Letting $r = 0$, $s = 1$, $t = 1$, and $u = 1$ gives

$$\left.\frac{\partial p}{\partial t}\right|_{(0,1,1,1)} = -\frac{0(1)(1)(2(0)(1) + (1)^2)}{(0(1)^2 + (1)^2(1))^2} = 0$$

Now Work Problem 31 ◁

PROBLEMS 17.1

In Problems 1–26, a function of two or more variables is given. Find the partial derivative of the function with respect to each of the variables.

1. $f(x, y) = 2x^2 + 3xy + 4y^2 + 5x + 6y - 7$

2. $f(x, y) = 2x^2 + 3xy$

3. $f(x, y) = 2y + 1$

4. $f(x, y) = e^{\pi} \ln 2$

5. $g(x, y) = 3x^4y + 2xy^2 - 5xy + 8x - 9y$

6. $g(x, y) = (x^2 + 1)^2 + (y^3 - 3)^3 + 5xy^3 - 2x^2y^2$

7. $g(p, q) = \sqrt{pq}$

8. $g(w, z) = \sqrt[3]{w^2 + z^2}$

9. $h(s, t) = \dfrac{s^2 + 1}{t^2 - 1}$

10. $h(u, v) = \dfrac{8uv^2}{u^2 + v^2}$

11. $u(q_1, q_2) = \ln \sqrt{q_1 + 2} + \ln \sqrt[3]{q_2 + 5}$

12. $Q(l, k) = 2l^{0.38}k^{1.79} - 3l^{1.03} + 2k^{0.13}$

13. $h(x, y) = \dfrac{x^2 + 3xy + y^2}{\sqrt{x^2 + y^2}}$

14. $h(x, y) = \dfrac{x + 4}{xy^2 - x^2y}$

15. $z = e^{5xy}$

16. $z = (x^3 + y^3)e^{xy+3x+3y}$

17. $z = 5x \ln(x^2 + y)$

18. $z = \ln(5x^3y^2 + 2y^4)^4$

19. $f(r, s) = \sqrt{r - s}(r^2 - 2rs + s^2)$

20. $f(r, s) = \sqrt{rs}\, e^{2+r}$

21. $f(r, s) = e^{3-r} \ln(7 - s)$

22. $f(r, s) = (5r^2 + 3s^3)(2r - 5s)$

23. $g(x, y, z) = 2x^3y^2 + 2xy^3z + 4z^2$

24. $g(x, y, z) = xy^2z^3 + x^3yz^2 + x^2y^3z$

25. $g(r, s, t) = e^{s+t}(r^2 + 7s^3)$

26. $g(r, s, t, u) = rs \ln(t)e^u$

In Problems 27–34, evaluate the given partial derivatives.

27. $f(x, y) = x^3y + 7x^2y^2$; $\quad f_x(1, -2)$

28. $z = \sqrt{2x^3 + 5xy + 2y^2}$; $\quad \left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=0\\y=1}}$

29. $g(x, y, z) = e^{x+y+z}\sqrt{x^2 + y^2 + z^2}$; $\quad g_z(0, 3, 4)$

30. $g(x, y, z) = \dfrac{3x^2y^2 + 2xy + x - y}{xy - yz + xz}$, $\quad g_y(1, 1, 5)$

31. $h(r, s, t, u) = (rst^2u) \ln(1 + rstu)$; $\quad h_t(1, 1, 0, 1)$

32. $h(r, s, t, u) = \dfrac{7r + 3s^2u^2}{s}$; $\quad h_t(4, 3, 2, 1)$

33. $f(r, s, t) = rst(r^2 + s^3 + t^4)$; $\quad f_s(1, -1, 2)$

34. $z = \dfrac{x^2 - y^2}{e^{x^2 - y^2}}$; $\quad \left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=0\\y=1}}, \quad \left.\dfrac{\partial z}{\partial y}\right|_{\substack{x=1\\y=0}}$

35. If $z = xe^{x-y} + ye^{y-x}$, show that

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^{x-y} + e^{y-x}$$

36. Stock Prices of a Dividend Cycle In a discussion of stock prices of a dividend cycle, Palmon and Yaari[1] consider the function f given by

$$u = f(t, r, z) = \frac{(1 + r)^{1-z} \ln(1 + r)}{(1 + r)^{1-z} - t}$$

where u is the instantaneous rate of ask-price appreciation, r is an annual opportunity rate of return, z is the fraction of a dividend cycle over which a share of stock is held by a midcycle seller, and t is the effective rate of capital gains tax. They claim that

$$\frac{\partial u}{\partial z} = \frac{t(1 + r)^{1-z} \ln^2(1 + r)}{[(1 + r)^{1-z} - t]^2}$$

Verify this.

37. Money Demand In a discussion of inventory theory of money demand, Swanson[2] considers the function

$$F(b, C, T, i) = \frac{bT}{C} + \frac{iC}{2}$$

and determines that $\dfrac{\partial F}{\partial C} = -\dfrac{bT}{C^2} + \dfrac{i}{2}$. Verify this partial derivative.

38. Interest Rate Deregulation In an article on interest rate deregulation, Christofi and Agapos[3] arrive at the equation

$$r_L = r + D\frac{\partial r}{\partial D} + \frac{dC}{dD} \tag{3}$$

where r is the deposit rate paid by commercial banks, r_L is the rate earned by commercial banks, C is the administrative cost of transforming deposits into return-earning assets, and D is the savings deposit level. Christofi and Agapos state that

$$r_L = r\left[\frac{1 + \eta}{\eta}\right] + \frac{dC}{dD} \tag{4}$$

where $\eta = \dfrac{r/D}{\partial r/\partial D}$ is the deposit elasticity with respect to the deposit rate. Express Equation (3) in terms of η to verify Equation (4).

[1] D. Palmon and U. Yaari, "Taxation of Capital Gains and the Behavior of Stock Prices over the Dividend Cycle," *The American Economist,* XXVII, no. 1 (1983), 13–22.

[2] P. E. Swanson, "Integer Constraints on the Inventory Theory of Money Demand," *Quarterly Journal of Business and Economics,* 23, no. 1 (1984), 32–37.

[3] A. Christofi and A. Agapos, "Interest Rate Deregulation: An Empirical Justification," *Review of Business and Economic Research,* XX (1984), 39–49.

39. Advertising and Profitability In an analysis of advertising and profitability, Swales[4] considers a function R given by

$$R = R(r, a, n) = \frac{r}{1 + a\left(\dfrac{n-1}{2}\right)}$$

where R is the adjusted rate of profit, r is the accounting rate of profit, a is a measure of advertising expenditures, and n is the number of years that advertising fully depreciates. In the analysis, Swales determines $\partial R/\partial n$. Find this partial derivative and the other two partial derivatives.

Objective

To develop the notions of partial marginal cost, marginal productivity, and competitive and complementary products.

17.2 Applications of Partial Derivatives

From Section 17.1, we know that if $z = f(x, y)$, then $\partial z/\partial x$ and $\partial z/\partial y$ can be geometrically interpreted as giving the slopes of the tangent lines to the surface $z = f(x, y)$ in the x- and y-directions, respectively. There are other interpretations: Because $\partial z/\partial x$ is the derivative of z with respect to x when y is held fixed, and because a derivative is a rate of change, we have

Here we have "rate of change" interpretations of partial derivatives.

$$\frac{\partial z}{\partial x} \text{ is the rate of change of } z \text{ with respect to } x \text{ when } y \text{ is held fixed.}$$

Similarly,

$$\frac{\partial z}{\partial y} \text{ is the rate of change of } z \text{ with respect to } y \text{ when } x \text{ is held fixed.}$$

We will now look at some applications in which the "rate of change" notion of a partial derivative is very useful.

Suppose a manufacturer produces x units of product X and y units of product Y. Then the total cost c of these units is a function of x and y and is called a **joint-cost function**. If such a function is $c = f(x, y)$, then $\partial c/\partial x$ is called the *(partial) marginal cost with respect to* x and is the rate of change of c with respect to x when y is held fixed. Similarly, $\partial c/\partial y$ is the *(partial) marginal cost with respect to* y and is the rate of change of c with respect to y when x is held fixed. It also follows that $\partial c/\partial x(x, y)$ is approximately the cost of producing one more unit of X when x units of X and y units of Y are produced. Similarly, $\partial c/\partial y(x, y)$ is approximately the cost of producing one more unit of Y when x units of X and y units of Y are produced.

For example, if c is expressed in dollars and $\partial c/\partial y = 2$, then the cost of producing an extra unit of Y when the level of production of X is fixed is approximately two dollars.

If a manufacturer produces n products, the joint-cost function is a function of n variables, and there are n (partial) marginal-cost functions.

