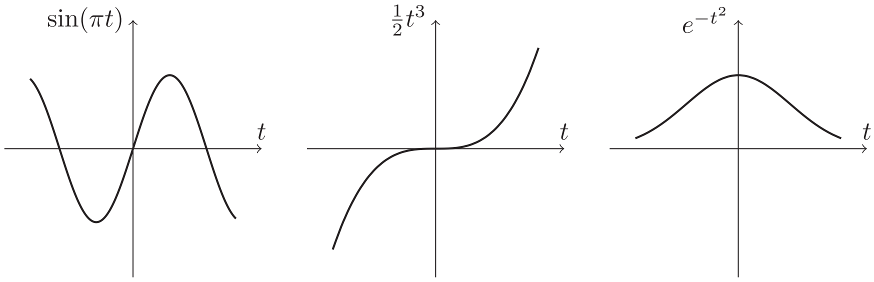

Figure 1: 1-dimensional continuous-time signals

1 Signals and systems

A **signal** is a function mapping an input variable to some output variable. For example

$$\sin(\pi t), \qquad \frac{1}{2}t^3, \qquad e^{-t^2}$$

all represent **signals** with input variable $t \in \mathbb{R}$, and they are plotted in Figure 1. If x is a signal and t an input variable we write $x(t)$ for the output variable. Signals can be multidimensional. This page is an example of a 2-dimensional signal, the independent variables are the horizontal and vertical position on the page, and the signal maps this position to a colour, in this case either black or white. A moving image such as seen on your television or computer monitor is an example of a 3-dimensional signal, the three independent variables being vertical and horizontal screen position and time. The signal maps each position and time to a colour on the screen. In this course we focus exclusively on 1-dimensional signals such as those in Figure 1 and we will only consider signals that are real or complex valued.

1.1 Properties of signals

A signal x is **bounded** if there exists a real number M such that

$$|x(t)| \leq M \qquad \text{for all } t \in \mathbb{R}$$

where $|\cdot|$ denotes the complex magnitude. Both $\sin(\pi t)$ and e^{-t^2} are examples of bounded signals because $|\sin(\pi t)| \leq 1$ and $|e^{-t^2}| \leq 1$ for all $t \in \mathbb{R}$. However, $\frac{1}{2}t^3$ is not bounded because its magnitude grows indefinitely as t moves away from the origin.

A signal x is **periodic** if there exists a real number T such that

$$x(t) = x(t + kT) \qquad \text{for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

For example, the signal $\sin(\pi t)$ is periodic with period $T = 2$. Neither $\frac{1}{2}t^3$ or e^{-t^2} are periodic.

A function $x : \mathbb{R} \to \mathbb{R}$ is called **locally integrable** if for all constants a and b,

$$\int_a^b |x(t)|\, dt$$

exists. In this course we always assume that signals are locally integrable, that is, signals are locally integrable functions mapping $\mathbb{R} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{C}$. Two signals x and y are equal, i.e. $x = y$ if $x(t) = y(t)$ for all $t \in \mathbb{R}$.

A signal x is called **absolutely integrable** if

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt \tag{1.1}$$

exists (evaluates to a finite number). Here we introduce the notation $\|x\|_1$ called the ℓ_1**-norm** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not absolutely integrable, but e^{-t^2} is because

$$\int_{-\infty}^{\infty} |e^{-t^2}| dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}. \tag{1.2}$$

The signal x is called is **square integrable** if

$$\|x\|_2 = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

exists. Square integrable signals are also called **energy signals**, and the value of $\|x\|_2$ is called the **energy** of x (it is also called the ℓ_2**-norm** of x). For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not energy signals, but e^{-t^2} is.

