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PhD Research Proposal

Lattice Theory and Signal Processing for Communications

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Contents

1 Introduction

Lattices provide an effective arrangement of discrete points in Euclidean space that makes them useful for neumerous applications in geometry of numbers, mathematics, signal processing for wireless communications, cryptography as well as in the areas of chemistry, materials science and solid-state physics. Important applications in geometric problems and mathematics include sphere packing, sphere covering, the kissing number problem, number theory and combinatorics. Applications in wireless communications include data detecion and channel coding for multiple input multiple output (MIMO) systems [1]. Recently lattices have also found applications for phase unwrapping problems in radar signal processing [2].

In numerous signal processing applications, such as speech, radar and medical imaging, the quantity of primary interest is the phase of a received signal [3, 4]. For example, in radar applications, the phase may provide information about the distance to a target. An inherent property of the phase is that only its principal component is observed, that is, the observed value of the phase is always in the range $[-\pi, \pi)$. In applications, this leads to ambiguities in the value of some parameter of interest (such as the distance to a target). The task of rectifying these ambiguities is called phase unwrapping. The accurate and error-free operation of systems employing phase dependent applications mainly rely upon the failure or success of this process. A typical distance measurement or an image extraction may contain hundreds of phase wraps. Out of these phase wraps, some are false that are caused either by phase noise or by the phase extraction algorithm itself while others are genuine. However, identifiying these false phase wraps out of the genuine ones is an extremely difficult process that makes phase unwrapping algorithm more complex.

Another important reason that makes the phase unwrapping problem more complicated is its accumulative nature. As in case of image processing, an image is scanned chronologically pixel by pixel. A single undetected genuine phase wrap between two neighbouring pixels can result in an error in unwrapping both pixels. This type of error accumulates throughout the whole image during scanning. This accumulative property of the phase unwrapping process enforces that the algorithms designed for this process must be highly efficient in terms of detecting the wrapped phase.

The Chinese remainder theorem (CRT) is a popular tool to solve the phase ambiguity problem in the phase-based target localization/distance measurement techniques. However, phase-based distance measurement methods using CRT for distance estimation suffer from problems such as low spectrum efficiency, special frequency requirements and noise sensitivity [5]. To overcome these problems there is a need to develop robust phase unwrapping algorithm that can overcome these issues. This project will pioneer a novel approach to phase unwrapping that is based on a fundamental task in algebraic number theory, called lattice reduction. This new approach promises to be both computationally simple, and statistically more accurate and robust than the current state-of-the-art phase unwrapping algorithms. This would immediately enhance existing medical imaging devices such as magnetic resonance imaging (MRI) and radar imaging devices such as synthetic aperture radar (SAR).

2 Background and Literature Review

2.1 Important Definitions

2.1.1 Lattice

If $\mathbf{b}_1, ..., \mathbf{b}_n$ are linearly independent vectors in m-dimensional Euclidean space $\mathbf{R}^m$ with $m \geq n$, the set of all vectors

$$\mathbf{\Lambda} = \{\mathbf{x} = u_1\mathbf{b_1} + ... + u_n\mathbf{b_n}\} = \{\sum u_i\mathbf{b_i}|u_i \in \mathbb{Z}\} \tag{2.1}$$

is called an n-dimensional *lattice* $\mathbf{\Lambda}$. Where vectors $\mathbf{b_1}, ..., \mathbf{b_n}$ are referred as a basis of the lattice $\mathbf{\Lambda}$ and $u_1, ..., u_n$ are arbitrary integers. Equivalently, if we define an $m \times n$ matrix $\mathbf{B}$, with columns $\mathbf{b_1}, ..., \mathbf{b_n}$, then the lattice generated by $\mathbf{B}$ is

$$\mathbf{\Lambda}\{\mathbf{x} = \mathbf{Bu}|\mathbf{u} \in \mathbb{Z}^n\} \tag{2.2}$$

2.1.2 Generator matrix

$\mathbf{B} \in \mathbb{R}^{m\times n}$ in Eq.(reflattice:def)is called the *basis or generator matrix*. The rank of the lattice is *m* and its dimension is *n*. The lattice is called a full rank lattice, if $m = n$. If $m > n$, then the lattice points lie in the n-dimensional subspace of $\mathbf{R}^m$. The set of integers $\mathbf{Z}^n$ is also a lattice with identity matrix as its generator matrix. We call it the **integer lattice**. The basis matrix of a lattice is not unique. If **M** is an $n \times n$ matrix with integer elements and $\det\mathbf{M} = \pm 1$ then $\mathbf{B}$ and $\mathbf{BM}$ are both generator matrices for $\mathbf{\Lambda}$. $\mathbf{M}$ is called a **unimodular matrix**.

2.1.3 The Voronoi cell

Around each lattice point *x* is its Voronoi region called the *Voronoi cell*. It consists of all points of $\mathbb{R}^n$ which are at least as close to *x* as to any other lattice point, or more formally [6]

$$Vor(x) = z \in \mathbb{R}^n : ||z - x|| \leq ||z - y||, \forall y \in \Lambda \tag{2.3}$$

The Voronoi regions of all the lattice points are identical due to the translation symmetry of the lattice. The Voronoi region $V(0)$ around the origin is simply called the Voronoi cell of the lattice and it is an *n*-dimensional convex polytope that is symmetric about the origin. Hence, we can write the Voronoi cell $V(0)$ of a lattice as the Voronoi region around origin as [7]

$$Vor(0) = z \in \mathbb{R}^n : ||z|| \leq ||z - y||, \forall y \in \Lambda \tag{2.4}$$

2.2 Lattice reduction techniques

Lattice reduction plays an important role in many signal processing applications in different fields. It provides efficient solutions for many classical problems in lattice theory. For example

lattice reduction is closely related to the search of a shortest vector in a lattice, which results in development of the famous sphere decoding algorithm [8]. Lattice reduction plays an important role in signal processing applications involving integer programming. An example is provided in [9] where lattice reduction is used to solve an integer least squares problem in an efficient manner. While authors in that paper applied lattice reduction to the global positioning system (GPS), they mentioned that their results could have impact in radar imaging and magnetic resonance imaging (MRI). Another important application of lattice reduction with regard to signal processing is color space estimation in JPEG images [10]. Here, the lattice points in three dimensional space correspond to the coefficients of quantized discrete cosine transform in three color planes.

Lattice reduction is the process of finding a good basis (as short as possible and reasonably orthogonal) for a given lattice based on some criterion. Most of the lattice reduction techniques are based on the orthogonal decomposition of a lattice basis matrix. In the following we first discuss QR decomposition and then some of the important lattice reduction techniques.

