

# Signals and Systems

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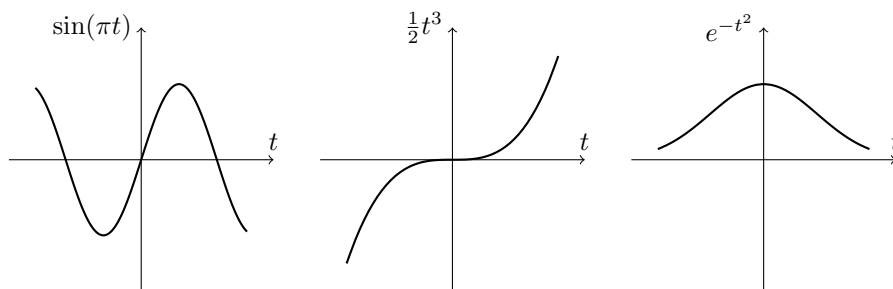


Figure 1: 1-dimensional continuous-time signals

## 1 Signals and systems

A **signal** is a function mapping an input variable to some output variable. For example

$$\sin(\pi t), \quad \frac{1}{2}t^3, \quad e^{-t^2}$$

all represent **signals** with input variable  $t \in \mathbb{R}$ , and they are plotted in Figure 1. If  $x$  is a signal and  $t$  an input variable we write  $x(t)$  for the output variable. Signals can be multidimensional. This page is an example of a 2-dimensional signal, the independent variables are the horizontal and vertical position on the page, and the signal maps this position to a colour, in this case either black or white. A moving image such as seen on your television or computer monitor is an example of a 3-dimensional signal, the three independent variables being vertical and horizontal screen position and time. The signal maps each position and time to a colour on the screen. In this course we focus exclusively on 1-dimensional signals such as those in Figure 1 and we will only consider signals that are real or complex valued. Many of the results presented here can be extended to deal with multidimensional signals.

### 1.1 Properties of signals

A signal  $x$  is **bounded** if there exists a real number  $M$  such that

$$|x(t)| \leq M \quad \text{for all } t \in \mathbb{R}$$

where  $|\cdot|$  denotes the (complex) magnitude. Both  $\sin(\pi t)$  and  $e^{-t^2}$  are examples of bounded signals because  $|\sin(\pi t)| \leq 1$  and  $|e^{-t^2}| \leq 1$  for all  $t \in \mathbb{R}$ . However,  $\frac{1}{2}t^3$  is not bounded because its magnitude grows indefinitely as  $t$  moves away from the origin.

A signal  $x$  is **periodic** if there exists a real number  $T$  such that

$$x(t) = x(t + kT) \quad \text{for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

For example, the signal  $\sin(\pi t)$  is periodic with period  $T = 2$ . Neither  $\frac{1}{2}t^3$  or  $e^{-t^2}$  are periodic.

A signal  $x$  is called **locally integrable** if for all constants  $a$  and  $b$ ,

$$\int_a^b |x(t)| dt$$

exists (evaluates to a finite number). An example of a signal that is not locally integrable is  $x(t) = \frac{1}{t}$  (Exercise 1.2). Two signals  $x$  and  $y$  are equal, i.e.  $x = y$  if  $x(t) = y(t)$  for all  $t \in \mathbb{R}$ .

A signal  $x$  is called **absolutely integrable** if

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)| dt \quad (1.1)$$

exists. Here we introduce the notation  $\|x\|_1$  called the  $\ell_1$ -**norm** of  $x$ . For example  $\sin(\pi t)$  and  $\frac{1}{2}t^3$  are not absolutely integrable, but  $e^{-t^2}$  is because [Nicholas and Yates, 1950]

$$\int_{-\infty}^{\infty} |e^{-t^2}| dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}. \quad (1.2)$$

The signal  $x$  is called is **square integrable** if

$$\|x\|_2 = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

exists. Square integrable signals are also called **energy signals**, and the value of  $\|x\|_2$  is called the **energy** of  $x$  (it is also called the  $\ell_2$ -**norm** of  $x$ ). For example  $\sin(\pi t)$  and  $\frac{1}{2}t^3$  are not energy signals, but  $e^{-t^2}$  is.

A signal  $x$  is **right sided** if there exists a  $T \in \mathbb{R}$  such that  $x(t) = 0$  for all  $t < T$ . Correspondingly  $x$  is **left sided** if  $x(t) = 0$  for all  $T > t$ . For example, the **step function**

$$u(t) = \begin{cases} 1 & t > 0, \\ 0 & t \leq 0 \end{cases} \quad (1.3)$$

is right-sided. Its reflection in time  $u(-t)$  is left sided (Figure 2). A signal  $x$  is called **finite in time** if it is both left and right sided, that is, if there exists a  $T \in \mathbb{R}$  such that  $x(t) = x(-t) = 0$  for all  $t > T$ . A signal is called **unbounded in time** if it is neither left nor right sided. For example, the continuous time signals  $\sin(\pi t)$  and  $e^{-t^2}$  are unbounded in time, but the **rectangular pulse**

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases} \quad (1.4)$$

is finite in time.

## 1.2 Systems (functions of signals)

A **system** (also known as an **operator** or **functional**) maps a signal to another signal. For example

$$x(t) + 3x(t-1), \quad \int_0^1 x(t-\tau) d\tau, \quad \frac{1}{x(t)}, \quad \frac{d}{dt}x(t)$$

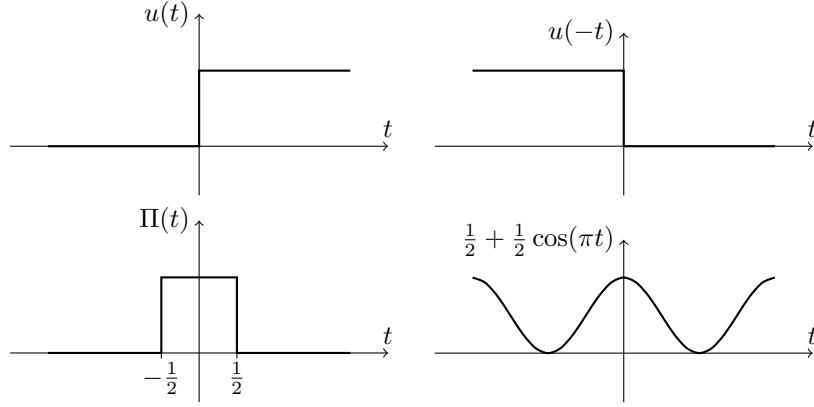


Figure 2: The right sided step function  $u(t)$ , its left sided reflection  $u(-t)$ , the finite in time rectangular pulse  $\Pi(t)$  and the unbounded in time signal  $\frac{1}{2} + \frac{1}{2} \cos(x)$ .