A signal x is **right sided** if there exists a $T \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T$. Correspondingly x is **left sided** if $x(t) = 0$ for all $T > t$. For example, the signal

$$u(t) = \begin{cases} 1 & t > 0, \\ 0 & t \leq 0 \end{cases} \tag{1.3}$$

called the **step function** is right-sided. Its reflection in time $u(-t)$ is left sided (Figure 2). A signal x is called **finite in time** if it is both left and right sided, that is, if there exits a $T \in \mathbb{R}$ such that $x(t) = x(-t) = 0$ for all $t > T$. A signal is called **unbounded in time** if it is neither left nor right sided. For example, the continuous time signals $\sin(\pi t)$ and e^{-t^2} are unbounded in time, but the signal

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise}, \end{cases} \tag{1.4}$$

called the rectangle function is finite in time.

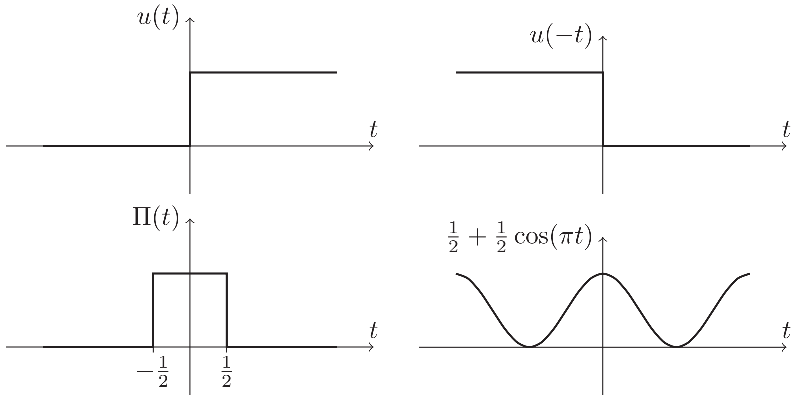

Figure 2: The right sided step function $u(t)$, its left sided reflection $u(-t)$, the finite in time rectangular pulse $\Pi(t)$ and the unbounded in time signal $\frac{1}{2} + \frac{1}{2}\cos(x)$.

1.2 Systems (functions of signals)

A **system** (also known as an **operator** or **functional**) maps a signal to another signal. For example

$$x(t) + 3x(t-1), \qquad \int_0^1 x(t-\tau)d\tau, \qquad \frac{1}{x(t)}, \qquad \frac{d}{dt}x(t)$$

represent systems, each mapping the signal x to another signal. Consider the electric circuit in Figure 3 called a **voltage divider**. If the voltage at time t is $x(t)$ then, by Ohm's law, the current at time t satisfies

$$i(t) = \frac{1}{R_1 + R_2}x(t).$$

and the voltage over the resistor R_2 is

$$y(t) = R_2 i(t) = \frac{R_2}{R_1 + R_2}x(t) \tag{1.5}$$

The circuit can be considered as a system mapping the signal x representing the voltage to the signal $i = \frac{1}{R_1+R_2}x$ representing the current, or a system mapping x to the signal $y = \frac{R_2}{R_1+R_2}x$ representing the voltage over resistor R_2.

We denote systems with capital letters such as H and G. A system H is a function that maps a signal x to another signal denoted $H(x)$. We call x the **input signal** and $H(x)$ the **output signal** or the **response** of system H to signal x. If we want to include the independent variable t we will write $H(x)(t)$ or $H(x,t)$ and do not distinguish between these [Curry and Feys, 1968]. It is sometimes useful to depict systems with a block diagram. Figure 4 is simple block diagram showing the input signal and output signals of a system H.

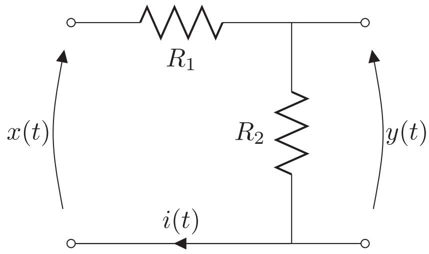

Figure 3: A **voltage divider** circuit.

