In the new integral, the exponent of t has been reduced to 1. We can match this integral with Formula (38),

$$\int ue^{au}du = \frac{e^{au}}{a^2}(au-1)+C$$

by letting $u = t$ and $a = -0.08$. Then $du = dt$, and

$$\begin{aligned} A &= \int_0^{10} t^2 e^{-0.08t}\,dt = \left.\frac{t^2e^{-0.08t}}{-0.08}\right|_0^{10} - \frac{2}{-0.08}\left.\left(\frac{e^{-0.08t}}{(-0.08)^2}(-0.08t-1)\right)\right|_0^{10} \\ &= \frac{100e^{-0.8}}{-0.08} - \frac{2}{-0.08}\left(\frac{e^{-0.8}}{(-0.08)^2}(-0.8-1) - \frac{1}{(-0.08)^2}(-1)\right) \\ &\approx 185 \end{aligned}$$

The present value is \$185.

Now Work Problem 59 ◁

PROBLEMS 15.1

In Problems 1 and 2, use Formula (19) in Appendix B to determine the integrals.

1. $\displaystyle\int \frac{dx}{(9-x^2)^{3/2}}$

2. $\displaystyle\int \frac{dx}{(25-4x^2)^{3/2}}$

In Problems 3 and 4, use Formula (30) in Appendix B to determine the integrals.

3. $\displaystyle\int \frac{dx}{x^2\sqrt{16x^2+3}}$

4. $\displaystyle\int \frac{3\,dx}{x^3\sqrt{x^4-9}}$

In Problems 5–38, find the integrals by using the table in Appendix B.

5. $\displaystyle\int \frac{dx}{x(6+7x)}$

6. $\displaystyle\int \frac{3x^2dx}{(1+2x)^2}$

7. $\displaystyle\int \frac{dx}{x\sqrt{x^2+9}}$

8. $\displaystyle\int \frac{dx}{(x^2+7)^{3/2}}$

9. $\displaystyle\int \frac{x\,dx}{(2+3x)(4+5x)}$

10. $\displaystyle\int 2^{5x}\,dx$

11. $\displaystyle\int \frac{dx}{3+e^{2x}}$

12. $\displaystyle\int x^2\sqrt{1+x}\,dx$

13. $\displaystyle\int \frac{7\,dx}{x(5+2x)^2}$

14. $\displaystyle\int \frac{dx}{x\sqrt{5-11x^2}}$

15. $\displaystyle\int_0^1 \frac{x\,dx}{2+x}$

16. $\displaystyle\int \frac{-2x^2dx}{3x-2}$

17. $\displaystyle\int \sqrt{x^2-3}\,dx$

18. $\displaystyle\int \frac{dx}{(1+5x)(2x+3)}$

19. $\displaystyle\int_0^{1/12} xe^{12x}\,dx$

20. $\displaystyle\int \sqrt{\frac{2+3x}{5+3x}}\,dx$

21. $\displaystyle\int x^2e^{2x}dx$

22. $\displaystyle\int_1^2 \frac{4\,dx}{x^2(1+x)}$

23. $\displaystyle\int \frac{\sqrt{5x^2+1}}{2x^2}\,dx$

24. $\displaystyle\int \frac{dx}{x\sqrt{2-x}}$

25. $\displaystyle\int \frac{x\,dx}{(1+3x)^2}$

26. $\displaystyle\int \frac{2dx}{\sqrt{(2+2x)(5+2x)}}$

27. $\displaystyle\int \frac{dx}{7-5x^2}$

28. $\displaystyle\int 7x^2\sqrt{3x^2-6}\,dx$

29. $\displaystyle\int 36x^5\ln(3x)\,dx$

30. $\displaystyle\int \frac{5\,dx}{x^2(3+2x)^2}$

31. $\displaystyle\int 2x\sqrt{1+3x}dx$

32. $\displaystyle\int 9x^2\ln x\,dx$

33. $\displaystyle\int \frac{dx}{\sqrt{4x^2-13}}$

34. $\displaystyle\int \frac{dx}{x\ln(2x)}$

35. $\displaystyle\int \frac{2\,dx}{x^2\sqrt{16-9x^2}}$

36. $\displaystyle\int \frac{\sqrt{5-x^2}}{x}dx$

37. $\displaystyle\int \frac{dx}{\sqrt{x}(\pi+7e^{4\sqrt{x}})}$

38. $\displaystyle\int_0^1 \frac{3x^2\,dx}{1+2x^3}$

In Problems 39–56, find the integrals by any method.

39. $\displaystyle\int \frac{x\,dx}{x^2+1}$

40. $\displaystyle\int 3x\sqrt{x}e^{x^{5/2}}\,dx$

41. $\displaystyle\int \frac{(\ln x)^2}{x}dx$

42. $\displaystyle\int \frac{5x^3-\sqrt{x}}{2x}\,dx$

43. $\displaystyle\int \frac{dx}{x^2-5x+6}$

44. $\displaystyle\int \frac{e^{2x}}{\sqrt{e^{2x}+3}}\,dx$

45. $\displaystyle\int x^3\ln x\,dx$

46. $\displaystyle\int (4x+2)e^{6x+3}dx$

47. $\displaystyle\int 4x^3e^{3x^2}\,dx$

48. $\displaystyle\int_1^2 35x^2\sqrt{3+2x}\,dx$

49. $\displaystyle\int \ln^2 x\,dx$

50. $\displaystyle\int_1^e 3x\ln x^2\,dx$

51. $\displaystyle\int_{-1}^1 \frac{2xdx}{\sqrt{5+2x}}$

52. $\displaystyle\int_2^3 x\sqrt{2+3x}\,dx$

53. $\int_0^1 \frac{2x\,dx}{\sqrt{8-x^2}}$

54. $\int_0^{\ln 2} x^2 e^{3x}\,dx$

55. $\int_1^2 x\ln(2x)\,dx$

56. $\int_{-1}^1 dX$

57. Biology In a discussion about gene frequency,[1] the integral

$$\int_{q_0}^{q_n} \frac{dq}{q(1-q)}$$

occurs, where the q's represent gene frequencies. Evaluate this integral.

58. Biology Under certain conditions, the number, n, of generations required to change the frequency of a gene from 0.3 to 0.1 is given by[2]

$$n = -\frac{1}{0.4}\int_{0.3}^{0.1} \frac{dq}{q^2(1-q)}$$

Find n (to the nearest integer).

59. Continuous Annuity Find the present value, to the nearest dollar, of a continuous annuity at an annual rate of r for T years if the payment at time t is at the annual rate of $f(t)$ dollars, given that

(a) $r = 0.04$ $\quad T = 9$ $\quad f(t) = 1000$
(b) $r = 0.06$ $\quad T = 10$ $\quad f(t) = 500t$

60. If $f(t) = k$, where k is a positive constant, show that the value of the integral in Equation (1) of this section is

$$k\left(\frac{1-e^{-rT}}{r}\right)$$

61. Continuous Annuity Find the accumulated amount, to the nearest dollar, of a continuous annuity at an annual rate of r for T years if the payment at time t is at an annual rate of $f(t)$ dollars, given that

(a) $r = 0.05$ $\quad T = 20$ $\quad f(t) = 100$
(b) $r = 0.06$ $\quad T = 25$ $\quad f(t) = 100$

62. Value of Business Over the next five years, the profits of a business at time t are estimated to be $50{,}000t$ dollars per year. The business is to be sold at a price equal to the present value of these future profits. To the nearest 10 dollars, at what price should the business be sold if interest is compounded continuously at the annual rate of 7%?

Objective

To estimate the value of a definite integral by using both the trapezoidal rule and Simpson's rule.

15.2 Approximate Integration

Trapezoidal Rule

We mentioned in the opening paragraphs for this chapter that there are seemingly simple functions that cannot be integrated in terms of elementary functions. No table contains, for example, a "formula" for

$$\int e^{-x^2}\,dx$$

even though, as we will see, *definite* integrals with integrands very similar to that above are extremely important in practical applications.

On the other hand, consider a function f that is continuous on a closed interval $[a, b]$ with $f(x) \geq 0$ for all x in $[a, b]$. The *definite integral* $\int_a^b f(x)dx$ is simply the *number* that gives the area of the region bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$. It is unsatisfying, and perhaps impractical, not to say anything about the number $\int_a^b f(x)dx$ because of an inability to "do" the integral $\int f(x)dx$. This also applies when the integral $\int f(x)dx$ is merely too difficult for the person who needs to find the number $\int_a^b f(x)dx$, and these remarks apply to any continuous function, not just those with $f(x) \geq 0$.

Since $\int_a^b f(x)dx$ is defined as a limit of sums of the form $\sum f(x)\Delta x$, any particular well-formed sum of the form $\sum f(x)\Delta x$ can be regarded as an approximation of $\int_a^b f(x)dx$. At least for nonnegative f, such sums can be regarded as sums of areas of thin rectangles. Consider, for example, Figure 14.11 in Section 14.6, in which two rectangles are explicitly shown. It is clear that the error that arises from such rectangles is associated with the small side at the top. The error would be reduced if we replaced

[1] W. B. Mather, *Principles of Quantitative Genetics* (Minneapolis: Burgess Publishing Company, 1964).

[2] E. O. Wilson and W. H. Bossert, *A Primer of Population Biology* (Stamford, CT: Sinauer Associates, Inc., 1971).

Life Table

	$l(x)$			$l(x)$	
Age = x	Males	Females	Age = x	Males	Females
0	100,000	100,000	45	93,717	96,582
5	99,066	99,220	50	91,616	95,392
10	98,967	99,144	55	88,646	93,562
15	98,834	99,059	60	84,188	90,700
20	98,346	98,857	65	77,547	86,288
25	97,648	98,627	70	68,375	79,926
30	96,970	98,350	75	56,288	70,761
35	96,184	97,964	80	42,127	58,573
40	95,163	97,398			

Solution: We want to estimate

$$\int_{20}^{35} l(t)dt$$

We have $h = \dfrac{b-a}{n} = \dfrac{35-20}{3} = 5$. The terms to be added under the trapezoidal rule are

$$\begin{aligned} l(20) &= 98,857 \\ 2l(25) = 2(98,627) &= 197,254 \\ 2l(30) = 2(98,350) &= 196,700 \\ l(35) &= \underline{97,964} \\ & 590,775 = \text{sum} \end{aligned}$$

By the trapezoidal rule,

$$\int_{20}^{35} l(t)dt \approx \frac{5}{2}(590,775) = 1,476,937.5$$

Now Work Problem 17 ◁

Formulas used to determine the accuracy of answers obtained with the trapezoidal rule or Simpson's rule can be found in standard texts on numerical analysis. For example, one such formula tells us that the error committed by using the trapezoidal rule to estimate $\int_a^b f(x)dx$ is given by

$$-\frac{(b-a)^3}{12n^2} f''(\bar{x}) \quad \text{for some } \bar{x} \text{ in } (a,b)$$

The point of such formulas is that, given a required degree of accuracy, we can determine how big n needs to be to ensure that the trapezoidal approximation is adequate.

PROBLEMS 15.2

In Problems 1 and 2, use the trapezoidal rule or Simpson's rule (as indicated) and the given value of n to estimate the integral.

