Signals and Systems

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Chapter 1

Signals and systems

A **signal** is a function mapping an input variable to some output variable. For example

$$\sin(\pi t), \qquad \frac{1}{2}t^3, \qquad e^{-t^2}$$

all represent **signals** with real input variable $t \in \mathbb{R}$ and real output variable. These signals are plotted in Figure 1.1. If x is a signal and t an input variable we write $x(t)$ for the output variable corresponding with t. Signals can be multidimensional. This page is an example of a 2-dimensional signal, the independent variables are the horizontal and vertical position on the page, and the signal maps this position to a colour, in this case either black or white. A moving image such as seen on your television or computer monitor is an example of a 3-dimensional signal, the three independent variables being vertical and horizontal screen position and time. The signal maps each position and time to a colour on the screen. In these notes we focus exclusively on 1-dimensional signals such as those in Figure 1.1 and we will only consider signals where the output variable is real or complex valued. Many of the results presented here can be extended to deal with multidimensional signals.

1.1 Properties of signals

A signal x is **bounded** if there exists a real number M such that

$$|x(t)| \leq M \qquad \text{for all } t \in \mathbb{R}$$

where $|\cdot|$ denotes the (complex) magnitude. Both $\sin(\pi t)$ and e^{-t^2} are examples of bounded signals because $|\sin(\pi t)| \leq 1$ and $|e^{-t^2}| \leq 1$ for all $t \in \mathbb{R}$. However, $\frac{1}{2}t^3$ is not bounded because its magnitude grows indefinitely as t moves away from the origin.

A signal x is **periodic** if there exists a positive real number T such that

$$x(t) = x(t + kT) \qquad \text{for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

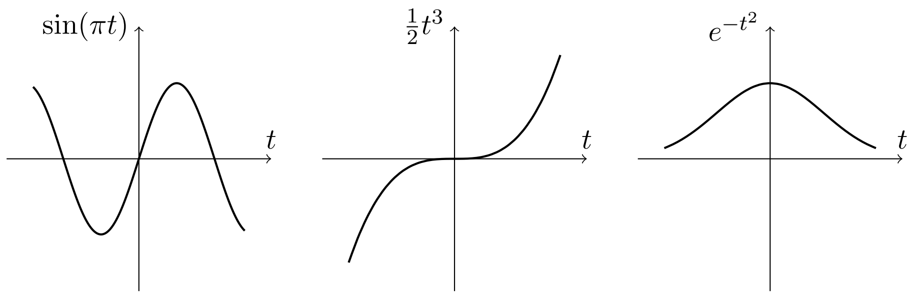

Figure 1.1: 1-dimensional signals

The smallest such positive T is called the **period**. For example, the signal $\sin(\pi t)$ is periodic with period $T = 2$. Neither $\frac{1}{2}t^3$ or e^{-t^2} are periodic.

A signal x is **right sided** if there exists a $T \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T$. Correspondingly x is **left sided** if $x(t) = 0$ for all $T > t$. For example, the **step function**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1.1.1}$$

is right-sided. Its reflection in time $u(-t)$ is left sided (Figure 1.2). A signal x is called **finite in time** if it is both left and right sided, that is, if there exits a $T \in \mathbb{R}$ such that $x(t) = x(-t) = 0$ for all $t > T$. A signal is called **unbounded in time** if it is neither left nor right sided. For example, the signals $\sin(\pi t)$ and e^{-t^2} are unbounded in time, but the **rectangular pulse**

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.1.2}$$

is finite in time.

A signal x is **even** (or **symmetric**) if

$$x(t) = x(-t) \qquad \text{for all } t \in \mathbb{R}$$

and **odd** (or **antisymmetric**) if

$$x(t) = -x(-t) \qquad \text{for all } t \in \mathbb{R}.$$

For example, $\sin(\pi t)$ and $\frac{1}{2}t^3$ are odd and e^{-t^2} is even.

A signal x is **locally integrable** if

$$\int_a^b |x(t)|\, dt < \infty$$

for all finite constants a and b, where by $< \infty$ we mean that the integral evaluates to a finite number. An example of a signal that is not locally

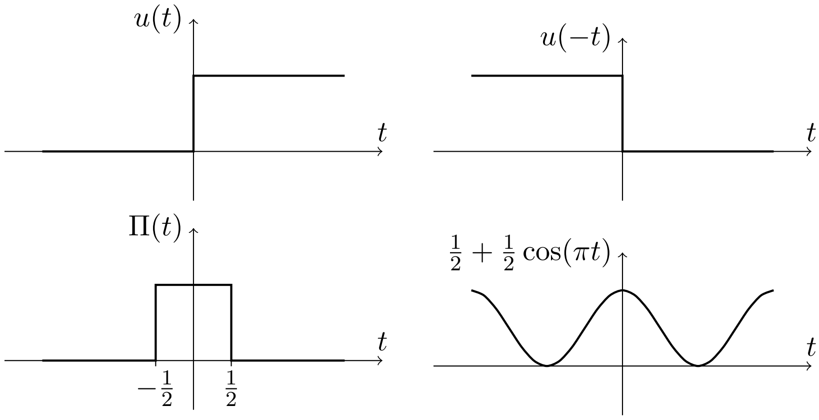

Figure 1.2: The right sided step function $u(t)$, its left sided reflection $u(-t)$, the finite in time rectangular pulse $\Pi(t)$ and the unbounded in time signal $\frac{1}{2} + \frac{1}{2}\cos(x)$.

integrable is $x(t) = \frac{1}{t}$ (Exercise 1.2). A signal x is **absolutely integrable** if

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt < \infty. \tag{1.1.3}$$

Here we introduce the notation $\|x\|_1$ called the **L^1-norm** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not absolutely integrable, but e^{-t^2} is because [Nicholas and Yates, 1950]

$$\int_{-\infty}^{\infty} |e^{-t^2}| dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}. \tag{1.1.4}$$

It is common to denote the set of absolutely integrable signals by L^1 or $L^1(\mathbb{R})$. So, $e^{-t^2} \in L^1$ and $\frac{1}{2}t^3 \notin L^1$. A signal x is called is **square integrable** if

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty.$$

The real number $\|x\|_2$ is called the **L^2-norm** of x. Square integrable signals are also called **energy signals**, and the squared L^2-norm $\|x\|_2^2$ is called the **energy** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not energy signals, but e^{-t^2} is (Exercise 1.5). The set of square integrable signals is often denoted by L^2 or $L^2(\mathbb{R})$.

We write $x = y$ to indicate that two signals x and y are **equal pointwise**, that is, $x(t) = y(t)$ for all $t \in \mathbb{R}$. This definition of equality is often stronger than we desire. For example, the step function u and the signal

$$z(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

are not equal pointwise because they are not equal at $t = 0$, that is, $u(0) = 1$ and $z(0) = 0$. It is useful to identify signals that differ only at isolated points and for this we use a weaker definition of equality. We say that two signals x and y are equal **almost everywhere** if

$$\int_a^b |x(t) - y(t)|\, dt = 0$$

for all finite constants a and b. So, in the previous example, while $u \neq z$ pointwise we do have $u = z$ almost everywhere. Typically the term almost everywhere is abbreviated to a.e. and one writes

$$x = y \;\text{ a.e.} \qquad \text{or} \qquad x(t) = y(t) \;\text{ a.e.}$$

to indicate that the signals x and y are equal almost everywhere.