Consider an $n \times m$ lattice generator matrix (basis) $\mathbf{B}$, where $n \geq m$ and rank$(\mathbf{B}) = m$. The QR decomposition of $\mathbf{B}$ gives $\mathbf{B} = \mathbf{QR}$, where $\mathbf{R}$ is an upper triangular matrix with +ve diagonal entries and $\mathbf{Q} = [\mathbf{q}_1 \dots \mathbf{q}_m]$ is an column-orthogonal matrix. Let $r_{k,l}$ denote the (k, l) entry in $\mathbf{R}$. We have $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_m$, since the orthogonal columns of $\mathbf{Q}$ have unit norm. The QR decomposition expresses the *lth* column of $\mathbf{Q}$ in terms of $\mathbf{q}_1 \dots \mathbf{q}_l$ i.e.

$$\mathbf{b}_l = \sum_{k=1}^{l} \mathbf{r}_{k,l}\mathbf{q}_k \tag{2.5}$$

Here, $\mathbf{r}_{l,l}$ defines the component of $\mathbf{b}_l$ that is orthogonal to the space spanned by $\mathbf{b}_1 \dots \mathbf{b}_{l-1}$ or equivalently by $\mathbf{q}_1 \dots \mathbf{q}_{l-1}$. The QR decomposition actually describes the orthogonality of a basis. If the absolute values of $\mathbf{r}_{1,l} \dots \mathbf{r}_{l-1,l}$ are close to zero, then the basis vector $\mathbf{b}_l$ is nearly orthogonal to the space spanned by $\mathbf{b}_1 \dots \mathbf{b}_{l-1}$. However, if $\mathbf{r}_{1,l} \dots \mathbf{r}_{l-1,l}$ are exactly zero then $\mathbf{b}_l$ is orthogonal to the space spanned by $\mathbf{b}_1 \dots \mathbf{b}_{l-1}$. However, for general lattices exactly orthogonal basis does not exist and one has to compromise for a basis satisfying less strict criteria.

As the columns of the matrix $\mathbf{B}$ define a lattice $\mathbf{\Lambda}$, the same lattice can also be generated by any matrix that is constructed from $\mathbf{B}$ by the three basic column operations i.e. Reflection, Swap and Translation. It is observed that the quality of the basis mainly depends on the translation operation, whereas column swap and reflection provide a systematic approach for appropriate translations. Any sequential combination of these three operations is equivalent to post multiplying $\mathbf{B}$ with a unimodular matrix $\mathbf{T}$. The actual goal of a lattice reduction algorithm is to find this unimodular matrix $\mathbf{T}$, or in other words the goal of a lattice reduction algorithm is to find a suitable sequence of these basic column operations that transform a given basis $\mathbf{B}$ to a reduced basis $\tilde{\mathbf{B}} = \mathbf{BT}$. Now we discuss some important lattice reduction algorithms. In the following the QR factorization of $\tilde{\mathbf{B}}$ will be denoted by $\tilde{\mathbf{B}} = \tilde{\mathbf{Q}}\tilde{\mathbf{R}}$.

2.2.1 Minkowski and Hermite-Korkine-Zolotareff reduction

H. Minkowski [11] proposed a robust reduction criterion that requires that the first vector $\tilde{\mathbf{b}}_1$ of ordered basis $\tilde{\mathbf{B}}$ is a shortest non-zero vector in $\mathbf{\Lambda}(\tilde{\mathbf{B}})$. All the remaining vectors $\tilde{\mathbf{b}}_l$ ($2 \leq l \leq m$) have to be shortest vectors such that the vectors $\tilde{\mathbf{b}}_1 \ldots \tilde{\mathbf{b}}_l$ form a basis of $\mathbf{\Lambda}(\tilde{\mathbf{B}})$. Thus, $\tilde{\mathbf{b}}_1$ is a shortest vector in $\mathbf{\Lambda}(\tilde{\mathbf{B}})$ and it is not a linear combination of $\tilde{\mathbf{b}}_1 \ldots \tilde{\mathbf{b}}_{l-1}$. Hermite-Korkine-Zolotareff-reduced basis [1] definition is related to that of Minkowski. It states that the projection of the basis vector $\tilde{\mathbf{b}}_l$ onto the orthogonal complement of the space spanned by $\tilde{\mathbf{b}}_1 \ldots \tilde{\mathbf{b}}_{l-1}$ are the shortest vectors of the resultant projected lattice. Hence, the first vector $\tilde{\mathbf{b}}_1$ is the shortest vector of $\mathbf{\Lambda}(\tilde{\mathbf{B}})$ as in case of Minkowski reduced basis. However, these two reductions are highly computationally complex.

2.2.2 Gauss reduction

The lattice reduction method introduced by C.F. Gauss [1] is restricted to the lattices of rank $m = 2$, i.e. $\mathbf{B} = (\mathbf{b}_1 \mathbf{b}_2) \in \mathbb{R}^{2\times 2}$. For such lattices, Gauss reduction constructs a basis satisfying the Minkowski and Hermite-Korkine-Zolotareff reduction criterion. Gauss reduction also includes column swapping in addition to the size reduction. A basis $\tilde{\mathbf{B}}$ is size reduced if the elements of the corresponding upper triangular matrix $\tilde{\mathbf{R}}$ satisfy the following condition.

$$|\tilde{r}_{k,l}| \leq \frac{1}{2}|\tilde{r}_{k,k}| \quad \text{for} \quad 1 \leq k < l \leq m \tag{2.6}$$

In Gauss reduction, after size reduction of a given basis $\mathbf{B}$, the columns of the resulting basis $\tilde{\mathbf{B}} = (\tilde{\mathbf{b}}_1 \tilde{\mathbf{b}}_2)$ are swaped if the length of $\tilde{\mathbf{b}}_2$ is smaller than that of $\tilde{\mathbf{b}}_1$ and the resulting basis is again size reduced. This iterative process of size reduction and column swapping is carried out until the length of $\tilde{\mathbf{b}}_1$ is smaller than that of $\tilde{\mathbf{b}}_2$. Gauss reduction results in the two shortest vectors in $\mathbf{\Lambda}$ that form the basis for $\mathbf{\Lambda}$.

2.2.3 LLL reduction algorithm

LLL (or L^3) reduction is named after A.K. Lenstra, H.W. Lenstra and L. Lovász who introduced the famous reduction criterion for any arbitrary lattice dimensions [12]. It is an extension to the Gauss reduction algorithms for lattices of rank $m > 2$. A basis $\tilde{\mathbf{B}} = \tilde{\mathbf{Q}}\tilde{\mathbf{R}}$ is called LLL reduced if

$$|\tilde{r}_{k,l}| \leq \frac{1}{2}|\tilde{r}_{k,k}| \quad \text{for} \quad 1 \leq k < l \leq m \tag{2.7}$$

$$\delta|\tilde{r}_{l-1,l-1}|^2 \leq |\tilde{r}_{l,l}|^2 + |\tilde{r}_{l-1,l}|^2 \quad \text{for} \quad l = 2, \ldots, m. \tag{2.8}$$

where the parameter in the above equation is $1/4 < \delta \leq 1$. The quality and the computational complexity of the reduced basis depends upon the choice of the parameter δ. Larger value of δ result in a better basis but more computational price. A typical choice for δ is 3/4. The inequality in Eq. (2.7) is the condition that must be satisfied for a size-reduced basis and Eq. (2.8) is called the Lovász condition.

