Mathematics for Business Analytics and Finance

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MSIS2402/2502

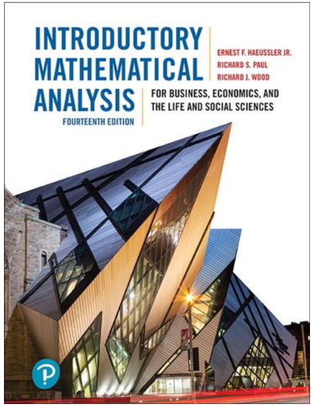
Module 1

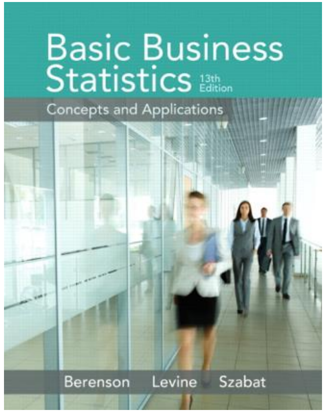
Introduction

- Sami Najafi
  - MS, Industrial Engineering
    - Sharif University of Technology, Tehran, Iran
  - MS and PhD, Management Science and Operations
    - London Business School (LBS), London, UK
  - Postdoctoral Fellow in Operations Management
    - University of Toronto , Toronto, Canada
  - Associate Professor of Operations Management at Leavey
  - Research interests:
    - Stochastic Modelling, Optimization
    - Pricing and Revenue Management, Service Operations
    - Operations-Marketing interface: Online Advertising

Course Material

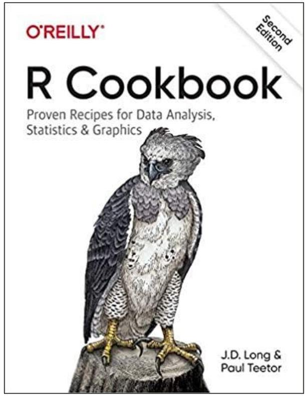
- Lecture slides, excel files, and all other teaching materials:
  - Posted on the course page
- Recommended Optional Textbooks:
  - Modules 1-4: Introductory mathematical analysis for business, economics and the life and social sciences (14th Ed.)
    - By by Ernest F Haeussler, Richard S. Paul, Richard J. Wood. Pearson (July 24th 2018)
  - Modules 5-9: Basic business statistics: Concepts and applications. Pearson higher education (13th Ed.).
    - Berenson, M., Levine, D., Szabat, K. A., & Krehbiel, T. C. (2014). Pearson (2014)





Course Material

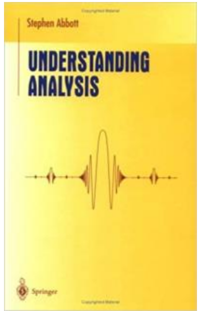
- Recommended book for R:
  - R Cookbook: Proven Recipes for Data Analysis, Statistics, and Graphics, 2nd edition
    - By J.D. Long and Paul Teetor (2019)

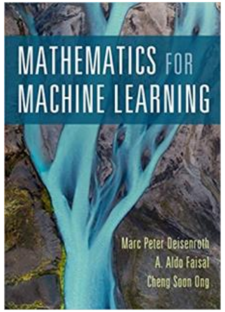


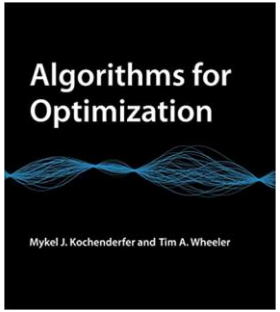
Additional References

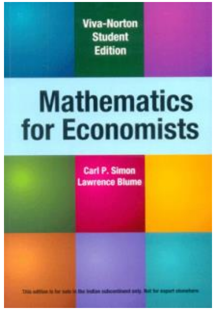
For Calculus, Matrix Algebra, and Optimization

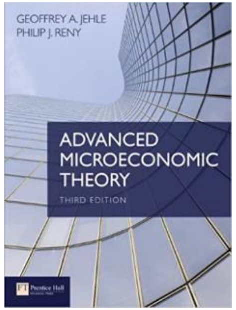
- Abbott, Stephen. *Understanding analysis*. Vol. 2. New York: Springer, 2001.
- Deisenroth, Marc Peter, A. Aldo Faisal, and Cheng Soon Ong. *Mathematics for machine learning*. Cambridge University Press, 2020.
- Kochenderfer, Mykel J., and Tim A. Wheeler. *Algorithms for optimization*. Mit Press, 2019.
- Simon Carl P, and Lawrence Blume. *Mathematics for Economists,* Norton & Company, Inc, 2010
- Jehle, Geoffrey, Philip Reny. *Advanced microeconomic theory*. Pearson, 2010.







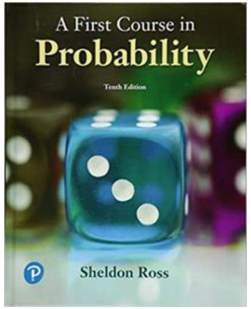


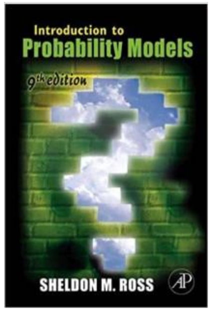


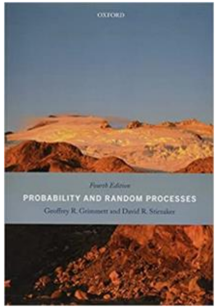
Additional References

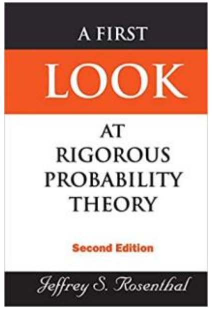
For Probability

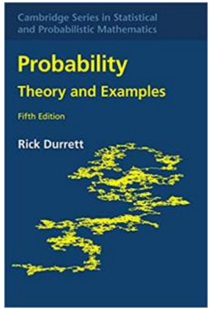
- Ross, Sheldon M. *A first course in probability*. Boston: Pearson, 2019.
- Ross, Sheldon M. *Introduction to probability models*. Academic press, 2014.
- Grimmett, Geoffrey, and David Stirzaker. *Probability and random processes*. Oxford university press, 2020.
- Durret, R. *Probability: Theory and Examples*, Cambridge University Press, 2020.
- Rosenthal, Jeffrey S. *First Look At Rigorous Probability Theory, A*. World Scientific Publishing Company, 2006.







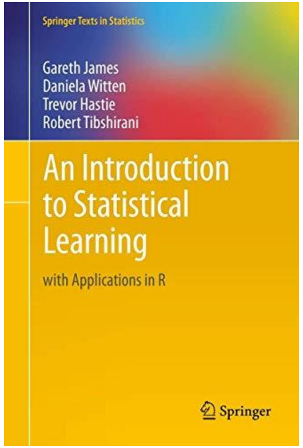


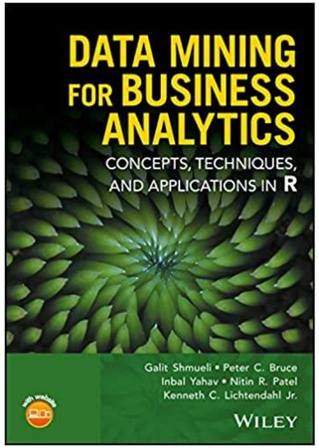


Additional References

For More Coding in R Especially for Data Science

- James, G., Witten, D., Hastie, T., & Tibshirani, R. (2021). An Introduction to Statistical Learning: with Applications in R (Springer Texts in Statistics) 2nd ed.
- Shmueli, Galit, et al. *Data mining for business analytics: concepts, techniques, and applications in R*. John Wiley & Sons, 2017.





Course Grading and Expectations

Fair grading, and communicating clearly the expectations of the course

Grading Breakdown:

• Attendance	10%
• Weekly Assignments	30%
• Group Project	20%
• Midterm Exam	20%
• Final Exam	20%

Attendance

- Accounts for 10% of your final grade.

Missed Classes:

- Absences will reduce your grade proportionally.
- Exceptions: Documented medical or extenuating circumstances.

Grace Period & Late Policy:

- 15-minute grace period allowed
- Arriving after grace period = “Late”
- 3 “Late” = 1 Absence
- Exceptions: Documented extenuating circumstances

Weekly Assignments

- Accounts for 30% of your final grade; 14 problem sets

Scoring:

- Attempt each set twice; higher score is retained
- Submit detailed solutions for partial credit

Due Dates:

- End of the day on Sundays

Late Submission Policy

- Daily Penalty:
    - 10% per day after due date
- Exceptions:
    - Medical emergencies
    - Extenuating circumstances
    - Subject to my discretion

Midterm/Final Exam

- Each accounts for 20% of your final grade

Coverage:

- Midterm: Modules 1-4
- Final: Modules 5-8

Format:

- In-class, closed-book, written
- Questions similar to weekly assignments

Permitted Materials:

- Handwritten notes (a maximum of 2 Letter-sized sheets)
- One calculator (any type is ok)

Not Permitted:

- Other electronic devices (laptops, cellphones, smartwatch, etc)
- Internet access

Group Project

- Constitutes 20% of your final grade.

