Chapter 6

Apply It 6.1

1. There are 3 rows, one for each source. There are two columns, one for each raw material. Thus, the size of the matrix is 3×2. Alternatively, she could use a 2×3 matrix.

2. The first column consists of 1's each representing the 1 hour needed for each phase of project 1. The second column consists of 2's for each phase of project 2 and so on. In general the nth column will consist of 2^n's, each representing the 2^n hours needed for each phase of project n. The time-analysis matrix is as follows.
$$\begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 2 & 4 & 8 & 16 \end{bmatrix}$$

Problems 6.1

1. **a.** The size is the number of rows by the columns. Thus A is 2×3, B is 3×3, C is 3×2, D is 2×2, E is 4×4, F is 1×2, G is 3×1, H is 3×3, and J is 1×1.

   b. A square matrix has the same number of rows as columns. Thus the square matrices are B, D, E, H, and J.

   c. An upper triangular matrix is a square matrix where all entries below the main diagonal are zeros. Thus H and J are upper triangular. A lower triangular matrix is a square matrix where all entries above the main diagonal are zeros. Thus D and J are lower triangular.

   d. A row vector (or row matrix) has only one row. Thus F and J are row vectors.

   e. A column vector (or column matrix) has only one column. Thus G and J are column vectors.

2. A has 4 rows and 4 columns. Thus the order of A is 4.

3. A_{21} is the entry in the 2nd row and 1st column, namely 6.

4. A_{42} is the entry in the 4th row and 2nd column, namely 0.

5. A_{32} is the entry in the 3rd row and 2nd column, namely 4.

6. A_{34} is the entry in the 3rd row and 4th column, namely 0.

7. A_{44} is the entry in the 4th row and 4th column, namely 0.

8. A_{55} is the entry in the 5th row and 5th column. But A has only 4 rows and 4 columns. Thus a_{55} does not exist.

9. The entries of the third row are the numbers arranged horizontally three rows down from the top of the matrix A: 5, 4, 1, 0

10. $$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

11. **a.** A is 2×3 and $A_{ij} = -i + 2j$.
$$\begin{bmatrix} -1+2(1) & -1+2(2) & -1+2(3) \\ -2+2(1) & -2+2(2) & -2+2(3) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix}$$

    b. C is 2×4 and $C_{ij} = (i+j)^2$.
$$\begin{bmatrix} (1+1)^2 & (1+2)^2 & (1+3)^2 & (1+4)^2 \\ (2+1)^2 & (2+2)^2 & (2+3)^2 & (2+4)^2 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \end{bmatrix}$$

12. **a.** B is 2×2 and $B_{ij} = (-1)^{i-j}(i^2 - j^2)$.
$$\begin{bmatrix} (-1)^{1-1}(1^2-1^2) & (-1)^{1-2}(1^2-2^2) \\ (-1)^{2-1}(2^2-1^2) & (-1)^{2-2}(2^2-2^2) \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

b. D is 2×3 and $D_{ij} = (-1)^i (j^3)$.

$$\begin{bmatrix} (-1)^1(1^3) & (-1)^1(2^3) & (-1)^1(3^3) \\ (-1)^2(1^3) & (-1)^2(2^3) & (-1)^2(3^3) \end{bmatrix} = \begin{bmatrix} -1 & -8 & -27 \\ 1 & 8 & 27 \end{bmatrix}$$

13. $12 \cdot 10 = 120$, so A has 120 entries. For a_{33}, $i = 3 = j$, so $a_{33} = 1$. Since $5 \neq 2$, $a_{52} = 0$. For $a_{10,10}$, $i = 10 = j$, so $a_{10,10} = 1$. Since $12 \neq 10$, $a_{12,10} = 0$.

14. The main diagonal is the diagonal extending from the upper left corner to the lower right corner.

a. 2, 5, −3, 1

b. $x^2, \sqrt{y}, 1$

15. A zero matrix is a matrix in which all entries are zeros.

a. $$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b. $$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

16. If A is 7×9, then A^T is 9×7.

17. $$A^T = \begin{bmatrix} 6 & -3 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} 6 & 2 \\ -3 & 4 \end{bmatrix}$$

18. $$A^T = \begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

19. $$A^T = \begin{bmatrix} 2 & 5 & -3 & 0 \\ 0 & 3 & 6 & 2 \\ 7 & 8 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 7 \\ 5 & 3 & 8 \\ -3 & 6 & -2 \\ 0 & 2 & 1 \end{bmatrix}$$

20. $$A^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

21. a. A and C are diagonal matrices.

b. All are them are triangular matrices.

22. $$A^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Since $A^T = A$, the matrix of Problem 20 is *symmetric*.

23. $$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 7 & 0 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 7 \\ 0 & 0 \\ -1 & 9 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} 1 & 7 \\ 0 & 0 \\ -1 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 \\ 7 & 0 & 9 \end{bmatrix} = A$$

24. Equating corresponding entries gives $3x = 9$, $2y - 1 = 6$, $z = 7$, and $5w = 15$. Thus $x = 3$, $y = \frac{7}{2}$, $z = 7$, and $w = 3$.

25. Equating corresponding entries gives $6 = 6$, $2 = 2$, $x = 6$, $7 = 7$, $3y = 2$, and $2z = 7$. Thus $x = 6$, $y = \frac{2}{3}$, $z = \frac{7}{2}$.

26. Equating entries in the 3rd row and 3rd column gives $7 = 8$, which is never true, so there is no solution.

27. Equating corresponding entries gives $2x = y$, $7 = 7$, $7 = 7$, and $2y = y$. Now $2y = y$ yields $y = 0$. Thus from $2x = y$ we get $2x = 0$, so $x = 0$. The solution is $x = 0$, $y = 0$.

28. $$\begin{bmatrix} 125 & 275 & 400 \end{bmatrix} \begin{bmatrix} 0.95 \\ 1.03 \\ 1.25 \end{bmatrix}$$

29. a. From J, the entry in row 3 (extreme) and column 2 (white) is 1. Thus in January, 1 white extreme model was sold.

b. From F, the entry in row 2 (deluxe) and column 3 (blue) is 4. Thus in February, 4 blue deluxe models were sold.

c. The entries in row 1 (regular) and column 4 (purple) give the number of purple regular models sold. For J the entry is 0 and for F the entry is 7. Thus more purple regular models were sold in February.

d. In January, there were
$1 + 4 + 5 + 0 = 10$ regular,
$3 + 5 + 2 + 7 = 17$ deluxe, and
$4 + 1 + 3 + 2 = 10$ extreme models sold. In February, there were
$2 + 5 + 7 + 7 = 21$ regular,
$2 + 4 + 4 + 6 = 16$ deluxe, and
$0 + 0 + 1 + 2 = 3$ extreme models sold.
Thus, no model sold the same number of units in both months.
In January, there were $1 + 3 + 4 = 8$ red, $4 + 5 + 1 = 10$ white, $5 + 2 + 3 = 10$ blue, and $0 + 7 + 2 = 9$ purple models sold. In February, there were $2 + 2 + 0 = 4$ red, $5 + 4 + 0 = 9$ white, $7 + 4 + 1 = 12$ blue, and $7 + 6 + 2 = 15$ purple models sold. Thus, no color sold the same number of units in both months.

e. In January a total of
$3 + 5 + 2 = 7 = 17$ deluxe models were sold.
In February a total of
$2 + 4 + 4 + 6 = 16$ deluxe models were sold.
Thus, more deluxe models were sold in January.

f. In January a total of $1 + 3 + 4 = 8$ red widgets were sold, while in February a total of $2 + 2 + 0 = 4$ red widgets were sold. Thus, more red widgets were sold in January.

g. Adding all entries in matrix J yields that a total of 37 widgets were sold in January.

30. The sums of the entries in the columns are 680, 710, 1510, and 6690. The sum of the entries in the rows are 680, 710, 1510, and 6690. The amount an industry consumes is equal to the amount of its output. Industry B has to increase output by (0.20)(90) = 18 units and industry C has to increase output by (0.20)(120) = 24 units. All other producers have to increase it by (0.20)(420) = 84 units.

31. By equating entries we find that x must satisfy
$x^2 + 2000x = 2001$ and $\sqrt{x^2} = -x$.
The second equation implies that $x < 0$. From the first equation, $x^2 + 2000x - 2001 = 0$,
$(x + 2001)(x - 1) = 0$, so $x = -2001$.

32. $\begin{bmatrix} 3 & -2 \\ -4 & 1 \\ 5 & 6 \end{bmatrix}$

33. $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 7 & 4 \\ 4 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}$

Apply It 6.2

3. $T = J + F = \begin{bmatrix} 120 & 80 \\ 105 & 130 \end{bmatrix} + \begin{bmatrix} 110 & 140 \\ 85 & 125 \end{bmatrix}$
$= \begin{bmatrix} 120+110 & 80+140 \\ 105+85 & 130+125 \end{bmatrix} = \begin{bmatrix} 230 & 220 \\ 190 & 255 \end{bmatrix}$

4. $0.8\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 40 \\ 30 \\ 60 \end{bmatrix} = 2\begin{bmatrix} 248 \\ 319 \\ 532 \end{bmatrix}$

$\begin{bmatrix} 0.8x_1 \\ 0.8x_2 \\ 0.8x_3 \end{bmatrix} - \begin{bmatrix} 40 \\ 30 \\ 60 \end{bmatrix} = \begin{bmatrix} 496 \\ 638 \\ 1064 \end{bmatrix}$

$\begin{bmatrix} 0.8x_1 - 40 \\ 0.8x_2 - 30 \\ 0.8x_3 - 60 \end{bmatrix} = \begin{bmatrix} 496 \\ 638 \\ 1064 \end{bmatrix}$

Solve $0.8x_1 - 40 = 496$ to get $x_1 = 670$.
Solve $0.8x_2 - 30 = 638$ to get $x_2 = 835$.
Solve $0.8x_3 - 60 = 1064$ to get $x_3 = 1405$.

Problems 6.2

1. $\begin{bmatrix} 2 & 0 & -3 \\ -1 & 4 & 0 \\ 1 & -6 & 5 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 4 \\ -1 & 6 & 5 \\ 9 & 11 & -2 \end{bmatrix} = \begin{bmatrix} 2+2 & 0+(-3) & -3+4 \\ -1+(-1) & 4+6 & 0+5 \\ 1+9 & -6+11 & 5+(-2) \end{bmatrix}$

$= \begin{bmatrix} 4 & -3 & 1 \\ -2 & 10 & 5 \\ 10 & 5 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -7 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 2+7+2 & -7+(-4)+7 \\ -6+(-2)+7 & 4+1+2 \end{bmatrix} = \begin{bmatrix} 11 & -4 \\ -1 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -3 \\ 5 & -9 \\ -4 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 1 \\ 9 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ -4 & -9 \\ -2 & 6 \end{bmatrix}$

4. $\frac{1}{2}\begin{bmatrix} 4 & -2 & 6 \\ 2 & 10 & -12 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cdot 4 & \frac{1}{2}(-2) & \frac{1}{2}\cdot 6 \\ \frac{1}{2}\cdot 2 & \frac{1}{2}\cdot 10 & \frac{1}{2}(-12) \\ \frac{1}{2}\cdot 0 & \frac{1}{2}\cdot 0 & \frac{1}{2}\cdot 8 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 5 & -6 \\ 0 & 0 & 4 \end{bmatrix}$

5. $2[2\ \ -1\ \ 3] + 4[-2\ \ 0\ \ 1] - 0[2\ \ 3\ \ 1]$
$= [4\ \ -2\ \ 6] + [-8\ \ 0\ \ 4] - [0\ \ 0\ \ 0]$
$= [4-8-0\quad -2+0-0\quad 6+4-0]$
$= [-4\ \ -2\ \ 10]$

6. $\begin{bmatrix} 7 & 7 \end{bmatrix}$ is a matrix and 66 is a number, so the sum is not defined.

7. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has size 2×2, and $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ has size 2×1. Thus the sum is not defined.

8. $\begin{bmatrix} 5 & 3 \\ -2 & 6 \end{bmatrix} + 7\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -2 & 6 \end{bmatrix}$

9. $-6\begin{bmatrix} 2 & -6 & 7 & 1 \\ 7 & 1 & 6 & -2 \end{bmatrix} = \begin{bmatrix} -6\cdot 2 & -6(-6) & -6\cdot 7 & -6\cdot 1 \\ -6\cdot 7 & -6\cdot 1 & -6\cdot 6 & -6(-2) \end{bmatrix} = \begin{bmatrix} -12 & 36 & -42 & -6 \\ -42 & -6 & -36 & 12 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -6 \\ 4 & 9 \end{bmatrix} - 3\begin{bmatrix} -6 & 9 \\ 2 & 6 \\ 1 & -2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -6 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} -18 & 27 \\ 6 & 18 \\ 3 & -6 \\ 12 & 15 \end{bmatrix} = \begin{bmatrix} 19 & -28 \\ -4 & -18 \\ 0 & 0 \\ -8 & -6 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -5 & 0 \\ -2 & 7 & 0 \\ 4 & 6 & 10 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 10 & 0 & 30 \\ 0 & 5 & 0 \\ 5 & 20 & 25 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 \\ -2 & 7 & 0 \\ 4 & 6 & 10 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 6 \\ 0 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 6 \\ -2 & 8 & 0 \\ 5 & 10 & 15 \end{bmatrix}$

12. $3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - 3\left(\begin{bmatrix}1 & 2 & 0\\ 0 & -2 & 1\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}4 & -2 & 2\\ -3 & 21 & -9\\ 0 & 1 & 0\end{bmatrix}\right) = \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix} - 3\begin{bmatrix}-3 & 4 & -2\\ 3 & -23 & 10\\ 0 & -1 & 1\end{bmatrix}$

$= \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix} - \begin{bmatrix}-9 & 12 & -6\\ 9 & -69 & 30\\ 0 & -3 & 3\end{bmatrix}$

$= \begin{bmatrix}12 & -12 & 6\\ -9 & 72 & -30\\ 0 & 3 & 0\end{bmatrix}$

13. $-2C = -2\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix} = \begin{bmatrix}4 & 2\\ 6 & -6\end{bmatrix}$

14. $-(A-B) = -\begin{bmatrix}2-(-6) & 1-(-5)\\ 3-2 & -3-(-3)\end{bmatrix} = -\begin{bmatrix}8 & 6\\ 1 & 0\end{bmatrix} = \begin{bmatrix}-8 & -6\\ -1 & 0\end{bmatrix}$

15. $2(0) = 2\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}2\cdot 0 & 2\cdot 0\\ 2\cdot 0 & 2\cdot 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0$

16. $A - B + C = \begin{bmatrix}2-(-6)+(-2) & 1-(-5)+(-1)\\ 3-2+(-3) & -3-(-3)+3\end{bmatrix} = \begin{bmatrix}6 & 5\\ -2 & 3\end{bmatrix}$

17. $3(2A - 3B) = 3\left\{2\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix} - 3\begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix}\right\} = 3\left\{\begin{bmatrix}4 & 2\\ 6 & -6\end{bmatrix} - \begin{bmatrix}-18 & -15\\ 6 & -9\end{bmatrix}\right\} = 3\begin{bmatrix}22 & 17\\ 0 & 3\end{bmatrix} = \begin{bmatrix}66 & 51\\ 0 & 9\end{bmatrix}$

18. $0(2A+3B-5C) = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

19. $3(A - C)$ is a 2×2 matrix and 6 is a number. Therefore $3(A - C) + 6$ is not defined.

