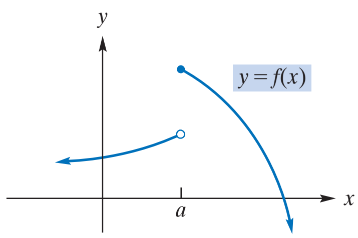

FIGURE 11.9 f is not continuous at a, so f is not differentiable at a.

As stated in Section 10.3, this condition means that f is continuous at a. The foregoing, then, proves that f is continuous at a when f is differentiable there. More simply, we say that **differentiability at a point implies continuity at that point**.

If a function is not continuous at a point, then it cannot have a derivative there. For example, the function in Figure 11.9 is discontinuous at a. The curve has no tangent at that point, so the function is not differentiable there.

EXAMPLE 7 Continuity and Differentiability

a. Let $f(x) = x^2$. The derivative, $2x$, is defined for all values of x, so $f(x) = x^2$ must be continuous for all values of x.

b. The function $f(p) = \dfrac{1}{2p}$ is not continuous at $p = 0$ because f is not defined there. Thus, the derivative does not exist at $p = 0$.

◁

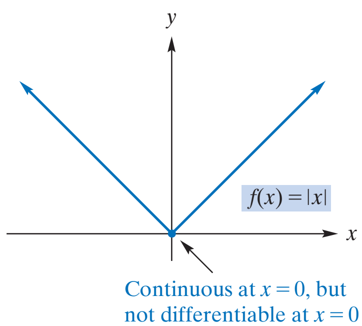

FIGURE 11.10 Continuity does not imply differentiability.

The converse of the statement that differentiability implies continuity is *false*. That is, continuity does not imply differentiability. In Example 8, we give a function that is continuous at a point, but not differentiable there.

EXAMPLE 8 Continuity Does Not Imply Differentiability

The function $y = f(x) = |x|$ is continuous at $x = 0$. (See Figure 11.10.) As we mentioned earlier, there is no tangent line at $x = 0$. Thus, the derivative does not exist there. This shows that continuity does *not* imply differentiability.

◁

Finally, we remark that while differentiability of f at a implies continuity of f at a, the derivative function, f', is not necessarily continuous at a. Unfortunately, the classic example is constructed from a function not considered in this book.

PROBLEMS 11.1

In Problems 1 and 2, a function f and a point P on its graph are given.

(a) *Find the slope of the secant line PQ for each point $Q = (x, f(x))$ whose x-value is given in the table. Round your answers to four decimal places.*

(b) *Use your results from part (a) to estimate the slope of the tangent line at P.*

1. $f(x) = x^3 + 3, P = (-2, -5)$

x-value of Q	-3	-2.5	-2.2	-2.1	-2.01	-2.001
m_{PQ}						

2. $f(x) = \ln x, P = (1, 0)$

x-value of Q	2	1.5	1.2	1.1	1.01	1.001
m_{PQ}						

In Problems 3–18, use the definition of the derivative to find each of the following.

3. $f'(x)$ if $f(x) = x$

4. $f'(x)$ if $f(x) = 4x - 1$

5. $\dfrac{dy}{dx}$ if $y = 3x + 5$

6. $\dfrac{dy}{dx}$ if $y = -5x$

7. $\dfrac{d}{dx}(5 - 7x)$

8. $\dfrac{d}{dx}\left(1 - \dfrac{x}{2}\right)$

9. $f'(x)$ if $f(x) = 3$

10. $f'(x)$ if $f(x) = 7.01$

11. $\dfrac{d}{dx}(x^2 + 4x - 8)$

12. y' if $y = x^2 + 5x + 7$

13. $\dfrac{dp}{dq}$ if $p = 3q^2 + 2q + 1$

14. $\dfrac{d}{dx}(x^2 - x - 3)$

15. y' if $y = \dfrac{6}{x}$

16. $\dfrac{dC}{dq}$ if $C = 7 + 2q - 3q^2$

17. $f'(x)$ if $f(x) = \sqrt{5x}$

18. $H'(x)$ if $H(x) = \dfrac{3}{x - 2}$

19. Find the slope of the curve $y = x^2 + 4$ at the point $(-2, 8)$.

20. Find the slope of the curve $y = 1 - x^2$ at the point $(1, 0)$.

21. Find the slope of the curve $y = 4x^2 - 5$ when $x = 0$.

22. Find the slope of the curve $y = \sqrt{5x}$ when $x = 20$.

In Problems 23–28, find an equation of the tangent line to the curve at the given point.

23. $y = x + 4$; $(3, 7)$

24. $y = 3x^2 - 4$; $(1, -1)$

25. $y = x^2 + 2x + 3$; $(1, 6)$

26. $y = (x - 7)^2$; $(6, 1)$

27. $y = \dfrac{5}{x+3}$; $(2, 1)$

28. $y = \dfrac{5}{1-3x}$; $(2, -1)$

29. Banking Equations may involve derivatives of functions. In an article on interest rate deregulation, Christofi and Agapos[1] solve the equation

$$r = \left(\frac{\eta}{1+\eta}\right)\left(r_L - \frac{dC}{dD}\right)$$

for η (the Greek letter "eta"). Here r is the deposit rate paid by commercial banks, r_L is the rate earned by commercial banks, C is the administrative cost of transforming deposits into return-earning assets, D is the savings deposits level, and η is the deposit elasticity with respect to the deposit rate. Find η.

In Problems 30 and 31, use the numerical derivative feature of your graphing calculator to estimate the derivatives of the functions at the indicated values. Round your answers to three decimal places.

30. $f(x) = \sqrt{2x^2 + 3x}$; $x = 1, x = 2$

31. $f(x) = e^x(4x - 7)$; $x = 0, x = 1.5$

In Problems 32 and 33, use the "limit of a difference quotient" definition to estimate $f'(x)$ at the indicated values of x. Round your answers to three decimal places.

32. $f(x) = x \ln x - x$; $x = 1, x = e$

33. $f(x) = \dfrac{x^2 + 4x + 2}{x^3 - 3}$; $x = 2, x = -4$

34. Find an equation of the tangent line to the curve $f(x) = x^2 + x$ at the point $(-2, 2)$. Graph both the curve and the tangent line. Notice that the tangent line is a good approximation to the curve near the point of tangency.

35. The derivative of $f(x) = x^3 - x + 2$ is $f'(x) = 3x^2 - 1$. Graph both the function f and its derivative f'. Observe that there are two points on the graph of f where the tangent line is horizontal. For the x-values of these points, what are the corresponding values of $f'(x)$? Why are these results expected? Observe the intervals where $f'(x)$ is positive. Notice that tangent lines to the graph of f have positive slopes over these intervals. Observe the interval where $f'(x)$ is negative. Notice that tangent lines to the graph of f have negative slopes over this interval.

In Problems 36 and 37, verify the identity $(z - x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) = z^n - x^n$ for the indicated values of n and calculate the derivative using the $z \to x$ form of the definition of the derivative in Equation (2).

36. $n = 4, n = 3, n = 2$; $f'(x)$ if $f(x) = 2x^4 + x^3 - 3x^2$

37. $n = 5, n = 3$; $f'(x)$ if $f(x) = 2x^5 - 5x^3$

Objective

To develop the basic rules for differentiating constant functions and power functions and the combining rules for differentiating a constant multiple of a function and a sum of two functions.

11.2 Rules for Differentiation

Differentiating a function by direct use of the definition of derivative can be tedious. However, if a function is constructed from simpler functions, then the derivative of the more complicated function can be constructed from the derivatives of the simpler functions. Usually, we need to know only the derivatives of a few basic functions and ways to assemble derivatives of constructed functions from the derivatives of their components. For example, if functions f and g have derivatives f' and g', respectively, then $f + g$ has a derivative given by $(f + g)' = f' + g'$. However, some *rules* are less intuitive. For example, if $f \cdot g$ denotes the function whose value at x is given by $(f \cdot g)(x) = f(x) \cdot g(x)$, then $(f \cdot g)' = f' \cdot g + f \cdot g'$. In this chapter we study most such combining rules and some basic rules for calculating derivatives of certain basic functions.

We begin by showing that the derivative of a constant function is zero. Recall that the graph of the constant function $f(x) = c$ is a horizontal line (see Figure 11.11), which has a slope of zero at each point. This means that $f'(x) = 0$ regardless of x. As a formal proof of this result, we apply the definition of the derivative to $f(x) = c$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

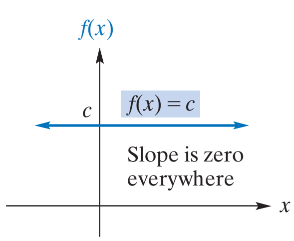

FIGURE 11.11 The slope of a constant function is 0.