Topic and Format:

- Focus on optimizing support staffing decisions at Tesla to minimize operational costs.
- Use R programming to solve this optimization problem.
- Teams may consist of 2 to 4 students.
- The case study story is not real but it simulates real-world complexities.

Learning Objectives: Tools and Techniques

Students will acquire proficiency in the following key areas:

- Single variable calculus
- Matrix algebra
- Multivariable calculus and optimization
- Probability and statistics

These competencies are essential for a comprehensive understanding of quantitative topics in:

- Analytics
- Finance
- Marketing
- Economics

Course Outline

Module	Topics Covered	Sessions (approx.)	Suggested Readings
1	Exponential and Logarithmic Functions, Differentiation, and Optimization	**1,2,3,4,5**	Haeussler: Chapters 4, 11, 12, 13
2	Integration and Its Applications	**5,6,7**	Haeussler: Chapters 14, 15
3	Matrix Algebra	**7,8,9**	Haeussler: Chapter 6
4	Multivariate Calculus and Optimization	**9,10,11,12**	Haeussler: Chapter 17
5	Basic Probability Concepts	**13,14**	Brenson: Chapter 4 (Sections 4.1-4.3)
Midterm Exam	Covers Modules 1- 4	**14 or 15**	Haeussler: Chapter 8
6	Discrete Random Variables and Probability Distributions	**16,17**	Brenson: Chapter 5
Group Project Assigned	—	**14 or 15**	Haeussler: Chapter 9 (Sections 9.1-9.2)
7	Continuous Random Variables and Distributions	**18, 19**	Brenson: Chapter 6; Haeussler: Chapter 16
8	Sampling Distributions and Confidence Intervals / Workshop - Tesla	**19, 20**	Brenson: Chapters 7, 8
Final Exam	Covers Modules 5 – 8 (Module 8 only if covered instead of workshop otherwise it will be optional)	**Finals week**	—

Key Reminders on Functions and Limits

Functions

Definition: Given two sets A and B, a function from A to B is a rule or mapping that takes each element $x \in A$ and associates it with a single element in B. In this case, we write $f: A \to B$. The set A is called domain and the subset of B that includes all possible outcomes is called range.

Example: Demand function: $D(p) = \frac{100}{p}$

Equal Functions: Functions f and g are equal $(f = g)$ if:

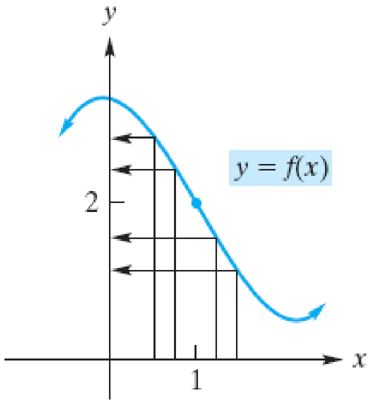
- Domain of f = Domain of g
- $f(x) = g(x)$.

Example: Determine if $f(x) = \frac{(x+2)(x-1)}{(x-1)}$ and $g(x) = x + 2$ are equal.

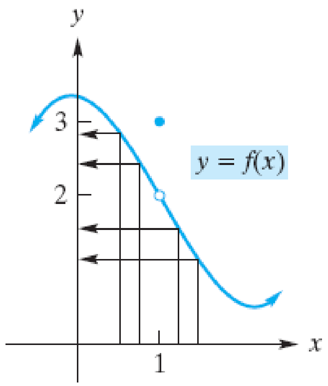
Limits

The limit of $f(x)$ as x approaches a is the value L that $f(x)$ approaches. We write this as $\lim_{x \to a} f(x) = L$. This means that as x gets closer and closer to a, $f(x)$ gets closer and closer to the number L.

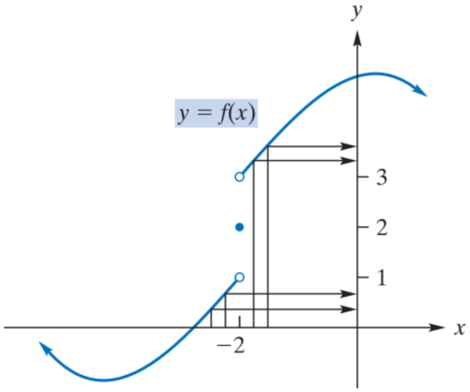
Example: Estimate $\lim_{x \to 1} f(x)$ from the graphs (a) and (b)



(a)



(b)



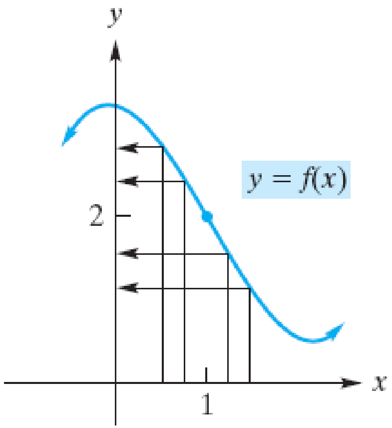
(c)

Continuity

$f(x)$ is said to be continuous at $x = a$ if the following three conditions are satisfied:

1. $f(a)$ is defined at $x = a$
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

(a)



(b)

Properties of Limits

During some of the proofs throughout the course we will use the following properties of limits:

Theorem 1 (Algebra of limits Theorem): Let f and g be real functions whose sum, difference, product, and quotient are defined, and suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

- $\lim_{x\to a}(f \pm g)(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
- $\lim_{x\to a}(f/g)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided that $\lim_{x\to a} g(x) \neq 0$

Properties of Limits

Theorem 2 (Composite Function Theorem): Let f and g be real functions. If $\lim_{x \to a} g(x)$ exists and f is continuous at $x = a$, then:

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Theorem 3 (Squeeze Theorem): Suppose that $g(x) \leq f(x) \leq h(x)$ for all x near a. If $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then:

$$\lim_{x \to a} f(x) = L$$

A Brief Guide to Programming in R

Entering Commands and Getting Help

Using R as A Calculator

```
> 1+1
[1] 2
> max(1,3,5)
[1] 5
> pi
[1] 3.141593
> sqrt(2)
[1] 1.414214
```

To know more about a function:

```
> help("mean") or  > ?mean
> example(mean)
```

Printing Something

Print

```
print(x,number of digits)
Print("expression")
```

```
> print(sqrt(2),4)
[1] 1.414
```

The only way to print multiple items is to print them one at a time

```
> print("The zero occurs at"); print(2*pi); print("radians")
[1] "The zero occurs at"
[1] 6.283185
[1] "radians"
```

Printing Something

Concatenate

cat(num1/expr1, num2/expr2,…) combines multiple items into a continuous output

```
> cat("The zero occurs at", 2*pi, "radians.", "\n")
The zero occurs at 6.283185 radians.

> fib = c(0,1,1,2,3,5,8,13,21,34)
> cat("The first few Fibonacci numbers are:", fib, "...\n")
The first few Fibonacci numbers are: 0 1 1 2 3 5 8 13 21 34 ...

> iter=1
> cat("iteration = ", iter = iter + 1, "\n")
iteration = 2
```

Variables

Defining Variables

Assignment operators: <- , <<- , = , -> , ->>.

```
> x <- 3
> z <- sqrt(x^2)
> print(z)
[1] 9
```

```
> f = 3
> print(f)
[1] 3
```

```
> 5 -> u
> print(u)
[1] 5
```

Variables

Deleting a Variable

`rm(x)` removes the variable

```
> x <- 2*pi
> x
[1] 6.283185
> rm(x)
> x
Error: object "x" not found
```

Functions

Defining a Function

```
function(param1, ...., paramN) expr

function(param1, ..., paramN) {
expr1
.
.
.
exprM
}
```

```
> cv = function(x) sd(x)/mean(x)
> cv(1:10)
[1] 0.5504819

s=function(n){
  if(n>=5) return (2*n)
  if((n>=0)&&(n<5))return (n)
  else return (3*n)
}
> s(3)
[1] 3
```

Exponential Functions

Exponential Functions

The function f defined by

$$f(x) = b^x, \quad b > 0, b \neq 1,$$

and the exponent x is any real number, is called an *exponential function* with base $\boldsymbol{b}$.

Properties of Exponential Functions

- $b^x b^y = b^{x+y}$
- $\frac{b^x}{b^y} = b^{x-y}$
- $b^0 = 1$
- $(b^x)^y = b^{xy}$
- $(bc)^x = b^x\, c^x$
- $\left(\frac{b}{c}\right)^x = \frac{b^x}{c^x}$
- $b^{-x} = \frac{1}{b^x}$

Example

The number of an app users after t units of time is given by $N(t) = 300\left(\frac{4}{3}\right)^t$

a. How many users are using the app initially?

b. Approximately how many users will use the app after 3 units of time?

Solution:

$$\mathrm{N}(0) = 300\left(\frac{4}{3}\right)^0 = 300(1) = 300$$

$$\mathrm{N}(3) = 300\left(\frac{4}{3}\right)^3 = 300\left(\frac{64}{27}\right) = \frac{6400}{9} \approx 711$$

Compound Interest

The compound amount S of the principal P at the end of n interest-periods at the rate of r compounded per period is

$$\boxed{S = P(1+r)^n}$$

Example: Suppose $\$1000$ is invested for 10 years at 6% compounded annually.

a) Find the compound amount.

b) Find the compound interest.