EXAMPLE 1 Marginal Costs

A company manufactures two types of skis, the Lightning and the Alpine models. Suppose the joint-cost function for producing x pairs of the Lightning model and y pairs of the Alpine model per week is

$$c = f(x, y) = 0.07x^2 + 75x + 85y + 6000$$

where c is expressed in dollars. Determine the marginal costs $\partial c/\partial x$ and $\partial c/\partial y$ when $x = 100$ and $y = 50$, and interpret the results.

---

[4]J. K. Swales, "Advertising as an Intangible Asset: Profitability and Entry Barriers: A Comment on Reekie and Bhoyrub," *Applied Economics,* 17, no. 4 (1985), 603–17.

Since p_A and p_B represent prices, they are both positive. Hence, $\partial q_A/\partial p_B > 0$ and $\partial q_B/\partial p_A > 0$. We conclude that A and B are competitive products.

Now Work Problem 19 ◁

PROBLEMS 17.2

For the joint-cost functions in Problems 1–3, find the indicated marginal cost at the given production level.

1. $c = 7x + 0.3y^2 + 2y + 900; \quad \dfrac{\partial c}{\partial y}, x = 20, y = 30$

2. $c = 2x\sqrt{x+y} + 6000; \quad \dfrac{\partial c}{\partial x}, x = 70, y = 74$

3. $c = 0.03(x+y)^3 - 0.6(x+y)^2 + 9.5(x+y) + 7700;$
$\dfrac{\partial c}{\partial x}, x = 50, y = 80$

For the production functions in Problems 4 and 5, find the marginal productivity functions $\partial P/\partial k$ and $\partial P/\partial l$.

4. $P = 15lk - 3l^2 + 5k^2 + 500$

5. $P = 2.527l^{0.314}k^{0.686}$

6. Cobb–Douglas Production Function In economics, a Cobb–Douglas production function is a production function of the form $P = Al^{\alpha}k^{\beta}$, where A, α, and β are constants and $\alpha + \beta = 1$. For such a function, show that

(a) $\partial P/\partial l = \alpha P/l$ **(b)** $\partial P/\partial k = \beta P/k$

(c) $l\dfrac{\partial P}{\partial l} + k\dfrac{\partial P}{\partial k} = P$. This means that summing the products of the marginal productivity of each factor and the amount of that factor results in the total product P.

In Problems 7–9, q_A and q_B are demand functions for products A and B, respectively. In each case, find $\partial q_A/\partial p_A$, $\partial q_A/\partial p_B$, $\partial q_B/\partial p_A$, and $\partial q_B/\partial p_B$, and determine whether A and B are competitive, complementary, or neither.

7. $q_A = 1500 - 40p_A + 3p_B; \quad q_B = 900 + 5p_A - 20p_B$

8. $q_A = 20 - p_A - 2p_B; \quad q_B = 50 - 2p_A - 3p_B$

9. $q_A = \dfrac{100}{p_A\sqrt{p_B}}; \quad q_B = \dfrac{500}{p_B\sqrt[3]{p_A}}$

10. Canadian Manufacturing The production function for the Canadian manufacturing industries for 1927 is estimated by[6] $P = 33.0l^{0.46}k^{0.52}$, where P is product, l is labor, and k is capital. Find the marginal productivities for labor and capital, and evaluate when $l = 1$ and $k = 1$.

11. Dairy Farming An estimate of the production function for dairy farming in Iowa (1939) is given by[7]

$$P = A^{0.27}B^{0.01}C^{0.01}D^{0.23}E^{0.09}F^{0.27}$$

where P is product, A is land, B is labor, C is improvements, D is liquid assets, E is working assets, and F is cash operating expenses. Find the marginal productivities for labor and improvements.

12. Production Function Suppose a production function is given by $P = \dfrac{kl}{3k + 5l}$.

(a) Determine the marginal productivity functions.

(b) Show that when $k = l$, the marginal productivities sum to $\dfrac{1}{8}$.

13. MBA Compensation In a study of success among graduates with master of business administration (MBA) degrees, it was estimated that for staff managers (which include accountants, analysts, etc.), current annual compensation (in dollars) was given by

$$z = 43{,}960 + 4480x + 3492y$$

where x and y are the number of years of work experience before and after receiving the MBA degree, respectively.[8] Find $\partial z/\partial x$ and interpret your result.

14. Status A person's general status S_g is believed to be a function of status attributable to education, S_e, and status attributable to income, S_i, where S_g, S_e, and S_i are represented numerically. If

$$S_g = 7\sqrt[3]{S_e}\sqrt{S_i}$$

determine $\partial S_g/\partial S_e$ and $\partial S_g/\partial S_i$ when $S_e = 125$ and $S_i = 100$, and interpret your results.[9]

15. Reading Ease Sometimes we want to evaluate the degree of readability of a piece of writing. Rudolf Flesch[10] developed a function of two variables that will do this, namely,

$$R = f(w, s) = 206.835 - (1.015w + 0.846s)$$

where R is called the *reading ease score*, w is the average number of words per sentence in 100-word samples, and s is the average number of syllables in such samples. Flesch says that an article for which $R = 0$ is "practically unreadable," but one with $R = 100$ is "easy for any literate person." **(a)** Find $\partial R/\partial w$ and $\partial R/\partial s$. **(b)** Which is "easier" to read: an article for which $w = w_0$ and $s = s_0$, or one for which $w = w_0 + 1$ and $s = s_0$?

[6] P. Daly and P. Douglas, "The Production Function for Canadian Manufactures," *Journal of the American Statistical Association*, 38 (1943), 178–86.

[7] G. Tintner and O. H. Brownlee, "Production Functions Derived from Farm Records," *American Journal of Agricultural Economics*, 26 (1944), 566–71.

[8] Adapted from A. G. Weinstein and V. Srinivasen, "Predicting Managerial Success of Master of Business Administration (M.B.A.) Graduates," *Journal of Applied Psychology*, 59, no. 2 (1974), 207–12.

[9] Adapted from R. K. Leik and B. F. Meeker, *Mathematical Sociology* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975).

[10] R. Flesch, *The Art of Readable Writing* (New York: Harper & Row Publishers, Inc., 1949).

16. Model for Voice The study of frequency of vibrations of a taut wire is useful in considering such things as an individual's voice. Suppose

$$\omega = \frac{1}{bL}\sqrt{\frac{\tau}{\pi\rho}}$$

where ω (a Greek letter read "omega") is frequency, b is diameter, L is length, ρ (a Greek letter read "rho") is density, and τ (a Greek letter read "tau") is tension.[11] Find $\partial\omega/\partial b$, $\partial\omega/\partial L$, $\partial\omega/\partial\rho$, and $\partial\omega/\partial\tau$.

17. Traffic Flow Consider the following traffic-flow situation. On a highway where two lanes of traffic flow in the same direction, there is a maintenance vehicle blocking the left lane. (See Figure 17.3.) Two vehicles (*lead* and *following*) are in the right lane with a gap between them. The *subject* vehicle can choose either to fill or not to fill the gap. That decision may be based not only on the distance x shown in the diagram but also on other factors (such as the velocity of the *following* vehicle). A *gap index* g has been used in analyzing such a decision.[12,13] The greater the g-value, the greater is the propensity for the *subject* vehicle to fill the gap. Suppose

$$g = \frac{x}{V_F} - \left(0.75 + \frac{V_F - V_S}{19.2}\right)$$

where x (in feet) is as before, V_F is the velocity of the *following* vehicle (in feet per second), and V_S is the velocity of the *subject* vehicle (in feet per second). From the diagram, it seems reasonable that if both V_F and V_S are fixed and x increases, then g should increase. Show that this is true by applying calculus to the function g. Assume that x, V_F, and V_S are positive.

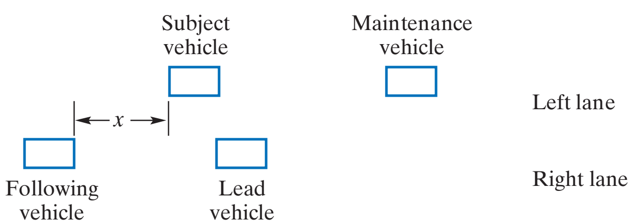

FIGURE 17.3

18. Demand Suppose the demand equations for related products A and B are

$$q_A = e^{-(p_A + p_B)} \quad \text{and} \quad q_B = \frac{16}{p_A^2 p_B^2}$$

where q_A and q_B are the number of units of A and B demanded when the unit prices (in thousands of dollars) are p_A and p_B, respectively.

