PROBLEMS 9.1

In Problems 1–4, the distribution of the random variable X is given. Determine μ, Var(X), and σ. In Problem 1, construct the probability histogram. In Problem 2, graph the distribution.

1. $f(0) = 0.2, f(1) = 0.3, f(2) = 0.3, f(3) = 0.2$

2. $f(4) = 0.4, f(5) = 0.6$

3. See Figure 9.4.

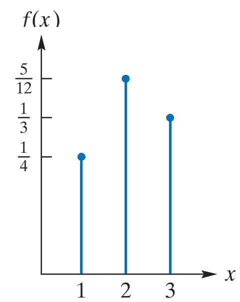


FIGURE 9.4

4. See Figure 9.5.

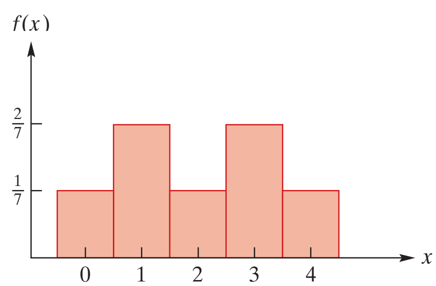


FIGURE 9.5

5. The random variable X has the following distribution:

x	$P(X = x)$
3	
5	0.3
6	0.2
7	0.4

(a) Find $P(X = 3)$; **(b)** Find μ; **(c)** Find σ^2

6. The random variable X has the following distribution:

x	$P(X = x)$
2	0.1
4	$5a$
6	4a

(a) Find $P(X = 4)$ and $P(X = 6)$; **(b)** Find μ.

In Problems 7–10, determine E(X), σ^2, and σ for the random variable X.

7. Coin Toss Three fair coins are tossed. Let X be the number of heads that occur.

8. Balls in a Basket A basket contains eight balls, each of which shows a number. Three balls show a 1; two show a 2; two show a 3; and one shows a 4. A ball is randomly selected and the number that shows, X, is observed.

9. Committee From a group of two women and three men, two persons are selected at random to form a committee. Let X be the number of men on the committee.

10. Jelly Beans in a Jar A jar contains two red and three green jelly beans. Two jelly beans are randomly withdrawn in succession with replacement, and the number of red jelly beans, X, is observed.

11. Marbles in a Bag A bag contains five red and three white marbles. Two marbles are randomly withdrawn in succession without replacement. Let X be the number of red marbles withdrawn. Find the distribution f for X.

12. Subcommittee From a state government committee consisting of four Whigs and six Tories, a subcommittee of three is to be randomly selected. Let X be the number of Whigs in the subcommittee. Find a general formula, in terms of combinations, that gives $P(X = x)$, where $x = 0, 1, 2, 3$.

13. Raffle A charitable organization is having a raffle for a single prize of \$7000. Each raffle ticket costs \$3, and 9000 tickets have been sold.

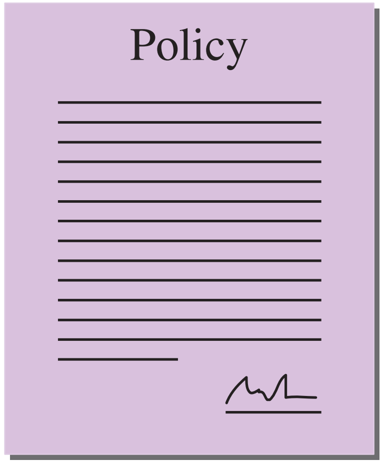
(a) Find the expected gain for the purchaser of a single ticket.
(b) Find the expected gain for the purchaser of two tickets.

14. Coin Game Consider the following game. You are to toss three fair coins. If three heads or three tails turn up, your friend pays you \$10. If either one or two heads turn up, you must pay your friend \$6. What are your expected winnings or losses per game?

15. Earnings A landscaper earns \$200 per day when working and loses \$30 per day when not working. If the probability of working on any day is $\frac{4}{7}$, find the landscaper's expected daily earnings.

16. Fast-Food Restaurant A fast-food chain estimates that if it opens a restaurant in a shopping center, the probability that the restaurant is successful is 0.72. A successful restaurant earns an annual profit of \$120,000; a restaurant that is not successful loses \$36,000. What is the expected gain to the chain if it opens a restaurant in a shopping center?