Solution:

a) $S = P(1+r)^n = 1000(1+0.06)^{10} = \$1790.85$

b) $\text{Interest} = S - P = 1790.85 - 1000 = \$790.85$

Compound Interest

The compound amount S of the principal P at the end of y years compounded k times a year at the rate of r is:

$$\boxed{S = P\left(1 + \frac{r}{k}\right)^{yk}}$$

Example: Suppose the principal of $\$1000$ is invested for 10 years at annual interest rate 6% compounded quarterly.

a) Find the compound amount.

b) Find the compound interest.

Solution

The annual (nominal) interest rate is $\mathrm{r} = 6\%$. So, the interest rate per interest period (quarter) would be $\frac{r}{k} = \frac{6\%}{4} = 1.5\%$. The number of interest periods also will be $n = yk = 10(4) = 40$ periods. So, we have:

a) $S = P(1 + \frac{r}{k})^{yk} = 1000(1 + 0.015)^{40} = \$1814.02$

b) $\text{Interest} = S - P = 1814.02 - 1000 = \$814.02$

Compound Amount under Continuous Interest

The Euler Number (e)

- Suppose that a single dollar is invested for one year with an APR of 100% compounded annually.

$$S = (1)(1+1)^1 = \$2$$

- Without changing any of the other data, we now consider the effect of increasing the number of interest periods per year. If there are n interest periods per year, then the compound amount is given by

$$S = \ (1+\frac{1}{n})^n = (\frac{n+1}{n})^n$$

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n \approx 2.718$$

- e is the most important base. A function with base e is called a natural exponential function

Approximations of e

n	$\left(\frac{n+1}{n}\right)^n$
1	$\left(\frac{2}{1}\right)^1 = 2.00000$
2	$\left(\frac{3}{2}\right)^2 = 2.25000$
3	$\left(\frac{4}{3}\right)^3 \approx 2.37037$
4	$\left(\frac{5}{4}\right)^4 \approx 2.44141$
5	$\left(\frac{6}{5}\right)^5 = 2.48832$
10	$\left(\frac{11}{10}\right)^{10} \approx 2.59374$
100	$\left(\frac{101}{100}\right)^{100} \approx 2.70481$
1000	$\left(\frac{1001}{1000}\right)^{1000} \approx 2.71692$
10,000	$\left(\frac{10{,}001}{10{,}000}\right)^{10{,}000} \approx 2.71815$
100,000	$\left(\frac{100{,}001}{100{,}000}\right)^{100{,}000} \approx 2.71827$
1,000,000	$\left(\frac{1{,}000{,}001}{1{,}000{,}000}\right)^{1{,}000{,}000} \approx 2.71828$

In R: $e^x = \exp(\mathrm{x})$

Compound Amount under Continuous Interest

If P is the principal and r is the annual interest rate, then the amount after y years compounded continuously is:

$$S = Pe^{ry}$$

In addition, the effective rate corresponding to an annual rate of r compounded continuously is

$$r_e = e^{ry} - 1$$

Compound Amount under Continuous Interest

$$S = \lim_{k\to\infty} P\left(1+\frac{r}{k}\right)^{ky} = \lim_{k\to\infty} P\left(1+\frac{1}{\left(\frac{k}{r}\right)}\right)^{\left(\frac{k}{r}\right)ry} = \lim_{k\to\infty} P\left((1+\frac{1}{\left(\frac{k}{r}\right)})^{\left(\frac{k}{r}\right)}\right)^{ry}$$ (Property of the exponential function)

$$= P\lim_{k\to\infty}\left((1+\frac{1}{\left(\frac{k}{r}\right)})^{\left(\frac{k}{r}\right)}\right)^{ry}$$ (Algebraic Limit Theorem)

$$= P\left(\lim_{k\to\infty}(1+\frac{1}{\left(\frac{k}{r}\right)})^{\left(\frac{k}{r}\right)}\right)^{ry}$$ (Composite limit theorem)

$$= P\left(\lim_{N\to\infty}(1+\frac{1}{N})^{N}\right)^{ry} = pe^{ry}$$ (Change of variable and using the definition of e)

Logarithmic Functions

Logarithmic Functions

- Each exponential function has an inverse. These functions, inverse to the exponential functions, are called the logarithmic functions.
- If $f(x) = b^x$, the exponential function base b, then the inverse function $f^{-1}(x)$ is called the logarithm function base b and is denoted $\log_b x$. This means that the logarithm tells us the power we need to raise b to in order to get x.

$$\boxed{y = b^x \text{ means } x = \log_b y}$$

In R: $\log_b x = \log(x, b)$

- $5^2 = 25$ means $\log_5 25 = 2$
- $3^4 = 81$ means $\log_3 81 = 4$
- $10^0 = 1$ means $\log_{10} 1 = 0$

- Some fundamental properties:

$$\log_b b^x = x, \qquad b^{\log_b x} = x$$

Properties of Logarithms

1. $\log_b(mn) = \log_b m + \log_b n$

2. $\log_b \dfrac{m}{n} = \log_b m - \log_b n$

3. $\log_b m^r = r \log_b m$

4. $\log_b \dfrac{1}{m} = -\log_b m$

5. $\log_b 1 = 0$

6. $\log_b b = 1$

7. $\log_b m = \dfrac{\log_a m}{\log_a b}$ (Change-of-Base Formula)

Example

Simplify the following terms:

a) $\ln \frac{x}{zw}$

b) $\ln \sqrt[3]{\frac{x^5(x-2)^8}{x-3}}$

Solution

a)

$$\ln\frac{x}{zw} = \ln x - \ln(zw) = \ln x - (\ln z + \ln w) = \ln x - \ln z - \ln w$$

b)

$$\ln\sqrt[3]{\frac{x^5(x-2)^8}{x-3}} = \ln\left[\frac{x^5(x-2)^8}{x-3}\right]^{1/3} = \frac{1}{3}\ln\frac{x^5(x-2)^8}{x-3}$$

$$= \frac{1}{3}\{\ln[x^5(x-2)^8] - \ln(x-3)\}$$

$$= \frac{1}{3}[\ln x^5 + \ln(x-2)^8 - \ln(x-3)]$$

$$= \frac{1}{3}[5\ln x + 8\ln(x-2) - \ln(x-3)]$$

Example

Simplify the following expressions:

- $\ln e^{3x}$
- $\log 1 + \log 1\,000$
- $\log_7 \sqrt[9]{7^8}$
- $\log_3 \left(\frac{27}{81}\right)$
- $\ln e + \log \frac{1}{10}$

Solution

- $\ln e^{3x} = 3x.$
- $\log 1 + \log 1000 = 0 + \log 10^3 = 0 + 3 = 3$
- $\log_7 \sqrt[9]{7^8} = \log_7 7^{8/9} = \frac{8}{9}$
- $\log_3 \left(\frac{27}{81}\right) = \log_3 \left(\frac{3^3}{3^4}\right) = \log_3 (3^{-1}) = -1$
- $\ln e + \log \frac{1}{10} = \ln e + \log 10^{-1} = 1 + (-1) = 0$

Example

Find the solution for:

a) $5^x = 2$

b) $5 + (3)4^{\mathrm{x}-1} = 12.$

c) $\log_2 x = 5 - \log_2(x + 4)$

Solution

a)

$$5^x = 2 \Rightarrow \log 5^x = \log 2 \Rightarrow x \log 5 = \log 2 \Rightarrow x = \frac{\log 2}{\log 5} \approx 0.4307$$

b)

$$5 + (3)4^{x-1} = 12 \Rightarrow 4^{x-1} = \frac{7}{3} \Rightarrow \ln 4^{x-1} = \ln\frac{7}{3} \Rightarrow \; x \approx 1.61120$$

c)

$$\log_2 x = 5 - \log_2(x+4) \Rightarrow \log_2 x + \log_2(x+4) = 5 \Rightarrow \log_2 x(x+4) = 5 \Rightarrow x(x+4) = 2^5$$

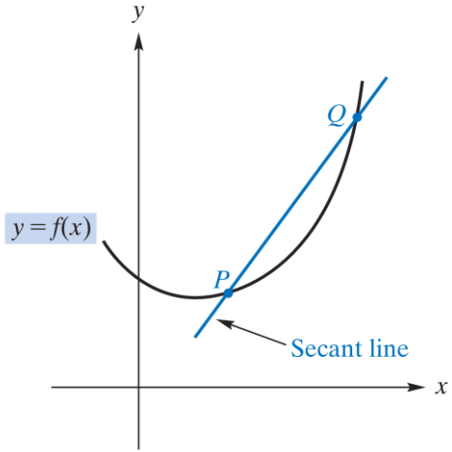
$$\Rightarrow x^2 + 4x - 32 = 0 \Rightarrow (x+8)(x-4) = 0$$

$$\Rightarrow x = -8 \; or \; x = +4$$

Definition of Derivatives as the Slope of the Tangent Line

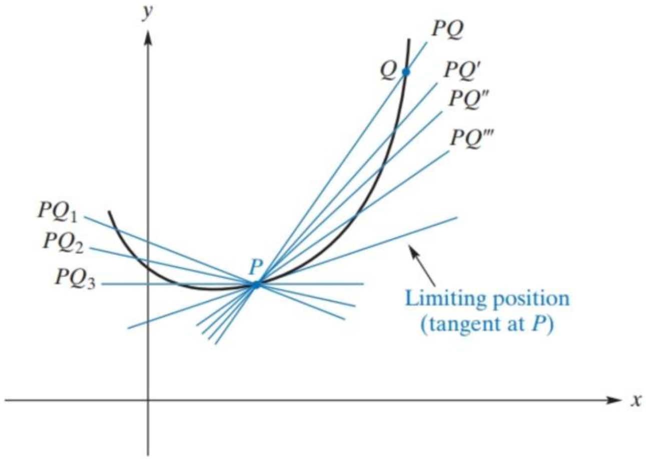
The Derivative

- A main objective of differential calculus is to find the slope of the tangent line to a curve at a specific point.
- Tangent Line: A line that touches a curve at exactly one point and has the same direction as the curve at that point.
- To obtain a suitable definition of tangent line, we use the limit concept and the geometric notion of a secant line.
- A secant line is a line that intersects a curve at two or more points.
- If Q is a different point on the curve, the line PQ is a different secant line.



