CECS 228 - HOMEWORK 8

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23

How many positive integers between 100 and 999 inclusive

a. are divisible by 7?
b. are odd?
c. have the same three decimal digits?
d. are not divisible by 4?
e. are divisible by 3 or 4?
f. are not divisible by either 3 or 4?
g. are divisible by 3 but not by 4?
h. are divisible by 3 and 4?

Solution:

a. 128. Every seventh number: 7, 14, and so on, is divisible by 7. Therefore the number of positive integers less than or equal to n and divisible by 7 is floor(n/7). So we find that there are floor(999/7) = 142 multiples of 7 not exceeding 999, of which floor(99/7) = 14 do not exceed 99. Therefore there are exactly 142 - 14 = 128 numbers in the desired range divisible by 7.
b. 450. We see that there are floor(999/2) = 499 even numbers not exceeding 999, and therefore 999 - 499 = 500 odd ones; there are similarly 99 - floor(99/2) = 50 odd numbers less than or equal to 99. Therefore there are 500 - 50 = 450 odd numbers between 100 and 999 inclusive.
c. 9. There are just 9 possible digits that a three-digit number can start with. If all of its digits are to be the same, then there is no choice after the leading digit has been specified. Therefore there are 9 such numbers.
d. 675. We find that there are 999 - floor(999/4) = 750 positive integers less than or equal to 999 not divisible by 4, and 99 - floor(99/4) = 75 such positive integers less than or equal to 99. Therefore there are 750 - 75 = 675 three-digit integers not divisible by 4.
e. 450. There are floor(999/3) - floor(99/3) = 300 three-digit numbers divisible by 3, and floor(999/4) - floor(999/4) = 225 three-digit numbers divisible by 4. There are floor(999/12) - floor(99/12) = 75 numbers divisible by both 3 and 4 (aka 12). In order to count each number divisible by 3 or 4 once and only once, we need to add the number of numbers divisible by 3 to the number of numbers divisible by 4, and

then subtract the number of numbers divisible by both 3 and 4 so as not to count them twice. Therefore the answer is 300 + 225 - 75 = 450.

f. 450. There are 450 three-digit integers that are divisible by either 3 or 4. Therefore there are 900 - 450 = 450 three-digit integers that are not divisible by either 3 or 4.
g. 225. There are 300 three-digit numbers divisible by 3 and that 75 of them are also divisible by 4. Therefore the remainder of those 300 numbers are not divisible by 4. Thus the answer is 300 - 75 = 225.
h. 75. There are floor(999/12) - floor(99/12) = 75 three-digit numbers divisible by both 3 and 4.

25

How many strings of four decimal digits

a. do not contain the same digit three times?
b. begin with an odd digit?
c. have exactly three digits that are 4s?

Solution:

a. 990. Subtract the number of strings that do not satisfy the condition from the total number of strings. There are 10 strings that consist of the same digit three times (000, 111, … , 999). Therefore, there are 1000 - 10 = 990 strings that do not contain the same digit three times.
b. 500. We have only five choices for the first digit if we have odd integers. We still have 10 choices for each of the remaining digits. Therefore, there are 5 * 10 * 10 = 500.
c. 27. We need to choose the position of the digit that is not a 4 (3 ways) and choose that digit (9 ways). Therefore, there are 3 * 9 = 27 strings that have exactly three digits that are 4s.

29

How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

Solution:

52,457,600. Sum Rule: we need to add the number of license plates of the first type and the number of license plates of the second type. Product Rule: there are 26 * 26 * 10 * 10 * 10 * 10 = 6,760,000 license plates consisting of 2 digits followed by 4 letters. Therefore, the answer is 6,760,000 + 45,697,600 = 52,457,600.

