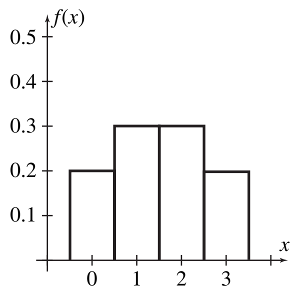
Chapter 9

Problems 9.1

1. $\mu = \sum_x x\,f(x) = 0(0.2) + 1(0.3) + 2(0.3) + 3(0.2) = 1.5$

$\text{Var}(X) = \sum_x x^2 f(x) - \mu^2 = [0^2(0.2) + 1^2(0.3) + 2^2(0.3) + 3^2(0.2)] - (1.5)^2 = 1.05$

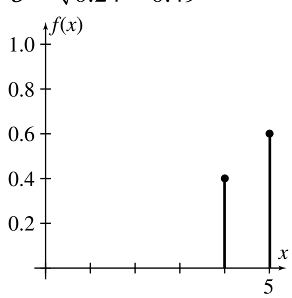
$\sigma = \sqrt{\text{Var}(X)} = \sqrt{1.05} \approx 1.02$



2. $\mu = \sum_x x\,f(x) = 4(0.4) + 5(0.6) = 4.6$

$\text{Var}(X) = [4^2(0.4) + 5^2(0.6)] - (4.6)^2 = 0.24$

$\sigma = \sqrt{0.24} \approx 0.49$



3. $\mu = \sum_x x\,f(x) = 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right) = \frac{9}{4} = 2.25$

$\text{Var}(X) = \sum_x x^2 f(x) - \mu^2 = \left[1^2\left(\frac{1}{4}\right) + 2^2\left(\frac{1}{4}\right) + 3^2\left(\frac{1}{2}\right)\right] - \left(\frac{9}{4}\right)^2 = \frac{11}{16} = 0.6875$

$\sigma = \sqrt{\frac{11}{16}} = \frac{\sqrt{11}}{4} \approx 0.83$

4. $\mu = \sum_x x\,f(x) = 0\left(\frac{1}{7}\right) + 1\left(\frac{2}{7}\right) + 2\left(\frac{1}{7}\right) + 3\left(\frac{2}{7}\right) + 4\left(\frac{1}{7}\right) = \frac{14}{7} = 2$

$\text{Var}(X) = \left[0^2\left(\frac{1}{7}\right) + 1^2\left(\frac{2}{7}\right) + 2^2\left(\frac{1}{7}\right) + 3^2\left(\frac{2}{7}\right) + 4^2\left(\frac{1}{7}\right)\right] - 2^2 = \frac{12}{7} \approx 1.71$

$\sigma = \sqrt{\frac{12}{7}} \approx 1.31$

5. a. $P(X = 3) = 1 - [P(X = 5) + P(X = 6) + P(X = 7)] = 1 - [0.3 + 0.2 + 0.4] = 0.1$

b. $\mu = \sum_x x\, f(x) = 3(0.1) + 5(0.3) + 6(0.2) + 7(0.4) = 5.8$

c. $\sigma^2 = \sum_x x^2 f(x) - \mu^2 = [3^2(0.1) + 5^2(0.3) + 6^2(0.2) + 7^2(0.4)] - (5.8)^2 = 1.56$

6. a. $0.1 + 5a + 4a = 1 \Rightarrow a = 0.1$

Thus $P(X = 4) = 5(0.1) = 0.5$, and $P(X = 6) = 4(0.1) = 0.4$.

b. $\mu = 2(0.1) + 4(0.5) + 6(0.4) = 4.6.$

7. Distribution of X:

$f(0) = \frac{1}{8},\ f(1) = \frac{3}{8},\ f(2) = \frac{3}{8},\ f(3) = \frac{1}{8}$

$E(X) = \sum_x x\, f(x) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = \frac{12}{8} = \frac{3}{2} = 1.5$

$\sigma^2 = \text{Var}(X) = \sum_x x^2 f(x) - [E(x)]^2$

$= \left[0^2\left(\frac{1}{8}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{3}{8}\right) + 3^2\left(\frac{1}{8}\right)\right] - \left(\frac{3}{2}\right)^2$

$= \frac{24}{8} - \frac{9}{4} = \frac{6}{8} = \frac{3}{4} = 0.75$

$\sigma = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \approx 0.87$

8. Distribution of X: $f(1) = \frac{4}{6} = \frac{2}{3},\ f(2) = \frac{2}{6} = \frac{1}{3}$

$E(X) = 1\left(\frac{2}{3}\right) + 2\left(\frac{1}{3}\right) = \frac{4}{3} \approx 1.33$

$\sigma^2 = \sum_x x^2 f(x) - [E(x)]^2 = \left[1^2\left(\frac{2}{3}\right) + 2^2\left(\frac{1}{3}\right)\right] - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9} \approx 0.22$

$\sigma = \sqrt{\frac{2}{9}} \approx 0.47$

9. The number of outcomes in the sample space is ${}_5C_2 = 10$.

Distribution of *X:*

$f(0) = \frac{{}_2C_2}{10} = \frac{1}{10}$, $f(1) = \frac{{}_2C_1 \cdot {}_3C_1}{10} = \frac{3}{5}$,

$f(2) = \frac{{}_3C_2}{10} = \frac{3}{10}$

$E(X) = \sum_x x\,f(x) = 0\left(\frac{1}{10}\right) + 1\left(\frac{3}{5}\right) + 2\left(\frac{3}{10}\right)$

$= \frac{6}{5} = 1.2$

$\sigma^2 = \sum_x x^2 f(x) - [E(x)]^2$

$= \left[0^2\left(\frac{1}{10}\right) + 1^2\left(\frac{3}{5}\right) + 2^2\left(\frac{3}{10}\right)\right] - \left(\frac{6}{5}\right)^2$

$= \frac{9}{5} - \frac{36}{25} = \frac{9}{25} = 0.36$

$\sigma = \sqrt{\frac{9}{25}} = \frac{3}{5} = 0.6$

10. Distribution of *X:*

$f(0) = \frac{9}{25}$, $f(1) = \frac{12}{25}$, $f(2) = \frac{4}{25}$

$E(X) = 0\left(\frac{9}{25}\right) + 1\left(\frac{12}{25}\right) + 2\left(\frac{4}{25}\right) = \frac{20}{25} = \frac{4}{5}$

$= 0.8$

$\sigma^2 = \left[0^2\left(\frac{9}{25}\right) + 1^2\left(\frac{12}{25}\right) + 2^2\left(\frac{4}{25}\right)\right] - \left(\frac{4}{5}\right)^2$

$= \frac{28}{25} - \frac{16}{25} = \frac{12}{25} = 0.48$

$\sigma = \sqrt{\frac{12}{25}} = \frac{2\sqrt{3}}{5} \approx 0.69$

11. $f(0) = P(X = 0) = \frac{{}_3C_2}{{}_8C_2} = \frac{3}{28}$

$f(1) = P(X = 1) = \frac{{}_3C_1 \cdot {}_5C_1}{{}_8C_2} = \frac{15}{28}$

$f(2) = P(X = 2) = \frac{{}_5C_2}{{}_8C_2} = \frac{10}{28} = \frac{5}{14}$

12. $P(X = x) = \frac{{}_4C_x \cdot {}_6C_{3-x}}{{}_{10}C_3}$

13. a. If *X* is the gain (in dollars), then $X = -2$ or 4998.

Distribution of *X:*

$f(-2) = \frac{7999}{8000}$, $f(4998) = \frac{1}{8000}$

$E(x) = \sum_x xf(x)$

$= -2 \cdot \frac{7999}{8000} + 4998 \cdot \frac{1}{8000}$

$= -\frac{11{,}000}{8000} \approx -\$1.38$ (a loss)

b. Here $X = -4$ or 4996. Distribution of *X:*

$f(-4) = \frac{7998}{8000}$, $f(4996) = \frac{2}{8000}$

$E(X) = \sum_x xf(x)$

$= -4 \cdot \frac{7998}{8000} + 4996 \cdot \frac{2}{8000}$

$= -\$2.75$ (a loss)

14. If *X* is the gain (in dollars) per game, then $X = 10$ or -6.

Distribution of *X:*

$f(10) = \frac{2}{8} = \frac{1}{4}$, $f(-6) = \frac{6}{8} = \frac{3}{4}$

$E(X) = \sum_x x\,f(x) = 10 \cdot \frac{1}{4} + (-6) \cdot \frac{3}{4}$

$= -\$2$ (a loss)

