Differentiating, we have

Note that $\ln b$ is just a constant!

$$\frac{d}{dx}(\log_b u) = \frac{d}{dx}\left(\frac{\ln u}{\ln b}\right) = \frac{1}{\ln b}\frac{d}{dx}(\ln u) = \frac{1}{\ln b}\cdot\frac{1}{u}\frac{du}{dx}$$

Summarizing,

$$\frac{d}{dx}(\log_b u) = \frac{1}{(\ln b)u}\cdot\frac{du}{dx} \quad \text{for } u > 0$$

Rather than memorizing this rule, we suggest remembering the procedure used to obtain it.

Procedure to Differentiate $\log_b u$

Convert $\log_b u$ to natural logarithms to obtain $\dfrac{\ln u}{\ln b}$, and then differentiate.

EXAMPLE 5 Differentiating a Logarithmic Function to the Base 2

Differentiate $y = \log_2 x$.

Solution: Following the foregoing procedure, we have

$$\frac{d}{dx}(\log_2 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 2}\right) = \frac{1}{\ln 2}\frac{d}{dx}(\ln x) = \frac{1}{(\ln 2)x}$$

It is worth mentioning that we can write our answer in terms of the original base. Because

$$\frac{1}{\ln b} = \frac{1}{\dfrac{\log_b b}{\log_b e}} = \frac{\log_b e}{1} = \log_b e$$

we can express $\dfrac{1}{(\ln 2)x}$ as $\dfrac{\log_2 e}{x}$. More generally, $\dfrac{d}{dx}(\log_b u) = \dfrac{\log_b e}{u}\cdot\dfrac{du}{dx}$.

Now Work Problem 15 ◁

APPLY IT ▶

2. The intensity of an earthquake is measured on the Richter scale. The reading is given by $R = \log \frac{I}{I_0}$, where I is the intensity and I_0 is a standard minimum intensity. If $I_0 = 1$, find $\frac{dR}{dI}$, the rate of change of the Richter-scale reading with respect to the intensity.

EXAMPLE 6 Differentiating a Logarithmic Function to the Base 10

If $y = \log(2x + 1)$, find the rate of change of y with respect to x.

Solution: The rate of change is dy/dx, and the base involved is 10. Therefore, we have

$$\frac{dy}{dx} = \frac{d}{dx}(\log(2x+1)) = \frac{d}{dx}\left(\frac{\ln(2x+1)}{\ln 10}\right)$$
$$= \frac{1}{\ln 10}\cdot\frac{1}{2x+1}(2) = \frac{2}{\ln 10(2x+1)}$$

◁

PROBLEMS 12.1

In Problems 1–44, differentiate the functions. If possible, first use properties of logarithms to simplify the given function.

1. $y = a\ln x$

2. $y = \dfrac{5\ln x}{9}$

3. $y = \ln(ax + b)$

4. $y = \ln(5x - 6)$

5. $y = \ln x^2$

6. $y = \ln(5x^3 + 3x^2 + 2x + 1)$

7. $y = \ln(1 - x^2)$

8. $y = \ln(ax^2 + bx + c)$

9. $f(X) = \ln(4X^6 + 2X^3)$

10. $f(r) = \ln(2r^4 - 3r^2 + 2r + 1)$

11. $f(t) = t\ln t - t$

12. $y = x^2\ln x$

13. $y = x^2\ln(ax + b)$

14. $y = (ax + b)^3\ln(ax + b)$

15. $y = \log_3(8x - 1)$

16. $f(w) = \log(w^2 + 2w + 1)$

17. $y = x^2 + \log_2(x^2 + 4)$

18. $y = x^a\log_b x$

19. $f(z) = \dfrac{\ln z}{z}$

20. $y = \dfrac{x^2}{\ln x}$

21. $y = \dfrac{x^4 + 3x^2 + x}{\ln x}$

22. $y = \ln x^{100}$

23. $y = \ln(ax^2 + bx + c)^d$

24. $y = 6 \ln \sqrt[3]{x}$

25. $y = 9 \ln \sqrt{1 + x^2}$

26. $f(t) = \ln \left(\dfrac{t^4}{1 + 6t + t^2} \right)$

27. $f(l) = \ln \left(\dfrac{1 + l}{1 - l} \right)$

28. $y = \ln \left(\dfrac{ax + b}{cx + d} \right)$

29. $y = \ln \sqrt[4]{\dfrac{1 + x^2}{1 - x^2}}$

30. $y = \ln \sqrt[3]{\dfrac{x^3 - 1}{x^3 + 1}}$

31. $y = \ln[(ax^2 + bx + c)^p (hx^2 + kx + l)^q]$

32. $y = \ln[(5x + 2)^4 (8x - 3)^6]$

33. $y = \ln(f(x)g(x))$

34. $y = 6 \ln \dfrac{x}{\sqrt{2x + 1}}$

35. $y = (x^2 + 1) \ln(2x + 1)$

36. $y = (ax^2 + bx + c) \ln(hx^2 + kx + l)$

37. $y = \ln x^3 + \ln^3 x$

38. $y = x^{\log_3 5}$

39. $y = \ln^4(ax)$

40. $y = \ln^2(2x + 11)$

41. $y = \ln \sqrt{f(x)}$

42. $y = \ln \left(x^3 \sqrt[4]{2x + 1} \right)$

43. $y = \sqrt{f(x) + \ln x}$

44. $y = \ln \left(x + \sqrt{1 + x^2} \right)$

45. Find an equation of the tangent line to the curve

$$y = \ln(x^2 - 3x - 3)$$

when $x = 4$.

46. Find an equation of the tangent line to the curve

$$y = x \ln x - x$$

at the point where $x = 1$.

47. Find the slope of the curve $y = \dfrac{x}{\ln x}$ when $x = 3$.

48. Marginal Revenue Find the marginal-revenue function if the demand function is $p = 50/\ln(q + 3)$.

49. Marginal Cost A total-cost function is given by

$$c = 25 \ln(q + 1) + 12$$

Find the marginal cost when $q = 6$.

50. Marginal Cost A manufacturer's average-cost function, in dollars, is given by

$$\bar{c} = \frac{500}{\ln(q + 20)}$$

Find the marginal cost (rounded to two decimal places) when $q = 50$.

51. Supply Change The supply of q units of a product at a price of p dollars per unit is given by $q(p) = 27 + 11 \ln(2p + 1)$. Find the rate of change of supply with respect to price, $\dfrac{dq}{dp}$.

52. Sound Perception The loudness of sound, L, measured in decibels, perceived by the human ear depends upon intensity levels, I, according to $L = 10 \log \dfrac{I}{I_0}$, where I_0 is the standard threshold of audibility. If $I_0 = 17$, find $\dfrac{dL}{dI}$, the rate of change of the loudness with respect to the intensity.

53. Biology In a certain experiment with bacteria, it is observed that the relative activeness of a given bacteria colony is described by

$$A = 6 \ln \left(\frac{T}{a - T} - a \right)$$

where a is a constant and T is the surrounding temperature. Find the rate of change of A with respect to T.

54. Show that the relative rate of change of $y = f(x)$ with respect to x is equal to the derivative of $y = \ln f(x)$.

55. Show that $\dfrac{d}{dx}(\log_b u) = \dfrac{1}{u}(\log_b e)\dfrac{du}{dx}$.

In Problems 56 and 57, use differentiation rules to find $f'(x)$. Then use your graphing calculator to find all roots of $f'(x) = 0$. Round your answers to two decimal places.

56. $f(x) = x^3 \ln x$

57. $f(x) = \dfrac{\ln(x^2)}{x^2}$

Objective

To develop a differentiation formula for $y = e^u$, to apply the formula, and to use it to differentiate an exponential function with a base other than e.

