Rules of Inference & Fallacies

Definitions:

- A *theorem is a valid logical assertion which* can be proved using
  - other theorems
  - *axioms (statements which are given to be true) and*
  - *rules of inference (logical rules which allow the* deduction of conclusions from premises).
- A *lemma is a 'pre-theorem' or a result* which is needed to prove a theorem.
- A *corollary is a 'post-theorem' or a result which follows* directly from a theorem.
- An argument is a sequence of statements end with a conclusion
- Valid the conclusion or final statement must follow from the truth of the preceding statements of promises of the argument

Rules of Inference

- Many of the tautologies in Chapter 1 are rules of inference.
- They have the form. $H_1 \wedge H_2 \wedge ..... \wedge H_n \rightarrow C$
  - H_i *are the hypotheses* and C *is the conclusion.*

$H_1 \wedge H_2 \wedge ..... \wedge H_n \rightarrow C$ as a rule of inference, has the symbolic form:

H_1

H_2

.

.

H_n

$\therefore C$

where ∴ means 'therefore' or 'it follows that'.

Modus ponens

The tautology $P \wedge (P \rightarrow Q) \rightarrow Q$ *becomes*

P

$P \rightarrow Q$

$\therefore Q$

- This means that whenever P *is true and* $P \rightarrow Q$ *is true we* can conclude logically that Q *is true.*
- This rule of inference is the most famous and has the name *modus ponens*
- Or the *law of detachment*

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Note:

- In Universal Generalization, x must be arbitrary.
- In Universal Instantiation, c need not be arbitrary but often is assumed to be.
- In Existential Instantiation, c must be an element of the universe which makes P(x) true.

Example:

Every man has two legs. John Smith is a man. Therefore, John Smith has two legs.

Define the predicates:

- M(x): x is a man
- L(x): x has two legs
- J: John Smith, a member of the universe

The argument becomes

1. $\forall x[M(x) \to L(x)]$
2. $M(J)$

$\therefore L(J)$

The proof is:

1. $\forall x[M(x) \to L(x)]$	*Hypothesis 1*
2. $M(J) \to L(J)$	*step 1 and UI*
3. $M(J)$	*Hypothesis 2*
4. $L(J)$	*steps 2 and 3 and modus ponens*

Using the rules of inference requires lots of practice.

Fallacies

- Fallacies are incorrect inferences.
- Some common fallacies: *The Fallacy of Affirming the Consequent*, *The Fallacy of Denying the Antecedent (or the hypothesis)*, *Begging the question or circular reasoning*

The Fallacy of Affirming the Consequent

If the butler did it he has blood on his hands.
The butler had blood on his hands.
Therefore, the butler did it.

This argument has the form

$P \to Q$

Q

$\therefore P$

Or $[(P \to Q) \land Q] \to P$

***The Fallacy of Denying the Antecedent (or the** hypothesis)*

If the butler is nervous, he did it.
The butler is really mellow.
Therefore, the butler didn't do it.

This argument has the form

$P \to Q$

$\neg P$

$\therefore \neg Q$

Or $[(P \to Q) \land \neg P] \to \neg Q$ which is also not a tautology and hence not a rule of inference.

Begging the question or circular reasoning

This occurs when we use the truth of statement being proved (or something equivalent) in the proof itself.

Example:

Conjecture: *if x^2 is even then x is even.*

Proof: If x^2 is even then $x^2 = 2k$ for some k. Then $x = 2l$ for some l. Hence, x must be even.