

On the Cramér-Rao bound for polynomial phase signals

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Abstract—Polynomial-phase signals have applications including radar, sonar, geophysics, and radio communication. Of practical importance is the estimation of the parameters of a polynomial phase signal from a sequence of noisy observations. We derive closed form formulae for the Cramér-Rao lower bounds of these parameters under the assumption that the noise is additive white and Gaussian. This is achieved by making use of a family of orthogonal polynomials.

Index Terms—Polynomial-phase signals, Cramér-Rao bound.

I. INTRODUCTION

A uniformly sampled, constant amplitude, polynomial-phase signal of order m has the model [1–3],

$$s_n(b) = \alpha \exp \left(j \sum_{k=0}^m b_k (\Delta n)^k \right) \quad (1)$$

where $n \in \mathbb{Z}$, α is the signal amplitude (a positive real number), the argument $b = (b_0, \dots, b_m) \in \mathbb{R}^{m+1}$ is a vector representing the polynomial coefficients, and Δ is the interval between consecutive samples. We will assume, without loss of generality, that $\Delta = 1$.

Polynomial-phase signals have applications including radar, sonar, geophysics, and radio communication [4]. The signals are also used to describe the sounds emitted by bats for echo-location [1]. Of practical importance is the estimation of the coefficients $\beta = (\beta_0, \dots, \beta_m) \in \mathbb{R}^{m+1}$ from N consecutive samples y_1, \dots, y_N where

$$y_n = s_n(\beta) + w_n \quad (2)$$

and $\{w_n, n \in \mathbb{Z}\}$ is a sequence of noise variables. Many estimators have been studied and implemented [1, 4–11].

We remark on our notation. We use lower case Latin letters for working variables and we use Greek letters for fixed numbers. Where possible we use the Latin letter and its Greek counterpart when there is a relationship between a working variable and a fixed number. For example we use $b = (b_0, \dots, b_m)$ as a working variable to represent the polynomial phase coefficients in (1), and we use $\beta = (\beta_0, \dots, \beta_m)$ to denote the fixed (true) coefficients that we desire to estimate in (2). It has been common in the literature to use a subscript, for example b_0 to denote the true parameter, but this notation is confusing in our case since b_0 is also an element in the vector b .

Under the assumption that $\{w_n\}$ are additive white and Gaussian, Peleg and Porat [2] derived the Cramér-Rao lower

bound for unbiased estimators of the coefficients β . Their approach requires inversion of an $m+1$ by $m+1$ Fisher information matrix that can be poorly conditioned. To avoid this, they provide an approximation that is valid when the number of observations N is sufficiently larger than the order of the polynomial phase signal m . Here we remove the need for this approximation. By converting the polynomial basis to one that is orthogonal we derive a closed form and numerically stable formula for the inverse Fisher information matrix. This leads to closed form formulae for the Cramér-Rao bounds.

Strictly speaking, closed form formulae for the Cramér-Rao bounds have been derived by Peleg and Porat [2], in the sense that inversion of an $m+1$ by $m+1$ matrix requires a number of arithmetic operations that does not depend on N . Nevertheless, we believe there is value in our closed form and numerically stable expressions, over, albeit closed form, but potentially numerically troublesome expressions involving the matrix inverse.

The paper is organised as follows. In Section II we derive the Fisher information matrix in a similar way to Peleg and Porat [2]. The matrix is poorly conditioned and inverting it to obtain the Cramér-Rao bound is difficult. In Section III we describe some general results regarding changing the basis in which a polynomial phase signal may be represented. We describe the Fisher information matrix corresponding with a chosen basis, and relate the matrices between two bases. In Section IV we introduce a family of discrete orthogonal polynomials. By transforming into this basis, the corresponding Fisher information matrix becomes diagonal, making matrix inversion simple. This leads directly to closed form expressions for the Cramér-Rao bounds. To keep our derivation simple we initially assume that the amplitude and the noise variance are fixed constants, and not parameters to be estimated. However, including the amplitude and noise variance is straightforward, and we do this in Section V.

