PROBLEMS 14.1

In Problems 1–10, find the differential of the function in terms of x and dx.

1. $y = ax + b$

2. $y = 2$

3. $f(x) = \sqrt{x^3 - 27}$

4. $f(x) = (4x^2 - 5x + 2)^3$

5. $u = \dfrac{1}{x^2}$

6. $u = \sqrt{x}$

7. $p = \ln(x^2 + 7)$

8. $p = e^{x^4+3x^2+1}$

9. $y = (9x + 3)e^{2x^2+3}$

10. $y = \ln\sqrt{x^2 + 12}$

In Problems 11–16, find Δy and dy for the given values of x and dx.

11. $y = ax + b$; for any x and any dx

12. $y = 5x^2$; $x = -1$, $dx = -0.02$

13. $y = x^2 + 3x + 5$; $x = 2$, $dx = 0.01$

14. $y = (3x + 2)^2$; $x = -1$, $dx = -0.03$

15. $y = \sqrt{32 - x^2}$; $x = 4$, $dx = -0.05$ Round your answer to three decimal places.

16. $y = \ln x$; $x = 1$, $dx = 0.01$

17. Let $f(x) = \dfrac{x+5}{x+1}$.

(a) Evaluate $f'(1)$.

(b) Use differentials to estimate the value of $f(1.1)$.

18. Let $f(x) = x^x$.

(a) Evaluate $f'(1)$.

(b) Use differentials to estimate the value of $f(1.001)$.

In Problems 19–26, approximate each expression by using differentials.

19. $\sqrt{288}$ (*Hint:* $17^2 = 289$.)

20. $\sqrt{122}$

21. $\sqrt[3]{9}$

22. $\sqrt[4]{16.3}$

23. $\ln(0.998)$

24. $\ln 1.01$

25. $e^{0.001}$

26. $e^{-0.002}$

In Problems 27–32, find dx/dy or dp/dq as makes sense.

27. $y = 2x - 1$

28. $y = 2x^3 + 2x + 3$

29. $q = (p^2 + 5)^3$

30. $q = \sqrt{p + 5}$

31. $q = \dfrac{1}{p^2}$

32. $q = e^{4-2p}$

33. If $y = 5x^3 - \dfrac{7}{2}x^2 + 3$, find $dx/dy\Big|_{x=1/3}$.

34. If $y = \ln x^2$, find the value of dx/dy when $x = 3$.

In Problems 35 and 36, find the rate of change of q with respect to p for the indicated value of q.

35. $p = \dfrac{500}{q+2}$; $q = 18$

36. $p = 60 - \sqrt{2q}$; $q = 50$

37. Profit Suppose that the profit (in dollars) of producing q units of a product is

$$P = 397q - 2.3q^2 - 400$$

Using differentials, find the approximate change in profit if the level of production changes from $q = 90$ to $q = 91$. Find the true change.

38. Revenue Given the revenue function

$$r = 200q + 40q^2 - q^3$$

use differentials to find the approximate change in revenue if the number of units increases from $q = 10$ $q = 11$. Find the true change.

39. Demand The demand equation for a product is

$$p = \frac{10}{\sqrt{q}}$$

Using differentials, approximate the price when 24 units are demanded.

40. Demand Given the demand function

$$p = \frac{200}{\sqrt{q+8}}$$

use differentials to estimate the price per unit when 40 units are demanded.

41. If $y = f(x)$, then the *proportional change in y* is defined to be $\Delta y/y$, which can be approximated with differentials by dy/y. Use this last form to approximate the proportional change in the cost function

$$c = f(q) = \frac{q^2}{2} + 5q + 300$$

when $q = 10$ and $dq = 2$. Round your answer to one decimal place.

42. Status/Income Suppose that S is a numerical value of status based on a person's annual income I (in thousands of dollars). For a certain population, suppose $S = 20\sqrt{I}$. Use differentials to approximate the change in S if annual income decreases from \$45,000 to \$44,500.

43. Biology The volume of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Estimate the change in volume when the radius changes from 5.40×10^{-4} cm to 5.45×10^{-4} cm.

44. Muscle Contraction The equation

$$(P + a)(v + b) = k$$

is called the "fundamental equation of muscle contraction."[1] Here P is the load imposed on the muscle, v is the velocity of the shortening of the muscle fibers, and a, b, and k are positive constants. Find P in terms of v, and then use the differential to approximate the change in P due to a small change in v.

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[1]R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

45. Demand The demand, q, for a monopolist's product is related to the price per unit, p, according to the equation

$$2 + \frac{q^2}{200} = \frac{4000}{p^2}$$

(a) Verify that 40 units will be demanded when the price per unit is \$20.

(b) Show that $\frac{dq}{dp} = -2.5$ when the price per unit is \$20.

(c) Use differentials and the results of parts (a) and (b) to approximate the number of units that will be demanded if the price per unit is reduced to \$19.20.

46. Profit The demand equation for a monopolist's product is

$$p = \frac{1}{2}q^2 - 66q + 7000$$

and the average-cost function is

$$\bar{c} = 500 - q + \frac{80{,}000}{2q}$$

(a) Find the profit when 100 units are demanded.

(b) Use differentials and the result of part (a) to estimate the profit when 101 units are demanded.

Objective

To define the antiderivative and the indefinite integral and to apply basic integration formulas.

14.2 The Indefinite Integral

Given a function f, if F is a function such that

$$F'(x) = f(x) \tag{1}$$

then F is called an *antiderivative* of f. Thus,

An *antiderivative* of f is simply a function whose derivative is f.

Multiplying both sides of Equation (1) by the differential dx gives $F'(x)dx = f(x)dx$. However, because $F'(x)dx$ is the differential of F, we have $dF = f(x)dx$. Hence, we can also think of an antiderivative of f as a function whose differential is $f(x)dx$.

Definition

An **antiderivative** of a function f is a function F such that

$$F'(x) = f(x)$$

Equivalently, in differential notation,

$$dF = f(x)dx$$

For example, because the derivative of x^2 is $2x$, x^2 is an antiderivative of $2x$. However, it is not the only antiderivative of $2x$: Since

$$\frac{d}{dx}(x^2 + 1) = 2x \qquad \text{and} \qquad \frac{d}{dx}(x^2 - 5) = 2x$$

both x^2+1 and x^2-5 are also antiderivatives of $2x$. In fact, it is obvious that because the derivative of a constant is zero, x^2+C is also an antiderivative of $2x$ for *any* constant C. Thus, $2x$ has infinitely many antiderivatives. More importantly, although not obviously, *every* antiderivative of $2x$ is a function of the form x^2+C, for some constant C. It can be shown that if a continuous function has a derivative of 0 on an interval then the function is constant on that interval. We note:

Any two antiderivatives of a function differ only by a constant.

Since $x^2 + C$ describes all antiderivatives of $2x$, we refer to it as being the *most general antiderivative* of $2x$, and denote it by $\int 2xdx$, which is read "the *indefinite integral* of $2x$ with respect to x." In fact, we write

$$\int 2xdx = x^2 + C$$

EXAMPLE 9 **Using Algebraic Manipulation to Find an Indefinite Integral**

a. Find $\int \frac{(2x-1)(x+3)}{6}dx.$

Solution: By factoring out the constant $\frac{1}{6}$ and multiplying the binomials, we get

$$\int \frac{(2x-1)(x+3)}{6}\,dx = \frac{1}{6}\int (2x^2+5x-3)\,dx$$
$$= \frac{1}{6}\left((2)\frac{x^3}{3}+(5)\frac{x^2}{2}-3x\right)+C$$
$$= \frac{x^3}{9}+\frac{5x^2}{12}-\frac{x}{2}+C$$

Another algebraic approach to part (b) is

$$\int \frac{x^3-1}{x^2}\,dx = \int (x^3-1)x^{-2}\,dx$$
$$= \int (x-x^{-2})\,dx$$

and so on.

b. Find $\int \frac{x^3-1}{x^2}dx.$

Solution: We can break up the integrand into fractions by dividing each term in the numerator by the denominator:

$$\int \frac{x^3-1}{x^2}\,dx = \int \left(\frac{x^3}{x^2}-\frac{1}{x^2}\right)dx = \int (x-x^{-2})\,dx$$
$$= \frac{x^2}{2}-\frac{x^{-1}}{-1}+C = \frac{x^2}{2}+\frac{1}{x}+C$$

Now Work Problem 49 ◁

PROBLEMS 14.2

In Problems 1–52, find the indefinite integrals.