12.2 Derivatives of Exponential Functions

As we pointed out in Section 12.1, the exponential functions cannot be constructed from power functions using multiplication by a constant, arithmetic operations, and composition. However, the functions b^x, for $b > 0$ and $b \neq 1$, are inverse to the functions $\log_b(x)$, and if an invertible function f is differentiable, it is fairly easy to see that its inverse is differentiable. The key idea is that the graph of the inverse of a function is obtained by reflecting the graph of the original function in the line $y = x$. This reflection process preserves smoothness so that if the graph of an invertible function is smooth, then so is the graph of its inverse. Differentiating $f(f^{-1}(x)) = x$, we have

$$\frac{d}{dx}(f(f^{-1}(x))) = \frac{d}{dx}(x)$$

$$f'(f^{-1}(x))\frac{d}{dx}(f^{-1}(x)) = 1 \qquad \text{Chain Rule}$$

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

applies. The third $(2^{\sqrt{x}})$ is a constant base to a variable power, so we must differentiate an exponential function. Taken all together, we have

$$\begin{aligned}\frac{d}{dx}\left(e^2 + x^e + 2^{\sqrt{x}}\right) &= 0 + ex^{e-1} + \frac{d}{dx}\left[e^{(\ln 2)\sqrt{x}}\right]\\ &= ex^{e-1} + \left[e^{(\ln 2)\sqrt{x}}\right](\ln 2)\left(\frac{1}{2\sqrt{x}}\right)\\ &= ex^{e-1} + \frac{2^{\sqrt{x}}\ln 2}{2\sqrt{x}}\end{aligned}$$

Now Work Problem 17 ◁

EXAMPLE 6 Differentiating Power Functions Again

We have often used the rule $d/dx(x^a) = ax^{a-1}$, but we have only *proved* it for a a positive integer and a few other special cases. At least for $x > 0$, we can now improve our understanding of power functions, using Equation (2).

For $x > 0$, we can write $x^a = e^{a \ln x}$. So we have

$$\frac{d}{dx}(x^a) = \frac{d}{dx}e^{a\ln x} = e^{a\ln x}\frac{d}{dx}(a \ln x) = x^a(ax^{-1}) = ax^{a-1}$$

Now Work Problem 19 ◁

PROBLEMS 12.2

In Problems 1–28, differentiate the functions.

1. $y = 5e^x$

2. $y = \dfrac{ae^x}{b}$

3. $y = e^{2x^2+3}$

4. $y = e^{3x^3+5x^2+7x+11}$

5. $y = e^{9-5x}$

6. $f(q) = e^{-q^3+6q-1}$

7. $f(r) = e^{4r^3+5r^2+2r+6}$

8. $y = e^{x^2+6x^3+1}$

9. $y = x^e e^x$

10. $y = 3x^4 e^{-x}$

11. $y = x^2 e^{-x^2}$

12. $y = xe^{ax}$

13. $y = \dfrac{e^x + e^{-x}}{3}$

14. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

15. $y = 5^{2x^3}$

16. $y = 2^x x^2$

17. $f(w) = \dfrac{e^{aw}}{w^2 + w + 1}$

18. $y = e^{x-\sqrt{x}}$

19. $y = e^{1-\sqrt{x}}$

20. $y = (e^{2x} + 1)^3$

21. $y = x^5 - 5^x$

22. $f(z) = e^{1/z}$

23. $y = \dfrac{e^x - 1}{e^x + 1}$

24. $y = e^{ax}(bx + c)$

25. $y = \ln e^x$

26. $y = e^{-x}\ln x$

27. $y = x^x$

28. $y = \ln e^{4x+1}$

29. If $f(x) = e^c e^{bx} e^{ax^2}$, find $f'(1)$.

30. If $f(x) = 5^{x^2 \ln x}$, find $f'(1)$.

31. Find an equation of the tangent line to the curve $y = e^x$ when $x = -2$.

32. Find an equation of the tangent line to the curve $y = e^x$ at the point $(1, e)$. Show that this tangent line passes through $(0, 0)$ and show that it is the only tangent line to $y = e^x$ that passes through $(0, 0)$.

For each of the demand equations in Problems 33 and 34, find the rate of change of price, p, with respect to quantity, q. What is the rate of change for the indicated value of q?

33. $p = 15e^{-0.001q}$; $q = 500$

34. $p = 5e^{-q/100}$; $q = 100$

In Problems 35 and 36, $\bar{c}$ is the average cost of producing q units of a product. Find the marginal-cost function and the marginal cost for the given values of q.

35. $\bar{c} = \dfrac{7000e^{q/700}}{q}$; $q = 350, q = 700$

36. $\bar{c} = \dfrac{850}{q} + 4000\dfrac{e^{(2q+6)/800}}{q}$; $q = 97, q = 197$

37. If $w = e^{x^2}$ and $x = \dfrac{t+1}{t-1}$, find $\dfrac{dw}{dt}$ when $t = 2$.

38. If $f'(x) = x^3$ and $u = e^x$, show that

$$\frac{d}{dx}[f(u)] = e^{4x}$$

39. If c is a positive constant and

$$\left.\frac{d}{dx}(c^x - x^c)\right|_{x=1} = 0$$

prove that $c = e$.

40. Calculate the relative rate of change of

$$f(x) = 10^{-x} + \ln(8 + x) + 0.01e^{x-2}$$

when $x = 2$. Round your answer to four decimal places.

41. Production Run For a firm, the daily output on the tth day of a production run is given by

$$q = 500(1 - e^{-0.2t})$$

Find the rate of change of output q with respect to t on the tenth day.

42. Normal-Density Function For the normal-density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

find $f'(-1)$.

43. Population The population, in millions, of the greater Seattle area t years from 1970 is estimated by $P = 1.92e^{0.0176t}$. Show that $dP/dt = kP$, where k is a constant. This means that the rate of change of population at any time is proportional to the population at that time.

44. Market Penetration In a discussion of diffusion of a new process into a market, Hurter and Rubenstein[1] refer to an equation of the form

$$Y = k\alpha^{\beta^t}$$

where Y is the cumulative level of diffusion of the new process at time t and k, α, and β are positive constants. Verify their claim that

$$\frac{dY}{dt} = k\alpha^{\beta^t}(\beta^t \ln \alpha) \ln \beta$$

45. Finance After t years, the value S of a principal of P dollars invested at the annual rate of r compounded continuously is given by $S = Pe^{rt}$. Show that the relative rate of change of S with respect to t is r.

46. Predator–Prey Relationship In an article concerning predators and prey, Holling[2] refers to an equation of the form

$$y = K(1 - e^{-ax})$$

where x is the prey density, y is the number of prey attacked, and K and a are constants. Verify his statement that

$$\frac{dy}{dx} = a(K - y)$$

47. Earthquakes According to Richter,[3] the number of earthquakes of magnitude M or greater per unit of time is given by $N = 10^A 10^{-bM}$, where A and b are constants. Find dN/dM.

48. Psychology Short-term retention was studied by Peterson and Peterson.[4] The two researchers analyzed a procedure in which an experimenter verbally gave a subject a three-letter consonant syllable, such as CHJ, followed by a three-digit number, such as 309. The subject then repeated the number and counted backward by 3's, such as 309, 306, 303, ... After a period of time, the subject was signaled by a light to recite the three-letter consonant syllable. The time between the experimenter's completion of the last consonant to the onset of the light was called the *recall interval*. The time between the onset of the light and the completion of a response was referred to as *latency*. After many trials, it was determined that, for a recall interval of t seconds, the approximate proportion of correct recalls with latency below 2.83 seconds was

$$p = 0.89[0.01 + 0.99(0.85)^t]$$

(a) Find dp/dt and interpret your result.
(b) Evaluate dp/dt when $t = 2$. Round your answer to two decimal places.