15. Let X = daily earnings (in dollars).

Distribution of *X:*

$f(200) = \frac{4}{7}$, $f(-30) = \frac{3}{7}$

$E(X) = \sum_x x\,f(x)$

$= 200 \cdot \frac{4}{7} + (-30) \cdot \frac{3}{7}$

$= \frac{710}{7} \approx \$101.43$

16. Let X = gain (in dollars) to the chain of a restaurant in a shopping center.

Distribution of *X:*

$f(120{,}000) = 0.72$, $f(-36{,}000) = 0.28$

$E(X) = 120{,}000(0.72) + (-36{,}600)(0.28)$

$= \$76{,}320.$

17. The probability that a person in the group is not hospitalized is
1 – (0.001 + 0.002 + 0.003 + 0.004 + 0.008) = 0.982.
Let X = gain (in dollars) to the company from a policy.
Distribution of X:
$f(10) = 0.982,\ f(-90) = 0.001,\quad f(-190) = 0.002,\quad f(-290) = 0.003,\quad f(-390) = 0.004,\ f(-490) = 0.008$
$E(X) = 10(0.982) + (-90)(0.001) + (-190)(0.002) + (-290)(0.003)\ + (-390)(0.004) + (-490)(0.008)$
$= \$3.00$

18. $E(X) = 0(0.05) + 1(0.10) + 2(0.15) + 3(0.20) + 4(0.15) + 5(0.15) + 6(0.10) + 7(0.05) + 8(0.05) = 3.70$

19. Let p = the annual premium (in dollars) per policy. If X = gain (in dollars) to the company from a policy, then either $X = p$ or $X = -(180{,}000 - p)$. We set $E(X) = 50$:

$$\begin{aligned} -(180{,}000 - p)(0.002) + p(0.998) &= 50 \\ -360 + 0.002p + 0.998p &= 50 \\ -360 + p &= 50 \\ p &= \$410 \end{aligned}$$

20. Let X = player's gain (in dollars) per play.
Distribution of X:

$$f(35) = \frac{1}{37},\ f(-1) = \frac{36}{37}$$

$$E(X) = 35 \cdot \frac{1}{37} + (-1) \cdot \frac{36}{37} = -\frac{1}{37} \approx -\$0.03 \text{ (a loss)}$$

21. Let X = gain (in dollars) on a play.

If 0 heads show, then $X = 0 - 2.50 = -\frac{5}{2}$.

If exactly 1 head shows, then $X = 2.00 - 2.50 = -\frac{1}{2}$.

If 2 heads show, then $X = 4.00 - 2.50 = \frac{3}{2}$.

Distribution of X:

$$f\left(-\frac{5}{2}\right) = \frac{1}{4},\ f\left(-\frac{1}{2}\right) = \frac{1}{2},\ f\left(\frac{3}{2}\right) = \frac{1}{4}$$

$$E(X) = \left(-\frac{5}{2}\right)\left(\frac{1}{4}\right) + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{2}\right)\left(\frac{1}{4}\right) = -\frac{1}{2} = -0.50$$

Thus there is an expected loss of \$0.50 on each play.
For a fair game, let p = amount (in dollars) paid to play.
Distribution of X:

$$f(-p) = \frac{1}{4},\ f(1-p) = \frac{1}{2},\ f(2-p) = \frac{1}{4}$$

We set $E(X) = 0$:

$$\begin{aligned} (-p)\frac{1}{4} + (2-p)\frac{1}{2} + (4-p)\frac{1}{4} &= 0 \\ -\frac{p}{4} + 1 - \frac{p}{2} + 1 - \frac{p}{4} &= 0 \\ 2 - p &= 0 \\ p &= 2 \end{aligned}$$

Thus you should pay \$2 for a fair game.

Apply It 9.2

1. Here $p = 0.30$, $q = 1 - p = 0.70$, and $n = 4$.

$P(X = x) = {}_nC_x p^x q^{n-x}$, $x = 0, 1, 2, 3, 4$

$P(X = 0) = {}_4C_0(0.3)^0(0.7)^4 = 0.2401$

$= \frac{2401}{10,000}$

$P(X = 1) = {}_4C_1(0.3)^1(0.7)^3 = 0.4116$

$= \frac{4116}{10,000}$

$P(X = 2) = {}_4C_2(0.3)^2(0.7)^2 = 0.2646$

$= \frac{2646}{10,000}$

$P(X = 3) = {}_4C_3(0.3)^3(0.7)^1 = 0.0756$

$= \frac{756}{10,000}$

$P(X = 4) = {}_4C_4(0.3)^4(0.7)^0 = 0.0081$

$= \frac{81}{10,000}$

Problems 9.2

1. $f(0) = {}_2C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^2 = \frac{2!}{0!\cdot 2!}\cdot 1\cdot\frac{16}{25}$

$= 1\cdot 1\cdot\frac{16}{25} = \frac{16}{25}$

$f(1) = {}_2C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^1 = \frac{2!}{1!\cdot 1!}\cdot\frac{1}{5}\cdot\frac{4}{5}$

$= 2\cdot\frac{1}{5}\cdot\frac{4}{5} = \frac{8}{25}$

$f(2) = {}_2C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^0 = \frac{2!}{2!\cdot 0!}\cdot\frac{1}{25}\cdot 1$

$= 1\cdot\frac{1}{25}\cdot 1 = \frac{1}{25}$.

$\mu = np = 2\cdot\frac{1}{5} = \frac{2}{5}$

$\sigma = \sqrt{npq} = \sqrt{2\cdot\frac{1}{5}\cdot\frac{4}{5}}$

$= \sqrt{\frac{8}{25}} = \frac{2\sqrt{2}}{5}$

2. $f(0) = {}_3C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = 1\cdot 1\cdot\frac{1}{8} = \frac{1}{8}$

$f(1) = {}_3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = 3\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{3}{8}$

$f(2) = {}_3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = 3\cdot\frac{1}{4}\cdot\frac{1}{2} = \frac{3}{8}$

$f(3) = {}_3C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = 1\cdot\frac{1}{8}\cdot 1 = \frac{1}{8}$

$\mu = np = 3\cdot\frac{1}{2} = \frac{3}{2}$

$\sigma = \sqrt{npq} = \sqrt{3\cdot\frac{1}{2}\cdot\frac{1}{2}} = \frac{\sqrt{3}}{2}$

3. $f(0) = {}_3C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3 = 1\cdot 1\cdot\frac{1}{27} = \frac{1}{27}$

$f(1) = {}_3C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 = \frac{3!}{1!\cdot 2!}\cdot\frac{2}{3}\cdot\frac{1}{9}$

$= 3\cdot\frac{2}{3}\cdot\frac{1}{9} = \frac{2}{9}$

$f(2) = {}_3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = \frac{3!}{2!\cdot 1!}\cdot\frac{4}{9}\cdot\frac{1}{3}$

$= 3\cdot\frac{4}{9}\cdot\frac{1}{3} = \frac{4}{9}$

$f(3) = {}_3C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = \frac{3!}{3!\cdot 0!}\cdot\frac{8}{27}\cdot 1$

$= 1\cdot\frac{8}{27}\cdot 1 = \frac{8}{27}$

$\mu = np = 3\cdot\frac{2}{3} = 2; \sigma = \sqrt{npq} = \sqrt{3\cdot\frac{2}{3}\cdot\frac{1}{3}}$

$= \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$

4. $f(0) = {}_4C_0(0.4)^0(0.6)^4 = \frac{4!}{0!\cdot 4!}\cdot 1\cdot(0.6)^4$

$= 1\cdot 1\cdot(0.6)^4 = 0.1296$

$f(1) = {}_4C_1(0.4)^1(0.6)^3 = \frac{4!}{1!\cdot 3!}(0.4)(0.6)^3$

$= 4(0.4)(0.6)^3 = 0.3456$

$$f(2) = {}_4C_2(0.4)^2(0.6)^2 = \frac{4!}{2!\cdot 2!}(0.4)^2(0.6)^2$$
$$= 6(0.4)^2(0.6)^2 = 0.3456$$
$$f(3) = {}_4C_3(0.4)^3(0.6)^1 = \frac{4!}{3!\cdot 1!}(0.4)^3(0.6)$$
$$= 4(0.4)^3(0.6) = 0.1536$$
$$f(4) = {}_4C_4(0.4)^4(0.6)^0 = \frac{4!}{4!\cdot 0!}(0.4)^4\cdot 1$$
$$= 1(0.4)^4\cdot 1 = 0.0256$$
$$\mu = np = 4(0.4) = 1.6$$
$$\sigma = \sqrt{npq} = \sqrt{4(0.4)(0.6)} \approx 0.98$$

5. $P(X=3) = {}_4C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 = \frac{8}{81}$

6. $P(X=2) = {}_5C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3$
$$= 10\cdot\frac{1}{9}\cdot\frac{8}{27} = \frac{80}{243} \approx 0.3292$$

7. $P(X=2) = {}_4C_2\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2 = 6\cdot\frac{16}{25}\cdot\frac{1}{25}$
$$= \frac{96}{625} = 0.1536$$

