Testable linear shift-invariant systems (Exercise Solutions)

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Contents

Chapter 1

Signals and systems

Exercises

1.1. How many distinct functions from the set $X = \{\text{Mario}, \text{Link}\}$ to the set $Y = \{\text{Freeman}, \text{Ryu}, \text{Sephiroth}\}$ exist? Write down each function, that is, write down all functions from the set $X \to Y$.

Solution: Each of the two elements in X can be mapped to one of the three elements of Y. There are thus $3^2 = 9$ distinct functions in $X \to Y$. They are

$$f_1(x) = \begin{cases} \text{Freeman} & x = \text{Mario} \\ \text{Freeman} & x = \text{Link} \end{cases} \qquad f_2(x) = \begin{cases} \text{Freeman} & x = \text{Mario} \\ \text{Ryu} & x = \text{Link} \end{cases}$$

$$f_3(x) = \begin{cases} \text{Ryu} & x = \text{Mario} \\ \text{Freeman} & x = \text{Link} \end{cases} \qquad f_4(x) = \begin{cases} \text{Freeman} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \end{cases}$$

$$f_5(x) = \begin{cases} \text{Sephiroth} & x = \text{Mario} \\ \text{Freeman} & x = \text{Link} \end{cases} \qquad f_6(x) = \begin{cases} \text{Ryu} & x = \text{Mario} \\ \text{Ryu} & x = \text{Link} \end{cases}$$

$$f_7(x) = \begin{cases} \text{Ryu} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \end{cases} \qquad f_8(x) = \begin{cases} \text{Sephiroth} & x = \text{Mario} \\ \text{Ryu} & x = \text{Link} \end{cases}$$

$$f_9(x) = \begin{cases} \text{Sephiroth} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \end{cases}$$

1.2. State whether the step function $u(t)$ is bounded, periodic, absolutely integrable, an energy signal. **Solution:** The magnitude of u is less than or equal to one and so the signal is bounded. The signal is not periodic, since for any hypothesised period $T > 0$ we have $u(T) = 1$ but $u(0) = 0$. The signal is not absolutely integrable, nor an energy signal since

$$\|u\|_1 = \|u\|_2 = \int_{-\infty}^{\infty} |u(t)|\, dt = \int_0^{\infty} dt$$

is not finite.

1.3. Show that the signal t^2 is locally integrable, but that the signal $\frac{1}{t^2}$ is not.

Solution: For any a and b

$$\int_a^b t^2 dt = \frac{b^3}{3} - \frac{a^3}{3}$$

is finite and so t^2 is locally integrable. Put $a = 0$ and $b > 0$ and

$$\int_0^b \frac{1}{t^2} dt = -\frac{1}{b} + \lim_{t \to 0} \frac{1}{t} = \infty.$$

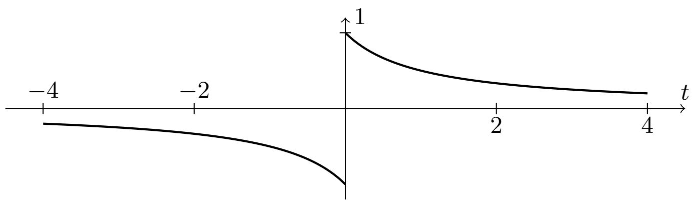
The limit above diverges and so $\frac{1}{t^2}$ is not locally integrable.

1.4. Plot the signal

$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \leq 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

Solution:



The signal is bounded since $|x(t)| < M$ for any $M > 1$. The signal is locally integrable because it is bounded, i.e., for any finite constants a and b

$$\int_a^b |x(t)|\, dt < \int_a^b M dt = (b - a)M < \infty.$$

The signal x is not absolutely integrable since

$$\begin{aligned} \|x\|_1 &= \int_{-\infty}^{\infty} |x(t)|\, dt \\ &= 2\int_0^{\infty} \frac{1}{t+1} dt \\ &= 2\int_1^{\infty} \frac{1}{t} dt \\ &= 2\log(1) + \lim_{t \to \infty} 2\log(t) \end{aligned}$$

and the limit diverges. The signal is square integrable since

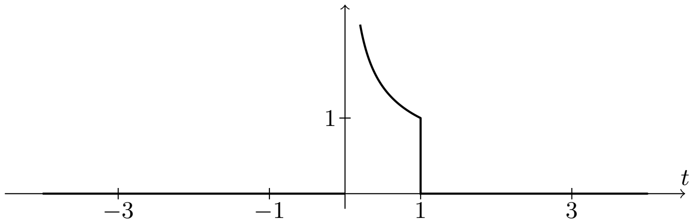
$$\begin{aligned} \|x\|_2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= 2\int_0^{\infty} \frac{1}{(t+1)^2} dt \\ &= 2\int_1^{\infty} \frac{1}{t^2} dt \\ &= 2 - \lim_{t \to \infty} \frac{2}{t} = 2. \end{aligned}$$

1.5. Plot the signal

$$x(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that x is absolutely integrable, but not square integrable.

Solution:



The integral

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt = \int_0^1 t^{-1/2} dt = [2\sqrt{t}]_0^1 = 2$$

and so x is absolutely integrable. The integral

$$\|x\|_2 = \int_{-\infty}^{\infty} |x(t)|\, dt = \int_0^1 t^{-1} dt = [\log(t)]_0^1 = \log(1) - \lim_{t \to 0} \log(t) = \infty$$

and so x is not square integrable.

1.6. Compute the energy of the signal $e^{-\alpha^2 t^2}$ (Hint: use equation (1.1.4) on page 4 and a change of variables). **Solution:** From (1.1.4) we the energy of e^{-t^2} is $\sqrt{\pi}$. Now

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} dt = \frac{1}{\alpha} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau = \frac{\sqrt{\pi}}{\alpha}$$

by the change of variables $\tau = \alpha t$.

1.7. Show that the signal t^2 is differentiable, but the step function u and rectangular pulse Π are not. **Solution:** We have

$$\lim_{h \to 0} \frac{(t+h)^2 - t^2}{h} = \lim_{h \to 0} \frac{2th + h^2}{h} = 2t.$$

$$\lim_{h \to 0} \frac{t^2 - (t-h)^2}{h} = \lim_{h \to 0} \frac{2th - h^2}{h} = 2t$$

and so t^2 is continuously differentiable with derivative $\frac{d}{dt} t^2 = 2t$. At $t = 0$ the corresponding limits for the step function are

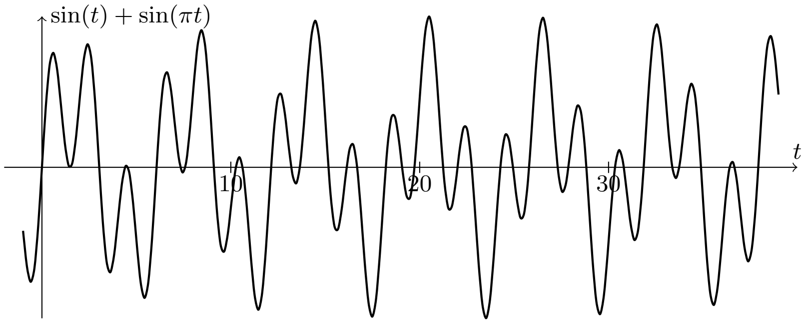
$$\lim_{h \to 0} \frac{u(h) - u(0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

but

$$\lim_{h \to 0} \frac{u(0) - u(-h)}{h} = \lim_{h \to 0} \frac{1}{h} = \infty$$

so the step function u is not differentiable at $t = 0$. A similar argument at $t = \frac{1}{2}$ or $t = -\frac{1}{2}$ shows that Π is not differentiable.

1.8. Plot the signal $\sin(t) + \sin(\pi t)$. Show that this signal is not periodic. **Solution:** A plot of the signal is below:



The following argument is due to Qiaochu Yuan. Suppose $\sin(t)+\sin(\pi t)$ is periodic. Then

$$\sin(t) + \sin(\pi t) = \sin(t + T) + \sin(\pi t + T)$$

for some $T > 0$. Differentiating both sides twice with respect to t gives

$$\sin(t) + \pi^2 \sin(\pi t) = \sin(t + T) + \pi^2 \sin(\pi t + T)$$

Subtracting the first equation from the second gives $\sin(t) = \sin(t + T)$ and substituting this into the second equation gives $\sin(\pi t) = \sin(\pi t + T)$. The equation $\sin(t) = \sin(t + T)$ implies that $T = 2\pi k$ for some integer $k \neq 0$. The equation $\sin(\pi t) = \sin(\pi t + T)$ implies that $T = 2\ell$ for some integer $\ell \neq 0$. We would thus have $2\pi k = 2\ell$ and so $\pi = \frac{\ell}{k}$. However, this is impossible because π is irrational. Thus $\sin(t) + \sin(\pi t)$ is not periodic.