II. THE CRAMÉR-RAO BOUND

We make the assumption that the noise sequence $\{w_n\}$ is independent and identically distributed complex Gaussian, each element w_n having independent real and imaginary parts with zero mean and variance σ^2 . Thus $\mathbb{E}|w_n|^2 = 2\sigma^2$ where \mathbb{E} denotes expectation. The probability density function of w_n is

$$p(x) = \frac{1}{2\pi\sigma^2} \exp \left(-\frac{|x|^2}{2\sigma^2} \right)$$

where $|x|^2$ is the squared magnitude of the complex number x . We wish to estimate β from y_1, \dots, y_N . The likelihood

function is

$$L(b) = \frac{1}{(2\pi\sigma^2)^N} \exp \left(-\frac{1}{2\sigma^2} \sum_{n=1}^N |y_n - s_n(b)|^2 \right).$$

For $i, k \in \{0, \dots, m\}$ define functions

$$f_{ik}(b) = f_{ki}(b) = -\mathbb{E} \frac{\partial^2 \log L(b)}{\partial b_i \partial b_k}.$$

The Fisher information matrix F corresponding with β is the $m+1$ by $m+1$ matrix with elements $F_{ik} = f_{ik}(\beta)$. The elements are

$$F_{ik} = F_{ki} = \frac{\alpha^2}{\sigma^2} \sum_{n=1}^N n^i n^k.$$

It is a convenient property of polynomial phase estimation that the Fisher information matrix does not depend on the value of the polynomial phase coefficients being estimated. That is, F does not depend on β .

Let $\hat{\beta}$ be an unbiased estimator of β . The Cramér-Rao bound asserts: If C is the covariance matrix of $\hat{\beta}$, then $C - F^{-1}$ is positive semidefinite, where F^{-1} is the inverse of F [12]. The diagonal elements of a positive semidefinite matrix are nonnegative so $C_{ii} \geq F_{ii}^{-1}$ and correspondingly,

$$C_{ii} = \text{var } \hat{\beta}_i \geq F_{ii}^{-1}, \quad i = 0, \dots, m.$$

Be aware that F_{ik}^{-1} is the ik th element of the inverse matrix F^{-1} . It is not the reciprocal of F_{ik} . Peleg and Porat [2] found that F can be poorly conditioned and for this reason F^{-1} can be difficult to compute. We will circumvent this problem by applying a change of basis to the polynomial coefficients β . This will lead to a closed form formula for F^{-1} .

III. CHANGING THE BASIS

For a polynomial $g(x) = b_0 + b_1x + \dots + b_mx^m$ of order at most m denote by $\text{coef}(g) = (b_0, b_1, \dots, b_m)$ the vector of length $m+1$ containing its coefficients. Let p_0, \dots, p_m be a family of linearly independent polynomials of orders at most m . By linearly independent it is meant that any polynomial of order at most m can be written as a linear combination $c_0p_0 + c_1p_1 + \dots + c_mp_m$ where c_1, \dots, c_m are real numbers. Observe that p_0, \dots, p_m are linearly independent if and only if the vectors $\text{coef}(p_0), \dots, \text{coef}(p_m)$ are linearly independent. We call p_0, \dots, p_m a *basis* for the space of polynomials of order m . We call the polynomials x^0, x^1, \dots, x^m the *standard basis*.

Let

$$P = \begin{pmatrix} \text{coef}(p_0) \\ \vdots \\ \text{coef}(p_m) \end{pmatrix}$$

be the $m+1$ by $m+1$ matrix with k th row $\text{coef}(p_k)$. If

$$b_0 + b_1x + \dots + b_mx^m = c_0p_0(x) + \dots + c_mp_m(x)$$

then the relationship between the vectors $b = (b_0, \dots, b_m)$ and $c = (c_0, \dots, c_m)$ is $b = cP$. Since P is invertible we also have $c = bP^{-1}$.

Put,

$$r_n(c) = \alpha \exp \left(j \sum_{k=0}^m c_k p_k(n) \right)$$

so that $r_n(c)$ represents a polynomial phase signal expressed in terms of the basis p_0, \dots, p_m . Correspondingly $s_n(b)$ represents a polynomial phase signal expressed in terms of the standard basis x^0, x^1, \dots, x^m . The two signals are related by

$$r_n(c) = s_n(cP) = s_n(b).$$

The observed samples y_1, \dots, y_N from (2) can be written using r_n as

$$y_n = r_n(\gamma) + w_n = s_n(\gamma P) + w_n$$

where $\beta = \gamma P$. We can now consider the problem of estimating γ instead of β . If $\hat{\beta}$ is an estimator of β then the corresponding estimator of γ is $\hat{\gamma} = \hat{\beta}P^{-1}$. If C is the covariance of $\hat{\beta}$ then the covariance of $\hat{\gamma}$ is $(P^{-1})'CP^{-1}$ where $'$ is the transpose.

