Mathematics for Analytics and Finance

Sami Najafi

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Module 3

Matrix Algebra

Vectors

Definition: A vector, in simple terms, is like a list of numbers arranged either in a single column or a single row.

Example: $\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $a_1 =$ number of products 1, $a_2 =$ number of products 2

Vector addition and scalar multiplication are component-wise. For $\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ we define:

- **Vector addition:** $\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} \in \mathbb{R}^2$
- **Scalar Multiplication:** $\lambda \boldsymbol{a} = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \end{bmatrix} \in \mathbb{R}^2$

(this is extended to n dimensions)

Remark: $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is also written as $\boldsymbol{a} = (1,2,3)$

Dot Products

Definition: The dot product or inner product of $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ is

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 a_2 + b_1 b_2 + \cdots + a_n b_n.$$

Example: The vectors $\boldsymbol{a} = (4,2)$ and $\boldsymbol{b} = (-1,2)$ have a zero dot product.

$$\boldsymbol{a} \cdot \boldsymbol{b} = (4)(-1) + (2)(2) = 0$$

Remark: Zero is a special case. It means that the two vectors are perpendicular.

Example: We have three products to buy. The prices are (p_1, p_2, p_3) and the quantities are (q_1, q_2, q_3). Then, our total cost is:

$$\text{Total Cost} = (p_1, p_2, p_3) \cdot (q_1, q_2, q_3) = p_1 q_1 + p_2 q_2 + p_3 q_3$$

Length and Unit Vectors

Definition: The length or norm of a vector $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ is the square root of $\boldsymbol{a} \cdot \boldsymbol{a}$:

$$\text{Length} = \|\boldsymbol{a}\| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_1^2 + \cdots + a_n^2}.$$

Example: The length of $\boldsymbol{a} = (3, 4)$ is $\sqrt{3^2 + 4^2} = 5$

Definition: A unit vector $\boldsymbol{u}$ is a vector whose length is equal to one. That is, $\boldsymbol{u} \cdot \boldsymbol{u} = 1$

Example: $\boldsymbol{i} = (1,0)$ and $\boldsymbol{j} = (0,1)$ and $\boldsymbol{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ are unit vectors.

Remark: Divide any nonzero vector $\boldsymbol{a}$ by its length. Then the vector $\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$ is a unit vector in the same direction as $\boldsymbol{a}$.

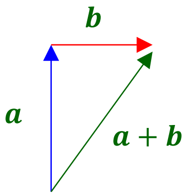
Angle between Two Vectors

If $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Proof: (For interested students)

By Pythagorean Theorem: $\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 = \|\boldsymbol{a} + \boldsymbol{b}\|^2 \Rightarrow \left(a_1^2 + b_2^2\right) + \left(b_1^2 + b_2^2\right) = (a_1 + b_1)^2 + (a_2 + b_2)^2$

$$\Rightarrow 2(a_1 b_1 + a_2 b_2) = 0 \Rightarrow a_1 b_1 + a_2 b_2 = 0$$



Vectors in R

Creating and Combining Vectors

```
c(first number or string,…,last number or character)
```

```
> c(1,1,2,3,5,8,13,21)
[1] 1 1 2 3 5 8 13 21
> c(1*pi, 2*pi, 3*pi, 4*pi)
[1] 3.141593 6.283185 9.424778 12.566371
> c("Everyone", "loves", "stats.")
[1] "Everyone" "loves" "stats."
```

Combining two vectors:

```
> v1 = c(1,2,3)
> v2 = c(4,5,6)
> c(v1,v2)
[1] 1 2 3 4 5 6
```

Vectors in R

Computing Basic Statistics

`mean(x), median(x), sd(x) var(x), cor(x,y), cov(x,y), range(x), quantile(x)`

`summary(x)` gives some of the summary statistics

```
> x = c(0,1,1,2,3,5,8,13,21,34)
> y = log(x+1)
> mean(x)
[1] 8.8
> median(x)
[1] 4
> sd(x)
[1] 11.03328

> summary(x)
Min.   1st Qu.   Median   Mean   3rd Qu.   Max.
0.00     1.25     4.00    8.80    11.75   34.00
```