1.9. Show that the set of locally integrable signals L_{loc}, the set of absolutely integrable signals L^1, and the set of square integrable signals L^2 are linear shift-invariant spaces. **Solution:** Let $x, y \in L^1$ and $a, b \in \mathbb{C}$. Now

$$\begin{aligned}
\|ax + by\|_1 &= \int_{-\infty}^{\infty} |ax(t) + by(t)|\, dt \\
&\leq \int_{-\infty}^{\infty} a\,|x(t)| + b\,|y(t)|\, dt \qquad \text{triangle inequality} \\
&= a\|x\|_1 + b\|y\|_1 < \infty
\end{aligned}$$

and so $ax + by \in L_1$ and L_1 is a linear space. Also

$$\begin{aligned}
\|T_\tau x\|_1 &= \int_{-\infty}^{\infty} |T_\tau x(t)|\, dt \\
&= \int_{-\infty}^{\infty} |x(t-\tau)|\, dt \\
&= \int_{-\infty}^{\infty} |x(k)|\, dk \qquad \text{change variable } k = t - \tau \qquad = \|x\|_1 < \infty
\end{aligned}$$

and so L_1 is a shift-invariant space.

Now

$$\begin{aligned}
\|ax + by\|_2^2 &= \int_{-\infty}^{\infty} |ax(t) + by(t)|^2\, dt \\
&\int_{-\infty}^{\infty} |ax(t)|^2 + |by(t)|^2 + 2\,\mathrm{Re}\big(a^* x(t)^* by(t)\big) dt
\end{aligned}$$

where $*$ denotes the complex cojugate and Re denotes the real part of a complex number. Now

$$\operatorname{Re}\left(a^* x(t)^* b y(t)\right) \leq |ax(t)|\,|by(t)| \leq \max(|ax(t)|^2, |by(t)|^2) \leq |ax(t)|^2 + |by(t)|^2$$

and so

$$\begin{aligned}\|ax + by\|_2^2 &\leq \int_{-\infty}^{\infty} 3\,|ax(t)|^2 + 3\,|by(t)|^2\,dt \\ &= \int_{-\infty}^{\infty} 3\,|a|^2\,|x(t)|^2 + 3\,|b|^2\,|y(t)|^2\,dt \\ &= 3\,|a|^2\,\|x\|_2^2 + 3\,|b|^2\,\|y\|_2^2 < \infty\end{aligned}$$

and L_2 is thus a linear space. Also

$$\|T_\tau x\|_2^2 = \int_{-\infty}^{\infty} |T_\tau x(t)|^2\,dt = \int_{-\infty}^{\infty} |x(t-\tau)|^2\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \|x\|_2^2 < \infty$$

and so L_2 is a shift-invariant space.

1.10. Show that the set of periodic signals is a shift-invariant space, but not a linear space. **Solution:** Let P be the set of periodic signals. If $x \in P$ then there exists $T > 0$ such that $x(t + kT) = x(t)$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. The shifted signal $T_\tau x \in P$ since, for the same T, we have

$$T_\tau x(t - kT) = x(t - \tau - kT) = x(t - \tau) = T_\tau x(t)$$

for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}$. Since $x \in P$ and $\tau \in \mathbb{R}$ are arbitrary, this holds for all signals $x \in P$ and all shifts $\tau \in reals$. Thus, the set of periodic signals is a shift invariant space.

The set of period signals is not a linear space. Consider the signal $x(t) = \sin(t)$ with period 2π and $y(t) = \sin(\pi t)$ with period 2. Both x and y are in P. However, exercise 1.8 shows that the sum $x(t) + y(t) = \sin(t) + \sin(\pi t)$ is not periodic, that is, $x + y \in P$.

1.11. Show that the set of bounded signals is a linear shift-invariant space. **Solution:** Let B be the set of bounded signals. If $x \in B$ there exists $M > 0$ such that $|x(t)| < M$ for all $t \in \mathbb{R}$ then the shift $T_\tau x(t)$ satisfies $|T_\tau x(t)| < M$ for all $t \in \mathbb{R}$. Since x and τ are arbitrary this holds for all $x \in B$ and $\tau \in \mathbb{R}$. Thus B is a shift invariant space.

Let $x \in B$ and $y \in B$ be bounded signals. There exists $M_x > 0$ and $M_y > 0$ such that

$$|x(t)| < M_x \qquad |y(t)| < M_y \qquad \text{for all } t \in \mathbb{R}.$$

Now for $a, b \in \mathbb{C}$ the signal $ax + by$ satisfies

$$|ax(t) + by(t)| \leq |a|\,|x(t)| + |b|\,|y(t)| < |a|\,M_x + |b|\,M_y$$

for all $t \in \mathbb{R}$. Thus the linear combination $ax + by$ is bounded. Since $a, b \in \mathbb{C}$ and $x, y \in B$ are arbtirary this holds for all $a, b \in \mathbb{C}$ and all $x, y \in B$ and so B is a linear space.

1.12. Let $K > 0$ be a fixed real number. Show that the set of signals bounded below K is a shift invariant space, but not a linear space. **Solution:** Let B_K be the set of signals bounded less than K, that is,

$$B_K = \{x \in \mathbb{R} \to \mathbb{C}\ ;\ |x(t)| < K \text{for all } t \in \mathbb{R}\}.$$

If $x \in B_K$ then $|T_\tau x(t)| < K$ for all $t \in \mathbb{R}$ and so B_K is $T_\tau x \in B_K$. Thus, B_K is a shift invariant space.

Consider constant signals $x(t) = K/2$ and $y(t) = 2K/3$. Both x and y are bounded less than K and so are in B_K However, the signal $x + y$ is such that

$$|x(t) + y(t)| = K/2 + 2K/3 = 7K/6 > K$$

and so $x + y \notin B_K$. Thus B_K is not a linear space.

1.13. Show that the set of even signals and the set of odd signals are not shift invariant spaces.

1.14. Show that the integrator I_c with finite $c \in \mathbb{R}$ is not stable. **Solution:** Put $M > 1$. The shifted step function $u(t + a)$ is locally integrable and bounded below M, i.e. $|u(t + a)| \leq 1 < M$ for all $t \in \mathbb{R}$. However, the response of the integrator I_a to $u(t + a)$ is

$$I_a u(t+a) = \int_{-a}^{t} u(\tau + a) d\tau = \begin{cases} \int_{-a}^{t} d\tau = t + a & t \geq -a \\ 0 & t < -a \end{cases},$$

and this is not a bounded signal, that is, for every K we have $t + a > K$ whenever $t > K - a$.

1.15. Show that if the signal x is locally integrable and $\int_{-\infty}^{0} |x(t)|\, dt < \infty$ then $I_\infty x(t) = \int_{-\infty}^{t} x(t)dt < \infty$ for all $t \in \mathbb{R}$. **Solution:** We have

$$\begin{aligned} I_\infty x(t) \leq |I_\infty x(t)| &= \left| \int_{-\infty}^{t} x(t)dt \right| \\ &\leq \int_{-\infty}^{t} |x(t)|\, dt \\ &= \int_{-\infty}^{0} |x(t)|\, dt + \int_{0}^{t} |x(t)|\, dt \end{aligned}$$

Now $\int_{-\infty}^{0} |x(t)|\, dt < \infty$ by assumption and $\int_{0}^{t} |x(t)|\, dt$ because x is locally integrable. It follows that

1.16. Show that the integrator I_∞ is not stable. **Solution:** By default the domain for I_∞ is the subset of locally integrable signals for which $\int_{-\infty}^{0} |x(t)|\, dt < \infty$. The step function $u(t)$ is in this domain. The argument now follows similiarly to Exercise 1.16.