1.2 Systems (functions of signals)

A **system** is a function that maps a signal to another signal. For example

$$x(t) + 3x(t-1), \qquad \int_0^1 x(t-\tau)d\tau, \qquad \frac{1}{x(t)}, \qquad \frac{d}{dt}x(t)$$

represent systems, each mapping the signal x to another signal. Consider the electric circuit in Figure 1.3 called a **voltage divider**. If the voltage at time t is $x(t)$ then, by Ohm's law, the current at time t satisfies

$$i(t) = \frac{1}{R_1 + R_2} x(t),$$

and the voltage over the resistor R_2 is

$$y(t) = R_2 i(t) = \frac{R_2}{R_1 + R_2} x(t). \tag{1.2.1}$$

The circuit can be considered as a system mapping the signal x representing the voltage to the signal $i = \frac{1}{R_1+R_2}x$ representing the current, or a system mapping x to the signal $y = \frac{R_2}{R_1+R_2}x$ representing the voltage over resistor R_2.

We denote systems with capital letters such as H and G. A system H is a function that maps a signal x to another signal denoted $H(x)$. We call x the **input signal** and $H(x)$ the **output signal** or the **response** of system H to signal x. The value of the signal $H(x)$ at t is denoted by $H(x,t)$ or $H(x,t)$ and we do not distinguish between these notations. It is sometimes useful to depict systems with a block diagram. Figure 1.4 is a simple block diagram showing the input and output signals of a system H.

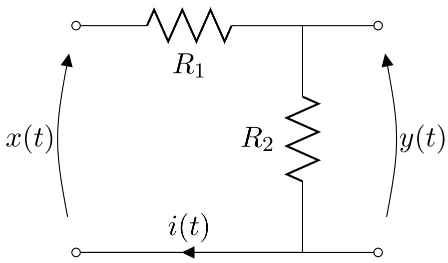

Figure 1.3: A **voltage divider** circuit.

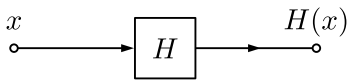

Figure 1.4: System block diagram with input signal x and output signal $H(x)$.

The electric circuit in Figure 1.3 corresponds with the system

$$H(x) = \frac{R_2}{R_1 + R_2} x = y.$$

This system multiplies the input signal x by $\frac{R_2}{R_1+R_2}$. This brings us to our first practical test.

Test 1 (**Voltage divider**) In this test we construct the voltage divider from Figure 1.3 on a breadboard with resistors $R_1 \approx 100\Omega$ and $R_2 \approx 470\Omega$ with values accurate to within 5%. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \sin(2\pi f_1 t) \qquad \text{with} \qquad f_1 = 100$$

is passed through the circuit. The approximation is generated by sampling $x(t)$ at rate $F = \frac{1}{P} = 44100\text{Hz}$ to generate samples

$$x(nP) \qquad n = 0, \dots, 2F$$

corresponding to approximately 2 seconds of signal. These samples are passed to the soundcard which starts playback. The voltage over resistor R_2 is recorded (also using the soundcard) that returns a list of samples $y_1, \dots, y_L$ taken at rate F. The voltage over R_2 can be (approximately) reconstructed from these samples as

$$\tilde{y}(t) = \sum_{\ell=1}^{L} y_\ell \operatorname{sinc}(Ft - \ell) \tag{1.2.2}$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{1.2.3}$$

is the called the **sinc function** and is plotted in Figure 5.1. We will justify this reconstruction in Section 8. Simultaneously the (stereo) soundcard is used to record the input voltage x producing samples $x_1, \dots, x_L$ taken at rate F. An approximation of the input signal is

$$\tilde{x}(t) = \sum_{\ell=1}^{L} x_\ell \,\mathrm{sinc}(Ft - \ell). \tag{1.2.4}$$

In view of (1.2.1) we would expect the approximate relationship

$$\tilde{y} \approx \frac{R_2}{R_1 + R_2}\tilde{x} = \frac{42}{57}\tilde{x}.$$

A plot of $\tilde{y}$, $\tilde{x}$ and $\frac{42}{57}\tilde{x}$ over a 20ms period from 1s to 1.02s is given in Figure 1.5. The hypothesised output signal $\frac{42}{57}\tilde{x}$ does not match the observed output signal $\tilde{y}$. A primary reason is that the circuitry inside the soundcard itself cannot be ignored. When deriving the equation for the voltage divider we implicitly assumed that current flows through the output of the soundcard without resistance (a short circuit), and that no current flows through the input device of the soundcard (an open circuit). These assumptions are not realistic. Modelling the circuitry in the sound card wont be attempted here. In Section 2.2 we will construct circuits that contain external sources of power (active circuits). These are less sensitive to the circuitry inside the soundcard.

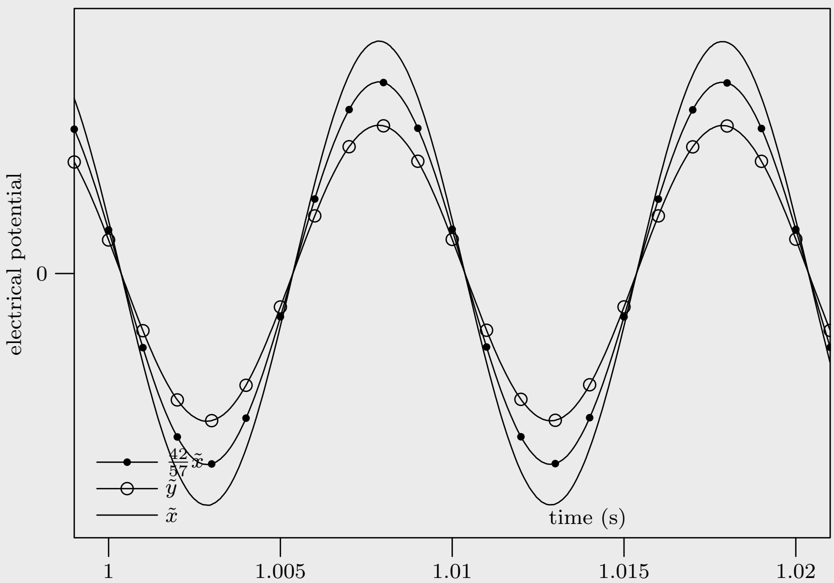

Figure 1.5: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $\frac{42}{57}\tilde{x}$ (solid line with dot) for the voltage divider circuit in Figure 1.3. The hypothesised signal does not match $\tilde{y}$. One reason is that the model does not take account of the circuitry inside the soundcard.

Not all signals can be input to all systems. For example, the system

$$H(x,t) = \frac{1}{x(t)}$$

is not defined at those t where $x(t) = 0$ because we cannot divide by zero. Another example is the system

$$I_\infty(x,t) = \int_{-\infty}^{t} x(\tau)d\tau, \tag{1.2.5}$$

called an **integrator**. The signal $x(t) = 1$ cannot be input to the integrator because the integral $\int_{-\infty}^{t} dt$ is not finite for any t.

When specifying a system it is necessary to also specify a set of signals that can be input. This is called a **domain** for the system. We are free to choose the domain at our convenience. For example, a domain for the system $H(x,t) = \frac{1}{x(t)}$ is a the set of signals $x(t)$ which are not zero for any t. An example of a domain for the integrator I_∞ is the set L^1 of absolutely integrable signals because, if x is absolutely integrable, then

$$|I_\infty(x,t)| \leq \left|\int_{-\infty}^{t} x(\tau)d\tau\right| \leq \int_{-\infty}^{t} |x(\tau)|\, d\tau < \int_{-\infty}^{\infty} |x(\tau)|\, d\tau = \|x\|_1 < \infty$$

and so, $I_\infty(x,t)$ is finite for all t. In this text, the domain used for a given system will usually be obvious from the context in which the system is defined. For this reason we will not usually state the domain explicitly. We will only do so if there is chance for confusion.