17. Insurance An insurance company offers a hospitalization policy to individuals in a certain group. For a one-year period, the company will pay \$100 per day, up to a maximum of five days, for each day the policyholder is hospitalized. The company estimates that the probability that any person in this group is hospitalized for exactly one day is 0.001; for exactly two days, 0.002; for exactly three days, 0.003; for exactly four days, 0.004; and for five or more days, 0.008. Find the expected gain per policy to the company if the annual premium is \$10.



18. Demand The following table for a small car rental company gives the probability that x cars are rented daily:

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.05	0.05	0.10	0.25	0.20	0.20	0.15	0.10	0.05

Determine the expected daily demand for their cars.

19. Insurance Premium In Example 3, if the company wants an expected gain of \$50 per policy, determine the annual premium.

20. Roulette In the game of roulette, there is a wheel with 37 slots numbered with the integers from 0 to 36, inclusive. A player bets \$1 (for example) and chooses a number. The wheel is spun and a ball rolls on the wheel. If the ball lands in the slot showing the chosen number, the player receives the \$1 bet plus \$35. Otherwise, the player loses the \$1 bet. Assume that all numbers are equally likely, and determine the expected gain or loss per play.

21. Coin Game Suppose that you pay \$2.50 to play a game in which two fair coins are tossed. If n heads occur, you receive $2n$ dollars. What is your expected gain (or loss) on each play? The game is said to be *fair* to you when your expected gain is \$0. What should you pay to play if this is to be a fair game?

Objective

To develop the binomial distribution and relate it to the binomial theorem.

9.2 The Binomial Distribution

Binomial Theorem

Later in this section we will see that the terms in the expansion of a power of a binomial are useful in describing the distributions of certain random variables. It is worthwhile, therefore, first to discuss the *binomial theorem,* which is a formula for expanding $(a + b)^n$, where n is a positive integer.

Regardless of n, there are patterns in the expansion of $(a + b)^n$. To illustrate, we consider the *cube* of the binomial $a + b$. By successively applying the distributive law, we have

$$\begin{aligned}(a + b)^3 &= [(a + b)(a + b)](a + b)\\ &= [a(a + b) + b(a + b)](a + b)\\ &= [aa + ab + ba + bb](a + b)\\ &= aa(a + b) + ab(a + b) + ba(a + b) + bb(a + b)\\ &= aaa + aab + aba + abb + baa + bab + bba + bbb \qquad (1)\end{aligned}$$

so that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad (2)$$

Three observations can be made about the right side of Equation (2). First, notice that the number of terms is four, which is one more than the power to which $a + b$ is raised (3). Second, the first and last terms are the *cubes* of a and b; the powers of a *decrease* from left to right (from 3 to 0); and the powers of b *increase* (from 0 to 3). Third, for each term, the sum of the exponents of a and b is 3, which is the power to which $a + b$ is raised.

Let us now focus on the coefficients of the terms in Equation (2). Consider the coefficient of the ab^2-term. It is the number of terms in Equation (1) that involve exactly two b's, namely, 3. But let us see *why* there are three terms that involve two b's. Notice

PROBLEMS 9.2

In Problems 1–4, determine the distribution f for the binomial random variable X if the number of trials is n and the probability of success on any trial is p. Also, find μ and σ.

1. $n = 2, p = \frac{1}{5}$

2. $n = 3, p = \frac{1}{2}$

3. $n = 3, p = \frac{2}{3}$

4. $n = 5, p = 0.3$

In Problems 5–10, determine the given probability if X is a binomial random variable, n is the number of trials, and p is the probability of success on any trial.

5. $P(X = 3)$; $n = 4, p = \frac{1}{3}$

6. $P(X = 2)$; $n = 5, p = \frac{1}{3}$

7. $P(X = 2)$; $n = 4, p = \frac{4}{5}$

8. $P(X = 4)$; $n = 7, p = 0.2$

9. $P(X > 3)$; $n = 5, p = 0.3$

10. $P(X \geq 3)$; $n = 4, p = \frac{4}{5}$

11. Coin A fair coin is tossed 11 times. What is the probability that exactly eight heads occur?

12. Multiple-Choice Quiz Each question in a six-question multiple-choice quiz has four choices, only one of which is correct. If a student guesses at all six questions, find the probability that exactly three will be correct.