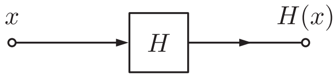

Figure 4: System block diagram with input signal x and output signal $H(x)$.

Using this notation the voltage divider circuit in Figure 3 corresponds with the system

$$H(x) = \frac{R_2}{R_1 + R_2} x = y$$

This system multiplies the input signal x by $\frac{R_2}{R_1+R_2}$. This brings us to our first practical test.

Test 1. (**Voltage divider**) In this test we construct the voltage divider from Figure 3 on a breadboard with resistors $R_1 = 100\Omega$ and $R_2 = 470\Omega$. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \sin(2\pi f_1 t) \qquad \text{with} \qquad f_1 = 100$$

is passed through the circuit. The approximation is generated by sampling $x(t)$ at rate $F_s = \frac{1}{T_s} = 44100$ to generate samples

$$x_n = x(nT_s) \qquad n = 0, \dots, 2F_s$$

corresponding to approximately 2 seconds of signal. These samples are passed to the soundcard which starts playback. The voltage over the resistor R_2 is recorded (also using the soundcard) that returns a lists of samples $y_1, \dots, y_L$ taken at rate F_s. The continuous-time voltage over the capacitor can be (approximately) reconstructed from these samples as

$$\tilde{y}(t) = \sum_{\ell=1}^{L} y_\ell \operatorname{sinc}(t - F_s \ell) \tag{1.6}$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{1.7}$$

is the called the **sinc function**. We will justify this reconstruction in Section 7. Simultaneously the (stereo) soundcard is used to record the input voltage $x(t)$ producing samples $x_1, \dots, x_L$ taken at rate F_s. An approximation of the continuous-time input signal is

$$\tilde{x}(t) = \sum_{\ell=1}^{L} x_\ell \operatorname{sinc}(t - F_s \ell). \tag{1.8}$$

In view of (1.5) we would expect the approximate relationship

$$\tilde{y} = \frac{R_2}{R_1 + R_2}\tilde{x} = \frac{47}{57}\tilde{x}.$$

However, the output resistance of the soundcard itself can not be ignored. Appendix A describes a method for estimating this resistance that, with this hardware used for this test, is approximately $R_o = 250\Omega$. This resistance is in series with R_1 and so, we expect the approximate relationship

$$\tilde{y} \approx \frac{R_2}{R_1 + R_o + R_2}\tilde{x} = \frac{47}{82}\tilde{x}.$$

A plot of $\tilde{y}$, $\tilde{x}$ and $\frac{47}{82}\tilde{x}$ over a 20ms period from 1s to 1.02s is given in Figure 5. Observe that $\tilde{y}$ is indeed close to $\frac{47}{82}\tilde{x}$.

Not all systems apply to all signals. For example the system

$$H(x,t) = \frac{1}{x(t)}$$

is not defined at those t where $x(t) = 0$ because we cannot divide by zero. Another example is the system

$$I_\infty(x,t) = \int_{-\infty}^{t} x(\tau)d\tau, \tag{1.9}$$

called an **integrator**, that is not defined for those signals where the integral above does not exist (is not finite). For example, the signal $x(t) = 1$ cannot be applied with the integrator since the integral $\int_{-\infty}^{t} dt$ does not exist.

Thus, when specifying a system it is necessary to also specify a set of signals to which the system can be applied. For example, the system $H(x,t) = \frac{1}{x(t)}$ can be applied only to those signals $x(t)$ which are not zero for any t. The integrator $I_\infty(x,t)$ can be applied only to those x where the integral $\int_{-\infty}^{t} x(\tau)d\tau$ exists. The set of signals associated with a given system is usually obvious from the specification of the system itself. For this reason we will not usually state the

