[8] Inner Product

Inner Product

DEFINITION: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. The inner product of $\vec{x}$ and $\vec{y}$, denoted $\langle \vec{x}, \vec{y} \rangle$, is a scalar defined by $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \vec{x} * \vec{y}$

Properties:

i. $\langle \vec{x}, \vec{x} \rangle \neq 0 \; if \; \vec{x} \neq \vec{0}$

ii. $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$ (commutative)

iii. $\langle \lambda \vec{x}, \vec{y} \rangle = \lambda \langle \vec{x}, \vec{y} \rangle$

iv. $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

v. $\langle \vec{0}, \vec{y} \rangle = 0$

Inner Product (Proof of properties)

i. $\langle\vec{x},\vec{x}\rangle \neq 0 \; if \; \vec{x} \neq \vec{0}$

- $\langle\vec{x},\vec{x}\rangle = x_1x_1 + x_2x_2 + \cdots + x_nx_n = x_1^2 + x_2^2 + \cdots + x_n^2$
- As long as $x_i \neq 0$ for any $i \in [1, n]$, this cannot equal 0

ii. $\langle\vec{x},\vec{y}\rangle = \langle\vec{y},\vec{x}\rangle$

- $\langle\vec{x},\vec{y}\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n = y_1x_1 + y_2x_2 + \cdots + y_nx_n$ (by commutativity of multiplication)
- $\langle\vec{y},\vec{x}\rangle = y_1x_1 + y_2x_2 + \cdots + y_nx_n = \langle\vec{x},\vec{y}\rangle$

iii. $\langle\lambda\vec{x},\vec{y}\rangle = \lambda\langle\vec{x},\vec{y}\rangle$

- $\langle\lambda\vec{x},\vec{y}\rangle = \lambda x_1y_1 + \lambda x_2y_2 + \cdots + \lambda x_ny_n = \lambda(x_1y_1 + x_2y_2 + \cdots + x_ny_n) = \lambda\langle\vec{x},\vec{y}\rangle$

Inner Product (Proof of properties)

iv. $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

- $\langle \vec{x} + \vec{z}, \vec{y} \rangle = (x_1 + z_1)y_1 + (x_2 + z_2)y_2 + \cdots + (x_n + z_n)y_n = x_1y_1 + z_1y_1 + x_2y_2 + z_2y_2 + \cdots + x_ny_n + z_ny_n = x_1y_1 + x_2y_2 + \cdots + x_ny_n + z_1y_1 + z_2y_2 + \cdots + z_ny_n = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

v. $\langle \vec{0}, \vec{y} \rangle = 0$

- $\langle \vec{0}, \vec{y} \rangle = 0y_1 + 0y_2 + \cdots + 0y_n = 0$

Inner Product (Example)

Find $\langle \vec{x}, \vec{x} \rangle$ and $\langle \vec{x}, \vec{y} \rangle$ for $\vec{x} = \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{,2}\right], \vec{y} = [1,2,3,-4]$

Solution:

$$\langle \vec{x}, \vec{x} \rangle = \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = 4\left(\frac{1}{2}\right) = 1$$

$$\langle \vec{x}, \vec{y} \rangle = \frac{1}{2} * 1 + \frac{1}{2} * 2 + \frac{1}{2} * 3 + \frac{1}{2} * (-4) = \frac{1}{2} + 1 + \frac{3}{2} - 2 = 1$$

Vector Norm

DEFINITION: The norm of a vector $\vec{x}$, denoted $||\vec{x}||$, is defined as $||\vec{x}||^2 = \langle\vec{x},\vec{x}\rangle \rightarrow$
$||\vec{x}|| = \sqrt{\langle\vec{x},\vec{x}\rangle}$
Properties:

- $||\vec{x}||$ is a nonnegative real number.
- $||\vec{x}||$ is zero if and only if $\vec{x}$ is a zero vector.
- For any scalar , $||\alpha\vec{x}|| = \alpha||\vec{x}||$.
- $||\vec{x} + \vec{y}|| \leq ||\vec{x}||$+ $||\vec{y}||$(triangle inequality).

DEFINITION: The distance between two vectors is defined as $||\vec{x} - \vec{y}||$

Unit Vector

DEFINITION: Let $\vec{x} \in \mathbb{F}^n$. If $\langle \vec{x}, \vec{x} \rangle = 1$ then $\vec{x}$ is called a unit vector (i.e. $\|\vec{x}\|^2 = 1 \rightarrow \|\vec{x}\| = 1$). Also, the unit vector $\mathrm{y} = \frac{\vec{\mathrm{x}}}{\|\vec{x}\|}$ is called a normalized vector, because it has magnitude 1 and was created from another vector

CLAIM: $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| * \|\vec{y}\| \cos\theta$ where θ is the angle between the vectors
For proof use the law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$

$$\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle - 2\|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$-\frac{1}{2}\langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle + \frac{1}{2}\langle \vec{x}, \vec{x} \rangle + \frac{1}{2}\langle \vec{y}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$-\frac{1}{2}\langle -2\vec{x}, -2\vec{y} \rangle = \|\vec{x}\|\|\vec{y}\|\cos\theta$$

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| * \|\vec{y}\|\cos\theta$$

[9] Orthogonality

Orthogonality

DEFINITION: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. We say that $\vec{x}$ and $\vec{y}$ are orthogonal, denoted $\vec{x} \perp \vec{y}$, if $\langle \vec{x}, \vec{y} \rangle = 0$

