Module 7: Supplementary Slides

Continuous Distributions:
Normal Distribution: Extra Details

The Normal Distribution

Remark:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}, \qquad \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$$

Note that the probability density function integrates to 1:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi} = 1$$

The Normal Mean and Variance

If X is a normal random variable with parameters μ and σ then

$$E(X) = \mu, \qquad var(X) = \sigma^2$$

Proof:

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_{-\infty}^{+\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{+\infty} \frac{(\mu + \sigma z)}{\sqrt{\pi}} e^{-\frac{1}{2}z^2} dz = \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z e^{-\frac{1}{2}z^2} dz$$

$$= \frac{\mu}{\sqrt{2\pi}} \sqrt{2\pi} + 0 = \mu$$

$$var(X) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x) dx = \int_{-\infty}^{+\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{+\infty} \frac{\sigma^2 z^2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

Integrating by parts ($u = z,\ dv = z e^{-\frac{1}{2}z^2} dz$) gives

$$var(X) = \frac{\sigma^2}{\sqrt{2\pi}} \left(z e^{-\frac{1}{2}z^2} \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2} dz \right) = \frac{\sigma^2}{\sqrt{2\pi}} \left(0 + \sqrt{2\pi}\right) = \sigma^2$$

The Normal Distribution

Remark: If X is normally distributed with parameters μ and σ then $Y = \alpha X + \beta$ is normally distributed with parameters $\alpha\mu + \beta$ and $\alpha\sigma$.

Proof:

Suppose $\alpha > 0$ (showing for $\alpha < 0$ is similar). Then the cumulative distribution of the random variable Y is:

$$F_Y(a) = P(\alpha X + \beta \leq a) = P\left(X \leq \frac{a-\beta}{\alpha}\right)$$

$$= \int_{-\infty}^{\frac{a-\beta}{\alpha}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad \text{(Taking } y = \alpha x + \beta\text{)}$$

$$= \int_{-\infty}^{\alpha} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\alpha\mu-\beta}{\alpha\sigma}\right)^2} dy$$

Implication: If X is normally distributed with parameters μ and σ then $Z = \frac{X-\mu}{\sigma}$ is normally distributed with parameters 0 and 1. Such a random variable is said to have the standard normal distribution.

Continuous Distributions:
Gamma Distribution

The Gamma Distribution

We say that X is a gamma random variable with parameters $\lambda > 0$ and $\alpha > 0$ if its probability density function is

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & \text{, if } x \geq 0 \\ 0 & \text{, if } x < 0 \end{cases}$$

The quantity $\Gamma(\alpha)$ is called the gamma function and is defined by:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

Remark:

$$\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

Remark: It is easy to show that when $\alpha = n$ then

$$\Gamma(\alpha) = (\mathrm{n}-1)!$$

Note that the probability density function integrates to 1:

$$\int_{-\infty}^{+\infty} f(x) dx = \int_0^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx = \frac{\Gamma(\alpha)\lambda^\alpha}{\Gamma(\alpha)\lambda^\alpha} = 1$$

The Gamma Mean and Variance

If X is an exponential random variable with parameter λ then

$$E(X) = \frac{\alpha}{\lambda}, \qquad var(X) = \frac{\alpha}{\lambda^2}$$

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_0^{\infty} x \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx = \frac{\alpha}{\lambda} \underbrace{\int_0^{\infty} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha}}{\Gamma(\alpha+1)} dx}_{1} = \frac{\alpha}{\lambda}$$

$$E\left(X^2\right) = \int_0^{\infty} x^2 \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} dx = = \frac{\alpha(\alpha+1)}{\lambda^2} \underbrace{\int_0^{\infty} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha+1}}{\Gamma(\alpha+2)} dx}_{1} = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$var(X) = E\left(X^2\right) - E(X)^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$