[1] A. Christofi and A. Agapos, "Interest Rate Deregulation: An Empirical Justification," *Review of Business and Economic Research,* XX, no. 1 (1984), 39–49.

Rewriting y as a difference of two functions, we have

$$y = \frac{3x^2}{x} - \frac{2}{x} = 3x - 2x^{-1}$$

Thus,

$$\frac{dy}{dx} = 3(1) - 2((-1)x^{-2}) = 3 + \frac{2}{x^2}$$

The slope of the tangent line to the curve when $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 + \frac{2}{1^2} = 5$$

To find the y-coordinate of the point on the curve where $x = 1$, we evaluate $y = \frac{3x^2 - 2}{x}$ at $x = 1$. This gives

To obtain the y-value of the point on the curve when $x = 1$, evaluate the *original* function at $x = 1$.

$$y = \frac{3(1)^2 - 2}{1} = 1$$

Hence, the point $(1, 1)$ lies on both the curve and the tangent line. Therefore, an equation of the tangent line is

$$y - 1 = 5(x - 1)$$

In slope-intercept form, we have

$$y = 5x - 4$$

Now Work Problem 81 ◁

PROBLEMS 11.2

In Problems 1–74, differentiate the functions.

1. $f(x) = \pi$

2. $f(x) = \left(\frac{6}{7}\right)^{2/3}$

3. $y = x^{17}$

4. $f(x) = x^{21}$

5. $y = x^{80}$

6. $y = x^{2.1}$

7. $f(x) = 9x^2$

8. $y = 7x^6$

9. $g(w) = 8w^7$

10. $v(x) = x^e$

11. $y = \frac{3}{5}x^6$

12. $f(p) = \sqrt{3}p^4$

13. $f(s) = \frac{s^5}{30}$

14. $y = \frac{x^7}{7}$

15. $f(x) = x + 3$

16. $f(x) = 5x - e$

17. $f(x) = 4x^2 - 2x + 3$

18. $G(x) = 7x^3 - 5x^2$

19. $g(p) = p^4 - 3p^3 - 1$

20. $f(t) = -13t^2 + 14t + 1$

21. $y = x^4 - \sqrt[3]{x}$

22. $y = -8x^4 + \ln 2$

23. $y = 11x^5 + 12x^3 - 5x$

24. $V(r) = r^8 - 7r^6 + 3r^2 + 1$

25. $f(x) = 2(13 - x^4)$

26. $\psi(t) = e(t^7 - 5^3)$

27. $g(x) = \frac{13 - x^4}{3}$

28. $f(x) = \frac{3(x^3 - 2x)}{4}$

29. $h(x) = 4x^4 + x^3 - \frac{9x^2}{2} + 8x$

30. $k(x) = -2x^2 + \frac{5}{3}x + 11$

31. $f(x) = \frac{5}{7}x^9 + \frac{3}{5}x^7$

32. $p(x) = \frac{x^7}{7} + \frac{2x}{3}$

33. $f(x) = x^{2/7}$

34. $f(x) = 2x^{-14/5}$

35. $y = x^{3/4} + 2x^{5/3}$

36. $y = 4x^2 - x^{-3/5}$

37. $y = 11\sqrt{x}$

38. $y = \sqrt[5]{x^3}$

39. $f(r) = 6\sqrt[3]{r}$

40. $y = 4\sqrt[8]{x^2}$

41. $f(x) = x^{-6}$

42. $f(s) = 2s^{-3}$

43. $f(x) = x^{-6} + x^{-4} + x^{-2}$

44. $f(x) = 100x^{-3} + 10x^{1/2}$

45. $y = \frac{1}{x}$

46. $f(x) = \frac{3}{x^4}$

47. $y = \frac{8}{x^5}$

48. $y = \frac{1}{2x^3}$

49. $g(x) = \frac{4}{3x^3}$

50. $y = \frac{1}{x^2}$

51. $f(t) = \frac{3}{5t^3}$

52. $g(x) = \frac{7}{9x}$

53. $f(x) = \frac{x^2}{2} + \frac{2}{x^2}$

54. $\Phi(x) = \frac{x^3}{3} - \frac{3}{x^3}$

55. $f(x) = -9x^{1/3} + 5x^{-2/5}$

56. $f(z) = 5z^{3/4} - 6^2 - 8z^{1/4}$

57. $q(x) = \frac{1}{\sqrt[3]{8x^2}}$

58. $f(x) = \dfrac{5}{\sqrt[6]{x^5}}$ **59.** $y = \dfrac{2}{\sqrt{x}}$ **60.** $y = \dfrac{1}{2\sqrt{x}}$

61. $y = x^3\sqrt[3]{x}$ **62.** $f(x) = (2x^3)(4x^2)$

63. $f(x) = 3x^2(2x^3 - 3x)$ **64.** $f(x) = x^3(3x^6 - 5x^2 + 4)$

65. $f(x) = x^3(3x)^2$ **66.** $s(x) = \sqrt{x}(\sqrt[5]{x} + 7x + 2)$

67. $v(x) = x^{-2/3}(x + 5)$ **68.** $f(x) = x^{2/7}(x^3 + 5x + 2)$

69. $f(q) = \dfrac{3q^2 + 4q - 2}{q}$ **70.** $f(w) = \dfrac{w - 5}{w^5}$

71. $f(x) = (x - 1)(x + 2)$ **72.** $f(x) = x^2(x - 2)(x + 4)$

73. $w(x) = \dfrac{x + x^2}{x}$ **74.** $f(x) = \dfrac{7x^3 + x}{6\sqrt{x}}$

For each curve in Problems 75–78, find the slopes at the indicated points.

75. $y = 3x^2 + 4x - 8$; $(0, -8)$, $(2, 12)$, $(-3, 7)$

76. $y = 3 + 5x - 3x^3$; $(0, 3)$, $(\frac{1}{2}, \frac{41}{8})$, $(2, -11)$

77. $y = 4$; when $x = -4, x = 7, x = 22$

78. $y = 2x - 2\sqrt{x}$; when $x = 9, x = 16, x = 25$

In Problems 79–82, find an equation of the tangent line to the curve at the indicated point.

79. $y = 4x^2 + 5x + 6$; $(1, 15)$ **80.** $y = \dfrac{1 - x^2}{5}$; $(4, -3)$

81. $y = \dfrac{1}{x^2}$; $(2, \frac{1}{4})$ **82.** $y = -\sqrt[3]{x}$; $(8, -2)$

83. Find an equation of the tangent line to the curve

$$y = 2 + 3x - 5x^2 + 7x^3$$

when $x = 1$.

84. Repeat Problem 83 for the curve

$$y = \frac{\sqrt{x}(2 - x^2)}{x}$$

when $x = 4$.

85. Find all points on the curve

$$y = \frac{5}{2}x^2 - x^3$$

where the tangent line is horizontal.

86. Repeat Problem 85 for the curve

$$y = \frac{x^6}{6} - \frac{x^2}{2} + 1$$

87. Find all points on the curve

$$y = x^2 - 5x + 3$$

where the slope is 1.

88. Repeat Problem 87 for the curve

$$y = x^5 - 4x + 13$$

89. If $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, evaluate the expression

$$\frac{x - 1}{2x\sqrt{x}} - f'(x)$$

90. Economics Eswaran and Kotwal[2] consider agrarian economies in which there are two types of workers, permanent and casual. Permanent workers are employed on long-term contracts and may receive benefits, such as holiday gifts and emergency aid. Casual workers are hired on a daily basis and perform routine and menial tasks, such as weeding, harvesting, and threshing. The difference z in the present-value cost of hiring a permanent worker over that of hiring a casual worker is given by

$$z = (1 + b)w_p - bw_c$$

where w_p and w_c are wage rates for permanent labor and casual labor, respectively, b is a constant, and w_p is a function of w_c. Eswaran and Kotwal claim that

$$\frac{dz}{dw_c} = (1 + b)\left[\frac{dw_p}{dw_c} - \frac{b}{1 + b}\right]$$

Verify this.

91. Find an equation of the tangent line to the graph of $y = x^3 - 2x + 1$ at the point $(1, 0)$. Graph both the function and the tangent line on the same screen.

92. Find an equation of the tangent line to the graph of $y = \sqrt[3]{x}$, at the point $(-8, -2)$. Graph both the function and the tangent line on the same screen. Notice that the line passes through $(-8, -2)$ and the line appears to be tangent to the curve.