(a) Classify A and B as competitive, complementary, or neither.
(b) If the unit prices of A and B are \$1000 and \$2000, respectively, estimate the change in the demand for A when the price of B is decreased by \$20 and the price of A is held constant.

19. Demand The demand equations for related products A and B are given by

$$q_A = 10\sqrt{\frac{p_B}{p_A}} \quad \text{and} \quad q_B = 3\sqrt[3]{\frac{p_A}{p_B}}$$

where q_A and q_B are the quantities of A and B demanded and p_A and p_B are the corresponding prices (in dollars) per unit.

(a) Find the values of the two marginal demands for product A when $p_A = 9$ and $p_B = 16$.
(b) If p_B were reduced to 14 from 16, with p_A fixed at 9, use part (a) to estimate the corresponding change in demand for product A.

20. Joint-Cost Function A manufacturer's joint-cost function for producing q_A units of product A and q_B units of product B is given by

$$c = \frac{q_A^2(q_B^3 + q_A)^{1/2}}{16} + q_A^{1/2}q_B^{1/3} + 500$$

where c is in dollars.

(a) Find the marginal-cost function with respect to q_A.
(b) Evaluate the marginal-cost function with respect to q_A when $q_A = 18$ and $q_B = 9$. Round your answer to two decimal places.
(c) Use your answer to part (a) to estimate the change in cost if production of product A is decreased from 18 to 17 units, while production of product B is held constant at 9 units.

21. Elections For the congressional elections of 1974, the Republican percentage, R, of the Republican–Democratic vote in a district is given (approximately) by[14]

$$\begin{aligned} R &= f(E_r, E_d, I_r, I_d, N) \\ &= 15.4725 + 2.5945E_r - 0.0804E_r^2 - 2.3648E_d \\ &\quad + 0.0687E_d^2 + 2.1914I_r - 0.0912I_r^2 \\ &\quad - 0.8096I_d + 0.0081I_d^2 - 0.0277E_rI_r \\ &\quad + 0.0493E_dI_d + 0.8579N - 0.0061N^2 \end{aligned}$$

Here E_r and E_d are the campaign expenditures (in units of \$10,000) by Republicans and Democrats, respectively; I_r and I_d are the number of terms served in Congress, *plus one,* for the Republican and Democratic candidates, respectively; and N is the percentage of the two-party presidential vote that Richard Nixon received in the district for 1968. The variable N gives a measure of Republican strength in the district.

(a) In the Federal Election Campaign Act of 1974, Congress set a limit of \$188,000 on campaign expenditures. By analyzing $\partial R/\partial E_r$, would you have advised a Republican candidate who served nine terms in Congress to spend \$188,000 on his or her campaign?

---

[11] R. M. Thrall, J. A. Mortimer, K. R. Rebman, and R. F. Baum, eds., *Some Mathematical Models in Biology,* rev. ed., Report No. 40241-R-7. Prepared at University of Michigan, 1967.

[12] P. M. Hurst, K. Perchonok, and E. L. Seguin, "Vehicle Kinematics and Gap Acceptance," *Journal of Applied Psychology,* 52, no. 4 (1968), 321–24.

[13] K. Perchonok and P. M. Hurst, "Effect of Lane-Closure Signals upon Driver Decision Making and Traffic Flow," *Journal of Applied Psychology,* 52, no. 5 (1968), 410–13.

[14] J. Silberman and G. Yochum, "The Role of Money in Determining Election Outcomes," *Social Science Quarterly,* 58, no. 4 (1978), 671–82.

(b) Find the percentage above which the Nixon vote had a negative effect on R; that is, find N when $\partial R/\partial N < 0$. Give your answer to the nearest percent.

22. Sales After a new product has been launched onto the market, its sales volume (in thousands of units) is given by

$$S = \frac{AT + 450}{\sqrt{A + T^2}}$$

where T is the time (in months) since the product was first introduced and A is the amount (in hundreds of dollars) spent each month on advertising.

(a) Verify that the partial derivative of sales volume with respect to time is given by

$$\frac{\partial S}{\partial T} = \frac{A^2 - 450T}{(A + T^2)^{3/2}}$$

(b) Use the result in part (a) to predict the number of months that will elapse before the sales volume begins to decrease if the amount allocated to advertising is held fixed at \$9000 per month.

Let q_A be demand for product A and suppose that $q_A = q_A(p_A, p_B)$, so that q_A is the quantity of A demanded when the price per unit of A is p_A and the price per unit of product B is p_B. The partial elasticity of demand for A with respect to p_A, denoted η_{p_A}, is defined as $\eta_{p_A} = (p_A/q_A)(\partial q_A/\partial p_A)$. The partial elasticity of demand for A with respect to p_B, denoted η_{p_B}, is defined as $\eta_{p_B} = (p_B/q_A)(\partial q_A/\partial p_B)$. Roughly speaking, η_{p_A} is the ratio of a percentage change in the quantity of A demanded to a percentage change in the price of A when the price of B is fixed. Similarly, η_{p_B} can be roughly interpreted as the ratio of a percentage change in the quantity of A demanded to a percentage change in the price of B when the price of A is fixed. In Problems 23–25, find η_{p_A} and η_{p_B} for the given values of p_A and p_B.

23. $q_A = 1000 - 50p_A + 2p_B$; $p_A = 2, p_B = 10$

24. $q_A = 60 - 3p_A - 2p_B$; $p_A = 5, p_B = 3$

25. $q_A = 1000/(p_A^2\sqrt{p_B})$; $p_A = 2, p_B = 9$

Objective

To compute higher-order partial derivatives.

17.3 Higher-Order Partial Derivatives

If $z = f(x, y)$, then not only is z a function of x and y, but also f_x and f_y are each functions of x and y, which may themselves have partial derivatives. If we can differentiate f_x and f_y, we obtain **second-order partial derivatives** of f. Symbolically,

$$f_{xx} \text{ means } (f_x)_x \qquad f_{xy} \text{ means } (f_x)_y$$
$$f_{yx} \text{ means } (f_y)_x \qquad f_{yy} \text{ means } (f_y)_y$$

In terms of ∂-notation,

$$\frac{\partial^2 z}{\partial x^2} \text{ means } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y\,\partial x} \text{ means } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} \text{ means } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \qquad \frac{\partial^2 z}{\partial y^2} \text{ means } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

For $z = f(x, y)$, $f_{xy} = \partial^2 z/\partial y \partial x$.

Note that to find f_{xy}, we first differentiate f with respect to x. For $\partial^2 z/\partial x\,\partial y$, we first differentiate with respect to y.

We can extend our notation beyond second-order partial derivatives. For example, f_{xxy} $(= \partial^3 z/\partial y \partial x^2)$ is a third-order partial derivative of f, namely, the partial derivative of f_{xx} $(= \partial^2 z/\partial x^2)$ with respect to y. The generalization of higher-order partial derivatives to functions of more than two variables should be obvious.

EXAMPLE 1 Second-Order Partial Derivatives

Find the four second-order partial derivatives of $f(x, y) = x^2y + x^2y^2$.

Solution: Since

$$f_x(x, y) = 2xy + 2xy^2$$

we have

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(2xy + 2xy^2) = 2y + 2y^2$$

and

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(2xy + 2xy^2) = 2x + 4xy$$

Also, since

$$f_y(x, y) = x^2 + 2x^2y$$

we have

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(x^2 + 2x^2y) = 2x^2$$

and

$$f_{yx}(x, y) = \frac{\partial}{\partial x}(x^2 + 2x^2y) = 2x + 4xy$$

Now Work Problem 1 ◁

The derivatives f_{xy} and f_{yx} are called **mixed partial derivatives**. Observe in Example 1 that $f_{xy}(x, y) = f_{yx}(x, y)$. Under suitable conditions, mixed partial derivatives of a function are equal; that is, the order of differentiation is of no concern. You may assume that this is the case for all the functions that we consider.

EXAMPLE 2 Mixed Partial Derivative

Find the value of $\left.\frac{\partial^3 w}{\partial z \partial y \partial x}\right|_{(1,2,3)}$ if $w = (2x + 3y + 4z)^3$.

Solution:

$$\begin{aligned}
\frac{\partial w}{\partial x} &= 3(2x + 3y + 4z)^2 \frac{\partial}{\partial x}(2x + 3y + 4z) \\
&= 6(2x + 3y + 4z)^2 \\
\frac{\partial^2 w}{\partial y \partial x} &= 6 \cdot 2(2x + 3y + 4z) \frac{\partial}{\partial y}(2x + 3y + 4z) \\
&= 36(2x + 3y + 4z) \\
\frac{\partial^3 w}{\partial z \partial y \partial x} &= 36 \cdot 4 = 144
\end{aligned}$$

Thus,

$$\left.\frac{\partial^3 w}{\partial z \partial y \partial x}\right|_{(1,2,3)} = 144$$

Now Work Problem 3 ◁

PROBLEMS 17.3

In Problems 1–10, find the indicated partial derivatives.