20. $A+(C+B) = \begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix} + \begin{bmatrix}-2+(-6) & -1+(-5)\\ -3+2 & 3+(-3)\end{bmatrix} = \begin{bmatrix}2+(-8) & 1+(-6)\\ 3+(-1) & -3+0\end{bmatrix} = \begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix}$

21. $2B - 3A + 2C = 2\begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix} - 3\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix} + 2\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix}$

$= \begin{bmatrix}-12 & -10\\ 4 & -6\end{bmatrix} - \begin{bmatrix}6 & 3\\ 9 & -9\end{bmatrix} + \begin{bmatrix}-4 & -2\\ -6 & 6\end{bmatrix}$

$= \begin{bmatrix}-18 & -13\\ -5 & 3\end{bmatrix} + \begin{bmatrix}-4 & -2\\ -6 & 6\end{bmatrix} = \begin{bmatrix}-22 & -15\\ -11 & 9\end{bmatrix}$

22. $3C - 2B = \begin{bmatrix}-6 & -3\\ -9 & 9\end{bmatrix} - \begin{bmatrix}-12 & -10\\ 4 & -6\end{bmatrix} = \begin{bmatrix}6 & 7\\ -13 & 15\end{bmatrix}$

23. $\frac{1}{3}A+3(2B+5C)=\frac{1}{3}A+6B+15C$

$$=\frac{1}{3}\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}+6\begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix}+15\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ 1 & -1\end{bmatrix}+\begin{bmatrix}-36 & -30\\ 12 & -18\end{bmatrix}+\begin{bmatrix}-30 & -15\\ -45 & 45\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{196}{3} & -\frac{134}{3}\\ -32 & 26\end{bmatrix}$$

24. $\frac{1}{2}A-5(B+C)=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & -\frac{3}{2}\end{bmatrix}-5\begin{bmatrix}-8 & -6\\ -1 & 0\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{2}\\ \frac{3}{2} & -\frac{3}{2}\end{bmatrix}+\begin{bmatrix}40 & 30\\ 5 & 0\end{bmatrix}=\begin{bmatrix}41 & \frac{61}{2}\\ \frac{13}{2} & -\frac{3}{2}\end{bmatrix}$

25. $3(A+B)=3\begin{bmatrix}-4 & -4\\ 5 & -6\end{bmatrix}=\begin{bmatrix}-12 & -12\\ 15 & -18\end{bmatrix}$

$3A+3B=\begin{bmatrix}6 & 3\\ 9 & -9\end{bmatrix}+\begin{bmatrix}-18 & -15\\ 6 & -9\end{bmatrix}=\begin{bmatrix}-12 & -12\\ 15 & -18\end{bmatrix}$

Thus $3(A+B)=3A+3B$.

26. $(2+3)A=5A=\begin{bmatrix}10 & 5\\ 15 & -15\end{bmatrix}$

$2A+3A=\begin{bmatrix}4 & 2\\ 6 & -6\end{bmatrix}+\begin{bmatrix}6 & 3\\ 9 & -9\end{bmatrix}=\begin{bmatrix}10 & 5\\ 15 & -15\end{bmatrix}$

Thus $(2+3)A=2A+3A$.

27. $k_1(k_2A)=k_1\begin{bmatrix}2k_2 & k_2\\ 3k_2 & -3k_2\end{bmatrix}=\begin{bmatrix}2k_1k_2 & k_1k_2\\ 3k_1k_2 & -3k_1k_2\end{bmatrix}$

$(k_1k_2)A=\begin{bmatrix}2k_1k_2 & k_1k_2\\ 3k_1k_2 & -3k_1k_2\end{bmatrix}$

Thus $k_1(k_2A)=(k_1k_2)A$.

28. $k(A-2B+C)=k\left(\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}-2\begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix}+\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix}\right)$

$$=k\left(\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}+\begin{bmatrix}12 & 10\\ -4 & 6\end{bmatrix}+\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix}\right)$$

$$=k\begin{bmatrix}2+12-2 & 1+10-1\\ 3-4-3 & -3+6+3\end{bmatrix}$$

$$=k\begin{bmatrix}12 & 10\\ -4 & 6\end{bmatrix}$$

$$=\begin{bmatrix}12k & 10k\\ -4k & 6k\end{bmatrix}$$

$$kA-2kB+kC = k\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix} - 2k\begin{bmatrix}-6 & -5\\ 2 & -3\end{bmatrix} + k\begin{bmatrix}-2 & -1\\ -3 & 3\end{bmatrix}$$
$$= \begin{bmatrix}2k & k\\ 3k & -3k\end{bmatrix} + \begin{bmatrix}12k & 10k\\ -4k & 6k\end{bmatrix} + \begin{bmatrix}-2k & -k\\ -3k & 3k\end{bmatrix}$$
$$= \begin{bmatrix}2k+12k-2k & k+10k-k\\ 3k-4k-3k & -3k+6k+3k\end{bmatrix}$$
$$= \begin{bmatrix}12k & 10k\\ -4k & 6k\end{bmatrix}$$

Thus $k(A-2B+C) = kA-2kB+kC$.

29. $$3A+D^{\mathrm{T}} = 3\begin{bmatrix}1 & 2\\ 0 & -1\\ 7 & 0\end{bmatrix} + \begin{bmatrix}1 & 1\\ 2 & 0\\ -1 & 2\end{bmatrix} = \begin{bmatrix}3 & 6\\ 0 & -3\\ 21 & 0\end{bmatrix} + \begin{bmatrix}1 & 1\\ 2 & 0\\ -1 & 2\end{bmatrix} = \begin{bmatrix}4 & 7\\ 2 & -3\\ 20 & 2\end{bmatrix}$$

30. $$(B-C)^{\mathrm{T}} = \left\{\begin{bmatrix}1 & 3\\ 4 & -1\end{bmatrix} - \begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix}\right\}^{\mathrm{T}} = \begin{bmatrix}0 & 3\\ 3 & -3\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}0 & 3\\ 3 & -3\end{bmatrix}$$

31. $$2B^{\mathrm{T}} - 3C^{\mathrm{T}} = 2\begin{bmatrix}1 & 4\\ 3 & -1\end{bmatrix} - 3\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix} = \begin{bmatrix}2 & 8\\ 6 & -2\end{bmatrix} - \begin{bmatrix}3 & 3\\ 0 & 6\end{bmatrix} = \begin{bmatrix}-1 & 5\\ 6 & -8\end{bmatrix}$$

32. $$2B+B^{\mathrm{T}} = 2\begin{bmatrix}1 & 3\\ 4 & -1\end{bmatrix} + \begin{bmatrix}1 & 4\\ 3 & -1\end{bmatrix} = \begin{bmatrix}2 & 6\\ 8 & -2\end{bmatrix} + \begin{bmatrix}1 & 4\\ 3 & -1\end{bmatrix} = \begin{bmatrix}3 & 10\\ 11 & -3\end{bmatrix}$$

33. $$A+D^T-B = \begin{bmatrix}1 & 2\\ 0 & -1\\ 7 & 0\end{bmatrix} + \begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 2\end{bmatrix}^T - \begin{bmatrix}1 & 3\\ 4 & -1\end{bmatrix}$$

The computation is undefined, since B is not the same size as A and D^T.

34. $$\left(D-2A^{\mathrm{T}}\right)^{\mathrm{T}} = \left\{\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 2\end{bmatrix} - 2\begin{bmatrix}1 & 0 & 7\\ 2 & -1 & 0\end{bmatrix}\right\}^{\mathrm{T}}$$
$$= \left\{\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 2\end{bmatrix} - \begin{bmatrix}2 & 0 & 14\\ 4 & -2 & 0\end{bmatrix}\right\}^{\mathrm{T}} = \begin{bmatrix}-1 & 2 & -15\\ -3 & 2 & 2\end{bmatrix}^{\mathrm{T}}$$
$$= \begin{bmatrix}-1 & -3\\ 2 & 2\\ -15 & 2\end{bmatrix}$$

35. $$x\begin{bmatrix}3\\ 2\end{bmatrix} - y\begin{bmatrix}-4\\ 7\end{bmatrix} = 3\begin{bmatrix}2\\ 4\end{bmatrix}$$
$$\begin{bmatrix}3x\\ 2x\end{bmatrix} - \begin{bmatrix}-4y\\ 7y\end{bmatrix} = \begin{bmatrix}6\\ 12\end{bmatrix} = \begin{bmatrix}3x+4y\\ 2x-7y\end{bmatrix} = \begin{bmatrix}6\\ 12\end{bmatrix}$$

Equating corresponding entries gives

$$\begin{cases}3x+4y=6\\ 2x-7y=12\end{cases}$$

Multiply the first equation by 2 and the second equation by –3 to get

$$\begin{cases} 6x + 8y = 12 \\ -6x + 21y = -36 \end{cases}$$

Now add the two equations to get

$$29y = -24$$
$$y = -\frac{24}{29}$$

Therefore

$$3x = 6 - 4\left(-\frac{24}{29}\right) = \frac{270}{29}$$
$$x = \frac{90}{29}$$

The solution is $x = \frac{90}{29},\ y = -\frac{24}{29}$.

36. $\begin{bmatrix} 2x-4y \\ 5x+7y \end{bmatrix} = \begin{bmatrix} 16 \\ -3 \end{bmatrix}$

$$\begin{bmatrix} 2x \\ 5x \end{bmatrix} + \begin{bmatrix} -4y \\ 7y \end{bmatrix} = \begin{bmatrix} 16 \\ -3 \end{bmatrix}$$
$$x\begin{bmatrix} 2 \\ 5 \end{bmatrix} + y\begin{bmatrix} -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 16 \\ -3 \end{bmatrix}$$

37. $3\begin{bmatrix} x \\ y \end{bmatrix} - 3\begin{bmatrix} -2 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 6 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 3x+6 \\ 3y-12 \end{bmatrix} = \begin{bmatrix} 24 \\ -8 \end{bmatrix}$$

$3x + 6 = 24$, $3x = 18$, or $x = 6$.

$3y - 12 = -8$, $3y = 4$, or $y = \frac{4}{3}$.

Thus $x = 6,\ y = \frac{4}{3}$.

38. $5\begin{bmatrix} x \\ 3 \end{bmatrix} - 6\begin{bmatrix} 2 \\ -2y \end{bmatrix} = \begin{bmatrix} -4x \\ 3y \end{bmatrix}$

$$\begin{bmatrix} 5x-12 \\ 15+12y \end{bmatrix} = \begin{bmatrix} -4x \\ 3y \end{bmatrix}$$

$5x - 12 = -4x$, $9x = 12$, or $x = \frac{4}{3}$.

$15 + 12y = 3y$, $9y = -15$, or $y = -\frac{5}{3}$.

Thus $x = \frac{4}{3},\ y = -\frac{5}{3}$.

39. $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + 2\begin{bmatrix} x \\ y \\ 4z \end{bmatrix} = \begin{bmatrix} -10 \\ -24 \\ 14 \end{bmatrix}$

$$\begin{bmatrix} 2+2x \\ 4+2y \\ 6+8z \end{bmatrix} = \begin{bmatrix} -10 \\ -24 \\ 14 \end{bmatrix}$$

$2 + 2x = -10$, $2x = -12$, or $x = -6$.
$4 + 2y = -24$, $2y = -28$, or $y = -14$.
$6 + 8z = 14$, $8z = 8$, or $z = 1$.
Thus $x = -6$, $y = -14$, $z = 1$.

40. $x\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} + y\begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 2x+12-5y \end{bmatrix}$

$$\begin{bmatrix} 2x-2 \\ 2y \\ 2x+12-5y \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 2x+12-5y \end{bmatrix}$$

$2x - 2 = 10$, $2x = 12$, or $x = 6$.
$2y = 6$ or $y = 3$.
$2x + 12 - 5y = 2x + 12 - 5y$, which is true for all values of x and y. Thus $x = 6$, $y = 3$.

41. $X + Y = \begin{bmatrix} 30 & 50 \\ 800 & 720 \\ 25 & 30 \end{bmatrix} + \begin{bmatrix} 15 & 25 \\ 960 & 800 \\ 10 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 30+15 & 50+25 \\ 800+960 & 720+800 \\ 25+10 & 30+5 \end{bmatrix} = \begin{bmatrix} 45 & 75 \\ 1760 & 1520 \\ 35 & 35 \end{bmatrix}$$

42. $2B - A = 2\begin{bmatrix} 380 & 330 & 220 \\ 460 & 320 & 750 \end{bmatrix} - \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot 380 & 2\cdot 330 & 2\cdot 220 \\ 2\cdot 460 & 2\cdot 320 & 2\cdot 750 \end{bmatrix} - \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix}$$
$$= \begin{bmatrix} 760 & 660 & 440 \\ 920 & 640 & 1500 \end{bmatrix} - \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix}$$
$$= \begin{bmatrix} 360 & 310 & 290 \\ 470 & 360 & 650 \end{bmatrix}$$

43. $P + 0.16P$

$$= [p_A \quad p_B \quad p_C \quad p_D] + 0.16[p_A \quad p_B \quad p_C \quad p_D]$$
$$= [1.16p_A \quad 1.16p_B \quad 1.16p_C \quad 1.16p_D]$$
$$= 1.16P$$

Thus P must be multiplied by 1.16.

44. $(A-B)^{\text{T}} = [A+(-1)B]^{\text{T}}$ [definition of subtraction]
$= A^{\text{T}} + [(-1)B]^{\text{T}}$ [transpose of a sum]
$= A^{\text{T}} + (-1)B^{\text{T}}$ [transpose of a scalar multiple]
$= A^{\text{T}} - B^{\text{T}}$ [definition of subtraction]

45. $\begin{bmatrix} 15 & -4 & 26 \\ 4 & 7 & 30 \end{bmatrix}$

46. $\begin{bmatrix} -16 & -11 & -24 \\ -16 & -3 & -36 \end{bmatrix}$

47. $\begin{bmatrix} -10 & 22 & 12 \\ 24 & 36 & -44 \end{bmatrix}$

Apply It 6.3

5. Represent the value of each book by $\begin{bmatrix} 28 & 22 & 16 \end{bmatrix}$ and the number of each book by $\begin{bmatrix} 100 \\ 70 \\ 90 \end{bmatrix}$.
The total value is given by the following matrix product.
$$\begin{bmatrix} 28 & 22 & 16 \end{bmatrix}\begin{bmatrix} 100 \\ 70 \\ 90 \end{bmatrix} = [2800+1540+1440]$$
$= [5780]$
The total value is \$5780.

6. The total cost is given by the matrix product PQ.
$$PQ = \begin{bmatrix} 26.25 & 34.75 & 28.50 \end{bmatrix}\begin{bmatrix} 250 \\ 325 \\ 175 \end{bmatrix}$$
$= \begin{bmatrix} 6562.5 + 11,293.75 + 4987.5 \end{bmatrix} = [22,843.75]$
The total cost is \$22,843.75.

7. First, write the equations with the variable terms on the left-hand side.
$$\begin{cases} y + \dfrac{8}{5}x = \dfrac{8}{5} \\ y + \dfrac{1}{3}x = \dfrac{5}{3} \end{cases}$$
Let $A = \begin{bmatrix} 1 & \frac{8}{5} \\ 1 & \frac{1}{3} \end{bmatrix}$, $X = \begin{bmatrix} y \\ x \end{bmatrix}$, and $B = \begin{bmatrix} \frac{8}{5} \\ \frac{5}{3} \end{bmatrix}$.
Then the pair of lines is equivalent to the matrix equation $AX = B$ or $\begin{bmatrix} 1 & \frac{8}{5} \\ 1 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ \frac{5}{3} \end{bmatrix}$.

