In a square matrix A of order n, the entries $A_{11}, A_{22}, A_{33}, \ldots, A_{nn}$ lie on the diagonal extending from the upper left corner to the lower right corner of the matrix and are said to constitute the **main diagonal**. Thus, in the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

the main diagonal (see the shaded region) consists of $A_{11} = 1$, $A_{22} = 5$, and $A_{33} = 9$.

A square matrix A is called a **diagonal matrix** if all the entries that are off the main diagonal are zero—that is, if $A_{ij} = 0$ for $i \neq j$. Examples of diagonal matrices are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

It follows that a matrix is diagonal if and only if it is both upper triangular and lower triangular.

A square matrix A is said to be an **upper triangular matrix** if all entries *below* the main diagonal are zero—that is, if $A_{ij} = 0$ for $i > j$. Similarly, a matrix A is said to be a **lower triangular matrix** if all entries *above* the main diagonal are zero—that is, if $A_{ij} = 0$ for $i < j$. When a matrix is either upper triangular or lower triangular, it is called a **triangular matrix**. Thus, the matrices

$$\begin{bmatrix} 5 & 1 & 1 \\ 0 & -3 & 7 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 7 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 6 & 5 & -4 & 0 \\ 1 & 6 & 0 & 1 \end{bmatrix}$$

are upper and lower triangular matrices, respectively, and are therefore triangular matrices.

PROBLEMS 6.1

1. Let

$$A = \begin{bmatrix} 1 & -6 & 2 \\ -4 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 6 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 6 & 2 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad J = [4]$$

(a) State the size of each matrix.
(b) Which matrices are square?
(c) Which matrices are upper triangular? lower triangular?
(d) Which are row vectors?
(e) Which are column vectors?

In Problems 2–9, let

$$A = [A_{ij}] = \begin{bmatrix} 7 & -2 & 14 & 6 \\ 6 & 2 & 3 & -2 \\ 5 & 4 & 1 & 0 \\ 8 & 0 & 2 & 0 \end{bmatrix}$$

2. What is the order of A?

Find the following entries.

3. A_{21}

4. A_{42}

5. A_{24}

6. A_{34}

7. A_{44}

8. A_{55}

9. What are the third row entries?

10. Write the lower triangular matrix A, of order 3, for which all entries *not required to be 0* satisfy $A_{ij} = i - j$.

11. **(a)** Construct the matrix $A = [A_{ij}]$ if A is 2×3 and $A_{ij} = -i + 2j$.
(b) Construct the 2×4 matrix $C = [(i + j)^2]$.

12. **(a)** Construct the matrix $B = [B_{ij}]$ if B is 2×2 and $B_{ij} = (-1)^{i-j}(i^2 - j^2)$.
(b) Construct the 2×3 matrix $D = [(-1)^i(j^3)]$.

13. If $A = [A_{ij}]$ is 12×10, how many entries does A have? If $A_{ij} = 1$ for $i = j$ and $A_{ij} = 0$ for $i \neq j$, find A_{33}, A_{52}, $A_{10,10}$, and $A_{12,10}$.

14. List the main diagonal of

(a) $\begin{bmatrix} 2 & 4 & -2 & 9 \\ 7 & 5 & 0 & -1 \\ -4 & 6 & -3 & 1 \\ 2 & 5 & 7 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} x^2 & 1 & 2y \\ 9 & \sqrt{y} & 3 \\ y & z & 1 \end{bmatrix}$

15. Write the zero matrix **(a)** of order 3 and **(b)** of size 2×4.

16. If A is a 7×9 matrix, what is the size of A^T?

In Problems 17–20, find A^T.

17. $A = \begin{bmatrix} 6 & -3 \\ 2 & 4 \end{bmatrix}$

18. $A = \begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}$

19. $A = \begin{bmatrix} 2 & 5 & -3 & 0 \\ 0 & 3 & 6 & 2 \\ 7 & 8 & -2 & 1 \end{bmatrix}$

20. $A = \begin{bmatrix} -2 & 3 & 0 \\ 3 & 4 & 5 \\ 0 & 5 & -6 \end{bmatrix}$

21. Let

$$A = \begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 10 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

(a) Which are diagonal matrices?
(b) Which are triangular matrices?

22. A matrix is *symmetric* if $A^T = A$. Is the matrix of Problem 19 symmetric?

23. If

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 7 & 0 & 9 \end{bmatrix}$$

verify the general property that $(A^T)^T = A$ by finding A^T and then $(A^T)^T$.

In Problems 24–27, solve the matrix equation.

24. $\begin{bmatrix} 3x & 2y-1 \\ z & 5w \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 7 & 15 \end{bmatrix}$

25. $\begin{bmatrix} x & 3 \\ 5 & 7 \\ z & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & y \\ -5 & 4 \end{bmatrix}$

26. $\begin{bmatrix} 4 & 2 & 1 \\ 3x & y & 3z \\ 0 & w & 7 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ 6 & 7 & 9 \\ 0 & 9 & 8 \end{bmatrix}$

27. $\begin{bmatrix} 2x & 7 \\ 7 & 2y \end{bmatrix} = \begin{bmatrix} y & 7 \\ 7 & y \end{bmatrix}$

28. Inventory A grocer sold 125 cans of tomato soup, 275 cans of beans, and 400 cans of tuna. Write a row vector that gives the number of each item sold. If the items sell for \$0.95, \$1.03, and \$1.25 each, respectively, write this information as a column vector.

29. Sales Analysis The Widget Company has its monthly sales reports given by means of matrices whose rows, in order, represent the number of regular, deluxe, and extreme models sold, and the columns, in order, give the number of red, white, blue, and purple units sold. The matrices for January and February are

$$J = \begin{bmatrix} 1 & 4 & 5 & 0 \\ 3 & 5 & 2 & 7 \\ 4 & 1 & 3 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & 5 & 7 & 7 \\ 2 & 4 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

respectively. **(a)** How many white extreme models were sold in January? **(b)** How many blue deluxe models were sold in February? **(c)** In which month were more purple regular models sold? **(d)** Which models and which colors sold the same number of units in both months? **(e)** In which month were more deluxe models sold? **(f)** In which month were more red widgets sold? **(g)** How many widgets were sold in January?

30. Input–Output Matrix Input–output matrices, which were developed by W. W. Leontief, indicate the interrelationships that exist among the various sectors of an economy during some period of time. A hypothetical example for a simplified economy is given by matrix M at the end of this problem. The consuming sectors are the same as the producing sectors and can be thought of as manufacturers, government, steel industry, agriculture, households, and so on. Each row shows how the output of a given sector is consumed by the four sectors. For example, of the total output of industry A, 50 went to industry A itself, 70 to B, 200 to C, and 360 to all others. The sum of the entries in row 1—namely, 680—gives the total output of A for a given period. Each column gives the output of each sector that is consumed by a given sector. For example, in producing 680 units, industry A consumed 50 units of A, 90 of B, 120 of C, and 420 from all other producers. For each column, find the sum of the entries. Do the same for each row. What do we observe in comparing these totals? Suppose sector A increases its output by 10%; namely, by 68 units. Assuming that this results in a uniform 10% increase of all its inputs, by how many units will sector B have to increase its output? Answer the same question for C and for "all other producers".

$$M = \begin{array}{c} \\ \\ \text{PRODUCERS} \\ \text{Industry A} \\ \text{Industry B} \\ \text{Industry C} \\ \text{All Other Producers} \end{array} \begin{array}{c} \overbrace{\hspace{14em}}^{\text{CONSUMERS}} \\ \begin{array}{cccc} \text{Industry A} & \text{Industry B} & \text{Industry C} & \text{All Other Consumers} \end{array} \\ \begin{bmatrix} 50 & 70 & 200 & 360 \\ 90 & 30 & 270 & 320 \\ 120 & 240 & 100 & 1050 \\ 420 & 370 & 940 & 4960 \end{bmatrix} \end{array}$$

31. Find all the values of x for which

$$\begin{bmatrix} x^2 + 2000x & \sqrt{x^2} \\ x^2 & \ln(e^x) \end{bmatrix} = \begin{bmatrix} 2001 & -x \\ 2001 - 2000x & x \end{bmatrix}$$

In Problems 32 and 33, find A^T.

