Testable linear shift-invariant systems
(Exercise Solutions)

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Contents

Chapter 1

Signals and systems

Exercises

1.1. How many distinct functions from the set $X = \{\text{Mario}, \text{Link}\}$ to the set $Y = \{\text{Freeman}, \text{Ryu}, \text{Sephiroth}\}$ exist? Write down each function, that is, write down all functions from the set $X \to Y$.

Solution: Each of the two elements in X can be mapped to one of the three elements of Y. There are thus $3^2 = 9$ distinct functions in $X \to Y$. They are

$$f_1(x) = \begin{cases} \text{Freeman} & x = \text{Mario} \\ \text{Freeman} & x = \text{Link} \end{cases} \qquad f_2(x) = \begin{cases} \text{Freeman} & x = \text{Mario} \\ \text{Ryu} & x = \text{Link} \end{cases}$$

$$f_3(x) = \begin{cases} \text{Ryu} & x = \text{Mario} \\ \text{Freeman} & x = \text{Link} \end{cases} \qquad f_4(x) = \begin{cases} \text{Freeman} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \end{cases}$$

$$f_5(x) = \begin{cases} \text{Sephiroth} & x = \text{Mario} \\ \text{Freeman} & x = \text{Link} \end{cases} \qquad f_6(x) = \begin{cases} \text{Ryu} & x = \text{Mario} \\ \text{Ryu} & x = \text{Link} \end{cases}$$

$$f_7(x) = \begin{cases} \text{Ryu} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \end{cases} \qquad f_8(x) = \begin{cases} \text{Sephiroth} & x = \text{Mario} \\ \text{Ryu} & x = \text{Link} \end{cases}$$

$$f_9(x) = \begin{cases} \text{Sephiroth} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \end{cases}$$

1.2. State whether the step function $u(t)$ is bounded, periodic, absolutely integrable, an energy signal. **Solution:** The magnitude of u is less than or equal to one and so the signal is bounded. The signal is not periodic, since for any hypothesised period $T > 0$ we have $u(T) = 1$ but $u(0) = 0$. The signal is not absolutely integrable, nor an energy signal since

$$\|u\|_1 = \|u\|_2 = \int_{-\infty}^{\infty} |u(t)|\, dt = \int_0^{\infty} dt$$

is not finite.

1.3. Show that the signal t^2 is locally integrable, but that the signal $\frac{1}{t^2}$ is not.

Solution: For any a and b

$$\int_a^b t^2 dt = \frac{b^3}{3} - \frac{a^3}{3}$$

is finite and so t^2 is locally integrable. Put $a = 0$ and $b > 0$ and

$$\int_0^b \frac{1}{t^2} dt = -\frac{1}{b} + \lim_{t \to 0} \frac{1}{t} = \infty.$$

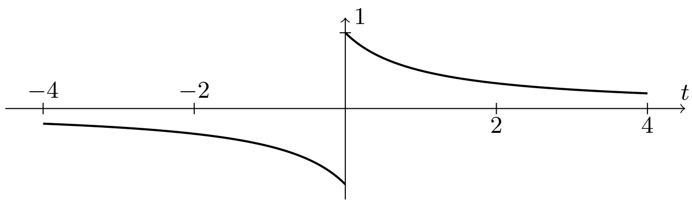
The limit above diverges and so $\frac{1}{t^2}$ is not locally integrable.

1.4. Plot the signal

$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \leq 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

Solution:



The signal is bounded since $|x(t)| < M$ for any $M > 1$. The signal is locally integrable because it is bounded, i.e., for any finite constants a and b

$$\int_a^b |x(t)| \, dt < \int_a^b M dt = (b - a)M < \infty.$$

The signal x is not absolutely integrable since

$$\begin{aligned} \|x\|_1 &= \int_{-\infty}^{\infty} |x(t)| \, dt \\ &= 2\int_0^{\infty} \frac{1}{t+1} dt \\ &= 2\int_1^{\infty} \frac{1}{t} dt \\ &= 2\log(1) + \lim_{t \to \infty} 2\log(t) \end{aligned}$$

and the limit diverges. The signal is square integrable since

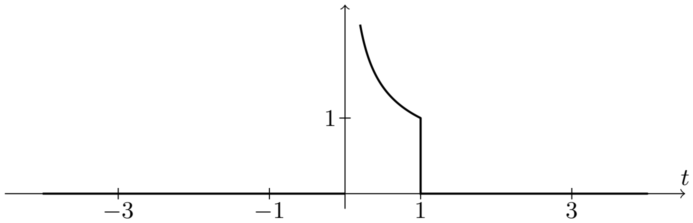
$$\begin{aligned} \|x\|_2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= 2\int_0^{\infty} \frac{1}{(t+1)^2} dt \\ &= 2\int_1^{\infty} \frac{1}{t^2} dt \\ &= 2 - \lim_{t \to \infty} \frac{2}{t} = 2. \end{aligned}$$

1.5. Plot the signal

$$x(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that x is absolutely integrable, but not square integrable.

