Module 2: Supplementary Slides

Using Substitution

An Integral Involving b^u

$$\boxed{\int b^u \, du = \frac{1}{\ln(b)} b^u + C}$$

$$y = b^u = e^{\ln(b^u)} = e^{u\ln(b)} \Rightarrow y' = \ln(b)e^{u\ln(b)} = b^u \ln(b)$$

Example: Find $\int 2^{3-x} \, dx$.

Solution: $u = 3 - x \Rightarrow \int 2^{3-x} \, dx = -\int 2^u \, du = -\frac{2^{3-x}}{\ln(2)} + C$

Example

Repeated Integration by Parts

Solve $\int x^2 e^{2x+1}\,dx$.

Solution

$$u = x^2 \qquad dv = e^{2x+1}\,dx$$

$$du = 2x\,dx \qquad v = e^{2x+1}/2$$

$$\int x^2 e^{2x+1}\,dx = \frac{x^2 e^{2x+1}}{2} - \int \frac{e^{2x+1}}{2}(2x)\,dx = \frac{x^2 e^{2x+1}}{2} - \int x e^{2x+1}\,dx$$

$$\int x e^{2x+1}\,dx = \frac{x e^{2x+1}}{2} - \int \frac{e^{2x+1}}{2}\,dx = \frac{x e^{2x+1}}{2} - \frac{e^{2x+1}}{4} + C_1$$

$$\int x^2 e^{2x+1}\,dx = \frac{x^2 e^{2x+1}}{2} - \frac{x e^{2x+1}}{2} + \frac{e^{2x+1}}{4} + C = \frac{e^{2x+1}}{2}\left(x^2 - x + \frac{1}{2}\right) + C$$

Creating Loops in R

Creating Loops

```
for (variable in vector) {
expressions
}

while (condition) {

}
```

```
g=function(n){
s=1;
if(n<0) s="give a positive value!";
if (n==0) {s=1};
if (n>=1){
  for (i in 1:n){s=s*i}
}
return(s);
}

s=1
while(s<=5){s=s+1;print(s);}
```

Recursive or Nested Functions

Creating Recursive Functions

```
f=function (n){
f(n-1) can be called within the function
}
```

```
f=function(n){
  if (n==0) {y=1};
  if (n>0){y=n*f(n-1)};
  if(n<0) {y="Give a positive number!"};
  return (y);}

f=function(n){
  if (n==1) {y=1};
  if (n>0){y=n^2+f(n-1)};
  if(n<0) {y="Give a positive number!"};
  return (y);}
```

Example

Creating Recursive Functions

Calculate the below sum in R by creating a recursive function:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \left(4 - \left(\frac{2k}{n}\right)^2\right)\frac{2}{n}$$

Solution:

```
n=1000;
f=function(k){
  if (k==0) {y=0};
  if (k>0){y=(4-(2*k/n)^2)*2/n+f(k-1)};
  if (k<0) {y="Give a positive number!"};
  return (y);}
f(1000)
```

The Definite Integral

Some Important Summations

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(k)$$

$$\sum_{k=1}^{n} [f(k) \pm g(k)] = \sum_{k=1}^{n} f(k) \pm \sum_{k=1}^{n} g(k)$$

$$\sum_{k=1}^{n} cf(k) = c\sum_{k=1}^{n} f(k)$$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{m} a^k n^k}{n^m} = \lim_{n\to\infty} \frac{a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1}{n^m} = a^m$$

$$\sum_{k=1}^{n} 1 = 1 + 1 + \cdots + 1 = n$$

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

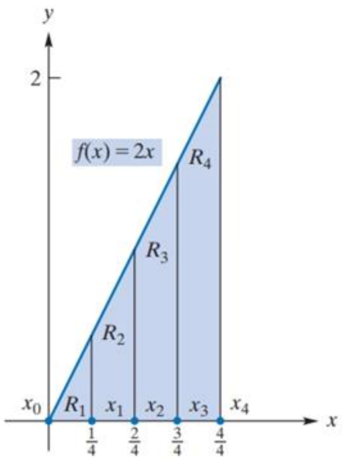
The Definite Integral

Recap: Definite Integral as a Limit of a Sum

$$S = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

$$\Delta x = \frac{b-a}{n}, \qquad x_k = k\Delta x, \qquad f(x_k) = f(k\Delta x)$$

Example: Find $S = \int_0^1 2x dx$



Four subregions of R.

