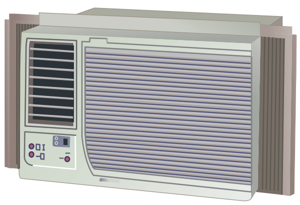
PROBLEMS 8.1

1. Production Process In a production process, a product goes through one of the assembly lines A, B, or C and then goes through one of the finishing lines D or E. Draw a tree diagram that indicates the possible production routes for a unit of the product. How many production routes are possible?

2. Air Conditioner Models A manufacturer produces air conditioners having 6000-, 8000-, and 10,000-BTU capacities. Each capacity is available with one- or two-speed fans. Draw a tree diagram that represents all types of models. How many types are there?



3. Dice Rolls A red die is rolled, and then a green die is rolled. Draw a tree diagram to indicate the possible results. How many results are possible?

4. Coin Toss A coin is tossed four times. Draw a tree diagram to indicate the possible results. How many results are possible?

In Problems 5–10, use the Basic Counting Principle.

5. Course Selection A student must take a mathematics course, a philosophy course, and a physics course. The available mathematics classes are category theory, measure theory, real analysis, and combinatorics. The philosophy possibilities are logical positivism, epistemology, and modal logic. The available physics courses are classical mechanics, electricity and magnetism, quantum mechanics, general relativity, and cosmology. How many three-course selections can the student make?

6. Auto Routes A person lives in city A and commutes by automobile to city B. There are five roads connecting A and B. **(a)** How many routes are possible for a round trip? **(b)** How many round-trip routes are possible if a different road is to be used for the return trip?

7. Dinner Choices At a restaurant, a complete dinner consists of an appetizer, an entree, a dessert, and a beverage. The choices for the appetizer are soup and salad; for the entree, the choices are chicken, fish, steak, and lamb; for the dessert, the choices are cherries jubilee, fresh peach cobbler, chocolate truffle cake, and blueberry roly-poly; for the beverage, the choices are coffee, tea, and milk. How many complete dinners are possible?

8. Multiple-Choice Exam In how many ways is it possible to answer a six-question multiple-choice examination if each question has four choices (and one choice is selected for each question)?

9. True–False Exam In how many ways is it possible to answer a 10-question true–false examination?

10. Canadian Postal Codes A Canadian postal code consists of a string of six characters, of which three are letters and three are digits, which begins with a letter and for which each letter is followed by a (single) digit. (For readability, the string is broken into strings of three. For example, M5W 1E6 is a valid postal code.) How many Canadian postal codes are possible? What percentage of these begin with M5W? What percentage end with 1E6?

In Problems 11–16, determine the values.

11. ${}_6P_3$ **12.** ${}_{95}P_1$ **13.** ${}_6P_6$

14. ${}_9P_4$ **15.** ${}_7P_4 \cdot {}_4P_2$ **16.** $\dfrac{{}_{99}P_5}{{}_{99}P_4}$

17. Compute 1000!/999! without using a calculator. Now try it with your calculator, using the factorial feature.

18. Determine $\dfrac{{}_nP_r}{n!}$.

In Problems 19–42, use any appropriate counting method.

19. Name of Firm Flynn, Peters, and Walters are forming an advertising firm and agree to name it by their three last names. How many names for the firm are possible?

20. Softball If a softball league has six teams, how many different end-of-the-season rankings are possible? Assume that there are no ties.

21. Contest In how many ways can a judge award first, second, and third prizes in a contest having eight contestants?

22. Matching-Type Exam On a history exam, each of six items in one column is to be matched with exactly one of eight items in another column. No item in the second column can be selected more than once. In how many ways can the matching be done?

23. Die Roll A die (with six faces) is rolled four times and the outcome of each roll is noted. How many results are possible?

24. Coin Toss A coin is tossed eight times. How many results are possible if the order of the tosses is considered?

25. Problem Assignment In a mathematics class with 12 students, the instructor wants homework problems 1, 3, 5, and 7 put on the board by four different students. In how many ways can the instructor assign the problems?

26. Combination Lock A combination lock has 26 different letters, and a sequence of three different letters must be selected for the lock to open. How many combinations are possible?

27. Student Questionnaire A university issues a questionnaire whereby each student must rank the four items with which he or she is most dissatisfied. The items are

tuition fees	professors
parking fees	cafeteria food
dormitory rooms	class sizes

The ranking is to be indicated by the numbers 1, 2, 3 and 4, where 1 indicates the item involving the greatest dissatisfaction and 4 the least. In how many ways can a student answer the questionnaire?

28. Die Roll A die is rolled three times. How many results are possible if the order of the rolls is considered and the second roll produces a number less than 3?

29. Letter Arrangements How many six-letter words from the letters in the word MEADOW are possible if no letter is repeated?

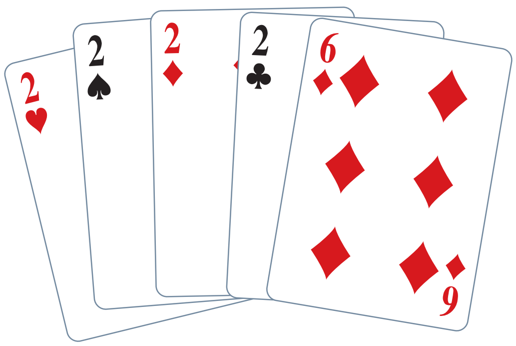
30. Letter Arrangements Using the letters in the word BREXIT, how many four-letter words are possible if no letter is repeated?

31. Book Arrangements In how many ways can five of seven books be arranged on a bookshelf? In how many ways can all seven books be arranged on the shelf?

32. Lecture Hall A lecture hall has five doors. In how many ways can a student enter the hall by one door and

(a) Exit by a different door?
(b) Exit by any door?

33. Poker Hand A poker hand consists of 5 cards drawn from a deck of 52 playing cards. The hand is said to be "four of a kind" if four of the cards have the same face value. For example, hands with four 10's or four jacks or four 2's are four-of-a-kind hands. How many such hands are possible?



34. Merchandise Choice In a merchandise catalog, a CD rack is available in the colors of black, red, yellow, gray, and blue. When placing an order for one CD rack, customers must indicate their first and second color choices. In how many ways can this be done?

35. Fast Food Order Four students go to a pizzeria and order a margharita, a diavalo, a Greek, and a meat-lover's — one item for each student. When the waiter brings the food to the table, she forgets which student ordered which item and simply places one before each student. In how many ways can she do this?

36. Group Photograph In how many ways can three men and two women line up for a group picture? In how many ways can they line up if a woman is to be at each end?

37. Club Officers A club has 12 members.

(a) In how many ways can the offices of president, vice president, secretary, and treasurer be filled if no member can serve in more than one office?
(b) In how many ways can the four offices be filled if the president and vice president must be different members?

38. Fraternity Names Suppose a fraternity is named by three Greek letters. (There are 24 letters in the Greek alphabet.)

(a) How many names are possible?
(b) How many names are possible if no letter can be used more than one time?

39. Basketball In how many ways can a basketball coach assign positions to her five-member team if two of the members are qualified for the center position and all five are qualified for all the other positions?

40. Car Names A European car manufacturer has three series of cars A, S, and R in sizes 3, 4, 5, 6, 7, and 8. Each car it makes potentially comes in a Komfort (K), Progressiv (P), or a Technik (T) trim package, with either an automatic (A) or a manual (M) transmission, and a 2-, 3-, or 5-litre engine. The manufacturer names its products with a string of letters and digits as suggested by these attributes in the order given. For example, the manufacturer speaks of the R4TA5. How many models can the manufacturer name using these criteria?

41. Baseball A baseball manager determines that, of his nine team members, three are strong hitters and six are weak. If the manager wants the strong hitters to be the first three batters in a batting order, how many batting orders are possible?

42. Signal Flags When at least one of four flags colored red, green, yellow, and blue is arranged vertically on a flagpole, the result indicates a signal (or message). Different arrangements give different signals.

(a) How many different signals are possible if all four flags are used?
(b) How many different signals are possible if at least one flag is used?

Objective

To discuss combinations, permutations with repeated objects, and assignments to cells.

8.2 Combinations and Other Counting Principles

Combinations

We continue our discussion of counting methods by considering the following. In a 20-member club the offices of president, vice president, secretary, and treasurer are to be filled, and no member may serve in more than one office. If these offices, in the order given, are filled by members A, B, C, and D, respectively, then we can represent this slate by

ABCD

A different slate is

BACD

These two slates represent different permutations of 20 members taken four at a time. Now, as a different situation, let us consider four-person *committees* that can be formed from the 20 members. In that case, the two arrangements

ABCD and BACD

From Equation (6), the number of ways this can be done is

$$\frac{20!}{4!4!3!9!} = 1{,}939{,}938{,}000$$

Method 2 We can handle the problem in terms of a two-stage procedure and use the Basic Counting Principle. First, 11 paintings are selected for exhibit. Then, these are split into three groups (cells) corresponding to the three galleries. We proceed as follows.

Selecting 11 of the 20 paintings for exhibit (order is of no concern) can be done in ${}_{20}C_{11}$ ways. Once a selection is made, four of the paintings go into one cell (gallery A), four go to a second cell (gallery B), and three go to a third cell (gallery C). By Equation (6), this can be done in $\frac{11!}{4!4!3!}$ ways. Applying the Basic Counting Principle gives the number of ways the artist can send the paintings to the galleries:

$${}_{20}C_{11} \cdot \frac{11!}{4!4!3!} = \frac{20!}{11!9!} \cdot \frac{11!}{4!4!3!} = 1{,}939{,}938{,}000$$

Method 3 Another approach to this problem is in terms of a three-stage procedure. First, 4 of the 20 paintings are selected for shipment to gallery A. This can be done in ${}_{20}C_4$ ways. Then, from the remaining 16 paintings, the number of ways 4 can be selected for gallery B is ${}_{16}C_4$. Finally, the number of ways 3 can be sent to gallery C from the 12 paintings that have not yet been selected is ${}_{12}C_3$. By the Basic Counting Principle, the entire procedure can be done in

$${}_{20}C_4 \cdot {}_{16}C_4 \cdot {}_{12}C_3 = \frac{20!}{4!16!} \cdot \frac{16!}{4!12!} \cdot \frac{12!}{3!9!} = \frac{20!}{4!4!3!9!}$$

ways, which gives the previous answer, as expected!

Now Work Problem 27 ◁

PROBLEMS 8.2

In Problems 1–6, determine the values.

1. ${}_7C_4$ **2.** ${}_6C_2$ **3.** ${}_{100}C_{100}$

4. ${}_{1,000,001}C_1$ **5.** ${}_5P_3 \cdot {}_4C_2$ **6.** ${}_5P_2 \cdot {}_6C_4$

7. Verify that ${}_nC_r = {}_nC_{n-r}$. **8.** Determine ${}_nC_n$.

9. Committee In how many ways can a five-member committee be formed from a group of 19 people?

10. Horse Race In a horse race, a horse is said to *finish in the money* if it finishes in first, second, or third place. For an eight-horse race, in how many ways can the horses finish in the money? Assume no ties.

11. Math Exam On a 12-question mathematics examination, a student must answer any 8 questions. In how many ways can the 8 questions be chosen (without regard to order)?

12. Cards From a deck of 52 playing cards, how many 4-card hands are comprised solely of red cards?

13. Quality Control A quality-control technician must select a sample of 10 dresses from a production lot of 74 couture dresses. How many different samples are possible? Express your answer in terms of factorials.