49. Medicine Suppose a tracer, such as a colored dye, is injected instantly into the heart at time $t = 0$ and mixes uniformly with blood inside the heart. Let the initial concentration of the tracer in the heart be C_0, and assume that the heart has constant volume V. Also assume that, as fresh blood flows into the heart, the diluted mixture of blood and tracer flows out at the constant positive rate r. Then the concentration of the tracer in the heart at time t is given by

$$C(t) = C_0 e^{-(r/V)t}$$

Show that $dC/dt = (-r/V)C(t)$.

50. Medicine In Problem 49, suppose the tracer is injected at a constant rate R. Then the concentration at time t is

$$C(t) = \frac{R}{r}\left[1 - e^{-(r/V)t}\right]$$

(a) Find $C(0)$.
(b) Show that $\frac{dC}{dt} = \frac{R}{V} - \frac{r}{V}C(t)$.

51. Schizophrenia Several models have been used to analyze the length of stay in a hospital. For a particular group of schizophrenics, one such model is[5]

$$f(t) = 1 - e^{-0.008t}$$

where $f(t)$ is the proportion of the group that was discharged at the end of t days of hospitalization. Find the rate of discharge (the proportion discharged per day) at the end of 100 days. Round your answer to four decimal places.

52. Savings and Consumption A country's savings S (in billions of dollars) is related to its national income I (in billions of dollars) by the equation

$$S = \ln \frac{3}{2 + e^{-I}}$$

(a) Find the marginal propensity to consume as a function of income.
(b) To the nearest million dollars, what is the national income when the marginal propensity to save is $\frac{1}{7}$?

In Problems 53 and 54, use differentiation rules to find $f'(x)$. Then use your graphing calculator to find all real zeros of $f'(x)$. Round your answers to two decimal places.

53. $f(x) = e^{2x^3+x^2-3x}$ **54.** $f(x) = xe^{-x}$

[1] A. P. Hurter, Jr., A. H. Rubenstein, et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change*, 11 (1978), 197–221.

[2] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist*, XCI, no. 7 (1959), 385–98.

[3] C. F. Richter, *Elementary Seismology* (San Francisco: W. H. Freeman and Company, Publishers, 1958).

[4] L. R. Peterson and M. J. Peterson, "Short-Term Retention of Individual Verbal Items," *Journal of Experimental Psychology*, 58 (1959), 193–98.

[5] Adapted from W. W. Eaton and G. A. Whitmore, "Length of Stay as a Stochastic Process: A General Approach and Application to Hospitalization for Schizophrenia," *Journal of Mathematical Sociology*, 5 (1977), 273–92.

To find the marginal revenue, dr/dq, we differentiate r by using the product rule:

$$\frac{dr}{dq} = p + q\frac{dp}{dq}. \tag{5}$$

Factoring the right side of Equation (5), we have

$$\frac{dr}{dq} = p\left(1 + \frac{q}{p}\frac{dp}{dq}\right)$$

But

$$\frac{q}{p}\frac{dp}{dq} = \frac{\frac{dp}{dq}}{\frac{p}{q}} = \frac{1}{\eta}$$

Thus,

$$\boxed{\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right)} \tag{6}$$

If demand is elastic, then $\eta < -1$, so $1 + \dfrac{1}{\eta} > 0$. If demand is inelastic, then $\eta > -1$, so $1 + \dfrac{1}{\eta} < 0$. We can assume that $p > 0$. From Equation (6) we can conclude that $dr/dq > 0$ on intervals for which demand is elastic. As we will soon see, a function is increasing on intervals for which its derivative is positive, and a function is decreasing on intervals for which its derivative is negative. Hence, total revenue r is increasing on intervals for which demand is elastic, and total revenue is decreasing on intervals for which demand is inelastic.

Thus, we conclude from the preceding argument that as more units are sold, a manufacturer's total revenue increases if demand is elastic but decreases if demand is inelastic. That is, if demand is elastic, a lower price will increase revenue. This means that a lower price will cause a large enough increase in demand to actually increase revenue. If demand is inelastic, a lower price will decrease revenue. For unit elasticity, a lower price leaves total revenue unchanged.

If we solve the demand equation to obtain the form $q = g(p)$, rather than $p = f(q)$, then a similar analysis gives

$$\boxed{\frac{dr}{dp} = q(1 + \eta)} \tag{7}$$

and the conclusions of the last paragraph follow even more directly.

PROBLEMS 12.3

In Problems 1–14, find the point elasticity of the demand equations for the indicated values of q or p, and determine whether demand is elastic, is inelastic, or has unit elasticity.

1. $p = 40 - 2q;\ q = 5$

2. $p = 10 - 0.04q;\ q = 100$

3. $p = \dfrac{3000}{q};\ q = 300$

4. $p = \dfrac{500}{q^2};\ q = 52$

5. $p = \dfrac{100}{q+1};\ q = 100$

6. $p = \dfrac{800}{2q+1};\ q = 24$

7. $p = 150 - e^{q/100};\ q = 100$

8. $p = 250e^{-q/50};\ q = 50$

9. $q = 1200 - 150p;\ p = 4$

10. $p = 100 - q;\ p = 50$

11. $q = \sqrt{500 - p};\ p = 400$

12. $q = \sqrt{2500 - p^2};\ p = 20$

13. $q = (p - 50)^2;\ p = 10$

14. $q = p^2 - 50p + 850;\ p = 20$

15. For the linear demand equation $p = 15 - q$, verify that demand is elastic when $p = 10$, is inelastic when $p = 5$, and has unit elasticity when $p = 7.5$.

16. For what value (or values) of q do the following demand equations have unit elasticity?
(a) $p = 36 - 0.25q$
(b) $p = 300 - q^2$

17. The demand equation for a product is

$$q = 500 - 40p + p^2$$

where p is the price per unit (in dollars) and q is the quantity of units demanded (in thousands). Find the point elasticity of demand when $p = 15$. If this price of 15 is increased by $\frac{1}{2}\%$, what is the approximate change in demand?

18. The demand equation for a certain product is

$$q = \sqrt{3000 - p^2}$$

where p is in dollars. Find the point elasticity of demand when $p = 40$, and use this value to compute the percentage change in demand if the price of \$40 is increased by 7%.

19. For the demand equation $p = 500 - 2q$, verify that demand is elastic and total revenue is increasing for $0 < q < 125$. Verify that demand is inelastic and total revenue is decreasing for $125 < q < 250$.

20. Show that if the demand equation can be written as $q = g(p)$ then $\frac{dr}{dp} = q(1 + \eta)$.

21. Repeat Problem 20 for $p = \frac{1000}{q^2}$.

22. Suppose $p = mq + b$ is a linear demand equation, where $m \neq 0$ and $b > 0$.
(a) Show that $\lim_{p \to b^-} \eta = -\infty$.
(b) Show that $\eta = 0$ when $p = 0$.

23. The demand equation for a manufacturer's product has the form

$$q = a\sqrt{b - cp^2}$$

where a, b, and c are positive constants.
(a) Show that elasticity does not depend on a.
(b) Determine the interval of prices for which demand is elastic.
(c) For which price is there unit elasticity?

