Linear Programming (LP)

Example

Formulating LP Models

1. Given a problem, first, determine the ***objective*** *or* ***goal***. Maximize (or minimize) what?

   Maximize profits

2. Identify & define the ***decision variables*** (unknowns).
   - What should they represent and how many do we need?

   X_1 = number of Aqua-Spas to produce
   X_2 = number of Hydro-Luxes to produce

   or abbreviate as follows

   X_i = no. of product *i* to make *i=1,2*

3. State the *objective* as a linear *function* of the *decision variables.*

   Max $350\, X_1 + 300\, X_2$

Formulating LP Models, Continued

4. Translate the requirements, restrictions, or wishes, that are in narrative form to *linear functions.*

$$1X_1 + 1X_2 <= 200 \quad \} \text{ pumps}$$
$$9X_1 + 6X_2 <= 1566 \quad \} \text{ labor}$$
$$12X_1 + 16X_2 <= 2880 \quad \} \text{ tubing}$$

5. Identify any lower or upper bounds on the decision variables (non-negativity constraints are very common).

$$X_1 >= 0$$
$$X_2 >= 0 \quad \text{or} \quad X_i >= 0 \ i=1,2$$

The Complete LP Model

MAX: $350X_1 + 300X_2$

S.T.: $1X_1 + 1X_2 <= 200$

$9X_1 + 6X_2 <= 1566$

$12X_1 + 16X_2 <= 2880$

$X_1 , X_2 >= 0$

The general form of an LP model:

MAX (or MIN): $c_1X_1 + c_2X_2 + \ldots + c_nX_n$

Subject to: $a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n <= b_1$

:

$a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n >= b_k$

:

$a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n = b_m$

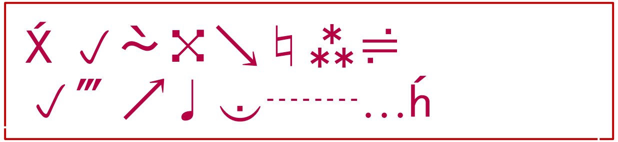
Solving LP Problems: A Random Guess Approach

X1	X2	Profit	

Can we gain more than 60,000?

MAX: $350X_1 + 300X_2$

S.T.:
$1X_1 + 1X_2 <= 200$ }pumps
$9X_1 + 6X_2 <= 1566$ } labor
$12X_1 + 16X_2 <= 2880$ } tubing.
$X_1, X_2 >= 0$



A painful process….

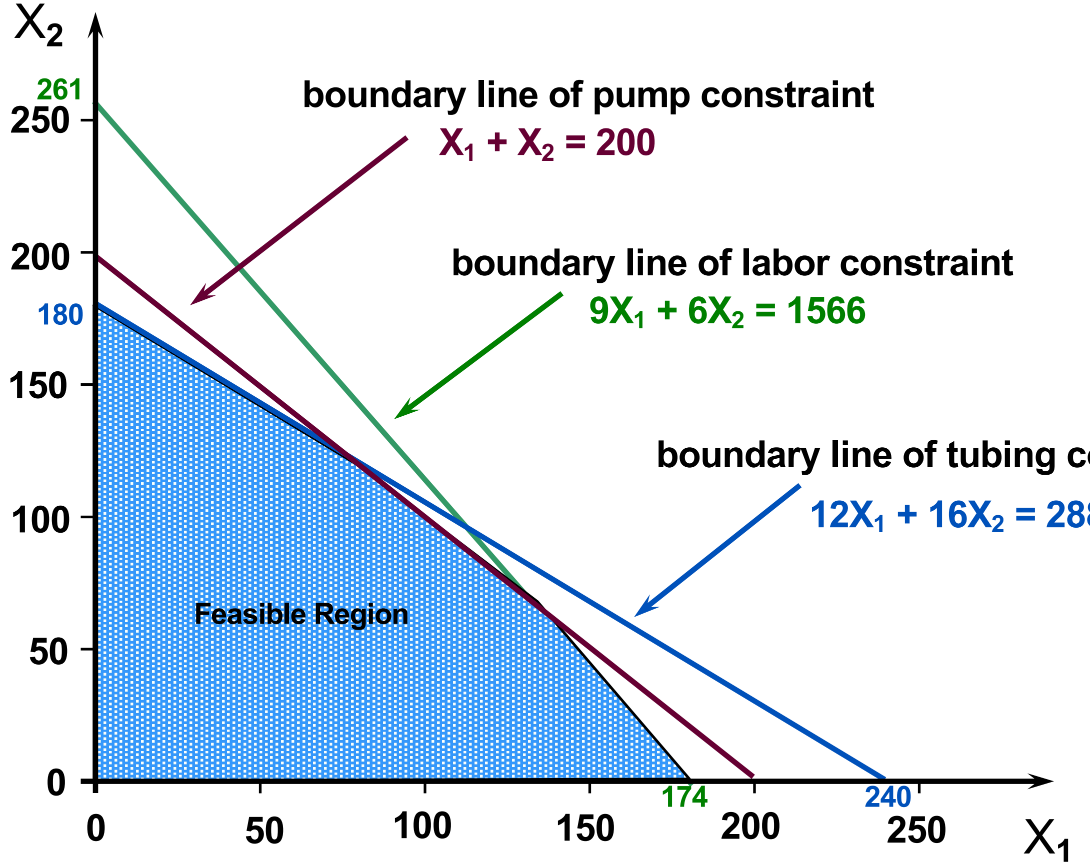
Solving LP Problems: An Intuitive Approach