Objective

To motivate the instantaneous rate of change of a function by means of velocity and to interpret the derivative as an instantaneous rate of change. To develop the "marginal" concept, which is frequently used in business and economics.

11.3 The Derivative as a Rate of Change

We have given a geometric interpretation of the derivative as being the slope of the tangent line to a curve at a point. Historically, an important application of the derivative involves the motion of an object traveling in a straight line. This gives us a convenient way to interpret the derivative as a *rate of change*.

To denote the change in a variable, such as x, the symbol Δx (read "delta x") is commonly used. For example, if x changes from 1 to 3, then the change in x is $\Delta x = 3 - 1 = 2$. The new value of $x(= 3)$ is the old value plus the change, which

[2]M. Eswaran and A. Kotwal, "A Theory of Two-Tier Labor Markets in Agrarian Economies," *The American Economic Review*, 75, no. 1 (1985), 162–77.

In general, for any function f, we have the following definition:

Definition

The **relative rate of change** of $f(x)$ is

$$\frac{f'(x)}{f(x)}$$

The **percentage rate of change** of $f(x)$ is

$$\frac{f'(x)}{f(x)} \cdot 100\%$$

APPLY IT ▶

5. The volume V enclosed by a capsule-shaped container with a cylindrical height of 4 feet and radius r is given by

$$V(r) = \frac{4}{3}\pi r^3 + 4\pi r^2$$

Determine the relative and percentage rates of change of volume with respect to the radius when the radius is 2 feet.

EXAMPLE 9 Relative and Percentage Rates of Change

Determine the relative and percentage rates of change of

$$y = f(x) = 3x^2 - 5x + 25$$

when $x = 5$.

Solution: Here,

$$f'(x) = 6x - 5$$

Since $f'(5) = 6(5) - 5 = 25$ and $f(5) = 3(5)^2 - 5(5) + 25 = 75$, the relative rate of change of y when $x = 5$ is

$$\frac{f'(5)}{f(5)} = \frac{25}{75} \approx 0.333$$

Multiplying 0.333 by 100% gives the percentage rate of change: $(0.333)(100) = 33.3\%$.

Now Work Problem 35 ◁

PROBLEMS 11.3

1. Suppose that the position function of an object moving along a straight line is $s = f(t) = 2t^2 + 3t$, where t is in seconds and s is in meters. Find the average velocity $\Delta s/\Delta t$ over the interval $[1, 1 + \Delta t]$, where Δt is given in the following table:

Δt	1	0.5	0.2	0.1	0.01	0.001
$\Delta s/\Delta t$						

From your results, estimate the velocity when $t = 1$. Verify your estimate by using differentiation.

2. If $y = f(x) = \sqrt{2x + 5}$, find the average rate of change of y with respect to x over the interval $[3, 3 + \Delta x]$, where Δx is given in the following table:

Δx	1	0.5	0.2	0.1	0.01	0.001
$\Delta y/\Delta x$						

From your result, estimate the rate of change of y with respect to x when $x = 3$.

In each of Problems 3–8, a position function is given, where t is in seconds and s is in meters.

(a) *Find the position at the given t-value.*

(b) *Find the average velocity over the given interval.*

(c) *Find the velocity at the given t-value.*

3. $s = 2t^2 - 4t$; $[7, 7.5]$; $t = 7$

4. $s = \frac{2}{3}t + 4$; $[3, 3.03]$; $t = 3$

5. $s = 5t^3 + 3t + 24$; $[1, 1.01]$; $t = 1$

6. $s = -3t^2 + 2t + 1$; $[1, 1.25]$; $t = 1$

7. $s = t^4 - 2t^3 + t$; $[2, 2.1]$; $t = 2$

8. $s = 3t^4 - t^{7/2}$; $[0, \frac{1}{4}]$; $t = 0$

9. Income–Education Sociologists studied the relation between income and number of years of education for members of a particular urban group. They found that a person with x years of education before seeking regular employment can expect to receive an average yearly income of y dollars per year, where

$$y = 6x^{9/4} + 5900 \quad \text{for } 4 \le x \le 16$$

Find the rate of change of income with respect to number of years of education. Evaluate the function at $x = 16$.

10. Find the rate of change of the volume V of a ball, with respect to its radius r, when $r = 1.5$ m. The volume V of a ball as a function of its radius r is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

11. Skin Temperature The approximate temperature T of the skin in terms of the temperature T_e of the environment is given by

$$T = 32.8 + 0.27(T_e - 20)$$

where T and T_e are in degrees Celsius.[3] Find the rate of change of T with respect to T_e.

12. Biology The volume V of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Find the rate of change of volume with respect to the radius when $r = 6.3 \times 10^{-4}$ cm.

In Problems 13–18, cost functions are given, where c is the cost of producing q units of a product. In each case, find the marginal-cost function. What is the marginal cost at the given value(s) of q?

13. $c = 500 + 10q; q = 100$

14. $c = 7500 + 5q; q = 24$

15. $c = 0.2q^2 + 4q + 50; q = 10$

16. $c = 0.1q^2 + 3q + 2; q = 3$

17. $c = q^2 + 50q + 1000; q = 15, q = 16, q = 17$

18. $c = 0.04q^3 - 0.5q^2 + 4.4q + 7500; q = 5, q = 25, q = 1000$

In Problems 19–22, $\bar{c}$ represents average cost per unit, which is a function of the number, q, of units produced. Find the marginal-cost function and the marginal cost for the indicated values of q.

19. $\bar{c} = 0.02q + 3 + \dfrac{600}{q}; q = 40, q = 80$

20. $\bar{c} = 5 + \dfrac{2000}{q}; q = 25, q = 250$

21. $\bar{c} = 0.00002q^2 - 0.01q + 6 + \dfrac{20{,}000}{q}; q = 100, q = 500$

22. $\bar{c} = 0.002q^2 - 0.5q + 60 + \dfrac{7000}{q}; q = 15, q = 25$

In Problems 23–26, r represents total revenue and is a function of the number, q, of units sold. Find the marginal-revenue function and the marginal revenue for the indicated i values of q.

23. $r = 0.8q; q = 9, q = 300, q = 500$

24. $r = q(25 - \frac{1}{20}q); q = 10, q = 20, q = 100$

25. $r = 240q + 40q^2 - 2q^3; q = 10; q = 15; q = 20$

26. $r = 2q(30 - 0.1q); q = 10, q = 20$

27. Hosiery Mill The total-cost function for a hosiery mill is estimated by Dean[4] to be

$$c = -10{,}484.69 + 6.750q - 0.000328q^2$$

where q is output in dozens of pairs and c is total cost in dollars. Find the marginal-cost function and the average cost function and evaluate each when $q = 2000$.

[3] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).

[4] J. Dean, "Statistical Cost Functions of a Hosiery Mill," *Studies in Business Administration*, XI, no. 4 (Chicago: University of Chicago Press, 1941).

28. Light and Power Plant The total-cost function for an electric light and power plant is estimated by Nordin[5] to be

$$c = 32.07 - 0.79q + 0.02142q^2 - 0.0001q^3 \quad 20 \le q \le 90$$

where q is the eight-hour total output (as a percentage of capacity) and c is the total fuel cost in dollars. Find the marginal-cost function and evaluate it when $q = 70$.

29. Urban Concentration Suppose the 100 largest cities in the United States in 1920 are ranked according to area. From Lotka,[6] the following relation holds, approximately:

$$PR^{0.93} = 5{,}000{,}000$$

where P is the population of the city having rank R. This relation is called the *law of urban concentration* for 1920. Determine P as a function of R and find how fast the population is changing with respect to rank.

30. Depreciation Under the straight-line method of depreciation, the value, v, of a certain machine after t years have elapsed is given by

$$v = 120{,}000 - 15{,}500t$$

where $0 \le t \le 6$. How fast is v changing with respect to t when $t = 2$? $t = 4$? at any time?

31. Winter Moth A study of the winter moth was made in Nova Scotia (adapted from Embree).[7] The prepupae of the moth fall onto the ground from host trees. At a distance of x ft from the base of a host tree, the prepupal density (number of prepupae per square foot of soil) was y, where

$$y = 59.3 - 1.5x - 0.5x^2 \quad 1 \le x \le 9$$

(a) At what rate is the prepupal density changing with respect to distance from the base of the tree when $x = 6$?
(b) For what value of x is the prepupal density decreasing at the rate of 6 prepupae per square foot per foot?

