Why might an individual be willing to deposit money in a bank and *pay* the bank to keep it there? Well, if the money is subject to a large enough tax bill then a bank account charging less than the tax bill and providing confidentiality might actually appear attractive. Of course many jurisdictions actively combat this sort of tax evasion.

The negative interest rates in the news in 2016 were those offered by central banks (of countries) to the ordinary commercial banks with which individuals and companies do their banking. Rates offered by central banks are usually considered to be a tool to implement economic policy. When commercial banks *lend* to the central bank they have virtually no risk. In difficult economic times very conservative banks will prefer the lower rates offered by the risk-free central bank over the higher rates they themselves offer to ordinary borrowers, often with considerable risk. Of course, if individuals and companies are unable to borrow money then the economy of their country stagnates. In 2016, central banks of several countries sought to put more money in the hands of individuals and businesses to boost their sluggish national economies. By offering negative interest rates to commercial banks, the central banks *encouraged* commercial banks to put their money in circulation by lending to individuals and companies rather than hoarding cash at the central bank.

PROBLEMS 5.1

In Problems 1 and 2, find **(a)** *the compound amount and* **(b)** *the compound interest for the given investment and rate.*

1. \$6000 for eight years at an effective rate of 8%

2. \$750 for 12 months at an effective rate of 7%

In Problems 3–6, find the effective rate, to three decimal places that corresponds to the given nominal rate.

3. 2.75% compounded monthly

4. 5% compounded quarterly

5. 3.5% compounded daily

6. 6% compounded daily

7. Find the effective rate of interest (rounded to three decimal places) that is equivalent to a nominal rate of 10% compounded

(a) yearly **(b)** semiannually
(c) quarterly **(d)** monthly
(e) daily

8. Find **(i)** the compound interest and **(ii)** the effective rate, to four decimal places, if \$1000 is invested for one year at an annual rate of 5% compounded

(a) quarterly **(b)** monthly
(c) weekly **(d)** daily

9. Over a five-year period, an original principal of \$2000 accumulated to \$2950 in an account in which interest was compounded quarterly. Determine the effective rate of interest, rounded to two decimal places.

10. Suppose that over a six-year period, \$1000 accumulated to \$1959 in an investment certificate in which interest was compounded quarterly. Find the nominal rate of interest, compounded quarterly, that was earned. Round your answer to two decimal places.

In Problems 11 and 12, find how many years it would take to double a principal at the given effective rate. Give your answer to one decimal place.

11. 9% **12.** 5%

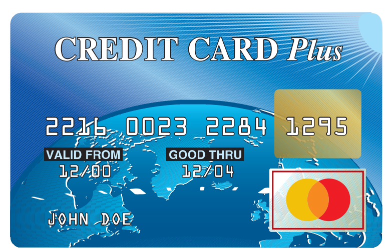
13. A \$4000 certificate of deposit is purchased for \$4000 and is held for eleven years. If the certificate earns an effective rate of 7%, what is it worth at the end of that period?

14. How many years will it take for money to triple at the effective rate of r?

15. University Costs Suppose attending a certain university cost \$25,500 in the 2009–2010 school year. This price included tuition, room, board, books, and other expenses. Assuming an effective 3% inflation rate for these costs, determine what the university costs were in the 2015–2016 school year.

16. University Costs Repeat Problem 15 for an inflation rate of 2% compounded quarterly.

17. Finance Charge A major credit-card company has a finance charge of $1\frac{1}{2}\%$ per month on the outstanding indebtedness. **(a)** What is the nominal rate compounded monthly? **(b)** What is the effective rate?



18. How long would it take for a principal of P to double if it is invested with an APR of 7%, compounded monthly.

19. To what sum will \$2000 amount in eight years if invested at a 6% effective rate for the first four years and at 6% compounded semiannually thereafter?

20. How long will it take for \$100 to amount to \$1000 if invested at 6% compounded monthly? Express the answer in years, rounded to two decimal places.

