Proofs

Dear students: Learning how to construct proofs is probably one of the most difficult things you will face in life. Few of us are gifted enough to do it with ease. One only learns how to do it by practicing……. practice practice practice practice

Formal Proofs:

- To prove an argument is valid or the **conclusion** follows *logically from the* ***hypotheses****:*
  - Assume the hypotheses are true
  - Use the rules of inference and logical equivalences to determine that the conclusion is true.

Example

Consider the following logical argument:

If horses fly or cows eat artichokes, then the mosquito is the national bird. If the mosquito is the national bird then peanut butter tastes good on hot dogs. But peanut butter tastes terrible on hot dogs. Therefore, cows don't eat artichokes.

1. Assign propositional variables to the component propositions in the argument

Let F Horses fly
A Cows eat artichokes
M The mosquito is the national bird
P Peanut butter tastes good on hot dogs

2. Represent the formal argument using the variables

$(F \lor A) \to M$
$M \to P$
$\neg P$
$\therefore \neg A$

3. Use the hypotheses and the rules of inference and any logical equivalences to construct the proof.

4. Assertion	Reasons
5. 1.$((F \lor A) \to M$	*Hypothesis 1.*
6. 2. $M \to P$	*Hypothesis 2.*
7. 3.$(F \lor A) \to P$`	*steps 1 and 2 and hypothetical syll.*
8. 4.$\neg P$	*Hypothesis 3.*
9. 5. $\neg(F \lor A)$	*steps 3 and 4 and modus tollens*
10. 6. $\neg F \land \neg A$	*step 5 and DeMorgan*
11. 7. $\neg A \land \neg F$	*step 6 and* commutativity of 'and'
12. 8. $\neg A$	*step 7 and simplification*

Methods of Proof

We wish to establish the truth of the 'theorem' *P→Q.*
P may be a conjunction of other hypotheses. P → Q is a conjecture until a proof is produced.

- ***Trivial proof***
  If we know *Q is true then P → Q is true.*
  If it's raining today then the void set is a subset of every set.
  The assertion is *trivially true independent of the truth of P.*
- ***Vacuous proof***
  If we know one of the hypotheses in *P is false then P → Q is vacuously true.(we used it before -the phi set)*
  If I am both rich and poor then hurricane Fran was a mild breeze.
  $(P \land \neg P) \to Q$
- ***Direct proof***
  assumes the hypotheses are true, uses the rules of inference, axioms and any logical equivalences to establish the truth of the conclusion.
  Note, in order to show that a conjunction of hypotheses is true it is suffices to show just one of the hypotheses is true.
  The cow example
  Example- Theorem: *If 6x + 9y = 101, then x or y is not an integer.*
  Direct Proof: Assume $6x + 9y = 101$ is true.
  Use the rules of algebra $3(2x + 3y) = 101$.
  But 101/3 is not an integer so it must be the case that one of 2x or 3y is not an integer (maybe both). Therefore, one of x or y must not be an integer.
- ***Indirect proof***
  A direct proof of the contrapositive:
    - assumes the conclusion of *P→Q is false (¬Q* is true)
    - uses the rules of inference, axioms and any logical equivalences to establish the premise *P is false.*

  Note, in order to show that a conjunction of hypotheses is false is suffices to show just one of the hypotheses is false.
  Example- Theorem: *A perfect number is not a prime.*
  "A *perfect number is one which is the sum of all its divisors* except itself. For example, 6 is perfect since 1 + 2 + 3 = 6. So is 28."
  Proof: (*Indirect). We assume the number p is a prime and* show it is not perfect.
  But the only divisors of a prime are 1 and itself.
  Hence the sum of the divisors less than p is 1 which is not equal to p. Hence p cannot be perfect.
- ***Proof by contradiction***
  assumes the conclusion *Q is false*, then derives a contradiction, usually of the form
  $P \land \neg P$ *which establishes* $\neg Q \to 0$.
  The contrapositive of this assertion is $1 \to Q$ *from which it* follows that Q must be true.

- ***Proof by Cases***

Break the premise of $P \rightarrow Q$ *into an equivalent* disjunction of the form $P1 \lor P2 \lor \ldots \lor Pn$.
Then use the tautology
$[(P1 \rightarrow Q) \lor (P2 \rightarrow Q) \lor \ldots \lor (Pn \rightarrow Q)] \leftrightarrow [(P1 \lor P2 \lor \ldots \lor Pn) \rightarrow Q]$
Each of the implications $Pi \rightarrow Q$ *is a case.*
You must

- Convince the reader that the cases are inclusive, i.e., they exhaust all possibilities

establish all implications

Existence Proofs
We wish to establish the truth of $\exists xP(x)$.
Constructive existence proof:

- Establish P(c) is true for some c in the universe.
- Then $\exists xP(x)$ *is true by Existential Generalization*
- *There exists an integer solution to the equation* $x^2 + y^2 = z^2$

Nonconstructive existence proof.

- Assume no c exists which makes P(c) true and derive a contradiction.

- We wish to establish the truth of $\neg \exists xP(x)$ *(which is equivalent to "* $\forall \neg P(x)$*).*
    - Use a proof by contradiction by assuming there is a c which makes P(c) true.

Universally Quantified Assertions
We wish to establish the truth of $\forall xP(x)$.

- We assume that x is an arbitrary member of the universe and show P(x) must be true. Using UG it follows that $\forall xP(x)$.