Testable linear shift invariant systems

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Chapter 1

Signals and systems

It is assumed the reader is familiar with the concept of a function! That is, a map from the elements in a set X to the elements in another set Y. Consider sets

$$X = \left\{ \begin{array}{c} \text{Mario} \\ \text{Link} \\ \text{Ness} \end{array} \right\} \qquad Y = \left\{ \begin{array}{c} \text{Freeman} \\ \text{Ryu} \\ \text{Sephiroth} \\ \text{Conker} \\ \text{Ness} \end{array} \right\}.$$

An example of function from X to Y is

$$f(x) = \begin{cases} \text{Conker} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \\ \text{Sephiroth} & x = \text{Ness}. \end{cases}$$

The function f maps Mario to Conker, Link to Sephiroth, and Ness to Sephiroth. The set X is called the **domain** of the function f and the set Y a **range**. The value of f for input x is denoted by $f(x)$ so, for example, $f(\text{Mario}) = \text{Conker}$ and $f(\text{Link}) = \text{Sephiroth}$. The set of all functions mapping X to Y is denoted by $X \to Y$ and so $f \in X \to Y$ in the example above.

A **signal** is a function with domain the set of real numbers $\mathbb{R}$ and range the set of complex numbers $\mathbb{C}$, that is, a signal is a function from the set $\mathbb{R} \to \mathbb{C}$. For example

$$\sin(\pi t), \qquad \frac{1}{2}t^3, \qquad e^{-t^2}$$

all represent signals with $t \in \mathbb{R}$. These signals are plotted in Figure 1.3. Many physical phenomena, such as sound, light, weather, and motion, can be modelled using signals. In this text we primarily focus on examples from

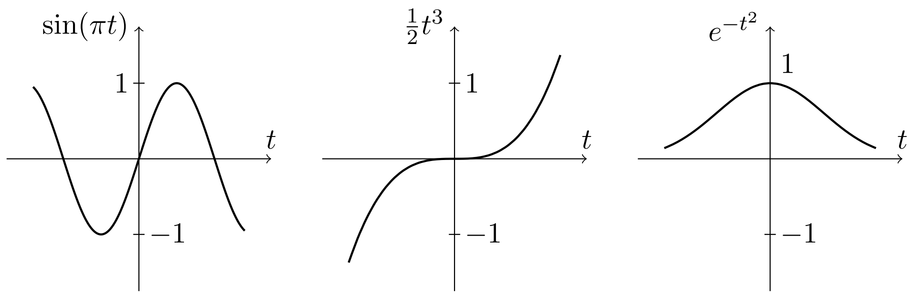

Figure 1.1: Plots of three signals.

electrical and mechanical engineering where signals are use to model changes in voltage, current, position, and angle over time. In these examples, the independent variable t represents "time". However, there is no fundamental reason for this and the techniques developed here can be applied equally well when t represents a quantity other than time. An example where this occurs is image processing.

1.1 Properties of signals

A signal x is **bounded** if there exists a real number M such that

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R}$$

where $|\cdot|$ denotes the (complex) magnitude. Both $\sin(\pi t)$ and e^{-t^2} are examples of bounded signals because $|\sin(\pi t)| \leq 1$ and $|e^{-t^2}| \leq 1$ for all $t \in \mathbb{R}$. However, $\frac{1}{2}t^3$ is not bounded because its magnitude grows indefinitely as t moves away from the origin.

A signal x is **periodic** if there exists a positive real number T such that

$$x(t) = x(t + kT) \qquad \text{for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

The smallest such positive T it is called the **fundamental period** or simply the **period**. For example, the signal $\sin(\pi t)$ is periodic with period $T = 2$. Neither $\frac{1}{2}t^3$ or e^{-t^2} are periodic.

A signal x is **right sided** if there exists a $T \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T$. Correspondingly x is **left sided** if $x(t) = 0$ for all $T > t$. For example, the **step function**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1.1.1}$$

is right-sided. Its horizontal reflection $u(-t)$ is left sided (Figure 1.2). A signal x is said to have **finite support** if it is both left and right sided, that

