Signals and Systems

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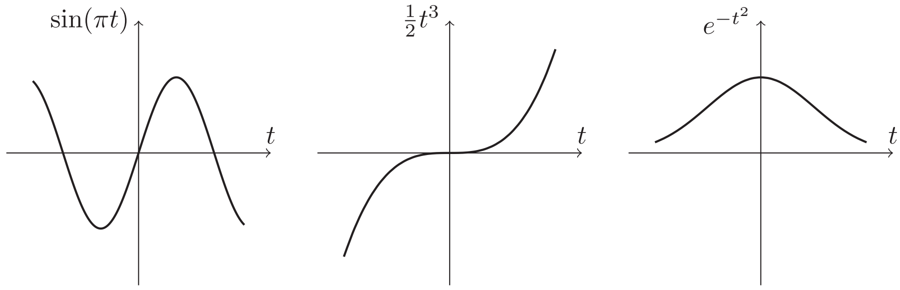

Figure 1: 1-dimensional continuous-time signals

1 Signals and systems

A **signal** is a function mapping an input variable to some output variable. For example

$$\sin(\pi t), \qquad \frac{1}{2}t^3, \qquad e^{-t^2}$$

all represent **signals** with input variable $t \in \mathbb{R}$, and they are plotted in Figure 1. If x is a signal and t an input variable we write $x(t)$ for the output variable. Signals can be multidimensional. This page is an example of a 2-dimensional signal, the independent variables are the horizontal and vertical position on the page, and the signal maps this position to a colour, in this case either black or white. A moving image such as seen on your television or computer monitor is an example of a 3-dimensional signal, the three independent variables being vertical and horizontal screen position and time. The signal maps each position and time to a colour on the screen. In this course we focus exclusively on 1-dimensional signals such as those in Figure 1 and we will only consider signals that are real or complex valued. Many of the results presented here can be extended to deal with multidimensional signals.

1.1 Properties of signals

A signal x is **bounded** if there exists a real number M such that

$$|x(t)| \leq M \qquad \text{for all } t \in \mathbb{R}$$

where $|\cdot|$ denotes the (complex) magnitude. Both $\sin(\pi t)$ and e^{-t^2} are examples of bounded signals because $|\sin(\pi t)| \leq 1$ and $|e^{-t^2}| \leq 1$ for all $t \in \mathbb{R}$. However, $\frac{1}{2}t^3$ is not bounded because its magnitude grows indefinitely as t moves away from the origin.

A signal x is **periodic** if there exists a real number T such that

$$x(t) = x(t + kT) \qquad \text{for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

For example, the signal $\sin(\pi t)$ is periodic with period $T = 2$. Neither $\frac{1}{2}t^3$ or e^{-t^2} are periodic.

A signal x is called **locally integrable** if for all constants a and b,

$$\int_a^b |x(t)|\, dt$$

exists (evaluates to a finite number). Every bounded signal is locally integrable (Exercise 1.3). An example of a signal that is not locally integrable is $x(t) = \frac{1}{t}$ (Exercise 1.2). Two signals x and y are equal, i.e. $x = y$ if $x(t) = y(t)$ for all $t \in \mathbb{R}$.

A signal x is called **absolutely integrable** if

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt \tag{1.1}$$

exists. Here we introduce the notation $\|x\|_1$ called the ℓ_1**-norm** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not absolutely integrable, but e^{-t^2} is because [nic]

$$\int_{-\infty}^{\infty} |e^{-t^2}| dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}. \tag{1.2}$$

The signal x is called is **square integrable** if

$$\|x\|_2 = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

exists. Square integrable signals are also called **energy signals**, and the value of $\|x\|_2$ is called the **energy** of x (it is also called the ℓ_2**-norm** of x). For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not energy signals, but e^{-t^2} is. A signal that is absolutely integrable is always also square integrable, but the reverse statement is not true (Exercise 1.4).