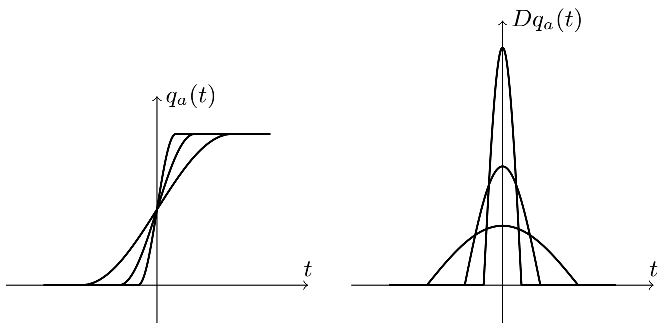
1.17. Show that the differentiator system D is not stable. **Solution:** Put $M > 2$. Define the signal

$$q_a(t) = \begin{cases} 0 & 2t < -a \\ 1 + \sin\left(\frac{\pi t}{a}\right) & -a < 2t < a \\ 2 & 2t > a, \end{cases}$$

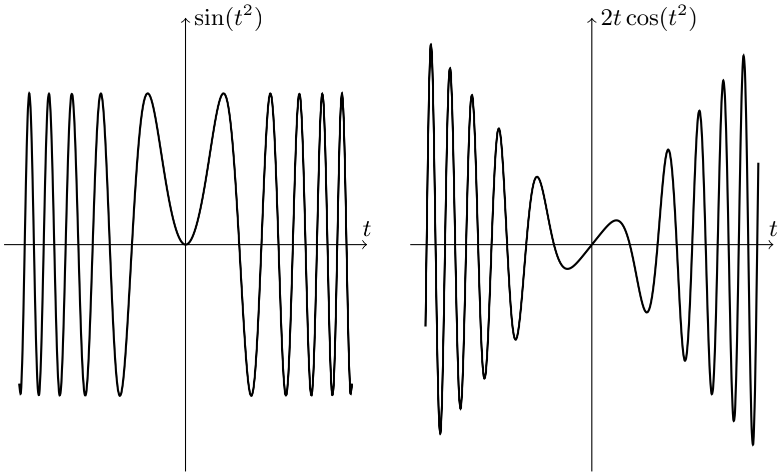
and observe that q_a is differentiable and bounded below M. The response of the differentiator D to q_a is

$$Dq_a(t) = \begin{cases} 0 & 2t < -a \\ \frac{\pi}{a} \cos\left(\frac{\pi t}{a}\right) & -a < 2t < a \\ 1 & 2t > a. \end{cases}$$

The signal p_a and the response Dp_a are plotted below for $a = \frac{1}{2}, 1$ and 2. The response Dp_a obtains a maximum amplitude of $\frac{\pi}{a}$ at $t = 0$. So D is not stable because for any K we can choose $a < \frac{\pi}{K}$ so that $\frac{\pi}{a} > K$.



Another solution was suggested by Badri Vellambi. Consider the signal $x(t) = \sin(t^2)$ plotted in the figure below. This signal is bounded below any $M > 1$. The response of the differentiator is $Dx(t) = 2t\cos(t^2)$ and this is not bounded.



1.18. Show that the shifter T_τ is linear and shift-invariant and that the time-scaler is linear, but not time invariant. **Solution:** The shifter T_τ is shift-invariant since

$$T_k T_\tau x = T_k x(t-\tau) = x(t-\tau-k) = T_\tau x(t-k) = T_\tau T_k x$$

for all signals x, that is, shifters commute with shifters. The shifter is linear because

$$T_\tau(ax+by) = ax(t-\tau) + by(t-\tau) = aT_\tau x + bT_\tau y.$$

The time-scaler $Hx = x(\alpha t)$ is linear because

$$H(ax+by) = ax(\alpha t) + by(\alpha t) = aHx + bHy.$$

The system is not shift-invariant because

$$HT_\tau x = Hx(t-\tau) = x(\alpha t - \tau)$$

but

$$T_\tau H x = T_\tau x(\alpha t) = x(\alpha(t-\tau)) = x(\alpha t - \alpha\tau),$$

and these signals are not equal in general. For example consider the rectangular pulse Π. With time-scaling parameter $\alpha = 2$ and shift $\tau = 1$,

$$HT_1\Pi = \Pi(2t-1) \neq \Pi(2t-2) = T_1 H \Pi.$$

1.19. Show that the integrator I_c with finite $c \in \mathbb{R}$ is linear, but not shift-invariant. **Solution:** The system is linear because, if $x, y \in L_{\mathrm{loc}}$, then

$$\begin{aligned}
I_c(ax+by) &= \int_{-c}^{t} ax(\tau) + by(\tau) d\tau \\
&= a\int_{-c}^{t} x(\tau)d\tau + b\int_{-c}^{t} y(\tau)d\tau \\
&= aI_c x + bI_c y.
\end{aligned}$$

The system is not shift-invariant because

$$T_k I_c x = I_c(x, t-k) = \int_{-c}^{t-k} x(\tau)d\tau$$

but

$$I_c T_k x = \int_{-c}^{t} x(\tau - k)d\tau.$$

We now need only find some signal $x \in L_{\mathrm{loc}}$ for which the integrals on the right hand side of the above equations are not equal. Choose the signal $x = 1$, i.e., the signal that is equal to 1 for all time. In this case

$$T_k I_c 1 = \int_{-c}^{t-k} d\tau = t - k + c \neq t + c = \int_{-c-k}^{t-k} d\tau = I_c T_k 1 \qquad \text{when } k \neq 0.$$

1.20. Show that the integrator I_∞ is linear and shift-invariant. **Solution:** The system is linear because

$$\begin{aligned}
I_\infty(ax+by) &= \int_{-\infty}^{t} ax(\tau) + by(\tau) d\tau \\
&= a\int_{-\infty}^{t} x(\tau)d\tau + b\int_{-\infty}^{t} y(\tau)d\tau \\
&= aI_\infty x + bI_\infty y.
\end{aligned}$$

The system is shift-invariant because

$$T_k I_\infty x = I_\infty x(t-k) = \int_{-\infty}^{t-k} x(\tau)d\tau,$$

and

$$I_\infty T_k x = \int_{-\infty}^{t} x(\tau - k)d\tau = \int_{-\infty}^{t-k} x(\tau)d\tau.$$

1.21. State whether the system $Hx = x + 1$ is linear, shift-invariant, stable.
Solution: It is not linear because for any signal x and real number $a \neq 1$,

$$H(ax) = ax + 1 \neq aHx = a\big(x + 1\big) = ax + a.$$

It is shift-invariant because

$$HT_\tau x = x(t - \tau) + 1 = T_\tau(x + 1) = T_\tau Hx.$$

It is stable because for any signal x with $x(t) < M$ for all $t \in \mathbb{R}$,

$$Hx(t) = x(t) + 1 < M + 1 \qquad \text{for all } t \in \mathbb{R}.$$

1.22. State whether the system $Hx = 0$ is linear, shift-invariant, stable.
Solution: It is linear because

$$H(ax + by) = 0 = aHx + bHy = 0.$$

It is shift-invariant because

$$HT_\tau x(t) = 0 = Hx(t - \tau).$$

It is stable because for any $K > 0$,

$$Hx(t) = 0 < K \qquad \text{for all } t \in \mathbb{R} \text{ and all signals } x.$$

1.23. State whether the system $Hx = 1$ is linear, shift-invariant, stable.
Solution: It is not linear because for any signal x and real number $a \neq 1$

$$H(ax) = 1 \neq aHx = a.$$

It is shift-invariant because

$$HT_\tau x = 1 = T_\tau(1) = T_\tau Hx.$$

It is stable because for any $K > 1$,

$$|Hx(t)| = 1 < K \qquad \text{for all } t \in \mathbb{R} \text{ and all signals } x.$$

1.24. Let x be a signal with period T that is not equal to zero almost everywhere. Show that x is neither absolutely integrable nor square integrabl. **Solution:** This is plain and does not really require further explanation, but I've found some students desire more rigour.

Since x does not equal to zero almost everywhere there exist some finite real numbers a and b such that $\int_a^b |x(t)|\, dt = C > 0$. Let k be an integer such $-kT < a$ and $kT > b$ so that the integral over $2k + 1$ periods

$$\int_{-kT}^{kT} |x(t)|\, dt \geq \int_a^b |x(t)|\, dt = C > 0.$$

Now, since x has period T

$$\int_{-ckT}^{ckT} |x(t)|\, dt = (2c + 1) \int_{-kT}^{kT} |x(t)|\, dt \geq (2c + 1)C > 0$$

for integers c and since this integral is increasing monotonically with c we have $\int_{-ckT}^{ckT} |x(t)|\, dt \geq \lfloor 2c+1 \rfloor C$ for all $c \in \mathbb{R}$ where $\lfloor 2c+1 \rfloor$ denotes the largest integer less than or equal to $2c+1$. Now,

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt = \lim_{c\to\infty} \int_{-ckT}^{ckT} |x(t)|\, dt \geq \lim_{c\to\infty} \lfloor 2c+1 \rfloor C = \infty,$$

and so, x is not absolutely integrable.