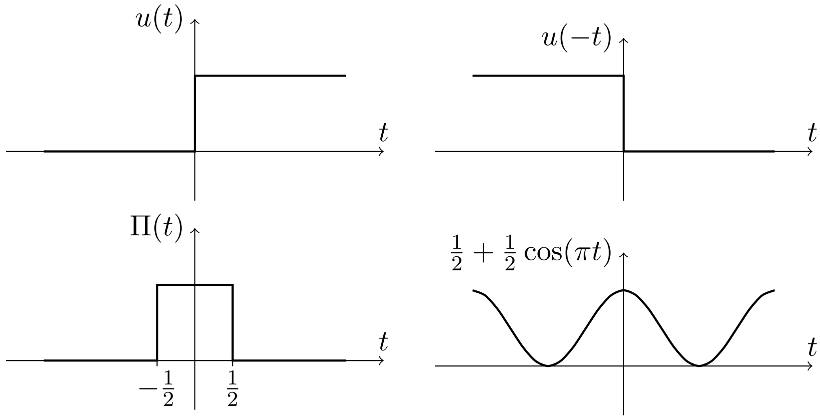

Figure 1.2: The right sided step function $u(t)$, its left sided reflection $u(-t)$, the finite rectangular pulse $\Pi(t)$ and the signal $\frac{1}{2}+\frac{1}{2}\cos(x)$ that is not finite.

is, if there exits a $T \in \mathbb{R}$ such that $x(t) = x(-t) = 0$ for all $t > T$. The signals $\sin(\pi t)$ and e^{-t^2} do not have finite support, but the **rectangular pulse**

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.1.2}$$

does.

A signal x is **even** (or **symmetric**) if $x(t) = x(-t)$ for all $t \in \mathbb{R}$ and **odd** (or **antisymmetric**) if $x(t) = -x(-t)$ for all $t \in \mathbb{R}$. For example, $\sin(\pi t)$ and $\frac{1}{2}t^3$ are odd and e^{-t^2} is even. A signal x is **conjugate symmetric** if $x(t) = x(-t)^*$ for all $t \in \mathbb{R}$ and **conjugate antisymmetric** if $x(t) = -x(-t)^*$ for all $t \in \mathbb{R}$, where $*$ denotes the complex conjugate of a complex number. Equivalently, x is conjugate symmetric if its real part $\mathrm{Re}(x)$ is an even signal and its imaginary part $\mathrm{Im}(x)$ is an odd signal, and x is conjugate antisymmetric if its real part is odd and its imaginary part is even. For example, the signal $e^{-t^2} + j\sin(\pi t)$ where $j = \sqrt{-1}$ is conjugate symmetric and the signal $\frac{1}{2}t^3 + je^{-t^2}$ is conjugate antisymmetric.

A signal x is **continuous at** $t \in \mathbb{R}$ if

$$\lim_{h\to 0} x(t+h) = \lim_{h\to 0} x(t-h)$$

and x is said to be **continuous** if it is continuous at all $t \in \mathbb{R}$. The signals $\sin(\pi t)$, $\frac{1}{2}t^3$, and e^{-t^2} are continuous, but the step function u is not continuous at zero because

$$\lim_{h\to 0} u(h) = 1 \neq 0 = \lim_{h\to 0} u(-h).$$

The set of continuous signals is denoted by $C^0(\mathbb{R})$ or just C^0. A signal x is

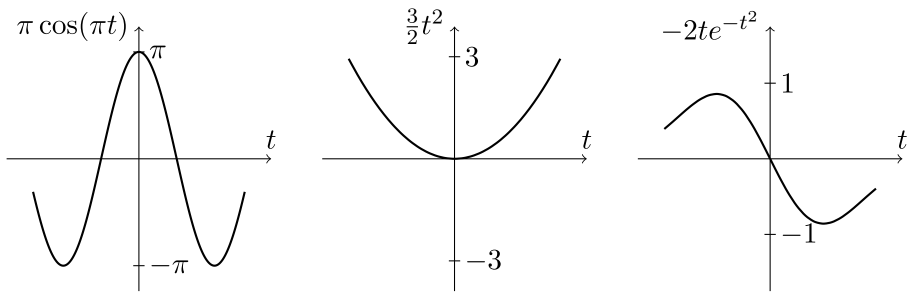

Figure 1.3: Derivatives of the signals $\sin(\pi t)$, $\frac{1}{2}t^3$, e^{-t^2} from Figure 1.3.

continuously differentiable or just **differentiable** if

$$\lim_{h\to 0}\frac{x(t+h)-x(t)}{h} = \lim_{h\to 0}\frac{x(t)-x(t-h)}{h} \qquad \text{for all } t\in\mathbb{R}.$$