The Derivative

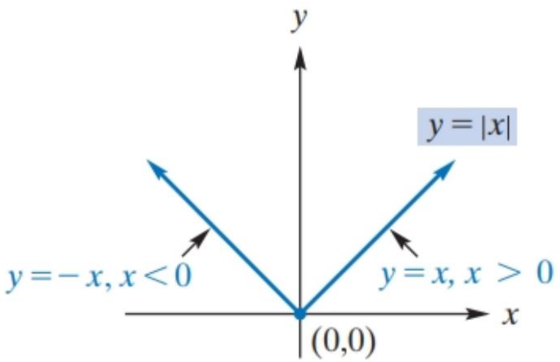
- If Q moves along the curve and approaches P from the right, typical secant lines are PQ, PQ'', and so on. As Q approaches P from the left, typical secant lines are PQ_1, PQ_2, and so on. In both cases, the secant lines approach the same limiting position. This common limiting position of the secant lines is defined to be the tangent line to the curve at P.



The tangent line is a limiting position of secant lines.

The Derivative

- A curve does not necessarily have a tangent line at each of its points.



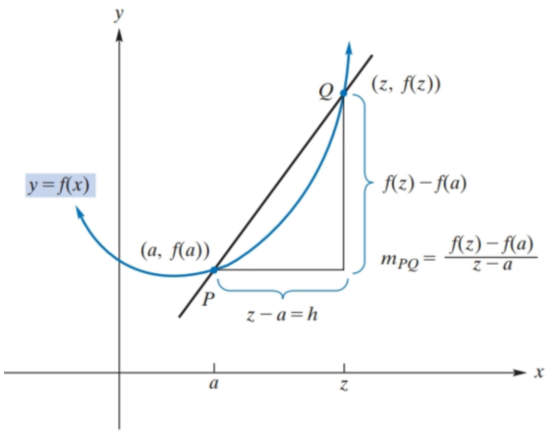
No tangent line to graph of $y = |x|$ at $(0, 0)$.

The Derivative

- The slope of a curve at P, if exits, is the slope of the tangent line at P.
- The slope of the tangent line at $(a, f(a))$ is:

$$\boxed{m_{tan} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

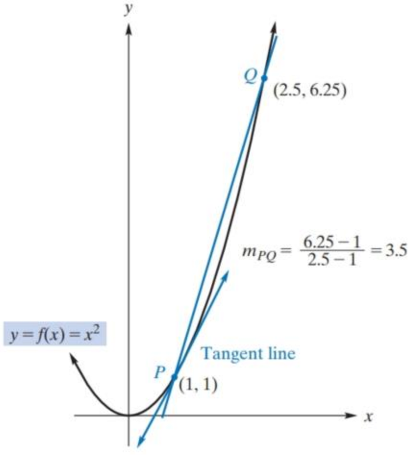
- If m_{tan} exists, it is called the derivative of $f(x)$ at $x = a$ and is denoted with $f'(a)$.



Secant line through P and Q.

The Derivative

Slopes of Secant Lines to the Curve $f(x) = x^2$ at $P = (1, 1)$	
Q	Slope of PQ
(2.5, 6.25)	(6.25 – 1)/(2.5 – 1) = 3.5
(2, 4)	(4 – 1)/(2 – 1) = 3
(1.5, 2.25)	(2.25 – 1)/(1.5 – 1) = 2.5
(1.25, 1.5625)	(1.5625 – 1)/(1.25 – 1) = 2.25
(1.1, 1.21)	(1.21 – 1)/(1.1 – 1) = 2.1
(1.01, 1.0201)	(1.0201 – 1)/(1.01 – 1) = 2.01



Secant line to $f(x) = x^2$ through (1, 1) and (2.5, 6.25).

Key Takeaways:

- **Secant Line vs. Tangent Line:** Secant line measures the average rate of change over an interval. Tangent line measures the instantaneous rate of change at a single point.
- **Role of Limits:** The concept of a limit allows us to rigorously define the tangent line by considering what happens as Q infinitely approaches P.

Example

Find the derivative of $f(x) = x^2$ at the point $x = 1$.

Solution

$$f'(1) = m = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{(1+h)^2-(1)^2}{h} = \lim_{h\to 0}\frac{h^2+2h}{h} = \lim_{h\to 0}(h+2) = 2$$

Example

Find $\frac{d}{dx}\left(2x^2 + 2x + 3\right)$.

Solution

$$f'(x) = \lim_{\mathrm{h}\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\mathrm{h}\to 0} \frac{\left(2(x+h)^2 + 2(x+h) + 3\right) - \left(2x^2 + 2x + 3\right)}{h} = 4x + 2$$

$$f'(1) = 4(1) + 2 = 6$$

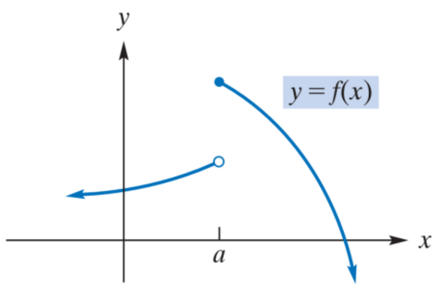
Differentiability and Continuity

- If $f(x)$ is differentiable at $x = a$, it is continuous at $x = a$.

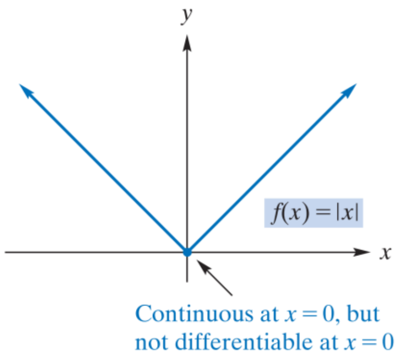
$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \Rightarrow \lim_{h\to 0} f(a+h)-f(a) = f'(a)\times\lim_{h\to 0} h = 0 \Rightarrow \lim_{h\to 0} f(a+h) = f(a)$$

- If $f(x)$ is continuous at $x = a$, it is might not be differentiable at $x = a$.

- If $f(x)$ is not continuous at $x = a$, it is not differentiable at $x = a$.



f is not continuous at a, so f is not differentiable at a.



Continuity does not imply differentiability.

Rules for Differentiation

Part 1

Rules for Differentiation

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}\big(cf(x)\big) = cf'(x)$
- $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$

Rules for Differentiation

Remark:

- $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-1} + y^{n-1})$

- $f(x) = c \Rightarrow f'(x) = \lim_{h\to 0} \frac{c-c}{h} = 0$
- $f(x) = x^n \Rightarrow f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{h\left[(x+h)^{n-1} + h(x+h)^{n-2} + \cdots + h^{n-1}\right]}{h} = nx^{n-1}$
- $g(x) = cf(x) \Rightarrow g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{cf(c+h) - cf(x)}{h} = c \lim_{h\to 0} \frac{f(c+h) - f(x)}{h} = cf'(x)$
- $F(x) = f(x) \pm g(x) \Rightarrow F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f(x+h) \pm g(x+h) - (f(x) \pm g(x))}{h} = \lim_{h\to 0} \left[\frac{f(x+h) \pm f(x)}{h} \pm \frac{g(x+h) \pm g(x)}{h}\right] = f'(x) \pm g'(x)$

Rules for Differentiation

Example: Find the following derivatives:

- $y = \sqrt{x}$
- $h(x) = \frac{1}{x\sqrt{x}}$
- $F(x) = 3x^5 + \sqrt{x}$

Solution

- $y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2\sqrt{x}}$

- $h(x) = \frac{1}{x\sqrt{x}} \Rightarrow h'^{(x)} = \frac{d}{dx}\left(x^{-\frac{3}{2}}\right) = -\frac{3}{2}x^{\left(-\frac{3}{2}\right)-1} = -\frac{3}{2}x^{-\frac{5}{2}}$

- $F(x) = 3x^5 + \sqrt{x} \Rightarrow F'(x) = \frac{d}{dx}(3x^5) + \frac{d}{dx}(x^{1/2}) = 3(5x^4) + \frac{1}{2}(x^{-1/2}) = 15x^4 + \frac{1}{2\sqrt{x}}$

Rules for Differentiation

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$ (Product Rule)

Product Rule

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x) \text{ (Product Rule)}$$

- $F(x) = f(x)g(x)$

$$f'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}\right] = f'(x)g(x) + f(x)g'(x)$$

Rules for Differentiation

Example: Take the following derivatives using the product rule

a) $f(x) = \left(x^2 + 3x\right)(4x + 5)$

b) $y = (x + 2)(x + 3)(x + 4)$

Solution

a)

$$f'(x) = \left(\frac{d}{dx}(x^2 + 3x)\right)(4x + 5) + (x^2 + 3x)\left(\frac{d}{dx}(4x + 5)\right)$$

$$= (2x + 3)(4x + 5) + (x^2 + 3x)(4) = 12x^2 + 34x + 15$$

b)

$$y' = \left(\frac{d}{dx}(x + 2)(x + 3)\right)(x + 4) + ((x + 2)(x + 3))\left(\frac{d}{dx}(x + 4)\right)$$

$$= 3x^2 + 18x + 26$$

Rules for Differentiation

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$ (Quotient Rule)

Rules for Differentiation

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$ (Quotient Rule)

- $F(x) = \frac{f(x)}{g(x)}$

$$Fg = f \Rightarrow F'g + Fg' = f' \Rightarrow F' = \frac{f' - Fg'}{g} = \frac{f' - \left(\frac{f}{g}\right)g'}{g} = \frac{f'g - fg'}{g^2}$$

Rules for Differentiation

Example: If $f(x) = \frac{4x^2+3}{2x-1}$, find $f'(x)$.