1.3 Some important systems

The system

$$T_\tau(x,t) = x(t-\tau)$$

is called a **time-shifter**. This system shifts the input signal along the t axis ('time' axis) by τ. When τ is positive T_τ delays the input signal by τ. The time-shifter will appear so regularly in this course that we use the special notation T_τ to represent it. Figure 1.6 depicts the action of time-shifters

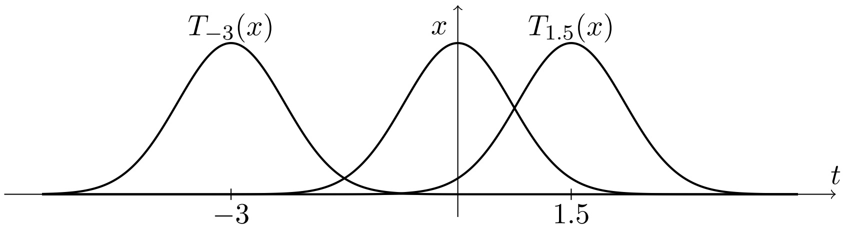

Figure 1.6: Time-shifter system $T_{1.5}(x,t) = x(t-1.5)$ and $T_{-3}(x,t) = x(t+3)$ acting on the signal $x(t) = e^{-t^2}$.

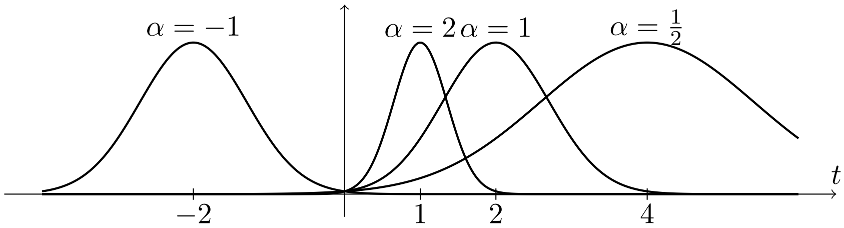

Figure 1.7: Time-scaler system $H(x,t) = x(\alpha t)$ for $\alpha = -1, \frac{1}{2}, 1$ and 2 acting on the signal $x(t) = e^{-(t-2)^2}$.

$T_{1.5}$ and T_{-3} on the signal $x(t) = e^{-t^2}$. When $\tau = 0$ the time-shifter is the **identity system**

$$T_0(x) = x$$

that maps the signal x to itself.

Another important system is the **time-scaler** that has the form

$$H(x,t) = x(\alpha t)$$

for $\alpha \in \mathbb{R}$. Figure 1.7 depicts the action of a time-scaler with a number of values for α. When $\alpha = -1$ the time-scaler reflects the input signal in the time axis. When $\alpha = 1$ the time-scaler is the identity system T_0.

Another system we regularly encounter is the **differentiator**

$$D(x,t) = \frac{d}{dt}x(t),$$

that returns the derivative of the input signal. We also define a kth differentiator

$$D^k(x,t) = \frac{d^k}{dt^k}x(t)$$

that returns the kth derivative of the input signal.

A related system is the **integrator**

$$I_a(x,t) = \int_{-a}^{t} x(\tau)d\tau.$$

The parameter a describes the lower bound of the integral. In this course it will often be that $a = \infty$. For example, the response of the integrator I_∞ to the signal $tu(t)$ where u is the step function (1.1.1) is

$$\int_{-\infty}^{t} \tau u(\tau) d\tau = \begin{cases} \int_0^t \tau d\tau = \frac{t^2}{2} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Observe that the integrator I_∞ cannot be applied to the signal $x(t) = t$ because $\int_{-\infty}^{t} \tau d\tau$ is not finite for any t. A domain for I_∞ would not contain the signal $x(t) = t$.

1.4 Properties of systems

In this section we define a number of important properties that systems can possess. In what follows H will be a system and the phrase "for all signals" will mean for all signals inside some domain for H.

A system H is called **memoryless** if the output signal $H(x)$ at time t depends only on the input signal x at time t. For example $\frac{1}{x(t)}$ and the identity system T_0 are memoryless, but

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau) d\tau$$

are not. A time-shifter T_τ with $\tau \neq 0$ is not memoryless.

A system H is **causal** if the output signal $H(x)$ at time t depends on the input signal only at times less than or equal to t. Memoryless systems such as $\frac{1}{x(t)}$ and T_0 are also causal. Time-shifters T_τ are causal when $\tau \geq 0$, but are not causal when $\tau < 0$. The systems

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau) d\tau$$

are causal, but the systems

$$x(t) + 3x(t+1) \qquad \text{and} \qquad \int_0^1 x(t+\tau) d\tau$$

are not causal.

A system H is called **bounded-input-bounded-output (BIBO) stable** or just **stable** if the output signal $H(x)$ is bounded whenever the input signal x is bounded. That is, H is stable if for every positive real number M there exists a positive real number K such that for all signals x satisfying

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R},$$

it also holds that

$$|H(x,t)| < K \qquad \text{for all } t \in \mathbb{R}.$$

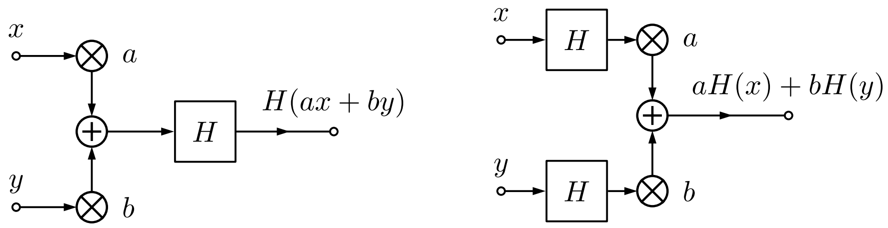

Figure 1.8: If H is a linear system the outputs of these two diagrams are the same signal, i.e. $H(ax+by) = aH(x) + bH(y)$.

For example, the system $x(t) + 3x(t-1)$ is stable with $K = 4M$ since if $|x(t)| < M$ then

$$|x(t) + 3x(t-1)| \leq |x(t)| + 3\,|x(t-1)| < 4M = K.$$

The integrator I_a for any $a \in \mathbb{R}$ and differentiator D are not stable (Exercises 1.6 and 1.7).

A system H is **linear** if

$$H(ax+by) = aH(x) + bH(y)$$

for all signals x and y and all complex numbers a and b. That is, a linear system has the property: If the input consists of a weighted sum of signals, then the output consists of the same weighted sum of the responses of the system to those signals. Figure 1.8 indicates the linearity property using a block diagram. For example, the differentiator is linear because

$$\begin{aligned} D(ax+by,t) &= \frac{d}{dt}\big(ax(t) + by(t)\big) \\ &= a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) \\ &= aD(x,t) + bD(y,t) \end{aligned}$$

whenever both x and y are differentiable. However, the system $H(x,t) = \frac{1}{x(t)}$ is not linear because

$$H(ax+by,t) = \frac{1}{ax(t)+by(t)} \neq \frac{a}{x(t)} + \frac{b}{y(t)} = aH(x,t) + bH(y,t)$$

in general.