Considered as a function of t this limit is called the **derivative** of x at t and is typically denoted by $\frac{d}{dt}x(t)$. For example, the signals $\sin(\pi t)$, $\frac{1}{2}t^3$, e^{-t^2}, and t^2 are differentiable with derivatives

$$\pi\cos(\pi t), \qquad \tfrac{3}{2}t^2, \qquad -2te^{-t^2}, \qquad 2t,$$

but the step function u and the rectangular pulse Π are not differentiable (Exercise 1.7). The set of differentiable signals is denoted by C^1 or $C^1(\mathbb{R})$. A signal is **k-times differentiable** if its $k-1$th derivative is differentiable. The set of k-times differentiable signals is denoted by C^k or $C^k(\mathbb{R})$.

A signal x is **locally integrable** if

$$\int_a^b |x(t)|\,dt < \infty$$

for all finite constants a and b, where $<\infty$ means that the integral evaluates to a finite number. The signals $\sin(\pi t)$, $\frac{1}{2}t^3$, and e^{-t^2} are all locally integrable. An example of a signal that is not locally integrable is $x(t)=\frac{1}{t}$ (Exercise 1.3). A signal x is **absolutely integrable** if

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\,dt < \infty. \qquad (1.1.3)$$

Here we introduce the notation $\|x\|_1$ called the **L^1-norm** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not absolutely integrable, but e^{-t^2} is because [Nicholas and Yates, 1950]

$$\int_{-\infty}^{\infty} |e^{-t^2}|dt = \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}. \qquad (1.1.4)$$

It is common to denote the set of absolutely integrable signals by L^1 or $L^1(\mathbb{R})$. So, $e^{-t^2} \in L^1$ and $\frac{1}{2}t^3 \notin L^1$. A signal x is **square integrable** if

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty.$$

The real number $\|x\|_2$ is called the **L^2-norm** of x. Square integrable signals are also called **energy signals** and the squared L^2-norm $\|x\|_2^2$ is called the **energy** of x. For example, $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not energy signals, but e^{-t^2} is. It has energy $\|e^{-t^2}\|_2^2 = \sqrt{\pi/2}$ (Exercise 1.6). The set of square integrable signals is denoted by L^2 or $L^2(\mathbb{R})$.

We write $x = y$ to indicate that two signals x and y are **equal pointwise**, that is, $x(t) = y(t)$ for all $t \in \mathbb{R}$. This definition of equality is often stronger than we desire. For example, the step function u and the signal

$$z(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

are not equal pointwise because they are not equal at $t = 0$ since $u(0) = 1$ and $z(0) = 0$. It is useful to identify signals that differ only at isolated points and for this we use a weaker definition of equality. We say that two signals x and y are equal **almost everywhere** if

$$\int_a^b |x(t) - y(t)| \, dt = 0$$

for all finite constants a and b. So, in the previous example, while $u \neq z$ pointwise we do have $u = z$ almost everywhere. Typically the term almost everywhere is abbreviated to a.e. and one writes

$$x = y \;\text{ a.e.} \qquad \text{or} \qquad x(t) = y(t) \;\text{ a.e.}$$

to indicate that the signals x and y are equal almost everywhere.

1.2 Systems (functions of signals)

A **system** is a function that maps a signal to another signal. For example,

$$x(t) + 3x(t-1), \qquad \int_0^1 x(t-\tau)d\tau, \qquad \frac{1}{x(t)}, \qquad \frac{d}{dt}x(t)$$

represent systems, each mapping the signal x to another signal. Consider the electric circuit in Figure 1.4 called a **voltage divider**. If the voltage at time t is $x(t)$ then, by Ohm's law, the current at time t satisfies

$$i(t) = \frac{1}{R_1 + R_2} x(t),$$

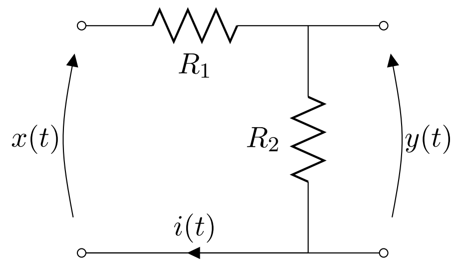

Figure 1.4: A **voltage divider** circuit.