33

How many strings of eight English letters are there

a. that contain no vowels, if letters can be repeated?
b. that contain no vowels, if letters cannot be repeated?
c. that start with a vowel, if letters can be repeated?
d. that start with a vowel, if letters cannot be repeated?
e. that contain at least one vowel, if letters can be repeated?
f. that contain exactly one vowel, if letters can be repeated?
g. that start with X and contain at least one vowel, if letters can be repeated?
h. that start and end with X and contain at least one vowel, if letters can be repeated?

Solution:

a. 37,822,859,361. There are 8 slots, each of which can be filled with one of the 26 - 5 = 21 consonants, so by the product rule the answer is 21^8 = 37,822,859,361.
b. 8,204,716,800. There are 21 choices for the first slot in our string, but only 20 choices for the second slot, 19 for the third, and so on. So the answer is $21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14$ = 8,204, 716,800.
c. 40,159,050,880. There are 26 choices for each slot except the first, for which there are 5 choices, so the answer is $5 \cdot 26^7$ = 40, 159,050,880.
d. 12,113,640,000. there are only five choices in the first slot, and we are free to choose from all the letters not used so far, rather than just the consonants. Thus the answer is $5 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19$ = 12,113,640,000.
e. 171,004,205,215. We subtract from the total number of strings (26^8) the number that do not contain at least one vowel (21^8) obtaining the answer $26^8 - 21^8$ = 208,827,064,576 - 37,822,859,361 = 171,004,205,215.
f. 72,043,541,640. First decide where the vowel goes (8 choices), then to decide what the vowel is to be (A, E, I, 0, or U-5 choices), and then to fill the remaining slots with any consonants (21^7 choices, since one slot has already been filled). Therefore the answer is $8 \cdot 5 \cdot 21^7$ = 72,043,541,640.
g. 6,230,721,635. We can ignore the first slot, since there is no choice. There are only 7 slots to fill. So the answer is $26^7 - 21^7$ = 8,031,810,176 - 1,801,088,541 = 6,230,721,635.
h. 223,149,655. There are only 6 slots to fill. So the answer is $26^6 - 21^6$ = 308,915,776 - 85,766,121 = 223,149,655.

35

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a. 4
b. 5
c. 6
d. 7

Solution:

a. 0. k(k - l)(k - 2)(k - 3)(k - 4) = 4 · 3 · 2 · 1 · 0 = 0
b. 120. k(k - l)(k - 2)(k - 3)(k - 4) = 5 · 4 · 3 · 2 · 1 = 120.
c. 720. k(k - l)(k - 2)(k - 3)(k - 4) = 6 · 5 · 4 · 3 · 2 = 720.
d. 2520. k(k - l)(k - 2)(k - 3)(k - 4) = 7 · 6 · 5 · 4 · 3 = 2520.

59

The name of a variable in the JAVA programming language is a string of between 1 and 65,535 characters, inclusive, where each character can be an uppercase or a lowercase letter, a dollar sign, an underscore, or a digit, except that the first character must not be a digit. Determine the number of different variable names in JAVA.

Solution:

$54(64^{65536} - 1) / 63$. Suppose the name has length k, where 1<= k <= 65535. There are 26 + 26 + 1+1+10 = 64 choices for each character, except that the first character cannot be a digit, so there are only 54 choices for the first character. By the product rule there are 54 · 54^{k-1} choices for such a string. To get the final answer, we need to sum this over all lengths, for a total of $\sum_{k=1}^{65535} 54 \cdot 64^{k-1}$ Applying the formula for the sum of a geometric series gives $54(64^{65535} - 1) / 63$ = 5.5 x 10118369.

69

Use a tree diagram to determine the number of subsets of {3, 7, 9, 11, 24} with the property that the sum of the elements in the subset is less than 28.