Vectors in R

Creating Sequences

```
seq(from, to) or from:to
seq(from, to, by= )
seq(from, to, length= )
rep(number, number of repetitions)
```

```
> 10:19
[1] 10 11 12 13 14 15 16 17 18 19
> 9:0
[1] 9 8 7 6 5 4 3 2 1 0
> seq(from=0, to=20)
[1] 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
> seq(from=0, to=20, by=2)
[1] 0 2 4 6 8 10 12 14 16 18 20
> seq(from=1.0, to=2.0, length=5)
[1] 1.00 1.25 1.50 1.75 2.00
```

Vectors in R

Comparing Numbers and Vectors

The comparison operators compare two values and return TRUE or FALSE.

For two vectors, the comparison is component-wise. The result is a logical vector.

```
> a = 3
> a == pi
[1] FALSE
> a != pi
[1] TRUE
> a < pi
[1] TRUE

> v = c( 3, pi, 4)
> w = c(pi, pi, pi)
> v == w
[1] FALSE TRUE FALSE
```

Vectors in R

Comparing Numbers and Vectors

```
any(v == w) Return TRUE if any element of v equals the same element in w
all(v == w) Return TRUE if all element of v equals the same element in w
any(v == num) Return TRUE if any element of v equals the value num
all(v == num) Return TRUE if all element of v equals the value num
```

```
> v = c(3,pi,4)
> w = c(pi,pi,pi)
> v != w
[1] TRUE FALSE TRUE
> v <= w
[1] TRUE TRUE FALSE
> v != 3
[1] FALSE TRUE TRUE
> any(v != 3)
[1] TRUE
```

Vectors in R

Choosing Vector Elements

```
vec[a] Select element a
vec[a:b] Select elements a through b
vec(c(a,b,c)) Select elements a,b, and c.
vec(-a:-b) or vec(-(a:b)) exclude elements a through b
```

```
> fib = c(0,1,1,2,3,5,8,13,21,34)
> fib[6]
[1] 05
> fib[4:9]
[1] 2 3 5 8 13 21
> fib[c(1,2,4,8)]
[1] 0 1 2 13
> fib[-1:-5]
[1] 5 8 13 21 34
```

Vectors in R

More Complex Element Selections

Vec[vec<=a] Select elements that are not greater than a

Vec[vec>=a&vec<=b] Select elements that are between a and b inclusively

More complex element selections can be coded similarly.

```
> fib = c(0,1,1,2,3,5,8,13,21,34)
> fib[fib < 10] # Use that vector to select elements less than 10
[1] 0 1 1 2 3 5 8
> fib[fib > mean(fib)] # Use that vector to select elements less than 10
[1] 13 21 34
> fib[fib<=5|fib>=21] # | means "or"
[1] 0 1 1 2 3 5 21 34
> fib[fib>5&fib<21] # | means "and"
[1] 8 13
> w=c(1,2,4,3,5,6,8,2,7,0)
> fib[w>5]
[1] 5 8 21
```

Vectors in R

Arithmetic Operations

+, -, *, /, ^: These arithmetic operations are component-wise

```
> v = c(11,12,13,14,15)
> w = c(1,2,3,4,5)
> v + w
[1] 12 14 16 18 20
> v - w
[1] 10 10 10 10 10
> v * w
[1] 11 24 39 56 75
> v / w
[1] 11.000000 6.000000 4.333333 3.500000 3.000000
> w ^ v
[1] 1 4096 1594323 268435456 30517578125
```

Vectors in R

Arithmetic Operations between a Vector and a Scalar

The operation is performed between every vector element and the scalar

```
> w + 2
[1] 3 4 5 6 7
> w - 2
[1] -1 0 1 2 3
> w * 2
[1] 2 4 6 8 10
> w / 2
[1] 0.5 1.0 1.5 2.0 2.5
> w ^ 2
[1] 1 4 9 16 25
> 2 ^ w
[1] 2 4 8 16 32
```