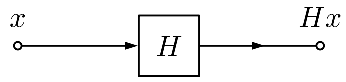

Figure 1.5: System block diagram with input signal x and output signal $H(x)$.

and the voltage over the resistor R_2 is

$$y(t) = R_2 i(t) = \frac{R_2}{R_1 + R_2} x(t). \tag{1.2.1}$$

The circuit can be considered as a system mapping the signal x representing the voltage to the signal $i = \frac{1}{R_1+R_2}x$ representing the current, or a system mapping x to the signal $y = \frac{R_2}{R_1+R_2}x$ representing the voltage over resistor R_2.

We denote systems with capital letters such as H and G. A system H is a function that maps a signal x to another signal denoted $H(x)$. We call x the **input signal** and $H(x)$ the **output signal** or the **response** of system H to signal x. We will often drop the brackets and write simply Hx for the response of H to x[1]. The value of the output signal Hx at t is denoted by $Hx(t)$ or $H(x)(t)$ or $H(x,t)$ and we do not distinguish between these notations. It is sometimes useful to depict systems with a block diagram as in Figure 1.5. The electric circuit in Figure 1.4 corresponds with the system

$$Hx = \frac{R_2}{R_1 + R_2} x = y.$$

This system multiplies the input signal x by $\frac{R_2}{R_1+R_2}$. This brings us to our first practical test.

Test 1 (**Voltage divider**) In this test we construct the voltage divider from Figure 1.4 on a breadboard with resistors $R_1 \approx 100\Omega$ and $R_2 \approx 470\Omega$ with

[1] In the literature it is customary to drop the brackets only when H is a **linear** system (Section 1.5). In this text the we occasionally drop the brackets even when H is not linear. Since we deal primarily with linear systems this Faux Pas will occur rarely.

values accurate to within 5%. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \sin(2\pi f_1 t) \qquad \text{with} \qquad f_1 = 100$$

is passed through the circuit. The approximation is generated by sampling $x(t)$ at rate $F = \frac{1}{P} = 44100$Hz to generate samples

$$x(nP) \qquad n = 0, \dots, 2F$$

corresponding to approximately 2 seconds of signal. These samples are passed to the soundcard which starts playback. The voltage over resistor R_2 is recorded (also using the soundcard) that returns a list of samples $y_1, \dots, y_L$ taken at rate F. The voltage over R_2 can be (approximately) reconstructed from these samples as

$$\tilde{y}(t) = \sum_{\ell=1}^{L} y_\ell \operatorname{sinc}(Ft - \ell) \tag{1.2.2}$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{1.2.3}$$

is the called the **sinc function** and is plotted in Figure 5.1. We will justify this reconstruction in Section 5.4. Simultaneously the (stereo) soundcard is used to record the input voltage x producing samples $x_1, \dots, x_L$ taken at rate F. An approximation of the input signal is

$$\tilde{x}(t) = \sum_{\ell=1}^{L} x_\ell \operatorname{sinc}(Ft - \ell). \tag{1.2.4}$$

In view of (1.2.1) we would expect the approximate relationship

$$\tilde{y} \approx \frac{R_2}{R_1 + R_2} \tilde{x} = \frac{47}{57} \tilde{x}.$$

A plot of $\tilde{y}$, $\tilde{x}$ and $\frac{47}{57}\tilde{x}$ over a 20ms period from 1s to 1.02s is given in Figure 1.6. The hypothesised output signal $\frac{47}{57}\tilde{x}$ does not match the observed output signal $\tilde{y}$. A primary reason is that the circuitry inside the soundcard itself cannot be ignored. When deriving the equation for the voltage divider we implicitly assumed that current flows through the output of the soundcard without resistance (a short circuit), and that no current flows through the input device of the soundcard (an open circuit). These assumptions are not realistic. Modelling the circuitry in the sound card wont be attempted here. In Section 2.2 we will construct circuits that contain external sources of power (active circuits). These are less sensitive to the circuitry inside the soundcard.

Let X and Y be set of signals, that is, $X \subseteq \mathbb{R} \to \mathbb{C}$ and $Y \subseteq \mathbb{R} \to \mathbb{C}$. A system is a function $H \in X \to Y$ that maps each signal from X to a signal from Y. We are free to choose the domain X and range Y at our convenience. In cases such as the voltage divider above it is feasible to choose the domain $X = \mathbb{R} \to \mathbb{C}$, that is, the domain can contain *all* signals. However, this is not always convenient or possible. For example, the system

$$Hx(t) = \frac{1}{x(t)}$$

is not defined at those t where $x(t) = 0$ because we cannot divide by zero. To avoid this we might choose the domain as the set of signals $x(t)$ which are not zero for any t.

