PROBLEMS 13.1

In Problems 1–4, the graph of a function is given (Figures 13.18–13.21). Find the open intervals on which the function is increasing, the open intervals on which the function is decreasing, and the coordinates of all relative extrema.

1.

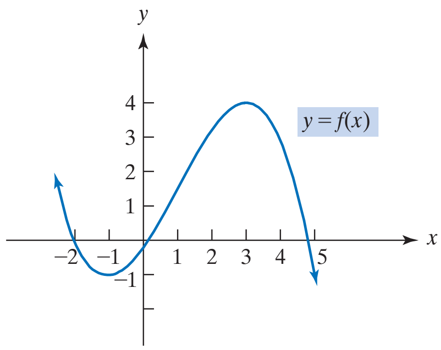

FIGURE 13.18

2.

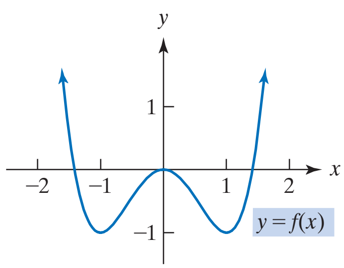

FIGURE 13.19

3.

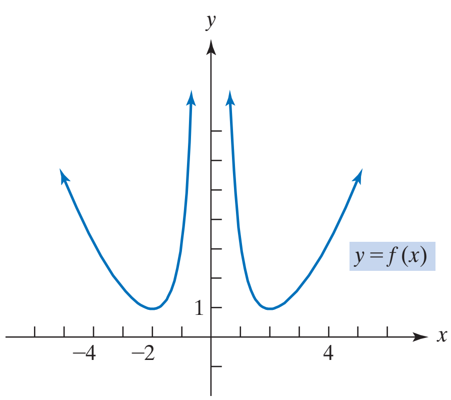

FIGURE 13.20

4.

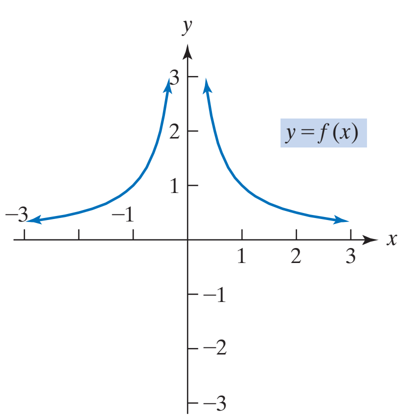

FIGURE 13.21

In Problems 5–8, the derivative of a differentiable function f is given. Find the open intervals on which f is **(a)** *increasing;* **(b)** *decreasing; and* **(c)** *find the x-values of all relative extrema.*

5. $f'(x) = (x+3)(x-1)(x-2)$

6. $f'(x) = x^2(x-2)^3$

7. $f'(x) = (x+1)(x-3)^2$

8. $f'(x) = \dfrac{x(x+2)}{x^2+1}$

In Problems 9–52, determine where the function is **(a)** *increasing;* **(b)** *decreasing; and* **(c)** *determine where relative extrema occur. Do not sketch the graph.*

9. $y = -x^3 - 1$

10. $y = x^2 + 4x + 3$

11. $y = 5 - 2x - x^2$

12. $y = x^3 - \dfrac{5}{2}x^2 - 2x + 6$

13. $y = -\dfrac{x^3}{3} - 2x^2 + 5x - 2$

14. $y = -\dfrac{x^4}{4} - x^3$

15. $y = x^4 - 2x^2$

16. $y = x^3 - \dfrac{3}{2}x^2 - 36x$

17. $y = x^3 - \dfrac{7}{2}x^2 + 2x - 5$

18. $y = x^3 - 6x^2 + 12x - 6$

19. $y = 2x^3 - \dfrac{19}{2}x^2 + 10x + 2$

20. $y = -5x^3 + x^2 + x - 7$

21. $y = 1 - 3x + 3x^2 - x^3$

22. $y = \dfrac{9}{5}x^5 - \dfrac{47}{3}x^3 + 10x$

23. $y = 3x^5 - 5x^3$

24. $y = 3x - \dfrac{x^6}{2}$ (Remark: $x^4 + x^3 + x^2 + x + 1 = 0$ has no real roots.)

25. $y = -x^5 - 5x^4 + 200$

26. $y = \dfrac{x^4}{4} - \dfrac{5x^3}{3} + \dfrac{7x^2}{2} - 3x$

27. $y = 8x^4 - x^8$

28. $y = \dfrac{4}{5}x^5 - \dfrac{13}{3}x^3 + 3x + 4$

29. $y = (x^2 - 4)^4$

30. $y = \sqrt[3]{x}(x-2)$

31. $y = \dfrac{3}{x+2}$

32. $y = \dfrac{3}{x}$

33. $y = \dfrac{10}{\sqrt{x}}$

34. $y = \dfrac{ax+b}{cx+d}$
(a) for $ad - bc > 0$
(b) for $ad - bc < 0$

35. $y = \dfrac{x^2}{2-x}$

36. $y = \dfrac{27x^2}{2} + \dfrac{1}{x}$

37. $y = \dfrac{x^2-3}{x+2}$

38. $y = \dfrac{2x^2}{4x^2-25}$

39. $y = \dfrac{ax^2+b}{cx^2+d}$ for $d/c < 0$
(a) for $ad - bc > 0$
(b) for $ad - bc < 0$

40. $y = \sqrt[3]{x^3 - 9x}$

41. $y = (x+1)^{2/3}$

42. $y = x^2(x+3)^4$

43. $y = x^3(x-6)^4$

44. $y = (1-x)^{2/3}$

45. $y = e^{-\pi x} + \pi$

46. $y = x^3 \ln x$

47. $y = x^2 - 9 \ln x$

48. $y = x^{-1} e^x$

49. $y = e^x - e^{-x}$

50. $y = e^{-x^2/2}$

51. $y = x^2 \ln x$

52. $y = (x^2 + 1)e^{-x}$

In Problems 53–64, determine intervals on which the function is increasing; intervals on which the function is decreasing; relative extrema; symmetry; and those intercepts that can be obtained conveniently. Then sketch the graph.

53. $y = x^2 - 3x - 10$

54. $y = 2x^2 + x - 10$

55. $y = 3x - x^3$

56. $y = x^4 - 81$

57. $y = 2x^3 - 9x^2 + 12x$

58. $y = 2x^3 - x^2 - 4x + 4$

59. $y = x^4 - 2x^2$

60. $y = x^6 - \frac{6}{5}x^5$

61. $y = (x - 1)^3(x - 2)^2$

62. $y = \sqrt{x}(x^2 - x - 2)$

63. $y = 2\sqrt{x} - x$

64. $y = x^{5/3} - 2x^{2/3}$

65. Sketch the graph of a continuous function f such that $f(2) = 2$, $f(4) = 6$, $f'(2) = f'(4) = 0$, $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $2 < x < 4$, f has a relative maximum at 4, and $\lim_{x\to\infty} f(x) = 0$.

66. Sketch the graph of a continuous function f such that $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 1, f'(0) = 0 = f'(2)$, there is a vertical tangent line when $x = 1$ and when $x = 3$, $f'(x) < 0$ for x in $(-\infty, 0)$ and x in $(2, 3)$, $f'(x) > 0$ for x in $(0, 1)$ and x in $(1, 2)$ and x in $(3, \infty)$.

67. Average Cost If $c_f = 25{,}000$ is a fixed-cost function, show that the average fixed-cost function $\bar{c}_f = c_f/q$ is a decreasing function for $q > 0$. Thus, as output q increases, each unit's portion of fixed cost declines.

68. Marginal Cost If $c = 3q - 3q^2 + q^3$ is a cost function, when is marginal cost increasing?

69. Marginal Revenue Given the demand function

$$p = 500 - 5q$$

find when marginal revenue is increasing.

70. Cost Function For the cost function $c = \sqrt{q}$, show that marginal and average costs are always decreasing for $q > 0$.

71. Revenue For a manufacturer's product, the revenue function is given by $r = 180q + 87q^2 - 2q^3$. Determine the output for maximum revenue.

72. Labor Markets Eswaran and Kotwal[1] consider agrarian economies in which there are two types of workers, permanent and casual. Permanent workers are employed on long-term contracts and may receive benefits such as holiday gifts and emergency aid. Casual workers are hired on a daily basis and perform routine and menial tasks such as weeding, harvesting, and threshing. The difference, z, in the present-value cost of hiring a permanent worker over that of hiring a casual worker is given by

$$z = (1 + b)w_p - bw_c$$

where w_p and w_c are wage rates for permanent labor and casual labor, respectively, b is a positive constant, and w_p is a function of w_c.

(a) Show that

$$\frac{dz}{dw_c} = (1 + b)\left[\frac{dw_p}{dw_c} - \frac{b}{1 + b}\right]$$

(b) If $dw_p/dw_c < b/(1 + b)$, show that z is a decreasing function of w_c.

73. Thermal Pollution In Shonle's discussion of thermal pollution,[2] the efficiency of a power plant is given by

$$E = 0.71\left(1 - \frac{T_c}{T_h}\right)$$

where T_h and T_c are the respective absolute temperatures of the hotter and colder reservoirs. Assume that T_c is a positive constant and that T_h is positive. Using calculus, show that as T_h increases, the efficiency increases.

74. Telephone Service In a discussion of the pricing of local telephone service, Renshaw[3] determines that total revenue r is given by

$$r = 2F + \left(1 - \frac{a}{b}\right)p - p^2 + \frac{a^2}{b}$$

where p is an indexed price per call, and a, b, and F are constants. Determine the value of p that maximizes revenue.

75. Storage and Shipping Costs In his model for storage and shipping costs of materials for a manufacturing process, Lancaster[4] derives the cost function

$$C(k) = 100\left(100 + 9k + \frac{144}{k}\right) \qquad 1 \le k \le 100$$

where $C(k)$ is the total cost (in dollars) of storage and transportation for 100 days of operation if a load of k tons of material is moved every k days.