Vectors in R

Operation of Functions on Vectors

Functions operate on every element. The result is a vector.

```
> w
[1] 1 2 3 4 5
> sqrt(w)
[1] 1.000000 1.414214 1.732051 2.000000 2.236068
> log(w)
[1] 0.0000000 0.6931472 1.0986123 1.3862944 1.6094379
> sin(w)
[1] 0.8414710 0.9092974 0.1411200 -0.7568025 -0.9589243
```

Matrices

Definition: A matrix with m rows and n columns is called an $m \times n$ *matrix* or a *matrix of size* $m \times n$.

$$\boldsymbol{A} = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{12} & \dots & a_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

- **Row Vector:** A matrix that has exactly one row
- **Column Vector:** A matrix that has exactly one column
- **Remark:** Scalars can be treated as 1×1 matrices.

Equal Matrices: For $\boldsymbol{A} = [a_{ij}]_{m\times n}$ and $\boldsymbol{B} = [b_{ij}]_{m\times n}$

$\boldsymbol{A} = \boldsymbol{B}$ if and only if $a_{ij} = b_{ij}$ for all $1 \leq i \leq m, 1 \leq j \leq n$

Special Matrices

Consider $\boldsymbol{A}_{m\times n} = [a_{ij}]$:

Zero matrix: If $a_{ij} = 0$ for all $1 \leq i \leq m, 1 \leq j \leq n$, $\boldsymbol{A}_{m\times n}$ is called a zero matrix and denoted with $\boldsymbol{O}_{m\times n}$.

Square Matrix: If $m = n$, then $\boldsymbol{A}_{m\times m}$ is called a square matrix and sometimes denoted with $\boldsymbol{A}_m$.

Diagonal Matrix: A square matrix $\boldsymbol{A}_{m\times m}$ is called a diagonal matrix if for all $i \neq j$, $a_{ij} = 0$.

Identity Matrix: A diagonal matrix with $a_{ii} = 1$. It is sometimes denoted with $\boldsymbol{I}_m$.

Matrices in R

Creating a Matrix

```
Mat=matrix(c(a1,…,an),nrow=,ncol=,byrow=)

dim(Mat) gives the matrix dimension
```

```
> mat=matrix(c(1,2,3,4),2,2,byrow=1)
> mat
     [,1] [,2]
[1,]    1    2
[2,]    3    4
```

```
Zero Matrix: z= matrix(0,nrow=,ncol=)
```

```
> matrix(0, ncol = 4, nrow = 6)
```

```
Diagonal Matrix: d=diag(c(a1,…,an))
```

```
> diag(c(1,4,6))
```

```
Identity Matrix: I=diag(n)
```

```
> diag(3)
```

Matrix Addition and Scalar Multiplication

For any two matrices $\boldsymbol{A} = [a_{ij}]_{m \times n}$ and $\boldsymbol{B} = [b_{ij}]_{m \times n}$:

- Matrix Addition: $\boldsymbol{A} + \boldsymbol{B} = [a_{ij} + b_{ij}]_{m \times n}$ (In R use: `A+B`)
- Scalar Multiplication: $k\boldsymbol{A} = [ka_{ij}]_{m \times n}$, $k \in R$ (In R use: `k*A`)

Example:

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$3 \times \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 0 & 3 \end{bmatrix}$$

Matrix Addition and Scalar Multiplication

Properties

Matrix Addition:

- Commutativity: $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$
- Associativity: $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$
- Identity: $\boldsymbol{A}+\boldsymbol{O}=\boldsymbol{O}+\boldsymbol{A}=\boldsymbol{A}$

Scalar Multiplication:

- Distributivity: $k(\boldsymbol{A}+\boldsymbol{B})=k\boldsymbol{B}+k\boldsymbol{A}$
- Distributivity: $(k+l)\boldsymbol{A}=k\boldsymbol{A}+l\boldsymbol{A}$
- Associativity: $k(l\boldsymbol{A})=(kl)\boldsymbol{A}$

- Remark: $\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-1)\boldsymbol{B}=\left[a_{ij}-b_{ij}\right]_{m\times n}$