32. Cost Function For the cost function

$$c = 0.4q^2 + 4q + 5$$

find the rate of change of c with respect to q when $q = 2$. Also, what is $\Delta c/\Delta q$ over the interval [2, 3]?

In Problems 33–38, find **(a)** *the rate of change of y with respect to x and* **(b)** *the relative rate of change of y. At the given value of x, find* **(c)** *the rate of change of y,* **(d)** *the relative rate of change of y, and* **(e)** *the percentage rate of change of y.*

33. $y = f(x) = x + 4; x = 5$

34. $y = f(x) = 9 - 5x; x = 3$

35. $y = 2x^2 + 5; x = 10$

36. $y = 5 - 3x^3; x = 1$

37. $y = 8 - x^3; x = 1$

38. $y = x^2 + 3x - 4; x = -1$

39. Cost Function For the cost function

$$c = 0.4q^2 + 3.2q + 11$$

how fast does c change with respect to q when $q = 20$? Determine the percentage rate of change of c with respect to q when $q = 20$.

[5] J. A. Nordin, "Note on a Light Plant's Cost Curves," *Econometrica,* 15 (1947), 231–35.

[6] A. J. Lotka, *Elements of Mathematical Biology* (New York: Dover Publications, Inc., 1956).

[7] D. G. Embree, "The Population Dynamics of the Winter Moth in Nova Scotia, 1954–1962," *Memoirs of the Entomological Society of Canada,* no. 46 (1965).

40. Organic Matters/Species Diversity In a discussion of contemporary waters of shallows seas, Odum[8] claims that in such waters the total organic matter, y (in milligrams per liter), is a function of species diversity, x (in number of species per thousand individuals). If $y = 100/x$, at what rate is the total organic matter changing with respect to species diversity when $x = 10$? What is the percentage rate of change when $x = 10$?

41. Revenue For a certain manufacturer, the revenue obtained from the sale of q units of a product is given by

$$r = 30q - 0.3q^2$$

(a) How fast does r change with respect to q? When $q = 10$, **(b)** find the relative rate of change of r, and **(c)** to the nearest percent, find the percentage rate of change of r.

42. Revenue Repeat Problem 41 for the revenue function given by $r = 10q - 0.2q^2$ and $q = 25$.

43. Weight of Limb The weight of a limb of a tree is given by $W = 2t^{0.432}$, where t is time. Find the relative rate of change of W with respect to t.

44. Response to Shock A psychological experiment[9] was conducted to analyze human responses to electrical shocks (stimuli). The subjects received shocks of various intensities. The response, R, to a shock of intensity, I (in microamperes), was to be a number that indicated the perceived magnitude relative to that of a "standard" shock. The standard shock was assigned a magnitude of 10. Two groups of subjects were tested under slightly different conditions. The responses R_1 and R_2 of the first and second groups to a shock of intensity I were given by

$$R_1 = \frac{I^{1.3}}{1855.24} \quad \text{for } 800 \le I \le 3500$$

and

$$R_2 = \frac{I^{1.3}}{1101.29} \quad \text{for } 800 \le I \le 3500$$

(a) For each group, determine the relative rate of change of response with respect to intensity.
(b) How do these changes compare with each other?
(c) In general, if $f_1(x) = C_1 f(x)$ and $f_2(x) = C_2 f(x)$, where C_1 and C_2 are constants, how do the relative rates of change of f_1 and f_2 compare?

45. Cost A manufacturer of mountain bikes has found that when 20 bikes are produced per day, the average cost is \$200 and the marginal cost is \$150. Based on that information, approximate the total cost of producing 21 bikes per day.

46. Marginal and Average Costs Suppose that the cost function for a certain product is $c = f(q)$. If the relative rate of change of c (with respect to q) is $\frac{1}{q}$, prove that the marginal-cost function and the average-cost function are equal.

In Problems 47 and 48, use the numerical derivative feature of your graphing calculator.

47. If the total-cost function for a manufacturer is given by

$$c = \frac{5q^2}{\sqrt{q^2 + 3}} + 5000$$

where c is in dollars, find the marginal cost when 10 units are produced. Round your answer to the nearest cent.

48. The population of a city t years from now is given by

$$P = 250{,}000e^{0.04t}$$

Find the rate of change of population with respect to time t three years from now. Round your answer to the nearest integer.

Objective

To find derivatives by applying the product and quotient rules, and to develop the concepts of marginal propensity to consume and marginal propensity to save.

11.4 The Product Rule and the Quotient Rule

The equation $F(x) = (x^2 + 3x)(4x + 5)$ expresses $F(x)$ as a product of two functions: $x^2 + 3x$ and $4x + 5$. To find $F'(x)$ by using only our previous rules, we first multiply the functions. Then we differentiate the result, term by term:

$$F(x) = (x^2 + 3x)(4x + 5) = 4x^3 + 17x^2 + 15x$$

$$F'(x) = 12x^2 + 34x + 15 \tag{1}$$

However, in many problems that involve differentiating a product of functions, the multiplication is not as simple as it is here. At times, it is not even practical to attempt it. Fortunately, there is a rule for differentiating a product, and the rule avoids such multiplications. Since the derivative of a sum of functions is the sum of their derivatives, one might expect a similar rule for products. There is a rule; however, the situation for products is more subtle than that for sums.

[8] H. T. Odum, "Biological Circuits and the Marine Systems of Texas," in *Pollution and Marine Biology*, eds T. A. Olsen and F. J. Burgess (New York: Interscience Publishers, 1967).

[9] H. Babkoff, "Magnitude Estimation of Short Electrocutaneous Pulses," *Psychological Research*, 39, no. 1 (1976), 39–49.

Solution:

$$\frac{dC}{dI} = 5\left(\frac{(I+10)\frac{d}{dI}(2I^{3/2}+3) - (2\sqrt{I^3}+3)\frac{d}{dI}(I+10)}{(I+10)^2}\right)$$

$$= 5\left(\frac{(I+10)(3I^{1/2}) - (2\sqrt{I^3}+3)(1)}{(I+10)^2}\right)$$

When $I = 100$, the marginal propensity to consume is

$$\left.\frac{dC}{dI}\right|_{I=100} = 5\left(\frac{1297}{12,100}\right) \approx 0.536$$

The marginal propensity to save when $I = 100$ is $1 - 0.536 = 0.464$. This means that if a current income of \$100 billion increases by \$1 billion, the nation consumes approximately 53.6% (536/1000) and saves 46.4% (464/1000) of that increase.

Now Work Problem 69 ◁

PROBLEMS 11.4

In Problems 1–48, differentiate the functions.