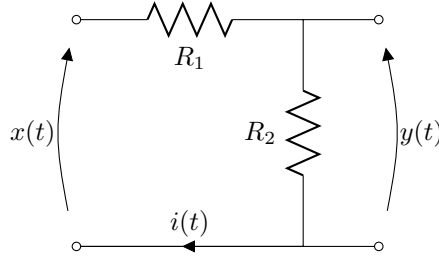


Figure 3: A **voltage divider** circuit.

represent systems, each mapping the signal  $x$  to another signal. Consider the electric circuit in Figure 3 called a **voltage divider**. If the voltage at time  $t$  is  $x(t)$  then, by Ohm's law, the current at time  $t$  satisfies

$$i(t) = \frac{1}{R_1 + R_2} x(t),$$

and the voltage over the resistor  $R_2$  is

$$y(t) = R_2 i(t) = \frac{R_2}{R_1 + R_2} x(t) \quad (1.5)$$

The circuit can be considered as a system mapping the signal  $x$  representing the voltage to the signal  $i = \frac{1}{R_1 + R_2} x$  representing the current, or a system mapping  $x$  to the signal  $y = \frac{R_2}{R_1 + R_2} x$  representing the voltage over resistor  $R_2$ .

We denote systems with capital letters such as  $H$  and  $G$ . A system  $H$  is a function that maps a signal  $x$  to another signal denoted  $H(x)$ . We call  $x$  the **input signal** and  $H(x)$  the **output signal** or the **response** of system  $H$  to signal  $x$ . If we want to include the independent variable  $t$  we will write  $H(x)(t)$

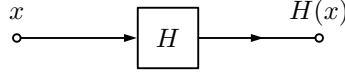


Figure 4: System block diagram with input signal  $x$  and output signal  $H(x)$ .

or  $H(x, t)$  and do not distinguish between these [Curry and Feys, 1968]. It is sometimes useful to depict systems with a block diagram. Figure 4 is a simple block diagram showing the input and output signals of a system  $H$ .

Using this notation the electric circuit in Figure 3 corresponds with the system

$$H(x) = \frac{R_2}{R_1 + R_2} x = y.$$

This system multiplies the input signal  $x$  by  $\frac{R_2}{R_1 + R_2}$ . This brings us to our first practical test.

**Test 1 (Voltage divider)** In this test we construct the voltage divider from Figure 3 on a breadboard with resistors  $R_1 \approx 100\Omega$  and  $R_2 \approx 470\Omega$  with values accurate to within 5%. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \sin(2\pi f_1 t) \quad \text{with} \quad f_1 = 100$$

is passed through the circuit. The approximation is generated by sampling  $x(t)$  at rate  $F_s = \frac{1}{T_s} = 44100\text{Hz}$  to generate samples

$$x_n = x(nT_s) \quad n = 0, \dots, 2F_s$$

corresponding to approximately 2 seconds of signal. These samples are passed to the soundcard which starts playback. The voltage over the resistor  $R_2$  is recorded (also using the soundcard) that returns a lists of samples  $y_1, \dots, y_L$  taken at rate  $F_s$ . The continuous-time voltage over  $R_2$  can be (approximately) reconstructed from these samples as

$$\tilde{y}(t) = \sum_{\ell=1}^L y_\ell \text{sinc}(t - F_s \ell) \quad (1.6)$$

where

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \quad (1.7)$$

is the called the **sinc function** and is plotted in Figure 6. We will justify this reconstruction in Section 6. Simultaneously the (stereo) soundcard is used to record the input voltage  $x(t)$  producing samples  $x_1, \dots, x_L$  taken at rate  $F_s$ . An approximation of the continuous-time input signal is

$$\tilde{x}(t) = \sum_{\ell=1}^L x_\ell \text{sinc}(t - F_s \ell). \quad (1.8)$$

In view of (1.5) we would expect the approximate relationship

$$\tilde{y} \approx \frac{R_2}{R_1 + R_2} \tilde{x} = \frac{42}{57} \tilde{x}$$

A plot of  $\tilde{y}$ ,  $\tilde{x}$  and  $\frac{42}{57}\tilde{x}$  over a 20ms period from 1s to 1.02s is given in Figure 5. The hypothesised output signal  $\frac{42}{57}\tilde{x}$  does not match the observed output signal  $\tilde{y}$ . A primary reason is that the circuitry inside the soundcard itself cannot be ignored. When deriving the equation for the voltage divider we implicitly assumed that current flows through the output of the soundcard without resistance (a short circuit), and that no current flows through the input device of the soundcard (an open circuit). These assumptions are not realistic. Modelling the circuitry in the sound card wont be attempted here. In the next section we will construct circuits that contain external sources of power (active circuits). These are less sensitive to the circuitry inside the soundcard.

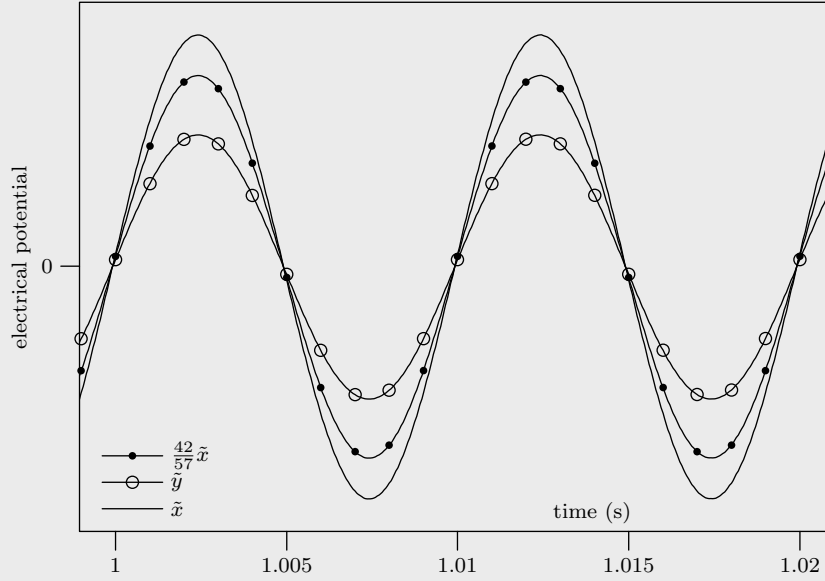


Figure 5: Plot of reconstructed input signal  $\tilde{x}$  (solid line), output signal  $\tilde{y}$  (solid line with circle) and hypothesised output signal  $\frac{42}{57}\tilde{x}$  (solid line with dot) for the voltage divider circuit in Figure 3. The hypothesised signal does not match  $\tilde{y}$ . One reason is that the model does not take account of the circuitry inside the soundcard.

