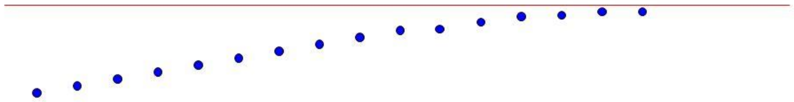
Supplementary Slides

Proof of The Convergence of $a_n = \left(1 + \frac{1}{n}\right)^n$

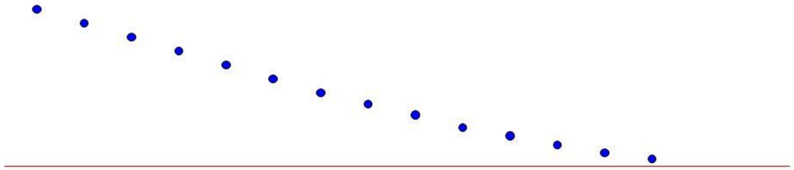
Monotone Convergence Theorem (MCT)

Monotone Convergence Theorem (MCT): If a sequence $\{a_n\}$ is monotone and bounded, it is convergent (i.e., it has a limit).

If a sequence $\{a_n\}$ is bounded above and non-decreasing, it has a limit



If a sequence $\{a_n\}$ is bounded below and non-increasing, it has a limit



Proof of The Convergence of $a_n = \left(1+\frac{1}{n}\right)^n$

Consider the sequence $\{a_n\}_{n\geq 1}$ where $a_n = \left(1+\frac{1}{n}\right)^n$. $a_1 = 2, a_2 = 2.25, a_3 = 2.3, \ldots$. To show that a_n is convergent, we use the MTC. First, we show that a_n is increasing in n:

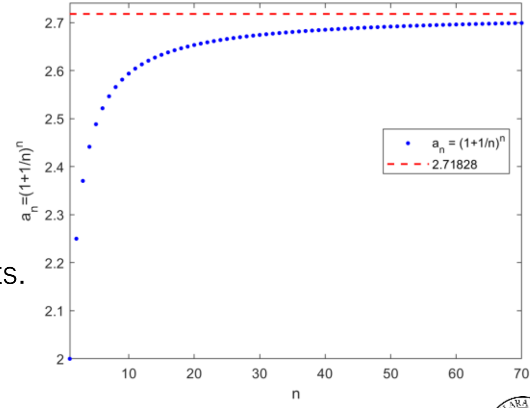
- $\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^{n}\left(\frac{n+2}{n+1}\right)}{\left(\frac{n+1}{n}\right)^n} = \left(\frac{n+2}{n+1}\frac{n}{n+1}\right)^n\left(\frac{n+2}{n+1}\right) = \left(\frac{(n+1)^2-1}{(n+1)^2}\right)^n\left(\frac{n+2}{n+1}\right) = \left(1-\frac{1}{(n+1)^2}\right)^n\left(\frac{n+2}{n+1}\right) \geq \left(1-\frac{n}{(n+1)^2}\right)\left(\frac{n+2}{n+1}\right) = \frac{((n+1)^2-n)(n+2)}{(n+1)^3} \geq \frac{(n+1)^2\,(n+1)}{(n+1)^3} = 1,$

in which we used the identity: $(1+x)^n \geq 1+nx \ (x \geq -1)$ (the proof is by induction)

Next, we show that a_n is bounded:

- $|a_n| = \left|\left(1+\frac{1}{n}\right)^n\right| = \sum_{i=0}^{n}\binom{n}{i}\left(\frac{1}{n}\right)^i = 2 + \sum_{i=2}^{n}\frac{n(n-1)\ldots(n-i+1)}{n^i} \times \frac{1}{i!} \leq 3$

Hence, by the MCT a_n converges to some constant as $n \to \infty$. That is, $e = \lim\limits_{n\to\infty} a_n$ exists.



Properties of Exponential Functions for General Real Numbers

Rationale for the Proof

We already know that $b^x b^y = b^{x+y}$ holds for integers x and y. The idea is to extend this intuition to real numbers by using limits. We first show that the idea can be extended to any two rational numbers, and then extend it to any two real numbers by using the property that any real number is the limit of a sequence of rational numbers.