1. $f(x, y) = 5x^3y$; $f_x(x, y), f_{xy}(x, y), f_{yx}(x, y)$

2. $f(x, y) = 2x^3y^2 + 6x^2y^3 - 3xy$; $f_x(x, y)$, $f_{xx}(x, y)$

3. $f(x, y) = 7x^2 + 3y$; $f_y(x, y)$, $f_{yy}(x, y)$, $f_{yyx}(x, y)$

4. $f(x, y) = (x^2 + xy + y^2)(xy + x + y)$; $f_x(x, y)$, $f_{xy}(x, y)$

5. $f(x, y) = 9e^{2xy}$; $f_y(x, y)$, $f_{yx}(x, y)$, $f_{yxy}(x, y)$

6. $f(x, y) = \ln(x^2 + y^3) + 5$; $f_x(x, y), f_{xx}(x, y), f_{xy}(x, y)$

7. $f(x, y) = (x + y)^2(xy)$; $f_x(x, y)$, $f_y(x, y)$, $f_{xx}(x, y)$, $f_{yy}(x, y)$

8. $f(x, y, z) = x^2y^3z^4$; $f_x(x, y, z)$, $f_{xz}(x, y, z)$, $f_{zx}(x, y, z)$

9. $z = \ln\sqrt{x^2 + y^2}$; $\frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial y^2}$

10. $z = \frac{\ln(x^2 + 5)}{y}$; $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial y\, \partial x}$

In Problems 11–16, find the indicated value.

11. If $f(x, y, z) = 5$, find $f_{yxxz}(4, 3, -2)$.

12. If $f(x, y, z) = z^2(3x^2 - 4xy^3)$, find $f_{xyz}(1, 2, 3)$.

13. If $f(l,k) = 3l^3k^6 - 2l^2k^7$, find $f_{klk}(2,1)$.

14. If $f(x,y) = x^3y^2 + x^2y - x^2y^2$, find $f_{xxy}(2,3)$ and $f_{xyx}(2,3)$.

15. If $f(x,y) = y^2e^x + \ln(xy)$, find $f_{xyy}(1,1)$.

16. If $f(x,y) = 2x^3 + 3x^2y + 5xy^2 + 7y^3$, find $f_{xy}(2,3)$.

17. Cost Function Suppose the cost, c, of producing q_A units of product A and q_B units of product B is given by

$$c = (3q_A^2 + q_B^3 + 4)^{1/3}$$

and the coupled demand functions for the products are given by

$$q_A = 10 - p_A + p_B^2$$

and

$$q_B = 20 + p_A - 11p_B$$

Find the value of

$$\frac{\partial^2 c}{\partial q_A\, \partial q_B}$$

when $p_A = 25$ and $p_B = 4$.

18. For $f(x,y) = x^4y^4 + 3x^3y^2 - 7x + 4$, show that

$$f_{xyx}(x,y) = f_{xxy}(x,y)$$

19. For $f(x,y) = e^{x^2+xy+y^2}$, show that

$$f_{xy}(x,y) = f_{yx}(x,y)$$

20. For $f(x,y) = e^{xy}$, show that

$$f_{xx}(x,y) + f_{xy}(x,y) + f_{yx}(x,y) + f_{yy}(x,y)$$
$$= f(x,y)((x+y)^2 + 2)$$

21. For $w = \ln(x^2+y^2)$, show that $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0$. For $w = \ln(x^2+y^2+z^2)$, show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{2}{x^2+y^2+z^2}$$

Objective

To discuss relative maxima and relative minima, to find critical points, and to apply the second-derivative test for a function of two variables.

17.4 Maxima and Minima for Functions of Two Variables

We now extend the notion of relative maxima and minima (relative extrema) to functions of two variables.

Definition

A function $z = f(x,y)$ is said to have a **relative maximum** at the point (a,b) if, for all points (x,y) in the plane that are sufficiently close to (a,b), we have

$$f(a,b) \geq f(x,y) \qquad (1)$$

For a **relative minimum**, we replace $\geq$ by $\leq$ in Inequality (1).

To say that $z = f(x,y)$ has a relative maximum at (a,b) means, geometrically, that the point $(a,b,f(a,b))$ on the graph of f is higher than or is as high as all other points on the surface that are "near" $(a,b,f(a,b))$. In Figure 17.4(a), f has a relative maximum at (a,b). Similarly, the function f in Figure 17.4(b) has a relative minimum when $x = y = 0$, which corresponds to a low point on the surface.

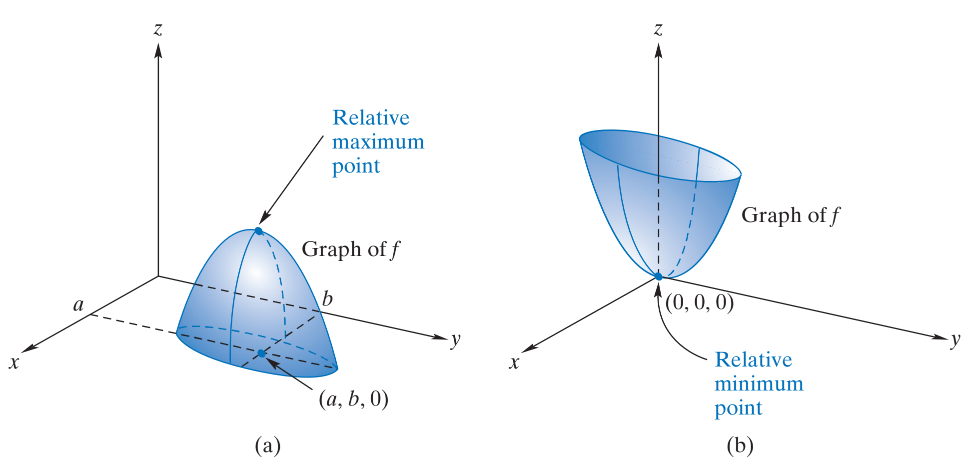

FIGURE 17.4 Relative extrema.

and

$$q_B = 400(9 + p_A - 2p_B)$$

where p_A and p_B are the selling prices (in dollars per pound) of A and B, respectively. Determine the selling prices that will maximize the company's profit, P.

Solution: The total profit is given by

$$P = \begin{pmatrix} \text{profit} \\ \text{per pound} \\ \text{of A} \end{pmatrix} \begin{pmatrix} \text{pounds} \\ \text{of A} \\ \text{sold} \end{pmatrix} + \begin{pmatrix} \text{profit} \\ \text{per pound} \\ \text{of B} \end{pmatrix} \begin{pmatrix} \text{pounds} \\ \text{of B} \\ \text{sold} \end{pmatrix}$$

For A and B, the profits per pound are $p_A - 2$ and $p_B - 3$, respectively. Thus,

$$\begin{aligned} P &= (p_A - 2)q_A + (p_B - 3)q_B \\ &= (p_A - 2)[400(p_B - p_A)] + (p_B - 3)[400(9 + p_A - 2p_B)] \end{aligned}$$

Notice that P is expressed as a function of two variables, p_A and p_B. To maximize P, we set its partial derivatives equal to 0:

$$\begin{aligned} \frac{\partial P}{\partial p_A} &= (p_A - 2)[400(-1)] + [400(p_B - p_A)](1) + (p_B - 3)[400(1)] \\ &= 0 \\ \frac{\partial P}{\partial p_B} &= (p_A - 2)[400(1)] + (p_B - 3)[400(-2)] + 400(9 + p_A - 2p_B)](1) \\ &= 0 \end{aligned}$$

Simplifying the preceding two equations gives

$$\begin{cases} -2p_A + 2p_B - 1 = 0 \\ 2p_A - 4p_B + 13 = 0 \end{cases}$$

whose solution is $p_A = 5.5$ and $p_B = 6$. Moreover, we find that

$$\frac{\partial^2 P}{\partial p_A^2} = -800 \qquad \frac{\partial^2 P}{\partial p_B^2} = -1600 \qquad \frac{\partial^2 P}{\partial p_B \partial p_A} = 800$$

Therefore,

$$D(5.5, 6) = (-800)(-1600) - (800)^2 > 0$$

Since $\partial^2 P/\partial p_A^2 < 0$, we indeed have a maximum, and the company should sell candy A at \$5.50 per pound and B at \$6.00 per pound.

Now Work Problem 23 ◁

PROBLEMS 17.4

In Problems 1–6, find the critical points of the functions.

