[1] The Field

Coding the Matrix by Phillip Klein

The Field: Introduction to complex numbers

Solutions to $x^2 = -1$?

Mathematicians invented **i** to be one solution

Guest Week: Bill Amend (excerpt, http://xkcd.com/824)

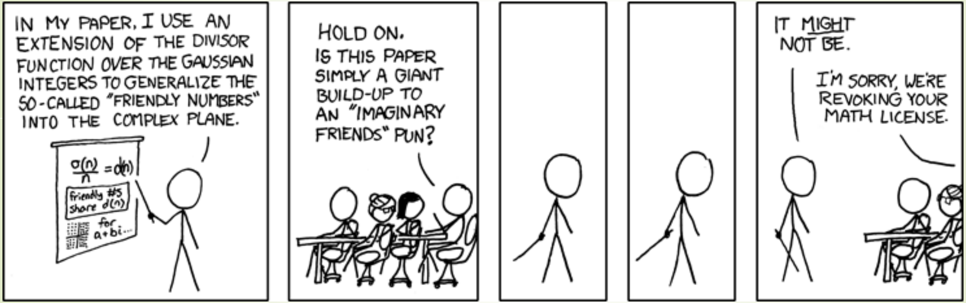
Can use **i** to solve other equations, e.g.:

$$x^2 = -9$$

Solution is $x = 3\mathbf{i}$

Introduction to complex numbers

Numbers such as $\mathbf{i}$, $-\mathbf{i}$, $3\mathbf{i}$, $2.17\mathbf{i}$ are called *imaginary* numbers.



Math Paper (http://xkcd.com/410)

The Field: Introduction to complex numbers

- Solution to $(x-1)^2 = -9$?
- One is $x = 1 + 3\mathbf{i}$.

- A real number plus an imaginary number is a *complex number*.
- A complex number has a *real part* and an *imaginary part*.

$$\text{complex number} = (\text{real part}) + (\text{imaginary part})\ \mathbf{i}$$

Field notation

When we want to refer to a field without specifying which field, we will use the notation F.

Field F is a set of values with defined addition and multiplication operations that satisfy the following:

i) Closure
ii) Commutativity
iii) Associativity
iv) Distributivity
v) Identity Element
vi) Inverse Element

Closure

- $\forall x, y \in F$
- $\rightarrow x + y \in F$
- $\rightarrow x * y \in F$
- Example: This is true for $\mathbb{Z}$

Commutativity

- $\forall x, y \in F$
- $\rightarrow x + y = y + x$
- $\rightarrow x * y = y * x$
- Example: This is true for $\mathbb{Z}$

Associativity

- $\forall x, y, z \in F$
- $\rightarrow (x + y) + z = x + (y + z)$
- $\rightarrow (x * y) * z = x * (y * z)$
- Example: This is true for $\mathbb{Z}$

Distributive Identity

- $\forall x, y, z \in F$
- $z(x + y) = zx + zy$
- Example: This is true for $\mathbb{Z}$

Identity Element

- $\forall x \in F\ \exists I \in F$
- $\rightarrow x + I_+ = x$
- $\rightarrow x * I_* = x$
- Example: This is true for $\mathbb{Z}$. $I_+ = 0$ and $I_* = 1$ for all x in $\mathbb{Z}$

Inverse Element

- $\forall x \in F \; \exists I \in F$
- $\rightarrow x + x_+ = I_+$
- $\rightarrow x * x_* = I_*$
- Example: This is not true for $\mathbb{Z}$. $x_+ = -x$ and $x_* = \frac{1}{x}$ for all x in $\mathbb{Z}$ but $\frac{1}{x} \notin \mathbb{Z}$
- So then $\mathbb{Z}$ is not a field

Abstracting over *Fields*

- We study three fields:
- ►The field R of real numbers
- ►The field C of complex numbers
- ►The finite field *GF*(2), which consists of 0 and 1 under mod 2 arithmetic.

