[3] The Vector Space

Linear Combinations

An expression

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$

is a *linear combination* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

The scalars $a_1, \ldots, a_n$ are the *coefficients* of the linear combination.

Example: One linear combination of [2, 3.5] and [4, 10] is

$$-5\,[2, 3.5] + 2\,[4, 10]$$

which is equal to $[-5 \cdot 2, -5 \cdot 3.5] + [2 \cdot 4, 2 \cdot 10]$

Another linear combination of the same vectors is

$$0\,[2, 3.5] + 0\,[4, 10]$$

which is equal to the zero vector [0, 0].

Definition: A linear combination is *trivial* if the coefficients are all zero.

Linear Combinations: JunkCo

The JunkCo factory makes five products:

using various resources.

	metal	concrete	plastic	water	electricity
garden gnome	0	1.3	0.2	0.8	0.4
hula hoop	0	0	1.5	0.4	0.3
slinky	0.25	0	0	0.2	0.7
silly putty	0	0	0.3	0.7	0.5
salad shooter	0.15	0	0.5	0.4	0.8

For each product, there is a vector specifying how much of each resource is used per unit of product.

For making one gnome:

$\mathbf{v}_1$ = {metal:0, concrete:1.3, plastic:0.2, water:.8, electricity:0.4}

Linear Combinations: JunkCo

For making one gnome:
$\mathbf{v}_1 = \{\text{metal:}0, \text{ concrete:}1.3, \text{ plastic:}0.2, \text{ water:}0.8, \text{ electricity:}0.4\}$
For making one hula hoop:
$\mathbf{v}_2 = \{\text{metal:}0, \text{ concrete:}0, \text{ plastic:}1.5, \text{ water:}0.4, \text{ electricity:}0.3\}$
For making one slinky:
$\mathbf{v}_3 = \{\text{metal:}0.25, \text{ concrete:}0, \text{ plastic:}0, \text{ water:}0.2, \text{ electricity:}0.7\}$
For making one silly putty:
$\mathbf{v}_4 = \{\text{metal:}0, \text{ concrete:}0, \text{ plastic:}0.3, \text{ water:}0.7, \text{ electricity:}0.5\}$
For making one salad shooter:
$\mathbf{v}_5 = \{\text{metal:}1.5, \text{ concrete:}0, \text{ plastic:}0.5, \text{ water:}0.4, \text{ electricity:}0.8\}$

Suppose the factory chooses to make α_1 gnomes, α_2 hula hoops, α_3 slinkies, α_4 silly putties, and α_5 salad shooters.

Total resource utilization is $\mathbf{b} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 + \alpha_5 \mathbf{v}_5$

Linear Combinations: JunkCo: Industrial espionage

Total resource utilization is $\mathbf{b} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4 + \alpha_5 \mathbf{v}_5$

Suppose I am spying on JunkCo.

I find out how much metal, concrete, plastic, water, and electricity are consumed by the factory.

That is, I know the vector **b**. Can I use this knowledge to figure out how many gnomes they are making?

Computational Problem: *Expressing a given vector as a linear combination of other given vectors*

- *input*: a vector **b** and a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- *output*: a list $[\alpha_1, \ldots, \alpha_n]$ of coefficients such that

$$\mathbf{b} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

or a report that none exists.

Question: Is the solution unique?

Lights Out

Button vectors for 2 × 2 *Lights Out*: $\begin{bmatrix}\bullet & \bullet\\ \bullet & \end{bmatrix}$ $\begin{bmatrix}\bullet & \bullet\\ & \bullet\end{bmatrix}$ $\begin{bmatrix}\bullet & \\ \bullet & \bullet\end{bmatrix}$ $\begin{bmatrix} & \bullet\\ \bullet & \bullet\end{bmatrix}$

For a given initial state vector $\mathbf{s} = \begin{bmatrix}\bullet & \\ \bullet & \end{bmatrix}$,

$\boxed{\text{Which subset of button vectors sum to } \mathbf{s}\text{?}}$

Reformulate in terms of linear combinations.