Not all signals can be input to all systems. For example, the system

$$H(x, t) = \frac{1}{x(t)}$$

is not defined at those  $t$  where  $x(t) = 0$  because we cannot divide by zero.

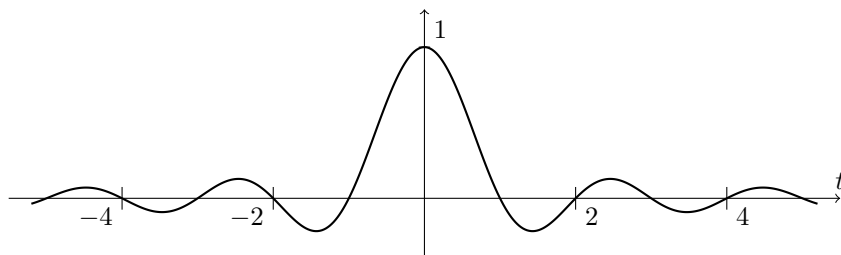


Figure 6: The **sinc function**  $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ .

Another example is the system

$$I_{\infty}(x, t) = \int_{-\infty}^t x(\tau) d\tau, \quad (1.9)$$

called an **integrator**, that is not defined for those signals where the integral above does not exist (is not finite). For example, the signal  $x(t) = 1$  cannot be input to the integrator since the integral  $\int_{-\infty}^t dt$  does not exist.

Thus, when specifying a system it is necessary to also specify a set of signals that can be input, called the **domain** of the system. For example, the domain of the system  $H(x, t) = \frac{1}{x(t)}$  is the set of signals  $x(t)$  which are not zero for any  $t$ . The domain of the integrator  $I_{\infty}(x, t)$  is the set of signals for which the integral  $\int_{-\infty}^t x(\tau) d\tau$  exists for all  $t \in \mathbb{R}$ . The domain of a system is usually obvious from the specification of the system itself. For this reason we will not usually state the domain explicitly. We will only do so if there is chance for confusion.

### 1.3 Some important systems

The system

$$T_{\tau}(x, t) = x(t - \tau)$$

is called the **time-shifter**. This system shifts the input signal along the  $t$  axis ('time' axis) by  $\tau$ . When  $\tau$  is positive  $T_{\tau}$  delays the input signal by  $\tau$ . The time-shifter will appear so regularly in this course that we use the special notation  $T_{\tau}$  to represent it. Figure 7 depicts the action of time-shifters  $T_{1.5}$  and  $T_{-3}$  on the signal  $x(t) = e^{-t^2}$ . When  $\tau = 0$  the time-shifter is the **identity system**

$$T_0(x) = x$$

that maps the signal  $x$  to itself.

Another important system is the **time-scaler** that has the form

$$H(x, t) = x(\alpha t)$$

for  $\alpha \in \mathbb{R}$ . Figure 8 depicts the action of a time-scaler with a number of values for  $\alpha$ . When  $\alpha = -1$  the time-scaler reflects the input signal in the time axis.



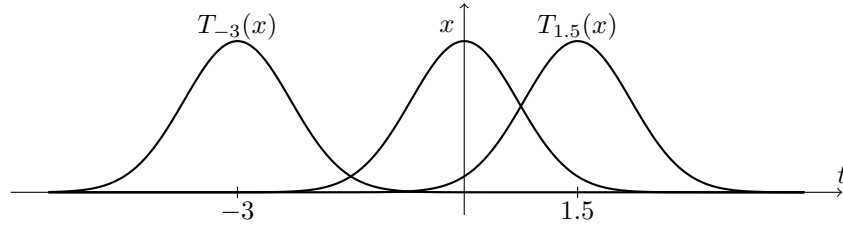


Figure 7: Time-shifter system  $T_{1.5}(x, t) = x(t - 1.5)$  and  $T_{-3}(x, t) = x(t + 3)$  acting on the signal  $x(t) = e^{-t^2}$ .

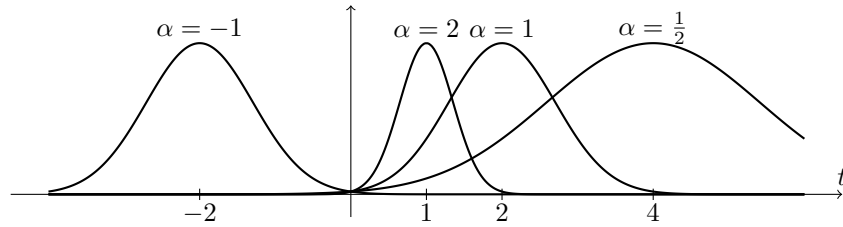


Figure 8: Time-scaler system  $H(x, t) = x(\alpha t)$  for  $\alpha = -1, \frac{1}{2}, 1$  and  $2$  acting on the signal  $x(t) = e^{-(t-2)^2}$ .

Another system we regularly encounter is the **differentiator**

$$D(x, t) = \frac{d}{dt}x(t),$$

that returns the derivative of the input signal. We also define a  $k$ th differentiator

$$D^k(x, t) = \frac{d^k}{dt^k}x(t)$$

that returns the  $k$ th derivative of the input signal.

Another important system is the **integrator**

$$I_a(x, t) = \int_{-a}^t x(\tau) d\tau.$$

The parameter  $a$  describes the lower bound of the integral. In this course it will often be that  $a = \infty$  or  $a = 0$ . The integrator can only be applied to those signals for which the integral above exists. For example, the integrator  $I_\infty$  can be applied to the signal  $tu(t)$  where  $u(t)$  is the step function (1.3). The output signal is

$$\int_{-\infty}^t \tau u(\tau) d\tau = \int_0^t \tau d\tau = \frac{t^2}{2}.$$

However, the integrator cannot be applied to the signal  $x(t) = t$  because  $\int_{-\infty}^t \tau d\tau$  does not exist.

## 1.4 Properties of systems

A system  $H$  is called **memoryless** if the output signal  $H(x)$  at time  $t$  depends only on the input signal  $x$  at time  $t$ . For example  $\frac{1}{x(t)}$  and the identity system  $T_0$  are memoryless, but

$$x(t) + 3x(t-1) \quad \text{and} \quad \int_0^1 x(t-\tau) d\tau$$

are not. A time-shifter system  $T_\tau$  with  $\tau \neq 0$  is not memoryless.