2.2.4 Seysen's lattice reduction algorithm

For any lattice $\mathbf{\Lambda}$, there is an associated lattice called the dual lattice defined as

$$\mathbf{\Lambda}^\star = \triangleq \{\mathbf{x}^\star \in span(\mathbf{B}) | \mathbf{x}^T\mathbf{x}^\star \in \mathbb{Z} \quad \text{for} \quad \text{all} \quad \mathbf{x} \in \mathbf{\Lambda}\} \tag{2.9}$$

If $\mathbf{B}$ is the basis for the lattice $\mathbf{\Lambda}$, then $\mathbf{B}^\star$ is a basis for the dual lattice $\mathbf{\Lambda}^\star$ that can be obtained by the right Moore-Penrose pseudoinverse, i.e.

$$\mathbf{B}^\star = \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1} \tag{2.10}$$

The basic concept of Seyen's lattice reduction algorithm [13] is the simultaneous reduction of the basis $\mathbf{B}$ and $\mathbf{B}^\star$. He defined the orthogonality criterion as

$$S(\mathbf{B}) = \sum_{l=1}^{m} \|\mathbf{b}_l\|^2 \|\mathbf{b}_l^\star\|^2$$

It achieves its minimum $S(\mathbf{B}) = m$ if and only if the basis $\mathbf{B}$ is orthogonal. If $S(\mathbf{B}) \leq S(\mathbf{BT})$ for all possible unimodular transformation matrices $\mathbf{T}$, then the basis $S(\mathbf{B})$ is called *S*-reduced. The determination of S-reduced basis is generally computationally too expensive.

2.3 Nearest lattice point problem

Another important problem in lattice theory is the **shortest vector problem**. This problem is closely related to the lattice reduction problem. Apparently the problem of finding the shortest vector is computationally infeasible, however, the LLL algorithm finds the shortest vectors in a reasonable amount of time. Another important problem in lattice theory is the **nearest lattice point problem**. For a given lattice Λ contained in $\mathbb{R}^n$ and a given input point $\mathbf{y} \in \mathbb{R}^n$, the nearest lattice point problem is to find the point $\mathbf{x} \in \mathbf{\Lambda}$ such that the distance, with respect to a given norm (e.g. Euclidean norm), between $\mathbf{y}$ and $\mathbf{x}$ is minimized [14].

$$\mathbf{x} = \text{NearestPt}(\mathbf{y}, \Lambda) \Leftrightarrow \mathbf{y} \in Vor(\Lambda) + \mathbf{x} \tag{2.11}$$

Again nearest lattice point problem appears computationally expensive. However, Babai's algorithm [15] which is based on the LLL algorithm is able to find an almost nearest lattice point in a reasonable amount of time.

Solutions to the nearest lattice point problem have numerous applications in signal processing and communications. Applications in communications theory involve both modulation and quantization. In channel coding, the nearest lattice point problem corresponds to the maximum-likelihood *decoding* in the demodulator [16], [17], [18]. Identically, when a lattice is used as a code-book for vector quantization employing mean-squared error criterion, then encoding of each input vector corresponds to nearest lattice point search [19]. Solutions to the problems of vector perturbation [20–22], minimum mean square error (MMSE) [23] and limited feedback beamforming [24, 25] for multiple-input multiple-out (MIMO) communications systems all involve nearest lattice point search. The unwrapping of phase data from the wrapped phase also corresponds to the nearest lattice point problem in many applications including magnetic resonance imaging (MRI) [26, 27], synthetic aperture radar (SAR) [28–30], range estimation [2] and global positioning system [31, 32].

2.4 Phase unwrapping

Phase ambiguity is a common and important problem in many measurement applications in mathematics and engineering. An inherent property of the phase is that only its principal component is observed, that is, the observed value of the phase is always in the range $[-\pi, \pi)$. In many signal processing applications, this leads to ambiguities in the value of some parameter of interest (such as the distance to a target). The task of rectifying these ambiguities is called **phase unwrapping**. It is considered to be one of the most difficult problems and different researchers have devoted a huge amount of effort to solve this problem and proposed a wide variety of solutions. Authors in [4] clearly explained the difficulties in solving the phase unwrapping problem and discussed some suggested solutions. A huge number of algorithms have been suggested in last two decades and they vary according to their application and approaches to the phase unwrapping problem. The proposed algorithms exist for a wide range of mathematical theory and engineering applications, making it difficult for any researcher to completely understand the wide spectrum of these algorithms.

Theoretical principles that have been used to propose solutions to the phase unwrapping problem include global and local optimization theory, signal and image processing algorithms such as dynamic programming, region growing, graph theory, number theory, Hilbert and cosine transforms, wavelets, the Fourier, network flow algorithms, probability and estimation theories, Bayesian approaches, fractals, statistical approaches, cellular automata, heuristic and exhaustive search algorithms, Greens functions, minimum spanning tree methods, cost functions, etc. The importance of the phase unwrapping problem is reflected by research activity in this area and the fact that over five hundred journal papers have been published proposing different algorithms to solve this problem. However, solutions to the phase unwrapping problem suffer from the lack of generality, i.e. a certain phase unwrapping algorithm is only limited to specific applications and will not work for other applications. There exists no unique phase unwrapping algorithm that can be used to unwrap any kind of phase wrap.

2.4.1 Phase unwrapping and image processing

There are many digital image processing techniques that can be used to extract the phase distribution from images generated by applications such as magnetic resonance imaging (MRI), synthetic aperture sonar and synthetic aperture radar (SAR) etc. In these applications extracted phase is related to the physical properties of the object under consideration. These digital image processing techniques employ **arctangent** function to extract the phase of the signal. The arctangent function produces outputs that are wrapped onto the range $-\pi$ to $+\pi$. Thus an unwrapping step must be added to the phase retrieval process to retrieve the true phase.

Phase unwrapping is a technique applied to the wrapped phase images to eliminate the 2π discontinuities implanted within the phase map. The phase unwrapping process detects 2π phase jumps and subtracts or adds an integer multiple of 2π from the successive pixels, based on a threshold mechanism, to retrieve contiguous form of the phase map. Phase unwrapping is not a simple process because of the presence of different kinds of noise. Until now numerous phase unwrapping algorithms have been developed to unwrap these wraps. However, different forms

of noise that can be found in real world applications make the problem of phase unwrapping difficult and complex to solve.