Write

$$\begin{bmatrix}\bullet & \\ \bullet & \end{bmatrix} = \alpha_1 \begin{bmatrix}\bullet & \bullet\\ \bullet & \end{bmatrix} + \alpha_2 \begin{bmatrix}\bullet & \bullet\\ & \bullet\end{bmatrix} + \alpha_3 \begin{bmatrix}\bullet & \\ \bullet & \bullet\end{bmatrix} + \alpha_4 \begin{bmatrix} & \bullet\\ \bullet & \bullet\end{bmatrix}$$

What values for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ make this equation true?

Solution: $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 1$

$\boxed{\text{Solve an instance of } \textit{Lights Out}}$ $\Rightarrow$ $\boxed{\text{Which set of button vectors sum to } \mathbf{s}\text{?}}$

$\Rightarrow$ $\boxed{\text{Find subset of } GF(2) \text{ vectors } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ whose sum equals } \mathbf{s}}$ $\Rightarrow$ $\boxed{\text{Express as a linear combination of } \mathbf{v}_1, \ldots, \mathbf{v}_n}$

Lights Out

We can solve the puzzle if we have an algorithm for

Computational Problem: *Expressing a given vector as a linear combination of other given vectors*

Span

Definition: The set of all linear combinations of some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called the *span* of these vectors

Written Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Span: Attacking the authentication scheme

If Eve knows the password satisfies

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= \beta_1 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= \beta_m \end{aligned}$$

Then she can calculate right response to any challenge in Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$:

Proof: Suppose $\mathbf{a} = \alpha_1 \mathbf{a}_1 + \cdots + \alpha_m \mathbf{a}_m$. Then

$$\begin{aligned} \mathbf{a} \cdot \mathbf{x} &= (\alpha_1 \mathbf{a}_1 + \cdots + \alpha_m \mathbf{a}_m) \cdot \mathbf{x} \\ &= \alpha_1 \mathbf{a}_1 \cdot \mathbf{x} + \cdots + \alpha_m \mathbf{a}_m \cdot \mathbf{x} && \text{by distributivity} \\ &= \alpha_1 (\mathbf{a}_1 \cdot \mathbf{x}) + \cdots + \alpha_m (\mathbf{a}_m \cdot \mathbf{x}) && \text{by homogeneity} \\ &= \alpha_1 \beta_1 + \cdots + \alpha_m \beta_m \end{aligned}$$

Question: Any others? Answer will come later.

Span: GF (2) vectors

Quiz: How many vectors are in Span $\{[1, 1], [0, 1]\}$ over the field $GF(2)$?

Answer: The linear combinations are

$$0\,[1, 1] + 0\,[0, 1] = [0, 0]$$
$$0\,[1, 1] + 1\,[0, 1] = [0, 1]$$
$$1\,[1, 1] + 0\,[0, 1] = [1, 1]$$
$$1\,[1, 1] + 1\,[0, 1] = [1, 0]$$

Thus there are four vectors in the span.

Span: $GF(2)$ vectors

Question: How many vectors in Span $\{[1, 1]\}$ over $GF(2)$?

Answer: The linear combinations are

$$0\,[1, 1] = [0, 0]$$
$$1\,[1, 1] = [1, 1]$$

Thus there are two vectors in the span.

Question: How many vectors in Span $\{\}$?

Answer: Only one: the zero vector

Question: How many vectors in Span $\{[2, 3]\}$ over $\mathbb{R}$?

Answer: An infinite number: $\{\alpha\,[2, 3] : \alpha \in \mathbb{R}\}$
Forms the line through the origin and $(2, 3)$.

Generators

Definition: Let $\mathcal{V}$ be a set of vectors. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors such that $\mathcal{V} = \text{Span}\ \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ then

- we say $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a *generating set* for $\mathcal{V}$;
- we refer to the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as *generators* for $\mathcal{V}$.