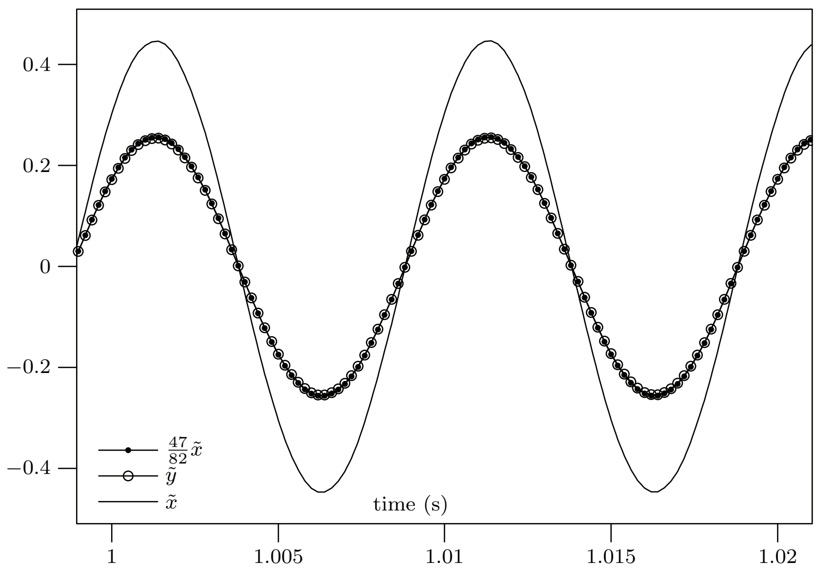

Figure 5: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $\frac{47}{82}\tilde{x}$ (solid line with dot) over a 20ms time window for the voltage divider circuit in Figure 3. The input signal is $x(t) = \sin(200\pi t)$.

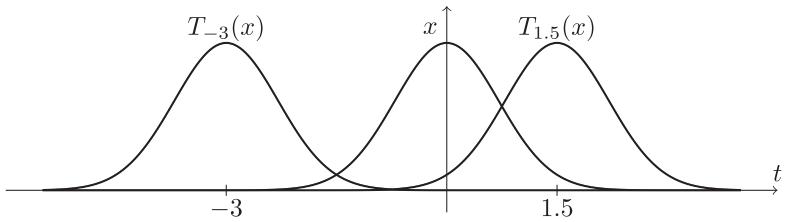

Figure 6: Time-shifter system $T_{1.5}(x,t) = x(t-1.5)$ and $T_{-3}(x,t) = x(t+3)$ acting on the signal $x(t) = e^{-t^2}$.

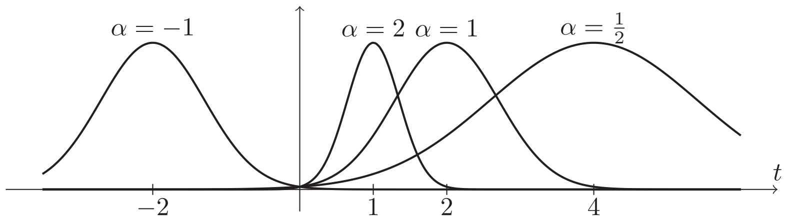

Figure 7: Time-scaler system $H(x,t) = x(\alpha t)$ for $\alpha = -1, \frac{1}{2}, 1$ and 2 acting on the signal $x(t) = e^{-(t-2)^2}$.

set of signals that apply to a given system. We will only do so if there is chance for confusion.

1.3 Some important systems

We call the system

$$T_\tau(x,t) = x(t-\tau)$$

a **time-shifter**. This system shifts the input signal along the t axis ('time' axis) by τ. When τ is positive T_τ delays the input signal by τ. The time-shifter will appear so regularly in this course that we use the special notation T_τ to represent it. Figure 6 depicts the action of time-shifters $T_{1.5}$ and T_{-3} on the signal $x(t) = e^{-t^2}$. When $\tau = 0$ the time-shifter is the **identity system**

$$T_0(x) = x$$

that maps the signal x to itself.