24. Given the demand equation $q^2(1 + p)^2 = p$, determine the point elasticity of demand when $p = 9$.

25. The demand equation for a product is

$$q = \frac{60}{p} + \ln(65 - p^3)$$

(a) Determine the point elasticity of demand when $p = 4$, and classify the demand as elastic, inelastic, or of unit elasticity at this price level.
(b) If the price is lowered by 2% (from \$4.00 to \$3.92), use the answer to part (a) to estimate the corresponding percentage change in quantity sold.
(c) Will the changes in part (b) result in an increase or decrease in revenue? Explain.

26. The demand equation for a manufacturer's product is

$$p = 50(151 - q)^{0.02\sqrt{q+19}}$$

(a) Find the value of dp/dq when 150 units are demanded.
(b) Using the result in part (a), determine the point elasticity of demand when 150 units are demanded. At this level, is demand elastic, inelastic, or of unit elasticity?
(c) Use the result in part (b) to approximate the price per unit if demand decreases from 150 to 140 units.
(d) If the current demand is 150 units, should the manufacturer increase or decrease price in order to increase revenue? (Justify your answer.)

27. A manufacturer of aluminum doors currently is able to sell 500 doors per week at a price of \$80 each. If the price were lowered to \$75 each, an additional 50 doors per week could be sold. Estimate the current elasticity of demand for the doors, and also estimate the current value of the manufacturer's marginal-revenue function.

28. Given the demand equation

$$p = 2000 - q^2$$

where $5 \le q \le 40$, for what value of q is $|\eta|$ a maximum? For what value is it a minimum?

29. Repeat Problem 28 for

$$p = \frac{200}{q + 5}$$

such that $5 \le q \le 95$.

Objective

To discuss the notion of a function defined implicitly and to determine derivatives by means of implicit differentiation.

12.4 Implicit Differentiation

Implicit differentiation is a technique for differentiating "functions" that are not given in the form $y = f(x)$ nor in the form $x = g(y)$. To introduce this technique, we will find the slope of a tangent line to a *circle*. Circles are smooth curves, and it is clear that at any point on any circle there is a tangent line. But, for any circle with positive radius, there will be some vertical lines that intersect the circle at more than one point. So we know that any such circle cannot be described as the graph of a *single* function. For definiteness in our discussion, let us take the circle of radius 2 whose center is at the origin (Figure 12.4). Its equation is

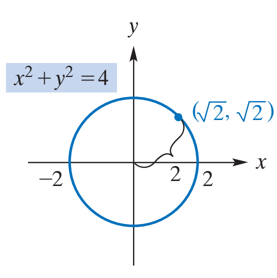

FIGURE 12.4 The circle $x^2 + y^2 = 4$.

$$x^2 + y^2 = 4$$

$$\text{equivalently} \quad x^2 + y^2 - 4 = 0 \tag{1}$$

The point $(\sqrt{2}, \sqrt{2})$ lies on the circle. To find the slope at this point, we need to find dy/dx there. Until now, we have always had y given explicitly (directly) in terms of x before determining y'; that is, our equation was always in the form $y = f(x)$ or in the form $x = g(y)$. In Equation (1), this is not so. We say that Equation (1) has the form $F(x, y) = 0$, where $F(x, y)$ denotes a function of two variables as introduced in Section 2.8. The obvious thing to do is solve Equation (1) for y in terms of x:

$$x^2 + y^2 - 4 = 0$$
$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2} \tag{2}$$

APPLY IT ►

6. A 10-foot ladder is placed against a vertical wall. Suppose the bottom of the ladder slides away from the wall at a constant rate of 3 ft/s. (That is, $\frac{dx}{dt} = 3$.) How fast is the top of the ladder sliding down the wall when the top of the ladder is 8 feet from the ground (that is, when $y = 8$)? (That is, what is $\frac{dy}{dt}$?) (Use the Pythagorean theorem for right triangles, $x^2 + y^2 = z^2$, where x and y are the legs of the triangle and z is the hypotenuse.)

EXAMPLE 4 Implicit Differentiation

If $q - p = \ln q + \ln p$, find dq/dp.

Solution: We assume that q is a function of p and differentiate both sides with respect to p:

$$\frac{d}{dp}(q-p) = \frac{d}{dp}(\ln q + \ln p)$$

$$\frac{dq}{dp} - \frac{dp}{dp} = \frac{d}{dp}(\ln q) + \frac{d}{dp}(\ln p)$$

$$\frac{dq}{dp} - 1 = \frac{1}{q}\frac{dq}{dp} + \frac{1}{p}$$

$$\frac{dq}{dp}\left(1 - \frac{1}{q}\right) = \frac{1}{p} + 1$$

$$\frac{dq}{dp}\left(\frac{q-1}{q}\right) = \frac{1+p}{p}$$

$$\frac{dq}{dp} = \frac{(1+p)q}{p(q-1)} \quad \text{for } p(q-1) \neq 0$$

Now Work Problem 19 ◁

PROBLEMS 12.4

In Problems 1–24, find dy/dx by implicit differentiation.

1. $x^2 - y^2 = 1$

2. $3x^2 + 6y^2 = 1$

3. $2y^3 - 7x^2 = 5$

4. $5y^2 - 2x^2 = 10$

5. $\sqrt[3]{x} + \sqrt[3]{y} = 3$

6. $\sqrt{x} - \sqrt{y} = 1$

7. $x^{3/4} + y^{3/4} = 5$

8. $y^3 = 4x$

9. $xy = 36$

10. $x^2 + xy - 2y^2 = 0$

11. $x + xy + y = 1$

12. $x^3 - y^3 = 3x^2y - 3xy^2$

13. $2x^3 + y^3 - 12xy = 0$

14. $5x^3 + 6xy + 7y^3 = 0$

15. $x = \sqrt{y} + \sqrt[4]{y}$

16. $x^2y^2 = 1$

17. $5x^3y^4 - x + y^2 = 25$

18. $y^2 + y = \ln x$

19. $\ln(xy) = e^{xy}$

20. $\ln(xy) + x = 4$

21. $xe^y + ye^x = 1$

22. $4x^2 + 9y^2 = 16$

23. $(1 + e^{3x})^2 = 3 + \ln(x + y)$

24. $e^{x-y} = \ln(x - y)$

25. If $x + xy + y^2 = 7$, find dy/dx at $(1, 2)$.

26. If $(x+1)\sqrt{y} = (y+1)\sqrt{x}$, find dy/dx at $(2, 2)$.

27. Find the slope of the curve $4x^2 + 9y^2 = 1$ at the point $\left(0, \frac{1}{3}\right)$; at the point (x_0, y_0).

28. Find the slope of the curve $(x^2 + y^2)^2 = 4y^2$ at the point $(0, 2)$.

29. Find equations of the tangent lines to the curve

$$x^3 + xy + y^3 = -1$$

at the points $(-1, -1)$, $(-1, 0)$, and $(-1, 1)$.

30. Repeat Problem 29 for the curve

$$y^2 + xy - x^2 = 5$$

at the point $(4, 3)$.

For the demand equations in Problems 31– 34, find the rate of change of q with respect to p.

31. $p = 100 - q^3$

32. $p = 400 - \sqrt{q}$

33. $p = \dfrac{20}{(q+5)^2}$

34. $p = \dfrac{3}{q^2 + 1}$

35. Radioactivity The relative activity I/I_0 of a radioactive element varies with elapsed time according to the equation

$$\ln\left(\frac{I}{I_0}\right) = -\lambda t$$

where λ (a Greek letter read "lambda") is the disintegration constant and I_0 is the initial intensity (a constant). Find the rate of change of the intensity, I, with respect to the elapsed time, t.

36. Earthquakes The magnitude, M, of an earthquake and its energy, E, are related by the equation[6]

$$1.5M = \log\left(\frac{E}{2.5 \times 10^{11}}\right)$$

Here M is given in terms of Richter's preferred scale of 1958 and E is in ergs. Determine the rate of change of energy with respect to magnitude and the rate of change of magnitude with respect to energy.