Solution:

17. In our first tree, we let each branching point represent a decision as to whether to include the next element in the set (starting with the largest element). At the top of the tree, for example, we can either choose to include 24 or to exclude it, ~24. We branch one way for each possibility. In the first figure below, the entire subtree to the right represents those sets that do not include 24, and the subtree to the left represents those that do. At the point below and to the left of the 24, we have only one branch, ~11, since after we

have included 24 in our set, we cannot include 11, because the sum would not be less than 28 if we did. At the point below and to the right of the 0 24, however, we again branch twice, since we can choose either to include 11 or to exclude it. To answer the question, we look at the points in the last row of the tree. Each represents a set whose sum is less than 28. For example, the sixth point from the right represents the set {11, 3}. There are 17 such points.

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11

What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Solution:

4951. The generalized pigeonhole principle applies here. The pigeons are the students, and the pigeonholes are the states, 50 in number. By the generalized pigeonhole principle if we want there to be at least 100 pigeons in at least one of the pigeonholes, then we need to have a total of N pigeons so that ceil(N/50) >= 100. This will be the case as long as N >= 99 · 50 + 1 = 4951. Therefore we need at least 4951 students to guarantee that at least 100 come from a single state.

21

Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.

a. Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.
b. Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.

Solution:

a. If there were fewer than 9 freshmen, fewer than 9 sophomores, and fewer than 9 juniors in the class, there would be no more than 8 with each of these three class standings, for a total of at most 24 students, contradicting the fact that there are 25 students in the class.
b. If there were fewer than 3 freshmen, fewer than 19 sophomores, and fewer than 5 juniors, then there would be at most 2 freshmen, at most 18 sophomores, and at

most 4 juniors, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class.

29

Show that in a group of 10 people (where any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends.

Solution:

First note that the role of "mutual friend" and "mutual enemy" is symmetric, so it is really enough to prove one of these statements; the other will follow by interchanging the roles. So let us prove that in every group of 10 people, either there are 3 mutual friends or 4 mutual enemies. Consider one person; call this person A. Of the 9 other people, either there must be 6 enemies of A, or there must be 4 friends of A (if there were 5 or fewer enemies and 3 or fewer friends, that would only account for 8 people). We need to consider the two cases separately. First suppose that A has 6 enemies. Apply the result of Example 13 to these 6 people: among them either there are 3 mutual friends or there are 3 mutual enemies. If there are 3 mutual friends, then we are done. If there are 3 mutual enemies, then these 3 people, together with A, form a group of 4 mutual enemies, and again we are done. That finishes the first case. The second case was that A had 4 friends. If some pair of these people are friends, then they, together with A, form the desired group of 3 mutual friends. Otherwise, these 4 people are the desired group of 4 mutual enemies. Thus in either case we have found either 3 mutual friends or 4 mutual enemies.

41

Find the least number of cables required to connect 100 computers to 20 printers to guarantee that every subset of 20 computers can directly access 20 different printers. (Here, the assumptions about cables and computers are the same as in Example 9.) Justify your answer.

Solution:

Label the computers C_1 through C100, and label the printers P_1 through P_{20}. If we connect C_k to P_k for k = 1, 2,…, 20 and connect each of the computers C_{21} through C_{100} to all the printers, then we have used a total of $20 + 80 \cdot 20 = 1620$ cables. This is sufficient, because if computers C_1 through C_{20} need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, because they are connected to all the printers. Now we must show that 1619 cables is not enough. Because there are 1619 cables and 20 printers, the average number of computers per printer is 1619⁄20, which is less than 81. Therefore, some printer must be connected to fewer than 81 computers. That means it is connected to 80 or fewer computers, so there are 20 computers that are not connected to it. If those 20 computers all needed a printer simultaneously, then they would be out of luck, because they are connected to at most the 19 other printers.

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19

A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a. are there in total?
b. contain exactly two heads?
c. contain at most three tails?
d. contain the same number of heads and tails?