1. $\int 7\,dx$

2. $\int \frac{1}{x}\,dx$

3. $\int x^8\,dx$

4. $\int 3x^{37}dx$

5. $\int 5x^{-7}\,dx$

6. $\int \frac{z^{-3}}{3}\,dz$

7. $\int \frac{5}{x^7}\,dx$

8. $\int \frac{7}{x^4}dx$

9. $\int \frac{1}{t^{5/2}}dt$

10. $\int \frac{7}{2x^{9/4}}\,dx$

11. $\int (4+t)\,dt$

12. $\int (7r^5+4r^2+1)\,dr$

13. $\int (y^5-5y)\,dy$

14. $\int (2-3w-5w^2)dw$

15. $\int (3t^2-4t+5)\,dt$

16. $\int (1+t^2+t^4+t^6)\,dt$

17. $\int (\sqrt{2}+e)\,dx$

18. $\int (5-2^{-1})dx$

19. $\left(\frac{x}{7}-\frac{2}{3}x^5\right)$

20. $\int \left(\frac{2x^2}{7}-\frac{8}{3}x^4\right)dx$

21. $\int \pi e^x\,dx$

22. $\int (e^x+3x^2+2x)\,dx$

23. $\int (x^{8.3}-9x^6+3x^{-4}+x^{-3})\,dx$

24. $\int (0.3y^4+2y^{-2})dy$

25. $\int \frac{-2\sqrt{x}}{3}\,dx$

26. $\int dz$

27. $\int \frac{5}{3\sqrt[3]{x^2}}\,dx$

28. $\int \frac{-4}{(3x)^3}\,dx$

29. $\int \left(\frac{x^4}{4}-\frac{4}{x^4}\right)dx$

30. $\int \left(\frac{1}{2x^3}-\frac{1}{x^4}\right)dx$

31. $\int \left(\frac{3w^2}{2}-\frac{2}{3w^2}\right)dw$

32. $\int 7e^{-s}\,ds$

33. $\int \frac{3u-4}{5}\,du$

34. $\int \frac{1}{e}\left(\frac{2}{3}e^x\right)dx$

35. $\int (u^e+e^u)\,du$

36. $\int \left(3y^3-2y^2+\frac{e^y}{6}\right)dy$

37. $\int \left(\frac{3}{\sqrt{x}}-12\sqrt[3]{x}\right)dx$

38. $\int 0\,dt$

39. $\int \left(\frac{\sqrt[5]{x^3}}{5}-\frac{2}{5\sqrt{x}}+711x\right)dx$

40. $\int \left(\sqrt[3]{u}+\frac{1}{\sqrt{u}}\right)du$

41. $\int (x^2+5)(x-3)\,dx$

42. $\int x^3(x^2 + 5x + 2)\,dx$

43. $\int \sqrt{x}(x + 3)\,dx$

44. $\int (z - 3)^3 dz$

45. $\int (3u + 2)^3\,du$

46. $\int \left(\frac{2}{\sqrt[5]{x}} - 1\right)^2 dx$

47. $\int x^{-2}(3x^4 + 4x^2 - 5)\,dx$

48. $\int (6e^u - u^3(\sqrt{u} + 1))\,du$

49. $\int \frac{z^5 + 7z^2}{3z} dz$

50. $\int \frac{x^4 - 5x^2 + 2x}{5x^2}\,dx$

51. $\int \frac{e^x + e^{2x}}{e^x}\,dx$

52. $\int \frac{(x^2 + 1)^3}{x}\,dx$

53. If $F(x)$ and $G(x)$ are such that $F'(x) = G'(x)$, is it true that $F(x) - G(x)$ must be zero?

54. **(a)** Find a function F such that $\int F(x)dx = x^2e^x + C$.
(b) How many functions F are there which satisfy the equation given in part (a)?

55. Find $\int \frac{d}{dx}\left(\frac{1}{\sqrt{x^2 + 1}}\right)\,dx$.

Objective

To find a particular antiderivative of a function that satisfies certain conditions. This involves evaluating constants of integration.

14.3 Integration with Initial Conditions

If we know the rate of change, f', of the function f, then the function f itself is an antiderivative of f' (since the derivative of f is f'). Of course, there are many antiderivatives of f', and the most general one is denoted by the indefinite integral. For example, if

$$f'(x) = 2x$$

then

$$f(x) = \int f'(x)dx = \int 2xdx = x^2C. \qquad (1)$$

That is, *any* function of the form $f(x) = x^2 + C$ has its derivative equal to $2x$. Because of the constant of integration, notice that we do not know $f(x)$ specifically. However, if f must assume a certain function value for a particular value of x, then we can determine the value of C and thus determine $f(x)$ specifically. For instance, if $f(1) = 4$, then, from Equation (1),

$$f(1) = 1^2 + C$$
$$4 = 1 + C$$
$$C = 3$$

Thus,

$$f(x) = x^2 + 3$$

That is, we now know the particular function $f(x)$ for which $f'(x) = 2x$ and $f(1) = 4$. The condition $f(1) = 4$, which gives a function value of f for a specific value of x, is called an **initial condition**.

APPLY IT ►

6. The rate of growth of a species of bacteria is estimated by $\frac{dN}{dt} = 800 + 200e^t$, where N is the number of bacteria (in thousands) after t hours. If $N(5) = 40{,}000$, find $N(t)$.

EXAMPLE 1 Initial-Condition Problem

If y is a function of x such that $y' = 8x - 4$ and $y(2) = 5$, find y. (Note: $y(2) = 5$ means that $y = 5$ when $x = 2$.) Also, find $y(4)$.

Solution: Here $y(2) = 5$ is the initial condition. Since $y' = 8x - 4$, y is an antiderivative of $8x - 4$,

$$y = \int (8x - 4)dx = 8 \cdot \frac{x^2}{2} - 4x + C = 4x^2 - 4x + C \qquad (2)$$

We can determine the value of C by using the initial condition. Because $y = 5$ when $x = 2$, from Equation (2), we have

$$5 = 4(2)^2 - 4(2) + C$$
$$5 = 16 - 8 + C$$
$$C = -3$$

Revenue is 0 when q is 0.

We assume that *when no units are sold, there is no revenue*; that is, $r = 0$ when $q = 0$. This is our initial condition. Putting these values into Equation (7) gives

$$0 = 2000(0) - 10(0)^2 - 0^3 + C$$

Although $q = 0$ gives $C = 0$, this is not true in general. It occurs in this section because the revenue functions are polynomials. In later sections, evaluating at $q = 0$ may produce a nonzero value for C.

Hence, $C = 0$, and

$$r = 2000q - 10q^2 - q^3$$

To find the demand function, we use the fact that $p = r/q$ and substitute for r:

$$p = \frac{r}{q} = \frac{2000q - 10q^2 - q^3}{q}$$

$$p = 2000 - 10q - q^2$$

Now Work Problem 11 ◁

EXAMPLE 5 Finding Cost from Marginal Cost

In the manufacture of a product, fixed costs per week are \$4000. Fixed costs are costs, such as rent and insurance, that remain constant at all levels of production during a given time period. If the marginal-cost function is

$$\frac{dc}{dq} = 0.000001(0.002q^2 - 25q) + 0.2$$

where c is the total cost (in dollars) of producing q kilograms of product per week, find the cost of producing 10,000 kg in 1 week.

Solution: Since dc/dq is the derivative of the total cost c,

$$c(q) = \int (0.000001(0.002q^2 - 25q) + 0.2)dq$$

$$= 0.000001 \int (0.002q^2 - 25q)dq + \int 0.2dq$$

$$c(q) = 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + C$$

When q is 0, total cost is equal to fixed cost.

Fixed costs are constant regardless of output. Therefore, when $q = 0$, $c = 4000$, which is our initial condition. Putting $c(0) = 4000$ in the last equation, we find that $C = 4000$, so

Although $q = 0$ gives C a value equal to fixed costs, this is not true in general. It occurs in this section because the cost functions are polynomials. In later sections, evaluating at $q = 0$ may produce a value for C that is different from fixed cost.

$$c(q) = 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + 4000 \qquad (8)$$

From Equation (8), we have $c(10,000) = 5416\frac{2}{3}$. Thus, the total cost for producing 10,000 pounds of product in 1 week is \$5416.67.

Now Work Problem 15 ◁

PROBLEMS 14.3

In Problems 1 and 2, find y, subject to the given conditions.

1. $dy/dx = 3x - 4; \quad y(-1) = \frac{13}{2}$

2. $dy/dx = x^2 - x; \quad y(3) = \frac{19}{2}$

In Problems 3 and 4, if y satisfies the given conditions, find y(x) for the given value of x.

3. $y' = \dfrac{9}{8\sqrt{x}}, y(16) = 10; \quad x = 9$

4. $y' = -x^2 + 2x, \ y(2) = 1; \quad x = 1$

In Problems 5–8, find y, subject to the given conditions.

5. $y'' = -5x^2 + 2x; \quad y'(1) = 0, \ y(0) = 3$

6. $y'' = x + 1; \quad y'(0) = 0, \ y(0) = 5$

7. $y''' = 2x; \quad y''(-1) = 3, \ y'(3) = 10, \ y(0) = 13$

8. $y''' = 2e^{-x} + 3; \quad y''(0) = 7, \ y'(0) = 5, \ y(0) = 1$

In Problems 9–12, dr/dq is a marginal-revenue function. Find the demand function.