1. $f(x, y) = x^2 - 3y^2 - 8x + 9y + 3xy$

2. $f(x, y) = x^2 + 3y^2 - 4x - 30y$

3. $f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7$

4. $f(x, y) = xy - x + y$

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

6. $f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3)$

In Problems 7–20, find the critical points of the functions. For each critical point, determine, by the second-derivative test, whether it corresponds to a relative maximum, to a relative minimum, or to neither, or whether the test gives no information.

7. $f(x, y) = x^2 + 4y^2 - 6x - 32y + 1$

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

9. $f(x, y) = y - y^2 - 3x - 6x^2$

10. $f(x, y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$

11. $f(x, y) = x^2 + 3xy + y^2 - 9x - 11y + 3$

12. $f(x, y) = 2x^3 + 3y^2 + 6xy + 6x + 6y$

13. $f(x, y) = \frac{1}{3}(x^3 + 8y^3) - 2(x^2 + y^2) + 1$

14. $f(x, y) = x^2 + y^2 - xy + x^3$

15. $f(l, k) = \frac{l^2}{2} + 2lk + 3k^2 - 69l - 164k + 17$

16. $f(l, k) = l^2 + 4k^2 - 4lk$

17. $f(x, y) = xy - \frac{1}{x} - \frac{1}{y}$

18. $f(x, y) = (x - 3)(y - 3)(x + y - 3)$

19. $f(x, y) = (y^2 - 4)(e^x - 1)$

20. $f(x, y) = \ln(xy) + 2x^2 - xy - 6x$

21. Maximizing Output Suppose

$$P = f(l, k) = 2.18l^2 - 0.02l^3 + 1.97k^2 - 0.03k^3$$

is a production function for a firm. Find the quantities of inputs l and k that maximize output P.

22. Maximizing Output In a certain office, computers C and D are utilized for c and d hours, respectively. If daily output Q is a function of c and d, namely,

$$Q = 10c + 20d - 3c^2 - 4d^2 - cd$$

find the values of c and d that maximize Q.

In Problems 23–35, unless otherwise indicated, the variables p_A and p_B denote selling prices of products A *and* B*, respectively. Similarly, q_A and q_B denote quantities of* A *and* B *that are produced and sold during some time period. In all cases, the variables employed will be assumed to be units of output, input, money, and so on.*

23. Profit A candy company produces two varieties of candy, A and B, for which the constant average costs of production are 60 and 70 (cents per lb), respectively. The demand functions for A and B are given by

$$q_A = 5(p_B - p_A) \quad \text{and} \quad q_B = 500 + 5(p_A - 2p_B)$$

Find the selling prices p_A and p_B that maximize the company's profit.

24. Profit Repeat Problem 23 if the constant costs of production of A and B are a and b (cents per lb), respectively.

25. Price Discrimination Suppose a monopolist is practicing price discrimination in the sale of a product by charging different prices in two separate markets. In market A the demand function is

$$p_A = 100 - q_A$$

and in B it is

$$p_B = 84 - q_B$$

where q_A and q_B are the quantities sold per week in A and B, and p_A and p_B are the respective prices per unit. If the monopolist's cost function is

$$c = 600 + 4(q_A + q_B)$$

how much should be sold in each market to maximize profit? What selling prices give this maximum profit? Find the maximum profit.

26. Profit A monopolist sells two competitive products, A and B, for which the demand functions are

$$q_A = 16 - p_A + p_B \quad \text{and} \quad q_B = 24 + 2p_A - 4p_B$$

If the constant average cost of producing a unit of A is 2 and a unit of B is 4, how many units of A and B should be sold to maximize the monopolist's profit?

27. Profit For products A and B, the joint-cost function for a manufacturer is

$$c = 9q_A^2 + 6q_B^2$$

and the demand functions are $p_A = 81 - q_A^2$ and $p_B = 90 - 2q_B^2$. Find the level of production that maximizes profit.

28. Profit For a monopolist's products A and B, the joint-cost function is $c = 2(q_A + q_B + q_A q_B)$, and the demand functions are $q_A = 20 - 2p_A$ and $q_B = 10 - p_B$. Find the values of p_A and p_B that maximize profit. What are the quantities of A and B that correspond to these prices? What is the total profit?

29. Cost An open-top rectangular box is to have a volume of 6 ft^3. The cost per square foot of materials is \$3 for the bottom, \$1 for the front and back, and \$0.50 for the other two sides. Find the dimensions of the box so that the cost of materials is minimized. (See Figure 17.7.)

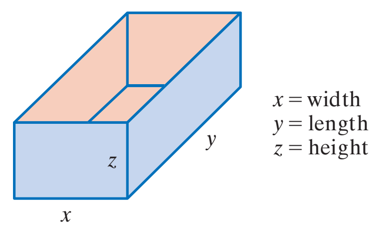

FIGURE 17.7

30. Collusion Suppose A and B are the only two firms in the market selling the same product. (We say that they are *duopolists*.) The industry demand function for the product is

$$p = 92 - q_A - q_B$$

where q_A and q_B denote the output produced and sold by A and B, respectively. For A, the cost function is $c_A = 10q_A$; for B, it is $c_B = 0.5q_B^2$. Suppose the firms decide to enter into an agreement on output and price control by jointly acting as a monopoly. In this case, we say they enter into *collusion*. Show that the profit function for the monopoly is given by

$$P = pq_A - c_A + pq_B - c_B$$

Express P as a function of q_A and q_B, and determine how output should be allocated so as to maximize the profit of the monopoly.

31. Suppose $f(x, y) = x^2 + 3y^2 + 9$, where x and y must satisfy the equation $x + y = 2$. Find the relative extrema of f, subject to the given condition on x and y, by first solving the second equation for y (or x). Substitute the result in the first equation. Thus, f is expressed as a function of one variable. Now find where relative extrema for f occur.

32. Repeat Problem 31 if $f(x, y) = x^2 + 4y^2 + 11$, subject to the condition that $x - y = 1$.

33. Suppose the joint-cost function

$$c = q_A^2 + 3q_B^2 + 2q_A q_B + aq_A + bq_B + d$$

has a relative minimum value of 15 when $q_A = 3$ and $q_B = 1$. Determine the values of the constants a, b, and d.

34. Suppose that the function $f(x, y)$ has continuous partial derivatives f_{xx}, f_{yy}, and f_{xy} at all points (x, y) near a critical point (a, b). Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$ and suppose that $D(a, b) > 0$.

(a) Show that $f_{xx}(a, b) < 0$ if and only if $f_{yy}(a, b) < 0$.
(b) Show that $f_{xx}(a, b) > 0$ if and only if $f_{yy}(a, b) > 0$.

35. Profit from Competitive Products A monopolist sells two competitive products, A and B, for which the demand equations are

$$p_A = 35 - 2q_A^2 + q_B$$

and

$$p_B = 20 - q_B + q_A$$

The joint-cost function is

$$c = -8 - 2q_A^3 + 3q_A q_B + 30q_A + 12q_B + \frac{1}{2}q_A^2$$

(a) How many units of A and B should be sold to obtain a relative maximum profit for the monopolist? Use the second-derivative test to justify your answer.
(b) Determine the selling prices required to realize the relative maximum profit. Also, find this relative maximum profit.

36. Profit and Advertising A retailer has determined that the number of TV sets he can sell per week is

$$\frac{7x}{2 + x} + \frac{4y}{5 + y}$$

where x and y represent his weekly expenditures (in dollars) on newspaper and radio advertising, respectively. The profit is \$300 per sale, less the cost of advertising, so the weekly profit is given by the formula

$$P = 300\left(\frac{7x}{2 + x} + \frac{4y}{5 + y}\right) - x - y$$

Find the values of x and y for which the profit is a relative maximum. Use the second-derivative test to verify that your answer corresponds to a relative maximum profit.

37. Profit from Tomato Crop The revenue (in dollars per square meter of ground) obtained from the sale of a crop of tomatoes grown in an artificially heated greenhouse is given by

$$r = 5T(1 - e^{-x})$$

where T is the temperature (in °C) maintained in the greenhouse and x is the amount of fertilizer applied per square meter. The cost of fertilizer is $20x$ dollars per square meter, and the cost of heating is given by $0.1T^2$ dollars per square meter.

(a) Find an expression, in terms of T and x, for the profit per square meter obtained from the sale of the crop of tomatoes.
(b) Verify that the pairs

$$(T, x) = (20, \ln 5) \quad \text{and} \quad (T, x) = (5, \ln \tfrac{5}{4})$$

are critical points of the profit function in part (a). (*Note:* You need not derive the pairs.)
(c) The points in part (b) are the only critical points of the profit function in part (a). Use the second-derivative test to determine whether either of these points corresponds to a relative maximum profit per square meter.