32. $A = \begin{bmatrix} 3 & -4 & 5 \\ -2 & 1 & 6 \end{bmatrix}$

33. $A = \begin{bmatrix} 3 & 1 & 4 & 2 \\ 1 & 7 & 3 & 6 \\ 1 & 4 & 1 & 2 \end{bmatrix}$

Objective

To define matrix addition and scalar multiplication and to consider properties related to these operations.

6.2 Matrix Addition and Scalar Multiplication

Matrix Addition

Consider a snowmobile dealer who sells two models, Deluxe and Super. Each is available in one of two colors, red and blue. Suppose that the sales for January and February are represented by the matrices

$$J = \begin{array}{c} \\ \text{red} \\ \text{blue} \end{array} \begin{array}{c} \begin{array}{cc} \text{Deluxe} & \text{Super} \end{array} \\ \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \end{array} \qquad F = \begin{array}{c} \\ \text{red} \\ \text{blue} \end{array} \begin{array}{c} \begin{array}{cc} \text{Deluxe} & \text{Super} \end{array} \\ \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \end{array}$$

PROBLEMS 6.2

In Problems 1–12, perform the indicated operations.

1. $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -7 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -3 \\ 5 & -9 \\ -4 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 1 \\ 9 & 0 \\ -2 & 3 \end{bmatrix}$

4. $\frac{1}{2}\begin{bmatrix} 4 & -2 & 6 \\ 2 & 10 & -12 \\ 0 & 0 & 7 \end{bmatrix}$

5. $2[2 \quad -1 \quad 3] + 4[-2 \quad 0 \quad 1] - 0[2 \quad 3 \quad 1]$

6. $[3 \; 5 \; 1] + 24$

7. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

8. $\begin{bmatrix} 5 & 3 \\ -2 & 6 \end{bmatrix} + 7\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

9. $-6\begin{bmatrix} 2 & -6 & 7 & 1 \\ 7 & 1 & 6 & -2 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -6 \\ 4 & 9 \end{bmatrix} - 3\begin{bmatrix} -6 & 9 \\ 2 & 6 \\ 1 & -2 \\ 4 & 5 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 7 & 1 \\ 3 & 0 & 3 \\ -1 & 0 & 5 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 & 4 \\ 1 & -2 & 3 \\ 1 & 3 & -5 \end{bmatrix}$

12. $3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 2 \\ -3 & 21 & -9 \\ 0 & 1 & 0 \end{bmatrix}\right)$

In Problems 13–24, compute the required matrices if

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix} \quad B = \begin{bmatrix} -6 & -5 \\ 2 & -3 \end{bmatrix} \quad C = \begin{bmatrix} -2 & -1 \\ -3 & 3 \end{bmatrix} \quad 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

13. $-2C$

14. $-(A - B)$

15. $2(0)$

16. $A + B - C$

17. $3(2A - 3B)$

18. $0(2A + 3B - 5C)$

19. $3(A - C) + 6$

20. $A + (C + B)$

21. $A - 2B + 3C$

22. $3C - 2B$

23. $\frac{1}{3}A + 3(2B + 5C)$

24. $\frac{1}{2}A - 5(B + C)$

In Problems 25–28, verify the equations for the preceding matrices A, B, and C.

25. $3(A + B) = 3A + 3B$

26. $(3 + 4)B = 3B + 4B$

27. $k_1(k_2 A) = (k_1 k_2)A$

28. $k(A - 2B + C) = kA - 2kB + kC$

In Problems 29–34, let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 7 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

Compute the indicated matrices, if possible.

29. $3A + D^T$

30. $(B - C)^T$

31. $3B^T + 4C^T$

32. $2B + B^T$

33. $A + D^T - B$

34. $(D - 2A^T)^T$

35. Express the matrix equation

$$x\begin{bmatrix} 3 \\ 2 \end{bmatrix} - y\begin{bmatrix} -4 \\ 7 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

as a system of linear equations and solve.

36. In the reverse of the manner used in Problem 35, write the system

$$\begin{cases} x + 2y = 7 \\ 3x + 4y = 14 \end{cases}$$

as a matrix equation.

In Problems 37–40, solve the matrix equations.

37. $3\begin{bmatrix} x \\ y \end{bmatrix} - 3\begin{bmatrix} -2 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 6 \\ -2 \end{bmatrix}$

38. $5\begin{bmatrix} x \\ 3 \end{bmatrix} - 6\begin{bmatrix} 2 \\ -2y \end{bmatrix} = \begin{bmatrix} -4x \\ 3y \end{bmatrix}$

39. $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + 2\begin{bmatrix} x \\ y \\ 4z \end{bmatrix} = \begin{bmatrix} -10 \\ -24 \\ 14 \end{bmatrix}$

40. $x\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} + y\begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 2x + 12 - 5y \end{bmatrix}$

41. Production An auto parts company manufactures distributors, sparkplugs, and magnetos at two plants, I and II. Matrix X represents the production of the two plants for retailer X, and matrix Y represents the production of the two plants for retailer Y. Write a matrix that represents the total production at the two plants for both retailers, where

$$X = \begin{array}{c} \\ \text{DIS} \\ \text{SPG} \\ \text{MAG} \end{array} \begin{array}{c} \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \begin{bmatrix} 35 & 60 \\ 850 & 700 \\ 35 & 50 \end{bmatrix} \end{array} \qquad Y = \begin{array}{c} \\ \text{DIS} \\ \text{SPG} \\ \text{MAG} \end{array} \begin{array}{c} \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \begin{bmatrix} 10 & 45 \\ 900 & 700 \\ 15 & 10 \end{bmatrix} \end{array}$$

42. Sales Let matrix A represent the sales (in thousands of dollars) of a toy company in 2007 in three cities, and let B represent the sales in the same cities in 2009, where

$$A = \begin{array}{c} \text{Action} \\ \text{Educational} \end{array} \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix}$$

$$B = \begin{array}{c} \text{Action} \\ \text{Educational} \end{array} \begin{bmatrix} 380 & 330 & 220 \\ 460 & 320 & 750 \end{bmatrix}$$

If the company buys a competitor and doubles its 2009 sales in 2010, what is the change in sales between 2003 and 2010?

43. Suppose the prices of products A, B, C, and D are given, in that order, by the price row vector

$$P = [p_A \quad p_B \quad p_C \quad p_D]$$

If the prices are to be increased by 16%, the vector for the new prices can be obtained by multiplying P by what scalar?

44. Prove that $(A - B)^T = A^T - B^T$. (*Hint:* Use the definition of *subtraction* and properties of the transpose operation.)

In Problems 45–47, compute the given matrices if

$$A = \begin{bmatrix} 3 & -4 & 5 \\ -2 & 1 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 & 3 \\ 2 & 6 & -6 \end{bmatrix}$$

45. $4A + 3B$

46. $2(3A + 4B) + 5C$

47. $2(3C - A) + 2B$

Objective

To define multiplication of matrices and to consider associated properties. To express a system as a single matrix equation by using matrix multiplication.