Another example is the system I_∞ defined by

$$I_\infty x(t) = \int_{-\infty}^{t} x(\tau) d\tau, \tag{1.2.5}$$

called an **integrator**. The signal $x(t) = 1$ cannot be input to the integrator because the integral $\int_{-\infty}^{t} dt$ is not finite for any t. However, the integrator I_∞ can operate on absolutely integrable signals because, if x is absolutely integrable, then

$$I_\infty x(t) = \int_{-\infty}^{t} x(\tau) d\tau \leq \int_{-\infty}^{t} |x(\tau)| \, d\tau < \int_{-\infty}^{\infty} |x(\tau)| \, d\tau = \|x\|_1 < \infty$$

for all $t \in \mathbb{R}$. We might then choose a domain for I_∞ as the set of absolutely integrable signals L^1. The integrator can also be applied to signals that are right sided and locally integrable because, for any right sided signal x there exists $T \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T$ and so,

$$I_\infty x(t) = \int_{-\infty}^{t} x(\tau) d\tau = \int_{T}^{t} x(\tau) d\tau < \infty$$

for all $t \in \mathbb{R}$ if x is locally integrable. So another possible domain for I_∞ is the set of right sided locally integrable signals. The union of L^1 and the set of right sided locally integrable signals is another possible domain.

The domain used for a given system will usually be obvious from the context in which the system is defined. For this reason we will not usually state the domain explicitly. We will only do so if there is chance for confusion.

1.3 Some important systems

The system

$$T_\tau x(t) = x(t - \tau)$$

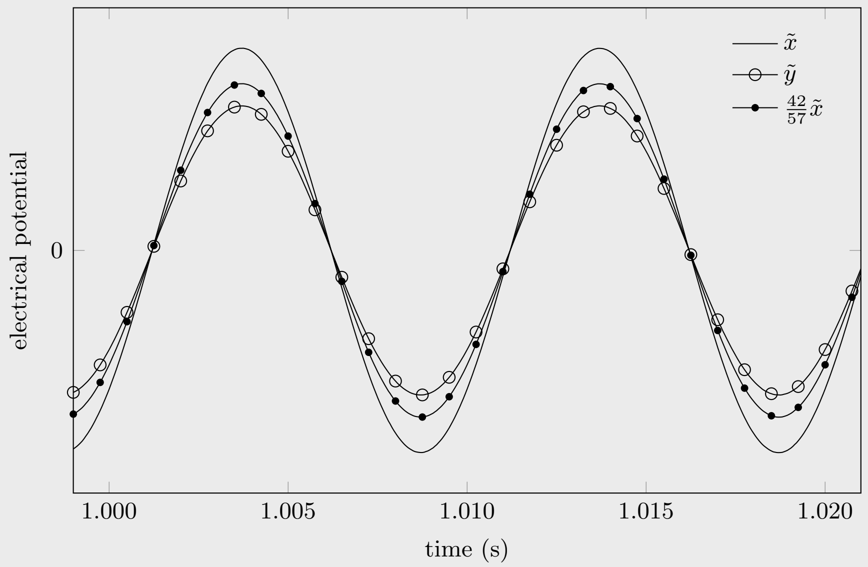

Figure 1.6: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $\frac{47}{57}\tilde{x}$ (solid line with dot) for the voltage divider circuit in Figure 1.4. The hypothesised signal does not match $\tilde{y}$. One reason is that the model does not take account of the circuitry inside the soundcard.

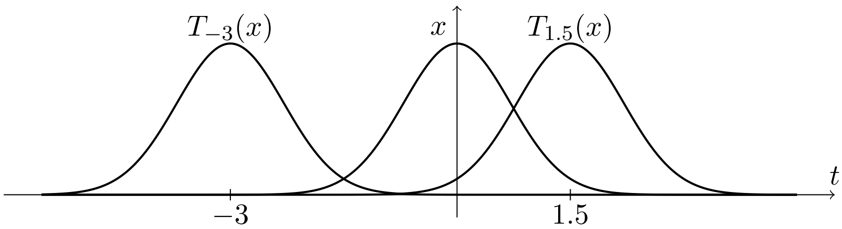

Figure 1.7: Shifter systems $T_{1.5}x(t) = x(t - 1.5)$ and $T_{-3}x(t) = x(t + 3)$ acting on the signal $x(t) = e^{-t^2}$.