Problems 6.3

1. $C_{11} = 1(0) + 3(-2) + (-2)(3) = -12$

2. $C_{21} = (-2)(0) + (1)(-2) + (-1)(3) = -5$

3. $C_{32} = 0(-2) + 4(4) + 3(1) = 19$

4. $C_{33} = 0(3) + 4(-2) + 3(-1) = -11$

5. $C_{31} = 0(0) + 4(-2) + 3(3) = 1$

6. $C_{12} = 1(-2) + 3(4) + (-2)(1) = 8$

7. A is 2×3 and B is 3×1, so AB is 2×1; $2 \cdot 1 = 2$ entries.

8. D is 4×3 and E is 3×2, so DE is 4×2; $4 \cdot 2 = 8$ entries.

9. E is 3×2 and C is 2×5, so EC is 3×5; $3 \cdot 5 = 15$ entries.

10. D is 4×3 and B is 3×1, so DB is 4×1; $4 \cdot 1 = 4$ entries.

11. F is 2×3 and B is 3×1, so FB is 2×1; $2 \cdot 1 = 2$ entries.

12. B is 3×1 and E is 3×2. Because the number of columns of B does not equal the number of rows of E, BE is not defined.

13. E is 3×2, E^{T} is 2×3, and B is 3×1, so $EE^{\text{T}}B$ is 3×1; $3 \cdot 1 = 3$ entries.

14. A is 2×3 and E is 3×2, so AE is 2×2. Thus $E(AE)$ is 3×2; $3 \cdot 2 = 6$ entries.

15. E is 3×2. F is 2×3 and B is 3×1, so FB is 2×1. Thus $E(FB)$ is 3×1; $3 \cdot 1 = 3$ entries.

16. Both F and A are 2×3, so $F + A$ is 2×3. Because B is 3×1, $(F + A)B$ is 2×1; $2 \cdot 1 = 2$ entries.

17. $I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

18. $I_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(4)+(-4)(-1) & 2(0)+(-4)(3) \\ 3(4)+2(-1) & 3(0)+2(3) \end{bmatrix} = \begin{bmatrix} 12 & -12 \\ 10 & 6 \end{bmatrix}$

20. $\begin{bmatrix} -1 & 1 \\ 0 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1(1)+1(3) & -1(-2)+1(4) \\ 0(1)+4(3) & 0(-2)+4(4) \\ 2(1)+1(3) & 2(-2)+1(4) \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 12 & 16 \\ 5 & 0 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2(1)+0(4)+3(7) \\ -1(1)+4(4)+5(7) \end{bmatrix} = \begin{bmatrix} 23 \\ 50 \end{bmatrix}$

22. $\begin{bmatrix} 2 & 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} = [2(0)+5(1)+0(0)+1(-2)] = [3]$

23. $\begin{bmatrix} 1 & 4 & -1 \\ 0 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1(2)+4(0)+(-1)1 & 1(1)+4(-1)+(-1)(1) & 1(0)+4(1)+(-1)(2) \\ 0(2)+0(0)+2(1) & 0(1)+0(-1)+2(1) & 0(0)+0(1)+2(2) \\ -2(2)+1(0)+1(1) & -2(1)+1(-1)+1(1) & -2(0)+1(1)+1(2) \end{bmatrix} = \begin{bmatrix} 1 & -4 & 2 \\ 2 & 2 & 4 \\ -3 & -2 & 3 \end{bmatrix}$$

24. $\begin{bmatrix} 4 & 2 & -2 \\ 3 & 10 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4(3)+2(0)+(-2)(0) & 4(1)+2(0)+(-2)(1) & 4(1)+2(0)+(-2)(0) & 4(0)+2(0)+(-2)(1) \\ 3(3)+10(0)+0(0) & 3(1)+10(0)+0(1) & 3(1)+10(0)+0(0) & 3(0)+10(0)+0(1) \\ 1(3)+0(0)+2(0) & 1(1)+0(0)+2(1) & 1(1)+0(0)+2(0) & 1(0)+0(0)+2(1) \end{bmatrix}$

$= \begin{bmatrix} 12 & 2 & 4 & -2 \\ 9 & 3 & 3 & 0 \\ 3 & 3 & 1 & 2 \end{bmatrix}$

25. $[1 \quad -2 \quad 5]\begin{bmatrix} 1 & 5 & -2 & -1 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 1 & -3 \end{bmatrix}$

$= [1+0-5 \quad 5+0+0 \quad -2-4+5 \quad -1-2-15]$

$= [-4 \quad 5 \quad -1 \quad -18]$

26. The first matrix is 1×2 and the second is 3×2, so the product is not defined.

27. $\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}[0 \quad 1 \quad -3] = \begin{bmatrix} 1(0) & 1(1) & 1(-3) \\ 4(0) & 4(1) & 4(-3) \\ -2(0) & -2(1) & -2(-3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 4 & -12 \\ 0 & -2 & 6 \end{bmatrix}$

28. $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\left\{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0(1)+1(1) & 0(1)+1(1) & 0(1)+1(1) \\ 2(1)+3(1) & 2(1)+3(1) & 2(1)+3(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 5 & 5 \end{bmatrix}$

29. $3\left\{\begin{bmatrix} -2 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} + 2\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \end{bmatrix}\right\}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$= 3\left\{\begin{bmatrix} -2 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 4 \\ 2 & 2 & -4 \end{bmatrix}\right\}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$= 3\left\{\begin{bmatrix} -4 & 0 & 6 \\ 5 & 1 & -3 \end{bmatrix}\right\}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -12 & 0 & 18 \\ 15 & 3 & -9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$= \begin{bmatrix} -12(1)+0(3)+18(5) & -12(2)+0(4)+18(6) \\ 15(1)+3(3)+(-9)(5) & 15(2)+3(4)+(-9)(6) \end{bmatrix} = \begin{bmatrix} 78 & 84 \\ -21 & -12 \end{bmatrix}$

30. $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1(-1)+(-1)(2) & 1(0)+(-1)(1) & 1(-1)+(-1)(2) & 1(0)+(-1)(1) & 1(0)+(-1)(1) \\ 0(-1)+3(2) & 0(0)+3(1) & 0(-1)+3(2) & 0(0)+3(1) & 0(0)+3(1) \end{bmatrix}$

$= \begin{bmatrix} -3 & 1 & -3 & -1 & -1 \\ 6 & 3 & 6 & 3 & 3 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left\{\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left\{\begin{bmatrix} 2+0+3 & -4+0+0 \\ 1+0-6 & -2+0+0 \end{bmatrix}\right\}$

$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ -5 & -2 \end{bmatrix} = \begin{bmatrix} 5-10 & -4-4 \\ 15-20 & -12-8 \end{bmatrix} = \begin{bmatrix} -5 & -8 \\ -5 & -20 \end{bmatrix}$

32. $2\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} - 5\left(\begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 6 & 2 \\ -4 & 0 \end{bmatrix} - 5\begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 10 & 20 \\ 30 & 10 \end{bmatrix} = \begin{bmatrix} -4 & -18 \\ -34 & -10 \end{bmatrix}$

33. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0\cdot x+0\cdot y+1\cdot z \\ 0\cdot x+1\cdot y+0\cdot z \\ 1\cdot x+0\cdot y+0\cdot z \end{bmatrix} = \begin{bmatrix} z \\ y \\ x \end{bmatrix}$

34. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$

35. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 9 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 + 3x_3 \\ 4x_1 + 9x_2 + 7x_3 \end{bmatrix}$

36. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 \\ x_2 \\ 2x_1 + x_2 \end{bmatrix}$

37. $F - \frac{1}{2}DI = F - \frac{1}{2}D = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & 0 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{1}{6} \end{bmatrix}$

38. $DD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+1 & 0+1+2 & 0+1+1 \\ 1+0+1 & 0+2+2 & 0+2+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$

39. $3A-2BC=3\begin{bmatrix}1 & -2\\0 & 3\end{bmatrix}-2\begin{bmatrix}-2 & 3 & 0\\1 & -4 & 1\end{bmatrix}\begin{bmatrix}-1 & 1\\0 & 3\\2 & 4\end{bmatrix}$

$=\begin{bmatrix}3 & -6\\0 & 9\end{bmatrix}-2\begin{bmatrix}2+0+0 & -2+9+0\\-1+0+2 & 1-12+4\end{bmatrix}$

$=\begin{bmatrix}3 & -6\\0 & 9\end{bmatrix}-\begin{bmatrix}4 & 14\\2 & -14\end{bmatrix}=\begin{bmatrix}-1 & -20\\-2 & 23\end{bmatrix}$

40. $B(D+E)=\begin{bmatrix}-2 & 3 & 0\\1 & -4 & 1\end{bmatrix}\begin{bmatrix}4 & 0 & 0\\0 & 7 & 1\\1 & 2 & 4\end{bmatrix}=\begin{bmatrix}-8+0+0 & 0+21+0 & 0+3+0\\4+0+1 & 0-28+2 & 0-4+4\end{bmatrix}$

$=\begin{bmatrix}-8 & 21 & 3\\5 & -26 & 0\end{bmatrix}$

41. $3I-\frac{2}{3}FE=3I-\frac{2}{3}\begin{bmatrix}\frac{1}{3} & 0 & 0\\0 & \frac{1}{6} & 0\\0 & 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}3 & 0 & 0\\0 & 6 & 0\\0 & 0 & 3\end{bmatrix}$

$=3I-\frac{2}{3}\begin{bmatrix}\frac{1}{3}\cdot 3+0+0 & 0+0+0 & 0+0+0\\0+0+0 & \frac{1}{6}\cdot 6+0 & 0+0+0\\0+0+0 & 0+0+0 & 0+0+\frac{1}{3}\cdot 3\end{bmatrix}$

$=3I-\frac{2}{3}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{bmatrix}-\begin{bmatrix}\frac{2}{3} & 0 & 0\\0 & \frac{2}{3} & 0\\0 & 0 & \frac{2}{3}\end{bmatrix}=\begin{bmatrix}\frac{7}{3} & 0 & 0\\0 & \frac{7}{3} & 0\\0 & 0 & \frac{7}{3}\end{bmatrix}$

42. $CB(D-I)=\begin{bmatrix}-1 & 1\\0 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}-2 & 3 & 0\\1 & -4 & 1\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\1 & 2 & 1\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)$

$=\begin{bmatrix}-1(-2)+1(1) & -1(3)+1(-4) & -1(0)+1(1)\\0(-2)+3(1) & 0(3)+3(-4) & 0(0)+3(1)\\2(-2)+4(1) & 2(3)+4(-4) & 2(0)+4(1)\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\1 & 2 & 2\end{bmatrix}$

$=\begin{bmatrix}3 & -7 & 1\\3 & -12 & 3\\0 & -10 & 4\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\1 & 2 & 0\end{bmatrix}$

$=\begin{bmatrix}3(0)-7(0)+1(1) & 3(0)-7(0)+1(2) & 3(0)-7(1)+1(0)\\3(0)-12(0)+3(1) & 3(0)-12(0)+3(2) & 3(0)-12(1)+3(0)\\0(0)-10(0)+4(1) & 0(0)-10(0)+4(2) & 0(0)-10(1)+4(0)\end{bmatrix}$

$=\begin{bmatrix}1 & 2 & -7\\3 & 6 & -12\\4 & 8 & -10\end{bmatrix}$

43. $(DC)A = \left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}\right\}A = \begin{bmatrix} -1+0+0 & 1+0+0 \\ 0+0+2 & 0+3+4 \\ -1+0+2 & 1+6+4 \end{bmatrix}A$

$= \begin{bmatrix} -1 & 1 \\ 2 & 7 \\ 1 & 11 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1+0 & 2+3 \\ 2+0 & -4+21 \\ 1+0 & -2+33 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 2 & 17 \\ 1 & 31 \end{bmatrix}$

44. $A(BC) = A\left\{\begin{bmatrix} -2 & 3 & 0 \\ 1 & -4 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}\right\} = A\begin{bmatrix} 2+0+0 & -2+9+0 \\ -1+0+2 & 1-12+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 1 & -7 \end{bmatrix}$

$= \begin{bmatrix} 2-2 & 7+14 \\ 0+3 & 0-21 \end{bmatrix} = \begin{bmatrix} 0 & 21 \\ 3 & -21 \end{bmatrix}$

45. Impossible: A is not a square matrix, so A^2 is not defined.

46. $A^{\mathrm{T}}A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

47. $B^4 = \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}B^2$

$= \begin{bmatrix} 0 & 0 & -2 \\ -2 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}B$

$= \begin{bmatrix} 0 & 0 & -4 \\ 2 & -1 & -2 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & -8 \\ -2 & 1 & -6 \\ 0 & 0 & 16 \end{bmatrix}$

48. $A(B^{\mathrm{T}})^2C = A\begin{bmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix}C$

$= A\begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 4 \end{bmatrix}C$

$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 4 \end{bmatrix}C$

$= \begin{bmatrix} 0 & -3 & 0 \\ -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -6 & 3 \\ -4 & 5 \end{bmatrix}$

49. $(AIC)^{\mathrm{T}} = \left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}\right)^{\mathrm{T}}$

$= \left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}\right)^{\mathrm{T}}$

$= \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}^{\mathrm{T}}$

$= \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$

50. $A^{\mathrm{T}}\left(2C^{\mathrm{T}}\right) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 0 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -2 & -6 & 2 \\ 0 & -2 & 2 \end{bmatrix}$

51. $\left(BA^{\mathrm{T}}\right)^{\mathrm{T}} = \left\{\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}\right\}^{\mathrm{T}} = \begin{bmatrix} 0 & -1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

52. $(3A)^{\mathrm{T}} = \left(3\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right)^{\mathrm{T}} = \begin{bmatrix} 3 & -3 & 0 \\ 0 & 3 & 3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 3 & 0 \\ -3 & 3 \\ 0 & 3 \end{bmatrix}$

53. $(2I)^2 - 2I^2 = (2I)^2 - 2I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^2 - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

54. A^T is 3×2, C^T is 2×3, and B is 3×3, so A^TC^TB is 3×3 and $(A^TC^TB)^0 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

55. $A(I - 0) = A(I) = AI$. Since I is 3×3 and A has three columns, $AI = A$. Thus $A(I - 0) = A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

56. $I^T 0 = I0 = 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

57. $(AC)(AC)^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}(AC)^T$

$= \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}^T$

$= \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$

58. $B^2 - 3B + 2I$

$= \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & -2 \\ -2 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -3 \\ 6 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ -8 & 4 & -2 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -8 & 6 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

59. $AX = B$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & -9 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$B = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

The system is represented by

$$\begin{bmatrix} 3 & 1 \\ 2 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}.$$

60. $AX = B$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & -1 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}$$

The system is represented by

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}.$$

61. $AX = B$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & -1 & 2 \\ 3 & -2 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$B = \begin{bmatrix} 9 \\ 5 \\ 11 \end{bmatrix}$$

The system is represented by

$$\begin{bmatrix} 2 & -1 & 3 \\ 5 & -1 & 2 \\ 3 & -2 & 2 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 11 \end{bmatrix}.$$

62. $E = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

play/it/again/sam
16, 12, 1, 25/9, 20/1, 7, 1, 9, 14/19, 1, 13

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 16 \\ 12 \end{bmatrix} = \begin{bmatrix} 28 \\ 44 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 25 \end{bmatrix} = \begin{bmatrix} 26 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 20 \end{bmatrix} = \begin{bmatrix} 29 \\ 38 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 14 \\ 19 \end{bmatrix} = \begin{bmatrix} 33 \\ 47 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 13 \end{bmatrix} = \begin{bmatrix} 14 \\ 15 \end{bmatrix}$$

The encoded message is 28, 44, 26, 27/29, 38/8, 9, 10, 11, 33/47, 14, 15.

63. $$\begin{bmatrix} 6 & 10 & 7 \end{bmatrix}\begin{bmatrix} 55 \\ 150 \\ 35 \end{bmatrix} = [6 \cdot 55 + 10 \cdot 150 + 7 \cdot 35]$$

$= [330 + 1500 + 245]$
$= [2075]$
The value of the inventory is \$2075.

64. $$\begin{bmatrix} 200 & 300 & 500 & 250 \end{bmatrix}\begin{bmatrix} 100 \\ 150 \\ 200 \\ 300 \end{bmatrix} = [240{,}000]$$

The total cost of the stocks is \$240,000.

65. $Q = [5 \quad 2 \quad 4]$

$$R = \begin{bmatrix} 5 & 20 & 16 & 7 & 17 \\ 7 & 18 & 12 & 9 & 21 \\ 6 & 25 & 8 & 5 & 13 \end{bmatrix}$$

$$C = \begin{bmatrix} 2500 \\ 1200 \\ 800 \\ 150 \\ 1500 \end{bmatrix}$$

$$QRC = Q(RC) = Q\begin{bmatrix} 5\cdot 2500+20\cdot 1200+16\cdot 800+7\cdot 150+17\cdot 1500 \\ 7\cdot 2500+18\cdot 1200+12\cdot 800+9\cdot 150+21\cdot 1500 \\ 6\cdot 2500+25\cdot 1200+8\cdot 800+5\cdot 150+13\cdot 1500 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 4 \end{bmatrix}\begin{bmatrix} 75,850 \\ 81,550 \\ 71,650 \end{bmatrix}$$

$$= [5(75,850)+2(81,550)+4(71,650)]$$

$$= [828,950]$$

The total cost of raw materials is $828,950.