A signal x is **right sided** if there exists a $T \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T$. Correspondingly x is **left sided** if $x(t) = 0$ for all $T > t$. For example, the **step function**

$$u(t) = \begin{cases} 1 & t > 0, \\ 0 & t \leq 0 \end{cases} \tag{1.3}$$

is right-sided. Its reflection in time $u(-t)$ is left sided (Figure 2). A signal x is called **finite in time** if it is both left and right sided, that is, if there exits a $T \in \mathbb{R}$ such that $x(t) = x(-t) = 0$ for all $t > T$. A signal is called **unbounded in time** if it is neither left nor right sided. For example, the continuous time signals $\sin(\pi t)$ and e^{-t^2} are unbounded in time, but the **rectangular pulse**

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise}, \end{cases} \tag{1.4}$$

is finite in time.

A signal x is said to converge to the limit ℓ at $t = p$ from below if for any $\epsilon > 0$ there exists a $t_0 < p$ such that $|x(t) - \ell| < \epsilon$ for all t in the open interval

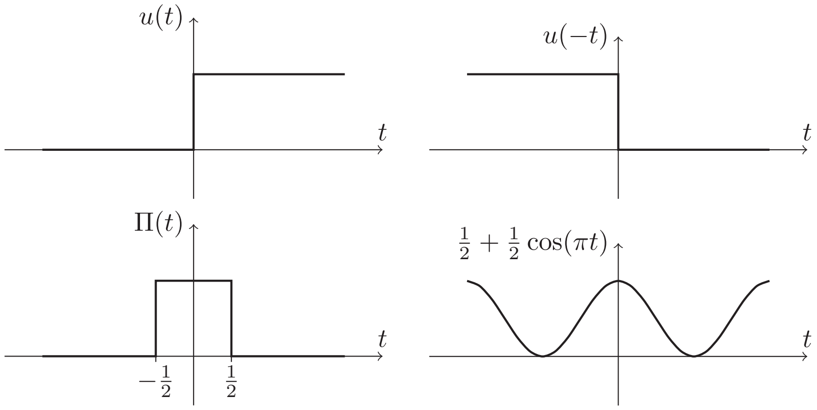

Figure 2: The right sided step function $u(t)$, its left sided reflection $u(-t)$, the finite in time rectangular pulse $\Pi(t)$ and the unbounded in time signal $\frac{1}{2} + \frac{1}{2}\cos(x)$.

(t_0, p). Similarly x is said to converge to the limit ℓ from above if for any $\epsilon > 0$ there exists a $t_0 > p$ such that $|x(t) - \ell|$ for all t in the open interval (p, t_0). As is customary, we write

$$\lim_{t \to p^-} x(t) = \ell, \qquad \lim_{t \to p^+} x(t) = \ell$$

to indicate limits from below and above respectively. In the case where $p = \pm\infty$ the definition of a limit is the same, for example, $\lim_{t\to\infty} x(t) = \ell$ means that for any $\epsilon > 0$ there is a t_0 such that $|x(t) - \ell| < \epsilon$ for all $t > t_0$.

A signal x is **continuous** at p if the limits of x as $t \to p$ from above and below are equal, i.e.,

$$\lim_{t \to p^-} x(t) = \lim_{t \to p^+} x(t) = x(p).$$

In other words, x is continuous at p if for any $\epsilon > 0$ there exists a δ such that $|x(t) - \ell| < \epsilon$ whenever $|t - p| < \delta$. Observe that δ in the previous definition may depend on p. A signal is said to be uniformly continuous if δ can be chosen indepdendently of p. That is, a signal x is **uniformly continuous** if for any $\epsilon > 0$ there exists a δ such that $|t - p| < \delta$ implies that $|x(t) - x(p)| < \epsilon$ for all $p \in \mathbb{R}$.