1. $\displaystyle\int_{-2}^{4} \frac{170}{1+x^2}\, dx$; trapezoidal rule, $n = 6$

2. $\displaystyle\int_{-2}^{4} \frac{170}{1+x^2} dx$; Simpson's rule, $n = 4$

In Problems 3–8, use the trapezoidal rule or Simpson's rule (as indicated) and the given value of n to estimate the integral. Compute each term to four decimal places, and round the answer to three decimal places. In Problems 3–6, also evaluate the integral by antidifferentiation (the Fundamental Theorem of Calculus).

3. $\displaystyle\int_0^1 x^3\, dx$; trapezoidal rule, $n = 5$

4. $\displaystyle\int_0^1 x^2\, dx$; Simpson's rule, $n = 4$

5. $\displaystyle\int_1^4 \frac{dx}{x^2}$; Simpson's rule, $n = 4$

6. $\displaystyle\int_1^4 \frac{dx}{x}$; trapezoidal rule, $n = 6$

7. $\displaystyle\int_0^2 \frac{xdx}{x+1}$; trapezoidal rule, $n = 8$

8. $\displaystyle\int_1^4 \frac{dx}{x}$; Simpson's rule, $n = 6$

In Problems 9 and 10, use the life table in Example 3 to estimate the given integrals by the trapezoidal rule.

9. $\displaystyle\int_{45}^{70} l(t)\,dt$, males, $n = 5$

10. $\displaystyle\int_{35}^{55} l(t)\,dt$, females, $n = 4$

In Problems 11 and 12, suppose the graph of a continuous function f, where $f(x) \geq 0$, contains the given points. Use Simpson's rule and all of the points to approximate the area between the graph and the x-axis on the given interval. Round the answer to one decimal place.

11. $(1, 0.4), (2, 0.6), (3, 1.2), (4, 0.8), (5, 0.5)$; $[1,5]$

12. $(2, 1), (2.5, 3), (3, 6), (3.5, 10), (4, 6), (4.5, 3), (5, 1)$; $[2, 5]$

13. Using all the information given in Figure 15.3, estimate $\int_1^3 f(x)dx$ by using Simpson's rule. Give the answer in fractional form.

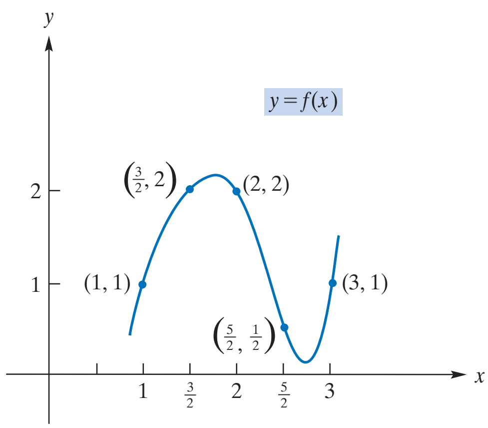


FIGURE 15.3

In Problems 14 and 15, use Simpson's rule and the given value of n to estimate the integral. Compute each term to four decimal places, and round the answer to three decimal places.

14. $\displaystyle\int_1^3 \frac{2}{\sqrt{1+x}}dx$; $n = 4$ Also, evaluate the integral by the Fundamental Theorem of Calculus.

15. $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$; $n = 4$

16. Revenue Use Simpson's rule to approximate the total revenue received from the production and sale of 80 units of a product if the values of the marginal-revenue function dr/dq are as follows:

q (units)	0	10	20	30	40	50	60	70	80
$\frac{dr}{dq}$ (\$ per unit)	10	9	8.5	8	8.5	7.5	7	6.5	7

17. Area of Pool Dexter Griffith, who is keen on mathematics, would like to determine the surface area of his family's curved, irregularly shaped, swimming pool. (All the tiles on the bottom of the pool need to be replaced and nobody has been able to determine how many boxes of tiles to buy.) There is a straight fence that runs along the side of the pool deck. Dexter marks off points a and b on the fence as shown in Figure 15.4. He notes that the distance from a to b is 8m and subdivides the interval into eight equal subintervals, naming the resulting points on the fence $x_1, x_2, x_3, x_4, x_5, x_6$, and x_7. Dexter (D) stands at point x_1, holds a tape measure, and has his little brother Remy (R) take the free end of the tape measure to the point P_1 on the far side of the pool. He asks his Mum, Lesley (L) to stand at point Q_1 on the near side of the pool and note the distance on the tape measure. See Figure 15.4.

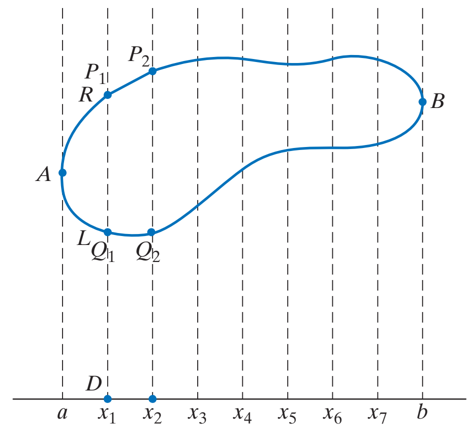


FIGURE 15.4

Dexter then moves to point x_2 and asks his brother to move to P_2, and his Mum to move to Q_2 and repeat the procedure. They do this for each of the remaining points x_3 to x_7. Dexter tabulates their measurements in the following table:

Distance along fence (m)	0	1	2	3	4	5	6	7	8
Distance across pool (m)	0	3	4	3	3	2	2	2	0

Dexter says that Simpson's rule now allows them to approximate the area of the pool as

$$\frac{1}{3}(4(3) + 2(4) + 4(3) + 2(3) + 4(2) + 2(2) + 4(2))$$

square meters. Remy says that this is not how he remembers Simpson's rule. Lesley thinks that some terms are missing, but Remy gets bored and goes for a swim. Is Dexter's calculation correct? Explain, calculate the area, and then determine how many tiles, each with area 6.25cm^2, are needed to tile the bottom of the pool.

18. Manufacturing A manufacturer estimated both marginal cost (MC) and marginal revenue (MR) at various levels of output (q). These estimates are given in the following table:

q(units)	0	20	40	60	80	100
MC ($ per unit)	260	250	240	200	240	250
MR ($ per unit)	410	350	300	250	270	250

(a) Using the trapezoidal rule, estimate the total variable costs of production for 100 units.
(b) Using the trapezoidal rule, estimate the total revenue from the sale of 100 units.
(c) If we assume that maximum profit occurs when MR = MC (that is, when $q = 100$), estimate the maximum profit if fixed costs are $2000.

Objective

To find the area of a region bounded by curves using integration over both vertical and horizontal strips.

15.3 Area Between Curves

In Sections 14.6 and 14.7 we saw that the area of a region bounded by the lines $x = a$, $x = b$, $y = 0$, and a curve $y = f(x)$ with $f(x) \geq 0$ for $a \leq x \leq b$ can be found by evaluating the definite integral $\int_a^b f(x)dx$. Similarly, for a function $f(x) \leq 0$ on an interval $[a, b]$, the area of the region bounded by $x = a$, $x = b$, $y = 0$, and $y = f(x)$ is given by $-\int_a^b f(x)dx = \int_a^b -f(x)dx$. Most of the functions, f, we have encountered, and will encounter, are continuous and have a finite number of roots of $f(x) = 0$. For such functions, the roots of $f(x) = 0$ partition the domain of f into a finite number of intervals on each of which we have either $f(x) \geq 0$ or $f(x) \leq 0$. For such a function we can determine the area bounded by $y = f(x)$, $y = 0$ and *any* pair of vertical lines $x = a$ and $x = b$, with a and b in the domain of f. We have only to find all the roots $c_1 < c_2 < \cdots < c_k$ with $a < c_1$ and $c_k < b$; calculate the integrals $\int_a^{c_1} f(x)\,dx$, $\int_{c_1}^{c_2} f(x)\,dx, \cdots, \int_{c_k}^{b} f(x)\,dx$; attach to each integral the correct sign to correspond to an area; and add the results. Example 1 will provide a modest example of this idea.

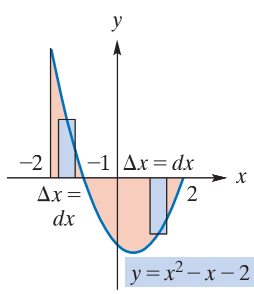


FIGURE 15.5 Diagram for Example 1.

For such an area determination, a rough sketch of the region involved is extremely valuable. To set up the integrals needed, a sample rectangle should be included in the sketch for each individual integral as in Figure 15.5. The area of the region is a limit of sums of areas of rectangles. A sketch helps to understand the integration process and it is indispensable when setting up integrals to find areas of complicated regions. Such a rectangle (see Figure 15.5) is called a **vertical strip**. In the diagram, the width of the vertical strip is Δx. We know from our work on differentials in Section 14.1 that we can consistently write $\Delta x = dx$, for x the independent variable. The height of the vertical strip is the y-value of the curve. Hence, the rectangle has area $y\Delta x = f(x)dx$. The area of the entire region is found by summing the areas of all such vertical strips between $x = a$ and $x = b$ and finding the limit of this sum, which is the definite integral. Symbolically, we have

$$\Sigma y \Delta x \to \int_a^b f(x)dx$$

For $f(x) \geq 0$ it is helpful to think of dx as a length differential and $f(x)dx$ as an area differential dA. Then, as we saw in Section 14.7, we have $\frac{dA}{dx} = f(x)$ for some area function A and

$$\int_a^b f(x)\,dx = \int_a^b dA = A(b) - A(a)$$

(If our area function A measures area starting at the line $x = a$, as it did in Section 14.7, then $A(a) = 0$ and the area under f (and over 0) from a to b is just $A(b)$.) It is important to understand here that we need $f(x) \geq 0$ in order to think of $f(x)$ as a length and, hence, $f(x)dx$ as a differential area. But if $f(x) \leq 0$ then $-f(x) \geq 0$ so that $-f(x)$ becomes a length and $-f(x)dx$ becomes a differential area.

PROBLEMS 15.3

In Problems 1–24, use a definite integral to find the area of the region bounded by the given curve, the x-axis, and the given lines. In each case, first sketch the region. Watch out for areas of regions that are below the x-axis.

1. $y = 5x + 2, \quad x = 1, \quad x = 4$

2. $y = x + 5, \quad x = 2, \quad x = 4$

3. $y = 5x^2, \quad x = 2, \quad x = 6$

4. $y = x^2, \quad x = 2, \quad x = 3$

5. $y = x + x^2 + x^3, \quad x = 1$

6. $y = x^2 - 2x, \quad x = -3, \quad x = -1$

7. $y = 3x^2 - 4x, \quad x = -2, \quad x = -1$

8. $y = 2 - x - x^2$

9. $y = \frac{4}{x}, \quad x = 1, \quad x = 2$

10. $y = 2 - x - x^3, \quad x = -3, \quad x = 0$

11. $y = e^x, \quad x = 1, \quad x = 3$

12. $y = \frac{1}{(x-1)^2}, \quad x = 2, \quad x = 3$

13. $y = -\frac{1}{x}, \quad x = -e, \quad x = -1$

14. $y = \sqrt{x + 9}, \quad x = -9, \quad x = 0$

15. $y = x^2 - 4x, \quad x = 2, \quad x = 6$

16. $y = \sqrt{2x - 1}, \quad x = 1, \quad x = 5$

17. $y = x^3 + 3x^2, \quad x = -2, \quad x = 2$

18. $y = \sqrt[3]{x}, \quad x = 8$

19. $y = e^x + 1, \quad x = 0, \quad x = 1$

20. $y = |x|, \quad x = -2, \quad x = 2$

21. $y = x + \frac{2}{x}, \quad x = 1, \quad x = 2$

22. $y = x^3, \quad x = -2, \quad x = 4$

23. $y = \sqrt{x - 3}, \quad x = 3, \quad x = 28$

24. $y = x^2 + 1, \quad x = 0, \quad x = 4$

25. Given that

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \le x < 2 \\ 16 - 2x & \text{if } x \ge 2 \end{cases}$$

determine the area of the region bounded by the graph of $y = f(x)$, the x-axis, and the line $x = 3$. Include a sketch of the region.