(a) Find $C(1)$.

(b) For what value of k does $C(k)$ have a minimum?

(c) What is the minimum value?

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[1] M. Eswaran and A. Kotwal, "A Theory of Two-Tier Labor Markets in Agrarian Economics," *The American Economic Review*, 75, no. 1 (1985), 162–77.

[2] J. I. Shonle, *Environmental Applications of General Physics* (Reading, MA: Addison-Wesley Publishing Company, Inc., 1975).

[3] E. Renshaw, "A Note on Equity and Efficiency in the Pricing of Local Telephone Services," *The American Economic Review*, 75, no. 3 (1985), 515–18.

[4] P. Lancaster, *Mathematics: Models of the Real World* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1976).

76. Physiology—The Bends When a deep-sea diver undergoes decompression or a pilot climbs to a high altitude, nitrogen may bubble out of the blood, causing what is commonly called *the bends*. Suppose the percentage P of people who suffer effects of the bends at an altitude of h thousand feet is given by[5]

$$P = \frac{100}{1 + 100{,}000e^{-0.36h}}$$

Is P an increasing function of h?

In Problems 77–80, from the graph of the function, find the coordinates of all relative extrema. Round your answers to two decimal places.

77. $y = 0.3x^2 + 2.3x + 5.1$

78. $y = 3x^4 - 4x^3 - 5x + 1$

79. $y = \dfrac{8.2x}{0.4x^2 + 3}$

80. $y = \dfrac{e^x(3 - x)}{7x^2 + 1}$

81. Graph the function

$$f(x) = (x(x-2)(2x-3))^2$$

in a calculator window with $-1 \le x \le 3$, $-1 \le y \le 3$. At first glance, it may appear that this function has two relative minimum points and one relative maximum point. However, in reality, it has three relative minimum points and two relative maximum points. Determine the x-values of all these points. Round answers to two decimal places.

82. If $f(x) = 3x^3 - 7x^2 + 4x + 2$, display the graphs of f and f' on the same screen. Notice that $f'(x) = 0$ where relative extrema of f occur.

83. Let $f(x) = 6 + 4x - 3x^2 - x^3$. **(a)** Find $f'(x)$. **(b)** Graph $f'(x)$. **(c)** Observe where $f'(x)$ is positive and where it is negative. Give the intervals (rounded to two decimal places) where f is increasing and where f is decreasing. **(d)** Graph f and f' on the same screen, and verify your results to part (c).

84. If $f(x) = x^4 - x^2 - (x + 2)^2$, find $f'(x)$. Determine the critical values of f. Round your answers to two decimal places.

Objective

To find extreme values on a closed interval.

13.2 Absolute Extrema on a Closed Interval

If a function f is *continuous* on a *closed* interval $[a, b]$, it can be shown that of *all* the function values $f(x)$ for x in $[a, b]$, there must be an absolute maximum value and an absolute minimum value. These two values are called **extreme values** of f on that interval. This important property of continuous functions is called the **extreme-value theorem**.

Extreme-Value Theorem

If a function is continuous on a closed interval, then the function has *both* a maximum value *and* a minimum value on that interval.

For example, each function in Figure 13.22 is continuous on the closed interval $[1, 3]$. Geometrically, the extreme-value theorem assures us that over this interval each graph has a highest point and a lowest point.

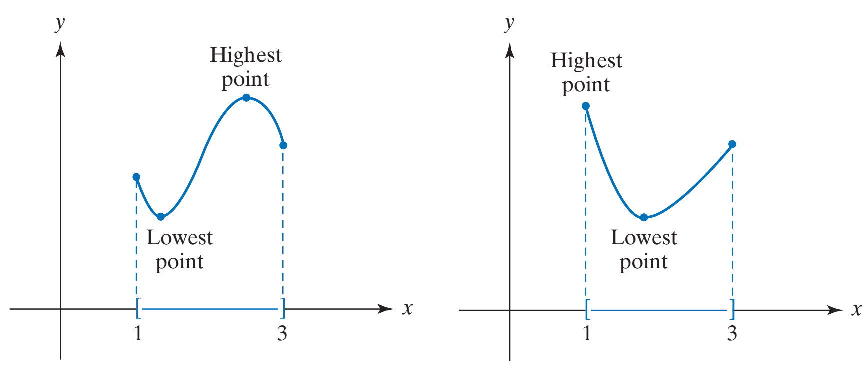


FIGURE 13.22 Illustrating the extreme-value theorem.

In the extreme-value theorem, it is important that we are dealing with

1. a closed interval, and
2. a function continuous on that interval.

[5]Adapted from G. E. Folk, Jr., *Textbook of Environmental Physiology,* 2nd ed. (Philadelphia: Lea & Febiger, 1974).

EXAMPLE 1 Finding Extreme Values on a Closed Interval

Find absolute extrema for $f(x) = x^2 - 4x + 5$ over the closed interval $[1, 4]$.

Solution: Since f is continuous on $[1, 4]$, the foregoing procedure applies.

Step 1. To find the critical values of f, we first find f':

$$f'(x) = 2x - 4 = 2(x - 2)$$

This gives the critical value $x = 2$.

Step 2. Evaluating $f(x)$ at the endpoints 1 and 4 and at the critical value 2, we have

$$\boxed{\begin{aligned} f(1) &= 2 \\ f(4) &= 5 \end{aligned}} \quad \text{values of } f \text{ at endpoints}$$

and

$$\boxed{f(2) = 1} \quad \text{value of } f \text{ at critical value 2 in } (1, 4)$$

Step 3. From the function values in Step 2, we conclude that the maximum is $f(4) = 5$ and the minimum is $f(2) = 1$. (See Figure 13.25.)

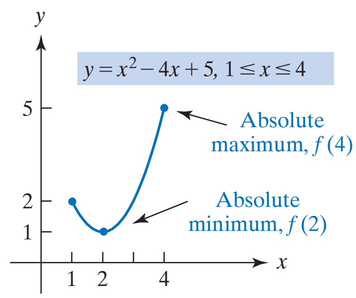


FIGURE 13.25 Extreme values for Example 1.

Now Work Problem 1 ◁

PROBLEMS 13.2

In Problems 1–14, find the absolute extrema of the given function on the given interval.

1. $f(x) = x^2 - 2x + 3$, $[0, 3]$
2. $f(x) = -3x^2 + 12x + 1$, $[1, 3]$
3. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 1$, $[-1, 0]$
4. $f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2$, $[0, 1]$
5. $f(x) = x^3 - 5x^2 - 8x + 50$, $[0, 5]$
6. $f(x) = x^{2/3}$, $[-8, 8]$
7. $f(x) = (1/6)x^6 - (3/4)x^4 - 2x^2$, $[-1, 1]$
8. $f(x) = \frac{7}{3}x^3 + 2x^2 - 3x + 1$, $[0, 3]$
9. $f(x) = 3x^4 - x^6$, $[-1, 2]$
10. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 2$, $[0, 4]$
11. $f(x) = x^4 - 9x^2 + 2$, $[-1, 3]$
12. $f(x) = \dfrac{x}{x^2 - 1}$, $[2, 3]$
13. $f(x) = (x - 1)^{2/3}$, $[-26, 28]$
14. $f(x) = 0.2x^3 - 3.6x^2 + 2x + 1$, $[-1, 2]$
15. Consider the function

$$f(x) = x^4 + 8x^3 + 21x^2 + 20x + 9$$

over the interval $[-4, 9]$.

(a) Determine the value(s) (rounded to two decimal places) of x at which f attains a minimum value.
(b) What is the minimum value (rounded to two decimal places) of f?
(c) Determine the value(s) of x at which f attains a maximum value.
(d) What is the maximum value of f?

Objective

To test a function for concavity and inflection points. To sketch curves with the aid of the information obtained from both first and second derivatives.

13.3 Concavity

The first derivative provides a lot of information for sketching curves. It is used to determine where a function is increasing, is decreasing, has relative maxima, and has relative minima. However, to be sure we know the true shape of a curve, we may need more information. For example, consider the curve $y = f(x) = x^2$. Since $f'(x) = 2x$, $x = 0$ is a critical value. If $x < 0$, then $f'(x) < 0$, and f is decreasing; if $x > 0$, then $f'(x) > 0$, and f is increasing. Thus, there is a relative minimum at $x = 0$. In Figure 13.26, both curves meet the preceding conditions. But which one truly describes the curve $y = x^2$? This question will be settled easily by using the second derivative and the notion of *concavity*.

PROBLEMS 13.3

In Problems 1–6, a function and its second derivative are given. Determine the concavity of f and find x-values where points of inflection occur.

1. $f(x) = x^4 - 3x^3 - 6x^2 + 6x + 1; f''(x) = 6(2x + 1)(x - 2)$

2. $f(x) = \dfrac{x^5}{20} + \dfrac{x^4}{4} - 2x^2; f''(x) = (x - 1)(x + 2)^2$

3. $f(x) = \dfrac{x^2 + 3x + 1}{x^2 + 2x + 1}; f''(x) = \dfrac{2x - 4}{(x + 1)^4}$

4. $f(x) = \dfrac{x^2}{(x - 1)^2}; f''(x) = \dfrac{2(2x + 1)}{(x - 1)^4}$

5. $f(x) = \dfrac{x^2 + 1}{x^2 - 2}; f''(x) = \dfrac{6(3x^2 + 2)}{(x^2 - 2)^3}$

6. $f(x) = x\sqrt{a^2 - x^2}; f''(x) = \dfrac{x(2x^2 - 3a^2)}{(a^2 - x^2)^{3/2}}$

In Problems 7–34, determine concavity and the x-values where points of inflection occur. Do not sketch the graphs.