21. An investor has a choice of investing a sum of money at 8% compounded annually or at 7.8% compounded semiannually. Which is the better of the two rates?

22. What nominal rate of interest, compounded monthly, corresponds to an effective rate of 4.5%?

23. Savings Account A bank advertises that it pays interest on savings accounts at the rate of $3\frac{1}{4}\%$ compounded daily. Find the effective rate if the bank assumes that a year consists of **(a)** 360 days or **(b)** 365 days in determining the *daily rate*. Assume that compounding occurs 365 times a year.

24. Savings Account Suppose that \$700 amounted to \$801.06 in a savings account after two years. If interest was compounded quarterly, find the nominal rate of interest, compounded quarterly, that was earned by the money.

25. Inflation As a hedge against inflation, an investor purchased a 1972 Gran Torino in 1990 for \$90,000. It was sold in 2000 for \$250,000. At what effective rate did the car appreciate in value? Express the answer as a percentage rounded to three decimal places.

26. Inflation If the rate of inflation for certain goods is $7\frac{1}{4}\%$ compounded daily, how many years will it take for the average price of such goods to double?

27. Zero-Coupon Bond A *zero-coupon bond* is a bond that is sold for less than its face value (that is, it is *discounted*) and has no periodic interest payments. Instead, the bond is redeemed for its face value at maturity. Thus, in this sense, interest is paid at maturity. Suppose that a zero-coupon bond sells for \$420 and can be redeemed in 14 years for its face value of \$1000. The bond earns interest at what nominal rate, compounded semiannually?

28. Misplaced Funds Suppose that \$1000 is misplaced in a non-interest-bearing checking account and forgotten. Each year, the bank imposes a service charge of 1.5%. After 20 years, how much remains of the \$1000? [*Hint:* Recall the notion of *negative interest rates*.]

29. General Solutions Equation (1) can be solved for each of the variables in terms of the other three. Find each of P, r, and n in this way. (There is no need to memorize any of the three new formulas that result. The point here is that by showing the general solutions exist, we gain confidence in our ability to handle any particular cases.)

5.2 Present Value

Objective

To study present value and to solve problems involving the time value of money by using equations of value. To introduce the net present value of cash flows.

Suppose that \$100 is deposited in a savings account that pays 6% compounded annually. Then at the end of two years, the account is worth

$$100(1.06)^2 = 112.36$$

To describe this relationship, we say that the compound amount of \$112.36 is the **future value** of the \$100, and \$100 is the **present value** of the \$112.36. Sometimes we know the future value of an investment and want to find the present value. To obtain a formula for doing this, we solve the equation $S = P(1 + r)^n$ for P. The result is $P = S/(1 + r)^n = S(1 + r)^{-n}$.

Present Value

The principal P that must be invested at the periodic rate of r for n interest periods so that the compound amount is S is given by

$$P = S(1 + r)^{-n} \tag{1}$$

and is called the **present value** of S.

EXAMPLE 1 Present Value

Find the present value of \$1000 due after three years if the interest rate is 9% compounded monthly.

Solution: We use Equation (1) with $S = 1000$, $r = 0.09/12 = 0.0075$, and $n = 3(12) = 36$:

$$P = 1000(1.0075)^{-36} \approx 764.15$$

This means that \$764.15 must be invested at 9% compounded monthly to have \$1000 in three years.

Now Work Problem 1 ◁

merely a rational number raised to what is often a positive integer, it really can't be calculated, if the exponent is large, without a "decent" calculator. A so-called "x^y" key is needed, and any such calculator has an "Exp" key that will calculate e^{rt} as Exp(rt). (There is never any need to enter a decimal approximation of the irrational number e.) On a typical "decent" calculator, $(1 + r/n)^{nt}$ requires about twice as many key strokes as e^{rt}. This paragraph does nothing to answer its question. We suggest that readers ask their bank managers why their banks do not compound interest continuously. ;-)

PROBLEMS 5.3

In Problems 1 and 2, find the compound amount and compound interest if \$4000 is invested for six years and interest is compounded continuously at the given annual rate.