Solution:

a. 1024. Each flip can be either heads or tails, so there are $2^{10} = 1024$ possible outcomes.
b. 45. To specify an outcome that has exactly two heads, we simply need to choose the two flips that came up heads. There are C(10, 2) = 45 such outcomes.
c. 176. To contain at most three tails means to contain three tails, two tails, one tail, or no tails. There are C(10, 3) + C(10, 2) + C(10, 1) + C(10, 0) = 120 + 45 + 10 + 1 = 176 such outcomes.
d. 252. To have an equal number of heads and tails in this case means to have five heads. Therefore the answer is C(10, 5) = 252.

21

How many permutations of the letters ABCDEFG contain

a. the string BCD?
b. the string CFGA?
c. the strings BA and GF?
d. the strings ABC and DE?
e. the strings ABC and CDE?
f. the strings CBA and BED?

Solution:

a. 120. If BCD is to be a substring, then we can think of that block of letters as one superletter, and the problem is to count permutations of five items~the letters A, E, F, and G, and the superletter BCD. Therefore the answer is P(5, 5) = 5! = 120.
b. 24. P(4,4) = 4! = 24.
c. 120. We put BA into one item and glue GF into one item. Therefore we need to permute five items, and there are P(5, 5) = 5! = 120 ways to do it.
d. 24. Glue ABC into one item and glue DE into one item, producing four items, so the answer is P(4,4) = 4! = 24.
e. 6. If both ABC and CDE are substrings, then ABCDE has to be a substring. So we are really just permuting three items: ABCDE, F, and G. Therefore the answer is P(3,3) = 3! = 6.
f. 0. There are no permutations with both of these substrings, since B cannot be followed by both A and E at the same time.

27

One hundred tickets, numbered 1, 2, 3,…, 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if

a. there are no restrictions?
b. the person holding ticket 47 wins the grand prize?
c. the person holding ticket 47 wins one of the prizes?
d. the person holding ticket 47 does not win a prize?
e. the people holding tickets 19 and 47 both win prizes?
f. the people holding tickets 19, 47, and 73 all win prizes?
g. the people holding tickets 19, 47, 73, and 97 all win prizes?
h. none of the people holding tickets 19, 47, 73, and 97 wins a prize?
i. the grand prize winner is a person holding ticket 19, 47, 73, or 97?
j. the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

Solution:

a. 94,109,400. Since the prizes are different, we want an ordered arrangement of four numbers from the set of the first 100 positive integers. Thus there are P(100, 4) = 94,109,400 ways to award the prizes.
b. 941,094. If the grand prize winner is specified, then we need to choose an ordered set of three tickets to win the other three prizes. This can be done is P(99, 3) = 941,094 ways.

c. 3,764,376. We can first determine which prize the person holding ticket 47 will win (this can be done in 4 ways), and then we can determine the winners of the other three prizes, exactly as in part (b). Therefore the answer is 4P(99, 3) = 3,764,376.
d. 90,345,024. This is the same calculation as in part (a), except that there are only 99 viable tickets. Therefore the answer is P(99, 4) = 90,345,024. Note that this answer plus the answer to part (c) equals the answer to part (a), since the person holding ticket 47 either wins a prize or does not win a prize.
e. 114,072. This is similar to part (c). There are $4 \cdot 3 = 12$ ways to determine which prizes these two lucky people will win, after which there are P(98, 2) = 9506 ways to award the other two prizes. Therefore the answer is 12 * 9506 = 114,072.
f. 2328. This is like part (e). There are P(4, 3) = 24 ways to choose the prizes for the three people mentioned, and then 97 ways to choose the other winner. This gives $24 \cdot 97 = 2328$ ways in all.
g. 24. Here it is just a matter of ordering the prizes for these four people, so the answer is P(4, 4) = 24.
h. 79,727,040. This is similar to part (d), except that this time the pool of viable numbers has only 96 numbers in it. Therefore the answer is P(96,4) = 79,727,040.
i. 3,764,376. There are four ways to determine the grand prize winner under these conditions. Then there are P(99,3) ways to award the remaining prizes. This gives an answer of 4P(99. 3) = 3, 764,376.
j. 109,440. First we need to choose the prizes for the holder of 19 and 47. Since there are four prizes, there are P(4, 2) = 12 ways to do this. Then there are 96 people who might win the remaining prizes, and there are P(96, 2) = 9120 ways to award these prizes. Therefore the answer is $12 \cdot 9120 = 109{,}440$.