8. $P(X=4) = {}_7C_4(0.2)^4(0.8)^3$
$$= 35(0.0016)(0.512) = 0.028672$$

9. $P(X<2) = P(X=0) + P(X=1)$
$$= {}_5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 + {}_5C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4$$
$$= 1\cdot 1\cdot\frac{1}{32} + 5\cdot\frac{1}{2}\cdot\frac{1}{16} = \frac{6}{32} = \frac{3}{16}$$

10. $P(X\ge 3) = P(X=3) + P(X=4)$
$$= {}_4C_3\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^1 + {}_4C_4\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^0$$
$$= 4\cdot\frac{64}{125}\cdot\frac{1}{5} + 1\cdot\frac{256}{625}\cdot 1$$
$$= \frac{512}{625}$$

11. Let X = number of heads that occurs.
$$p = \frac{1}{2}, n = 11$$
$$P(X=8) = {}_{11}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^3$$
$$= 165\cdot\frac{1}{256}\cdot\frac{1}{8}$$
$$= \frac{165}{2048} \approx 0.081$$

12. Let X = number of correct answers. $p = \frac{1}{4}, n = 6$
$$P(X=3) = {}_6C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^3 = 20\cdot\frac{27}{4^6}$$
$$= \frac{540}{4096} \approx 0.132$$

13. Let X = number of green marbles drawn. The probability of selecting a green marble on any draw is $\frac{7}{12}$, $n = 4$.
$$P(X=2) = {}_4C_2\left(\frac{7}{12}\right)^2\left(\frac{5}{12}\right)^2$$
$$= 6\cdot\frac{49}{144}\cdot\frac{25}{144} = \frac{1225}{3456} \approx 0.3545$$

14. Let X = number of aces selected. The probability of selecting an ace on any draw is $p = \frac{4}{52} = \frac{1}{13}$.
$n = 3$
$$P(X=2) = {}_3C_2\left(\frac{1}{13}\right)^2\left(\frac{12}{13}\right)^1 = 3\cdot\frac{1}{169}\cdot\frac{12}{13}$$
$$= \frac{36}{2197} \approx 0.016$$

15. Let X = number of defective switches selected. The probability that a switch is defective is $p = 0.03$, $n = 5$.
$$P(X=3) = {}_5C_3(0.03)^3(0.97)^2 = 2.54043\times 10^{-4}$$

16. $p = 0.2$, $n = 3$
$$P(X=x) = {}_3C_x(0.2)^x(0.8)^{3-x}$$

17. Let X = number of heads that occurs. $p=\frac{1}{4}, n=3$

a. $P(X=2)={}_3C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1=3\cdot\frac{1}{16}\cdot\frac{3}{4}=\frac{9}{64}$

b. $P(X=3)={}_3C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0=1\cdot\frac{1}{64}\cdot 1=\frac{1}{64}$

Thus

$$P(X=2)+P(X=3)=\frac{9}{64}+\frac{1}{64}=\frac{10}{64}=\frac{5}{32}$$

18. Let X = number of hearts selected.

$p=\frac{13}{52}=\frac{1}{4}, n=7$

a. $P(X=4)={}_7C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^3=35\cdot\frac{1}{256}\cdot\frac{27}{64}=\frac{945}{16,384}\approx 0.058$

b.
$$\begin{aligned}P(X\ge 4)&=P(X=4)+P(X=5)+P(X=6)+P(X=7)\\&=\frac{945}{16,384}+{}_7C_5\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^2+{}_7C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^1+{}_7C_7\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^0\\&=\frac{945}{16,384}+21\cdot\frac{1}{1024}\cdot\frac{9}{16}+7\cdot\frac{1}{4096}\cdot\frac{3}{4}+1\cdot\frac{1}{16,384}\cdot 1\\&=\frac{1156}{16,384}=\frac{289}{4096}\approx 0.071\end{aligned}$$

19. Let X = number of defective in sample.

$p=\frac{1}{5},\ n=6$

$$\begin{aligned}P(X\le 1)&=P(X=0)+P(X=1)\\&={}_6C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^6+{}_6C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^5\\&=1\cdot 1\cdot\frac{4096}{15,625}+6\cdot\frac{1}{5}\cdot\frac{1024}{3125}\\&=\frac{10,240}{15,625}=\frac{2048}{3125}\approx 0.655\end{aligned}$$

20. Let X = number of persons with high-speed Internet.

$p=0.8, n=4$

$$\begin{aligned}P(X\ge 3)&=P(X=3)+P(X=4)\\&={}_4C_3(0.8)^3(0.2)^1+{}_4C_4(0.8)^4(0.2)^0\\&=4(0.512)(0.2)+1(0.4096)(1)\\&=0.8192\end{aligned}$$

21. Let X = number of hits in four at-bats.
$p = 0.300$, $n = 4$
$P(X \geq 1) = 1 - P(X = 0) = 1 - {}_4C_0(0.300)^0(0.700)^4 = 1 - 1 \cdot 1 \cdot (0.2401) = 0.7599$

22. Let X = number of stocks that increase in value. The probability that a stock increases in value is $p = 0.6$. Here $n = 4$. We must find
$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$.
$P(X = 0) = {}_4C_0(0.6)^0(0.4)^4 = 1 \cdot 1 \cdot (0.0256) = 0.0256$
$P(X = 1) = {}_4C_1(0.6)^1(0.4)^3 = 4(0.6)(0.064) = 0.1536$
$P(X \geq 2) = 1 - P(X < 2) = 1 - [0.0256 + 0.1536] = 1 - 0.1792 \approx 0.82$

23. Let X = number of girls. The probability that a child is a girl is $p = \frac{1}{2}$. Here $n = 5$. We must find
$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$.

$$P(X = 0) = {}_5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = 1 \cdot 1 \cdot \frac{1}{32} = \frac{1}{32}$$

$$P(X = 1) = {}_5C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 = 5 \cdot \frac{1}{2} \cdot \frac{1}{16} = \frac{5}{32}$$

Thus,

$$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)] = 1 - \left[\frac{1}{32} + \frac{5}{32}\right] = 1 - \frac{3}{16} = \frac{13}{16}$$

24. $p = \frac{2}{5}, n = 50,\ q = 1 - p = 1 - \frac{2}{5} = \frac{3}{5}$

$$\sigma^2 = npq = 50 \cdot \frac{2}{5} \cdot \frac{3}{5} = 12$$

25. $\mu = 2, \sigma^2 = \frac{3}{2}$

Since $\mu = np$, then $np = 2$. Since $\sigma^2 = npq$, then $(np)q = \frac{3}{2}$, or $2q = \frac{3}{2}$, so $q = \frac{3}{4}$. Thus, $p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$.

Since $np = 2$, then $n \cdot \frac{1}{4} = 2$, or $n = 8$. Thus

$$P(X = 2) = {}_8C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6 = \frac{5103}{16{,}384} \approx 0.31146.$$

26. **a.** $E(X) = \mu = np = 15(0.06) = 0.9$

b. $\text{Var}(X) = \sigma^2 = npq = 15(0.06)(0.94) = 0.846$

c.
$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= {}_{15}C_0(0.06)^0(0.94)^{15} + {}_{15}C_1(0.06)^1(0.94)^{14} \\ &= 1 \cdot 1 \cdot (0.94)^{15} + 15(0.06)(0.94)^{14} \approx 0.77 \end{aligned}$$

Problems 9.3

1. $\begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{3}{2} & \frac{1}{3} \end{bmatrix}$

No, since the entry at row 2 column 1 is negative.

2. $\begin{bmatrix} 0.1 & 1 \\ 0.9 & 0 \end{bmatrix}$

Yes, since all entries are nonnegative and the sum of the entries in each column is 1.

3. $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{3} \\ -\frac{1}{4} & \frac{5}{8} & \frac{1}{3} \\ \frac{3}{4} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$

No, since there is a negative entry.

4. $\begin{bmatrix} 0.2 & 0.6 & 0.5 \\ 0.7 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}$

No, since the sum of the entries in column 3 is not 1.

5. $\begin{bmatrix} 0.4 & 0 & 0.5 \\ 0.2 & 0.1 & 0.3 \\ 0.4 & 0.9 & 0.2 \end{bmatrix}$

Yes, since all entries are nonnegative and the sum of the entries in each column is 1.

6. $\begin{bmatrix} 0.5 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.4 \end{bmatrix}$

No, since the sum of the entries in column 1 is not 1.