9. $dr/dq = 0.7$

10. $dr/dq = 12 - \frac{1}{15}q$

11. $dr/dq = 275 - q - 0.3q^2$

12. $dr/dq = 5{,}000 - 3(2q + 2q^3)$

In Problems 13–16, dc/dq is a marginal-cost function and fixed costs are indicated in braces. For Problems 13 and 14, find the total-cost function. For Problems 15 and 16, find the total cost for the indicated value of q.

13. $dc/dq = 2.47; \quad \{159\}$

14. $dc/dq = 2q + 75; \quad \{2000\}$

15. $dc/dq = 0.09q^2 - 1.4q + 6.7; \quad \{8500\}; \quad q = 20$

16. $dc/dq = 0.000204q^2 - 0.046q + 6; \quad \{15{,}000\}; \quad q = 200$

17. Diet for Rats A group of biologists studied the nutritional effects on rats that were fed a diet containing 10% protein.[2] The protein consisted of yeast and corn flour.

Over a period of time, the group found that the (approximate) rate of change of the average weight gain G (in grams) of a rat with respect to the percentage P of yeast in the protein mix was

$$\frac{dG}{dP} = -\frac{P}{25} + 2 \qquad 0 \le P \le 100$$

If $G = 38$ when $P = 10$, find G.

18. Winter Moth A study of the winter moth was made in Nova Scotia.[3] The prepupae of the moth fall onto the ground from host trees. It was found that the (approximate) rate at which prepupal density, y (the number of prepupae per square foot of soil), changes with respect to distance, x (in feet), from the base of a host tree is

$$\frac{dy}{dx} = -1.5 - x \qquad 1 \le x \le 9$$

If $y = 59.6$ when $x = 1$, find y.

19. Fluid Flow In the study of the flow of fluid in a tube of constant radius R, such as blood flow in portions of the body, one can think of the tube as consisting of concentric tubes of radius r, where $0 \le r \le R$. The velocity, v, of the fluid is a function of r and is given by[4]

$$v = \int -\frac{(P_1 - P_2)r}{2l\eta}\, dr$$

where P_1 and P_2 are pressures at the ends of the tube, η (a Greek letter read "eta") is fluid viscosity, and l is the length of the tube. If $v = 0$ when $r = R$, show that

$$v = \frac{(P_1 - P_2)(R^2 - r^2)}{4l\eta}$$

20. Elasticity of Demand The sole producer of a product has determined that the marginal-revenue function is

$$\frac{dr}{dq} = 800 - 6q^2$$

Determine the point elasticity of demand for the product when $q = 5$. (*Hint:* First find the demand function.)

21. Average Cost A manufacturer has determined that the marginal-cost function is

$$\frac{dc}{dq} = 0.003q^2 - 0.4q + 40$$

where q is the number of units produced. If marginal cost is \$27.50 when $q = 50$ and fixed costs are \$5000, what is the *average* cost of producing 100 units?

22. If $f''(x) = 30x^4 + 12x$ and $f'(1) = 10$, evaluate

$$f(965.335245) - f(-965.335245)$$

Objective

To learn and apply the formulas for $\int u^a\,du$, $\int e^u x\,du$, and $\int \frac{1}{u}\,du$.

14.4 More Integration Formulas

Power Rule for Integration

The formula

$$\int x^a\,dx = \frac{x^{a+1}}{n+1} + C \qquad \text{if } a \ne -1$$

which applies to a power of x, can be generalized to handle a power of a *function* of x. Let u be a differentiable function of x. By the power rule for differentiation, if $a \ne -1$, then

$$\frac{d}{dx}\left(\frac{(u(x))^{a+1}}{a+1}\right) = \frac{(a+1)(u(x))^a \cdot u'(x)}{a+1} = (u(x))^a \cdot u'(x)$$

[2] Adapted from R. Bressani, "The Use of Yeast in Human Foods," in *Single-Cell Protein,* eds. R. I. Mateles and S. R. Tannenbaum (Cambridge, MA: MIT Press, 1968).

[3] Adapted from D. G. Embree, "The Population Dynamics of the Winter Moth in Nova Scotia, 1954–1962," *Memoirs of the Entomological Society of Canada,* no. 46 (1965).

[4] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

Since $x^2 + 5$ is always positive, we can omit the absolute-value bars:

$$\int \frac{2x}{x^2+5}dx = \ln(x^2+5) + C$$

Now Work Problem 31 ◁

EXAMPLE 7 **An Integral Involving $\frac{1}{u}du$**

Find $\displaystyle\int \frac{(2x^3+3x)dx}{x^4+3x^2+7}$.

Solution: If $u = x^4 + 3x^2 + 7$, then $du = (4x^3 + 6x)dx$, which is two times the numerator giving $(2x^3+3x)dx = \frac{du}{2}$. To apply Equation (3), we write

$$\begin{aligned}
\int \frac{2x^3+3x}{x^4+3x^2+7}dx &= \frac{1}{2}\int \frac{1}{u}du \\
&= \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln|x^4+3x^2+7| + C \qquad \text{Rewrite } u \text{ in terms of } x. \\
&= \frac{1}{2}\ln(x^4+3x^2+7) + C \qquad x^4+3x^2+7>0 \quad \text{for all } x
\end{aligned}$$

Now Work Problem 51 ◁

EXAMPLE 8 **An Integral Involving Two Forms**

Find $\displaystyle\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right)dw$.

Solution:

$$\begin{aligned}
\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right)dw &= \int (1-w)^{-2}dw + \int \frac{1}{w-1}dw \\
&= -1\int (1-w)^{-2}(-dw) + \int \frac{1}{w-1}dw
\end{aligned}$$

The first integral has the form $\int u^{-2}\,du$, and the second has the form $\displaystyle\int \frac{1}{v}dv$. Thus,

$$\begin{aligned}
\int \left(\frac{1}{(1-w)^2} + \frac{1}{w-1}\right)dw &= -\frac{(1-w)^{-1}}{-1} + \ln|w-1| + C \\
&= \frac{1}{1-w} + \ln|w-1| + C
\end{aligned}$$

◁

PROBLEMS 14.4

In Problems 1–80, find the indefinite integrals.