- Idea: Each Aqua-Spa (X_1) generates the highest unit profit (\$350), so let's make as many of them as possible!
- How many would that be?
  - Let $X_2 = 0$
    - 1st constraint: $1X_1 <= 200$
    - 2nd constraint: $9X_1 <=1566$ or $X_1 <=174$
    - 3rd constraint: $12X_1 <= 2880$ or $X_1 <= 240$
- If X_2=0, the maximum value of X_1 is 174 and the total profit is \$350*174 + \$300*0 = \$60,900
- This solution is feasible, but is it optimal?
  - No

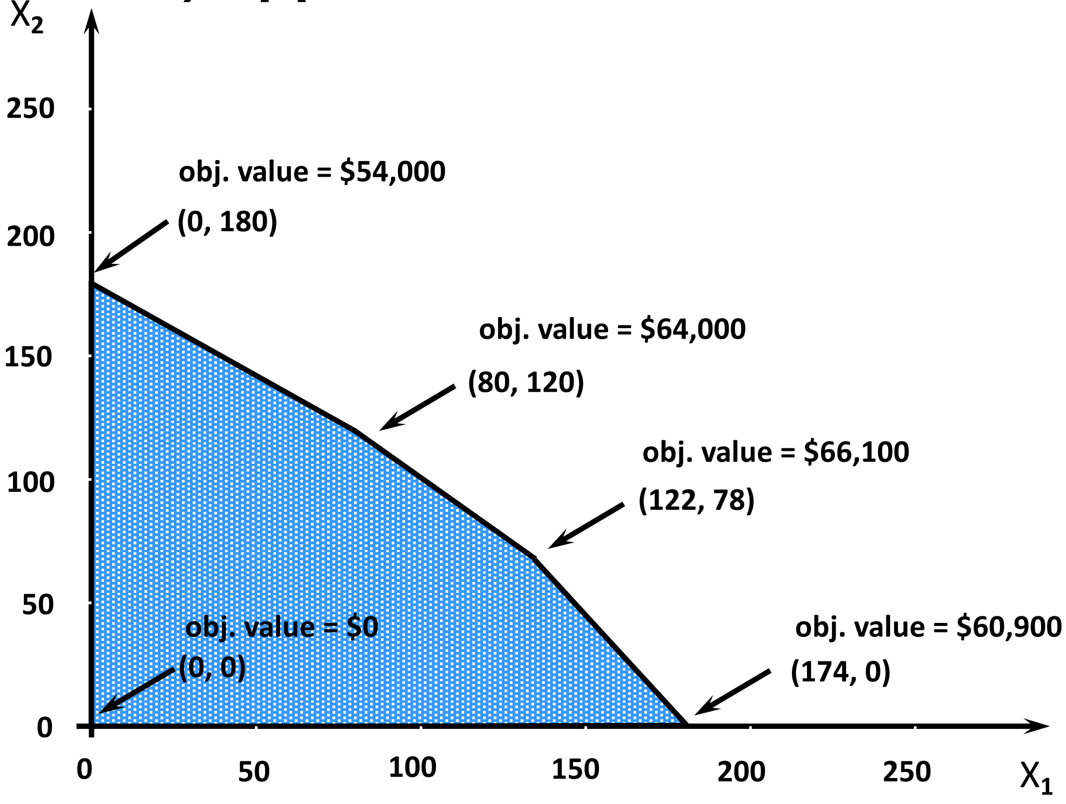
Solving LP Problems: A Graphical Approach