13. Marbles A jar contains five red and seven green marbles. Four marbles are randomly withdrawn in succession with replacement. Determine the probability that exactly two of the marbles withdrawn are green.

14. Cards From a deck of 52 playing cards, 4 cards are randomly selected in succession *with replacement*. Determine the probability that at least two cards are jacks.

15. Quality Control A manufacturer produces electrical switches, of which 3% are defective. From a production run of 60,000 switches, five are randomly selected and each is tested. Determine the probability that the sample contains exactly three defective switches. Round your answer to three decimal places. Assume that the four trials are independent and that the number of defective switches in the sample has a binomial distribution.

16. Coin A coin is biased so that $P(H) = 0.2$ and $P(T) = 0.8$. If X is the number of heads in three tosses, determine a formula for $P(X = x)$.

17. Coin A biased coin is tossed three times in succession. The probability of heads on any toss is $\frac{1}{4}$. Find the probability that **(a)** exactly two heads occur and **(b)** two or three heads occur.

18. Cards From an ordinary deck of 52 playing cards, 7 cards are randomly drawn in succession with replacement. Find the probability that there are **(a)** exactly four hearts and **(b)** at least four hearts.

19. Quality Control In a large production lot of smartphones, it is believed that 0.015 are defective. If a sample of 10 is randomly selected, find the probability that less than 2 will be defective.

20. High-Speed Internet For a certain large population, the probability that a randomly selected person has access to high-speed Internet is 0.8. If four people are selected at random, find the probability that at least three have access to high-speed Internet.

21. Baseball The probability that a certain baseball player gets a hit is 0.300. Find the probability that if he goes to bat four times, he will get at least one hit.

22. Stocks A financial advisor claims that 60% of the stocks that he recommends for purchase increase in value. From a list of 200 recommended stocks, a client selects 4 at random. Determine the probability, rounded to two decimal places, that at least 2 of the chosen stocks increase in value. Assume that the selections of the stocks are independent trials and that the number of stocks that increase in value has a binomial distribution.

23. Genders of Children If a family has five children, find the probability that at least two are girls. (Assume that the probability that a child is a girl is $\frac{1}{2}$.)

24. If X is a binomially distributed random variable with $n = 100$ and $p = \frac{1}{3}$, find $\text{Var}(X)$.

25. Suppose X is a binomially distributed random variable such that $\mu = 2$ and $\sigma^2 = \frac{3}{2}$. Find $P(X = 2)$.

26. Quality Control In a production process, the probability of a defective unit is 0.06. Suppose a sample of 15 units is selected at random. Let X be the number of defectives.

(a) Find the expected number of defective units.
(b) Find $\text{Var}(X)$.
(c) Find $P(X \leq 1)$. Round your answer to two decimal places.

Objective

To develop the notions of a Markov chain and the associated transition matrix. To find state vectors and the steady-state vector.

9.3 Markov Chains

We conclude this chapter with a discussion of a special type of stochastic process called a *Markov chain*, after the Russian mathematician Andrei Markov (1856–1922).

Markov Chain

A **Markov chain** is a sequence of trials of an experiment in which the possible outcomes of each trial remain the same from trial to trial, are finite in number, and have probabilities that depend only upon the outcome of the previous trial.

To illustrate a Markov chain, we consider the following situation. Imagine that a small town has only two service stations—say, stations 1 and 2—that handle the servicing needs of the town's automobile owners. (These customers form the population under consideration.) Each time a customer needs car servicing, he or she must make a *choice* of which station to use.

Equations (4) and (5) can always be combined into a single matrix equation:

$$T^*Q = 0^*$$

where T^* is the $(k+1)\times k$ matrix obtained by pasting the row $[1 \quad 1 \quad \cdots \quad 1]$ to the top of the $k\times k$ matrix $T-I_k$ (where I_k is the $k\times k$ identity matrix) and 0^* is the $k+1$-column vector obtained by pasting a 1 to the top of the zero k-column vector. We can then find Q by reducing the augmented matrix $[T^* \mid 0^*]$. The next example will illustrate.

EXAMPLE 2 Steady-State Vector

For the demography problem of Example 1, in the long run, what percentage of county residents will live in each region?