6.3 Matrix Multiplication

Besides the operations of matrix addition and scalar multiplication, the product AB of matrices A and B can be defined under a certain condition, namely, that *the number of columns of A is equal to the number of rows of B*. Although the following definition of *matrix multiplication* might not appear to be a natural one, a thorough study of matrices shows that the definition makes sense and is extremely practical for applications.

Definition

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then the product AB is the $m \times p$ matrix with entry $(AB)_{ik}$ given by

$$(AB)_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk} = A_{i1}B_{1k} + A_{i2}B_{2k} + \cdots + A_{in}b_{nk}$$

In words, $(AB)_{ik}$ is obtained by summing the products formed by multiplying, in order, each entry in row i of A by the corresponding entry in column k of B. If the number of columns of A is not equal to the number of rows of B, then the product AB is not defined.

Observe that the definition applies when A is a row vector with n entries and B is a column vector with n entries. In this case, A is $1 \times n$, B is $n \times 1$, and AB is 1×1. (We noted in Section 6.1 that a 1×1 matrix is just a *number*.) In fact,

$$\text{if} \quad A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

$$\text{then} \quad AB = \sum_{j=1}^{n} A_jB_j = A_1B_1 + A_2B_2 + \cdots + A_nB_n$$

Returning to our general definition, it now follows that the *number* $(AB)_{ik}$ is the product of the ith row of A and the kth column of B. This is very helpful when real computations are performed.

Three points must be completely understood concerning this definition of AB. First, the number of columns of A must be equal to the number of rows of B. Second, the product AB has as many rows as A and as many columns as B.

$$\begin{array}{cccc} A & B & = & C \\ m \times n & n \times p & & m \times p \end{array}$$

must be the same

size of product

Third, the definition refers to the product AB, *in that order:* A is the left factor and B is the right factor. For AB, we say that B is *premultiplied* by A or A is *postmultiplied* by B.

PROBLEMS 6.3

If $A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & 1 & -1 \\ 0 & 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 & 3 \\ -2 & 4 & -2 \\ 3 & 1 & -1 \end{bmatrix}$, *and* $AB = C = [C_{ij}]$, *find each of the following.*

1. C_{11} **2.** C_{22} **3.** C_{32}

4. C_{33} **5.** C_{31} **6.** C_{12}

If A is 2×3, *B is* 3×1, *C is* 2×5, *D is* 4×3, *E is* 3×2, *and F is* 2×3, *find the size and number of entries of each of the following.*

7. AE **8.** DE **9.** EC

10. DB **11.** FB **12.** EB

13. $EE^{\mathrm{T}}B$ **14.** $E(AE)$ **15.** $E(FB)$

16. $(F + A)B$

Write the identity matrix that has the following order:

17. 4 **18.** 6

In Problems 19–36, perform the indicated operations.

19. $\begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -1 & 3 \end{bmatrix}$

20. $\begin{bmatrix} -1 & 1 \\ 0 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$

21. $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

22. $[1\ 2\ 3]\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 4 & -1 \\ 0 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 4 & 2 & -2 \\ 3 & 10 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

25. $[1\ \ -2\ \ 5]\begin{bmatrix} 1 & 5 & -2 & -1 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 1 & -3 \end{bmatrix}$

26. $[1\ \ -4]\begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 5 & 0 \end{bmatrix}$

27. $\begin{bmatrix} 7 \\ 8 \end{bmatrix}[9\ 10]$

28. $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

29. $3\left(\begin{bmatrix} -2 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} + 2\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \end{bmatrix}\right)\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

30. $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}\right)$

32. $2\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} + 7\left(\begin{bmatrix} 8 & 9 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right)$

33. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

34. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

35. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 9 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

36. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

In Problems 37–44, compute the required matrices if

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -4 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad F = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

37. $6F + EI$ **38.** DD **39.** $3A - 2BC$

40. $B(D + E)$ **41.** $3I - \frac{2}{3}FE$ **42.** $E(D + I)$

43. $(DC)A$ **44.** $A(BC)$

In Problems 45–58, compute the required matrix, if it exists, given that

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

45. A^2 **46.** $A^{\mathrm{T}}A$ **47.** B^3

48. $A(B^{\mathrm{T}})^2C$ **49.** $(AIC)^{\mathrm{T}}$ **50.** $A^{\mathrm{T}}(2C^{\mathrm{T}})$

51. $(BA^{\mathrm{T}})^{\mathrm{T}}$ **52.** $(2I)^{\mathrm{T}}$ **53.** $(2I)^2 - 2I^2$

54. $(A^{\mathrm{T}}C^{\mathrm{T}}B)^0$ **55.** $A(I - 0)$ **56.** $I^{\mathrm{T}}0$

57. $(AB)(AB)^{\mathrm{T}}$ **58.** $B^2 - 3B + 2I$

In Problems 59–61, represent the given system by using matrix multiplication.

59. $\begin{cases} 3x + y = 6 \\ 2x - 9y = 5 \end{cases}$

60. $\begin{cases} 3x + y + z = 2 \\ x - y + z = 4 \\ 5x - y + 2z = 12 \end{cases}$

61. $\begin{cases} 2r - s + 3t = 9 \\ 5r - s + 2t = 5 \\ 3r - 2s + 2t = 11 \end{cases}$

62. Secret Messages Secret messages can be encoded by using a code and an encoding matrix. Suppose we have the following code:

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*
1	2	3	4	5	6	7	8	9	10	11	12	13
n	*o*	*p*	*q*	*r*	*s*	*t*	*u*	*v*	*w*	*x*	*y*	*z*
14	15	16	17	18	19	20	21	22	23	24	25	26

Let the encoding matrix be $E = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$. Then we can encode a message by taking every two letters of the message, converting them to their corresponding numbers, creating a 1×2 matrix, and then multiplying each matrix on the right by E. Use this code and matrix to encode the message "winter/is/coming", leaving the slashes to separate words.

63. Inventory A pet store has 6 kittens, 10 puppies, and 7 parrots in stock. If the value of each kitten is \$55, each puppy is \$150, and each parrot is \$35, find the total value of the pet store's inventory using matrix multiplication.

64. Stocks A stockbroker sold a customer 200 shares of stock A, 300 shares of stock B, 500 shares of stock C, and 250 shares of stock D. The prices per share of A, B, C, and D are \$100, \$150, \$200, and \$300, respectively. Write a row vector representing the number of shares of each stock bought. Write a column vector representing the price per share of each stock. Using matrix multiplication, find the total cost of the stocks.

65. Construction Cost In Example 9, assume that the contractor is to build five ranch-style, two Cape Cod–style, and four colonial-style houses. Using matrix multiplication, compute the total cost of raw materials.

66. Costs In Example 9, assume that the contractor wishes to take into account the cost of transporting raw materials to the building site as well as the purchasing cost. Suppose the costs are given in the following matrix:

$$C = \begin{array}{c} \begin{array}{cc} \text{Purchase} & \text{Transport} \end{array} \\ \begin{bmatrix} 3500 & 50 \\ 1500 & 50 \\ 1000 & 100 \\ 250 & 10 \\ 3500 & 0 \end{bmatrix} \end{array} \begin{array}{l} \text{Steel} \\ \text{Wood} \\ \text{Glass} \\ \text{Paint} \\ \text{Labor} \end{array}$$

(a) By computing RC, find a matrix whose entries give the purchase and transportation costs of the materials for each type of house.

(b) Find the matrix QRC whose first entry gives the total purchase price and whose second entry gives the total transportation cost.

(c) Let $Z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and then compute $QRCZ$, which gives the total cost of materials and transportation for all houses being built.