66. a. $$RC = \begin{bmatrix} 5 & 20 & 16 & 7 & 17 \\ 7 & 18 & 12 & 9 & 21 \\ 6 & 25 & 8 & 5 & 13 \end{bmatrix}\begin{bmatrix} 3500 & 50 \\ 1500 & 50 \\ 1000 & 100 \\ 250 & 10 \\ 3500 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 17,500+30,000+16,000+1750+59,500 & 250+1000+1600+70+0 \\ 24,500+27,000+12,000+2250+73,500 & 350+900+1200+90+0 \\ 21,000+37,500+8000+1250+45,500 & 300+1250+800+50+0 \end{bmatrix}$$

$$= \begin{bmatrix} 124,750 & 2920 \\ 139,250 & 2540 \\ 113,250 & 2400 \end{bmatrix}$$

b. $$QRC = Q(RC) = \begin{bmatrix} 5 & 7 & 12 \end{bmatrix}\begin{bmatrix} 124,750 & 2920 \\ 139,250 & 2540 \\ 113,250 & 2400 \end{bmatrix}$$

$$= \begin{bmatrix} 623,750+974,750+1,359,000 & 14,600+17,780+28,800 \end{bmatrix}$$

$$= \begin{bmatrix} 2,957,500 & 61,180 \end{bmatrix}$$

c. $$QRCZ = (QRC)Z = \begin{bmatrix} 2,957,500 & 61,180 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = [2,957,500 + 61,180] = [3,018,680]$$

67. a. Amount spent on goods:

coal industry: $D_C P = \begin{bmatrix} 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 10,000 \\ 20,000 \\ 40,000 \end{bmatrix} = [180,000]$

elec. industry: $D_E P = \begin{bmatrix} 20 & 0 & 8 \end{bmatrix}\begin{bmatrix} 10,000 \\ 20,000 \\ 40,000 \end{bmatrix} = [520,000]$

steel industry: $D_S P = \begin{bmatrix} 30 & 5 & 0 \end{bmatrix}\begin{bmatrix} 10,000 \\ 20,000 \\ 40,000 \end{bmatrix} = [400,000]$

The coal industry spends $180,000, the electric industry spends $520,000, and the steel industry spends $400,000.

consumer 1: $D_1P = \begin{bmatrix}3 & 2 & 5\end{bmatrix}\begin{bmatrix}10,000\\20,000\\40,000\end{bmatrix} = [270,000]$

consumer 2: $D_2P = \begin{bmatrix}0 & 17 & 1\end{bmatrix}\begin{bmatrix}10,000\\20,000\\40,000\end{bmatrix} = [380,000]$

consumer 3: $D_3P = \begin{bmatrix}4 & 6 & 12\end{bmatrix}\begin{bmatrix}10,000\\20,000\\40,000\end{bmatrix} = [640,000]$

Consumer 1 pays $270,000, consumer 2 pays $380,000, and consumer 3 pays $640,000.

b. From Example 3 of Sec. 6.2, the number of units sold of coal, electricity, and steel are 57, 31, and 30, respectively. Thus the profit for coal is 10,000(57) – 180,000 = $390,000, the profit for elec. is 20,000(31) – 520,000 = $100,000, and the profit for steel is 40,000(30) – 400,000 = $800,000.

c. From (a), the total amount of money that is paid out by all the industries and consumers is 180,000 + 520,000 + 400,000 + 270,000 + 380,000 + 640,000 = $2,390,000.

d. The proportion of the total amount in (c) paid out by the industries is

$\dfrac{180,000+520,000+400,000}{2,390,000} = \dfrac{110}{239}.$

The proportion of the total amount in (c) paid by consumers is

$\dfrac{270,000+380,000+640,000}{2,390,000} = \dfrac{129}{239}.$

68. $(A + B)(A - B) = A(A - B) + B(A - B)$ [dist. prop.]

$= A^2 - AB + BA - B^2$ [dist prop.]

$= A^2 - BA + BA - B^2$ [$AB = BA$, given]

$= A^2 - B^2$

69. $\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}\begin{bmatrix}2 & -3\\-1 & \frac{3}{2}\end{bmatrix} = \begin{bmatrix}1(2)+(2)(-1) & 1(-3)+2\left(\frac{3}{2}\right)\\1(2)+2(-1) & 1(-3)+2\left(\frac{3}{2}\right)\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

70. Let $D_1 = \begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{bmatrix}$ and $D_2 = \begin{bmatrix}d & 0 & 0\\0 & e & 0\\0 & 0 & f\end{bmatrix}$.

a. $D_1D_2 = \begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{bmatrix}\begin{bmatrix}d & 0 & 0\\0 & e & 0\\0 & 0 & f\end{bmatrix} = \begin{bmatrix}ad & 0 & 0\\0 & be & 0\\0 & 0 & cf\end{bmatrix}$

$D_2D_1 = \begin{bmatrix}d & 0 & 0\\0 & e & 0\\0 & 0 & f\end{bmatrix}\begin{bmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{bmatrix} = \begin{bmatrix}ad & 0 & 0\\0 & be & 0\\0 & 0 & cf\end{bmatrix}$

Both D_1D_2 and D_2D_1 are diagonal matrices.

b. From part (a), $D_1D_2 = D_2D_1$. Thus D_1 and D_2 commute. [In fact, all $n \times n$ diagonal matrices commute.]

71. $\begin{bmatrix} 72.82 & -9.8 \\ 51.32 & -36.32 \end{bmatrix}$

72. $\begin{bmatrix} 23.994 & -20.832 & -12.648 \\ 26.164 & 7.44 & -168.64 \end{bmatrix}$

73. $\begin{bmatrix} 15.606 & 64.08 \\ -739.428 & 373.056 \end{bmatrix}$

74. $\begin{bmatrix} 11.952 & 54.06 \\ 86.496 & 278.648 \end{bmatrix}$

Apply It 6.4

8. The corresponding system is

$$\begin{cases} 6A+B+3C=35 \\ 3A+2B+3C=22 \\ A+5B+3C=18 \end{cases}$$

Reduce the augmented coefficient matrix of the system.

$$\left[\begin{array}{ccc|c} 6 & 1 & 3 & 35 \\ 3 & 2 & 3 & 22 \\ 1 & 5 & 3 & 18 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 18 \\ 3 & 2 & 3 & 22 \\ 6 & 1 & 3 & 35 \end{array}\right]$$

$$\xrightarrow[-6R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 18 \\ 0 & -13 & -6 & -32 \\ 0 & -29 & -15 & -73 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{13}R_2} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 18 \\ 0 & 1 & \frac{6}{13} & \frac{32}{13} \\ 0 & -29 & -15 & -73 \end{array}\right]$$

$$\xrightarrow[29R_2+R_3]{-5R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{9}{13} & \frac{74}{13} \\ 0 & 1 & \frac{6}{13} & \frac{32}{13} \\ 0 & 0 & -\frac{21}{13} & -\frac{21}{13} \end{array}\right]$$

$$\xrightarrow{-\frac{13}{21}R_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{9}{13} & \frac{74}{13} \\ 0 & 1 & \frac{6}{13} & \frac{32}{13} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{6}{13}R_3+R_2]{-\frac{9}{13}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus there should be 5 blocks of A, 2 blocks of B, and 1 block of C suggested.

9. Let x be the number of tablets of X, y be the number of tablets of Y, and z be the number of tablets of Z. The system is

$40x + 10y + 10z = 180$
$20x + 10y + 50z = 200$
$10x + 30y + 20z = 190$

Reduce the augmented coefficient matrix of the system.

$$\left[\begin{array}{ccc|c} 40 & 10 & 10 & 180 \\ 20 & 10 & 50 & 200 \\ 10 & 30 & 20 & 190 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 10 & 30 & 20 & 190 \\ 20 & 10 & 50 & 200 \\ 40 & 10 & 10 & 180 \end{array}\right]$$

$$\xrightarrow[\substack{\frac{1}{10}R_2 \\ \frac{1}{10}R_3}]{\frac{1}{10}R_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 19 \\ 2 & 1 & 5 & 20 \\ 4 & 1 & 1 & 18 \end{array}\right]$$

$$\xrightarrow[-4R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 19 \\ 0 & -5 & 1 & -18 \\ 0 & -11 & -7 & -58 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 19 \\ 0 & 1 & -\frac{1}{5} & \frac{18}{5} \\ 0 & -11 & -7 & -58 \end{array}\right]$$

$$\xrightarrow[11R_2+R_3]{-3R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{5} & \frac{41}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{18}{5} \\ 0 & 0 & -\frac{46}{5} & -\frac{92}{5} \end{array}\right]$$

$$\xrightarrow{-\frac{5}{46}R_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{5} & \frac{41}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{18}{5} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[\frac{1}{5}R_3+R_2]{-\frac{13}{5}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

She should take 3 tablets of X, 4 tablets of Y, and 2 tablets of Z.

10. Let a, b, c, and d be the number of bags of foods A, B, C, and D, respectively. The corresponding system is

$$\begin{cases} 5a+5b+10c+5d=10{,}000 \\ 10a+5b+30c+10d=20{,}000 \\ 5a+15b+10c+25d=20{,}000 \end{cases}$$

Reduce the augmented coefficient matrix of the system.

$$\left[\begin{array}{cccc|c} 5 & 5 & 10 & 5 & 10{,}000 \\ 10 & 5 & 30 & 10 & 20{,}000 \\ 5 & 15 & 10 & 25 & 20{,}000 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2000 \\ 10 & 5 & 30 & 10 & 20{,}000 \\ 5 & 15 & 10 & 25 & 20{,}000 \end{array}\right]$$

$$\xrightarrow[-5R_1+R_3]{-10R_1+R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2000 \\ 0 & -5 & 10 & 0 & 0 \\ 0 & 10 & 0 & 20 & 10{,}000 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2000 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 10 & 0 & 20 & 10{,}000 \end{array}\right]$$

$$\xrightarrow[-10R_2+R_3]{-R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 4 & 1 & 2000 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 20 & 20 & 10{,}000 \end{array}\right]$$

$$\xrightarrow{\frac{1}{20}R_3} \left[\begin{array}{cccc|c} 0 & 0 & 4 & 0 & 2000 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

$$\xrightarrow[2R_3+R_2]{-4R_3+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 1000 \\ 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

This reduced matrix corresponds to the system

$$\begin{cases} a-3d=0 \\ b+2d=1000 \\ c+d=500 \end{cases}$$

Letting $d = r$, we get the general solution of the system:
$a = 3r$
$b = -2r + 1000$
$c = -r + 500$
$d = r$
Note that a, b, c, and d cannot be negative, given the context, hence $0 \le r \le 500$. One specific solution is when $r = 250$, then $a = 750$, $b = 500$, $c = 250$, and $d = 250$.

Problems 6.4

1. The first nonzero entry in row 2 is not to the right of the first nonzero entry in row 1, hence not reduced.

2. Reduced.

3. Reduced.

4. In row 2, the first nonzero entry is in column 2, but not all other entries in column 2 are zeros, hence not reduced.

5. The first row consists entirely of zeros and is not below each row containing a nonzero entry, hence not reduced.

6. The first nonzero entry of row 2 is to the left of the first nonzero entry of row 1, hence not reduced.

7. $$\begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} \xrightarrow{-4R_1+R_2} \begin{bmatrix} 1 & 3 \\ 0 & -12 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{12}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-3R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8. $$\begin{bmatrix} 0 & -3 & 0 & 2 \\ 1 & 5 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & -3 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -\frac{2}{3} \end{bmatrix}$$

$$\xrightarrow{-5R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{16}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \end{bmatrix}$$

9. $$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

10. $\begin{bmatrix} 2 & 3 \\ 1 & -6 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -6 \\ 2 & 3 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \xrightarrow[-R_1+R_4]{\substack{-2R_1+R_2 \\ -4R_1+R_3}} \begin{bmatrix} 1 & -6 \\ 0 & 15 \\ 0 & 32 \\ 0 & 13 \end{bmatrix} \xrightarrow{\frac{1}{15}R_2} \begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 0 & 32 \\ 0 & 13 \end{bmatrix} \xrightarrow[-13R_2+R_4]{\substack{6R_2+R_1 \\ -32R_2+R_3}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 7 & 2 & 3 \\ -1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 7 & 2 & 3 \\ 2 & 3 & 4 & 1 \\ -1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

$\xrightarrow{R_4 \leftrightarrow R_2} \begin{bmatrix} 1 & 7 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ -1 & 4 & 2 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix} \xrightarrow{\substack{R_1+R_3 \\ -2R_1+R_4}} \begin{bmatrix} 1 & 7 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 11 & 4 & 3 \\ 0 & -11 & 0 & -5 \end{bmatrix}$

$\xrightarrow{\substack{R_3+R_4 \\ -11R_2+R_3 \\ -7R_2+R_1}} \begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -7 & 3 \\ 0 & 0 & 4 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{7}R_3} \begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{7} \\ 0 & 0 & 4 & -2 \end{bmatrix}$

$\xrightarrow{\substack{5R_3+R_1 \\ -R_3+R_2 \\ -4R_3+R_4}} \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{3}{7} \\ 0 & 0 & 0 & -\frac{2}{7} \end{bmatrix} \xrightarrow{-\frac{7}{2}R_4} \begin{bmatrix} 1 & 0 & 0 & \frac{6}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{3}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{\substack{-\frac{6}{7}R_4+R_1 \\ -\frac{3}{7}R_4+R_2 \\ \frac{3}{7}R_4+R_3}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix}$

$\xrightarrow{-R_2} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{-4R_2+R_4} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_3+R_4]{-\frac{3}{2}R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

13. $\left[\begin{array}{cc|c} 2 & -7 & 50 \\ 1 & 3 & 10 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 10 \\ 2 & -7 & 50 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & -13 & 30 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & 1 & -\frac{30}{13} \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 0 & \frac{220}{13} \\ 0 & 1 & -\frac{30}{13} \end{array}\right]$

Thus $x = \dfrac{220}{13}$ and $y = -\dfrac{30}{13}$.

14. $\left[\begin{array}{cc|c} 1 & -3 & -11 \\ 4 & 3 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -11 \\ 0 & 15 & 53 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -11 \\ 0 & 1 & \frac{53}{15} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{53}{15} \end{array}\right]$

Thus $x = -\dfrac{2}{5}$, $y = \dfrac{53}{15}$.

15. $\left[\begin{array}{cc|c} 3 & 1 & 4 \\ 12 & 4 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 4 \\ 0 & 0 & -14 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & -14 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{array}\right]$

The last row indicates 0 = 1, which is never true, so there is no solution.

16. $\left[\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ -1 & -2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & -2 & -3 & 1 \\ 3 & 2 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 3 & 2 & -1 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & -4 & -10 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 1 & \frac{5}{2} & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{5}{2} & -1 \end{array}\right],$

which gives $\begin{cases} x - 2z = 1 \\ y + \dfrac{5}{2}z = -1 \end{cases}$.

Thus, $x = 2r + 1$, $y = -\dfrac{5}{2}r - 1$, $z = r$, where r is any real number.

17. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -6 & -1 & -7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{1}{6} & \frac{7}{6} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & 1 & \frac{1}{6} & \frac{7}{6} \end{array}\right],$

which gives $\begin{cases} x + \frac{2}{3}z = \frac{5}{3} \\ y + \frac{1}{6}z = \frac{7}{6} \end{cases}$.

Thus, $x = -\dfrac{2}{3}r + \dfrac{5}{3}$, $y = -\dfrac{1}{6}r + \dfrac{7}{6}$, $z = r$, where r is any real number.

18. $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 1 & 1 & 5 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -2 & 3 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{9}{2} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{2} & \frac{29}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{9}{2} \end{array}\right]$

Thus $x = -\dfrac{13}{2}r + \dfrac{29}{2}$, $y = \dfrac{3}{2}r - \dfrac{9}{2}$, $z = r$, where r is any real number.

19. $\left[\begin{array}{rr|r} 1 & -3 & 0 \\ 2 & 2 & 3 \\ 5 & -1 & 1 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -3 & 0 \\ 0 & 8 & 3 \\ 0 & 14 & 1 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -3 & 0 \\ 0 & 1 & \frac{3}{8} \\ 0 & 14 & 1 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 0 & \frac{9}{8} \\ 0 & 1 & \frac{3}{8} \\ 0 & 0 & -\frac{17}{4} \end{array}\right]$

From the third row, $0 = -\dfrac{17}{4}$, which is never true, so there is no solution.

20. $\left[\begin{array}{rr|r} 1 & 4 & 9 \\ 3 & -1 & 6 \\ 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 4 & 9 \\ 0 & -13 & -21 \\ 0 & -5 & -7 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 4 & 9 \\ 0 & 1 & \frac{21}{13} \\ 0 & -5 & -7 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 0 & \frac{33}{13} \\ 0 & 1 & \frac{21}{13} \\ 0 & 0 & \frac{14}{13} \end{array}\right]$

The last row indicates that $0 = \dfrac{14}{13}$, which is never true. There is no solution.

21. $\left[\begin{array}{rrr|r} 1 & 3 & 0 & 2 \\ 2 & 7 & 0 & 4 \\ 1 & 5 & 1 & 5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Thus, $x = 2$, $y = 0$, and $z = 3$.