26. Under conditions of a continuous uniform distribution (a topic in statistics), the proportion of persons with incomes between a and t, where $a \le t \le b$, is the area of the region between the curve $y = 1/(b - a)$ and the x-axis from $x = a$ to $x = t$. Sketch the graph of the curve and determine the area of the given region.

27. Suppose $f(x) = x/8$, where $0 \le x \le 4$. If f is a density function (refer to Example 2), find each of the following.

(a) $P(0 \le x \le 1)$ **(b)** $P(2 \le x \le 4)$ **(c)** $P(x \ge 3)$

28. Suppose $f(x) = \frac{1}{3}(1 - x)^2$, where $0 \le x \le 3$. If f is a density function (refer to Example 2), find each of the following.

(a) $P(1 \le x \le 3)$ **(b)** $P\left(1 \le x \le \frac{3}{2}\right)$ **(c)** $P(x \le 2)$

(d) $P(x \ge 2)$ using the result from part (c)

29. Suppose $f(x) = 1/x$, where $e \le x \le e^2$. If f is a density function (refer to Example 2), find each of the following.

(a) $P(3 \le x \le 7)$ **(b)** $P(x \le 5)$ **(c)** $P(x \ge 4)$

(d) Verify that $P(e \le x \le e^2) = 1$.

30. (a) Let r be a real number, where $r > 1$. Evaluate

$$\int_1^r \frac{1}{x^2}\,dx$$

(b) Your answer to part (a) can be interpreted as the area of a certain region of the plane. Sketch this region.

(c) Evaluate $\lim\limits_{r \to \infty} \left(\int_1^r \frac{1}{x^2}\,dx \right)$.

(d) Your answer to part (c) can be interpreted as the area of a certain region of the plane. Sketch this region.

In Problems 31–34, use definite integration to estimate the area of the region bounded by the given curve, the x-axis, and the given lines. Round the answers to two decimal places.

31. $y = \frac{1}{x^2 + 1}, \quad x = -2, \quad x = 1$

32. $y = \frac{x}{\sqrt{x + 5}}, \quad x = 2, \quad x = 7$

33. $y = x^4 - 2x^3 - 2, \quad x = 1, \quad x = 4$

34. $y = 1 + 3x - x^4$

In Problems 35–38, express the area of the shaded region in terms of an integral (or integrals). Do not evaluate your expression.

35. See Figure 15.14.

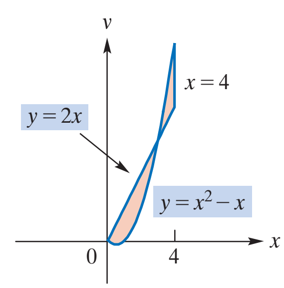


FIGURE 15.14

36. See Figure 15.15.

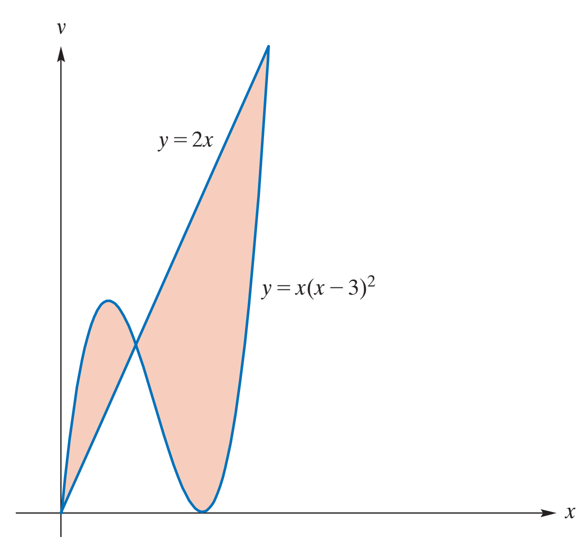


FIGURE 15.15

37. See Figure 15.16.

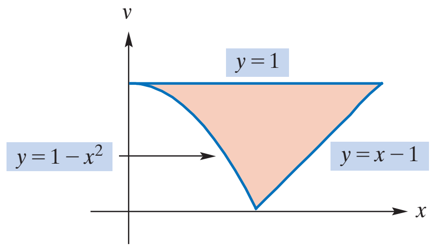


FIGURE 15.16

38. See Figure 15.17.

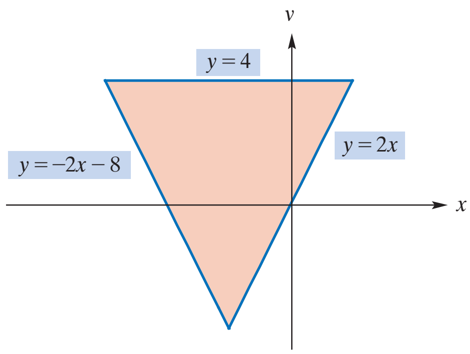


FIGURE 15.17

39. Express, in terms of a single integral, the total area of the region to the right of the line $x = 1$ that is between the curves $y = x^2 - 5$ and $y = 7 - 2x^2$. Do *not* evaluate the integral.

40. Express, in terms of a single integral, the total area of the region in the first quadrant bounded by the x-axis and the graphs of $y^2 = x$ and $2y = 3 - x$. Do *not* evaluate the integral.

In Problems 41–56, find the area of the region bounded by the graphs of the given equations. Be sure to find any needed points of intersection. Consider whether the use of horizontal strips makes the integral simpler than when vertical strips are used.

41. $y = x^2, \quad y = 2x$ **42.** $y = x, \quad y = -x + 3, \quad y = 0$

43. $y = 12 - x^2, \quad y = 3$ **44.** $y^2 = x + 1, \quad x = 1$

45. $x = 8 + 2y, \quad x = 0, \quad y = -1, \quad y = 3$

46. $y = x - 6, \quad y^2 = x$ **47.** $y^2 = 4x, \quad y = 2x - 4$

48. $y = x^3, \quad y = 6x + 9, \quad x = 0$

49. $2y = 4x - x^2, \quad 2y = x - 4$

50. $y = \sqrt{x}, \quad y = x^2$

51. $y = 8 - x^2, \quad y = x^2, \quad x = -1, \quad x = 1$

52. $y = x^3 + x, y = 0, x = -1, x = 2$

53. $y = x^3, \quad y = x$ **54.** $y = x^3, \quad y = \sqrt{x}$

55. $4x + 4y + 17 = 0, \quad y = \dfrac{1}{x}$

56. $y^2 = -x - 2, \quad x - y = 5, \quad y = -1, \quad y = 1$

57. Find the area of the region that is between the curves

$$y = x - 1 \quad \text{and} \quad y = 5 - 2x$$

from $x = 0$ to $x = 4$.

58. Find the area of the region that is between the curves

$$y = x^2 - 4x + 4 \quad \text{and} \quad y = 10 - x^2$$

from $x = 1$ to $x = 5$.

59. Lorenz Curve A *Lorenz curve* is used in studying income distributions. If x is the cumulative percentage of income recipients, ranked from poorest to richest, and y is the cumulative percentage of income, then equality of income distribution is given by the line $y = x$ in Figure 15.18, where x and y are expressed as decimals. For example, 10% of the people receive 10% of total income, 20% of the people receive 20% of the income, and so on. Suppose the actual distribution is given by the Lorenz curve defined by

$$y = \frac{14}{15}x^2 + \frac{1}{15}x$$

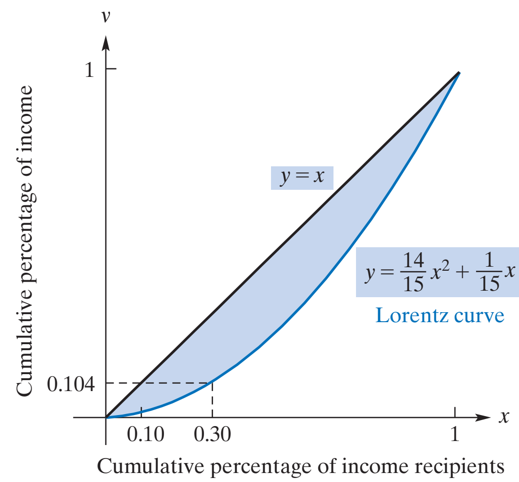


FIGURE 15.18

Note, for example, that 30% of the people receive only 10.4% of total income. The degree of deviation from equality is measured by the *coefficient of inequality*[4] for a Lorenz curve. This coefficient is defined to be the area between the curve and the diagonal, divided by the area under the diagonal:

$$\frac{\text{area between curve and diagonal}}{\text{area under diagonal}}$$

For example, when all incomes are equal, the coefficient of inequality is zero. Find the coefficient of inequality for the Lorenz curve just defined.

60. Lorenz curve Find the coefficient of inequality as in Problem 59 for the Lorenz curve defined by $y = \frac{11}{12}x^2 + \frac{1}{12}x$.

61. Find the area of the region bounded by the graphs of the equations $y^2 = 3x$ and $y = mx$, where m is a positive constant.

62. **(a)** Find the area of the region bounded by the graphs of $y = x^2 - 1$ and $y = 2x + 2$.
(b) What percentage of the area in part (a) lies above the x-axis?

63. The region bounded by the curve $y = x^2$ and the line $y = 1$ is divided into two parts of equal area by the line $y = k$, where k is a constant. Find the value of k.

[4] G. Stigler, *The Theory of Price,* 3rd ed. (New York: The Macmillan Company, 1966), pp. 293–94.

In Problems 64–68, estimate the area of the region bounded by the graphs of the given equations. Round your answer to two decimal places.

64. $y = x^2 - 4x + 1, \quad y = -\dfrac{6}{x}$

65. $y = \sqrt{25 - x^2}, \quad y = 7 - 2x - x^4$

66. $y = x^3 - 8x + 1, \quad y = x^2 - 5$

67. $y = x^5 - 3x^3 + 2x, \quad y = 3x^2 - 4$

68. $y = x^4 - 3x^3 - 15x^2 + 19x + 30, \quad y = x^3 + x^2 - 20x$

Objective

To develop the economic concepts of consumers' surplus and producers' surplus, which are represented by areas.

15.4 Consumers' and Producers' Surplus

Determining the area of a region has applications in economics. Figure 15.19 shows a supply curve for a product. The curve indicates the price, p, per unit at which the manufacturer will sell (supply) q units. The diagram also shows a demand curve for the product. This curve indicates the price, p, per unit at which consumers will purchase (demand) q units. The point (q_0, p_0) where the two curves intersect is called the *point of equilibrium*. Here p_0 is the price per unit at which consumers will purchase the same quantity, q_0, of a product that producers wish to sell at that price. In short, p_0 is the price at which stability in the producer–consumer relationship occurs.