[6] K. E. Bullen, *An Introduction to the Theory of Seismology* (Cambridge, U.K.: Cambridge at the University Press, 1963).

37. Physical Scale The relationship among the speed (v), frequency (f), and wavelength (λ) of any wave is given by

$$v = f\lambda$$

Find $df/d\lambda$ by differentiating implicitly. (Treat v as a constant.) Then show that the same result is obtained if you first solve the equation for f and then differentiate with respect to λ.

38. Biology The equation $(P + a)(v + b) = k$ is called the "fundamental equation of muscle contraction."[7] Here P is the load imposed on the muscle; v is the velocity of the shortening of the muscle fibers; and a, b, and k are positive constants. Use implicit differentiation to show that, in terms of P,

$$\frac{dv}{dP} = -\frac{k}{(P+a)^2}$$

39. Marginal Propensity to Consume A country's savings, S, is defined implicitly in terms of its national income, I, by the equation

$$S^2 + \frac{1}{4}I^2 = SI + I$$

where both S and I are in billions of dollars. Find the marginal propensity to consume when $I = 16$ and $S = 12$.

40. Technological Substitution New products or technologies often tend to replace old ones. For example, today most commercial airlines use jet engines rather than prop engines. In discussing the forecasting of technological substitution, Hurter and Rubenstein[8] refer to the equation

$$\ln \frac{f(t)}{1 - f(t)} + \sigma \frac{1}{1 - f(t)} = C_1 + C_2 t$$

where $f(t)$ is the market share of the substitute over time t and C_1, C_2, and σ (a Greek letter read "sigma") are constants. Verify their claim that the rate of substitution is

$$f'(t) = \frac{C_2 f(t)[1 - f(t)]^2}{\sigma f(t) + [1 - f(t)]}$$

Objective

To describe the method of logarithmic differentiation and to show how to differentiate a function of the form u^v.

12.5 Logarithmic Differentiation

A technique called **logarithmic differentiation** often simplifies the differentiation of $y = f(x)$ when $f(x)$ involves products, quotients, or powers. The procedure is as follows:

Logarithmic Differentiation

To differentiate $y = f(x)$,

1. Take the natural logarithm of both sides. This results in
$$\ln y = \ln(f(x))$$
2. Simplify $\ln(f(x))$ by using properties of logarithms.
3. Differentiate both sides with respect to x.
4. Solve for $\frac{dy}{dx}$.
5. Express the answer in terms of x only. This requires substituting $f(x)$ for y.

There are a couple of points worth noting. First, irrespective of any simplification, the procedure produces

$$\frac{y'}{y} = \frac{d}{dx}(\ln(f(x))$$

so that

$$\frac{dy}{dx} = y\frac{d}{dx}(\ln(f(x))$$

[7] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).

[8] A. P. Hurter, Jr., A. H. Rubenstein et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change,* 11 (1978), 197–221.

For type (a), use the power rule. For type (b), use the differentiation formula for exponential functions. (If $b \neq e$, first convert $b^{g(x)}$ to an e^u function.) For type (c), use logarithmic differentiation or first convert to an e^u function. Do not apply a rule in a situation where the rule does not apply. For example, the power rule does not apply to x^x.

PROBLEMS 12.5

In Problems 1–12, find y' by using logarithmic differentiation.

1. $y = (x+1)^2(x-2)(x^2+3)$

2. $y = (2x-3)(5x-7)^2(11x-13)^3$

3. $y = (3x^3-1)^2(2x+5)^3$

4. $y = (2x^2+1)\sqrt{8x^2-1}$

5. $y = \sqrt{x+1}\sqrt{x-1}\sqrt{x^2+1}$

6. $y = (2x+1)\sqrt{x^3+2}\sqrt[3]{2x+5}$

7. $y = \dfrac{\sqrt[3]{1+x^2}}{1+x}$

8. $y = \sqrt{\dfrac{x^2+5}{x+9}}$

9. $y = \dfrac{(2x^2+2)^2}{(x+1)^2(3x+2)}$

10. $y = \dfrac{x^2(1+x^2)}{\sqrt{x^2+4}}$

11. $y = \sqrt{\dfrac{(x+3)(x-2)}{2x-1}}$

12. $y = \sqrt[5]{\dfrac{(x^2+1)^2}{x^2e^{-x}}}$

In Problems 13–20, find y'.

13. $y = x^{x^2+1}$

14. $y = (2x)^{\sqrt{x}}$

15. $y = x^{\sqrt{x}}$

16. $y = \left(\dfrac{3}{x^2}\right)^x$

17. $y = (2x+3)^{5x}$

18. $y = (x^2+1)^{x+1}$

19. $y = 4e^x x^{3x}$

20. $y = (\sqrt{x})^x$

21. If $y = (4x-3)^{2x+1}$, find dy/dx when $x = 1$.

22. If $y = (e^x)^{(e^x)}$, find dy/dx when $x = 0$.

23. Find an equation of the tangent line to

$$y = (x+1)(x+2)^2(x+3)^2$$

at the point where $x = 0$.

24. Find an equation of the tangent line to the graph of

$$y = x^x$$

at the point where $x = 1$.

25. Find an equation of the tangent line to the graph of

$$y = x^x$$

at the point where $x = e$.

26. If $y = x^x$, find the relative rate of change of y with respect to x when $x = 1$.

27. If $y = x^x$, find the value of x for which the *percentage* rate of change of y with respect to x is 50%.

28. Suppose $f(x)$ is a positive differentiable function and g is a differentiable function and $y = (f(x))^{g(x)}$. Use logarithmic differentiation to show that

$$\frac{dy}{dx} = (f(x))^{g(x)}\left(f'(x)\frac{g(x)}{f(x)} + g'(x)\ln(f(x))\right)$$

29. The demand equation for a DVD is

$$q = 500 - 40p + p^2$$

If the price of \$15 is increased by 1/2%, find the corresponding percentage change in revenue.

30. Repeat Problem 29 with the same information except for a 5% *decrease* in price.

Objective

To approximate real roots of an equation by using calculus. The method shown is suitable for calculators.

12.6 Newton's Method

It is easy to solve equations of the form $f(x) = 0$ when f is a linear or quadratic function. For example, we can solve $x^2+3x-2 = 0$ by the quadratic formula. However, if $f(x)$ has a degree greater than 2 or is not a polynomial, it may be difficult, or even impossible, to find solutions of $f(x) = 0$, in terms of known functions, even when it is proveable that at least one solution exists. For this reason, we may settle for approximate solutions, which can be obtained in a variety of efficient ways. For example, a graphing calculator can be used to estimate the real roots of $f(x) = 0$. In this section, we will study how the derivative can be so used (provided that f is differentiable). The procedure we will develop, called *Newton's method,* is well suited to a calculator or computer.

Newton's method requires an initial estimate for a root of $f(x) = 0$. One way of obtaining this estimate is by making a rough sketch of the graph of $y = f(x)$. A point on the graph where $y = 0$ is an x-intercept, and the x-value of this point is a solution of $f(x) = 0$. Another way of estimating a root is based on the following fact:

> If f is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one real root between a and b.

EXAMPLE 2 Approximating a Root by Newton's Method

Approximate the root of $x^3 = 3x - 1$ that lies between -1 and -2. Continue the approximation procedure until two successive approximations differ by less than 0.0001.