1. $\int (x+3)^5 dx$

2. $\int 15(x+2)^4\,dx$

3. $\int 2x(x^2+3)^5\,dx$

4. $\int (4x+3)(2x^2+3x+1)\,dx$

5. $\int (3y^2+6y)(y^3+3y^2+1)^{2/3}\,dy$

6. $\int (12t^2-4t+3)(4t^3-2t^2+3t)^8 dt$

7. $\displaystyle\int \frac{5}{(3x-1)^3}dx$

8. $\displaystyle\int \frac{4x}{(2x^2-7)^{10}}dx$

9. $\int \sqrt{7x+3}\,dx$

10. $\displaystyle\int \frac{1}{\sqrt{x-5}}dx$

11. $\int (5x-2)^5 dx$

12. $\int x^2(3x^3+7)^3\,dx$

13. $\int u(5u^2-9)^{14}\,du$

14. $\int x\sqrt{3+5x^2}\,dx$

15. $\int 4x^4(27+x^5)^{1/3}\,dx$

16. $\int (3-2x)^7\,dx$

17. $\int 3e^{3x}\,dx$

18. $\int 5e^{3t+7}\,dt$

19. $\int (3t+1)e^{3t^2+2t+1}\,dt$

20. $\int -3w^2e^{-w^3}\,dw$

21. $\int 3xe^{5x^2}\,dx$

22. $\int x^3e^{4x^4}\,dx$

23. $\int 4e^{-3x}\,dx$

24. $\int 24x^5e^{-2x^6+7}\,dx$

25. $\int \frac{1}{x+5}\,dx$

26. $\int \frac{30^2+8x+6}{3x+2x^2+5x^3}dx$

27. $\int \frac{3x^2+4x^3}{x^3+x^4}\,dx$

28. $\int \frac{6x^2-6x}{1-3x^2+2x^3}\,dx$

29. $\int \frac{8z}{(z^2-5)^7}\,dz$

30. $\int \frac{3}{(5v-1)^4}\,dv$

31. $\int \frac{7}{x}dx$

32. $\int \frac{3}{1+2y}\,dy$

33. $\int \frac{s^2}{s^3+5}\,ds$

34. $\int \frac{32x^3}{4x^4+9}\,dx$

35. $\int \frac{5}{4-2x}\,dx$

36. $\int \frac{4t}{3t^2+1}dt$

37. $\int \sqrt{5x}\,dx$

38. $\int \frac{1}{(3x)^6}\,dx$

39. $\int \frac{x}{\sqrt{ax^2+b}}\,dx$

40. $\int \frac{9}{1-3x}\,dx$

41. $\int 2y^3e^{y^4+1}dy$

42. $\int 2\sqrt{2x-1}\,dx$

43. $\int v^2e^{-2v^3+1}\,dv$

44. $\int \frac{x^2+x+1}{\sqrt[3]{x^3+\frac{3}{2}x^2+3x}}\,dx$

45. $\int (e^{-5x}+2e^x)\,dx$

46. $\int 7\sqrt[5]{y+3}dy$

47. $\int (8x+10)(7-2x^2-5x)^3\,dx$

48. $\int 2ye^{3y^2}\,dy$

49. $\int \frac{6x^2+8}{x^3+4x}\,dx$

50. $\int (e^x+2e^{-3x}-e^{5x})\,dx$

51. $\int \frac{24s+16}{1+4s+3s^2}ds$

52. $\int (6t^2+4t)(t^3+t^2+1)^6\,dt$

53. $\int x(2x^2+1)^{-1}\,dx$

54. $\int (45w^4+18w^2+12)(3w^5+2w^3+4)^{-4}\,dw$

55. $\int -(x^2-2x^5)(x^3-x^6)^{-10}\,dx$

56. $\int \frac{2}{7}(v+4)e^{2+8v+v^2}dv$

57. $\int (2x^3+x)(x^4+x^2)\,dx$

58. $\int (e^{3.1})^2\,dx$

59. $\int \frac{9+18x}{(5-x-x^2)^4}\,dx$

60. $\int (e^x-e^{-x})^2\,dx$

61. $\int \left(\frac{9}{2}x^3+5x\right)e^{3x^3+5x^2+2}dx$

62. $\int (u^3-ue^{6-3u^2})\,du$

63. $\int x\sqrt{(8-5x^2)^3}\,dx$

64. $\int e^{ax}\,dx$

65. $\int \left(\sqrt{2x}-\frac{1}{\sqrt{2x}}\right)\,dx$

66. $\int 4\frac{x^7}{e^{x^8}}dx$

67. $\int (x^2+1)^2\,dx$

68. $\int \left[x(x^2-16)^2-\frac{1}{2x+5}\right]\,dx$

69. $\int \left(\frac{x}{x^2+1}+\frac{x}{(x^2+1)^2}\right)\,dx$

70. $\int \left[\frac{3}{x-1}+\frac{1}{(x-1)^2}\right]dx$

71. $\int \left(\frac{3}{5x+2}-(5x^2+10x^5)(x^3+x^6)^{-5}\right)\,dx$

72. $\int (r^3+5)^2\,dr$

73. $\int \left[\sqrt{3x+1}-\frac{x}{x^2+3}\right]\,dx$

74. $\int \left(\frac{x}{7x^2+2}-\frac{x^2}{(x^3+2)^4}\right)\,dx$

75. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

76. $\int (e^7-7^e)dx$

77. $\int \frac{1+e^{2x}}{4e^x}\,dx$

78. $\int \frac{2}{t^2}\sqrt{\frac{1}{t}+9}\,dt$

79. $\int \frac{4x+3}{2x^2+3x}\ln(2x^2+3x)dx$

80. $\int \sqrt[3]{x}e^{\sqrt[3]{8x^4}}\,dx$

In Problems 81–84, find y, subject to the given conditions.

81. $y'=(5-7x)^3;\quad y(0)=2$

82. $y'=\frac{x}{x^2+6};\quad y(1)=0$

83. $y''=\frac{1}{x^2};\quad y'(-2)=3, y(1)=2$

84. $y''=(x+1)^{1/2};\quad y'(8)=19, y(24)=\frac{2572}{3}$

85. Real Estate The rate of change of the value of a house that cost \$350,000 to build can be modeled by $\frac{dV}{dt}=8e^{0.05t}$, where t is the time in years since the house was built and V is the value (in thousands of dollars) of the house. Find $V(t)$.

86. Life Span If the rate of change of the expected life span, l, at birth of people born in Canada can be modeled by $\frac{dl}{dt}=\frac{12}{2t+50}$, where t is the number of years after 1940 and the expected life span was 63 years in 1940, find the expected life span for people born in 2000.

87. Oxygen in Capillary In a discussion of the diffusion of oxygen from capillaries,[5] concentric cylinders of radius r are used as a model for a capillary. The concentration C of oxygen in the capillary is given by

$$C = \int \left(\frac{Rr}{2K} + \frac{B_1}{r} \right) dr$$

where R is the constant rate at which oxygen diffuses from the capillary, and K and B_1 are constants. Find C. (Write the constant of integration as B_2.)

88. Find $f(2)$ if $f\left(\frac{1}{3}\right) = 2$ and $f'(x) = e^{3x+2} - 3x$.

Objective

To discuss techniques of handling more challenging integration problems, namely, by algebraic manipulation and by fitting the integrand to a familiar form. To integrate an exponential function with a base different from e and to find the consumption function, given the marginal propensity to consume.

14.5 Techniques of Integration

We turn now to some more difficult integration problems.

When integrating fractions, sometimes a **preliminary division** is needed to get familiar integration forms, as the next example shows.

EXAMPLE 1 Preliminary Division before Integration

a. Find $\displaystyle\int \frac{x^3 + x}{x^2}\,dx$.

Solution: A familiar integration form is not apparent. However, we can break up the integrand into two fractions by dividing each term in the numerator by the denominator. We then have

Here we split up the integrand.

$$\int \frac{x^3 + x}{x^2}\,dx = \int \left(\frac{x^3}{x^2} + \frac{x}{x^2} \right) dx = \int \left(x + \frac{1}{x} \right) dx$$

$$= \frac{x^2}{2} + \ln|x| + C$$

b. Find $\displaystyle\int \frac{2x^3 + 3x^2 + x + 1}{2x + 1}\,dx$.

Solution: Here the integrand is a quotient of polynomials in which the degree of the numerator is greater than or equal to that of the denominator. In such a situation we first use long division. Recall that if f and g are polynomials, with the degree of f greater than or equal to the degree of g, then long division allows us to find, uniquely, polynomials q and r, where either r is the zero polynomial or the degree of r is strictly less than the degree of g, satisfying

Here we used long division to rewrite the integrand.

$$\frac{f}{g} = q + \frac{r}{g}$$

Using an obvious, abbreviated notation, we see that

$$\int \frac{f}{g} = \int \left(q + \frac{r}{g} \right) = \int q + \int \frac{r}{g}$$

Since integrating a polynomial is easy, we see that integrating rational functions reduces to the task of integrating *proper rational functions*—those for which the degree of the numerator is strictly less than the degree of the denominator. In the case here we obtain

$$\int \frac{2x^3 + 3x^2 + x + 1}{2x + 1}\,dx = \int \left(x^2 + x + \frac{1}{2x + 1} \right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + \int \frac{1}{2x + 1}\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{2} \int \frac{1}{2x + 1}\,d(2x + 1)$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{2} \ln|2x + 1| + C$$

Now Work Problem 1 ◁

[5] W. Simon, *Mathematical Techniques for Physiology and Medicine* (New York: Academic Press, Inc., 1972).

EXAMPLE 4 **Finding a Consumption Function from Marginal Propensity to Consume**

For a certain country, the marginal propensity to consume is given by

$$\frac{dC}{dI} = \frac{3}{4} - \frac{1}{2\sqrt{3I}}$$

where consumption C is a function of national income I. Here, I is expressed in large denominations of money. Determine the consumption function for the country if it is known that consumption is 10 ($C = 10$) when $I = 12$.

Solution: Since the marginal propensity to consume is the derivative of C, we have

$$C = C(I) = \int\left(\frac{3}{4} - \frac{1}{2\sqrt{3I}}\right) dI = \int \frac{3}{4}dI - \frac{1}{2}\int (3I)^{-1/2} dI$$

$$= \frac{3}{4}I - \frac{1}{2}\int (3I)^{-1/2} dI$$

If we let $u = 3I$, then $du = 3dI = d(3I)$, and

$$C = \frac{3}{4}I - \left(\frac{1}{2}\right)\frac{1}{3}\int (3I)^{-1/2} d(3I)$$

$$= \frac{3}{4}I - \frac{1}{6}\frac{(3I)^{1/2}}{\frac{1}{2}} + K$$

$$C = \frac{3}{4}I - \frac{\sqrt{3I}}{3} + K$$

This is an example of an initial-value problem.

When $I = 12$, $C = 10$, so

$$10 = \frac{3}{4}(12) - \frac{\sqrt{3(12)}}{3} + K$$

$$10 = 9 - 2 + K$$

Thus, $K = 3$, and the consumption function is

$$C = \frac{3}{4}I - \frac{\sqrt{3I}}{3} + 3$$

Now Work Problem 61 ◁

PROBLEMS 14.5

In Problems 1–56, determine the indefinite integrals.