7. $\begin{bmatrix} \frac{2}{3} & b \\ a & \frac{1}{4} \end{bmatrix}$

$\frac{2}{3}+a=1, \text{so } a=\frac{1}{3}.$

$b+\frac{1}{4}=1, \text{so } b=\frac{3}{4}.$

8. $\begin{bmatrix} a & b \\ \frac{5}{12} & a \end{bmatrix}$

$a+\frac{5}{12}=1, \text{ so } a=1-\frac{5}{12}=\frac{7}{12}.$

$b+a=1, \text{ so } b=1-\frac{7}{12}=\frac{5}{12}.$

9. $\begin{bmatrix} 0.1 & a & a \\ a & 0.2 & b \\ 0.2 & b & c \end{bmatrix}$

$0.1 + a + 0.2 = 1$, so $a = 0.7$.
$a + 0.2 + b = 1$, so $b = 0.1$.
$a + b + c = 1$, so $c = 0.2$.

10. $\begin{bmatrix} a & a & a \\ a & b & b \\ a & \frac{1}{4} & c \end{bmatrix}$

$a+a+a=1, 3a=1, a=\frac{1}{3}$

$a+b+\frac{1}{4}=1, \frac{1}{3}+b+\frac{1}{4}=1, b=\frac{5}{12}$

$a+b+c=1, \frac{1}{3}+\frac{5}{12}+c=1, c=\frac{1}{4}$

11. $\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$

Yes, all entries are nonnegative and their sum is 1.

12. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Yes, all entries are nonnegative and their sum is 1.

13. $\begin{bmatrix} 0.2 \\ 0.7 \\ 0.5 \end{bmatrix}$

No, the sum of the entries is not 1.

14. $\begin{bmatrix} 0.1 \\ 1.1 \\ 0.2 \end{bmatrix}$

No, the sum of the entries is not 1.

15. $X_1 = TX_0 = \begin{bmatrix} \frac{2}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{11}{12} \\ \frac{1}{12} \end{bmatrix}$

$X_2 = TX_1 = \begin{bmatrix} \frac{2}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} \frac{11}{12} \\ \frac{1}{12} \end{bmatrix} = \begin{bmatrix} \frac{25}{36} \\ \frac{11}{36} \end{bmatrix}$

$X_3 = TX_2 = \begin{bmatrix} \frac{2}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} \frac{25}{36} \\ \frac{11}{36} \end{bmatrix} = \begin{bmatrix} \frac{83}{108} \\ \frac{25}{108} \end{bmatrix}$

16. $X_1 = TX_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix}$

$X_2 = TX_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix} = \begin{bmatrix} \frac{11}{32} \\ \frac{21}{32} \end{bmatrix}$

$X_3 = TX_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} \frac{11}{32} \\ \frac{21}{32} \end{bmatrix} = \begin{bmatrix} \frac{43}{128} \\ \frac{85}{128} \end{bmatrix}$

17. $X_1 = TX_0 = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.42 \\ 0.58 \end{bmatrix}$

$X_2 = TX_1 = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}\begin{bmatrix} 0.42 \\ 0.58 \end{bmatrix} = \begin{bmatrix} 0.416 \\ 0.584 \end{bmatrix}$

$X_3 = TX_2 = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}\begin{bmatrix} 0.416 \\ 0.584 \end{bmatrix} = \begin{bmatrix} 0.4168 \\ 0.5832 \end{bmatrix}$

18. $X_1 = TX_0 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$

$X_2 = TX_1 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}\begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \begin{bmatrix} 0.308 \\ 0.692 \end{bmatrix}$

$X_3 = TX_2 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}\begin{bmatrix} 0.308 \\ 0.692 \end{bmatrix} = \begin{bmatrix} 0.6536 \\ 0.3464 \end{bmatrix}$

19. $X_1 = TX_0 = \begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.7 & 0.4 & 0.4 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.1 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.33 \\ 0.21 \\ 0.46 \end{bmatrix}$

$X_2 = TX_1 = \begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.7 & 0.4 & 0.4 \end{bmatrix}\begin{bmatrix} 0.33 \\ 0.21 \\ 0.46 \end{bmatrix} = \begin{bmatrix} 0.271 \\ 0.23 \\ 0.499 \end{bmatrix}$

$X_3 = TX_2 = \begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.7 & 0.4 & 0.4 \end{bmatrix}\begin{bmatrix} 0.271 \\ 0.23 \\ 0.499 \end{bmatrix} = \begin{bmatrix} 0.2768 \\ 0.2419 \\ 0.4813 \end{bmatrix}$

20. $X_1 = TX_0 = \begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.1 \\ 0 & 0.1 & 0.3 & 0.3 \\ 0.4 & 0.7 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \\ 0.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.17 \\ 0.21 \\ 0.49 \\ 0.13 \end{bmatrix}$

$$X_2 = TX_1 = \begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.1 \\ 0 & 0.1 & 0.3 & 0.3 \\ 0.4 & 0.7 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.17 \\ 0.21 \\ 0.49 \\ 0.13 \end{bmatrix} = \begin{bmatrix} 0.200 \\ 0.207 \\ 0.463 \\ 0.130 \end{bmatrix}$$

$$X_3 = TX_2 = \begin{bmatrix} 0.4 & 0.1 & 0.2 & 0.1 \\ 0 & 0.1 & 0.3 & 0.3 \\ 0.4 & 0.7 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.200 \\ 0.207 \\ 0.463 \\ 0.130 \end{bmatrix} = \begin{bmatrix} 0.2063 \\ 0.1986 \\ 0.4621 \\ 0.1330 \end{bmatrix}$$

21. a. $T^2 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}$

$$T^3 = T^2T = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{7}{16} & \frac{9}{16} \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix}.$$

b. Entry in row 2, column 1, of T^2 is $\frac{3}{8}$.

c. Entry in row 1, column 2 of T^3 is $\frac{9}{16}$.

22. a. $T^2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{5}{12} \\ \frac{5}{9} & \frac{7}{12} \end{bmatrix}$

$$T^3 = T^2T = \begin{bmatrix} \frac{4}{9} & \frac{5}{12} \\ \frac{5}{9} & \frac{7}{12} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{23}{54} & \frac{31}{72} \\ \frac{31}{54} & \frac{41}{72} \end{bmatrix}$$

b. Entry in row 2, column 1, of T^2 is $\frac{5}{9}$.

c. Entry in row 1, column 2 of T^3 is $\frac{31}{72}$.

23. a. $T^2 = \begin{bmatrix} 0 & 0.5 & 0.3 \\ 1 & 0.4 & 0.3 \\ 0 & 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.3 \\ 1 & 0.4 & 0.3 \\ 0 & 0.1 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.50 & 0.23 & 0.27 \\ 0.40 & 0.69 & 0.54 \\ 0.10 & 0.08 & 0.19 \end{bmatrix}$

$$T^3 = T^2T = \begin{bmatrix} 0.50 & 0.23 & 0.27 \\ 0.40 & 0.69 & 0.54 \\ 0.10 & 0.08 & 0.19 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.3 \\ 1 & 0.4 & 0.3 \\ 0 & 0.1 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.230 & 0.369 & 0.327 \\ 0.690 & 0.530 & 0.543 \\ 0.080 & 0.101 & 0.130 \end{bmatrix}$$

b. Entry in row 2, column 1, of T^2 is 0.40.

c. Entry in row 1, column 2 of T^3 is 0.369.

24. a. $T^2 = \begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.7 & 0.4 & 0.4 \end{bmatrix}\begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.7 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.33 & 0.23 & 0.26 \\ 0.21 & 0.34 & 0.22 \\ 0.46 & 0.43 & 0.52 \end{bmatrix}$

$T^3 = T^2T = \begin{bmatrix} 0.33 & 0.23 & 0.26 \\ 0.21 & 0.34 & 0.22 \\ 0.46 & 0.43 & 0.52 \end{bmatrix}\begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.1 & 0.5 & 0.2 \\ 0.7 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.271 & 0.252 & 0.282 \\ 0.230 & 0.279 & 0.240 \\ 0.499 & 0.469 & 0.478 \end{bmatrix}$

b. Entry in row 2, column 1, of T^2 is 0.21.

c. Entry in row 1, column 2 of T^3 is 0.252.