29

A club has 25 members.

a. How many ways are there to choose four members of the club to serve on an executive committee?
b. How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

Solution:

a. 12,650. Since the order of choosing the members is not relevant (the offices are not differentiated), we need to use a combination. The answer is clearly C(25, 4) = 12,650.

b. 303,600. In contrast, here we need a permutation, since the order matters (we choose first a president, then a vice president, then a secretary, then a treasurer). The answer is clearly P(25,4) = 303,600.

37

How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

Solution:

45. To implement the condition that every 0 be immediately followed by a 1, let us think of "gluing" a 1 to the right of each 0. Then the objects we have to work with are eight blocks consisting of the string 01 and two l's. The question is, then, how many strings are there consisting of these ten objects? This is easy to calculate, for we simply have to choose two of the "positions" in the string to contain the l's and fill the remaining "positions" with the 01 blocks. Therefore the answer is C(10, 2) = 45.

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11

Use the binomial theorem to expand $(3x^4 - 2y^3)^5$ into a sum of terms of the form $c^x a^y b$, where c is a real number and a and b are nonnegative integers.

Solution:

$$\sum_{j=0}^{5}\left(\frac{5!}{j!\cdot(5-j)!}\right)(3x^4)^{5-j}(-2y^3)^j = 243x^{20} - 810x^{16}y^3 + 1080x^{12}y^6 - 720x^8y^9 + 240x^4y^{12} - 32y^{15}$$

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13

A book publisher has 3000 copies of a discrete mathematics book. How many ways are there to store these books in their three warehouses if the copies of the book are indistinguishable?

Solution:

4,504,501. Assuming that the warehouses are distinguishable, let w, be the number of books stored in warehouse i. Then we are asked for the number of solutions to the equation w1 + w2 + w3 = 3000. By Theorem 2 there are C(3 + 3000 - 1, 3000) = C(3002, 3000) = C(3002, 2) = 4,504,501 of them.

25

How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes so that two objects are placed in each box?

Solution:

7,484,400. There are several ways to count this. We can first choose the two objects to go into box #1 (C(12, 2) ways), then choose the two objects to go into box #2 (C(10, 2) ways, since only 10 objects remain), then choose the two objects to go into box #3 (C(8, 2) ways), and so on. So the answer is C(12. 2) · C(10, 2) · C(8, 2) · C(6, 2) · C(4. 2) · C(2, 2) = (12·11/2)(10 · 9/2)(8 · 7 /2)(6 · 5/2)(4·3/2)(2·1/2) = 12!/26 = 7.484,400. Alternatively, just line up the 12 objects in a row (12! ways to do that), and put the first two into box #1, the next two into box #2, and so on. This overcounts by a factor of 26, since there are that many ways to swap objects in the permutation without affecting the result (swap the first and second objects or not, and swap the third and fourth objects or not, and so on). So this results in the same answer. Here is a third way to get this answer. First think of pairing the objects. Think of the objects as ordered (a first, a second, and so on). There are 11 ways to choose a mate for the first object, then 9 ways to choose a mate for the first unused object, then 7 ways to choose a mate for the first still unused object, and so on. This gives 11 · 9 · 7 · 5 · 3 ways to do the pairing. Then there are 6! ways to choose the boxes for the pairs. So the answer is the product of these two quantities. which is again 7,484,400.

31

How many different bit strings can be transmitted if the string must begin with a 1 bit, must include three additional 1 bits (so that a total of four 1 bits is sent), must include a total of 12 0 bits, and must have at least two 0 bits following each 1 bit?