A system  $H$  is **causal** if the output signal  $H(x)$  at time  $t$  depends on the input signal only at times less than or equal to  $t$ . Memoryless systems such as  $\frac{1}{x(t)}$  and  $T_0$  are also causal. Time-shifters  $T_\tau(x, t) = x(t-\tau)$  are causal when  $\tau \geq 0$ , but are not causal when  $\tau < 0$ . The systems

$$x(t) + 3x(t-1) \quad \text{and} \quad \int_0^1 x(t-\tau) d\tau$$

are causal, but the systems

$$x(t) + 3x(t+1) \quad \text{and} \quad \int_0^1 x(t+\tau) d\tau$$

are not causal.

A system  $H$  is called **bounded-input-bounded-output (BIBO) stable** or just **stable** if the output signal  $H(x)$  is bounded whenever the input signal  $x$  is bounded. That is,  $H$  is stable if for every real number  $M$  there exists a real number  $K$  such that for all signals  $x$  satisfying

$$|x(t)| < M \quad \text{for all } t \in \mathbb{R},$$

it also holds that

$$|H(x, t)| < K \quad \text{for all } t \in \mathbb{R}.$$

For example, the system  $x(t) + 3x(t-1)$  is stable with  $K = 4M$  since if  $|x(t)| < M$  then

$$|x(t) + 3x(t-1)| \leq |x(t)| + 3|x(t-1)| < 4M = K.$$

The integrator  $I_a$  for any  $a \in \mathbb{R}$  and differentiator  $D$  are not stable (Exercises 1.5 and 1.6).

A system  $H$  is **linear** if

$$H(ax + by) = aH(x) + bH(y)$$

for all signals  $x$  and  $y$ , and for all complex numbers  $a$  and  $b$ . That is, a linear system has the property: If the input consists of a weighted sum of signals, then the output consists of the same weighted sum of the responses of the system to

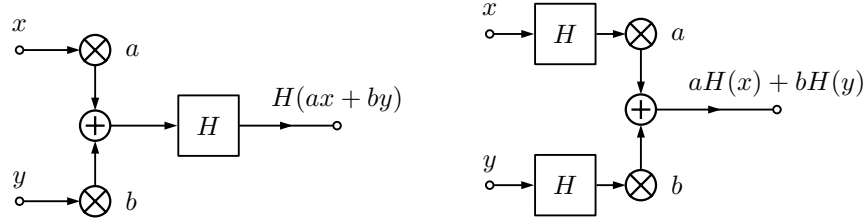


Figure 9: If  $H$  is a linear system the outputs of these two diagrams are the same signal, i.e.  $H(ax + by) = aH(x) + bH(y)$ .

those signals. Figure 9 indicates the linearity property using a block diagram. For example, the differentiator is linear because

$$\begin{aligned} D(ax + by, t) &= \frac{d}{dt}(ax(t) + by(t)) \\ &= a \frac{d}{dt}x(t) + b \frac{d}{dt}y(t) \\ &= aD(x, t) + bD(y, t), \end{aligned}$$

but the system  $H(x, t) = \frac{1}{x(t)}$  is not linear because

$$H(ax + by, t) = \frac{1}{ax(t) + by(t)} \neq \frac{a}{x(t)} + \frac{b}{y(t)} = aH(x, t) + bH(y, t)$$

in general.

The property of linearity trivially generalises to more than two signals. For example if  $x_1, \dots, x_k$  are signals and  $a_1, \dots, a_k$  are complex numbers for some finite  $k$ , then

$$H(a_1x_1 + \dots + a_kx_k) = a_1H(x_1) + \dots + a_kH(x_k).$$

A system  $H$  is **time-invariant** if

$$H(T_\tau(x), t) = H(x, t - \tau)$$

for all signals  $x$  and all time-shifts  $\tau \in \mathbb{R}$ . That is, a system is time-invariant if time-shifting the input signal results in the same time-shift of the output signal. Equivalently,  $H$  is time-invariant if  $H$  commutes the time-shifter  $T_\tau$ , that is, if

$$H(T_\tau(x)) = T_\tau(H(x))$$

for all  $\tau \in \mathbb{R}$  and all signals  $x$ . Figure 10 represents the property of time-invariance with a block diagram.

Let  $S$  be a set of signals. A system  $H$  is said to be **invertible** on  $S$  if each signal  $x \in S$  is mapped to a unique signal  $H(x)$ . That is, for all signals  $x, y \in S$  then  $H(x) = H(y)$  if and only if  $x = y$ . If a system  $H$  is invertible on  $S$  then there exists an inverse system  $H^{-1}$  such that

$$x = H^{-1}(H(x)) \quad \text{for all } x \in S.$$

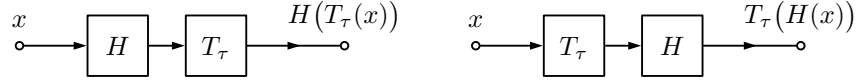


Figure 10: If  $H$  is a time-invariant system the outputs of these two diagrams are the same signal, i.e.  $H(T_\tau(x)) = T_\tau(H(x))$ .

For example, let  $S$  be a any set of signals. The time-shifter  $T_\tau$  is invertible on  $S$ . The inverse system is  $T_{-\tau}$  since

$$T_{-\tau}(T_\tau(x), t) = x(t - \tau + \tau) = x(t).$$

As another example, let  $S$  be the set of differentiable signals. The differentiator system  $D$  is **not** invertible on  $S$  because if  $x \in S$  and if  $y(t) = x(t) + c$  for any constant  $c$  then  $D(y) = D(x)$ . However, if we restrict  $S$  to those differentiable signals for which  $x(0) = c$  is fixed, then  $D$  is invertible on  $S$ . The inverse system in this case is

$$D^{-1}(x, t) = I_0(x, t) + c = \int_0^t x(t) dt + c$$

because

$$D^{-1}(D(x), t) = \int_0^t D(x, t) dt + c = \int_0^t \frac{d}{dt} x(t) dt + x(0) = x(t)$$

by the fundamental theorem of calculus.