Another important system is the **time-scaler** that has the form

$$H(x,t) = x(\alpha t)$$

for $\alpha \in \mathbb{R}$. Figure 7 depicts the action of a time-scaler with a number of values for α. When $\alpha = -1$ the time-scaler reflects the input signal in the time axis.

Another system we regularly encounter is the **differentiator**

$$D(x, t) = \frac{d}{dt}x(t).$$

the returns the derivative of the input signal. We also define a kth differentiator

$$D^k(x, t) = \frac{d^k}{dt^k}x(t)$$

that returns the kth derivative of the input signal.

Another important system is the **integrator**

$$I_a(x, t) = \int_{-a}^{t} x(\tau)d\tau.$$

The parameter a describes the lower bound of the integral. In this course it will often be that $a = \infty$ or $a = 0$. The integrator can only be applied to those signals for which the integral above exists. For example, the integrator $I_{-\infty}$ can be applied to the signal $tu(t)$ where $u(t)$ is the step function (1.3). The output signal is

$$\int_{-\infty}^{t} \tau u(\tau)d\tau = \int_{0}^{t} \tau d\tau = \frac{t^2}{2}.$$

However, the integrator cannot be applied to the signal $x(t) = t$ because $\int_{-\infty}^{t} \tau d\tau$ does not exist.

1.4 Properties of systems

A system H is called **memoryless** if the output signal $H(x)$ at time t depends only on the input signal x at time t. For example $\frac{1}{x(t)}$ and the identity system T_0 are memoryless, but

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau)d\tau$$

are not. A time-shifter system T_τ with $\tau \neq 0$ is not memoryless.

A system H is **causal** if the output signal $H(x)$ at time t depends on the input signal only at times less than or equal to t. Memoryless systems such as $\frac{1}{x(t)}$ and T_0 are also causal. Time-shifters $T_\tau(x, t) = x(t - \tau)$ are causal when $\tau \geq 0$, but are not causal when $\tau < 0$. The systems

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau)d\tau$$

are causal, but the systems

$$x(t) + 3x(t+1) \qquad \text{and} \qquad \int_0^1 x(t+\tau)d\tau$$

are not causal.

A system H is called **bounded-input-bounded-output stable** or **stable** if the output signal $H(x)$ is bounded whenever the input signal x is bounded. That is, H is stable if for every real number M there exists a real number K such that for all signals x satisfying

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R},$$

it also holds that

$$|H(x,t)| < K \qquad \text{for all } t \in \mathbb{R}.$$

For example, the system $x(t) + 3x(t-1)$ is stable with $K = 4M$ since if $|x(t)| < M$ then

$$|x(t) + 3x(t-1)| \leq |x(t)| + 3\,|x(t-1)| < 4M = K.$$

The integrator I_a for any $a \in \mathbb{R}$ and differentiator D are not stable (Exercises 4 and 5).

A system H is **linear** if

$$H(ax + by) = aH(x) + bH(y)$$

for all signals x and y, and for all complex numbers a and b. That is, a linear system has the property: If the input consists of a weighted sum of signals, then the output consists of the same weighted sum of the responses of the system to those signals. Figure 21 indicates the linearity property using a block diagram. For example, the differentiator is linear because

$$\begin{aligned} D(ax + by, t) &= \frac{d}{dt}\big(ax(t) + by(t)\big) \\ &= a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) \\ &= aD(x,t) + bD(y,t), \end{aligned}$$

but the system $H(x,t) = \frac{1}{x(t)}$ is not linear because

$$H(ax + by, t) = \frac{1}{ax(t) + by(t)} \neq \frac{a}{x(t)} + \frac{b}{y(t)} = aH(x,t) + bH(y,t)$$

in general.