67. Perform the following calculations for Example 6.

(a) Compute the amount that each industry and each consumer have to pay for the goods they receive.
(b) Compute the profit earned by each industry.
(c) Find the total amount of money that is paid out by all the industries and consumers.
(d) Find the proportion of the total amount of money found in part (c) paid out by the industries. Find the proportion of the total amount of money found in part (c) that is paid out by the consumers.

68. Prove that if $AB = BA$, then $(A + B)(A - B) = A^2 - B^2$.

69. Show that if

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -3 \\ -1 & \frac{3}{2} \end{bmatrix}$$

then $AB = 0$. Observe that since neither A nor B is the zero matrix, the algebraic rule for real numbers, "If $ab = 0$, then either $a = 0$ or $b = 0$", does not hold for matrices. It can also be shown that the cancellation law is not true for matrices; that is, if $AB = AC$, then it is not necessarily true that $B = C$.

70. Let D_1 and D_2 be two arbitrary 3×3 diagonal matrices. By computing D_1D_2 and D_2D_1, show that

(a) Both D_1D_2 and D_2D_1 are diagonal matrices.
(b) D_1 and D_2 *commute,* meaning that $D_1D_2 = D_2D_1$.

In Problems 71–74, compute the required matrices, given that

$$A = \begin{bmatrix} 3.2 & -4.1 & 5.1 \\ -2.6 & 1.2 & 6.8 \end{bmatrix} \quad B = \begin{bmatrix} 1.1 & 4.8 \\ -2.3 & 3.2 \\ 4.6 & -1.4 \end{bmatrix} \quad C = \begin{bmatrix} -1.2 & 1.5 \\ 2.4 & 6.2 \end{bmatrix}$$

71. $A(2B)$ **72.** $2.6(BC)$ **73.** $3CA(-B)$ **74.** C^3

Objective

To show how to reduce a matrix and to use matrix reduction to solve a linear system.

6.4 Solving Systems by Reducing Matrices

In this section we illustrate a method by which matrices can be used to solve a *system* of linear equations. It is important here to recall, from Section 3.4, that two systems of equations are *equivalent* if they have the same set of solutions. It follows that in attempting to solve a linear system, call it S_1, we can do so by solving any system S_2 that is equivalent to S_1. If the solutions of S_2 are more easily found than those of S_1, then replacing S_1 by S_2 is a useful step in solving S_1. In fact, the method we illustrate amounts to finding a sequence of equivalent systems, $S_1, S_2, S_3, \cdots, S_n$ for which the solutions of S_n are *obvious*. In our development of this method, known as the *method of reduction*, we will first solve a system by the usual method of elimination. Then we will obtain the same solution by using matrices.

Let us consider the system

$$\begin{cases} 3x - y = 1 & (1) \\ x + 2y = 5 & (2) \end{cases}$$

consisting of two linear equations in two unknowns, x and y. Although this system can be solved by various algebraic methods, we will solve it by a method that is readily adapted to matrices.

A system of linear equations has zero, one, or infinitely many solutions.

Examples 3–5 illustrate the fact that a system of linear equations may have a unique solution, no solution, or infinitely many solutions. It can be shown that these are the *only* possibilities.

PROBLEMS 6.4

In Problems 1–6, determine whether the matrix is reduced or not reduced.

1. $\begin{bmatrix} 1 & 2 \\ 7 & 0 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ **5.** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In Problems 7–12, reduce the given matrix.

7. $\begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}$ **8.** $\begin{bmatrix} 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & 4 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 3 \\ 1 & -6 \\ 4 & 8 \\ 1 & 7 \end{bmatrix}$ **11.** $\begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 7 & 2 & 3 \\ -1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ **12.** $\begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

Solve the systems in Problems 13–26 by the method of reduction.

13. $\begin{cases} 3x + 5y = 25 \\ x - 2y = 1 \end{cases}$ **14.** $\begin{cases} x - 3y = -11 \\ 4x + 3y = 9 \end{cases}$

15. $\begin{cases} 3x + y = 4 \\ 12x + 4y = 2 \end{cases}$ **16.** $\begin{cases} 3x + 2y - z = 1 \\ -x - 2y - 3z = 1 \end{cases}$

17. $\begin{cases} x + 2y + z - 4 = 0 \\ 3x + 2z - 5 = 0 \end{cases}$ **18.** $\begin{cases} x + y - 5z - 8 = 0 \\ 2x - y - z - 1 = 0 \end{cases}$

19. $\begin{cases} x_1 - 3x_2 = 0 \\ 2x_1 + 2x_2 = 3 \\ 5x_1 - x_2 = 1 \end{cases}$ **20.** $\begin{cases} x_1 + 4x_2 = 9 \\ 3x_1 - x_2 = 6 \\ x_1 - x_2 = 2 \end{cases}$

21. $\begin{cases} x + 3y = 2 \\ 2x + 7y = 4 \\ x + 5y + z = 5 \end{cases}$ **22.** $\begin{cases} x + y - z = 7 \\ 2x - 3y - 2z = 4 \\ x - y - 5z = 23 \end{cases}$

23. $\begin{cases} 3x - y + z = 12 \\ x + y + z = 2 \\ x + 2y - z = -2 \\ 2x + y - 3z = 1 \end{cases}$ **24.** $\begin{cases} x + 3z = -1 \\ 3x + 2y + 11z = 1 \\ x + y + 4z = 1 \\ 2x - 3y + 3z = -8 \end{cases}$

25. $\begin{cases} x_1 - x_2 - x_3 - x_4 - x_5 = 0 \\ x_1 + x_2 - x_3 - x_4 - x_5 = 0 \\ x_1 + x_2 + x_3 - x_4 - x_5 = 0 \\ x_1 + x_2 + x_3 + x_4 - x_5 = 0 \end{cases}$

26. $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 - x_3 + x_4 = 0 \end{cases}$

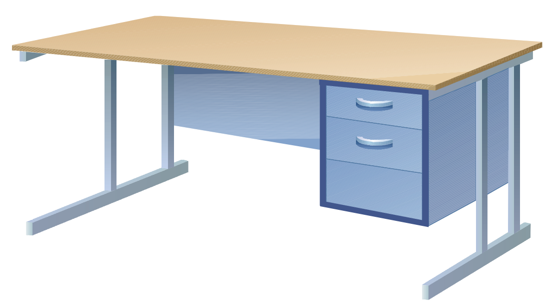
Solve Problems 27–33 by using matrix reduction.

27. Taxes A company has taxable income of \$312,000. The federal tax is 25% of that portion that is left after the state tax has been paid. The state tax is 10% of that portion that is left after the federal tax has been paid. Find the company's federal and state taxes.

28. Decision Making A manufacturer produces two products, A and B. For each unit of A sold, the profit is \$10, and for each unit of B sold, the profit is \$12. From experience, it has been found that 50% more of A can be sold than of B. Next year the manufacturer desires a total profit of \$54,000. How many units of each product must be sold?

29. Production Scheduling A manufacturer produces three products: A, B, and C. The profits for each unit of A, B, and C sold are \$1, \$2, and \$3, respectively. Fixed costs are \$17,000 per year, and the costs of producing each unit of A, B, and C are \$4, \$5, and \$7, respectively. Next year, a total of 11,000 units of all three products is to be produced and sold, and a total profit of \$25,000 is to be realized. If total cost is to be \$80,000, how many units of each of the products should be produced next year?

30. Production Allocation National Desk Co. has plants for producing desks on both the East Coast and West Coast. At the East Coast plant, fixed costs are \$20,000 per year and the cost of producing each desk is \$90. At the West Coast plant, fixed costs are \$18,000 per year and the cost of producing each desk is \$95. Next year the company wants to produce a total of 800 desks. Determine the production order for each plant for the forthcoming year if the total cost for each plant is to be the same.