Solution:



The integral

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt = \int_0^1 t^{-1/2} dt = [2\sqrt{t}]_0^1 = 2$$

and so x is absolutely integrable. The integral

$$\|x\|_2 = \int_{-\infty}^{\infty} |x(t)|\, dt = \int_0^1 t^{-1} dt = [\log(t)]_0^1 = \log(1) - \lim_{t\to 0} \log(t) = \infty$$

and so x is not square integrable.

1.6. Compute the energy of the signal $e^{-\alpha^2 t^2}$ (Hint: use equation (1.1.4) on page 4 and a change of variables). **Solution:** From (1.1.4) we the energy of e^{-t^2} is $\sqrt{\pi}$. Now

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} dt = \frac{1}{\alpha} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau = \frac{\sqrt{\pi}}{\alpha}$$

by the change of variables $\tau = \alpha t$.

1.7. Show that the signal t^2 is differentiable, but the step function u and rectangular pulse Π are not. **Solution:** We have

$$\lim_{h\to 0} \frac{(t+h)^2 - t^2}{h} = \lim_{h\to 0} \frac{2th + h^2}{h} = 2t.$$

$$\lim_{h\to 0} \frac{t^2 - (t-h)^2}{h} = \lim_{h\to 0} \frac{2th - h^2}{h} = 2t$$

and so t^2 is continuously differentiable with derivative $\frac{d}{dt}t^2 = 2t$. At $t = 0$ the corresponding limits for the step function are

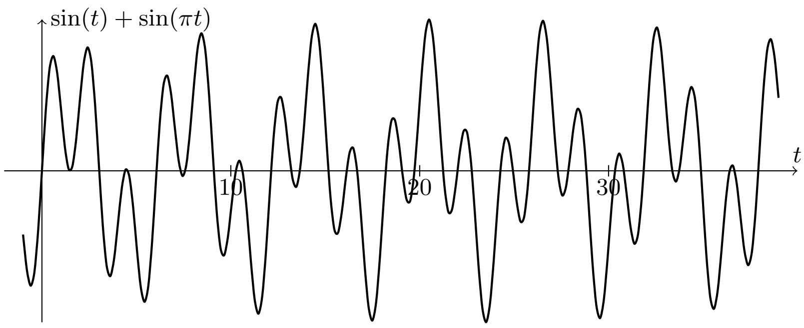
$$\lim_{h\to 0} \frac{u(h) - u(0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

but

$$\lim_{h\to 0} \frac{u(0) - u(-h)}{h} = \lim_{h\to 0} \frac{1}{h} = \infty$$

so the step function u is not differentiable at $t = 0$. A similar argument at $t = \frac{1}{2}$ or $t = -\frac{1}{2}$ shows that Π is not differentiable.

1.8. Plot the signal $\sin(t) + \sin(\pi t)$. Show that this signal is not periodic. **Solution:** A plot of the signal is below:



The following argument is due to Qiaochu Yuan (http://math.stackexchange.com/questions/1079/sum-of-two-periodic-functions). Suppose $\sin(t) + \sin(\pi t)$ is periodic. Then

$$\sin(t) + \sin(\pi t) = \sin(t + T) + \sin(\pi t + T)$$

for some $T > 0$. Differentiating both sides twice with respect to t gives

$$\sin(t) + \pi^2 \sin(\pi t) = \sin(t + T) + \pi^2 \sin(\pi t + T)$$

Subtracting the first equation from the second gives $\sin(t) = \sin(t + T)$ and substituting this into the second equation gives $\sin(\pi t) = \sin(\pi t + T)$. The equation $\sin(t) = \sin(t + T)$ implies that $T = 2\pi k$ for some integer $k \neq 0$. The equation $\sin(\pi t) = \sin(\pi t + T)$ implies that $T = 2\ell$ for some integer $\ell \neq 0$. We would thus have $2\pi k = 2\ell$ and so $\pi = \frac{\ell}{k}$. However, this is impossible because π is irrational. Thus $\sin(t) + \sin(\pi t)$ is not periodic .

1.9. Show that the set of locally integrable signals L_{loc}, the set of absolutely integrable signals L^1, and the set of square integrable signals L^2 are linear shift-invariant spaces. **Solution:** Let $x, y \in L^1$ and $a, b \in \mathbb{C}$. Now

$$\begin{aligned}\|ax + by\|_1 &= \int_{-\infty}^{\infty} |ax(t) + by(t)|\, dt \\ &\leq \int_{-\infty}^{\infty} a\,|x(t)| + b\,|y(t)|\, dt \qquad \text{triangle inequality} \\ &= a\|x\|_1 + b\|y\|_1 < \infty\end{aligned}$$

and so $ax + by \in L_1$ and L_1 is a linear space. Also

$$\begin{aligned}\|T_\tau x\|_1 &= \int_{-\infty}^{\infty} |T_\tau x(t)|\, dt \\ &= \int_{-\infty}^{\infty} |x(t-\tau)|\, dt \\ &= \int_{-\infty}^{\infty} |x(k)|\, dk \qquad \text{change variable } k = t - \tau \qquad = \|x\|_1 < \infty\end{aligned}$$

and so L_1 is a shift-invariant space.

