[12] Eigenvectors

Eigenvalues/Eigenvectors

DEFINITION: Let $A \in \mathbb{F}_{n\times n}$ and $x \in \mathbb{F}^{\mathrm{n}}$. The vector $\vec{x}$ is an eigenvector of A if and only if $\exists\lambda \in \mathbb{F}$ such that $A\vec{x} = \lambda\vec{x}$. In such case λ is called the eigenvalue corresponding to eigen vector $\vec{x}$

Example:

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \vec{x}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \rightarrow A\vec{x}_1 = \begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix} = 2\vec{x}_1 \rightarrow \lambda_1 = 2$$

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \rightarrow A\vec{x}_2 = \begin{bmatrix} 9 \\ 6 \\ 6 \end{bmatrix} = 3\vec{x}_2 \rightarrow \lambda_2 = 3$$

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \rightarrow A\vec{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 1\vec{x}_3 \rightarrow \lambda_3 = 1$$

Nullspace

DEFINITION: Let $A \in \mathbb{F}_{n\times n}$. The nullspace of A, $\mathrm{Null}(A)$ is the set of vectors $\vec{n}_i \in \mathbb{F}^n$ such that $A\vec{n}_i = 0$. i.e. $\mathrm{Null}(A) = \vec{n}_i | A\vec{n}_i = \vec{0}$. Hence $\vec{x}$ is the nullspace of matrix $A - \lambda I$

Q: How do we find a vector in the nullspace of a matrix?

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = 0$$

$$A\vec{x} - \lambda I_n\vec{x} = 0$$

$$(A - \lambda I_n)\vec{x} = 0$$

Note:

1. $\vec{x} \in NullSp(A - \lambda \mathrm{I})$
2. Let $B = A - \lambda\mathrm{I}$ and $\vec{x} \in \mathbb{F}^n \rightarrow B\vec{x} = 0$ – solve as system of equations

Inverse

DEFINITION: Let $A \in \mathbb{F}_{n \times n}$. The inverse of A, A^{-1} is an $n \times n$ matrix such that

$$A * A^{-1} = A^{-1}A = I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

Note:

1. Let $B * \vec{x} = 0$
2. If B is invertible then $B^{-1} \in \mathbb{F}_{n \times n}$ such that $B * B^{-1} * \vec{x} = \vec{0}$
3. Since $\vec{0}$ is not an eigenvector, A will only have an eigenvector if B is not invertible

When is matrix not invertible?

Determinant

DEFINITION: Let $A \in \mathbb{F}_{n\times n}$. If $n = 1, \det(A) = a_n$ and if $n = 2$ then

$$\det(A) = a_{11} * a_{22} - a_{12} * a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \det(A) = a_{11} * a_{22} - a_{12} * a_{21}$$

Example – Find the determinant of the following:

1. $A = [2] \to \det(A) = 2$

2. $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \to \det(A) = 2 * 4 - (-1) * 3 = 11$

3. $A = \begin{bmatrix} 5 & -6 \\ 7 & 8 \end{bmatrix} \to \det(A) = 5 * (8) - 7 * (-6) = 82$

Minor

DEFINITION: Let $A \in \mathbb{F}_{n \times n}$. The minor of a_{ij} is the determinant of the submatrix obtained by ignoring the i-th row j-th column

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

1. Minor of $a_{11} = \det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right) = 45 - 48 = -3$
2. Minor of $a_{22} = \det\left(\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}\right) = 9 - 21 = -12$
3. Minor of $a_{21} = \det\left(\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}\right) = 18 - 24 = -6$

Co-Factor

DEFINITION: Let $A \in \mathbb{F}_{n\times n}$. The co-factor of a_{ij} is the scalar $(-1)^{i+j} \times minor(a_{ij})$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

1. Cofactor of $a_{11} = (-1)^{1+1} \times \det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right) = 1 \times (-3) = -3$
2. Cofactor of $a_{22} = (-1)^{2+2} \times \det\left(\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}\right) = 1 \times (-12) = -12$
3. Cofactor of $a_{21} = (-1)^{2+1} \times \det\left(\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}\right) = (-1)(-6) = 6$

Determinant

DEFINITION: Let $A \in \mathbb{F}_{n \times n}$. The $\det(A)$ is given by the following steps

1. Pick any row or column
2. Find the cofactor of each element in that row or column
3. Multiply each element in the row/col by its cofactor
4. $\det(A)$ s the sum of the products in step 3

Example:

$$A = \begin{bmatrix} 3 & 5 & 0 \\ -1 & 2 & 1 \\ 3 & -6 & 4 \end{bmatrix}$$

$$\det(A) = a_{13} * cofact(a_{13}) + a_{23} * cofact(a_{23}) + a_{33} * cofact(a_{33})$$

$$\det(A) = 0 * (-1)^4(6-6) + 1 * (-1)^5(-18-15) + 4 * (-1)^6\big(6-(-5)\big) = 77$$

Invertibility

Theorem: A square matrix $A \in \mathbb{F}_{n\times n}$ is invertible if and only if $\det(A) \neq 0$

Recall:

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = 0$$

$$A\vec{x} - \lambda I_n\vec{x} = 0$$

$$(A - \lambda I)\vec{x} = 0$$

$$\vec{x} = B^{-1}\vec{0}$$

$$\vec{x} = \vec{0}$$

$$\Longrightarrow \det(B) = 0$$

$$\Longrightarrow \det(A - \lambda I) = 0$$

Characteristic Equation

Definition: Let $A \in \mathbb{F}_{n \times n}$. The characteristic equation of A is given by

$$\det(A - \lambda I) = 0$$

Note: Solving the characteristic equation yields the eigen values of A

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 2 \\ 4 & 3 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1 - \lambda)(3 - \lambda) - 8$$

Characteristic equation: $(1 - \lambda)(3 - \lambda) - 8 = 0$

$$3 - 4\lambda + \lambda^2 - \ 8 = 0$$

$$4\lambda + \lambda^2 - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0 \rightarrow \lambda_1 = 5 \; or \; \lambda_2 = -1$$

Finding Eigenvectors

Q: How do we find eigenvectors?

$$A\vec{x} = \lambda\vec{x}$$

$$(A - \lambda)\vec{x} = 0$$

A: $\vec{x}$ is in the nullspace of $A - \lambda I$

Example- Find the eigenvectors:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 5 \; or \; \lambda_2 = -1$

$$A - \lambda_1 I = \begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$$

$$(A - \lambda_1)\vec{x}_1 = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -4x_1 + 2x_2 = 0 \rightarrow x_2 = 2x_1$$

$$\vec{x}_1 = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigenvalues/Eigenvectors

Properties:

1. If $\vec{x}$ is an eigenvector of $A \in \mathbb{F}_{n\times n}$ with associated eigenvalue λ, then $\alpha\vec{x}$ is also an eigenvector
2. If λ is an eigenvalue of $A \in \mathbb{F}_{n\times n}$, then λ is also an eigenvalue of A^T

Proof: $\forall\alpha \in \mathbb{F} \to A(\alpha\vec{x}) = \alpha(A\vec{x}) = \alpha\lambda\vec{x} \Longrightarrow \alpha\vec{x}$ is an eigenvector

Lemma: For a matrix A, for any set T of distinct eigenvalues, the corresponding eigenvectors are linearly independent