- The constraints of an LP problem define its feasible region.
- The best point in the feasible region is the optimal solution to the problem.
- For LP problems with two variables, it is easy to plot the feasible region and find the optimal solution.

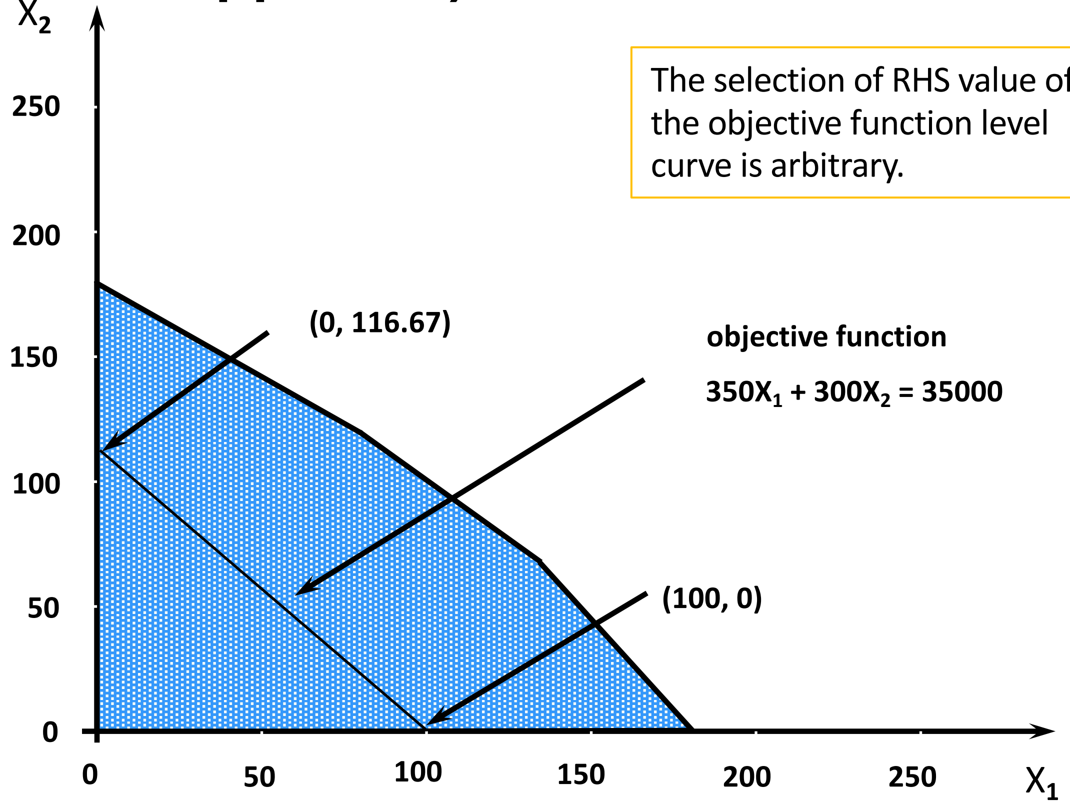
Graphical Solution Approach – To Develop an Understanding of the "Constrained Optimization" Environment