Solution

$$f'(x) = \frac{(2x-1)\frac{d}{dx}\left(4x^2+3\right) - \left(4x^2+3\right)\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$= \frac{(2x-1)(8x) - \left(4x^2+3\right)(2)}{(2x-1)^2}$$

$$= \frac{2(2x+1)(2x-3)}{(2x-1)^2}$$

Rules for Differentiation

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ (The Chain Rule)

Example: Find $\frac{dy}{dx}$:

a) $y = 2u^2 - 3u - 2$ and $u = x^2 + 4$

b) $y = w^{1/2}$ and $w = 7 - x^3$

c) $y = \left(x^3 - 1\right)^7$

Solution

a)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left(2u^2 - 3u - 2\right) \cdot \frac{d}{dx}\left(x^2 + 4\right) = (4u - 3)(2x) = \left(4x^2 + 13\right)(2x)$$

$$\frac{dy}{dx} = \left[4\left(x^2 + 4\right) - 3\right](2x) = \left[\left(4x^2 + 13\right)\right](2x) = 8x^3 + 26x$$

b)

$$\frac{dy}{dx} = \frac{d}{dw}\left(w^{1/2}\right) \cdot \frac{d}{dt}\left(7 - x^3\right) = -\frac{3x^2}{2\sqrt{w}} = -\frac{3x^2}{2\sqrt{7-x^3}}$$

c)

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 1\right) \times 7\left(x^3 - 1\right)^{7-1} = \left(3x^2\right)\left[7\left(x^3 - 1\right)^6\right] = 21x^2\left(x^2 - 1\right)^6$$

Derivative as a Rate of Change

Applications in Economics:

- Total-cost function $c(q)$: The total cost incurred in producing q units
  - Marginal cost $c'(q) = \frac{dc}{dq} \approx$ Additional cost incurred by producing one more unit.
- Total-revenue function $r(q)$: The total revenue from selling q units
  - Marginal revenue $r'(q) = \frac{dr}{dq} \approx$ Additional revenue generated by selling one more unit.

Example: A manufacturer's average-cost for producing q is given by:

$$\bar{c}(q) = 0.0001q^2 - 0.02q + 5 + \frac{5000}{q}.$$

Find the marginal cost when $\mathrm{q} = 50$.

Solution

$$\bar{c}(q) = \frac{c(q)}{q} \Rightarrow c(q) = q\bar{c}(q) = q\left(0.0001q^2 - 0.02q + 5 + \frac{5000}{q}\right)$$

$$= 0.0001q^3 - 0.02q^2 + 5q + 5000$$

$$\Rightarrow c'(q) = 0.0003q^2 - 0.04q + 5 \Rightarrow c'(50) = 0.0003(50)^2 - 0.04(50) + 5 = 3.75$$

Rules for Differentiation

Part 2

Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x}\,, \qquad x > 0$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}, \qquad u > 0$$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x}\,, \qquad x > 0$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx}, \qquad u > 0$$

- $f(x) = \ln x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) = \lim_{h\to 0}\left[\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right] = \ln\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right]$$

Note that limit can go inside because of the Compositive Function Theorem for Limits (f is a continuous function and the limit of the function inside f is defined- see the previous slides).

Now take $N = \frac{1}{h}$. Clearly when $h \to 0$, then $N \to \infty$. Hence, from the definition of Euler number, we get:

$$f'(x) = \ln\left[\lim_{N\to\infty}\left(1+\frac{\left(\frac{1}{x}\right)}{N}\right)^{N}\right] = \ln\left(e^{\frac{1}{x}}\right) = \frac{1}{x}.$$

Derivatives of Logarithmic Functions

Example: Differentiate the following functions

a) $y = \frac{\ln x}{x^2}$

b) $y = \ln(2x + 5)^3$

c) $f(p) = \ln\left((p+1)^2(p+2)^3(p+3)^4\right)$

Solution

a) $y = \frac{\ln x}{x^2} \Rightarrow y' = \frac{x^2 \frac{d}{dx}(\ln x) - (\ln x)\frac{d}{dx}(x^2)}{(x^2)^2} = \frac{x^2\left(\frac{1}{x}\right) - (\ln x)(2x)}{x^4} = \frac{1-2\ln x}{x^3}$ for $x > 0$

b) $y = \ln(2x+5)^3 = 3\ln(2x+5) \Rightarrow \frac{dy}{dx} = 3(2)\left(\frac{1}{2x+5}\right) = \frac{6}{2x+5}$ for $x > -\frac{5}{2}$

c) $f(p) = \ln\left((p+1)^2(p+2)^3(p+3)^4\right)$

$$\Rightarrow f'(p) = 2\left(\frac{1}{p+1}\right)(1) + 3\left(\frac{1}{p+2}\right)(1) + 4\left(\frac{1}{p+3}\right)(1) = \frac{2}{p+1} + \frac{3}{p+2} + \frac{4}{p+3}$$

Derivatives of Logarithmic Functions

$$\frac{d}{du}(\log_b u) = \frac{d}{du}\left(\frac{\ln u}{\ln b}\right) = \frac{1}{\ln b} \times \frac{d}{du}(\ln u)$$

Example: Find $\frac{d}{dx}(\log_2 x)$.

Solution:

$$\frac{d}{dx}(\log_2 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 2}\right) = \frac{1}{(\ln 2)x}$$

Derivatives of Exponential Functions

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^u) = \frac{du}{dx}e^u$
- $\frac{d}{dx}(b^u) = \frac{du}{dx}b^u(\ln b)$
- $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Derivatives of Exponential Functions

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^u) = \frac{du}{dx}e^u$
- $\frac{d}{dx}(b^u) = \frac{du}{dx}b^u(\ln b)$
- $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

- $y = e^x \Rightarrow \ln y = x \Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = y = e^x$
- $y = b^u \Rightarrow y = e^{\ln(b^u)} = e^{u(\ln b)} = e^{u_1}, u_1 = u(\ln b) \Rightarrow \frac{dy}{dx} = \frac{dy}{du_1} \times \frac{du_1}{du} \times \frac{du}{dx}$ $(chain\ rule) \Rightarrow \frac{dy}{dx} = e^{u(\ln b)}(\ln b)\frac{du}{dx}$
- $y = f(x) \Rightarrow f^{-1}(y) = x \Rightarrow \frac{df^{-1}(y)}{dy} \times \frac{dy}{dx} = 1 \Rightarrow \frac{dx}{dy} \times \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Derivatives of Exponential Functions

Example: Find $\frac{dy}{dx}$ when

a) $y = 3e^x$.

b) $y = \frac{x}{e^x}$,

c) $y = e^2 + e^x + \ln 3$

Solution

a)

$$\frac{d}{dx}(3e^x) = 3\frac{d}{dx}(e^x) = 3e^x$$

b)

$$\frac{dy}{dx} = e^{-x}\frac{d}{dx}x + x\frac{d}{dx}(e^{-x}) = \frac{1-x}{e^x}$$

c)

$$y = e^2 + e^x + \ln 3$$

$$y' = 0 + e^x + 0 = e^x$$

Implicit Differentiation

Sometimes y is not expressed as an explicit function of x. To determine $\frac{dy}{dx}$ follow these steps:

1. Differentiate both sides with respect to x.
2. Collect all $\frac{dy}{dx}$ terms on one side and solve for $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$ from

a) $y + y^3 - x = 7.$

b) $x^3 = \left(y - x^2\right)^2$ at $(1,2)$.