Now

$$\begin{aligned}\|ax + by\|_2^2 &= \int_{-\infty}^{\infty} |ax(t) + by(t)|^2\, dt \\ &\int_{-\infty}^{\infty} |ax(t)|^2 + |by(t)|^2 + 2\operatorname{Re}\big(a^* x(t)^* by(t)\big) dt\end{aligned}$$

where $*$ denotes the complex cojugate and Re denotes the real part of a complex number. Now

$$\operatorname{Re}\left(a^* x(t)^* b y(t)\right) \leq |ax(t)|\,|by(t)| \leq \max(|ax(t)|^2, |by(t)|^2) \leq |ax(t)|^2 + |by(t)|^2$$

and so

$$\begin{aligned}\|ax+by\|_2^2 &\leq \int_{-\infty}^{\infty} 3\,|ax(t)|^2 + 3\,|by(t)|^2\,dt \\ &= \int_{-\infty}^{\infty} 3\,|a|^2\,|x(t)|^2 + 3\,|b|^2\,|y(t)|^2\,dt \\ &= 3\,|a|^2\,\|x\|_2^2 + 3\,|b|^2\,\|y\|_2^2 < \infty\end{aligned}$$

and L_2 is thus a linear space. Also

$$\|T_\tau x\|_2^2 = \int_{-\infty}^{\infty} |T_\tau x(t)|^2\,dt = \int_{-\infty}^{\infty} |x(t-\tau)|^2\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \|x\|_2^2 < \infty$$

and so L_2 is a shift-invariant space.

1.10. Show that the set of periodic signals is a shift-invariant space, but not a linear space.

1.11. Show that the set of bounded signals is a linear shift-invariant space.

1.12. Let $K > 0$ be a fixed real number. Show that the set of signals bounded below K is a shift invariant space, but not a linear space.

1.13. Show that the set of even signals and the set of odd signals are not shift invariant spaces.

1.14. Show that the integrator I_c with finite $c \in \mathbb{R}$ is not stable. **Solution:** Put $M > 1$. The shifted step function $u(t+a)$ is locally integrable and bounded below M, i.e. $|u(t+a)| \leq 1 < M$ for all $t \in \mathbb{R}$. However, the response of the integrator I_a to $u(t+a)$ is

$$I_a u(t+a) = \int_{-a}^{t} u(\tau + a)d\tau = \begin{cases} \int_{-a}^{t} d\tau = t + a & t \geq -a \\ 0 & t < -a \end{cases},$$

and this is not a bounded signal, that is, for every K we have $t + a > K$ whenever $t > K - a$.

1.15. Show that if the signal x is locally integrable and $\int_{-\infty}^{0} |x(t)|\,dt < \infty$ then $I_\infty x(t) = \int_{-\infty}^{t} x(t)dt < \infty$ for all $t \in \mathbb{R}$. **Solution:** We have

$$\begin{aligned}I_\infty x(t) \leq |I_\infty x(t)| &= \left|\int_{-\infty}^{t} x(t)dt\right| \\ &\leq \int_{-\infty}^{t} |x(t)|\,dt \\ &= \int_{-\infty}^{0} |x(t)|\,dt + \int_{0}^{t} |x(t)|\,dt\end{aligned}$$

Now $\int_{-\infty}^{0} |x(t)|\,dt < \infty$ by assumption and $\int_{0}^{t} |x(t)|\,dt$ because x is locally integrable. It follows that

1.16. Show that the integrator I_∞ is not stable. **Solution:** By default the domain for I_∞ is the subset of locally integrable signals for which $\int_{-\infty}^{0} |x(t)|\, dt < \infty$. The step function $u(t)$ is in this domain. The argument now follows similiarly to Exercise 1.16.

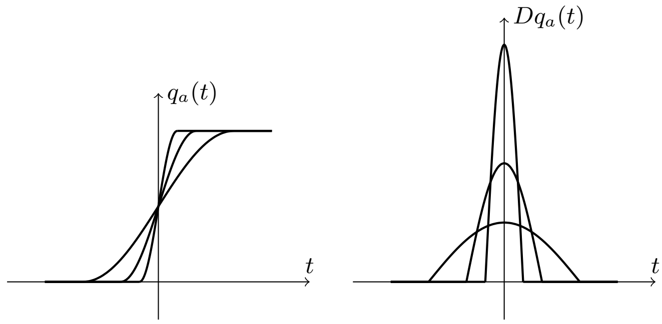
1.17. Show that the differentiator system D is not stable. **Solution:** Put $M > 2$. Define the signal

$$q_a(t) = \begin{cases} 0 & 2t < -a \\ 1 + \sin\left(\frac{\pi t}{a}\right) & -a < 2t < a \\ 2 & 2t > a, \end{cases}$$