Solution: The population distribution in the long run is given by the steady-state vector Q, which we now proceed to find. The matrix T for this example was shown to be

$$\begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$$

so that $T-I$ is

$$\begin{bmatrix} -0.3 & 0.1 & 0.2 \\ 0.2 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.3 \end{bmatrix}$$

and $[T^* \mid 0^*]$ is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ -0.3 & 0.1 & 0.2 & 0 \\ 0.2 & -0.2 & 0.1 & 0 \\ 0.1 & 0.1 & -0.3 & 0 \end{array}\right]$$

which reduces to

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 5/16 \\ 0 & 1 & 0 & 7/16 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

showing that the steady-state vector $Q = \begin{bmatrix} 5/16 \\ 7/16 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0.3125 \\ 0.4375 \\ 0.2500 \end{bmatrix}$. Thus, in the long run, the percentages of county residents living in regions 1, 2, and 3 are 31.25%, 43.75%, and 25%, respectively.

Now Work Problem 37 ◁

PROBLEMS 9.3

In Problems 1–6, can the given matrix be a transition matrix for a Markov chain?

1. $\begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{3}{2} & \frac{1}{3} \end{bmatrix}$

2. $\begin{bmatrix} 0.1 & 1 \\ 0.9 & 0 \end{bmatrix}$

3. $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{3} \\ -\frac{1}{4} & \frac{5}{8} & \frac{1}{3} \\ \frac{3}{4} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$

4. $\begin{bmatrix} 0.2 & 0.6 & 0.5 \\ 0.7 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}$

5. $\begin{bmatrix} 0.2 & 0.1 & 0.7 \\ 0.1 & 0.2 & 0 \\ 0.7 & 0.7 & 0.3 \end{bmatrix}$

6. $\begin{bmatrix} 0.5 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.4 \end{bmatrix}$

In Problems 7–10, a transition matrix for a Markov chain is given. Determine the values of the letter entries.

7. $\begin{bmatrix} \frac{2}{3} & b \\ a & \frac{1}{4} \end{bmatrix}$

8. $\begin{bmatrix} a & b \\ \frac{5}{12} & a \end{bmatrix}$

9. $\begin{bmatrix} 0.1 & a & a \\ a & 0.2 & b \\ 0.2 & b & c \end{bmatrix}$

10. $\begin{bmatrix} a & b & c \\ a & \frac{1}{4} & b \\ a & a & a \end{bmatrix}$

In Problems 11–14, determine whether the given vector could be a state vector for a Markov chain.

11. $\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$

12. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

13. $\begin{bmatrix} 0.2 \\ 0.7 \\ 0.5 \end{bmatrix}$ **14.** $\begin{bmatrix} 0.1 \\ 1.1 \\ 0.2 \end{bmatrix}$

In Problems 15–20, a transition matrix T and an initial state vector X_0 are given. Compute the state vectors X_1, X_2, and X_3.

15. $T = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{3}{4} & 1 \end{bmatrix}$ $X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

16. $T = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$ $X_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

17. $T = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$ $X_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$

18. $T = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}$ $X_0 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$

19. $T = \begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.7 & 0.4 & 0.4 \end{bmatrix}$ $X_0 = \begin{bmatrix} 0.2 \\ 0.1 \\ 0.7 \end{bmatrix}$

20. $T = \begin{bmatrix} 0 & 0.1 & 0.2 & 0.7 \\ 0.1 & 0.2 & 0.7 & 0 \\ 0.2 & 0.7 & 0 & 0.1 \\ 0.7 & 0 & 0.1 & 0.2 \end{bmatrix}$ $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

In Problems 21–24, a transition matrix T is given.

(a) Compute T^2 and T^3.
(b) What is the probability of going to state 2 from state 1 after two steps?
(c) What is the probability of going to state 1 from state 2 after three steps?

21. $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ **22.** $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix}$

23. $\begin{bmatrix} 0 & 0.5 & 0.3 \\ 1 & 0.4 & 0.3 \\ 0 & 0.1 & 0.4 \end{bmatrix}$ **24.** $\begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.7 & 0.4 & 0.4 \end{bmatrix}$

In Problems 25–30, find the steady-state vector for the given transition matrix.