The property of linearity trivially generalises to more than two signals. For example, if $x_1, \dots, x_k$ are signals and $a_1, \dots, a_k$ are complex numbers for some finite k, then

$$H(a_1x_1 + \cdots + a_kx_k) = a_1H(x_1) + \cdots + a_kH(x_k).$$

A system H is **time-invariant** if

$$H\big(T_\tau(x)\big)\big(t\big) = H(x, t-\tau)$$

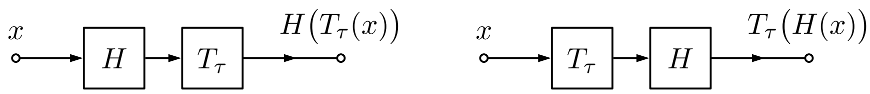

Figure 1.9: If H is a time-invariant system the outputs of these two diagrams are the same signal, i.e. $H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$.

for all signals x and all time-shifts $\tau \in \mathbb{R}$. That is, a system is time-invariant if time-shifting the input signal results in the same time-shift of the output signal. Equivalently, H is time-invariant if it commutes with the time-shifter T_τ, that is, if

$$H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$$

for all $\tau \in \mathbb{R}$ and all signals x. Figure 1.9 represents the property of time-invariance with a block diagram.

1.5 Exercises

1.1. State whether the step function $u(t)$ is bounded, periodic, absolutely integrable, an energy signal.

1.2. Show that the signal t^2 is locally integrable, but that the signal $\frac{1}{t^2}$ is not.

1.3. Plot the signal

$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \leq 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

1.4. Plot the signal

$$x(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that x is absolutely integrable, but not square integrable.

1.5. Compute the energy of the signal $e^{-\alpha^2 t^2}$ (Hint: use equation (1.1.4) on page 3 and a change of variables).

1.6. Show that the integrator I_a for any $a \in \mathbb{R}$ is not stable.

1.7. Show that the differentiator system D is not stable.

1.8. Show that the time-shifter is linear and time-invariant, and that the time-scaler is linear, but not time invariant

1.9. Show that the integrator I_c with c finite is linear, but not time-invariant.

1.10. Show that the integrator I_∞ is linear and time invariant.

1.11. State whether the system $H(x,t) = x(t) + 1$ is linear, time-invariant, stable.

1.12. State whether the system $H(x,t) = 0$ is linear, time-invariant, stable.

1.13. State whether the system $H(x,t) = 1$ is linear, time-invariant, stable.

1.14. Let x be a signal with period T that is not equal to zero almost everywhere. Show that x is not absolutely integrable.

Chapter 2

Systems modelled by differential equations

Systems of particular interest in this text are those where the input signal x and output signal y are related by a linear differential equation with constant coefficients, that is, an equation of the form

$$\sum_{\ell=0}^{m} a_\ell \frac{d^\ell}{dt^\ell} x(t) = \sum_{\ell=0}^{k} b_\ell \frac{d^\ell}{dt^\ell} y(t),$$

where $a_0, \dots, a_m$ and $b_0, \dots, b_k$ are real or complex numbers. In what follows we use the differentiator system D rather than the notation $\frac{d}{dt}$ to represent differentiation. To represent the ℓth derivative we write D^ℓ instead of $\frac{d^\ell}{dt^\ell}$. Using this notation the differential equation above is

$$\sum_{\ell=0}^{m} a_\ell D^\ell(x) = \sum_{\ell=0}^{k} b_\ell D^\ell(y). \tag{2.0.1}$$

Equations of this form can be used to model a large number of electrical, mechanical and other real world devices. For example, consider the resistor and capacitor (RC) circuit in Figure 2.1. Let the signal v_R represent the voltage over the resistor and i the current through both resistor and capacitor. The voltage signals satisfy

$$x = y + v_R,$$

and the current satisfies both

$$v_R = Ri \qquad \text{and} \qquad i = CD(y).$$

Combining these equations,

$$x = y + RCD(y) \tag{2.0.2}$$

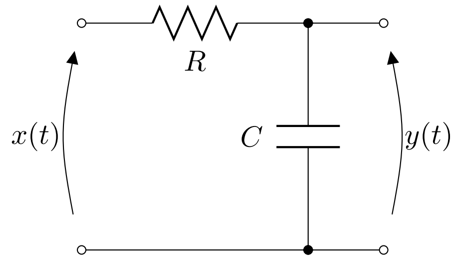

Figure 2.1: An electrical circuit with resistor and capacitor in series, otherwise known as an **RC circuit**.

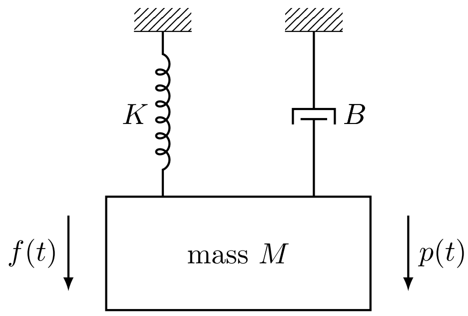

Figure 2.2: A mechanical mass-spring-damper system

that is in the form of (2.0.1).

As another example, consider the mass-spring-damper in Figure 2.2. A force represented by the signal f is externally applied to the mass, and the position of the mass is represented by the signal p. The spring exerts force $-Kp$ that is proportional to the position of the mass, and the damper exerts force $-BD(p)$ that is proportional to the velocity of the mass. The cumulative force exerted on the mass is

$$f_m = f - Kp - BD(p)$$

and by Newton's law the acceleration of the mass $D^2(p)$ satisfies

$$MD^2(p) = f_m = f - Kp - BD(p).$$

We obtain the differential equation

$$f = Kp + BD(p) + MD^2(p) \tag{2.0.3}$$

that is in the form of (2.0.1) if we put $x = f$ and $y = p$. Given p we can readily solve for the corresponding force f. As a concrete example, let the spring constant, damping constant and mass be $K = B = M = 1$. If the position satisfies $p(t) = e^{-t^2}$, then the corresponding force satisfies

$$f(t) = e^{-t^2}(4t^2 - 2t - 1).$$

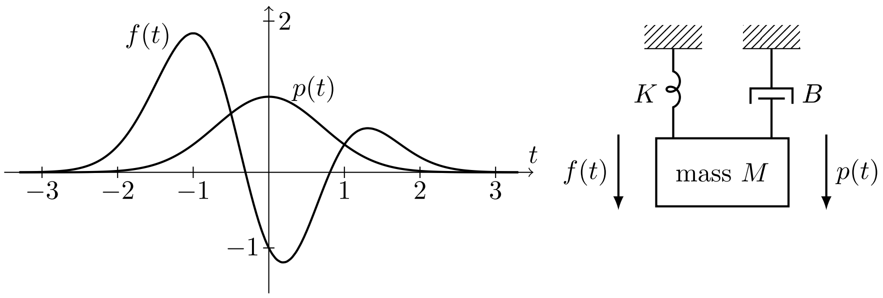

Figure 2.3: A solution to the mass-spring-damper system with $K = B = M = 1$. The position is $p(t) = e^{-t^2}$ with corresponding force $f(t) = e^{-t^2}(4t^2 - 2t - 1)$.

Figure 2.3 depicts these signals.

What happens if a particular force signal f is applied to the mass? For example, say we apply the force

$$f(t) = \Pi(t - \tfrac{1}{2}) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the corresponding position signal p? We are not yet ready to answer this question, but will be later (Exercise 4.11).

In both the mechanical mass-spring-damper system in Figure 2.2 and the electrical RC circuit in Figure 2.1 we obtain a differential equation relating the input signal x with the output signal y. The equations do not specify the output signal y explicitly in terms of the input signal x, that is, they do not explicitly define a system H such $y = H(x)$. As they are, the differential equations do not provide as much information about the behaviour of the system as we would like. For example, is the system stable? We will be able to obtain much more information about these systems when the **Laplace transform** is introduced in Chapter 4. The remainder of this chapter details the construction of differential equations that model various mechanical, electrical, and electro-mechanical systems. We will use the systems constructed here as examples throughout the course.