Chapter 2

Systems modelled by differential equations

Exercises

2.1. Analyse the inverting amplifier circuit in Figure 2.7 to obtain the relationship between input voltage x and output voltage y given by (2.2.1). You may wish to use a symbolic programming language (for example Maxima, Sage, Mathematica, or Maple). **Solution:** We provide two solutions. Let v_i, v_o, v_1 and v_2 be the voltages over the input resistor R_i, the output resistor R_o, and resistors R_1 and R_2 respectively. We have 8 unknown voltages $x, y, v_i, v_1, v_2, v_o, v_+, v_-$. We will need 7 independent equations to find an equation relating x and y. All currents are considered to be flowing either downwards or to the right in the circuit diagram. The first 4 equations are given by voltages over each resitor,

$$\begin{aligned} x &= v_- + v_1 \\ v_- &= y + v_2 \\ v_- &= v_+ + v_i \\ y &= v_o + A(v_+ - v_-) \end{aligned}$$

The next two equations apply Kirchoff's current law to each node betweeen resistors. The currents into the 3 way connection between R_i, R_1 and R_2 sum to zero, and so

$$\frac{v_1}{R_1} + \frac{v_i}{R_i} = \frac{v_2}{R_2}$$

by Ohm's law. Finally the currents through R_o and R_2 are the same, and so

$$\frac{v_o}{R_o} = \frac{v_2}{R_2}.$$

The final equation simply observes that the non-inverting terminal v_+ is connected to ground

$$v_+ = 0.$$

We now have 7 linearly independent equations for the 8 unknowns $x, y, v_i, v_1, v_2, v_o, v_+, v_-$. We can use these to find an equation that describes y in terms of x. The Mathematica command

```
Simplify[Solve[{x == vm + v1,
   vm == y + v2,
   vm == vp + vi,
   y == vo + A*(vp - vm),
   v1/r1 + vi/ri == v2/r2,
   vo/ro == v2/r2,
   vp == 0,
   r1 > 0, r2 > 0, ro > 0, ri > 0, A > 0},
  {y,vi,vo,v2,v1,vp,vm}, Reals]]
```

or Maxima command

```
linsolve([x=vm+v1,
          vm=y+v2,
          vm=vp+vi,
          y=vo+A(vp-vm),
          v1/R1=v2/R2+vi/Ri,
          v2/R2=vo/Ro,
          vp=0],
          [y,vp,vm,v1,v2,vo,vi]);
```

readily obtains

$$y = \frac{R_i(R_o - AR_2)}{R_i(R_2 + R_o) + R_1(R_2 + R_i + AR_i + R_o)}x.$$

The second solution is thanks to Badri Vellambi. Badri sets $v_i = v_+ - v_-$ so that the voltage over the dependent voltage source is Av_i. Consider the operational amplifier circuit with feedback presented in Fig. 2.1. Suppose that the voltage signal fed into the circuit is $x(t)$ and the voltage signal measured at the output of the opamp is $y(t)$.

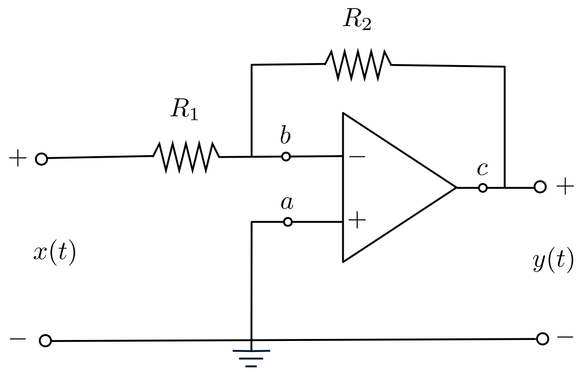


Figure 2.1: The circuit

To simplify the circuit, one has to use the model for the opamp given in Fig. 2.2 which involves the voltage-controlled voltage-source (VCVS) at the output side (indicated in green). While replacing the operational amplifier with its model, it must be noted that the positive terminal of the operational amplifier is connected to the ground.

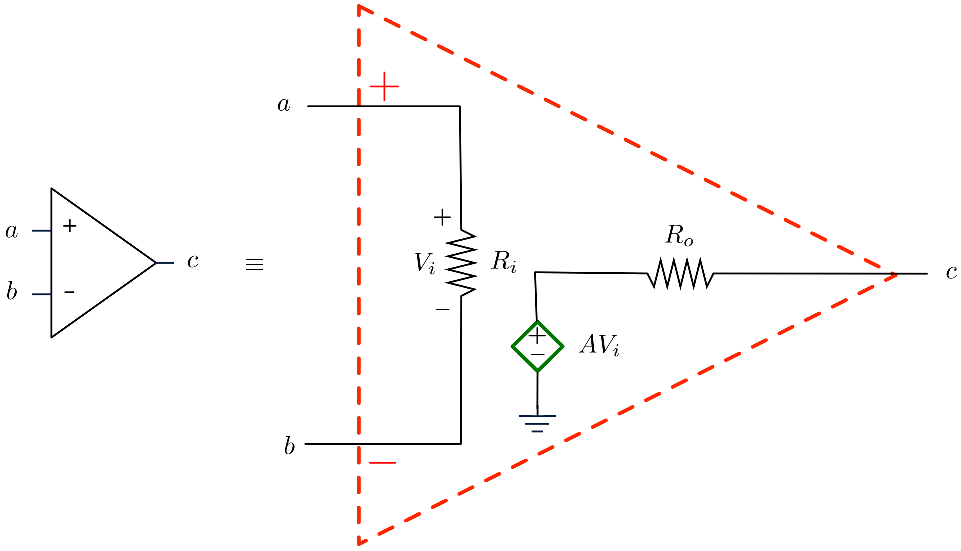


Figure 2.2: The model for an operational amplifier

Upon replacement, we obtain the following equivalent circuit. Again notice that since the positive terminal of the opamp was connected to the ground, the voltage output by the VCVS is AV_i where V_i is the voltage between the ground and the top of the resistance R_i, and is measured against the flow of the current $i - i_1$ as is indicated in the figure.

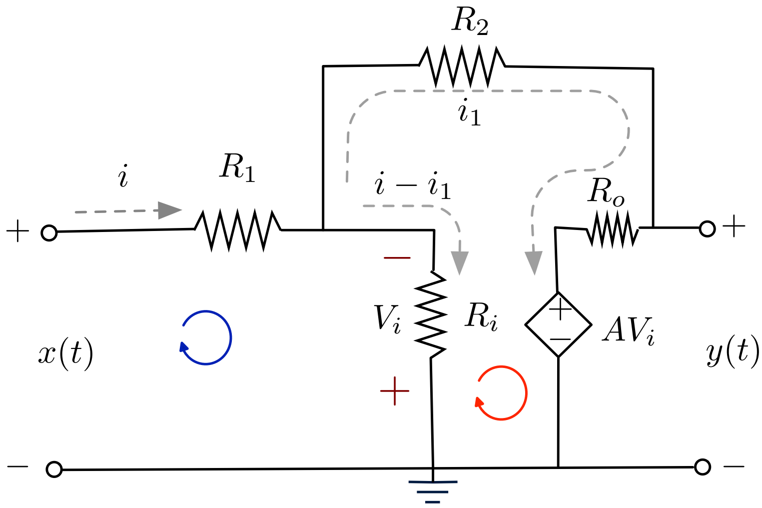


Figure 2.3: The operational amplifier circuit with the model

Applying Kirchoff's law to the outer loop indicated in blue in Fig. 2.3, we obtain the following equation.

$$x(t) = iR_1 + (i - i_1)R_i = i(R_1 + R_i) - i_1 R \tag{2.0.1}$$

Note that by definition, the voltage V_i that controls the VCVS is the voltage across R_i measured against the indicated direction of the current $i - i_1$, and is given by

$$V_i = -(i - i_1)R_i. \tag{2.0.2}$$

Next, writing out the Kirchoff's law for the inner loop indicated in red, we obtain the following.

$$0 = i_1 R_2 + i_2 R_0 + AV_i - (i - i_1)R_i \tag{2.0.3}$$

Substituting V_i in the above equation with the RHS of (2.0.2), we obtain the following.

$$\begin{aligned} 0 &= i_1(R_2 + R_0) - A(i - i_1)R_i - (i - i_1)R_i && (2.0.4)\\ &= -i(1 + A)R_i + i_1((1 + A)R_i + R_0 + R_2) && (2.0.5) \end{aligned}$$

Combining (2.0.5) and (2.0.1), we obtain the following linear system of equations governing the electrical circuit.

$$\begin{bmatrix} R_1 + R_i & -R_i \\ -(1 + A)R_i & (1 + A)R_i + R_0 + R_2 \end{bmatrix} \begin{bmatrix} i \\ i_1 \end{bmatrix} = \begin{bmatrix} x(t) \\ 0 \end{bmatrix} \tag{2.0.6}$$

Solving the above linear system, we identify the current in the different branches to be

$$\begin{bmatrix} i \\ i_1 \end{bmatrix} = x(t) \begin{bmatrix} \frac{(1+A)R_i + R_0 + R_2}{(1+A)R_i R_1 + R_0 R_1 + R_2 R_1 + R_0 R_i + R_2 R_i} \\ \frac{(1+A)R_i}{(1+A)R_i R_1 + R_0 R_1 + R_2 R_1 + R_0 R_i + R_2 R_i} \end{bmatrix}. \tag{2.0.7}$$

Lastly, notice that

$$\begin{aligned} y(t) &= i_1 R_0 + AV_i && (2.0.8)\\ &= i_1 R_0 - (i - i_1)R_i. && (2.0.9) \end{aligned}$$

Substituting the solutions for i and i_1 in terms of $x(t)$, we obtain the following.

$$y(t) = \left(\frac{R_i R_0 - R_2 R_i A}{(1 + A)R_i R_1 + R_0 R_1 + R_2 R_1 + R_0 R_i + R_2 R_i} \right) x(t) \tag{2.0.10}$$

2.2. Figure 2.4 depicts a mechanical system involving two masses, two springs, and a damper connected between two walls. Suppose that the spring K_2 is at rest when the mass M_2 is at position $p(t) = 0$. A force, represented by the signal f, is applied to mass M_1. Derive a differential equation relating the force f and the position p of mass M_2.