Objective

To find critical points for a function, subject to constraints, by applying the method of Lagrange multipliers.

17.5 Lagrange Multipliers

We will now find relative maxima and minima for a function on which certain *constraints* are imposed. Such a situation could arise if a manufacturer wished to minimize a joint-cost function and yet obtain a particular production level.

Suppose we want to find the relative extrema of

$$w = x^2 + y^2 + z^2 \tag{1}$$

subject to the constraint that x, y, and z must satisfy

$$x - y + 2z = 6 \tag{2}$$

We can transform w, which is a function of three variables, into a function of two variables such that the new function reflects constraint (2). Solving Equation (2) for x, we get

$$x = y - 2z + 6 \tag{3}$$

which, when substituted for x in Equation (1), gives

$$w = (y - 2z + 6)^2 + y^2 + z^2 \tag{4}$$

Since w is now expressed as a function of two variables, to find relative extrema we follow the usual procedure of setting the partial derivatives of w equal to 0:

$$\frac{\partial w}{\partial y} = 2(y - 2z + 6) + 2y = 4y - 4z + 12 = 0 \tag{5}$$

$$\frac{\partial w}{\partial z} = -4(y - 2z + 6) + 2z = -4y + 10z - 24 = 0 \tag{6}$$

This appears to be a challenging system to solve. Some ingenuity will come into play. Here is one sequence of operations that will allow us to find the critical points. We can write the system as

$$\begin{cases} \dfrac{y}{2x} = \lambda_1 & (30) \\ x + z - 2y\lambda_1 - z\lambda_2 = 0 & (31) \\ \lambda_2 = 1 & (32) \\ x^2 + y^2 = 8 & (33) \\ z = \dfrac{8}{y} & (34) \end{cases}$$

In deriving Equation (30) we assumed $x \neq 0$. This is permissible because if $x = 0$, then by Equation (25) we have also $y = 0$, which is impossible because the second constraint, $yz = 8$, provides $y \neq 0$. We also used $y \neq 0$ to derive Equations (32) and (34).

Substituting $\lambda_2 = 1$ from Equation (32) into Equation (31) and simplifying gives the equation $x - 2y\lambda_1 = 0$, so

$$\lambda_1 = \frac{x}{2y}$$

Substituting into Equation (30) gives

$$\frac{y}{2x} = \frac{x}{2y}$$

$$y^2 = x^2 \qquad (35)$$

Substituting into Equation (33) gives $x^2 + x^2 = 8$, from which it follows that $x = \pm 2$. If $x = 2$, then, from Equation (35), we have $y = \pm 2$. Similarly, if $x = -2$, then $y = \pm 2$. Thus, if $x = 2$ and $y = 2$, then, from Equation (34), we obtain $z = 4$. Continuing in this manner, we obtain four critical points:

$$(2, 2, 4) \quad (2, -2, -4) \quad (-2, 2, 4) \quad (-2, -2, -4)$$

Now Work Problem 9 ◁

PROBLEMS 17.5

In Problems 1–12, find, by the method of Lagrange multipliers, the critical points of the functions, subject to the given constraints.

1. $f(x, y) = x^2 + 4y^2 + 6; \quad 2x - 8y = 20$

2. $f(x, y) = 3x^2 - 2y^2 + 9; \quad x + y = 1$

3. $f(x, y, z) = x^2 + y^2 + z^2; \quad x + y + z = 1$

4. $f(x, y, z) = x + y + z; \quad xyz = 8$

5. $f(x, y, z) = 2x^2 + xy + y^2 + z; \quad x + 2y + 4z = 3$

6. $f(x, y, z) = xyz^2; \quad x - y + z = 20 \ (xyz^2 \neq 0)$

7. $f(x, y, z) = xyz; \quad x + y + z = 1 \ (xyz \neq 0)$

8. $f(x, y, z) = x^2 + 4y^2 + 9z^2; \quad x + y + z = 3$

9. $f(x, y, z) = x^2 + 2y - z^2; \quad 2x - y = 0, \ y + z = 0$

10. $f(x, y, z) = x^2 + y^2 + z^2; \quad x + y + z = 4, \ x - y + z = 4$

11. $f(x, y, z) = xy^2z; \quad x + y + z = 1, \ x - y + z = 0 \ (xyz \neq 0)$

12. $f(x, y, z, w) = x^2 + 2y^2 + 3z^2 - w^2; \ 4x + 3y + 2z + w = 10$

13. Production Allocation To fill an order for 100 units of its product, a firm wishes to distribute production between its two plants, plant 1 and plant 2. The total-cost function is given by

$$c = f(q_1, q_2) = q_1^2 + 3q_1 + 25q_2 + 1000$$

where q_1 and q_2 are the numbers of units produced at plants 1 and 2, respectively. How should the output be distributed in order to minimize costs? (Assume that the critical point obtained corresponds to the minimum cost.)

14. Production Allocation Repeat Problem 13 if the cost function is

$$c = 3q_1^2 + q_1q_2 + 2q_2^2$$

and a total of 200 units are to be produced.

15. Maximizing Output The production function for a firm is

$$f(l, k) = 12l + 20k - l^2 - 2k^2$$

The cost to the firm of l and k is 4 and 8 per unit, respectively. If the firm wants the total cost of input to be 88, find the greatest output possible, subject to this budget constraint. (You may assume that the critical point obtained does correspond to the maximum output.)

16. Maximizing Output Repeat Problem 15, given that

$$f(l, k) = 20l + 25k - l^2 - 3k^2$$

and the budget constraint is $2l + 4k = 50$.

17. Advertising Budget A computer company has a monthly advertising budget of \$20,000. Its marketing department estimates that if x dollars are spent each month on advertising in newspapers and y dollars per month on television advertising, then the monthly sales will be given by $S = 80x^{1/4}y^{3/4}$ dollars. If the profit is 10% of sales, less the advertising cost, determine how to allocate the advertising budget in order to maximize the monthly profit. (You may assume that the critical point obtained does correspond to the maximum profit.)

18. Maximizing Production When l units of labor and k units of capital are invested, a manufacturer's total production, q, is given by the Cobb-Douglas production function, $P = 8l^{3/5}k^{2/5}$. Each unit of labor costs \$50, and each unit of capital costs \$39. If exactly \$30,750 is to be spent on production, determine the numbers of units of labor and capital that should be invested to maximize production. (Assume that the maximum occurs at the critical point obtained.)

19. Political Advertising Newspaper advertisements for political parties always have some negative effects. The recently elected party assumed that the three most important election issues, X, Y, and Z, had to be mentioned in each ad, with space x, y, and z units, respectively, allotted to each. The combined bad effect of this coverage was estimated by the party's backroom operative as

$$B(x, y, z) = x^2 + y^2 + 2z^2$$

Aesthetics dictated that the total space for X and Y together must be 20, and realism suggested that the total space allotted to Y and Z together must also be 20 units. What values of x, y, and z in each ad would produce the lowest negative effect? (You may assume that any critical point obtained provides the minimum effect.)

20. Maximizing Profit Suppose a manufacturer's production function is given by

$$16q = 65 - 4(l - 4)^2 - 2(k - 5)^2$$

and the cost to the manufacturer is \$8 per unit of labor and \$16 per unit of capital, so that the total cost (in dollars) is $8l + 16k$. The selling price of the product is \$64 per unit.

(a) Express the profit as a function of l and k. Give your answer in expanded form.

(b) Find all critical points of the profit function obtained in part (a). Apply the second-derivative test at each critical point. If the profit is a relative maximum at a critical point, compute the corresponding relative maximum profit.

(c) The profit may be considered a function of l, k, and q (that is, $P = 64q - 8l - 16k$), subject to the constraint

$$16q = 65 - 4(l - 4)^2 - 2(k - 5)^2$$

Use the method of Lagrange multipliers to find all critical points of $P = 64q - 8l - 16k$, subject to the constraint.

Problems 21–24 refer to the following definition. A utility function *is a function that attaches a measure to the satisfaction or utility a consumer gets from the consumption of products per unit of time. Suppose $U = f(x, y)$ is such a function, where x and y are the amounts of two products,* X *and* Y. *The* marginal utility *of* X *is $\partial U/\partial x$ and approximately represents the change in total utility resulting from a one-unit change in consumption of product* X *per unit of time. We define the marginal utility of* Y *similarly. If the prices of* X *and* Y *are p_X and p_Y, respectively, and the consumer has an income or budget of I to spend, then the budget constraint is:*

$$xp_X + yp_Y = I$$

In Problems 21–23, find the quantities of each product that the consumer should buy, subject to the budget, that will allow maximum satisfaction. That is, in Problems 21 and 22, find values of x and y that maximize $U = f(x, y)$, subject to $xp_X + yp_Y = I$. Perform a similar procedure for Problem 23. Assume that such a maximum exists.