Solution: Letting $f(x) = x^3 - 3x + 1$, so that the equation becomes $f(x) = 0$, we find that

$$f(-1) = (-1)^3 - 3(-1) + 1 = 3$$

and

$$f(-2) = (-2)^3 - 3(-2) + 1 = -1$$

(Note the change in sign.) Since $f(-2)$ is closer to 0 than is $f(-1)$, we choose -2 to be our first approximation, x_1. Now,

$$f'(x) = 3x^2 - 3$$

so

$$N(x) = x - \frac{x^3 - 3x + 1}{3x^2 - 3} = \frac{3x^3 - 3x - x^3 + 3x - 1}{3x^2 - 3} = \frac{2x^3 - 1}{3x^2 - 3}$$

and

$$x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 3}$$

Since $x_1 = -2$, letting $n = 1$ gives

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 3} = \frac{2(-2)^3 - 1}{3(-2)^2 - 3} \approx 1.88889$$

Continuing in this way, we obtain Table 12.2. Because the values of x_3 and x_4 differ by less than 0.0001, we take our approximation of the root to be $x_4 \approx -1.87939$.

Table 12.2

n	x_n	$x_n + 1$
1	-2.00000	-1.88889
2	-1.88889	-1.87945
3	-1.87945	-1.87939

Now Work Problem 3 ◁

If our choice of x_1 has $f'(x_1) = 0$, then Newton's method will fail to produce a value for x_2. *When this happens, the choice of* x_1 *must be rejected and a different number, close to the desired root, must be chosen for* x_1. A graph of f can be helpful in this situation. Finally, there are times when the sequence of approximations does not approach the root. A discussion of such situations is beyond the scope of this book.

PROBLEMS 12.6

In Problems 1–10, use Newton's method to approximate the indicated root of the given equation. Continue the approximation procedure until the difference of two successive approximations is less than 0.0001.

1. $x^3 - 5x + 1 = 0$; root between 0 and 1

2. $x^3 + 2x^2 - 1 = 0$; root between 0 and 1

3. $x^3 + x + 1 = 0$; root between -1 and 2.

4. $x^3 - 9x + 6 = 0$; root between 2 and 3

5. $x^3 + x + 1 = 0$; root between -1 and 0

6. $x^3 = 2x + 6$; root between 2 and 3

7. $x^4 = 3x - 1$; root between 0 and 1

8. $x^4 + x^3 - 1 = 0$; root between 0 and 1

9. $x^4 - 2x^3 + x^2 - 3 = 0$; root between 1 and 2

10. $x^4 - x^3 + x - 2 = 0$; root between 1 and 2

11. Estimate, to three-decimal-place accuracy, the cube root of 73. [*Hint:* Show that the problem is equivalent to finding a root of $f(x) = x^3 - 73 = 0$.] Choose 4 as the initial estimate. Continue the iteration until two successive approximations, rounded to three decimal places, are the same.

12. Estimate $\sqrt[4]{19}$, to two-decimal-place accuracy. Use 2 as your initial estimate.

13. Find, to three-decimal-place accuracy, all *positive* real solutions of the equation $e^x = x + 3$. (*Hint:* A rough sketch of the graphs of $y = e^x$ and $y = x + 3$ in the same plane makes it clear how many such solutions there are. Use the integer values suggested by the graph to choose the initial values.)

14. Find, to three-decimal-place accuracy, all real solutions of the equation $\ln x = 5 - x$.

15. Break-Even Quantity The cost of manufacturing q tons of a certain product is given by

$$c = 250 + 2q - 0.1q^3$$

and the revenue obtained by selling the q tons is given by

$$r = 3q$$

Approximate, to two-decimal-place accuracy, the break-even quantity. (*Hint:* Approximate a root of $r - c = 0$ by choosing 13 as your initial estimate.)

16. Break-Even Quantity The total cost of manufacturing q hundred pencils is c dollars, where

$$c = 50 + 4q + \frac{q^2}{1000} + \frac{1}{q}$$

Pencils are sold for \$8 per hundred.

(a) Show that the break-even quantity is a solution of the equation

$$f(q) = \frac{q^3}{1000} - 4q^2 + 50q + 1 = 0$$

(b) Use Newton's method to approximate the solution of $f(q) = 0$, where $f(q)$ is given in part (a). Use 10 as your initial approximation, and give your answer to two-decimal-place accuracy.

17. Equilibrium Given the supply equation $p = 2q + 5$ and the demand equation $p = \dfrac{100}{q^2 + 1}$, use Newton's method to estimate the market equilibrium quantity. Give your answer to three-decimal-place accuracy.

18. Equilibrium Given the supply equation

$$p = 0.2q^3 + 0.6q + 2$$

and the demand equation $p = 9 - q$, use Newton's method to estimate the market equilibrium quantity, and find the corresponding equilibrium price. Use 2 as an initial estimate for the required value of q, and give the answer to two-decimal-place accuracy.

19. Use Newton's method to approximate (to two-decimal-place accuracy) a critical value of the function

$$f(x) = \frac{x^3}{3} - x^2 - 5x + 1$$

on the interval [3, 4].

Objective

To find higher-order derivatives both directly and implicitly.

12.7 Higher-Order Derivatives

We know that the derivative of a function $y = f(x)$ is itself a function, $f'(x)$. If we differentiate $f'(x)$, the resulting function $(f')'(x)$ is called the **second derivative** of f at x. It is denoted $f''(x)$, which is read "f double prime of x." Similarly, the derivative of the second derivative is called the **third derivative**, written $f'''(x)$. Continuing in this way, we get **higher-order derivatives**. Some notations for higher-order derivatives are given in Table 12.3. To avoid clumsy notation, primes are not used beyond the third derivative.

The symbol d^2y/dx^2 represents the second derivative of y. It is not the same as $(dy/dx)^2$, the square of the first derivative of y.

Table 12.3

First derivative:	y'	$f'(x)$	$\dfrac{dy}{dx}$	$\dfrac{d}{dx}(f(x))$	$D_x y$
Second derivative:	y''	$f''(x)$	$\dfrac{d^2y}{dx^2}$	$\dfrac{d^2}{dx^2}(f(x))$	$D_x^2 y$
Third derivative:	y'''	$f'''(x)$	$\dfrac{d^3y}{dx^3}$	$\dfrac{d^3}{dx^3}(f(x))$	$D_x^3 y$
Fourth derivative:	$y^{(4)}$	$f^{(4)}(x)$	$\dfrac{d^4y}{dx^4}$	$\dfrac{d^4}{dx^4}(f(x))$	$D_x^4 y$
nth derivative:	$y^{(n)}$	$f^{(n)}(x)$	$\dfrac{d^ny}{dx^n}$	$\dfrac{d^n}{dx^n}(f(x))$	$D_x^n y$

The Leibniz notation for higher derivatives is a little less mysterious when we note that $\dfrac{d^ny}{dx^n}$ is a convention for $\left(\dfrac{d}{dx}\right)^n(y)$, which is to say, differentiation with respect to x, $\dfrac{d}{dx}$, applied n times to y.

In Example 5, the simplification of d^2y/dx^2 by making use of the original equation is not unusual.

We can further simplify the answer. Since $x^2 + 4y^2 = 4$ (the original equation),

$$\frac{d^2y}{dx^2} = -\frac{4}{16y^3} = -\frac{1}{4y^3}$$

Now Work Problem 23 ◁

EXAMPLE 6 Higher-Order Implicit Differentiation

Find $\dfrac{d^2y}{dx^2}$ if $y^2 = e^{x+y}$.