1.2 Systems (functions of signals)

A **system** (also known as an **operator** or **functional**) maps a signal to another signal. For example

$$x(t) + 3x(t-1), \qquad \int_0^1 x(t-\tau)d\tau, \qquad \frac{1}{x(t)}, \qquad \frac{d}{dt}x(t)$$

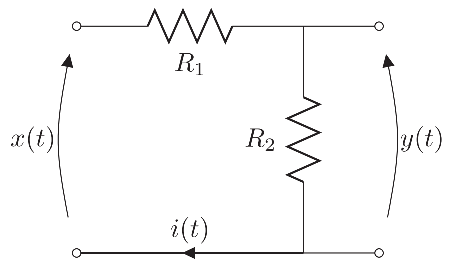

Figure 3: A **voltage divider** circuit.

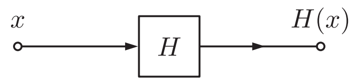

Figure 4: System block diagram with input signal x and output signal $H(x)$.

represent systems, each mapping the signal x to another signal. Consider the electric circuit in Figure 3 called a **voltage divider**. If the voltage at time t is $x(t)$ then, by Ohm's law, the current at time t satisfies

$$i(t) = \frac{1}{R_1 + R_2} x(t),$$

and the voltage over the resistor R_2 is

$$y(t) = R_2 i(t) = \frac{R_2}{R_1 + R_2} x(t) \tag{1.5}$$

The circuit can be considered as a system mapping the signal x representing the voltage to the signal $i = \frac{1}{R_1+R_2} x$ representing the current, or a system mapping x to the signal $y = \frac{R_2}{R_1+R_2} x$ representing the voltage over resistor R_2.

We denote systems with capital letters such as H and G. A system H is a function that maps a signal x to another signal denoted $H(x)$. We call x the **input signal** and $H(x)$ the **output signal** or the **response** of system H to signal x. If we want to include the independent variable t we will write $H(x)(t)$ or $H(x, t)$ and do not distinguish between these [Curry and Feys, 1968]. It is sometimes useful to depict systems with a block diagram. Figure 4 is a simple block diagram showing the input and output signals of a system H.

Using this notation the electric circuit in Figure 3 corresponds with the system

$$H(x) = \frac{R_2}{R_1 + R_2} x = y.$$

This system multiplies the input signal x by $\frac{R_2}{R_1+R_2}$. This brings us to our first practical test.

Test 1 (**Voltage divider**) In this test we construct the voltage divider from Figure 3 on a breadboard with resistors $R_1 \approx 820\Omega$ and $R_2 \approx 6800\Omega$ with values accurate to within 5%. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \sin(2\pi f_1 t) \qquad \text{with} \qquad f_1 = 100$$

is passed through the circuit. The approximation is generated by sampling $x(t)$ at rate $F_s = \frac{1}{T_s} = 44100\text{Hz}$ to generate samples

$$x_n = x(nT_s) \qquad n = 0, \dots, 2F_s$$

corresponding to approximately 2 seconds of signal. These samples are passed to the soundcard which starts playback. The voltage over the resistor R_2 is recorded (also using the soundcard) that returns a lists of samples $y_1, \dots, y_L$ taken at rate F_s. The continuous-time voltage over R_2 can be (approximately) reconstructed from these samples as

$$\tilde{y}(t) = \sum_{\ell=1}^{L} y_\ell \operatorname{sinc}(t - F_s \ell) \tag{1.6}$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{1.7}$$

is the called the **sinc function** and is plotted in Figure 5. We will justify this reconstruction in Section 6. Simultaneously the (stereo) soundcard is used to record the input voltage $x(t)$ producing samples $x_1, \dots, x_L$ taken at rate F_s. An approximation of the continuous-time input signal is

$$\tilde{x}(t) = \sum_{\ell=1}^{L} x_\ell \operatorname{sinc}(t - F_s \ell). \tag{1.8}$$

In view of (1.5) we would expect the approximate relationship

$$\tilde{y} \approx \frac{R_2}{R_1 + R_2} \tilde{x} = \frac{340}{381} \tilde{x}$$