The field of two-dimensional phase unwrapping has been extensively studied in the last three decades and numerous research articles have been publish providing solutions to this problem. These solutions have varying accuracy and computational requirements depending on the application. Mostly, more accurate algorithms are computationally expensive. These algorithms can be broadly classified into two categories; local and global phase unwrapping algorithms. A complete review of two-dimensional phase unwrapping algorithms can be found in [4]

Local phase unwrapping algorithms are also called path following algorithms. These algorithms employ local integration technique where the phase gradients over a certain path are integrated. The choice of the integration path affects the unwrapped phase, i.e. if two different paths are followed from one pixel to another, it may produce two different results for the unwrapped phase. Local phase unwrapping algorithms try to isolate unreliable regions from the reliable regions. These algorithms are mainly divided into two main types. The first type is the quality guided algorithms. These algorithms unwrap the high quality pixels first and the low quality pixels in the last [33,34].The second type of algorithms are called residue-balancing algorithms . These algorithms attempt to prevent error propagation by identifying the source of noise in the wrapped phase i.e residues. These residues are isolated by using branch cuts that act as barriers. The unwrapping path should not pass these branch cuts once they are identified. This results in the prevention of error propagation [35, 36].

Global phase unwrapping algorithms are also called "minimum norm methods". These algorithms follow a completely different approach to phase unwrapping. These algorithms rely on the minimization of a global function to predict the phase gradient and hence do not rely on an integration path to unwrap a wrapped phase. Global phase unwrapping algorithms try to approximate the unwrapped phase by minimizing a a global error function

$$\varepsilon^p = \|solution - problem\|^p \tag{2.12}$$

Some of the important global phase unwrapping algorithms can be found in [4]. A more advanced algorithm developed in [37] is the minimum L^p-norm method. This algorithm finds solution to the the unwrapped phase that minimizes Eq. (2.12) for an arbitrary value of p.

2.4.2 Phase unwrapping and range estimation

Range estimation is an important parameter in solving many real world engineering applications involving tracking and localizing moving targets [38], wireless indoor positioning [39], ultrasonic systems [40] and radio navigation [41]. Various signal parameters can be estimated for range estimation, based on the system constraints and accuracy requirements of the application under consideration. Common signal parameters considered for range estimation include power, direction and/or time of arrival of the received signal, which are briefly described in the following paragraphs.

Range estimation from received signal strength (RSS): The energy of a signal traveling between the transmitter and the receiver is a signal parameter that contains information about the distance or range between them. It is commonly referred to as RSS. It can be used together with the shadowing model and path loss to estimate a distance. A signal traveling between two nodes experiences multi-path fading, path loss and shadowing. However, over a long time interval, averaging RSS does not includes the effects of shadowing and multi-path fading and the path loss model can be represented as

$$\bar{P}(d) = P_0 - 10nlog_{10}(d/d_0) \tag{2.13}$$

Where $\bar{P}(d)$ is the received power in dB at a distance d whereas P_0 is the received power in dB at a small reference distance d_0. n is the path-loss exponent. However, in practical scenarios, the observation interval is long enough to include the shadowing effects. Therefore, the received power is generally modeled to include both the shadowing effects and path loss. Shadowing is commonly modeled as a Gaussian random variable with zero mean and a variance of σ_{sh}^2. Therefore, the received power can be expressed as [42]

$$P(d) \sim \mathcal{N}(\bar{P}(d), \sigma_{sh}^2) \tag{2.14}$$

where $P(d)$ is defined in Eq.(2.13). This model is valid for both line-of-sight and non-line-of-sight scenarios with a suitable choice of channel parameter. From Eq.(2.14), the Cramer-Rao lower bounds (CRLBs) for the unbaised distance estimator can be given as [43]

$$\sqrt{Var\{\hat{d}\}} \geq \frac{(\ln 10)\sigma_{sh}d}{10n} \tag{2.15}$$

where $\hat{d}$ represents the unbiased estimate for d. It is observed that the accuracy of the RSS method deteriorates as the distance from the target increases.

Range estimation from angle of arrival (AOA): Another signal parameter that can be used to estimate the distance between two nodes is the angle of arrival (AOA) of the received signal. Commonly, antenna array are used to measure the AOA of a signal. If the antenna array geometry is known, the angle information of the received signal is obtained from the difference in arrival times of an incoming signal at these antenna arrays. Consider a uniform linear array (ULA) with N_a antenna elements. Assume the fading coefficient α is the same for all signals received at the array elements, then the CRLB on the variance of unbiased AOA estimators can be expressed as [42]

$$\sqrt{Var\{\hat{\psi}\}} \geq 1\frac{\sqrt{3}c}{\sqrt{2}\pi\sqrt{SNR}\beta \triangle \sqrt{N_a(N_a^2-1)}\sin\psi} \tag{2.16}$$

where c is the speed of light, ψ is the angle of arrival, SNR is the signal to noise ration for each element, $\triangle$ is the spacing between the array elements and β is the effective bandwidth. From 2.16, it is observed that the accuracy of AOA estimation increases by an increase in SNR, β and/or $\triangle$. It is also observed that the maximum AOA estimation accuracy is achieved when the ULA line and the signal direction are perpendicular to each other.

Range estimation fom time of arrival (TOA): Similar to the RSS, the time taken by a signal for traveling from one node to the other node i.e. the time-of-arrival (TOA) is another important

parameter that conveys information about the distance between two nodes. In order to calculate this parameter, the two nodes must either be synchronized or exchange timing information by certain protocols. Conventionally, matched filter (MF) receiver or correlator is used for the estimation of TOA. Suppose that a signal $s(t)$ is transmitted from one node towards the another, then the received signal can be expressed as [42]

$$r(t) = s(t - \tau) + n(t) \tag{2.17}$$

where $n(t)$ represents the white Gaussian noise with zero mean and a spectral density of $\mathcal{N}_0/2$ and τ is the TOA. For the signal model in Eq. (2.17), both MF and correlator-based approaches optimal in the are maximum likelihood (ML) sense. However, in practical scenarios, the signal follows multiple paths to arrive at the receiver and in such environment both MF and correlator-based approaches become sub-optimal. High-resolution time delay estimation techniques **??** are employed for accurate TOA estimation in multi-path environments. In order to understand the relation between the theoretical limits for TOA estimates and the signal bandwidth, consider the CRLB for the model in Eq. (2.17), which is given as [42]

$$\sqrt{Var(\hat{\tau})} \geq \frac{1}{2\sqrt{2}\pi\sqrt{SNR}\beta} \tag{2.18}$$

where $\hat{\tau}$ denotes the unbiased TOA estimate, and β represents the effective signal bandwidth. The accuracy of the TOA estimation can be enhanced by increasing the effective signal bandwidth and/or the SNR. Hence, TOA estimation can provide an accurate range measurement for ultra-wide-band (UWB) systems.