25. $T - I = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & -\frac{2}{3} \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{1}{2} & \frac{2}{3} & 0 \\ \frac{1}{2} & -\frac{2}{3} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{4}{7} \\ 0 & 1 & \frac{3}{7} \\ 0 & 0 & 0 \end{array}\right]$$

$Q = \begin{bmatrix} \frac{4}{7} \\ \frac{3}{7} \end{bmatrix}$

26. $T - I = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{array}\right]$$

$Q = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \end{bmatrix}$

27. $T - I = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{2}{5} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ \frac{4}{5} & -\frac{3}{5} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{7} \\ 0 & 1 & \frac{4}{7} \\ 0 & 0 & 0 \end{array}\right]$$

$Q = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}$

28. $T-I=\begin{bmatrix}\frac{1}{4} & \frac{1}{3}\\ \frac{3}{4} & \frac{2}{3}\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{3}{4} & \frac{1}{3}\\ \frac{3}{4} & -\frac{1}{3}\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & 1 & 1\\ -\frac{3}{4} & \frac{1}{3} & 0\\ \frac{3}{4} & -\frac{1}{3} & 0\end{array}\right]\to\cdots\to\left[\begin{array}{cc|c}1 & 0 & \frac{4}{13}\\ 0 & 1 & \frac{9}{13}\\ 0 & 0 & 0\end{array}\right]$$

$$Q=\begin{bmatrix}\frac{4}{13}\\ \frac{9}{13}\end{bmatrix}$$

29. $T-I=\begin{bmatrix}0.2 & 0.1 & 0.4\\ 0.1 & 0.5 & 0.2\\ 0.7 & 0.4 & 0.4\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-0.8 & 0.1 & 0.4\\ 0.1 & -0.5 & 0.2\\ 0.7 & 0.4 & -0.6\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ -0.8 & 0.1 & 0.4 & 0\\ 0.1 & -0.5 & 0.2 & 0\\ 0.7 & 0.4 & -0.6 & 0\end{array}\right]\to\cdots\to\left[\begin{array}{ccc|c}1 & 0 & 0 & \frac{22}{81}\\ 0 & 1 & 0 & \frac{20}{81}\\ 0 & 0 & 1 & \frac{13}{27}\\ 0 & 0 & 0 & 0\end{array}\right]$$

30. $T-I=\begin{bmatrix}0.1 & 0.4 & 0.3\\ 0.2 & 0.2 & 0.3\\ 0.7 & 0.4 & 0.4\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-0.9 & 0.4 & 0.3\\ 0.2 & -0.8 & 0.3\\ 0.7 & 0.4 & -0.6\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ -0.9 & 0.4 & 0.3 & 0\\ 0.2 & -0.8 & 0.3 & 0\\ 0.7 & 0.4 & -0.6 & 0\end{array}\right]\to\cdots\to\left[\begin{array}{ccc|c}1 & 0 & 0 & 0.2707\\ 0 & 1 & 0 & 0.2481\\ 0 & 0 & 1 & 0.4812\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$Q\approx\begin{bmatrix}0.2707\\ 0.2481\\ 0.4812\end{bmatrix}$$

31. a. $T=\begin{array}{c c}& \begin{array}{cc}\text{Flu} & \text{No flu}\end{array}\\ \begin{array}{c}\text{Flu}\\ \text{No flu}\end{array} & \begin{bmatrix}0.1 & 0.2\\ 0.9 & 0.8\end{bmatrix}\end{array}$

b. $X_0=\begin{bmatrix}\frac{120}{200}\\ \frac{80}{200}\end{bmatrix}=\begin{bmatrix}0.6\\ 0.4\end{bmatrix}.$

If a period is 4 days, then 8 days corresponds to 2 periods, and 12 days corresponds to 3 periods. The state vector corresponding to 8 days from now is

$$X_2=T^2X_0=\begin{bmatrix}0.19 & 0.18\\ 0.81 & 0.82\end{bmatrix}\begin{bmatrix}0.6\\ 0.4\end{bmatrix}=\begin{bmatrix}0.186\\ 0.814\end{bmatrix}.$$

Thus 0.186(200) $\approx$ 37 students can be expected to have the flu 8 days from now.
The state vector corresponding to 12 days from now is

$$X_3 = T^3 X_0 = \begin{bmatrix} 0.181 & 0.182 \\ 0.819 & 0.818 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.1814 \\ 0.8186 \end{bmatrix}.$$

Thus 0.1814(200) $\approx$ 36 students can be expected to have the flu 12 days from now.

32. $T = \begin{array}{c|cc} & H & L \\ \hline H & 0.55 & 0.25 \\ L & 0.45 & 0.75 \end{array}$

$$X_0 = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

$$X_2 = T^2 X_0 = \begin{bmatrix} 0.415 & 0.325 \\ 0.585 & 0.675 \end{bmatrix} \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.3835 \\ 0.6165 \end{bmatrix}$$

38.35% of the members will be performing high-impact exercising.

33. a. $T = \begin{array}{c|cc} & A & B \\ \hline A & 0.7 & 0.4 \\ B & 0.3 & 0.6 \end{array}$

b. Wednesday corresponds to step 2.

$$T^2 = \begin{bmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{bmatrix}.$$

The probability is 0.61.

34. a. $X_1 = TX_0 = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.9 & 0.1 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.28 \\ 0.37 \\ 0.35 \end{bmatrix}$

28% to location 1, 37% to location 2, 35% to location 3

b. $X_2 = TX_1 = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.9 & 0.1 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 0.28 \\ 0.37 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.268 \\ 0.424 \\ 0.308 \end{bmatrix}$

26.8% to location 1, 42.4% to location 2, 30.8% to location 3

35. a. $T = \begin{array}{c|ccc} & D & R & O \\ \hline D & 0.8 & 0.1 & 0.3 \\ R & 0.1 & 0.8 & 0.2 \\ O & 0.1 & 0.1 & 0.5 \end{array}$

b. $T^2 = \begin{bmatrix} 0.68 & 0.19 & 0.41 \\ 0.18 & 0.67 & 0.29 \\ 0.14 & 0.14 & 0.30 \end{bmatrix}$

The probability is 0.19.

c. $X_1 = TX_0$

$$= \begin{bmatrix} 0.8 & 0.1 & 0.3 \\ 0.1 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.5 \end{bmatrix} \begin{bmatrix} 0.40 \\ 0.40 \\ 0.20 \end{bmatrix} = \begin{bmatrix} 0.42 \\ 0.40 \\ 0.18 \end{bmatrix}$$

40% are expected to be Republican.

36. $T = \begin{array}{c|ccc} & U & S & R \\ \hline U & 0.7 & 0.1 & 0.1 \\ S & 0.1 & 0.8 & 0.1 \\ R & 0.2 & 0.1 & 0.8 \end{array}$

a. 15 years corresponds to step 3.

$$T^3 = \begin{bmatrix} 0.412 & 0.196 & 0.196 \\ 0.219 & 0.562 & 0.219 \\ 0.369 & 0.242 & 0.585 \end{bmatrix}$$

The entry in row 3, column 2 of T^3 is 0.242, so the probability is 0.242.

b. $X_3 = T^3 X_0$

$$= \begin{bmatrix} 0.412 & 0.196 & 0.196 \\ 0.219 & 0.562 & 0.219 \\ 0.369 & 0.242 & 0.585 \end{bmatrix} \begin{bmatrix} 0.50 \\ 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.304 \\ 0.30475 \\ 0.39125 \end{bmatrix}$$

The population is expected to be 30.4% urban, 30.475% suburban, 39.125% rural.

37. a. $T = \begin{array}{c} \\ \text{A} \\ \text{Compet.} \end{array}\!\!\begin{array}{c} \text{A Compet.} \\ \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \end{array}$

b. $X_1 = TX_0 = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.70 \\ 0.30 \end{bmatrix}$

$= \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$

A is expected to control 65% of the market.

c. $T - I = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.2 & 0.3 & 0 \\ 0.2 & -0.3 & 0 \end{array}\right] \to \ldots \to \left[\begin{array}{cc|c} 1 & 0 & 0.6 \\ 0 & 1 & 0.4 \\ 0 & 0 & 0 \end{array}\right]$

$Q = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

In the long run, A can expect to control 60% of the market.

38. a. $T = \begin{array}{c} \\ \text{Fords} \\ \text{Non-fords} \end{array}\!\!\begin{array}{c} \text{Fords} \quad \text{Non-Fords} \\ \begin{bmatrix} 0.75 & 0.35 \\ 0.25 & 0.65 \end{bmatrix} \end{array}$

b. $T - I = \begin{bmatrix} 0.75 & 0.35 \\ 0.25 & 0.65 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -0.25 & 0.35 \\ 0.25 & -0.35 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.25 & 0.35 & 0 \\ 0.25 & -0.35 & 0 \end{array}\right] \to \ldots$

$\to \left[\begin{array}{cc|c} 1 & 0 & 0.5833 \\ 0 & 1 & 0.4167 \\ 0 & 0 & 0 \end{array}\right]$

$Q \approx \begin{bmatrix} 0.5833 \\ 0.4167 \end{bmatrix}$

In the long run, 58.33% of car purchases in the region are expected to be Fords.

39. a. $T = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} 1 \quad 2 \\ \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix} \end{array}$

b. $X_2 = T^2X_0 = \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

60% in compartment 1 and 40% in compartment 2

c. $T - I = \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{2}{5} & \frac{3}{5} & 0 \\ \frac{2}{5} & -\frac{3}{5} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{array}\right]$

$Q = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

In the long run, there will be 60% in compartment 1 and 40% in compartment 2.