31. Vitamins A person is ordered by a doctor to take 10 units of vitamin A, 9 units of vitamin D, and 19 units of vitamin E each day. The person can choose from three brands of vitamin pills. Brand X contains 2 units of vitamin A, 3 units of vitamin D, and 5 units of vitamin E; brand Y has 1, 3, and 4 units, respectively; and brand Z has 1 unit of vitamin A, none of vitamin D, and 1 of vitamin E.

(a) Find all possible combinations of pills that will provide exactly the required amounts of vitamins.
(b) If brand X costs 1 cent a pill, brand Y 6 cents, and brand Z 3 cents, are there any combinations in part (a) costing exactly 15 cents a day?
(c) What is the least expensive combination in part (a)? the most expensive?

32. Production A firm produces three products, A, B, and C, that require processing by three machines, I, II, and III. The time in hours required for processing one unit of each product by the three machines is given by the following table:

	A	B	C
I	3	1	2
II	1	2	1
III	2	4	1

Machine I is available for 440 hours, machine II for 310 hours, and machine III for 560 hours. Find how many units of each product should be produced to make use of all the available time on the machines.

33. Investments An investment company sells three types of pooled funds, Standard (S), Deluxe (D), and Gold Star (G).
Each unit of S contains 12 shares of stock A, 16 of stock B, and 8 of stock C.
Each unit of D contains 20 shares of stock A, 12 of stock B, and 28 of stock C.
Each unit of G contains 32 shares of stock A, 28 of stock B, and 36 of stock C.
Suppose an investor wishes to purchase exactly 220 shares of stock A, 176 shares of stock B, and 264 shares of stock C by buying units of the three funds.
(a) Set up equations in s, for units of S, d, for units of D, and g, for units of G whose solution would provide the number of units of S, D, and G that will meet the investor's requirements exactly.
(b) Solve the system set up in (a) and show that it has infinitely many solutions, if we naively assume that s, d, and g can take on arbitrary real values.
(c) Pooled funds can be bought only in units that are non-negative integers. In the solution to (b) above, it follows that we must require each of s, d, and g to be non-negative integers. Enumerate the solutions in (b) that remain after we impose this new constraint.
(d) Suppose the investor pays \$300 for each unit of S, \$400 for each unit of D, and \$600 for each unit of G. Which of the possible solutions from part (c) will minimize the total cost to the investor?

Objective

To focus our attention on nonhomogeneous systems that involve more than one parameter in their general solution; and to solve, and consider the theory of, homogeneous systems.

6.5 Solving Systems by Reducing Matrices (Continued)

As we saw in Section 6.4, a system of linear equations may have a unique solution, no solution, or infinitely many solutions. When there are infinitely many, the general solution is expressed in terms of at least one parameter. For example, the general solution in Example 5 was given in terms of the parameter r:

$$\begin{aligned} x_1 &= 4 - \tfrac{5}{2}r \\ x_2 &= 0 \\ x_3 &= 1 - \tfrac{1}{2}r \\ x_4 &= r \end{aligned}$$

At times, more than one parameter is necessary. In fact, we saw a very simple example in Example 7 of Section 3.4. Example 1 illustrates further.

EXAMPLE 1 Two-Parameter Family of Solutions

Using matrix reduction, solve

$$\begin{cases} x_1 + 2x_2 + 5x_3 + 5x_4 = -3 \\ x_1 + x_2 + 3x_3 + 4x_4 = -1 \\ x_1 - x_2 - x_3 + 2x_4 = 3 \end{cases}$$

Solution: The augmented coefficient matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 5 & 5 & -3 \\ 1 & 1 & 3 & 4 & -1 \\ 1 & -1 & -1 & 2 & 3 \end{array}\right]$$

Since the reduced coefficient matrix corresponds to

$$\begin{cases} x + 3z = 0 \\ y + z = 0 \end{cases}$$

the solution may be given in parametric form by

$$x = -3r$$

$$y = -r$$

$$z = r$$

where r is any real number.

b. $\begin{cases} 3x + 4y = 0 \\ x - 2y = 0 \\ 2x + y = 0 \\ 2x + 3y = 0 \end{cases}$

Solution: Reducing the coefficient matrix, we have

$$\begin{bmatrix} 3 & 4 \\ 1 & -2 \\ 2 & 1 \\ 2 & 3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The number of nonzero-rows (2) in the reduced coefficient matrix equals the number of unknowns in the system. By the theorem, the system must have a unique solution, namely, the trivial solution $x = 0, y = 0$.

Now Work Problem 13 ◁

PROBLEMS 6.5

In Problems 1–8, solve the systems by using matrix reduction.

1. $\begin{cases} w + x - y - 9z = -3 \\ 2w + 3x + 2y + 15z = 12 \\ 2w + x + 2y + 5z = 8 \end{cases}$

2. $\begin{cases} 2w + x + 10y + 15z = -5 \\ w - 5x + 2y + 15z = -10 \\ w + x + 6y + 12z = 9 \end{cases}$

3. $\begin{cases} 3w - x - 3y - z = -2 \\ 2w - 2x - 6y - 6z = -4 \\ 2w - x - 3y - 2z = -2 \\ 3w + x + 3y + 7z = 2 \end{cases}$

4. $\begin{cases} 3w + x - 10y - 2z = 5 \\ 2w - 6y - 2z = 2 \\ w - x - 2y - 2z = -1 \\ w + x - 4y = 3 \end{cases}$

5. $\begin{cases} w - 3x + y - z = 5 \\ w - 3x - y + 3z = 1 \\ 3w - 9x + y + z = 11 \\ 2w - 6x - y + 4z = 4 \end{cases}$

6. $\begin{cases} w + x + y + 2z = 4 \\ 2w + x + 2y + 2z = 7 \\ w + 2x + y + 4z = 5 \\ 3w - 2x + 3y - 4z = 7 \\ 4w - 3x + 4y - 6z = 9 \end{cases}$

7. $\begin{cases} 4x_1 - 3x_2 + 5x_3 - 10x_4 + 11x_5 = -8 \\ 2x_1 + x_2 + 5x_3 + 3x_5 = 6 \end{cases}$

8. $\begin{cases} x_1 + 3x_3 + x_4 + 4x_5 = 1 \\ x_2 + x_3 - 2x_4 = 0 \\ 2x_1 - 2x_2 + 3x_3 + 10x_4 + 15x_5 = 10 \\ x_1 + 2x_2 + 3x_3 - 2x_4 + 2x_5 = -2 \end{cases}$

For Problems 9–14, determine whether the system has infinitely many solutions or only the trivial solution. Do not solve the systems.

9. $\begin{cases} 2.17x - 5.3y + 0.27z = 0 \\ 3.51x - 1.4y + 0.01z = 0 \end{cases}$

10. $\begin{cases} 5w + 7x - 2y - 5z = 0 \\ 7w - 6x + 9y - 5z = 0 \end{cases}$

11. $\begin{cases} 3x - 4y = 0 \\ x + 5y = 0 \\ 4x - y = 0 \end{cases}$

12. $\begin{cases} 2x + 3y + 12z = 0 \\ 3x - 2y + 5z = 0 \\ 4x + y + 14z = 0 \end{cases}$

13. $\begin{cases} x + y + z = 0 \\ x - z = 0 \\ x - 2y - 5z = 0 \end{cases}$

14. $\begin{cases} 3x + 2y + z = 0 \\ 2x + 2y + z = 0 \\ 4x + y + z = 0 \end{cases}$

Solve each of the following systems.

