THE UNIVERSITY OF SOUTH AUSTRALIA

PRACTICE EXAM

EEET3041

Signals and Systems

PART OPEN BOOK

TIME: for working

for perusal before examination begins

Each question worth 8 marks

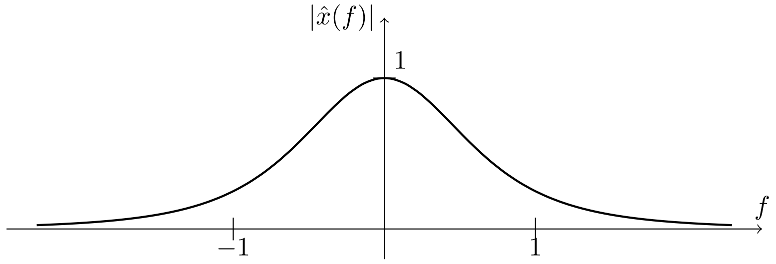
1. Suppose that x is the signal with Fourier transform

$$\hat{x}(f) = \frac{1}{(f-j)^4}.$$

Sketch the magnitude of $\hat{x}$ and find and sketch the time domain signal x. Is x absolutely integrable? Is x square integrable? **Solution:** The magnitude of $\hat{x}$ is

$$|\hat{x}(f)| = \frac{1}{|f-j|^4} = \frac{1}{(f^2+1^2)^2}.$$

This is plotted below



To find the time domain signal first realise that this is in the form of the only Laplace transform pair you need to remember (see (4.2.3) from the notes)! Recall that

$$\mathcal{L}\big(e^{\alpha t}t^n u(t)\big) = \frac{n!}{(s-\alpha)^{n+1}} \qquad \operatorname{Re} s > \operatorname{Re} \alpha.$$

If $\operatorname{Re}\alpha < 0$ then the region of convergence contains the imaginary axis, the signal $e^{\alpha t}t^n u(t)$ is absolutely integrable and therefore has a Fourier transform. Putting $s = 2\pi j f$ we have

$$\mathcal{F}\big(e^{\alpha t}t^n u(t)\big) = \frac{n!}{(2\pi j f-\alpha)^{n+1}} \qquad \operatorname{Re} s > \alpha.$$

Now, putting $\alpha = -2\pi$ and $n = 3$ and multiplying through by $\frac{8}{3}\pi^4$ we have

$$\mathcal{F}\big(\tfrac{8}{3}\pi^4 e^{-2\pi t}t^3 u(t)\big) = \frac{8}{3}\pi^4\frac{3!}{(2\pi j f+2\pi)^4} = \frac{1}{(jf+1)^4}$$

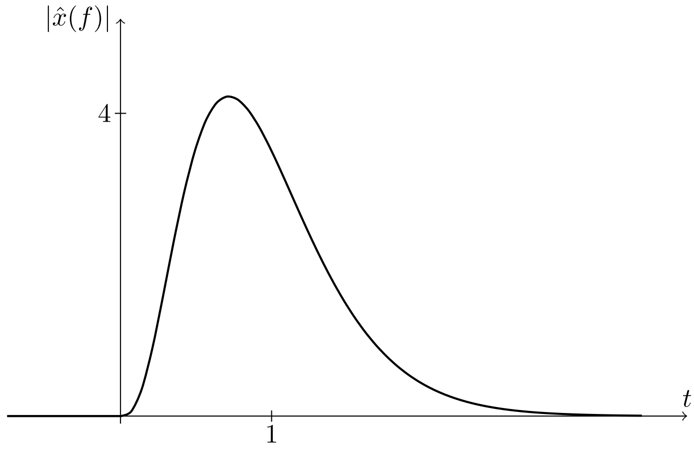
Now

$$(jf+1)^4 = \left(\frac{j^2 f + j}{j}\right)^4 = \frac{(-f+j)^4}{j^4} = (f-j)^4$$

and so

$$\mathcal{F}x = \mathcal{F}\big(\tfrac{8}{3}\pi^4 e^{-2\pi t}t^3 u(t)\big) = \frac{1}{(f-j)^4}$$

and so the time domain signal is $x(t) = \frac{8}{3}\pi^4 e^{-2\pi t}t^3 u(t)$. This signal is both absolutely integrable and square integrable. A sketch is below

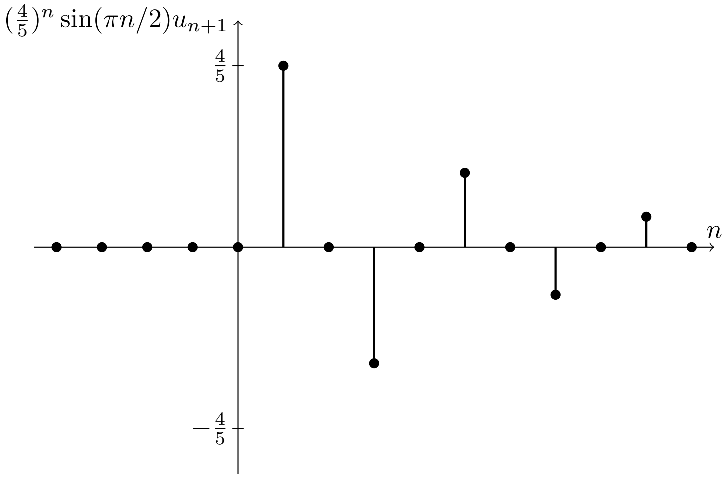


2. Sketch the sequence

$$x_n = (\tfrac{4}{5})^n \sin(\pi n/2) u_n$$

where u_n is the step sequence. Find the discrete time Fourier transform of x.

Solution: A plot of the sequence is below.