40. a. $T = \begin{array}{r} \\ \\ \text{Works} \\ \text{Doesn't Work} \end{array}\!\!\begin{array}{c} \qquad\quad \text{Doesn't} \\ \text{Works} \quad \text{Work} \\ \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \end{array}$

b. $T^3 = \begin{bmatrix} 0.562 & 0.219 \\ 0.438 & 0.781 \end{bmatrix}$

The probability is 0.562.

c. $T - I = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.2 & 0.1 & 0 \\ 0.2 & -0.1 & 0 \end{array}\right] \to \ldots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{array}\right]$

$Q = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

In the long run, the number of machines working properly is $\left(\frac{1}{3}\right)(42) = 14$.

41. a. $T - I = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{array}\right]$$

$$Q = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

b. Presently, A accounts for 50% of sales and in long run A will account for $\frac{2}{3}$, or $66\frac{2}{3}\%$, of sales. Thus the percentage increase in sales above the present level is $\frac{66\frac{2}{3} - 50}{50} \cdot 100\% = \frac{16\frac{2}{3}}{50} \cdot 100\% = 33\frac{1}{3}\%$.

42. a. $T = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \end{array}$

b. $T^2 = \begin{bmatrix} 0.68 & 0.32 & 0.32 \\ 0.16 & 0.52 & 0.16 \\ 0.16 & 0.16 & 0.52 \end{bmatrix}$

The probability is 0.52.

c. Initially 500 customers are to be considered. The probability that a customer goes to branch A is $\frac{200}{500} = 0.4$; to branch B, $\frac{200}{500} = 0.4$; and to branch C, $\frac{100}{500} = 0.2$. Thus $X_0 = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix}$.

$$X_1 = TX_0 = \begin{bmatrix} 0.8 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.44 \\ 0.34 \\ 0.22 \end{bmatrix}$$

Thus 0.44(500) = 220 customers can be expected to go to A on their next visit,
0.34(500) = 170 to B, and 0.22(500) = 110 to C.

d. $T - I = \begin{bmatrix} 0.8 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.2 & 0.2 \\ 0.1 & -0.3 & 0.1 \\ 0.1 & 0.1 & -0.3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -0.2 & 0.2 & 0.2 & 0 \\ 0.1 & -0.3 & 0.1 & 0 \\ 0.1 & 0.1 & -0.3 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.50 \\ 0 & 1 & 0 & 0.25 \\ 0 & 0 & 1 & 0.25 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In the long run, 0.50(500) = 250 can be expected to go to A, 0.25(500) = 125 to B, and 0.25(500) to C.

43. $T^2 = TT = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

Since all entries of T^2 are positive, T is regular.

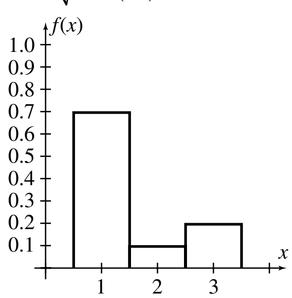
44. For the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $T^2 = I$ (the 3×3 identity matris). Thus $T^n = I$ if n is even, and $T^n = T$ if n is odd. In either case there are nonpositive entries, and thus T is not regular.

Chapter 9 Review Problems

1. $\mu = \sum_x xf(x) = 1 \cdot f(1) + 2 \cdot f(2) + 3 \cdot f(3) = 1(0.7) + 2(0.1) + 3(0.2) = 1.5$

$\text{Var}(X) = \sum_x x^2 f(x) - \mu^2 = \left[1^2(0.7) + 2^2(0.1) + 3^2(0.2)\right] - (1.5)^2 = 0.65$

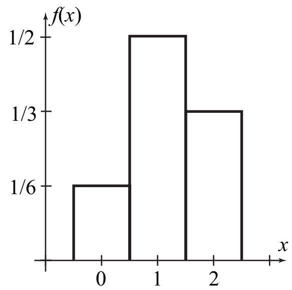
$\sigma = \sqrt{\text{Var}(X)} = \sqrt{0.65} \approx 0.81$



2. $\mu = \sum_x xf(x) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} = \frac{7}{6}$

$$\begin{aligned} \text{Var}(X) &= \sum_x x^2 f(x) - \mu^2 \\ &= \left[0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{3}\right] - \left(\frac{7}{6}\right)^2 \\ &= \frac{11}{6} - \frac{49}{36} = \frac{17}{36} \end{aligned}$$

$\sigma = \sqrt{\frac{17}{36}} = \frac{\sqrt{17}}{6} \approx 0.69$



3. a. $n(S) = 2 \cdot 6 = 12$

The event $X = 1$ is T1, so $f(1) = \frac{1}{12}$. The event $X = 2$ is H1 or T2, so $f(2) = \frac{2}{12} = \frac{1}{6}$.

Similarly, $f(3) = f(4) = f(5) = f(6) = \frac{1}{6}$ and $f(7) = \frac{1}{12}$.

b. $$\begin{aligned}E(X) &= \sum_x xf(x) \\ &= 1 \cdot \frac{1}{12} + \frac{2+3+4+5+6}{6} + 7 \cdot \frac{1}{12} \\ &= \frac{1}{12} + \frac{20}{6} + \frac{7}{12} \\ &= \frac{48}{12} \\ &= 4\end{aligned}$$

4. a. $n(S) = {}_{52}C_2 = \frac{52!}{2!50!} = \frac{52 \cdot 51}{2} = 1326$. In a deck there are 4 aces and 48 non-aces. Thus $n(E_{0\text{ aces}}) = {}_{48}C_2 = \frac{48!}{2!46!} = \frac{48 \cdot 47}{2}$ $= 1128$.

For $E_{1\text{ ace}}$ to occur, one card is an ace and the other is non-ace. Thus $n(E_{1\text{ ace}}) = 4 \cdot 48 = 192$.

$n(E_{2\text{ aces}}) = {}_4C_2 = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2} = 6$.

Therefore,

$f(0) = P(E_{0\text{ aces}}) = \frac{1128}{1326} = \frac{188}{221}$,

$f(1) = P(E_{1\text{ ace}}) = \frac{192}{1326} = \frac{32}{221}$,

$f(2) = P(E_{2\text{ aces}}) = \frac{6}{1326} = \frac{1}{221}$.

b. $$E(X) = \sum_x xf(x) = 0 \cdot \frac{188}{221} + 1 \cdot \frac{32}{221} + 2 \cdot \frac{1}{221} = \frac{34}{221} = \frac{2}{13}$$

5. Let X = gain (in dollars) on a play. If no 10 appears, then $X = 0 - \frac{1}{4} = -\frac{1}{4}$; if exactly one 10 appears, then $X = 1 - \frac{1}{4} = \frac{3}{4}$; if two 10's appear, then $X = 2 - \frac{1}{4} = \frac{7}{4}$.

$n(S) = 52 \cdot 52$. In a deck, there are 4 10's and 48 non 10's. Thus $n(E_{\text{no }10}) = 48 \cdot 48$. The event $E_{\text{one }10}$ occurs if the first card is a 10 and the second is a non-10, or vice versa. Thus $n(E_{\text{one }10}) = 4 \cdot 48 + 48 \cdot 4 = 2 \cdot 4 \cdot 48$.

$n(E_{\text{two }10\text{'s}}) = 4 \cdot 4$.

Dist. of X:

$f\left(-\frac{1}{4}\right) = \frac{48 \cdot 48}{52 \cdot 52} = \frac{144}{169}$,

$f\left(\frac{3}{4}\right) = \frac{2 \cdot 4 \cdot 48}{52 \cdot 52} = \frac{24}{169}$,

$f\left(\frac{7}{4}\right) = \frac{4 \cdot 4}{52 \cdot 52} = \frac{1}{169}$.

$$\begin{aligned}E(X) &= -\frac{1}{4} \cdot \frac{144}{169} + \frac{3}{4} \cdot \frac{24}{169} + \frac{7}{4} \cdot \frac{1}{169} \\ &= \frac{-144 + 72 + 7}{4 \cdot 169} = -\frac{65}{676} = -\frac{5}{52} \approx -0.10\end{aligned}$$

There is a loss of \$0.10 per play.

6. Let X = gain (in dollars) to company.
Dist. of X: $f(40{,}000) = 0.45$,
$f(-10{,}000) = 1 - 0.45 = 0.55$
$E(X) = (40{,}000)(0.45) + (-10{,}000)(0.55)$
$= 18{,}000 - 5500 = \$12{,}500$ per station

7. a. Let X = gain (in dollars) on each unit shipped. Then $P(X = -100) = 0.08$ and $P(X = 200) = 1 - 0.08 = 0.92$.
$E(X) = -100f(-100) + 200f(200)$
$= -100(0.08) + 200(0.92)$
$= \$176$ per unit

b. Since the expected gain per unit is \$176 and 4000 units are shipped per year, then expected annual profit is $4000(176) = \$704{,}000$.

