CECS 228 - HOMEWORK 5
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3

Use a proof by cases to show that 100 is not the cube of a positive integer. [Hint: Consider two cases: (i) $1 \leq x \leq 4$ (ii) $x \geq 5$.]

Solution:

For all positive integers x it is not true that $x^3 = 100$. case (i): If $x \leq 4$, then $x^3 \leq 64$, so $x^3 \neq 100$. case (ii): If $x \geq 5$, then $x^3 \geq 125$, so $x^3 \neq 100$.

9

Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals -x if $x < 0$).

Solution:

Let a be a nonnegative real number. $|x| \leq a$ if and only if $-a \leq x \leq a$

Since the absolute value x is equal to x or -x. $x = |x|$ or $x = -|x|$. We can then say that x is between $-|x|$ and $|x|$ (inclusive):

$-|x| \leq x \leq |x|$

Since the absolute value of y is equal to y or -y. $y = |y|$ or $y = -|y|$. We can then say that y is between $-|y|$ and $|y|$(inclusive):

$-|y|| \leq y \leq |y|$

After combining the two previous inequalities, we have:

$-|x|-|y| \leq x+y \leq |x|+|y|$

Rewrite the difference:

$-(|x|+|y|) \leq x+y \leq |x| +|y|$

Using property of an absolute value:

$|x+y| \leq |x| + |y|$

13

Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

Solution:
$8 = 2^3$ and $9 = 3^2$

15
Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.

Solution:
Let $x = 2$ and $y = \sqrt{2}$. If $x^y = 2^{\sqrt{2}}$ is irrational, we are done. If not, then let $x = 2^{\sqrt{2}}$ and $y = \sqrt{2}/4$. Then $x^y = (2^{\sqrt{2}})^{\sqrt{2}/4} = 2^{\sqrt{2} * (\sqrt{2})/4} = 2^{1/2} = \sqrt{2}$.

21
Show that if n is an odd integer, then there is a unique integer k such that n is the sum of $k - 2$ and $k + 3$.

Solution:
We are being asked to solve $n = (k - 2) + (k + 3)$ for k. Using the usual, reversible, rules of algebra, we see that this equation is equivalent to $k = (n - 1)/2$. In other words, this is the one and only value of k that makes our equation true. Because n is odd, $n - 1$ is even, so k is an integer.

25
The harmonic mean of two real numbers x and y equals $2xy/(x + y)$. By computing the harmonic and geometric means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

Solution:
The harmonic mean of distinct positive real numbers x and y is always less than their geometric mean. To prove $2xy/(x + y) < \sqrt{xy}$, multiply both sides by $(x + y)/(2\sqrt{xy})$ to obtain the equivalent inequality $\sqrt{xy} < (x + y)/2$.

29
Formulate a conjecture about the decimal digits that appear as the final decimal digit of the fourth power of an integer. Prove your conjecture using a proof by cases.

Solution:
Without loss of generality we can assume that n is nonnegative, because the fourth power of an integer and the fourth power of its negative are the same. We divide an arbitrary positive integer n by 10, obtaining a quotient k and remainder l, whence $n = 10k + l$, and l is an integer between 0

and 9, inclusive. Then we compute n4 in each of these 10 cases. We get the following values, where X is some integer that is a multiple of 10, whose exact value we do not care about.

$(10k + 0)^4 = 10{,}000k^4 = 10{,}000k^4 + 0$, $(10k + 1)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 1$,

$(10k + 2)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 16$,

$(10k + 3)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 81$,

$(10k + 4)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^3 + X \cdot k + 256$,

$(10k + 5)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 625$,

$(10k + 6)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 1296$,

$(10k + 7)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 2401$,

$(10k + 8)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 4096$,

$(10k + 9)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 6561$.

Because each coefficient indicated by X is a multiple of 10, the corresponding term has no effect on the ones digit of the answer. Therefore, the ones digits are 0, 1, 6, 1, 6, 5, 6, 1, 6, 1, respectively, so it is always a 0, 1, 5, or 6.

33

Prove that there are no solutions in positive integers x and y to the equation $x^4 + y^4 = 625$.

Solution:

Because 54 = 625, both x and y must be less than 5. Then $x^4 + y^4 \leq 4^4 + 4^4 = 512 < 625$.

47

Use a proof by exhaustion to show that a tiling using dominoes of a 4×4 checkerboard with opposite corners removed does not exist. [Hint: First show that you can assume that the squares in the upper left and lower right corners are removed. Number the squares of the original checkerboard from 1 to 16, starting in the first row, moving right in this row, then starting in the leftmost square in the second row and moving right, and so on. Remove squares 1 and 16. To begin the proof, note that square 2 is covered either by a domino laid horizontally, which covers squares 2 and 3, or vertically, which covers square 2 and 6. Consider each of these cases separately, and work through all the subcases that arise.]

Solution:

We can rotate the board if necessary to make the removed squares be 1 and 16. Square 2 must be covered by a domino. If that domino is placed to cover squares 2 and 6, then the following domino placements are forced in succession: 5-9, 13-14, and 10-11, at which point there is no way to cover square 15. Otherwise, square 2 must be covered by a domino placed at 2-3. Then the following domino placements are forced: 4-8, 11-12, 6-7, 5-9, and 10-14, and again there is no way to cover square 15.