[Review Copy Only]

Linear Algebra and its Applications 438 (2013) 3001–3010

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Two matrix-based lattice construction techniques

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ARTICLE INFO

Article history:
Received 12 June 2012
Accepted 22 October 2012
Available online 4 January 2013

Submitted by R.A. Brualdi

AMS classification:
52C17
11H31
15A15

Keywords:
Sphere packings
Geometry of numbers
Lattices
Generator matrices

ABSTRACT

Let m and n be integers greater than 1. Given lattices A and B of dimensions m and n, respectively, a technique for constructing a lattice from them of dimension $m+n-1$ is introduced. Furthermore, if A and B possess bases satisfying certain conditions, then a second technique yields a lattice of dimension $m+n-2$. The relevant parameters of the new lattices are given in terms of the respective parameters of A, B, and a lattice C isometric to a sublattice of A and B. Denser sphere packings than previously known ones in dimensions 52, 68, 84, 248, 520, and 4098 are obtained.

1. Introduction

In this work a lattice will always mean a point lattice, that is, a discrete subgroup of $\mathbb{R}^t$, and A and B will denote full lattices of dimensions m and n, respectively, with t, m, and n positive integers. Obtaining lattices of high center density is one of the main objectives of this paper. This will be achieved via the new construction techniques being proposed and apparently not found in the literature. The techniques are solely based on matrix algebra, and as it will be illustrated, new lattices having record-breaking center densities are obtained. More importantly, the new techniques help fill out gaps in tables of densest packings [6] with lattices satisfying the Minkowski bound [3].

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0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2012.10.031

The paper is organized as follows. In Section 2, we provide a background on lattices. In Section 3, we recall some key results and prove others that will be necessary for the development of the theory. In Section 4, A and B will be “glued together" along a “common edge," namely, lattice $\mathbb{Z}$. The resulting $(m+n-1)$-dimensional lattice is denoted by $A \times_{\mathbb{Z}} B$ and we prove that its center density is equal to $\frac{\delta_A \cdot \delta_B}{\delta_{\mathbb{Z}}}$, where δ_L denotes center density of lattice L. Already one will see that this is enough to obtain denser sphere packings than previously known ones in certain dimensions. In Section 5, under the hypothesis that A and B share a certain common two-dimensional face, that is, A and B contain lattices isometric to a two-dimensional lattice C, it is shown that the technique can be modified to construct an $(m+n-2)$-dimensional lattice $A \times_C B$ whose center density equals $\frac{\delta_A \cdot \delta_B}{\delta_C}$. In this case, we will say that the new lattice was obtained by “gluing together" A and B along a “common face," namely, lattice C. Lastly, in Section 6, we compare the two new construction techniques, namely, gluing along $\mathbb{Z}$ and gluing along a two-dimensional sublattice.

2. Background on lattices

This section presents the definitions and properties of lattices that are necessary for the rest of the paper. For a complete account on the theory, the interested reader is referred to the books [1,3] from which much of the material in this section was borrowed.

A lattice L is a discrete subgroup of $\mathbb{R}^t$. Equivalently, L is a finitely generated free $\mathbb{Z}$-module with positive definite symmetric bilinear form. More explicitly, L can be described as

$$\left\{ \sum_{i=1}^{d} a_i v_i \mid a_i \in \mathbb{Z} \quad \text{for } i = 1, \ldots, d \right\},$$

where $d \leqslant t$, v_i, $i = 1, \ldots, d$, is a $1 \times t$ vector with entries in $\mathbb{R}$, and the set $\{v_i\}_{i=1}^{d}$, called a basis for L, is linearly independent over $\mathbb{R}$. In this case, L is said to be a lattice of dimension $\dim L = d$ or a d-dimensional lattice. If $d = t$, then L is said to be a full lattice. Simple examples of lattices include $\mathbb{Z}^d$ and its subgroups.