## 1.5 Exercises

- 1.1. State whether the step function  $u(t)$  is bounded, periodic, absolutely summable, an energy signal.
- 1.2. Show that the signal  $t^2$  is locally integrable, but that the signal  $\frac{1}{t^2}$  is not.
- 1.3. Plot the signal

$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \leq 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

- 1.4. Compute the energy of the signal  $e^{-\alpha^2 t^2}$  (Hint: use (1.2) and a change of variables).
- 1.5. Show that the integrator  $I_a$  for any  $a \in \mathbb{R}$  is not stable.
- 1.6. Show that the differentiator system  $D$  is not stable.
- 1.7. Show that the time-shifter is linear and time-invariant, and that the time-scaler is linear, but not time invariant

- 1.8. Show that the integrator  $I_c$  with  $c$  finite is linear, but not time-invariant.
- 1.9. Show that the integrator  $I_\infty$  is linear and time invariant.
- 1.10. State whether the system  $H(x, t) = x(t) + 1$  is linear, time-invariant, stable.
- 1.11. State whether the system  $H(x, t) = 0$  is linear, time-invariant, stable.
- 1.12. State whether the system  $H(x, t) = 1$  is linear, time-invariant, stable.

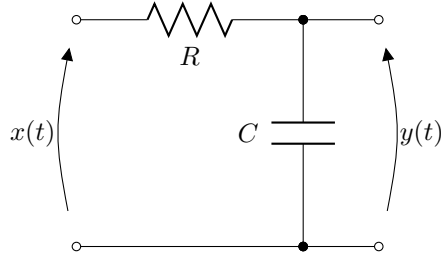


Figure 11: An electrical circuit with resistor and capacitor in series, otherwise known as an **RC circuit**.

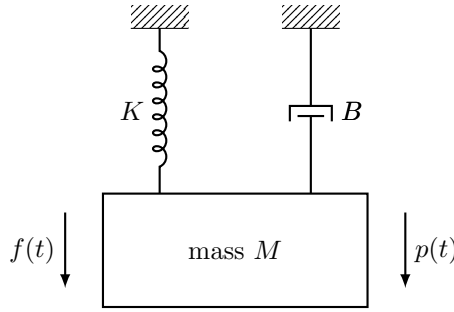


Figure 12: A mechanical mass-spring-damper system

## 2 Systems modelled by differential equations

Let  $H$  be a system and let  $y = H(x)$  be the response of the system to input signal  $x$ . Systems of significant interest in this course are those where the input signal  $x$  and output signal  $y$  are related by a linear differential equation with constant coefficients, that is, an equation of the form

$$\sum_{\ell=0}^m a_{\ell} \frac{d^{\ell}}{dt^{\ell}} x(t) = \sum_{\ell=0}^k b_{\ell} \frac{d^{\ell}}{dt^{\ell}} y(t)$$

where  $a_0, \dots, a_m$  and  $b_0, \dots, b_k$  are constant real numbers. In what follows we will use the differentiator system  $D(x)$  rather than the notation  $\frac{d}{dt}x(t)$  to represent differentiation of the signal  $x$ . To represent the  $\ell$ th derivative we write  $D^{\ell}(x)$ . Using this notation the differential equation above is

$$\sum_{\ell=0}^m a_{\ell} D^{\ell}(x) = \sum_{\ell=0}^k b_{\ell} D^{\ell}(y). \quad (2.1)$$

Equations of this form can be used to model a large number of electrical, mechanical and other real world devices. For example, consider the resistor and capacitor (RC) circuit in Figure 11. Let the signal  $v_R$  represent the voltage over

the resistor and  $i$  the current through both resistor and capacitor. The voltage signals satisfy

$$x = y + v_R,$$

and the current satisfies both

$$v_R = Ri, \quad \text{and} \quad i = CD(y).$$

Combining these equations,

$$x = y + RCD(y) \tag{2.2}$$

that is in the form of (2.1).

As another example, consider the mass, spring and damper in Figure 12. A force represented by the signal  $f$  is externally applied to the mass, and the position of the mass is represented by the signal  $p$ . The spring exerts force  $-Kp$  that is proportional to the position of the mass, and the damper exerts force  $-BD(p)$  that is proportional to the velocity of the mass. The cumulative force exerted on the mass is

$$f_m = f - Kp - BD(p)$$

and by Newton's law the acceleration of the mass  $D^2(p)$  satisfies

$$MD^2(p) = f_m = f - Kp - BD(p),$$

from which we obtain the differential equation

$$f = Kp + BD(p) + MD^2(p) \tag{2.3}$$

that is in the form of (2.1) if we put  $x = f$  and  $y = p$ . Given  $p$  we can readily solve for the corresponding force  $f$ . As a concrete example, let the spring constant, damping constant and mass be  $K = B = M = 1$ . If the position satisfies  $p(t) = e^{-t^2}$ , then the corresponding force satisfies

$$f(t) = e^{-t^2}(4t^2 - 2t - 1).$$

Figure 13 depicts these signals.

What happens if a particular force signal  $f$  is applied to the mass? For example, say we apply the force

$$f(t) = \Pi(t - \tfrac{1}{2}) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the corresponding position signal  $p$ ? We are not yet ready to answer this question, but will be later (Exercise 4.6).

In both the mechanical mass-spring-damper system in Figure 12 and the electrical RC circuit in Figure 11 we obtain a differential equation relating the input signal  $x$  with the output signal  $y$ . The equations do not specify the output

Figure 13: A solution to the mass spring damper system with  $K = B = M = 1$ . The position is  $p(t) = e^{-t^2}$  with corresponding force  $f(t) = e^{-t^2}(4t^2 - 2t - 1)$ .

signal  $y$  explicitly in terms of the input signal  $x$ , that is, they do not explicitly define a system  $H$  such  $y = H(x)$ . As they are, the differential equations, do not provide as much information about the behaviour of the system as we would like. For example, is the system stable? Is it invertible? The **Laplace transform**, described in Section 4, is a useful tool for finding answering these questions. A key property enabling the Laplace transform is that systems described by differential equations of the form (2.1) are linear and time-invariant (Exercise 2.1). We further study linear, time-invariant systems in Section 3. The remainder of this section details the construction of differential equations that model various mechanical, electrical, and electro-mechanical systems. We will use the systems constructed here as examples throughout the course.