Transpose of A Matrix

For $\boldsymbol{A} \in \boldsymbol{R}^{m\times n}$, the matrix $\boldsymbol{B} \in \boldsymbol{R}^{n\times m}$ with $b_{ij} = a_{ji}$ is called the transpose of $\boldsymbol{A}$. We write $\boldsymbol{B} = \boldsymbol{A}^{\boldsymbol{T}}$. (In R, use: `t(A)`)

Example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow \boldsymbol{A}^{\boldsymbol{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties:

- $\left(\boldsymbol{A}^{\boldsymbol{T}}\right)^{\boldsymbol{T}} = \boldsymbol{A}$
- $(\boldsymbol{A} + \boldsymbol{B})^{\boldsymbol{T}} = \boldsymbol{A}^{\boldsymbol{T}} + \boldsymbol{B}^{\boldsymbol{T}}$
- $\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{A}$, whenever $\boldsymbol{A}$ symmetric

Example: If $\boldsymbol{A} = \begin{bmatrix} 6 & 0 \\ 2 & -1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 3 & -3 \\ 1 & 2 \end{bmatrix}$, find $\boldsymbol{A}^{\boldsymbol{T}} - 2\boldsymbol{B}$

Solution:

$$\boldsymbol{A}^{\boldsymbol{T}} - 2\boldsymbol{B} = \begin{bmatrix} 6 & 2 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 6 & -6 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ -2 & -5 \end{bmatrix}$$

Matrix Multiplication

Matrix Multiplication: For any two matrices $\boldsymbol{A} = [a_{ij}]_{m\times n}$ and $\boldsymbol{B} = [b_{ij}]_{n\times p}$, the product $\boldsymbol{AB}$ is the $m \times p$ matrix with the entry defined as

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = (row\ i\ of\ \boldsymbol{A}) \cdot (column\ j\ of\ \boldsymbol{B})$$

In R use: `A %*% B`

Properties of Matrix Multiplication:

- $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$ (Associative property)
- $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$ (Distributive property)
- $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$ (Distributive property)
- $k(\boldsymbol{AB}) = \boldsymbol{A}(k\boldsymbol{B})$ (for any scalar k)
- $(\boldsymbol{AB})^{\boldsymbol{T}} = \boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}}$
- Often $\boldsymbol{AB} \neq \boldsymbol{BA}$ (Not Communicative Law)

Example

Matrix Multiplication

a. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} =$

b. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} =$

c. $\begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & 1 \\ 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 5 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix} =$

Solution

a. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$

b. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 12 \\ 3 & 18 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & 1 \\ 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 5 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 16 & -3 & 11 \\ 10 & -1 & 0 \\ -7 & 4 & 10 \end{bmatrix}$

Example

The Cost Vector

Given the price and the quantities, calculate the total cost.

$$\boldsymbol{P} = [2\ 3\ 4], \quad \boldsymbol{Q} = \begin{array}{l} \begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix} \end{array} \begin{array}{l} \text{units of A} \\ \text{units of B} \\ \text{units of C} \end{array}$$

Solution: The cost vector is

$$\boldsymbol{PQ} = [2\ 3\ 4]\begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix} = [2 \times 7 + 3 \times 5 + 4 \times 11] = [73]$$

Example

Linear Equation

Write the system of equations in matrix form by using matrix multiplication.

$$\begin{cases} 2x_1 + 5x_2 = 12 \\ 8x_1 + 3x_2 = 14 \end{cases}$$

Solution

Let

$$\boldsymbol{A} = \begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$$

then the single matrix equation is

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$$

Matrix Reduction

Solving Systems by Reducing Matrices

Elementary Row Operations

1. Interchanging two rows of a matrix
2. Multiplying a row of a matrix by a nonzero number
3. Adding a multiple of one row of a matrix to a different row of that matrix

Notation	Corresponding Row Operation
$\boldsymbol{R_i} \leftrightarrow \boldsymbol{R_j}$	Interchange rows $\boldsymbol{R_i}$ and $\boldsymbol{R_j}$.
$k\boldsymbol{R_i}$	Multiply row $\boldsymbol{R_i}$ by a nonzero constant k.
$k\boldsymbol{R_i} + \boldsymbol{R_j}$	Add k times row $\boldsymbol{R_i}$ to row $\boldsymbol{R_j}$ (but leave $\boldsymbol{R_j}$ unchanged).