7. $y = -2x^2 + 4x$

8. $y = 4x^2 - 375x + 947$

9. $y = 4x^3 + 12x^2 - 12x$

10. $y = x^3 - 6x^2 + 9x + 1$

11. $y = ax^3 + bx^2 + cx + d$

12. $y = x^4 - 8x^2 - 6$

13. $y = x^5 - 10x^4 + \frac{110}{3}x^3 - 60x^2$

14. $y = -\dfrac{x^4}{4} + \dfrac{9x^2}{2} + 2x$

15. $y = 2x^{1/5}$

16. $y = \dfrac{a}{x^3}$

17. $y = \dfrac{x^4}{2} + \dfrac{19x^3}{6} - \dfrac{7x^2}{2} + x + 5$

18. $y = \dfrac{2}{4}x^4 + \dfrac{11}{6}x^3 + \dfrac{3}{2}x^2 + \dfrac{7}{5}x + \dfrac{3}{5}$

19. $y = \dfrac{1}{20}x^5 - \dfrac{1}{4}x^4 + \dfrac{1}{6}x^3 - \dfrac{1}{2}x - \dfrac{2}{3}$

20. $y = \dfrac{1}{10}x^5 - 3x^3 + 17x + 43$

21. $y = \dfrac{1}{30}x^6 - \dfrac{7}{12}x^4 + 6x^2 + 5x - 4$

22. $y = x^6 - 3x^4$

23. $y = \dfrac{x - 1}{x + 1}$

24. $y = 1 - \dfrac{1}{x^2}$

25. $y = \dfrac{x^2}{x^2 + 1}$

26. $y = \dfrac{ax^2}{x + b}$

27. $y = \dfrac{21x + 40}{6(x + 3)^2}$

28. $y = (x^2 - 12)^2$

29. $y = 5e^x$

30. $y = e^x - e^{-x}$

31. $y = axe^x$

32. $y = xe^{x^2}$

33. $y = \ln x$

34. $y = \dfrac{x^2 + 1}{3e^x}$

In Problems 35–62, determine intervals on which the function is increasing, decreasing, concave up, and concave down; relative maxima and minima; inflection points; symmetry; and those intercepts that can be obtained conveniently. Then sketch the graph.

35. $y = x^2 - x - 6$

36. $y = x^2 + a$ for $a > 0$

37. $y = 5x - 2x^2$

38. $y = -1 - x^2 + 2x$

39. $y = x^3 - 9x^2 + 24x - 19$

40. $y = x^3 - 25x^2$

41. $y = \dfrac{x^3}{3} - 5x$

42. $y = x^3 - 6x^2 + 9x$

43. $y = x^3 + 3x^2 + 3x + 1$

44. $y = 2x^3 + \dfrac{5}{2}x^2 + 2x$

45. $y = 4x^3 - 3x^4$

46. $y = -x^3 + 8x^2 - 5x + 3$

47. $y = -2 + 12x - x^3$

48. $y = -(3x + 2)^3$

49. $y = 2x^3 - 6x^2 + 6x - 2$

50. $y = \dfrac{x^5}{100} - \dfrac{x^4}{20}$

51. $y = 16x - x^5$

52. $y = x^2(x - 1)^2$

53. $y = 6x^4 - 8x^3 + 3$

54. $y = 3x^5 - 5x^3$

55. $y = 4x^2 - x^4$

56. $y = x^2e^x$

57. $y = x^{1/3}(x - 8)$

58. $y = (x + 1)^2(x - 2)^2$

59. $y = 4x^{1/3} + x^{4/3}$

60. $y = (x + 1)\sqrt{x + 4}$

61. $y = 2x^{2/3} - x$

62. $y = 5x^{2/3} - x^{5/3}$

63. Sketch the graph of a continuous function f such that $f(0) = 0 = f(3), f'(1) = 0 = f'(3), f''(x) < 0$ for $x < 2$, and $f''(x) > 0$ for $x > 2$.

64. Sketch the graph of a continuous function f such that $f(4) = 4, f'(4) = 0, f''(x) < 0$ for $x < 4$, and $f''(x) > 0$ for $x > 4$.

65. Sketch the graph of a continuous function f such that $f(1) = 1, f'(1) = 0$, and $f''(x) < 0$ for all x.

66. Sketch the graph of a continuous function f such that $f(1) = 1$, both $f'(x) < 0$ and $f''(x) < 0$ for $x < 1$, and both $f(x) > 0$ and $f''(x) < 0$ for $x > 1$.

67. Demand Equation Show that the graph of the demand equation $p = \dfrac{100}{q + 2}$ is decreasing and concave up for $q > 0$.

68. Average Cost For the cost function

$$c = q^2 + 3q + 2$$

show that the graph of the average-cost function $\bar{c}$ is concave up for all $q > 0$.

69. Species of Plants The number of species of plants on a plot may depend on the size of the plot. For example, in Figure 13.42, we see that on 1-m^2 plots there are three species (A, B, and C on the left plot, A, B, and D on the right plot), and on a 2-m^2 plot there are four species (A, B, C, and D).

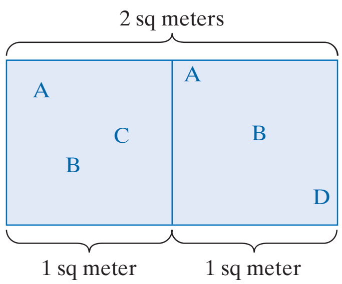


FIGURE 13.42

In a study of rooted plants in a certain geographic region,[6] it was determined that the average number of species, S, occurring on plots of size A (in square meters) is given by

$$S = f(A) = 12\sqrt[4]{A} \quad 0 \leq A \leq 625$$

Sketch the graph of f. (*Note:* Your graph should be rising and concave down. Thus, the number of species is increasing with respect to area, but at a decreasing rate.)

70. Inferior Good In a discussion of an inferior good, Persky[7] considers a function of the form

$$g(x) = e^{(U_0/A)}e^{-x^2/(2A)}$$

where x is a quantity of a good, U_0 is a constant that represents utility, and A is a positive constant. Persky claims that the graph of g is concave down for $x < \sqrt{A}$ and concave up for $x > \sqrt{A}$. Verify this.

71. Psychology In a psychological experiment involving conditioned response,[8] subjects listened to four tones, denoted 0, 1, 2, and 3. Initially, the subjects were conditioned to tone 0 by receiving a shock whenever this tone was heard. Later, when each of the four tones (stimuli) was heard without shocks, the subjects' responses were recorded by means of a tracking device that measures galvanic skin reaction. The average response to each stimulus (without shock) was determined, and the results were plotted on a coordinate plane where the x- and y-axes represent the stimuli (0, 1, 2, 3) and the average galvanic responses, respectively. It was determined that the points fit a curve that is approximated by the graph of

$$y = 12.5 + 5.8(0.42)^x$$

Show that this function is decreasing and concave up.

72. Entomology In a study of the effects of food deprivation on hunger,[9] an insect was fed until its appetite was completely satisfied. Then it was deprived of food for t hours (the deprivation period). At the end of this period, the insect was re-fed until its appetite was again completely satisfied. The weight H (in grams) of the food that was consumed at this time was statistically found to be a function of t, where

$$H = 1.00[1 - e^{-(0.0464t+0.0670)}]$$

Here H is a measure of hunger. Show that H is increasing with respect to t and is concave down.

73. Insect Dispersal In an experiment on the dispersal of a particular insect,[10] a large number of insects are placed at a release point in an open field. Surrounding this point are traps that are placed in a concentric circular arrangement at a distance of 1 m, 2 m, 3 m, and so on from the release point. Twenty-four hours after the insects are released, the number of insects in each trap is counted. It is determined that at a distance of r meters from the release point, the average number of insects contained in a trap is

$$n = f(r) = 0.1\ln(r) + \frac{7}{r} - 0.8 \quad 1 \leq r \leq 10$$

(a) Show that the graph of f is always falling and concave up. **(b)** Sketch the graph of f. (c) When $r = 5$, at what rate is the average number of insects in a trap decreasing with respect to distance?

74. Graph $y = -0.35x^3 + 4.1x^2 + 8.3x - 7.4$, and from the graph determine the number of **(a)** relative maximum points, **(b)** relative minimum points, and **(c)** inflection points.

75. Graph $y = x^5(x - 2.3)$, and from the graph determine the number of inflection points. Now, prove that for any $a \neq 0$, the curve $y = x^5(x - a)$ has two points of inflection.

76. Graph $y = xe^{-x}$ and determine the number of inflection points, first using a graphing calculator and then using the techniques of this chapter. If a demand equation has the form $q = q(p) = Qe^{-Rp}$ for constants Q and R, relate the graph of the resulting revenue function to that of the function graphed above, by taking $Q = 1 = R$.

77. Graph the curve $y = x^3 - 2x^2 + x + 3$, and also graph the tangent line to the curve at $x = 2$. Around $x = 2$, does the curve lie above or below the tangent line? From your observation determine the concavity at $x = 2$.

78. Let f be a function for which both $f'(x)$ and $f''(x)$ exist. Suppose that f' has a a relative minimum at a. Show that f changes its direction of bending at a. This means that the concavity of f changes at $x = a$ which means that the direction of bending of the graph of f changes at $x = a$.

79. If $f(x) = x^6 + 3x^5 - 4x^4 + 2x^2 + 1$, find the x-values (rounded to two decimal places) of the inflection points of f.

80. If $f(x) = \dfrac{x+1}{x^2+1}$, find the x-values (rounded to two decimal places) of the inflection points of f.

[6] Adapted from R. W. Poole, *An Introduction to Quantitative Ecology* (New York: McGraw-Hill Book Company, 1974).

[7] A. L. Persky, "An Inferior Good and a Novel Indifference Map," *The American Economist* XXIX, no. 1 (1985), 67–69.

[8] Adapted from C. I. Hovland, "The Generalization of Conditioned Responses: I. The Sensory Generalization of Conditioned Responses with Varying Frequencies of Tone," *Journal of General Psychology,* 17 (1937), 125–48.

[9] C. S. Holling, "The Functional Response of Invertebrate Predators to Prey Density," *Memoirs of the Entomological Society of Canada,* no. 48 (1966).