14. Packaging An energy drink producer makes five types of energy drinks. The producer packages "3-paks" containing three drinks, no two of which are of the same type. To reflect the three national chains through which the drinks are distributed, the producer uses three colors of cardboard bands that hold the drinks together. How many different 3-paks are possible?

15. Scoring on Exam In a 10-question examination, each question is worth 10 points and is graded right or wrong. Considering the individual questions, in how many ways can a student score 80 or better?

16. Team Results A sports team plays 13 games. In how many ways can the outcomes of the games result in three wins, eight losses, and two ties?

17. Letter Arrangements How many distinguishable arrangements of all the letters in the word MISSISSAUGA are possible?

18. Letter Arrangements How many distinguishable arrangements of all the letters in the word STREETSBORO are possible?

19. Coin Toss If a coin is tossed six times and the outcome of each toss is noted, in how many ways can two heads and four tails occur?

20. Die Roll A die is rolled six times and the order of the rolls is considered. In how many ways can two 2's, three 3's, and one 4 occur?

21. Scheduling Patients A physician's secretary must schedule eight office consultations. In how many ways can this be done?

22. Baseball A Little League baseball team has 12 members and must play an away game. Three cars will be used for transportation. In how many ways can the manager assign the members to specific cars if each car can accommodate four members?

23. Project Assignment The director of research and development for a company has nine scientists who are equally qualified to work on projects A, B, and C. In how many ways can the director assign three scientists to each project?

24. Identical Siblings A set of identical quadruplets, a set of identical triplets, and three sets of identical twins pose for a group photograph. In how many ways can these 13 individuals line up in ways that are distinguishable in a photograph?

25. True–False Exam A biology instructor includes several true–false questions on quizzes. From experience, a student believes that half of the questions are true and half are false. If there are 10 true–false questions on the next quiz, in how many ways can the student answer half of them "true" and the other half "false"?

26. Food Order A waiter takes the following order from a lunch counter with nine people: three baconburgers, two veggieburgers, two tofuburgers, and two porkbelly delights. Upon returning with the food, he forgets who ordered what item and simply places an item in front of each person. In how many ways can the waiter do this?

27. Caseworker Assignment A social services office has 15 new clients. The supervisor wants to assign 5 clients to each of three specific caseworkers. In how many ways can this be done?

28. Hockey There are 11 members on a hockey team, and all but one, the goalie, are qualified for the other five positions. In how many ways can the coach form a starting lineup?

29. Large Families Large families give rise to an enormous number of *relationships* within the family that make growing up with many siblings qualitatively different from life in smaller families. Any two siblings within any family will have a relationship of some sort that affects the life of the whole family. In larger families, any three siblings or any four siblings will tend to have a three-way or four-way relationship, respectively, that affects the dynamics of the family, too. Janet Braunstein is third in a family of 12 siblings: Claire, Barbie, Janet, Paul, Glenn, Mark, Martha, Laura, Julia, Carrie, Emily, and Jim. If we define a sibling relationship to be any subset of the set of siblings of size greater than or equal to two, how many sibling relationships are there in Janet's family? How many sibling relationships are there in a family of three siblings?

30. Hiring A company personnel director must hire six people: four for the assembly department and two for the shipping department. There are 10 applicants who are equally qualified to work in each department. In how many ways can the personnel director fill the positions?

31. Financial Portfolio A financial advisor wants to create a portfolio consisting of nine stocks and five bonds. If ten stocks and twelve bonds are acceptable for the portfolio, in how many ways can the portfolio be created?

32. World Series A baseball team wins the World Series if it is the first team in the series to win four games. Thus, a series could range from four to seven games. For example, a team winning the first four games would be the champion. Likewise, a team losing the first three games and winning the last four would be champion. In how many ways can a team win the World Series?

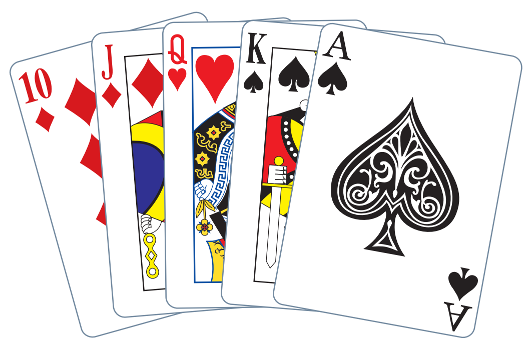
33. Subcommittee A committee has seven members, three of whom are male and four female. In how many ways can a subcommittee be selected if it is to consist exactly of

(a) three males?
(b) four females?
(c) two males and two females?

34. Subcommittee A committee has three male and five female members. In how many ways can a subcommittee of four be selected if at least two females are to serve on it?

35. Poker Hand A poker hand consists of 5 cards from a deck of 52 playing cards. The hand is a "full house" if there are 3 cards of one denomination and 2 cards of another. For example, three 10's and two jacks form a full house. How many full-house hands are possible?

36. Euchre Hand In euchre, only the denominations 9, 10, J, Q, K, and A from a standard 52-card deck are used. A euchre hand consists of 5 cards from this reduced deck. **(a)** How many possible euchre hands are there? **(b)** How many euchre hands contain exactly four cards of the same suit? **(c)** How many euchre hands contain exactly three cards of the same suit?



37. Tram Loading At a tourist attraction, two trams carry sightseers up a picturesque mountain. One tram can accommodate seven people and the other eight. A busload of 18 tourists arrives, and both trams are at the bottom of the mountain. Obviously, only 15 tourists can initially go up the mountain. In how many ways can the attendant load 15 tourists onto the two trams?

38. Discussion Groups A history instructor wants to split a class of 10 students into three discussion groups. One group will consist of four students and discuss topic A. The second and third groups will discuss topics B and C, respectively, and consist of three students each.

(a) In how many ways can the instructor form the groups?
(b) If the instructor designates a group leader and a secretary (different students) for each group, in how many ways can the class be split?

Definition

Events E and F are said to be **mutually exclusive events** if and only if $E \cap F = \emptyset$.

When two events are mutually exclusive, the occurrence of one event means that the other event cannot occur; that is, the two events cannot occur simultaneously. An event and its complement are mutually exclusive, since $E \cap E' = \emptyset$.

EXAMPLE 8 Mutually Exclusive Events

If E, F, and G are events for an experiment and F and G are mutually exclusive, show that events $E \cap F$ and $E \cap G$ are also mutually exclusive.

Solution: Given that $F \cap G = \emptyset$, we must show that the intersection of $E \cap F$ and $E \cap G$ is the empty set. Using the properties in Table 8.1, we have

$$\begin{aligned}
(E \cap F) \cap (E \cap G) &= (E \cap F \cap E) \cap G && \text{property 15} \\
&= (E \cap E \cap F) \cap G && \text{property 11} \\
&= (E \cap F) \cap G && \text{property 2} \\
&= E \cap (F \cap G) && \text{property 15} \\
&= E \cap \emptyset && \text{given} \\
&= \emptyset && \text{property 9}
\end{aligned}$$

Now Work Problem 31 ◁

PROBLEMS 8.3

In Problems 1–6, determine a sample space for the given experiment.

1. Card Selection A card is drawn from a four-card deck consisting of the 9 of diamonds, 9 of hearts, 9 of clubs, and 9 of spades.

2. Euchre Deck A card is drawn from a euchre deck as described in Problem 36 of Section 8.2.

3. Die Roll and Coin Tosses A die is rolled and then a coin is tossed twice in succession.

4. Dice Roll Two dice are rolled, and the sum of the numbers that turns up is observed.

5. Digit Selection Two different digits are selected, in succession, from those in the number "64901".

6. Genders of Children The genders of the first, second, third, and fourth children of a four-child family are noted. (Let, for example, BGGB denote that the first, second, third, and fourth children are boy, girl, girl, boy, respectively.)

7. Jelly Bean Selection A bag contains three colored jelly beans: one red, one white, and one blue. Determine a sample space if **(a)** three jelly beans are selected with replacement and **(b)** three jelly beans are selected without replacement.

8. Manufacturing Process A company makes a product that goes through three processes during its manufacture. The first is an assembly line, the second is a finishing line, and the third is an inspection line. There are four assembly lines (A, B, C, and D), two finishing lines (E and F), and two inspection lines (G and H). For each process, the company chooses a line at random. Determine a sample space.

In Problems 9–14, describe the nature of a sample space for the given experiment, and determine the number of sample points.

9. Coin Toss A coin is tossed six times in succession, and the faces showing are observed.

10. Dice Roll Five dice are rolled, and the numbers that turn up are observed.

11. Card and Die A card is drawn from an ordinary deck of 52 cards, and then a die is rolled.

12. Rabbit Selection From a hat containing eight distinguishable rabbits, five rabbits are pulled successively without replacement.

13. Card Deal A four-card hand is dealt from a deck of 52 cards.

14. Letter Selection A four-letter "word" is formed by successively choosing any four letters from the alphabet with replacement.

Suppose that $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is the sample space for an experiment with events

$$E = \{1, 3, 5\} \quad F = \{3, 5, 7, 9\} \quad G = \{2, 4, 6, 8\}$$

In Problems 15–22, determine the indicated events.

15. $E \cup F$

16. G'

17. $F' \cap G$

18. $E' \cup G'$

19. F'

20. $(E \cup F)'$

21. $(F \cap G)'$

22. $(F \cup G) \cap E'$

23. Of the following events, which pairs are mutually exclusive?

$$E_1 = \{1, 2, 3\} \quad E_2 = \{3, 4, 5\}$$
$$E_3 = \{1, 2\} \quad E_4 = \{6, 7\}$$

24. Card Selection From a standard deck of 52 playing cards, 2 cards are drawn without replacement. Suppose E_J is the event that both cards are jacks, E_C is the event that both cards are clubs, and E_3 is the event that both cards are 3's. Which pairs of these events are mutually exclusive?

25. Card Selection From a standard deck of 52 playing cards, 1 card is selected. Which pairs of the following events are mutually exclusive?

$$E = \{\text{diamond}\}$$
$$F = \{\text{face card}\}$$
$$G = \{\text{black}\}$$
$$H = \{\text{red}\}$$
$$I = \{\text{ace of diamonds}\}$$

26. Dice A red and a green die are thrown, and the numbers on each are noted. Which pairs of the following events are mutually exclusive?

$$E = \{\text{both are even}\}$$
$$F = \{\text{both are odd}\}$$
$$G = \{\text{sum is 2}\}$$
$$H = \{\text{sum is 4}\}$$
$$I = \{\text{sum is greater than 10}\}$$

27. Coin Toss A coin is tossed three times in succession, and the results are observed. Determine each of the following:

(a) The usual sample space S
(b) The event E that at least two heads occur
(c) The event F that at least one tail occurs
(d) $E \cup F$
(e) $E \cap F$
(f) $(E \cup F)'$
(g) $(E \cap F)'$

28. Genders of Children A husband and wife have three children. The outcome of the first child being a boy, the second a girl, and the third a girl can be represented by BGG. Determine each of the following:

(a) Sample space that describes all the orders of the possible genders of the children
(b) The event that at least one child is a girl
(c) The event that at least one child is a boy
(d) Is the event in part (c) the complement of the event in part (b)?