21. $U = x^3y^3$; $p_X = 2, p_Y = 3, I = 48$ $(x^3y^3 \neq 0)$

22. $U = 40x - 8x^2 + 2y - y^2$; $p_X = 4, p_Y = 6, I = 100$

23. $U = f(x, y, z) = xyz$; $p_X = 1$; $p_Y = 2$; $p_Z = 3$; $I = 100$; $(xyz \neq 0)$

24. Let $U = f(x, y)$ be a utility function subject to the budget constraint $xp_X + yp_Y = I$, where p_X, p_Y, and I are constants. Show that, to maximize satisfaction, it is necessary that

$$\lambda = \frac{f_x(x, y)}{p_X} = \frac{f_y(x, y)}{p_Y}$$

where $f_x(x, y)$ and $f_y(x, y)$ are the marginal utilities of X and Y, respectively. Show that $f_x(x, y)/p_X$ is the marginal utility of one dollar's worth of X. Hence, maximum satisfaction is obtained when the consumer allocates the budget so that the marginal utility of a dollar's worth of X is equal to the marginal utility per dollar's worth of Y. Performing the same procedure as that for $U = f(x, y)$, verify that this is true for $U = f(x, y, z, w)$, subject to the corresponding budget equation. In each case, λ is called the *marginal utility of income*.

Objective

To compute double and triple integrals.

17.6 Multiple Integrals

Recall that the definite integral of a function of one variable is concerned with integration over an *interval*. There are also definite integrals of functions of two variables, called (definite) **double integrals**. These involve integration over a *region* in the plane.

For example, the symbol

$$\int_0^2 \int_3^4 xy\,dx\,dy = \int_0^2 \left(\int_3^4 xy\,dx \right) dy$$

A double integral can be interpreted in terms of the volume of a region between the x, y-plane and a surface $z = f(x, y)$ if $z \geq 0$. In Figure 17.10 is a region whose volume we will consider. The element of volume for this region is a vertical column with height approximately $x = f(x, y)$ and base area dyx. Thus, its volume is approximately $f(x, y)dydx$. The volume of the entire region can be found by summing the volumes of all such elements for $a \leq x \leq b$ and $c \leq y \leq d$ via a double integral:

$$\text{volume} = \int_a^b \int_c^d f(x, y)dydx$$

Triple integrals are handled by successively evaluating three integrals, as the next example shows.

EXAMPLE 3 Evaluating a Triple Integral

Find $\int_0^1 \int_0^x \int_0^{x-y} xdzdydx$.

Solution:

$$\begin{aligned}
\int_0^1 \int_0^x \int_0^{x-y} xdzdydx &= \int_0^1 \int_0^x \left(\int_0^{x-y} xdz \right) dydx \\
&= \int_0^1 \int_0^x (xz)\Big|_0^{x-y} dydx = \int_0^1 \int_0^x (x(x-y) - 0)dydx \\
&= \int_0^1 \int_0^x (x^2 - xy)dydx = \int_0^1 \left(\int_0^x (x^2 - xy)dy \right) dx \\
&= \int_0^1 \left(x^2y - \frac{xy^2}{2} \right)\Big|_0^x dx = \int_0^1 \left[\left(x^3 - \frac{x^3}{2} \right) - 0 \right] dx \\
&= \int_0^1 \frac{x^3}{2} dx = \frac{x^4}{8}\Big|_0^1 = \frac{1}{8}
\end{aligned}$$

Now Work Problem 21 ◁

PROBLEMS 17.6

In Problems 1–22, evaluate the multiple integrals.

1. $\int_0^3 \int_0^4 x\,dy\,dx$

2. $\int_1^4 \int_0^3 y\,dy\,dx$

3. $\int_0^1 \int_0^1 xy\,dx\,dy$

4. $\int_0^1 \int_0^2 xy^2 dydx$

5. $\int_1^3 \int_1^2 (x^2 - y)\,dx\,dy$

6. $\int_{-2}^3 \int_0^2 (y^2 - 2xy)\,dy\,dx$

7. $\int_0^1 \int_0^2 (x + y)\,dy\,dx$

8. $\int_0^3 \int_0^x (x^2 + y^2)\,dy\,dx$

9. $\int_1^2 \int_0^{x^2} ydydx$

10. $\int_1^2 \int_0^{x-1} 2y\,dy\,dx$

11. $\int_0^1 \int_{3x}^{x^2} 14x^2y\,dy\,dx$

12. $\int_0^2 \int_0^{x^2} xy\,dy\,dx$

13. $\int_0^3 \int_0^{\sqrt{9-x^2}} y\,dy\,dx$

14. $\int_0^1 \int_{y^3}^{y^2} dxdy$

15. $\int_{-1}^1 \int_x^{1-x} 3(x + y)\,dy\,dx$

16. $\int_0^3 \int_{y^2}^{3y} 5x\,dx\,dy$

17. $\int_0^1 \int_0^y e^{x+y} dx\,dy$

18. $\int_0^1 \int_0^1 e^{y-x} dx\,dy$

19. $\int_0^1 \int_0^2 \int_0^3 x^3y^2zdxdydz$

20. $\int_0^1 \int_0^x \int_0^{x+y} x^2\,dz\,dy\,dx$

21. $\int_0^1 \int_{x^2}^x \int_0^{xy} dz\,dy\,dx$

22. $\int_1^e \int_{\ln x}^x \int_0^y dz\,dy\,dx$

23. Statistics In the study of statistics, a joint density function $z = f(x, y)$ defined on a region in the x,y-plane is represented by a surface in space. The probability that

$$a \leq x \leq b \quad \text{and} \quad c \leq y \leq d$$

is given by

$$P(a \leq x \leq b, c \leq y \leq d) = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

and is represented by the volume between the graph of f and the rectangular region given by

$$a \le x \le b \quad \text{and} \quad c \le y \le d$$

If $f(x, y) = e^{-(x+y)}$ is a joint density function, where $x \ge 0$ and $y \ge 0$, find

$$P(0 \le x \le 2, 1 \le y \le 2)$$

and give your answer in terms of e.

24. Statistics In Problem 23, let $f(x, y) = 6e^{-(2x+3y)}$ for $x, y \ge 0$. Find

$$P(1 \le x \le 3, 2 \le y \le 4)$$

and give your answer in terms of e.

25. Statistics In Problem 23, let $f(x, y) = 1$, where $0 \le x \le 1$ and $0 \le y \le 1$. Find $P(x \ge 1/2, y \ge 1/3)$.

26. Statistics In Problem 23, let f be the uniform density function $f(x, y) = 1/8$ defined over the rectangle $0 \le x \le 4, 0 \le y \le 2$. Determine the probability that $0 \le x \le 1$ and $0 \le y \le 1$.

Chapter 17 Review

Important Terms and Symbols

		Examples
Section 17.1	**Partial Derivatives**	
	partial derivative $\dfrac{\partial z}{\partial x} = f_x(x, y)$ $\quad \left.\dfrac{\partial z}{\partial x}\right\vert_{(a,b)} = f_x(a, b)$	Ex. 2, p. 735
Section 17.2	**Applications of Partial Derivatives**	
	joint-cost function production function marginal productivity	Ex. 3, p. 740
	competitive products complementary products	Ex. 4, p. 741
Section 17.3	**Higher-Order Partial Derivatives**	
	$\dfrac{\partial^2 z}{\partial y \partial x} = f_{xy}$ $\quad \dfrac{\partial^2 z}{\partial x \partial y} = f_{yx}$ $\quad \dfrac{\partial^2 z}{\partial x^2} = f_{xx}$ $\quad \dfrac{\partial^2 z}{\partial y^2} = f_{yy}$	Ex. 1, p. 744
Section 17.4	**Maxima and Minima for Functions of Two Variables**	
	relative maximum and minimum critical point	Ex. 1, p. 747
	second-derivative test for functions of two variables	Ex. 3, p. 749
Section 17.5	**Lagrange Multipliers**	
	Lagrange multipliers	Ex. 1, p. 756
Section 17.6	**Multiple Integrals**	
	double integral triple integral	Ex. 3, p. 764

Summary

For a function of n variables, we can consider n partial derivatives. For example, if $w = f(x, y, z)$, we have the partial derivatives of f with respect to x, with respect to y, and with respect to z, denoted either f_x, f_y, and f_z, or $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$, respectively. To find $f_x(x, y, z)$, we treat y and z as constants and differentiate f with respect to x in the usual way. The other partial derivatives are found similarly. We can interpret $f_x(x, y, z)$ as the approximate change in w that results from a one-unit change in x when y and z are held fixed. There are similar interpretations for the other partial derivatives.

