-1}$ with the n and $r-1$ being understood. There are constants $\alpha_1 \leqq 1$ and $\alpha_2 \geqq 1$, which depend only on r, such that

$$N_i(x) \geqq \alpha_1 n, \qquad t_i + n^{-1} \leqq x \leqq t_{i+r-1} - n^{-1}, \tag{2.1}$$

$$N_i(x) \leqq \alpha_2 n, \qquad -\infty < x < \infty. \tag{2.2}$$

The B-splines N_i form a basis for $\mathcal{S}(r-1, n)$. If $S \in \mathcal{S}(r-1, n)$, then there are unique constants $(a_i)_{-r+2}^{n-1}$, such that $S = \sum a_i N_i$. Also, there is a constant $\alpha_3 \geqq 1$, which depends only on r, such that

$$(2.3) \qquad |a_i| \leqq \alpha_3 n^{-1} \sup_{t_i \leqq x \leqq t_{i+r-1}} |S(x)|.$$

This last inequality follows from the C. deBoor–G. Fix formula for quasi-interpolation [1] which gives a formula for a_i in terms of derivatives of S on (t_i, t_{i+r-1}). We use Markov's inequality to replace derivatives of S by the supremum of S over (t_i, t_{i+r-1}).

3. Interpolation techniques. There is a very useful idea in approximation that to prove a result like (1.1), it is frequently enough to prove the result for only the end point, which in our case is when f has a bounded rth derivative. This is accomplished by using an interpolation argument to derive the general result from the end point result. The argument relies on replacing the arbitrary function f by a function which has a bounded rth derivative (controlled by the smoothness of f) and approximates f well. More precisely, if $\varepsilon > 0$, $f \in C^{(k)}[0, 1]$, and $k < r$, then there is a function g_ε with the following properties:

$$(3.1) \qquad \|f - g_\varepsilon\| \leqq C_1 \varepsilon^k \omega(f^{(k)}, \varepsilon),$$

$$(3.2) \qquad \|g_\varepsilon^{(r)}\| \leqq C_1 \varepsilon^{k-r} \omega(f^{(k)}, \varepsilon)$$

where $C_1 \geqq 1$ is a constant depending only on r (see e.g. G Freud and V. Popov [4]).

In the case of monotone approximation, in order to use this technique directly, we would need to know that when $f\uparrow$, then the functions g_ε can also be chosen to be nondecreasing. This does not follow from the Freud–Popov construction and it is not known whether this is actually the case. However, it still will be useful to use this interpolation technique in some of our proofs. We will also use the fact that when $k \geqq 1$, g_ε can be chosen to satisfy

$$(3.3) \qquad \|f' - g_\varepsilon'\| \leqq C_1 \varepsilon^{k-1} \omega(f^{(k)}, \varepsilon).$$

In fact, the g_ε given by Freud–Popov already satisfies (3.3).

Let us point out how (3.1) and (3.2) can be used in the proof of the unrestricted version of (1.1), since some of our later arguments are based on this approach. The idea follows V. Popov and Bl. Sendov [8]. We prove first that if g is any function with $\|g^{(r)}\| \leqq M$, then there is a spline $S \in \mathcal{S}(r, n)$, such that

$$(3.4) \qquad \|g - S\| \leqq CMn^{-r},$$

with C depending only on r. This is proved by establishing the more general statement that for each $j = 1, 2, \cdots, r$, there is an $S \in \mathcal{S}(j, n)$, with $\|g^{(r-j)} - S_j\| \leqq 2^j r^j M n^{-j}$. For $j = 1$, the function S_1 can be taken as $S_1(x) = g^{(r-1)}(i/n)$, $x \in [in^{-1}, (i+1)n^{-1}$, $i = 0, 1, \cdots, n-1$.

Suppose then that we have shown the existence of a spline $S_j \in \mathscr{S}(j, n)$ with $\|g^{(r-j)} - S_j\| \leqq 2^j r^j M n^{-j}$. Let $y_i = rin^{-1}$, $i = 0, 1, \cdots, \lambda$, with $\lambda = [n/r]$. Define

$$a_{i,j} = \int_{y_i}^{y_{i+1}} \{g^{(r-j)}(t) - S_j(t)\}\, dt, \qquad i = 0, 1, \cdots, \lambda - 1.$$

We can take

$$S_{j+1}(x) = \int_0^x \left\{ S_j(t) + \sum_{i=0}^{\lambda-1} a_{i,j} N_{ir,n,j}(t) \right\} dt + g^{(r-j-1)}(0).$$

The spline S_{j+1} is in $\mathscr{S}(j+1, n)$ and $S_{j+1}(y_i) = g^{(r-j-1)}(y_i)$, $0 \leqq i \leqq \lambda$. Hence,

$$\begin{aligned} |g^{(r-j-1)}(x) - S_{j+1}(x)| &= \left| \int_{y_i}^x (g^{(r-j)}(t) - S_j(t))\, dt \right| + |a_{i,j}| \\ &\leqq 2^{j+1} r^j M n^{-j} |y_{i+1} - y_i| \leqq 2^{j+1} r^{j+1} M n^{-j-1}, \qquad x \in [y_i, y_{i+1}). \end{aligned}$$

The same estimate holds on $[y_\lambda, 1]$. This shows (3.4).

Now, we take $\varepsilon = n^{-1}$ and g_ε as a function which satisfies (3.1) and (3.2). Let S be the spline which satisfies (3.4) for $g = g_\varepsilon$. Then,

$$\begin{aligned} \|f - S\| \leqq \|f - g_\varepsilon\| + \|g_\varepsilon - S\| &\leqq (C_1 + CC_1) n^{-k} \omega(f^{(k)}, n^{-1}) \\ &\leqq C_2 n^{-k} \omega(f^{(k)}, n^{-1}) \end{aligned} \tag{3.5}$$

with $C_2 > C_1$ a constant depending only on r. This is the unrestricted analogue of (1.1).