Enumerating the Corner Points: A Brute Force (Non-Recommended) Approach

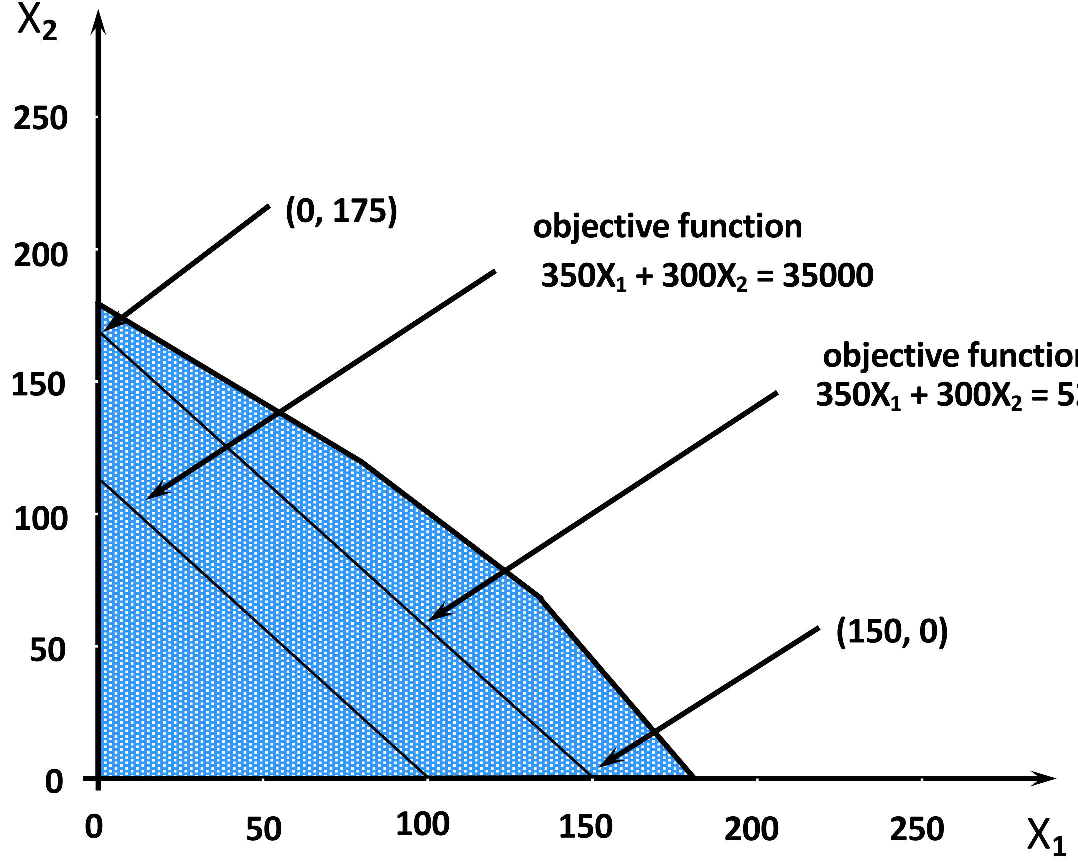


Plotting A Level Curve of the Objective Function (A Recommended Approach)

