Nested Quantifiers

Think about it as a **nested loop**

If one is within the scope of the other

$\forall x \exists y(x + y = 0)$ is the additive inverse

- The same as $\forall x$ $Q(x)$ where $Q(x)$ $\exists y$ $P(x,y)$ where $P(x,y) = (x + y = 0)$

$\forall x \forall y(x + y = y + x)$ commutative

$\forall x \forall y \forall z(x + (y+z) = (x+y) + z)$ associative

- **But what is the domain, does the order of the quantifier change the value??**

Nested Loops

- $\forall x \forall y$ $P(x, y)$ loop for every x and for each x we loop for y.
- $\forall x \exists y$ $P(x, y)$ loop for every x, we loop through the values of y till we find a y where $P(x,y)$ is true

Attention:

Assume that $Q(x, y) = x+y = 0$, determine the truth value of (domain is real numbers):

- $\exists y \forall x Q(x,y)$
- $\forall x \exists y Q(x,y)$

Assume $Q(x, y, z)$ is $x + y = z$, determine the truth value of (domain is real numbers):

- $\forall x \forall y \exists z$ $Q(x, y, z)$
- $\exists z \forall x \forall y$ $Q(x, y, z)$

Statement	When True?	When False?
$\forall x \forall y\, P(x, y)$ $\forall y \forall x\, P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y\, P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for all y.
$\exists x \forall y\, P(x, y)$	There is an x for which $P(x, y)$ is true for all y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y\, P(x, y)$ $\exists y \exists x\, P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

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Negating Nested Quantifiers

Let us learn it by example:

Negate: $\forall x \exists y\, (xy = 1)$

Nested Quantifiers ***<u>Dangerous situations</u>***

Commutativity of quantifiers

- *$\forall x \forall y\ p(x,y) \Leftrightarrow \forall y \forall x\ p(x,y)$?*
  - *Yes*
- *$\forall x \exists y\ p(x,y) \Leftrightarrow \exists y \forall x\ p(x,y)$?*
  - ***No***

Distributivity of quantifiers over operators

- *$\forall x\ [P(x) \land Q(x)] \Leftrightarrow \forall x\ P(x) \land \forall x\ Q(x)$?*
  - *Yes*
- *$\forall x\ [P(x) \rightarrow Q(x)] \Leftrightarrow [\forall x\ P(x) \rightarrow \forall x\ Q(x)]$?*
  - *No*

Remember:

- We can distribute a **universal quantifier** over a conjunction (and)
- We **cannot** distribute a **universal quantifier** over a **disjunction** (or)
- We can distribute an **existential quantifier** over a disjunction
- We **cannot** distribute an **existential quantifier** over a conjunction

Converting from English
☺

Translate **Every student in this class has studied calculus** into logical expressions

- Rewrite to identify appropriate quantifier
  - **For every** student in this class, that student has studied calculus
- Introduce a variable x
  - For every **student x** in this class, x has studied calculus
  - For every person x, if person x is a student in this class then x has studied calculus
  - $\forall x\ (s(x) \rightarrow Q(x, \text{calculus}))$
- Read using quantifiers in System specifications

Examples

Assume that:

- F(x): x is a fun
- S(x): x is a sun
- T(x): x is a tum
- U={funs, suns, tums}

Translate:

1. Everything is (a) fun.
2. Nothing is a sun.
3. All funs are suns.
4. Some funs are tums.
5. No sun is a tum.
6. If any fun is a sun then it's also a tum.