- Reasons for studying the field C of complex numbers:
  - ►C is similar enough to R to be familiar but different enough to illustrate the idea of a field.
- ►Complex numbers are built into Python.
- ►Complex numbers are the intellectual ancestors of vectors.
  - ►In more advanced parts of linear algebra (to be covered in a follow-on course), complex numbers play an important role.

Example 1

- Verify that $\mathbb{R}$ is a field

Example 1 (Solution)

- Verify that $\mathbb{R}$ is a field

i) Closure: $x + y \in \mathbb{R}$ and $x * y \in \mathbb{R}$

ii) Commutativity: $x + y = y + x$ and $x * y = y * x$

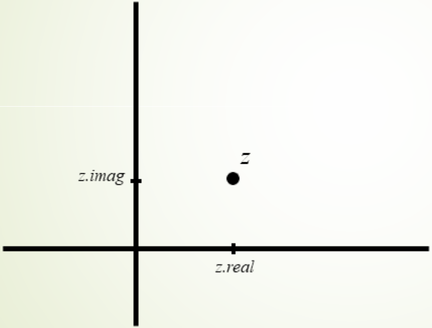
iii) Associativity: $(x + y) + z = x + (y + z)$ and $(x * y) * z = x * (y * z)$

iv) Distributivity: $\mathrm{z}(x + y) = zx + zy$

v) Identity Element: $I_{+} = 0$ and $I_{*} = 1$ for all x in $\mathbb{R}$

vi) Inverse Element: $x_{+} = -x$ and $x_{*} = \frac{1}{x}$ for all x in $\mathbb{R}$

Complex numbers as points in the complex plane

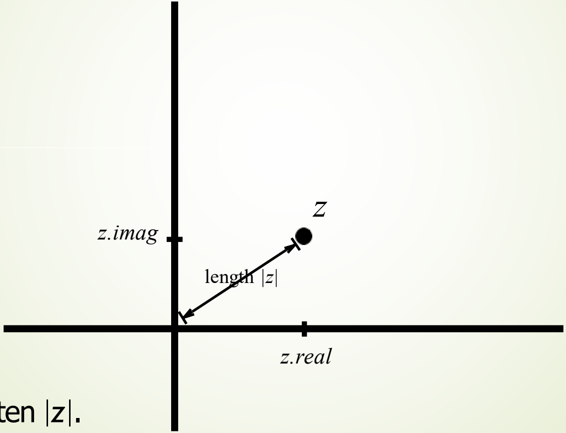


- Can interpret *real* and *imaginary* parts of a complex number as *x* and *y* coordinates. Thus can interpret a complex number as a *point* in the plane

(the *complex plane*)

Playing with C: The absolute value of a complex number

Absolute value of z = distance from the origin to the point z in the complex plane.

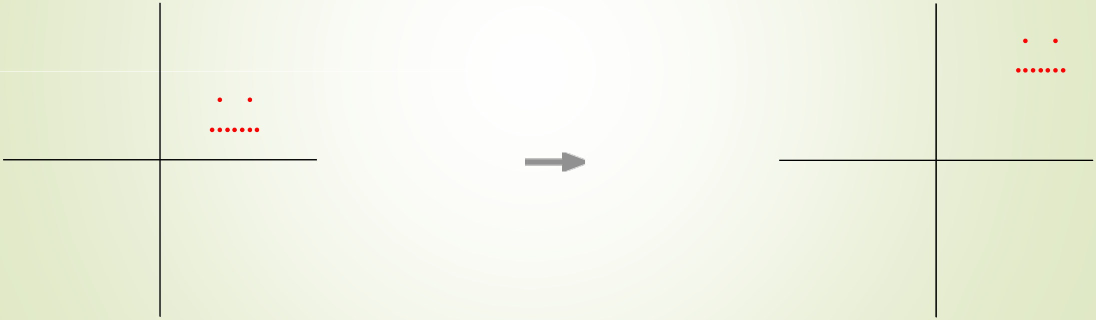


- In Mathese, written $|z|$.
- In Python, written `abs(z)`.