15. $\begin{cases} 2x + 3y = 0 \\ 5x - 7y = 0 \end{cases}$

16. $\begin{cases} 2x - 5y = 0 \\ 8x - 20y = 0 \end{cases}$

17. $\begin{cases} x + 6y - 2z = 0 \\ 2x - 3y + 4z = 0 \end{cases}$

18. $\begin{cases} 4x + 7y = 0 \\ 2x + 3y = 0 \end{cases}$

19. $\begin{cases} x + y = 0 \\ 7x - 5y = 0 \\ 9x - 4y = 0 \end{cases}$

20. $\begin{cases} 2x + y + z = 0 \\ x - y + 2z = 0 \\ x + y + z = 0 \end{cases}$

21. $\begin{cases} x + y + z = 0 \\ -7y - 14z = 0 \\ -2y - 4z = 0 \\ -5y - 10z = 0 \end{cases}$

22. $\begin{cases} x + y + 7z = 0 \\ x - y - z = 0 \\ 2x - 3y - 6z = 0 \\ 3x + y + 13z = 0 \end{cases}$

23. $\begin{cases} w + x + y + 4z = 0 \\ w + x + 5z = 0 \\ 2w + x + 3y + 4z = 0 \\ w - 3x + 2y - 9z = 0 \end{cases}$

24. $\begin{cases} w + x - 2y - 2z = 0 \\ w - x = 0 \\ 2w + x - 3y - 3z = 0 \\ w + 2x - 3y - 3z = 0 \end{cases}$

Objective

To determine the inverse of an invertible matrix and to use inverses to solve systems.

6.6 Inverses

We have seen how useful the method of reduction is for solving systems of linear equations. But it is by no means the only method that uses matrices. In this section, we will discuss a different method that applies to *certain* systems of n linear equations in n unknowns.

In Section 6.3, we showed how a system of linear equations can be written in matrix form as the single matrix equation $AX = B$, where A is the coefficient matrix. For example, the system

$$\begin{cases} x_1 + 2x_2 = 3 \\ x_1 - x_2 = 1 \end{cases}$$

can be written in the matrix form $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Motivation for what we now have in mind is provided by looking at the procedure for solving the algebraic equation $ax = b$. The latter equation is solved by simply multiplying both sides by the multiplicative inverse of a, if it exists. (Recall that the multiplicative inverse of a nonzero number a is denoted a^{-1} (which is $1/a$) and has the property that $a^{-1}a = 1$.) For example, if $3x = 11$, then

$$3^{-1}(3x) = 3^{-1}(11) \quad \text{so} \quad x = \frac{11}{3}$$

If we are to apply a similar procedure to the *matrix* equation

$$AX = B \tag{1}$$

then we need a multiplicative inverse of A—that is, a matrix C such that $CA = I$. If we have such a C, then we can simply multiply both sides of Equation (1) by C and get

$$\begin{aligned} C(AX) &= CB \\ (CA)X &= CB \\ IX &= CB \\ X &= CB \end{aligned}$$

This shows us that ***if*** there is a solution of $AX = B$, *then* the only possible solution is the matrix CB. Since we know that a matrix equation can have no solutions, a unique solution, or infinitely many solutions, we see immediately that this strategy cannot possibly work unless the matrix equation has a unique solution. For CB to actually be a solution, we require that $A(CB) = B$, which is the same as requiring that $(AC)B = B$. However, since matrix multiplication is not commutative, our assumption that $CA = I$ does not immediately give us $AC = I$. Consider the matrix products below, for example:

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1] = I_1 \qquad \text{but} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq I_2$$

However, if A and C are square matrices of the same order n, then it can be proved that $AC = I_n$ follows from $CA = I_n$, so *in this case* we can finish the argument above and conclude that CB is a solution, necessarily the only one, of $AX = B$. For a square matrix A, when a matrix C exists satisfying $CA = I$, necessarily C is also square of the same size as A and we say that it is an **inverse matrix** (or simply an *inverse*) of A.

The method of reduction in Sections 6.4 and 6.5 is a faster computation than that of finding a matrix inverse.

While the solution of a system using a matrix inverse is very elegant, we must provide a caution. Given $AX = B$, the computational work required to find A^{-1} is greater than that required to reduce the augmented matrix of the system, namely $[A \mid B]$. If there are several equations to solve, all with the same matrix of coefficients but variable right-hand sides, say $AX = B_1, AX = B_2, \ldots, AX = B_k$, then for suitably large k it *might* be faster to compute A^{-1} than to do k reductions, but a numerical analyst will in most cases still advocate in favour of the reductions. For even with A^{-1} in hand, one still has to compute $A^{-1}B$ and, if the order of A is large, this too takes considerable time.

EXAMPLE 6 A Coefficient Matrix That Is Not Invertible

Solve the system

$$\begin{cases} x - 2y + z = 0 \\ 2x - y + 5z = 0 \\ x + y + 4z = 0 \end{cases}$$

Solution: The coefficient matrix is

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 5 \\ 1 & 1 & 4 \end{bmatrix}$$

Since

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

the coefficient matrix is not invertible. Hence, the system *cannot* be solved by inverses. Instead, another method must be used. In Example 4(a) of Section 6.5, the solution was found to be $x = -3r$, $y = -r$, and $z = r$, where r is any real number (thus, providing infinitely many solutions).

Now Work Problem 31 ◁

PROBLEMS 6.6

In Problems 1–18, if the given matrix is invertible, find its inverse.

1. $\begin{bmatrix} 6 & 1 \\ 7 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ a in $(-\infty, \infty)$

5. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 & 8 \\ -1 & 4 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & 1 \\ 4 & -6 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

16. $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

18. $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

19. Solve $AX = B$ if

$$A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

20. Solve $AX = B$ if

$$A^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

For Problems 21–34, if the coefficient matrix of the system is invertible, solve the system by using the inverse. If not, solve the system by the method of reduction.

21. $\begin{cases} 6x + 5y = 2 \\ x + y = -3 \end{cases}$

22. $\begin{cases} 2x + 4y = 5 \\ -x + 3y = -2 \end{cases}$

23. $\begin{cases} 3x + y = 5 \\ 3x - y = 7 \end{cases}$

24. $\begin{cases} 6x + y = 2 \\ 7x + y = 7 \end{cases}$

25. $\begin{cases} x + 3y = 7 \\ 3x - y = 1 \end{cases}$

26. $\begin{cases} 2x + 6y = 8 \\ 3x + 9y = 7 \end{cases}$

27. $\begin{cases} x + 2y + z = 4 \\ 3x + z = 2 \\ x - y + z = 1 \end{cases}$

28. $\begin{cases} x + y + z = 6 \\ x - y + z = -1 \\ x - y - z = 4 \end{cases}$

29. $\begin{cases} x + y + z = 3 \\ x + y - z = 4 \\ x - y - z = 5 \end{cases}$

30. $\begin{cases} x + y + z = 6 \\ x + y - z = 0 \\ x - y + z = 2 \end{cases}$

31. $\begin{cases} x + 3y + 3z = 7 \\ 2x + y + z = 4 \\ x + y + z = 4 \end{cases}$

32. $\begin{cases} x + 3y + 3z = 7 \\ 2x + y + z = 4 \\ x + y + z = 3 \end{cases}$

33. $\begin{cases} w + 2y + z = 4 \\ w - x + 2z = 12 \\ 2w + x + z = 12 \\ w + 2x + y + z = 12 \end{cases}$

34. $\begin{cases} x - 3y - z = -1 \\ w + y + = 0 \\ -w + 2x - 2y - z = 6 \\ y + z = 4 \end{cases}$

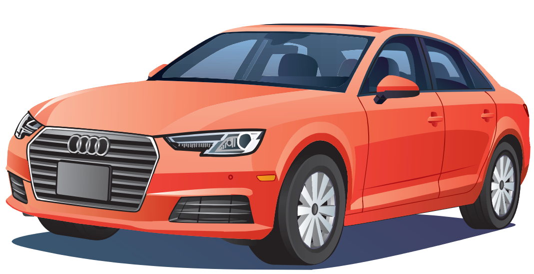
For Problems 35 and 36, find $(I - A)^{-1}$ for the given matrix A.