1. Case for Rational Numbers

Rational numbers can be expressed in terms of fractions. Let, $x = \frac{m}{n}, y = \frac{p}{q}$ where m, n, p, q are integer values. we can prove $b^x b^y = b^{x+y}$ based on the integer case:

$$b^{\left(\frac{m}{n}\right)} = (b^m)^{\frac{1}{n}}$$

Then using properties for integer exponents, we can write:

$$b^x b^y = b^{\frac{m}{n}} b^{\frac{p}{q}} = b^{\frac{mq}{nq}} b^{\frac{pn}{nq}} = (b^{mq})^{\frac{1}{nq}} (b^{pn})^{\frac{1}{nq}} = (b^{mq} b^{pn})^{\frac{1}{nq}} = (b^{mq+pn})^{\frac{1}{nq}} = b^{\left(\frac{mq}{nq}+\frac{pn}{nq}\right)} = b^{\frac{m}{n}+\frac{p}{q}} = b^{x+y}$$

2. Case for Real Numbers as Limits of Rational Numbers

Any real number can be approximated by a sequence of rational numbers. Let x_n, y_n be sequences of rational numbers such that $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y$. By continuity of the exponential function, we can then write:

$$b^x = b^{\lim_{n\to\infty} x_n} = \lim_{n\to\infty} b^{x_n}$$

$$b^y = b^{\lim_{n\to\infty} y_n} = \lim_{n\to\infty} b^{y_n}$$

Therefore:

$$b^x b^y = \left(\lim_{n\to\infty} b^{x_n}\right)\left(\lim_{n\to\infty} b^{y_n}\right) = \lim_{n\to\infty} \left(b^{x_n} b^{y_n}\right) = \lim_{n\to\infty} \left(b^{x_n+y_n}\right) = b^{\left(\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n\right)} = b^{x+y}$$

and the proof is complete. ■

L'Hôpital’s Rule for Fractional Limits

L'Hôpital's Rule

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Then:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Proof:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\frac{f(x)-0}{x-a}}{\frac{g(x)-0}{x-a}} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example:

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$$

With L'Hopital:

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{2x}{1} = 2$$

More on R Programming

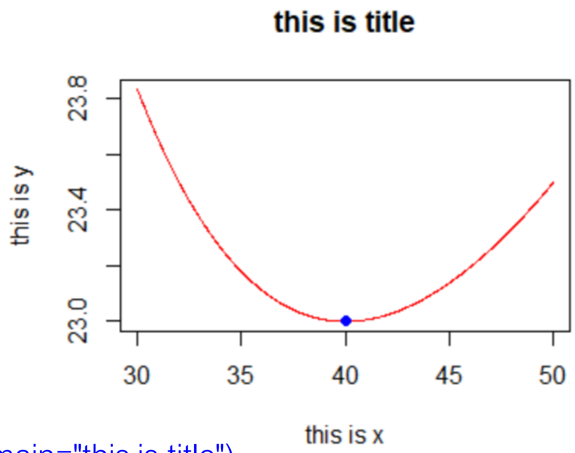
Optimizing a Function in R

The following code finds a local minimum value of a one-variable function within a given interval [a, b]:

optimize(f,c(a,b)) or: optimize(f,lower=a,upper=b)

Example:

```
f=function(q) q/4+3+400/q
y=optimize(f,c(0,100))
print (y)
$minimum
[1] 40.00001
$objective
[1] 23
plot(f,from=30,to=50,n=1000,col="red",xlab="this is x",ylab="this is y",main="this is title")
points(y$minimum,y$objective,col="blue",type="o",pch=19)
```



Plotting a Function in R

plot(f, from = 0, to = 1, n = 101, col="red", lwd=1, lty=1)

- f: function
- from, to: the interval for the graph
- n: the number of points to evaluate betweee “from” and “to” values.
- lty, lwd: set line types and thickness

Example:

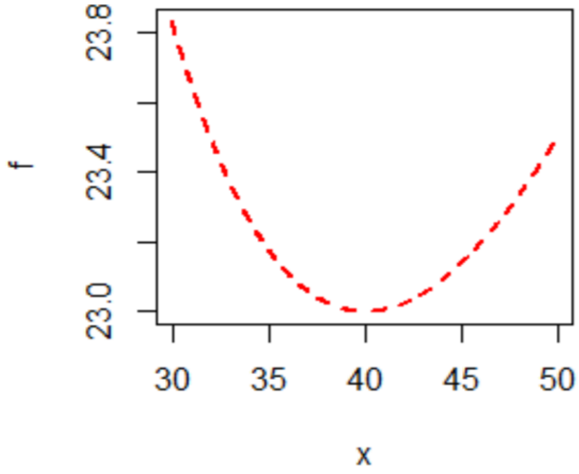
```
f=function(q) q/4+3+400/q
plot(f,from=30,to=50,n=1000,col="red“)
```

or just type:

```
plot(f,30,50,col="red")
```

You can also determine the line type and thickness:

```
plot(f,from=30,to=50,n=1000,col="red",lwd=2, lty = 8)
```