22. $\left[\begin{array}{rrr|r} 1 & 1 & -1 & 7 \\ 2 & -3 & -2 & 4 \\ 1 & -1 & -5 & 23 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & -1 & 7 \\ 0 & -5 & 0 & -10 \\ 0 & -2 & -4 & 16 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & -2 & -4 & 16 \end{array}\right]$

$\to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & 20 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right]$

Thus $x = 0$, $y = 2$, $z = -5$.

23. $\left[\begin{array}{rrr|r} 2 & 0 & -4 & 8 \\ 1 & -2 & -2 & 14 \\ 1 & 1 & -2 & -1 \\ 3 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -2 & 4 \\ 1 & -2 & -2 & 14 \\ 1 & 1 & -2 & -1 \\ 3 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -2 & 4 \\ 0 & -2 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 1 & 7 & -12 \end{array}\right]$

$\to \left[\begin{array}{rrr|r} 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & -2 & 0 & 10 \\ 0 & 1 & 7 & -12 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Thus $x = 2$, $y = -5$, $z = -1$.

24. $\left[\begin{array}{rrr|r} 1 & 0 & 3 & -1 \\ 3 & 2 & 11 & 1 \\ 1 & 1 & 4 & 1 \\ 2 & -3 & 3 & -8 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 3 & -1 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & -3 & -6 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & -3 & -6 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Thus $x = -3r - 1$, $y = -r + 2$, $z = r$, where r is any real number.

25. $\left[\begin{array}{ccccc|c} 1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$

Thus, $x_1 = r$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, and $x_5 = r$, where *r* is any number.

26. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

Thus, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 0$.

27. Let *x* = federal tax and *y* = state tax. Then *x* = 0.25(312,000 – *y*) and *y* = 0.10(312,000 – *x*). Equivalently,

$\begin{cases} x + 0.25y = 78{,}000 \\ 0.10x + y = 31{,}200. \end{cases}$

$\left[\begin{array}{cc|c} 1 & 0.25 & 78{,}000 \\ 0.10 & 1 & 31{,}200 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0.25 & 78{,}000 \\ 0 & 0.975 & 23{,}400 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0.25 & 78{,}000 \\ 0 & 1 & 24{,}000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 72{,}000 \\ 0 & 1 & 24{,}000 \end{array}\right].$

Thus *x* = 72,000 and *y* = 24,000, so the federal tax is $72,000 and the state tax is $24,000.

28. *x* = no. of units of A to be sold and *y* = no. of units of B to be sold. Then *x* = 1.25*y* and 8*x* + 11*y* = 42,000. Equivalently,

$\begin{cases} x - 1.25y = 0, \\ 8x + 11y = 42{,}000. \end{cases}$

$\left[\begin{array}{cc|c} 1 & -1.25 & 0 \\ 8 & 11 & 42{,}000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1.25 & 0 \\ 0 & 21 & 42{,}000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1.25 & 0 \\ 0 & 1 & 2000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2500 \\ 0 & 1 & 2000 \end{array}\right].$

Thus *x* = 2500 and *y* = 2000, so 2500 units of A and 2000 units of B must be sold.

29. Let x = number of units of A produced, y = number of units of B produced, and z = number of units of C produced. Then
no. of units: $x + y + z = 11,000$
total cost: $4x + 5y + 7z + 17,000 = 80,000$
total profit: $x + 2y + 3z = 25,000$
Equivalently,

$$\begin{cases} x+y+z=11,000 \\ 4x+5y+7z=63,000 \\ x+2y+3z=25,000 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 11,000 \\ 4 & 5 & 7 & 63,000 \\ 1 & 2 & 3 & 25,000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11,000 \\ 0 & 1 & 3 & 19,000 \\ 0 & 1 & 2 & 14,000 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8,000 \\ 0 & 1 & 3 & 19,000 \\ 0 & 0 & -1 & -5,000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8,000 \\ 0 & 1 & 3 & 19,000 \\ 0 & 0 & 1 & 5,000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2000 \\ 0 & 1 & 0 & 4000 \\ 0 & 0 & 1 & 5000 \end{array}\right]$$

Thus $x = 2000$, $y = 4000$, and $z = 5000$, so 2000 units of A, 4000 units of B and 5000 units of C should be produced.

30. Let x = number of desks to be produced at the East Coast plant and y = number of desks to be produced at the West Coast plant. Then $x + y = 800$ and $90x + 20,000 = 95y + 18,000$.
Equivalently,

$$\begin{cases} x+y=800 \\ 90x-95y=-2000. \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 800 \\ 90 & -95 & -2000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 800 \\ 0 & -185 & -74,000 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 800 \\ 0 & 1 & 400 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 400 \\ 0 & 1 & 400 \end{array}\right]$$

$x = 400$ and $y = 400$
Thus the production order is 400 units at the East Coast plant and 400 units at the West Coast plant.

31. Let x = number of brand X pills, y = number of brand Y pills, and z = number of brand Z pills. Considering the unit requirements gives the system

$$\begin{cases} 2x+1y+1z=10 & \text{(vitamin A)} \\ 3x+3y+0z=9 & \text{(vitamin D)} \\ 5x+4y+1z=19 & \text{(vitamin E)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 3 & 0 & 9 \\ 5 & 4 & 1 & 19 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \\ 3 & 3 & 0 & 9 \\ 5 & 4 & 1 & 19 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \\ 0 & \frac{3}{2} & -\frac{3}{2} & -6 \\ 0 & \frac{3}{2} & -\frac{3}{2} & -6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\begin{cases} x = 7 - r \\ y = r - 4 \\ z = r \end{cases}$ where $r = 4, 5, 6, 7$.

The only solutions for the problem are $z = 4$, $x = 3$, and $y = 0$; $z = 5$, $x = 2$, and $y = 1$; $z = 6$, $x = 1$, and $y = 2$; $z = 7$, $x = 0$, and $y = 3$. Their respective costs (in cents) are 15, 23, 31, and 39.

a. The possible combinations are 3 of X, 4 of Z; 2 of X, 1 of Y, 5 of Z; 1 of X, 2 of Y, 6 of Z; 3 of Y, 7 of Z.

b. The combination 3 of X, 4 of Z costs 15 cents a day.

c. The least expensive combination is 3 of X, 4 of Z; the most expensive is 3 of Y, 7 of Z.

32. Let x, y, and z be the numbers of units of A, B, and C, respectively.

$$\begin{cases} 3x+1y+2z=490 & \text{(machine I)} \\ 1x+2y+1z=310 & \text{(machine (II)} \\ 2x+4y+1z=560 & \text{(machine III)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 490 \\ 1 & 2 & 1 & 310 \\ 2 & 4 & 1 & 560 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 310 \\ 3 & 1 & 2 & 490 \\ 2 & 4 & 1 & 560 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 310 \\ 0 & -5 & -1 & -440 \\ 0 & 0 & -1 & -60 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 310 \\ 0 & 1 & \frac{1}{5} & 88 \\ 0 & 0 & -1 & -60 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & 134 \\ 0 & 1 & \frac{1}{5} & 88 \\ 0 & 0 & -1 & -60 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & 134 \\ 0 & 1 & \frac{1}{5} & 88 \\ 0 & 0 & 1 & 60 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 98 \\ 0 & 1 & 0 & 76 \\ 0 & 0 & 1 & 60 \end{array}\right]$$

$x = 98$, $y = 76$, $z = 60$

Thus, 98 units of A, 76 units of B, and 60 units of C should be produced.

33. a. Let s, d, and g represent the number of units of S, D, and G, respectively. Then

$$\begin{cases} 12s+20d+32g=220 & \text{(stock A)} \\ 16s+12d+28g=176 & \text{(stock B)} \\ 8s+28d+36g=264 & \text{(stock C)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 12 & 20 & 32 & 220 \\ 16 & 12 & 28 & 176 \\ 8 & 28 & 36 & 264 \end{array}\right] \xrightarrow[\substack{\left(\frac{1}{4}\right)R_2 \\ \left(\frac{1}{8}\right)R_3}]{\left(\frac{1}{4}\right)R_1} \left[\begin{array}{ccc|c} 3 & 5 & 8 & 55 \\ 4 & 3 & 7 & 44 \\ 1 & \frac{7}{2} & \frac{9}{2} & 33 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & \frac{7}{2} & \frac{9}{2} & 33 \\ 4 & 3 & 7 & 44 \\ 3 & 5 & 8 & 55 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_3]{-4R_1+R_2} \left[\begin{array}{ccc|c} 1 & \frac{7}{2} & \frac{9}{2} & 33 \\ 0 & -11 & -11 & -88 \\ 0 & -\frac{11}{2} & -\frac{11}{2} & -44 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{11}R_2} \left[\begin{array}{ccc|c} 1 & \frac{7}{2} & \frac{9}{2} & 33 \\ 0 & 1 & 1 & 8 \\ 0 & -\frac{11}{2} & -\frac{11}{2} & -44 \end{array}\right]$$

$$\xrightarrow[\frac{11}{2}R_2+R_3]{-\frac{7}{2}R_2+R_1}\left[\begin{array}{ccc|c}1&0&1&5\\0&1&1&8\\0&0&0&0\end{array}\right]$$

Thus $s = 5 - r$, $d = 8 - r$, and $g = r$, where $r = 0, 1, 2, 3, 4, 5$.
The six possible combinations are given by

COMBINATION						
r	0	1	2	3	4	5
S	5	4	3	2	1	0
D	8	7	6	5	4	3
G	0	1	2	3	4	5

b. Computing the cost of each combination, we find that they are 4700, 4600, 4500, 4400, 4300, and 4200 dollars, respectively. Buying 3 units of Deluxe and 5 units of Gold Star ($s = 0$, $d = 3$, $g = 5$) minimizes the cost.

Apply It 6.5

11. Write the coefficient matrix and reduce.

$$\begin{bmatrix}5&3&4\\6&8&7\\3&1&2\end{bmatrix}\xrightarrow{\frac{1}{5}R_1}\begin{bmatrix}1&\frac{3}{5}&\frac{4}{5}\\6&8&7\\3&1&2\end{bmatrix}\xrightarrow[-3R_1+R_3]{-6R_1+R_2}\begin{bmatrix}1&\frac{3}{5}&\frac{4}{5}\\0&\frac{22}{5}&\frac{11}{5}\\0&-\frac{4}{5}&-\frac{2}{5}\end{bmatrix}$$

$$\xrightarrow{\frac{5}{22}R_2}\begin{bmatrix}1&\frac{3}{5}&\frac{4}{5}\\0&1&\frac{1}{2}\\0&-\frac{4}{5}&-\frac{2}{5}\end{bmatrix}\xrightarrow[\frac{4}{5}R_2+R_3]{-\frac{3}{5}R_2+R_1}\begin{bmatrix}1&0&\frac{1}{2}\\0&1&\frac{1}{2}\\0&0&0\end{bmatrix}$$

The system has infinitely many solutions since there are two nonzero rows in the reduced coefficient matrix.

$$x+\frac{1}{2}z=0$$

$$y+\frac{1}{2}z=0$$

Let $z = r$, so $x=-\frac{1}{2}r$ and $y=-\frac{1}{2}r$, where r is any real number.

Problems 6.5

1. $$\left[\begin{array}{cccc|c}1&1&-1&-9&-3\\2&3&2&15&12\\2&1&2&5&8\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&-1&-9&3\\0&1&4&33&18\\0&-1&4&23&14\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&-1&-9&-3\\0&1&4&33&18\\0&0&8&56&32\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&-1&-9&-3\\0&1&4&33&18\\0&0&1&7&4\end{array}\right]$$

$$\to\left[\begin{array}{cccc|c}1&1&0&-2&1\\0&1&0&5&2\\0&0&1&7&4\end{array}\right]\to\left[\begin{array}{cccc|c}1&0&0&-7&-1\\0&1&0&5&2\\0&0&1&7&4\end{array}\right]$$

Thus $w = -1 + 7r$, $x = 2 - 5r$, $y = 4 - 7r$, $z = r$ (where r is any real number).

2. $\left[\begin{array}{cccc|c} 2 & 1 & 10 & 15 & -5 \\ 1 & -5 & 2 & 15 & -10 \\ 1 & 1 & 6 & 12 & 9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -5 & 2 & 15 & -10 \\ 2 & 1 & 10 & 15 & -5 \\ 1 & 1 & 6 & 12 & 9 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & -5 & 2 & 15 & -10 \\ 0 & 11 & 6 & -15 & 15 \\ 0 & 6 & 4 & -3 & 19 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -5 & 2 & 15 & -10 \\ 0 & 1 & \frac{6}{11} & -\frac{15}{11} & \frac{15}{11} \\ 0 & 6 & 4 & -3 & 19 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & 0 & \frac{52}{11} & \frac{90}{11} & -\frac{35}{11} \\ 0 & 1 & \frac{6}{11} & -\frac{15}{11} & \frac{15}{11} \\ 0 & 0 & \frac{8}{11} & \frac{57}{11} & \frac{119}{11} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & \frac{52}{11} & \frac{90}{11} & -\frac{35}{11} \\ 0 & 1 & \frac{6}{11} & -\frac{15}{11} & \frac{15}{11} \\ 0 & 0 & 1 & \frac{57}{8} & \frac{119}{8} \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{51}{2} & -\frac{147}{2} \\ 0 & 1 & 0 & -\frac{21}{4} & -\frac{27}{4} \\ 0 & 0 & 1 & \frac{57}{8} & \frac{119}{8} \end{array}\right]$

Thus, $w = \frac{51}{2}r - \frac{147}{2}$, $x = \frac{21}{4}r - \frac{27}{4}$, $y = -\frac{57}{8}r + \frac{119}{8}$, $z = r$ (where r is any real number).

3. $\left[\begin{array}{cccc|c} 3 & -1 & -3 & -1 & -2 \\ 2 & -2 & -6 & -6 & -4 \\ 2 & -1 & -3 & -2 & -2 \\ 3 & 1 & 3 & 7 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & -1 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & -2 & -6 & -6 & -4 \\ 2 & -1 & -3 & -2 & -2 \\ 3 & 1 & 3 & 7 & 2 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & -1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & -\frac{4}{3} & -4 & -\frac{16}{3} & -\frac{8}{3} \\ 0 & -\frac{1}{3} & -1 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & 2 & 6 & 8 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & -1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 3 & 4 & 2 \\ 0 & -\frac{1}{3} & -1 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & 2 & 6 & 8 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Thus, $w = -s$, $x = -3r - 4s + 2$, $y = r$, $z = s$ (where r and s are any real numbers).

4. $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & -3 & 4 & -7 & 1 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & -4 & 4 & -12 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -4 & 4 & -12 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Thus, $w = -r - 2s + 1$, $x = r - 3s$, $y = r$, $z = s$ (where r and s are any real numbers).

5. $\left[\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 5 \\ 1 & -3 & -1 & 3 & 1 \\ 3 & -9 & 1 & 1 & 11 \\ 2 & -6 & -1 & 4 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 5 \\ 0 & 0 & -2 & 4 & -4 \\ 0 & 0 & -2 & 4 & -4 \\ 0 & 0 & -3 & 6 & -6 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 5 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 3 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Thus, $w = 3r - s + 3$, $y = 2s + 2$, $x = r$, $z = s$ (where r and s are any real numbers).

6. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 4 \\ 2 & 1 & 2 & 2 & 7 \\ 1 & 2 & 1 & 4 & 5 \\ 3 & -2 & 3 & -4 & 7 \\ 4 & -3 & 4 & -6 & 9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 4 \\ 0 & -1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & -5 & 0 & -10 & -5 \\ 0 & -7 & 0 & -14 & -7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & -5 & 0 & -10 & -5 \\ 0 & -7 & 0 & -14 & -7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Thus, $w = -r + 3$, $x = -2s + 1$, $y = r$, $z = s$ (where r and s are any real numbers).

7. $\left[\begin{array}{ccccc|c} 4 & -3 & 5 & -10 & 11 & -8 \\ 2 & 1 & 5 & 0 & 3 & 6 \end{array}\right] \to \left[\begin{array}{ccccc|c} 0 & -5 & -5 & -10 & 5 & -20 \\ 2 & 1 & 5 & 0 & 3 & 6 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 2 & 1 & 5 & 0 & 3 & 6 \\ 0 & -5 & -5 & -10 & 5 & -20 \end{array}\right] \to \left[\begin{array}{ccccc|c} 2 & 1 & 5 & 0 & 3 & 6 \\ 0 & 1 & 1 & 2 & -1 & 4 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 2 & 0 & 4 & -2 & 4 & 2 \\ 0 & 1 & 1 & 2 & -1 & 4 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 2 & -1 & 2 & 1 \\ 0 & 1 & 1 & 2 & -1 & 4 \end{array}\right]$

Thus, $x_1 = -2r + s - 2t + 1$, $x_2 = -r - 2s + t + 4$, $x_3 = r$, $x_4 = s$, $x_5 = t$ (where r, s, and t are any real numbers).