## 2.1 Passive circuits

Passive electrical circuits require no sources of power other than the input signal itself. For example, the voltage divider in Figure 3 and the RC circuit in Figure 11 are passive circuits. Another common passive electrical circuit is the resistor, capacitor and inductor (RLC) circuit depicted in Figure 14. In this circuit we let the output signal  $y$  be the voltage over the resistor. Let  $v_C$  represent the voltage over the capacitor and  $v_L$  the voltage over the inductor and let  $i$  be the current. We have

$$y = Ri, \quad i = CD(v_C), \quad v_L = LD(i),$$

leading to the following relationships between  $y$ ,  $v_C$  and  $v_L$ ,

$$y = RCD(v_C), \quad Rv_L = LD(y).$$

Kirchhoff's voltage law gives  $x = y + v_C + v_L$  and by differentiating both sides

$$D(x) = D(y) + D(v_C) + D(v_L).$$

Substituting the equations relating  $y$ ,  $v_C$  and  $v_L$  leads to

$$RCD(x) = y + RCD(y) + LCD^2(y). \quad (2.4)$$

We can similarly find equations relating the input voltage with  $v_C$  and  $v_L$ .



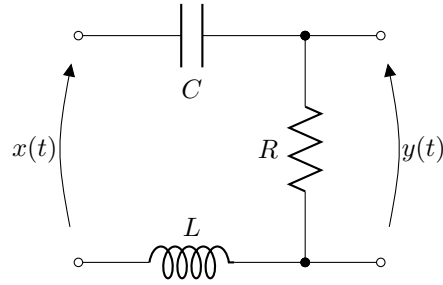


Figure 14: An electrical circuit with resistor, capacitor and inductor in series, otherwise known as an **RLC circuit**.

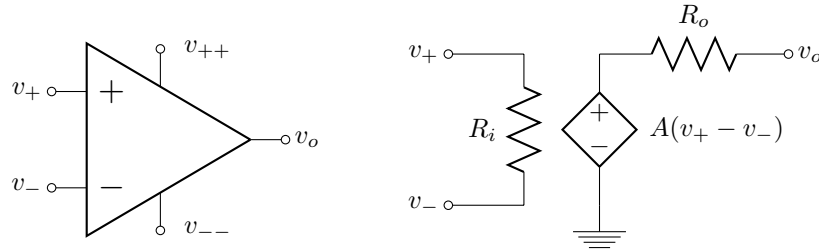


Figure 15: Left: triangular component diagram of an **operational amplifier**. The  $v_{++}$  and  $v_{--}$  connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. Right: model for an operational amplifier including input resistance  $R_i$ , output resistance  $R_o$ , and open loop gain  $A$ . The diamond shaped component is a dependent voltage source. This model is only useful when the operational amplifier is in a negative feedback circuit.

## 2.2 Active circuits

Unlike passive electrical circuits, an **active circuit** requires a source of power external to the input signal. In this course active circuits will be modelled using **operational amplifiers** as depicted in Figure 15. The left hand side of Figure 15 shows a triangular circuit diagram for an operational amplifier, and the right hand side of Figure 15 shows a circuit that can be used to model the behaviour of the amplifier. The  $v_{++}$  and  $v_{--}$  connectors indicate where an external voltage source can be connected to the amplifier, and will normally not be drawn. The diamond shaped component is a dependent voltage source with voltage  $A(v_+ - v_-)$  that depends on the difference between the voltage at the **non-inverting input**  $v_+$  and the voltage at the **inverting input**  $v_-$ . The dimensionless constant  $A$  is called the **open loop gain**. Most operational amplifiers have large open loop gain  $A$ , large input resistance  $R_i$  and small output resistance  $R_o$ . As we will see, it can be convenient to consider the behaviour as  $A \rightarrow \infty$ ,  $R_i \rightarrow \infty$  and  $R_o \rightarrow 0$ , resulting in an **ideal operational amplifier**.

As an example, an operational amplifier configured as a **multiplier** is de-

picted in Figure 16. This circuit is an example of an operation amplifier configured with **negative feedback**, meaning that the output of the amplifier is connected (in this case by a resistor) to the inverting input. The horizontal wire at the bottom of the plot is consider to be ground (zero volts) and is connected to the negative terminal of the depedent voltage source of the operational amplifier depicted in Figure 15. An equivalent circuit for the multiplier using the model in Figure 15 is shown in Figure 17. Solving this circuit (Excercise 2.2) yields the following relationship between the input voltage signal  $x$  and the output voltage signal  $y$ ,

$$y = \frac{R_i(AR_2 + R_o)}{R_i(R_2 + R_o) + R_1(R_2 + R_i - AR_i + R_o)}x. \quad (2.5)$$

For an ideal operational amplifier we let  $A \rightarrow \infty$ ,  $R_i \rightarrow \infty$  and  $R_o \rightarrow 0$ . In this case terms involving the product  $AR_i$  dominate and we are left with the simpler equation

$$y = -\frac{R_2}{R_1}x. \quad (2.6)$$

Thus, assuming an ideal operational amplifier, the circuit acts as a multiplier with constant  $-\frac{R_2}{R_1}$ .

The equation relating  $x$  and  $y$  is much simpler for the ideal operational amplifier. Fortunately this equation can be obtained directly using the following two rules:

1. the voltage at the inverting and non-inverting inputs are equal,
2. no current flows through the inverting and non-inverting inputs.

These rules are only useful for analysing circuits with negative feedback. Let us now rederive (2.6) using these rules. Since the non-inverting input is connected to ground, the voltage at the inverting input is zero. So, the voltage over resistor  $R_2$  is  $y = R_2i$ . Since no current flows through the inverting input the current through  $R_1$  is also  $i$  and  $x = -R_1i$ . Combing these results, the input voltage  $x$  and the output voltage  $y$  are related by

$$y = -\frac{R_2}{R_1}x.$$

In Test 2 the inverting amplifier circuit is constructed and the relationship above is tested using a computer soundcard.

We now consider another circuit consisiting of an operational amplifier, two resistors and a capcitor depicted in Figure 18. Assuming an ideal operational amplifier, the voltage at the inverting terminal is zero because the non-inverting terminal is connected to ground. Thus, the voltage over capacitor  $C_2$  and resistor  $R_2$  is equal to  $y$  and, by Kirchoff's current law

$$i = \frac{y}{R_2} + C_2D(y).$$

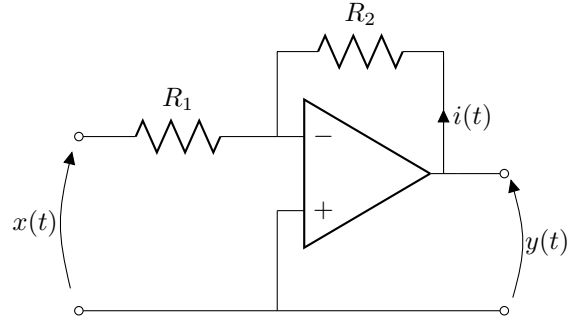


Figure 16: Inverting amplifier

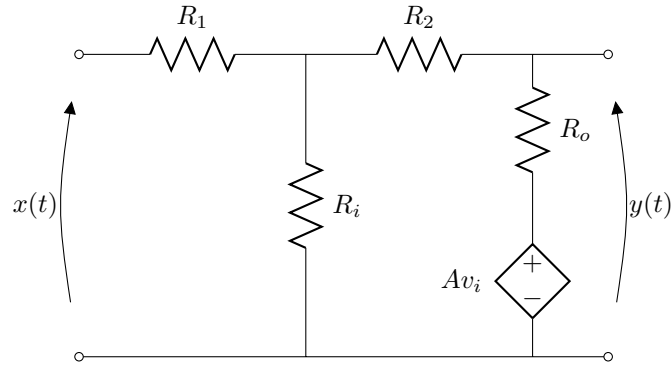


Figure 17: An equivalent circuit for the inverting amplifier from Figure 16 using the model for an operational amplifier in Figure 15. The symbol  $v_i$  is the voltage over resistor  $R_i$ .