8. There are 41 million combinations from which to choose. Let x = gain (in dollars) per play. If the player wins, then
$x = 50{,}000{,}000 - 4 = 49{,}999{,}996$ and
$P(X = 49{,}999{,}996) = \dfrac{1}{41{,}000{,}000}$. If the player loses, then $X = -4$ and
$P(X = -4) = 1 - \dfrac{1}{41{,}000{,}000} = \dfrac{40{,}999{,}999}{41{,}000{,}000}$.
$E(X) = 49{,}999{,}996 f(49{,}999{,}996) - 4f(-4)$
$= 49{,}999{,}996\left(\dfrac{1}{41{,}000{,}000}\right) - 4\left(\dfrac{40{,}999{,}999}{41{,}000{,}000}\right) \approx -2.78$
There is a loss of about $2.78 per play.

9. $f(0) = {}_4C_0(0.15)^0(0.85)^4 \approx \dfrac{4!}{0!4!}\cdot 1(0.522) = 0.522$

$f(1) = {}_4C_1(0.15)^1(0.85)^3 \approx \dfrac{4!}{1!3!}\cdot(0.15)(0.614) = 0.368$

$f(2) = {}_4C_2(0.15)^2(0.85)^2 = \dfrac{4!}{2!2!}\cdot(0.0225)(0.7225) \approx 0.098$

$f(3) = {}_4C_3(0.15)^3(0.85)^1 = \dfrac{4!}{3!1!}\cdot(0.003375)(0.85) \approx 0.011$

$f(4) = {}_4C_4(0.15)^4(0.85)^0 \approx \dfrac{4!}{4!0!}\cdot(0.000506)1 = 0.0005$

$\mu = np = 4(0.15) = 0.6$

$\sigma = \sqrt{npq} = \sqrt{4(0.15)(0.85)} \approx 0.71$

10. $f(0) = {}_5C_0\left(\dfrac{1}{3}\right)^0\left(\dfrac{2}{3}\right)^5 = 1\cdot 1\cdot\dfrac{32}{243} = \dfrac{32}{243}$

$f(1) = {}_5C_1\left(\dfrac{1}{3}\right)^1\left(\dfrac{2}{3}\right)^4 = 5\cdot\dfrac{1}{3}\cdot\dfrac{16}{81} = \dfrac{80}{243}$

$f(2) = {}_5C_2\left(\dfrac{1}{3}\right)^2\left(\dfrac{2}{3}\right)^3 = 10\cdot\dfrac{1}{9}\cdot\dfrac{8}{27} = \dfrac{80}{243}$

$f(3) = {}_5C_3\left(\dfrac{1}{3}\right)^3\left(\dfrac{2}{3}\right)^2 = 10\cdot\dfrac{1}{27}\cdot\dfrac{4}{9} = \dfrac{40}{243}$

$f(4) = {}_5C_4\left(\dfrac{1}{3}\right)^4\left(\dfrac{2}{3}\right)^1 = 5\cdot\dfrac{1}{81}\cdot\dfrac{2}{3} = \dfrac{10}{243}$

$f(5) = {}_5C_5\left(\dfrac{1}{3}\right)^5\left(\dfrac{2}{3}\right)^0 = 1\cdot\dfrac{1}{243}\cdot 1 = \dfrac{1}{243}$

$\mu = np = 5\cdot\dfrac{1}{3} = \dfrac{5}{3}$

$\sigma = \sqrt{npq} = \sqrt{5\cdot\dfrac{1}{3}\cdot\dfrac{2}{3}} = \sqrt{\dfrac{10}{9}} = \dfrac{\sqrt{10}}{3} \approx 1.05$

11. $P(X \le 1) = P(X = 0) + P(X = 1)$

$= {}_5C_0\left(\dfrac{3}{4}\right)^0\left(\dfrac{1}{4}\right)^5 + {}_5C_1\left(\dfrac{3}{4}\right)^1\left(\dfrac{1}{4}\right)^4$

$= 1\cdot 1\cdot\dfrac{1}{1024} + 5\cdot\dfrac{3}{4}\cdot\dfrac{1}{256} = \dfrac{16}{1024} = \dfrac{1}{64}$

12. $P(X = 0) = {}_6C_0\left(\dfrac{2}{3}\right)^0\left(\dfrac{1}{3}\right)^6 = \dfrac{6!}{0!\cdot 6!}(1)\left(\dfrac{1}{729}\right)$

$= 1(1)\left(\dfrac{1}{729}\right) = \dfrac{1}{729}$

$P(X = 1) = {}_6C_1\left(\dfrac{2}{3}\right)^1\left(\dfrac{1}{3}\right)^5 = \dfrac{6!}{1!\cdot 5!}\left(\dfrac{2}{3}\right)\left(\dfrac{1}{243}\right)$

$= 6\left(\dfrac{2}{3}\right)\left(\dfrac{1}{243}\right) = \dfrac{12}{729}$

$P(X = 2) = {}_6C_2\left(\dfrac{2}{3}\right)^2\left(\dfrac{1}{3}\right)^4 = \dfrac{6!}{2!\cdot 4!}\left(\dfrac{4}{9}\right)\left(\dfrac{1}{81}\right)$

$= \dfrac{6\cdot 5\cdot 4!}{2\cdot 1\cdot 4!}\left(\dfrac{4}{9}\right)\left(\dfrac{1}{81}\right) = 15\left(\dfrac{4}{9}\right)\left(\dfrac{1}{81}\right) = \dfrac{60}{729}$

$P(X > 2) = 1 - P(X \le 2)$
$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$
$= 1 - \left[\dfrac{1}{729} + \dfrac{12}{729} + \dfrac{60}{729}\right] = 1 - \dfrac{73}{729} = \dfrac{656}{729}$

13. The probability that a 7 (1 and 6, 2 and 5, 3 and 4) results on one roll is $\frac{6}{36}=\frac{1}{6}$. Let X = number of 7's that appear on 5 rolls. Then X is binomial with $p=\frac{1}{6}$ and $n = 5$.

$$P(X=3)={}_5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$
$$=10\cdot\frac{1}{216}\cdot\frac{25}{36}$$
$$=\frac{125}{3888}$$

14. Let X = number of bushes that live. Then X is binomial.

$$P(X=0)={}_4C_0(0.9)^0(0.1)^4=0.0001$$

15. Let X = number of heads that occur. Then X is binomial.

$$P(X=0)={}_5C_0\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^5=1\cdot1\cdot\frac{243}{3125}=\frac{243}{3125}$$
$$P(X=1)={}_5C_1\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^4=5\cdot\frac{2}{5}\cdot\frac{81}{625}=\frac{810}{3125}$$
$$P(X\geq 2)=1-[P(X=0)+P(X=1)]$$
$$=1-\left[\frac{243}{3125}+\frac{810}{3125}\right]=1-\frac{1053}{3125}=\frac{2072}{3125}$$

16. On any draw, the probability of selecting a red jelly bean is $\frac{2}{10}=\frac{1}{5}$. Let X = number of red jelly beans selected in five draws. Then X is binomial with $p=\frac{1}{5}$ and $n = 5$.

$$P(X=0)={}_5C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^5=1\cdot1\cdot\frac{1024}{3125}$$
$$=\frac{1024}{3125}$$
$$P(X=1)={}_5C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^4=5\cdot\frac{1}{5}\cdot\frac{256}{625}$$
$$=\frac{1280}{3125}$$
$$P(X=2)={}_5C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3=10\cdot\frac{1}{25}\cdot\frac{64}{125}$$
$$=\frac{640}{3125}$$
$$P(X\leq 2)=P(X=0)+P(X=1)+P(X=2)$$
$$=\frac{1024}{3125}+\frac{1280}{3125}+\frac{640}{3125}=\frac{2944}{3125}=0.94208$$

17. From column 1, $0.1 + a + 0.6 = 1$, so $a = 0.3$.
From column 2, $2a + b + b = 1$, so $2b = 1 - 2a$,
or $b=\frac{1-2a}{2}=\frac{1-2(0.3)}{2}=0.2$.
From column 3, $a + b + c = 1$, so $c = 1 - a - b$, or
$c = 1 - 0.3 - 0.2 = 0.5$.

18. From column 1, $a + b + 0.4 = 1$. (1)
From column 2, $a + b + c = 1$. (2)
From column 3, $a + a + b = 1 = 2a + b$. (3)
From (1), $a + b = 0.6$. Subtracting this result from (3) gives $a = 0.4$. From (1), we have $0.4 + b + 0.4 = 1$, so $b = 0.2$. From (2), we have $0.4 + 0.2 + c = 1$, so $c = 0.4$.