Example: Determine if the following vectors are orthogonal:

$$\vec{x} = [3,3], \vec{y} = [1,-1], \vec{z} = [2,3]$$

$$\langle \vec{x}, \vec{y} \rangle = 3 * 1 + 3 * (-1) = 0 \rightarrow not\ orthogonal$$

$$\langle \vec{x}, \vec{z} \rangle = 3 * 2 + 3 * 3 = 15 \rightarrow not\ orthogonal$$

$$\langle \vec{y}, \vec{z} \rangle = 1 * 2 + (-1) * 3 = -1 \rightarrow not\ orthogonal$$

Orthonormal

DEFINITION: A set $V = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is said to be orthonormal if:

i. $\forall \vec{v}_i, \vec{v}_j : i \neq j \rightarrow \vec{v}_i \perp \vec{v}_j$ (all vectors are orthogonal to each other)

ii. $\|\vec{v}_i\| = 1$ for $i = 1,2, \dots, k$

DEFINITION: Kronecker Delta - $S_{ij} = \{1\ if\ i = j, 0\ if\ i \neq j$

Notes:

i. V is orthonormal iff $\langle \vec{v}_i, \vec{v}_j \rangle = S_{ij}$

- E.g. $S_{11} = \langle \vec{v}_1, \vec{v}_1 \rangle = \|\vec{v}_1\|^2$
- $S_{12} = \langle \vec{v}_1, \vec{v}_2 \rangle = 0$

ii. An orthogonal set can be made orthonormal by normalizing each vector in the set

iii. Any orthonormal set that spans V is a basis for V

Projection

DEFINITION: Let $\vec{x}, \vec{y} \in \mathbb{F}^n$. The projection of $\vec{x}$ onto $\vec{y}$ is defined as $\mathrm{proj}_{\vec{y}}(\vec{x}) = \frac{\langle \vec{x},\vec{y} \rangle}{\langle \vec{y},\vec{y} \rangle}\vec{y}$

Remarks:

i. $\vec{x}^{||\vec{u}} = \mathrm{proj}_{\vec{u}}(\vec{x})$

ii. $\vec{x}^{\perp\vec{u}} = \vec{x} - \mathrm{proj}_{\vec{u}}(\vec{x})$

iii. Given basis $B = \{\vec{x}, \vec{u}\}$ of $V \in \mathbb{R}^2$

- $T = \{\vec{x} - \mathrm{proj}_{\vec{u}}(\vec{x}), \vec{u}\}$ is an orthogonal basis for V
- $N = \left\{\frac{\vec{x} - \mathrm{proj}_{\vec{u}}(\vec{x})}{\|\vec{x} - \mathrm{proj}_{\vec{u}}(\vec{x})\|}, \frac{\vec{u}}{\|\vec{u}\|}\right\}$ is an orthonormal basis for V

Gram-Schmidt Algorithm

Input: $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ linearly independent; possibly a basis for vector space V

Output: $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ orthonormal set/basis

Step 1: $\vec{y}_1 = \vec{x}_1$ and $\vec{u}_1 = \frac{\vec{y}_1}{\|\vec{y}_1\|}$

Step k: for $i = 2,3, \dots, m$: $\vec{y}_i = \vec{x}_i - \sum_{k=1}^{i-1} \frac{\langle \vec{x}_i, \vec{y}_k \rangle}{\langle \vec{y}_k, \vec{y}_k \rangle} \vec{y}_k$ and $\vec{u}_i = \frac{\vec{y}_i}{\|\vec{y}_i\|}$

Gram-Schmidt Algorithm (Example)

Use the Gram-Schmidt algorithm to construct an orthonormal set of vectors from

$\{\vec{x}_1 = [1,1,0], \vec{x}_2 = [0,1,1], \vec{x}_3 = [1,0,1]\}$

Step 1: $\vec{y}_1 = \vec{x}_1$ and $\vec{u}_1 = \frac{\vec{y}_1}{\|\vec{y}_1\|} = \frac{1}{\sqrt{2}}[1,1,0] = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]$

Step 2: $\vec{y}_2 = \vec{x}_2 - \sum_{k=1}^{1} \frac{\langle \vec{x}_2, \vec{y}_k \rangle}{\langle \vec{y}_k, \vec{y}_k \rangle} \vec{y}_k = \vec{x}_2 - \frac{\langle \vec{x}_2, \vec{y}_1 \rangle}{\langle \vec{y}_1, \vec{y}_1 \rangle} \vec{y}_1 = [0,1,1] - \frac{1}{2}[1,1,0] = \left[-\frac{1}{2}, \frac{1}{2}, 1\right]$

$$\vec{u}_2 = \frac{\vec{y}_2}{\|\vec{y}_2\|} = \frac{\vec{y}_2}{\sqrt{6}/2} = \frac{\sqrt{6}}{3}\left[-\frac{1}{2}, \frac{1}{2}, 1\right] = \left[-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right]$$

Step 3: $\vec{y}_3 = \vec{x}_3 - \sum_{k=1}^{2} \frac{\langle \vec{x}_3, \vec{y}_k \rangle}{\langle \vec{y}_k, \vec{y}_k \rangle} \vec{y}_k = \vec{x}_3 - \left(\frac{\langle \vec{x}_3, \vec{y}_1 \rangle}{\langle \vec{y}_1, \vec{y}_1 \rangle} \vec{y}_1 + \frac{\langle \vec{x}_3, \vec{y}_2 \rangle}{\langle \vec{y}_2, \vec{y}_2 \rangle} \vec{y}_2\right) = [1,0,1] - \left(\frac{1}{2}[1,1,0] + \frac{1}{3}\left[-\frac{1}{2}, \frac{1}{2}, 1\right]\right)$

$= \left[\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right] \rightarrow \vec{u}_2 = \frac{\vec{y}_2}{\|\vec{y}_2\|} = \frac{\vec{y}_2}{2/\sqrt{3}} = \frac{\sqrt{3}}{2}\left[\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right] = \left[\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]$