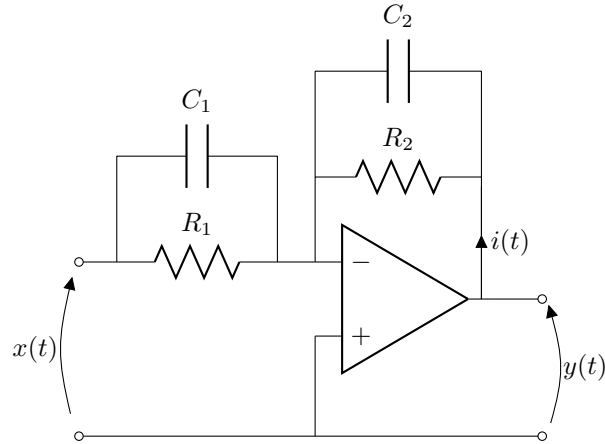


Figure 18: Operational amplifier configured with a resistor at the output, and a feed back loop with resistor and capacitor in parallel.

**Test 2 (Inverting amplifier)** In this test we construct the inverting amplifier circuit from Figure 16 with  $R_2 \approx 22\text{k}\Omega$  and  $R_1 \approx 12\text{k}\Omega$  that are accurate to within 5% of these values. The operational amplifier used is the Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \frac{1}{3} \sin(2\pi f_1 t) + \frac{1}{3} \sin(2\pi f_2 t)$$

with  $f_1 = 100$  and  $f_2 = 233$  is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal  $x$  and the output signal  $y$ . Approximate reconstructions of the input signal  $\tilde{x}$  and output signal  $\tilde{y}$  are given according to (1.8), and (1.6). According to (2.4) we expect the approximate relationship

$$\tilde{y} \approx -\frac{R_2}{R_1} \tilde{x} = -\frac{11}{6} \tilde{x}.$$

Each of  $\tilde{y}$ ,  $\tilde{x}$  and  $-\frac{11}{6} \tilde{x}$  are plotted in Figure 19. Observe that the amplitude of the hypothesised output signal  $-\frac{11}{6} \tilde{x}$  is slightly larger than the observed output signal  $\tilde{y}$ . One explanation is that the ideal model we have used for the operational amplifier is only an approximation.

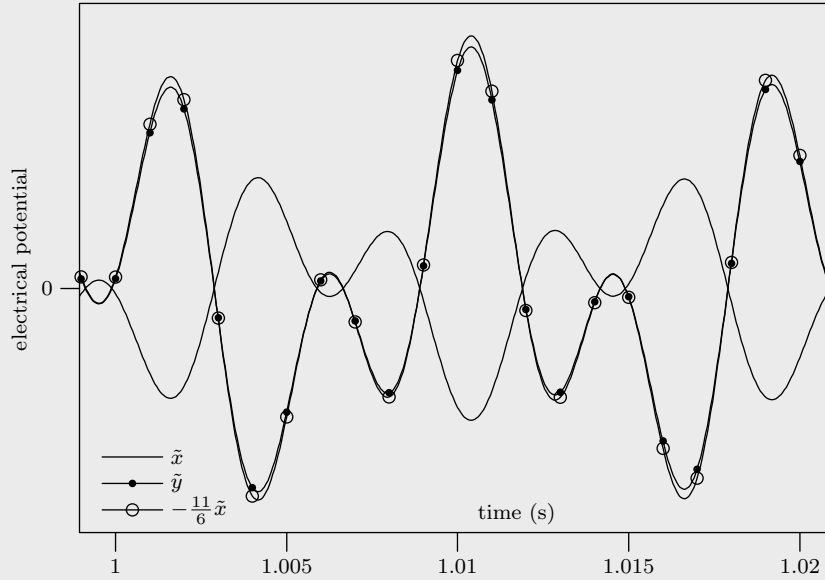


Figure 19: Plot of reconstructed input signal  $\tilde{x}$  (solid line), output signal  $\tilde{y}$  (solid line with circle) and hypothesised output signal  $-\frac{11}{6} \tilde{x}$  (solid line with dot).

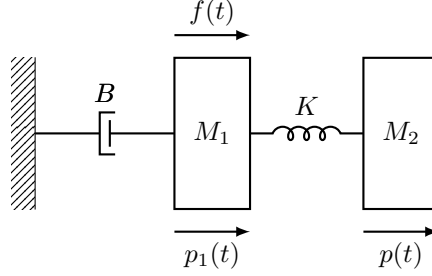


Figure 19: Two masses, a spring and a damper

Similarly, since no current flows through the inverting terminal,

$$i = -\frac{x}{R_1} - C_1 D(x).$$

Combining these equations yields

$$-\frac{x}{R_1} - C_1 D(x) = \frac{y}{R_2} + CD(y). \quad (2.7)$$

Observe the similarity between this equation, and that for the passive RC circuit (2.2) if we put  $R_1 = R_2$  and  $C_1 = 0$  (and open circuit). In this case

$$x = -y - R_1 C_2 D(y). \quad (2.8)$$

### 2.3 Masses, springs and dampers

A mechanical mass, spring, damper system was described in Section 2 and Figure 12. We now consider another mechanical system involving a different configuration of masses, a spring and a damper depicted in Figure 19. A mass  $M_1$  is connected to a wall by a damper with constant  $B$ , and to another mass  $M_2$  by a spring with constant  $K$ . A force represented by the signal  $f$  is applied to the first mass. We will derive a differential equation relating  $f$  with the position  $p$  of the second mass. We assume that the spring applies no force (is in equilibrium) when masses are distance  $d$  apart. The forces due to the spring satisfy