19. $$X_1=TX_0=\begin{bmatrix}0.1&0.3&0.1\\0.2&0.4&0.1\\0.7&0.3&0.8\end{bmatrix}\begin{bmatrix}0.5\\0\\0.5\end{bmatrix}=\begin{bmatrix}0.10\\0.15\\0.75\end{bmatrix}$$
$$X_2=TX_1=\begin{bmatrix}0.1&0.3&0.1\\0.2&0.4&0.1\\0.7&0.3&0.8\end{bmatrix}\begin{bmatrix}0.10\\0.15\\0.75\end{bmatrix}=\begin{bmatrix}0.130\\0.155\\0.715\end{bmatrix}$$
$$X_3=TX_2=\begin{bmatrix}0.1&0.3&0.1\\0.2&0.4&0.1\\0.7&0.3&0.8\end{bmatrix}\begin{bmatrix}0.130\\0.155\\0.715\end{bmatrix}$$
$$=\begin{bmatrix}0.1310\\0.1595\\0.7095\end{bmatrix}$$

20. $X_1 = TX_0 = \begin{bmatrix} 0.4 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.5 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.3 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.13 \\ 0.50 \\ 0.37 \end{bmatrix}$

$X_2 = TX_1 = \begin{bmatrix} 0.4 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.5 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 0.13 \\ 0.50 \\ 0.37 \end{bmatrix} = \begin{bmatrix} 0.139 \\ 0.511 \\ 0.350 \end{bmatrix}$

$X_3 = TX_2 = \begin{bmatrix} 0.4 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.5 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 0.139 \\ 0.511 \\ 0.350 \end{bmatrix}$

$= \begin{bmatrix} 0.1417 \\ 0.5094 \\ 0.3489 \end{bmatrix}$

21. a. $T^2 = TT = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{6}{7} & \frac{4}{7} \end{bmatrix}\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{6}{7} & \frac{4}{7} \end{bmatrix} = \begin{bmatrix} \frac{19}{49} & \frac{15}{49} \\ \frac{30}{49} & \frac{34}{49} \end{bmatrix}$

$T^3 = T^2T = \begin{bmatrix} \frac{19}{49} & \frac{15}{49} \\ \frac{30}{49} & \frac{34}{49} \end{bmatrix}\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{6}{7} & \frac{4}{7} \end{bmatrix}$

$= \begin{bmatrix} \frac{109}{343} & \frac{117}{343} \\ \frac{234}{343} & \frac{226}{343} \end{bmatrix}$

b. From T^2, entry in row 1, column 2, is $\frac{15}{49}$.

c. From T^3, entry in row 2, column 1, is $\frac{234}{343}$.

22. a. $T^2 = TT$

$= \begin{bmatrix} 0 & 0.4 & 0.3 \\ 0 & 0.3 & 0.5 \\ 1 & 0.3 & 0.2 \end{bmatrix}\begin{bmatrix} 0 & 0.4 & 0.3 \\ 0 & 0.3 & 0.5 \\ 1 & 0.3 & 0.2 \end{bmatrix}$

$= \begin{bmatrix} 0.3 & 0.21 & 0.26 \\ 0.5 & 0.24 & 0.25 \\ 0.2 & 0.55 & 0.49 \end{bmatrix}$

$T^3 = T^2T$

$= \begin{bmatrix} 0.3 & 0.21 & 0.26 \\ 0.5 & 0.24 & 0.25 \\ 0.2 & 0.55 & 0.49 \end{bmatrix}\begin{bmatrix} 0 & 0.4 & 0.3 \\ 0 & 0.3 & 0.5 \\ 1 & 0.3 & 0.2 \end{bmatrix}$

$= \begin{bmatrix} 0.26 & 0.261 & 0.247 \\ 0.25 & 0.347 & 0.32 \\ 0.49 & 0.392 & 0.433 \end{bmatrix}$

b. From T^2, entry in row 1, column 2, is 0.21.

c. From T^3, entry in row 2, column 1, is 0.25.

23. $T - I = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{3} \\ \frac{3}{4} & -\frac{1}{3} \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -\frac{3}{4} & \frac{1}{3} & 0 \\ \frac{3}{4} & -\frac{1}{3} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{4}{13} \\ 0 & 1 & \frac{9}{13} \\ 0 & 0 & 0 \end{array}\right]$

$Q = \begin{bmatrix} \frac{4}{13} \\ \frac{9}{13} \end{bmatrix}$

24. $T - I = \begin{bmatrix} 0.4 & 0.4 & 0.3 \\ 0.3 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -0.6 & 0.4 & 0.3 \\ 0.3 & -0.8 & 0.3 \\ 0.3 & 0.4 & -0.6 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -0.6 & 0.4 & 0.3 & 0 \\ 0.3 & -0.8 & 0.3 & 0 \\ 0.3 & 0.4 & -0.6 & 0 \end{array}\right] \to \ldots$

$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.36 \\ 0 & 1 & 0 & 0.27 \\ 0 & 0 & 1 & 0.36 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$Q \approx \begin{bmatrix} 0.36 \\ 0.27 \\ 0.36 \end{bmatrix}$

25. $T = \begin{array}{r} \\ \text{Japanese} \\ \text{Non-Japanese} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \text{Japanese} & \text{Non-Japanese} \end{array} \\ \begin{bmatrix} 0.8 & \quad\quad 0.6 \\ 0.2 & \quad\quad 0.4 \end{bmatrix} \end{array}$

a. $T^2 = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$

$= \begin{bmatrix} 0.76 & 0.72 \\ 0.24 & 0.28 \end{bmatrix}$

From row 1, column 1, the probability that a person who currently owns a Japanese car will buy a Japanese car two cars later is 0.76. Thus 76% of people who currently own Japanese cars will own Japanese cars two cars later.

b. $X_2 = T^2 X_0 = \begin{bmatrix} 0.76 & 0.72 \\ 0.24 & 0.28 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.744 \\ 0.256 \end{bmatrix}$

Two cars from now, we expect 74.4% Japanese, 25.6% non-Japanese.

c. $T - I = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -0.2 & 0.6 & 0 \\ 0.2 & -0.6 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cc|c} 1 & 0 & 0.75 \\ 0 & 1 & 0.25 \\ 0 & 0 & 0 \end{array}\right]$$

$Q = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$

In the long run, 75% Japanese cars, 25% non-Japanese cars.

26. a. $X_1 = TX_0 = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.49 \\ 0.27 \\ 0.24 \end{bmatrix}$

49% are expected to vote for party 1, 27% for party 2, 24% for party 3.

b. $T - I = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.3 & 0.4 & 0.1 \\ 0.2 & -0.5 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -0.3 & 0.4 & 0.1 & 0 \\ 0.2 & -0.5 & 0.1 & 0 \\ 0.1 & 0.1 & -0.2 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{7} \\ 0 & 1 & 0 & \frac{5}{21} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$Q \approx \begin{bmatrix} 0.429 \\ 0.238 \\ 0.333 \end{bmatrix}$

In the long run, 43% will vote for party 1, 24% for party 2, and 33% for party 3.

Explore and Extend—Chapter 9

1. For $X_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, the first entry of the state vector is greater than 0.5 for $n = 7$ or greater. If $X_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, then

$T^7 X_0 \approx \begin{bmatrix} 0.5217 \\ 0.0000 \\ 0.4783 \\ 0.0000 \end{bmatrix}$.

2. $T - I = \begin{bmatrix} 1 & 0.1 & 0.1 & 0.01 \\ 0 & 0 & 0.9 & 0.09 \\ 0 & 0.9 & 0 & 0.09 \\ 0 & 0 & 0 & 0.81 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 0.1 & 0.01 \\ 0 & -1 & 0.9 & 0.09 \\ 0 & 0.9 & -1 & 0.09 \\ 0 & 0 & 0 & -0.19 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0.1 & 0.1 & 0.01 & 0 \\ 0 & -1 & 0.9 & 0.09 & 0 \\ 0 & 0.9 & -1 & 0.09 & 0 \\ 0 & 0 & 0 & -0.19 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

3. Against Always Defect,

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0.1 & 1 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0.9 \end{bmatrix} \end{array}.$$

Against Always Cooperate,

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0.1 & 1 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0.9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

Against regular Tit-for-tat,

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.9 \end{bmatrix} \end{array}.$$

4. With Player 2 always defecting, after one round the game is in a stable pattern of Player 1 cooperating with probability 0.1 and defecting with probability 0.9. The steady state vector in this case is $\begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0.9 \end{bmatrix}$.

With Player 2 always cooperating, after one round the game settles into steady mutual cooperation.

With Player 2 playing standard Tit-for-tat, the probabilities gradually tilt toward mutual cooperation: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is the steady state vector. In this case, it takes only one "forgiving" Tit-for-tat-er to guarantee mutual cooperation in the long run.