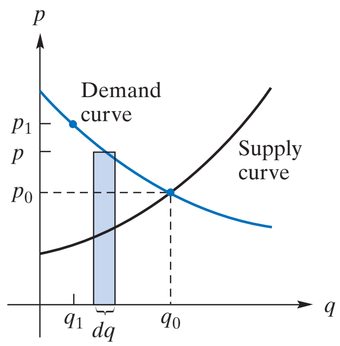


FIGURE 15.19 Supply and demand curves.

Let us assume that the market is at equilibrium and the price per unit of the product is p_0. According to the demand curve, there are consumers who would be willing to pay *more* than p_0. For example, at the price per unit of p_1, consumers would buy q_1 units. These consumers are benefiting from the lower equilibrium price p_0.

The vertical strip in Figure 15.19 has area pdq. This expression can also be thought of as the total amount of money that consumers would spend by buying dq units of the product if the price per unit were p. Since the price is actually p_0, these consumers spend only p_0dq for the dq units and, thus, benefit by the amount $pdq - p_0dq$. This expression can be written $(p - p_0)dq$, which is the area of a rectangle of width dq and height $p - p_0$. (See Figure 15.20.) Summing the areas of all such rectangles from $q = 0$ to $q = q_0$ by definite integration, we have

$$\int_0^{q_0} (p - p_0)dq$$

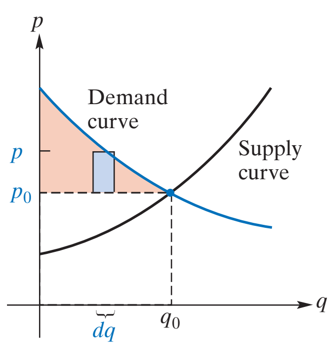


FIGURE 15.20 Benefit to consumers for dq units.

This integral, under certain conditions, represents the total gain to consumers who are willing to pay more than the equilibrium price. This total gain is called **consumers' surplus**, abbreviated CS. If the demand function is given by $p = f(q)$, then

$$\boxed{\text{CS} = \int_0^{q_0} (f(q) - p_0)dq}$$

Geometrically (see Figure 15.21), consumers' surplus is represented by the area between the line $p = p_0$ and the demand curve $p = f(q)$ from $q = 0$ to $q = q_0$.

Some of the producers also benefit from the equilibrium price, since they are willing to supply the product at prices *less* than p_0. Under certain conditions, the total gain to the producers is represented geometrically in Figure 15.22 by the area between the

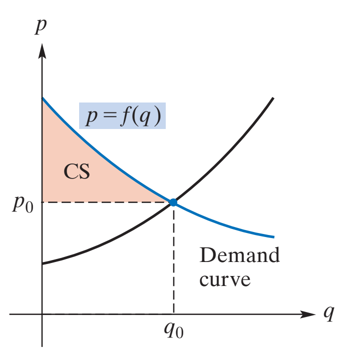


FIGURE 15.21 Consumers' surplus.

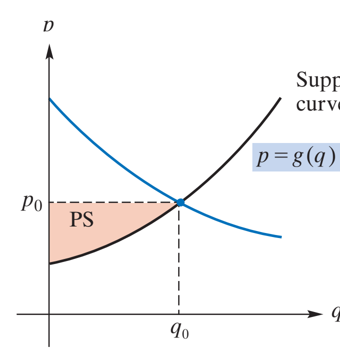


FIGURE 15.22 Producers' surplus.

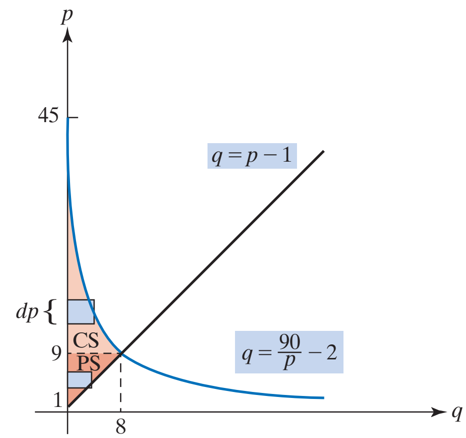


FIGURE 15.23 Diagram for Example 2.

Thus, $p_0 = 9$, so $q_0 = 9 - 1 = 8$. (See Figure 15.23.) Note that the demand equation expresses q as a function of p and that when $q = 0$, $p = 45$. Since consumers' surplus can be considered an area, this area can be determined by means of horizontal strips of width dp and length $q = f(p)$. The areas of these strips are summed from $p = 9$ to $p = 45$ by integrating with respect to p:

$$\begin{aligned}\text{CS} &= \int_9^{45} \left(\frac{90}{p} - 2\right) dp = (90 \ln |p| - 2p)\Big|_9^{45} \\ &= 90 \ln 5 - 72 \approx 72.85\end{aligned}$$

Using horizontal strips for producers' surplus, we have

$$\text{PS} = \int_1^9 (p-1)\, dp = \frac{(p-1)^2}{2}\Big|_1^9 = 32$$

Now Work Problem 5 ◁

PROBLEMS 15.4

In Problems 1–6, the first equation is a demand equation and the second is a supply equation of a product. In each case, determine consumers' surplus and producers' surplus under market equilibrium.

1. $p = 22 - 0.8q$
$p = 6 + 1.2q$

2. $p = 2200 - q^2$
$p = 400 + q^2$

3. $p = \dfrac{50}{q+5}$
$p = \dfrac{q}{10} + 4.5$

4. $p = 1000 - q^2$
$p = 10q + 400$

5. $q = 100(10 - 2p)$
$q = 50(2p - 1)$

6. $q = \sqrt{100 - p}$
$q = \dfrac{p}{2} - 10$

7. The demand equation for a product is

$$q = 10\sqrt{100 - p}$$

Calculate consumers' surplus under market equilibrium, which occurs at a price of \$84.

8. The demand equation for a product is

$$q = 400 - p^2$$

and the supply equation is

$$p = \frac{q}{60} + 5$$

Find producers' surplus and consumers' surplus under market equilibrium.

9. The demand equation for a product is $p = 2^{9-q}$, and the supply equation is $p = 2^{q+3}$, where p is the price per unit (in hundreds of dollars) when q units are demanded or supplied. Determine, to the nearest thousand dollars, consumers' surplus under market equilibrium.

10. The demand equation for a product is

$$(p + 10)(q + 20) = 1000$$

and the supply equation is

$$q - 4p + 10 = 0$$

(a) Verify, by substitution, that market equilibrium occurs when $p = 10$ and $q = 30$.
(b) Determine consumers' surplus under market equilibrium.

11. The demand equation for a product is

$$p = 60 - \frac{50q}{\sqrt{q^2 + 3600}}$$

and the supply equation is

$$p = 10\ln(q + 20) - 26$$

Determine consumers' surplus and producers' surplus under market equilibrium. Round the answers to the nearest integer.

12. Producers' Surplus The supply function for a product is given by the following table, where p is the price per unit (in dollars) at which q units are supplied to the market:

q	0	10	20	30	40	50
p	25	49	59	71	80	94

Use the trapezoidal rule to estimate the producers' surplus if the selling price is \$80.

Objective

To develop the concept of the average value of a function.

15.5 Average Value of a Function

If we are given the three numbers 1, 2, and 9, then their average value, also known as their *mean*, is their sum divided by 3. Denoting this average by $\bar{y}$, we have

$$\bar{y} = \frac{1 + 2 + 9}{3} = 4$$

Similarly, suppose we are given a function f defined on the interval $[a, b]$, and the points $x_1, x_2, \ldots, x_n$ are in the interval. Then the average value of the n corresponding function values $f(x_1), f(x_2), \ldots, f(x_n)$ is

$$\bar{y} = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{\sum_{i=1}^{n} f(x_i)}{n} \tag{1}$$

We can go a step further. Let us divide the interval $[a, b]$ into n subintervals of equal length. We will choose x_i to be the right-hand endpoint of the ith subinterval. Because $[a, b]$ has length $b - a$, each subinterval has length $\frac{b-a}{n}$, which we will call dx. Thus, Equation (1) can be written

$$\bar{y} = \frac{\sum_{i=1}^{n} f(x_i)\left(\frac{dx}{dx}\right)}{n} = \frac{\frac{1}{dx}\sum_{i=1}^{n} f(x_i)dx}{n} = \frac{1}{n\,dx}\sum_{i=1}^{n} f(x_i)dx \tag{2}$$

Since $dx = \frac{b-a}{n}$, it follows that $ndx = b - a$. So the expression $\frac{1}{ndx}$ in Equation (2) can be replaced by $\frac{1}{b-a}$. Moreover, as $n \to \infty$, the number of function values used in computing $\bar{y}$ increases, and we get the so-called *average value of the function f*, denoted by $\bar{f}$:

$$\bar{f} = \lim_{n\to\infty}\left(\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)dx\right) = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)dx$$

But the limit on the right is just the definite integral $\int_a^b f(x)dx$. This motivates the following definition:

Definition

The **average value of a function** $f(x)$ over the interval $[a, b]$ is denoted $\bar{f}$ and is given by

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

PROBLEMS 15.5

In Problems 1–8, find the average value of the function over the given interval.

1. $f(x) = x^2$; $[-1, 3]$

2. $f(x) = 2x + 1$; $[0, 1]$

3. $f(x) = 2 - 3x^2$; $[-1, 2]$

4. $f(x) = x^2 + x + 1$; $[1, 3]$

5. $f(t) = 3t^4$; $[-1, 2]$

6. $f(t) = t\sqrt{t^2 + 9}$; $[0, 4]$

7. $f(x) = \sqrt{x}$; $[0, 1]$

8. $f(x) = 5/x^2$; $[1, 3]$

9. Profit The profit (in dollars) of a business is given by

$$P = P(q) = 369q - 2.1q^2 - 400$$

where q is the number of units of the product sold. Find the average profit on the interval from $q = 0$ to $q = 100$.

10. Cost Suppose the cost (in dollars) of producing q units of a product is given by

$$c = 5000 + 12q + 0.3q^2$$

Find the average cost on the interval from $q = 200$ to $q = 500$.

11. Investment An investment of \$3000 earns interest at an annual rate of 5% compounded continuously. After t years, its value, S (in dollars), is given by $S = 3000e^{0.05t}$. Find the average value of a two-year investment.

12. Medicine Suppose that colored dye is injected into the bloodstream at a constant rate, R. At time t, let

$$C(t) = \frac{R}{F(t)}$$

be the concentration of dye at a location distant (distal) from the point of injection, where $F(t)$ is as given in Example 2. Show that the average concentration on $[0, T]$ is

$$\overline{C} = \frac{R\left(1 + \alpha T + \frac{1}{3}\alpha^2 T^2\right)}{F_1}$$

13. Revenue Suppose a manufacturer receives revenue, r, from the sale of q units of a product. Show that the average value of the marginal-revenue function over the interval $[0, q_0]$ is the price per unit when q_0 units are sold.

14. Find the average value of $f(x) = \dfrac{1}{x^2 - 4x + 5}$ over the interval $[0, 1]$ using an approximate integration technique. Round your answer to two decimal places.

Objective

To solve a differential equation by using the method of separation of variables. To discuss particular solutions and general solutions. To develop interest compounded continuously in terms of a differential equation. To discuss exponential growth and decay.