Properties of a Reduced Matrix

A matrix is said to be a reduced matrix if all the following are true:

- All zero-rows at the bottom.
- For each nonzero-row, leading entry is 1 and all other entries in the column of the leading entry are 0.
- Leading entry in each row is to the right of the leading entry in any row above it.

Remark:

- A leading entry is the first nonzero entry in a nonzero-row
- Each matrix is equivalent to a unique reduced matrix.

Example

Reduced Matrices

For each of the following matrices, determine whether it is reduced or not reduced.

a. $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution

a. Not reduced because the leading entry in the second row is not 1

b. Reduced

c. Not reduced as the leading entry in the second row is not to the right of the leading entry in the first row

d. Reduced

e. Not reduced because the second row, which is a zero-row, is not at the bottom of the matrix

f. Reduced

Reducing a Matrix

The Strategy for Reducing a Matrix

- Get the leading entry to be a 1 in the first row, the leading entry a 1 in the second row, and so on, until we arrive at a zero-row, if there are any.
- Work from left to right, because the leading entry in each row must be to the left of all other leading entries in the rows below it.

Example: Reduce the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 3 & -6 & -3 & 0 \\ 6 & -12 & 2 & 11 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 3 & -6 & -3 & 0 \\ 6 & -12 & 2 & 11 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & -6 & -3 & 0 \\ 0 & 0 & 1 & 2 \\ 6 & -12 & 2 & 11 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 2 \\ 6 & -12 & 2 & 11 \end{bmatrix} \xrightarrow{-6R_1+R_3} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 8 & 11 \end{bmatrix}$$

$$\xrightarrow{R_2+R_1} \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -4 & 11 \end{bmatrix} \xrightarrow{-8R_2+R_3} \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_3} \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_3+R_2} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solving a System by Reduction

Elimination does the same row operation to $\boldsymbol{A}$ and to $\boldsymbol{b}$. We can include $\boldsymbol{b}$ as an extra column and follow it through elimination. The matrix $\boldsymbol{A}$ is augmented by an extra column $\boldsymbol{b}$:

$$[\boldsymbol{A}|\boldsymbol{b}] \to \cdots \to [\boldsymbol{R}|\boldsymbol{b}']$$

Example: By using matrix reduction, solve the system

$$\begin{cases} 2x + 3y = -1 \\ 2x + y = 5 \\ x + y = 1 \end{cases}$$

Solution

Reducing the augmented coefficient matrix of the system,

$$\left[\begin{array}{cc|c} 2 & 3 & -1 \\ 2 & 1 & 5 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 5 \\ 2 & 3 & -1 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 2 & 3 & -1 \end{array}\right] \xrightarrow{-2R_1 + R_3} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{array}\right] \xrightarrow{-R_2 + R_3} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

We have $\begin{cases} x = 4 \\ y = -3 \end{cases}$

Example

Parametric Form of a Solution

Using matrix reduction, solve

$$\begin{cases} 2x_1 + 3x_2 + 2x_3 + 6x_4 = 10 \\ x_2 + 2x_3 + x_4 = 2 \\ 3x_1 - 3x_3 + 6x_4 = 9 \end{cases}$$

Solution

Reducing the augmented coefficient matrix of the system,

$$\left[\begin{array}{cccc|c} 2 & 3 & 2 & 6 & 10 \\ 0 & 1 & 2 & 1 & 2 \\ 3 & 0 & -3 & 6 & 9 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & 3/2 & 1 & 3 & 5 \\ 0 & 1 & 2 & 1 & 2 \\ 3 & 0 & -3 & 6 & 9 \end{array}\right] \xrightarrow{-3R_1+R_3} \left[\begin{array}{cccc|c} 1 & 3/2 & 1 & 3 & 5 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & -9/2 & -6 & -3 & -6 \end{array}\right]$$

$$\xrightarrow{-\frac{3}{2}R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3/2 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & -9/2 & -6 & -3 & -6 \end{array}\right] \xrightarrow{\frac{9}{2}R_2+R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3/2 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 3 & 3/2 & 3 \end{array}\right] \xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3/2 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1/2 & 1 \end{array}\right]$$

$$\xrightarrow{2R_3+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5/2 & 4 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1/2 & 1 \end{array}\right] \xrightarrow{-2R_3+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5/2 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1 \end{array}\right]$$