Because of (3.3) and the way S is constructed, we also have the estimate

$$\|f' - S'\| \leqq C_2 n^{-k+1} \omega(f^{(k)}, n^{-1}). \tag{3.6}$$

The spline S has a piecewise $(r-1)$st derivative which is of course a step function. The jumps in this step function are controlled because of (3.2) and the construction of S. Namely,

$$\left| \text{jump } S^{(r-1)}\left(\frac{j}{n}\right) \right| \leqq C_2 n^{r-k-1} \omega(f^{(k)}, n^{-1}) \tag{3.7}$$

for $j = 1, \cdots, n-1$. In some sense, we have come full circle since the spline S can be used as the function g_ε when $\varepsilon = n^{-1}$ except that S does not have a continuous rth derivative, but instead we have (3.7). These results about the spline S in unrestricted approximation will be used later.

4. Decomposition of monotone functions. As we have observed in the Introduction, the estimate (1.1) is already known for $k = 0, 1$ and so we will assume from here on that $k \geqq 2$ and $r > 2$. In particular, f' is then continuous.

Note that if f' is strictly positive on $[0, 1]$ then the spline S introduced in the previous section will be monotone nondecreasing when n is sufficiently large because of (3.6). This spline also approximates with the correct order to give (1.1) because of (3.5). Thus, we see that the real difficulty in proving Theorem 1 will be when f' has zeros. The idea then is to decompose f in such a way that we have good control over the derivative of f. This is done by isolating certain kinds of intervals on which f' is small, while on the remaining intervals, f' is large at a suitable number of places. We begin by introducing four types of intervals. The function f' will be small on the intervals of type 1, 2, and 3, while on the type 4 intervals f' will be large at least some of the time.

Let $A_1 = 100r^2 2^{r^4} C_2 \alpha_1^{-1} \alpha_2^2$, where $C_2 \geqq 1$ is the constant that appears in (3.5)–(3.7) and $\alpha_1 \leqq 1$ and $\alpha_2 \geqq 1$ are the constants that appear in (2.1) and (2.2). The constant A_1 depends only on r. Let $\varepsilon_n = n^{-k}\omega(f^{(k)}, n^{-1})$. An interval I is said to be of type 1 if

$$I = [i_1 n^{-1}, i_2 n^{-1}], \textit{ with } i_1, i_2 \textit{ integers}, i_2 - i_1 \geqq r^2, \textit{ and } I \subseteq [0, 1], \tag{4.1}$$

$$f'(x) \leqq A_1 n \varepsilon_n, \qquad x \in I, \tag{4.2}$$

$$\textit{if } J = [j_1 n^{-1}, j_2 n^{-1}], \textit{ with } j_1, j_2 \textit{ integers}, I \subseteq J \textit{ and } f'(x) \leqq A_1 n \varepsilon_n, x \in J, \textit{ then } J = I. \tag{4.3}$$

The condition (4.3) guarantees that I is a maximal interval on which (4.2) holds.

Let $A_2 = \alpha_3 A_1^2$, where α_3 is the constant of (2.3). Then $A_2 > A_1$. If $I = [i_1 n^{-1}, i_2 n^{-1}]$ is an interval of type 1, then let $j_1 \leqq i_1$ be the smallest integer such that

$$f'(x) \leqq A_2 n \varepsilon_n, \qquad x \in [j_1 n^{-1}, i_1 n^{-1}]. \tag{4.4}$$

Similarly, let $j_2 \geqq i_2$ be the largest integer such that

$$f'(x) \leqq A_2 n \varepsilon_n, \qquad x \in [i_2 n^{-1}, j_2 n^{-1}]. \tag{4.5}$$

We call the intervals $[j_1 n^{-1}, i_1 n^{-1}]$ and $[i_2 n^{-1}, j_2 n^{-1}]$, intervals of type 2.

When we remove the intervals of type 1 and type 2 from $[0, 1]$, then we are left with a finite number of intervals. Such a left over interval is said to be of type 3 or type 4 according to whether the length of the interval is less than $2r^2 n^{-1}$ or greater than or equal to $2r^2 n^{-1}$ respectively. It turns out that f' is also small on intervals of type 3. This follows from our first lemma which shows that if f' is small on an interval it is also small on adjacent intervals.

LEMMA 1. *Suppose* $0 \leqq a \leqq 1 - n^{-1}$, *and*

$$f'(x) \leqq M, \qquad x \in [a, a + n^{-1}]. \tag{4.6}$$

Then, for any integer $l > 1$, *we have*

$$f'(x) \leqq 2^{lk^2}(M + n\varepsilon_n), \qquad x \in [a - ln^{-1}, a + (l+1)n^{-1}] \cap [0, 1]. \tag{4.7}$$

Proof. We derive the estimate for $[a + n^{-1}, a + (l+1)n^{-1}] \cap [0, 1]$. The other case is proved in the same way. Suppose first that $x \in [a + n^{-1}, a + n^{-1} + k^{-1}n^{-1}]$. We use standard notation $\Delta_h^k(g, x)$ to denote the kth difference of g with step size h at the point x. Take $h = k^{-1}(x - a)$. So, $a + kh = x$ and $a + (k-1)h \leqq a + n^{-1}$. Hence, with $g = f'$, we have

$$\begin{aligned} g(x) = g(a + kh) &\leqq |\Delta_h^k(g, a)| + |\Delta_h^k(g, a) - g(a + kh)| \\ &\leqq |\Delta_h^k(g, a)| + (2^k - 1)M, \end{aligned} \tag{4.8}$$

where we used the fact that the second term in absolute values only involves values of g on $[a, a + n^{-1}]$, where (4.6) holds.