Solution: Let p_1 be a signal representing the position of mass M_1. Suppose that the spring K_1 connecting masses M_1 and M_2 is a rest when the masses are distance d_1 apart, i.e., $p - p_1 = d_1$. The force applied by the spring on M_2 is by spring K_1 is

$$f_1 = -K_1(p - p_1 - d_1) = -K_1(p - g)$$

where $g = p_1 + d_1$. The force applied by spring K_1 on mass M_1 is then $-f_1$. The force applied by the damper on M_1 is

$$f_d = -BDp_1 = -BD(g - d_1) = -BDg.$$

The total force applied to M_1 is $f + f_d - f_1$ and by Newton's law

$$M_1 D^2 p_1 = M_1 D^2 g = f + f_d - f_1 = f - BDg + K_1(p - g).$$

The force applied to M_2 by the spring K_2 is

$$f_2 = -K_2 p$$

because the spring is assumed to be at rest when $p = 0$. The total force applied to M_2 is $f_1 + f_2$ and by Newton's law

$$M_2 D^2 p = f_1 + f_2 = -K_1(p - g) - K_2 p.$$

Rearranging gives

$$-K_1 g = (K_1 + K_2)p + M_2 D^2 p$$

and

$$-K_1(p - g) = M_2 D^2 p + K_2 p.$$

Now,

$$M_1 D^2 g + BDg + M_2 D^2 p + K_2 p = f$$

and so

$$K_1 K_2 p + B(K_1 + K_2)Dp + (M_1 K_1 + M_1 K_2 + K_1 M_2)D^2 p + BM_2 D^3 p + M_1 M_2 D^4 p = K_1 f.$$

In the case that $M_1 = K_1 = K_2 = B = 1$ and $M_2 = 2$ we have

$$p + 2Dp + 4D^2 p + 2D^3 p + 2D^4 p = f.$$

2.3. Consider the electromechanical system in Figure 2.5. A direct current motor is connected to a potentiometer in such a way that the voltage at the output of the potentiometer is equal to the angle of the motor θ. This voltage is fed back to the input terminal of the motor. An input voltage v is applied to the other terminal on the motor. Find the differential equation relating v and θ. What is the input voltage v if the motor angle satisfies $\theta(t) = \frac{\pi}{2}(1 + \mathrm{erf}(t))$? Plot θ and v in this case when the motor coefficients satisfy $L = 0$, $R = \frac{3}{4}$, and $K_b = K_\tau = B = J = 1$.

Solution: The input voltage to the DC motor is $v - \theta$. From (2.4.1) of the lecture notes the relationship between the input voltage and motor angle is

$$v - \theta = \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ}{K_\tau} D^2\theta$$

and so

$$v = \theta + \left(\frac{RB}{K_\tau} + K_b\right) D\theta + \frac{RJ}{K_\tau} D^2\theta.$$

If $\theta(t) = \frac{\pi}{2}(1 + \mathrm{erf}(t))$ then

$$D\theta(t) = \sqrt{\pi}e^{-t^2}, \qquad D^2\theta(t) = -2t\sqrt{\pi}e^{-t^2}$$

and so

$$v(t) = \frac{\pi\,(\mathrm{erf}\,(t) + 1)}{2} - 2\sqrt{\pi}\,t\,e^{-t^2} + 2\sqrt{\pi}\,e^{-t^2}$$

The signals v and θ are plotted in the figure below. Observe that as $t \to \infty$ both $\theta(t)$ and $v(t)$ converge to π.

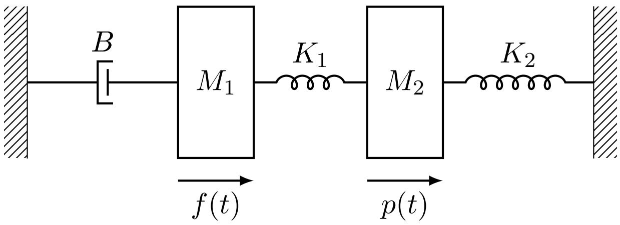


Figure 2.4: Two masses, a spring, and a damper connect between two walls for Exercise 2.2.

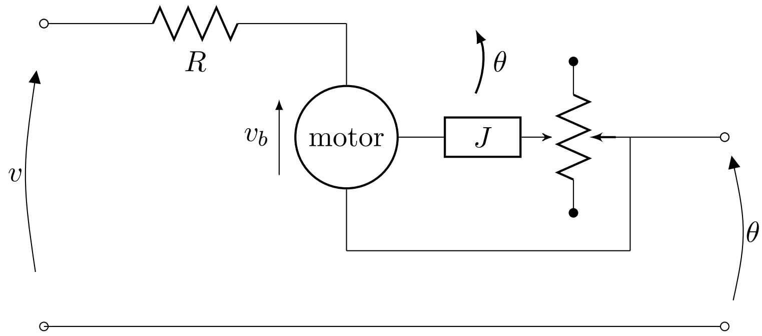


Figure 2.5: Diagram for a rotary direct current (DC) with potentiometer feedback for Exercise 2.3.

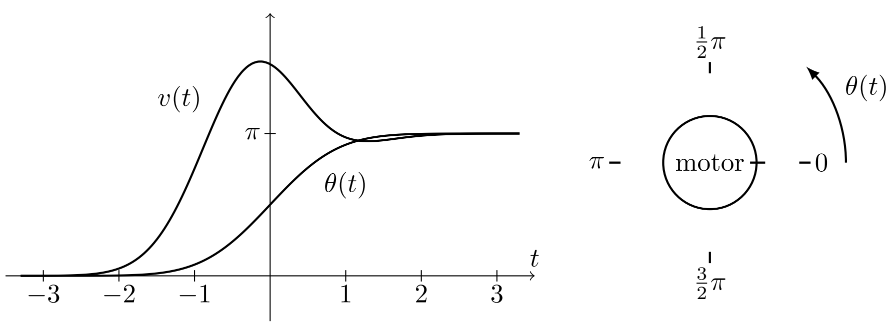


Figure 2.6: Voltage and corresponding angle for the dc motor with potentiometer in Figure 2.5 with constants $L = 0$, $R = \frac{3}{4}$, and$K_b = K_\tau = B = J = 1$.

Chapter 3

Linear time-invariant systems

Exercises

3.1. Let h be a locally integrable signal. Show that the set $\mathrm{dom}(h)$ defined in Section 3.1 on page 33 is a linear shift-invariant space.

3.2. Show that $\mathrm{dom}(u)$ where u is the step function is the subset of locally integrable signals such that $\int_{-\infty}^{0} |x(t)|\, dt < \infty$. **Solution:** By definition $\mathrm{dom}(u)$ is the set of signals x such that

$$\int_{-\infty}^{\infty} |u(\tau)x(t-\tau)|\, d\tau < \infty \qquad \text{for all } t \in \mathbb{R}.$$

Denote by B the subset of locally integrable signals such that $\int_{-\infty}^{0} |x(t)|\, dt < \infty$. We first show that $\mathrm{dom}(u)$ is a subset of B, that is $\mathrm{dom}(u) \subseteq B$. We do so by contraposition, that is, we show that if $x \notin B$ then $x \notin \mathrm{dom}(u)$. Suppose that x is not locally integrable, that is, suppose there exists $a, b \in \mathbb{R}$ such that $\int_a^b |x(t)|\, d\tau$ is not finite. Then $x \notin B$. Now

$$\int_{-\infty}^{\infty} |u(\tau)x(t-\tau)|\, d\tau = \int_{-\infty}^{\infty} |u(t-k)x(k)|\, dk = \int_{-\infty}^{t} |x(k)|\, dk$$

the second equation following from the change of variable $k = t - \tau$. Choosing $k > b$ we have

$$\int_{-\infty}^{\infty} |u(\tau)x(t-\tau)|\, d\tau = \int_{-\infty}^{t} |x(\tau)|\, d\tau \geq \int_{a}^{b} |x(\tau)|\, d\tau$$

which, by assumption, is not finite, and so $x \notin \mathrm{dom}(u)$.