Extra Examples
(Strongly Recommended for Exam)

Example

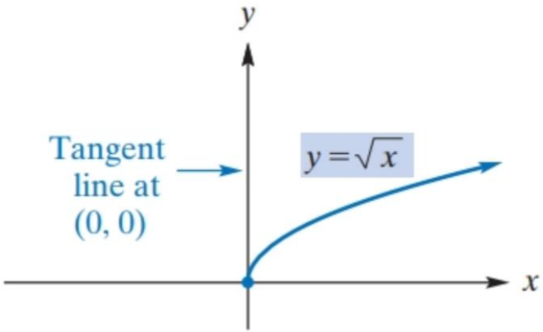
Find $\frac{d}{dx}\left(\sqrt{x}\right)$.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{2x}}$$

Remark: $\sqrt{x}$ is not differentiable at (0,0). It is defined only when $x > 0$.



Example

Suppose that $\$500$ amounted to $\$588.38$ in a savings account after three years. If interest was compounded semiannually, find the nominal rate of interest, compounded semiannually, that was earned by the money.

Solution:

The semiannual rate was 2.75%, so the nominal rate was 5.5% compounded semiannually.

$$588.38 = 500(1+\frac{r}{2})^6 \Rightarrow \frac{r}{2} = \sqrt[6]{\frac{588.38}{500}} - 1 \Rightarrow \frac{r}{2} = 0.0275 \Rightarrow r = 0.055$$

Example

How long will it take for $\$600$ to amount to $\$900$ at an annual rate of 6% compounded quarterly?

Solution

$$S = P\left(1 + \frac{r}{k}\right)^{ky} \Rightarrow 900 = 600\left(1 + \frac{0.06}{4}\right)^{4y}$$

$$\Rightarrow (1.015)^{4y} = 1.5$$

$$\Rightarrow \ln(1.015)^{4y} = \ln(1.5)$$

$$\Rightarrow (4y)\ln 1.015 = \ln(1.5)$$

$$\Rightarrow y = \frac{1}{4}\frac{\ln(1.5)}{\ln(1.015)} \approx 6.8083 \text{ years}$$

$$= 6 \text{ years}, 9\frac{1}{2} \text{ months}$$

Second Derivative Tests

Example: Test the following for relative maxima and minima for: $y = 18x - \frac{2}{3}x^3$.

Solution

$y' = 2(3 + x)(3 - x)$

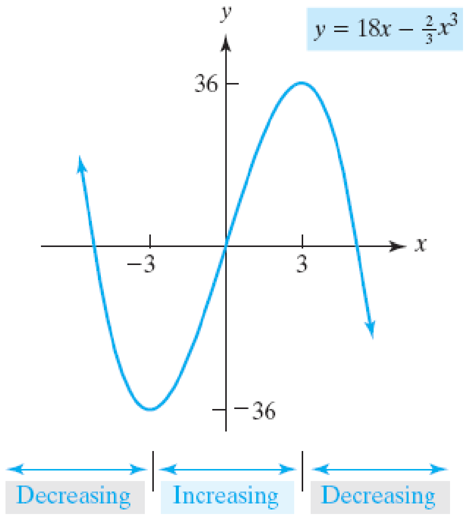
$y'' = -4x$

When $y' = 0,$ we have $x = \pm 3$

When $x = +3,$ $\quad y'' = -4(3) = -12 < 0$

When $x = -3,$ $\quad y'' = -4(-3) = 12 > 0$

Relative minimum when $x = -3$.



Example: Minimizing Average Costs

A manufacturer's total-cost function is given by $c = c(q) = \frac{q^2}{4} + 3q + 400$ where c is the total cost of producing q units. At what level of output will average cost per unit be a minimum? What is this minimum?

Solution

Average-cost function is

$$\bar{c} = \bar{c}(q) = \frac{c}{q} = \frac{\frac{q^2}{4} + 3q + 400}{q} = \frac{q}{4} + 3 + \frac{400}{q}$$

To find critical values, we set

$$\frac{d\bar{c}}{dq} = 0 = \frac{q^2 - 1600}{4q^2} \Rightarrow q = 40 \text{ since } q > 0$$

$\frac{d^2\bar{c}}{dq^2} = \frac{800}{q^3}$ is positive when $q = 40$, which is the only relative extremum. The minimum average cost is

$$\bar{c}(40) = \frac{40}{4} + 3 + \frac{400}{40} = 23$$