8. $\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 2 & -2 & 3 & 10 & 15 & 10 \\ 1 & 2 & 3 & -2 & 2 & -2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & -2 & -3 & 8 & 7 & 8 \\ 0 & 2 & 0 & -3 & -2 & -3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 4 & 7 & 8 \\ 0 & 0 & -2 & 1 & -2 & -3 \end{array}\right]$

$\to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -4 & -7 & -8 \\ 0 & 0 & 0 & -7 & -16 & -19 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & \frac{12}{7} & -\frac{12}{7} \\ 0 & 1 & 1 & 0 & \frac{32}{7} & \frac{38}{7} \\ 0 & 0 & 1 & 0 & \frac{15}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 1 & \frac{16}{7} & \frac{19}{7} \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -\frac{33}{7} & -\frac{72}{7} \\ 0 & 1 & 0 & 0 & \frac{17}{7} & \frac{18}{7} \\ 0 & 0 & 1 & 0 & \frac{15}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 1 & \frac{16}{7} & \frac{19}{7} \end{array}\right]$

Thus $x_1 = -\frac{72}{7} + \frac{33}{7}r$, $x_2 = \frac{18}{7} - \frac{17}{7}r$, $x_3 = \frac{20}{7} - \frac{15}{7}r$, $x_4 = \frac{19}{7} - \frac{16}{7}r$, and $x_5 = r$, where r is any real number.

9. The system is homogeneous with fewer equations than unknowns (2 < 3), so there are infinitely many solutions.

10. The system is homogeneous with fewer equations than unknowns (2 < 4), so there are infinitely many solutions.

11. $\begin{bmatrix} 3 & -4 \\ 1 & 5 \\ 4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 3 & -4 \\ 4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 0 & -19 \\ 0 & -21 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 0 & -21 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = A$

A has $k = 2$ nonzero rows. Number of unknowns is $n = 2$. Thus $k = n$, so the system has the trivial solution only.

12. $\begin{bmatrix} 2 & 3 & 12 \\ 3 & -2 & 5 \\ 4 & 1 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 6 \\ 3 & -2 & 5 \\ 4 & 1 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 6 \\ 0 & -\frac{13}{2} & -13 \\ 0 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 6 \\ 0 & 1 & 2 \\ 0 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = A$

A has $k = 2$ nonzero rows. Number of unknowns is $n = 3$. Thus $k < n$, so the system has infinitely many solutions.

13. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = A$

A has $k = 2$ nonzero rows. Number of unknowns is $n = 3$. Thus $k < n$, so the system has infinitely many solutions.

14. $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & -2 \\ 0 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 2 & 2 & -2 \\ 0 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{2}{3} \\ 0 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$

A has $k = 3$ nonzero rows. Number of unknowns is $n = 3$. Thus $k = n$, so the system has the trivial solution only.

15. $\begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} \\ 5 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{29}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The solution is $x = 0$, $y = 0$.

16. $\begin{bmatrix} 2 & -5 \\ 8 & -20 \end{bmatrix} \to \begin{bmatrix} 2 & -5 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{5}{2} \\ 0 & 0 \end{bmatrix}$

The solution is $x = \dfrac{5}{2}r$, $y = r$.

17. $\begin{bmatrix} 1 & 6 & -2 \\ 2 & -3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & -2 \\ 0 & -15 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & -2 \\ 0 & 1 & -\frac{8}{15} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{6}{5} \\ 0 & 1 & -\frac{8}{15} \end{bmatrix}$

The solution is $x = -\dfrac{6}{5}r$, $y = \dfrac{8}{15}r$, $z = r$.

18. $\begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{7}{4} \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The solution is $x = 0$, $y = 0$.

19. $\begin{bmatrix} 1 & 1 \\ 3 & -4 \\ 5 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & -7 \\ 0 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

The solution is $x = 0$, $y = 0$.

20. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The solution is $x = 0$, $y = 0$, $z = 0$.

21. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -7 & -14 \\ 0 & -2 & -4 \\ 0 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \\ 0 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The solution is $x = r$, $y = -2r$, $z = r$.

22. $\begin{bmatrix} 1 & 1 & 7 \\ 1 & -1 & -1 \\ 2 & -3 & -6 \\ 3 & 1 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 7 \\ 0 & -2 & -8 \\ 0 & -5 & -20 \\ 0 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 7 \\ 0 & 1 & 4 \\ 0 & -5 & -20 \\ 0 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The solution is $x = -3r$, $y = -4r$, $z = r$.

23. $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & 0 & 5 \\ 2 & 1 & 3 & 4 \\ 1 & -3 & 2 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & -4 \\ 0 & -4 & 1 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & -1 & 1 \\ 0 & -4 & 1 & -13 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & -4 & 1 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The solution is $w = -2r$, $x = -3r$, $y = r$, $z = r$.

24. $\begin{bmatrix} 1 & 1 & 2 & 7 \\ 1 & -2 & -1 & 1 \\ 1 & 2 & 3 & 9 \\ 2 & -3 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 7 \\ 0 & -3 & -3 & -6 \\ 0 & 1 & 1 & 2 \\ 0 & -5 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -5 & -5 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The solution is $w = -r - 5s$, $x = -r - 2s$, $y = r$, $z = s$.

Apply It 6.6

12. $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Yes, they are inverses.

13. $\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 28 \\ 46 \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \end{bmatrix} = \begin{bmatrix} \text{M} \\ \text{E} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 65 \\ 90 \end{bmatrix} = \begin{bmatrix} 5 \\ 20 \end{bmatrix} = \begin{bmatrix} \text{E} \\ \text{T} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 61 \\ 82 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} \text{A} \\ \text{T} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 59 \\ 88 \end{bmatrix} = \begin{bmatrix} 14 \\ 15 \end{bmatrix} = \begin{bmatrix} \text{N} \\ \text{O} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 57 \\ 86 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \end{bmatrix} = \begin{bmatrix} \text{O} \\ \text{N} \end{bmatrix}$

$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 60 \\ 84 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix} = \begin{bmatrix} \text{F} \\ \text{R} \end{bmatrix}$

$$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix}\begin{bmatrix} 21 \\ 34 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} \text{I} \\ \text{D} \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix}\begin{bmatrix} 76 \\ 102 \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \end{bmatrix} = \begin{bmatrix} \text{A} \\ \text{Y} \end{bmatrix}$$

The message is "MEET AT NOON FRIDAY."

14. $$[E \mid I] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-2R_1 + R_3]{-3R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & -2 & -1 & 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & -2 & -1 & 1 & -\frac{3}{2} & 0 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & -2 & -1 & 1 & -\frac{3}{2} & 0 \end{array}\right]$$

$$\xrightarrow[2R_2 + R_3]{-R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & -3 & 1 & \frac{1}{2} & -2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}\right]$$

$$\xrightarrow[R_3 + R_2]{-2R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{5}{6} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}\right]$$

$$E^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

$$[F \mid I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 4 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 4 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-4R_1 + R_3]{-3R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_2 + R_3]{-\frac{1}{2}R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

F does not reduce to I so F is not invertible.

15. Let x be the number of shares of A, y be the number of shares of B, and z be the number of shares of C. We get the following equations from the given conditions.

$50x + 20y + 80z = 500{,}000$

$x = 2z$

$0.13(50x) + 0.15(20y) + 0.10(80z)$
$\quad = 0.12(50x + 20y + 80z)$

Simplify the first equation.

$5x + 2y + 8z = 50{,}000$

Simplify the second equation.

$x - 2z = 0$

Simplify the third equation.

$6.5x + 3y + 8z = 6x + 2.4y + 9.6z$

$0.5x + 0.6y - 1.6z = 0$

$5x + 6y - 16z = 0$

Thus, we solve the following system of equations.

$x - 2z = 0$

$5x + 6y - 16z = 0$

$5x + 2y + 8z = 50{,}000$

The coefficient matrix is $A=\begin{bmatrix}1 & 0 & -2\\ 5 & 6 & -16\\ 5 & 2 & 8\end{bmatrix}$.

$$[A\,|\,I]=\left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 5 & 6 & -16 & 0 & 1 & 0\\ 5 & 2 & 8 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow[-5R_1+R_3]{-5R_1+R_2}\left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 0 & 6 & -6 & -5 & 1 & 0\\ 0 & 2 & 18 & -5 & 0 & 1\end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_2}\left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 0 & 1 & -1 & -\frac{5}{6} & \frac{1}{6} & 0\\ 0 & 2 & 18 & -5 & 0 & 1\end{array}\right]$$

$$\xrightarrow{-2R_2+R_3}\left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 0 & 1 & -1 & -\frac{5}{6} & \frac{1}{6} & 0\\ 0 & 0 & 20 & -\frac{10}{3} & -\frac{1}{3} & 1\end{array}\right]$$

$$\xrightarrow{\frac{1}{20}R_3}\left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 0 & 1 & -1 & -\frac{5}{6} & \frac{1}{6} & 0\\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{60} & \frac{1}{20}\end{array}\right]$$

$$\xrightarrow[R_3+R_2]{2R_3+R_1}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{2}{3} & -\frac{1}{30} & \frac{1}{10}\\ 0 & 1 & 0 & -1 & \frac{3}{20} & \frac{1}{20}\\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{60} & \frac{1}{20}\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}\frac{2}{3} & -\frac{1}{30} & \frac{1}{10}\\ -1 & \frac{3}{20} & \frac{1}{20}\\ -\frac{1}{6} & -\frac{1}{60} & \frac{1}{20}\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\frac{2}{3} & -\frac{1}{30} & \frac{1}{10}\\ -1 & \frac{3}{20} & \frac{1}{20}\\ -\frac{1}{6} & -\frac{1}{60} & \frac{1}{20}\end{bmatrix}\begin{bmatrix}0\\ 0\\ 50,000\end{bmatrix}=\begin{bmatrix}5000\\ 2500\\ 2500\end{bmatrix}$$

They should buy 5000 shares of Company A, 2500 shares of Company B, and 2500 shares of Company C.

Problems 6.6

1. $\left[\begin{array}{cc|cc}6 & 1 & 1 & 0\\ 7 & 1 & 0 & 1\end{array}\right]\rightarrow\left[\begin{array}{cc|cc}1 & \frac{1}{6} & \frac{1}{6} & 0\\ 7 & 1 & 0 & 1\end{array}\right]\rightarrow\left[\begin{array}{cc|cc}1 & \frac{1}{6} & \frac{1}{6} & 0\\ 0 & -\frac{1}{6} & -\frac{7}{6} & 1\end{array}\right]\rightarrow\left[\begin{array}{cc|cc}1 & 0 & -1 & 1\\ 0 & -\frac{1}{6} & -\frac{7}{6} & 1\end{array}\right]\rightarrow\left[\begin{array}{cc|cc}1 & 0 & -1 & 1\\ 0 & 1 & 7 & -6\end{array}\right]$

The inverse is $\begin{bmatrix}-1 & 1\\ 7 & -6\end{bmatrix}$.

2. $\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 1 & 2 & 0 & \frac{1}{3} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{3} \end{array}\right]$

The given matrix is not invertible.

3. $\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]$

The given matrix is not invertible.

4. $\left[\begin{array}{cc|cc} 1 & a & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -a \\ 0 & 1 & 0 & 1 \end{array}\right]$

The inverse is $\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$.

5. $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \end{array}\right]$

The inverse is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$.

6. $\left[\begin{array}{ccc|ccc} 2 & 0 & 8 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 4 & 4 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -8 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 1 & -8 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & -9 & -\frac{9}{8} & -\frac{1}{4} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & 1 & 0 & 0 & \frac{2}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right]$

The inverse is $\begin{bmatrix} 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{bmatrix}$.

7. $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{5}{4} & 1 \end{array}\right]$

The given matrix is not invertible.

8. $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$

The given matrix is not invertible.

9. The matrix is not square, so it is not invertible.

10. For any 3×3 matrix B, $B\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I.$

Thus the matrix is not invertible.

11. $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

The inverse is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

12. $\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -3 & 3 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 15 & -1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{15} & \frac{1}{5} & \frac{1}{15} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & 0 & \frac{4}{15} & \frac{1}{5} & -\frac{4}{15} \\ 0 & 0 & 1 & -\frac{1}{15} & \frac{1}{5} & \frac{1}{15} \end{array}\right]$

The inverse is $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \\ \frac{4}{15} & \frac{1}{5} & -\frac{4}{15} \\ -\frac{1}{15} & \frac{1}{5} & \frac{1}{15} \end{bmatrix}$.

13. $\left[\begin{array}{ccc|ccc} 7 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{2}{7} & \frac{1}{7} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{2}{7} & \frac{1}{7} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} & \frac{3}{7} & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} & \frac{3}{7} & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 7 \end{array}\right]$

The inverse is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 7 \end{bmatrix}$.

14. $\left[\begin{array}{ccc|ccc} 2 & -3 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 4 & -6 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 4 & -6 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 3 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$

The inverse is $\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{3} & 0 \\ 2 & 0 & -1 \end{bmatrix}$.

15. $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 4 & -1 & 5 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 4 & -1 & 5 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -3 & 0 & 1 & -4 \\ 0 & 3 & -4 & 1 & 0 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & \frac{1}{3} & -\frac{4}{3} \\ 0 & 3 & -4 & 1 & 0 & -2 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & -1 & 0 & \frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & -1 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & -1 & 0 & \frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{2}{3} & \frac{5}{3} \\ 0 & 1 & 0 & -1 & \frac{4}{3} & -\frac{10}{3} \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right].$

The inverse is $\begin{bmatrix} 1 & -\frac{2}{3} & \frac{5}{3} \\ -1 & \frac{4}{3} & -\frac{10}{3} \\ -1 & 1 & -2 \end{bmatrix}$.

16. $\left[\begin{array}{rrr|rrr} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & -2 & 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -\frac{6}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{6}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{8}{5} & -\frac{6}{5} & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{6}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$

The inverse is $\begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$.

17. $\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 1 & 5 & 12 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 9 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 3 & 2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & -1 & \frac{1}{3} \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{11}{3} & -3 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{7}{3} & 3 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -1 & \frac{1}{3} \end{array}\right]$

The inverse is $\begin{bmatrix} \frac{11}{3} & -3 & \frac{1}{3} \\ -\frac{7}{3} & 3 & -\frac{2}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{bmatrix}$.

18. $\left[\begin{array}{rrr|rrr} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

The inverse is $\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

19. $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ 38 \end{bmatrix} \Rightarrow x_1 = 27,\ x_2 = 38$

20. $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 16 \end{bmatrix} \Rightarrow x_1 = 9,\ x_2 = 6,\ x_3 = 16$

21. $\left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 6 & 5 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -6 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 6 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -5 \\ 0 & 1 & -1 & 6 \end{array}\right]$

$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} 1 & -5 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 17 \\ -20 \end{bmatrix} \Rightarrow x = 17,\ y = -20$

22. $\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 5 & \frac{1}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{10} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{10} & \frac{1}{5} \end{array}\right]$

$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{3}{10} & -\frac{2}{5} \\ \frac{1}{10} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{23}{10} \\ \frac{1}{10} \end{bmatrix} \Rightarrow x = \frac{23}{10},\ y = \frac{1}{10}$

23. $\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow x = 2,\ y = -1$

24. $\left[\begin{array}{cc|cc} 6 & 1 & 1 & 0 \\ 7 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{6} & \frac{1}{6} & 0 \\ 7 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & -\frac{1}{6} & -\frac{7}{6} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 1 & 7 & -6 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 7 & -6 \end{array}\right]$

$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} -1 & 1 \\ 7 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -28 \end{bmatrix} \Rightarrow x = 5,\ y = -28$

25. The coefficient matrix is not invertible. The method of reduction yields
$\left[\begin{array}{cc|c} 2 & 6 & 2 \\ 3 & 9 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 3 & 9 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right].$
Thus $x = -3r + 1$, $y = r$.

26. The coefficient matrix is not invertible. The method of reduction yields
$\left[\begin{array}{cc|c} 2 & 6 & 8 \\ 3 & 9 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 3 & 9 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 0 & -5 \end{array}\right].$
Second row indicates $0 = -5$, which is never true, so there is no solution.