Note that a basis for L is not unique. A generator matrix for L is a $d \times t$ matrix G whose rows are $v_1, \ldots, v_d$. The set

$$\mathcal{P} = \{r_1 \mathbf{v}_1 + \cdots + r_d \mathbf{v}_d \mid 0 \leqslant r_i < 1 \text{ for } i = 1, \ldots, d\}$$

is called a fundamental parallelotope of L. Let G^T be the transpose of G. The volume of $\mathcal{P}$ (or its Lebesgue measure) is equal to $\sqrt{|\det(G \cdot G^T)|}$ and it is independent of the choice of the basis for L. It is denoted by $\operatorname{vol} L$.

The minimal distance between lattice points in L is denoted by $d_{\min}(L)$. A minimal vector in L is a nonzero vector $\mathbf{v} \in L$ such that $||\mathbf{v}|| = d_{\min}$, where $||\cdot||$ denotes the ℓ^2-norm. The packing radius of L, denoted by ρ, is equal to half the minimal distance between lattice points.

If we center a sphere of radius ρ at each lattice point in L, then we obtain the lattice (or sphere) packing associated to L. An important parameter for describing the packing properties of L is its packing density

$$\Delta_L = \frac{\text{volume of one sphere}}{\operatorname{vol} L} = \frac{V_d \rho^d}{\operatorname{vol} L},$$

where V_d is the volume of a d-dimensional sphere of radius 1. For comparison purposes among different lattice packings, the simpler number

$$\delta_L = \frac{\Delta_L}{V_d} = \frac{\rho^d}{\operatorname{vol} L},$$

called the center density of L, is more frequently used.

The set $\{\mathbf{v} \in L \mid ||\mathbf{v}|| = d_{\min}(L)\}$ is finite and its cardinality, denoted by kiss L, is called the kissing number of the associated sphere packing. Equivalently, kiss L equals the number of spheres in the associated packing that touch one sphere. The d-dimensional version of the kissing number problem asks for the greatest value of the kissing number attained by any packing of d-dimensional spheres.

Record values for the center densities and kissing numbers of lattices in several dimensions can be found in [6]. An important family of lattices that will be invoked in later sections is $\{\Lambda_n\}_{n\geqslant 1}$, that is, the family of laminated lattices described in [3, Ch. 6]. Λ_1 is simply the lattice $\mathbb{Z}$ and Λ_2 is the two-dimensional lattice with basis $\left\{(1,0), \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)\right\}$. Well-studied members of the family of laminated lattices are Λ_8 and the Leech lattice Λ_{24}. Both are associated to the single densest packings in their dimensions, namely, eight and twenty four, respectively.

3. Preliminary results

We begin this section with the observation that by dilating a lattice, that is, by multiplying each of its vectors by a nonzero constant in $\mathbb{R}$, we obtain another lattice with the same packing density, the same center density, and the same kissing number. This follows from the corresponding formulas presented in Section 2. Thus, for the purposes of this work, given a lattice L, it is no loss of generality to assume that $d_{\min}(L) = 1$. Finally, if a lattice is rotated, the aforementioned parameters remain unchanged.

Let $L \subset \mathbb{R}^d$ be a lattice of minimum distance 1 and with basis $\mathbf{b}_1, \ldots, \mathbf{b}_d$, $d \geqslant 2$. In Section 4 we will need to form a basis for L containing a minimal vector in L. Our ability to do that is an immediate consequence of

Lemma 1 [1, Ch. I.2, Lemma 2]. *Notation as above, let*

$$\mathbf{a}_i = \sum_{j=1}^{d} v_{ij}\mathbf{b}_j \qquad (1 \leqslant i \leqslant e < d)$$

be vectors in L. A necessary and sufficient condition that $\mathbf{a}_1, \ldots, \mathbf{a}_e$ be extendable to a basis $\mathbf{a}_1, \ldots, \mathbf{a}_d$ of L is that the $e \times e$ determinants formed by taking e columns of the array

$$(v_{ij}) \qquad (1 \leqslant i \leqslant e,\ 1 \leqslant j \leqslant d)$$

shall not have a common factor.