$$f_{s1} = -f_{s2} = K(p - p_1 - d)$$

where  $f_{s1}$  and  $f_{s2}$  are signals representing the force due to the spring on mass  $M_1$  and  $M_2$  respectively. It is convenient to define the signal  $g(t) = p_1(t) + d$  so that forces due to spring satisfy the simpler equation

$$f_{s1} = -f_{s2} = K(p - g).$$

The only force applied to  $M_2$  is by the spring and so, by Newton's law, the acceleration of  $M_2$  satisfies

$$M_2 D^2(p) = f_{s2}.$$

Substituting this into the previous equation gives a differential equation relating  $g$  and  $p$ ,

$$Kg = Kp + M_2 D^2(p). \quad (2.9)$$

The force applied by the damper on mass  $M_1$  is given by the signal

$$f_d = -BD(p_1) = -BD(g)$$

where the replacement of  $p_1$  by  $g$  is justified because differentiation will remove the constant  $d$ . The cumulative force on  $M_1$  is given by the signal

$$\begin{aligned} f_1 &= f + f_d + f_{s1} \\ &= f - Kg + Kp - BD(g), \end{aligned} \quad (2.10)$$

and by Newton's law the acceleration of  $M_1$  satisfies

$$M_1 D(p_1) = M_1 D(g) = f_1.$$

Substituting this into (2.10) and using (2.9) we obtain a fourth order differential equation relating  $p$  and  $f$ ,

$$f = BD(p) + (M_1 + M_2)D^2(p) - \frac{BM_2}{K}D^3(p) + \frac{M_1M_2}{K}D^4(p). \quad (2.11)$$

Given the position of the second mass  $p$  we can readily solve for the corresponding force  $f$  and position of the first mass  $p$ . For example, if the constants  $B = K = 1$  and  $M_1 = M_2 = \frac{1}{2}$  and  $d = \frac{5}{2}$ , and if the position of the second mass satisfies

$$p(t) = e^{-t^2}$$

then, by application of (2.11) and (2.9),

$$f(t) = e^{-t^2}(1 - 8t - 8t^2 + 4t^3 + 4t^4), \quad \text{and} \quad p_1(t) = 2e^{-t^2}t^2 - \frac{5}{2}.$$

This solution is plotted in Figure 20.

## 2.4 Direct current motors

Direct current (DC) motors convert electric energy, in the form of a voltage, into rotary kinetic energy [Nise, 2007, page 76]. We derive a differential equation relating the input voltage  $v$  to the angular position of the motor  $\theta$ . Figure 21 depicts the components of a DC motor.

The voltages over the resistor and inductor satisfy

$$v_R = Ri, \quad v_L = LD(i),$$

and the motion of the motor induces a voltage called the back electromotive force (EMF),

$$v_b = K_b D(\theta)$$

Figure 20: Solution of the system describing two masses with a spring and damper where  $B = K = 1$  and  $M_1 = M_2 = \frac{1}{2}$  and the position of the seconds mass is  $p(t) = e^{-t^2}$ .

that we model as being propotional to the angular velocity of the motor. The input voltage now satisfies

$$v = v_R + v_L + v_b = Ri + LD(i) + K_b D(\theta).$$

The torque  $\tau$  applied by the motor is modelled as being proportional to the current  $i$ ,

$$\tau = K_\tau i.$$

A load with inertia  $J$  is attached to the motor. Two forces are assumed to act on the load, the torque  $\tau$  applied by the current, and a torque  $\tau_d = BD(\theta)$  modelling a damper that acts proportionally against the angular velocity of the motor. By Newton's law, the angular acceleration of the load satisfies

$$JD(\theta) = \tau - \tau_d = K_\tau i - BD(\theta).$$

Combining these equation we obtain the 3rd order differential equation

$$v = \left( \frac{RB}{K_\tau} + K_b \right) D(\theta) + \frac{RJ + LB}{K_\tau} D^2(\theta) + \frac{LJ}{K_\tau} D^3(\theta)$$

relating voltage and motor position. In many DC motors the inductance  $L$  is small and can be ignored, leaving the simpler second order equation

$$v = \left( \frac{RB}{K_\tau} + K_b \right) D(\theta) + \frac{RJ}{K_\tau} D^2(\theta). \quad (2.12)$$

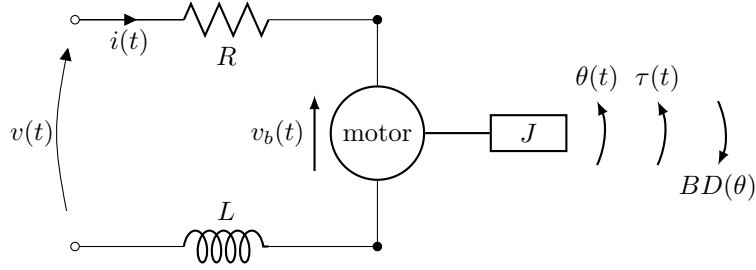


Figure 21: Diagram for a rotary direct current (DC) motor

Figure 22: Voltage and corresponding angle for a DC motor with constants  $K_b = K_\tau = B = R = J = 1$ .

Given the position signal  $\theta$  we can find the corresponding voltage signal  $v$ . For example, put the constants  $K_b = K_\tau = B = R = J = 1$  and assume that

$$\theta(t) = 2\pi(1 + \text{erf}(t))$$

where  $\text{erf}(t) = \frac{2}{\pi} \int_{-\infty}^t e^{-\tau^2} d\tau$  is the **error function**. The corresponding angular velocity  $D(\theta)$  and voltage  $v$  satisfy

$$D(\theta, t) = 4\sqrt{\pi}e^{-t^2}, \quad v(t) = 8\sqrt{\pi}e^{-t^2}(1 - t).$$

These signals are depicted in Figure 22. This voltage signal is sufficient to make the motor perform two revolutions and then come to rest.

## 2.5 Exercises

2.1. Show that systems described by differential equations of the form (2.1) are linear and time invariant. That is, show:

- (a) if  $x, y$  are solutions of the equation, then  $T_\tau(x), T_\tau(y)$  are solutions,
- (b) if  $x_1, y_1$  and  $x_2, y_2$  are solutions, then  $cx_1 + dx_2, cy_1 + dy_2$  are solutions for complex numbers  $c$  and  $d$ .



- 2.2. Analyse the inverting amplifier circuit in Figure 17 to obtain the relationship between input voltage  $x$  and output voltage  $y$  given by (2.5). You may wish to use a symbolic programming language, for example Mathematica.