1. $\int \frac{2x^6 + 8x^4 - 4x}{2x^2} dx$

2. $\int \frac{4x^2 + 3}{2x} dx$

3. $\int (3x^2 + 2)\sqrt{2x^3 + 4x + 1}\, dx$

4. $\int \frac{x}{\sqrt[4]{x^2 + 1}} dx$

5. $\int \frac{3}{\sqrt{4 - 5x}} dx$

6. $\int \frac{2xe^{x^2}\, dx}{e^{x^2} - 2}$

7. $\int 2^{5x} dx$

8. $\int 5^t\, dt$

9. $\int 2x(7 - e^{x^2/4})\, dx$

10. $\int \frac{e^x + 1}{e^x} dx$

11. $\int \frac{6x^2 - 11x + 5}{3x - 1} dx$

12. $\int \frac{(3x + 1)(x + 3)}{x + 2} dx$

13. $\int \frac{5e^{2x}}{7e^{2x} + 4} dx$

14. $\int 6(e^{4-3x})^2\, dx$

15. $\int \frac{5e^{13/x}}{x^2} dx$

16. $\int \frac{2x^4 - 6x^3 + x - 2}{x - 2} dx$

17. $\int \frac{2x^3}{x^2 + 1} dx$

18. $\int \frac{5 - 4x^2}{3 + 2x} dx$

19. $\int \frac{(\sqrt{x} + 2)^2}{3\sqrt{x}} dx$

20. $\int \frac{5e^s}{1+3e^s}\,ds$

21. $\int \frac{5(x^{1/3}+2)^4}{\sqrt[3]{x^2}}\,dx$

22. $\int \frac{\sqrt{a+\sqrt{x}}}{\sqrt{x}}\,dx$

23. $\int \frac{\ln x}{x}\,dx$

24. $\int \sqrt{t}(3-t\sqrt{t})^{0.6}\,dt$

25. $\int \frac{r\sqrt{\ln(r^2+1)}}{r^2+1}\,dr$

26. $\int \frac{9x^5-6x^4-ex^3}{7x^2}\,dx$

27. $\int \frac{(11)^{\ln x}}{x}\,dx$

28. $\int \frac{4}{x\ln(2x^2)}\,dx$

29. $\int x^2\sqrt{e^{x^3+1}}\,dx$

30. $\int \frac{ax+b}{cx+d}\,dx \quad c \neq 0$

31. $\int \frac{8}{(x+3)\ln(x+3)}\,dx$

32. $\int (e^{e^e}+x^e+e^x)dx$

33. $\int \frac{x^3+x^2-x-3}{x^2-3}\,dx$

34. $\int \frac{4x\ln\sqrt{1+x^2}}{1+x^2}\,dx$

35. $\int \frac{12x^3\sqrt{\ln(x^4+1)^3}}{x^4+1}\,dx$

36. $\int 3(x^2+2)^{-1/2}xe^{\sqrt{x^2+2}}\,dx$

37. $\int \left(\frac{x^3-1}{\sqrt{x^4-4x}}-\ln 2\right)dx$

38. $\int \frac{x-x^{-2}}{x^2+2x^{-1}}\,dx$

39. $\int \frac{2x^4-8x^3-6x^2+4}{x^3}\,dx$

40. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

41. $\int \frac{x}{x+1}\,dx$

42. $\int \frac{4x^3+2x}{(x^4+x^2)\ln(x^4+x^2)}\,dx$

43. $\int \frac{xe^{x^2}}{\sqrt{e^{x^2}+2}}\,dx$

44. $\int \frac{5}{(3x+1)[1+\ln(3x+1)]^2}\,dx$

45. $\int \frac{(e^{-x}+5)^3}{e^x}\,dx$

46. $\int \left[\frac{1}{8x+1}-\frac{1}{e^x(8+e^{-x})^2}\right]dx$

47. $\int (x^2+\sqrt{2}x)\sqrt{x^2+\sqrt{2}}\,dx$

48. $\int 3^{x\ln x}(1+\ln x)\,dx \quad$ [*Hint:* $\frac{d}{dx}(x\ln x) = 1+\ln x$]

49. $\int \sqrt{x}\sqrt{(8x)^{3/2}+3}\,dx$

50. $\int \frac{7}{x(\ln x)^\pi}\,dx$

51. $\int \frac{\sqrt{s}}{e^{\sqrt{s^3}}}\,ds$

52. $\int \frac{\ln^5 x}{7x}\,dx$

53. $\int e^{\ln(x^2+1)}\,dx$

54. $\int dx$

55. $\int \frac{\ln\left(\frac{e^x}{x}\right)}{x}\,dx$

56. $\int e^{f(x)+\ln(f'(x))}\,dx \quad$ assuming $f' > 0$

In Problems 57 and 58, dr/dq is a marginal-revenue function. Find the demand function.

57. $\frac{dr}{dq} = \frac{300}{(q+3)^2}$

58. $\frac{dr}{dq} = \frac{900}{(2q+3)^3}$

In Problems 59 and 60, dc/dq is a marginal-cost function. Find the total-cost function if fixed costs in each case are 2000.

59. $\frac{dc}{dq} = \frac{20}{q+5}$

60. $\frac{dc}{dq} = 4e^{0.005q}$

In Problems 61–63, dC/dI represents the marginal propensity to consume. Find the consumption function subject to the given condition.

61. $\frac{dC}{dI} = \frac{1}{\sqrt{I}}; \quad C(9) = 8$

62. $\frac{dC}{dI} = \frac{1}{3} - \frac{1}{2\sqrt{3I}}; \quad C(25/3) = 2$

63. $\frac{dC}{dI} = \frac{3}{4} - \frac{1}{6\sqrt{I}}; \quad C(25) = 23$

64. Cost Function The marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = 10 - \frac{100}{q+10}$$

where c is the total cost in dollars when q units are produced. When 100 units are produced, the average cost is \$50 per unit. To the nearest dollar, determine the manufacturer's fixed cost.

65. Cost Function Suppose the marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = \frac{100q^2-3998q+60}{q^2-40q+1}$$

where c is the total cost in dollars when q units are produced.

(a) Determine the marginal cost when 40 units are produced.
(b) If fixed costs are \$10,000, find the total cost of producing 40 units.
(c) Use the results of parts (a) and (b) and differentials to approximate the total cost of producing 42 units.

66. Cost Function The marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = \frac{9}{10}\sqrt{q}\sqrt{0.04q^{3/4}+4}$$

where c is the total cost in dollars when q units are produced. Fixed costs are \$360.

(a) Determine the marginal cost when 25 units are produced.
(b) Find the total cost of producing 25 units.
(c) Use the results of parts (a) and (b) and differentials to approximate the total cost of producing 23 units.

67. Value of Land It is estimated that t years from now the value, V (in dollars), of an acre of land near the ghost town of Lonely Falls, B.C., will be increasing at the rate of $\frac{8t^3}{\sqrt{0.2t^4+8000}}$ dollars per year. If the land is currently worth \$600 per acre, how much will it be worth in 15 years? Give the answer to the nearest dollar.

68. Revenue Function The marginal-revenue function for a manufacturer's product is of the form

$$\frac{dr}{dq} = \frac{a}{e^q + b}$$

for constants a and b, where r is the total revenue received (in dollars) when q units are produced and sold. Find the demand function, and express it in the form $p = f(q)$. (*Hint:* Rewrite dr/dq by multiplying both numerator and denominator by e^{-q}.)

69. Savings A certain country's marginal propensity to save is given by

$$\frac{dS}{dI} = \frac{5}{(I+2)^2}$$

where S and I represent total national savings and income, respectively, and are measured in billions of dollars. If total national consumption is \$7.5 billion when total national income is \$8 billion, for what value(s) of I is total national savings equal to zero?

70. Consumption Function A certain country's marginal propensity to save is given by

$$\frac{dS}{dI} = \frac{2}{5} - \frac{1.6}{\sqrt[3]{2I^2}}$$

where S and I represent total national savings and income, respectively, and are measured in billions of dollars.

(a) Determine the marginal propensity to consume when total national income is \$16 billion.

(b) Determine the consumption function, given that savings are \$10 billion when total national income is \$54 billion.

(c) Use the result in part (b) to show that consumption is $\$\frac{82}{5} = 16.4$ billion when total national income is \$16 billion (a deficit situation).

(d) Use differentials and the results in parts (a) and (c) to approximate consumption when total national income is \$18 billion.

Objective

To motivate, by means of the concept of area, the definite integral as a limit of a special sum; to evaluate simple definite integrals by using a limiting process.

14.6 The Definite Integral

Figure 14.2 shows the region, R, bounded by the lines $y = f(x) = 2x$, $y = 0$ (the x-axis), and $x = 1$. The region is simply a right triangle. If b and h are the lengths of the base and the height, respectively, then, from geometry, the area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(1)(2) = 1$ square unit. (Henceforth, we will treat areas as pure numbers and write *square unit* only if it seems necessary for emphasis.) We will now find this area by another method, which, as we will see later, applies to more complex regions. This method involves the summation of areas of rectangles.