The next lemma will be used shortly to show that if $\mathbf{b}_1$ is a minimal vector in L and $\mathbf{c}$ is a shortest vector in L linearly independent from $\mathbf{b}_1$, then there is a basis of L containing $\mathbf{b}_1$ and $\mathbf{c}$.

Lemma 2. *Notation as above, suppose that $\mathbf{b}_1$ is a minimal vector in L, and let $\mathbf{c}$ be a shortest vector in L linearly independent from $\mathbf{b}_1$. Write $\mathbf{c} = a_1\mathbf{b}_1 + z\mathbf{u}$ where $\mathbf{u}$ is a $\mathbb{Z}$-linear combination of $\mathbf{b}_2, \ldots, \mathbf{b}_d$, and $a_1, z \in \mathbb{Z}$ with $z > 0$. Then $z = 1$.*

Proof. Without loss of generality, we may assume that the angle θ between $\mathbf{b}_1$ and $\mathbf{c}$ satisfies $\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{2}$. Note that

$$||\mathbf{c}||^2 = a_1^2 + 2a_1 z(\mathbf{b}_1 \cdot \mathbf{u}) + z^2\,||\mathbf{u}||^2 = (a_1 + z\,||\mathbf{u}||\cos\theta)^2 + z^2\,||\mathbf{u}||^2 \sin^2\theta,$$

where $\cdot$ denotes the dot product. If $a_1 + z\,||\mathbf{u}||\cos\theta \notin \left[-\frac{1}{2}, \frac{1}{2}\right]$, then we may take $\mathbf{c}' = b\mathbf{b}_1 + z\mathbf{u}$ with either $b = a_1 + 1$ or $b = a_1 - 1$ in order that $(b + z\,||\mathbf{u}||\cos\theta)^2 < (a_1 + z\,||\mathbf{u}||\cos\theta)^2$. This inequality implies that $||b\mathbf{b}_1 + z\mathbf{u}|| < ||a\mathbf{v}_1 + z\mathbf{u}||$, contradicting the minimality of $||\mathbf{c}||$. Therefore, $a_1 + z\,||\mathbf{u}||\cos\theta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Let $\mathbf{c}' = a\mathbf{b}_1 + (z-1)\mathbf{u}$. Then

$$\left|\left|\mathbf{c}'\right|\right|^2 = ||\mathbf{c}||^2 - 2(a_1 + z\,||\mathbf{u}||\cos\theta)\,||\mathbf{u}||\cos\theta + ||\mathbf{u}||^2\,(-2z\sin^2\theta + 1).$$

By way of contradiction, assume that $z \geqslant 2$. Since $-\frac{1}{2} \leqslant a_1 + z\,||\mathbf{u}||\cos\theta \leqslant \frac{1}{2}$ and $\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{2}$, it follows that $||\mathbf{c}'|| < ||\mathbf{c}||$, a contradiction, whence $z = 1$. $\square$

With the notation as in the beginning of this section, without loss of generality suppose that $\mathbf{b}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$ is a minimal vector in L. Let $\mathbf{c}$ be a shortest vector in L that is linearly independent from $\mathbf{b}_1$. There exist integers $v_{2j}, j = 1, \ldots, d$, such that

$$\mathbf{c} = v_{21}\mathbf{b}_1 + v_{22}\mathbf{b}_2 + \cdots + v_{2d}\mathbf{b}_d.$$

By Lemma 2, $v_{22}, \ldots, v_{2d}$ are relatively prime. Together with Lemma 1, these statements prove the next result, which will be needed in Section 5.