Now for any y, we have $\Delta_h^{k-1}(g, y) = h^{k-1} g^{(k-1)}(\xi) = h^{k-1} f^{(k)}(\xi)$, with $y < \xi < y + kh$. Thus,

$$|\Delta_h^k(g, a)| = |\Delta_h^{k-1}(g, a+h) - \Delta_h^{k-1}(g, a)| \leqq h^{k-1} |f^{(k)}(\xi_1) - f^{(k)}(\xi_2)|,$$

with ξ_1 and ξ_2 in $[a, a + n^{-1} + k^{-1} n^{-1}]$. Hence, because $h = k^{-1} n^{-1}$, we have

$$\begin{aligned} |\Delta_h^k(g, a)| &\leqq k^{-k+1} n^{-k+1} \omega(f^{(k)}, n^{-1} + k^{-1} n^{-1}) \\ &\leqq 2k^{-k+1} n^{-k+1} \omega(f^{(k)}, n^{-1}) \leqq n\varepsilon_n. \end{aligned}$$

Here we have used the fact that $k \geqq 2$. If we use this inequality back in (4.8), we find that

$$f'(x) = g(x) \leqq (2^k - 1)M + n\varepsilon_n \leqq 2^k M + n\varepsilon_n, \qquad x \in [a, a + n^{-1} + k^{-1} n^{-1}].$$

This extends our original inequality (4.6) to the larger interval $[a, a + n^{-1} + k^{-1} n^{-1}]$. Now, we repeat this procedure lk times to find

$$f'(x) \leqq 2^{lk^2} M + (2^{lk^2 - 1} + \cdots + 2^k + 1) n\varepsilon_n \leqq 2^{lk^2}(M + n\varepsilon_n),$$

as desired.

As an immediate consequence of Lemma 1, we have the following lemma which shows that f' is small on intervals of type 3.

LEMMA 2. *There is a constant $A_3 > 0$, which depends only on r, such that for any interval I of type 3, we have*

$$f'(x) \leqq A_3 n\varepsilon_n, \qquad x \in I. \tag{4.9}$$

Proof. Since I is an interval of type 3, it must be adjacent to either an interval of type 1 or of type 2. Suppose that $[a, b]$ is this adjacent interval and assume that $[a, b]$ is to the right of I, so that $a \in \bar{I}$. The other case is handled in the same way. On the interval $[a, a + n^{-1}]$, we have

$$f'(x) \leqq A_2 n\varepsilon_n, \qquad x \in [a, a + n^{-1}],$$

because of (4.2) if $[a, b]$ is of type 1 and because of (4.4) if $[a, b]$ is of type 2. We know from Lemma 1 that this inequality can be extended to adjacent intervals. Indeed, Lemma 1 gives

$$f'(x) \leqq 2^{2r^2k^2}(A_2 n\varepsilon_n + n\varepsilon_n) \leqq 2^{2r^4}(A_2 + 1) n\varepsilon_n,$$

because I has length $< 2r^2 n^{-1}$. This proves the lemma with $A_3 = 2^{2r^4}(A_2 + 1)$.

While f' is small on intervals of type 3, on intervals of type 4 there are always places where f' is suitably large as our next lemma shows.

LEMMA 3. *Let $B_2 = 2^{-r^4} A_2$. Suppose $I = [i_1 n^{-1}, i_2 n^{-1}]$ is an interval of type 4. Consider the intervals $J_\nu = [(i_1 + \nu r) n^{-1}, (i_1 + (\nu + 1) r) n^{-1}]$, $\nu = 0, 1, \cdots, r - 1$. If $i_1 > 0$, then for some value of ν, we have*

$$f'(x) \geqq B_2 n\varepsilon_n, \qquad x \in J_\nu. \tag{4.10}$$

Similarly, consider the intervals $J'_\nu = [(i_2 - (\nu + 1) r) n^{-1}, (i_2 - \nu r) n^{-1}]$, $\nu = 0, 1, \cdots, r - 1$. If $i_2 < n$, then there is a value of ν such that

$$f'(x) \geqq B_2 n\varepsilon_n, \qquad x \in J'_\nu. \tag{4.11}$$

Proof. We will prove (4.10). The proof of (4.11) is the same. First observe that we cannot have $f'(x) \leqq B_2 n\varepsilon_n$, $x \in J_\nu$, for any $\nu = 0, 1, \cdots, r-1$. Otherwise, we could use Lemma 1 to extend this inequality to the left and find

$$f'(x) \leqq 2^{r^2k^2}(B_2 n\varepsilon_n + n\varepsilon_n) \leqq 2^{r^4} B_2 n\varepsilon_n, \qquad x \varepsilon J_0. \tag{4.12}$$

Here, we used the facts that $B_2 \geqq 1$ and $k \leqq r-1$. Now, (4.12) and the fact that $2^{r^4}B_2 = A_2$ shows that $f'(x) \leqq A_2 n\varepsilon_n$ which means J_0 should be contained in the type 2 interval immediately to the left of J_0 (here is where we need $i_1 > 0$). This is a contradiction.

So, now we know that for each ν, there are points $\xi_\nu \in J_\nu$ for which $f'(\xi_\nu) \geqq B_2 n\varepsilon_n$. On the other hand, suppose that for each ν, there are points $\xi'_\nu \in J_\nu$, with $f'(\xi'_\nu) < B_2 n\varepsilon_n$. Again, we must find a contradiction. This is done as follows. By the continuity of f', there is for each ν a point $x_\nu \in J_\nu$ such that $f'(x_\nu) = B_2 n\varepsilon_n$. Also on J_0, there is a point $x_0 \in J_0$ with $f'(x_0) = A_2 n\varepsilon_n$. Otherwise we would have $f'(x) < A_2 n\varepsilon_n$, $x \in J_0$, which again would put J_0 in the interval of type 2 immediately to the left of J_0.