Let us divide the interval [0, 1] on the x-axis into four subintervals of equal length by means of the equally spaced points $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{4}, x_3 = \frac{3}{4}$, and $x_4 = \frac{4}{4} = 1$. (See Figure 14.3.) Each subinterval has length $\Delta x = \dfrac{1}{4}$. These subintervals determine four subregions of R: R_1, R_2, R_3, and R_4, as indicated.

With each subregion, we can associate a *circumscribed* rectangle (Figure 14.4)—that is, a rectangle whose base is the corresponding subinterval and whose height is the *maximum* value of $f(x)$ on that subinterval. Since f is an increasing function, the maximum value of $f(x)$ on each subinterval occurs when x is the right-hand endpoint. Thus, the areas of the circumscribed rectangles associated with regions R_1, R_2, R_3, and

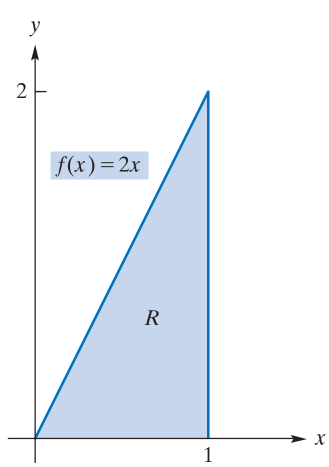

FIGURE 14.2 Region bounded by $f(x) = 2x$, $y = 0$, and $x = 1$.

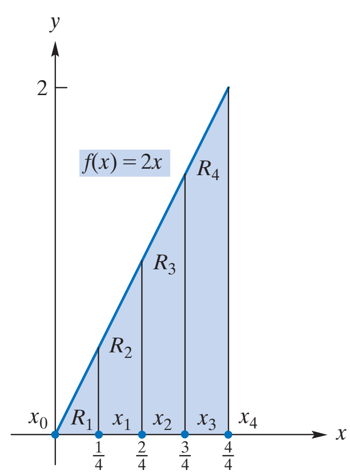

FIGURE 14.3 Four subregions of R.

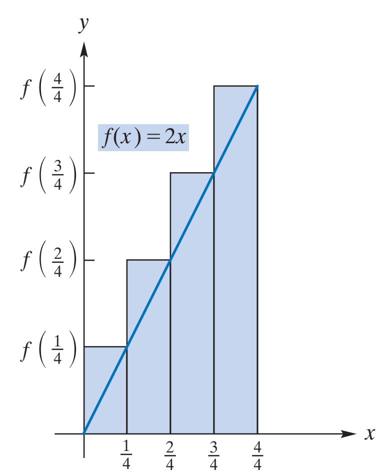

FIGURE 14.4 Four circumscribed rectangles.

PROBLEMS 14.7

In Problems 1–43, evaluate the definite integral.

1. $\int_0^3 5\,dx$

2. $\int_1^5 (e+3e)\,dx$

3. $\int_1^2 5x\,dx$

4. $\int_{-1}^6 -3x\,dx$

5. $\int_{-3}^1 (2x-3)\,dx$

6. $\int_{-1}^1 (4-9y)\,dy$

7. $\int_1^4 (y^2+4y+4)\,dy$

8. $\int_4^1 (2t-3t^2)\,dt$

9. $\int_{-4}^{-2} (3w^2-2w+3)\,dw$

10. $\int_8^9 dt$

11. $\int_1^3 3t^{-3}\,dt$

12. $\int_2^3 \frac{3}{x^2}\,dx$

13. $\int_{-8}^8 \sqrt[3]{x^4}\,dx$

14. $\int_0^1 (x^4+x^3+x^2+x+1)\,dx$

15. $\int_{1/2}^3 \frac{1}{x^2}\,dx$

16. $\int_9^{36} (\sqrt{x}-2)\,dx$

17. $\int_{-2}^2 (z+1)^4\,dz$

18. $\int_1^8 (x^{1/3}-x^{-1/3})\,dx$

19. $\int_0^1 3x^3(x^4-1)^4\,dx$

20. $\int_2^3 (x+2)^3\,dx$

21. $\int_1^8 \frac{4}{y}\,dy$

22. $\int_{-e^\pi}^{-1} \frac{2}{x}\,dx$

23. $\int_0^1 e^5\,dx$

24. $\int_2^e \frac{1}{x+1}\,dx$

25. $\int_0^1 5x^2e^{x^3}\,dx$

26. $\int_0^1 (3x^2+4x)(x^3+2x^2)^4\,dx$

27. $\int_3^4 \frac{3}{(x+3)^2}\,dx$

28. $\int_{-1/3}^{20/3} \sqrt{3x+5}\,dx$

29. $\int_1^6 \sqrt{10-p}\,dp$

30. $\int_{-1}^1 q\sqrt{q^2+3}\,dq$

31. $\int_0^1 x^2\sqrt[3]{7x^3+1}\,dx$

32. $\int_0^{\sqrt{2}} \left(2x-\frac{x}{(x^2+1)^{2/3}}\right)dx$

33. $\int_0^1 \frac{2x^3+x}{x^2+x^4+1}\,dx$

34. $\int_a^b (my+ny^2)\,dy$

35. $\int_0^1 \frac{e^x-e^{-x}}{2}\,dx$

36. $\int_{-2}^1 8|x|\,dx$

37. $\int_e^{\sqrt{2}} 3(x^{-2}+x^{-3}-x^{-4})\,dx$

38. $\int_1^2 \left(6\sqrt{x}-\frac{1}{\sqrt{2x}}\right)dx$

39. $\int_1^2 (x+1)e^{3x^2+6x}\,dx$

40. $\int_1^{95} \frac{x}{\ln e^x}\,dx$

41. $\int_0^2 \frac{x^6+6x^4+x^3+8x^2+x+5}{x^3+5x+1}\,dx$

42. $\int_1^2 \frac{1}{1+e^x}\,dx$ (*Hint:* Multiply the integrand by $\frac{e^{-x}}{e^{-x}}$.)

43. $\int_0^2 f(x)\,dx$, where $f(x)=\begin{cases}4x^2 & \text{if } 0\le x<\frac{1}{2}\\ 2x & \text{if } \frac{1}{2}\le x\le 2\end{cases}$

44. Evaluate $\left(\int_1^2 x\,dx\right)^2-\int_1^2 x^2\,dx$.

45. Suppose $f(x)=\int_1^x 3\frac{1}{t^2}\,dt$. Evaluate $\int_e^1 f(x)\,dx$.

46. Evaluate $\int_7^7 e^{x^2}\,dx+\int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}}\,dx$.

47. If $\int_1^2 f(x)\,dx=5$ and $\int_3^1 f(x)\,dx=2$, find $\int_2^3 f(x)\,dx$.

48. If $\int_1^4 f(x)\,dx=6$, $\int_2^4 f(x)\,dx=5$, and $\int_1^3 f(x)\,dx=2$, find $\int_2^3 f(x)\,dx$.

49. Evaluate $\int_0^1 \left(\frac{d}{dx}\int_0^1 e^{x^2}\,dx\right)dx$ (*Hint:* It is not necessary to find $\int_0^1 e^{x^2}\,dx$.)

50. Suppose that $f(x)=\int_e^x \frac{e^t-e^{-t}}{e^t+e^{-t}}\,dt$ where $x>e$. Find $f'(x)$.

51. Severity Index In discussing traffic safety, Shonle[6] considers how much acceleration a person can tolerate in a crash so that there is no major injury. The *severity index* is defined as

$$\text{S.I.}=\int_0^T \alpha^{5/2}\,dt$$

where α (a Greek letter read "alpha") is considered a constant involved with a weighted average acceleration, and T is the duration of the crash. Find the severity index.

[6] J. I. Shonle, *Environmental Applications of General Physics* (Reading, MA: Addison-Wesley Publishing Company, Inc., 1975).

52. Statistics In statistics, the mean μ (a Greek letter read "mu") of the continuous probability density function f defined on the interval $[a, b]$ is given by

$$\mu = \int_a^b xf(x)\,dx$$

and the variance σ^2 (σ is a Greek letter read "sigma") is given by

$$\sigma^2 = \int_a^b (x-\mu)^2 f(x)\,dx$$

Compute μ and then σ^2 if $a = 0$, $b = 1$, and $f(x) = 6(x - x^2)$.

53. Distribution of Incomes The economist Pareto[7] has stated an empirical law of distribution of higher incomes that gives the number, N, of persons receiving x or more dollars. If

$$\frac{dN}{dx} = -Ax^{-B}$$

where A and B are constants, set up a definite integral that gives the total number of persons with incomes between a and b, where $a < b$.

54. Biology In a discussion of gene mutation,[8] the following integral occurs:

$$\int_0^{10^{-4}} x^{-1/2}dx$$

Evaluate this integral.

55. Continuous Income Flow The present value (in dollars) of a continuous flow of income of \$2000 a year for five years at 6% compounded continuously is given by

$$\int_0^5 2000e^{-0.06t}\,dt$$

Evaluate the present value to the nearest dollar.