25. $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}$ **26.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

27. $\begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{2}{5} \end{bmatrix}$ **28.** $\begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix}$

29. $\begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.7 & 0.4 & 0.4 \end{bmatrix}$ **30.** $\begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.4 & 0.3 & 0.5 \\ 0.5 & 0.5 & 0.4 \end{bmatrix}$

31. Spread of Flu A flu has attacked a college dorm that has 200 students. Suppose the probability that a student having the flu will still have it 4 days later is 0.1. However, for a student who does not have the flu, the probability of having the flu 4 days later is 0.2.

(a) Find a transition matrix for this situation.
(b) If 120 students now have the flu, how many students (to the nearest integer) can be expected to have the flu 8 days from now? 12 days from now?

32. Physical Fitness A physical-fitness center has found that, of those members who perform high-impact exercising on one visit, 55% will do the same on the next visit and 45% will do low-impact exercising. Of those who perform low-impact exercising on one visit, 75% will do the same on the next visit and 25% will do high-impact exercising. On the last visit, suppose that 65% of members did high-impact exercising and 35% did low-impact exercising. After two more visits, what percentage of members will be performing high-impact exercising?

33. Newspapers In a certain area, two daily newspapers are available. It has been found that if a customer buys newspaper A on one day, then the probability is 0.3 that he or she will change to the other newspaper the next day. If a customer buys newspaper B on one day, then the probability is 0.6 that he or she will buy the same newspaper the next day.

(a) Find the transition matrix for this situation.
(b) Find the probability that a person who buys A on Monday will buy A on Wednesday.

34. Video Rentals A video rental store has three locations in a city. A video can be rented from any of the three locations and returned to any of them. Studies show that videos are rented from one location and returned to a location according to the probabilities given by the following matrix:

$$\begin{array}{cc} \text{Returned to} & \text{Rented from} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.9 & 0.1 \\ 0.1 & 0 & 0.8 \end{bmatrix} \end{array}$$

Suppose that 30% of the videos are initially rented from location 1, 30% from 2, and 40% from 3. Find the percentages of videos that can be expected to be returned to each location:

(a) After this rental
(b) After the next rental

35. Voting In a certain city, voter preference was analyzed according to party affiliation: Liberal, Conservative, and other. It was found that on a year-to-year basis, the probability that a voter switches to Conservative from Liberal is 0; to other from Liberal, 0.3; to Liberal from Conservative, 0.1; to other from Conservative, 0.2; to Liberal from other, 0.3; and to Conservative from other, 0.1.

(a) Find a transition matrix for this situation.
(b) What is the probability that a current Conservative voter will be Liberal two years from now?
(c) If 40% of the present voters are Liberal and 30% are Conservative, what percentage can be expected to be Conservative one year from now?

36. Demography The residents of a certain region are classified as urban (U), suburban (S), or rural (R). A marketing firm has found that over successive 5-year periods, residents shift from one classification to another according to the probabilities given by the following matrix:

$$\begin{array}{c} \\ U \\ S \\ R \end{array}\begin{array}{c} \begin{array}{ccc} U & S & R \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.8 \end{bmatrix}\end{array}$$

(a) Find the probability that a suburban resident will be a rural resident in 15 years.
(b) Suppose the initial population of the region is 50% urban, 25% suburban, and 25% rural. Determine the expected population distribution in 15 years.

37. Long-Distance Telephone Service A major long-distance telephone company (company A) has studied the tendency of telephone users to switch from one carrier to another. The company believes that over successive six-month periods, the probability that a customer who uses A's service will switch to a competing service is 0.2 and the probability that a customer of any competing service will switch to A is 0.3.
(a) Find a transition matrix for this situation.
(b) If A presently controls 70% of the market, what percentage can it expect to control six months from now?
(c) What percentage of the market can A expect to control in the long run?

38. Automobile Purchases In a certain region, a study of car ownership was made. It was determined that if a person presently owns a Ford, then the probability that the next car the person buys is also a Ford is 0.75. If a person does not presently own a Ford, then the probability that the person will buy a Ford on the next car purchase is 0.35.
(a) Find the transition matrix for this situation.
(b) In the long run, what proportion of car purchases in the region can be expected to be Fords?