Solution

a)

$$\frac{d}{dx}\left(y + y^3 - x\right) = \frac{d}{dx}(7) \Rightarrow \frac{dy}{dx} + 3y^2\frac{dy}{dx} - 1 = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{1 + 3y^2}$$

b)

$$\frac{d}{dx}\left(x^3\right) = \frac{d}{dx}\left[\left(y - x^2\right)^2\right] \Rightarrow 3x^2 = 2\left(y - x^2\right)\left(\frac{dy}{dx} - 2x\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 + 4xy - 4x^3}{2(y - x^2)} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{7}{2}$$

Logarithmic Differentiation

Sometimes to differentiate a function $y = f(x)$, it is easier first to take natural log from both sides and take the derivative from the natural log function. In that case, follow these steps:

1. Take the natural logarithm of both sides which gives $\ln y = \ln\big(f(x)\big)$.
2. Differentiate both sides with respect to x.
3. Solve for $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$ for $y = \frac{(2x-5)^3}{x^2\sqrt[4]{x^2+1}}$

Solution

$$\ln y = \ln \frac{(2x-5)^3}{x^2\sqrt[4]{x^2+1}} \Rightarrow \ln y = \ln(2x-5)^3 - \ln x^2 - \ln\sqrt[4]{x^2+1} = 3\ln(2x-5) - 2\ln x - \frac{1}{4}\ln(x^2+1)$$

$$\Rightarrow \frac{y'}{y} = 3(2)\left(\frac{1}{2x-5}\right) - 2\left(\frac{1}{x}\right) - \frac{1}{4}(2x)\left(\frac{1}{x^2+1}\right) = \frac{6}{2x-5} - \frac{2}{x} - \frac{x}{2(x^2+1)}$$

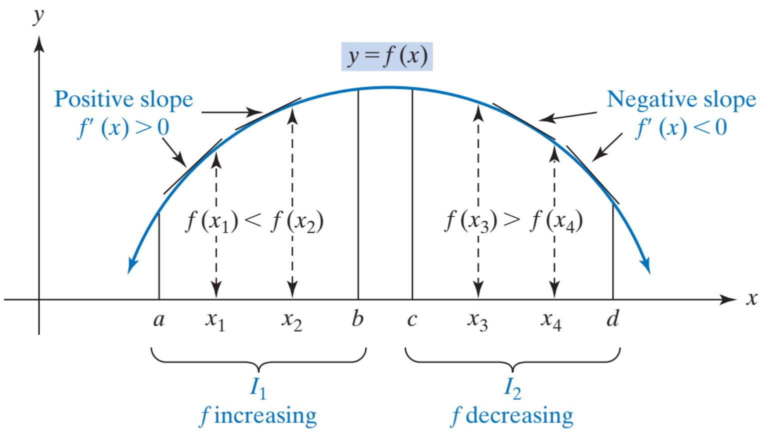
$$\Rightarrow y' = \frac{(2x-5)^3}{x^2\sqrt[4]{x^2+1}}\left[\frac{6}{2x-5} - \frac{2}{x} - \frac{x}{2(x^2+1)}\right]$$

Unconstrained Optimization

Monotonic Function

Definition:

- $f(x)$ is increasing if for any $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- $f(x)$ is decreasing if for any $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.



If $f(x)$ is increasing, then: $f'(x) \geq 0$.

If $f(x)$ is decreasing, then: $f'(x) \leq 0$.

If $f(x)$ is increasing, then by the definition: $f(x+h) \geq f(x)$ for all $h > 0$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \geq 0$$

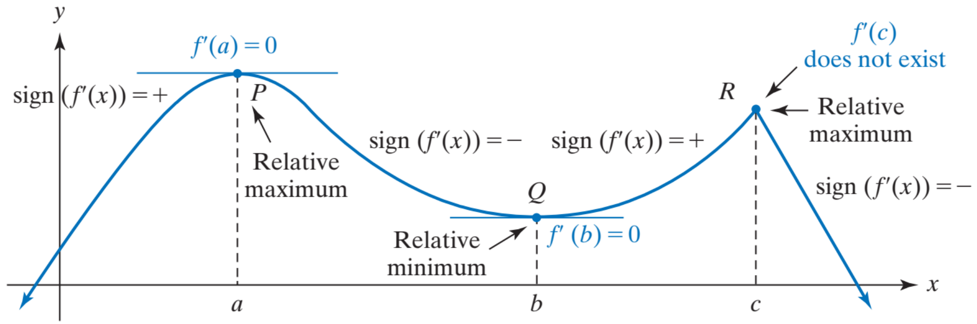
If $f(x)$ is decreasing, then by the definition: $f(x+h) \leq f(x)$ for all $h > 0$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \leq 0$$

Extrema

Definition:

- A function f has a relative (or local) maximum at a if there is an open interval containing a on which $f(a) \geq f(x)$ for all x in the interval. The relative maximum value is $f(a)$. A function f has a relative (or local) minimum at a if there is an open interval containing a on which $f(a) \leq f(x)$ for all x in the interval. The relative minimum value is $f(a)$.
- A function f has an absolute (or global) maximum at a if $f(a) \geq f(x)$ in the domain of f. The absolute maximum value is $f(a)$. A function f has an absolute (or global) minimum at a if $f(a) \leq f(x)$ in the domain of f. The absolute minimum value is $f(a)$.



First Order Necessary Condition (FONC)

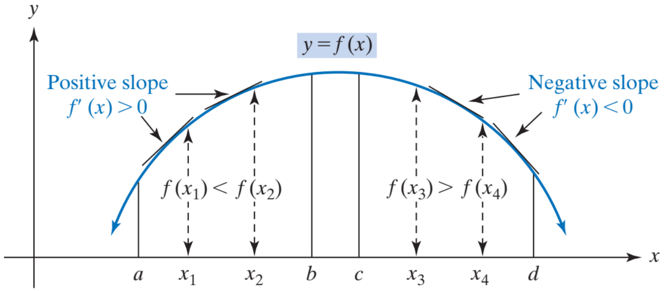
If f is differentiable and $(x^*, f(x^*))$ is a relative extrema, then: $f'(x^*) = 0$.

Suppose, for example, that x^* is a relative minimum. Then:

$h > 0 \Rightarrow f'(x^*) = \lim_{h \to 0} \frac{f(x^*+h) - f(x^*)}{h} \geq 0$ (right derivative)

$h < 0 \Rightarrow f'(x^*) = \lim_{h \to 0} \frac{f(x^*+h) - f(x^*)}{h} \leq 0$ (left derivative)

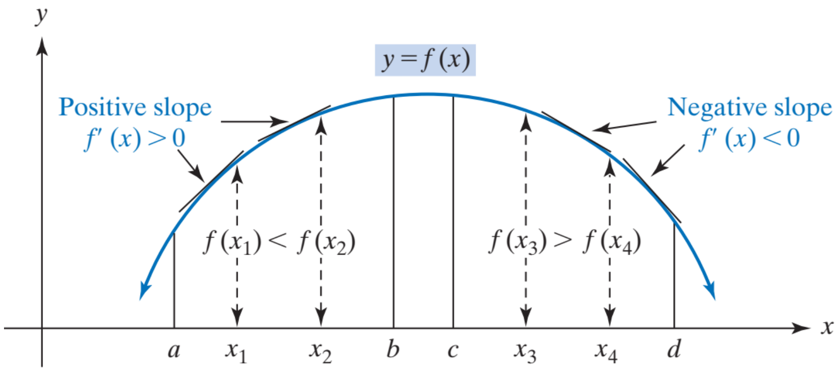
The left and right derivatives must be equal (otherwise, the limit would not exist). So, $f'(x^*) = 0$.

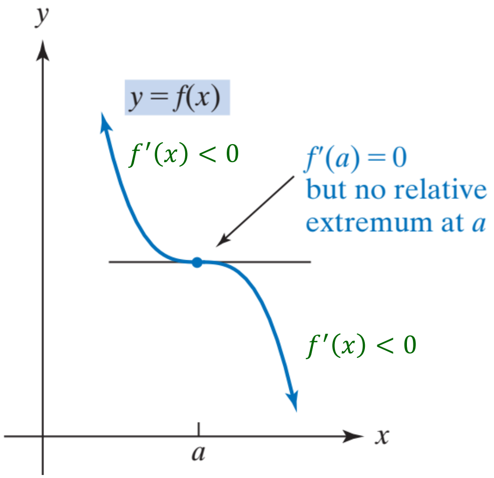


First Order Necessary Condition (FONC)

Suppose f is continuous on an open interval I that contains the critical value a and f is differentiable on I, except possibly at a. Then:

1. If $f'(x)$ changes from positive to negative as x increases through a, then f has a relative maximum at a.
2. If $f'(x)$ changes from negative to positive as x increases through a, then f has a relative minimum at a.





Relative (Local) Extrema

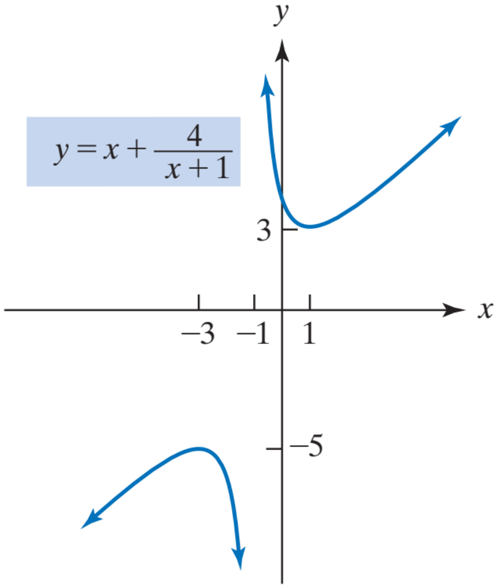
Example: If $y = f(x) = x + \frac{4}{x+1}$ for $x \neq -1$, determine the critical points and whether each is a relative minimum or maximum.