56. Biology In biology, problems frequently arise involving the transfer of a substance between compartments. An example is a transfer from the bloodstream to tissue. Evaluate the following integral, which occurs in a two-compartment diffusion problem:[9]

$$\int_0^t (e^{-a\tau} - e^{-b\tau})d\tau$$

Here, τ (read "tau") is a Greek letter; a and b are constants.

57. Demography For a certain small population, suppose l is a function such that $l(x)$ is the number of persons who reach the age of x in any year of time. This function is called a *life table function*. Under appropriate conditions, the integral

$$\int_a^b l(t)\,dt$$

gives the expected number of people in the population between the exact ages of a and b, inclusive. If

$$l(x) = 1000\sqrt{110-x} \quad \text{for } 0 \le x \le 110$$

determine the number of people between the exact ages of 10 and 29, inclusive. Give your answer to the nearest integer, since fractional answers make no sense. What is the size of the population?

58. Mineral Consumption If C is the yearly consumption of a mineral at time $t = 0$, then, under continuous consumption, the total amount of the mineral used in the interval $[0, t]$ is

$$\int_0^t Ce^{k\tau}\,d\tau$$

where k is the rate of consumption. For a rare-earth mineral, it has been determined that $C = 3000$ units and $k = 0.05$. Evaluate the integral for these data.

59. Marginal Cost A manufacturer's marginal-cost function is

$$\frac{dc}{dq} = 0.1q + 9$$

If c is in dollars, determine the cost involved to increase production from 71 to 82 units.

60. Marginal Cost Repeat Problem 59 if

$$\frac{dc}{dq} = 0.004q^2 - 0.5q + 50$$

and production increases from 90 to 180 units.

61. Marginal Revenue A manufacturer's marginal-revenue function is

$$\frac{dr}{dq} = \frac{2000}{\sqrt{300q}}$$

If r is in dollars, find the change in the manufacturer's total revenue if production is increased from 500 to 800 units.

62. Marginal Revenue Repeat Problem 61 if

$$\frac{dr}{dq} = 100 + 50q - 3q^2$$

and production is increased from 5 to 10 units.

63. Crime Rate A sociologist is studying the crime rate in a certain city. She estimates that t months after the beginning of next year, the total number of crimes committed will increase at the rate of $8t + 10$ crimes per month. Determine the total number of crimes that can be expected to be committed next year. How many crimes can be expected to be committed during the last six months of that year?

64. Hospital Discharges For a group of hospitalized individuals, suppose the discharge rate is given by

$$f(t) = \frac{81 \times 10^6}{(300+t)^4}$$

where $f(t)$ is the proportion of the group discharged per day at the end of t days. What proportion has been discharged by the end of 500 days?

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[7] G. Tintner, *Methodology of Mathematical Economics and Econometrics* (Chicago: University of Chicago Press, 1967), p. 16.

[8] W. J. Ewens, *Population Genetics* (London: Methuen & Company Ltd., 1969).

[9] W. Simon, *Mathematical Techniques for Physiology and Medicine* (New York: Academic Press, Inc., 1972).

65. Production Imagine a one-dimensional country of length $2R$. (See Figure 14.21.[10]) Suppose the production of goods for this country is continuously distributed from border to border. If the amount produced each year per unit of distance is $f(x)$, then the country's total yearly production is given by

$$G = \int_{-R}^{R} f(x)\,dx$$

Evaluate G if $f(x) = i$, where i is constant.

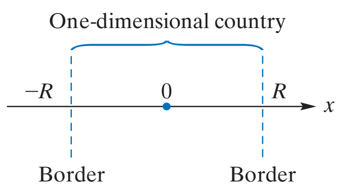

FIGURE 14.21

66. Exports For the one-dimensional country of Problem 65, under certain conditions the amount of the country's exports is given by

$$E = \int_{-R}^{R} \frac{i}{2}\left[e^{-k(R-x)} + e^{-k(R+x)}\right] dx$$

where i and k are constants ($k \neq 0$). Evaluate E.

67. Average Delivered Price In a discussion of a delivered price of a good from a mill to a customer, DeCanio[11] claims that the average delivered price paid by consumers is given by

$$A = \frac{\displaystyle\int_0^R (m+x)[1-(m+x)]\,dx}{\displaystyle\int_0^R [1-(m+x)]\,dx}$$

where m is mill price, and x is the maximum distance to the point of sale. DeCanio determines that

$$A = \frac{m + \dfrac{R}{2} - m^2 - mR - \dfrac{R^2}{3}}{1 - m - \dfrac{R}{2}}$$

Verify this.

In Problems 68–70, use the Fundamental Theorem of Integral Calculus to determine the value of the definite integral.

68. $\displaystyle\int_{2.5}^{3.5} (1 + 2x + 3x^2)\,dx$

69. $\displaystyle\int_0^1 \frac{1}{(x+1)^2}\,dx$

70. $\displaystyle\int_0^1 e^{3t}\,dt$ Round the answer to two decimal places.

In Problems 71–74, estimate the value of the definite integral by using an approximating sum. Round the answer to two decimal places.

71. $\displaystyle\int_{-1}^{5} \frac{x^2+1}{x^2+4}\,dx$

72. $\displaystyle\int_3^4 \frac{1}{x\ln x}\,dx$

73. $\displaystyle\int_0^3 2\sqrt{t^2+3}\,dt$

74. $\displaystyle\int_{-1}^{1} \frac{6\sqrt{q+1}}{q+3}\,dq$

Chapter 14 Review

Important Terms and Symbols — Examples

Section 14.1 Differentials
differential, dy, dx — Ex. 1, p. 620

Section 14.2 The Indefinite Integral
antiderivative indefinite integral $\int f(x)\,dx$ integral sign — Ex. 1, p. 626
integrand variable of integration constant of integration — Ex. 2, p. 627

Section 14.3 Integration with Initial Conditions
initial condition — Ex. 1, p. 631

Section 14.4 More Integration Formulas
power rule for integration — Ex. 1, p. 636

Section 14.5 Techniques of Integration
preliminary division — Ex. 1, p. 642

Section 14.6 The Definite Integral
definite integral $\int_a^b f(x)\,dx$ bounds of integration — Ex. 2, p. 651

Section 14.7 The Fundamental Theorem of Calculus
Fundamental Theorem of Integral Calculus $F(x)\big|_a^b = F(b) - F(a)$ — Ex. 1, p. 655

[10] R. Taagepera, "Why the Trade/GNP Ratio Decreases with Country Size," *Social Science Research,* 5 (1976), 385–404.

[11] S. J. DeCanio, "Delivered Pricing and Multiple Basing Point Equationilibria: A Reevaluation," *The Quarterly Journal of Economics,* XCIX, no. 2 (1984), 329–49.

Summary

If $y = f(x)$ is a differentiable function of x, we define the differential dy by

$$dy = f'(x)dx$$

where $dx = \Delta x$ is a change in x, which can be any real number. (Thus, dy is a function of two variables, namely x and dx.) If dx is close to zero, then dy is an approximation to $\Delta y = f(x + dx) - f(x)$.

$$\Delta y \approx dy$$

Moreover, dy can be used to approximate a function value using

$$f(x + dx) \approx f(x) + dy$$

An antiderivative of a function f is a function F such that $F'(x) = f(x)$. Any two antiderivatives of f differ at most by a constant. The most general antiderivative of f is called the indefinite integral of f and is denoted $\int f(x)dx$. Thus,

$$\int f(x)dx = F(x) + C$$

where C is called the constant of integration, if and only if $F' = f$.

It is important to remember that $\int (\ )dx$ is an operation, like $\frac{d}{dx}(\ )$, that applies to functions to produce new functions. The aptness of these strange notations becomes apparent only after considerable study.

Some elementary integration formulas are as follows:

$$\int k\,dx = kx + C \qquad k \text{ a constant}$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \qquad a \neq -1$$

$$\int \frac{1}{x}dx = \ln x + C \qquad \text{for } x > 0$$

$$\int e^x\,dx = e^x + C$$

$$\int kf(x)\,dx = k\int f(x)\,dx \qquad k \text{ a constant}$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Another formula is the power rule for integration:

$$\int u^a du = \frac{u^{a+1}}{a+1} + C, \quad \text{if } a \neq -1$$

Here u represents a differentiable function of x, and du is its differential. In applying the power rule to a given integral, it is important that the integral be written in a form that precisely matches the power rule. Other integration formulas are

$$\int e^u du = e^u + C$$

$$\text{and} \quad \int \frac{1}{u}du = \ln|u| + C \qquad u \neq 0$$

If the rate of change of a function f is known—that is, if f' is known—then f is an antiderivative of f'. In addition, if we know that f satisfies an initial condition, then we can find the particular antiderivative. For example, if a marginal-cost function dc/dq is given to us, then by integration we can find the most general form of c. That form involves a constant of integration. However, if we are also given fixed costs (that is, costs involved when $q = 0$), then we can determine the value of the constant of integration and, thus, find the particular cost function, c. Similarly, if we are given a marginal-revenue function dr/dq, then by integration and by using the fact that $r = 0$ when $q = 0$, we can determine the particular revenue function, r. Once r is known, the corresponding demand equation can be found by using the equation $p = r/q$.