A plot of $\tilde{y}$, $\tilde{x}$ and $\frac{340}{381}\tilde{x}$ over a 20ms period from 1s to 1.02s is given in Figure 6. The hypothesised output signal $0.89\tilde{x}$ does not match the observed output signal $\tilde{y}$. A primary reason is that the circuitry inside the soundcard itself cannot be ignored. When deriving the equation for the voltage divider we implicitly assumed that current flows through the output of the soundcard without resistance (a short circuit), and that no current flows through the input device of the soundcard (an open circuit). These assumptions are not realistic. Modelling the circuitry in the sound card wont be attempted here. In the next section we will construct circuits that contain external sources of power (active circuits). These are less sensitive to the circuitry inside the soundcard.

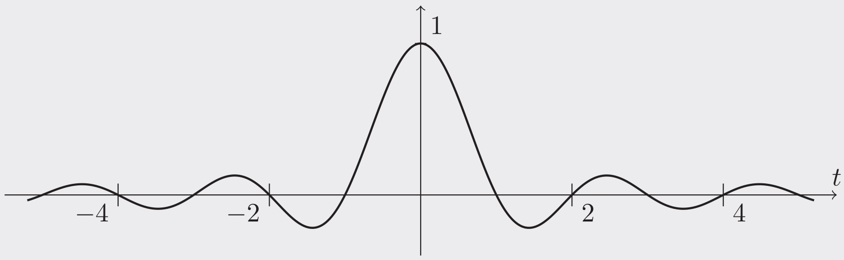

Figure 5: The **sinc function** $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

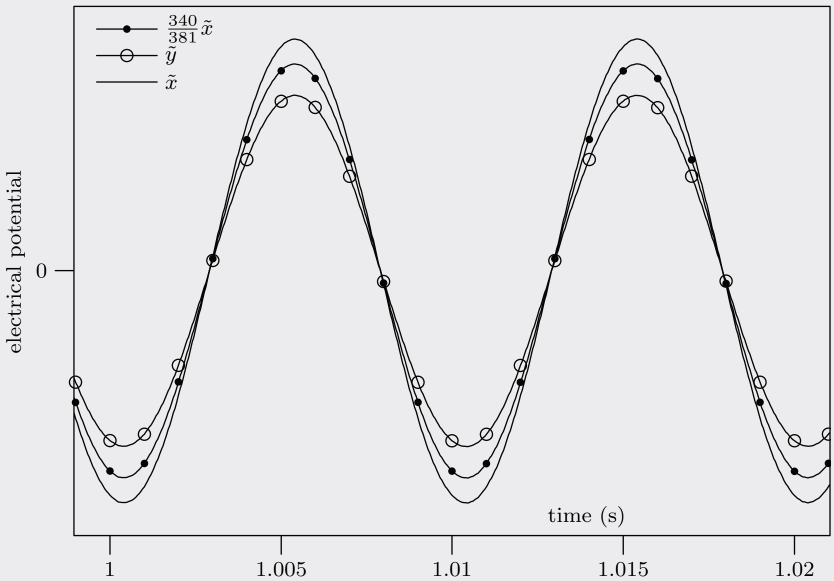

Figure 6: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $0.89\tilde{x}$ (solid line with dot) for the voltage divider circuit in Figure 3. The hypothesised signal does not match $\tilde{y}$ because the model does not take account of the circuitry inside the soundcard.