15.6 Differential Equations

Frequently, we have to solve an equation that involves the derivative of an unknown function. Such an equation is called a **differential equation**. An example is

$$y' = xy^2 \tag{1}$$

More precisely, Equation (1) is called a **first-order differential equation**, because it involves a derivative of the first order and none of any higher order. A solution of Equation (1) is any function $y = f(x)$ that is defined on an interval and satisfies the equation for all x in the interval.

To solve $y' = xy^2$, equivalently,

$$\frac{dy}{dx} = xy^2 \tag{2}$$

we think of dy/dx as a quotient of differentials, and algebraically we "separate the variables" by rewriting the equation so that each side contains only one variable and all differentials appear as numerators:

$$\frac{dy}{y^2} = xdx$$

Integrating both sides and combining the constants of integration, we obtain

$$\int \frac{1}{y^2}dy = \int xdx$$

$$-\frac{1}{y} = \frac{x^2}{2} + C_1$$

$$-\frac{1}{y} = \frac{x^2 + 2C_1}{2}$$

Since $2C_1$ is an arbitrary constant, we can replace it by C.

$$-\frac{1}{y} = \frac{x^2 + C}{2} \tag{3}$$

Setting $t = 40$ gives the expected population in 2010:

$$N = N(40) = 125,000(1.12)^2 = 156,800$$

We remark that from $e^{20k} = 1.12$ we have $20k = \ln(1.12)$ and hence $k = \frac{\ln(1.12)}{20} \approx 0.0057$, which can be placed in $N = 125,000e^{kt}$ to give

$$N \approx 125,000e^{0.0057t} \tag{12}$$

Now Work Problem 23 ◁

In Chapter 4, radioactive decay was discussed. Here we will consider this topic from the perspective of a differential equation. The rate at which a radioactive element decays at any time is found to be proportional to the amount of that element present. If N is the amount of a radioactive substance at time t, then the rate of decay is given by

$$\frac{dN}{dt} = -\lambda N. \tag{13}$$

The positive constant λ (a Greek letter read "lambda") is called the **decay constant**, and the minus sign indicates that N is decreasing as t increases. Thus, we have exponential decay. From Equation (10), the solution of this differential equation is

$$N = N_0e^{-\lambda t} \tag{14}$$

If $t = 0$, then $N = N_0 \cdot 1 = N_0$, so N_0 represents the amount of the radioactive substance present when $t = 0$.

The time for one-half of the substance to decay is called the **half-life** of the substance. In Section 4.2, it was shown that the half-life is given by

$$\text{half-life} = \frac{\ln 2}{\lambda} \approx \frac{0.69315}{\lambda} \tag{15}$$

Note that the half-life depends on λ. In Chapter 4, Figure 4.13 shows the graph of radioactive decay.

EXAMPLE 3 Finding the Decay Constant and Half-Life

If 60% of a radioactive substance remains after 50 days, find the decay constant and the half-life of the element.

Solution: From Equation (14),

$$N = N_0e^{-\lambda t}$$

where N_0 is the amount of the element present at $t = 0$. When $t = 50$, then $N = 0.6N_0$, and we have

$$\begin{aligned}
0.6N_0 &= N_0e^{-50\lambda} \\
0.6 &= e^{-50\lambda} \\
-50\lambda &= \ln(0.6) && \text{logarithmic form} \\
\lambda &= -\frac{\ln(0.6)}{50} \approx 0.01022
\end{aligned}$$

Thus, $N \approx N_0e^{-0.01022t}$. The half-life, from Equation (15), is

$$\frac{\ln 2}{\lambda} \approx 67.82\text{days}$$

Now Work Problem 27 ◁

Radioactivity is useful in dating such things as fossil plant remains and archaeological remains made from organic material. Plants and other living organisms contain a small amount of radioactive carbon 14 (^{14}C) in addition to ordinary carbon (^{12}C). The ^{12}C atoms are stable, but the ^{14}C atoms are decaying exponentially. However, ^{14}C is formed in the atmosphere due to the effect of cosmic rays. This ^{14}C is taken up by plants during photosynthesis and replaces what has decayed. As a result, the ratio of ^{14}C atoms to ^{12}C atoms is considered constant in living tissues over a long period of time. When a plant dies, it stops absorbing ^{14}C, and the remaining ^{14}C atoms decay. By comparing the proportion of ^{14}C to ^{12}C in a fossil plant to that of plants found today, we can estimate the age of the fossil. The half-life of ^{14}C is approximately 5730 years. Thus, if a fossil is found to have a ^{14}C-to-^{12}C ratio that is half that of a similar substance found today, we would estimate the fossil to be 5730 years old.

EXAMPLE 4 Estimating the Age of an Ancient Tool

A wood tool found in a Middle East excavation site is found to have a ^{14}C-to-^{12}C ratio that is 0.6 of the corresponding ratio in a present-day tree. Estimate the age of the tool to the nearest hundred years.

Solution: Let N be the amount of ^{14}C present in the wood t years after the tool was made. Then $N = N_0e^{-\lambda t}$, where N_0 is the amount of ^{14}C when $t = 0$. Since the ratio of ^{14}C to ^{12}C is 0.6 of the corresponding ratio in a present-day tree, this means that we want to find the value of t for which $N = 0.6N_0$. Thus, we have

$$\begin{aligned}
0.6N_0 &= N_0e^{-\lambda t} \\
0.6 &= e^{-\lambda t} \\
-\lambda t &= \ln(0.6) \qquad \text{logarithmic form} \\
t &= -\frac{1}{\lambda}\ln(0.6)
\end{aligned}$$

From Equation (15), the half-life is $(\ln 2)/\lambda$, which is approximately 5730, so $\lambda \approx (\ln 2)/5730$. Consequently,

$$\begin{aligned}
t &\approx -\frac{1}{(\ln 2)/5730}\ln(0.6) \\
&\approx -\frac{5730\ln(0.6)}{\ln 2} \\
&\approx 4200 \text{ years}
\end{aligned}$$

Now Work Problem 29 ◁

PROBLEMS 15.6

In Problems 1–8, solve the differential equations.

1. $y' = 3xy^2$

2. $y' = x^2y^2$

3. $\dfrac{dy}{dx} - 2x\ln(x^2+1) = 0$

4. $\dfrac{dy}{dx} = \dfrac{x}{y}$

5. $\dfrac{dy}{dx} = y, y > 0$

6. $y' = e^xy^2$

7. $y' = \dfrac{y}{x}, x, y > 0$

8. $\dfrac{dy}{dx} - x\ln x = 0$

In Problems 9–18, solve each of the differential equations, subject to the given conditions.

9. $y' = \dfrac{1}{y^2}$; $y(1) = 1$

10. $y' = e^{x-y}$; $y(0) = 0$ (*Hint:* $e^{x-y} = e^x/e^y$.)

11. $e^yy' - x^3 = 0$; $y = 1$ when $x = 0$

12. $x^2y' + \dfrac{1}{y^2} = 0$; $y(1) = 2$

13. $(3x^2+2)^3y' - xy^2 = 0$; $y(0) = 2$

14. $y' + x^3y = 0$; $y = e$ when $x = 0$

15. $\dfrac{dy}{dx} = \dfrac{3x\sqrt{1+y^2}}{y}$; $y > 0, y(1) = \sqrt{8}$

16. $2y(x^3+x+1)\dfrac{dy}{dx} = \dfrac{3x^2+1}{\sqrt{y^2+4}}$; $y(0) = 0$

17. $2\dfrac{dy}{dx} = \dfrac{xe^{-y}}{\sqrt{x^2+3}}$; $y(1) = 0$

18. $dy = 2xye^{x^2}\,dx$, $y > 0$; $y(0) = e$

19. Cost Find the manufacturer's cost function $c = f(q)$ given that

$$(q+1)^2\frac{dc}{dq} = cq$$

and fixed cost is e.

20. Find $f(2)$, given that $f(1) = 0$ and that $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = xe^{x-y}$$

21. Circulation of Money A country has 1.00 billion dollars of paper money in circulation. Each week 25 million dollars is brought into the banks for deposit, and the same amount is paid out. The government decides to issue new paper money; whenever the old money comes into the banks, it is destroyed and replaced by new money. Let y be the amount of old money (in millions of dollars) in circulation at time t (in weeks). Then y satisfies the differential equation

$$\frac{dy}{dt} = -0.025y$$

How long will it take for 95% of the paper money in circulation to be new? Round your answer to the nearest week. (*Hint:* If 95% of money is new, then y is 5% of 1000.)

22. Marginal Revenue and Demand Suppose that a monopolist's marginal-revenue function is given by the differential equation

$$\frac{dr}{dq} = (50 - 4q)e^{-r/5}$$

Find the demand equation for the monopolist's product.

23. Population Growth In a certain town, the population at any time changes at a rate proportional to the population. If the population in 1990 was 60,000 and in 2000 was 64,000, find an equation for the population at time t, where t is the number of years past 1990. What is the expected population in 2010?

24. Population Growth The population of a town increases by natural growth at a rate proportional to the number, N, of persons present. If the population at time $t = 0$ is 50,000, find two expressions for the population N, t years later, if the population doubles in 50 years. Assume that $\ln 2 = 0.69$. Also, find N for $t = 100$.

25. Population Growth Suppose that the population of the world in 1930 was 2 billion and in 1960 was 3 billion. If the exponential law of growth is assumed, what is the expected population in 2015? Give your answer in terms of e.

26. Population Growth If exponential growth is assumed, in approximately how many years will a population double if it triples in 81 years?

27. Radioactivity If 30% of the initial amount of a radioactive sample remains after 100 seconds, find the decay constant and the half-life of the element.

28. Radioactivity If 20% of the initial amount of a radioactive sample has *decayed* after 100 seconds, find the decay constant and the half-life of the element.

29. Carbon Dating An Egyptian scroll was found to have a ^{14}C-to-^{12}C ratio 0.7 of the corresponding ratio in similar present-day material. Estimate the age of the scroll, to the nearest hundred years.

30. Carbon Dating A recently discovered archaeological specimen has a ^{14}C-to-^{12}C ratio 0.1 of the corresponding ratio found in present-day organic material. Estimate the age of the specimen, to the nearest hundred years.

31. Population Growth Suppose that a population follows exponential growth given by $dN/dt = kN$ for $t \geq t_0$. Suppose also that $N = N_0$ when $t = xt_0$. Find N, the population size at time t.

32. Radioactivity Polonium-210 has a half-life of about 140 days. **(a)** Find the decay constant in terms of ln 2. **(b)** What fraction of the original amount of a sample of polonium-210 remains after one year?

33. Radioactivity Radioactive isotopes are used in medical diagnoses as tracers to determine abnormalities that may exist in an organ. For example, if radioactive iodine is swallowed, after some time it is taken up by the thyroid gland. With the use of a detector, the rate at which it is taken up can be measured, and a determination can be made as to whether the uptake is normal. Suppose radioactive technetium-99m, which has a half-life of six hours, is to be used in a brain scan two hours from now. What should be its activity now if the activity when it is used is to be 12 units? Give your answer to one decimal place. [*Hint:* In Equation (14), let N = activity t hours from now and N_0 = activity now.]

34. Radioactivity A radioactive substance that has a half-life of eight days is to be temporarily implanted in a hospital patient until three-fifths of the amount originally present remains. How long should the implant remain in the patient?