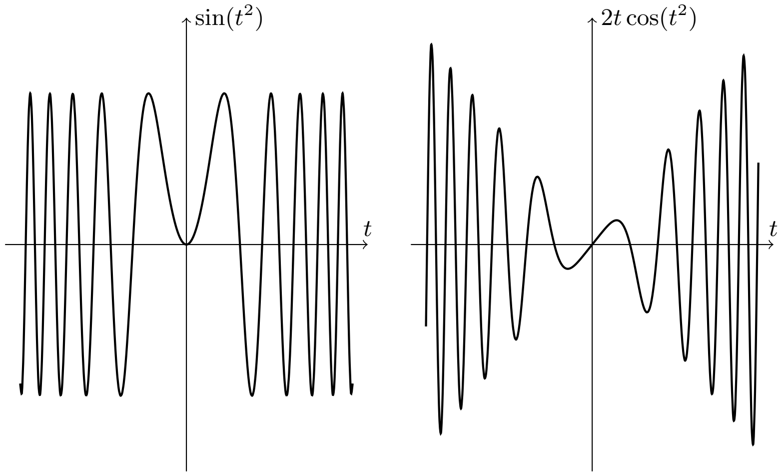
and observe that q_a is differentiable and bounded below M. The response of the differentiator D to q_a is

$$Dq_a(t) = \begin{cases} 0 & 2t < -a \\ \frac{\pi}{a} \cos\left(\frac{\pi t}{a}\right) & -a < 2t < a \\ 1 & 2t > a. \end{cases}$$

The signal p_a and the response Dp_a are plotted below for $a = \frac{1}{2}, 1$ and 2. The response Dp_a obtains a maximum amplitude of $\frac{\pi}{a}$ at $t = 0$. So D is not stable because for any K we can choose $a < \frac{\pi}{K}$ so that $\frac{\pi}{a} > K$.



Another solution was suggested by Badri Vellambi. Consider the signal $x(t) = \sin(t^2)$ plotted in the figure below. This signal is bounded below any $M > 1$. The response of the differentiator is $Dx(t) = 2t\cos(t^2)$ and this is not bounded.



1.18. Show that the shifter T_τ is linear and shift-invariant and that the time-scaler is linear, but not time invariant. **Solution:** The shifter T_τ is shift-invariant since

$$T_k T_\tau x = T_k x(t-\tau) = x(t-\tau-k) = T_\tau x(t-k) = T_\tau T_k x$$

for all signals x, that is, shifters commute with shifters. The shifter is linear because

$$T_\tau(ax+by) = ax(t-\tau) + by(t-\tau) = aT_\tau x + bT_\tau y.$$

The time-scaler $Hx = x(\alpha t)$ is linear because

$$H(ax+by) = ax(\alpha t) + by(\alpha t) = aHx + bHy.$$

The system is not shift-invariant because

$$HT_\tau x = Hx(t-\tau) = x(\alpha t - \tau)$$

but

$$T_\tau Hx = T_\tau x(\alpha t) = x(\alpha(t-\tau)) = x(\alpha t - \alpha\tau),$$

and these signals are not equal in general. For example consider the rectangular pulse Π. With time-scaling parameter $\alpha = 2$ and shift $\tau = 1$,

$$HT_1\Pi = \Pi(2t-1) \neq \Pi(2t-2) = T_1 H\Pi.$$

1.19. Show that the integrator I_c with finite $c \in \mathbb{R}$ is linear, but not shift-invariant. **Solution:** The system is linear because, if $x, y \in L_{\mathrm{loc}}$, then

$$\begin{aligned} I_c(ax+by) &= \int_{-c}^{t} ax(\tau) + by(\tau)d\tau \\ &= a\int_{-c}^{t} x(\tau)d\tau + b\int_{-c}^{t} y(\tau)d\tau \\ &= aI_c x + bI_c y. \end{aligned}$$

The system is not shift-invariant because

$$T_k I_c x = I_c(x, t-k) = \int_{-c}^{t-k} x(\tau)d\tau$$

but

$$I_c T_k x = \int_{-c}^{t} x(\tau-k)d\tau.$$

We now need only find some signal $x \in L_{\mathrm{loc}}$ for which the integrals on the right hand side of the above equations are not equal. Choose the signal $x = 1$, i.e., the signal that is equal to 1 for all time. In this case

$$T_k I_c 1 = \int_{-c}^{t-k} d\tau = t-k+c \neq t+c = \int_{-c-k}^{t-k} d\tau = I_c T_k 1 \qquad \text{when } k \neq 0.$$