Corollary 1. *Notation as before, if* $\mathbf{u}$ *is a minimal vector in* L *and* $\mathbf{v}$ *a shortest vector in* L *linearly independent from* $\mathbf{u}$*, then there exists a basis of* L *containing* $\mathbf{u}$ *and* $\mathbf{v}$*.*

4. Gluing along a one-dimensional lattice

In the Introduction we assumed A and B to be m and n-dimensional lattices, respectively. In Section 3, we observed that we may dilate and rotate A and B without changing their densities and kissing numbers. We also recalled that given any minimal vector in a lattice L, there is always a basis of L containing it. In conclusion, without loss of generality, we may assume that there are bases of A and B containing the vectors $(0, \ldots, 0, 1) \in \mathbb{R}^m$ and $(1, 0, \ldots, 0) \in \mathbb{R}^n$, respectively. Having this in mind, let

$$G_1 = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m-1} & a_{1,m} \\ & & \vdots & & \\ a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,m-1} & a_{m-1,m} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

and

$$G_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n-1} & b_{2,n} \\ & & \vdots & & \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n-1} & b_{n,n} \end{bmatrix}$$

be generator matrices for A and B, respectively.

Theorem 1. *Notation as above, let $D = A \times_{\mathbb{Z}} B$ be the lattice having*

$$G = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m-1} & a_{1,m} & 0 & \cdots & 0 \\ & \vdots & & & & \vdots & \\ a_{m-1,1} & \cdots & a_{m-1,m-1} & a_{m-1,m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ & \vdots & & & & \vdots & \\ 0 & \cdots & 0 & b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} \tag{1}$$

as a generator matrix. Then:

(1) $\dim D = \dim A + \dim B - 1$.
(2) $\operatorname{vol} D = \operatorname{vol} A \cdot \operatorname{vol} B$.
(3) $d_{\min}(D) = 1$.
(4) $\delta_D = 2 \cdot \delta_A \cdot \delta_B$.
(5) $\operatorname{kiss} D = \operatorname{kiss} A + \operatorname{kiss} B - 2$.

Proof. The proof of (1) and (2) follows from the fact that G is a square matrix of order $m + n - 1$ and $\det G = \det G_1 \cdot \det G_2 \neq 0$. To prove (3), we begin by observing that $d_{\min}(D) \leqslant 1$. If we show that no nontrivial linear combination of the rows of G has norm less than 1, we are done. Indeed, any nonzero vector in D that can be written as a linear combination of the first m rows of G has norm greater than or equal to 1. Similarly, any nonzero vector in D that can be written as a linear combination of the last n rows of G has norm greater than or equal to 1. By way of contradiction, suppose $\mathbf{w} \in D$, $\mathbf{w} \neq \mathbf{0}$, and $||\mathbf{w}|| < 1$. Then

$$\mathbf{w} = \sum_{i=1}^{m-1} x_i \mathbf{u}_i + k\mathbf{e} + \sum_{i=2}^{n} y_i \mathbf{v}_i$$

where

$$\begin{cases} \mathbf{u}_i &= (a_{i,1}, \ldots, a_{i,m}, 0, \ldots, 0), \quad i = 1, \ldots, m-1, \\ \mathbf{e} &= (0, \ldots, 0, 1, 0, \ldots, 0), \\ \mathbf{v}_i &= (0, \ldots, 0, b_{i,1}, \ldots, b_{i,n}), \quad i = 2, \ldots, n, \end{cases}$$

and $x_i, y_i, k \in \mathbb{Z}$ with at least one of the x_i and at least one of the y_i being different from zero. We want to find a lower bound on $||\mathbf{w}||^2$. Let

$$\begin{cases} \mathbf{u}_i' &= (a_{i,1}, \ldots, a_{i,m-1}), \quad i = 1, \ldots, m-1, \\ \mathbf{v}_i' &= (b_{i,2}, \ldots, b_{i,n}), \quad i = 2, \ldots, n. \end{cases}$$