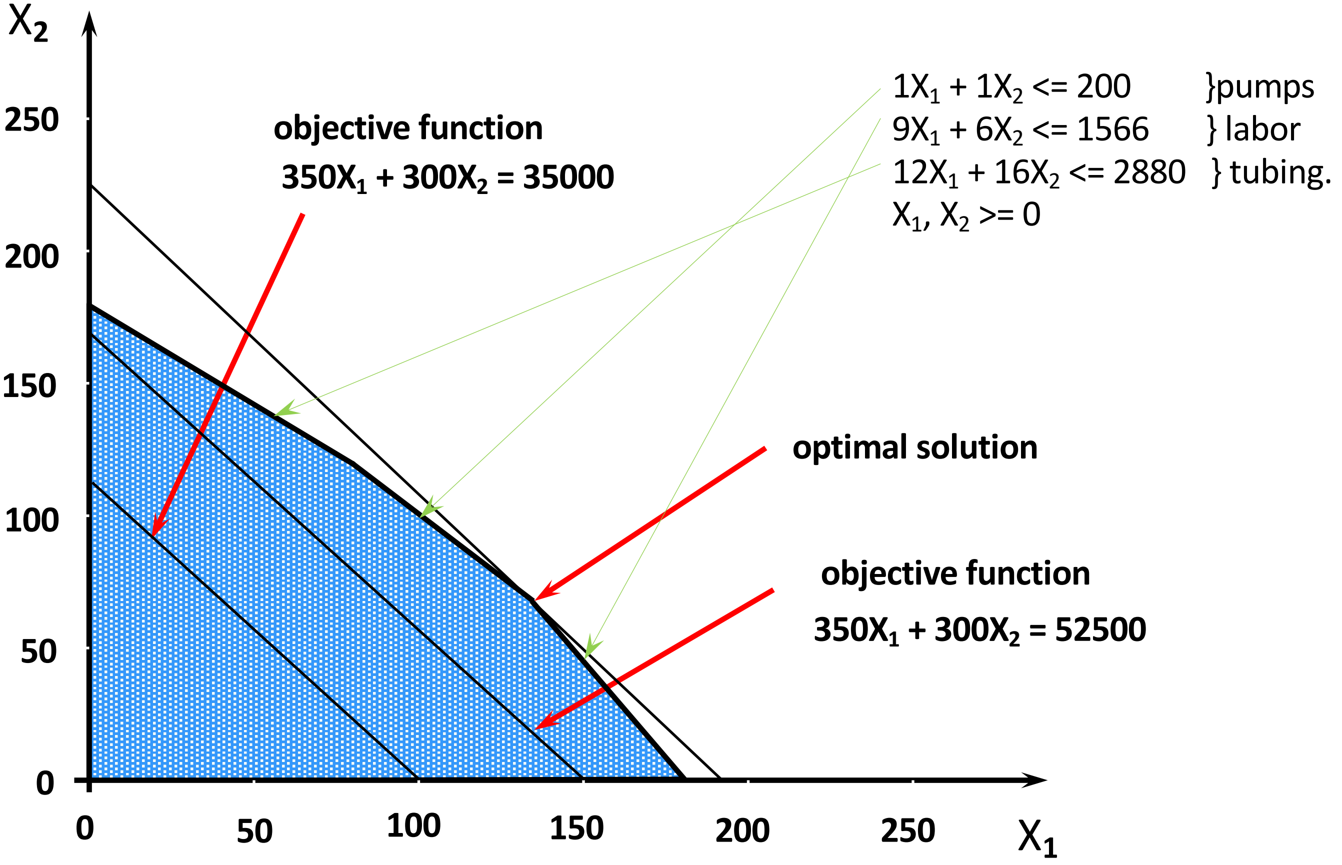


The selection of RHS value of the objective function level curve is arbitrary.

A Second Level Curve of the Objective Function



Using A Level Curve to Locate the Optimal Solution



Calculating the Optimal Solution

- The optimal solution occurs where the "pumps" and "labor" constraints intersect.
- This occurs where:
  - $X_1 + X_2 = 200$ (1) and
  - $9X_1 + 6X_2 = 1566$ (2)
- From (1) we have, $X_2 = 200 - X_1$ (3)
- Substituting (3) for X_2 in (2) we have,
  - $9X_1 + 6(200 - X_1) = 1566$
  - which reduces to $X_1 = 122$
- So, the optimal solution is,
  - $X_1=122$, $X_2=200-X_1=78$
- Optimal value:
  - Total Profit = \$350*122 + \$300*78 = \$66,100

Warning

Optimal value means the final (optimal) value of the objective function.

Optimal solution means the final (optimal) solution of the decision variables.

But, in the textbook and HWs, "optimal solution" is used for both of the above two concepts.

Extra Thinking: Add a New Constraint

Suppose that management believes that the optimal product mix from this model relies too heavily on the Aqua-Spa hot tub. They would prefer a solution that limits the production of Aqua-Spas to no more than 55% of the total number of hot tubs to be produced.

What changes would you need to make to the mathematical model to implement this policy?

A Minimization Problem