Now, we want to compute the divided difference of $g = f'$ at the points x_ν. There exist points $\eta_1, \eta_2 \in [i_1 n^{-1}, (i_1 + r^2)n^{-1}]$ with

$$\begin{aligned} g^{(k-1)}(\eta_1) &= (k-1)!g[x_0, x_1, \cdots, x_{k-1}] \\ &= (k-1)!(A_2 - B_2)n\varepsilon_n \prod_{i=1}^{k-1} (x_0 - x_i)^{-1}, \end{aligned}$$

and likewise

$$g^{(k-1)}(\eta_2) = (k-1)!g[x_1, x_2, \cdots, x_k] = 0.$$

Hence,

$$|A_2 - B_2| \leqq ((k-1)!n\varepsilon_n)^{-1} |g^{(k-1)}(\eta_1) - g^{(k-1)}(\eta_2)| \prod_{i=1}^{k-1} (x_i - x_0). \tag{4.13}$$

On the other hand,

$$\begin{aligned} |g^{(k-1)}(\eta_1) - g^{(k-1)}(\eta_2)| &= |f^{(k)}(\eta_1) - f^{(k)}(\eta_2)| \\ &\leqq \omega(f^{(k)}, r^2 n^{-1}) \leqq r^2 \omega(f^{(k)}, n^{-1}) = r^2 n^k \varepsilon_n \end{aligned}$$

and

$$\prod_{i=1}^{k-1} (x_i - x_0) \leqq (r^2)^{k-1} n^{-k+1}.$$

Putting this back in (4.13) gives the estimate

$$|A_2 - B_2| \leqq ((k-1)!)^{-1} r^{2k} \leqq r^{2r}. \tag{4.14}$$

From the very definition of the constants A_1, A_2 and B_2, we find

$$|A_2 - B_2| \geqq A_1 \geqq r^2 2^{r^4} > r^{2r}, \tag{4.15}$$

since $r \geqq 2$. The estimate (4.14) and (4.15) contradict one another and therefore we have proved the lemma.

Besides guaranteeing that f' is large on parts of intervals of type 4, we can also show that f' is large on parts of any interval that does not intersect a type 1 interval.

LEMMA 4. *Let* $B_1 = 2^{-r^4}A_1$. *If* $J = [jn^{-1}, (j+r^2)n^{-1}]$ *is any interval in* $[0, 1]$ *that does not intersect any interval of type* 1, *then for one of the intervals* $J_\nu = [(j+\nu r)n^{-1}, (j+(\nu+1)r)n^{-1}]$, $\nu = 0, 1, \cdots, r-1$, *we have*

$$f'(x) \geqq B_1 n\varepsilon_n, \qquad x \in J_\nu. \tag{4.16}$$

Proof. This proof is almost identical to that of Lemma 3. If $0 < \nu < r-1$, then we cannot have

$$f'(x) \leqq B_1 n\varepsilon_n, \qquad x \in J_\nu,$$

since otherwise this inequality could be extended by using Lemma 1 to give $f'(x) \leqq A_1 n\varepsilon_n$, $x \in J$, which means that J is already an interval of type 1, as is not the case. Thus, for each $\nu = 0, 1, \cdots, r-1$ there is a point $x_\nu \in J$, and $f'(x_\nu) = B_1 n\varepsilon_n$. This means that with $g = f'$, there is an $\eta_1 \in J$, with

$$g^{(k-1)}(\eta_1) = g[x_0, x_1, \cdots, x_{k-1}] = 0.$$

Assume that (4.16) does not hold. Since J does not intersect any type 1 interval, there must be a point $y \in J$ with $f'(y) = A_1 n\varepsilon_n$. This means we can find points $y_1 < y_2 < \cdots < y_k$ in J with $f'(y_i) = B_1 n\varepsilon_n$, $i \neq i_0$ and $f'(y_{i_0}) = A_1 n\varepsilon_n$. So, there is an $\eta_2 \in J$, with

$$g^{(k-1)}(\eta_2) = g[y_1, \cdots, y_k] = (k-1)!(A_1 - B_1)n\varepsilon_n \prod_{\substack{i=1\\ i\neq i_0}}^{k} (y_{i_0} - y_i)^{-1}.$$

Arguing as in the proof of Lemma 3, we get

$$\begin{aligned} |A_1 - B_1| &\leqq ((k-1)!n\varepsilon_n)^{-1} |g^{(k-1)}(\eta_1) - g^{(k-1)}(\eta_2)| \prod_{\substack{i=1\\ i\neq i_0}}^{k} (y_i - y_{i_0})^{-1} \\ &\leqq r^{2k} \leqq r^{2r}. \end{aligned}$$

While on the other hand from the value of the constants, we have

$$|A_1 - B_1| \geqq \tfrac{1}{2}A_1 \geqq \tfrac{1}{2}r^2 2^{r^4} \geqq r^{2r}.$$

This is the desired contradiction and the lemma is proved.

Now that we have given the important properties of intervals of types 1–4, we can give our decomposition for f. Let $I_1, \cdots, I_m$ be the intervals of type 4. For each $j = 1, \cdots, m$, let I_j^* denote the closure of the union of I_j with I_j^l and I_j^r where I_j^l is the interval of type 2 adjacent and to the left of I_j and I_j^r is the interval of type 2 adjacent and to the right of I_j. The intervals I_j^l and I_j^r may be empty. If we delete $I_1^*, \cdots, I_m^*$ from $[0, 1]$, then we are left with a finite number of open intervals which we denote by $J_0^*, \cdots, J_m^*$ where J_0^* or J_m^* may be empty.