We now claim that any nonzero linear combination of the $\mathbf{u}_i'$ has quadratic norm greater than or equal to $\frac{3}{4}$. Indeed, given

$$\mathbf{z} = \sum_{i=1}^{m-1} x_i \mathbf{u}_i',$$

there exists $x_m \in \mathbb{Z}$ such that ℓ, the mth coordinate of the vector

$$\widetilde{\mathbf{z}} = \sum_{i=1}^{m} x_i \mathbf{u}_i,$$

belongs to $\left[-\frac{1}{2}, \frac{1}{2}\right]$, where $\mathbf{u}_m = \mathbf{e}$. Since $||\widetilde{\mathbf{z}}||^2 = ||\mathbf{z}||^2 + \ell^2 \geqslant 1$, it follows that $||\mathbf{z}||^2 \geqslant \frac{3}{4}$. Analogously, any nonzero linear combination of the $\mathbf{v}_i'$ has quadratic norm greater than or equal to $\frac{3}{4}$. Thus, $||\mathbf{w}||^2 \geqslant \frac{3}{2}$, a contradiction. This proves that $d_{\min}(D) = 1$. Statement 4) is an immediate consequence of 2) and 3). Finally, the above argument also shows that the number of vectors in D of norm 1 is equal to kiss A + kiss $B - 2$, which proves 5). $\square$

Theorem 1 yields the following lattice construction technique, which we refer to as *Construction I*:

1. Let G_1 be an $m \times m$ generator matrix for A whose last row is $(0, \ldots, 0, 1)$. Let G_2 be an $n \times n$ generator matrix for B whose first row is $(1, 0, \ldots, 0)$.
2. The new lattice D of dimension $m + n - 1$ has matrix G in (1) as a generator matrix.

Example 1. Using Construction I, we construct an $(m+1)$-dimensional lattice D from an m-dimensional lattice A. The procedure is to glue A and a two-dimensional lattice B together along the lattice $\mathbb{Z}$. To maximize the center density of D, we must choose B as the two-dimensional lattice with the highest center density. In this case, our only choice is $B = \Lambda_2$, [3, p. 12]. Since $\delta_B = \frac{1}{2\sqrt{3}}$ and kiss $B = 6$, see [3, p. 15], Theorem 1 gives $\delta_D = \frac{\delta_A}{\sqrt{3}}$ and kiss D = kiss $A + 4$.

Example 2. In dimension 246, the densest known lattice packing satisfies $\log_2 \delta \cong 249.28$, see [2]. By gluing the lattice A associated to this packing and Λ_3 together along $\mathbb{Z}$, we obtain a lattice in dimension 248 with center density δ such that $\log_2 \delta \cong 247.78$. The densest known packing in dimension 248 is associated to a lattice whose center density δ satisfies $\log_2 \delta \cong 227.09$, see [2].

Example 3. In dimension 512, the densest known lattice packing satisfies $\log_2 \delta \cong 797.31$, see [2]. By gluing the lattice A associated to this packing and Λ_{10} together along $\mathbb{Z}$, we obtain a lattice in dimension 520 with center density δ such that $\log_2 \delta \cong 793.51$. The densest known packing in dimension 520 is associated to a lattice whose center density δ satisfies $\log_2 \delta \cong 767.46$, see [3, p. 17].

Example 4. In dimension 4096, the densest known lattice packing satisfies $\log_2 \delta \cong 11527$, see [3, p. xviii]. By gluing the lattice A associated to this packing and Λ_3 together along $\mathbb{Z}$, we obtain a lattice in dimension 4098 with center density δ such that $\log_2 \delta \cong 11525.50$. The densest known packing in dimension 4098 is associated to a lattice whose center density δ satisfies $\log_2 \delta \cong 11281.80$, see [4].

In the next section we will extend Construction I. Again, a new lattice will be constructed starting from two given lattices A and B. When the extended construction technique can be applied, the resulting lattices will have center densities higher than the ones obtained in this section.

5. Gluing along a two-dimensional lattice

Throughout this section we will assume that kiss A, kiss B, m, and n are all greater than 2 and that there are two sets of linearly independent minimal vectors $\{\mathbf{r}, \mathbf{s}\}$ and $\{\mathbf{r}', \mathbf{s}'\}$ in A and B, respectively, such that the two-dimensional lattices generated by $\{\mathbf{r}, \mathbf{s}\}$ and $\{\mathbf{r}', \mathbf{s}'\}$ are isometric. Two lattices L and M are isometric if there exists a bijective linear map from L into M which preserves the inner product. By the definitions presented in Section 2, isometric lattices have the same packing density, the same center density, and the same kissing number.