Range estimation from phase measurements: Specifically, range estimation using the phase measurements uses the wavelength of the signal to estimate the distance. Since the wave length of the radio signal is usually very small, range estimation using the phase of a signal provides high accuracy. However, when the distance to be measured is longer than the signal wavelength, an unknown ambiguity is introduced in the form of an integer multiple of the signal wavelength [44]. As a result, the estimated distance is also ambiguous and its "integral" wavelength information is lost. The phase shift, due to the propagation characteristics of electromagnetic wave, for a signal after traveling a distance d at a frequency f can be expressed as [5]

$$\phi_0 \equiv \frac{df}{c} 2\pi \pmod{2\pi} \tag{2.19}$$

where c is the velocity of the light. The subscript "0" in Eq.(2.19) represents the ideal value of the phase in the absence of noise. Because the phase always lies in the interval $[0,\ 2\pi)$, the distance δ_0 is d modulo the wavelength $\lambda = c/f$

$$\delta_0 = \frac{\phi_0}{2\pi}\lambda \equiv d \pmod{\lambda} \tag{2.20}$$

where the integral part of the wavelength is ignored. Range estimation using phase measurements provide higher accuracy as compared to the RSS, AOA and TOA methods. The use of phase only signal to measure the distance of a target is a common example where ambiguous phase is introduced [38]. Specially, in sensor networks involving ranging and positioning applications, the ranging techniques involve two or more frequencies known as multiple frequency continuous wave (MFCW). MFCW ultimately increases the maximum unambiguous detection

range [45]. A more complex example is the Radio Interferometric Positioning System (RIPS) where sum of signal phase differences are measured for wireless mote localization. A single RIPS measurement consists of a free mote and three anchor motes. Two motes, acting as transmitters, transmit pure sinusoidal waves at two close frequencies. The two other motes, acting as receivers, receive an interference signal at a low beat frequency. Phase difference between the received interference signals at two receiving motes can be used to describe the sum of range difference between the four motes. The measurement of phase difference is ambiguous in a feasible RIPS setup due to hardware constraints [46].

Ahmad *et al.* [47] used dual-frequency radars to address the ranging and localization problem. In RIPS measurements, they actually extended the dual-frequency to multiple frequency to improve the measurable range and accuracy of the system. Total distance is reconstructed from the phases by measuring the same distance with multiple frequencies (wavelengths). It then searches for the ambiguous integral parts of these wavelengths and calculates the final estimate by using the minimum mean squared error criterion. However, the computational complexity of this method increases exponentially with the number of wavelengths used as the search requires a large number of computations.

Phase ambiguity problem also occurs in single frequency estimation and is know as phase wrap problem [48]. Authors in this paper formulated the maximum likelihood estimation of frequency as a nearest lattice point problem. They showed that the unwrapping of the input phase data can be interpreted as the nearest lattice point problem, in a certain sense. Based on Babai's nearest lattice point algorithm, they have presented an algorithm for the frequency estimation that was able to estimate the frequency for a moderately low SNR values of less than 5dB. However, the algorithm was proved to be computationally expensive but they suggested that a similar approach can be applied for the related problems and it may result in efficient algorithms.

One way to address the phase ambiguity problem is to use multiple frequencies for the transmitted signal. When multiple frequencies are used to measure the distance between the transmitter and the target, multiple values of the received signal phase shifts modulo 2π are observed. The integer ambiguities in the phase shifts corresponding to the distance measurements can now be described by a class of linear Diophantine Equations [49]. A Diophantine Equation is an indeterminate polynomial that allows the variables to be integers only. Solution to the linear Diophantine Equations is not trivial as the number of equations available is smaller than the number of unknown variables. Solutions to these equations require finding integers that satisfy all these equations [50].

A number of techniques are available in the literature to address this problem. **Chinese Remainder Theorem (CRT)** is the most popular among them. In the absence of phase noise, CRT method can be used to exactly calculate the unknown distance. One condition for the CRT method is that the least common multiple of the moduli must be large enough to resolve the ambiguity. However, in the presence of phase noise, CRT method for distance measurement also suffers from the phase ambiguity which is exhibited by the wrapped error distribution. Therefore, this problem must be addressed using a statistical estimation technique. A CRT based method is proposed in [51] and [52] to resolve the phase measurement ambiguity in the presence of phase noise via the least square type search process. CRT methods employing max-

imum likelihood estimation approaches [53] are also studied recently and when these methods are applied to node localization using RIPS measurements [54], it is observed that the computational complexity of these algorithms increases exponentially with the number of the moduli used. Recently a "closed form CRT algorithm" is presented in [55] where the difference between the noise amplitudes of any two measurements is assumed to be less than a half. As a result the difference between any two measurements is directly rounded to an integer to simplify the extensive search process.

The phase measurement ambiguity problem can also be resolved using **data association techniques** as presented in [38]. They have estimated the trajectory/location of a target using phase-only measurements by three sensors in different locations. A space look-up table is used to map the phase wrapped measurement into a target location and then the target location ambiguity is resolved in two cases. The technique of stochastic filtering employing data association is proposed for the targets in significant motion, whereas a distribution discrimination method is proposed for the targets with micro motion and probability distribution function (pdf) of phase measurement is derived for such targets.

Another important approach to solve the ambiguous phase measurement is in the context of frequency estimation where the **least square phase unwrapping estimator** is used to find the frequency and phase of an unknown signal [56]. Least square phase unwrapping estimator is actually a solution to the nearest lattice point problem in lattice theory. Sphere decoder algorithm [8, 57] is commonly used to solve the nearest lattice point problem but it is computationally infeasible. Some of the computationally efficient least square phase unwrapping estimator are presented in [58, 59].

Recently, authors in [2] proposed a new method based on lattice theoretic concepts to solve the problem of range estimation involving ambiguous phase measurements. Their method consists of two parts. Firstly, a lattice theoretic technique is applied to convert the unwrapped phase measurement error distribution problem to the nearest lattice point problem. Using some reasonable assumptions they have showed that the nearest point can be found by directly solving corresponding Diophantine equations in a closed form. Secondly, a maximum likelihood estimator based on unwrapped phase measurements distribution is used to estimate the unknown distance. Their algorithm is efficient as compared to the existing algorithms for phase unwrapping. They have also derived an upper bound for the scaling of the wavelength dependent measurement noise variance. This ensures that the exact number of whole wavelengths can be found, with a probability close to unity, in the unknown distance.

In this thesis we will examine the application of the nearest lattice point problem to the phase unwrapping problem. The primary aim of this research is to investigate the ability of lattice reduction algorithms for the purpose of phase-unwrapping. There are two components to this aim. Firstly, we aim to develop computationally efficient lattice reduction algorithms tailored for the purpose of phase unwrapping. Secondly, we aim to construct a rigorous statistical framework for understanding and analyzing phase unwrapping algorithms. This analysis will pave the way for future developments in signal processing applications, such as radar and medical imaging, where phase is the key parameter.