35. $\mathbf{A} = \begin{bmatrix} -3 & -1 \\ -2 & 4 \end{bmatrix}$

36. $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 4 & 3 \end{bmatrix}$

37. Auto Production Solve the following problems by using the inverse of the matrix involved.

(a) An automobile factory produces two models, A and B. Model A requires 1 labor hour to paint and $\frac{1}{2}$ labor hour to polish; model B requires 1 labor hour for each process. During each hour that the assembly line is operating, there are 100 labor hours available for painting and 80 labor hours for polishing. How many of each model can be produced each hour if all the labor hours available are to be utilized?



(b) Suppose each model A requires 10 widgets and 14 shims and each model B requires 7 widgets and 10 shims. The factory can obtain 800 widgets and 1130 shims each hour. How many cars of each model can it produce while using all the parts available?

38. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where $a, b, c \neq 0$, show that

$$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$$

39. (a) If A and B are invertible matrices with the same order, show that $(AB)^{-1} = B^{-1}A^{-1}$. [*Hint:* Consider $(B^{-1}A^{-1})(AB)$.]
(b) If

$$A^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

find $(AB)^{-1}$.

40. If A is invertible, it can be shown that $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$. Verify this identity for

$$A = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}$$

41. A matrix P is said to be *orthogonal* if $P^{-1} = P^{\mathrm{T}}$. Is the matrix $P = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ orthogonal?

42. Secret Message A friend has sent a friend a secret message that consists of three row matrices of numbers as follows:

$$R_1 = [33 \quad 87 \quad 70] \qquad R_2 = [57 \quad 133 \quad 20]$$
$$R_3 = [38 \quad 90 \quad 33]$$

Both friends have committed the following matrix to memory (the first friend used it to code the message):

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ -1 & -2 & 2 \end{bmatrix}$$

Decipher the message by proceeding as follows:
(a) Calculate the three matrix products R_1A^{-1}, R_2A^{-1}, and R_3A^{-1}.
(b) Assume that the letters of the alphabet correspond to the numbers 1 through 26, replace the numbers in the preceding three matrices by letters, and determine the message.

43. Investing A group of investors decides to invest \$500,000 in the stocks of three companies. Company D sells for \$60 a share and has an expected growth of 16% per year. Company E sells for \$80 per share and has an expected growth of 12% per year. Company F sells for \$30 a share and has an expected growth of 9% per year. The group plans to buy four times as many shares of company F as of company E. If the group's goal is 13.68% growth per year, how many shares of each stock should the investors buy?

44. Investing The investors in Problem 43 decide to try a new investment strategy with the same companies. They wish to buy twice as many shares of company F as of company E, and they have a goal of 14.52% growth per year. How many shares of each stock should they buy?

The result of evaluating $(I - A)^{-1}D$ on a TI-83 Plus is

$$\begin{bmatrix} 692.5 \\ 380 \\ 495 \end{bmatrix}$$

Now Work Problem 7 ◁

PROBLEMS 6.7

1. A very simple economy consists of two sectors: agriculture and forestry. To produce one unit of agricultural products requires $\frac{1}{4}$ of a unit of agricultural products and $\frac{1}{12}$ of a unit of forestry products. To produce one unit of forestry products requires $\frac{2}{3}$ of a unit of agricultural products and no units of forestry products. Determine the production levels needed to satisfy an external demand for 400 units of agriculture and 600 units of forestry products.

2. An economy consists of three sectors: coal, steel, and railroads. To produce one unit of coal requires $\frac{1}{10}$ of a unit of coal, $\frac{1}{10}$ of a unit of steel, and $\frac{1}{10}$ of a unit of railroad services. To produce one unit of steel requires $\frac{1}{3}$ of a unit of coal, $\frac{1}{10}$ of a unit of steel, and $\frac{1}{10}$ of a unit of railroad services. To produce one unit of railroad services requires $\frac{1}{4}$ of a unit of coal, $\frac{1}{3}$ of a unit of steel, and $\frac{1}{10}$ of a unit of railroad services. Determine the production levels needed to satisfy an external demand for 300 units of coal, 200 units of steel, and 500 units of railroad services.

3. Suppose that a simple economy consists of three sectors: agriculture (A), manufacturing (M), and transportation (T). Economists have determined that to produce one unit of A requires $\frac{1}{18}$ units of A, $\frac{1}{9}$ units of B, and $\frac{1}{9}$ units of C, while production of one unit of M requires $\frac{3}{16}$ units of A, $\frac{1}{4}$ units of M, and $\frac{3}{16}$ units of T, and production of one unit of T requires $\frac{1}{15}$ units of A, $\frac{1}{3}$ units of M, and $\frac{1}{6}$ units of T. There is an external demand for 40 units of A, 30 units of M, and no units of T. Determine the production levels necessary to meet the external demand.

4. Given the input–output matrix

	Industry		*Final*
	Steel	*Coal*	*Demand*
Industry: Steel	200	500	500
Coal	400	200	900
Other	600	800	—

find the output matrix if final demand changes to 600 for steel and 805 for coal. Find the total value of the other production costs that this involves.

5. Given the input–output matrix

	Industry		*Final*
	Education	*Government*	*Demand*
Industry: Education	40	120	40
Government	120	90	90
Other	40	90	—

find the output matrix if final demand changes to **(a)** 200 for education and 300 for government; **(b)** 64 for education and 64 for government.

6. Given the input–output matrix

	Industry			*Final*
	Grain	*Fertilizer*	*Cattle*	*Demand*
Industry: Grain	15	30	45	10
Fertilizer	25	30	60	5
Cattle	50	40	60	30
Other	10	20	15	—

(a) find the Leontief matrix. **(b)** If external demand changes to 15 for grain, 10 for fertilizer, and 35 for cattle, write the augmented matrix whose reduction will give the necessary production levels needed to meet these new demands.

7. Given the input–output matrix

	Industry			*Final*
	Water	*Electric Power*	*Agriculture*	*Demand*
Industry: Water	100	400	240	260
Electric Power	100	80	480	140
Agriculture	300	160	240	500
Other	500	160	240	—

find the output matrix if final demand changes to 500 for water, 150 for electric power, and 700 for agriculture. Round the entries to two decimal places.

8. Given the input–output matrix

	Industry			*Final*
	Government	*Agriculture*	*Manufacturing*	*Demand*
Industry: Government	400	200	200	200
Agriculture	200	400	100	300
Manufacturing	200	100	300	400
Other	200	300	400	—

with entries in billions of dollars, find the output matrix for the economy if the final demand changes to 300 for government, 350 for agriculture, and 450 for manufacturing. Round the entries to the nearest billion dollars.

9. Given the input–output matrix in Problem 8, find the output matrix for the economy if the final demand changes to 250 for government, 300 for agriculture, and 350 for manufacturing. Round the entries to the nearest billion dollars.

10. Given the input–output matrix in Problem 8, find the output matrix for the economy if the final demand changes to 300 for government, 400 for agriculture, and 500 for manufacturing. Round the entries to the nearest billion dollars.