Playing with **C**: Adding complex numbers

Geometric interpretation of $f(z) = z + (1 + 2\mathbf{i})$?

Increase each real coordinate by 1 and increases each imaginary coordinate by 2.



$f(z) = z + (1 + 2\mathbf{i})$ is called a *translation*.

Playing with **C**: Adding complex numbers

- *Translation in general:*

$$f(z) = z + z_0$$

where z_0 is a complex number.

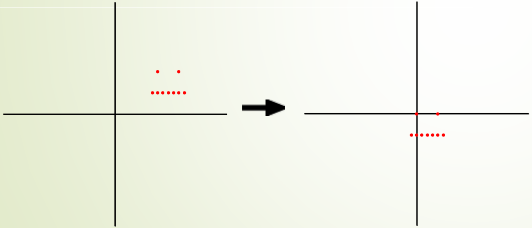
- A translation can "move" the picture anywhere in the complex plane.

Playing with $\mathbb{C}$: Adding complex numbers

- *Quiz:* The "left eye" of the list L of complex numbers is located at $2 + 2\mathbf{i}$. For what complex number z_0 does the translation

$$f(z) = z + z_0$$

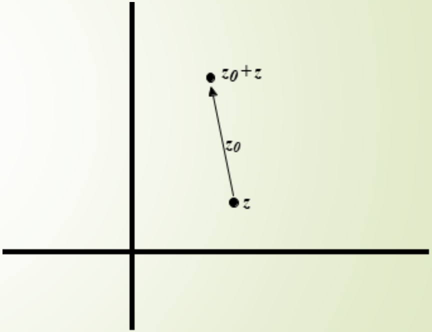
move the left eye to the origin $0 + 0\mathbf{i}$?



- *Answer:* $z_0 = -2 - 2\mathbf{i}$

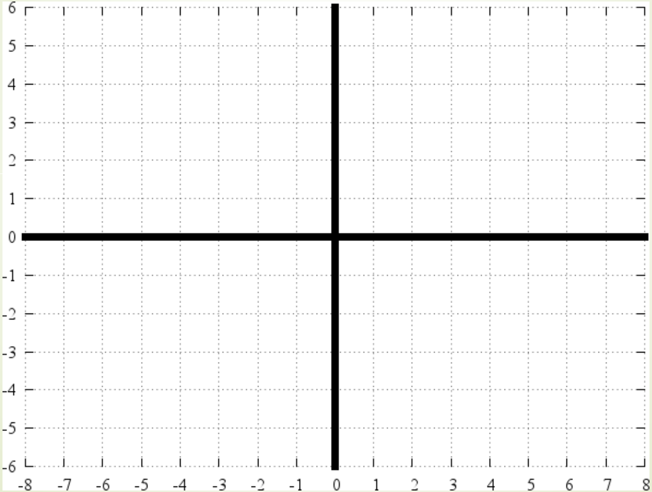
Playing with **C**: Adding complex numbers: Complex numbers as arrows

- Interpret z_0 as representing the translation $f(z) = z + z_0$.
- ► Visualize a complex number z_0 as an arrow.
- ► Arrow's tail located an any point z
- ► Arrow's head located at $z + z_0$
- ► Shows an example of what the translation $f(z) = z + z_0$ does



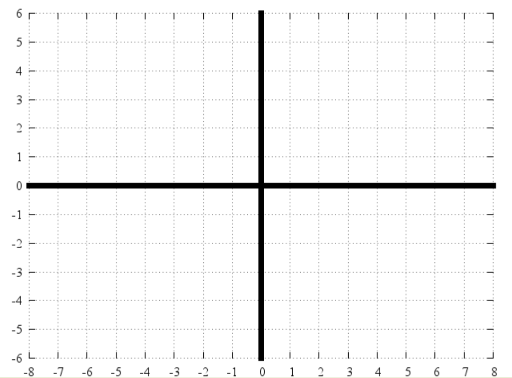
Playing with C: Adding complex numbers: Complex numbers as arrows

Example: Represent $-6 + 5\mathbf{i}$ as an arrow.