1.20. Show that the integrator I_∞ is linear and shift-invariant. **Solution:** The system is linear because

$$\begin{aligned} I_\infty(ax+by) &= \int_{-\infty}^{t} ax(\tau) + by(\tau)d\tau \\ &= a\int_{-\infty}^{t} x(\tau)d\tau + b\int_{-\infty}^{t} y(\tau)d\tau \\ &= aI_\infty x + bI_\infty y. \end{aligned}$$

The system is shift-invariant because

$$T_k I_\infty x = I_\infty x(t-k) = \int_{-\infty}^{t-k} x(\tau) d\tau,$$

and

$$I_\infty T_k x = \int_{-\infty}^{t} x(\tau - k) d\tau = \int_{-\infty}^{t-k} x(\tau) d\tau.$$

1.21. State whether the system $Hx = x + 1$ is linear, shift-invariant, stable.
Solution: It is not linear because for any signal x and real number $a \neq 1$,

$$H(ax) = ax + 1 \neq aHx = a\big(x + 1\big) = ax + a.$$

It is shift-invariant because

$$HT_\tau x = x(t-\tau) + 1 = T_\tau(x+1) = T_\tau Hx.$$

It is stable because for any signal x with $x(t) < M$ for all $t \in \mathbb{R}$,

$$Hx(t) = x(t) + 1 < M + 1 \qquad \text{for all } t \in \mathbb{R}.$$

1.22. State whether the system $Hx = 0$ is linear, shift-invariant, stable.
Solution: It is linear because

$$H(ax + by) = 0 = aHx + bHy = 0.$$

It is shift-invariant because

$$HT_\tau x(t) = 0 = Hx(t-\tau).$$

It is stable because for any $K > 0$,

$$Hx(t) = 0 < K \qquad \text{for all } t \in \mathbb{R} \text{ and all signals } x.$$

1.23. State whether the system $Hx = 1$ is linear, shift-invariant, stable.
Solution: It is not linear because for any signal x and real number $a \neq 1$

$$H(ax) = 1 \neq aHx = a.$$

It is shift-invariant because

$$HT_\tau x = 1 = T_\tau(1) = T_\tau Hx.$$

It is stable because for any $K > 1$,

$$|Hx(t)| = 1 < K \qquad \text{for all } t \in \mathbb{R} \text{ and all signals } x.$$

1.24. Let x be a signal with period T that is not equal to zero almost everywhere. Show that x is neither absolutely integrable nor square integrabl. **Solution:** This is plain and does not really require further explanation, but I've found some students desire more rigour.

Since x does not equal to zero almost everywhere there exist some finite real numbers a and b such that $\int_a^b |x(t)|\,dt = C > 0$. Let k be an integer such $-kT < a$ and $kT > b$ so that the integral over $2k+1$ periods

$$\int_{-kT}^{kT} |x(t)|\,dt \geq \int_a^b |x(t)|\,dt = C > 0.$$

Now, since x has period T

$$\int_{-ckT}^{ckT} |x(t)|\,dt = (2c+1)\int_{-kT}^{kT} |x(t)|\,dt \geq (2c+1)C > 0$$

for integers c and since this integral is increasing monotonically with c we have $\int_{-ckT}^{ckT} |x(t)|\,dt \geq \lfloor 2c+1 \rfloor C$ for all $c \in \mathbb{R}$ where $\lfloor 2c+1 \rfloor$ denotes the largest integer less than or equal to $2c+1$. Now,

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\,dt = \lim_{c\to\infty} \int_{-ckT}^{ckT} |x(t)|\,dt \geq \lim_{c\to\infty} \lfloor 2c+1 \rfloor C = \infty,$$

and so, x is not absolutely integrable.

Chapter 2

Systems modelled by differential equations

Exercises

2.1. Analyse the inverting amplifier circuit in Figure 2.7 to obtain the relationship between input voltage x and output voltage y given by (2.2.1). You may wish to use a symbolic programming language (for example Maxima, Sage, Mathematica, or Maple). **Solution:** We provide two solutions. Let v_i, v_o, v_1 and v_2 be the voltages over the input resistor R_i, the output resistor R_o, and resistors R_1 and R_2 respectively. Observe that $v_+ - v_i = v_i$ and so the voltage over the dependent source is Av_i. The voltages satisfy,

$$x = v_1 - v_i$$
$$y = -v_i - v_2$$
$$y = v_o + Av_i$$

The currents into the 3 way connection between R_i, R_1 and R_2 sum to zero, and so

$$\frac{v_1}{R_1} + \frac{v_i}{R_i} = \frac{v_2}{R_2}$$

by Ohm's law, the direction of current moving from positive to negative voltage. Finally the currents through R_o and R_2 are the same, and so