We now show that $B \subseteq \mathrm{dom}(u)$. Suppose that $x \in \mathrm{dom}(u)$, that is, suppose that

$$\int_{-\infty}^{\infty} |u(\tau)x(t-\tau)|\, d\tau < \infty$$

for all t. Then

$$\int_{-\infty}^{\infty} |u(\tau)x(t-\tau)|\, d\tau = \int_{-\infty}^{t} |x(\tau)|\, d\tau = \int_{-\infty}^{a} |x(\tau)|\, d\tau + \int_{a}^{t} |x(\tau)|\, d\tau$$

for all $a, t \in \mathbb{R}$ and so, the two integrals on the right are finite for all $a, t \in \mathbb{R}$. In particular

$$\int_a^t |x(\tau)|\, d\tau < \infty$$

for all $a, t \in \mathbb{R}$ and so x is locally integrabls and putting $a = 0$ we have that

$$\int_{-\infty}^0 |x(\tau)|\, d\tau < \infty.$$

It follows that $x \in B$. We have now show that $\operatorname{dom}(u) \subseteq B$ and that $B \subseteq \operatorname{dom}(u)$ and so it must be that $B = \operatorname{dom}(u)$.

3.3. Show that convolution distributes with addition and commutes with scalar multiplication, that is, show that $a(x*w)+b(y*w) = (ax+by)*w$.
Solution:

$$\begin{aligned} a(x*w) + b(y*w) &= a\int_{-\infty}^{\infty} x(\tau)w(t-\tau)d\tau + b\int_{-\infty}^{\infty} y(\tau)w(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} \big(ax(\tau) + by(\tau)\big)w(t-\tau)d\tau \\ &= (ax+by)*w. \end{aligned}$$

3.4. Show that convolution is associative. That is, if x, y, z are signals then $x*(y*z) = (x*y)*z$. **Solution:**

$$\begin{aligned} (x*y)*z &= \int_{-\infty}^{\infty} (x*y)(\tau)z(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\kappa)y(\tau-\kappa)z(t-\tau)d\kappa d\tau \\ &= \int_{-\infty}^{\infty} x(\kappa)\int_{-\infty}^{\infty} y(\tau-\kappa)z(t-\tau)d\tau d\kappa \qquad \text{(swap order of integration)} \\ &= \int_{-\infty}^{\infty} x(\kappa)\int_{-\infty}^{\infty} y(\nu)z(t-\kappa-\nu)d\tau d\kappa \qquad \text{(change variable } \nu = \tau-\kappa\text{)} \\ &= \int_{-\infty}^{\infty} x(\kappa)(y*z)(t-\kappa)d\kappa \\ &= x*(y*z). \end{aligned}$$

The exchange of integration order can be justified using Fubini's theorem whenever the all of the convolutions involved in $x*(y*z) = (x*y)*z$ exist.

3.5. Show that the convolution of two absolutely integrable signals is absolutely integrable. **Solution:** Let x and y be absolutely integrable. We want to show that the convolution

$$(x*y)(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

is absolutely integrable. Write

$$\begin{aligned}
\|x * y\|_1 &= \int_{-\infty}^{\infty} |(x*y)(t)|\, dt \\
&= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \right| dt \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x(\tau)y(t-\tau)|\, d\tau dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x(\tau)|\, |y(t-\tau)|\, dt d\tau \qquad \text{(change order of integration (Tonelli's theorem))} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y(t-\tau)|\, dt\, |x(\tau)|\, d\tau \\
&= \|y\|_1 \int_{-\infty}^{\infty} |x(\tau)|\, d\tau \\
&= \|y\|_1 \|x\|_1
\end{aligned}$$

which is finite by our assumption that x and y are absolutely integrable. This result also follows as a special case of Young's Theorem Rudin [1986].

3.6. Show that a regular system is stable if and only if its impulse response is absolutely integrable. **Solution:** Let H be a regular system and h its impulse response. If h is absolutely integrable then for all signals x such that $|x(t)| < M$ for all t,

$$\begin{aligned}
H(x,t) &= h * x \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&\leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|d\tau \\
&\leq \int_{-\infty}^{\infty} M|h(\tau)|d\tau \\
&= M\|h\|_1
\end{aligned}$$

for all t, and so $H(x,t)$ is bounded. On the other hand if h is not absolutely integrable then the bounded signal

$$s(t) = \begin{cases} 1 & h(-t) > 0 \\ -1 & h(-t) \leq 0 \end{cases}$$

is such that

$$H(x,0) = \int_{-\infty}^{\infty} h(\tau)s(-\tau)d\tau = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \infty,$$

and so the signal $H(x)$ is not bounded at $t = 0$.

3.7. Show that the system $H(x) = \int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau$ is linear time invariant and regular. Find and sketch the impulse response and the step response. **Solution:** The easy way is to spot the impulse response directly.

Observe that

$$
\begin{aligned}
H(x)(t) &= \int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau \\
&= \int_{-\infty}^{\infty} \Pi(\tau/2)\sin(\pi\tau)x(t+\tau)d\tau \\
&= -\int_{\infty}^{-\infty} \Pi(-\tau/2)\sin(-\pi\tau)x(t-\tau)d\tau \qquad \text{(ch. var. } \tau \to -\tau) \\
&= -\int_{-\infty}^{\infty} \Pi(\tau/2)\sin(\pi\tau)x(t-\tau)d\tau \\
&= (h * x)(t),
\end{aligned}
$$

where we put $h(t) = -\Pi(t/2)\sin(\pi t)$. It follows that h is the impulse response of H. Since h has an impulse resposne it is regular, and since it is regular its also linear and time invariant.

The hard way is to first show linear, then show time invariance, and then find this impulse response as the limit

$$h = \lim_{\gamma\to\infty} Hp_\gamma.$$

where the function

$$p_\gamma(t) = \begin{cases} \gamma, & 0 < t \leq \frac{1}{\gamma} \\ 0, & \text{otherwise,} \end{cases}$$

is introduced in Section 3.1. We have

$$
\begin{aligned}
H(ax+by) &= \int_{-1}^{1} \sin(\pi\tau)\big(ax(t+\tau) + by(t+\tau)\big)d\tau \\
&= a\int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau + b\int_{-1}^{1} \sin(\pi\tau)y(t+\tau)\big)d\tau \\
&= aH(x) + bH(y),
\end{aligned}
$$

and so, H is linear. We also have

$$
\begin{aligned}
H\big(T_k(x)\big) &= \int_{-1}^{1} \sin(\pi\tau)T_k(x)(t+\tau)d\tau \\
&= \int_{-1}^{1} \sin(\pi\tau)x(t+\tau-k)d\tau \\
&= T_k\left(\int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau\right) \\
&= T_k\big(H(x)\big),
\end{aligned}
$$

and so, H is time invariant. Now, if H is regular then its impulse response is $h = \lim_{\gamma\to\infty} H(p_\gamma)$. Let h_γ be the signal

$$h_\gamma(t) = \int_{-1}^{1} \sin(\pi\tau)p_\gamma(t+\tau)d\tau.$$

The impulse response exists if h_γ converges for each fixed t as $\gamma \to \infty$. Now, $p_\gamma(t+\tau) = \gamma$ for $t+\tau \in [0, \frac{1}{\gamma})$, i.e. $\tau \in [-t, \frac{1}{\gamma} - t)$, and zero otherwise. The integral ranges from -1 to 1 so we are also interested in those $\tau \in [-1, 1]$. When $t > \frac{1}{\gamma} + 1$ or $t < -1$ the intervals $[-1, 1]$ and $[-t, \frac{1}{\gamma} - t)$ are disjoint and we obtain $h(t) = 0$.