29. Arrivals Persons A, B, and C enter a building at different times. The outcome of A arriving first, B second, and C third can be indicated by ABC. Determine each of the following:

(a) The sample space involved for the arrivals
(b) The event that A arrives first
(c) The event that A does not arrive first

30. Supplier Selection A grocery store can order fruits and vegetables from suppliers U, V, and W; meat from suppliers U, V, X, and Y; and dry goods from suppliers V, W, X, and Z. The grocery store selects one supplier for each type of item. The outcome of U being selected for fruits and vegetables, V for meat, and W for dry goods can be represented by UVW.

(a) Determine a sample space.
(b) Determine the event E that one supplier supplies all the grocery store's requirements.
(c) Determine E' and give a verbal description of this event.

31. If E and F are events for an experiment, prove that events $E \cap F$ and $E \cap F'$ are mutually exclusive.

32. If E and F are events for an experiment, show that

$$(E \cap F) \cup (E \cap F') = E$$

Note that from Problem 31, $E \cap F$ and $E \cap F'$ are mutually exclusive events. Thus, the foregoing equation expresses E as a union of mutually exclusive events.

Objective

To define what is meant by the probability of an event. To develop formulas that are used in computing probabilities. Emphasis is placed on equiprobable spaces.

8.4 Probability

Equiprobable Spaces

We now introduce the basic concepts underlying the study of probability. Consider tossing a well-balanced die and observing the number that turns up. The usual sample space for the experiment is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Before the experiment is performed, we cannot predict with certainty which of the six possible outcomes (sample points) will occur. But it does seem reasonable that each outcome has the same chance of occurring; that is, the outcomes are **equally likely**. This does not mean that in six tosses each number must turn up once. Rather, it means that if the experiment were performed a large number of times, each outcome would occur about $\frac{1}{6}$ of the time.

To be more specific, let the experiment be performed n times. Each performance of an experiment is called a **trial**. Suppose that we are interested in the event of obtaining

22. Baseball A Little League baseball team has 12 members and must play an away game. Three cars will be used for transportation. In how many ways can the manager assign the members to specific cars if each car can accommodate four members?

23. Project Assignment The director of research and development for a company has nine scientists who are equally qualified to work on projects A, B, and C. In how many ways can the director assign three scientists to each project?

24. Identical Siblings A set of identical quadruplets, a set of identical triplets, and three sets of identical twins pose for a group photograph. In how many ways can these 13 individuals line up in ways that are distinguishable in a photograph?

25. True–False Exam A biology instructor includes several true–false questions on quizzes. From experience, a student believes that half of the questions are true and half are false. If there are 10 true–false questions on the next quiz, in how many ways can the student answer half of them "true" and the other half "false"?

26. Food Order A waiter takes the following order from a lunch counter with nine people: three baconburgers, two veggieburgers, two tofuburgers, and two porkbelly delights. Upon returning with the food, he forgets who ordered what item and simply places an item in front of each person. In how many ways can the waiter do this?

27. Caseworker Assignment A social services office has 15 new clients. The supervisor wants to assign 5 clients to each of three specific caseworkers. In how many ways can this be done?

28. Hockey There are 11 members on a hockey team, and all but one, the goalie, are qualified for the other five positions. In how many ways can the coach form a starting lineup?

29. Large Families Large families give rise to an enormous number of *relationships* within the family that make growing up with many siblings qualitatively different from life in smaller families. Any two siblings within any family will have a relationship of some sort that affects the life of the whole family. In larger families, any three siblings or any four siblings will tend to have a three-way or four-way relationship, respectively, that affects the dynamics of the family, too. Janet Braunstein is third in a family of 12 siblings: Claire, Barbie, Janet, Paul, Glenn, Mark, Martha, Laura, Julia, Carrie, Emily, and Jim. If we define a sibling relationship to be any subset of the set of siblings of size greater than or equal to two, how many sibling relationships are there in Janet's family? How many sibling relationships are there in a family of three siblings?

30. Hiring A company personnel director must hire six people: four for the assembly department and two for the shipping department. There are 10 applicants who are equally qualified to work in each department. In how many ways can the personnel director fill the positions?

31. Financial Portfolio A financial advisor wants to create a portfolio consisting of nine stocks and five bonds. If ten stocks and twelve bonds are acceptable for the portfolio, in how many ways can the portfolio be created?

32. World Series A baseball team wins the World Series if it is the first team in the series to win four games. Thus, a series could range from four to seven games. For example, a team winning the first four games would be the champion. Likewise, a team losing the first three games and winning the last four would be champion. In how many ways can a team win the World Series?

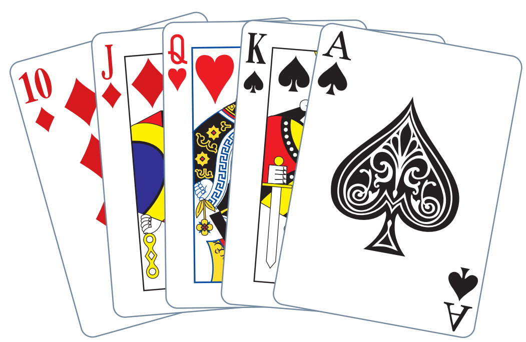
33. Subcommittee A committee has seven members, three of whom are male and four female. In how many ways can a subcommittee be selected if it is to consist exactly of

(a) three males?
(b) four females?
(c) two males and two females?

34. Subcommittee A committee has three male and five female members. In how many ways can a subcommittee of four be selected if at least two females are to serve on it?

35. Poker Hand A poker hand consists of 5 cards from a deck of 52 playing cards. The hand is a "full house" if there are 3 cards of one denomination and 2 cards of another. For example, three 10's and two jacks form a full house. How many full-house hands are possible?

36. Euchre Hand In euchre, only the denominations 9, 10, J, Q, K, and A from a standard 52-card deck are used. A euchre hand consists of 5 cards from this reduced deck. **(a)** How many possible euchre hands are there? **(b)** How many euchre hands contain exactly four cards of the same suit? **(c)** How many euchre hands contain exactly three cards of the same suit?



37. Tram Loading At a tourist attraction, two trams carry sightseers up a picturesque mountain. One tram can accommodate seven people and the other eight. A busload of 18 tourists arrives, and both trams are at the bottom of the mountain. Obviously, only 15 tourists can initially go up the mountain. In how many ways can the attendant load 15 tourists onto the two trams?

38. Discussion Groups A history instructor wants to split a class of 10 students into three discussion groups. One group will consist of four students and discuss topic A. The second and third groups will discuss topics B and C, respectively, and consist of three students each.

(a) In how many ways can the instructor form the groups?
(b) If the instructor designates a group leader and a secretary (different students) for each group, in how many ways can the class be split?

Objective

To determine a sample space and to consider events associated with it. To represent a sample space and events by means of a Venn diagram. To introduce the notions of complement, union, and intersection.

8.3 Sample Spaces and Events

Sample Spaces

Inherent in any discussion of probability is the performance of an experiment (a procedure) in which a particular result, or *outcome,* involves chance. For example, consider the experiment of tossing a coin. There are only two ways the coin can fall, a head (H) or a tail (T), but the actual outcome is determined by chance. (We assume that the coin does not land on its edge.) The set of possible outcomes,

$$\{\text{H}, \text{T}\}$$

is called a *sample space* for the experiment, and H and T are called *sample points*.

Definition

A **sample space** S for an experiment is the set of all possible outcomes of the experiment. The elements of S are called **sample points**. If there is a finite number of sample points, that number is denoted $\#(S)$, and S is said to be a **finite sample space**.

The order in which sample points are listed in a sample space is of no concern.

When determining "possible outcomes" of an experiment, we must be sure that they reflect the situation about which we are concerned. For example, consider the experiment of rolling a die and observing the top face. We could say that a sample space is

$$S_1 = \{1, 2, 3, 4, 5, 6\}$$

where the possible outcomes are the number of dots on the top face. However, other possible outcomes are

	odd number of dots appear	(odd)
and	even number of dots appear	(even)

Thus, the set

$$S_2 = \{\text{odd}, \text{even}\}$$

is also a sample space for the experiment, so an experiment can have more than one sample space.

If an outcome in S_1 occurred, then we know which outcome in S_2 occurred, but the reverse is not true. To describe this asymmetry, we say that S_1 is a *more primitive* sample space than S_2. Usually, the more primitive a sample space is, the more questions pertinent to the experiment it allows us to answer. For example, with S_1, we can answer such questions as

"Did a 3 occur?"
"Did a number greater than 2 occur?"
"Did a number less than 4 occur?"

But with S_2, we cannot answer these questions. As a rule of thumb, the more primitive a sample space is, the more elements it has and the more detail it indicates. Unless otherwise stated, when an experiment has more than one sample space, it will be our practice to consider only a sample space that gives sufficient detail to answer all pertinent questions relative to the experiment. For example, for the experiment of rolling a die and observing the top face, it will be tacitly understood that we are observing the number of dots. Thus, we will consider the sample space to be

$$S_1 = \{1, 2, 3, 4, 5, 6\}$$

and will refer to it as the *usual* sample space for the experiment.

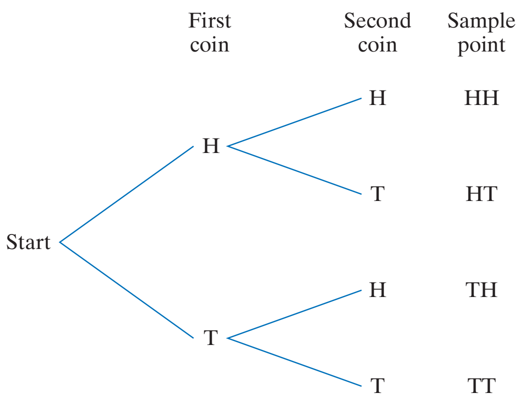


FIGURE 8.6 Tree diagram for toss of two coins.

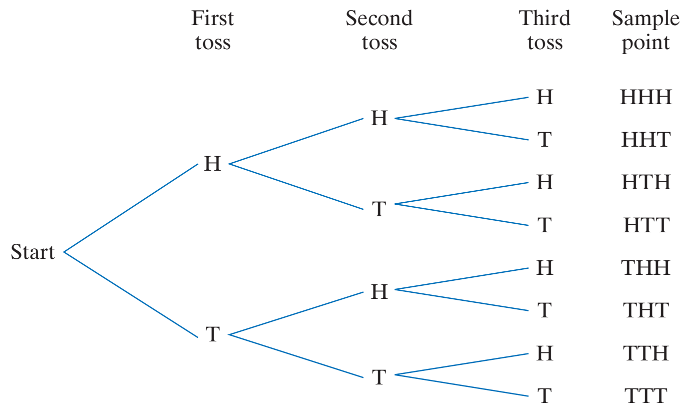


FIGURE 8.7 Tree diagram for three tosses of a coin.

EXAMPLE 1 Sample Space: Toss of Two Coins

Two different coins are tossed, and the result (*H* or *T*) for each coin is observed. Determine a sample space.

Solution: One possible outcome is a head on the first coin and a head on the second, which we can indicate by the ordered pair (H, H) or, more simply, HH. Similarly, we indicate a head on the first coin and a tail on the second by HT, and so on. A sample space is

$$S = \{\text{HH, HT, TH, TT}\}$$

A tree diagram is given in Figure 8.6 which illustrates further structure of this sample space. We remark that S is also a sample space for the experiment of tossing a single coin twice in succession. In fact, these two experiments can be considered one and the same. Although other sample spaces can be contemplated, we take S to be the *usual* sample space for these experiments.