Example: $\{[3, 0, 0], [0, 2, 0], [0, 0, 1]\}$ is a generating set for $\mathbb{R}^3$.

Proof: Must show two things:

1. Every linear combination is a vector in $\mathbb{R}^3$.
2. Every vector in $\mathbb{R}^3$ is a linear combination.

First statement is easy: every linear combination of 3-vectors over $\mathbb{R}$ is a 3-vector over $\mathbb{R}$, and $\mathbb{R}^3$ contains all 3-vectors over $\mathbb{R}$.

Proof of second statement: Let $[x, y, z]$ be any vector in $\mathbb{R}^3$. I must show it is a linear combination of my three vectors....

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

Generators

Claim: Another generating set for R^3 is $\{[1, 0, 0], [1, 1, 0], [1, 1, 1]\}$

Another way to prove that every vector in R^3 is in the span:

- We already know $R^3 = \text{Span}\ \{[3, 0, 0], [0, 2, 0], [0, 0, 1]\}$,
- so just show $[3, 0, 0]$, $[0, 2, 0]$, and $[0, 0, 1]$ are in Span $\{[1, 0, 0], [1, 1, 0], [1, 1, 1]\}$

$$
\begin{aligned}
[3, 0, 0] &= 3\,[1, 0, 0] \\
[0, 2, 0] &= -2\,[1, 0, 0] + 2\,[1, 1, 0] \\
[0, 0, 1] &= -1\,[1, 0, 0] - 1\,[1, 1, 0] + 1\,[1, 1, 1]
\end{aligned}
$$

Why is that sufficient?

- We already know any vector in R^3 can be written as a linear combination of the old vectors.
- We know each old vector can be written as a linear combination of the new vectors.
- We can convert *a linear combination of linear combination of new vectors* into *a linear combination of new vectors*.

Generators

We can convert *a linear combination of linear combination of new vectors* into *a linear combination of new vectors*.

- Write $[x, y, z]$ as a linear combination of the old vectors:

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

- Replace each old vector with an equivalent linear combination of the new vectors:

$$[x, y, z] = \left(\frac{x}{3}\right)(3\,[1, 0, 0]) + \left(\frac{y}{2}\right)(-\ 2[1, 0, 0] + 2[1, 1, 0])$$
$$+z(-1\,[1, 0, 0] - 1[1, 1, 0] + 1[1, 1, 1])$$

- Multiply through, using distributivity and associativity:

$$[x, y, z] = x\,[1, 0, 0] - y\,[1, 0, 0] + y\,[1, 1, 0] - z\,[1, 0, 0] - z\,[1, 1, 0] + z\,[1, 1, 1$$

- Collect like terms, using distributivity:

$$[x, y, z] = (x - y - z)\,[1, 0, 0] + (y - z)\,[1, 1, 0] + z\,[1, 1, 1]$$

Generators

Question: How to write each of the old vectors [3, 0, 0], [0, 2, 0], and [0, 0, 1] as a linear combination of new vectors [2, 0, 1], [1, 0, 2], [2, 2, 2], and [0, 1, 0]?

Answer:

$$[3,0,0] = 2\,[2,0,1] - 1\,[1,0,2] + 0\,[2,2,2]$$

$$[0,2,0] = -\frac{2}{3}[2,0,1] - \frac{2}{3}[1,0,2] + 1\,[2,2,2]$$

$$[0,0,1] = -\frac{1}{3}[2,0,1] + \frac{2}{3}[1,0,2] + 0\,[2,2,2]$$

Standard generators

Writing $[x, y, z]$ as a linear combination of the vectors $[3, 0, 0]$, $[0, 2, 0]$, and $[0, 0, 1]$ is simple.

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

Even simpler if instead we use $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$:

$$[x, y, z] = x\,[1, 0, 0] + y\,[0, 1, 0] + z\,[0, 0, 1]$$

These are called *standard generators* for R^3.
Written $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$

Standard generators

Question: Can 2×2 *Lights Out* be solved from every starting configuration?