1. $f(x) = (4x+1)(6x+3)$

2. $f(x) = (3x-1)(7x+2)$

3. $s(t) = (5-3t)(t^3-2t^2)$

4. $Q(x) = (x^2+3x)(7x^2-5)$

5. $f(r) = (2r^7-3r^5)(5r^2-2r+7)$

6. $C(I) = (2I^2-3)(3I^2-4I+1)$

7. $f(x) = x^2(2x^2-5)$

8. $f(x) = 3x^3(x^2-2x+2)$

9. $y = (x^2+5x-7)(6x^2-5x+4)$

10. $\phi(x) = (2+3x-5x^2)(7+11x-13x^2)$

11. $f(w) = (w^2+3w-7)(2w^3-4)$

12. $f(x) = (3x-x^2)(3-x-x^2)$

13. $y = (x^2-1)(3x^3-6x+5) - 4(4x^2+2x+1)$

14. $h(x) = 5(x^7+4) + 4(5x^3-2)(4x^2+7x)$

15. $F(p) = \frac{5}{7}(2\sqrt{p}-3)(11p+2)$

16. $g(x) = (\sqrt{x}+5x-2)(\sqrt[3]{x}-3\sqrt{x})$

17. $y = 7 \cdot \frac{2}{3}$

18. $y = (x-1)(x-2)(x-3)$

19. $y = (5x+3)(2x-5)(7x+9)$

20. $y = \dfrac{3x-5}{7x+11}$

21. $f(x) = \dfrac{5x}{x-1}$

22. $H(x) = \dfrac{-5x}{5-x}$

23. $f(x) = \dfrac{-13}{3x^5}$

24. $f(x) = \dfrac{3(5x^2-7)}{4}$

25. $y = \dfrac{ax+b}{cx+d}$

26. $h(w) = \dfrac{3w^2+5w-1}{w-3}$

27. $h(z) = \dfrac{6-2z}{z^2-4}$

28. $z = \dfrac{2x^2+5x-2}{3x^2+5x+3}$

29. $y = \dfrac{4x^2+3x+2}{3x^2-2x+1}$

30. $f(x) = \dfrac{x^3+x^2+1}{x^2+1}$

31. $y = \dfrac{x^2-4x+3}{2x^2-3x+2}$

32. $F(z) = \dfrac{z^4+4}{3z}$

33. $g(x) = \dfrac{1}{x^{100}+7}$

34. $y = \dfrac{-8}{7x^6}$

35. $u(v) = \dfrac{v^3+1}{v}$

36. $y = \dfrac{x-5}{8\sqrt{x}}$

37. $y = \dfrac{3x^2-x-1}{\sqrt[3]{x}}$

38. $y = \dfrac{x^{0.3}-2}{2x^{2.1}+1}$

39. $y = 1 - \dfrac{5}{2x+5} + \dfrac{2x}{3x+1}$

40. $q(x) = x^3 + \dfrac{2x+3}{5x+7} - \dfrac{11}{x^3}$

41. $y = \dfrac{x-5}{(x+2)(x-4)}$

42. $y = \dfrac{(9x-1)(3x+2)}{4-5x}$

43. $s(t) = \dfrac{t^2+3t}{(t^2-1)(t^3+7)}$

44. $f(s) = \dfrac{17}{s(4s^3+5s-23)}$

45. $y = 2x - \dfrac{\frac{3}{x} - \frac{5}{x-1}}{x-2}$

46. $y = 3 - 12x^3 + \dfrac{1 - \frac{5}{x^2+2}}{x^2+5}$

47. $f(x) = \dfrac{a+x}{a-x}$

48. $f(x) = \dfrac{x^{-1}+a^{-1}}{x^{-1}-a^{-1}}$, where a is a constant

49. Find the slope of the curve $y = (2x^2-x+3)(x^3+x+1)$ at $(1, 12)$.

50. Find the slope of the curve $y = \dfrac{1}{x^2+1}$ at $(-1, \frac{1}{2})$.

In Problems 51–54, find an equation of the tangent line to the curve at the given point.

51. $y = \dfrac{6}{x-1}$; (3, 3) **52.** $y = \dfrac{x+5}{x^2}$; (1, 6)

53. $y = (2x+3)[2(x^4 - 5x^2 + 4)]$; (0, 24)

54. $y = \dfrac{x-1}{x(x^2+1)}$; $(2, \frac{1}{10})$

In Problems 55 and 56, determine the relative rate of change of y with respect to x for the given value of x.

55. $y = \dfrac{x}{x+1}$; $x = 1$ **56.** $y = \dfrac{1-x}{1+x}$; $x = 5$

57. Motion The position function for an object moving in a straight line is

$$s = \frac{2}{t^3+1}$$

where t is in seconds and s is in meters. Find the position and velocity of the object at $t = 1$.

58. Motion The position function for an object moving in a straight-line path is

$$s = \frac{t+3}{t^2+7}$$

where t is in seconds and s is in meters. Find the positive value(s) of t for which the velocity of the object is 0.

In Problems 59–62, each equation represents a demand function for a certain product, where p denotes the price per unit for q units. Find the marginal-revenue function in each case. Recall that revenue = pq.

59. $p = 80 - 0.02q$ **60.** $p = 300/q$

61. $p = \dfrac{108}{q+2} - 3$ **62.** $p = \dfrac{q+750}{q+50}$

63. Consumption Function For the United States (1922–1942), the consumption function is estimated by[10]

$$C = 0.672I + 113.1$$

Find the marginal propensity to consume.

64. Consumption Function Repeat Problem 63 for $C = 0.836I + 127.2$.

In Problems 65–68, each equation represents a consumption function. Find the marginal propensity to consume and the marginal propensity to save for the given value of I.

65. $C = 2 + 3\sqrt{I} + 5\sqrt[3]{I}$ for $40 \le I \le 70$; $I = 64$

66. $C = 6 + \dfrac{3I}{4} - \dfrac{\sqrt{I}}{3}$; $I = 25$

67. $C = \dfrac{16\sqrt{I} + 0.8\sqrt{I^3} - 0.2I}{\sqrt{I}+4}$; $I = 36$

68. $C = \dfrac{20\sqrt{I} + 0.5\sqrt{I^3} - 0.4I}{\sqrt{I}+5}$; $I = 100$

69. Consumption Function Suppose that a country's consumption function is given by

$$C = \frac{9\sqrt{I} + 0.8\sqrt{I^3} - 0.3I}{\sqrt{I}}$$

where C and I are expressed in billions of dollars.
(a) Find the marginal propensity to save when income is \$25 billion.
(b) Determine the relative rate of change of C with respect to I when income is \$25 billion.

70. Marginal Propensities to Consume and to Save Suppose that the savings function of a country is

$$S = \frac{I + \sqrt{I} - 6}{\sqrt{I}+3}$$

where the national income (I) and the national savings (S) are measured in billions of dollars. Find the country's marginal propensity to consume and its marginal propensity to save when the national income is \$121 billion. (*Hint:* It is helpful to first factor the numerator of S.)

71. Marginal Cost If the total-cost function for a manufacturer is given by

$$c = \frac{6q^2}{q+2} + 6000$$

find the marginal-cost function.

72. Marginal and Average Costs Given the cost function $c = f(q)$, show that if $\dfrac{d}{dq}(\bar{c}) = 0$, then the marginal-cost function and average-cost function are equal.

73. Host–Parasite Relation For a particular host–parasite relationship, it is determined that when the host density (number of hosts per unit of area) is x, the number of hosts that are parasitized is y, where

$$y = \frac{900x}{10+45x}$$

At what rate is the number of hosts parasitized changing with respect to host density when $x = 2$?

74. Acoustics The persistence of sound in a room after the source of the sound is turned off is called *reverberation*. The *reverberation time* RT of the room is the time it takes for the intensity level of the sound to fall 60 decibels. In the acoustical design of an auditorium, the following formula may be used to compute the RT of the room:[11]

$$\text{RT} = \frac{0.05V}{A + xV}$$

Here, V is the room volume, A is the total room absorption, and x is the air absorption coefficient. Assuming that A and x are positive constants, show that the rate of change of RT with respect to V is always positive. If the total room volume increases by one unit, does the reverberation time increase or decrease?

[10] T. Haavelmo, "Methods of Measuring the Marginal Propensity to Consume," *Journal of the American Statistical Association*, XLII (1947), 105–22.

[11] L. L. Doelle, *Environmental Acoustics* (New York: McGraw-Hill Book Company, 1972).

75. Predator–Prey In a predator-prey experiment,[12] it was statistically determined that the number of prey consumed, y, by an individual predator was a function of the prey density, x (the number of prey per unit of area), where

$$y = \frac{0.7355x}{1 + 0.02744x}$$

Determine the rate of change of prey consumed with respect to prey density.

76. Social Security Benefits In a discussion of social security benefits, Feldstein[13] differentiates a function of the form

$$f(x) = \frac{a(1+x) - b(2+n)x}{a(2+n)(1+x) - b(2+n)x}$$

where a, b, and n are constants. He determines that

$$f'(x) = \frac{-1(1+n)ab}{(a(1+x) - bx)^2(2+n)}$$

Verify this. (*Hint:* For convenience, let $2 + n = c$.) Next, observe that Feldstein's function f is of the form

$$g(x) = \frac{A + Bx}{C + Dx}, \quad \text{where } A, B, C, \text{ and } D \text{ are constants}$$

Show that $g'(x)$ is a constant divided by a nonnegative function of x. What does this mean?

77. Business The manufacturer of a product has found that when 20 units are produced per day, the average cost is \$150 and the marginal cost is \$125. What is the relative rate of change of average cost with respect to quantity when $q = 20$?

78. Use the result $(fgh)' = f'gh + fg'h + fgh'$ to find dy/dx if

$$y = (3x+1)(2x-1)(x-4)$$

Objective

To introduce and apply the chain rule, to derive a special case of the chain rule, and to develop the concept of the marginal-revenue product as an application of the chain rule.

11.5 The Chain Rule

Our next rule, the **chain rule**, is ultimately the most important rule for finding derivatives. It involves a situation in which y is a function of the variable u, but u is a function of x, and we want to find the derivative of y with respect to x. For example, the equations

$$y = u^2 \quad \text{and} \quad u = 2x + 1$$

define y as a function of u and u as a function of x. If we substitute $2x + 1$ for u in the first equation, we can consider y to be a function of x:

$$y = (2x+1)^2$$

To find dy/dx, we first expand $(2x + 1)^2$:

$$y = 4x^2 + 4x + 1$$

Then

$$\frac{dy}{dx} = 8x + 4$$

From this example, we can see that finding dy/dx by first performing a substitution *could* be quite involved. For instance, if originally we had been given $y = u^{100}$ instead of $y = u^2$, we wouldn't even want to try substituting. Fortunately, the chain rule will allow us to handle such situations with ease.