[10] Adapted from Poole, op. cit.

and

$$\text{if } x = 1, \quad \text{then } y'' > 0$$

By the second-derivative test, there is a relative minimum when $x = 1$. We cannot apply the test when $x = 0$ because $y'' = 0$ there. To analyze what is happening at 0, we turn to the first-derivative test:

$$\text{If } x < 0, \quad \text{then } y' < 0.$$
$$\text{If } 0 < x < 1, \quad \text{then } y' < 0.$$

Thus, no maximum or minimum exists when $x = 0$. (Refer to Figure 13.35.)

Now Work Problem 5 ◁

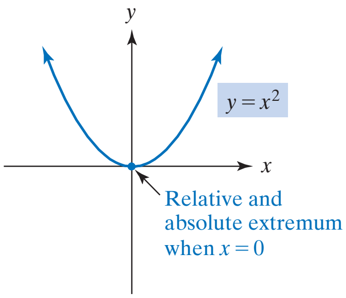


FIGURE 13.44 Exactly one relative extremum implies an absolute extremum.

If a continuous function has *exactly one* relative extremum on an interval, it can be shown that the relative extremum must also be an *absolute* extremum on the interval. To illustrate, in Figure 13.44 the function $y = x^2$ has a relative minimum when $x = 0$, and there are no other relative extrema. Since $y = x^2$ is continuous, this relative minimum is also an absolute minimum for the function.

EXAMPLE 2 Absolute Extrema

If $y = f(x) = x^3 - 3x^2 - 9x + 5$, determine when absolute extrema occur on the interval $(0, \infty)$.

Solution: We have

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3)$$
$$= 3(x + 1)(x - 3)$$

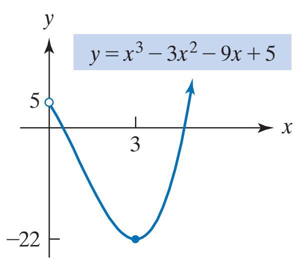


FIGURE 13.45 On $(0, \infty)$, there is an absolute minimum at 3.

The only critical value on the interval $(0, \infty)$ is 3. Applying the second-derivative test at this point gives

$$f''(x) = 6x - 6$$
$$f''(3) = 6(3) - 6 = 12 > 0$$

Thus, there is a relative minimum at 3. Since this is the only relative extremum on $(0, \infty)$ and f is continuous there, we conclude by our previous discussion that there is an *absolute* minimum value at 3; this value is $f(3) = -22$. (See Figure 13.45.)

Now Work Problem 3 ◁

PROBLEMS 13.4

In Problems 1–14, test for relative maxima and minima. Use the second-derivative test, if possible. In Problems 1–4, state whether the relative extrema are also absolute extrema.

1. $y = x^2 - 5x + 6$

2. $y = 3x^2 + 12x + 14$

3. $y = -4x^2 + 2x - 8$

4. $y = -5x^2 + 11x - 7$

5. $y = \frac{1}{3}x^3 + 2x^2 - 5x + 1$

6. $y = x^3 - 12x + 1$

7. $y = 2x^3 - 3x^2 - 36x + 17$

8. $y = x^4 - 2x^2 + 4$

9. $y = 3 + 5x^4$

10. $y = -2x^7$

11. $y = 81x^5 - 5x$

12. $y = 15x^3 + x^2 - 15x + 2$

13. $y = (x^2 + 7x + 10)^2$

14. $y = 2x^3 - 9x^2 - 60x + 42$

x	0	1	$\sqrt{3}$	-1	$-\sqrt{3}$
y	0	2	$\sqrt{3}$	-2	$-\sqrt{3}$

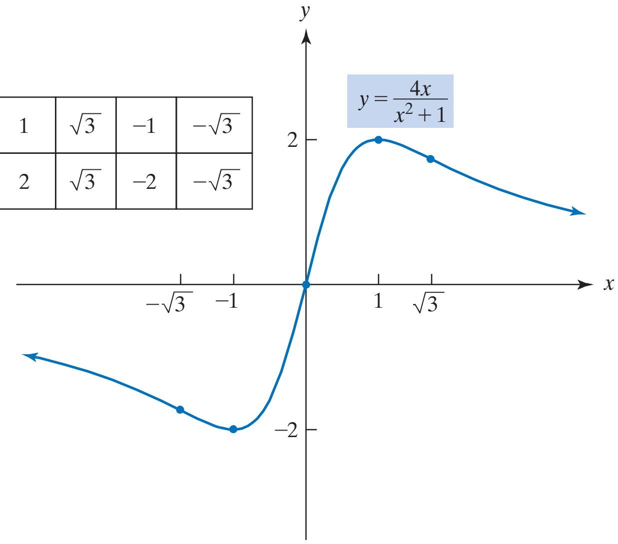


FIGURE 13.60 Graph of $y = \dfrac{4x}{x^2+1}$.

Discussion After consideration of all of the preceding information, the graph of $y = 4x/(x^2 + 1)$ is given in Figure 13.60, together with a table of important points.

Now Work Problem 39 ◁

PROBLEMS 13.5

In Problems 1–24, find the vertical asymptotes and the nonvertical asymptotes for the graphs of the functions. Do not sketch the graphs.

1. $y = \dfrac{x}{x-1}$

2. $y = \dfrac{x+1}{x}$

3. $f(x) = \dfrac{x+5}{2x+7}$

4. $y = \dfrac{2x+1}{2x+1}$

5. $y = \dfrac{3}{x^3}$

6. $y = 1 - \dfrac{2}{x^2}$

7. $y = \dfrac{1}{x^2-1}$

8. $y = \dfrac{x}{x^2-9}$

9. $y = x^2 - 5x + 5$

10. $y = \dfrac{x^3}{x^2-1}$

11. $f(x) = \dfrac{2x^2}{x^2+x-6}$

12. $f(x) = \dfrac{x^3}{5}$

13. $y = \dfrac{15x^2+31x+1}{x^2-7}$

14. $y = \dfrac{2x^3+1}{3x(2x-1)(4x-3)}$

15. $y = \dfrac{3}{x-5} + 7$

16. $f(x) = \dfrac{x^2-1}{2x^2-9x+4}$

17. $f(x) = \dfrac{3-x^4}{x^3+x^2}$

18. $y = \dfrac{5x^2+7x^3+9x^4}{3x^2}$

19. $y = \dfrac{x^2-3x-4}{1+4x+4x^2}$

20. $y = \dfrac{x^3+1}{1-x^3}$

21. $y = \dfrac{9x^2-16}{2(3x+4)^2}$

22. $y = \dfrac{2}{5} + \dfrac{2x}{12x^2+5x-2}$

23. $y = 5e^{x-3} - 2$

24. $f(x) = 12e^{-x}$

In Problems 25–46, determine intervals on which the function is increasing, decreasing, concave up, and concave down; relative maxima and minima; inflection points; symmetry; vertical and nonvertical asymptotes; and those intercepts that can be obtained conveniently. Then sketch the curve.

25. $y = \dfrac{1}{x^3}$

26. $y = \dfrac{2}{2x-3}$

27. $y = \dfrac{x}{x-1}$

28. $y = \dfrac{50}{\sqrt{3x}}$

29. $y = x^2 + \dfrac{1}{x^2}$

30. $y = \dfrac{x^2+x+1}{x-2}$

31. $y = \dfrac{1}{x^2-1}$

32. $y = \dfrac{1}{x^2+1}$

33. $y = \dfrac{2+x}{3-x}$

34. $y = \dfrac{1+x}{x^2}$

35. $y = \dfrac{x^2}{x-1}$

36. $y = \dfrac{x^3+1}{x}$

37. $y = \dfrac{9}{9x^2-6x-8}$

38. $y = \dfrac{4x^2+2x+1}{2x^2}$

39. $y = \dfrac{3x+1}{(3x-2)^2}$

40. $y = \dfrac{3x+5}{(7x+11)^2}$

41. $y = \dfrac{x^2-1}{x^3}$

42. $y = \dfrac{3x}{(x-2)^2}$

43. $y = 2x + 1 + \dfrac{1}{x-1}$

44. $y = \dfrac{3x^4+1}{x^3}$

45. $y = \dfrac{1-x^2}{x^2-1}$

46. $y = 3x + 2 + \dfrac{1}{3x+2}$

47. Sketch the graph of a function f such that $f(0) = 0$, there is a horizontal asymptote $y = 1$ for $x \to \pm\infty$, there is a vertical asymptote $x = 2$, both $f'(x) < 0$ and $f''(x) < 0$ for $x < 2$, and both $f'(x) < 0$ and $f''(x) > 0$ for $x > 2$.

48. Sketch the graph of a function f such that $f(0) = -4$ and $f(4) = -2$, there is a horizontal asymptote $y = -3$ for $x \to \pm\infty$, there is a vertical asymptote $x = 2$, both $f'(x) < 0$ and $f''(x) < 0$ for $x < 2$, and both $f'(x) < 0$ and $f''(x) > 0$ for $x > 2$.

49. Sketch the graph of a function f such that $f(0) = 0$, there is a horizontal asymptote $y = 0$ for $x \to \pm\infty$, there are vertical asymptotes $x = -1$ and $x = 2$, $f'(x) < 0$ for $x < -1$ and $-1 < x < 2$, and $f''(x) < 0$ for $x > 2$.

50. Sketch the graph of a function f such that $f(0) = 0$, there are vertical asymptotes $x = -1$ and $x = 1$, there is a horizontal asymptote $y = 0$ for $x \to \pm\infty$. $f'(x) < 0$ for x in $(-\infty, -1)$, in $(-1, 1)$, and in $(1, \infty)$. $f''(0) = 0$; $f''(x) < 0$ for x in $(-\infty, -1)$ and in $(0, 1)$; $f''(x) > 0$ for x in $(-1, 0)$ and in $(1, \infty)$.

51. Purchasing Power In discussing the time pattern of purchasing, Mantell and Sing[11] use the curve

$$y = \frac{x}{a + bx}$$

as a mathematical model. Find the asymptotes for their model.

52. Sketch the graphs of $y = 6 - 3e^{-x}$ and $y = 6 + 3e^{-x}$. Show that they are asymptotic to the same line. What is the equation of this line?