Not all signals can be input to all systems. For example, the system

$$H(x,t) = \frac{1}{x(t)}$$

is not defined at those t where $x(t) = 0$ because we cannot divide by zero. Another example is the system

$$I_\infty(x,t) = \int_{-\infty}^{t} x(\tau) d\tau, \tag{1.9}$$

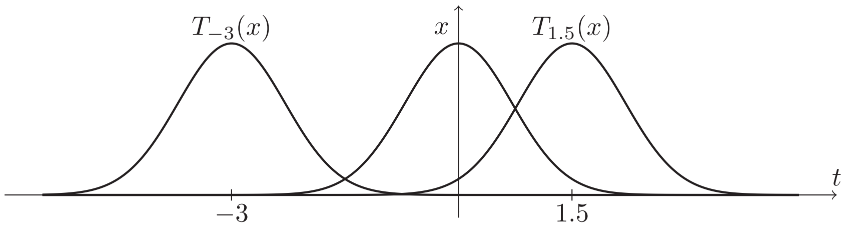

Figure 7: Time-shifter system $T_{1.5}(x,t) = x(t-1.5)$ and $T_{-3}(x,t) = x(t+3)$ acting on the signal $x(t) = e^{-t^2}$.

called an **integrator**, that is not defined for those signals where the integral above does not exist (is not finite). For example, the signal $x(t) = 1$ cannot be input to the integrator since the integral $\int_{-\infty}^{t} dt$ does not exist.

Thus, when specifying a system it is necessary to also specify a set of signals that can be input, called the **domain** of the system. For example, the domain of the system $H(x,t) = \frac{1}{x(t)}$ is the set of signals $x(t)$ which are not zero for any t. The domain of the integrator $I_\infty(x,t)$ is the set of signals for which the integral $\int_{-\infty}^{t} x(\tau)d\tau$ exists for all $t \in \mathbb{R}$. The domain of a system is usually obvious from the specification of the system itself. For this reason we will not usually state the domain explicitly. We will only do so if there is chance for confusion.

1.3 Some important systems

The system

$$T_\tau(x,t) = x(t-\tau)$$

is called the **time-shifter**. This system shifts the input signal along the t axis ('time' axis) by τ. When τ is positive T_τ delays the input signal by τ. The time-shifter will appear so regularly in this course that we use the special notation T_τ to represent it. Figure 7 depicts the action of time-shifters $T_{1.5}$ and T_{-3} on the signal $x(t) = e^{-t^2}$. When $\tau = 0$ the time-shifter is the **identity system**

$$T_0(x) = x$$

that maps the signal x to itself.

Another important system is the **time-scaler** that has the form

$$H(x,t) = x(\alpha t)$$

for $\alpha \in \mathbb{R}$. Figure 8 depicts the action of a time-scaler with a number of values for α. When $\alpha = -1$ the time-scaler reflects the input signal in the time axis.

Another system we regularly encounter is the **differentiator**

$$D(x,t) = \frac{d}{dt}x(t),$$

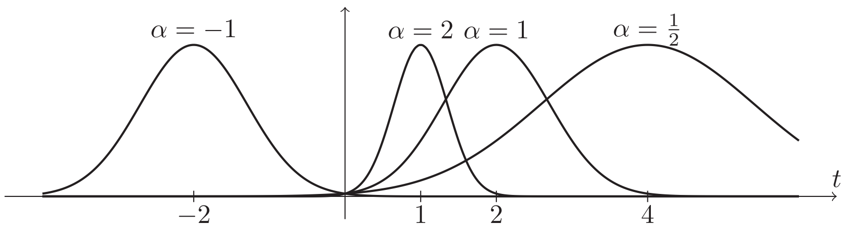

Figure 8: Time-scaler system $H(x,t) = x(\alpha t)$ for $\alpha = -1, \frac{1}{2}, 1$ and 2 acting on the signal $x(t) = e^{-(t-2)^2}$.

that returns the derivative of the input signal. We also define a kth differentiator

$$D^k(x,t) = \frac{d^k}{dt^k} x(t)$$

that returns the kth derivative of the input signal.