Solution

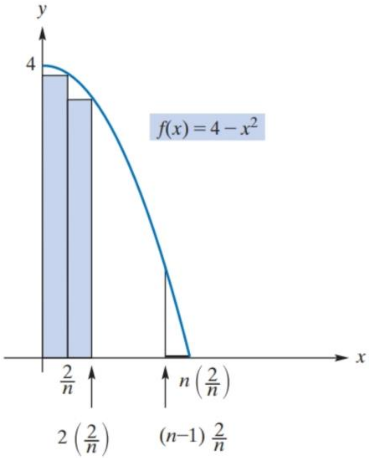
$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}, \qquad x_k = k\Delta x = \frac{k}{n}, \qquad f(x_k) = 2x_k = \frac{2k}{n}$$

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2k}{n}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{2}{n^2} \sum_{k=1}^{n} k = \lim_{n\to\infty} \left[\frac{2}{n^2}\frac{n(n+1)}{2}\right] = 1$$

Example

Definite Integral as a Limit of a Sum

Find the area of the region bounded by $f(x) = 4 - x^2$ between $x = 0$ and $x = 2$.



n subintervals and corresponding rectangles

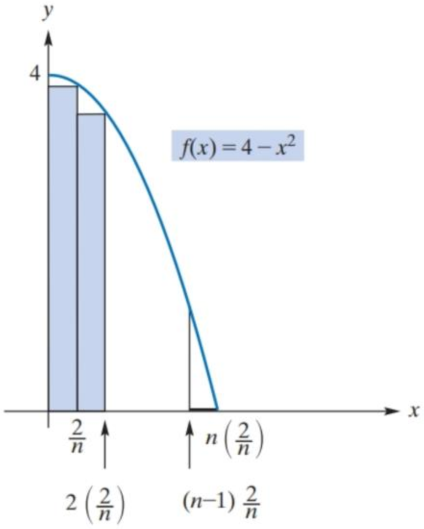
Solution

Since the length of $[0,2]$ is 2, $\Delta x = \frac{2}{n}$. Considering inscribed rectangles, $x_k = \frac{2}{n}k$.

Summing the areas, we get

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n}\left(4 - \left(\frac{2k}{n}\right)^2\right)\frac{2}{n}$$

$$= \frac{8}{n}n - \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} = 8 - \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)$$



n subintervals and corresponding rectangles

We take the limit of S_n as $n \to \infty$:

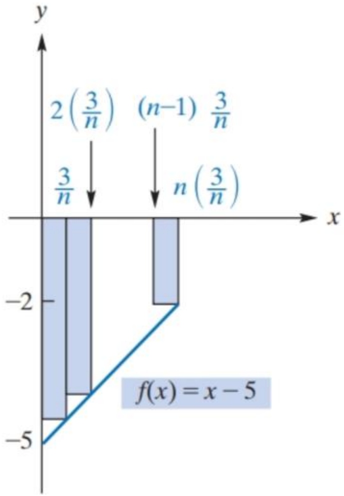
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(8 - \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)\right) = 8 - \frac{8}{3} = \frac{16}{3}$$

Hence, the area of the region is $\frac{16}{3}$.

Example

Definite Integral as a Limit of a Sum

Integrate $f(x) = x - 5$ from $x = 0$ to $x = 3$



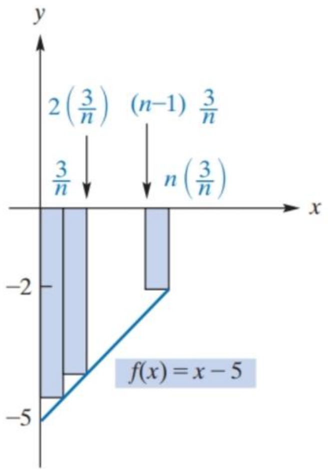
Solution

Since the length of $[0,3]$ is 3, $\Delta x = \frac{3}{n}$. Considering inscribed rectangles, $x_k = \frac{3}{n}k$. Summing the areas, we get

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n}\left(\frac{3k}{n} - 5\right)\left(\frac{3}{n}\right) = \frac{9(n+1)}{2n} - 15 = \frac{9}{2}\left(1 + \frac{1}{n}\right) - 15$$

We take the limit of S_n as $n \to \infty$:

$$\int_0^3 (x-5)dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{9}{2}\left(1+\frac{1}{n}\right) - 15\right) = \frac{9}{2} - 15 = -\frac{21}{2}$$



Remark: Since $f(x) \le 0$ for all x, the definite integral has become a negative number.

The Fundamental Theorem of Integral Calculus: Proof

The Fundamental Theorem of Integral Calculus

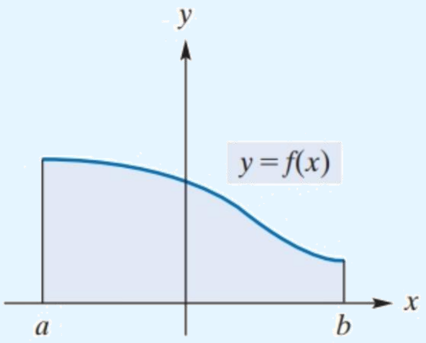
Fundamental Theorem of Integral Calculus

If f is continuous on the interval $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

a) $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$

b) $\int_a^b f(x)dx = F(b) - F(a)$

If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x)dx$ represents the area under the curve.