Solution:

35. There are at least two good ways to do this problem. First we present a solution in the spirit of this section. Let us place the l's and some gaps in a row. A 1 will come first, followed by a gap, followed by another 1, another gap, a third 1, a third gap, a fourth 1, and a fourth gap. Into the gaps we must place the 12 0's that are in this string. Let g_1 , g_2 , g_3 and g_4 be the numbers of 0's placed in gaps 1 through 4, respectively. The only restriction is that each $g_i >= 2$. Thus we want to count the number of solutions to the equation $g_1 + g_2 + g_3 + g_4 = 12$, with $g_i >= 2$ for each i. Letting $g_i = g'_i + 2$, we want to count, equivalently, the number of nonnegative solutions to $g'_1 + g'_2 + g'_3 + g'_4 = 4$. By Theorem 2 there are C(4+4-1, 4) = C(7, 4) = C(7, 3) = 35.

33

How many different strings can be made from the letters in ABRACADABRA, using all the letters?

Solution:

83,160. This is a direct application of Theorem 3, with $n = 11$, $n_1 = 5$, $n_2 = 2$, $n_3 = n_4 = 1$, and $n_5 = 2$ (where n_1 represents the number of A's, ...). Thus the answer is $11! / (5!\ 2!\ 1!\ 1!\ 2!) = 83{,}160$.

45

How many ways are there to deal hands of five cards to each of six players from a deck containing 48 different cards?

Solution:

6.5×10^{32}. We assume that we are to care about which player gets which cards. The order in which a player receives his or her cards is not relevant, however, so we are dealing with combinations. We can look at one player at a time. There are C(48, 5) ways to choose the cards for the first player, then C(43, 5) ways to choose the cards for the second player (because five of the cards are gone), and so on. So the answer, by the multiplication principle, is $C(48,5) \cdot C(43,5) \cdot C(38,5) \cdot C(33, 5) \cdot C(28, 5) \cdot C(23, 5) = 6.5 \text{ x } 10^{32}$.

47

How many ways can n books be placed on k distinguishable shelves

a. if the books are indistinguishable copies of the same title?
b. if no two books are the same, and the positions of the books on the shelves matter?

Solution:

a. $C(k + n - 1, n)$. Letting x, be the number of copies of the book placed on shelf i, we are asking for the number of solutions to the equation $x_1 + x_2 + \cdot \cdot \cdot + x_K = n$, with each x_i a nonnegative integer. By Theorem 2 this is $C(k + n - 1, n)$.
b. $(k + n - 1)!/(k - 1)!$ No generality is lost if we number the books $b_1, b_2, \ldots, b_n$ and think of placing book b_1, then placing b_2, and so on. There are clearly k ways to place b_1, since we can put it as the first book on any of the shelves. After b_1 is placed, there are $k + 1$ ways to place b_2, since it can go to the right of b_1 or it can be the first book on any of the shelves. We continue in this way: there are $k + 2$ ways to place b_3 (to the right of b_1, to the right of b_2, or as the first book on some shelf), $k + 3$ ways to place $b_4, \ldots, k + n - 1$ ways to place b_n. Therefore the answer

is the product of these numbers, which can more easily be expressed as $(k + n - 1)!/(k - 1)!$.

59

How many ways are there to pack nine identical DVDs into three indistinguishable boxes so that each box contains at least two DVDs?

Solution:

3. Since each box has to contain at least two DVDs, we might as well put two DVDs into each box to begin with. This leaves us with just three more DVDs, and there are only three choices: we can put them all into the same box (so that the partition we end up with is $9 = 5 + 2 + 2$), or we can put two into one box and one into another (so that the partition we end up with is $9 = 4 + 3 + 2$), or we can put them all into different boxes (so that the partition we end up with is $9 = 3 + 3 + 3$). So the answer is 3.