3 Research Methodology

The success or failure of phase unwrapping algorithms affects the overall performance of important imaging devices including synthetic aperture radar (SAR) and magnetic resonance imaging (MRI). This research will pioneer a novel approach to phase unwrapping algorithm that is based on lattice reduction. This research will produce computationally efficient and statistically accurate and robust phase unwrapping algorithms, enhancing important imaging tools such SAR and MRI. The primary aim of this research is to investigate the ability of lattice reduction algorithms for the purpose of phase-unwrapping. The main objectives of this thesis are as follows:

- **Computation of Bit Error Rate for the E8 Lattice**.
- **A novel approach to phase unwrapping based on lattice reduction**.
- **Development of efficient lattice reduction algorithms for phase unwrapping problem**.
- **Performance analysis of phase unwrapping algorithms based on lattice reduction**.
- **Comparison of lattice and non-lattice based algorithms for phase unwrapping**.

3.1 Computation of bit error rate for the E_8 lattice

The first task of this research is to understand the structure of some important lattices and based on that understanding compute the bit error rate for the most important lattice E_8. The Voronoi cell of a lattice completely describes the geometric structure of a lattice. The knowledge of the structure of the Voronoi cell of a lattice leads to the solution of some important parameters of a lattice including packing radius, kissing number, covering radius, thickness, center density and normalized second moment (quantizing constant). A number of desirable properties of the E_8 lattice such as the densest packing, highest kissing number and being the best quantizer in eight dimensions make it an attractive choice for numerous signal processing applications. It is well known that the Voronoi cell of A_n is equivalent to the projection of Voronoi cell of Z^{n+1} lattice into the zero sum plane [60]. It is also known that the Vorornoi cell of E_8 lattice can be obtained from the Voronoi cell of the A_8 lattice by a systematic approach [Reference to be added]. This approach and the corresponding results for the error computation of the Voronoi cell of E_8 lattice are discussed in detail in the following section "Preliminary Results". The error performance of the codes constructed from the A_n lattice is already presented in [60]. Based on these result we computed the error performance of the codes constructed from the E_8 lattice.

3.2 A novel approach to phase unwrapping based on lattice reduction

Two important problems in lattice theory are the lattice reduction problem and the nearest lattice point problem. Solutions to these problems have numerous applications in signal and image processing applications. However, these problems have not been explored with reference to the phase unwrapping problem, specially in the presence of phase noise. In this thesis we will implement a novel approach to phase unwrapping based on lattice reduction. The initial implementation will make use of standard algorithms for lattice reduction that are already used in a variety of applications. As a first task these lattice reduction techniques will be used in range estimation to unwrap the phase of a reflected signal from a stationary target. After successful completion of this task, it will be extended for the range estimation problem for the moving targets.

3.3 Development of efficient lattice reduction algorithms for phase unwrapping problem

Although there are a number of lattice reduction algorithms available in the literature and can be employed for the phase unwrapping problem. However, these lattice reduction algorithms are not specifically designed for the phase unwrapping problem and can be computationally expensive for this problem. Hence, there is a need to develop simple and computationally feasible algorithms for the phase unwrapping problem. The phase unwrapping problem leads to lattices with specific properties. By exploiting these properties, computationally simple lattice reduction algorithms, tailored for the purpose of phase unwrapping, can be developed. We will investigate these specific lattice properties with reference to the phase unwrapping problem and will develop and implement lattice reduction algorithms for the purpose of phase unwrapping that are computationally feasible.

3.4 Performance analysis of phase unwrapping algorithms based of lattice reduction

We will opportunistically develop techniques for the analysis of existing phase unwrapping algorithms whenever these techniques do not already exist in the literature. We will also derive analytical expressions to describe the performance of lattice reduction algorithms tailored for the phase unwrapping problem. It will result in the construction of a rigorous statistical framework to solve the phase unwrapping problem employing information theoretic concepts from the lattice theory.

3.5 Comparison of lattice and non-lattice based algorithms for phase unwrapping

We will also compare the lattice reduction based phase unwrapping algorithms against existing phase unwrapping algorithms to verify the usefulness and computational feasibility of our algorithms. This comparison will be based on both the simulated data and real data obtained from magnetic resonance imaging (MRI) and synthetic aperture radar (SAR). The simulation tools that we will use for this comparison include different programming softwares such as Sage, Java, MATLAB etc. These experiments will also be used to validate the statistical analysis performed in the last step.

4 Preliminary Results

The root lattices A_n, D_n, E_6, E_7, E_8, due to their highly symmetric structure, provide extremely efficient structured codes for the additive white Gaussian noise (AWGN) channel [61]. The Voronoi cell of a lattice inherits useful properties such as covering radius, packing radius, kissing number, thickness, center density and the normalized second moment. The Voronoi cell of a lattice can also be used to evaluate the error probability of a lattice code. A number of desirable properties of the E_8 lattice such as the densest packing, highest kissing number and being the best quantizer in eight dimensions [61] make it an attractive choice for numerous signal processing applications [62–64]. It has a triangular generator matrix, which makes it suitable for encoding. It also possesses an efficient decoding algorithm. In the even coordinate system E_8 consists of the points

$$\{(x_1, ..., x_8) : \text{ all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + 1/2 , \sum x_i \equiv 0 \pmod 2\} \tag{4.1}$$

The odd coordinate system is obatined by changing the sign of any coordinate: then points are

$$\{(x_1, ..., x_8) : \text{ all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + 1/2 , \sum x_i \equiv 2x_8 \pmod 2\} \tag{4.2}$$

Authors in [60] derived formulas for accurately predicting the performance of codes constructed from the root lattice A_n. They derived formulas for the moments of the Voronoi cell of A_n that enabled the accurate prediction of the error probability of the lattice codes constructed from A_n lattice. In this thesis, as a preliminary step, we evaluate the error probability of codes constructed from the E_8 lattice employing the results in [60] for the A_8 lattice.