35. Ecology In a forest, natural litter occurs, such as fallen leaves and branches, dead animals, and so on.[6] Let $A = A(t)$ denote the amount of litter present at time t, where $A(t)$ is expressed in grams per square meter and t is in years. Suppose that there is no litter at $t = 0$. Thus, $A(0) = 0$. Assume that

(i) Litter falls to the ground continuously at a constant rate of 200 grams per square meter per year.
(ii) The accumulated litter decomposes continuously at the rate of 50% of the amount present per year (which is $0.50A$).
The difference of the two rates is the rate of change of the amount of litter present with respect to time:

$$\begin{pmatrix}\text{rate of change}\\ \text{of litter present}\end{pmatrix} = \begin{pmatrix}\text{rate of falling}\\ \text{to ground}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\ \text{decomposition}\end{pmatrix}$$

Therefore,

$$\frac{dA}{dt} = 200 - 0.50A$$

Solve for A. To the nearest gram, determine the amount of litter per square meter after one year.

[6] R. W. Poole, *An Introduction to Quantitative Ecology* (New York: McGraw-Hill Book Company, 1974).

36. Profit and Advertising A company determines that the rate of change of monthly net profit P, as a function of monthly advertising expenditure x, is proportional to the difference between a fixed amount, \$250,000, and $2P$; that is, dP/dx is proportional to $250,000 - 2P$. Furthermore, if no money is spent on monthly advertising, the monthly net profit is \$10,000; if \$1000 is spent on monthly advertising, the monthly net profit is \$50,000. What would the monthly net profit be if \$5000 were spent on advertising each month?

37. Value of a Machine The value of a certain machine depreciates 25% in the first year after the machine is purchased. The rate at which the machine subsequently depreciates is proportional to its value. Suppose that such a machine was purchased new on July 1, 1995, for \$80,000 and was valued at \$38,900 on January 1, 2006.

(a) Determine a formula that expresses the value V of the machine in terms of t, the number of years after July 1, 1996.

(b) Use the formula in part (a) to determine the year and month in which the machine has a value of exactly \$14,000.

Objective

To develop the logistic function as a solution of a differential equation. To model the spread of a rumor. To discuss and apply Newton's law of cooling.

15.7 More Applications of Differential Equations

Logistic Growth

In the previous section, we found that if the number N of individuals in a population at time t follows an exponential law of growth, then $N = N_0e^{kt}$, where $k > 0$ and N_0 is the population when $t = 0$. This law assumes that at time t the rate of growth, dN/dt, of the population is proportional to the number of individuals in the population. That is, $dN/dt = kN$.

Under exponential growth, a population would get "infinitely large" as time goes on, meaning that $\lim_{t\to\infty} N_0e^{kt} = \infty$. In reality, however, when the population gets large enough, environmental factors slow down the rate of growth. Examples are food supply, predators, overcrowding, and so on. These factors cause dN/dt to decrease eventually. It is reasonable to assume that the size of a population is limited to some maximum number M, where $0 < N < M$, and as $N \to M$, $dN/dt \to 0$, and the population size tends to be stable.

In summary, we want a population model that has exponential growth initially but also includes the effects of environmental resistance to large population growth. Such a model is obtained by multiplying the right side of $dN/dt = kN$ by the factor $(M-N)/M$:

$$\frac{dN}{dt} = kN\left(\frac{M-N}{M}\right)$$

Notice that if N is small, then $(M-N)/M$ is close to 1, and we have growth that is approximately exponential. As $N \to M$, then $M - N \to 0$ and $dN/dt \to 0$, as we wanted in our model. Because k/M is a constant, we can replace it by K. Thus,

$$\frac{dN}{dt} = KN(M-N) \tag{1}$$

This states that the rate of growth is proportional to the product of the size of the population and the difference between the maximum size and the actual size of the population. We can solve for N in the differential Equation (1) by the method of separation of variables:

$$\frac{dN}{N(M-N)} = K\,dt$$

$$\int \frac{1}{N(M-N)}\,dN = \int K\,dt \tag{2}$$

The integral on the left side can be found by using Formula (5) in the table of integrals in Appendix B. Thus, Equation (2) becomes

$$\frac{1}{M}\ln\left|\frac{N}{M-N}\right| = Kt + C$$

so

$$\ln\left|\frac{N}{M-N}\right| = MKt + MC$$

Separating variables, we have

$$\frac{dT}{T-22} = kdt$$

$$\int \frac{dT}{T-22} = \int kdt$$

$$\ln|T-22| = kt + C$$

Because $T - 22 > 0$,

$$\ln(T-22) = kt + C$$

When $t = 0$, then $T = 31$. Therefore,

$$\ln(31-22) = k \cdot 0 + C$$

$$C = \ln 9$$

Hence,

$$\ln(T-22) = kt + \ln 9$$

$$\ln(T-22) - \ln 9 = kt$$

$$\ln\frac{T-22}{9} = kt \qquad \ln a - \ln b = \ln\frac{a}{b}$$

When $t = 1$, then $T = 30$, so

$$\ln\frac{30-22}{9} = k \cdot 1$$

$$k = \ln\frac{8}{9}$$

Thus,

$$\ln\frac{T-22}{9} = t\ln\frac{8}{9}$$

Now we find t when $T = 37$:

$$\ln\frac{37-22}{9} = t\ln\frac{8}{9}$$

$$t = \frac{\ln(15/9)}{\ln(8/9)} \approx -4.34$$

Accordingly, the murder occurred about 4.34 hours *before* the time of discovery of the body (11:00 P.M.). Since 4.34 hours is (approximately) 4 hours and 20 minutes, the industrialist was murdered about 6:40 P.M.

Now Work Problem 9 ◁

PROBLEMS 15.7

1. Population The population of a city follows logistic growth and is limited to 100,000. If the population in 1995 was 50,000 and in 2000 was 60,000, what will the population be in 2005? Give your answer to the nearest hundred.

2. Production A company believes that the production of its product in present facilities will follow logistic growth. Presently, 300 units per day are produced, and production will increase to 500 units per day in one year. If production is limited to 900 units per day, what is the anticipated daily production in two years? Give the answer to the nearest unit.

3. Spread of Rumor In a university of 40,000 students, the administration holds meetings to discuss the idea of bringing in a major rock band for homecoming weekend. Before the plans are officially announced, student representatives on the administrative council spread information about the event as a rumor. At the end of one week, 100 people know the rumor. Assuming logistic growth, how many people know the rumor after two weeks? Give your answer to the nearest hundred.

4. Spread of a Fad At a university with 50,000 students, it is believed that the number of students with a particular ring tone on their mobile phones is following a logistic growth pattern. The student newspaper investigates when a survey reveals that 500 students have the ring tone. One week later, a similar survey reveals that 1500 students have it. The newspaper writes a story about it and includes a formula predicting the number $N = N(t)$ of students who will have the ring tone t weeks after the first survey. What is the formula that the newspaper publishes?

5. Flu Outbreak In a city whose population is 100,000, an outbreak of flu occurs. When the city health department begins its recordkeeping, there are 500 infected persons. One week later, there are 1000 infected persons. Assuming logistic growth, estimate the number of infected persons two weeks after recordkeeping begins.

6. Sigmoid Function A very special case of the logistic function defined by Equation (3) is the *sigmoid function,* obtained by taking $M = b = c = 1$ so that we have

$$N(t) = \frac{1}{1 + e^{-t}}$$

(a) Show directly that the sigmoid function is the solution of the differential equation

$$\frac{dN}{dt} = N(1 - N)$$

and the initial condition $N(0) = 1/2$.
(b) Show that $(0, 1/2)$ is an inflection point on the graph of the sigmoid function.
(c) Show that the function

$$f(t) = \frac{1}{1 + e^{-t}} - \frac{1}{2}$$

is symmetric about the origin.
(d) Explain how (c) above shows that the sigmoid function is *symmetric about the point* $(0, 1/2)$, explaining at the same time what this means.
(e) Sketch the graph of the sigmoid function.

7. Biology In an experiment,[7] five *Paramecia* were placed in a test tube containing a nutritive medium. The number N of *Paramecia* in the tube at the end of t days is given approximately by

$$N = \frac{375}{1 + e^{5.2-2.3t}}$$

(a) Show that this equation can be written as

$$N = \frac{375}{1 + 181.27e^{-2.3t}}$$

and, hence, is a logistic function.
(b) Find $\lim_{t\to\infty} N$.
(c) How many days will it take for the number of *Paramecia* to exceed 370?

8. Biology In a study of the growth of a colony of unicellular organisms,[8] the equation

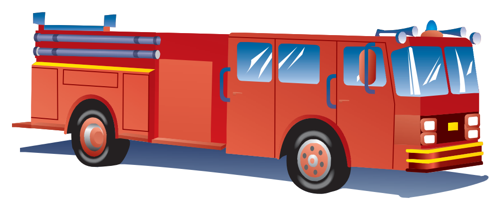
$$N = \frac{0.2524}{e^{-2.128x} + 0.005125} \qquad 0 \le x \le 5$$

was obtained, where N is the estimated area of the growth in square centimeters and x is the age of the colony in days after being first observed.
(a) Put this equation in the form of a logistic function.
(b) Find the area when the age of the colony is 0.

9. Time of a Murder A murder was committed in an abandoned warehouse, and the victim's body was discovered at 3:17 A.M. by the police. At that time, the temperature of the body was 27°C and the temperature in the warehouse was −5°C. One hour later, the body temperature was 19°C and the warehouse temperature was unchanged. The police forensic mathematician calculates using Newton's law of cooling. What is the time she reports as the time of the murder?

10. Enzyme Formation An enzyme is a protein that acts as a catalyst for increasing the rate of a chemical reaction that occurs in cells. In a certain reaction, an enzyme A is converted to another enzyme, B. Enzyme B acts as a catalyst for its own formation. Let p be the amount of enzyme B at time t and I be the total amount of both enzymes when $t = 0$. Suppose the rate of formation of B is proportional to $p(I - p)$. Without directly using calculus, find the value of p for which the rate of formation will be a maximum.

11. Fund-Raising A small town decides to conduct a fund-raising drive for a fire engine, the cost of which is \$200,000. The initial amount in the fund is \$50,000. On the basis of past drives, it is determined that t months after the beginning of the drive, the rate, dx/dt, at which money is contributed to such a fund is proportional to the difference between the desired goal of \$200,000 and the total amount, x, in the fund at that time. After one month, a total of \$100,000 is in the fund. How much will be in the fund after three months?



12. Birthrate In a discussion of unexpected properties of mathematical models of population, Bailey[9] considers the case in which the birthrate per *individual* is proportional to the population size N at time t. Since the growth rate per individual is $\frac{1}{N}\frac{dN}{dt}$, this means that

$$\frac{1}{N}\frac{dN}{dt} = kN$$

so that

$$\frac{dN}{dt} = kN^2 \qquad \text{subject to } N = N_0 \text{ at } t = 0$$

where $k > 0$. Show that

$$N = \frac{N_0}{1 - kN_0 t}$$

Use this result to show that

$$\lim N = \infty \quad \text{as} \quad t \to \left(\frac{1}{kN_0}\right)^-$$

This means that over a finite interval of time, there is an infinite amount of growth. Such a model might be useful only for rapid growth over a short interval of time.