Solution: Differentiating both sides with respect to x gives

$$2y\frac{dy}{dx} = e^{x+y}\left(1 + \frac{dy}{dx}\right)$$

Solving for dy/dx, we obtain

$$2y\frac{dy}{dx} = e^{x+y} + e^{x+y}\frac{dy}{dx}$$

$$2y\frac{dy}{dx} - e^{x+y}\frac{dy}{dx} = e^{x+y}$$

$$(2y - e^{x+y})\frac{dy}{dx} = e^{x+y}$$

$$\frac{dy}{dx} = \frac{e^{x+y}}{2y - e^{x+y}}$$

Since $y^2 = e^{x+y}$ (the original equation),

$$\frac{dy}{dx} = \frac{y^2}{2y - y^2} = \frac{y}{2-y}$$

$$\frac{d^2y}{dx^2} = \frac{(2-y)\dfrac{dy}{dx} - y\left(-\dfrac{dy}{dx}\right)}{(2-y)^2} = \frac{2\dfrac{dy}{dx}}{(2-y)^2}$$

Now we express our answer without dy/dx. Since $\dfrac{dy}{dx} = \dfrac{y}{2-y}$,

$$\frac{d^2y}{dx^2} = \frac{2\left(\dfrac{y}{2-y}\right)}{(2-y)^2} = \frac{2y}{(2-y)^3}$$

Now Work Problem 31 ◁

PROBLEMS 12.7

In Problems 1–20, find the indicated derivatives.

1. $y = 4x^3 - 12x^2 + 6x + 2$, y'''
2. $y = x^5 + x^4 + x^3 + x^2 + x + 1$, y'''
3. $y = 8 - x$, $\dfrac{d^2y}{dx^2}$
4. $y = ax^2 + bx + c$, $\dfrac{d^3y}{dx^3}$
5. $y = x^3 + e^x$, $y^{(4)}$
6. $F(q) = \ln(q+1)$, $\dfrac{d^3F}{dq^3}$
7. $f(x) = x^3 \ln x$, $f'''(x)$
8. $y = \dfrac{1}{x}$, y'''
9. $f(q) = \dfrac{1}{3q^3}$, $f'''(q)$
10. $f(x) = \sqrt{x}$, $f''(x)$
11. $f(r) = \sqrt{9-r}$, $f''(r)$
12. $y = e^{ax^2}$, y''
13. $y = \dfrac{1}{2x+3}$, $\dfrac{d^2y}{dx^2}$
14. $y = (ax+b)^6$, y'''
15. $y = \dfrac{x+1}{x-1}$, y''
16. $y = 2x^{1/2} + (2x)^{1/2}$, y''
17. $y = \ln[x(x+a)]$, y''
18. $y = \ln \dfrac{(2x+5)(5x-2)}{x+1}$, y''

19. $f(z) = z^3 e^z, f'''(z)$

20. $y = \dfrac{x}{e^x}, \dfrac{d^2y}{dx^2}$

21. If $y = e^{2x} + e^{3x}$, find $\left.\dfrac{d^5y}{dx^5}\right|_{x=0}$.

22. If $y = e^{2\ln(x^2+1)}$, find y'' when $x = 1$.

In Problems 23–32, find y''.

23. $x^2 + 4y^2 - 16 = 0$

24. $x^2 + y^2 = 1$

25. $y^2 = 4x$

26. $9x^2 + 16y^2 = 25$

27. $a\sqrt{x} + b\sqrt{y} = c$

28. $y^2 - 6xy = 4$

29. $x + xy + y = 1$

30. $x^2 + 2xy + y^2 = 1$

31. $y = e^{x+y}$

32. $e^x + e^y = x^2 + y^2$

33. If $x^2 + 3x + y^2 = 4y$, find d^2y/dx^2 when $x = 0$ and $y = 0$.

34. Show that the equation

$$f''(x) + 2f'(x) + f(x) = 0$$

is satisfied if $f(x) = (x+1)e^{-x}$.

35. Find the rate of change of $f'(x)$ if $f(x) = (5x-3)^4$.

36. Find the rate of change of $f''(x)$ if

$$f(x) = 6\sqrt{x} + \frac{1}{6\sqrt{x}}$$

37. Marginal Cost If $c = 0.2q^2 + 2q + 500$ is a cost function, how fast is marginal cost changing when $q = 97.357$?

38. Marginal Revenue If $p = 400 - 40q - q^2$ is a demand equation, how fast is marginal revenue changing when $q = 4$?

39. If $f(x) = \frac{1}{12}x^4 - \frac{1}{2}x^3 + x^2 - 4$, determine the values of x for which $f''(x) = 0$.

40. Suppose that $e^y = y^2e^x$. **(a)** Determine dy/dx, and express your answer in terms of y only. **(b)** Determine d^2y/dx^2, and express your answer in terms of y only.

In Problems 41 and 42, determine $f''(x)$. Then use your graphing calculator to find all real roots of $f''(x) = 0$. Round your answers to two decimal places.

41. $f(x) = 6e^x - x^3 - 15x^2$

42. $f(x) = \dfrac{x^5}{20} + \dfrac{x^4}{12} + \dfrac{5x^3}{6} + \dfrac{x^2}{2}$

Chapter 12 Review

Important Terms and Symbols — Examples

Section 12.1 **Derivatives of Logarithmic Functions**
derivative of $\ln x$ and of $\log_b u$ — Ex. 5, p. 536

Section 12.2 **Derivatives of Exponential Functions**
derivative of e^x and of b^u — Ex. 4, p. 540

Section 12.3 **Elasticity of Demand**
point elasticity of demand, η elastic unit elasticity inelastic — Ex. 2, p. 546

Section 12.4 **Implicit Differentiation**
implicit differentiation — Ex. 1, p. 551

Section 12.5 **Logarithmic Differentiation**
logarithmic differentiation relative rate of change of $f(x)$ — Ex. 3, p. 556

Section 12.6 **Newton's Method**
recursion formula, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = \dfrac{x_nf'(x_n) - f(x_n)}{f'(x_n)}$ — Ex. 1, p. 560

Section 12.7 **Higher-Order Derivatives**
higher-order derivatives, $f''(x)$, $\dfrac{d^3y}{dx^3}$, $\dfrac{d^4}{dx^4}[f(x)], \ldots$ — Ex. 1, p. 563

Summary

The derivative formulas for natural logarithmic and exponential functions are

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

and

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

To differentiate logarithmic and exponential functions in bases other than e, first transform the function to base e and then differentiate the result. Alternatively, differentiation formulas can be applied:

$$\frac{d}{dx}(\log_b u) = \frac{1}{(\ln b)u} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(b^u) = b^u(\ln b) \cdot \frac{du}{dx}$$

Point elasticity of demand is a function that measures how consumer demand is affected by a change in price. It is given by

$$\eta = \frac{p}{q}\frac{dq}{dp}$$

where p is the price per unit at which q units are demanded. The three categories of elasticity are as follows:

$$\begin{aligned} |\eta(p)| &> 1 \quad \text{demand is elastic} \\ |\eta(p)| &= 1 \quad \text{unit elasticity} \\ |\eta(p)| &< 1 \quad \text{demand is inelastic} \end{aligned}$$

For a given percentage change in price, if there is a greater (respectively, lesser) percentage change in quantity demanded, then demand is elastic (respectively, inelastic).

Two relationships between elasticity and the rate of change of revenue are given by

$$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right) \qquad \frac{dr}{dp} = q(1 + \eta)$$

If an equation implicitly defines y as a function of x (rather than defining it explicitly in the form $y = f(x)$), then dy/dx can be found by implicit differentiation. With this method, we treat y as a differentiable function of x and differentiate both sides of the equation with respect to x. When doing this, remember that

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$

and, more generally, that

$$\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$$

Finally, we solve the resulting equation for dy/dx.

