3.5.6 *Projected Normal Distributions*

Distributions on the circle can be obtained by radial projection of distributions on the plane. Let $\mathbf{x}$ be a random two-dimensional vector such that $\Pr(\mathbf{x} = \mathbf{0}) = 0$. Then $||\mathbf{x}||^{-1}\mathbf{x}$ is a random point on the unit circle. (For more on this construction, see the discussion of projected distributions in Section 9.3.3.) An important instance is that in which $\mathbf{x}$ has a bivariate normal distribution $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, in which case $||\mathbf{x}||^{-1}\mathbf{x}$ is said to have a *projected normal* (or *angular Gaussian* or *offset normal*) distribution $PN_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. A typical application occurs in meteorology: when wind velocity is modelled by a bivariate normal distribution, the resulting marginal distribution for wind direction is a projected normal distribution.

A tedious calculation shows that the probability density function of the projected normal distribution $PN_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is (Mardia, 1972a, p. 52)

$$p(\theta; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{\phi(\theta; \mathbf{0}, \boldsymbol{\Sigma}) + |\boldsymbol{\Sigma}|^{-1/2} D(\theta)\Phi(D(\theta))\phi(|\boldsymbol{\Sigma}|^{-1/2}(\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x})^{-1/2}\boldsymbol{\mu} \wedge \mathbf{x})}{\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x}}, \tag{3.5.48}$$

where $\phi(\cdot; \mathbf{0}, \boldsymbol{\Sigma})$ denotes the probability density function of $N_2(\mathbf{0}, \boldsymbol{\Sigma})$, ϕ and Φ denote the probability density function and cumulative density function of $N(0, 1)$, $\mathbf{x} = (\cos\theta, \sin\theta)^T$,

$$D(\theta) = \frac{\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x}}{(\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x})^{1/2}}$$

and $\boldsymbol{\mu} \wedge \mathbf{x} = \mu_1 \sin\theta - \mu_2 \cos\theta$ with $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$. In particular,

$$p(\theta; (\mu, 0), \mathbf{I}_2) = \frac{1}{\sqrt{2\pi}}\phi(\mu) + \mu\cos\theta\,\phi(\mu\sin\theta)\Phi(\mu\cos\theta), \tag{3.5.49}$$

where $\mathbf{I}_2$ denotes the 2×2 identity matrix. The distribution $PN_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ reduces to the uniform distribution if and only if $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}_2$. Projected normal distributions can be bimodal and/or asymmetrical.

Projected normal distributions with $\boldsymbol{\mu} = \mathbf{0}$ are called *angular central Gaussian distributions*. The probability density function of the angular central Gaussian distribution $PN_2(\mathbf{0}, \boldsymbol{\Sigma})$ is

$$p(\theta; \boldsymbol{\Sigma}) = \frac{(1 - b^2)^{1/2}}{2\pi(1 - b\cos 2(\theta - \mu))}, \tag{3.5.50}$$

where

$$b = \frac{2(\mathrm{tr}(\boldsymbol{\Sigma}) - 2|\boldsymbol{\Sigma}|^{1/2})^{1/2}(\mathrm{tr}(\boldsymbol{\Sigma}) + 2|\boldsymbol{\Sigma}|^{1/2})^{3/2}}{(2\mathrm{tr}\boldsymbol{\Sigma})^2 + 4|\boldsymbol{\Sigma}|}, \qquad \tan\mu = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}$$

with

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

Note that $p(\theta; c\boldsymbol{\Sigma}) = p(\theta; \boldsymbol{\Sigma})$ for any non-zero c, so we may assume without loss of generality that $|\boldsymbol{\Sigma}| = 1$. A connection between angular central Gaussian distributions and wrapped Cauchy distributions is given in (3.5.72). Because $p(\theta; \boldsymbol{\Sigma}) = p(\theta + \pi; \boldsymbol{\Sigma})$, the angular central Gaussian distributions provide useful models for axial data.

Each invertible linear transformation $\mathbf{A}$ of the plane gives rise to an invertible transformation $\varphi_{\mathbf{A}}$ of the unit circle by

$$\varphi_{\mathbf{A}}(\mathbf{x}) = \frac{1}{||\mathbf{A}\mathbf{x}||}\mathbf{A}\mathbf{x}. \tag{3.5.51}$$

Then

$$\theta \sim PN_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad \Rightarrow \quad \mathbf{A}\mathbf{x} \sim PN_2(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T). \tag{3.5.52}$$

Thus the set of projected normal distributions is closed under these transformations. Since $\varphi_{\mathbf{A}} = \varphi_{|\mathbf{A}|^{-1}\mathbf{A}}$, we may assume without loss of generality that $|\mathbf{A}| = 1$, i.e. that $\mathbf{A}$ is a unimodular matrix. Thus the projected normal distributions form a composite transformation model with group $SL_2(\mathbb{R})$, the set of 2×2 unimodular matrices. This was exploited by Cairns (1975) and Fraser (1979, pp. 219–231). By (3.5.72), maximum likelihood estimation of the parameters in angular central Gaussian distributions (with $|\boldsymbol{\Sigma}| = 1$) is equivalent to maximum likelihood estimation in wrapped Cauchy distributions. This is considered in Section 5.4.

It follows from (3.5.52) that

$$\theta \sim PN_2(\mathbf{0}, \boldsymbol{\Sigma}) \quad \Rightarrow \quad \mathbf{A}\mathbf{x} \sim PN_2(\mathbf{0}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T), \tag{3.5.53}$$

so that the angular central Gaussian distributions form a transformation model with group $SL_2(\mathbb{R})$. In particular, each angular central Gaussian distribution can be obtained from the uniform distribution by a suitable transformation $\varphi_{\mathbf{A}}$. This is why the angular central Gaussian distributions occur in image analysis, when 'textures' (fields of axes) are projected from one plane to another (see Blake & Marinos, 1990).

3.5.7 *Wrapped Distributions*

Definition

Given a distribution on the line, we can wrap it around the circumference of the circle of unit radius. That is, if x is a random variable on the line, the corresponding random variable x_w of the wrapped distribution is given by

$$x_w = x \,(\mathrm{mod}\, 2\pi). \tag{3.5.54}$$

If the circle is identified with the set of complex numbers with unit modulus then the wrapping map $x \mapsto x_w$ can be written as

$$x \mapsto e^{2\pi i x}. \tag{3.5.55}$$