13. Population Suppose that the rate of growth of a population is proportional to the difference between some maximum size M and the number N of individuals in the population at time t. Suppose that when $t = 0$, the size of the population is N_0. Find a formula for N.

[7] G. F. Gause, *The Struggle for Existence* (New York: Hafner Publishing Co., 1964).

[8] A. J. Lotka, *Elements of Mathematical Biology* (New York: Dover Publications, Inc., 1956).

[9] N. T. J. Bailey, *The Mathematical Approach to Biology and Medicine* (New York: John Wiley & Sons, Inc., 1967).

Objective

To define and evaluate improper integrals.

15.8 Improper Integrals

Suppose $f(x)$ is continuous and nonnegative for $a \leq x < \infty$. (See Figure 15.27.) We know that the integral $\int_a^b f(x)dx$ is the area of the region between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$. As $b \to \infty$, we can think of

$$\lim_{b\to\infty} \int_a^b f(x)dx$$

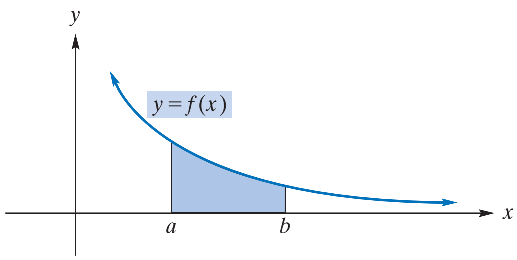


FIGURE 15.27 Area from a to b.

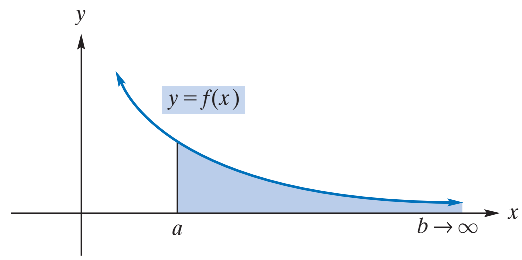


FIGURE 15.28 Area from a to b as $b \to \infty$.

as the area of the unbounded region that is shaded in Figure 15.28. This limit is abbreviated by

$$\int_a^\infty f(x)dx \tag{1}$$

and called an **improper integral**. If the limit exists, $\int_a^\infty f(x)dx$ is said to be **convergent** and the improper integral *converges* to that limit. In this case the unbounded region is considered to have a finite area, and this area is represented by $\int_a^\infty f(x)dx$. If the limit does not exist, the improper integral is said to be **divergent**, and the region does not have a finite area.

We can remove the restriction that $f(x) \geq 0$. In general, the improper integral $\int_a^\infty f(x)dx$ is defined by

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

Other types of improper integrals are

$$\int_{-\infty}^b f(x)\,dx \tag{2}$$

and

$$\int_{-\infty}^\infty f(x)dx \tag{3}$$

APPLY IT ►

4. The rate at which the human body eliminates a certain drug from its system may be approximated by $R(t) = 3e^{-0.1t} - 3e^{-0.3t}$, where $R(t)$ is in milliliters per minute and t is the time in minutes since the drug was taken. Find $\int_0^\infty (3e^{-0.1t} - 3e^{-0.3t})\,dt$, the total amount of the drug that is eliminated.

In each of the three types of improper integrals [(5), (2), and (3)], the interval over which the integral is evaluated has infinite length. The improper integral in (2) is defined by

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

If this limit exists, $\int_{-\infty}^b f(x)dx$ is said to be convergent. Otherwise, it is divergent. We will define the improper integral in (3) after the following example.

EXAMPLE 3 Density Function

In statistics, a function, f, is called a density function if $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Suppose

$$f(x) = \begin{cases} ke^{-x} & \text{for } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

is a density function. Find k.

Solution: We write the equation $\int_{-\infty}^{\infty} f(x)dx = 1$ as

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$

Since $f(x) = 0$ for $x < 0$, $\int_{-\infty}^{0} f(x)dx = 0$. Thus,

$$\int_{0}^{\infty} ke^{-x}dx = 1$$

$$\lim_{b\to\infty} \int_{0}^{b} ke^{-x}dx = 1$$

$$\lim_{b\to\infty} -ke^{-x}\Big|_{0}^{b} = 1$$

$$\lim_{b\to\infty} (-ke^{-b} + k) = 1$$

$$0 + k = 1 \qquad \lim_{b\to\infty} e^{-b} = 0$$

$$k = 1$$

Now Work Problem 13 ◁

PROBLEMS 15.8

In Problems 1–12, determine the integrals if they exist. Indicate those that are divergent.

1. $\displaystyle\int_{3}^{\infty} \frac{1}{x^3}\,dx$

2. $\displaystyle\int_{1}^{\infty} \frac{1}{(3x-1)^2}\,dx$

3. $\displaystyle\int_{e^{1000}}^{\infty} \frac{1}{x}dx$

4. $\displaystyle\int_{2}^{\infty} \frac{1}{\sqrt[3]{(x+2)^2}}\,dx$

5. $\displaystyle\int_{37}^{\infty} e^{-x}\,dx$

6. $\displaystyle\int_{0}^{\infty} (5 + e^{-x})\,dx$

7. $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt{x}}\,dx$

8. $\displaystyle\int_{5}^{\infty} \frac{xdx}{\sqrt{(x^2-9)^3}}$

9. $\displaystyle\int_{-\infty}^{-3} \frac{1}{(x+1)^2}\,dx$

10. $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt[3]{x-1}}\,dx$

11. $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

12. $\displaystyle\int_{-\infty}^{\infty} (5-3x)\,dx$

13. Density Function The density function for the life x, in hours, of an electronic component in a radiation meter is given by

$$f(x) = \begin{cases} \dfrac{k}{x^2} & \text{for } x \geq 500 \\ 0 & \text{for } x < 500 \end{cases}$$

(a) If k satisfies the condition that $\int_{500}^{\infty} f(x)dx = 1$, find k.
(b) The probability that the component will last at least 1000 hours is given by $\int_{1000}^{\infty} f(x)dx$. Evaluate this integral.

14. Density Function Given the density function

$$f(x) = \begin{cases} ke^{-2x} & \text{for } x \geq 1 \\ 0 & \text{elsewhere} \end{cases}$$

find k. (*Hint:* See Example 3.)

15. Future Profits For a business, the present value of all future profits at an annual interest rate, r, compounded continuously is given by

$$\int_{0}^{\infty} p(t)e^{-rt}\,dt$$

where $p(t)$ is the profit per year in dollars at time t. If $p(t) = 500{,}000$ and $r = 0.02$, evaluate this integral.

16. Psychology In a psychological model for signal detection,[10] the probability α (a Greek letter read "alpha") of reporting a signal when no signal is present is given by

$$\alpha = \int_{x_c}^{\infty} e^{-x}\,dx \quad x \geq 0$$

The probability β (a Greek letter read "beta") of detecting a signal when it is present is

$$\beta = \int_{x_c}^{\infty} ke^{-kx}\,dx \quad x \geq 0$$

In both integrals, x_c is a constant (called a criterion value in this model). Find α and β if $k = \frac{1}{8}$.

17. Find the area of the region in the third quadrant bounded by the curve $y = e^{3x}$ and the x-axis.

18. Economics In discussing entrance of a firm into an industry, Stigler[11] uses the equation

$$V = \pi_0 \int_0^{\infty} e^{\theta t} e^{-\rho t}\,dt$$

where π_0, θ (a Greek letter read "theta"), and ρ (a Greek letter read "rho") are constants. Show that $V = \pi_0/(\rho - \theta)$ if $\theta < \rho$.

19. Population The predicted rate of growth per year of the population of a certain small city is given by

$$\frac{40,000}{(t+2)^2}$$

where t is the number of years from now. In the long run (that is, as $t \to \infty$), what is the expected change in population from today's level?

Chapter 15 Review

Important Terms and Symbols

Section	Terms	Examples
Section 15.1	**Integration by Tables**	
	present value and accumulated amount of a continuous annuity	Ex. 8, p. 670
Section 15.2	**Approximate Integration**	
	trapezoidal rule Simpson's rule	Ex. 2, p. 675
Section 15.3	**Area Between Curves**	
	vertical strip of area	Ex. 1, p. 679
	horizontal strip of area	Ex. 6, p. 683
Section 15.4	**Consumers' and Producers' Surplus**	
	consumers' surplus producers' surplus	Ex. 1, p. 688
Section 15.5	**Average Value of a Function**	
	average value of a function	Ex. 1, p. 691
Section 15.6	**Differential Equations**	
	first-order differential equation separation of variables	Ex. 1, p. 693
	exponential growth and decay decay constant half-life	Ex. 3, p. 696
Section 15.7	**More Applications of Differential Equations**	
	logistic function	Ex. 1, p. 701
	Newton's law of cooling	Ex. 3, p. 703
Section 15.8	**Improper Integrals**	
	improper integral convergent divergent	Ex. 1, p. 707
	$\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx$	Ex. 2, p. 707

Summary

An integral that does not have a familiar form may have been done by others and recorded in a table of integrals. However, it may be necessary to transform the given integral into an equivalent form before the matching can occur.

An annuity is a series of payments over a period of time. Suppose payments are made continuously for T years such that a payment at time t is at the rate of $f(t)$ per year. If the annual rate of interest is r compounded continuously, then the present value of the continuous annuity is given by

$$\boxed{A = \int_0^T f(t)e^{-rt}\,dt}$$

[10] D. Laming, *Mathematical Psychology* (New York: Academic Press, Inc., 1973).

[11] G. Stigler, *The Theory of Price,* 3rd ed. (New York: Macmillan Publishing Company, 1966), p. 344.

and the accumulated amount is given by

$$S = \int_0^T f(t)e^{r(T-t)}\,dt$$

If the integrand of a definite integral, $\int_a^b f(x)dx$, does not have an elementary antiderivative, or even if the antiderivative is merely daunting, the required number can be found, approximately, with either the Trapezoidal Rule:

$$\frac{h}{2}(f(a) + 2f(a+h) + \cdots + 2f(a+(n-1)h) + f(b))$$

or Simpson's Rule:

$$\frac{h}{3}(f(a)+4f(a+h)+2f(a+2h)+\cdots+4f(a+(n-1)h)+f(b))$$

where for both rules we have $= (b-a)/n$, but in the case of Simpson's Rule n must be even.

If $f(x) \geq 0$ is continuous on $[a, b]$, then the definite integral can be used to find the area of the region bounded by $y = f(x)$, the x-axis, $x = a$, and $x = b$. The definite integral can also be used to find areas of more complicated regions. In these situations, a strip of area should be drawn in the region. This allows us to set up the proper definite integral. In this regard, both vertical strips and horizontal strips have their uses.

One application of finding areas involves consumers' surplus and producers' surplus. Suppose the market for a product is at equilibrium and (q_0, p_0) is the equilibrium point (the point of intersection of the supply curve and the demand curve for the product). Then consumers' surplus, CS, corresponds to the area from $q = 0$ to $q = q_0$, bounded above by the demand curve and below by the line $p = p_0$. Thus,

$$\mathrm{CS} = \int_0^{q_0} (f(q) - p_0)\,dq$$

where f is the demand function. Producers' surplus, PS, corresponds to the area from $q = 0$ to $q = q_0$, bounded above by the line $p = p_0$ and below by the supply curve. Therefore,

$$\mathrm{PS} = \int_0^{q_0} (p_0 - g(q))\,dq$$

where g is the supply function.