53. Market for Product For a new product, the yearly number of thousand packages sold, y, t years after its introduction is predicted to be given by

$$y = f(t) = 250 - 83e^{-t}$$

Show that $y = 250$ is a horizontal asymptote for the graph. This reveals that after the product is established with consumers, the market tends to be constant.

54. Graph $y = \dfrac{x^2-2}{x^3 + \frac{7}{2}x^2 + 12x + 1}$. From the graph, locate any horizontal or vertical asymptotes.

55. With a graphing utility, graph $y = \dfrac{2x^3 - 2x^2 + 6x - 1}{x^3 - 6x^2 + 11x - 6}$. From the graph, locate any horizontal or vertical asymptotes.

56. Graph $y = \dfrac{\ln(x+4)}{x^2 - 8x + 5}$ in the standard window. The graph suggests that there are two vertical asymptotes of the form $x = k$, where $k > 0$. Also, it appears that the graph "begins" near $x = -4$. As $x \to -4^+$, $\ln(x+4) \to -\infty$ and $x^2 - 8x + 5 \to 53$. Thus, $\lim_{x \to 4^+} y = -\infty$. This gives the vertical asymptote $x = -4$. So, in reality, there are *three* vertical asymptotes. Use the zoom feature to make the asymptote $x = -4$ apparent from the display.

57. Graph $y = \dfrac{0.34e^{0.7x}}{4.2 + 0.71e^{0.7x}}$, where $x > 0$. From the graph, determine an equation of the horizontal asymptote by examining the y-values as $x \to \infty$. To confirm this equation algebraically, find $\lim_{x \to \infty} y$ by first dividing both the numerator and denominator by $e^{0.7x}$.

Objective

To model situations involving maximizing or minimizing a quantity.

13.6 Applied Maxima and Minima

By using techniques from this chapter, we can solve problems that involve maximizing or minimizing a quantity. For example, we might want to maximize profit or minimize cost. *The crucial part is expressing the quantity to be maximized or minimized as a function of some variable in the problem.* Then we differentiate and test the resulting critical values. For this, the first-derivative test or the second-derivative test can be used, although it may be obvious from the nature of the problem whether or not a critical value represents an appropriate answer. Because our interest is in *absolute* maxima and minima, sometimes we must examine endpoints of the domain of the function. Very often the function used to model the situation of a problem will be the restriction to a closed interval of a function that has a large natural domain. Such real-world limitations tend to generate endpoints.

The aim of this example is to set up a cost function from which cost is minimized.

EXAMPLE 1 Minimizing the Cost of a Fence

For insurance purposes, a manufacturer plans to fence in a 10,800-ft^2 rectangular storage area adjacent to a building by using the building as one side of the enclosed area. The fencing parallel to the building faces a highway and will cost \$3 per foot, installed, whereas the fencing for the other two sides costs \$2 per foot, installed. Find the amount

[11] L. H. Mantell and F. P. Sing, *Economics for Business Decisions* (New York: McGraw-Hill Book Company, 1972), p. 107.

to the consumer, and the monopolist must bear the cost of the balance. The profit now is \$15,495, which is less than the former profit.

Now Work Problem 13 ◁

This discussion leads to the economic principle that when profit is maximum, marginal revenue is equal to marginal cost.

We conclude this section by using calculus to develop an important principle in economics. Suppose $p = f(q)$ is the demand function for a firm's product, where p is price per unit and q is the number of units produced and sold. Then the total revenue is given by $r = qp = qf(q)$, which is a function of q. Let the total cost of producing q units be given by the cost function $c = g(q)$. Thus, the total profit, which is total revenue minus total cost, is also a function of q, namely,

$$P(q) = r - c = qf(q) - g(q)$$

Let us consider the most profitable output for the firm. Ignoring special cases, we know that profit is maximized when $dP/dq = 0$ and $d^2P/dq^2 < 0$. We have

$$\frac{dP}{dq} = \frac{d}{dq}(r - c) = \frac{dr}{dq} - \frac{dc}{dq}$$

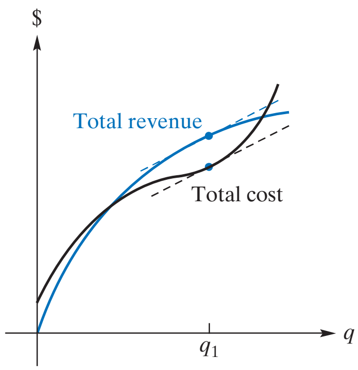


FIGURE 13.63 At maximum profit, marginal revenue equals marginal cost.

Consequently, $dP/dq = 0$ when

$$\frac{dr}{dq} = \frac{dc}{dq}$$

That is, at the level of maximum profit, the slope of the tangent to the total-revenue curve must equal the slope of the tangent to the total-cost curve (Figure 13.63). But dr/dq is the marginal revenue MR, and dc/dq is the marginal cost MC. Thus, under typical conditions, to maximize profit, it is necessary that

$$\text{MR} = \text{MC}$$

For this to indeed correspond to a maximum, it is necessary that $d^2P/dq^2 < 0$:

$$\frac{d^2P}{dq^2} = \frac{d^2}{dq^2}(r - c) = \frac{d^2r}{dq^2} - \frac{d^2c}{dq^2} < 0 \quad \text{equivalently} \quad \frac{d^2r}{dq^2} < \frac{d^2c}{dq^2}$$

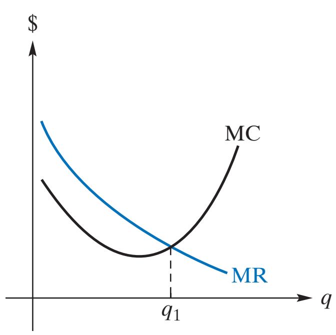


FIGURE 13.64 At maximum profit, the marginal-cost curve cuts the marginal-revenue curve from below.

That is, when MR = MC, in order to ensure maximum profit, the slope of the marginal-revenue curve must be less than the slope of the marginal-cost curve.

The condition that $d^2P/dq^2 < 0$ when $dP/dq = 0$ can be viewed another way. Equivalently, to have MR = MC correspond to a maximum, dP/dq must go from $+$ to $-$; that is, it must go from $dr/dq - dc/dq > 0$ to $dr/dq - dc/dq < 0$. Hence, as output increases, we must have MR > MC and then MR < MC. This means that at the point q_1 of maximum profit, *the marginal-revenue curve must cut the marginal-cost curve from above* (Figure 13.64). For production up to q_1, the revenue from additional output would be greater than the cost of such output, and the total profit would increase. For output beyond q_1, MC > MR, and each unit of output would add more to total costs than to total revenue. Hence, total profits would decline.

PROBLEMS 13.6

In this set of problems, unless otherwise specified, p is price per unit (in dollars) and q is output per unit of time. Fixed costs refer to costs that remain constant at all levels of production during a given time period. (An example is rent.)

1. Find two numbers whose sum is 96 and whose product is as big as possible.

2. Find two nonnegative numbers whose sum is 20 and for which the product of twice one number and the square of the other number will be a maximum.

3. Fencing A company has set aside \$9000 to fence in a rectangular portion of land adjacent to a stream by using the stream for one side of the enclosed area. The cost of the fencing parallel to the stream is \$15 per foot, installed, and the fencing for the remaining two sides costs \$9 per foot, installed. Find the dimensions of the maximum enclosed area.

4. Fencing The owner of the Laurel Nursery Garden Center wants to fence in 1400 ft^2 of land in a rectangular plot to be used for different types of shrubs. The plot is to be divided into six equal plots with five fences parallel to the same pair of sides, as shown in Figure 13.65. What is the least number of feet of fence needed?

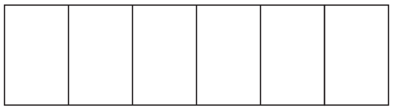


FIGURE 13.65

5. Average Cost A manufacturer finds that the total cost, c, of producing a product is given by the cost function

$$c = 0.05q^2 + 5q + 500$$

At what level of output will average cost per unit be a minimum?

6. Automobile Expense The cost per hour (in dollars) of operating an automobile is given by

$$C = 0.0015s^2 - 0.24s + 1 \qquad 0 \le s \le 100$$

where s is the speed in kilometers per hour. At what speed is the cost per hour a minimum?

7. Revenue The demand equation for a monopolist's product is

$$p = -5q + 30$$

At what price will revenue be maximized?

8. Revenue Suppose that the demand function for a monopolist's product is of the form

$$q = Ae^{-Bp}$$

for positive constants A and B. In terms of A and B, find the value of p for which maximum revenue is obtained. Can you explain why your answer does not depend on A?

9. Weight Gain A group of biologists studied the nutritional effects on rats that were fed a diet containing 10% protein.[12] The protein consisted of yeast and cottonseed flour. By varying the percent, p, of yeast in the protein mix, the group found that the (average) weight gain (in grams) of a rat over a period of time was

$$f(p) = 170 - p - \frac{1600}{p + 15} \qquad 0 \le p \le 100$$

Find **(a)** the maximum weight gain and **(b)** the minimum weight gain.

10. Drug Dose The severity of the reaction of the human body to an initial dose, D, of a drug is given by[13]

$$R = f(D) = D^2\left(\frac{C}{2} - \frac{D}{3}\right)$$

where the constant C denotes the maximum amount of the drug that may be given. Show that R has a maximum *rate of change* when $D = C/2$.

11. Profit For a monopolist's product, the demand function is

$$p = 75 - 0.05q$$

and the cost function is

$$c = 500 + 40q$$

At what level of output will profit be maximized? At what price does this occur, and what is the profit?

12. Profit For a monopolist, the cost per unit of producing a product is \$3, and the demand equation is

$$p = \frac{10}{\sqrt{q}}$$

What price will give the greatest profit?

13. Profit For a monopolist's product, the demand equation is

$$p = 42 - 4q$$

and the average-cost function is

$$\bar{c} = 2 + \frac{80}{q}$$

Find the profit-maximizing price.