Observe that the sequence is zero when either $n \leqslant 0$ or when n is even. So,

$$\mathcal{D}x(f) = \sum_{n \in \mathbb{Z}} x_n e^{-j2\pi n f} = \sum_{n=0}^{\infty} (\tfrac{4}{5})^n \sin(\pi n/2) e^{-j2\pi n f} = \sum_{n>0, n \text{ odd}}^{\infty} (\tfrac{4}{5})^n \sin(\pi n/2) e^{-j2\pi n f}.$$

The sum over the odd integers can be written by replacing n by $2n+1$ so that

$$\mathcal{D}x(f) = \sum_{n=0}^{\infty} (\tfrac{4}{5})^{2n+1} \sin(\pi(2n+1)/2) e^{-j2\pi(2n+1)f} = \sum_{n=0}^{\infty} (\tfrac{4}{5})^{2n+1} (-1)^n e^{-j2\pi(2n+1)f}$$

becuase

$$\sin(\pi(2n+1)/2) = \sin(\pi n + \pi/2) = \cos(\pi n) = (-1)^n.$$

We want to write this as a geometric sum. To do this write

$$(\tfrac{4}{5})^{2n+1} (-1)^n e^{-j2\pi(2n+1)f} = \tfrac{4}{5}(-\tfrac{16}{25})^n e^{-j4\pi n f} e^{-j2\pi f} = \tfrac{4}{5} e^{-j2\pi f} (-\tfrac{16}{25} e^{-j4\pi f})^n = ar^n$$

where $a = \frac{4}{5}e^{-j2\pi f}$ and $r = -\frac{16}{25}e^{-j4\pi f}$. Now

$$\mathcal{D}x(f) = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = \frac{\frac{4}{5}e^{-j2\pi f}}{1 + \frac{16}{25}e^{-j4\pi f}} = \frac{20e^{-j2\pi f}}{25 + 16e^{-j4\pi f}}$$

because $\left|\frac{16}{25}e^{-j4\pi f}\right| = \frac{16}{25} < 1$.

3. Show that the differential equation relating the force f and position p of the mass-spring-damper in Figure 1 is

$$f = Kp + BDp + MD^2p.$$

Find the transfer function of a linear time invariant system H that maps the input force signal f to the output position signal p. Assuming that the constants satisfy

$$M = 1 \qquad K = \frac{\pi^2}{16} \qquad B = \frac{\pi}{2},$$

draw a pole zero plot for this system and find and sketch the impulse response. Comment on whether the system is underdamped, overdamped, or critically damped.

Solution: The cumulative force exerted on the mass is

$$f - Kp - BDp$$

and by Newton's law the acceleration of the mass D^2p satisfies

$$MD^2p = f - Kp - BDp.$$

We obtain the differential equation

$$f = Kp + BDp + MD^2p.$$

The transfer function is

$$\lambda H = \frac{1}{K + Bs + Ms^2}.$$

The roots are

$$\beta - \alpha, \qquad -\beta - \alpha,$$

where

$$\alpha = \frac{B}{2M} = \frac{\pi}{4} \qquad \beta = \frac{\sqrt{B^2 - 4KM}}{2M}.$$

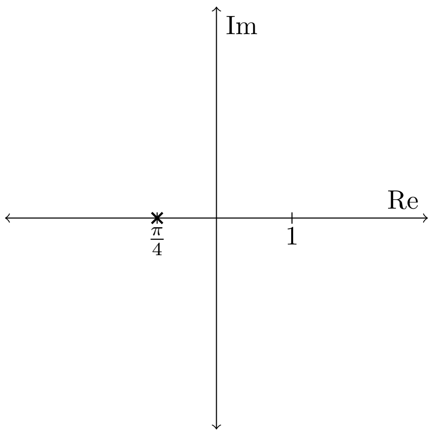
Now

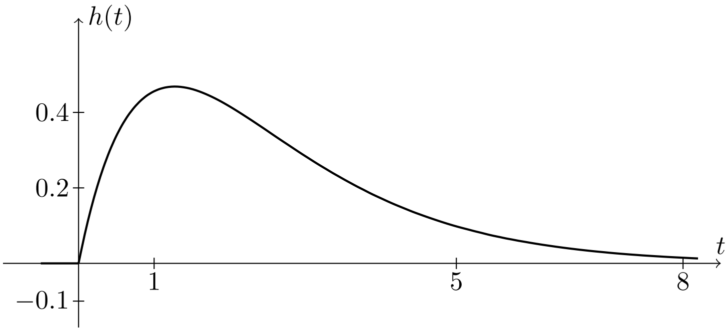
$$B^2 - 4KM = \left(\frac{\pi}{2}\right)^2 - 4\frac{\pi^2}{16} = \frac{\pi^2}{4} - \frac{\pi^2}{4} = 0$$

and so, $\beta = 0$. The system has two identical real roots at $-\pi/4$ and it therefore critically damped. The impulse response is (see Section 4.5 of the notes)

$$h(t) = \frac{1}{M}te^{-\alpha t}u(t) = te^{-\pi t/4}u(t)$$

There are two repeated real poles.


Im
Re
$\frac{\pi}{4}$
1


$h(t)$
0.4
0.2
-0.1
1
5
8
t

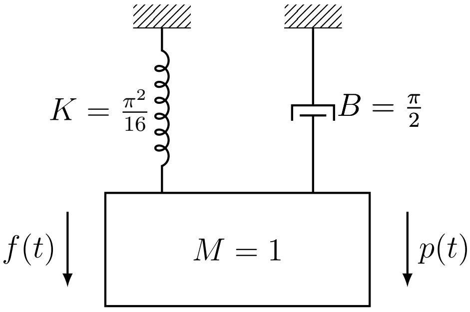


Figure 1: A mechanical mass-spring-damper system