Another important system is the **integrator**

$$I_a(x,t) = \int_{-a}^{t} x(\tau) d\tau.$$

The parameter a describes the lower bound of the integral. In this course it will often be that $a = \infty$ or $a = 0$. The integrator can only be applied to those signals for which the integral above exists. For example, the integrator I_∞ can be applied to the signal $tu(t)$ where $u(t)$ is the step function (1.3). The output signal is

$$\int_{-\infty}^{t} \tau u(\tau) d\tau = \int_{0}^{t} \tau d\tau = \frac{t^2}{2}.$$

However, the integrator cannot be applied to the signal $x(t) = t$ because $\int_{-\infty}^{t} \tau d\tau$ does not exist.

1.4 Properties of systems

A system H is called **memoryless** if the output signal $H(x)$ at time t depends only on the input signal x at time t. For example $\frac{1}{x(t)}$ and the identity system T_0 are memoryless, but

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau) d\tau$$

are not. A time-shifter system T_τ with $\tau \neq 0$ is not memoryless.

A system H is **causal** if the output signal $H(x)$ at time t depends on the input signal only at times less than or equal to t. Memoryless systems such as $\frac{1}{x(t)}$ and T_0 are also causal. Time-shifters $T_\tau(x,t) = x(t-\tau)$ are causal when

$\tau \geq 0$, but are not causal when $\tau < 0$. The systems

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau)d\tau$$

are causal, but the systems

$$x(t) + 3x(t+1) \qquad \text{and} \qquad \int_0^1 x(t+\tau)d\tau$$

are not causal.

A system H is called **bounded-input-bounded-output (BIBO) stable** or just **stable** if the output signal $H(x)$ is bounded whenever the input signal x is bounded. That is, H is stable if for every real number M there exists a real number K such that for all signals x satisfying

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R},$$

it also holds that

$$|H(x,t)| < K \qquad \text{for all } t \in \mathbb{R}.$$

For example, the system $x(t)+3x(t-1)$ is stable with $K = 4M$ since if $|x(t)| < M$ then

$$|x(t) + 3x(t-1)| \leq |x(t)| + 3\,|x(t-1)| < 4M = K.$$

The integrator I_a for any $a \in \mathbb{R}$ and differentiator D are not stable (Exercises 1.7 and 1.8).

A system H is **linear** if

$$H(ax + by) = aH(x) + bH(y)$$

for all signals x and y, and for all complex numbers a and b. That is, a linear system has the property: If the input consists of a weighted sum of signals, then the output consists of the same weighted sum of the responses of the system to those signals. Figure 9 indicates the linearity property using a block diagram. For example, the differentiator is linear because

$$\begin{aligned} D(ax+by,t) &= \frac{d}{dt}\big(ax(t) + by(t)\big) \\ &= a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) \\ &= aD(x,t) + bD(y,t), \end{aligned}$$

but the system $H(x,t) = \frac{1}{x(t)}$ is not linear because

$$H(ax+by,t) = \frac{1}{ax(t) + by(t)} \neq \frac{a}{x(t)} + \frac{b}{y(t)} = aH(x,t) + bH(y,t)$$

in general.

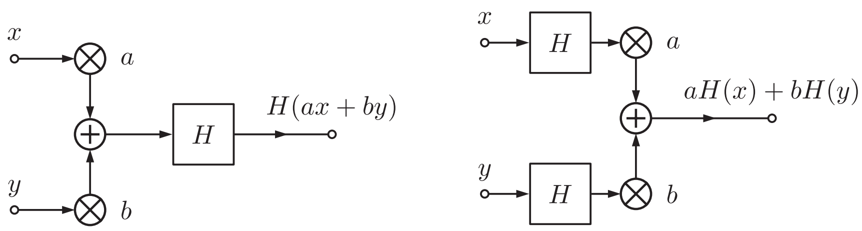

Figure 9: If H is a linear system the outputs of these two diagrams are the same signal, i.e. $H(ax + by) = aH(x) + bH(y)$.