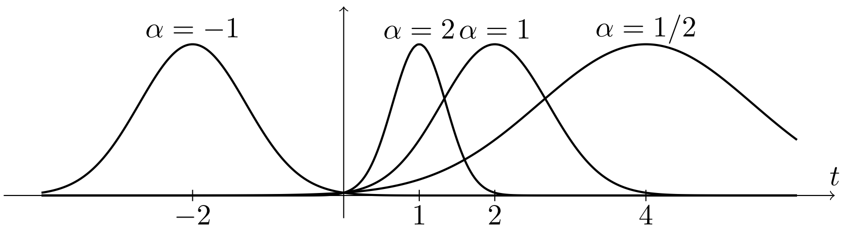

Figure 1.8: Time-scaler system $Hx(t) = x(\alpha t)$ for $\alpha = -1, \frac{1}{2}, 1$ and 2 acting on the signal $x(t) = e^{-(t-2)^2}$.

is called a **time-shifter** or simply **shifter**. This system shifts the input signal along the t axis ("time" axis) by τ. When τ is positive T_τ delays the input signal by τ. The shifter will appear so regularly that we use the special notation T_τ to represent it. Figure 1.7 depicts the action of shifters $T_{1.5}$ and T_{-3} on the signal $x(t) = e^{-t^2}$. When $\tau = 0$ the shifter is the **identity system** $T_0 x = x$ that maps a signal to itself. Another important system is the **time-scaler** that has the form

$$Hx(t) = x(\alpha t), \qquad \alpha \in \mathbb{R}.$$

Figure 1.8 depicts the action of time-scalers with different values for α. When $\alpha = -1$ the time-scaler reflects the input signal in the t axis. When $\alpha = 1$ the time-scaler is the identity system T_0. Both the shifter and time-scaler are well defined for all signals and so it is reasonable to choose their domains as the entire set of signals $\mathbb{R} \to \mathbb{C}$.

Another regularly encountered system is the **differentiator**

$$Dx(t) = \frac{d}{dt}x(t)$$

that returns the derivative of the input signal. We also define a kth differentiator

$$D^k x(t) = \frac{d^k}{dt^k}x(t)$$

that returns the kth derivative of the input signal. The differentiator is only defined for differentiable signals. The largest possible domain for D is the set of differentiable signals C^1 and the largest possible domain for D^k is the set of k-times differentiable signals C^k.

Another important system is the **integrator**

$$I_a(x, t) = \int_{-a}^{t} x(\tau)d\tau.$$

The parameter a describes the lower bound of the integral. In this course it will often be that $a = \infty$. For example, the response of the integrator I_∞ to the signal $tu(t)$ where u is the step function (1.1.1) is

$$\int_{-\infty}^{t} \tau u(\tau)d\tau = \begin{cases} \int_0^t \tau d\tau = \frac{t^2}{2} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Observe that the integrator I_∞ cannot be applied to the signal $x(t) = t$ because $\int_{-\infty}^{t} \tau d\tau$ is not finite for any t. A domain for I_∞ cannot contain the signal $x(t) = t$. As described in Section 1.2, possible domains for I_∞ are the set of absolutely integrable signals L^1 and the set of signals that are both right sided and locally integrable.

1.4 Linear spaces and shift-invariant sets of signals

The domain and range of a system are subsets of the set of signals $\mathbb{R} \to \mathbb{C}$. Before further describing the properties of systems we describe two important types of subsets. These are the **linear spaces** and those subsets with the property of **shift-invariance**.

Let x and y be signals. We denote by $x + y$ the signal that takes the value $x(t) + y(t)$ for each $t \in \mathbb{R}$, that is, the signal that results from adding x and y. For a a complex number we denote by ax the signal that takes the value $ax(t)$ for each $t \in \mathbb{R}$, that is, the signal that results from multiplying x by a (Figure 1.9). For signals x and y and complex numbers a and b the signal

$$ax + by$$

is called a **linear combination** of x and y.

Let $X \subseteq \mathbb{R} \to \mathbb{C}$ be a set of signals. The set X is a **linear space** (or **vector space**) if for all signals x and y from X and all complex numbers a and b the signal formed by the linear combination $ax + by$ is also in X. The set of all signals $\mathbb{R} \to \mathbb{C}$ is a linear space. Another example is the set of differentiable signals, because, if x and y are differentiable, then the linear combination $ax + by$ is differentiable. The derivative is $aDx + bDy$. The set of even signals is another example of a linear space because if x and y are even then

$$ax(t) + by(t) = ax(-t) + by(-t)$$

and so the linear combination $ax + by$ is even. The set of absolutely integrable signals L^1 and the set of square integrable signals L^2 are linear spaces (Exercise 1.8). The set of periodic signals is not a linear space (Exercise 1.9).