Equivalent to asking whether the 2×2 button vectors

$$\boxed{\begin{matrix}\bullet & \bullet \\ \bullet & \end{matrix}} \quad \boxed{\begin{matrix}\bullet & \bullet \\ & \bullet\end{matrix}} \quad \boxed{\begin{matrix}\bullet & \\ \bullet & \bullet\end{matrix}} \quad \boxed{\begin{matrix} & \bullet \\ \bullet & \bullet\end{matrix}}$$

are generators for $GF(2)^D$, where $D = \{(0,0), (0,1), (1,0), (1,1)\}$.

Yes! For proof, we show that each standard generator can be written as a linear combination of the button vectors:

$$\boxed{\begin{matrix}\bullet & \\ & \end{matrix}} = 1 \boxed{\begin{matrix}\bullet & \bullet \\ \bullet & \end{matrix}} + 1 \boxed{\begin{matrix}\bullet & \bullet \\ & \bullet\end{matrix}} + 1 \boxed{\begin{matrix}\bullet & \\ \bullet & \bullet\end{matrix}} + 0 \boxed{\begin{matrix} & \bullet \\ \bullet & \bullet\end{matrix}}$$

$$\boxed{\begin{matrix} & \bullet \\ & \end{matrix}} = 1 \boxed{\begin{matrix}\bullet & \bullet \\ \bullet & \end{matrix}} + 1 \boxed{\begin{matrix}\bullet & \bullet \\ & \bullet\end{matrix}} + 0 \boxed{\begin{matrix}\bullet & \\ \bullet & \bullet\end{matrix}} + 1 \boxed{\begin{matrix} & \bullet \\ \bullet & \bullet\end{matrix}}$$

$$\boxed{\begin{matrix} & \\ \bullet & \end{matrix}} = 1 \boxed{\begin{matrix}\bullet & \bullet \\ \bullet & \end{matrix}} + 0 \boxed{\begin{matrix}\bullet & \bullet \\ & \bullet\end{matrix}} + 1 \boxed{\begin{matrix}\bullet & \\ \bullet & \bullet\end{matrix}} + 1 \boxed{\begin{matrix} & \bullet \\ \bullet & \bullet\end{matrix}}$$

$$\boxed{\begin{matrix} & \\ & \bullet\end{matrix}} = 0 \boxed{\begin{matrix}\bullet & \bullet \\ \bullet & \end{matrix}} + 1 \boxed{\begin{matrix}\bullet & \bullet \\ & \bullet\end{matrix}} + 1 \boxed{\begin{matrix}\bullet & \\ \bullet & \bullet\end{matrix}} + 1 \boxed{\begin{matrix} & \bullet \\ \bullet & \bullet\end{matrix}}$$

Geometry of sets of vectors: span of vectors over R

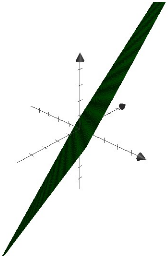
Span of a single nonzero vector **v**:

$$\text{Span}\,\{\mathbf{v}\} = \{\alpha\,\mathbf{v} : \alpha \in \mathbb{R}\}$$

This is the line through the origin and **v**. *One-dimensional*

Span of the empty set:just the origin. *Zero-dimensional* Span {[1, 2], [3, 4]}: all points in the plane. *Two-dimensional*

Span of two 3-vectors? Span {[1, 0, 1.65], [0, 1, 1]} is a plane in three dimensions:



Two-dimensional

Geometry of sets of vectors: span of vectors over R

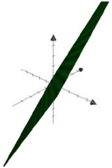
Is the span of k vectors always k-dimensional?
No.

- Span {[0, 0]} is 0-dimensional.
- Span {[1, 3], [2, 6]} is 1-dimensional.
- Span {[1, 0, 0], [0, 1, 0], [1, 1, 0]} is 2-dimensional.

Fundamental Question: How can we predict the dimensionality of the span of some vectors?