Chapter 6 Review

Important Terms and Symbols

		Examples
Section 6.1	**Matrices**	
	matrix size entry, A_{ij} row vector column vector	Ex. 1, p. 242
	equality of matrices transpose of matrix, A^T zero matrix, 0	Ex. 3, p. 244
Section 6.2	**Matrix Addition and Scalar Multiplication**	
	addition and subtraction of matrices scalar multiplication	Ex. 4, p. 249
Section 6.3	**Matrix Multiplication**	
	matrix multiplication identity matrix, I power of a matrix	Ex. 12, p. 261
	matrix equation, $AX = B$	Ex. 13, p. 262
Section 6.4	**Solving Systems by Reducing Matrices**	
	coefficient matrix augmented coefficient matrix	Ex. 3, p. 269
	elementary row operation equivalent matrices reduced matrix	Ex. 4, p. 270
	parameter	Ex. 5, p. 271
Section 6.5	**Solving Systems by Reducing Matrices (Continued)**	
	homogeneous system nonhomogeneous system trivial solution	Ex. 4, p. 277
Section 6.6	**Inverses**	
	inverse matrix invertible matrix	Ex. 1, p. 280
Section 6.7	**Leontief's Input–Output Analysis**	
	input–output matrix Leontief matrix	Ex. 1, p. 287

Summary

A matrix is a rectangular array of numbers enclosed within brackets. There are a number of special types of matrices, such as zero matrices, identity matrices, square matrices, and diagonal matrices. Besides the operation of scalar multiplication, there are the operations of matrix addition and subtraction, which apply to matrices of the same size. The product AB is defined when the number of columns of A is equal to the number of rows of B. Although matrix addition is commutative, matrix multiplication is not. By using matrix multiplication, we can express a system of linear equations as the matrix equation $AX = B$.

A system of linear equations may have a unique solution, no solution, or infinitely many solutions. The main method of solving a system of linear equations using matrices is by applying the three elementary row operations to the augmented coefficient matrix of the system until an equivalent reduced matrix is obtained. The reduced matrix makes any solutions to the system obvious and allows the detection of nonexistence of solutions. If there are infinitely many solutions, the general solution involves at least one parameter.

Occasionally, it is useful to find the inverse of a (square) matrix. The inverse (if it exists) of a square matrix A is found by augmenting A with I and applying elementary row operations to $[A \mid I]$ until A is reduced resulting in $[R \mid B]$ (with R reduced). If $R = I$, then A is invertible and $A^{-1} = B$. If $R \neq I$, then A is not invertible, meaning that A^{-1} does not exist. If the inverse of an $n \times n$ matrix A exists, then the unique solution to $AX = B$ is given by $X = A^{-1}B$. If A is not invertible, the system has either no solution or infinitely many solutions.

Our final application of matrices dealt with the interrelationships that exist among the various sectors of an economy and is known as Leontief's input–output analysis.

Review Problems

In Problems 1–8, simplify.

1. $2\begin{bmatrix} 3 & 4 \\ -5 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$

2. $2\begin{bmatrix} -2 & -3 \\ 6 & 8 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 7 \\ 2 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 6 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 5 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}\left(\begin{bmatrix} 2 & 3 \\ 7 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 8 \\ 4 & 4 \end{bmatrix}\right)$

6. $-\left(\begin{bmatrix} 2 & 0 \\ 7 & 8 \end{bmatrix} + 2\begin{bmatrix} 0 & -5 \\ 6 & -4 \end{bmatrix}\right)$

7. $2\begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}^2 [4 \quad 2]^{\mathrm{T}}$

8. $\frac{1}{3}\begin{bmatrix} 3 & 0 \\ 3 & 6 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}^{\mathrm{T}}\right)^2$

In Problems 9–12, compute the required matrix if

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

9. $(2A)^{\mathrm{T}} - 3I^2$ **10.** $A(2I) - A0^{\mathrm{T}}$ **11.** $B^3 + I^5$

12. $(AB)^{\mathrm{T}} - B^{\mathrm{T}}A^{\mathrm{T}}$

In Problems 13 and 14, solve for x and y.

13. $\begin{bmatrix} 5 \\ 7 \end{bmatrix}[x] = \begin{bmatrix} 15 \\ y \end{bmatrix}$ **14.** $\begin{bmatrix} 1 & x \\ 2 & y \end{bmatrix}\begin{bmatrix} 2 & 1 \\ x & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & y \end{bmatrix}$

In Problems 15–18, reduce the given matrices.

15. $\begin{bmatrix} 1 & 4 \\ 5 & 8 \end{bmatrix}$ **16.** $\begin{bmatrix} 0 & 0 & 7 \\ 0 & 5 & 9 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 1 & 6 \end{bmatrix}$ **18.** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

In Problems 19–22, solve each of the systems by the method of reduction.

19. $\begin{cases} 2x - 5y = 0 \\ 4x + 3y = 0 \end{cases}$ **20.** $\begin{cases} x - y + 2z = 3 \\ 3x + y + z = 5 \end{cases}$

21. $\begin{cases} x + y + 2z = 1 \\ 3x - 2y - 4z = -7 \\ 2x - y - 2z = 2 \end{cases}$ **22.** $\begin{cases} x + 2y + 3z = 1 \\ x + 4y + 6z = 2 \\ x + 6y + 9z = 3 \end{cases}$

In Problems 23–26, find the inverses of the matrices.

23. $\begin{bmatrix} 1 & 5 \\ 3 & 9 \end{bmatrix}$ **24.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 1 & 0 \\ 3 & -2 & 2 \end{bmatrix}$ **26.** $\begin{bmatrix} 5 & 0 & 0 \\ -5 & 2 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

In Problems 27 and 28, solve the given system by using the inverse of the coefficient matrix.

27. $\begin{cases} x + y = 3 \\ y + z = 4 \\ x + z = 5 \end{cases}$ **28.** $\begin{cases} 5x = 3 \\ -5x + 2y + z = 0 \\ -5x + y + 3z = 2 \end{cases}$

29. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Find the matrices A^2, A^3, A^{1000}, and A^{-1} (if the inverse exists).

30. $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$. Show that $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$.

31. A consumer wishes to supplement his vitamin intake by *exactly* 13 units of vitamin A, 22 units of vitamin B, and 31 units of vitamin C per week. There are three brands of vitamin capsules available. Brand I contains 1 unit each of vitamins A, B, and C per capsule; brand II contains 1 unit of vitamin A, 2 of B, and 3 of C; and brand III contains 4 units of A, 7 of B, and 10 of C.
(a) What combinations of capsules of brands I, II, and III will produce *exactly* the desired amounts?
(b) If brand I capsules cost 5 cents each, brand II 7 cents each, and brand III 20 cents each, what combination will minimize the consumer's weekly cost?

32. Suppose that A is an invertible $n \times n$ matrix.
(a) Prove that A^k is invertible, for any integer k, where we employ the convention that $A^0 = I$.
(b) Prove that if B and C are $n \times n$ matrices such that $ABA^{-1} = ACA^{-1}$, then $B = C$.
(c) If $A^2 = A$, find A.

33. If $A = \begin{bmatrix} 10 & -3 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 6 \\ -7 & -3 \end{bmatrix}$, find $3AB - 4B^2$.

34. Solve the system

$$\begin{cases} 7.9x - 4.3y + 2.7z = 11.1 \\ 3.4x + 5.8y - 7.6z = 10.8 \\ 4.5x - 6.2y - 7.4z = 15.9 \end{cases}$$

by using the inverse of the coefficient matrix. Round the answers to two decimal places.

35. Given the input–output matrix

	Industry A	*Industry* B	*Final Demand*
Industry: A	10	20	4
B	15	14	10
Other	9	5	—

find the output matrix if final demand changes to 10 for A and 5 for B. (Data are in tens of billions of dollars.)