The Voronoi cells of the root lattices A_n $(n \geq 1), D_n$ $(n \geq 3), E_6, E_7, E_8$ may be obtained by finding a fundamental simplex S_Λ for the affine Weyl group $W(\Lambda)$ of these lattices. For any root lattice Λ, the Voronoi cell around the origin $V(0)$ is the union of the images of the fundamental simplex S_Λ under the Weyl group $W(\Lambda)$ [61]. For A_n lattice, $|W(A_n)| = (n+1)!$. Hence for A_8, $|W(A_8)| = 362880$ i.e. Voronoi cell of A_8 lattice $V(A_8)$ consists of the union of 362880 images of the fundamental simplex S_{A_8}. It is know that the Voronoi cell of E_8 can be obtained from the Voronoi cell of A_8 by a systemic approach. To obtain the Voronoi cell of E_8, fundamental simplex S_{A_8} of the A_8 lattice is first chopped by two hyper planes P_1 and P_2 (described below). The two planes P_1 and P_2 are obtained from the following equation (Here I have to add reference)

$$\mathbf{x}'(\mathbf{Q}\sum_{i=1}^{km}\mathbf{e}_i) = \frac{1}{2}(km - \frac{(km)^2}{n+1}) \quad \text{for all } k = 1, 2, \ldots, \frac{n+1}{m} - 1 \tag{4.3}$$

where

$$\mathbf{Q} = \mathbf{I} - \frac{\mathbf{1}\mathbf{1}'}{\mathbf{1}'\mathbf{1}} \tag{4.4}$$

is the projection matrix orthogonal to $\mathbf{1}$, the all one vector, and $\mathbf{I}$ is the $n+1$ by $n+1$ identity matrix. Now using $n = 8$ and $m = 3$ in Eq. (4.3) and solving, we get two intersecting planes for the fundamental simplex of A_8 as follows

$$P_1 = \mathbf{x}'(\mathbf{Q}\sum_{i=1}^{3}\mathbf{e}_i) - 1 = 0 \qquad \text{and} \qquad P_2 = \mathbf{x}'(\mathbf{Q}\sum_{i=1}^{6}\mathbf{e}_i) - 1 = 0 \tag{4.5}$$

where $\mathbf{e}_i$ is a vector with *ith* element equal to 1 and the remaining elements equal to zero.

Let S_{A_8} is chopped by planes P_1 and P_2 respectively and the corresponding chopped pieces are denoted by $\breve{S}_{A_8P1}$ and $\breve{S}_{A_8P2}$.

Let the combined chopped piece of S_{A_8} excluding origin is denoted as $\breve{S}_{A_8P1P2}$.

The polytope, including origin, left after removing the chopped part the fundamental simplex S_{A_8} is denoted by $\tilde{S}_{A_8P1P2}$ i.e. $\tilde{S}_{A_8P1P2} = S_{A_8} - \breve{S}_{A_8P1P2}$

The overall chopped region of the Voronoi cell of A_8 is denoted by $\breve{V}(A_8)$ i.e. $\breve{V}(A_8) = |W(A_8)|\breve{S}_{A_8P1P2}$

The polytope, including origin, left after truncating the overall chopped region of the Voronoi cell of A_8 is denoted by $\tilde{V}(A_8)$ i.e. $\tilde{V}(A_8) = |W(A_8)|\tilde{S}_{A_8P1P2}$ or

$$V(E_8) \equiv \tilde{V}(A_8) = V(A_8) - \breve{V}(A_8) \tag{4.6}$$

Now the Voronoi cell of E_8 is equivalent to $\tilde{V}(A_8)$. The probability of correct decoding for E_8 lattice can be given as

$$P_C(E_8) \equiv \tilde{P_C}(A_8) = P_C(A_8) - \breve{P_C}(A_8) \tag{4.7}$$

From [60], we know the probability of correct decoding $P_C(A_8)$ for the Voronoi cell $V(A_8)$ of the A_8 lattice. Hence, to calculate the probability of correct decoding $P_C(E_8)$ for the Voronoi cell of E_8, we need to calculate the probability of correct decoding $\breve{P}_c(A_8)$ for the truncated region $\breve{V}(A_8)$ of the Voronoi cell of the A_8 lattice. Finally, the probability of error $P_E(E_8)$ for the E_8 lattice can be expressed as

$$P_E(E_8) = 1 - P_C(E_8) \tag{4.8}$$

$$P_E(E_8) = 1 - P_C(A_8) + \breve{P_C}(A_8) \tag{4.9}$$

$$P_E(E_8) = 1 - \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{V(A_8)} e^{-\|\mathbf{x}\|^2/2\sigma^2} + \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\breve{V}(A_8)} e^{-\|\mathbf{x}\|^2/2\sigma^2} \tag{4.10}$$

The only unknown term in Eq.(4.10) is $\breve{P_C}(A_8) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\breve{V}(A_8)} e^{-\|\mathbf{x}\|^2/2\sigma^2}$ i.e. the probability of correct decoding over the truncated region $\breve{V}(A_8)$ of the Voronoi of the A_8 lattice. As $\breve{V}(A_8)$ is the union of $|W(A_8)| = 362880$ images of $\breve{S}_{A_8P1P2}$, we calculate the probability of correct decoding for $\breve{S}_{A_8P1P2}$ and then multiply it by $|W(A_8)|$ to get $P_C(\breve{V}(A_8))$ i.e.

$$P_C(\breve{V}(A_8)) = |W(A_8)|\breve{S}_{A_8P1P2} \tag{4.11}$$

We found that the best approach to find the probability of correct decoding over the $\breve{S}_{A_8P1P2}$ is to first find probability of correct decoding over $\breve{S}_{A_8P1}$, $\breve{S}_{A_8P2}$ and $\breve{S}_{A_8P1} \bigcap \breve{S}_{A_8P2}$ respectively and then find the probability of correct decoding over $\breve{S}_{A_8P1P2}$ using the following equation

$$\breve{S}_{A_8P1P2} = \breve{S}_{A_8P1} + \breve{S}_{A_8P2} - \breve{S}_{A_8P1} \bigcap \breve{S}_{A_8P2} \tag{4.12}$$