2.1 Passive electrical circuits

Passive electrical circuits require no sources of power other than the input signal itself. For example, the voltage divider in Figure 1.3 and the RC circuit in Figure 2.1 are passive circuits. Another common passive electrical circuit is the resistor, capacitor and inductor (RLC) circuit depicted in Figure 2.4. In this circuit we let the output signal y be the voltage over the resistor. Let v_C represent the voltage over the capacitor and v_L the voltage over the inductor and let i be the current. We have

$$y = Ri, \qquad i = CD(v_C), \qquad v_L = LD(i),$$

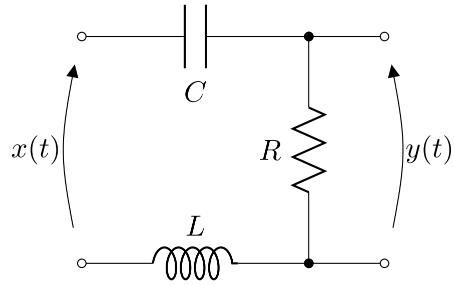

Figure 2.4: An electrical circuit with resistor, capacitor and inductor in series, otherwise known as an **RLC circuit**.

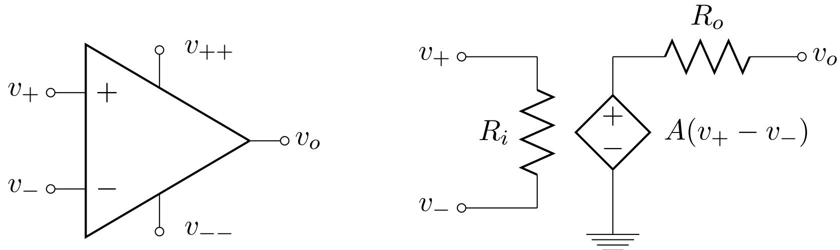

Figure 2.5: Left: triangular component diagram of an **operational amplifier**. The v_{++} and v_{--} connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. Right: model for an operational amplifier including input resistance R_i, output resistance R_o, and open loop gain A. The diamond shaped component is a dependent voltage source. This model is usually only useful when the operational amplifier is in a negative feedback circuit.

leading to the following relationships between y, v_C and v_L,

$$y = RCD(v_C), \qquad Rv_L = LD(y).$$

Kirchhoff's voltage law gives $x = y + v_C + v_L$ and by differentiating both sides

$$D(x) = D(y) + D(v_C) + D(v_L).$$

Substituting the equations relating y, v_C and v_L leads to

$$RCD(x) = y + RCD(y) + LCD^2(y). \tag{2.1.1}$$

We can similarly find equations relating the input voltage with v_C and v_L.

2.2 Active electrical circuits

Unlike passive electrical circuits, an **active electrical circuit** requires a source of power external to the input signal. Active circuits can be modelled and constructed using **operational amplifiers** as depicted in Figure 2.5. The left hand side of Figure 2.5 shows a triangular circuit diagram for an operational amplifier, and the right hand side of Figure 2.5 shows a circuit that can be used to model the behaviour of the amplifier. The v_{++} and v_{--} connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. The diamond shaped component is a dependent voltage source with voltage $A(v_+ - v_-)$ that depends on the difference between the voltage at the **non-inverting input** v_+ and the voltage at the **inverting input** v_-. The dimensionless constant A is called the **open loop gain**. Most operational amplifiers have large open loop gain A, large **input resistance** R_i and small **output resistance** R_o. As we will see, it can be convenient to consider the behaviour as $A \to \infty$, $R_i \to \infty$ and $R_0 \to 0$, resulting in an **ideal operational amplifier**.

As an example, an operational amplifier configured as a **multiplier** is depicted in Figure 2.6. This circuit is an example of an operational amplifier configured with **negative feedback**, meaning that the output of the amplifier is connected (in this case by a resistor) to the inverting input v_-. The horizontal wire at the bottom of the plot is considered to be ground (zero volts) and is connected to the negative terminal of the dependent voltage source of the operational amplifier depicted in Figure 2.5. An equivalent circuit for the multiplier using the model in Figure 2.5 is shown in Figure 2.7. Solving this circuit (Exercise 2.1) yields the following relationship between the input voltage signal x and the output voltage signal y,

$$y = \frac{R_i(AR_2 + R_o)}{R_i(R_2 + R_o) + R_1(R_2 + R_i - AR_i + R_o)}x. \tag{2.2.1}$$

For an ideal operational amplifier we let $A \to \infty$, $R_i \to \infty$ and $R_o \to 0$. In this case terms involving the product AR_i dominate and we are left with the simpler equation

$$y = -\frac{R_2}{R_1}x. \tag{2.2.2}$$

Thus, assuming an ideal operational amplifier, the circuit acts as a multiplier with constant $-\frac{R_2}{R_1}$.

The equation relating x and y is much simpler for the ideal operational amplifier. Fortunately this equation can be obtained directly using the following two rules:

1. the voltage at the inverting and non-inverting inputs are equal,
2. no current flows through the inverting and non-inverting inputs.

These rules are only useful for analysing circuits with negative feedback. Let us now rederive (2.2.2) using these rules. Because the non-inverting input is connected to ground, the voltage at the inverting input is zero. So, the voltage over resistor R_2 is $y = R_2 i$. Because no current flows through the inverting input the current through R_1 is also i and $x = -R_1 i$. Combining these results, the input voltage x and the output voltage y are related by

$$y = -\frac{R_2}{R_1}x.$$

In Test 2 the inverting amplifier circuit is constructed and the relationship above is tested using a computer soundcard.

We now consider another circuit consisting of an operational amplifier, two resistors and two capacitors depicted in Figure 2.8. Assuming an ideal operational amplifier, the voltage at the inverting terminal is zero because the non-inverting terminal is connected to ground. Thus, the voltage over capacitor C_2 and resistor R_2 is equal to y and, by Kirchoff's current law,

$$i = \frac{y}{R_2} + C_2 D(y).$$

Similarly, since no current flows through the inverting terminal,

$$i = -\frac{x}{R_1} - C_1 D(x).$$

Combining these equations yields

$$-\frac{x}{R_1} - C_1 D(x) = \frac{y}{R_2} + C_2 D(y). \tag{2.2.3}$$

Observe the similarity between this equation and that for the passive RC circuit (2.0.2) when $R_1 = R_2$ and $C_1 = 0$ (an open circuit). In this case

$$x = -y - R_1 C_2 D(y). \tag{2.2.4}$$

We call this the **active RC circuit**. This circuit is tested in Test 3.