14. Profit For a monopolist's product, the demand function is

$$p = \frac{50}{\sqrt{q}}$$

and the average-cost function is

$$\bar{c} = \frac{1}{4} + \frac{2500}{q}$$

Find the profit-maximizing price and output.

15. Profit A manufacturer can produce at most 120 units of a certain product each year. The demand equation for the product is

$$p = q^2 - 100q + 3200$$

and the manufacturer's average-cost function is

$$\bar{c} = \frac{2}{3}q^2 - 40q + \frac{10{,}000}{q}$$

Determine the profit-maximizing output q and the corresponding maximum profit.

[12] Adapted from R. Bressani, "The Use of Yeast in Human Foods," in *Single-Cell Protein,* eds. R. I. Mateles and S. R. Tannenbaum (Cambridge, MA: MIT Press, 1968).

[13] R. M. Thrall, J. A. Mortimer. K. R. Rebman, and R. F. Baum, eds., *Some Mathematical Models in Biology,* rev. ed., Report No. 40241-R-7. Prepared at University of Michigan, 1967.

16. Cost A manufacturer has determined that, for a certain product, the average cost (in dollars per unit) is given by

$$\bar{c} = 2q^2 - 48q + 210 + \frac{200}{q}$$

where $2 \leq q \leq 7$.
(a) At what level within the interval [2, 7] should production be fixed in order to minimize total cost? What is the minimum total cost?
(b) If production were required to lie in the interval [3, 7], what value of q would minimize total cost?

17. Profit For XYZ Manufacturing Co., total fixed costs are \$1200, material and labor costs combined are \$2 per unit, and the demand equation is

$$p = \frac{100}{\sqrt{q}}$$

What level of output will maximize profit? Show that this occurs when marginal revenue is equal to marginal cost. What is the price at profit maximization?

18. Revenue A real-estate firm owns 100 garden-type apartments. At \$400 per month, each apartment can be rented. However, for each \$10-per-month increase, there will be two vacancies with no possibility of filling them. What rent per apartment will maximize monthly revenue?

19. Revenue A TV cable company has 6400 subscribers who are each paying \$24 per month. It can get 160 more subscribers for each \$0.50 decrease in the monthly fee. What rate will yield maximum revenue, and what will this revenue be?

20. Profit A manufacturer of a product finds that, for the first 600 units that are produced and sold, the profit is \$40 per unit. The profit on each of the units beyond 600 is decreased by \$0.05 times the number of additional units produced. For example, the total profit when 602 units are produced and sold is $600(40) + 2(39.90)$. What level of output will maximize profit?

21. Container Design A container manufacturer is designing a rectangular box, open at the top and with a square base, that is to have a volume of 13.5 ft^3. If the box is to require the least amount of material, what must be its dimensions?

22. Container Design An open-top box with a square base is to be constructed from 192 ft^2 of material. What should be the dimensions of the box if the volume is to be a maximum? What is the maximum volume?

23. Container Design An open box is to be made by cutting equal squares from each corner of a L-inch-square piece of cardboard and then folding up the sides. Find the length of the side of the square (in terms of L) that must be cut out if the volume of the box is to be maximized. What is the maximum volume? (See Figure 13.66.)

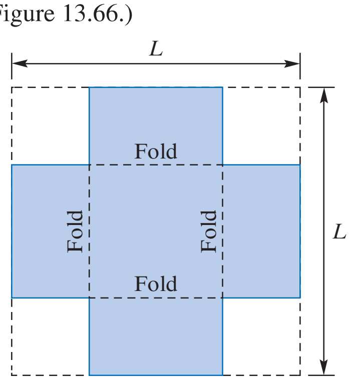


FIGURE 13.66

24. Poster Design A rectangular cardboard poster is to have 720 in^2 for printed matter. It is to have a 5-in. margin on each side and a 4-in. margin at the top and bottom. Find the dimensions of the poster so that the amount of cardboard used is minimized. (See Figure 13.67.)

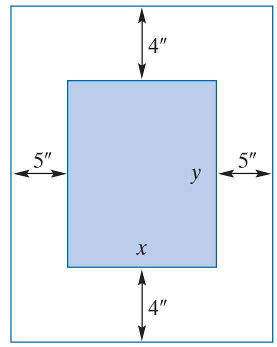


FIGURE 13.67

25. Container Design A cylindrical can, open at the top, is to have a fixed volume of K. Show that if the least amount of material is to be used, then both the radius and height are equal to $\sqrt[3]{K/\pi}$. (See Figure 13.68.)

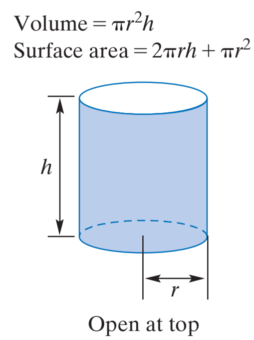


FIGURE 13.68

26. Container Design A cylindrical can, including both top and bottom, is to be made from a fixed amount of material, S. If the volume is to be a maximum, show that the radius is equal to $\sqrt{\frac{S}{6\pi}}$. Try also to show that $h = 2r$. (See Figure 13.68.)

27. Profit The demand equation for a monopolist's product is

$$p = 600 - 2q$$

and the total-cost function is

$$c = 0.2q^2 + 28q + 200$$

Find the profit-maximizing output and price, and determine the corresponding profit. If the government were to impose a tax of \$22 per unit on the manufacturer, what would be the new profit-maximizing output and price? What is the profit now?

28. Profit Use the *original* data in Problem 27, and assume that the government imposes a license fee of \$1000 on the manufacturer. This is a lump-sum amount without regard to output. Show that the profit-maximizing price and output remain the same. Show, however, that there will be less profit.

29. Economic Lot Size A manufacturer has to produce 3000 units annually of a product that is sold at a uniform rate during the year. The production cost of each unit is \$12, and carrying costs (insurance, interest, storage, etcetera) are estimated to be 19.2% of the value of average inventory. Setup costs per production run are \$54. Find the economic lot size.

30. Profit For a monopolist's product, the cost function is

$$c = 0.004q^3 + 20q + 5000$$

and the demand function is

$$p = 450 - 4q$$

Find the profit-maximizing output.

31. Workshop Attendance Imperial Educational Services (I.E.S.) is considering offering a workshop in resource allocation to key personnel at Acme Corp. To make the offering economically feasible, I.E.S. says that at least 40 persons must attend at a cost of \$200 each. Moreover, I.E.S. will agree to reduce the charge for *everybody* by \$2.50, for each person over the committed 40, who attends. How many people should be in the group for I.E.S. to maximize revenue? Assume that the maximum allowable number in the group is 70.

32. Cost of Leasing Motor The Kiddie Toy Company plans to lease an electric motor that will be used 80,000 horsepower-hours per year in manufacturing. One horsepower-hour is the work done in 1 hour by a 1-horsepower motor. The annual cost to lease a suitable motor is \$200, plus \$0.40 per horsepower. The cost per horsepower-hour of operating the motor is $\$0.008/N$, where N is the horsepower. What size motor, in horsepower, should be leased in order to minimize cost?

33. Transportation Cost The cost of operating a truck on a thruway (excluding the salary of the driver) is

$$0.165 + \frac{s}{200}$$

dollars per mile, where s is the (steady) speed of the truck in miles per hour. The truck driver's salary is \$18 per hour. At what speed should the truck driver operate the truck to make a 700-mile trip most economical?

34. Cost For a manufacturer, the cost of making a part is \$30 per unit for labor and \$10 per unit for materials; overhead is fixed at \$20,000 per week. If more than 5000 units are made each week, labor is \$45 per unit for those units in excess of 5000. At what level of production will average cost per unit be a minimum?

35. Profit Ms. Jones owns a small insurance agency that sells policies for a large insurance company. For each policy sold, Ms. Jones, who does not sell policies herself, is paid a commission of \$50 by the insurance company. From previous experience, Ms. Jones has determined that, when she employs m salespeople,

$$q = m^3 - 15m^2 + 92m$$

policies can be sold per week. She pays each of the m salespeople a salary of \$1000 per week, and her weekly fixed cost is \$3000. Current office facilities can accommodate at most eight salespeople. Determine the number of salespeople that Ms. Jones should hire to maximize her weekly profit. What is the corresponding maximum profit?

36. Profit A manufacturing company sells high-quality jackets through a chain of specialty shops. The demand equation for these jackets is

$$p = 1000 - 50q$$

where p is the selling price (in dollars per jacket) and q is the demand (in thousands of jackets). If this company's marginal-cost function is given by

$$\frac{dc}{dq} = \frac{1000}{q+5}$$

show that there is a maximum profit, and determine the number of jackets that must be sold to obtain this maximum profit.

37. Chemical Production Each day, a firm makes x tons of chemical A ($x \leq 4$) and

$$y = \frac{24 - 6x}{5 - x}$$

tons of chemical B. The profit on chemical A is \$2000 per ton, and on B it is \$1000 per ton. How much of chemical A should be produced per day to maximize profit? Answer the same question if the profit on A is P per ton and that on B is $P/2$ per ton.

38. Rate of Return To erect an office building, fixed costs are \$1.44 million and include land, architect's fees, a basement, a foundation, and so on. If x floors are constructed, the cost (excluding fixed costs) is

$$c = 10x[120{,}000 + 3000(x - 1)]$$

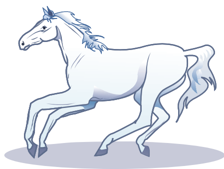
The revenue per month is \$60,000 per floor. How many floors will yield a maximum rate of return on investment? (Rate of return = total revenue/total cost.)

39. Gait and Power Output of an Animal In a model by Smith,[14] the power output of an animal at a given speed as a function of its movement or *gait*, j, is found to be

$$P(j) = Aj\frac{L^4}{V} + B\frac{V^3L^2}{1+j}$$

where A and B are constants, j is a measure of the "jumpiness" of the gait, L is a constant representing linear dimension, and V is a constant forward speed.