1. $6\frac{1}{4}\%$ **2.** 9%

In Problems 3 and 4, find the present value of \$2500 due eight years from now if interest is compounded continuously at the given annual rate.

3. $1\frac{1}{2}\%$ **4.** 8%

In Problems 5–8, find the effective rate of interest that corresponds to the given annual rate compounded continuously.

5. 2% **6.** 8% **7.** 3% **8.** 11%

9. Investment If \$100 is deposited in a savings account that earns interest at an annual rate of $4\frac{1}{2}\%$ compounded continuously, what is the value of the account at the end of two years?

10. Investment If \$1500 is invested at an annual rate of 4% compounded continuously, find the compound amount at the end of ten years.

11. Stock Redemption The board of directors of a corporation agrees to redeem some of its callable preferred stock in five years. At that time, \$1,000,000 will be required. If the corporation can invest money at an annual interest rate of 5% compounded continuously, how much should it presently invest so that the future value is sufficient to redeem the shares?

12. Trust Fund A trust fund is being set up by a single payment so that at the end of 30 years there will be \$50,000 in the fund. If interest is compounded continuously at an annual rate of 6%, how much money should be paid into the fund initially?

13. Trust Fund As a gift for their newly born daughter's 21st birthday, the Smiths want to give her at that time a sum of money that has the same buying power as does \$21,000 on the date of her birth. To accomplish this, they will make a single initial payment into a trust fund set up specifically for the purpose.
(a) Assume that the annual effective rate of inflation is 3.5%. In 21 years, what sum will have the same buying power as does \$21,000 at the date of the Smiths' daughter's birth?
(b) What should be the amount of the single initial payment into the fund if interest is compounded continuously at an annual rate of 3.5%?

14. Investment Currently, the Smiths have \$50,000 to invest for 18 months. They have two options open to them:
(a) Invest the money in a certificate paying interest at the nominal rate of 5% compounded quarterly;
(b) Invest the money in a savings account earning interest at the annual rate of 4.5% compounded continuously.
How much money will they have in 18 months with each option?

15. What annual rate compounded continuously is equivalent to an effective rate of 3%?

16. What annual rate r compounded continuously is equivalent to a nominal rate of 6% compounded semiannually?

17. If interest is compounded continuously at an annual rate of 0.07, how many years would it take for a principal P to triple? Give your answer to the nearest year.

18. If interest is compounded continuously, at what annual rate will a principal double in 20 years? Give the answer as a percentage correct to two decimal places.

19. Savings Options On July 1, 2001, Mr. Green had \$1000 in a savings account at the First National Bank. This account earns interest at an annual rate of 3.5% compounded continuously. A competing bank was attempting to attract new customers by offering to add \$20 immediately to any new account opened with a minimum \$1000 deposit, and the new account would earn interest at the annual rate of 3.5% compounded semiannually. Mr. Green decided to choose one of the following three options on July 1, 2001:
(a) Leave the money at the First National Bank;
(b) Move the money to the competing bank;
(c) Leave half the money at the First National Bank and move the other half to the competing bank.

For each of these three options, find Mr. Green's accumulated amount on July 1, 2003.

20. Investment **(a)** On April 1, 2006, Ms. Cheung invested \$75,000 in a 10-year certificate of deposit that paid interest at the annual rate of 3.5% compounded continuously. When the certificate matured on April 1, 2016, she reinvested the entire accumulated amount in corporate bonds, which earn interest at the rate of 4.5% compounded annually. What will be Ms. Cheung's accumulated amount on April 1, 2021?
(b) If Ms. Cheung had made a single investment of \$75,000 in 2006 that matures in 2021 and has an effective rate of interest of 4%, would her accumulated amount be more or less than that in part (a) and by how much?