COMBINING RULE 5 The Chain Rule

If y is a differentiable function of u and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

[12] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist,* XCI, no. 7 (1959), 385–98.

[13] M. Feldstein, "The Optimal Level of Social Security Benefits," *The Quarterly Journal of Economics*, C, no. 2 (1985), 303–20.

Hence,

$$\left.\frac{dr}{dq}\right|_{m=9} = \left.\frac{dr}{dq}\right|_{q=81} = \frac{8100}{(81+9)^2} = 1$$

Now we turn to dq/dm. From the quotient and power rules, we have

$$\begin{aligned}\frac{dq}{dm} &= \frac{d}{dm}\left(\frac{10m^2}{\sqrt{m^2+19}}\right)\\ &= \frac{(m^2+19)^{1/2}\frac{d}{dm}(10m^2) - (10m^2)\frac{d}{dm}[(m^2+19)^{1/2}]}{[(m^2+19)^{1/2}]^2}\\ &= \frac{(m^2+19)^{1/2}(20m) - (10m^2)[\frac{1}{2}(m^2+19)^{-1/2}(2m)]}{m^2+19}\end{aligned}$$

so

$$\begin{aligned}\left.\frac{dq}{dm}\right|_{m=9} &= \frac{(81+19)^{1/2}(20\cdot 9) - (10\cdot 81)[\frac{1}{2}(81+19)^{-1/2}(2\cdot 9)]}{81+19}\\ &= 10.71\end{aligned}$$

A direct formula for the marginal-revenue product is

$$\frac{dr}{dm} = \frac{dq}{dm}\left(p + q\frac{dp}{dq}\right)$$

Therefore, from the chain rule,

$$\left.\frac{dr}{dm}\right|_{m=9} = (1)(10.71) = 10.71$$

This means that if a tenth employee is hired, revenue will increase by approximately \$10.71 per day.

Now Work Problem 80 ◁

PROBLEMS 11.5

In Problems 1–8, use the chain rule.

1. If $y = u^3 + 3u^2$ and $u = x^2 + 1$, find dy/dx.

2. If $y = 2u^3 - 8u$ and $u = 7x - x^3$, find dy/dx.

3. If $y = \frac{1}{w}$ and $w = 3x - 5$, find dy/dx.

4. If $y = \sqrt[4]{z}$ and $z = x^5 - x^4 + 3$, find dy/dx.

5. If $w = u^3$ and $u = \frac{t-1}{t+1}$, find dw/dt when $t = 1$.

6. If $z = u^3 + \sqrt{u} + 5$ and $u = 2s^2 + 1$, find dz/ds when $s = 2$.

7. If $y = 3w^2 - 8w + 4$ and $w = 2x^2 + 1$, find dy/dx when $x = 0$.

8. If $y = 2u^3 + 3u^2 + 5u - 1$ and $u = 3x + 1$, find dy/dx when $x = 1$.

In Problems 9–52, find y'.

9. $y = (3x+2)^6$

10. $y = (x^2-4)^4$

11. $y = (2+3x^5)^7$

12. $y = (x^2+x)^4$

13. $y = 5(x^3-3x^2+2x)^{100}$

14. $y = \frac{(2x^2+1)^4}{2}$

15. $y = (x^2-2)^{-3}$

16. $y = (3x^3-2x^2)^{-10}$

17. $y = 2(x^2+5x-2)^{-5/7}$

18. $y = 3(5x-2x^3)^{-5/3}$

19. $y = \sqrt{5x^2-x}$

20. $y = \sqrt{3x^2-7}$

21. $y = \sqrt[3]{5x+7}$

22. $y = \sqrt[3]{8x^2-1}$

23. $y = 4\sqrt[7]{(x^2+1)^3}$

24. $y = 7\sqrt[3]{(x^5-3)^5}$

25. $y = \frac{6}{2x^2-x+1}$

26. $y = \frac{2}{x^3+5}$

27. $y = \frac{1}{(x^2-3x)^2}$

28. $y = \frac{1}{(3+5x)^3}$

29. $y = \frac{4}{\sqrt{9x^2+1}}$

30. $y = \frac{3}{(3x^2-x)^{2/3}}$

31. $y = \sqrt[5]{5x} + \sqrt[5]{5}x$

32. $y = \sqrt{2x} + \frac{1}{\sqrt{2x}}$

33. $y = x^3(2x+3)^7$

34. $y = x(x+4)^4$

35. $y = 4x^2\sqrt{5x+1}$

36. $y = 2x^3\sqrt{1-x^5}$

37. $y = (x^2+2x-1)^3(5x)$

38. $y = x^4(x^4-1)^5$

39. $y = (8x-1)^3(2x+1)^4$

40. $y = (3x+2)^5(4x-5)^2$

41. $y = \left(\dfrac{ax+b}{cx+d}\right)^{11}$

42. $y = \left(\dfrac{2x}{x+2}\right)^{4}$

43. $y = \sqrt{\dfrac{x+1}{x-5}}$

44. $y = \sqrt[3]{\dfrac{8x^2-3}{x^2+2}}$

45. $y = \dfrac{2x-5}{(x^2+4)^3}$

46. $y = \dfrac{(2x+3)^5}{2x^4+8}$

47. $y = \dfrac{(8x-1)^5}{(3x-1)^3}$

48. $y = \sqrt[3]{(x-3)^3(x+5)}$

49. $y = 6(5x^2+2)\sqrt{x^4+5}$

50. $y = 6+3x-4x(7x+1)^2$

51. $y = 2t + \dfrac{t+1}{t+3} - \left(\dfrac{2t+3}{5}\right)^7$

52. $y = \dfrac{(2x^3+6)(7x-5)}{(2x+4)^2}$

In Problems 53 and 54, use the quotient rule and power rule to find y'. Do not simplify your answer.

53. $y = \dfrac{(3x+2)^3(x+1)^4}{(x^2-7)^3}$

54. $y = \dfrac{\sqrt{x+2}(4x^2-1)^2}{9x-3}$

55. If $y = (5u+6)^3$ and $u = (x^2+1)^4$, find dy/dx when $x = 0$.

56. If $z = 3y^3 + 2y^2 + y$, $y = 2x^2 + x$, and $x = t+1$, find dz/dt when $t = 1$.

57. Find the slope of the curve $y = (x^2-7x-8)^3$ at the point $(8, 0)$.

58. Find the slope of the curve $y = \sqrt{x+2}$ at the point $(7, 3)$.

In Problems 59–62, find an equation of the tangent line to the curve at the given point.

59. $y = \sqrt[3]{(x^2-8)^2}$; $(3, 1)$

60. $y = (x+3)^3$; $(-1, 8)$

61. $y = \dfrac{\sqrt{5x+5}}{x+1}$; $(4, 1)$

62. $y = \dfrac{-3}{(3x^2+1)^3}$; $(0, -3)$

In Problems 63 and 64, determine the percentage rate of change of y with respect to x for the given value of x.

63. $y = (x^2+1)^4$; $x = 1$

64. $y = \dfrac{1}{(x^2-1)^3}$; $x = 2$

In Problems 65–68, q is the total number of units produced per day by m employees of a manufacturer, and p is the price per unit at which the q units are sold. In each case, find the marginal-revenue product for the given value of m.

65. $q = 5m$, $p = -0.4q + 50$; $m = 6$

66. $q = (100m - m^2)/10$, $p = -0.1q + 50$; $m = 10$

67. $q = 10m^2/\sqrt{m^2+9}$, $p = 525/(q+3)$; $m = 4$

68. $q = 50m/\sqrt{m^2+11}$, $p = 100/(q+10)$; $m = 5$

69. Demand Equation Suppose $p = 100 - \sqrt{q^2+20}$ is a demand equation for a manufacturer's product.
(a) Find the rate of change of p with respect to q.
(b) Find the relative rate of change of p with respect to q.
(c) Find the marginal-revenue function.

70. Marginal-Revenue Product If $p = k/q$, where k is a constant, is the demand equation for a manufacturer's product and $q = f(m)$ defines a function that gives the total number of units produced per day by m employees, show that the marginal-revenue product is always zero.