The likelihood function for γ is

$$L_\gamma(c) = L(cP) = L(b)$$

and we can compute the corresponding Fisher information matrix. It is found to be the $m+1$ by $m+1$ matrix H with elements

$$H_{ik} = H_{ki} = \frac{\alpha^2}{\sigma^2} \sum_{n=1}^N p_i(n)p_k(n).$$

The Cramér-Rao bound for γ is found by taking the inverse of H . It follows, for example from [12, Sec. 3.8], that the Fisher information matrices H and F are related by,

$$H = PFP'.$$

The utility of this relationship is as follows: Computation of the inverse matrix F^{-1} may be difficult, but with a carefully chosen change of basis matrix P , computation of H^{-1} might be easy (for example H could be diagonal). We then have

$$F^{-1} = P'H^{-1}P.$$

We will now apply this approach to compute the inverse Fisher information matrix for polynomial phase signals. The most useful change of basis is described by the following family of polynomials.

IV. THE DISCRETE ORTHOGONAL POLYNOMIALS

Definition 1. The discrete orthogonal polynomial of order k , denoted by d_k , is

$$d_k(x) = \frac{k!}{\binom{2k}{k}} \sum_{s=0}^k (-1)^{s+k} \binom{s+k}{s} \binom{N-s-1}{N-k-1} q_s(x),$$

where q_s is the polynomial

$$q_s(x) = \binom{x-1}{s} = \frac{(x-1)(x-2)\dots(x-s)}{s!}$$

and we define $q_0(x) = 1$.

The discrete orthogonal polynomials (as we have defined them) are monic, i.e. the coefficient of the highest order term

is equal to one, and the k th discrete orthogonal polynomial d_k has order k . This family of polynomials is closely related to what are called the *Gram polynomials* or *discrete Chebyshev polynomials* [13, p. 323][14, 15]. The specific family given by Definition 1 also appears in [16]. In this paper, we refer to d_0, d_1, \dots simply as the discrete orthogonal polynomials, but the reader should be aware that many other families of polynomials also go by this name.

The d_k are orthogonal in the sense that [16],

$$\sum_{n=1}^N d_i(n) d_k(n) = \begin{cases} 0 & k \neq i \\ (k!)^2 \binom{2k}{k}^{-1} \binom{N+k}{2k+1} & k = i, \end{cases}$$

and this will be a useful property for our purpose. Let

$$P = \begin{pmatrix} \text{coef}(d_0) \\ \vdots \\ \text{coef}(d_m) \end{pmatrix}$$

and put $\beta = \gamma P$. The Fisher information matrix corresponding to γ is the $m+1$ by $m+1$ matrix H with elements

$$\begin{aligned} H_{ik} &= \frac{\alpha^2}{\sigma^2} \sum_{n=1}^N d_i(n) d_k(n) \\ &= \begin{cases} 0 & k \neq i \\ \frac{\alpha^2}{\sigma^2} (k!)^2 \binom{2k}{k}^{-1} \binom{N+k}{2k+1} & k = i. \end{cases} \end{aligned}$$

Thus H is diagonal and the inverse has elements

$$H_{ik}^{-1} = \begin{cases} 0 & k \neq i \\ \frac{\sigma^2}{\alpha^2} (k!)^{-2} \binom{2k}{k} \binom{N+k}{2k+1}^{-1} & k = i. \end{cases}$$

The inverse Fisher information matrix in the standard basis is $F^{-1} = P' H^{-1} P$. It remains to find closed form expressions for the elements of P , i.e. for the coefficients of d_0, \dots, d_m .