If I is one of the intervals $I_1^*, \cdots, I_m^*$ or $J_0^*, \cdots, J_m^*$ then we define f_I by

$$f_I(x) = \int_0^x f'(t)\chi_I(t)\, dt,$$

where χ_I is the characteristic function of I. Since the intervals are disjoint,

$$f(x)=f(0)+\sum_1^m f_{I_j^*}(x)+\sum_0^m f_{J_j^*}(x)$$

which is our decomposition of f.

5. Proof of Theorem 1. We will prove Theorem 1 by showing the existence of splines $S_{J_j^*}, S_{I_j^*}\in\mathcal{S}^*(r,n)$ with the properties

(5.1) *if* $J_j^*=(a,b)$, *then* $S_{J_j^*}(x)=f_{J_j^*}(x)$, *for* $x\notin[a-rn^{-1}, b+rn^{-1}]$,

(5.2) $\|f_{J_j^*}-S_{J_j^*}\|\leqq C\varepsilon_n$ *with* C *depending only on* r,

(5.3) *if* $I_j^*=[a,b]$, *then* $S_{I_j^*}(x)=f_{I_j^*}(x)$, *for* $x\notin[a-2r^2n^{-1}, b+2r^2n^{-1}]$,

(5.4) $\|f_{I_j^*}-S_{I_j^*}\|\leqq C\varepsilon_n$ *with* C *depending only on* r.

Before proceeding to the proofs of (5.1)–(5.4), let's first see how they give the theorem. Recall first that each of the intervals I_j^* and J_j^* has length $\geqq r^2/n$. Therefore, for a given $x\in[0,1]$, it follows from (5.1) and (5.3) that there are at most five intervals I among $I_1^*,\cdots,I_m^*$ and $J_0^*,\cdots,J_m^*$ with $S_I(x)\neq f_I(x)$. Hence, if $S=f(0)+\sum S_{I_j^*}+\sum S_{J_j^*}$ then $S\uparrow$ and

$$\begin{aligned}\|f-S\|&\leqq 5\max\left(\sup_j\|f_{I_j^*}-S_{I_j^*}\|,\ \sup_j\|f_{J_j^*}-S_{J_j^*}\|\right)\\&\leqq 5C\varepsilon_n\end{aligned}$$

which gives the estimate (1.1).

Proof of (5.1)–(5.2). This is the easier of the two. Let $I=(i_1n^{-1}, i_2n^{-1})$ be one of the intervals J_j^* and write $i_2-i_1=\lambda(r-1)+\mu$, where λ and μ are integers with $\lambda\geqq1$ and $0\leqq\mu<r-1$. Define $x_\nu=(i_1+\nu(r-1))n^{-1}$, $\nu=0,1,\cdots,\lambda+1$, and

$$b_\nu=f_I(x_\nu)-f_I(x_{\nu-1}),\qquad \nu=1,2,\cdots,\lambda+1.$$

Let N_ν be the B-splines of *order* $r-1$ (*degree* $r-2$). Recall that N_ν is normalized to have integral one. Define

$$S_I(x)=\int_0^x\left[\sum_{\nu=0}^{\lambda}b_{\nu+1}N_{i_1+\nu(r-1)}(t)\right]dt.$$

Since each B-spline is nonnegative and the numbers $b_\nu\geqq0$, we have $S_I\uparrow$. Since $f_I'=0$ for x not in I, we have $S_I(x)=0=f_I(x)$, for $x\leqq x_0$ and $S_I(x)=S_I(x_{\lambda+1})=f_I(x_{\lambda+1})=f_I(x)$ for $x\geqq x_{\lambda+1}$. This shows that property (5.1) is satisfied for S_I.

For the estimate (5.2), we need only observe that $S_I(x_\nu)=f_I(x_\nu)$, $\nu=0,1,\cdots,\lambda+1$. Thus, if $x_\nu\leqq x\leqq x_{\nu+1}$, then

(5.5) $$|f_I(x)-S_I(x)|\leqq|f_I(x)-f_I(x_\nu)|+|S_I(x)-S_I(x_\nu)|\leqq 2b_{\nu+1}$$

because both f_I and S_I are nondecreasing. Now, the interval I is a union of intervals of type 1, type 2, and type 3. On any such interval, we have the estimate

$$f'(t)\leqq An\varepsilon_n,$$

where A is the maximum of the three constants A_1, A_2, and A_3 appearing in (4.2), (4.4), and (4.9), respectively. Integrating this last inequality gives

$$b_{\nu+1} \leqq \int_{x_\nu}^{x_{\nu+1}} f'(t)\,dt \leqq (r-1)n^{-1}An\varepsilon_n \leqq (r-1)A\varepsilon_n,$$

where we have used the fact that $x_{\nu+1}-x_\nu=(r-1)n^{-1}$. Using this inequality back in (5.5) gives (5.2) provided that $C \geqq 2(r-1)A$.

Proof of (5.3)–(5.4). Now let $I=[i_1n^{-1}, i_2n^{-1}]$ be one of the intervals I_j^*, $j=1, 2, \cdots, m$. The construction of the spline S_I is more complicated in this case and it may be beneficial to sketch the idea of the construction before actually embarking on details.

We start with a spline S_1 which approximates f according to (3.5)–(3.7). Since we want to approximate f_I and f_I' vanishes outside of I, we need to modify S_1. We do this by working with the B-spline representation of S_1' and deleting all terms that do not contribute to S_1' on I. In this way, we get a new spline S_2' which agrees with S_1' on I but vanishes once we get a little away from I. Integrating S_2' gives a new spline S_2. Fortunately, the intervals immediately to the right and left of I are of type 1 and so f' is small on these intervals according to (4.2). This means that S_2' will be small on these intervals and so S_2 is a good approximation to f_I.

Unfortunately, the spline S_2 is not necessarily monotone nondecreasing. However, we do know that S_2' is not too negative. On I, $S_2'=S_1'$ which is controlled by (3.6). Outside of I, S_2' is small as mentioned above. What we do then is add a spline to S_2' to pull it up so it can not be negative. We then integrate to get a new spline S_3 which is monotone nondecreasing, sure enough, but we may have introduced too much error to still have (5.4).