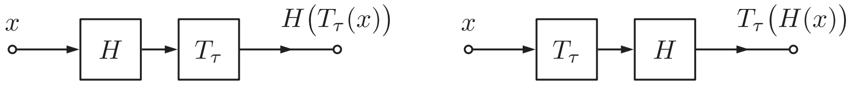

Figure 10: If H is a time-invariant system the outputs of these two diagrams are the same signal, i.e. $H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$.

The property of linearity trivially generalises to more than two signals. For example if $x_1, \dots, x_k$ are signals and $a_1, \dots, a_k$ are complex numbers for some finite k, then

$$H(a_1 x_1 + \cdots + a_k x_k) = a_1 H(x_1) + \cdots + a_k H(x_k).$$

A system H is **time-invariant** if

$$H\big(T_\tau(x), t\big) = H(x, t - \tau)$$

for all signals x and all time-shifts $\tau \in \mathbb{R}$. That is, a system is time-invariant if time-shifting the input signal results in the same time-shift of the output signal. Equivalently, H is time-invariant if H commutes the time-shifter T_τ, that is, if

$$H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$$

for all $\tau \in \mathbb{R}$ and all signals x. Figure 10 represents the property of time-invariance with a block diagram.

Let S be a set of signals. A system H is said to be **invertible** on S if each signal $x \in S$ is mapped to a unique signal $H(x)$. That is, for all signals $x, y \in S$ then $H(x) = H(y)$ if and only if $x = y$. If a system H is invertible on S then there exists an inverse system H^{-1} such that

$$x = H^{-1}\big(H(x)\big) \qquad \text{for all } x \in S.$$

For example, let S be a any set of signals. The time-shifter T_τ is invertible on S. The inverse system is $T_{-\tau}$ since

$$T_{-\tau}\big(T_\tau(x), t\big) = x(t - \tau + \tau) = x(t).$$

As another example, let S be the set of differentiable signals. The differentiator system D is **not** invertible on S because if $x \in S$ and if $y(t) = x(t) + c$ for any constant c then $D(y) = D(x)$. However, if we restrict S to those differentiable signals for which $x(0) = c$ is fixed, then D *is* invertible on S. The inverse system in this case is

$$D^{-1}(x,t) = I_0(x,t) + c = \int_0^t x(t)dt + c$$

because

$$D^{-1}(D(x),t) = \int_0^t D(x,t)dt + c = \int_0^t \frac{d}{dt}x(t)dt + x(0) = x(t)$$

by the fundamental theorem of calculus.

1.5 Exercises

1.1. State whether the step function $u(t)$ is bounded, periodic, continuous, differentiable, absolutely summable, an energy signal.

1.2. Show that the function t^2 is locally integrable, but that the function $\frac{1}{t^2}$ is not.

1.3. Show that every bounded function is locally integrable.

1.4. Show that an absolutely integrable function is square integrable, but a square integrable function is not always absolutely integrable.

1.5. Plot the signal

$$x(t) = \begin{cases} \frac{1}{x+1} & x > 0 \\ \frac{1}{x-1} & x \leq 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

1.6. Compute the energy of the signals e^{-t^2} and $e^{-t^2/4}$ (Hint: use (1.2) and a change of variables).

1.7. Show that the integrator I_a for any $a \in \mathbb{R}$ is not stable.

1.8. Show that the differentiator system D is not stable.

1.9. Show that the time-shifter is linear and time-invariant system, but that the time-scaler is neither linear or time invariant

1.10. Show that the kth differentiator $D^k(x,t) = \frac{d^k}{dt^k}x(t)$ is linear and time-invariant

1.11. Show that the integrator I_c with c finite is linear, but not time-invariant.

1.12. Show that the integrator I_∞ is linear and time invariant.

1.13. State whether the system $H(x,t) = x(t) + 1$ is linear, time-invariant, stable.

1.14. State whether the system $H(x, t) = 0$ is linear, time-invariant, stable.

1.15. State whether the system $H(x, t) = 1$ is linear, time-invariant, stable.