Otherwise, when $[-t, \frac{1}{\gamma} - t) \subset [-1, 1]$, i.e, $-t > -1$ and $\frac{1}{\gamma} - t < 1$ we obtain

$$\begin{aligned}
h_\gamma(t) &= \int_{-1}^{1} \sin(\pi\tau) p_\gamma(t+\tau) d\tau \\
&= \gamma \int_{-t}^{1/\gamma - t} \sin(\pi\tau) d\tau \\
&= -\frac{\gamma}{\pi}\Big(\cos\big(\pi(1/\gamma - t)\big) - \cos(-\pi t)\Big) \\
&= -\frac{\gamma}{\pi}\Big(\cos\big(\pi(t - \tfrac{1}{\gamma})\big) - \cos(\pi t)\Big).
\end{aligned}$$

Put $\Delta = -\frac{1}{\gamma}$ and

$$h_\gamma(t) = \frac{1}{\pi}\frac{\cos\big(\pi(t+\delta)\big) - \cos(\pi t)}{\delta}.$$

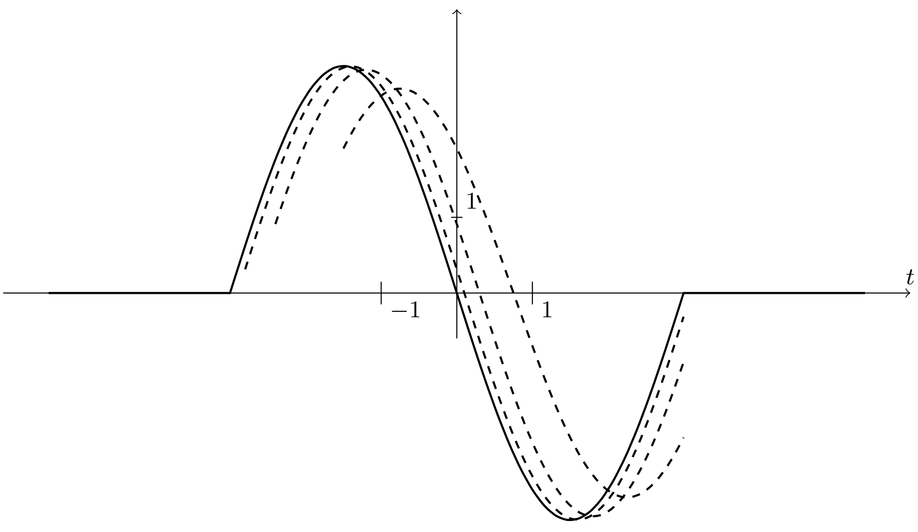
Recognising the limit as $\gamma \to \infty$, or equivalently as $\Delta \to 0$ as

$$\lim_{\delta \to 0} \frac{\cos\big(\pi(t+\delta)\big) - \cos(\pi t)}{\delta} = \frac{d}{dt}\cos(\pi t)$$

we immediately have

$$\lim_{\gamma\to\infty} h_\gamma(t) = h(t) = \frac{1}{\pi}\frac{d}{dt}\cos(\pi t) = -\sin(\pi t).$$

on the interval $t \in [\frac{1}{\gamma} - 1, 1)$. It remains to show what happens on the interval $[-1, \frac{1}{\gamma} - 1)$ that shrinks as $\gamma \to \infty$.



The step response can be found directly by inputing the step function u to the system. That is

$$H(u,t) = \int_{-1}^{1} \sin(\pi\tau) u(t+\tau) d\tau.$$

To find an explicit expression for this integral 3 cases must be considered separately. Observe that $u(t+\tau)$ is nozero only when $\tau > -t$. If $t < -1$ then $u(t+\tau) = 0$ for all $\tau \in [-1, 1]$ and so

$$H(u,t) = \int_{-1}^{1} \sin(\pi\tau) u(t+\tau) d\tau = 0 \qquad t < -1.$$

If $t > 1$ then $u(t+\tau) = 1$ for all $\tau \in [-1, 1]$ and so

$$H(u,t) = \int_{-1}^{1} \sin(\pi\tau) d\tau = -\frac{\cos(\pi\tau)}{\pi}\Big|_{-1}^{1} = \frac{-\cos(\pi) + \cos(-\pi)}{\pi} = 0 \qquad t > 1.$$

Finally, if $-1 \le t \le 1$ then $u(t+\tau)$ is 1 for $\tau \in [-t, 1]$ and 0 for $\tau \in [-1, -t)$ and so

$$\begin{aligned} H(u,t) &= \int_{-t}^{1} \sin(\pi\tau) d\tau \\ &= -\frac{\cos(\pi\tau)}{\pi}\Big|_{-t}^{1} \\ &= \frac{-\cos(\pi) + \cos(-\pi t)}{\pi} = \frac{\cos(\pi t) + 1}{\pi} \qquad 1 \le t \le 1. \end{aligned}$$

An alternative way to find the step response is to apply the integrator system I_∞ to the impulse response $h(t) = -\Pi(t/2)\sin(\pi t)$ we derived earlier. We have

$$H(u) = I_\infty(h) = -\int_{-\infty}^{t} \Pi(\tau/2)\sin(\pi\tau) d\tau.$$

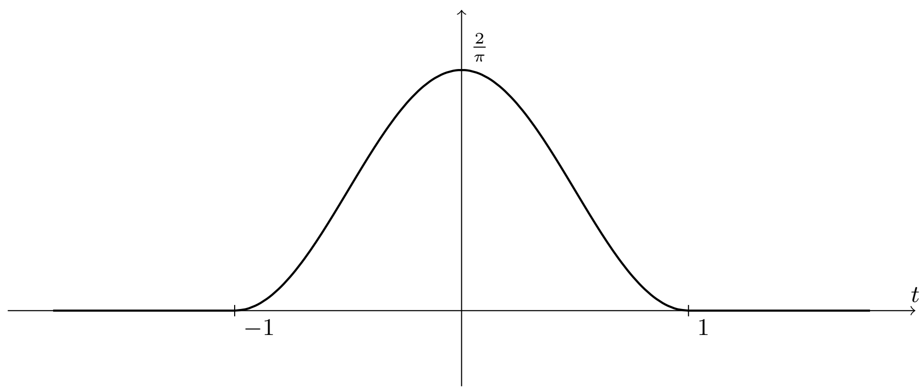
Again the integral needs to be split into cases. When $t < -1$ the $\Pi(\tau/2)$ occuring inside the integral is always zero and so $H(u,t) = 0$ for $t < -1$. When $t > 1$

$$H(u) = -\int_{-1}^{1} \sin(\pi\tau) d\tau = 0.$$

Finally, when $-1 \le t \le 1$ we have

$$H(u) = -\int_{-1}^{t} \sin(\pi\tau) d\tau = \frac{\cos(\pi\tau)}{\pi}\Big|_{-1}^{t} = \frac{\cos(\pi t) + 1}{\pi}.$$

Observe that this is the same as previously. The step response is plotted below.



3.8. Show that $\sum_{\ell=1}^{L} e^{\beta\ell} = \frac{e^{\beta(L+1)} - e^{\beta}}{e^{\beta} - 1}$ (Hint: sum a geometric progression).
Solution: Put $r = e^{\beta}$ and put

$$S_L = \sum_{\ell=1}^{L} e^{\beta\ell} = \sum_{\ell=1}^{L} r^{\ell}.$$

This is the sum of the first L terms of a geometric progression. We have

$$rS_L - S_L = r^{L+1} - r$$

and so

$$S_L = \frac{r^{L+1} - r}{r - 1} = \frac{e^{\beta(L+1)} - e^{\beta}}{e^{\beta} - 1}$$

as required.

3.9. Show that

$$\frac{2j}{L} \sum_{\ell=1}^{L} \sin(\gamma\ell - \theta) e^{-j\gamma\ell} = \alpha + \alpha^* C$$

where $\alpha = e^{-j\theta}$ and $C = e^{-j\gamma(L+1)} \frac{\sin(\gamma L)}{L \sin(\gamma)}$. (Hint: solve Exercise 3.8 first and then use the formula $2j\sin(x) = e^{jx} - e^{-jx}$). **Solution:** We have

$$2j\sin(\gamma\ell - \theta) = e^{j(\gamma\ell-\theta)} - e^{-j(\gamma\ell-\theta)}$$

and so the sum becomes

$$\begin{aligned} \frac{1}{L}\sum_{\ell=1}^{L}(e^{j(\gamma\ell-\theta)} - e^{-j(\gamma\ell-\theta)})e^{-j\gamma\ell} &= \frac{1}{L}\sum_{\ell=1}^{L} e^{-j\theta} - \frac{1}{L}\sum_{\ell=1}^{L} e^{-2j\gamma}e^{j\theta} \\ &= \alpha - \frac{\alpha^*}{L}\sum_{\ell=1}^{L} e^{-2j\gamma}. \end{aligned}$$

The sum is a geometric progression and, using the answer to Exercise 3.8, we have

$$\sum_{\ell=1}^{L} e^{-2j\gamma} = \frac{e^{-2j\gamma(L+1)} - e^{-2j\gamma})}{e^{-2j\gamma} - 1}.$$

The denominator satisfies

$$e^{-2j\gamma} - 1 = e^{-j\gamma}(e^{-j\gamma} - e^{j\gamma}) = -2je^{-j\gamma}\sin(\gamma).$$

The numerator satisfies

$$\begin{aligned} e^{-2j\gamma(L+1)} - e^{-2j\gamma} &= e^{-2j\gamma}(e^{-2j\gamma L} - 1) \\ &= e^{-2j\gamma}e^{-j\gamma L}(e^{-j\gamma L} - e^{j\gamma L}) \\ &= -2je^{-j\gamma(L+2)}\sin(\gamma L). \end{aligned}$$

Thus

$$\sum_{\ell=1}^{L} e^{-2j\gamma} = \frac{-2je^{-j\gamma(L+2)}\sin(\gamma L)}{-2je^{-j\gamma}\sin(\gamma)} = \frac{e^{-j\gamma(L+1)}\sin(\gamma L)}{\sin(\gamma)} = LC$$

where C is defined in the question statement. Now

$$\frac{2j}{L}\sum_{\ell=1}^{L}\sin(\gamma\ell - \theta)e^{-j\gamma\ell} = \alpha - \frac{\alpha^*}{L}LC = \alpha - \alpha^* C$$

as required.