It is helpful at this point to review summation notation from Section 1.5. This notation is especially useful in determining areas. For continuous $f \geq 0$, to find the area of the region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$, we divide the interval $[a, b]$ into n subintervals of equal length $dx = (b-a)/n$. If x_i is the right-hand endpoint of an arbitrary subinterval, then the product $f(x_i)\,dx$ is the area of a rectangle. Denoting the sum of all such areas of rectangles for the n subintervals by S_n, we define the limit of S_n as $n \to \infty$ as the area of the entire region:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,dx = \text{area}$$

If the restriction that $f(x) \geq 0$ is omitted, this limit is defined as the definite integral of f over $[a, b]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,dx = \int_a^b f(x)\,dx$$

Instead of evaluating definite integrals by using limits, we *may* be able to employ the Fundamental Theorem of Calculus. In symbols,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

where F is any antiderivative of f.

Some properties of the definite integral are

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \qquad k \text{ a constant}$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

It must be stressed that $\int f(x)dx$ is a *number*, which if $f(x) \geq 0$ on $[a, b]$ gives the area of the region bounded by $y = f(x)$, $y = 0$ and the vertical lines $x = a$ and $x = b$.

Review Problems

In Problems 1–40, determine the integrals.

1. $\int (x^3 + 2x - 7)\,dx$

2. $\int dx$

3. $\int_0^{12} (9\sqrt{3x} + 3x^2)\,dx$

4. $\int \frac{4}{5-3x}\,dx$

5. $\int \frac{3}{(x+2)^4}dx$

6. $\int_3^9 (y-6)^{301}\,dy$

7. $\int \frac{6x^2-12}{x^3-6x+1}\,dx$

8. $\int_0^3 2xe^{5-x^2}\,dx$

9. $\int_0^1 \sqrt[3]{3t+8}\,dt$

10. $\int \frac{3-4x}{2}dx$

11. $\int y(y+1)^2\,dy$

12. $\int_0^1 10^{-8}\,dx$

13. $\int \frac{\sqrt[7]{t}-\sqrt{t}}{\sqrt[3]{t}}\,dt$

14. $\int \frac{(0.5x-0.1)^4}{0.4}\,dx$

15. $\int_0^2 \frac{6t^2}{5+2t^3}dt$

16. $\int \frac{4x^2-x}{x}\,dx$

17. $\int x^2\sqrt{3x^3+2}\,dx$

18. $\int (6x^2+4x)(x^3+x^2)^{3/2}\,dx$

19. $\int (e^{2y} - e^{-2y})\,dy$

20. $\int \frac{4x}{5\sqrt[4]{7-x^2}}dx$

21. $\int \left(\frac{1}{x} + \frac{2}{x^2}\right)\,dx$

22. $\int_0^2 \frac{3e^{3x}}{1+e^{3x}}\,dx$

23. $\int_{-2}^2 (y^4+y^3+y^2+y)\,dy$

24. $\int_7^{70} dx$

25. $\int_0^1 4x\sqrt{5-x^2}dx$

26. $\int_0^1 (2x+1)(x^2+x)^4\,dx$

27. $\int_0^1 \left[2x - \frac{1}{(x+1)^{2/3}}\right]\,dx$

28. $\int_0^{18} (2x - 3\sqrt{2x} + 1)\,dx$

29. $\int \frac{\sqrt{t}-3}{t^2}\,dt$

30. $\int \frac{z^2}{z-1}dz$

31. $\int_{-1}^0 \frac{x^2+4x-1}{x+2}\,dx$

32. $\int \frac{(x^2+4)^2}{x^2}\,dx$

33. $\int \frac{e^{\sqrt{x}}+x}{2\sqrt{x}}\,dx$

34. $\int \frac{e^{\sqrt{5x}}}{\sqrt{3x}}\,dx$

35. $\int_1^2 \frac{e^{\ln x}}{x^3}dx$

36. $\int \frac{6x^2+4}{e^{x^3+2x}}\,dx$

37. $\int \frac{(1+e^{2x})^3}{e^{-2x}}\,dx$

38. $\int \frac{c}{e^{bx}(a+e^{-bx})^n}\,dx$ for $n \neq 1$ and $b \neq 0$

39. $\int 3\sqrt{10^{3x}}\,dx$

40. $\int \frac{3x^3+6x^2+17x+2}{x^2+2x+5}dx$

In Problems 41 and 42, find y, subject to the given condition.

41. $y' = e^{2x} + 3, \quad y(0) = -\frac{1}{2}$

42. $y' = \frac{x+5}{x}, \quad y(1) = 3$

In Problems 43–50, determine the area of the region bounded by the given curve, the x-axis, and the given lines.

43. $y = x^3, \quad x = 0, \ x = 2$

44. $y = 4e^x, \quad x = 0, \ x = 3$

45. $y = \sqrt{x+1}, \quad x = 0$

46. $y = x^2 - x - 6, \quad x = -4, \ x = 3$

47. $y = 5x - x^2$

48. $y = \sqrt[3]{x}, \quad x = 8, \ x = 16$

49. $y = \frac{1}{x} + 2, \quad x = 1, \ x = 4$

50. $y = x^3 - 8, \quad x = 0$

51. Marginal Revenue If marginal revenue is given by

$$\frac{dr}{dq} = 100 - \frac{3}{2}\sqrt{2q}$$

determine the corresponding demand equation.

52. Marginal Cost If marginal cost is given by

$$\frac{dc}{dq} = q^2 + 7q + 6$$

and fixed costs are 2500, determine the total cost of producing six units. Assume that costs are in dollars.

53. Marginal Revenue A manufacturer's marginal-revenue function is

$$\frac{dr}{dq} = 250 - q - 0.2q^2$$

If r is in dollars, find the increase in the manufacturer's total revenue if production is increased from 15 to 25 units.

54. Marginal Cost A manufacturer's marginal-cost function is

$$\frac{dc}{dq} = \frac{1000}{\sqrt{3q + 70}}$$

If c is in dollars, determine the cost involved to increase production from 10 to 33 units.

55. Hospital Discharges For a group of hospitalized individuals, suppose the discharge rate is given by

$$f(t) = 0.008e^{-0.008t}$$

where $f(t)$ is the proportion discharged per day at the end of t days of hospitalization. What proportion of the group is discharged at the end of 100 days?

56. Business Expenses The total expenditures (in dollars) of a business over the next five years are given by

$$\int_0^5 4000e^{0.05t}\, dt$$

Evaluate the expenditures.

57. Biology In a discussion of gene mutation,[12] the equation

$$\int_{q_0}^{q_n} \frac{dq}{q - \widehat{q}} = -(u + v)\int_0^n dt$$

occurs, where u and v are gene mutation rates, the q's are gene frequencies, and n is the number of generations. Assume that all letters represent constants, except q and t. Integrate both sides and then use your result to show that

$$n = \frac{1}{u + v}\ln\left|\frac{q_0 - \widehat{q}}{q_n - \widehat{q}}\right|$$

58. Fluid Flow In studying the flow of a fluid in a tube of constant radius R, such as blood flow in portions of the body, we can think of the tube as consisting of concentric tubes of radius r, where $0 \le r \le R$. The velocity v of the fluid is a function of r and is given by[13]

$$v = \frac{(P_1 - P_2)(R^2 - r^2)}{4\eta l}$$

where P_1 and P_2 are pressures at the ends of the tube, η (a Greek letter read "eta") is the fluid viscosity, and l is the length of the tube. The volume rate of flow through the tube, Q, is given by

$$Q = \int_0^R 2\pi r v\, dr$$

Show that $Q = \dfrac{\pi R^4(P_1 - P_2)}{8\eta l}$. Note that R occurs as a factor to the fourth power. Thus, doubling the radius of the tube has the effect of increasing the flow by a factor of 16. The formula that you derived for the volume rate of flow is called *Poiseuille's law,* after the French physiologist Jean Poiseuille.

59. Inventory In a discussion of inventory, Barbosa and Friedman[14] refer to the function

$$g(x) = \frac{1}{k}\int_1^{1/x} ku^r\, du$$

where k and r are constants, $k > 0$ and $r > -2$, and $x > 0$. Verify the claim that

$$g'(x) = -\frac{1}{x^{r+2}}$$

(*Hint:* Consider two cases: when $r \neq -1$ and when $r = -1$.)

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[12] W. B. Mather, *Principles of Quantitative Genetics* (Minneapolis: Burgess Publishing Company, 1964).

[13] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

[14] L. C. Barbosa and M. Friedman, "Deterministic Inventory Lot Size Models—a General Root Law," *Management Science,* 24, no. 8 (1978), 819–26.