Solution

$$f'(x) = 1 - \frac{4}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2} \text{ for } x \neq -1$$

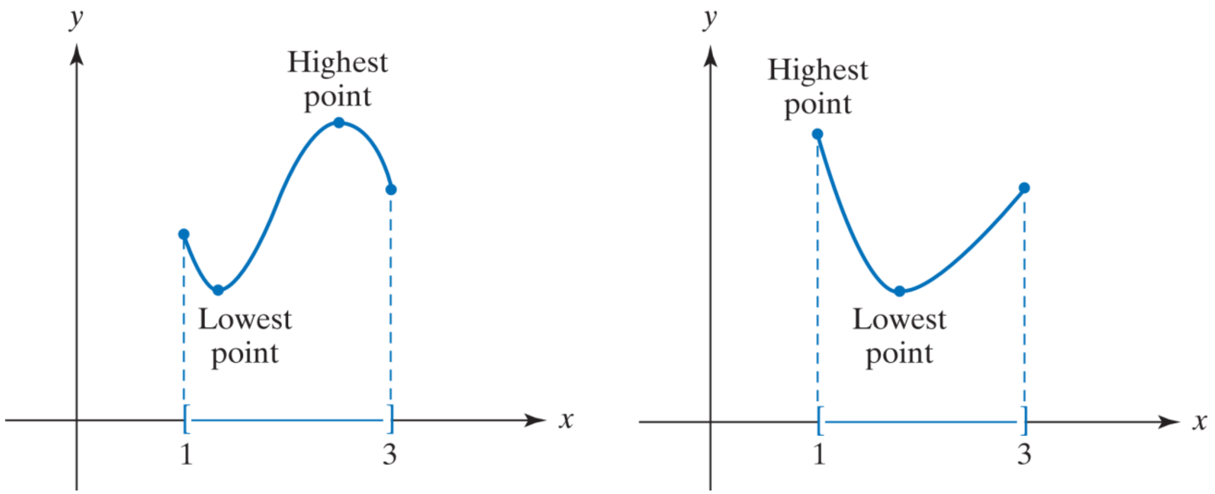
Setting $f'(x) = 0$ gives $x = -3, 1.$

	$-\infty$ to -3	-3	-3 to -1	-1	-1 to 1	1	1 to ∞
$x+3$	$-$	0	$+$		$+$		$+$
$(x+1)^{-2}$	$+$		$+$		$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$f'(x)$	$+$	0	$-$		$-$	0	$+$
$f(x)$							



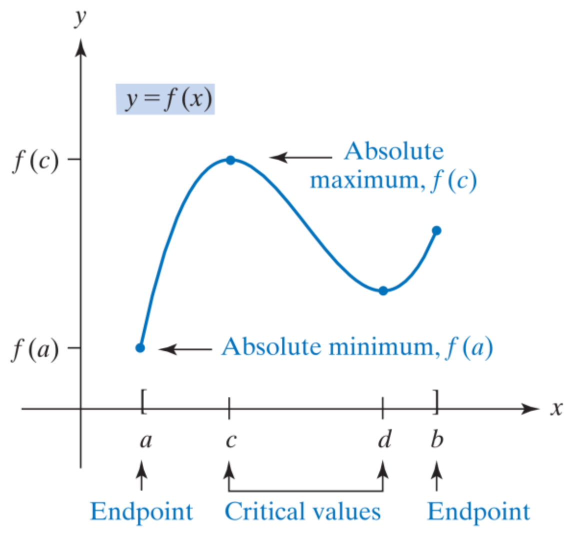
Absolute Extrema on a Closed Interval

Extreme-Value Theorem: If a function is continuous on a closed interval, then the function has a maximum value and a minimum value on that interval.



Absolute Extrema on a Closed Interval

1. Find the critical values of $f(x)$.
2. Evaluate $f(x)$ at the endpoints a and b and at the critical values in (a, b).
3. The maximum value of f is the greatest value found in step 2. The minimum value is the least value found in step 2.



Example: Find absolute extrema for $f(x) = x^2 - 4x + 5$ over the closed interval $[1,4]$.

Solution:

Step 1: $f'(x) = 2x - 4x = 2(x - 2)$

Step 2: $f(1) = 2, f(4) = 5$ (values of f at the endpoints)

$f(2) = 1$ (value of f at the critical value 2)

Max is $f(4) = 5$ and Min is $f(2) = 1$

Taylor Expansion

Analytic Function: An analytic function is one that can be written as an infinite sum (a power series) of terms around a certain point a in its domain. a is called the point of expansion or starting point:

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

Where c_k are coefficients.

Question: How can we determine the coefficients $c_0, c_1, \ldots$? (The main point is the use of derivatives!)

$$f(a) = c_0$$

$$f'(a) = c_1$$

$$f''(a) = \frac{1}{2} c_2$$

Taylor Expansion:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^2 \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots$$

if $f^{(k)}$ exists for all k.

Taylor Expansion

Example: Write the Taylor expansion of e^x around the point $a = 0$:

$$f(x) = e^x \Rightarrow f^{(k)}(x) = e^x \Rightarrow f^{(k)}(0) = 1$$

$$f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \cdots$$

$$\Rightarrow \; e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$e^x \approx 1 + x$ (linear approximation)

$e^x \approx 1 + x + \frac{x^2}{2!}$ (quadratic approximation)

$e^{0.25} = 1.2840$

$$e^{0.25} \approx 1 + 0.25 = 1.25$$

$$e^{0.25} \approx 1 + 0.25 + \frac{(0.25)^2}{2!} = 1.2812$$

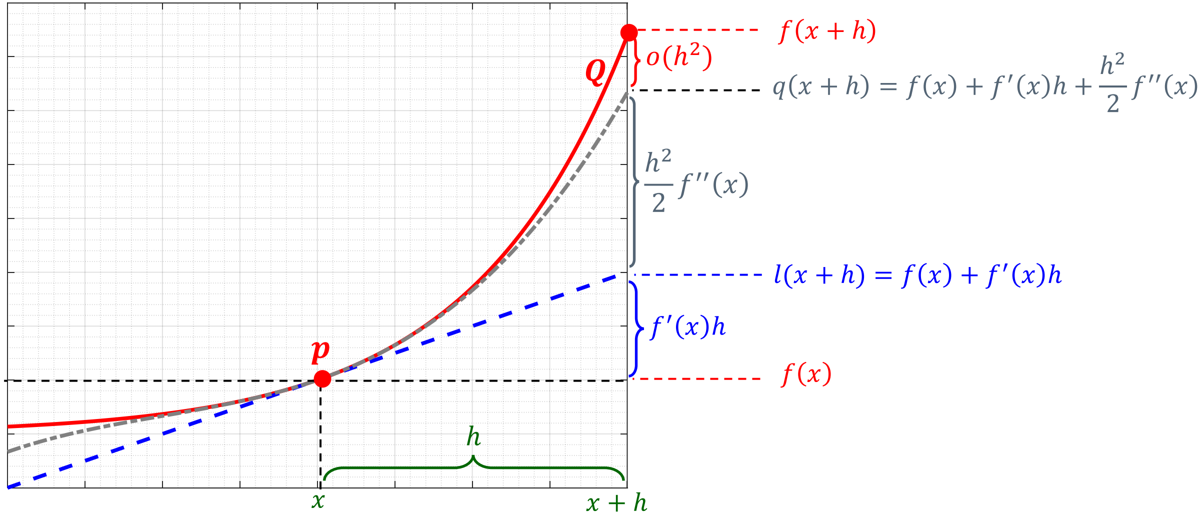
Taylor Expansion for a Single Variable Function

Main Use: approximating a function at some point like x (or a) by linearization or a quadratic form

$$f(x+h) = f(x) + f'(x)h + o(h)$$

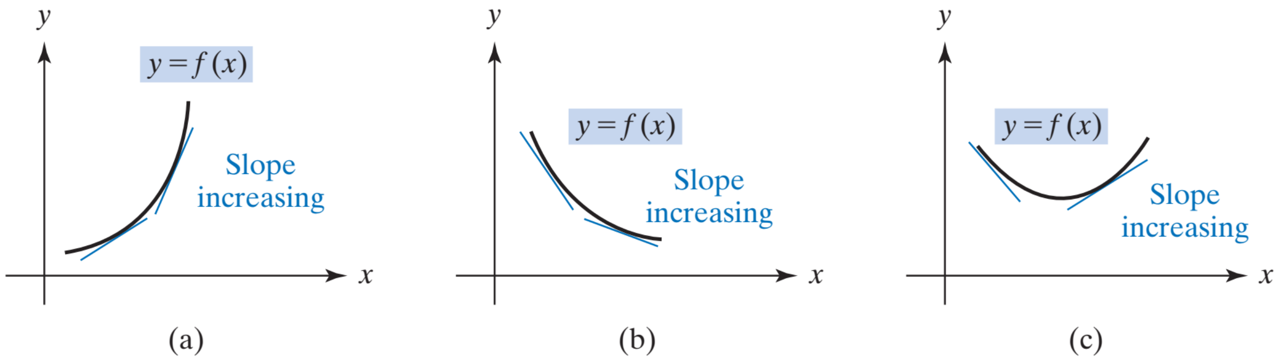
$$f(x+h) = f(x) + f'(x)h + \frac{h^2}{2}f''(x) + o(h^2)$$

$o(h^k)$ = Remainder with terms $h^{k+1}, h^{k+2}, \ldots$, i.e., $\lim_{h\to 0} \frac{o(h^k)}{h^k} = 0$.