[14] J. M. Smith, *Mathematical Ideas in Biology* (London: Cambridge University Press, 1968).



Assume that P is a minimum when $dP/dj = 0$. Show that when this occurs,

$$(1+j)^2 = \frac{BV^4}{AL^2}$$

As a passing comment, Smith indicates that "at top speed, j is zero for an elephant, 0.3 for a horse, and 1 for a greyhound, approximately."

40. Traffic Flow In a model of traffic flow on a lane of a freeway, the number of cars the lane can carry per unit time is given by[15]

$$N = \frac{-2a}{-2at_r + v - \dfrac{2al}{v}}$$

where a is the acceleration of a car when stopping $(a < 0)$, t_r is the reaction time to begin braking, v is the average speed of the cars, and l is the length of a car. Assume that a, t_r, and l are constant. To find how many cars a lane can carry at most, we want to find the speed v that maximizes N. To maximize N, it suffices to minimize the denominator

$$-2at_r + v - \frac{2al}{v}$$

(a) Find the value of v that minimizes the denominator.
(b) Evaluate your answer in part (a) when $a = -19.6$ (ft/s^2), $l = 20$ (ft), and $t_r = 0.5$ (s). Your answer will be in feet per second.
(c) Find the corresponding value of N to one decimal place. Your answer will be in cars per second. Convert your answer to cars per hour.
(d) Find the relative change in N that results when l is reduced from 20 ft to 15 ft, for the maximizing value of v.

41. Average Cost During the Christmas season, a promotional company purchases cheap red felt stockings, glues fake white fur and sequins onto them, and packages them for distribution. The total cost of producing q cases of stockings is given by

$$c = 3q^2 + 50q - 18q\ln q + 120$$

Find the number of cases that should be processed in order to minimize the average cost per case. Determine (to two decimal places) this minimum average cost.

42. Profit A monopolist's demand equation is given by

$$p = q^2 - 20q + 160$$

where p is the selling price (in thousands of dollars) per ton when q tons of product are sold. Suppose that fixed cost is \$50,000 and that each ton costs \$30,000 to produce. If current equipment has a maximum production capacity of 12 tons, use the graph of the profit function to determine at what production level the maximum profit occurs. Find the corresponding maximum profit and selling price per ton.

Chapter 13 Review

Important Terms and Symbols

Important Terms and Symbols		Examples
Section 13.1	**Relative Extrema**	
	increasing function decreasing function	Ex. 1, p. 575
	relative maximum relative minimum	Ex. 2, p. 576
	relative extrema absolute extrema	Ex. 3, p. 577
	critical value critical point first-derivative test	Ex. 4, p. 577
Section 13.2	**Absolute Extrema on a Closed Interval**	
	extreme-value theorem	Ex. 1, p. 583
Section 13.3	**Concavity**	
	concave up concave down inflection point	Ex. 1, p. 585
Section 13.4	**The Second-Derivative Test**	
	second-derivative test	Ex. 1, p. 591
Section 13.5	**Asymptotes**	
	vertical asymptote horizontal asymptote	Ex. 1, p. 594
	oblique asymptote	Ex. 3, p. 596
Section 13.6	**Applied Maxima and Minima**	
	economic lot size	Ex. 5, p. 607

[15] J. I. Shonle, *Environmental Applications of General Physics* (Reading, MA: Addison-Wesley Publishing Co., 1975).

Summary

Calculus provides the best way of understanding the graphs of functions. Even the best electronic computational aids need the judgement added by calculus to tell the user *where* to look at a graph.

The first derivative is used to determine where a function is increasing or decreasing and to locate relative maxima and minima. If $f'(x)$ is positive throughout an interval, then over that interval, f is increasing and its graph rises (from left to right). If $f'(x)$ is negative throughout an interval, then over that interval, f is decreasing and its graph is falling.

A point $(a, f(a))$ on the graph at which $f'(a)$ is 0 or is not defined is a candidate for a relative extremum, and a is called a critical value. For a relative extremum to occur at a, the first derivative must change sign around a. The following procedure is the first-derivative test for the relative extrema of $y = f(x)$:

First-Derivative Test for Relative Extrema

Step 1. Find $f'(x)$.

Step 2. Determine all values a where $f'(a) = 0$ or $f'(a)$ is not defined.

Step 3. On the intervals defined by the values in Step 2, determine whether f is increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$).

Step 4. For each critical value a at which f is continuous, determine whether $f'(x)$ changes sign as x increases through a. There is a relative maximum at a if $f'(x)$ changes from $+$ to $-$, and a relative minimum if $f'(x)$ changes from $-$ to $+$. If $f'(x)$ does not change sign, there is no relative extremum at a.

Under certain conditions, a function is guaranteed to have absolute extrema. The extreme-value theorem states that if f is continuous on a closed interval, then f has an absolute maximum value and an absolute minimum value over the interval. To locate absolute extrema, the following procedure can be used:

Procedure to Find Absolute Extrema for a Function f Continuous on $[a, b]$

Step 1. Find the critical values of f.

Step 2. Evaluate $f(x)$ at the endpoints a and b and at the critical values in (a, b).

Step 3. The maximum value of f is the greatest of the values found in Step 2. The minimum value of f is the least of the values found in Step 2.

The second derivative is used to determine concavity and inflection points. If $f''(x) > 0$ throughout an interval, then f is concave up over that interval, meaning that its graph bends upward. If $f''(x) < 0$ over an interval, then f is concave down throughout that interval, and its graph bends downward. A point on the graph where f is continuous and its concavity changes is an inflection point. The point $(a, f(a))$ on the graph is a possible inflection point if either $f''(a) = 0$ or $f''(a)$ is not defined and f is continuous at a.

The second derivative also provides a means for testing certain critical values for relative extrema:

Second-Derivative Test for Relative Extrema

Suppose $f'(a) = 0$. Then

If $f''(a) < 0$, then f has a relative maximum at a.

If $f''(a) > 0$, then f has a relative minimum at a.

Asymptotes are also aids in curve sketching. Graphs "blow up" near vertical asymptotes, and they "settle" near horizontal asymptotes and oblique asymptotes. The line $x = a$ is a vertical asymptote for the graph of a function f if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$. For the case of a rational function, $f(x) = P(x)/Q(x)$ in lowest terms, we can find vertical asymptotes without evaluating limits. If $Q(a) = 0$ but $P(a) \neq 0$, then the line $x = a$ is a vertical asymptote.

The line $y = b$ is a horizontal asymptote for the graph of a function f if at least one of the following is true:

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

The line $y = mx + b$ is an oblique asymptote for the graph of a function f if at least one (note $\pm$) of the following is true:

$$\lim_{x\to\pm\infty} (f(x) - (mx + b)) = 0$$

In particular, a polynomial function of degree greater than 1 has no asymptotes. Moreover, a rational function whose numerator has degree greater than that of the denominator does not have a horizontal asymptote, and a rational function whose numerator has degree more than one greater than that of the denominator does not have an oblique asymptote.

Applied Maxima and Minima

In applied work, calculus is very important in maximization and minimization problems. For example, in the area of economics, we can use it to maximize profit or minimize cost. Some important relationships that are used in economics problems are the following:

$$\bar{c} = \frac{c}{q} \qquad \text{average cost per unit} = \frac{\text{total cost}}{\text{quantity}}$$

$$r = pq \qquad \text{revenue} = (\text{price})(\text{quantity})$$

$$P = r - c \qquad \text{profit} = \text{total revenue} - \text{total cost}$$

Review Problems

In Problems 1–4, find horizontal and vertical asymptotes.

1. $y = \dfrac{3x^2}{x^2 - 16}$

2. $y = \dfrac{x+2}{5x - x^2}$

3. $y = \dfrac{5x^2 - 3}{(3x+2)^2}$

4. $y = \dfrac{4x+1}{3x-5} - \dfrac{3x+1}{2x-11}$

In Problems 5–8, find critical values.

5. $f(x) = \dfrac{3x^2}{9 - x^2}$

6. $f(x) = 8(x-1)^2(x+6)^4$

7. $f(x) = \dfrac{\sqrt[3]{x-1}}{2-3x}$

8. $f(x) = \dfrac{13xe^{-5x/6}}{6x+5}$

In Problems 9–12, find intervals on which the function is increasing or decreasing.

9. $f(x) = -\frac{5}{3}x^3 + 15x^2 + 35x + 10$

10. $f(x) = \dfrac{3x^2}{(x+2)^2}$

11. $f(x) = \dfrac{6x^4}{x^2 - 3}$

12. $f(x) = 5\sqrt[3]{2x^3 - 3x}$

In Problems 13–18, find intervals on which the function is concave up or concave down.

13. $f(x) = x^4 - x^3 - 14$

14. $f(x) = \dfrac{x-2}{x+2}$

15. $f(x) = \dfrac{1}{3x+2}$

16. $f(x) = x^3 + 2x^2 - 5x + 2$

17. $f(x) = (3x-1)(2x-5)^3$

18. $f(x) = (x^2 - x - 1)^2$

In Problems 19–24, test for relative extrema.

19. $f(x) = 2x^3 - 9x^2 + 12x + 7$

20. $f(x) = \dfrac{ax+b}{x^2}$ for $a > 0$ and $b > 0$

21. $f(x) = \dfrac{x^{10}}{10} + \dfrac{x^5}{5}$

22. $f(x) = \dfrac{2x^2}{x^2 - 1}$

23. $f(x) = x^{2/3}(x+1)$

24. $f(x) = x^3(x-2)^4$

In Problems 25–30, find the x-values where inflection points occur.

25. $y = 3x^5 + 20x^4 - 30x^3 - 540x^2 + 2x + 3$

26. $y = \dfrac{x^2 + 2}{5x}$

27. $y = 2(x-3)(x^4 + 1)$

28. $y = x^2 + 2\ln(-x)$

29. $y = \dfrac{x^3}{e^x}$

30. $y = (x^2 - 5)^3$

In Problems 31–34, test for absolute extrema on the given interval.