Therefore, without loss of generality, we may assume that

$$\mathbf{r} = (0, \ldots, 0, a, b), \quad \mathbf{s} = (0, \ldots, 0, 1),$$

$$\mathbf{r}' = (a, b, 0, \ldots, 0), \text{ and } \mathbf{s}' = (0, 1, 0, \ldots, 0).$$

By Corollary 1, there is a basis of A containing $\mathbf{r}$ and $\mathbf{s}$, and there is a basis of B containing $\mathbf{r}'$ and $\mathbf{s}'$. So let

$$G_1 = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m-1} & a_{1,m} \\ & & \vdots & & \\ a_{m-2,1} & a_{m-2,2} & \cdots & a_{m-2,m-1} & a_{m-2,m} \\ 0 & 0 & \cdots & a & b \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

and

$$G_2 = \begin{bmatrix} a & b & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ b_{3,1} & b_{3,2} & \cdots & b_{3,n-1} & b_{3,n} \\ & & \vdots & & \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n-1} & b_{n,n} \end{bmatrix}$$

be generator matrices for A and B, respectively. We will isometrically immerse A and B in $\mathbb{R}^{m+n-2}$ (in the obvious way) and then glue them together along the common two-dimensional lattice C having

$$(0, \ldots, 0, a, b, 0, \ldots, 0) \text{ and } (0, \ldots, 0, 1, 0, \ldots, 0),$$

both in $\mathbb{R}^{m+n-2}$, as its basis vectors. The basic parameters of the resulting lattice are given by the next theorem.

Theorem 2. *Notation as above, let $D = A \times_C B$ be the lattice having*

$$G = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m-2} & a_{1,m-1} & a_{1,m} & 0 & \cdots & 0 \\ & \cdots & & & & 0 & \cdots & 0 \\ a_{m-2,1} & \cdots & a_{m-2,m-2} & a_{m-2,m-1} & a_{m-2,m} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a & b & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{3,1} & b_{3,2} & b_{3,3} & \cdots & b_{3,n} \\ & \cdots & & & & & \cdots & \\ 0 & \cdots & 0 & b_{n,1} & b_{n,2} & b_{n,3} & \cdots & b_{n,n} \end{bmatrix} \tag{2}$$

as a generator matrix. Then:

(1) $\dim D = \dim A + \dim B - 2$.
(2) $\operatorname{vol} D = \dfrac{\operatorname{vol} A \cdot \operatorname{vol} B}{\operatorname{vol} C}$.
(3) $d_{\min}(D) = 1$.
(4) $\delta_D = \dfrac{\delta_A \cdot \delta_B}{\delta_C}$.

Proof. The proof of (1) and (2) follows from the fact that G is a square matrix of order $m+n-2$ and $\det G = \frac{\det G_1 \cdot \det G_2}{a} \neq 0$. To prove (3), we begin by observing that $d_{\min}(D) \leqslant 1$. If we show that no nontrivial linear combination of the rows of G has norm less than 1, we are done. Indeed, any nonzero vector in D that can be written as a linear combination of the first m rows of G has norm greater than or equal to 1. Similarly, any nonzero vector in D that can be written as a linear combination of the last n rows of G has norm greater than or equal to 1. By way of contradiction, suppose $\mathbf{w} \in D$, $\mathbf{w} \neq \mathbf{0}$, and $||\mathbf{w}|| < 1$. Then

$$\mathbf{w} = \sum_{i=1}^{m-2} x_i \mathbf{u}_i + k\mathbf{e} + \ell\mathbf{f} + \sum_{i=3}^{n} y_i \mathbf{v}_i$$

where

$$\begin{cases} \mathbf{u}_i &= (a_{i,1}, \ldots, a_{i,m}, 0, \ldots, 0), \quad i = 1, \ldots, m-2, \\ \mathbf{e} &= (0, \ldots, 0, a, b, 0, \ldots, 0), \\ \mathbf{f} &= (0, \ldots, 0, 0, 1, 0, \ldots, 0), \\ \mathbf{v}_i &= (0, \ldots, 0, b_{i,1}, \ldots, b_{i,n}), \quad i = 3, \ldots, n, \end{cases}$$