$$\frac{v_o}{R_o} = \frac{v_2}{R_2}.$$

We now have 5 linearly independent equations for the six unknowns v_1, v_2, v_o, v_i, x, y. We can use these to find an equation that describes y in terms of x. The Mathematica command

```
Simplify[Solve[{x == v1 - vi,
   y == vo + A*vi,
   y == -vi - v2,
   v1/r1 + vi/ri == v2/r2,
   vo/ro == v2/r2,
   r1 > 0, r2 > 0, ro > 0, ri > 0, A > 0},
  {y, vi, vo, v2, v1}, Reals]]
```

or Maxima command

```
linsolve([x = v1 - vi,
   y = vo + A*vi,
   y = -vi - v2,
   v1/r1 + vi/ri = v2/r2,
   vo/ro = v2/r2],
   [y, vi, vo, v2, v1]);
```

readily obtains

$$y = \frac{R_i(R_o - AR_2)}{R_i(R_2 + R_o) + R_1(R_2 + R_i + AR_i + R_o)} x.$$

The second solution is thanks to Badri Vellambi. Consider the operational amplifier circuit with feedback presented in Fig. 2.1. Suppose that the voltage signal fed into the circuit is $x(t)$ and the voltage signal measured at the output of the opamp is $y(t)$.

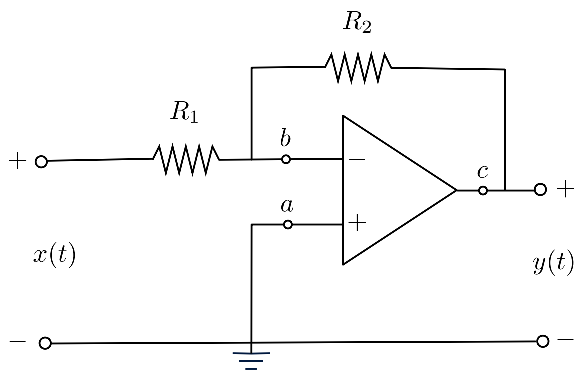


Figure 2.1: The circuit

To simplify the circuit, one has to use the model for the opamp given in Fig. 2.2 which involves the voltage-controlled voltage-source (VCVS) at the output side (indicated in green). While replacing the operational amplifier with its model, it must be noted that the positive terminal of the operational amplifier is connected to the ground.

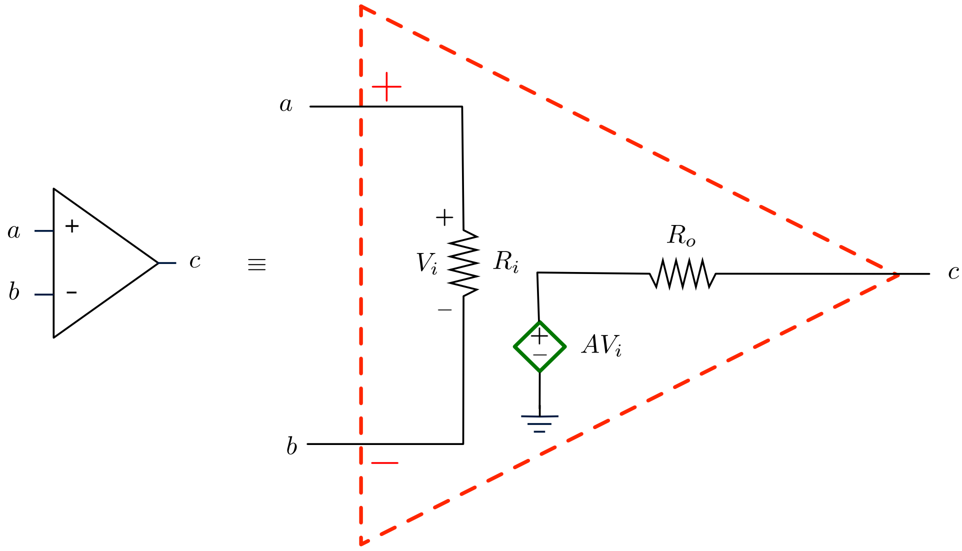


Figure 2.2: The model for an operational amplifier

Upon replacement, we obtain the following equivalent circuit. Again notice that since the positive terminal of the opamp was connected to the ground, the voltage output by the VCVS is AV_i where V_i is the voltage between the ground and the top of the resistance R_i, and is measured against the flow of the current $i - i_1$ as is indicated in the figure.