Now Work Problem 3 ◁

APPLY IT ▶

1. In 2016, on March 26, Netflix had 1197 TV shows in its US catalogue. A viewer wanted to select two shows. How many random choices did she have?

EXAMPLE 2 Sample Space: Three Tosses of a Coin

A coin is tossed three times, and the result of each toss is observed. Describe a sample space and determine the number of sample points.

Solution: Because there are three tosses, we choose a sample point to be an ordered *triple,* such as HHT, where each component is either H or T. By the Basic Counting Principle, the total number of sample points is $2 \cdot 2 \cdot 2 = 8$. A sample space (the *usual* one) is

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

and a tree diagram appears in Figure 8.7. Note that it is not necessary to list the entire sample space to determine the number of sample points in it.

Now Work Problem 9 ◁

FIGURE 8.8 Four colored jelly beans in a bag.

EXAMPLE 3 Sample Space: Jelly Beans in a Bag

A bag contains four jelly beans: one red, one pink, one black, and one white. (See Figure 8.8.)

a. A jelly bean is withdrawn at random, its color is noted, and it is put back in the bag. Then a jelly bean is again randomly withdrawn and its color noted. Describe a sample space and determine the number of sample points.

PROBLEMS 8.5

1. Given the equiprobable sample space

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

and events

$$E = \{1, 3\}$$
$$F = \{1, 2, 4, 5, 6\}$$
$$G = \{5, 6, 7, 8, 9\}$$

find each of the following.

(a) $P(E|F)$ **(b)** $P(E'|F)$ **(c)** $P(E|F')$
(d) $P(F|E)$ **(e)** $P(E|F \cap G)$

2. Given the equiprobable sample space

$$S = \{1, 2, 3, 4, 5\}$$

and events

$$E = \{1, 2\}$$
$$F = \{3, 4\}$$
$$G = \{1, 2, 3\}$$

find each of the following.

(a) $P(E)$ **(b)** $P(E|F)$ **(c)** $P(E|G)$
(d) $P(G|E)$ **(e)** $P(G|F')$ **(f)** $P(E'|F')$

3. If $P(E) > 0$, find $P(E|E)$.

4. If $P(E) > 0$, show $P(\emptyset|E) = 0$.

5. If $P(E'|F) = 0.62$, find $P(E|F)$.

6. If F and G are mutually exclusive events with positive probabilities, find $P(F|G)$.

7. If $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{3}$, and $P(E \cap F) = \frac{1}{6}$, find each of the following:

(a) $P(E|F)$ **(b)** $P(F|E)$

8. If $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{3}$, and $P(E|F) = \frac{3}{4}$, find $P(E \cup F)$. [*Hint:* Use the addition law to find $P(E \cup F)$.]

9. If $P(E) = \frac{1}{3}$, $P(E \cup F) = \frac{7}{12}$, and $P(E \cap F) = \frac{1}{12}$, find each of the following:
(a) $P(F|E)$ **(b)** $P(F)$ **(c)** $P(E|F)$
(d) $P(E|F')$ [*Hint:* Find $P(E \cap F')$ by using the identity $P(E) = P(E \cap F) + P(E \cap F')$.]

10. If $P(E) = \frac{4}{5}$, $P(F) = \frac{3}{10}$, and $P(E \cup F) = \frac{7}{10}$, find $P(F|E)$.

11. Gypsy Moth Because of gypsy moth infestation of three large areas that are densely populated with trees, consideration is being given to aerial spraying to destroy larvae. A survey was made of the 200 residents of these areas to determine whether or not they favor the spraying. The resulting data are shown in Table 8.7. Suppose that a resident is randomly selected. Let I be the event "the resident is from Area I" and so on. Find each of the following:

(a) $P(F)$ **(b)** $P(F|\text{II})$ **(c)** $P(O|\text{I})$
(d) $P(\text{III})$ **(e)** $P(\text{III}|O)$ **(f)** $P(\text{II}|N')$

Table 8.7

	Area I	Area II	Area III	Total
Favor (F)	46	35	44	125
Opposed (O)	22	15	10	47
No opinion (N)	10	8	10	28
Total	78	58	64	200

12. College Selection and Family Income A survey of 175 students resulted in the data shown in Table 8.8, which shows the type of college the student attends and the income level of the student's family. Suppose a student in the survey is randomly selected.

(a) Find the probability that the student attends a public college, given that the student comes from a middle-income family.
(b) Find the probability that the student is from a high-income family, given that the student attends a private college.
(c) If the student comes from a high-income family, find the probability that the student attends a private college.
(d) Find the probability that the student attends a public college or comes from a low-income family.

Table 8.8

	College		
Income	Private	Public	Total
High	14	11	25
Middle	25	55	80
Low	10	60	70
Total	49	126	175

13. Cola Preference A survey was taken among cola drinkers to see which of two popular brands people preferred. It was found that 45% liked brand A, 40% liked brand B, and 20% liked both brands. Suppose that a person in the survey is randomly selected.

(a) Find the probability that the person liked brand A, given that he or she liked brand B.
(b) Find the probability that the person liked brand B, given that he or she liked brand A.

14. Quality Control Of the smartphones produced by a well-known firm, 17% have poor sound quality and 11% have both poor sound quality and scratched screens. If a smartphone is randomly selected from a shipment and it has poor sound quality what is the probability that it has a scratched screen?

In Problems 15 and 16, assume that a child of either gender is equally likely and that, for example, having a girl first and a boy second is just as likely as having a boy first and a girl second.

15. Genders of Offspring If a family has two children, what is the probability that one child is a boy, given that at least one child is a girl?

16. Genders of Offspring If a family has three children, find each of the following.

(a) The probability that it has two girls, given that at least one child is a boy
(b) The probability that it has at least two girls, given that the oldest child is a girl

17. Coin Toss If a fair coin is tossed three times in succession, find each of the following.

(a) The probability of getting exactly two tails, given that the second toss is a tail
(b) The probability of getting exactly two tails, given that the second toss is a head

18. Coin Toss If a fair coin is tossed four times in succession, find the odds of getting four tails, given that the first toss is a tail.

19. Die Roll A fair die is rolled. Find the probability of getting a number greater than 4, given that the number is even.

20. Cards If a card is drawn randomly from a standard deck, find the probability of getting a spade, given that the card is black.

21. Dice Roll If two fair dice are rolled, find the probability that two 1's occur, given that at least one die shows a 1.

22. Dice Roll If a fair red die and a fair green die are rolled, find the probability that the sum is greater than 9, given that a 5 shows on the red die.

23. Dice Roll If a fair red die and a fair green die are rolled, find the probability of getting a total of 7, given that the green die shows an even number.

24. Dice Roll A fair die is rolled two times in succession.

(a) Find the probability that the sum is 7, given that the second roll is neither a 3 nor a 5.
(b) Find the probability that the sum is 7 and that the second roll is neither a 3 nor a 5.

25. Die Roll If a fair die is rolled two times in succession, find the probability of getting a total greater than 8, given that the first roll is greater than 2.

26. Coin and Die If a fair coin and a fair die are thrown, find the probability that the coin shows tails, given that the number on the die is odd.

27. Cards If a card is randomly drawn from a deck of 52 cards, find the probability that the card is a king, given that it is a heart.

28. Cards If a card is randomly drawn from a deck of 52 cards, find the probability that the card is a heart, given that it is a face card (a jack, queen, or king).

29. Cards If two cards are randomly drawn without replacement from a standard deck, find the probability that the second card is not a heart, given that the first card is a heart.

In Problems 30–35, consider the experiment to be a compound experiment.

30. Cards If two cards are randomly drawn from a standard deck, find the probability that both cards are aces if,

(a) the cards are drawn without replacement.
(b) the cards are drawn with replacement.

31. Cards If three cards are randomly drawn without replacement from a standard deck, find the probability of getting a king, a queen, and a jack, in that order.

32. Cards If three cards are randomly drawn without replacement from a standard deck, find the probability of getting the ace of spades, the ace of hearts, and the ace of diamonds, in that order.

33. Cards If three cards are randomly drawn without replacement from a standard deck, find the probability that all three cards are jacks.

34. Cards If two cards are randomly drawn without replacement from a standard deck of cards, find the probability that the second card is a face card.

35. Cards If two cards are randomly drawn without replacement from a standard deck, find the probability of getting two jacks, given that the first card is a face card.

36. Wake-Up Call Barbara Smith, a sales representative, is staying overnight at a hotel and has a breakfast meeting with an important client the following morning. She asked the desk to give her a 7 A.M. wake-up call so she can be prompt for the meeting. The probability that she will get the call is 0.9. If she gets the call, the probability that she will be on time is 0.9. If the call is not given, the probability that she will be on time is 0.4. Find the probability that she will be on time for the meeting.

37. Taxpayer Survey In a certain school district, a questionnaire was sent to all property taxpayers concerning whether or not a new high school should be built. Of those that responded, 60% favored its construction, 30% opposed it, and 10% had no opinion. Further analysis of the data concerning the area in which the respondents lived gave the results in Table 8.9.

Table 8.9

	Urban	Suburban
Favor	45%	55%
Oppose	55%	45%
No opinion	35%	65%

(a) If one of the respondents is selected at random, what is the probability that he or she lives in an urban area?
(b) If a respondent is selected at random, use the result of part (a) to find the probability that he or she favors the construction of the school, given that the person lives in an urban area.

38. Marketing A travel agency has a computerized telephone that randomly selects telephone numbers for advertising suborbital space trips. The telephone automatically dials the selected number and plays a prerecorded message to the recipient of the call. Experience has shown that 2% of those called show interest and contact the agency. However, of these, only 1.4% actually agree to purchase a trip.

(a) Find the probability that a person called will contact the agency and purchase a trip.
(b) If 100,000 people are called, how many can be expected to contact the agency and purchase a trip?

39. Rabbits in a Tall Hat A tall hat contains four yellow and three red rabbits.

(a) If two rabbits are randomly pulled from the hat without replacement, find the probability that the second rabbit pulled is yellow, given that the first rabbit pulled is red.
(b) Repeat part (a), but assume that the first rabbit is replaced before the second rabbit is pulled.

40. Jelly Beans in a Bag Bag 1 contains five green and two red jelly beans, and Bag 2 contains two green, two white, and three red jelly beans. A jelly bean is randomly taken from Bag 1 and placed into Bag 2. If a jelly bean is then randomly taken from Bag 2, find the probability that the jelly bean is green.

41. Balls in a Box Box 1 contains three red and two white balls. Box 2 contains two red and two white balls. A box is chosen at random and then a ball is chosen at random from it. What is the probability that the ball is white?

42. Balls in a Box Box 1 contains two red and three white balls. Box 2 contains three red and four white balls. Box 3 contains two red, two white, and two green balls. A box is chosen at random, and then a ball is chosen at random from it.

(a) Find the probability that the ball is white.
(b) Find the probability that the ball is red.
(c) Find the probability that the ball is green.

43. Jelly Beans in a Bag Bag 1 contains one green and one red jelly bean, and Bag 2 contains one white and one red jelly bean. A bag is selected at random. A jelly bean is randomly taken from it and placed in the other bag. A jelly bean is then randomly drawn from that bag. Find the probability that the jelly bean is white.