Functions of several variables occur frequently in business and economic analysis, as well as in other areas of study. If a manufacturer produces x units of product X and y units of product Y, then the total cost, c, of these units is a function of x and y and is called a joint-cost function. The partial derivatives $\partial c/\partial x$ and $\partial c/\partial y$ are called the marginal costs with respect to x and y, respectively. We can interpret, for example, $\partial c/\partial x$ as the approximate cost of producing an extra unit of X while the level of production of Y is held fixed.

If l units of labor and k units of capital are used to produce P units of a product, then the function $P = f(l, k)$ is called a production function. The partial derivatives of P are called marginal productivity functions.

Suppose two products, A and B, are such that the quantity demanded of each is dependent on the prices of both. If q_A and q_B are the quantities of A and B demanded when the prices of A and B are p_A and p_B, respectively, then q_A and q_B are each functions of p_A and p_B. When $\partial q_A/\partial p_B > 0$ and $\partial q_B/\partial p_A > 0$, then A and B are called competitive products (or substitutes). When $\partial q_A/\partial p_B < 0$ and $\partial q_B/\partial p_A < 0$, then A and B are called complementary products.

A partial derivative of a function of n variables is itself a function of n variables. By successively taking partial derivatives of partial derivatives, we obtain higher-order partial derivatives. For example, if f is a function of x and y, then f_{xy} denotes the partial derivative of f_x with respect to y; f_{xy} is called the second-partial derivative of f, first with respect to x and then with respect to y.

If the function $f(x, y)$ has a relative extremum at (a, b), then (a, b) must be a solution of the system

$$\begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases}$$

Any solution of this system is called a critical point of f. Thus, critical points are the candidates at which a relative extremum *may* occur. The second-derivative test for functions of two variables gives conditions under which a critical point corresponds to a relative maximum or a relative minimum. The test states that if (a, b) is a critical point of f and

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2$$

then

1. if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b);
2. if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b);
3. if $D(a, b) < 0$, then f has a saddle point at (a, b);
4. if $D(a, b) = 0$, no conclusion about an extremum at (a, b) can yet be drawn, and further analysis is required.

To find critical points of a function of several variables, subject to a constraint, we can sometimes use the method of Lagrange multipliers. For example, to find the critical points of $f(x, y, z)$, subject to the constraint $g(x, y, z) = 0$, we first form the function

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$$

By solving the system

$$\begin{cases} F_x = 0 \\ F_y = 0 \\ F_z = 0 \\ F_\lambda = 0 \end{cases}$$

we obtain the critical points of F. If (a, b, c, λ_0) is such a critical point, then (a, b, c) is a critical point of f, subject to the constraint. It is important to write the constraint in the form $g(x, y, z) = 0$. For example, if the constraint is $2x+3y-z = 4$, then $g(x, y, z) = 2x+3y-z-4$. If $f(x, y, z)$ is subject to two constraints, $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, then we would form the function $F = f - \lambda_1 g_1 - \lambda_2 g_2$ and solve the system

$$\begin{cases} F_x = 0 \\ F_y = 0 \\ F_z = 0 \\ F_{\lambda_1} = 0 \\ F_{\lambda_2} = 0 \end{cases}$$

When working with functions of several variables, we can consider their multiple integrals. These are determined by successive integration. For example, the double integral

$$\int_1^2 \int_0^y (x + y)dxdy$$

is determined by first treating y as a constant and integrating $x + y$ with respect to x. After evaluating between the bounds 0 and y, we integrate that result with respect to y from $y = 1$ to $y = 2$. Thus,

$$\int_1^2 \int_0^y (x + y)dxdy = \int_1^2 \left(\int_0^y (x + y)dx\right) dy$$

Triple integrals involve functions of three variables and are also evaluated by successive integration.

Review Problems

In Problems 1–12, find the indicated partial derivatives.

1. $f(x, y) = \ln(x^2 + y^2)$; $f_x(x, y), f_y(x, y)$

2. $P = l^3 + k^3 - lk$; $\partial P/\partial l, \partial P/\partial k$

3. $z = \dfrac{x}{x + y}$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

4. $f(p_A, p_B) = 4(p_A - 10) + 5(p_B - 15)$; $f_{p_B}(p_A, p_B)$

5. $f(x, y) = e^{\sqrt{x^2+y^2}}$; $\dfrac{\partial}{\partial y}(f(x, y))$

6. $w = \sqrt{x^2 + y^2}$; $\dfrac{\partial w}{\partial y}$

7. $w = e^{x^2yz}$; $w_{xy}(x, y, z)$

8. $f(x, y) = xy\ln(xy)$; $f_{xy}(x, y)$

9. $f(x, y, z) = (x + y + z)(x^2 + y^2 + z^2)$; $\dfrac{\partial^2}{\partial z^2}(f(x, y, z))$

10. $z = (x^2 - y^2)^2$; $\partial^2 z/\partial y\partial x$

11. $w = e^{x+y+z}\ln(xyz)$; $\partial^3 w/\partial z\partial y\partial x$

12. $P = 100l^{0.11}k^{0.89}$; $\partial^2 P/\partial k\partial l$

13. If $f(x, y, z) = \dfrac{x + y}{xz}$, find $f_{xyz}(2, 7, 4)$.

14. If $f(x, y, z) = (6x + 1)e^{y^2 \ln(z+1)}$, find $f_{xyz}(0, 1, 0)$.

15. Production Function If a manufacturer's production function is defined by $P = 100l^{0.8}k^{0.2}$, determine the marginal productivity functions.

16. Joint-Cost Function A manufacturer's cost for producing x units of product X and y units of product Y is given by

$$c = 3x + 0.05xy + 9y + 500$$

Determine the (partial) marginal cost with respect to x when $x = 50$ and $y = 100$.

17. Competitive/Complementary Products If $q_A = 100 - p_A + 2p_B$ and $q_B = 150 - 3p_A - 2p_B$, where q_A and q_B are the number of units demanded of products A and B, respectively, and p_A and p_B are their respective prices per unit, determine whether A and B are competitive products or complementary products or neither.

18. Innovation For industry, the following model describes the rate α (a Greek letter read "alpha") at which an innovation substitutes for an established process:[15]

$$\alpha = Z + 0.530P - 0.027S$$

Here, Z is a constant that depends on the particular industry, P is an index of profitability of the innovation, and S is an index of the extent of the investment necessary to make use of the innovation. Find $\partial\alpha/\partial P$ and $\partial\alpha/\partial S$.

19. Examine $f(x, y) = x^2 + 2y^2 - 2xy - 4y + 3$ for relative extrema.

20. Examine $f(w, z) = w^3 + z^3 - 3wz + 5$ for relative extrema.

21. Minimizing Material An open-top rectangular cardboard box is to have a volume of 32 cubic feet. Find the dimensions of the box so that the amount of cardboard used is minimized.

22. The function

$$f(x, y) = ax^2 + by^2 + cxy - x + y$$

has a critical point at $(x, y) = (0, 1)$, and the second-derivative test is inconclusive at this point. Determine the values of the constants a, b, and c.

23. Maximizing Profit A dairy produces two types of cheese, A and B, at constant average costs of 50 cents and 60 cents per pound, respectively. When the selling price per pound of A is p_A cents and of B is p_B cents, the demands (in pounds) for A and B, are, respectively,

$$q_A = 250(p_B - p_A)$$

and

$$q_B = 32{,}000 + 250(p_A - 2p_B)$$

Find the selling prices that yield a relative maximum profit. Verify that the profit has a relative maximum at these prices.

24. Find all critical points of $f(x, y, z) = xy^2z$, subject to the condition that

$$x + y + z - 1 = 0 \ (xyz \neq 0)$$

25. Find all critical points of $f(x, y) = \sqrt{x^2 + y^2}$, subject to the constraint $5x + y = 1$. Explain the answer geometrically.

In Problems 26–29, evaluate the double integrals.

26. $\displaystyle\int_1^2 \int_0^y x^2y^2\,dx\,dy$

27. $\displaystyle\int_0^1 \int_0^{y^2} xy\,dx\,dy$

28. $\displaystyle\int_1^4 \int_{x^2}^{2x} y\,dy\,dx$

29. $\displaystyle\int_0^1 \int_{\sqrt{x}}^{x^2} 7(x^2 + 2xy - 3y^2)\,dy\,dx$

[15] A. P. Hurter, Jr., A. H. Rubenstein, et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change,* 11 (1978), 197–221.