A set of signals X is **shift-invariant** if for all signals $x \in X$ and all real numbers τ the shifted signal $T_\tau x$ is also in X. The set of differentiable signals is shift invariant because, if x is differentiable, then $T_\tau x$ is differentiable. The derivative is $T_\tau Dx$. The set of periodic signals is shift invariant (Exercise 1.9) as are L^1 and L^2 (1.8). The set of even signals and the set of odd signals are not shift invariant.

1.5 Properties of systems

A system $H \in X \to Y$ is called **memoryless** if, for all input signals $x \in X$, the output signal Hx at time t depends only on x at time t. For example $\frac{1}{x(t)}$ and the identity system T_0 are memoryless, but

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau)d\tau$$

are not. A shifter T_τ with $\tau \neq 0$ is not memoryless.

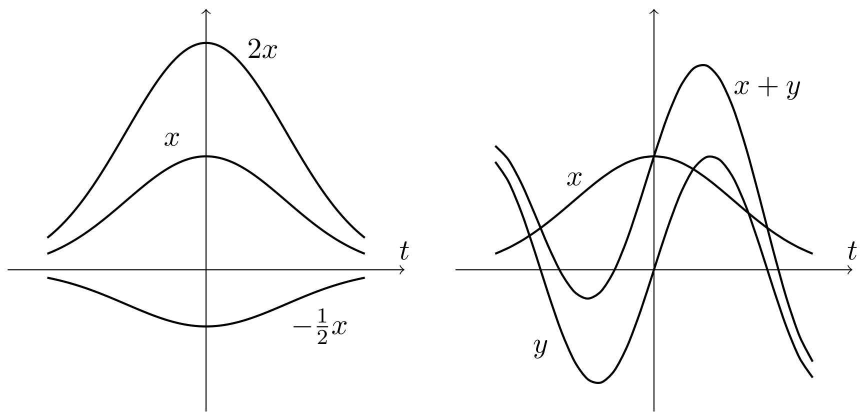

Figure 1.9: The signal $x(t) = e^{-t^2}$ and the signals $2x$ and $-\frac{1}{2}x$ (left). The signals $x(t) = e^{-t^2}$ and $y(t) = \sin(\pi t)$ and the signal $x + y$.

A system $H \in X \to Y$ is **causal** if, for all input signals $x \in X$, the output signal Hx at time t depends on x at times less than or equal to t. Memoryless systems such as $\frac{1}{x(t)}$ and T_0 are also causal. The shifter T_τ is causal when $\tau \geq 0$, but is not causal when $\tau < 0$. The systems

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau)d\tau$$

are causal, but the systems

$$x(t) + 3x(t+1) \qquad \text{and} \qquad \int_0^1 x(t+\tau)d\tau$$

are not causal.

A system $H \in X \to Y$ is called **bounded-input-bounded-output (BIBO) stable** or just **stable** if the output signal Hx is bounded whenever the input signal x is bounded. That is, H is stable if for every positive real number M there exists a positive real number K such that for all input signals $x \in X$ bounded below M, that is,

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R},$$

it holds that the output signal Hx is bounded below K, that is,

$$|Hx(t)| < K \qquad \text{for all } t \in \mathbb{R}.$$

For example, the system $x(t) + 3x(t-1)$ is stable with $K = 4M$ since if $|x(t)| < M$, then

$$|x(t) + 3x(t-1)| \leq |x(t)| + 3\,|x(t-1)| < 4M = K.$$

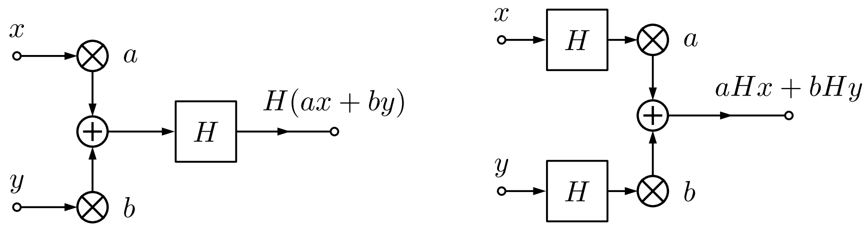

Figure 1.10: If H is a linear system the outputs of these two diagrams are the same signal, i.e. $H(ax + by) = aHx + bHy$.