44. Dead Batteries Ms. Wood's lights went out in a recent storm and she reached in the kitchen drawer for 4 batteries for her flashlight. There were 10 batteries in the drawer, but 5 of them were dead. Fortunately, the flashlight worked with the ones she randomly chose. Later, Ms Wood and Mr Wood discussed the question of the probability of choosing 4 dead batteries. She argued, with obvious notation $P(D_1 \cap D_2 \cap D_3 \cap D_4) = P(D_1)P(D_2|D_1)P(D_3|D_1 \cap D_2)P(D_4|D_1 \cap D_2 \cap D_3) = \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{1}{42}$. He said the answer is $\frac{{}_5C_4}{{}_{10}C_4}$. Who was right?

45. Quality Control An energy drink producer requires the use of a can filler on each of its two product lines. The Yellow Cow line produces 36,000 cans per day, and the Half Throttle line produces 60,000 cans per day. Over a period of time, it has been found that the Yellow Cow filler underfills 2% of its cans, whereas the Half Throttle filler underfills 1% of its cans. At the end of a day, a can was selected at random from the total production. Find the probability that the can was underfilled.

46. Game Show A TV game show host presents the following situation to a contestant. On a table are three identical boxes. One of them contains two identical envelopes. In one is a check for \$5000, and in the other is a check for \$1. Another box contains two envelopes with a check for \$5000 in each and six envelopes with a check for \$1 in each. The remaining box contains one envelope with a check for \$5000 inside and five envelopes with a check for \$1 inside each. If the contestant must select a box at random and then randomly draw an envelope, find the probability that a check for \$5000 is inside.

47. Quality Control A company uses one computer chip in assembling each unit of a product. The chips are purchased from suppliers A, B, and C and are randomly picked for assembling a unit. Ten percent come from A, 20% come from B, and the remainder come from C. The probability that a chip from A will prove to be defective in the first 24 hours of use is 0.06, and the corresponding probabilities for B and C are 0.04 and 0.05, respectively. If an assembled unit is chosen at random and tested for 24 continuous hours, what is the probability that the chip in it will prove to be defective?

48. Quality Control A manufacturer of widgets has four assembly lines: A, B, C, and D. The percentages of output produced by the lines are 30%, 20%, 35%, and 15%, respectively, and the percentages of defective units they produce are 6%, 3%, 2%, and 5%. If a widget is randomly selected from stock, what is the probability that it is defective?

49. Voting In a certain town, 45% of eligible voters are Liberals, 30% are Conservatives, and the remainder are Social Democrats. In the last provincial election, 20% of the Liberals, 35% of the Conservatives, and 40% of the Social Democrats voted.

(a) If an eligible voter is chosen at random, what is the probability that he or she is a Social Democrat who voted?
(b) If an eligible voter is chosen at random, what is the probability that he or she voted?

50. Job Applicants A restaurant has four openings for waiters. Suppose Allison, Lesley, Alan, Tom, Alaina, Bronwen, Ellie, and Emmy are the only applicants for these jobs, and all are equally qualified. If four are hired at random, find the probability that Allison, Lesley, Tom, and Bronwen were chosen, given that Ellie and Emmy were not hired.

51. Committee Selection Suppose six female and five male students wish to fill three openings on a campus committee on cultural diversity. If three of the students are chosen at random for the committee, find the probability that all three are female, given that at least one is female.

8.6 Independent Events

Objective

To develop the notion of independent events and apply the special multiplication law.

In our discussion of conditional probability, we saw that the probability of an event can be affected by the knowledge that another event has occurred. In this section, we consider the situation where the additional information has no effect. That is, the conditional probability $P(E|F)$ and the unconditional probability $P(E)$ are the same. In this discussion we assume that $P(E) \neq 0 \neq P(F)$.

When $P(E|F) = P(E)$, we say that E is *independent* of F. If E is independent of F, it follows that F is independent of E (and vice versa). To prove this, assume that $P(E|F) = P(E)$. Then

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{P(F)P(E \mid F)}{P(E)} = \frac{P(F)P(E)}{P(E)} = P(F)$$

Independence of two events is defined by probabilities, not by a causal relationship.

which means that F is independent of E. Thus, to prove independence, it suffices to show that either $P(E|F) = P(E)$ or $P(F|E) = P(F)$, and when one of these is true, we simply say that E and F are *independent events*.

are mutually exclusive, they cannot occur simultaneously. Although these two concepts are not the same, we can draw some conclusions about their relationship. If E and F are mutually exclusive events *with positive probabilities,* then

$$P(E \cap F) = 0 \neq P(E)P(F) \qquad \text{since } P(E) > 0 \text{ and} P(F) > 0$$

which shows that E and F are dependent. In short, *mutually exclusive events with positive probabilities must be dependent*. Another way of saying this is that *independent events with positive probabilities are not mutually exclusive*.

PROBLEMS 8.6

1. If events E and F are independent with $P(E) = \frac{1}{3}$ and $P(F) = \frac{3}{4}$, find each of the following.

(a) $P(E \cap F)$ **(b)** $P(E \cup F)$ **(c)** $P(E \mid F)$
(d) $P(E' \mid F)$ **(e)** $P(E \cap F')$ **(f)** $P(E \cup F')$
(g) $P(E \mid F')$

2. If events E, F, and G are independent with $P(E) = 0.1, P(F) = 0.3$, and $P(G) = 0.6$, find each of the following.

(a) $P(E \cap F)$ **(b)** $P(F \cap G)$
(c) $P(E \cap F \cap G)$ **(d)** $P(E \mid (F \cap G))$
(e) $P(E' \cap F \cap G')$

3. If events E and F are independent with $P(E) = \frac{2}{7}$ and $P(E \cap F) = \frac{1}{9}$, find $P(F)$.

4. If events E and F are independent with $P(E' \mid F') = \frac{1}{4}$, find $P(E)$.

In Problems 5 and 6, events E and F satisfy the given conditions. Determine whether E and F are independent or dependent.

5. $P(E) = \frac{2}{3}, P(F) = \frac{6}{7}, P(E \cap F) = \frac{4}{7}$

6. $P(E) = 0.28, P(F) = 0.15, P(E \cap F) = 0.038$

7. Stockbrokers Six hundred investors were surveyed to determine whether a person who uses a full-service stockbroker has better performance in his or her investment portfolio than one who uses a discount broker. In general, discount brokers usually offer no investment advice to their clients, whereas full-service brokers usually offer help in selecting stocks but charge larger fees. The data, based on the past 12 months, are given in Table 8.11. Determine whether the event of having a full-service broker and the event of having an increase in portfolio value are independent or dependent.

Table 8.11 Portfolio Value

	Increase	Decrease	Total
Full service	320	80	400
Discount	160	40	200
Total	480	120	600

8. Cinema Offenses An observation of 175 patrons in a theater resulted in the data shown in Table 8.12. The table shows three types of cinema offenses committed by male and female patrons. Crunchers include noisy eaters of popcorn and other morsels, as well as cold-drink slurpers. Determine whether the event of being a male and the event of being a cruncher are independent or dependent. (See page 5D of the July 21, 1991, issue of *USA TODAY* for the article "Pests Now Appearing at a Theater Near You".)

Table 8.12 Theater Patrons

	Male	Female	Total
Talkers	60	10	70
Crunchers	55	25	80
Seat kickers	15	10	25
Total	130	45	175

9. Dice Two fair dice are rolled, one red and one green, and the numbers on the top faces are noted. Let event E be "number on red die is neither 1 nor 2 nor 3" and event F be "sum is 7". Determine whether E and F are independent or dependent.

10. Cards A card is randomly drawn from an ordinary deck of 52 cards. Let E and F be the events "red card drawn" and "face card drawn" respectively. Determine whether E and F are independent or dependent.

11. Coins If two fair coins are tossed, let E be the event "at most one head" and F be the event "exactly one head". Determine whether E and F are independent or dependent.

12. Coins If three fair coins are tossed, let E be the event "at most one head" and F be the event "at least one head and one tail". Determine whether E and F are independent or dependent.

13. Chips in a Bowl A bowl contains seven chips numbered from 1 to 7. Two chips are randomly withdrawn with replacement. Let E, F, and G be the events

$$E = 3 \text{ on first withdrawal}$$
$$F = 3 \text{ on second withdrawal}$$
$$G = \text{sum is odd}$$

(a) Determine whether E and F are independent or dependent.
(b) Determine whether E and G are independent or dependent.
(c) Determine whether F and G are independent or dependent.
(d) Are E, F, and G independent?

14. Chips in a Bowl A bowl contains six chips numbered from 1 to 6. Two chips are randomly withdrawn. Let E be the event of withdrawing two even-numbered chips and let F be the event of withdrawing two odd-numbered chips.

(a) Are E and F mutually exclusive?
(b) Are E and F independent?

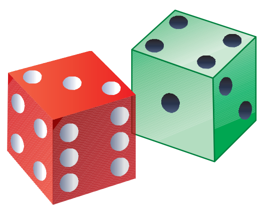
In Problems 15 and 16, events E and F satisfy the given conditions. Determine whether E and F are independent or dependent.

15. $P(E|F) = 0.6, P(E \cap F) = 0.2, P(F|E) = 0.4$

16. $P(E \mid F) = \frac{2}{3}, P(E \cup F) = \frac{17}{18}, P(E \cap F) = \frac{5}{9}$

In Problems 17–37, you may make use of your intuition concerning independent events if nothing to that effect is specified.

17. Dice Two fair dice are rolled, one red and one green. Find the probability that the red die is a 4 and the green die is a number greater than 4.



18. Die If a fair die is rolled three times, find the probability that a 2 or 3 comes up each time.

19. Fitness Classes At a certain fitness center, the probability that a member regularly attends an aerobics class is $\frac{1}{4}$. If two members are randomly selected, find the probability that both attend the class regularly. Assume independence.

20. Monopoly In the game of Monopoly, a player rolls two fair dice. One special situation that can arise is that the numbers on the top faces of the dice are the same (such as two 3's). This result is called a "double", and when it occurs, the player continues his or her turn and rolls the dice again. The pattern continues, unless the player is unfortunate enough to throw doubles three consecutive times. In that case, the player goes to jail. Find the probability that a player goes to jail in this way given that he has already rolled doubles twice in a row.

21. Cards Three cards are randomly drawn, with replacement, from an ordinary deck of 52 cards. Find the probability that the cards drawn, in order, are an ace, a face card (a jack, queen, or king), and a spade.

22. Die If a fair die is rolled seven times, find each of the following.

(a) The probability of getting a number greater than 4 each time
(b) The probability of getting a number less than 4 each time

23. Exam Grades In a sociology course, the probability that Bill gets an A on the final exam is $\frac{3}{4}$, and for Jim and Linda, the probabilities are $\frac{1}{2}$ and $\frac{4}{5}$, respectively. Assume independence and find each of the following.

(a) The probability that all three of them get an A on the exam
(b) The probability that none of them get an A on the exam
(c) The probability that, of the three, only Linda gets an A

24. Die If a fair die is rolled four times, find the probability of getting at least one 1.

25. Survival Rates The probability that person A survives 15 more years is $\frac{3}{4}$, and the probability that person B survives 15 more years is $\frac{4}{5}$. Find the probability of each of the following. Assume independence.

(a) A and B both survive 15 years.
(b) B survives 15 years, but A does not.
(c) Exactly one of A and B survives 15 years.
(d) At least one of A and B survives 15 years.
(e) Neither A nor B survives 15 years.