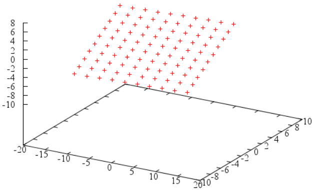
Geometry of sets of vectors: span of vectors over R

Span of two 3-vectors? Span $\{[1, 0, 1.65], [0, 1, 1]\}$ is a plane in three dimensions:



Two-dimensional

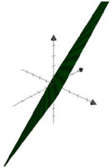
Useful for plotting the plane



$$\{\alpha\, [1, 0.1.65] + \beta\, [0, 1, 1] :$$
$$\alpha \in \{-5, -4, \ldots, 3, 4\},$$
$$\beta \in \{-5, -4, \ldots, 3, 4\}\}$$

Geometry of sets of vectors: span of vectors over R

Span of two 3-vectors? Span $\{[1, 0, 1.65], [0, 1, 1]\}$ is a plane in three dimensions:



Two-dimensional

Perhaps a more familiar way to specify a plane:

$$\{(x, y, z) : ax + by + cz = 0\}$$

Using dot-product, we could rewrite as

$$\{[x, y, z] : [a, b, c] \cdot [x, y, z] = 0\}$$

Set of vectors satisfying a linear equation with right-hand side *zero*.

We can similarly specify a line in three dimensions:

$$\{[x, y, z] : \mathbf{a}_1 \cdot [x, y, z] = 0, \mathbf{a}_2 \cdot [x, y, z] = 0\}$$

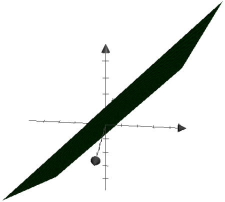
Two ways to represent a geometric object (line, plane, etc.) containing the origin:

- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

Geometry of sets of vectors: Two representations

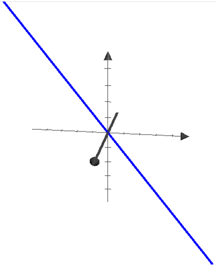
Two ways to represent a geometric object (line, plane, etc.) containing the origin:

- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides



Span $\{[4, -1, 1], [0, 1, 1]\}$

$$\{[x, y, z] :\ [4, -1, 1] \cdot [x, y, z] = 0,\ [0, 1, 1] \cdot [x, y, z] = 0\}$$



Span $\{[1, 2, -2]\}$

$$\{[x, y, z] : [1, 2, -2] \cdot [x, y, z] = 0\}$$

Geometry of sets of vectors: Two representations

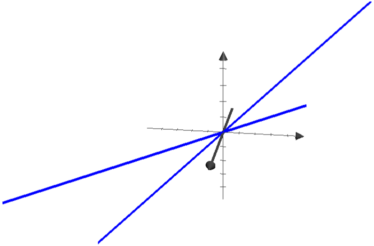
Two ways to represent a geometric object (line, plane, etc.) containing the origin:

- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

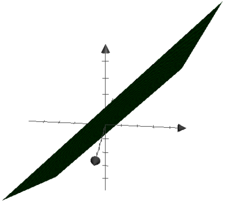
Each representation has its uses.

Suppose you want to find the plane containing two given lines

- First line is Span $\{[4, -1, 1]\}$.
- Second line is Span $\{[0, 1, 1]\}$.



- The plane containing these two lines is Span $\{[4, -1, 1], [0, 1, 1]\}$



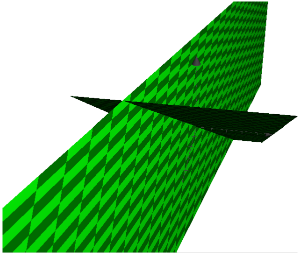
Geometry of sets of vectors: Two representations

Two ways to represent a geometric object (line, plane, etc.) containing the origin:

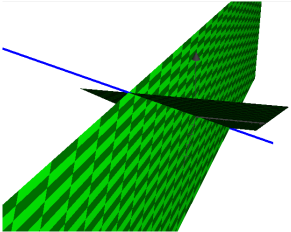
- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

Each representation has its uses.