27. $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -6 & -2 & -3 & 1 & 0 \\ 0 & -3 & 0 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & -3 & 0 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} -\frac{1}{6} & \frac{1}{2} & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Thus, $x = 0$, $y = 1$, $z = 2$.

28. $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{7}{2} \\ -\frac{5}{2} \end{bmatrix}$$

Thus, $x = 5,\ y = \frac{7}{2},\ z = -\frac{5}{2}.$

29. $\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Thus, $x = 4$, $y = -\dfrac{1}{2}$, $z = -\dfrac{1}{2}$.

30. $\left[\begin{array}{rrr|rrr} 2 & 0 & 8 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & -4 & 0 & 0 & -1 & 0 \\ 2 & 0 & 8 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & -4 & 0 & 0 & -1 & 0 \\ 0 & 8 & 8 & 1 & 2 & 0 \\ 0 & 9 & 0 & 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & -4 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 9 & 0 & 0 & 2 & 1 \end{array}\right]$

$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & -9 & -\frac{9}{8} & -\frac{1}{4} & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & 1 & 0 & 0 & \frac{2}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right]$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{bmatrix} \begin{bmatrix} 8 \\ 36 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix}$$

Thus, $x = 0$, $y = 9$, $z = 1$.

31. The coefficient matrix is not invertible. The method of reduction yields

$\left[\begin{array}{rrr|r} 1 & 3 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & 3 & 7 \\ 0 & -5 & -5 & -10 \\ 0 & -2 & -2 & -3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & 3 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right].$

The third row indicates that $0 = 1$, which is never true, so there is no solution.

32. The coefficient matrix is not invertible. The method of reduction yields

$\left[\begin{array}{rrr|r} 1 & 3 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & 3 & 7 \\ 0 & -5 & -5 & -10 \\ 0 & -2 & -2 & -4 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & 3 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$

Thus, $x = 1$, $y = -r + 2$, $z = r$.

33. $$\left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -4 & -1 & -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -6 & 0 & -3 & 1 & 1 & 0 \\ 0 & 0 & -5 & 2 & -3 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & -5 & 2 & -3 & 2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -1 & 0 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 2 & -\frac{1}{2} & \frac{7}{6} & -\frac{5}{6} & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -1 & 0 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & \frac{7}{12} & -\frac{5}{12} & \frac{1}{2} \end{array}\right]$$

$$\to \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & \frac{7}{12} & -\frac{5}{12} & \frac{1}{2} \end{array}\right]$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{12} & -\frac{1}{12} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ -\frac{1}{4} & \frac{7}{12} & -\frac{5}{12} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 7 \end{bmatrix}$$

Thus, $w = 1$, $x = 3$, $y = -2$, $z = 7$.

34. $\left[\begin{array}{cccc|cccc} 0 & 1 & -3 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 & 0 & 0 \\ -1 & 2 & -2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 1 & -2 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & -2 & 1 & 1 & -5 \end{array}\right]$

$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & \frac{5}{4} \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & \frac{5}{4} \end{array}\right]$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \\ 3 \end{bmatrix}$$

Thus $w = -1$, $x = 5$, $y = 1$, $z = 3$.

35. $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -1 & -1 \end{bmatrix}$

$\left[\begin{array}{cc|cc} -4 & 2 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & -1 & 0 & 1 \\ -4 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & -1 \\ -4 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 0 & -1 \\ 0 & 6 & 1 & -4 \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & 1 & 0 & -1 \\ 0 & 1 & \frac{1}{6} & -\frac{2}{3} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & \frac{1}{6} & -\frac{2}{3} \end{array}\right]$

Thus, $(I - A)^{-1} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{2}{3} \end{bmatrix}$.

36. $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -4 & -2 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 4 & -2 & 1 & 0 \\ -4 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ -4 & -2 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -4 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{8} & -\frac{1}{8} \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$

Thus $(I - A)^{-1} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

37. Let x = number of model A and y = number of model B.

a. The system is
$$\begin{cases} x+y=100 & \text{(painting)} \\ \frac{1}{2}x+y=80 & \text{(polishing)} \end{cases}$$
Let $A=\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{bmatrix}$.
$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ \frac{1}{2} & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 2 \end{array}\right]$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 100 \\ 80 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 100 \\ 80 \end{bmatrix} = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$$
Thus 40 of model A and 60 of model B can be produced.

b. The system is
$$\begin{cases} 10x+7y=800 & \text{(widgets)} \\ 14x+10y=1130 & \text{(shims)} \end{cases}$$
Let $A=\begin{bmatrix} 10 & 7 \\ 14 & 10 \end{bmatrix}$.
$$\left[\begin{array}{cc|cc} 10 & 7 & 1 & 0 \\ 14 & 10 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{7}{10} & \frac{1}{10} & 0 \\ 14 & 10 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 1 & \frac{7}{10} & \frac{1}{10} & 0 \\ 0 & \frac{1}{5} & -\frac{7}{5} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{7}{10} & \frac{1}{10} & 0 \\ 0 & 1 & -7 & 5 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 5 & -\frac{7}{2} \\ 0 & 1 & -7 & 5 \end{array}\right]$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 800 \\ 1130 \end{bmatrix} = \begin{bmatrix} 5 & -\frac{7}{2} \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 800 \\ 1130 \end{bmatrix} = \begin{bmatrix} 45 \\ 50 \end{bmatrix}$$
Thus 45 of model A and 50 of model B can be produced.

38. $\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

39. a. $\left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}B = I$

Since an invertible matrix has exactly one inverse, $B^{-1}A^{-1}$ is the inverse of AB.

b. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 11 & 23 \end{bmatrix}$

40. Left side: $A^{\mathrm{T}}=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$. We find that

$(A^{\mathrm{T}})^{-1}=\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix}$.

Right side: $A^{-1}=\begin{bmatrix}1 & 0\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$, so

$(A^{-1})^{\mathrm{T}}=\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix}$.

Thus $(A^{\mathrm{T}})^{-1}=(A^{-1})^{\mathrm{T}}$.

41. $P^{\mathrm{T}}P=\begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\ -\frac{4}{5} & \frac{3}{5}\end{bmatrix}\begin{bmatrix}\frac{3}{5} & -\frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I,$ so

$P^{\mathrm{T}}=P^{-1}$. Yes, P is orthogonal.

42. a. $A^{-1}=\begin{bmatrix}14 & -2 & 9\\ -6 & 1 & -4\\ 1 & 0 & 1\end{bmatrix}$

$$R_1A^{-1}=\begin{bmatrix}33 & 87 & 70\end{bmatrix}\begin{bmatrix}14 & -2 & 9\\ -6 & 1 & -4\\ 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}10 & 21 & 19\end{bmatrix}$$

$$R_2A^{-1}=\begin{bmatrix}57 & 133 & 20\end{bmatrix}\begin{bmatrix}14 & -2 & 9\\ -6 & 1 & -4\\ 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}20 & 19 & 1\end{bmatrix}$$

$$R_3A^{-1}=\begin{bmatrix}38 & 90 & 33\end{bmatrix}\begin{bmatrix}14 & -2 & 9\\ -6 & 1 & -4\\ 1 & 0 & 1\end{bmatrix}=\begin{bmatrix}25 & 14 & 15\end{bmatrix}$$

b. Just say no.

43. Let x be the number of shares of D, y be the number of shares of E, and z be the number of shares of F. We get the following equations.
$60x+80y+30z=500{,}000$
$0.16(60x)+0.12(80y)+0.09(30z)$
$=0.1368(60x+80y+30z)$
$z=4y$
Simplify the first equation.
$6x+8y+3z=50{,}000$
Simplify the second equation.
$9.6x+9.6y+2.7z=8.208x+10.944y+4.104z$
$1.392x-1.344y-1.404z=0$
$1392x-1344y-1404z=0$
$116x-112y-117z=0$
Simplify the third equation.
$4y-z=0$
Thus we solve the following system of equations.
$6x+8y+3z=50{,}000$
$116x-112y-117z=0$
$4y-z=0$

The coefficient matrix is $A=\begin{bmatrix}6 & 8 & 3\\ 116 & -112 & -117\\ 0 & 4 & -1\end{bmatrix}$.

$$\left[A\,|\,I\right]=\left[\begin{array}{ccc|ccc}6 & 8 & 3 & 1 & 0 & 0\\ 116 & -112 & -117 & 0 & 1 & 0\\ 0 & 4 & -1 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_1}\left[\begin{array}{ccc|ccc}1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0\\ 116 & -112 & -117 & 0 & 1 & 0\\ 0 & 4 & -1 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{-116R_1+R_2}\left[\begin{array}{ccc|ccc}1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0\\ 0 & -\frac{800}{3} & -175 & -\frac{58}{3} & 1 & 0\\ 0 & 4 & -1 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow[-\frac{1}{4}R_3]{-\frac{3}{800}R_2}\left[\begin{array}{ccc|ccc}1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0\\ 0 & 1 & \frac{21}{32} & \frac{29}{400} & -\frac{3}{800} & 0\\ 0 & -1 & \frac{1}{4} & 0 & 0 & -\frac{1}{4}\end{array}\right]$$

$$\xrightarrow{R_2+R_3}\left[\begin{array}{ccc|ccc}1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0\\ 0 & 1 & \frac{21}{32} & \frac{29}{400} & -\frac{3}{800} & 0\\ 0 & 0 & \frac{29}{32} & \frac{29}{400} & -\frac{3}{800} & -\frac{1}{4}\end{array}\right]$$

$$\xrightarrow{\frac{32}{29}R_3}\left[\begin{array}{ccc|ccc}1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0\\ 0 & 1 & \frac{21}{32} & \frac{29}{400} & -\frac{3}{800} & 0\\ 0 & 0 & 1 & \frac{2}{25} & -\frac{3}{725} & -\frac{8}{29}\end{array}\right]$$

$$\xrightarrow[-\frac{21}{32}R_3+R_2]{-\frac{1}{2}R_3+R_1}\left[\begin{array}{ccc|ccc}1 & \frac{4}{3} & 0 & \frac{19}{150} & \frac{3}{1450} & \frac{4}{29}\\ 0 & 1 & 0 & \frac{1}{50} & -\frac{3}{2900} & \frac{21}{116}\\ 0 & 0 & 1 & \frac{2}{25} & -\frac{3}{725} & -\frac{8}{29}\end{array}\right]$$

$$\xrightarrow{-\frac{4}{3}R_2+R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{10} & \frac{1}{290} & -\frac{3}{29} \\ 0 & 1 & 0 & \frac{1}{50} & -\frac{3}{2900} & \frac{21}{116} \\ 0 & 0 & 1 & \frac{2}{25} & -\frac{3}{725} & -\frac{8}{29} \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{290} & -\frac{3}{29} \\ \frac{1}{50} & -\frac{3}{2900} & \frac{21}{116} \\ \frac{2}{25} & -\frac{3}{725} & -\frac{8}{29} \end{bmatrix}\begin{bmatrix} 50{,}000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5000 \\ 1000 \\ 4000 \end{bmatrix}$$

They should buy 5000 shares of company D, 1000 shares of company E, and 4000 shares of company F.

44. Let x be the number of shares of D, y be the number of shares of E, and z be the number of shares of F. We get the following conditions.
$60x + 80y + 30z = 500{,}000$
$0.16(60x) + 0.12(80y) + 0.09(30z)$
$= 0.1452(60x + 80y + 30z)$
$z = 2y$
Simplify the first equation.
$6x + 8x + 3z = 50{,}000$
Simplify the second equation.
$9.6x + 9.6y + 2.7z = 8.712x + 11.616y + 4.356z$
$0.888x - 2.016y - 1.656z = 0$
$888x - 2016y - 1656z = 0$
$111x - 252y - 207z = 0$
Simplify the third equation.
$2y - z = 0$
Thus we solve the following system of equations.
$6x + 8y + 3z = 50{,}000$
$111x - 252y - 207z = 0$
$2y - z = 0$

The coefficient matrix is $A = \begin{bmatrix} 6 & 8 & 3 \\ 111 & -252 & -207 \\ 0 & 2 & -1 \end{bmatrix}$.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 6 & 8 & 3 & 1 & 0 & 0 \\ 111 & -252 & -207 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_1}\left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 111 & -252 & -207 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-111R_1+R_2}\left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & -400 & -\frac{525}{2} & -\frac{37}{2} & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{400}R_2}\left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{21}{32} & \frac{37}{800} & -\frac{1}{400} & 0 \\ 0 & -1 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_2+R_3}\left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{21}{32} & \frac{37}{800} & -\frac{1}{400} & 0 \\ 0 & 0 & \frac{37}{32} & \frac{37}{800} & -\frac{1}{400} & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{\frac{32}{37}R_3}\left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{21}{32} & \frac{37}{800} & -\frac{1}{400} & 0 \\ 0 & 0 & 1 & \frac{1}{25} & -\frac{2}{925} & -\frac{16}{37} \end{array}\right]$$

$$\xrightarrow[-\frac{21}{32}R_3+R_2]{-\frac{1}{2}R_3+R_1}\left[\begin{array}{ccc|ccc} 1 & \frac{4}{3} & 0 & \frac{11}{75} & \frac{1}{925} & \frac{8}{37} \\ 0 & 1 & 0 & \frac{1}{50} & -\frac{1}{925} & \frac{21}{74} \\ 0 & 0 & 1 & \frac{1}{25} & -\frac{2}{925} & -\frac{16}{37} \end{array}\right]$$

$$\xrightarrow{-\frac{4}{3}R_2+R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{25} & \frac{7}{2775} & -\frac{6}{37} \\ 0 & 1 & 0 & \frac{1}{50} & -\frac{1}{925} & \frac{21}{74} \\ 0 & 0 & 1 & \frac{1}{25} & -\frac{2}{925} & -\frac{16}{37} \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{25} & \frac{7}{2775} & -\frac{6}{37} \\ \frac{1}{50} & -\frac{1}{925} & \frac{21}{74} \\ \frac{1}{25} & -\frac{2}{925} & -\frac{16}{37} \end{bmatrix}\begin{bmatrix} 50{,}000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6000 \\ 1000 \\ 2000 \end{bmatrix}$$

They should buy 6000 shares of company D, 1000 shares of company E, and 2000 shares of company F.

Problems 6.7

1. $A = \begin{bmatrix} \frac{1}{3} & \frac{3}{4} \\ \frac{1}{4} & 0 \end{bmatrix}$

$D = \begin{bmatrix} 300 \\ 500 \end{bmatrix}$

$(I - A)X = D$

Reducing $\left[\begin{array}{cc|c} \frac{2}{3} & -\frac{3}{4} & 300 \\ -\frac{1}{4} & 1 & 500 \end{array}\right]$ with a calculator results in $\left[\begin{array}{cc|c} 1 & 0 & 1408.70 \\ 0 & 1 & 852.17 \end{array}\right]$.

Thus 1408.70 units of agriculture and 852.17 units of milling need to be produced.

2. $A = \begin{bmatrix} \frac{1}{10} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{3} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \end{bmatrix}$

$D = \begin{bmatrix} 300 \\ 200 \\ 500 \end{bmatrix}$

$(I - A)X = D$

Reducing $\left[\begin{array}{ccc|c} \frac{9}{10} & -\frac{1}{3} & -\frac{1}{4} & 300 \\ -\frac{1}{10} & \frac{9}{10} & -\frac{1}{3} & 200 \\ -\frac{1}{10} & -\frac{1}{10} & \frac{9}{10} & 500 \end{array}\right]$ with a calculator results in $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 736.39 \\ 0 & 1 & 0 & 563.29 \\ 0 & 0 & 1 & 699.96 \end{array}\right]$.

Thus 736.39 units of coal, 563.29 units of steel, and 699.96 units of railroad services need to be produced.

3. $A = \begin{bmatrix} \frac{1}{18} & \frac{3}{16} & \frac{1}{15} \\ \frac{1}{9} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{9} & \frac{3}{16} & \frac{1}{6} \end{bmatrix}$

$D = \begin{bmatrix} 40 \\ 30 \\ 0 \end{bmatrix}$

$(I - A)X = D$

Reducing $\left[\begin{array}{ccc|c} \frac{17}{18} & -\frac{3}{16} & -\frac{1}{15} & 40 \\ -\frac{1}{9} & \frac{3}{4} & -\frac{1}{3} & 30 \\ -\frac{1}{9} & -\frac{3}{16} & \frac{5}{6} & 0 \end{array}\right]$ with a calculator results in $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 55.13 \\ 0 & 1 & 0 & 57.15 \\ 0 & 0 & 1 & 20.21 \end{array}\right]$.