21. Investment Strategy Suppose that you have \$9000 to invest.
(a) If you invest it with the First National Bank at the nominal rate of 5% compounded quarterly, find the accumulated amount at the end of one year.
(b) The First National Bank also offers certificates on which it pays 5.5% compounded continuously. However, a minimum investment of \$10,000 is required. Because you have only \$9000, the bank is willing to give you a 1-year loan for the extra \$1000 that you need. Interest for this loan is at an effective rate of 8%, and both principal and interest are payable at the end of the year. Determine whether or not this strategy of investment is preferable to the strategy in part (a).

22. If interest is compounded continuously at an annual rate of 3%, in how many years will a principal double? Give the answer correct to two decimal places.

23. General Solutions In Problem 29 of Section 5.1 it was pointed out that the *discretely* compounded amount formula, $S = P(1 + r)^n$, can be solved for each of the variables in terms of the other three. Carry out the same derivation for the continuously compounded amount formula, $S = Pe^{rt}$. (Again, there is no need to memorize any of the three other formulas that result, although we have met one of them already. By seeing that the general solutions are easy, we are informed that all particular solutions are easy, too.)

Objective

To introduce the notions of ordinary annuities and annuities due. To use geometric series to model the present value and the future value of an annuity. To determine payments to be placed in a sinking fund.

5.4 Annuities

Annuities

It is best to define an **annuity** as any finite sequence of payments made at fixed periods of time over a given interval. The fixed periods of time that we consider will always be of equal length, and we refer to that length of time as the **payment period**. The given interval is the **term** of the annuity. The payments we consider will always be of equal value. An example of an annuity is the depositing of \$100 in a savings account every three months for a year.

The word *annuity* comes from the Latin word *annus*, which means “year,” and it is likely that the first usage of the word was to describe a sequence of annual payments. We emphasize that the payment period can be of any agreed-upon length. The informal definitions of *annuity* provided by insurance companies in their advertising suggest that an annuity is a sequence of payments in the nature of pension income. However, a sequence of rent, car, or mortgage payments fits the mathematics we wish to describe, so our definition is silent about the purpose of the payments.

When dealing with annuities, it is convenient to mark time in units of payment periods on a line, with time *now*, in other words the present, taken to be 0. Our generic annuity will consist of n payments, each in the amount R. With reference to such a timeline (see Figure 5.4), suppose that the n payments (each of amount R) occur at times $1, 2, 3, \ldots, n$. In this case we speak of an **ordinary annuity**. Unless otherwise specified, an annuity is assumed to be an ordinary annuity. Again with reference to our timeline (see Figure 5.5), suppose now that the n equal payments occur at times $0, 1, 2, \ldots, n - 1$. In this case we speak of an **annuity due**. Observe that in any event, the $n + 1$ different times $0, 1, 2, \ldots, n - 1, n$ define n consecutive time intervals (each of payment period length). We can consider that an ordinary annuity’s payments are at the *end* of each payment period while those of an annuity due are at the *beginning* of each payment period. A sequence of rent payments is likely to form an annuity due because most landlords demand the first month’s rent when the tenant moves in. By contrast, the sequence of wage payments that an employer makes to a regular full-time employee is likely to form an ordinary annuity because usually wages are for work *done* rather than for work *contemplated*.

We henceforth assume that interest is at the rate of r per payment period. For either kind of annuity, a payment of amount R made at time k, for k one of the times $0, 1, 2, \ldots, n - 1, n$, has a value at time 0 and a value at time n. The value at time 0 is the *present value* of the payment made at time k. From Section 5.2 we see that the

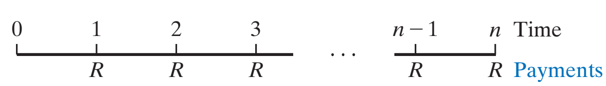


FIGURE 5.4 Ordinary annuity.

FIGURE 5.5 Annuity due.