The final step is to make a correction on each interval of length r^2n^{-1} to prevent the error from building up. This is done by using the fact that I is of type 4 and hence f' and so S_3' will be big at suitable places because of Lemma 3. What we do is pull down S_3' however much is necessary but still keep a positive derivative. The resulting spline when integrated will satisfy (5.3) and (5.4).

Now, to the actual details. We consider the case when I is strictly interior to $[0, 1]$ and so $r^2 \leqq i_1$ and $i_2 \leqq n-r^2$. When I contains one of the end points, the proof is similar and in fact somewhat easier, so we do not repeat the details. Let $S_1 \in \mathcal{S}(r, n)$ be a spline which satisfies (3.5)–(3.7) and let $S_1'=\sum a_\nu N_\nu$ be the B-spline representation of S_1'. We define

$$S_2(x)=\int_0^x \sum_{i_1-r+2}^{i_2-1} a_\nu N_\nu(t)\,dt. \tag{5.6}$$

We want to see that S_2 approximates f_I well and so we must show that S_2' is small outside of I. The interval $E_1=[(i_1-2r)n^{-1}, i_1n^{-1}]$ is contained in an interval of type 1 because each of the intervals I_j^* is the union of I_j with the left and right adjacent intervals of type 2. Also recall that intervals of type 1 have length $\geqq r^2n^{-1}>2rn^{-1}$.

From (4.2) and (3.6), we find that

$$|S_1'(x)| \leqq |f'(x)|+|f'(x)-S_1'(x)| \leqq (A_1+C_2)n\varepsilon_n, \qquad x \in E_1. \tag{5.7}$$

Similarly, for $E_2=[i_2, (i_2+2r)n^{-1}]$,

$$|S_1'(x)| \leqq (A_1+C_2)n\varepsilon_n, \qquad x \in E_2. \tag{5.8}$$

These last two estimates can be used to estimate a_ν when $\nu \in F = \{\mu : i_1-2r+4 \leqq \mu \leqq i_1-r+1 \text{ or } i_2 \leqq \mu \leqq i_2+r-3\}$. For these values of ν, $(t_\nu, t_{\nu+r-1}) \subseteq E_1 \cup E_2$ and so from (5.7), (5.8), and (2.3), we find

$$|a_\nu| \leqq \alpha_3(A_1+C_2)\varepsilon_n, \qquad \nu \in F. \tag{5.9}$$

Now, we use (5.9) with (2.2) to get the following estimate on $E_3 = [(i_1-r+2)n^{-1}, i_1 n^{-1}] \cup [i_2 n^{-1}, (i_2+r-2)n^{-1}]$:

$$|S_2'(x)-S_1'(x)| = |\sum_{\nu \in F} a_\nu N_\nu(x)| \leqq 2r\alpha_2\alpha_3(A_1+C_2)n\varepsilon_n, \qquad x \in E_3,$$

where we have used the fact that there are less than $2r$ integers ν in F. This last inequality together with (5.7) and (5.8) gives

$$\begin{aligned} |S_2'(x)| &\leqq |S_1'(x)| + |S_2'(x)-S_1'(x)| \\ &\leqq 3r\alpha_2\alpha_3(A_1+C_2)n\varepsilon_n, \qquad x \in E_3 \end{aligned} \tag{5.10}$$

where we have also used the facts that $\alpha_2, \alpha_3 \geqq 1$.

The estimate (5.10) shows that S_2' is small when $x \in E_3$. When $x \in I$, then $S_2'(x) = S_1'(x)$ and for all other values of x, $S_2'(x)=0$. Therefore, since $f_I'=0$, $x \notin I$ and $f_I = f$, $x \in I$, we have

$$\begin{aligned} \|f_I - S_2\| &\leqq 2\|f-S_1\| + \int_{E_3} |S_2'(x)|\,dx \\ &\leqq [2C_2 + 6r^2\alpha_2\alpha_3(A_1+C_2)]\varepsilon_n, \end{aligned} \tag{5.11}$$

where we have used the fact that $|E_3| \leqq 2rn^{-1}$.

The estimate (5.11) shows that S_2 is a good approximation to f_I. Unfortunately, S_2 is not necessarily nondecreasing and so we must make a correction. We know that S_2' is not too negative because of (3.6) and (5.10). Namely,

$$\begin{aligned} S_2'(x) = S_1'(x) &\geqq f'(x) - |S_1'(x)-f'(x)| \geqq f'(x) - C_2 n\varepsilon_n \\ &\geqq -C_2 n\varepsilon_n, \qquad x \in I. \end{aligned} \tag{5.12}$$

$$S_2'(x) \geqq -3r\alpha_2\alpha_3(A_1+C_2)n\varepsilon_n, \qquad x \in E_3. \tag{5.13}$$

Of course, $S_2'(x)=0$, for all other values of x.

Let's define

$$S_3(x) = \int_0^x (S_2'(t) + T_2(t))\,dt$$

where

$$T_2 = \gamma_1(N_{i_1-r+1} + N_{i_1-r+2} + N_{i_2-1} + N_{i_2}) + \gamma_2 \sum_{i_1-1}^{i_2-r+2} N_\nu$$

with $\gamma_1 = 3r\alpha_1^{-1}\alpha_2\alpha_3(A_1+C_2)\varepsilon_n$ and $\gamma_2 = \alpha_1^{-1}C_2\varepsilon_n$. Because of (5.12), (5.13) and

(2.1), we see that $S_3'(x) \geqq 0$, $x \in [0, 1]$. Thus, S_3 is nondecreasing but we may have added too much error to still have a good approximation.