Observe that the polynomials q_0, q_1, \dots satisfy the recursion

$$q_s(x) = \frac{x-s}{s} q_{s-1}(x), \quad q_0(x) = 1.$$

If $q_s(x) = q_{s0} + q_{s1}x + \dots, q_{ss}x^s$ so that $q_{s0}, q_{s1}, \dots, q_{ss}$ are the coefficients of q_s , then

$$q_{si} = \frac{1}{s} q_{s-1, i-1} - q_{s-1, i}$$

where $q_{00} = 1$ and $q_{0i} = 0$ for $i \neq 0$ and $q_{si} = 0$ if any of s or i is negative.

Now the i th coefficient of d_k is

$$P_{ki} = \frac{k!}{\binom{2k}{k}} \sum_{s=0}^k (-1)^{s+k} \binom{s+k}{s} \binom{N-s-1}{N-k-1} q_{si}$$

which is closed form in the sense that the number of operations required to compute P_{ki} does not depend on N . The inverse matrix $F^{-1} = P' H^{-1} P$ can now be computed in closed form. Evaluating the diagonal terms gives bounds,

$$\begin{aligned} \text{var } \hat{\beta}_i &\geq F_{ii}^{-1} = \sum_{s=0}^m \sum_{t=0}^m P_{si} P_{ti} H_{st}^{-1} \\ &= \sum_{k=0}^m P_{ki}^2 H_{kk}^{-1} \\ &= \frac{\sigma^2}{\alpha^2} \sum_{k=0}^m (k!)^{-2} \binom{2k}{k} \binom{N+k}{2k+1}^{-1} P_{ki}^2, \end{aligned}$$

where the second line follows because H^{-1} is diagonal.

V. INCLUDING THE AMPLITUDE AND NOISE VARIANCE

Including the amplitude α and noise variance σ^2 as estimation parameters and deriving the corresponding Cramér-Rao bounds is straightforward. Our model for a polynomial phase signal is now

$$s_n(b, a) = a \exp \left(j \sum_{k=0}^m b_k (\Delta n)^k \right)$$

where the argument $b = (b_0, \dots, b_m)$ represents the polynomial phase coefficients and the argument a represents the amplitude. We now consider estimation of $\beta = (\beta_0, \dots, \beta_m)$ and α and σ^2 from observations y_1, \dots, y_N where $y_n = s_n(\beta, \alpha) + w_n$. The likelihood function is now

$$L(b, a, v) = \frac{1}{(2\pi v)^N} \exp \left(-\frac{1}{2v} \sum_{n=1}^N |y_n - s_n(b, a)|^2 \right).$$

where the argument v represents the noise variance.

Pack the parameters $(\beta_0, \dots, \beta_m, \alpha, \sigma^2)$ into a vector of length $m+3$ and the corresponding Fisher information matrix can be shown to take the form

$$G = \begin{pmatrix} F & \vdots & \vdots \\ \cdots & \frac{N}{\sigma^2} & 0 \\ \cdots & 0 & \frac{N}{\sigma^4} \end{pmatrix}$$

where \cdots 's indicate zeros. The inverse is

$$G^{-1} = \begin{pmatrix} F^{-1} & \vdots & \vdots \\ \cdots & \frac{\sigma^2}{N} & 0 \\ \cdots & 0 & \frac{\sigma^4}{N} \end{pmatrix}.$$

So, the Cramér-Rao bounds for the polynomial phase coefficients β_0, \dots, β_m are unchanged by considering the amplitude and noise variance as parameters to be estimated. The bound for an unbiased variance estimator $\hat{\sigma}^2$ is

$$\text{var } \hat{\sigma}^2 \geq G_{m+2, m+2}^{-1} = \frac{\sigma^4}{N}.$$

The bound for an unbiased amplitude estimator $\hat{\alpha}$ is

$$\text{var } \hat{\alpha} \geq G_{m+1, m+1}^{-1} = \frac{\sigma^2}{N}.$$

This is in agreement with Peleg and Porat [2]. Note that in [2] the variance of the real and imaginary parts of w_n is $\sigma^2/2$ rather than σ^2 . This explains the factor of 2 difference between some of the formulae in [2] and our own.

VI. CONCLUSION

Using a family of discrete orthogonal polynomials we provide closed form formulae for the Cramér-Rao lower bounds for unbiased estimators of polynomial phase signals.

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