71. Cost Function The cost c of producing q units of a product is given by

$$c = 5000 + 10q + 0.1q^2$$

If the price per unit p is given by the equation

$$q = 1000 - 2p$$

use the chain rule to find the rate of change of cost with respect to price per unit when $p = 100$.

72. Hospital Discharges A governmental health agency examined the records of a group of individuals who were hospitalized with a particular illness. It was found that the total proportion that had been discharged at the end of t days of hospitalization was given by

$$f(t) = 1 - \left(\frac{250}{250+t}\right)^3$$

Find $f'(100)$ and interpret your answer.

73. Marginal Cost If the total-cost function for a manufacturer is given by

$$c = \frac{4q^2}{\sqrt{q^2+2}} + 6000$$

find the marginal-cost function.

74. Salary/Education For a certain population, if E is the number of years of a person's education and S represents average annual salary in dollars, then for $E \geq 7$,

$$S = 340E^2 - 4360E + 42{,}800$$

(a) How fast is salary changing with respect to education when $E = 16$?
(b) At what level of education does the rate of change of salary equal \$5000 per year of education?

75. Biology The volume of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. At time t seconds, the radius (in centimeters) is given by

$$r = 10^{-8}t^2 + 10^{-7}t$$

Use the chain rule to find dV/dt when $t = 10$.

76. Pressure in Body Tissue Under certain conditions, the pressure, p, developed in body tissue by ultrasonic beams is given as a function of the beam's intensity, I, via the equation[14]

$$p = (2\rho VI)^{1/2}$$

where ρ (a Greek letter read "rho") is density of the affected tissue and V is the velocity of propagation of the beam. Here ρ and V are constants. **(a)** Find the rate of change of p with respect to I. **(b)** Find the relative rate of change of p with respect to I.

[14] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).

77. Demography Suppose that, for a certain group of 20,000 births, the number of people surviving to age x years is

$$l_x = -0.000354x^4 + 0.00452x^3 + 0.848x^2 - 34.9x + 20{,}000$$
$$0 \le x \le 95.2$$

(a) Find the rate of change of l_x with respect to x, and evaluate your answer for $x = 65$.
(b) Find the relative rate of change and the percentage rate of change of l_x when $x = 65$. Round your answers to three decimal places.

78. Muscle Contraction A muscle has the ability to shorten when a load, such as a weight, is imposed on it. The equation

$$(P + a)(v + b) = k$$

is called the "fundamental equation of muscle contraction."[15] Here P is the load imposed on the muscle, v is the velocity of the shortening of the muscle fibers, and a, b, and k are positive constants. Express v as a function of P. Use your result to find dv/dP.

79. Economics Suppose $pq = 100$ is the demand equation for a manufacturer's product. Let c be the total cost, and assume that the marginal cost is 0.01 when $q = 200$. Use the chain rule to find dc/dp when $q = 200$.

80. Marginal-Revenue Product A monopolist who employs m workers finds that they produce

$$q = 2m(2m + 1)^{3/2}$$

units of product per day. The total revenue, r (in dollars), is given by

$$r = \frac{50q}{\sqrt{1000 + 3q}}$$

(a) What is the price per unit (to the nearest cent) when there are 12 workers?
(b) Determine the marginal revenue when there are 12 workers.
(c) Determine the marginal-revenue product when $m = 12$.

81. Suppose $y = f(x)$, where $x = g(t)$. Given that $g(2) = 3$, $g'(2) = 4$, $f(2) = 5$, $f'(2) = 6$, $g(3) = 7$, $g'(3) = 8$, $f(3) = 9$, and $\left.\frac{dy}{dt}\right|_{t=2} = 40$; determine $f'(3)$.

82. Business A manufacturer has determined that, for his product, the daily average cost (in hundreds of dollars) is given by

$$\bar{c} = \frac{324}{\sqrt{q^2 + 35}} + \frac{5}{q} + \frac{19}{18}$$

(a) As daily production increases, the average cost approaches a constant dollar amount. What is this amount?
(b) Determine the manufacturer's marginal cost when 17 units are produced per day.
(c) The manufacturer determines that if production (and sales) were increased to 18 units per day, revenue would increase by \$275. Should this move be made? Why?

83. If

$$y = (u + 2)\sqrt{u + 3}$$

and

$$u = x(x^2 + 3)^3$$

find dy/dx when $x = 0.1$. Round your answer to two decimal places.

84. If

$$y = \frac{2u + 3}{u^3 - 2}$$

and

$$u = \frac{x + 4}{(2x + 3)^3}$$

find dy/dx when $x = -1$. Round your answer to two decimal places.

Chapter 11 Review

Important Terms and Symbols | Examples

Section 11.1 The Derivative

secant line tangent line slope of a curve	Ex. 1, p. 485
derivative $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $\lim\limits_{z\to x}\frac{f(z)-f(x)}{z-x}$	Ex. 2, p. 486
difference quotient $f'(x)$ y' $\frac{d}{dx}(f(x))$ $\frac{dy}{dx}$	Ex. 4, p. 487

Section 11.2 Rules for Differentiation

power function constant factor rule sum or difference rule	Ex. 5, p. 496

Section 11.3 The Derivative as a Rate of Change

position function Δx velocity rate of change	Ex. 1, p. 501
total-cost function marginal cost average cost per unit	Ex. 7, p. 505
total-revenue function marginal revenue	Ex. 8, p. 506
relative rate of change percentage rate of change	Ex. 9, p. 507

[15] Ibid.

Summary

The tangent line (or tangent) to a curve at point P is the limiting position of secant lines PQ as Q approaches P along the curve. The slope of the tangent at P is called the slope of the curve at P.

If $y = f(x)$, the derivative of f at x is $f'(x)$ defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Geometrically, the derivative gives the slope of the curve $y = f(x)$ at the point $(x, f(x))$. At a particular point $(a, f(a))$ the slope of the tangent line is $f'(a)$, thus the point-slope form of the tangent line at $(a, f(a))$ is $y - f(a) = f'(a)(x-a)$. Any function that is differentiable at a point is also continuous at that point.

The rules for finding derivatives, discussed so far, are as follows, where all functions are assumed to be differentiable:

$$\frac{d}{dx}(c) = 0, \text{ where } c \text{ is any constant}$$

$$\frac{d}{dx}(x^a) = ax^{a-1}, \text{ where } a \text{ is any real number}$$

$$\frac{d}{dx}(cf(x)) = cf'(x), \text{ where } c \text{ is a constant}$$

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$$

$$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$$

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x)+f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}, \text{ where } y \text{ is a function of } u \text{ and } u \text{ is a function of } x$$

$$\frac{d}{dx}(u^a) = au^{a-1}\frac{du}{dx}, \text{ where } u \text{ is a function of } x \text{ and } a \text{ is any real number}$$

The derivative dy/dx can also be interpreted as giving the (instantaneous) rate of change of y with respect to x:

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\text{change in } y}{\text{change in } x}$$

In particular, if $s = f(t)$ is a position function, where s is position at time t, then

$$\frac{ds}{dt} = \text{velocity at time } t$$

In economics, the term *marginal* is used to describe derivatives of specific types of functions. If $c = f(q)$ is a total-cost function (c is the total cost of q units of a product), then the rate of change

$$\frac{dc}{dq} \text{ is called marginal cost}$$

We interpret marginal cost as the approximate cost of one additional unit of output. (Average cost per unit, $\bar{c}$, is related to total cost c by $\bar{c} = c/q$; equivalently, $c = \bar{c}q$.)

A total-revenue function $r = f(q)$ gives a manufacturer's revenue r for selling q units of product. (Revenue r and price p are related by $r = pq$.) The rate of change

$$\frac{dr}{dq} \text{ is called marginal revenue}$$

which is interpreted as the approximate revenue obtained from selling one additional unit of output.

If r is the revenue that a manufacturer receives when the total output of m employees is sold, then the derivative $dr/dm = dr/dq \cdot dq/dm$ is called the marginal-revenue product and gives the approximate change in revenue that results when the manufacturer hires an extra employee.

If $C = f(I)$ is a consumption function, where I is national income and C is national consumption, then

$$\frac{dC}{dI} \text{ is marginal propensity to consume}$$

and

$$1 - \frac{dC}{dI} \text{ is marginal propensity to save}$$

For any function, the relative rate of change of $f(x)$ is

$$\frac{f'(x)}{f(x)}$$

which compares the rate of change of $f(x)$ with $f(x)$ itself. The percentage rate of change is

$$\frac{f'(x)}{f(x)} \cdot 100\%$$

Review Problems

In Problems 1–4, use the definition of the derivative to find $f'(x)$.