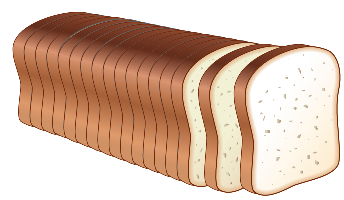
39. Laboratory Mice Suppose 100 mice are in a two-compartment cage and are free to move between the compartments. At regular time intervals, the number of mice in each compartment is observed. It has been found that if a mouse is in compartment 1 at one observation, then the probability that the mouse will be in compartment 1 at the next observation is $\frac{3}{5}$. If a mouse is in compartment 2 at one observation, then the probability that the mouse will be in compartment 2 at the next observation is $\frac{2}{5}$. Initially, suppose that 50 mice are placed into each compartment.
(a) Find the transition matrix for this situation.
(b) After two observations, what percentage of mice (rounded to two decimal places) can be expected to be in each compartment?
(c) In the long run, what percentage of mice can be expected in each compartment?

40. Vending Machines If a pop machine fails to deliver, people often warn bystanders, "Don't put your money in that thing; I tried it and it didn't work!" Suppose that if a vending machine is working properly one time, then the probability that it will work properly the next time is 0.85. On the other hand, suppose that if the machine is not working properly one time, then the probability that it will not work properly the next time is 0.95.
(a) Find a transition matrix for this situation.
(b) Suppose that four people line up at a pop machine that is known to have worked just before they arrived. What is the probability that the fourth person will receive a pop? (Assume nobody makes more than one attempt.)
(c) If there are 40 such pop machines on a university campus and they are not getting regular maintenance, how many, in the long run, do you expect to work properly?

41. Advertising A supermarket chain sells bread from bakeries A and B. Presently, A accounts for 50% of the chain's daily bread sales. To increase sales, A launches a new advertising campaign. The bakery believes that the change in bread sales at the chain will be based on the following transition matrix:

$$\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}\end{array}$$

(a) Find the steady-state vector.
(b) In the long run, by what percentage can A expect to increase present sales at the chain? Assume that the total daily sales of bread at the chain remain the same.



42. Bank Branches A bank with three branches, A, B, and C, finds that customers usually return to the same branch for their banking needs. However, at times a customer may go to a different branch because of a changed circumstance. For example, a person who usually goes to branch A may sometimes deviate and go to branch B because the person has business to conduct in the vicinity of branch B. For customers of branch A, suppose that 80% return to A on their next visit, 10% go to B, and 10% go to C. For customers of branch B, suppose that 70% return to B on their next visit, 20% go to A, and 10% go to C. For customers of branch C, suppose that 70% return to C on their next visit, 20% go to A, and 10% go to B.

(a) Find a transition matrix for this situation.
(b) If a customer most recently went to branch B, what is the probability that the customer returns to B on the second bank visit from now?
(c) Initially, suppose 200 customers go to A, 200 go to B, and 100 go to C. On their next visit, how many can be expected to go to A? To B? To C?
(d) Of the initial 500 customers, in the long run how many can be expected to go to A? To B? To C?

43. Show that the transition matrix $T = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$ is regular.
(*Hint:* Examine the entries in T^2.)

44. Show that the transition matrix $T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is not regular.

Chapter 9 Review

Important Terms and Symbols

		Examples
Section 9.1	**Discrete Random Variables and Expected Value**	
	discrete random variable probability function histogram	Ex. 2, p. 425
	mean, μ expected value, $E(X)$	Ex. 3, p. 427
	variance, $\mathrm{Var}(X)$ standard deviation, σ	Ex. 4, p. 429
Section 9.2	**The Binomial Distribution**	
	binomial theorem binomial coefficients	Ex. 1, p. 433
	Bernoulli trials binomial experiment binomial distribution	Ex. 2, p. 435
Section 9.3	**Markov Chains**	
	Markov chain transition matrix, T state vector, X_n	Ex. 1, p. 441
	regular transition matrix steady-state vector, Q	Ex. 2, p. 444

Summary

If X is a discrete random variable and f is the function such that $f(x) = P(X = x)$, then f is called the probability function, or distribution, of X. In general,

$$\sum_x f(x) = 1$$

The mean, or expected value, of X is the long-run average of X and is denoted μ or $E(X)$:

$$\mu = E(X) = \sum_x xf(x)$$

The mean can be interpreted as a measure of the central tendency of X in the long run. A measure of the dispersion of X is the variance, denoted $\mathrm{Var}(X)$ and given by

$$\mathrm{Var}(X) = \sum_x (x-\mu)^2 f(x)$$

equivalently, by

$$\mathrm{Var}(X) = \left(\sum_x x^2 f(x)\right) - \mu^2$$

Another measure of dispersion of X is the standard deviation σ:

$$\sigma = \sqrt{\mathrm{Var}(X)}$$

If an experiment is repeated several times, then each performance of the experiment is called a trial. The trials are independent when the outcome of any single trial does not affect the outcome of any other. If there are only two possible outcomes (success and failure) for each independent trial, and the probabilities of success and failure do not change from trial to trial, then the experiment is called a binomial experiment. For such an experiment, if X is the number of successes in n trials, then the distribution f of X is called a binomial distribution, and

$$f(x) = P(X = x) = {}_nC_x p^x q^{n-x}$$

where p is the probability of success on any trial and $q = 1-p$ is the probability of failure. The mean μ and standard deviation σ of *this* X are given by

$$\mu = np \quad \text{and} \quad \sigma = \sqrt{npq}$$

A binomial distribution is intimately connected with the binomial theorem, which is a formula for expanding the nth power of a binomial, namely,

$$(a+b)^n = \sum_{i=0}^{n} {}_nC_i a^{n-i} b^i$$

for n a positive integer.

A Markov chain is a sequence of trials of an experiment in which the possible outcomes of each trial, which are called states, remain the same from trial to trial, are finite in number, and have probabilities that depend only upon the outcome of the previous trial. For a k-state Markov chain, if the probability of moving to state i from state j from one trial to the next is written T_{ij}, then the $k \times k$ matrix $T = [T_{ij}]$ is called the transition matrix for the Markov chain. The entries in the nth power of T also represent probabilities; the entry in the ith row and jth column of T^n gives the probability of moving to state i from state j in n steps. A k-entry column vector in which the entry x_j is the probability of being in state j after the nth trial is called a state vector and is denoted X_n. The initial state probabilities are represented by the initial state vector X_0. The state vector X_n can be found by multiplying the previous state vector X_{n-1} on the left by the transition matrix T:

$$X_n = TX_{n-1}$$

Alternatively, X_n can be found by multiplying the initial state vector X_0 by T^n:

$$X_n = T^n X_0$$

If the transition matrix T is regular, that is, if there is a positive integer n such that all entries of T^n are strictly positive, then, as the number of trials n increases, X_n gets closer and closer to a vector Q, called the steady-state vector of T. If

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{bmatrix}$$

then the entries of Q indicate the long-run probability distribution of the states. The vector Q can be found by solving the matrix equation

$$T^*Q = 0^*$$

where T^* is the $(k+1)\times k$ matrix obtained by pasting the row $[1 \quad 1 \quad \cdots \quad 1]$ to the top of the $k \times k$ matrix $T - I_k$ (where I_k is the $k \times k$ identity matrix) and 0^* is the $k+1$-column vector obtained by pasting a 1 to the top of the zero k-column vector. Thus, we construct and reduce

$$\left[\begin{array}{c|c} 1 \cdots 1 & 1 \\ T - I & 0 \end{array}\right]$$

which, if T is regular, will result in

$$\left[\begin{array}{c|c} I & Q \\ 0 & 0 \end{array}\right]$$

Review Problems

In Problems 1 and 2, the distribution for the random variable X is given. Construct the probability histogram and determine μ, Var(X), and σ.

1. $f(1) = 0.2, f(2) = 0.5, f(3) = 0.3$

2. $f(0) = \frac{1}{6}$, $f(1) = \frac{1}{2}$, $f(2) = \frac{1}{3}$

3. Coin and Die A fair coin and a fair die are tossed. Let X be the number of dots that show plus the number of heads. Determine **(a)** the distribution f for X and **(b)** $E(X)$.

4. Cards Two cards from a standard deck of 52 playing cards are randomly drawn in succession without replacement, and the number of aces, X, is observed. Determine **(a)** the distribution f for X and **(b)** $E(X)$.

5. Card Game In a game, a player pays $0.25 to randomly draw 2 cards, with replacement, from a standard deck of 52 playing cards. For each ten that appears, the player receives $1. What is the player's expected gain or loss? Give your answer to the nearest cent.