We want to modify S_3 to prevent the error from building up. First, let's see how much S_3 differs from S_2. Since $S_3' = S_2' + T_2$, what we have added in T_2 is the cause of the error in S_3 and we will have to subtract this at suitable places to prevent a buildup of error.

We can write $T_2 = T_{2,1} + T_{2,2}$, where $T_{2,1}$ is the sum of the B-splines in T_2 which have coefficients γ_1 (there are four of these) and $T_{2,2}$ is the remaining sum, which consists of all the B-splines with coefficients γ_2.

Let's first correct for $T_{2,1}$. By Lemma 3, (4.10), there is an interval $[l_0 n^{-1}, (l_0+r)n^{-1}]$, with l_0 an integer, on which $f'(x) \geqq B_2 n^{-k+1} \omega(f^{(k)}, n^{-1})$. This interval is contained in $[a, a+r^2 n^{-1}]$ where $[a, b]$ is the interval of type 4 that makes up part of I. Recall that intervals of type 4 necessarily have length $\geqq 2r^2 n^{-1}$. We will subtract $4\gamma_1 N_{l_0}$ as our correction for $T_{2,1}$.

For the spline $T_{2,2}$, we have no control over the number of terms that appear and so we do not have the luxury of making just one correction. Instead, we will have to make a correction on each interval of length $r^2 n^{-1}$. This is done as follows. Let $i_2 = i_1 + \lambda r^2 + \mu$, with $\lambda > 1$ and $0 \leqq \mu < r^2$ and define $x_\nu = (i_1 + \nu r^2)n^{-1}$, $\nu = -1, 0, \cdots, \lambda+2$. From Lemma 4, it follows that for each $\nu = 1, 2, \cdots, \lambda - 1$, there is an interval $[l_\nu n^{-1}, (l_\nu + r)n^{-1}] \subseteq [x_{\nu-1}, x_\nu]$, on which $f'(x) \geqq B_1 n \varepsilon_n$. Our correction on $[x_{\nu-1}, x_\nu]$ will be the spline $r^2 \gamma_2 N_{l_\nu}$, when $1 \leqq \nu \leqq \lambda - 2$.

Now, γ_2 appears a total of $\lambda r^2 + \mu + 4 - r$ times in the definition of $T_{2,2}$. We have already taken care of $(\lambda - 1)r^2$ of these terms. So, we have yet to take care of $s = r^2 + \mu - r + 4 \leqq 2r^2 - r + 4$ terms. We correct for these with the spline $s\gamma_2 N_{l_{\lambda-1}}$.

This corrects for all the error from T_2; however, we would also like our new spline to agree with f_I when $x \geqq x_{\lambda+2}$. To do this, let $[l_\lambda n^{-1}, (l_\lambda + r)n^{-1}]$ be the interval guaranteed by Lemma 3, (4.11). This interval is contained in $[(i_2 - r^2)n^{-1}, i_2 n^{-1}]$. The spline $\gamma_3 N_{l_\lambda}$, with $\gamma_3 = S_2(x_{\lambda+2}) - f_I(x_{\lambda+2})$ is our correction in this case

Thus, our total correction will be the spline

$$T_3 = 4\gamma_1 N_{l_0} + r^2 \gamma_2 \sum_1^{\lambda-2} N_{l_\nu} + s\gamma_2 N_{l_{\lambda-1}} + \gamma_3 N_{l_\lambda}.$$

So, we define

$$S_I(x) = S_3(x) - \int_0^x T_3(t)\, dt.$$

This is the spline that will satisfy (5.3) and (5.4).

The verification of (5.3) is quite easy. If $x \notin [x_{-1}, x_{\lambda+2}]$, then $f_I'(x) = 0 = S_I'(x)$. Also, in substracting T_3, we have accomplished two things, taking away exactly the error introduced by T_2 and then forcing an interpolation at $x_{\lambda+2}$. Thus, when $x \leqq x_{-1}$, $S_I(x) = 0 = f_I(x)$, and when $x \geqq x_{\lambda+2}$, $S_I(x) = S_I(x_{\lambda+2}) = f_I(x_{\lambda+2}) = f_I(x)$. Therefore, (5.3) is satisfied.

In order to check (5.4), we must check that indeed we have prevented the error from building up. Now, $S_I' = S_2' + T_2 - T_3$, and we already know by (5.11) that

$$\left\| f_I(x) - \int_0^x S_2'(t)\, dt \right\| = \|f_I - S_2\| \leqq C\varepsilon_n,$$

with C a constant depending only on r. Hence, we need only show that

$$\left|\int_0^x (T_2(t)-T_3(t))\,dt\right| \leqq C\varepsilon_n, \qquad x\in[x_{-1}, x_{\lambda+2}], \tag{5.14}$$

again with C a constant depending only on r. Now, for $x\in[x_{-1}, x_{\lambda+2}]$, the two integrals in (5.14) can differ by at most $4\gamma_1+2r^2\gamma_2+|\gamma_3|$, because any of the terms involving γ_2 in T_2 are taken care of by a corresponding term with coefficient γ_2 in T_3, within a length of $2r^2n^{-1}$. Therefore,

$$\left|\int_0^x (T_2(t)-T_3(t))\,dt\right| \leqq 4\gamma_1+2r^2\gamma_2+|\gamma_3| \leqq C\varepsilon_n, \qquad x\in[x_{-1}, x_{\lambda+2}],$$

where we have used the definitions of the constants γ_1 and γ_2, and the fact that $|\gamma_3|\leqq\|f_I-S_2\|$, which in turn is estimated by (5.11). Thus, we have shown (5.14) and so property (5.4) is verified.