Consider the circuit in Figure 2.9. Assuming an ideal operational amplifier, the input voltage x satisfies

$$-i = \frac{x}{R_1} + C_1 D(x).$$

The voltage over the capacitor C_2 is $y - R_2 i$ and so the current satisfies

$$i = C_2 D(y - R_2 i).$$

Combining these equations gives

$$-\frac{x}{R_1} - C_1 D(x) = C_2 D(y) + \frac{R_2 C_2}{R_1} D(x) + R_2 C_2 C_1 D^2(x),$$

and after rearranging,

$$D(y) = -\frac{1}{R_1 C_1} x - \left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right) D(x) - R_2 C_1 D^2(x).$$

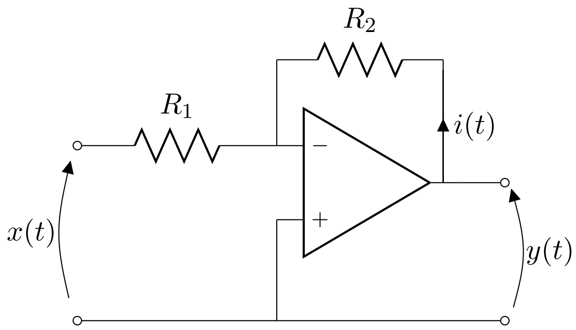

Figure 2.6: Inverting amplifier

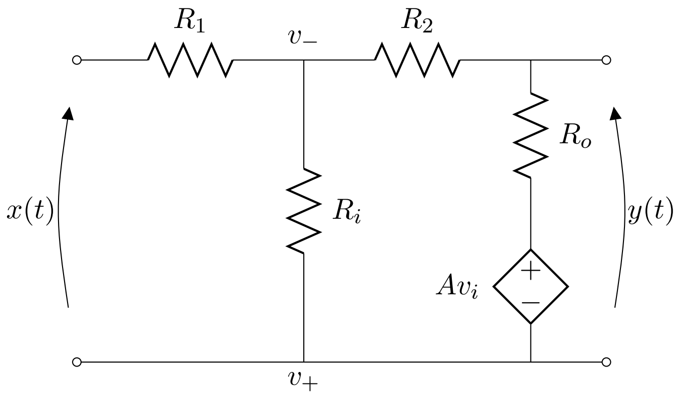

Figure 2.7: An equivalent circuit for the inverting amplifier from Figure 2.6 using the model for an operational amplifier in Figure 2.5. The symbol $v_i = v_+ - v_-$ is the voltage over resistor R_i.

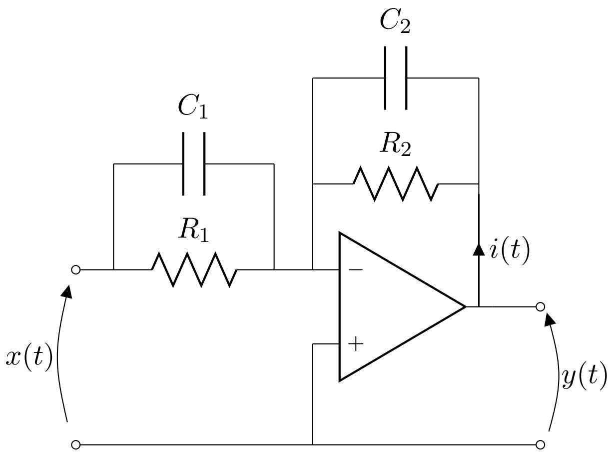

Figure 2.8: Operational amplifier configured with two capacitors and two resistors.

Test 2 (**Inverting amplifier**) In this test we construct the inverting amplifier circuit from Figure 2.6 with $R_2 \approx 22\text{k}\Omega$ and $R_1 \approx 12\text{k}\Omega$ that are accurate to within 5% of these values. The operational amplifier used is the Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \tfrac{1}{3}\sin(2\pi f_1 t) + \tfrac{1}{3}\sin(2\pi f_2 t)$$

with $f_1 = 100$ and $f_2 = 233$ is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal x and the output signal y. Approximate reconstructions of the input signal $\tilde{x}$ and output signal $\tilde{y}$ are given according to (1.2.4) and (1.2.2). According to (2.1.1) we expect the approximate relationship

$$\tilde{y} \approx -\frac{R_2}{R_1}\tilde{x} = -\frac{11}{6}\tilde{x}.$$

Each of $\tilde{y}$, $\tilde{x}$ and $-\frac{11}{6}\tilde{x}$ are plotted in Figure 2.9.

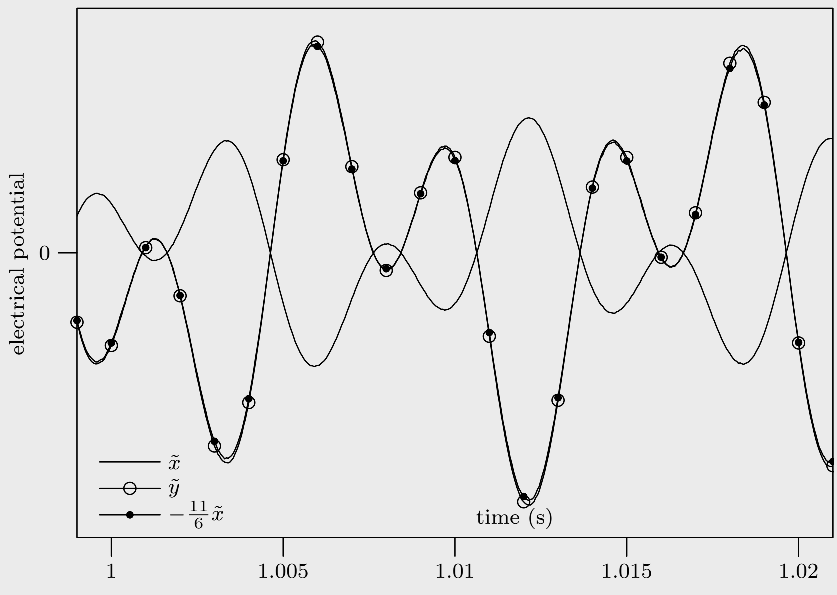

Figure 2.9: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $-\frac{11}{6}\tilde{x}$ (solid line with dot).

Test 3 (**Active RC circuit**) In this test we construct the circuit from Figure 2.8 with $R_1 \approx R_2 \approx 27\text{k}\Omega$ and $C_2 \approx 10\text{nF}$ accurate to within 5% of these values and $C_1 = 0$ (an open circuit). The operational amplifier used is a Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \tfrac{1}{3}\sin(2\pi f_1 t) + \tfrac{1}{3}\sin(2\pi f_2 t)$$

with $f_1 = 500$ and $f_2 = 1333$ is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal x and the output signal y and approximate reconstructions $\tilde{x}$ and $\tilde{y}$ are given according to (1.2.4) and (1.2.2). According to (2.2.4) we expect the approximate relationship

$$\tilde{x} \approx -\frac{R_1}{R_2}\tilde{y} - R_1 C D(\tilde{y}) = -\tilde{y} - \frac{27}{10000}D(\tilde{y}).$$

The derivative of the sinc function is

$$D(\text{sinc}, t) = \frac{1}{\pi t^2}\big(\pi t\cos(\pi t) - \sin(\pi t)\big), \tag{2.2.5}$$

and so,

$$D(\tilde{y}, t) = \frac{d}{dt}\left(\sum_{\ell=1}^{L} y_\ell \,\text{sinc}(Ft - \ell)\right) = F\sum_{\ell=1}^{L} y_\ell D(\text{sinc}, Ft - \ell). \tag{2.2.6}$$

Each of $\tilde{y}$, $\tilde{x}$ and $-\tilde{y} - \frac{27}{10000}D(\tilde{y})$ are plotted in Figure 2.9.