To differentiate $y = f(x)$, where $f(x)$ consists of products, quotients, or powers, the method of logarithmic differentiation may be used. In that method, we take the natural logarithm of both sides of $y = f(x)$ to obtain $\ln y = \ln(f(x))$. After simplifying $\ln(f(x))$ by using properties of logarithms, we differentiate both sides of $\ln y = \ln(f(x))$ with respect to x and then solve for y'. Logarithmic differentiation can also be used to differentiate $y = u^v$, where both u and v are differentiable functions of x.

Newton's method is the name given to the following formula, which is used to approximate the roots of the equation $f(x) = 0$, provided that f is differentiable:

$$x_{n+1} = \frac{x_n f'(x_n) - f(x_n)}{f'(x_n)}, \qquad n = 1, 2, 3, \ldots$$

In many cases encountered, the approximation improves as n increases.

Because the derivative $f'(x)$ of a function $y = f(x)$ is itself a function, it can be successively differentiated to obtain the second derivative $f''(x)$, the third derivative $f'''(x)$, and other higher-order derivatives.

Review Problems

In Problems 1–30, differentiate.

1. $y = 3e^x + e^2 + e^{x^2} + x^{e^2}$

2. $f(w) = we^w + w^2$

3. $f(r) = \ln(7r^2 + 4r + 5)$

4. $y = e^{\ln x}$

5. $y = e^{3x^2+5x+7}$

6. $f(t) = \log_6 \sqrt{t^2 + 1}$

7. $y = e^x(x^2 + 2)$

8. $y = 2^{3x^2}$

9. $y = \sqrt{(x - 6)(x + 5)(9 - x)}$

10. $f(t) = e^{\sqrt{t}}$

11. $y = \dfrac{\ln x}{e^x}$

12. $y = \dfrac{e^x + e^{-x}}{x^2}$

13. $f(q) = \ln[(q + a)^m(q + b)^n]$

14. $y = (x + 2)^3(x + 1)^4(x - 2)^2$

15. $y = 3^{5x^2+3x+2}$

16. $y = (e + e^2)^0$

17. $y = \dfrac{4e^{3x}}{xe^{x-1}}$

18. $y = \dfrac{\ln x}{e^x}$

19. $y = \log_2(8x + 5)^2$

20. $y = \ln\left(\dfrac{3x - 7}{x^2 + 5x - 2}\right)$

21. $f(l) = \ln(1 + l + l^2 + l^3)$

22. $y = (x^2)^{x^2}$

23. $y = (x^2 + 1)^{x+1}$

24. $y = \dfrac{1 + e^x}{1 - e^x}$

25. $\phi(t) = \ln(t^2\sqrt{5 - t^3})$

26. $y = (x + 3)^{\ln x}$

27. $y = \dfrac{(x^2 + 1)^{1/2}(x^2 + 2)^{1/3}}{(2x^3 + 6x)^{2/5}}$

28. $y = (\ln x)\sqrt{x}$

29. $y = (x^x)^x$

30. $y = x^{(x^x)}$

In Problems 31–34, evaluate y' at the given value of x.

31. $y = (x + 1)\ln x^2, x = 1$

32. $y = \dfrac{e^{x^2+1}}{\sqrt{x^2 + 1}}, x = 1$

33. $y = (1/x)^x, x = e$

34. $y = \left[\dfrac{2^{5x}(x^2 - 3x + 5)^{1/3}}{(x^2 - 3x + 7)^3}\right]^{-1}, x = 0$

In Problems 35 and 36, find an equation of the tangent line to the curve at the point corresponding to the given value of x.

35. $y = 2e^x, x = \ln 2$

36. $y = x + x^2 \ln x, x = 1$

37. Find the y-intercept of the tangent line to the graph of $y = x(2^{2-x^2})$ at the point where $x = 1$.

38. If $w = 2^x + \ln(1 + x^2)$ and $x = \ln(1 + t^2)$, find w and dw/dt when $t = 0$.

In Problems 39–42, find the indicated derivative at the given point.

39. $y = e^{x^2-2x+1}, y'', (1, 1)$

40. $y = x^3e^x, y''', (1, e)$

41. $y = \ln(2x), y''', (1, \ln 2)$

42. $y = x\ln x, y'', (1, 0)$

In Problems 43–46, find dy/dx.

43. $x^2 + 2xy + y^2 = 4$

44. $x^3y^3 = 3$

45. $\ln(xy) = xy$

46. $y^2e^{y\ln x} = e^2$

In Problems 47 and 48, find d^2y/dx^2 at the given point.

47. $x + xy + y = 5, (2, 1)$

48. $x^2 + xy + y^2 = 1, (0, -1)$

49. If y is defined implicitly by $e^y = (y + 1)e^x$, determine both dy/dx and d^2y/dx^2 as explicit functions of y only.

50. If $e^x + e^y = 1$, find $\dfrac{d^2y}{dx^2}$.

51. Schizophrenia Several models have been used to analyze the length of stay in a hospital. For a particular group of schizophrenics, one such model is[9]

$$f(t) = 1 - (0.8e^{-0.01t} + 0.2e^{-0.0002t})$$

where $f(t)$ is the proportion of the group that was discharged at the end of t days of hospitalization. Determine the discharge rate (proportion discharged per day) at the end of t days.

52. Earthquakes According to Richter,[10] the number N of earthquakes of magnitude M or greater per unit of time is given by $\log N = A - bM$, where A and b are constants. Richter claims that

$$\log\left(-\frac{dN}{dM}\right) = A + \log\left(\frac{b}{q}\right) - bM$$

where $q = \log e$. Verify this statement.

53. If $f(x) = e^{x^4-10x^3+36x^2-2x}$, find all real roots of $f'(x) = 0$. Round your answers to two decimal places.

54. If $f(x) = \dfrac{x^5}{10} + \dfrac{x^4}{6} + \dfrac{2x^3}{3} + x^2 + 1$, find all roots of $f''(x) = 0$. Round your answers to two decimal places.

For the demand equations in Problems 55–57, determine whether demand is elastic, is inelastic, or has unit elasticity for the indicated value of q.

55. $p = \dfrac{100}{q}; \quad q = 100$

56. $p = 900 - q^2; \quad q = 10$

57. $p = 18 - 0.02q; \quad q = 600$

58. The demand equation for a product is

$$q = \left(\frac{20 - p}{2}\right)^2 \quad \text{for } 0 \le p \le 20$$

(a) Find the point elasticity of demand when $p = 8$.
(b) Find all values of p for which demand is elastic.

59. The demand equation of a product is

$$q = \sqrt{2500 - p^2}$$

Find the point elasticity of demand when $p = 30$. If the price of 30 decreases $\frac{2}{3}\%$, what is the approximate change in demand?

60. The demand equation for a product is

$$q = \sqrt{144 - p}, \quad \text{where } 0 < p < 144$$

(a) Find all prices that correspond to elastic demand.
(b) Compute the point elasticity of demand when $p = 100$. Use the answer to estimate the percentage increase or decrease in demand when price is increased by 5% to $p = 105$.

61. The equation $x^3 - 2x - 2 = 0$ has a root between 1 and 2. Use Newton's method to estimate the root. Continue the approximation procedure until the difference of two successive approximations is less than 0.0001. Round your answer to four decimal places.

62. Find, to an accuracy of three decimal places, all real solutions of the equation $e^x = 3x$.

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[9] Adapted from W. W. Eaton and G. A. Whitmore, "Length of Stay as a Stochastic Process: A General Approach and Application to Hospitalization for Schizophrenia," *Journal of Mathematical Sociology,* 5 (1977) 273–92.

[10] C. F. Richter, *Elementary Seismology* (San Francisco: W. H. Freeman and Company, Publishers, 1958).