and $x_i, y_i, k, \ell \in \mathbb{Z}$ with at least one of the x_i and at least one of the y_i being different from zero. We want to find a lower bound on $||\mathbf{w}||^2$. Let

$$\begin{cases} \mathbf{u}_i' &= (a_{i,1}, \ldots, a_{i,m-2}), \quad i = 1, \ldots, m-2, \\ \mathbf{v}_i' &= (b_{i,3}, \ldots, b_{i,n}), \quad i = 3, \ldots, n. \end{cases}$$

Recalling that $|a| \leqslant 1$, then by an argument similar to that used in the proof of Theorem 1, one has

$$\left|\left|\sum_{i=1}^{m-2} x_i \mathbf{u}_i'\right|\right|^2 \geqslant \frac{1}{2}$$

for suitable choices of $\tilde{k}$, $\tilde{\ell}$. Analogously,

$$\left|\left|\sum_{i=3}^{n} y_i \mathbf{v}_i'\right|\right|^2 \geqslant \frac{1}{2}.$$

and thus, $||\mathbf{w}||^2 \geqslant 1$, a contradiction. This proves that $d_{\min}(D) = 1$. In view of 2) and 3), the proof of 4) is immediate. $\square$

Theorem 2 yields the following lattice construction technique, which we refer to as *Construction II*. The technique is based on the hypothesis that A and B each have a sublattice generated by two of the their minimal vectors and isometric to a lattice with basis $\{(a, b), (0, 1)\}$ where $a^2 + b^2 = 1$:

Table 1
$\log_2 \delta$ for densest known packings vs. $\log_2 \delta$ for new packings.

Dimension	Currently densest packing	New packing	$\log_2$ of center density for Current packing	New packing
52	G_{52} [5]	$P_{48p} \times_{\mathbb{Z}^2} \Lambda_6$	8.37	12.24
54	MW_{54} [3]	$P_{48p} \times_{\mathbb{Z}^2} \Lambda_8$	15.88	12.03
60	P_{60q} [6]	$L_{56,2}(\mathfrak{M}) \times_{\mathbb{Z}^2} \Lambda_6$	17.54	14.58
68	HKO_{68} [6]	$MW_{64} \times_{\mathbb{Z}^2} \Lambda_6$	19.88	22.92
84	[2]	$MW_{80} \times_{\mathbb{Z}^2} \Lambda_6$	37.00	38.34
86	[2]	$MW_{80} \times_{\mathbb{Z}^2} \Lambda_8$	39.32	36.14

Table 2
Dimension n, Minkowski bound, Lattice D, $\frac{1}{n}\log_2 \Delta_D$, and $\log_2 \delta_D$.

Dimension n	$\frac{1}{n}\log_2 \Delta$	D	$\frac{1}{n}\log_2 \Delta_D$	$\log_2 \delta_D$
49	−0.9795	$P_{48p} \times_{\mathbb{Z}^2} \Lambda_3$	−0.5581	13.5391
50	−0.9799	$P_{48p} \times_{\mathbb{Z}^2} \Lambda_4$	−0.5871	13.0391
51	−0.9803	$P_{48p} \times_{\mathbb{Z}^2} \Lambda_5$	−0.6151	12.5391
52	−0.9807	$P_{48p} \times_{\mathbb{Z}^2} \Lambda_6$	−0.6383	12.2466
53	−0.9811	$P_{48p} \times_{\mathbb{Z}^2} \Lambda_7$	−0.6594	12.0391
65	−0.9846	$MW_{64} \times_{\mathbb{Z}^2} \Lambda_3$	−0.6505	24.2188
66	−0.9848	$MW_{64} \times_{\mathbb{Z}^2} \Lambda_4$	−0.6740	23.7188
67	−0.9850	$MW_{64} \times_{\mathbb{Z}^2} \Lambda_5$	−0.6970	23.2188

1. Let G_1 be an $m \times m$ generator matrix for A whose last rows are $(a, b, 0, \ldots, 0)$ and $(0, \ldots, 0, 1)$. Let G_2 be an $n \times n$ generator matrix for B whose first two rows are $(a, b, 0, \ldots, 0)$ and $(0, \ldots, 0, 1)$.
2. The new lattice D of dimension $m + n - 2$ has matrix G in (2) as a generator matrix.