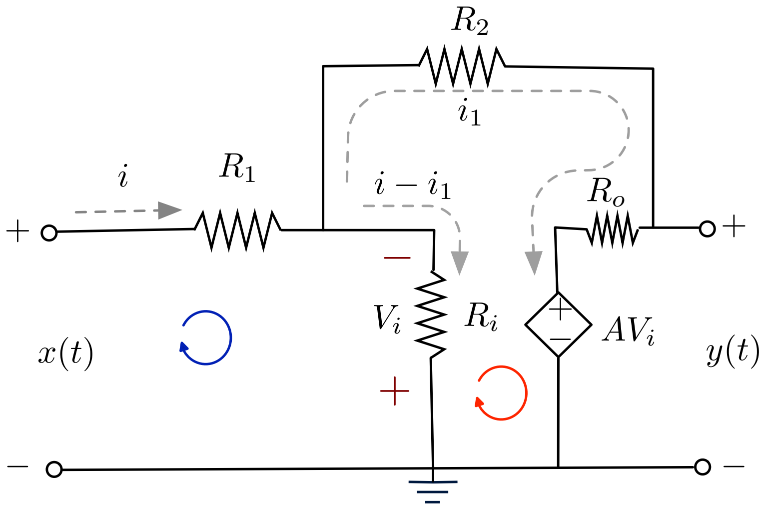


Figure 2.3: The operational amplifier circuit with the model

Applying Kirchoff's law to the outer loop indicated in blue in Fig. 2.3, we obtain the following equation.

$$x(t) = iR_1 + (i - i_1)R_i = i(R_1 + R_i) - i_1 R \tag{2.0.1}$$

Note that by definition, the voltage V_i that controls the VCVS is the voltage across R_i measured against the indicated direction of the current $i - i_1$, and is given by

$$V_i = -(i - i_1)R_i. \tag{2.0.2}$$

Next, writing out the Kirchoff's law for the inner loop indicated in red, we obtain the following.

$$0 = i_1 R_2 + i_2 R_0 + AV_i - (i - i_1)R_i \tag{2.0.3}$$

Substituting V_i in the above equation with the RHS of (2.0.2), we obtain the following.

$$\begin{aligned} 0 &= i_1(R_2 + R_0) - A(i - i_1)R_i - (i - i_1)R_i &(2.0.4)\\ &= -i(1 + A)R_i + i_1((1 + A)R_i + R_0 + R_2) &(2.0.5)\end{aligned}$$

Combining (2.0.5) and (2.0.1), we obtain the following linear system of equations governing the electrical circuit.

$$\begin{bmatrix} R_1 + R_i & -R_i \\ -(1 + A)R_i & (1 + A)R_i + R_0 + R_2 \end{bmatrix} \begin{bmatrix} i \\ i_1 \end{bmatrix} = \begin{bmatrix} x(t) \\ 0 \end{bmatrix} \tag{2.0.6}$$

Solving the above linear system, we identify the current in the different branches to be

$$\begin{bmatrix} i \\ i_1 \end{bmatrix} = x(t) \begin{bmatrix} \frac{(1+A)R_i+R_0+R_2}{(1+A)R_iR_1+R_0R_1+R_2R_1+R_0R_i+R_2R_i} \\ \frac{(1+A)R_i}{(1+A)R_iR_1+R_0R_1+R_2R_1+R_0R_i+R_2R_i} \end{bmatrix}. \tag{2.0.7}$$

Lastly, notice that

$$\begin{aligned} y(t) &= i_1 R_0 + AV_i &(2.0.8)\\ &= i_1 R_0 - (i - i_1)R_i. &(2.0.9)\end{aligned}$$

Substituting the solutions for i and i_1 in terms of $x(t)$, we obtain the following.

$$y(t) = \left(\frac{R_i R_0 - R_2 R_i A}{(1 + A)R_i R_1 + R_0 R_1 + R_2 R_1 + R_0 R_i + R_2 R_i} \right) x(t) \tag{2.0.10}$$

2.2. Figure 2.4 depicts a mechanical system involving two masses, two springs, and a damper connected between two walls. Suppose that the spring K_2 is at rest when the mass M_2 is at position $p(t) = 0$. A force, represented by the signal f, is applied to mass M_1. Derive a differential equation relating the force f and the position p of mass M_2.

Solution: Let p_1 be a signal representing the position of mass M_1. Suppose that the spring K_1 connecting masses M_1 and M_2 is a rest when the masses are distance d_1 apart, i.e., $p - p_1 = d_1$. The force applied by the spring on M_2 is by spring K_1 is

$$f_1 = -K_1(p - p_1 - d_1) = -K_1(p - g)$$

where $g = p_1 + d_1$. The force applied by spring K_1 on mass M_1 is then $-f_1$. The force applied by the damper on M_1 is

$$f_d = -BDp_1 = -BD(g - d_1) = -BDg.$$

The total force applied to M_1 is $f + f_d - f_1$ and by Newton's law

$$M_1 D^2 p_1 = M_1 D^2 g = f + f_d - f_1 = f - BDg + K_1(p - g).$$

The force applied to M_2 by the spring K_2 is

$$f_2 = -K_2 p$$

because the spring is assumed to be at rest when $p = 0$. The total force applied to M_2 is $f_1 + f_2$ and by Newton's law

$$M_2 D^2 p = f_1 + f_2 = -K_1(p - g) - K_2 p.$$

Rearranging gives

$$-K_1 g = (K_1 + K_2)p + M_2 D^2 p$$

and

$$-K_1(p - g) = M_2 D^2 p + K_2 p.$$

Now,

$$M_1 D^2 g + BDg + M_2 D^2 p + K_2 p = f$$

and so

$$K_1K_2p + B(K_1+K_2)Dp + (M_1K_1 + M_1K_2 + K_1M_2)D^2p + BM_2D^3p + M_1M_2D^4p = K_1 f.$$