Playing with **C**: Adding complex numbers: Composing translations, adding arrows

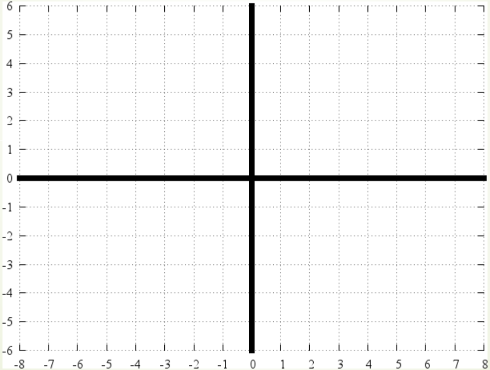
- Consider two complex numbers z_1 and z_2.
- They correspond to translations $f_1(z) = z + z_1$ and $f_2(z) = z + z_2$
- Functional composition: $(f_1 \circ f_2)(z) = z + z_1 + z_2$
- Represent functional composition by adding arrows.
- *Example:* $z_1 = 2 + 3\mathbf{i}$ and $z_2 = 3 + 1\mathbf{i}$



Playing with C: Multiplying complex numbers by a positive real number

Multiply each complex number by 0.5

$f(z) = 0.5z$



Arrow in same direction but half the length.

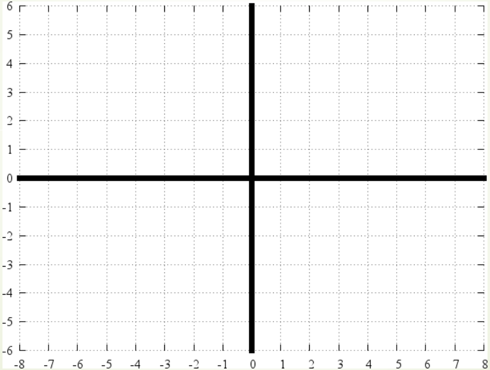
Scaling

Fields

Playing with C: Multiplying complex numbers by a negative number

Multiply each complex number by -1

$$f(z) = (-1)z$$



Arrow in opposite direction

Rotation by 180 degrees

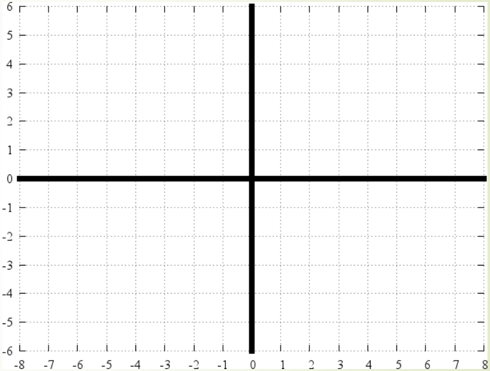
Playing with $\mathbf{C}$: Multiplying by $\mathbf{i}$: rotation by 90 degrees

How to rotate counterclockwise by 90°?

Need $x + y\,\mathbf{i} \mapsto -y + x\mathbf{i}$

Use $\mathbf{i}(x + y\mathbf{i}) = x\,\mathbf{i} + y\,\mathbf{i}^2 = x\,\mathbf{i} - y$