1. $f(x) = 2 - x^2$

2. $f(x) = 7x^4 + 5x^2 + 3$

3. $f(x) = \sqrt{3x}$

4. $f(x) = \dfrac{2}{1 + 4x}$

In Problems 5–38, differentiate.

5. $y = 7^4$

6. $y = ex$

7. $y = ex^3 + \sqrt[3]{3}x^2 + 7x^2 + 5x + 2$

8. $y = 4(x^2 + 5) - 7x$

9. $f(s) = s^2(s^2 + 2)$

10. $y = \sqrt{x + 3}$

11. $y = \dfrac{x^2 + 1}{5}$

12. $y = \dfrac{1}{x^n}$

13. $y = (x^3 + 7x^2)(x^3 - x^2 + 5)$

14. $y = (x^2 + 1)^{100}(x - 6)$

15. $f(x) = (2x^2 + 4x)^{100}$

16. $f(w) = w\sqrt{w} + w^2$

17. $y = \dfrac{(ax + b)^2}{(cx + d)^2}$

18. $y = \dfrac{5x^2 - 8x}{2x}$

19. $y = (8 + 2x)(x^2 + 1)^4$

20. $g(z) = (2z)^{3/5} + 5$

21. $f(z) = \dfrac{z^2 - 1}{z^2 + 4}$

22. $y = \sqrt{a^2 - x^2}$

23. $y = \sqrt[3]{4x - 1}$

24. $f(x) = (1 + 2^3)^{12}$

25. $y = \dfrac{1}{\sqrt{1 - x^2}}$

26. $y = \dfrac{x(x + 1)}{2x^2 + 3}$

27. $y = (x + a)^m(x + b)^n(x + c)^p$

28. $y = \dfrac{(x + 3)^5}{x}$

29. $y = \dfrac{5x - 4}{x + 6}$

30. $f(x) = 5x^3\sqrt{3 + 2x^4}$

31. $y = 2x^{-3/8} + (2x)^{-3/8}$

32. $y = \dfrac{x}{a} + \dfrac{a}{x}$

33. $y = \dfrac{x^2 + 6}{\sqrt{x^2 + 5}}$

34. $y = \sqrt[3]{(7 - 3x^2)^2}$

35. $y = (x^3 + 6x^2 + 9)^{3/5}$

36. $z = 0.4x^2(x + 1)^{-3} + 0.5$

37. $g(z) = \dfrac{(2z + 3)^2}{(5z + 7)^{-3}}$

38. $g(z) = \dfrac{-3}{4(z^5 + 2z - 5)^4}$

In Problems 39–42, find an equation of the tangent line to the curve at the point corresponding to the given value of x.

39. $y = x^2 - 6x + 4,\ x = 1$

40. $y = -2x^3 + 6x + 1,\ x = 2$

41. $y = \sqrt[3]{x},\ x = 8$

42. $y = \dfrac{x^3}{x^2 - 3},\ x = 2$

43. If $f(x) = 4x^2 + 2x + 8$, find the relative and percentage rates of change of $f(x)$ when $x = 1$.

44. If $f(x) = x/(x + 4)$, find the relative and percentage rates of change of $f(x)$ when $x = 1$.

45. Marginal Revenue If $r = q(20 - 0.1q)$ is a total-revenue function, find the marginal-revenue function.

46. Marginal Cost If

$$c = 0.0001q^3 - 0.02q^2 + 3q + 6000$$

is a total-cost function, find the marginal cost when $q = 100$.

47. Consumption Function If

$$C = 5 + 0.6I - 0.4\sqrt{I}$$

is a consumption function, find the marginal propensity to consume and the marginal propensity to save when $I = 25$.

48. Demand Equation If $p = \dfrac{q + 12}{q + 5}$ is a demand equation, find the rate of change of price, p, with respect to quantity, q.

49. Demand Equation If $p = -0.1q + 500$ is a demand equation, find the marginal-revenue function.

50. Average Cost If $\bar{c} = 0.03q + 1.2 + \dfrac{3}{q}$ is an average-cost function, find the marginal cost when $q = 100$.

51. Power-Plant Cost Function The total-cost function of an electric light and power plant is estimated by[16]

$$c = 16.68 + 0.125q + 0.00439q^2 \qquad 20 \le q \le 90$$

where q is the eight-hour total output (as a percentage of capacity) and c is the total fuel cost in dollars. Find the marginal-cost function and evaluate it when $q = 70$.

52. Marginal-Revenue Product A manufacturer has determined that m employees will produce a total of q units of product per day, where

$$q = m(50 - m)$$

If the demand function is given by

$$p = -0.05q + 10$$

find the marginal-revenue product when $m = 2$.

53. Winter Moth In a study of the winter moth in Nova Scotia,[17] it was determined that the average number of eggs, y, in a female moth was a function of the female's abdominal width, x (in millimeters), where

$$y = f(x) = 14x^3 - 17x^2 - 16x + 34$$

and $1.5 \le x \le 3.5$. At what rate does the number of eggs change with respect to abdominal width when $x = 2$?

54. Host–Parasite Relation For a particular host–parasite relationship, it is found that when the host density (number of hosts per unit of area) is x, the number of hosts that are parasitized is

$$y = 12\left(1 - \frac{1}{1 + 3x}\right) \qquad x \ge 0$$

For what value of x does dy/dx equal $\frac{1}{3}$?

[16] J. A. Nordin, "Note on a Light Plant's Cost Curves," *Econometrica,* 15 (1947), 231–55.

[17] D. G. Embree, "The Population Dynamics of the Winter Moth in Nova Scotia, 1954–1962," *Memoirs of the Entomological Society of Canada,* no. 46 (1965).

55. Bacteria Growth Bacteria are growing in a culture. The time, t (in hours), for the number of bacteria to double in number (the generation time) is a function of the temperature, T (in degrees Celsius), of the culture and is given by

$$t = f(T) = \begin{cases} \frac{1}{24}T + \frac{11}{4} & \text{if } 30 \le T \le 36 \\ \frac{4}{3}T - \frac{175}{4} & \text{if } 36 < T \le 39 \end{cases}$$

Find dt/dT when (a) $T = 38$ and (b) $T = 35$.

56. Motion The position function of a particle moving in a straight line is

$$s = \frac{9}{2t^2 + 3}$$

where t is in seconds and s is in meters. Find the velocity of the particle at $t = 1$.

57. Rate of Change The volume of an inflatable ball is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Find the rate of change of V with respect to r when $r = 10$ cm.

58. Motion The position function for a ball thrown vertically upward from the ground is

$$s = 218t - 16t^2$$

where s is the height in feet above the ground after t seconds. For what value(s) of t is the velocity 64 ft/s?

59. Find the marginal-cost function if the average-cost function is

$$\bar{c} = 2q + \frac{10{,}000}{q^2}$$

60. Find an equation of the tangent line to the curve

$$y = \frac{(x^3 + 2)\sqrt{x + 1}}{x^4 + 2x}$$

at the point on the curve where $x = 1$.

61. A manufacturer has found that when m employees are working, the number of units of product produced per day is

$$q = 10\sqrt{m^2 + 4900} - 700$$

The demand equation for the product is

$$8q + p^2 - 19{,}300 = 0$$

where p is the selling price when the demand for the product is q units per day.
(a) Determine the manufacturer's marginal-revenue product when $m = 240$.
(b) Find the relative rate of change of revenue with respect to the number of employees when $m = 240$.
(c) Suppose it would cost the manufacturer \$400 more per day to hire an additional employee. Would you advise the manufacturer to hire the 241st employee? Why or why not?

62. If $f(x) = e^x$, use the definition of the derivative ("limit of a difference quotient") to estimate $f'(0)$ correct to three decimal places.

63. If $f(x) = \sqrt[3]{x^2 + 3x - 4}$, use the numerical derivative feature of your graphing calculator to estimate the derivative when $x = 10$. Round your answer to three decimal places.

64. The total-cost function for a manufacturer is given by

$$c = \frac{5q^2 + 4}{\sqrt{q^2 + 6}} + 2500$$

where c is in dollars. Use the numerical derivative feature of your graphing calculator to estimate the marginal cost when 15 units are produced. Round your answer to the nearest cent.

65. Show that Basic Rule 1 is actually a consequence of Combining Rule 1 and the $a = 0$ case of Basic Rule 2.

66. Show that Basic Rule 2 *for positive integers* is a consequence of Combining Rule 3 (the Product Rule) and the $a = 1$ case of Basic Rule 2.