31. $f(x) = 3x^4 - 4x^3;\ [0, 2]$

32. $f(x) = x^3 - (9/2)x^2 - 12x + 2;\ [0, 5]$

33. $f(x) = \dfrac{x}{(5x-6)^2};\ [-2, 0]$

34. $f(x) = (x+1)^2(x-1)^{2/3};\ [2, 3]$

35. Let $f(x) = x \ln x$.

(a) Determine the values of x at which relative maxima and relative minima, if any, occur.
(b) Determine the interval(s) on which the graph of f is concave up, and find the coordinates of all points of inflection, if any.

36. Let $f(x) = \dfrac{x}{x^2 - 1}$.

(a) Determine whether the graph of f is symmetric about the x-axis, y-axis, or origin.
(b) Find the interval(s) on which f is increasing.
(c) Find the coordinates of all relative extrema of f.
(d) Determine $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.
(e) Sketch the graph of f.
(f) State the absolute minimum and absolute maximum values of $f(x)$ (if they exist).

In Problems 37–48, indicate intervals on which the function is increasing, decreasing, concave up, or concave down; indicate relative maximum points, relative minimum points, inflection points, horizontal asymptotes, vertical asymptotes, symmetry, and those intercepts that can be obtained conveniently. Then sketch the graph.

37. $y = x^2 - 4x - 21$

38. $y = 2x^3 + 15x^2 + 36x + 9$

39. $y = x^3 - 12x + 20$

40. $y = e^{1/x}$

41. $y = x^3 - x$

42. $y = \dfrac{x+1}{x-1}$

43. $f(x) = \dfrac{100(x+5)}{x^2}$

44. $y = \dfrac{x^2 - 4}{x^2 - 1}$

45. $y = \dfrac{x}{(x-1)^3}$

46. $y = 6x^{1/3}(2x - 1)$

47. $f(x) = \dfrac{e^x - e^{-x}}{2}$

48. $f(x) = 1 - \ln(x^3)$

49. Are the following statements true or false?

(a) If $f'(x_0) = 0$, then f must have a relative extremum at x_0.
(b) Since the function $f(x) = 1/x$ is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$, it is impossible to find x_1 and x_2 in the domain of f such that $x_1 < x_2$ and $f(x_1) < f(x_2)$.
(c) On the interval $(-1, 1]$, the function $f(x) = x^4$ has an absolute maximum and an absolute minimum.
(d) If $f''(x_0) = 0$, then $(x_0, f(x_0))$ must be a point of inflection.
(e) A function f defined on the interval $(-2, 2)$ with exactly one relative maximum must have an absolute maximum.

50. An important function in probability theory is the standard normal-density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(a) Determine whether the graph of f is symmetric about the x-axis, y-axis, or origin.
(b) Find the intervals on which f is increasing and those on which it is decreasing.
(c) Find the coordinates of all relative extrema of f.
(d) Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

(e) Find the intervals on which the graph of f is concave up and those on which it is concave down.
(f) Find the coordinates of all points of inflection.
(g) Sketch the graph of f.
(h) Find all absolute extrema.

51. Marginal Cost If $c = q^3 - 6q^2 + 12q + 18$ is a total-cost function, for what values of q is marginal cost increasing?

52. Marginal Revenue If $r = 200q^{3/2} - 3q^2$ is the revenue function for a manufacturer's product, determine the intervals on which the marginal-revenue function is increasing.

53. Revenue Function The demand equation for a manufacturer's product is

$$p = 200 - \frac{\sqrt{q}}{5} \quad \text{where } q > 0$$

Show that the graph of the revenue function is concave down wherever it is defined.

54. Contraception In a model of the effect of contraception on birthrate,[16] the equation

$$R = f(x) = \frac{x}{4.4 - 3.4x} \quad 0 \le x \le 1$$

gives the proportional reduction R in the birthrate as a function of the efficiency x of a contraception method. An efficiency of 0.2 (or 20%) means that the probability of becoming pregnant is 80% of the probability of becoming pregnant without the contraceptive. Find the reduction (as a percentage) when efficiency is **(a)** 0, **(b)** 0.5, and **(c)** 1. Find dR/dx and d^2R/dx^2, and sketch the graph of the equation.

55. Learning and Memory If you were to recite members of a category, such as four-legged animals, the words that you utter would probably occur in "chunks," with distinct pauses between such chunks. For example, you might say the following for the category of four-legged animals:

dog, cat, mouse, rat,
(pause)
horse, donkey, mule,
(pause)
cow, pig, goat, lamb,
etc.

The pauses may occur because you must mentally search for subcategories (animals around the house, beasts of burden, farm animals, etc.).

The elapsed time between onsets of successive words is called *interresponse time*. A function has been used to analyze the length of time for pauses and the chunk size (number of words in a chunk).[17] This function f is such that

$$f(t) = \begin{cases} \text{the average number of words} \\ \text{that occur in succession with} \\ \text{interresponse times less than } t \end{cases}$$

The graph of f has a shape similar to that in Figure 13.69 and is best fit by a third-degree polynomial, such as

$$f(t) = At^3 + Bt^2 + Ct + D$$

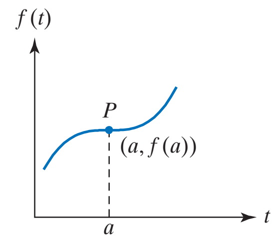


FIGURE 13.69

The point P has special meaning. It is such that the value a separates interresponse times *within* chunks from those *between* chunks. Mathematically, P is a critical point that is also a point of inflection. Assume these two conditions, and show that **(a)** $a = -B/(3A)$ and **(b)** $B^2 = 3AC$.

56. Market Penetration In a model for the market penetration of a new product, sales S of the product at time t are given by[18]

$$S = g(t) = \frac{m(p+q)^2}{p}\left[\frac{e^{-(p+q)t}}{\left(\frac{q}{p}e^{-(p+q)t} + 1\right)^2}\right]$$

where p, q, and m are nonzero constants.

(a) Show that

$$\frac{dS}{dt} = \frac{\frac{m}{p}(p+q)^3 e^{-(p+q)t}\left[\frac{q}{p}e^{-(p+q)t} - 1\right]}{\left(\frac{q}{p}e^{-(p+q)t} + 1\right)^3}$$

(b) Determine the value of t for which maximum sales occur. You may assume that S attains a maximum when $dS/dt = 0$.

In Problems 57–60, where appropriate, round the answers to two decimal places.

57. From the graph of $y = 3.9x^3 + 5.2x^2 - 7x + 3$, using a graphing utility, find the coordinates of all relative extrema.

58. From the graph of $f(x) = x^4 - 2x^3 + 3x - 1$, determine the absolute extrema of f over the interval $[-1, 1]$.

59. The graph of a function f has exactly one inflection point. If

$$f''(x) = \frac{x^3 + 3x + 2}{5x^2 - 2x + 4}$$

use the graph of f'' to determine the x-value of the inflection point of f.

[16] R. K. Leik and B. F. Meeker, *Mathematical Sociology* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975).

[17] A. Graesser and G. Mandler, "Limited Processing Capacity Constrains the Storage of Unrelated Sets of Words and Retrieval from Natural Categories," *Human Learning and Memory*, 4, no. 1 (1978), 86–100.

[18] A. P. Hurter, Jr., A. H. Rubenstein et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change*, vol. 11 (1978), 197–221.

60. Graph $y = \dfrac{5x^2 + 2x}{x^3 + 2x + 1}$. From the graph, locate any horizontal or vertical asymptotes.

61. Maximization of Production A manufacturer determined that m employees on a certain production line will produce q units per month, where

$$q = 80m^2 - 0.1m^4$$

To obtain maximum monthly production, how many employees should be assigned to the production line?

62. Revenue The demand function for a manufacturer's product is given by $p = 80e^{-0.05q}$. For what value of q does the manufacturer maximize total revenue?

63. Revenue The demand function for a monopolist's product is

$$p = \sqrt{500 - q}$$

If the monopolist wants to produce at least 100 units, but not more than 200 units, how many units should be produced to maximize total revenue?

64. Average Cost If $c = 0.01q^2 + 5q + 100$ is a cost function, find the average-cost function. At what level of production q is there a minimum average cost?

65. Profit The demand function for a monopolist's product is

$$p = 700 - 2q$$

and the average cost per unit for producing q units is

$$\bar{c} = q + 100 + \frac{1000}{q}$$

where p and $\bar{c}$ are in dollars per unit. Find the maximum profit that the monopolist can achieve.

66. Container Design A rectangular box is to be made by cutting out equal squares from each corner of a piece of cardboard 10 in. by 16 in. and then folding up the sides. What must be the length of the side of the square cut out if the volume of the box is to be maximum?

67. Fencing A rectangular portion of a field is to be enclosed by a fence and divided equally into four parts by three fences parallel to one pair of the sides. If a total of M meter of fencing is to be used, find the dimensions (in terms of M) that will maximize the fenced area.

68. Poster Design A rectangular poster having an area of 500 in^2 is to have a 4-in. margin at each side and at the bottom and a 6-in. margin at the top. The remainder of the poster is for printed matter. Find the dimensions of the poster so that the area for the printed matter is maximized.

69. Cost A furniture company makes personal-computer stands. For a certain model, the total cost (in thousands of dollars) when q *hundred* stands are produced is given by

$$c = 2q^3 - 9q^2 + 12q + 20$$

(a) The company is currently capable of manufacturing between 75 and 600 stands (inclusive) per week. Determine the number of stands that should be produced per week to minimize the total cost, and find the corresponding average cost per stand.
(b) Suppose that between 300 and 600 stands must be produced. How many should the company now produce in order to minimize total cost?

70. Bacteria In a laboratory, an experimental antibacterial agent is applied to a population of 100 bacteria. Data indicate that the number of bacteria t hours after the agent is introduced is given by

$$N = \frac{12{,}100 + 110t + 100t^2}{121 + t^2}$$

For what value of t does the maximum number of bacteria in the population occur? What is this maximum number?