Concavity

Definition: $f(x)$ is convex when the tangent line lies below the graph at all points. (The graph is bending upward.) This means that the linear approximation around any point underestimates the value of f at that point.



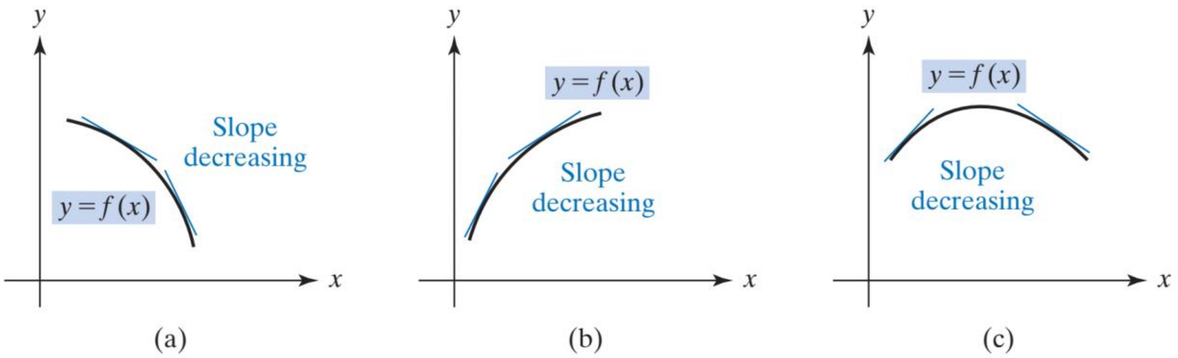
Remark: When $f(x)$ is convex, then $f''(x) \geq 0$. This means that $f'(x)$ (the slope of f) is increasing.

$$f(x+h) - f(x) - f'(x)h = \frac{f''(x)}{2}h^2 + o(h^2) \geq 0$$

$$\Rightarrow \frac{f''(x)}{2} + \frac{o(h^2)}{h^2} \geq 0 \Rightarrow \lim_{h \to 0}\left(\frac{f''(x)}{2} + \frac{o(h^2)}{h^2}\right) \geq 0 \Rightarrow f''(x) \geq 0.$$

Concavity

Definition: $f(x)$ is concave when the tangent line lies above the graph at all points. (The graph is bending downward.) This means that the linear approximation around any point overestimates the value of f at that point.



Remark: When $f(x)$ is convex, then $f''(x) \leq 0$. This means that $f'(x)$ (the slope of f) is decreasing.

$$f(x+h) - f(x) - f'(x)h = \frac{f''(x)}{2}h^2 + o(h^2) \leq 0$$

$$\Rightarrow \frac{f''(x)}{2} + \frac{o(h^2)}{h^2} \leq 0 \Rightarrow \lim_{h\to 0}\left(\frac{f''(x)}{2} + \frac{o(h^2)}{h^2}\right) \leq 0 \Rightarrow f''(x) \leq 0.$$

Second Order Necessary Conditions for Optimality

If f is ***twice differentiable***, then the following hold:

(i) If x^* is a local minimum of f, then: $f'(x^*) = 0,\ f''(x^*) > 0.$

(ii) If x^* is a local maximum of f, then : $f'(x^*) = 0,\ f''(x^*) < 0.$

Intuitively, if it is a local minimum, it is bending upward so $f''(x^*) > 0$ and if it is a local maximum, it is bending downward so $f''(x^*) < 0$. (More formal proof next slide for interested students).

Second Order Necessary Conditions

Proof for local minimum: Suppose $h > 0$. From the FOC, we already know that $f'(x^*) = 0$. Now, using Taylor expansion:

$$f(x^* + h) - f(x^*) = \frac{f''(x^*)}{2}h^2 + o(h^2) \geq 0 \Rightarrow \lim_{h \to 0}\left(\frac{f''(x^*)}{2} + \frac{o(h^2)}{h^2}\right) \geq 0 \Leftrightarrow f''(x^*) > 0$$

Proof for local maximum is also similar.

Second Order Sufficient Optimality Conditions

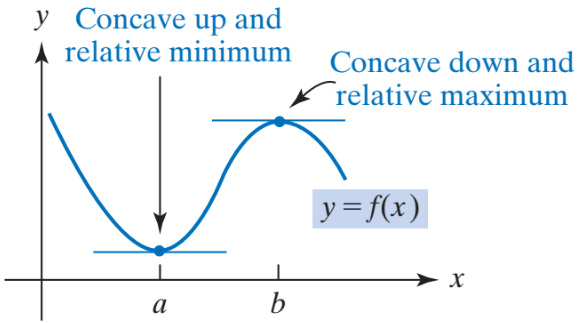
If f is twice differentiable and x^* is a point in the domain of f. Then:

(i) If $f'(x^*) = 0,\ \ f''(x^*) > 0,$ then x^* is a local minimum of f.

(ii) If $f'(x^*) = 0,\ \ f''(x^*) < 0,$ then x^* is a local maximum of f.

Intuitively, if x^* is a critical point at which f is bending upward it is a local minimum of f. If x^* is a critical point at which f is bending downward it is a local maximum of f.

(More formal proof next slide for interested students).



Second Order Sufficient Conditions

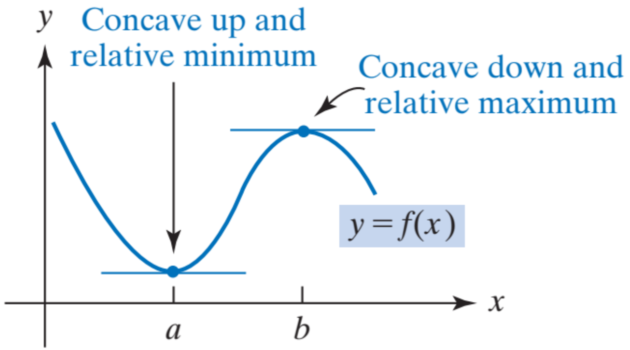
Proof:

(i): Suppose $h > 0$. If $f'(x^*) = 0,\ \ f''(x^*) > 0$, we apply the Taylor expansion to approximate the function near x^*:

$$f(x^* + h) - f(x^*) = \frac{f''(x^*)}{2}h^2 + o\left(h^2\right) > 0$$

Since $\lim_{h\to 0} \frac{o(h^2)}{h^2} = 0$, we know that for sufficiently small h, the term $o\left(h^2\right)$ is negligible compared to $\frac{f''(x^*)}{2}h^2$. This implies that for small h, $f(x^*) < f(x^* + h)$.

(ii): The argument is similar to part (i), but this time we only consider $h < 0$.



Example

Test the following for relative maxima and minima: $y = 6x^4 - 8x^3 + 1$

Solution

$y' = 24x^3 - 24x^2 = 24x^2(x-1)$

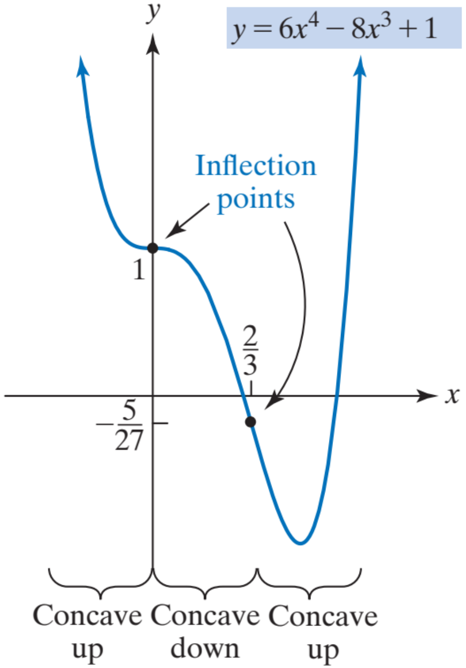
$y'' = 72x^2 - 48x$

When $y' = 0$, we have $x = 0,1$

When $x = 0,$ $y'' = 0$

When $x = 1,$ $y'' > 0$

No maximum or minimum exists at $x = 0$.



Finding a Local Minimum Using R

The following code finds a local minimum value within a given starting point:

`optim(starting-point,f,method = "BFGS",hessian=1)`

- `method:` choose one of the optimization methods:
  - `"Nelder-Mead":` A popular method for non-linear optimization.
  - `"BFGS":` Broyden–Fletcher–Goldfarb–Shanno algorithm (used for smooth functions).
  - `"CG":` Conjugate gradients method.
  - `"L-BFGS-B":` Limited-memory BFGS with box constraints.
  - `"SANN":` Simulated annealing.
  - `"Brent":` Brent's method for one-dimensional optimization.
- `hessian=1 :` This returns the Hessian matrix, which is the matrix of second derivatives. It provides information about the curvature of the function, helping determine if a point is a minimum, maximum, or saddle point.

Example: Code the example in the previous slide:

```
R 4.3.1 · ~/
> f = function(x) 6*x^4-8*x^3+1
> optim(90,f,method="BFGS")
$par
[1] 0.9999997

$value
[1] -1
```

```
> library(numDeriv)
> grad(f,3)
[1] 432
> hessian(f,1/3)
     [,1]
[1,]   -8
```