Predicate logic

Predicates P(x), M(x): statements involving **variables** x, y, z.
Using propositional logic alone **cannot adequately express the meaning of all statements** in mathematics.

- Neither true nor false statements
  - Every computer connected to the university is functioning properly.
  - CS2 is under attack by an intruder
  - Mr. x is not here

if x > 3 then x := x + 1, when this statement encountered, the value of x is entered into P(x) " x > 3 "

- If x is >3 then x := x + 1 will be executed.
- If P(x) is false, x will not change.

x > 7

x > 3. **x** is the **subject** , **>7** is the **predicate**, **P(x)= x > 7** is the **propositional function** (a generalization of propositions, contains variables that are replaced by elements from their domain)

- If we have a value x, then P(x) -the propositional function- becomes a proposition and has a truth value.

Predicates used to verify that program produces desired output (postconditions) when given valid input (preconditions)

n-place predicate, n-ary predicate

A statement involving the n variable $x_1, x_2, ..x_n$ Can be denoted by $P(x_1, x_2, ..x_n)$
iCliker through 5

Compound Expressions

Connectives (AND, OR, NOT, -> ..) from propositional logic can be used with predicate logic
$P(1) \vee P(3)$
iCliker through 10

Predicates, Propositions and Quantifiers

Predicates: propositions which contain variables.
Predicates become propositions once every variable is ***bound- by***

- **assigning it a value** from the *Universe of Discourse* U (Domain of the Discourse- Domain)
- Or **quantifying** it (Quantity)
  - Universal quantification
  - Existential quantification

Quantifiers

The quantifiers bind the variable *x* in the expressions.

Quantifications: all, some, many, none, one, and few.

- **Universal quantification**
  - Tells us that a predicate is **true for every element** under consideration (in the domain)
  - Needs domain of discourse or universe of discourse
- For all x P(x), For every x P(x)
- U={1,2,3}
- $\forall x\, P(x) = P(1) \wedge P(2) \wedge P(3)$
- **Counterexample** of $\forall x\, P(x)$ is an element for which P(x) is false
- When you can list the element then
  $\forall x\, P(x)$ is $P(x_1) \wedge P(x_2) \wedge ..... \wedge P(x_n)$
- for all, for every, all of, for each, for any…

- **Avoid for any x** (it could mean every or some) we want to avoid ambiguity.

Implicit Assumption

- Domain of discourse for quantifiers are nonempty.
- If domain is empty, $\forall x\, P(x)$ is true for any propositional function P(x)

- **Existential quantification**
  - Tells us that there is one or more elements under consideration for which the predicate is true it is **true** for **some element** in the domain
- There exist an element x in the domain such that P(X)
- $\exists x\, P(x)$ Existential Quantifier of P(x)
- There is an x such that P(x)
- There is at least one x such that P(x)
- True for at least one value of x in the domain
- For some xP(x)
- U={1,2,3}
- $\exists x\, P(x) = P(1) \vee P(2) \vee P(3)$
- When you can list the element then
  $\exists x\, P(x)$ is $P(x_1) \vee P(x_2) \vee \ldots.. P(x_n)$

Implicit Assumption

- Domain of discourse for quantifiers are nonempty.
- If the domain is empty, then $\exists x\, P(x)$ is false. There is no element where P(x) is true.

- **Unique Existential**
- P(x) is true for **one and only one x** in the universe of discourse.
- Notation: *unique existential quantifier*
- $\exists! x P(x)$ or $\exists 1 P(x)$
  - $\exists! x\, P(x)$ or $\exists 1x\, P(x)$ there exists a unique x such that P(x) is true
- 'There is a unique x such that P(x),' 'There is one and only one x such that P(x),' 'One can find only one x such that P(x).'

P(1)	P(2)	P(3)	$\exists! x P(x)$	$\exists x\, P(x)$	$\forall x\, P(x)$
0	0	0	0	0	0
0	0	1	1	1	0
0	1	0	1	1	0
0	1	1	0	1	0
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	0	1	0
1	1	1	0	1	1

Remember

A <u>predicate is not a proposition </u>until *all variables have* been bound either by quantification or assignment of a value!

Statement	***When True?***	***When False?***
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Examples:

1- $\forall y \neq 0$ ($y^3 \neq 0$), the domain: Real number
- For every real y with $y \neq 0$ and $y^3 \neq 0$
- The cube of every nonzero real number is nonzero
- $\forall y$ ($y \neq 0 \rightarrow y^3 \neq 0$)

2- $\exists$ z > 0 ($z^2 = 2$), Domain, real numbers
- There exist a real number z with z > 0 such that $z^2 = 2$ (there is a positive square root of 2)
- The Square of every nonzero real number is nonzero
- $\exists$ z (z > 0 $\wedge$ $z^2 = 2$)

Precedence of Quantifiers

$\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus
- $\forall$ x P(x) $\vee$ Q(x) = ($\forall$ x P(x))$\vee$ Q(x) and not $\forall$ x (P(x) $\vee$ Q(x))

Bound & Free variables

$\exists$x(x + y = 1)
- X is bound by $\exists$x
  - Bound when a quantifier is used on the variable
- Y is free
  - if it is outside the scope of all quantifiers in the formula that specifies this variable
- x + y = 1 is the scope of the quantifier $\exists$x
  - the part of a logical expression to which a quantifier is applied

REMEMBER A <u>predicate is not a proposition </u>until *all variables have* been bound either by quantification or assignment of a value!

Logical Equivalences

iff they have the <u>same truth value</u> no matter what predicates are substituted into those statements S $\equiv$ T

<u>Example:</u>

$\forall$ x(P(x) $\wedge$ Q(x)) $\equiv$ $\forall$ x P(x) $\wedge$ $\forall$ x Q(x)
- No matter what the predicates P and Q are
- No matter which domain of discourse is used
  - We show if $\forall$ x(P(x) $\wedge$ Q(x)) is true then $\forall$ x P(x) $\wedge$ $\forall$ x Q(x)
    - $\forall$ x(P(x) $\wedge$ Q(x)) is true, if a in the domain then P(a) $\wedge$ Q(a) is true, hence P(a) is true and Q(a) is true for every element in the domain, and this means that $\forall$ x P(x) $\wedge$ $\forall$ x Q(x) is true

– And we show if $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true then $\forall x(P(x) \wedge Q(x))$

Negating Quantified Expressions

Example:

Every student in the class has taken a course in calculus $\forall x\, P(x)$, the domain (students in my class)

- Negate it using "It is not the case"
- It is not the case that every student in the class… (equivalent to..)
  Ξ there is a student who has not taken..(translate back..)
- Existential quantification $\exists x \neg P(x)$

De Morgan's Laws for Quantifiers

Negation	**Equivalent statement**	**When true?**	**When False?**
$\neg \exists x\, P(x)$	$\forall x \neg P(x)$	For every x, P(x) is false	There is an x for which P(x) is true
$\neg \forall x\, P(x)$	$\exists x \neg P(x)$	There is an x for which P(x) is false	P(x) is true for every x

Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

Examples

Negate $\forall x(x^2 > x)$

Negate $\exists x\, (x^2 = 2)$

Negate $\neg \forall x\, (P(x) \rightarrow Q(x))$. (or show that … are logically equivalent)