Suppose you want to find the intersection of two given planes:



- First plane is $\{[x, y, z] : [4, -1, 1] \cdot [x, y, z] = 0\}$.
- Second plane is $\{[x, y, z] : [0, 1, 1] \cdot [x, y, z] = 0\}$.



- The intersection is $\{[x, y, z] : [4, -1, 1] \cdot [x, y, z] = 0, [0, 1, 1] \cdot [x, y, z] = 0\}$

Two representations: What's common?

Subset of F^D that satisfies three properties:

Property V1 Subset contains the zero vector $\mathbf{0}$

Property V2 If subset contains $\mathbf{v}$ then it contains $\alpha\,\mathbf{v}$ for every scalar α

Property V3 If subset contains $\mathbf{u}$ and $\mathbf{v}$ then it contains $\mathbf{u} + \mathbf{v}$

Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ satisfies

- Property V1 because

$$0\,\mathbf{v}_1 + \cdots + 0\,\mathbf{v}_n$$

- Property V2 because

$$\text{if } \mathbf{v} = \beta_1\,\mathbf{v}_1 + \cdots + \beta_n\,\mathbf{v}_n \text{ then } \alpha\,\mathbf{v} = \alpha\,\beta_1\mathbf{v}_1 + \cdots + \alpha\,\beta_n\,\mathbf{v}_n$$

- Property V3 because

$$\text{if } \mathbf{u} = \alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n$$
$$\text{and } \mathbf{v} = \beta_1\,\mathbf{v}_1 + \cdots + \beta_n\,\mathbf{v}_n$$
$$\text{then } \mathbf{u} + \mathbf{v} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n + \beta_n)\,\mathbf{v}_n$$

Abstract vector spaces

In traditional, abstract approach to linear algebra:

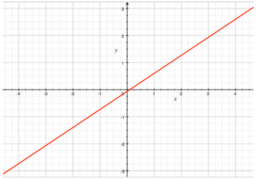
- We don't define vectors as sequences [1,2,3] or even functions `{a:1, b:2, c:3}`.
- DEFINITION: a vector space over a field $\mathbb{F}$ is any set $\mathcal{V}$ that is equipped with
  - an *addition* operation, and
  - a *scalar-multiplication* operation
  - satisfying certain axioms (commutate and distributive laws) and Properties V1, V2, V3.

Abstract approach has the advantage that it avoids committing to specific structure for vectors.

I avoid abstract approach in this class because more concrete notion of vectors is helpful in developing intuition.

Geometric objects that exclude the origin

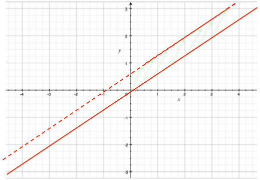
How to represent a line that does *not* contain the origin?

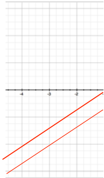


Start with a line that *does* contain the origin.

We know that points of such a line form a vector space V.

Translate the line by adding a vector **c** to every vector in V:





$$\{\mathbf{c} + \mathbf{v} : \mathbf{v} \in V\}$$

(abbreviated $\mathbf{c} + V$)

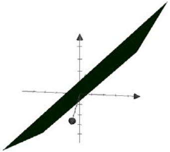
Result is line through **c** instead of through origin.

Geometric objects that exclude the origin

How to represent a plane that does *not* contain the origin?

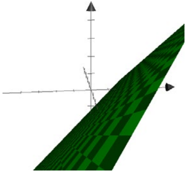
► Start with a plane that *does* contain the origin.

We know that points of such a plane form a vector space $\mathcal{V}$.



► Translate it by adding a vector **c** to every vector in $\mathcal{V}$

$\{\mathbf{c} + \mathbf{v} : \mathbf{v} \in \mathcal{V}\}$ (abbreviated $\mathbf{c} + \mathcal{V}$)



► Result is plane containing **c**.