26. Matching In his desk, a secretary has a drawer containing a mixture of two sizes of paper (A and B) and another drawer containing a mixture of envelopes of two corresponding sizes. The percentages of each size of paper and envelopes in the drawers are given in Table 8.13. If a piece of paper and an envelope are randomly drawn, find the probability that they are the same size.

Table 8.13 Paper and Envelopes

	Drawers	
Size	Paper	Envelopes
A	63%	57%
B	37%	43%

27. Jelly Beans in a Bag A bag contains five red, seven white, and six green jelly beans. If two jelly beans are randomly taken out with replacement, find each of the following.

(a) The probability that the first jelly bean is white and the second is green.
(b) The probability that one jelly bean is red and the other one is white

28. Dice Suppose two fair dice are rolled twice. Find the probability of getting a total of 7 on one of the rolls and a total of 12 on the other one.

29. Jelly Beans in a Bag A bag contains three red, two white, four blue, and two green jelly beans. If two jelly beans are randomly withdrawn with replacement, find the probability that they have the same color.

30. Die Find the probability of rolling three consecutive numbers in three throws of a fair die.

31. Tickets in Hat Twenty tickets numbered from 1 to 20 are placed in a hat. If two tickets are randomly drawn with replacement, find the probability that the sum is 35.

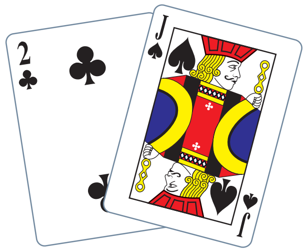
32. Coins and Dice Suppose two fair coins are tossed and then two fair dice are rolled. Find each of the following.

(a) The probability that two tails and two 3's occur
(b) The probability that two heads, one 4, and one 6 occur

33. Carnival Game In a carnival game, a well-balanced roulette-type wheel has 12 equally spaced slots that are numbered from 1 to 12. The wheel is spun, and a ball travels along the rim of the wheel. When the wheel stops, the number of the slot in which the ball finally rests is considered the result of the spin. If the wheel is spun three times, find each of the following.

(a) The probability that the first number will be 4 and the second and third numbers will be 5
(b) The probability that there will be one even number and two odd numbers

34. Cards Three cards are randomly drawn, with replacement, from an ordinary deck of 52 cards. Find each of the following.



(a) The probability of drawing, in order, a heart, a spade, and a red queen
(b) The probability of drawing exactly three aces
(c) The probability that one red queen, one spade and one red ace are drawn
(d) The probability of drawing exactly one ace

35. Multiple-Choice Exam A quiz contains 10 multiple-choice problems. Each problem has five choices for the answer, but only one of them is correct. Suppose a student randomly guesses the answer to each problem. Find each of the following by assuming that the guesses are independent.

(a) The probability that the student gets exactly three correct answers
(b) The probability that the student gets at most three correct answers
(c) The probability that the student gets four or more correct answers

36. Shooting Gallery At a shooting gallery, suppose Bill, Jim, and Linda each take one shot at a moving target. The probability that Bill hits the target is 0.5, and for Jim and Linda the probabilities are 0.4 and 0.7, respectively. Assume independence and find each of the following.

(a) The probability that none of them hit the target
(b) The probability that Linda is the only one of them that hits the target
(c) The probability that exactly one of them hits the target
(d) The probability that exactly two of them hit the target
(e) The probability that all of them hit the target

37. Decision Making[1] The president of Zeta Construction Company must decide which of two actions to take, namely, to rent or to buy expensive excavating equipment. The probability that the vice president makes a faulty analysis and, thus, recommends the wrong decision to the president is 0.04. To be thorough, the president hires two consultants, who study the problem independently and make their recommendations. After having observed them at work, the president estimates that the first consultant is likely to recommend the wrong decision with probability 0.05, the other with probability 0.1. He decides to take the action recommended by a majority of the three recommendations he receives. What is the probability that he will make the wrong decision?

Objective

To solve a Bayes' problem. To develop Bayes' formula.

8.7 Bayes' Formula

In this section, we will be dealing with a two-stage experiment in which we know the outcome of the second stage and are interested in the probability that a particular outcome has occurred in the first stage.

To illustrate, suppose it is believed that of the total population (our sample space), 8% have a particular disease. Imagine also that there is a new blood test for detecting the disease and that researchers have evaluated its effectiveness. Data from extensive testing show that the blood test is not perfect: Not only is it positive for only 95% of those who have the disease, but it is also positive for 3% of those who do not. Suppose a person from the population is selected at random and given the blood test. If the result is positive, what is the probability that the person has the disease?

To analyze this problem, we consider the following events:

$$D = \text{(having the disease)}$$

$$T = \text{(testing positive)}$$

and their complements:

$$D' = \text{(not having the disease)}$$

$$T' = \text{(testing negative)}$$

We are given:

$$P(D) = 0.08 \qquad P(T|D) = 0.95 \qquad P(T|D') = 0.03$$

[1]Samuel Goldberg, *Probability, an Introduction* (Prentice-Hall, Inc., 1960, Dover Publications, Inc., 1986), p. 113. Adapted by permission of the author.

Note that the unconditional probability of choosing Bag II, namely, $P(B_2) = \frac{1}{2}$, increases to $\frac{2}{3}$, given that a red jelly bean was taken. An increase is reasonable: Since there are only red jelly beans in Bag II, choosing a red jelly bean should make it more likely that it came from Bag II.

Method 2: Bayes' Formula Because B_1 and B_2 partition the sample space, by Bayes' formula we have

$$P(B_2|R) = \frac{P(B_2)P(R|B_2)}{P(B_1)P(R|B_1) + P(B_2)P(R|B_2)}$$

$$= \frac{\left(\frac{1}{2}\right)(1)}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)(1)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

Now Work Problem 7 ◁

PROBLEMS 8.7

1. Suppose events E and F partition a sample space S, where E and F have probabilities

$$P(E) = \frac{4}{7} \quad P(F) = \frac{3}{7}$$

If D is an event such that

$$P(D|E) = \frac{2}{9} \quad P(D|F) = \frac{1}{3}$$

find the probabilities **(a)** $P(E|D)$ and **(b)** $P(F|D')$.

2. A sample space is partitioned by events E_1, E_2, and E_3, whose probabilities are $\frac{1}{5}$, $\frac{3}{10}$, and $\frac{1}{2}$, respectively. Suppose S is an event such that the following conditional probabilities hold:

$$P(S\,|\,E_1) = \frac{2}{5} \quad P(S\,|\,E_2) = \frac{7}{10} \quad P(S\,|\,E_3) = \frac{1}{2}$$

Find the probabilities $P(E_1\,|\,S)$ and $P(E_3\,|\,S')$.

3. Voting In a certain precinct, 42% of the eligible voters are registered Democrats, 33% are Republicans, and the remainder are Independents. During the last primary election, 45% of the Democrats, 37% of the Republicans, and 35% of the Independents voted. Find the probability that a person who voted is a Democrat.

4. Imported versus Domestic Tires Out of 3000 tires in the warehouse of a tire distributor, 2000 tires are domestic and 1000 are imported. Among the domestic tires, 40% are all-season; of the imported tires, 10% are all-season. If a tire is selected at random and it is an all-season, what is the probability that it is imported?

5. Disease Testing A new test was developed for detecting Gamma's disease, which is believed to affect 3% of the population. Results of extensive testing indicate that 86% of persons who have this disease will have a positive reaction to the test, whereas 7% of those who do not have the disease will also have a positive reaction.
(a) What is the probability that a randomly selected person who has a positive reaction will actually have Gamma's disease?
(b) What is the probability that a randomly selected person who has a negative reaction will actually have Gamma's disease?

6. Earnings and Dividends Of the companies in a particular sector of the economy, it is believed that 1/4 will have an increase in quarterly earnings. Of those that do, 2/3 will declare a dividend. Of those that do not have an increase, 1/10 will declare a dividend. What percentage of companies that declare a dividend will have an increase in quarterly earnings?

7. Jelly Beans in a Bag A bag contains four red and two green jelly beans, and a second bag contains two red and three green jelly beans. A bag is selected at random and a jelly bean is randomly taken from it. The jelly bean is red. What is the probability that it came from the first bag?

8. Balls in a Bowl Bowl I contains three red, two white, and five green balls. Bowl II contains three red, six white, and nine green balls. Bowl III contains six red, two white, and two green balls. A bowl is chosen at random, and then a ball is chosen at random from it. The ball is red. Find the probability that it came from Bowl II.

9. Quality Control A manufacturing process requires the use of a robotic welder on each of two assembly lines, A and B, which produce 300 and 500 units of product per day, respectively. Based on experience, it is believed that the welder on A produces 2% defective units, whereas the welder on B produces 5% defective units. At the end of a day, a unit was selected at random from the total production and was found to be defective. What is the probability that it came from line A?

10. Quality Control An automobile manufacturer has four plants: A, B, C, and D. The percentages of total daily output that are produced by the four plants are 35%, 20%, 30%, and 15%, respectively. The percentages of defective units produced by the plants are estimated to be 2%, 5%, 3%, and 4%, respectively. Suppose that a car on a dealer's lot is randomly selected and found to be defective. What is the probability that it came from plant **(a)** A? **(b)** B? **(c)** C? **(d)** D?

11. Wake-Up Call Barbara Smith, a sales representative, is staying overnight at a hotel and has a breakfast meeting with an important client the following morning. She asked the front desk to give her a 6 A.M. wake-up call so she can be prompt for the meeting. The probability that the desk makes the call is 0.9. If the call is made, the probability that she will be on time is 0.9, but if the call is not made, the probability that she will be on time is 0.7. If she is on time for the meeting, what is the probability that the call was made?

12. Candy Snatcher On a high shelf are two identical opaque candy jars containing 50 raisin clusters each. The clusters in one of the jars are made with dark chocolate. In the other jar, 20 are made with dark chocolate and 30 are made with milk chocolate. (They are mixed well, however.) Bob Jones, who has a sudden craving for chocolate, reaches up and randomly takes a raisin cluster from one of the jars. If it is made with dark chocolate, what is the probability that it was taken from the jar containing only dark chocolate?

13. Physical Fitness Activity During the week of National Employee Health and Fitness Day, the employees of a large company were asked to exercise a minimum of three times that week for at least 20 minutes per session. The purpose was to generate “exercise miles”. All participants who completed this requirement received a certificate acknowledging their contribution. The activities reported were power walking, cycling, and running. Of all who participated, $\frac{1}{3}$ reported power walking, $\frac{1}{2}$ reported cycling, and $\frac{1}{6}$ reported running. Suppose that the probability that a participant who power walks will complete the requirement is $\frac{9}{10}$, and for cycling and running it is $\frac{2}{3}$ and $\frac{1}{3}$, respectively. What percentage of persons who completed the requirement do you expect reported power walking? (Assume that each participant got his or her exercise from only one activity.)

14. Battery Reliability When the weather is extremely frigid, a motorist must charge his car battery during the night in order to improve the likelihood that the car will start early the following morning. If he does not charge it, the probability that the car will not start is $\frac{4}{5}$. If he does charge it, the probability that the car will not start is $\frac{1}{8}$. Past experience shows that the probability that he remembers to charge the battery is $\frac{9}{10}$. One morning, during a cold spell, he cannot start his car. What is the probability that he forgot to charge the battery?