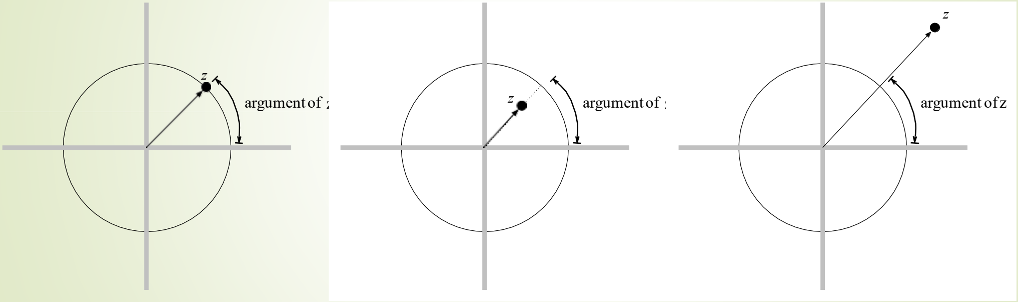
$f(z) = \mathbf{i}\,z$



Playing with **C**: The unit circle in the complex plane: *argument* and angle

What about rotating by another angle?

Definition: *Argument* of z is the angle in radians between z arrow and $1 + 0\mathbf{i}$ arrow.



Rotating a complex number z means *increasing its argument*.

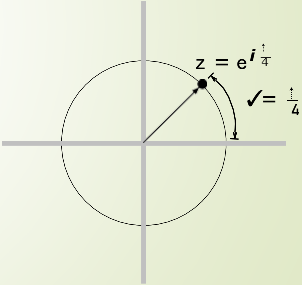
Playing with C: Euler's formula

"He calculated just as men breathe, as eagles sustain themselves in the air."
Said of Leonhard Euler

Euler's formula: For any real number θ,

$$e^{\theta \mathbf{i}}$$

is the point z on the unit circle with argument θ.



$e = 2.718281828...$

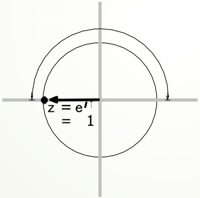
Playing with C: Euler's formula

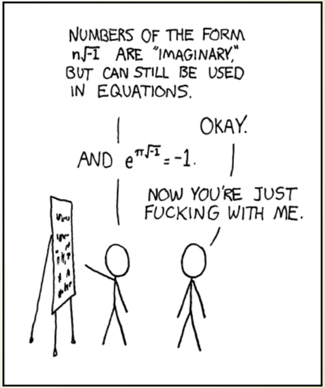
Euler's formula: For any real number θ,

$$e^{\theta\mathbf{i}}$$

is the point z on the unit circle with argument θ.

Plug in $\theta = \pi$ ....





Playing with C: Euler's formula

Plot

$$e^{0\cdot\frac{2\pi}{20}}, e^{1\cdot\frac{2\pi}{20}}, e^{2\cdot\frac{2\pi}{20}}, \dots, e^{19\cdot\frac{2\pi}{20}}$$

Playing with C: Rotation by τ radians

Back to question of rotation by any angle τ.

- Every complex number can be written in the form $z = re^{\theta \mathbf{i}}$
  - r is the absolute value of z
  - θ is the argument of z
- Need to increase the argument of z
- Use exponentiation law $e^a \cdot e^b = e^{a+b}$
- $re^{\theta \mathbf{i}} \cdot e^{\tau \mathbf{i}} = re^{\theta \mathbf{i} + \tau \mathbf{i}} = re^{(\theta + \tau)\mathbf{i}}$
- $f(z) = z \cdot e^{\tau \mathbf{i}}$ does rotation by angle τ.

Playing with C: Rotation by τ radians

Rotation by $3\pi/4$

Playing with *GF* (2)

Galois Field 2
has just two elements: 0 and 1

Addition is like exclusive-or:

+	0	1
0	0	1
1	1	0

Multiplication is like ordinary multiplication

$\times$	0	1
0	0	0
1	0	1

Evariste Galois, 1811-1832

Usual algebraic laws still hold, e.g. multiplication distributes over addition
$a \cdot (b + c) = a \cdot b + a \cdot c$

$GF(2)$ in Python

We provide a module GF2 that defines a value one.
This value acts like 1 in $GF(2)$:

```
>>> from GF2 import one
>>> one + one
0
>>> one * one
one
>>> one * 0
0
>>> one/one
one
```

We will use one in coding with $GF(2)$.