6. Gas Station Profits An oil company determines that the probability that a gas station located along the Trans-Canada Highway is successful is 0.55. A successful station earns an annual profit of $160,000; a station that is not successful loses $15,000 annually. What is the expected gain to the company if it locates a station along the Trans-Canada Highway

7. Mail-Order Computers A mail-order computer company offers a 30-day money-back guarantee to any customer who is not completely satisfied with its product. The company realizes a profit of $200 for each computer sold, but assumes a loss of $100 for shipping and handling for each unit returned. The probability that a unit is returned is 0.08.
(a) What is the expected gain for each unit shipped?
(b) If the distributor ships 4000 units per year, what is the expected annual profit?

8. Lottery In a certain lottery, you pay $4.00 to choose one of 41 million number combinations. If that combination is drawn, you win $50 million. What is your expected gain (or loss) per play?

In Problems 9 and 10, determine the distribution f for the binomial random variable X if the number of trials is n and the probability of success on any trial is p. Also, find μ and σ.

9. $n = 4, p = 0.15$

10. $n = 5, p = \dfrac{1}{3}$

In Problems 11 and 12, determine the given probability if X is a binomial random variable, n is the number of trials, and p is the probability of success on any trial.

11. $P(X > 4); n = 6, p = \dfrac{2}{3}$

12. $P(X > 2); n = 6, p = \dfrac{2}{3}$

13. Die A pair of fair dice is rolled five times. Find the probability that exactly three of the rolls result in a face sum of 7.

14. Planting Success The probability that a certain type of bush survives planting is 0.9. If four bushes are planted, what is the probability that all of them die?

15. Coin A biased coin is tossed five times. The probability that a head occurs on any toss is $\frac{2}{5}$. Find the probability that at least two heads occur.

16. Jelly Beans A bag contains three red, four green, and thee black jelly beans. Five jelly beans are randomly withdrawn in succession with replacement. Find the probability that at least four of the withdrawn jelly beans are black.

In Problems 17 and 18, a transition matrix for a Markov chain is given. Determine the values of a, b, and c.

17. $\begin{bmatrix} 0.1 & 2a & a \\ a & b & b \\ 0.6 & b & c \end{bmatrix}$

18. $\begin{bmatrix} a & a & a \\ b & b & a \\ 0.4 & c & b \end{bmatrix}$

In Problems 19 and 20, a transition matrix T and an initial state vector X_0 for a Markov chain are given. Compute the state vectors X_1, X_2, and X_3.

19. $T = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.7 & 0.3 & 0.8 \end{bmatrix}$

$X_0 = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}$

20. $T = \begin{bmatrix} 0.4 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.5 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}$

$X_0 = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.6 \end{bmatrix}$

In Problems 21 and 22, a transition matrix T for a Markov chain is given.

(a) Compute T^2 and T^3.
(b) What is the probability of going to state 1 from state 2 after two steps?
(c) What is the probability of going to state 2 from state 1 after three steps?

21. $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

22. $\begin{bmatrix} 0 & 0.4 & 0.3 \\ 0 & 0.3 & 0.5 \\ 1 & 0.3 & 0.2 \end{bmatrix}$

In Problems 23 and 24, find the steady-state vector for the given transition matrix for a Markov chain.

23. $\begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix}$

24. $\begin{bmatrix} 0.4 & 0.4 & 0.3 \\ 0.3 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.4 \end{bmatrix}$

25. Automobile Market For a particular segment of the automobile market, the results of a survey indicate that 80% of people who own a Japanese car would buy a Japanese car the next time and 20% would buy a non-Japanese car. Of owners of non-Japanese cars, 40% would buy a non-Japanese car the next time and 60% would buy a Japanese car.

(a) Of those who currently own a Japanese car, what percentage will buy a Japanese car two cars later?
(b) If 60% of this segment currently own Japanese cars and 40% own non-Japanese cars, what will be the distribution for this segment of the market two cars from now?
(c) How will this segment be distributed in the long run?

26. Voting Suppose that the probabilities of voting for particular parties in a future election depend on the voting patterns in the previous election. For a certain province where there is a three-party political system, assume that these probabilities are contained in the matrix

$$T = [T_{ij}] = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 \\ 0.3 & 0.2 & 0.8 \end{bmatrix}$$

where T_{ij} is the probability that a voter will vote for party i in the next election if he or she voted for party j in the last election.

(a) At the last election, 50% of the electorate voted for party 1, 30% for party 2, and 20% for party 3. What is the expected percentage distribution of votes for the next election?
(b) In the long run, what is the percentage distribution of votes?