Example 5. In dimension 48, the densest known lattice packing satisfies $\log_2 \delta \cong 14.04$, see [3, p. xx]. This packing is derived from lattice $A = P_{48p}$. By gluing A and Λ_6 together along $\mathbb{Z}^2$, we obtain a lattice in dimension 52 such that $\log_2 \delta \cong 12.24$. The densest known packing in dimension 52 has a center density satisfying $\log_2 \delta \cong 8.37$, see [6].

Table 1 next presents new packings and their center densities. The references for the Mordell-Weil lattices MW_n and the cyclo-quaternionic lattice $L_{56,2}(\mathfrak{M})$ can be found in [3, pp. xviii–xx].

The Minkowski bound guarantees the existence of n-dimensional lattices with packing density Δ satisfying $\Delta \geqslant \dfrac{\zeta(n)}{2^{n-1}}$ where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann zeta-function, see [3, p. 14]. Therefore, for large n,

$$\frac{1}{n}\log_2 \Delta \geqslant -1.$$

In Table 2, for each dimension n where $49 \leqslant n \leqslant 53$, we list $\frac{1}{n}\log_2 \Delta$ (the Minkowski bound for dimension n), $\frac{1}{n}\log_2 \Delta_D$, and $\log_2 \delta_D$ where Δ_D is the density achieved by the lattices obtained by gluing P_{48p} and Λ_i together along $\mathbb{Z}^2$, $i = 3, \ldots, 7$. The same parameters are displayed for each dimension n where $65 \leqslant n \leqslant 67$. Here, however, the new lattices are obtained by gluing MW_{64} and Λ_i together along $\mathbb{Z}^2$, $i = 3, 4, 5$.

6. Conclusion

There is a simple geometric interpretation for the techniques presented in the paper. The new lattice D furnished by Construction I arises as follows: Isometrically immerse A and B in $\mathbb{R}^{m+n-1}$, obtaining lattices A' and B', respectively. Then glue A' and B' together along their common minimal sublattice, namely, $\mathbb{Z}$, to get D. Similarly, the new lattice D furnished by Construction II arises as follows: Isometrically immerse A and B in $\mathbb{R}^{m+n-2}$, obtaining lattices A' and B', respectively. If A' and B' share a common minimal sublattice isometric to a two-dimensional lattice C, then glue A' and B' together along C to get D. Note that the presented techniques can be used to reproduce known lattices: For example, $\Lambda_i = \Lambda_j \times_{\mathbb{Z}^2} \Lambda_k$ for all triples (i, j, k) in $\{(4, 3, 3), (5, 4, 3), (9, 8, 3), (25, 24, 3)\}$.

Although Construction I may be used to obtain lattice packings denser than previously known ones in several dimensions, Construction II will yield lattice packings of even higher densities. Construction I is used when either A' and B' do not share a common two-dimensional sublattice or it is difficult to ascertain that they do. In any case, several examples were provided to illustrate that not only are some density records broken, but also gaps in tables of lattice packings can be filled with lattices satisfying the Minkowski bound. Finally, it would be worth investigating gluings along sublattices of dimension higher than two to determine whether new dense packings can be obtained.

Acknowledgment

The authors thank the editor and the reviewer for handling the manuscript, for carefully reading it, and for the suggestions. The latter have greatly improved the presentation of the paper. The work of T. P. da Nóbrega Neto has been supported by Fundação para o Desenvolvimento da UNESP (FUNDUNESP), Grant CCP 1649/09. The work of J. C. Interlando was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Grant No. 11/51654-5.

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