15. Automobile Satisfaction Survey In a customer satisfaction survey, $\frac{3}{5}$ of those surveyed had a Japanese-made car, $\frac{1}{10}$ a European-made car, and $\frac{3}{10}$ an American-made car. Of the first group, 85% said they would buy the same make of car again, and for the other two groups the corresponding percentages are 50% and 40%. What is the probability that a person who said he or she would buy the same make again had a Japanese-made car?

16. Mineral Test Borings A geologist believes that the probability that the rare earth mineral dalhousium occurs in the Greater Toronto region is 0.001. If dalhousium is present in that region, the geologist's test borings will have a positive result 90% of the time. However, if dalhousium is not present, a negative result will occur 80% of the time.

(a) If a test is positive on a site in the region, find the probability that dalhousium is there.

(b) If a test is negative on such a site, find the probability that dalhousium is there.

17. Physics Exam After a physics exam was given, it turned out that only 75% of the class answered every question. Of those who did, 80% passed, but of those who did not, only 50% passed. If a student passed the exam, what is the probability that the student answered every question? (P.S.: The instructor eventually reached the conclusion that the test was too long and curved the exam grades, to be fair and merciful.)

18. Giving Up Smoking In a 2004 survey of smokers, 50% predicted that they would still be smoking five years later. Five years later, 80% of those who predicted that they would be smoking did not smoke, and of those who predicted that they would not be smoking, 95% did not smoke. What percentage of those who were not smoking after five years had predicted that they would be smoking?

19. Alien Communication B. G. Cosmos, a scientist, believes that the probability is $\frac{2}{5}$ that aliens from an advanced civilization on Planet X are trying to communicate with us by sending high-frequency signals to Earth. By using sophisticated equipment, Cosmos hopes to pick up these signals. The manufacturer of the equipment, Trekee, Inc., claims that if aliens are indeed sending signals, the probability that the equipment will detect them is $\frac{3}{5}$. However, if aliens are not sending signals, the probability that the equipment will seem to detect such signals is $\frac{1}{10}$. If the equipment detects signals, what is the probability that aliens are actually sending them?

20. Calculus Grades In an honors Calculus I class, 60% of students had an A average at midterm. Of these, 70% ended up with a course grade of A, and of those who did not have an A average at midterm, 60% ended up with a course grade of A. If one of the students is selected at random and is found to have received an A for the course, what is the probability that the student did not have an A average at midterm?

21. Movie Critique A well-known pair of highly influential movie critics have a popular TV show on which they review new movie releases and recently released videos. Over the past 10 years, they gave a “Two Thumbs Up” to 60% of movies that turned out to be box-office successes; they gave a “Two Thumbs Down” to 90% of movies that proved to be unsuccessful. A new movie, *Math Guru*, whose release is imminent, is considered favorably by others in the industry who have previewed it; in fact, they give it a prior probability of success of 90%. Find the probability that it will be a success, given that the pair of TV critics give it a “Two Thumbs Down” after seeing it. Assume that all films are given either “Two Thumbs Up” or “Two Thumbs Down”.

22. Balls in a Bowl Bowl 1 contains five green and four red balls, and Bowl 2 contains three green, one white, and three red balls. A ball is randomly taken from Bowl 1 and placed in Bowl 2. A ball is then randomly taken from Bowl 2. If the ball is green, find the probability that a green ball was taken from Bowl 1.

23. Risky Loan In the loan department of The Bank of Montreal, past experience indicates that 25% of loan requests are considered by bank examiners to fall into the “substandard” class and should not be approved. However, the bank's loan reviewer, Mr. Blackwell, is lax at times and concludes that a request is not in the substandard class when it is, and vice versa. Suppose that 15% of requests that are actually substandard are not considered substandard by Blackwell and that 10% of requests that are not substandard are considered by Blackwell to be substandard and, hence, not approved.

(a) Find the probability that Blackwell considers that a request is substandard.

(b) Find the probability that a request is substandard, given that Blackwell considers it to be substandard.

(c) Find the probability that Blackwell makes an error in considering a request. (An error occurs when the request is not substandard but is considered substandard, or when the request is substandard but is considered to be not substandard.)

24. Coins in Chests Each of three identical chests has two drawers. The first chest contains a gold coin in each drawer. The second chest contains a silver coin in each drawer, and the third contains a silver coin in one drawer and a gold coin in the other. A chest is chosen at random and a drawer is opened. There is a gold coin in it. What is the probability that the coin in the other drawer of that chest is silver?

Chapter 8 Review

Important Terms and Symbols

	Important Terms and Symbols	Examples
Section 8.1	**Basic Counting Principle and Permutations**	
	tree diagram Basic Counting Principle	Ex. 1, p. 350
	permutation, ${}_nP_r$	Ex. 5, p. 352
Section 8.2	**Combinations and Other Counting Principles**	
	combination, ${}_nC_r$	Ex. 2, p. 357
	permutation with repeated objects cells	Ex. 6, p. 363
Section 8.3	**Sample Spaces and Events**	
	sample space sample point finite sample space	Ex. 1, p. 368
	event certain event impossible event simple event	Ex. 6, p. 370
	Venn diagram complement, E' union, $\cup$ intersection, $\cap$	Ex. 7, p. 371
	mutually exclusive events	Ex. 8, p. 373
Section 8.4	**Probability**	
	equally likely outcomes trial relative frequency	Ex. 1, p. 376
	equiprobable space probability of event, $P(E)$	Ex. 2, p. 377
	addition law for mutually exclusive events empirical probability	Ex. 5, p. 379
	odds	Ex. 9, p. 384
Section 8.5	**Conditional Probability and Stochastic Processes**	
	conditional probability, $P(E \mid F)$ reduced sample space	Ex. 1, p. 390
	general multiplication law trial compound experiment	Ex. 5, p. 393
	probability tree	Ex. 6, p. 394
Section 8.6	**Independent Events**	
	independent events dependent events	Ex. 1, p. 402
	special multiplication law	Ex. 3, p. 403
Section 8.7	**Bayes' Formula**	
	partition prior probability posterior probability	Ex. 1, p. 414
	Bayes' formula Bayes' probability tree	Ex. 2, p. 416

Summary

It is important to know the number of ways a procedure can occur. Suppose a procedure involves a sequence of k stages. Let n_1 be the number of ways the first stage can occur, and n_2 the number of ways the second stage can occur, and so on, with n_k the number of ways the kth stage can occur. Then the number of ways the procedure can occur is

$$n_1 \cdot n_2 \cdots n_k$$

This result is called the Basic Counting Principle.

An ordered selection of r objects, without repetition, taken from n distinct objects is called a permutation of the n objects taken r at a time. The number of such permutations is denoted ${}_nP_r$ and is given by

$${}_nP_r = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ factors}} = \frac{n!}{(n-r)!}$$

If the selection is made without regard to order, then it is simply an r-element subset of an n-element set and is called a combination of n objects taken r at a time. The number of such combinations is denoted ${}_nC_r$ and is given by

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

When some of the objects are repeated, the number of distinguishable permutations of n objects, such that n_1 are of one type, n_2 are of a second type, and so on, and n_k are of a kth type, is

$$\frac{n!}{n_1!n_2!\cdots n_k!} \tag{5}$$

where $n_1 + n_2 + \cdots + n_k = n$.

The expression in Equation (5) can also be used to determine the number of assignments of objects to cells. If n distinct objects are placed into k ordered cells, with n_i objects in cell i, for $i = 1, 2, \ldots, k$, then the number of such assignments is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

where $n_1 + n_2 + \cdots + n_k = n$.

A sample space for an experiment is a set S of all possible outcomes of the experiment. These outcomes are called sample points. A subset E of S is called an event. Two special events are the sample space itself, which is a certain event, and the empty set, which is an impossible event. An event consisting of a single sample point is called a simple event. Two events are said to be mutually exclusive when they have no sample point in common.

A sample space whose outcomes are equally likely is called an equiprobable space. If E is an event for a finite equiprobable space S, then the probability that E occurs is given by

$$P(E) = \frac{\#(E)}{\#(S)}$$

If F is also an event in S, we have

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
$$P(E \cup F) = P(E) + P(F) \quad \text{for } E \text{ and } F \text{ mutually exclusive}$$
$$P(E') = 1 - P(E)$$
$$P(S) = 1$$
$$P(\emptyset) = 0$$

For an event E, the ratio

$$\frac{P(E)}{P(E')} = \frac{P(E)}{1 - P(E)}$$

gives the odds that E occurs. Conversely, if the odds that E occurs are $a : b$, then

$$P(E) = \frac{a}{a+b}$$

The probability that an event E occurs, given that event F has occurred, is called a conditional probability. It is denoted by $P(E \mid F)$ and can be computed either by considering a reduced equiprobable sample space and using the formula

$$P(E \mid F) = \frac{\#(E \cap F)}{\#(F)}$$

or from the formula

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

which involves probabilities with respect to the original sample space.

To find the probability that two events both occur, we can use the general multiplication law:

$$P(E \cap F) = P(E)P(F \mid E) = P(F)P(E \mid F)$$

Here we multiply the probability that one of the events occurs by the conditional probability that the other one occurs, given that the first has occurred. For more than two events, the corresponding law is

$$\begin{aligned} &P(E_1 \cap E_2 \cap \cdots \cap E_n) \\ &\quad = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1 \cap E_2)\cdots \\ &\qquad P(E_n \mid E_1 \cap E_2 \cap \cdots \cap E_{n-1}) \end{aligned}$$

The general multiplication law is also called the law of compound probability, because it is useful when applied to a compound experiment—one that can be expressed as a sequence of two or more other experiments, called trials or stages.

When we analyze a compound experiment, a probability tree is extremely useful in keeping track of the possible outcomes for each trial of the experiment. A path is a complete sequence of branches from the start to a tip of the tree. Each path represents an outcome of the compound experiment, and the probability of that path is the product of the probabilities for the branches of the path.

Events E and F are independent when the occurrence of one of them does not affect the probability of the other; that is,

$$P(E \mid F) = P(E) \quad \text{or} \quad P(F \mid E) = P(F)$$

Events that are not independent are dependent.

If E and F are independent, the general multiplication law simplifies to the special multiplication law:

$$P(E \cap F) = P(E)P(F)$$

Here the probability that E and F both occur is the probability of E times the probability of F. The preceding equation forms the basis of an alternative definition of independence: Events E and F are independent if and only if

$$P(E \cap F) = P(E)P(F)$$

Three or more events are independent if and only if for each set of two or more of the events, the probability of the intersection of the events in that set is equal to the product of the probabilities of those events.

A partition divides a sample space into mutually exclusive events. If E is an event and $F_1, F_2, \ldots, F_n$ is a partition, then, to find the conditional probability of event F_i, given E, when prior and conditional probabilities are known, we can use Bayes' formula:

$$P(F_i \mid E) = \frac{P(F_i)P(E \mid F_i)}{P(F_1)P(E \mid F_1) + P(F_2)P(E \mid F_2) + \cdots + P(F_n)P(E \mid F_n)}$$

A Bayes-type problem can also be solved with the aid of a Bayes probability tree.

Review Problems

In Problems 1–4, determine the values.

1. ${}_8P_3$ **2.** ${}_rP_1$ **3.** ${}_9C_7$ **4.** ${}_{12}C_5$

5. License Plate A six-character license plate consists of three letters followed by three numbers. How many different license plates are possible?