We have one last task and that is to show that S_I is nondecreasing or, what amounts to the same thing, that $S_I'\geqq 0$. Now, $S_I'=S_3'-T_3$ and we already know that $S_3'\geqq 0$ and so we need only check where T_3 is not zero, namely, the intervals $[l_\nu n^{-1}, (l_\nu+r-1)n^{-1}]$, $\nu=0, 1, \cdots, \lambda$.

First, when $\nu=0$, we have

$$\begin{aligned} S_3'(x) &\geqq S_2'(x) \geqq f'(x)-C_2n\varepsilon_n \\ &\geqq (B_2-C_2)n\varepsilon_n, \qquad x\in[l_0n^{-1}, (l_0+r^{-1})n^{-1}], \end{aligned} \tag{5.15}$$

where in the second inequality we used (5.12) and in the last inequality, we used the fact that $[l_0n^{-1}, (l_o+r)n^{-1}]$ is the interval guaranteed by Lemma 3 to satisfy (4.10). We know that on $[l_0n^{-1}(l_0+r)n^{-1}]$, we have a contribution in T_3 from N_{l_0} but we may also have a contribution from some N_{l_ν}, $1\leqq\nu\leqq\lambda-1$. However, we have at most two such contributions, and so

$$\begin{aligned} |T_3(x)| &\leqq (4\gamma_1+r^2\gamma_2)\alpha_2 n \\ &\leqq (B_2-C_2)n\varepsilon_n \leqq S_3'(x), \qquad x\in[l_0n^{-1}, (l_0+r-1)n^{-1}], \end{aligned}$$

where the first inequality uses (2.2), the second inequality uses the values of the various constants, in particular that $C_2<A_1$, $r\geqq 3$, $\alpha_1<1$, and $\alpha_2, \alpha_3\geqq 1$, and the last inequality uses (5.15). This shows that $S_I'(x)=S_3'(x)-T_3(x)\geqq 0$ on $[l_0n^{-1}, (l_0+r-1)n^{-1}]$.

When $\nu=\lambda$, we argue as we did when $\nu=0$ to find that

$$\begin{aligned} |T_3(x)| &\leqq (|\gamma_3|+r^2\gamma_2)\alpha_2 n \leqq (B_2-C_2)n\varepsilon_n \\ &\leqq S_3'(x), \qquad x\in[l_\lambda n^{-1}, (l_\lambda+r-1)n^{-1}], \end{aligned}$$

because $|\gamma_3|\leqq\|f-S_2\|$, which in turn is estimated by (5.11).

When $\nu=1, \cdots, \lambda-2$, we argue in the same way. Now,

$$S_3'(x)\geqq(B_1-C_2)n\varepsilon_n, \qquad [x\in l_\nu n^{-1}, (l_\nu+r-1)n^{-1}],$$

because of (5.12) and Lemma 4. Also, we need only check T_3 on that part of $[l_\nu n^{-1}, (l_\nu+r-1)n^{-1}]$ which does not intersect either $[l_0n^{-1}, (l_0+r-1)n^{-1}]$ or

$[l_\lambda n^{-1}, (l_\lambda + r - 1)n^{-1}]$ and on this part we have

$$|T_3(x)| = r^2\gamma_2 N_{l_\nu}(x) \leqq r^2\gamma_2\alpha_2 n \leqq (B_1 - C_2)n\varepsilon_n$$
$$\leqq S_3'(x),$$

again because of the values of the various constants. This shows that $S_I'(x) \geqq 0$, for $x \in [l_\nu n^{-1}, (l_\nu + r - 1)n^{-1}]$, when $\nu = 1, \cdots, \lambda - 2$.

The proof that $S_I'(x) \geqq 0$, $x \in [l_{\lambda-1}n^{-1}, (l_{\lambda-1} + r - 1)n^{-1}]$ is exactly the same except that we use the additional fact that $s \leqq 3r^2$.

6. Remarks. We have mentioned in the Introduction that it is preferable to get the Jackson estimates of the form

$$E_n^*(f, r) \leqq C\omega_r(f, n^{-1}) \tag{6.1}$$

where ω_r is the rth order modulus of smoothness of f. The reason (6.1) is preferable is that then we would have the inverse theorem that if ω is an rth order modulus of smoothness and $E_n^*(f, r) = O(\omega(n^{-1}))$, $n \to \infty$, then $f\uparrow$ and $\omega_r(f, t) = O(\omega(t))$. This is because of inverse theorems for approximation by splines with equally spaced knots (see e.g. K. Scherer [9] for the case $\omega_r(t) = t^\theta$).

The reason that we do not get this result with our technique is that we approximate f' and then integrate to get our approximation to f. We could put our estimates in the form

$$E_n^*(f, r) \leqq Cn^{-1}\omega_{r-1}(f', n^{-1}) \tag{6.2}$$

when f' is continuous but even this does not reduce to (6.1) in this case.

In order to prove (6.1), it would be enough to show that whenever $f\uparrow$, and $\varepsilon > 0$, then there is a function $g_\varepsilon \uparrow$ with

$$\|f - g_\varepsilon\| \leqq C\omega_r(f, \varepsilon)$$

and

$$\|g_\varepsilon^{(r)}\| \leqq C\varepsilon^{-r}\omega_r(f, \varepsilon).$$

The key is that the functions g_ε be monotone nondecreasing, since otherwise the existence of the functions g_ε is already known as we have used in our proof. Once the existence of the functions g_ε are known then we would only need to use the fact that we can approximate g_ε with $\varepsilon = n^{-1}$, by a spline S in $\mathscr{S}^*(r, n)$ with error

$$\|g_\varepsilon - S\| \leqq Cn^{-r}\|g_\varepsilon^{(r)}\| \leqq C\omega_r(f, n^{-1})$$

because of our Theorem 1. This would give

$$\|f - S\| \leqq \|f - g_\varepsilon\| + \|g_\varepsilon - S\| \leqq C\omega_r(f, n^{-1}).$$

This in turn gives (6.1).

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