In the case that $M_1 = K_1 = K_2 = B = 1$ and $M_2 = 2$ we have

$$p + 2Dp + 4D^2p + 2D^3p + 2D^4p = f.$$

2.3. Consider the electromechanical system in Figure 2.5. A direct current motor is connected to a potentiometer in such a way that the voltage at the output of the potentiometer is equal to the angle of the motor θ. This voltage is fed back to the input terminal of the motor. An input voltage v is applied to the other terminal on the motor. Find the differential equation relating v and θ. What is the input voltage v if the motor angle satisfies $\theta(t) = \frac{\pi}{2}(1 + \mathrm{erf}(t))$? Plot θ and v in this case when the motor coefficients satisfy $L = 0$, $R = \frac{3}{4}$, and $K_b = K_\tau = B = J = 1$.

Solution: The input voltage to the DC motor is $v - \theta$. From (2.4.1) of the lecture notes the relationship between the input voltage and motor angle is

$$v - \theta = \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ}{K_\tau} D^2\theta$$

and so

$$v = \theta + \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ}{K_\tau} D^2\theta.$$

If $\theta(t) = \frac{\pi}{2}(1 + \mathrm{erf}(t))$ then

$$D\theta(t) = \sqrt{\pi}e^{-t^2}, \qquad D^2\theta(t) = -2t\sqrt{\pi}e^{-t^2}$$

and so

$$v(t) = \frac{\pi\,(\mathrm{erf}\,(t) + 1)}{2} - 2\sqrt{\pi}\,t\,e^{-t^2} + 2\sqrt{\pi}\,e^{-t^2}$$

The signals v and θ are plotted in the figure below. Observe that as $t \to \infty$ both $\theta(t)$ and $v(t)$ converge to π.

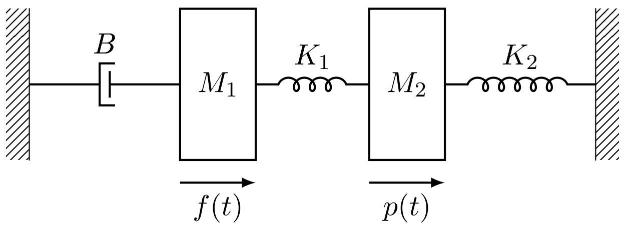


Figure 2.4: Two masses, a spring, and a damper connect between two walls for Exercise 2.2.

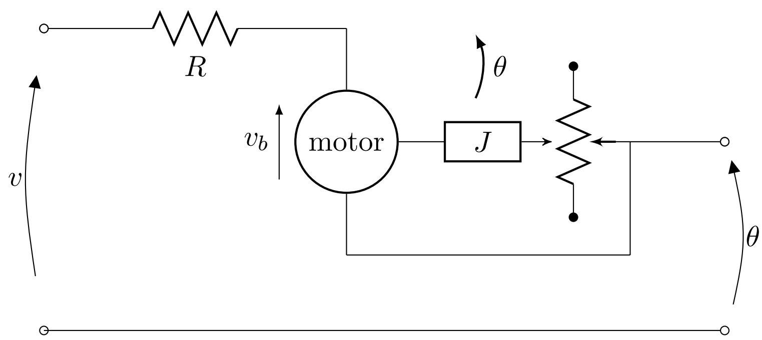


Figure 2.5: Diagram for a rotary direct current (DC) with potentiometer feedback for Exercise 2.3.

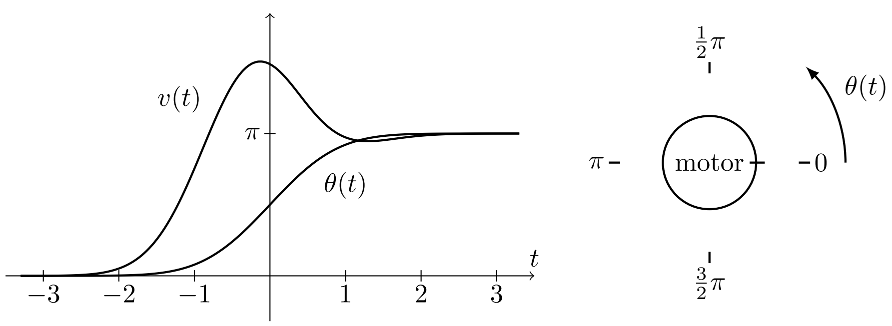


Figure 2.6: Voltage and corresponding angle for the dc motor with potentiometer in Figure 2.5 with constants $L = 0$, $R = \frac{3}{4}$, and$K_b = K_\tau = B = J = 1$.

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