The integrator I_a for any $a \in \mathbb{R}$ and differentiator D are not stable (Exercises 1.13 and 1.14).

Let $H \in X \to Y$ be a system with domain X and range Y being linear spaces. The system H is **linear** if

$$H(ax + by) = aHx + bHy$$

for all signals $x, y \in X$ and all complex numbers a and b. That is, a linear system has the property: If the input consists of a weighted sum of signals, then the output consists of the same weighted sum of the responses of the system to those signals. Figure 1.10 indicates the linearity property using a block diagram. For example, the differentiator is linear because

$$D(ax + by)(t) = \frac{d}{dt}\big(ax(t) + by(t)\big) \quad = a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) = aDx(t) + bDy(t)$$

whenever both x and y are differentiable. However, the system $Hx(t) = \frac{1}{x(t)}$ is not linear because

$$H(ax + by)(t) = \frac{1}{ax(t) + by(t)} \neq \frac{a}{x(t)} + \frac{b}{y(t)} = aHx(t) + bHy(t)$$

in general.

Let $H \in X \to Y$ be a system with shift-invariant domain X and shift-invariant range Y. The system H is **shift-invariant** if

$$HT_\tau x(t) = Hx(t - \tau)$$

for all signals $x \in X$ and all shifts $\tau \in \mathbb{R}$. That is, a system is shift-invariant if shifting the input signal results in the same shift of the output signal. Equivalently, H is shift-invariant if it commutes with the shifter T_τ, that is, if

$$HT_\tau x = T_\tau Hx$$

for all $\tau \in \mathbb{R}$ and all signals $x \in X$. Figure 1.11 represents the property of shift-invariance with a block diagram.

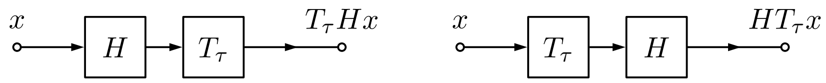

Figure 1.11: If H is a shift-invariant system the outputs of these two diagrams are the same signal, i.e. $HT_\tau x = T_\tau H x$.

1.6 Exercises

1.1. How many distinct functions from the set $X = \{\text{Mario}, \text{Link}\}$ to the set $Y = \{\text{Freeman}, \text{Ryu}, \text{Sephiroth}\}$ exist? Write down each function, that is, write down all functions from the set $X \to Y$.

1.2. State whether the step function $u(t)$ is bounded, periodic, absolutely integrable, an energy signal.

1.3. Show that the signal t^2 is locally integrable, but that the signal $\frac{1}{t^2}$ is not.

1.4. Plot the signal

$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \leq 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

1.5. Plot the signal

$$x(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that x is absolutely integrable, but not square integrable.

1.6. Compute the energy of the signal $e^{-\alpha^2 t^2}$ (Hint: use equation (1.1.4) on page 4 and a change of variables).

1.7. Show that the signal t^2 is differentiable, but the step function u and rectangular pulse Π are not.

1.8. Show that the set of absolutely integrable signals L^1 and the set of square integrable signals L^2 are linear spaces and that they are shift-invariant.

1.9. Show that the set of periodic signals is shift-invariant, but is not a linear space.

1.10. Show that the set of bounded signals is shift-invariant and is a linear space.

1.11. Let $K > 0$ be a fixed real number. Show that the set of signals bounded below K is shift invariant, but is not a linear space.

1.12. Show that the set of even signals and the set of odd signals are not shift invariant.

1.13. Show that the integrator I_a for any $a \in \mathbb{R}$ is not stable.

1.14. Show that the differentiator system D is not stable.

1.15. Show that the shifter T_τ is linear and shift-invariant and that the time-scaler is linear, but not time invariant.

1.16. Show that the integrator I_c with c finite is linear, but not shift-invariant.

1.17. Show that the integrator I_∞ is linear and shift-invariant.

1.18. State whether the system $Hx = x + 1$ is linear, shift-invariant, stable.

1.19. State whether the system $Hx = 0$ is linear, shift-invariant, stable.

1.20. State whether the system $Hx = 1$ is linear, shift-invariant, stable.

1.21. Let x be a signal with period T that is not equal to zero almost everywhere. Show that x is not absolutely integrable.

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