The property of linearity trivially generalises to more than two signals. For example if $x_1, \dots, x_k$ are signals and $a_1, \dots, a_k$ are complex numbers for some finite k, then

$$H(a_1 x_1 + \cdots + a_k x_k) = a_1 H(x_1) + \cdots + a_k H(x_k).$$

A system H is **time-invariant** if

$$H\big(T_\tau(x), t\big) = H(x, t - \tau)$$

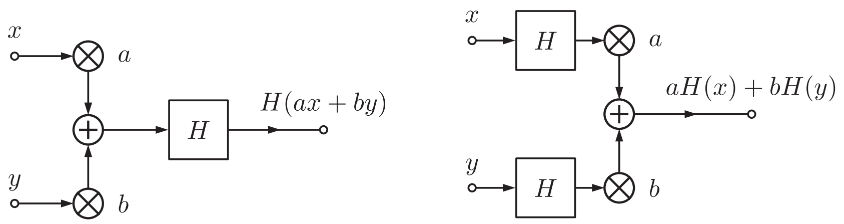

Figure 8: If H is a linear system the outputs of these two diagrams are the same signal, i.e. $H(ax + by) = aH(x) + bH(y)$.

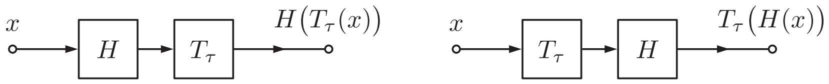

Figure 9: If H is a time-invariant system the outputs of these two diagrams are the same signal, i.e. $H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$.

for all signals x and all time-shifts $\tau \in \mathbb{R}$. That is, a system is time-invariant if time-shifting the input signal results in the same time-shift of the output signal. Equivalently, H is time-invariant if H commutes the time-shifter T_τ, that is, if

$$H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$$

for all $\tau \in \mathbb{R}$ and all signals x. Figure 9 represents the property of time-invariance with a block diagram.

Let S be a set of signals. A system H is said to be **invertible** on S if each signal $x \in S$ is mapped to a unique signal $H(x)$. That is, for all signals $x, y \in S$ then $H(x) = H(y)$ if and only if $x = y$. If a system H is invertible on S then there exists an inverse system H^{-1} such that

$$x = H^{-1}\big(H(x)\big) \qquad \text{for all } x \in S.$$

For example, let S be the set of differentiable signals. The differentiator system D is **not** invertible on S because if $x \in S$ and if $y(t) = x(t) + c$ for any constant c then $D(y) = D(x)$. However, if we restrict S to those differentiable signals for which $x(0) = c$ is fixed, then D *is* invertible on S. The inverse system in this case is

$$D^{-1}(x, t) = I_0(x, t) + c = \int_0^\infty x(t)dt + c$$

because

$$D^{-1}(D(x), t) = \int_0^\infty D(x, t)dt + c = \int_0^\infty \frac{d}{dt}x(t)dt + x(0) = x(t)$$

by the fundamental theorem of calculus.

1.5 Exercises

1. State whether the step function $u(t)$ is bounded, periodic, continuous, differentiable, absolutely summable, an energy signal.
2. Show that the function t^2 is locally integrable, but that the function $\frac{1}{t^2}$ is not.
3. Compute the energy of the signals e^{-t^2} and $e^{-t^2/4}$ (Hint: use (1.2) and a change of variables).
4. Show that the integrator I_a for any $a \in \mathbb{R}$ is not bibo stable.
5. Show that the differentiator system D is not bibo stable.
6. Show that the time-shifter is linear and time-invariant system, but that the time-scaler is neither linear or time invariant
7. Show that the kth differentiator $D^k(x,t) = \frac{d^k}{dt^k}x(t)$ is linear and time-invariant
8. State whether the system $H(x,t) = x(t) + 1$ is linear, time-invariant, bibo stable.
9. State whether the system $H(x,t) = 0$ is linear, time-invariant, bibo stable.
10. State whether the system $H(x,t) = 1$ is linear, time-invariant, bibo stable.