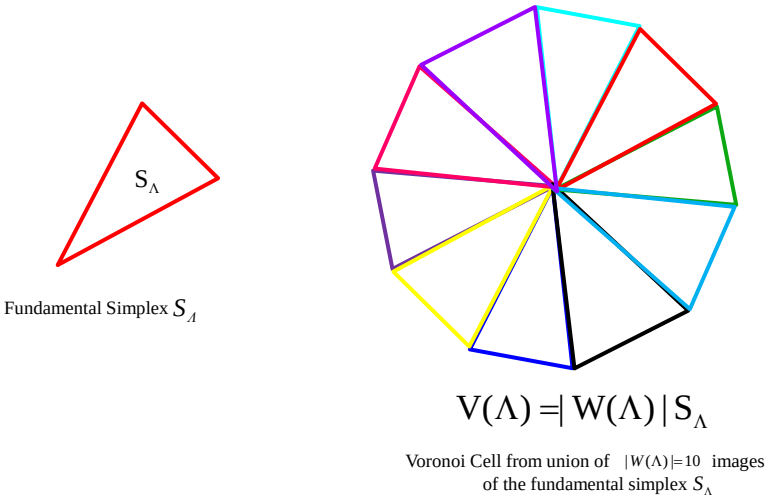

Figure 1: Voronoi cell and its fundamental simplex

In the following, we develop a two dimensional analogy of our eight dimensional problem as it is hard to visualize an eight dimensional object. We use $V(\Lambda)$ to describe the two dimensional Voronoi cell. Suppose that the Voronoi cell of A_8 is the same as shown in Fig. 1. It consists of union of ten images of the fundamental simplex S_Λ. We found that the best approach to truncate the fundamental simplex is as shown in Fig. 2, where S_Λ is chopped by two lines (hyper planes in case of eight dimension) P_1 and P_2 respectively to produce two arbitrary polytopes $\breve{S}_{\Lambda P_1}$ and $\breve{S}_{\Lambda P_2}$. The overall chopped region of the fundamental simplex is as shown in Fig. 2(c). Now taking the union of the ten images of Fig. 2(c) results in the overall chopped region, excluding origin, $\breve{V}(S_\Lambda)$ of the Voronoi cell Fig. 3(b). The remaining part, including origin, of $V(\Lambda)$ after truncating $\breve{V}(S_\Lambda)$ is denoted as $\tilde{V}(S_\Lambda)$ Fig. 3(d). Now returning to the original problem, as $\breve{S}_{A_8P1}$, $\breve{S}_{A_8P2}$ and $\breve{S}_{A_8P1} \bigcap \breve{S}_{A_8P2}$ in Eq.(4.12) are arbitrary polytopes in eight dimensions and it is known that to find integrals of polynomial functions over arbitrary polytopes is NP-hard. However, it is well known that we can integrate any polynomial over a simplex. Therefore, we first triangulate each of these polytopes to obtain corresponding simplices. The triangulation of $\breve{S}_{A_8P1}$, $\breve{S}_{A_8P2}$ and $\breve{S}_{A_8P1} \bigcap \breve{S}_{A_8P2}$ results in 35, 35 and 162 simplices respectively totalling 232 simplices. These 232 simplices are actually not standard simplices. Each of these simplices are first reflected using house-holder reflection and then translated to the origin to convert them to standard simplices. This process makes the integration of a polynomial function simple and easy. Integration term over the *ith* simplex is given as

$$\frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{S_i} e^{-\|\mathbf{x}\|^2/2\sigma^2} \tag{4.13}$$

Here we used multi-variable Taylor series expansion to approximate Eq.(4.13). This equa-

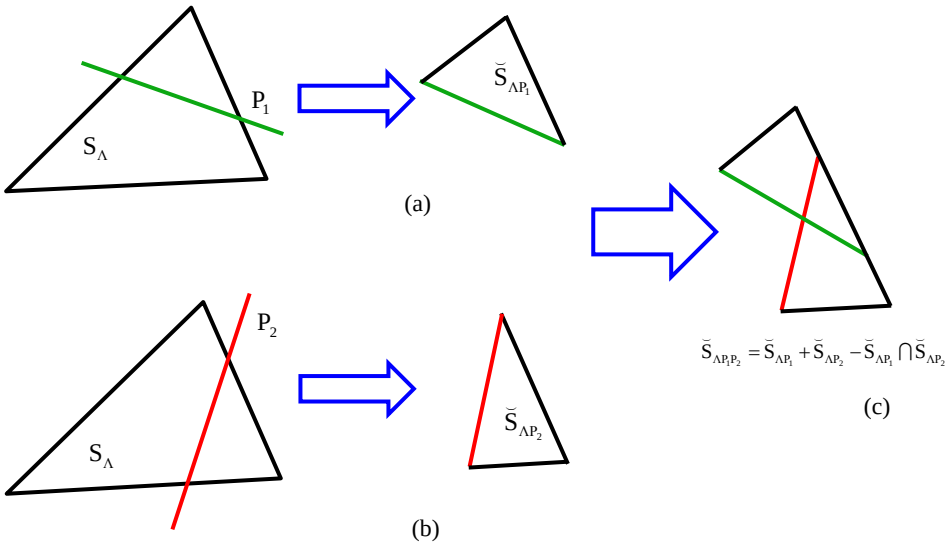

Figure 2: Simplex Chopped by two planes

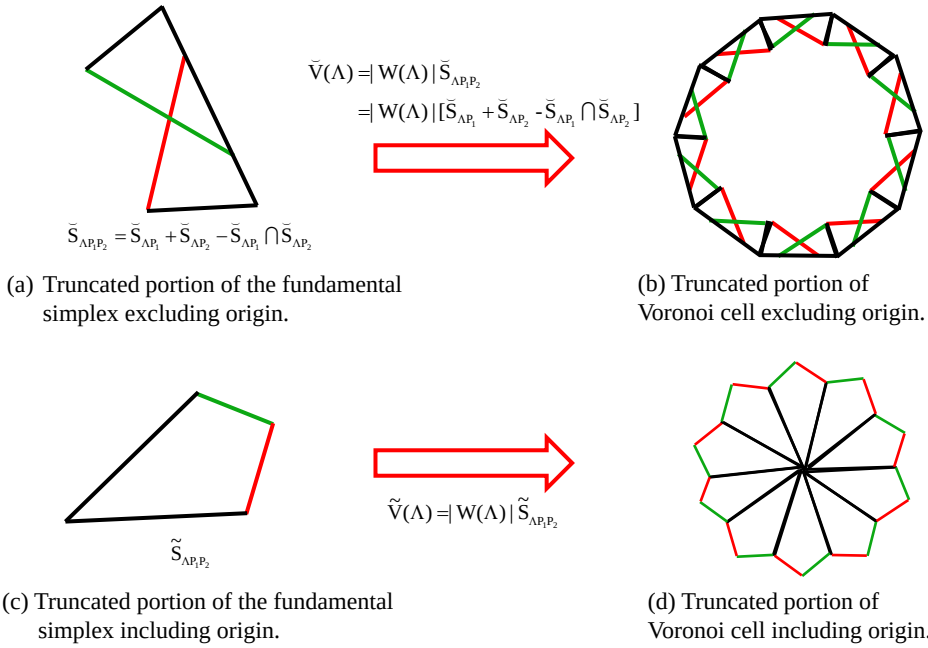

Figure 3: Truncated Voronoi Cell

tion is used to find the probability of correct decoding over the regions $\breve{S}_{A_8P1}$, $\breve{S}_{A_8P2}$ and $\breve{S}_{A_8P1} \bigcap \breve{S}_{A_8P2}$ and then using the relation in Eq.(4.12) we calculate the probability of correct decoding over the truncated region of the fundamental simplex of the A_8 lattice. When the probability of correct decoding over the truncated region of the fundamental simplex of the A_8 lattice is known, the probability of correct decoding for the overall truncated region of the voronoi cell of A_8 lattice is calculated by the relation $\breve{P}_C(A_8) = |W(A_8)|\breve{S}_{A_8P1P2} = |W(A_8)|[\breve{S}_{A_8P1} + \breve{S}_{A_8P2} - \breve{S}_{A_8P1} \bigcap \breve{S}_{A_8P2}]$. And, hence, using this result in Eq.(4.9) gives the probability of error for the Voronoi cell of the E_8 lattice.

NOTE: BER figure and research timeline to be added later.

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