The average value, $\bar{f}$, of a function, f, over the interval $[a, b]$ is given by

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

An equation that involves the derivative of an unknown function is called a differential equation. If the highest-order derivative that occurs is the first, the equation is called a first-order differential equation. Some first-order differential equations can be solved by the method of separation of variables. In that method, by considering the derivative to be a quotient of differentials, we rewrite the equation so that each side contains only one variable and a single differential in the numerator. Integrating both sides of the resulting equation gives the solution. This solution involves a constant of integration and is called the general solution of the differential equation. If the unknown function must satisfy the condition that it has a specific function value for a given value of the independent variable, then a particular solution can be found.

Differential equations arise when we know a relation involving the rate of change of a function. For example, if a quantity, N, at time t is such that it changes at a rate proportional to the amount present, then

$$\frac{dN}{dt} = kN, \qquad \text{where } k \text{ is a constant}$$

The solution of this differential equation is

$$N = N_0 e^{kt}$$

where N_0 is the quantity present at $t = 0$. The value of k may be determined when the value of N is known for a given value of t other than $t = 0$. If k is positive, then N follows an exponential law of growth; if k is negative, N follows an exponential law of decay. If N represents a quantity of a radioactive element, then

$$\frac{dN}{dt} = -\lambda N, \qquad \text{where } \lambda \text{ is a positive constant}$$

Thus, N follows an exponential law of decay, and hence,

$$N = N_0 e^{-\lambda t}$$

The constant λ is called the decay constant. The time for one-half of the element to decay is the half-life of the element:

$$\text{half-life} = \frac{\ln 2}{\lambda} \approx \frac{0.69315}{\lambda}$$

A quantity, N, may follow a rate of growth given by

$$\frac{dN}{dt} = KN(M - N), \qquad \text{where } K, M \text{ are constants}$$

Solving this differential equation gives a function of the form

$$N = \frac{M}{1 + be^{-ct}}, \qquad \text{where } b, c \text{ are constants}$$

which is called a logistic function. Many population sizes can be described by a logistic function. In this case, M represents the limit of the size of the population. A logistic function is also used in analyzing the spread of a rumor.

Newton's law of cooling states that the temperature, T, of a cooling body at time t changes at a rate proportional to the difference $T - a$, where a is the ambient temperature. Thus,

$$\frac{dT}{dt} = k(T - a), \qquad \text{where } k \text{ is a constant}$$

The solution of this differential equation can be used to determine, for example, the time at which a homicide was committed.

An integral of the form

$$\int_a^{\infty} f(x)\,dx \qquad \int_{-\infty}^{b} f(x)\,dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x)\,dx$$

is called an improper integral. The first two integrals are defined as follows:

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

and

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

If $\int_a^{\infty} f(x)dx$ $[\int_{-\infty}^{b} f(x)\,dx]$ is a finite number, we say that the integral is convergent; otherwise, it is divergent. The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined by

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$$

If both integrals on the right side are convergent, $\int_{-\infty}^{\infty} f(x)dx$ is said to be convergent; otherwise, it is divergent.

Review Problems

In Problems 1–18, determine the integrals.

1. $\int x^2 \ln x\,dx$

2. $\int \frac{1}{\sqrt{4x^2+1}}\,dx$

3. $\int_0^2 \sqrt{9x^2+16}\,dx$

4. $\int \frac{5x-13}{x-3}dx$

5. $\int \frac{15x-2}{(3x+1)(x-2)}\,dx$

6. $\int_{e^a}^{e^b} \frac{1}{x\ln x}\,dx$

7. $\int \frac{dx}{x(x+2)^2}$

8. $\int \frac{dx}{x^2-1}$

9. $\int \frac{dx}{x^2\sqrt{4-9x^2}}$

10. $\int x^3 \ln x^2\,dx$

11. $\int \frac{dx}{x^2-a^2}$

12. $\int \frac{x}{\sqrt{2+5x}}\,dx$

13. $\int 49xe^{7x}\,dx$

14. $\int \frac{dx}{5+2e^{3x}}$

15. $\int \frac{dx}{2x\ln x^2}$

16. $\int \frac{dx}{x(x+a)}$

17. $\int \frac{2x}{3+2x}\,dx$

18. $\int \frac{dx}{x^2\sqrt{4x^2-9}}$

In Problems 19–24, find the area of the region bounded by the given curves.

19. $y = -x(x-a)$, $\quad y = 0$ for $0 < a$

20. $y = 2x^2$, $\quad y = x^2 + 9$

21. $y = x^2 - x$, $\quad y = 10 - x^2$

22. $y = \sqrt{x}$, $\quad x = 0$, $\quad y = 3$

23. $y = \ln x$, $\quad x = 0$, $\quad y = 0$, $\quad y = 1$

24. $y = 3 - x$, $\quad y = x - 4$, $\quad y = 0$, $\quad y = 3$

25. Show that $\ln b = \int_1^b \frac{dx}{x}$. Use the trapezoidal rule, with $n = 8$ to approximate $\ln 2$. Express just those digits which agree with the true value of $\ln 2$.

26. Repeat Problem 25 using Simpson's rule with $n = 8$.

27. Consumers' and Producers' Surplus The demand equation for a product is

$$p = 0.01q^2 - 1.1q + 30$$

and the supply equation is

$$p = 0.01q^2 + 8$$

Determine consumers' surplus and producers' surplus when market equilibrium has been established.

28. Consumers' Surplus The demand equation for a product is

$$p = (q-4)^2$$

and the supply equation is

$$p = q^2 + q + 7$$

where p (in thousands of dollars) is the price per 100 units when q hundred units are demanded or supplied. Determine consumers' surplus under market equilibrium.

29. Find the average value of $f(x) = x^3 - 3x^2 + 2x + 1$ over the interval $[0, 5]$.

30. Find the average value of $f(t) = t^2e^t$ over the interval $[0, 1]$.

In Problems 31 and 32, solve the differential equations.

31. $y' = 3x^2y + 2xy \qquad y > 0$

32. $y' - f'(x)e^{f(x)-y} = 0 \qquad y(0) = f(0)$

In Problems 33–36, determine the improper integrals if they exist.

33. $\int_1^{\infty} \frac{1}{x^{2.5}}\,dx$

34. $\int_{-\infty}^{0} e^{3x}dx$

35. $\int_1^{\infty} \frac{1}{2x}\,dx$

36. $\int_{-\infty}^{\infty} xe^{1-x^2}\,dx$

37. Population The population of a fast-growing city was 500,000 in 1980 and 1,000,00 in 2000. Assuming exponential growth, project the population in 2020.

38. Population The population of a city doubles every 10 years due to exponential growth. At a certain time, the population is 40,000. Find an expression for the number of people, N, at time t years later. Give your answer in terms of $\ln 2$.

39. Radioactive If 98% of a radioactive substance remains after 1000 years, find the decay constant, and, to the nearest percent, give the percentage of the original amount present after 5000 years.

40. Medicine Suppose q is the amount of penicillin in the body at time t, and let q_0 be the amount at $t = 0$. Assume that the rate of change of q with respect to t is proportional to q and that q decreases as t increases. Then we have $dq/dt = -kq$, where $k > 0$. Solve for q. What percentage of the original amount present is there when $t = 7/k$?

41. Biology Two organisms are initially placed in a medium and begin to multiply. The number, N, of organisms that are present after t days is recorded on a graph with the horizontal axis labeled t and the vertical axis labeled N. It is observed that the points lie on a logistic curve. The number of organisms present after 6 days is 300, and beyond 10 days the number approaches a limit of 450. Find the logistic equation.

42. College Enrollment A university believes that its enrollment follows logistic growth. Last year enrollment was 10,000, and this year it is 11,000. If the university can accommodate a maximum of 20,000 students, what is the anticipated enrollment next year?

43. Time of Murder A coroner is called in on a murder case. He arrives at 6:00 P.M. and finds that the victim's temperature is 35°C. One hour later the body temperature is 34°C. The temperature of the room is 25°C. About what time was the murder committed? (Assume that normal body temperature is 37°C.)

44. Annuity Find the present value, to the nearest dollar, of a continuous annuity at an annual rate of 5% for 10 years if the payment at time t is at the annual rate of $f(t) = 100t$ dollars.

45. Hospital Discharges For a group of hospitalized individuals, suppose the proportion that has been discharged at the end of t days is given by

$$\int_0^t f(x)\,dx$$

where $f(x) = 0.007e^{-0.01x} + 0.00005e^{-0.0002x}$. Evaluate

$$\int_0^\infty f(x)\,dx$$

46. Integration by Parts Let f and g be differentiable functions. Show that if either $f'g$ or fg' has an antiderivative then the other one does. It suffices to show it in one case, so for definiteness, assume that $H'(x) = f'(x)g(x)$ (equivalently $\int f'(x)g(x)dx = H(x) + C$) and show that $\int f(x)g'(x) = f(x)g(x) - H(c) + C$. This is often written as

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Writing $u = f(x)$ and $v = g(x)$ we have, equivalently,

$$\int udv = uv - \int vdu$$

47. Product Consumption Suppose that $A(t)$ is the amount of a product that is consumed at time t and that A follows an exponential law of growth. If $t_1 < t_2$ and at time t_2 the amount consumed, $A(t_2)$, is double the amount consumed at time t_1, $A(t_1)$, then $t_2 - t_1$ is called a doubling period. In a discussion of exponential growth, Shonle[12] states that under exponential growth, "the amount of a product consumed during one doubling period is equal to the total used for all time up to the beginning of the doubling period in question." To justify this statement, reproduce his argument as follows. The amount of the product used up to time t_1 is given by

$$\int_{-\infty}^{t_1} A_0e^{kt}\,dt \quad k > 0$$

where A_0 is the amount when $t = 0$. Show that this is equal to $(A_0/k)e^{kt_1}$. Next, the amount used during the time interval from t_1 to t_2 is

$$\int_{t_1}^{t_2} A_0e^{kt}\,dt$$

Show that this is equal to

$$\frac{A_0}{k}e^{kt_1}[e^{k(t_2-t_1)} - 1] \tag{5}$$

If the interval $[t_1, t_2]$ is a doubling period, then

$$A_0e^{kt_2} = 2A_0e^{kt_1}$$

Show that this relationship implies that $e^{k(t_2-t_1)} = 2$. Substitute this value into Equation (5); your result should be the same as the total used during all time up to t_1, namely, $(A_0/k)e^{kt_1}$.

48. Revenue, Cost, and Profit The following table gives values of a company's marginal-revenue (MR) and marginal-cost (MC) functions:

q	0	3	6	9	12	15	18
MR	25	22	18	13	7	3	0
MC	15	14	12	10	7	4	2

The company's fixed cost is 25. Assume that profit is a maximum when MR = MC and that this occurs when $q = 12$. Moreover, assume that the output of the company is chosen to maximize the profit. Use the trapezoidal rule and Simpson's rule for each of the following parts.

(a) Estimate the total revenue by using as many data values as possible.

(b) Estimate the total cost by using as few data values as possible.

(c) Determine how the maximum profit is related to the area enclosed by the line $q = 0$ and the MR and MC curves, and use this relation to estimate the maximum profit as accurately as possible.

[12] J. I. Shonle, *Environmental Applications of General Physics* (Reading, MA: Addison-Wesley Publishing Company, Inc., 1975).