3.10. State whether each of the following systems are: causal, linear, shift-invariant, or stable. Plot the impulse and step response of the systems whenever they exist.

(a) $Hx(t) = 3x(t-1) - 2x(t+1)$

(b) $Hx(t) = \sin\big(2\pi x(t)\big)$

(c) $Hx(t) = t^2 x(t)$

(d) $Hx(t) = \int_{-1/2}^{1/2} \cos(\pi\tau) x(t+\tau) d\tau$

Solution:

(a) The system can be written as $H(x) = 3T_1(x) - 2T_{-1}(x)$ which is a linear combination of shifters. Since the shifter is linear and shift-invariant H will be also (Section 3.3 of the notes). Linearity can also be shown directly

$$\begin{aligned} H(ax+by) &= 3\big(ax(t-1) + by(t-1)\big) - 2\big(ax(t+1) + by(t+1)\big) \\ &= a\big(3x(t-1) - 2x(t+1)\big) + b\big(3y(t-1) - 2y(t+1)\big) \\ &= aHx + bHy. \end{aligned}$$

Shift-invariance can also be shown directly

$$\begin{aligned} T_\tau Hx(t) &= Hx(t-\tau) \\ &= 3x(t-1-\tau) - 2x(t+1-\tau) \\ &= 3T_\tau x(t-1) - 2T_\tau x(t+1) \\ &= HT_\tau x(t). \end{aligned}$$

The system is stable because for every input signal bounded less than $M > 0$, that is, for all input signals x such that $|x(t)| < M$ for all $t \in \mathbb{R}$, we can choose $K = 5M$ and

$$|Hx(t)| = |3x(t-1-\tau) - 2x(t+1-\tau)| \leq 3\,|x(t-1-\tau)| + 2\,|x(t+1-\tau)| < 5M = K,$$

i.e., the output signal is bounded less than K. The system is not causal, because it depends on the input signal x at time $t+1$, i.e., in the 'future'. The system is not regular because it is a linear combination of time shifters, and these are not regular, they don't formally have an impulse response (Section 3.1 of the notes). The system does have a step response equal to $H(u,t) = 3u(t-1) - 2u(t+1)$ that is plotted in the figures below.

(b) The system is causal, in fact it is memoryless since it only depends on the input signal x at time t, i.e. the 'present' time. The system $Hx(t) = \sin\big(2\pi x(t)\big)$ is shift-invariant because

$$HT_\tau x(t) = \sin\big(2\pi T_\tau x(t)\big) = \sin\big(2\pi x(t-\tau)\big) = T_\tau Hx(t).$$

The system is not linear since $H(ax,t) = \sin\big(2\pi a x(t)\big) \neq a \sin\big(2\pi x(t)\big)$ in general. The system is stable, for every input signal x bounded by M, the output is bounded by any $K > 1$. Because the system is not linear, it is not regular. It does have a step response equal to $Hu(t) = \sin\big(2\pi u(t)\big) = 0$. Also acceptable is that is doesn't have a step response because this is a feature we developed for linear shift-invariant systems.

(c) The system is causal and also memoryless. The system $Hx(t) = t^2 x(t)$ is linear because

$$H(ax+by) = t^2(ax+by) = at^2x + bt^2y = aHx + bHy.$$

The system is not shift-invariant because

$$T_\tau Hx(t) = (t-\tau)^2 x(t-\tau) \neq HT_\tau x(t) = t^2 x(t-\tau)$$

in general. The system is not regular because it is not shift-invariant. The system does not have an impulse response. It does have a step response equal to $Hu(t) = t^2 u(t)$. This is plotted in the figures below. Also acceptable is that is doesn't have a step response because this is a feature we developed for linear shift-invariant systems. The system is not stable, for example the input step u is bounded below $M > 1$ but the output Hu is not bounded, it grows indefinitely as $t \to \infty$.

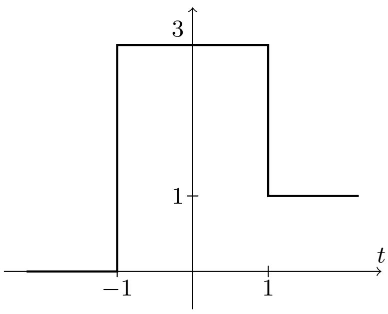
(d) Put $h(t) = \cos(\pi t)\Pi(t)$ where h is the rectangle function (see (1.1.2) of the lecture notes). Now

$$\begin{aligned}
Hx(t) &= \int_{-1/2}^{1/2} \cos(\pi\tau)x(t+\tau)d\tau \\
&= \int_{-1/2}^{1/2} \cos(-\pi\tau)x(t-\tau)d\tau && \text{(change var } \tau = -\tau) \\
&= \int_{-\infty}^{\infty} \cos(-\pi\tau)\Pi(t)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= (h * x)(t).
\end{aligned}$$

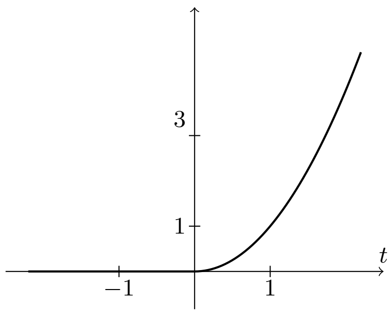
Thus, H is the regular system with impulse response $h(t) = \cos(\pi t)\Pi(t)$. A plot of h is given below. Since H is regular it is also linear and shift-invariant. The impulse response h is absolutely integrable with $\|h\|_1 = \frac{2}{\pi}$ and so H is stable. The system H is not causal because h is nonzero with some $t < 0$, specifically those $t \in (-\frac{1}{2}, \frac{1}{2})$. The step response is given by applying the integrator system I_∞ to h, that is,

$$I_\infty h(t) = \int_{-\infty}^{t} h(\tau)d\tau = \begin{cases} 0 & t \leq -\frac{1}{2} \\ \int_{-1/2}^{t} \cos(\pi\tau)d\tau = \big(\sin(\pi t) + 1\big)/\pi & -\frac{1}{2} < t \leq \frac{1}{2} \\ \frac{2}{\pi} & t > \frac{1}{2}. \end{cases}$$

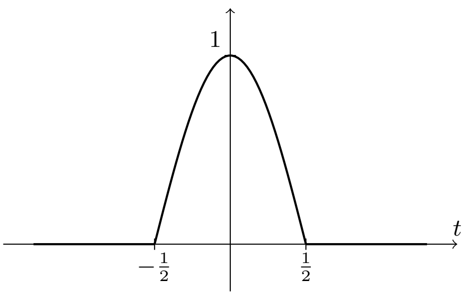
$Hu(t) = 3u(t-1) - 2u(t+1)$

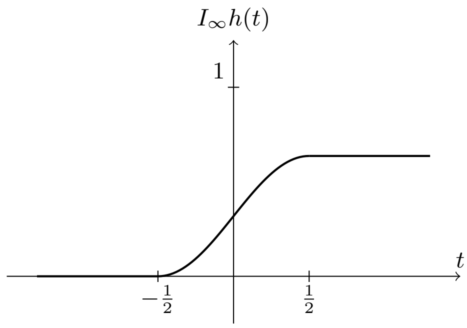


$Hu(t) = t^2u(t)$



$h(t) = \cos(\pi t)\Pi(t)$





$I_\infty h(t)$
1
t
$-\frac{1}{2}$
$\frac{1}{2}$

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