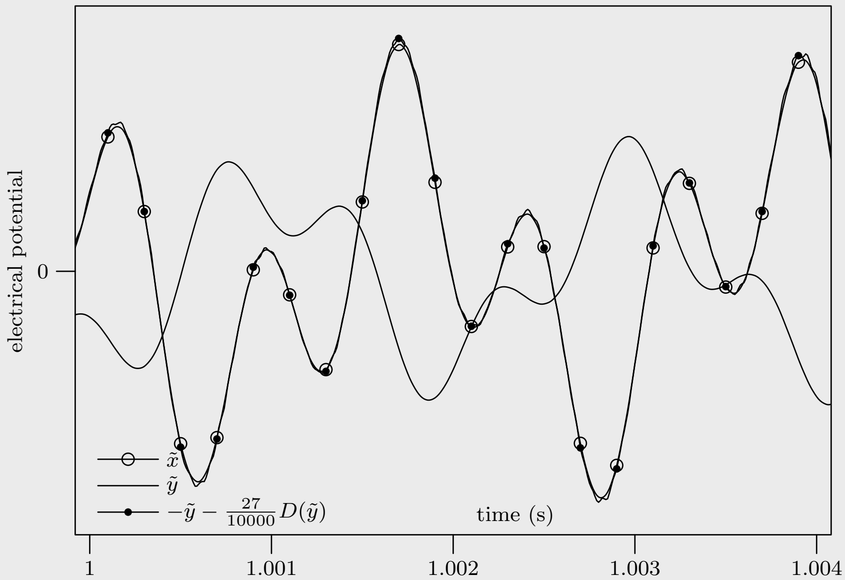

Figure 2.9: Plot of reconstructed input signal $\tilde{x}$ (solid line with circle), output signal $\tilde{y}$ (solid line), and hypothesised input signal $-\tilde{y} - \frac{27}{10000}D(\tilde{y})$ (solid line with dot).

Put
$$K_i = \frac{1}{R_1 C_2}, \qquad K_p = \frac{R_2}{R_1} + \frac{C_1}{C_2}, \qquad K_d = R_2 C_1$$
and now
$$D(y) = -K_i x - K_p D(x) - K_d D^2(x). \tag{2.2.7}$$
This equation models what is called a **proportional-integral-derivative controller** or **PID controller**. The coefficients K_i, K_p and K_d are called the **integral gain**, **proportional gain**, and **derivative gain**.

The final active circuit we consider is called a **Sallen-Key** [Sallen and Key, 1955] and is depicted in Figure 2.10. Observe that the output of the amplifier is connected directly to the inverting input and is also connected to the noninverting input by a capacitor and resistor. This circuit has both negative *and* positive feedback. It is not immediately apparent that we can use the simplifying assumptions for an ideal operational amplifier with negative feedback. However, we will do so, and will find that it works in this case.

Let v_{R1}, v_{R2}, v_{C1}, and v_{C2} be the voltages over the components R_1, R_2, C_1, and C_2. Kirchoff's voltage law leads to the equations
$$x = v_{R1} + v_{R2} + v_{C2}, \qquad y = v_{C1} + v_{R2} + v_{C2}.$$
The voltage at the inverting and noninverting terminals is y and so the voltage over the capacitor C_2 is y, that is, $y = v_{C2}$. Using this, the equations above simplify to
$$x = v_{R1} + v_{R2} + y, \qquad v_{C1} = -v_{R2}.$$
The current i_2 through capacitor C_2 satisfies $i_2 = C_2 D(v_{C2}) = C_2 D(y)$. Because no current flows into the inverting terminal of the amplifier the current through R_2 is also i_2 and so $v_{R2} = R_2 i_2 = R_2 C_2 D(y)$. Substituting this into the equations above gives
$$x = v_{R1} + R_2 C_2 D(y) + y, \qquad v_{C1} = -R_2 C_2 D(y). \tag{2.2.8}$$
Kirchoff's current law asserts that $i + i_1 = i_2$. The current i through capacitor C_1 satisfies $i = C_1 D(v_{C1}) = -R_2 C_1 C_2 D^2(y)$ and the current through resistor R_1 satisfies
$$v_{R1} = R_1 i_1 = R_1(i_2 - i) = R_1 C_2 D(y) + R_1 R_2 C_1 C_2 D^2(y).$$
Substituting this into the equation on the left of (2.2.8) gives
$$x = y + C_2(R_1 + R_2)D(y) + R_1 R_2 C_1 C_2 D^2(y). \tag{2.2.9}$$
The Sallen-Key will be useful when we consider the design of analogue electrical filters in Section 5.3.

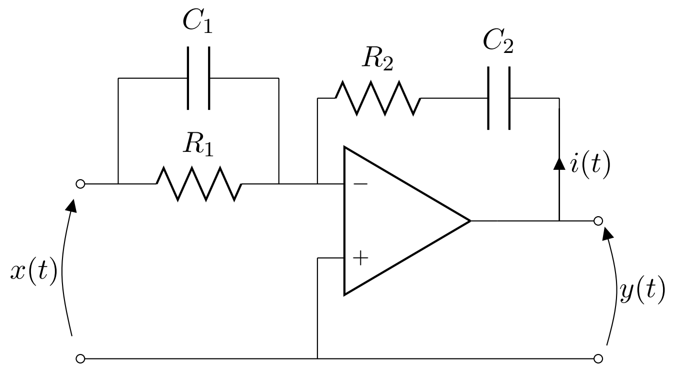

Figure 2.9: Operational amplifier implementing a **proportional-integral-derivative controller**.

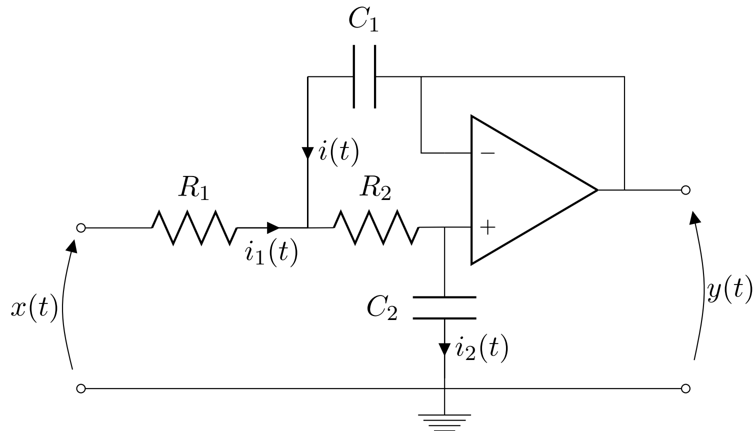

Figure 2.10: Operational amplifier implementing a **Sallen-Key**.

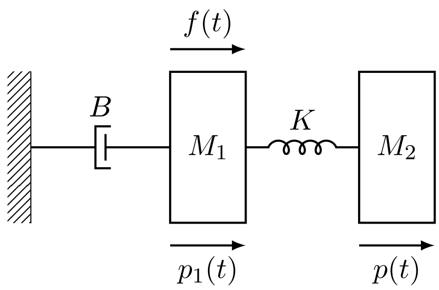

Figure 2.11: Two masses, a spring and a damper

2.3 Masses, springs, and dampers

A mechanical mass-spring-damper system was described in Section 2 and Figure 2.2. We now consider another mechanical system involving a different configuration of masses, a spring and a damper depicted in Figure 2.11. A mass M_1 is connected to a wall by a damper with constant B, and to another mass M_2 by a spring with constant K. A force represented by the signal f is applied to the first mass. We will derive a differential equation relating f with the position p of the second mass. Assume that the spring applies no force (is in equilibrium) when the masses are distance d apart. The forces due to the spring satisfy

$$f_{s1} = -f_{s2} = K(p - p_1 - d)$$

where f_{s1} and f_{s2} are signals representing the force due to the spring on mass M_1 and M_2 respectively. It is convenient to define the signal $g(t) = p_1(t) + d$ so that forces due to spring satisfy the simpler equation

$$f_{s1} = -f_{s2} = K(p - g).$$

The only force applied to M_2 is by the spring and so, by Newton's law, the acceleration of M_2 satisfies

$$M_2 D^2(p) = f_{s2}.$$

Substituting this into the previous equation gives a differential equation relating g and p,

$$Kg = Kp + M_2 D^2(p). \tag{2.3.1}$$

The force applied by the damper on mass M_1 is given by the signal

$$f_d = -BD(p_1) = -BD(g)$$

where the replacement of p_1 by g is justified because differentiation will remove the constant d. The cumulative force on M_1 is given by the signal

$$\begin{aligned} f_1 &= f + f_d + f_{s1} \\ &= f - Kg + Kp - BD(g), \end{aligned} \tag{2.3.2}$$

and by Newton's law the acceleration of M_1 satisfies

$$M_1 D^2(p_1) = M_1 D^2(g) = f_1.$$

Substituting this into (2.3.2) and using (2.3.1) we obtain a fourth order differential equation relating p and f,

$$f = BD(p) + (M_1 + M_2)D^2(p) - \frac{BM_2}{K} D^3(p) + \frac{M_1 M_2}{K} D^4(p). \tag{2.3.3}$$