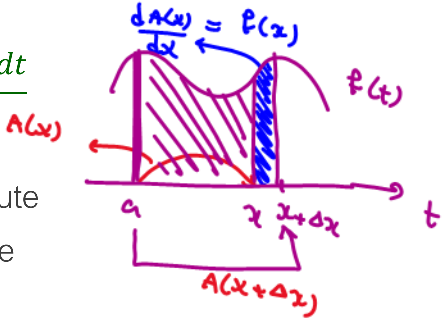


Properties of The Definite Integrals:

- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_b^c f(x)dx$

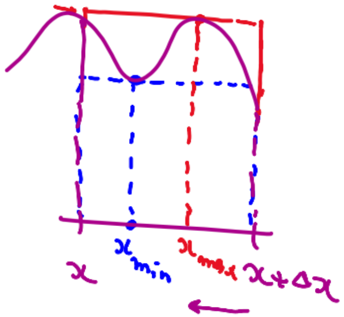
Part a) $A(x) = \int_a^x f(t)dt \Rightarrow \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = \frac{dA(x)}{dx} = \lim_{\Delta x\to 0}\frac{A(x+\Delta x)-A(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x} = \lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x}$$



Since $f(t)$ is within the closed interval $[x, x+\Delta x]$, by Extreme Value Theorem, it has at least one absolute maximum and at least one absolute minimum in the interval $[x, x+\Delta x]$. Suppose x_{max} and x_{min} are the maximizer and minimizer points. So, the area $\int_x^{x+\Delta x} f(t)dt$ is between the areas of the two rectangles shown in the picture. Then, we have:

$$f(x_{min})\Delta x \le \int_x^{x+\Delta x} f(t)dt \le f(x_{max})\Delta x$$



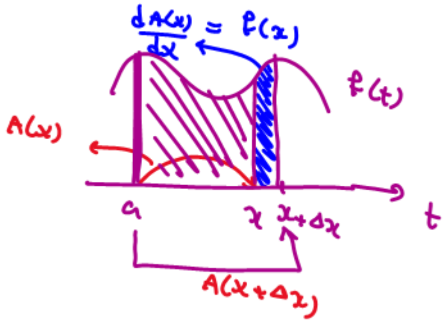
$$\Rightarrow \lim_{\Delta x\to 0} f(x_{min}) \le \lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x} \le \lim_{\Delta x\to 0} f(x_{max})$$

Noticing that since $x \le x_{min} \le x_{max} \le x+\Delta x$, then $\lim_{\Delta x\to 0} f(x_{min}) = \lim_{\Delta x\to 0} f(x_{max}) = f(x)$. Hence:

$$f(x) \le \lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x} \le f(x) \Rightarrow \frac{dA(x)}{dx} = \lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x} = f(x)$$

Part b)

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x) \Rightarrow \int_a^x f(t)dt = F(x) + C$$

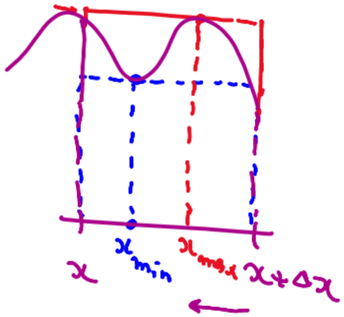


Choosing the initial value $x = a$ gives

$$\int_a^a f(t)dt = 0 = F(a) + C \Rightarrow C = -F(a)$$

Hence, $\int_a^x f(t)dt = F(x) - F(a)$ or with a more common notation:

The area under the curve in the interval $[a, b] = \int_a^b f(x)dx = F(b) - F(a)$



Remark: The above result means that any continuous function $f(x)$ is integrable (i.e., its antiderivative does exist). The reason is that the area under $f(x)$ always exists, which necessitates the existence of the value $F(b) - F(a)$, and by so, the existence of $F(x)$.

Leibniz Integral Rule

Leibniz Integral Rule

$$\frac{d}{dx}\left(\int_{u_1(x)}^{u_2(x)} f(t,x)dt\right) = f(u_2)\frac{du_2}{dx} - f(u_1)\frac{du_1}{dx} + \int_{u_1(x)}^{u_2(x)} f_x'(t,x)dt$$

Example: Find the derivative of $F(x) = \int_x^{x^2} xe^t dt$.

Solution:

$$\frac{dF}{dx} = \frac{d}{dx}\left(\int_x^{x^2} xe^t dt\right) = xe^{x^2}(2x) - xe^x + \int_x^{x^2} e^t dt = 2x^2 e^{x^2} - xe^x + e^{x^2} - e^x$$

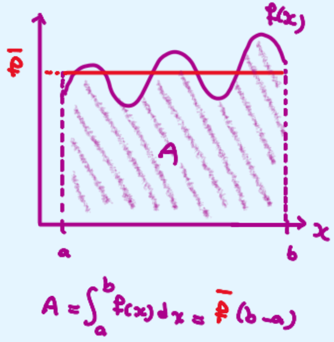
$$= e^{x^2}(2x^2+1) - e^x(x+1) = e^x\left[e^x(2x^2+1) - (x+1)\right]$$

Average Value of a Function

Average Value of a Function

The average value of a function $f(x)$ is given by:

$$\bar{f} = \frac{\int_a^b f(x)dx}{b-a}$$



Example.

Find the average value of the function $f(x) = x^2$ over the interval $[1,2]$.

Solution.

$$\bar{f} = \frac{1}{2-1}\int_1^2 x^2 dx = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$