Thus, to meet external demand, 55.13 units of agriculture, 57.15 units of manufacturing, and 20.21 units of transportation are required.

4. $A = \begin{bmatrix} \frac{200}{1200} & \frac{500}{1500} \\ \frac{400}{1200} & \frac{200}{1500} \end{bmatrix}$

$D = \begin{bmatrix} 600 \\ 805 \end{bmatrix}$

$X = (I - A)^{-1} D = \begin{bmatrix} 1290 \\ 1425 \end{bmatrix}$

The total value of other production costs is

$P_A + P_B = \frac{600}{1200}(1290) + \frac{800}{1500}(1425) = 1405$

5. $A = \begin{bmatrix} \frac{40}{200} & \frac{120}{300} \\ \frac{120}{200} & \frac{90}{300} \end{bmatrix}$

a. $D = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$

$X = (I - A)^{-1} D = \begin{bmatrix} 812.5 \\ 1125 \end{bmatrix}$

b. $D = \begin{bmatrix} 64 \\ 64 \end{bmatrix}$

$X = (I - A)^{-1} D = \begin{bmatrix} 220 \\ 280 \end{bmatrix}$

6. $A = \begin{bmatrix} \frac{15}{100} & \frac{30}{120} & \frac{45}{180} \\ \frac{25}{100} & \frac{30}{120} & \frac{60}{180} \\ \frac{50}{100} & \frac{40}{120} & \frac{60}{180} \end{bmatrix}$

a. $D = \begin{bmatrix} 15 \\ 10 \\ 35 \end{bmatrix}$

$X = (I - A)^{-1} D = \begin{bmatrix} 134.29 \\ 162.25 \\ 234.35 \end{bmatrix}$

b. $D = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$

$X = (I - A)^{-1} D = \begin{bmatrix} 68.59 \\ 84.50 \\ 108.69 \end{bmatrix}$

7. $A = \begin{bmatrix} \frac{100}{1000} & \frac{400}{800} & \frac{240}{1200} \\ \frac{100}{1000} & \frac{80}{800} & \frac{480}{1200} \\ \frac{300}{1000} & \frac{160}{800} & \frac{240}{1200} \end{bmatrix}$

$D = \begin{bmatrix} 500 \\ 150 \\ 700 \end{bmatrix}$

$X = (I - A)^{-1} D = \begin{bmatrix} 1559.81 \\ 1112.44 \\ 1738.04 \end{bmatrix}$

8. $A = \begin{bmatrix} \frac{400}{1000} & \frac{200}{1000} & \frac{200}{1000} \\ \frac{200}{1000} & \frac{400}{1000} & \frac{100}{1000} \\ \frac{200}{1000} & \frac{100}{1000} & \frac{300}{1000} \end{bmatrix}$

$D = \begin{bmatrix} 300 \\ 350 \\ 450 \end{bmatrix}$

$X = (I - A)^{-1} D = \begin{bmatrix} 1301 \\ 1215 \\ 1188 \end{bmatrix}$

9. $A = \begin{bmatrix} \frac{400}{1000} & \frac{200}{1000} & \frac{200}{1000} \\ \frac{200}{1000} & \frac{400}{1000} & \frac{100}{1000} \\ \frac{200}{1000} & \frac{100}{1000} & \frac{300}{1000} \end{bmatrix}$

$D = \begin{bmatrix} 250 \\ 300 \\ 350 \end{bmatrix}$

$X = (I - A)^{-1} D = \begin{bmatrix} 1073 \\ 1016 \\ 952 \end{bmatrix}$

10. $A = \begin{bmatrix} \frac{400}{1000} & \frac{200}{1000} & \frac{200}{1000} \\ \frac{200}{1000} & \frac{400}{1000} & \frac{100}{1000} \\ \frac{200}{1000} & \frac{100}{1000} & \frac{300}{1000} \end{bmatrix}$

$D = \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix}$

$X = (I - A)^{-1} D = \begin{bmatrix} 1382 \\ 1344 \\ 1301 \end{bmatrix}$

Chapter 6 Review Problems

1. $$2\begin{bmatrix} 3 & 4 \\ -5 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ -10 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 6 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -16 & -10 \end{bmatrix}$$

2. $$5\begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix} - 3\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -15 & 5 \\ 0 & 20 \end{bmatrix} - \begin{bmatrix} 6 & -3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -21 & 8 \\ -3 & 20 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & 7 \\ 2 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+42 & -2+7 \\ 2+0 & 0-18 & -4-3 \\ 1+0 & 0+0 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 & 42 & 5 \\ 2 & -18 & -7 \\ 1 & 0 & -2 \end{bmatrix}$$

4. $$[2 \quad 3 \quad 7]\begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 5 & 2 \end{bmatrix}$$
$$= [2(2) + 3(0) + 7(5) \quad 2(3) + 3(-1) + 7(2)]$$
$$= [39 \quad 17]$$

5. $$\begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}\left(\begin{bmatrix} 2 & 3 \\ 7 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 8 \\ 4 & 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 11 & -4 \\ 8 & 11 \end{bmatrix}$$

6. $-\left\{\begin{bmatrix}2 & 0\\7 & 8\end{bmatrix}+2\begin{bmatrix}0 & -5\\6 & -4\end{bmatrix}\right\}=-\left\{\begin{bmatrix}2 & 0\\7 & 8\end{bmatrix}+\begin{bmatrix}0 & -10\\12 & -8\end{bmatrix}\right\}=-\begin{bmatrix}2 & -10\\19 & 0\end{bmatrix}=\begin{bmatrix}-2 & 10\\-19 & 0\end{bmatrix}$

7. $3\begin{bmatrix}2 & 0\\2 & 1\end{bmatrix}^2[3\ \ 4]^{\mathrm{T}}=3\begin{bmatrix}4 & 0\\6 & 1\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=3\begin{bmatrix}12\\22\end{bmatrix}=\begin{bmatrix}36\\66\end{bmatrix}$

8. $\frac{1}{3}\begin{bmatrix}3 & 0\\3 & 6\end{bmatrix}\left\{\begin{bmatrix}1 & 0\\1 & 3\end{bmatrix}^{\mathrm{T}}\right\}^2=\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 3\end{bmatrix}^2=\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}\begin{bmatrix}1 & 4\\0 & 9\end{bmatrix}=\begin{bmatrix}1 & 4\\1 & 22\end{bmatrix}$

9. $(2A)^{\mathrm{T}}-3I^2=2A^{\mathrm{T}}-3I=2\begin{bmatrix}1 & -1\\1 & 2\end{bmatrix}-\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$

$=\begin{bmatrix}2 & -2\\2 & 4\end{bmatrix}-\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}=\begin{bmatrix}-1 & -2\\2 & 1\end{bmatrix}$

10. $A(2I)-A0^{\mathrm{T}}=2(AI)-A0=2A-0=2A=\begin{bmatrix}2 & 2\\-2 & 4\end{bmatrix}$

11. $B^3+I^5=\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}^3+\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^5=\begin{bmatrix}1 & 0\\0 & 8\end{bmatrix}+\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\0 & 9\end{bmatrix}$

12. $(ABBA)^{\mathrm{T}}-A^{\mathrm{T}}B^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}=A^{\mathrm{T}}B^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}-A^{\mathrm{T}}B^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}=0$

13. $\begin{bmatrix}5x\\7x\end{bmatrix}=\begin{bmatrix}15\\y\end{bmatrix}$

$5x = 15$, or $x = 3$
$7x = y$, $7 \cdot 3 = y$, or $y = 21$

14. $\begin{bmatrix}2+x^2 & 1+3x\\4+xy & 2+3y\end{bmatrix}=\begin{bmatrix}3 & 4\\3 & y\end{bmatrix}$

$2 + 3y = y$, $2y = -2$, or $y = -1$
$1 + 3x = 4$, $3x = 3$, or $x = 1$
For these values of x and y, $2+x^2=3$ is true, and $4 + xy = 3$ is true. Thus $x = 1$, $y = -1$.

15. $\begin{bmatrix}1 & 4\\5 & 8\end{bmatrix}\to\begin{bmatrix}1 & 4\\0 & -12\end{bmatrix}\to\begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}\to\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

16. $\begin{bmatrix}0 & 0 & 7\\0 & 5 & 9\end{bmatrix}\to\begin{bmatrix}0 & 5 & 9\\0 & 0 & 7\end{bmatrix}\to\begin{bmatrix}0 & 1 & \frac{9}{5}\\0 & 0 & 1\end{bmatrix}\to\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

17. $\begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 5 \\ 4 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -13 \\ 0 & -8 & -19 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & \frac{13}{5} \\ 0 & -8 & -19 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{13}{5} \\ 0 & 0 & \frac{9}{5} \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{13}{5} \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

19. $\left[\begin{array}{cc|c} 2 & -5 & 0 \\ 4 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -5 & 0 \\ 0 & 13 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{5}{2} & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

Thus $x = 0$, $y = 0$.

20. $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 3 & 1 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 4 & -5 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -\frac{5}{4} & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{4} & 2 \\ 0 & 1 & -\frac{5}{4} & -1 \end{array}\right]$

Thus $x = -\dfrac{3}{4}r + 2$, $y = \dfrac{5}{4}r - 1$, $z = r$.

21. $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & -2 & -4 & -7 \\ 2 & -1 & -2 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -5 & -10 & -10 \\ 0 & -3 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 6 \end{array}\right]$

Row three indicates that 0 = 6, which is never true, so there is no solution.

22. $\left[\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 1 & 2 & 5 & 1 \\ 4 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 3 & 1 & 2 & 0 \\ 4 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 0 & -5 & -13 & -3 \\ 0 & -8 & -19 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 0 & 1 & \frac{13}{5} & \frac{3}{5} \\ 0 & -8 & -19 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{13}{5} & \frac{3}{5} \\ 0 & 0 & \frac{9}{5} & \frac{4}{5} \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{13}{5} & \frac{3}{5} \\ 0 & 0 & 1 & \frac{4}{9} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{9} \\ 0 & 1 & 0 & -\frac{5}{9} \\ 0 & 0 & 1 & \frac{4}{9} \end{array}\right]$

Thus $x = -\dfrac{1}{9}$, $y = -\dfrac{5}{9}$, $z = \dfrac{4}{9}$.

23. $\left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & -6 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & 0 & -\frac{3}{2} & \frac{5}{6} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -\frac{3}{2} & \frac{5}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$

24. $\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

25. $\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -11 & 8 & -4 & 1 & 0 \\ 0 & -11 & 8 & -3 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -11 & 8 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{8}{11} & \frac{4}{11} & -\frac{1}{11} & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{11} & -\frac{1}{11} & \frac{3}{11} & 0 \\ 0 & 1 & -\frac{8}{11} & \frac{4}{11} & -\frac{1}{11} & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right] \Rightarrow$ no inverse exists

26. $\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & 0 \\ -5 & 2 & 1 & 0 & 1 & 0 \\ -5 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}.$

27. $\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & 2 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right]$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 1 & -1 & -1 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$

Thus $x = -3$, $y = 2$, $z = 2$.

28. We found A^{-1} in Exercise 26, so

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ \frac{7}{5} \end{bmatrix}$

29. $A^2 = AA = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^3 = A^2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

Since $A^3 = 0$, every higher power of *A* is also 0, so $A^{1000} = 0$.

Looking at $\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$, it is clear that there is no way of transforming the left side into I_3, since there is no way to get a nonzero entry in the first column. Thus *A* does not have an inverse.

30. $A^{\mathrm{T}} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

$\left(A^{\mathrm{T}}\right)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

$\left(A^{-1}\right)^{\mathrm{T}} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

Thus $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$.

31. a. Let *x*, *y*, and *z* represent the weekly doses of capsules of brand I, II, and III, respectively. Then

$$\begin{cases} x + y + 4z = 13 & \text{(vitamin A)} \\ x + 2y + 7z = 22 & \text{(vitamin B)} \\ x + 3y + 10z = 31 & \text{(vitamin C)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 13 \\ 1 & 2 & 7 & 22 \\ 1 & 3 & 10 & 31 \end{array}\right] \xrightarrow[-R_1 + R_3]{-R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 13 \\ 0 & 1 & 3 & 9 \\ 0 & 2 & 6 & 18 \end{array}\right]$$

$$\xrightarrow[-2R_2 + R_3]{-R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $x = 4 - r$, $y = 9 - 3r$, and $z = r$, where $r = 0, 1, 2, 3$.
The four possible combinations are

Combination	*x*	*y*	*z*
1	4	9	0
2	3	6	1
3	2	3	2
4	1	0	3

b. Computing the cost of each combination, we find that they are 83, 77, 71, and 65 cents, respectively. Thus combination 4, namely $x = 1$, $y = 0$, $z = 3$, minimizes weekly cost.

32. a.
$$\begin{aligned} A^n(A^{-1})^n &= A^{n-1}(AA^{-1})(A^{-1})^{n-1} \\ &= A^{n-1}I(A^{-1})^{n-1} \\ &= A^{n-2}(AA^{-1})(A^{-1})^{n-2} \\ &= A^{n-2}(A^{-1})^{n-2} \\ &\;\vdots \\ &= AA^{-1} \\ &= I \end{aligned}$$

b. $ABA = ACA$; so,
$$\begin{aligned} A^{-1}(ABA) &= A^{-1}(ACA) \\ (A^{-1}A)BA &= (A^{-1}A)CA \\ IBA &= ICA \\ BA &= CA \\ (BA)A^{-1} &= (CA)A^{-1} \\ B(AA^{-1}) &= C(AA^{-1}) \\ BI &= CI \\ B &= C \end{aligned}$$

c. $AA = A \Rightarrow A^{-1}AA = A^{-1}A$, $IA = I$, $A = I$. Thus $A = I_n$.

33. $\begin{bmatrix} 215 & 87 \\ 89 & 141 \end{bmatrix}$

34. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7.9 & -4.3 & 2.7 \\ 3.4 & 5.8 & -7.6 \\ 4.5 & -6.2 & -7.4 \end{bmatrix}^{-1} \begin{bmatrix} 11.1 \\ 10.8 \\ 15.9 \end{bmatrix} = \begin{bmatrix} 1.57 \\ -0.30 \\ -0.95 \end{bmatrix}$$
Thus $x = 1.57$, $y = -0.30$, $z = -0.95$.

35. $A = \begin{bmatrix} \frac{10}{34} & \frac{20}{39} \\ \frac{15}{34} & \frac{14}{39} \end{bmatrix}$; $D = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$;
$$X = (I - A)^{-1}D = \begin{bmatrix} 39.7 \\ 35.1 \end{bmatrix}$$

Explore and Extend—Chapter 6

1. $A = \begin{bmatrix} 20 & 40 & 30 & 10 \\ 30 & 0 & 10 & 10 \\ 10 & 0 & 30 & 50 \end{bmatrix}$

$$T = \begin{bmatrix} 7 \\ 10 \\ 7 \\ 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 9 \\ 8 \\ 10 \end{bmatrix}$$

$$\begin{aligned} C^{\mathrm{T}}(AT) &= C^{\mathrm{T}} \left\{ \begin{bmatrix} 20 & 40 & 30 & 10 \\ 30 & 0 & 10 & 10 \\ 10 & 0 & 30 & 50 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \\ 7 \\ 5 \end{bmatrix} \right\} \\ &= C^{\mathrm{T}} \begin{bmatrix} 800 \\ 330 \\ 530 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 8 & 10 \end{bmatrix} \begin{bmatrix} 800 \\ 330 \\ 530 \end{bmatrix} = [15{,}140] \end{aligned}$$

The cost is \$151.40.

2. To the linear system, add $x_1 + x_2 + x_3 + x_4 = 52$.

$$A = \begin{bmatrix} 20 & 40 & 30 & 10 \\ 30 & 0 & 10 & 10 \\ 10 & 0 & 30 & 50 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1180 \\ 580 \\ 1500 \\ 52 \end{bmatrix}$$

$$T = A^{-1}B = \begin{bmatrix} 8 \\ 10 \\ 14 \\ 20 \end{bmatrix}$$

Guest 1: 8 days; guest 2: 10 days; guest 3: 14 days; guest 4: 20 days

3. It is not possible. Different combinations of lengths of stays can cost the same. For example, guest 1 staying for 20 days and guest 3 staying for 17 days costs the same as guest 1 staying for 15 days and guest 3 staying for 21 days (each costs \$214.50).