6. Dinner In a restaurant, a complete dinner consists of one appetizer, one entree, and one dessert. The choices for the appetizer are soup and salad; for the entree, chicken, steak, lobster, and veal; and for the dessert, ice cream, pie, and pudding. How many complete dinners are possible?

7. Garage-Door Opener The transmitter for an electric garage-door opener transmits a coded signal to a receiver. The code is determined by 10 switches, each of which is either in an "on" or "off" position. Determine the number of different codes that can be transmitted.

8. Baseball A baseball manager must determine a batting order for his nine-member team. How many batting orders are possible?

9. Softball A softball league has seven teams. In terms of first, second, and third place, in how many ways can the season end? Assume that there are no ties.

10. Trophies In a trophy case, nine different trophies are to be placed—two on the top shelf, three on the middle, and four on the bottom. Considering the order of arrangement on each shelf, in how many ways can the trophies be placed in the case?

11. Groups Eleven stranded wait-listed passengers surge to the counter for boarding passes. But there are only six boarding passes available. How many different groups of passengers can board?

12. Cards From a 52-card deck of playing cards, a five-card hand is dealt. In how many ways can exactly two of the cards be of one denomination and exactly two be of another denomination? (Such a hand is called *two pairs*.)

13. Light Bulbs A carton contains 24 light bulbs, one of which is defective. **(a)** In how many ways can three bulbs be selected? **(b)** In how many ways can three bulbs be selected if one is defective?

14. Multiple-Choice Exam Each question of a 10-question multiple-choice examination is worth 10 points and has four choices, only one of which is correct. By guessing, in how many ways is it possible to receive a score of 90 or better?

15. Letter Arrangement How many distinguishable horizontal arrangements of the letters in MISSISSIPPI are possible?

16. Flag Signals Colored flags arranged vertically on a flagpole indicate a signal (or message). How many different signals are possible if two red, three green, and four white flags are all used?

17. Personnel Agency A mathematics professor personnel agency provides mathematics professors on a temporary basis to universities that are short of staff. The manager has a pool of 17 professors and must send four to Dalhousie University, seven to St. Mary's, and three to Mount Saint Vincent. In how many ways can the manager make assignments?

18. Tour Operator A tour operator has three vans, and each can accommodate seven tourists. Suppose 14 people arrive for a city sightseeing tour and the operator will use only two vans. In how many ways can the operator assign the people to the vans?

19. Suppose $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is the sample space and $E_1 = \{1, 2, 3, 4, 5, 6\}$ and $E_2 = \{4, 5, 6, 7\}$ are events for an experiment. Find **(a)** $E_1 \cup E_2$, **(b)** $E_1 \cap E_2$, **(c)** $E_1{}' \cup E_2$, **(d)** $E_1 \cap E_1{}'$, and **(e)** $(E_1 \cap E_2{}')'$. **(f)** Are E_1 and E_2 mutually exclusive?

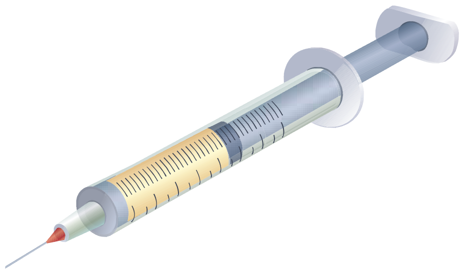
20. Die and Coin A die is rolled and then a coin is tossed. **(a)** Determine a sample space for this experiment. Determine the events that **(b)** a 2 shows and **(c)** a head and an even number show.

21. Bags of Jelly Beans Three bags, labeled 1, 2, and 3, each contain two jelly beans, one red and the other green. A jelly bean is selected at random from each bag. **(a)** Determine a sample space for this experiment. Determine the events that **(b)** exactly two jelly beans are red and **(c)** the jelly beans are the same color.

22. Suppose that E and F are events for an experiment. If $P(E) = 0.5$, $P(E \cup F) = 0.6$, and $P(E \cap F) = 0.1$, find $P(E' \cap F')$.

23. Quality Control A manufacturer of computer chips packages 10 chips to a box. For quality control, two chips are selected at random from each box and tested. If any one of the tested chips is defective, the entire box of chips is rejected for sale. For a box that contains exactly one defective chip, what is the probability that the box is rejected?

24. Drugs Each of 100 white rats was injected with one of four drugs, A, B, C, or D. Drug A was given to 35%, B to 25%, and C to 15%. If a rat is chosen at random, determine the probability that it was injected with either C or D. If the experiment is repeated on a larger group of 300 rats but with the drugs given in the same proportion, what is the effect on the previous probability?



25. Multiple-Choice Exam Each question on a five-question multiple-choice examination has four choices, only one of which is correct. If a student answers each question in a random fashion, what is the probability that the student answers exactly two questions incorrectly?

26. Cola Preference To determine the national preference of cola drinkers, an advertising agency conducted a survey of 200 of them. Two cola brands, A and B, were involved. The results of the survey are indicated in Table 8.14. If a cola drinker is selected at random, determine the (empirical) probability that the person

(a) Likes both A and B
(b) Likes A, but not B

Table 8.14 Cola Preference

Like A only	70
Like B only	80
Like both A and B	35
Like neither A nor B	15
Total	200

27. Jelly Beans in a Bag A bag contains six red and six green jelly beans.

(a) If two jelly beans are randomly selected in succession with replacement, determine the probability that both are red.
(b) If the selection is made without replacement, determine the probability that both are red.

28. Dice A pair of fair dice is rolled. Determine the probability that the sum of the numbers is **(a)** 2 or 7, **(b)** a multiple of 3, and **(c)** no less than 7.

29. Cards Three cards from a standard deck of 52 playing cards are randomly drawn in succession with replacement. Determine the probability that **(a)** all three cards are black and **(b)** two cards are black and the other is a diamond.

30. Cards Two cards from a standard deck of 52 playing cards are randomly drawn in succession without replacement. Determine the probability that **(a)** both are hearts and **(b)** one is an ace and the other is a red king.

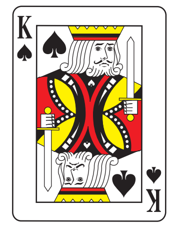
In Problems 31 and 32, for the given value of P(E), find the odds that E will occur.

31. $P(E) = \frac{3}{8}$ **32.** $P(E) = 0.93$

In Problems 33 and 34, the odds that E will occur are given. Find P(E).

33. 6 : 1 **34.** 3 : 4

35. Cards If a card is randomly drawn from a fair deck of 52 cards, find the probability that it is not a face card (a jack, queen, or king), given that it is a heart.



36. Dice If two fair dice are rolled, find the probability that the sum is less than 7, given that a 6 shows on at least one of the dice.

37. Movie and Sequel The probability that a particular movie will be successful is 0.55, and if it is successful, the probability that a sequel will be made is 0.60. Find the probability that the movie will be successful and followed by a sequel.

38. Cards Three cards are drawn from a standard deck of cards. Find the probability that the cards are, in order, a queen, a heart, and the ace of clubs if the cards are drawn with replacement.

39. Dice If two dice are thrown, find each of the following.

(a) The probability of getting a total of 7, given that a 4 occurred on at least one die
(b) The probability of getting a total of 7 and that a 4 occurred on at least one die

40. Die A fair die is tossed two times in succession. Find the probability that the first toss is less than 4, given that the total is greater than 8.

41. Die If a fair die is tossed two times in succession, find the probability that the first number is less than or equal to the second number, given that the second number is less than 3.

42. Cards Three cards are drawn without replacement from a standard deck of cards. Find the probability that the third card is a club.

43. Seasoning Survey A survey of 600 adults was made to determine whether or not they liked the taste of a new seasoning. The results are summarized in Table 8.15.

Table 8.15 Seasoning Survey

	Like	Dislike	Total
Male	80	40	120
Female	320	160	480
Total	400	200	600

(a) If a person in the survey is selected at random, find the probability that the person dislikes the seasoning (L'), given that the person is a female (F).
(b) Determine whether the events $L = \{\text{liking the seasoning}\}$ and $M = \{\text{being a male}\}$ are independent or dependent.

44. Chips A bowl contains six chips numbered from 1 to 6. Two chips are randomly withdrawn with replacement. Let E be the event of getting a 4 the first time and F be the event of getting a 4 the second time.

(a) Are E and F mutually exclusive?
(b) Are E and F independent?

45. College and Family Income A survey of 175 students resulted in the data shown in Table 8.16. The table shows the type of college the student attends and the income level of the student's family. If a student is selected at random, determine whether the event of attending a public college and the event of coming from a middle-class family are independent or dependent.

Table 8.16 Student Survey

	College		
Income	Private	Public	Total
High	15	10	25
Middle	25	55	80
Low	10	60	70
Total	50	125	175

46. If $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{3}$, and $P(E \mid F) = \frac{1}{6}$, find $P(E \cup F)$.

47. Shrubs When a certain type of shrub is planted, the probability that it will take root is 0.7. If four shrubs are planted, find each of the following. Assume independence.

(a) The probability that all of them take root
(b) The probability that exactly three of them take root
(c) The probability that at least three of them take root

48. Antibiotic A certain antibiotic is effective for 75% of the people who take it. Suppose four persons take this drug. What is the probability that it will be effective for at least three of them? Assume independence.

49. Bags of Jelly Beans Bag I contains three green and two red jelly beans, and Bag II contains four red, two green, and two white jelly beans. A jelly bean is randomly taken from Bag I and placed in Bag II. If a jelly bean is then randomly taken from Bag II, find the probability that the jelly bean is red.

50. Bags of Jelly Beans Bag I contains four red and two white jelly beans. Bag II contains two red and three white jelly beans. A bag is chosen at random, and then a jelly bean is randomly taken from it.

(a) What is the probability that the jelly bean is white?
(b) If the jelly bean is white, what is the probability that it was taken from Bag II?

51. Grade Distribution Last semester, the grade distribution for a certain class taking an upper-level college course was analyzed. It was found that the proportion of students receiving a grade of A was 0.4 and the proportion getting an A and being a graduate student was 0.1. If a student is randomly selected from this class and is found to have received an A, find the probability that the student is a graduate student.

52. Alumni Reunion At the most recent alumni day at Alpha University, 735 persons attended. Of these, 603 lived within the state, and 43% of them were attending for the first time. Among the alumni who lived out of the state, 72% were attending for the first time. That day a raffle was held, and the person who won had also won it the year before. Find the probability that the winner was from out of state.

53. Quality Control A music company burns CDs on two shifts. The first shift produces 3000 discs per day, and the second produces 5000. From past experience, it is believed that of the output produced by the first and second shifts, 1% and 2% are scratched, respectively. At the end of a day, a disc was selected at random from the total production.

(a) Find the probability that the CD is scratched.
(b) If the CD is scratched, find the probability that it came from the first shift.

54. Aptitude Test In the past, a company has hired only experienced personnel for its word-processing department. Because of a shortage in this field, the company has decided to hire inexperienced persons and will provide on-the-job training. It has supplied an employment agency with a new aptitude test that has been designed for applicants who desire such a training position. Of those who recently took the test, 35% passed. In order to gauge the effectiveness of the test, everyone who took the test was put in the training program. Of those who passed the test, 80% performed satisfactorily, whereas of those who failed, only 30% did satisfactorily. If one of the new trainees is selected at random and is found to be satisfactory, what is the probability that the person passed the exam?