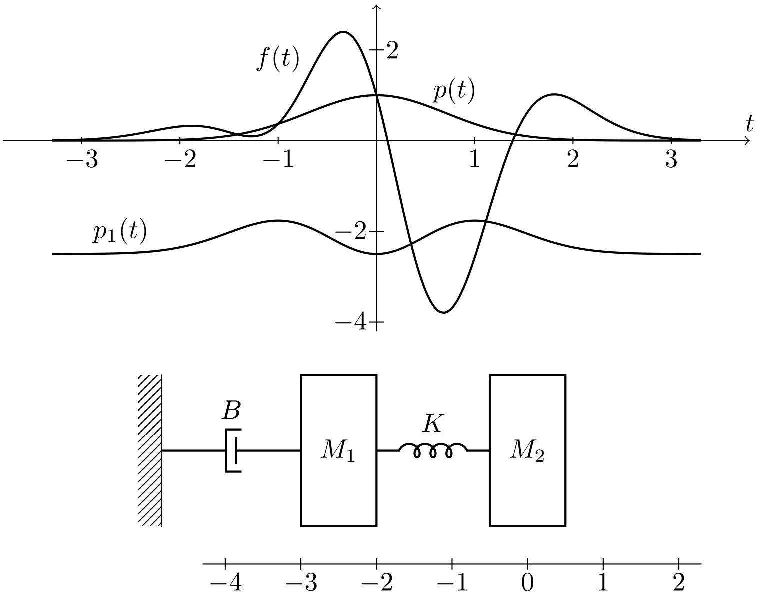

Figure 2.12: Solution of the system describing two masses with a spring and damper where $B = K = 1$ and $M_1 = M_2 = \frac{1}{2}$ and the position of the second mass is $p(t) = e^{-t^2}$.

Given the position of the second mass p we can readily solve for the corresponding force f and position of the first mass p. For example, if the constants $B = K = 1$ and $M_1 = M_2 = \frac{1}{2}$ and $d = \frac{5}{2}$, and if the position of the second mass satisfies

$$p(t) = e^{-t^2}$$

then, by application of (2.3.3) and (2.3.1),

$$f(t) = e^{-t^2}(1 - 8t - 8t^2 + 4t^3 + 4t^4), \qquad \text{and} \qquad p_1(t) = 2e^{-t^2}t^2 - \tfrac{5}{2}.$$

This solution is plotted in Figure 2.12.

2.4 Direct current motors

Direct current (DC) motors convert electrical energy, in the form of a voltage, into rotary kinetic energy [Nise, 2007, page 76]. We derive a differential equation relating the input voltage v to the angular position of the motor θ. Figure 2.13 depicts the components of a DC motor.

The voltages over the resistor and inductor satisfy

$$v_R = Ri, \qquad v_L = LD(i),$$

and the motion of the motor induces a voltage called the **back electromotive force** (EMF),

$$v_b = K_bD(\theta)$$

that we model as being proportional to the angular velocity of the motor. The input voltage now satisfies

$$v = v_R + v_L + v_b = Ri + LD(i) + K_b D(\theta).$$

The torque τ applied by the motor is modelled as being proportional to the current i,

$$\tau = K_\tau i.$$

A load with inertia J is attached to the motor. Two forces are assumed to act on the load, the torque τ applied by the current, and a torque $\tau_d = -BD(\theta)$ modelling a damper that acts proportionally against the angular velocity of the motor. By Newton's law, the angular acceleration of the load satisfies

$$JD^2(\theta) = \tau + \tau_d = K_\tau i - BD(\theta).$$

Combining these equations we obtain the 3rd order differential equation

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D(\theta) + \frac{RJ + LB}{K_\tau} D^2(\theta) + \frac{LJ}{K_\tau} D^3(\theta)$$

relating voltage and motor position. In many DC motors the inductance L is small and can be ignored, leaving the simpler second order equation

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D(\theta) + \frac{RJ}{K_\tau} D^2(\theta). \qquad (2.4.1)$$

Given the position signal θ we can find the corresponding voltage signal v. For example, put the constants $K_b = K_\tau = B = R = J = 1$ and assume that

$$\theta(t) = 2\pi(1 + \operatorname{erf}(t))$$

where $\operatorname{erf}(t) = \frac{2}{\pi}\int_{-\infty}^{t} e^{-\tau^2} d\tau$ is the **error function**. The corresponding angular velocity $D(\theta)$ and voltage v satisfy

$$D(\theta, t) = 4\sqrt{\pi} e^{-t^2}, \qquad v(t) = 8\sqrt{\pi} e^{-t^2}(1 - t).$$

These signals are depicted in Figure 2.14. This voltage signal is sufficient to make the motor perform two revolutions and then come to rest.

2.5 Exercises

2.1. Analyse the inverting amplifier circuit in Figure 2.7 to obtain the relationship between input voltage x and output voltage y given by (2.2.1). You may wish to use a symbolic programming language (for example Sage, Mathematica, or Maple).

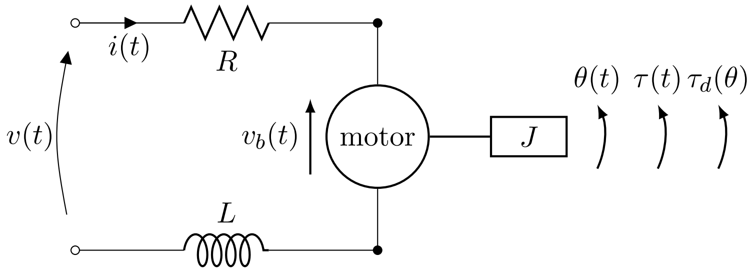

Figure 2.13: Diagram for a rotary direct current (DC) motor

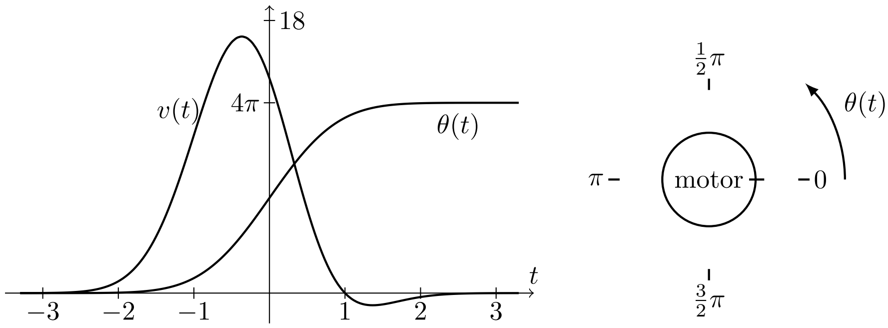

Figure 2.14: Voltage and corresponding angle for a DC motor with constants $K_b = K_\tau = B = R = J = 1$.

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