Solution (cont'd)

This matrix is reduced and corresponds to the system with x_4 being a free variable. The answer can be written as:

$$\begin{cases} x_1 + \frac{5}{2}x_4 = 4 \\ x_2 = 0 \\ x_3 + \frac{1}{2}x_4 = 1 \end{cases} \Rightarrow \boldsymbol{x} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{5}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Definition: The variables whose columns do not have a leading entry are called free variables. The variables whose columns have a leading entry are called pivot variables.

Matrix Inverse

Inverses

The rank of a matrix $\boldsymbol{A}$ is the number of its leading entries, k. It is equal to the number of nonzero rows of $\boldsymbol{A}$. It is also equal to the number of linearly independent columns as well as the number of the linearly independent rows of $\boldsymbol{A}$.

$\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is called full rank if $k = n$. If $\boldsymbol{A}$ is full-rank, there exists another matrix $\boldsymbol{A}^{-1}$ such that

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

$\boldsymbol{A}^{-1}$ is called the inverse of $\boldsymbol{A}$ and $\boldsymbol{A}$ is called an invertible matrix.

Example: Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ and $\boldsymbol{C} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$. Determine whether $\boldsymbol{C}$ is an inverse of $\boldsymbol{A}$.

Solution:

$$\boldsymbol{C}\boldsymbol{A} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

Thus, matrix $\boldsymbol{C}$ is an inverse of $\boldsymbol{A}$.

Method to Find the Inverse of a Matrix

If $\boldsymbol{A}$ is an $n \times n$ matrix, form the $n \times (2n)$ matrix $[\boldsymbol{A}|\boldsymbol{I}]$ and perform elementary row operations until the first n columns form a reduced matrix. Assume that the result is $[\boldsymbol{R}|\boldsymbol{B}]$ so that we have

$$[\boldsymbol{A}|\boldsymbol{I}] \to \cdots \to [\boldsymbol{R}|\boldsymbol{B}]$$

When matrix is reduced,

- If $\boldsymbol{R} = \boldsymbol{I}$, $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1} = \boldsymbol{B}$.
- If $\boldsymbol{R} \neq \boldsymbol{I}$, $\boldsymbol{A}$ is not invertible, meaning that $\boldsymbol{A}^{-1}$ does not exist.

Remark: To determine the inverse matrix in R you can use `solve(A)`

Example

Finding The Inverse by Reducing

Find the inverse of $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ if it exists.

Solution

$$[\boldsymbol{A}|\boldsymbol{I}] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1/2 \end{array}\right]$$

Matrix $\boldsymbol{A}$ is invertible where

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$$

Remark:

To create the segmented matrix $[\boldsymbol{A}|\boldsymbol{I}]$ in R type this code:

```
A=matrix(c(1,0,2,2),2,2,byrow=1)

I=diag(2)

AA=cbind(A,I)
```

To find $\boldsymbol{A}^{-1}$ directly type: `solve(A)`

Example 5

Using the Inverse to Solve a System

Solve the system by finding the inverse of the coefficient matrix.

$$\begin{cases} x_1 - 2x_3 = 1 \\ 4x_1 - 2x_2 + x_3 = 2 \\ x_1 + 2x_2 - 10x_3 = -1 \end{cases}$$

Solution

We have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -2 \\ 4 & -2 & 1 \\ 1 & 2 & -10 \end{bmatrix}$$

For inverse,

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -9 & 2 & 2 \\ -\frac{41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{bmatrix}$$

The solution is given by $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{B}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9 & 2 & 2 \\ -\frac{41}{2} & 4 & \frac{9}{2} \\ -5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -17 \\ -4 \end{bmatrix}$$

Remark: To find the solution $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{B}$ in R use: `solve(A,b)`