### EXAMPLE 11 Radioactive Decay

A radioactive element decays such that after $t$ days the number of milligrams present is given by

$$N = 100e^{-0.062t}$$

**a.** How many milligrams are initially present?

**Solution:** This equation has the form of Equation (3), $N = N_0e^{-\lambda t}$, where $N_0 = 100$ and $\lambda = 0.062$. $N_0$ is the initial amount and corresponds to $t = 0$. Thus, 100 milligrams are initially present. (See Figure 4.13.)

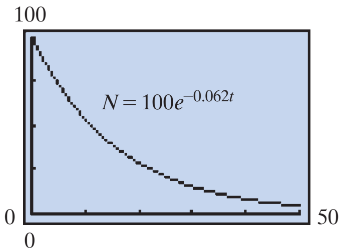


**FIGURE 4.13** Graph of radioactive decay function $N = 100e^{-0.062t}$.

**b.** How many milligrams are present after 10 days?

**Solution:** When $t = 10$,

$$N = 100e^{-0.062(10)} = 100e^{-0.62} \approx 53.8$$

Therefore, approximately 53.8 milligrams are present after 10 days.

Now Work Problem 47 ◁

## PROBLEMS 4.1

*In Problems 1–12, graph each function.*

**1.** $y = f(x) = 4^x$

**2.** $y = f(x) = 3^x$

**3.** $y = f(x) = \left(\frac{1}{5}\right)^x$

**4.** $y = f(x) = \left(\frac{1}{4}\right)^x$

**5.** $y = f(x) = 2^{(x-1)^2}$

**6.** $y = f(x) = 3(2)^x$

**7.** $y = f(x) = 3^{x+2}$

**8.** $y = f(x) = 3^{x+2}$

**9.** $y = f(x) = 3^x - 2$

**10.** $y = f(x) = 3^{x-1} - 1$

**11.** $y = f(x) = 3^{-x}$

**12.** $y = f(x) = \frac{1}{2}(2^{x/2})$

*Problems 13 and 14 refer to Figure 4.14, which shows the graphs of $y = 0.4^x$, $y = 2^x$, and $y = 5^x$.*

**13.** Of the curves $A$, $B$, and $C$, which is the graph of $y = 0.4^x$?

**14.** Of the curves $A$, $B$, and $C$, which is the graph of $y = 2^x$?

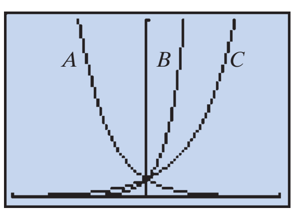


**FIGURE 4.14**

**15. Population** The projected population of a city is given by $P = 125,000(1.11)^{t/20}$, where $t$ is the number of years after 1995. What is the projected population in 2015?

**16. Population** For a certain city, the population $P$ grows at the rate of 1.5% per year. The formula $P = 1{,}527{,}000(1.015)^t$ gives the population $t$ years after 1998. Find the population in **(a)** 1999 and **(b)** 2000.

**17. Paired-Associate Learning** In a psychological experiment involving learning,[2] subjects were asked to give particular responses after being shown certain stimuli. Each stimulus was a pair of letters, and each response was either the digit 1 or 2. After each response, the subject was told the correct answer. In this so-called *paired-associate* learning experiment, the theoretical probability $P$ that a subject makes a correct response on the $n$th trial is given by

$$P = 1 - \frac{1}{2}(1-c)^{n-1}, \quad n \geq 1, \quad 0 < c < 1$$

where $c$ is a constant. Take $c = \dfrac{1}{2}$ and find $P$ when $n = 1$, $n = 2$, and $n = 3$.

**18.** Express $y = 3^{4x}$ as an exponential function in base 81.

*In Problems 19–27, find (a) the compound amount and (b) the compound interest for the given investment and annual rate.*

**19.** \$2000 for 5 years at 3% compounded annually

**20.** \$5000 for 20 years at 5% compounded annually

**21.** \$700 for 15 years at 7% compounded semiannually

**22.** \$4000 for 12 years at $7\frac{1}{2}$% compounded semiannually

**23.** \$3000 for 22 years at $8\frac{1}{4}$% compounded monthly

**24.** \$6000 for 2 years at 8% compounded quarterly

**25.** \$5000 for $2\frac{1}{2}$ years at 9% compounded monthly

**26.** \$500 for 5 years at 11% compounded semiannually

**27.** \$8000 for 3 years at $6\frac{1}{4}$% compounded daily. (Assume that there are 365 days in a year.)

**28. Investment** Suppose \$1300 is placed in a savings account that earns interest at the rate of 3.25% compounded monthly.
**(a)** What is the value of the account at the end of three years?
**(b)** If the account had earned interest at the rate of 3.5% compounded annually, what would be the value after three years?


**29. Investment** A certificate of deposit is purchased for \$6500 and is held for three years. If the certificate earns 2% compounded quarterly, what is it worth at the end of three years?

[2] D. Laming, *Mathematical Psychology* (New York: Academic Press, Inc., 1973).

**30. Population Growth** The population of a town of 5000 grows at the rate of 3% per year. **(a)** Determine an equation that gives the population $t$ years from now. **(b)** Find the population three years from now. Give your answer to (b) to the nearest integer.

**31. Bacteria Growth** Bacteria are growing in a culture, and their number is increasing at the rate of 5% an hour. Initially, 400 bacteria are present. **(a)** Determine an equation that gives the number, $N$, of bacteria present after $t$ hours. **(b)** How many bacteria are present after one hour? **(c)** After four hours? Give your answers to (b) and (c) to the nearest integer.

**32. Bacteria Reduction** A certain medicine reduces the bacteria present in a person by 10% each hour. Currently, 100,000 bacteria are present. Make a table of values for the number of bacteria present each hour for 0 to 4 hours. For each hour, write an expression for the number of bacteria as a product of 100,000 and a power of $\frac{9}{10}$. Use the expressions to make an entry in your table for the number of bacteria after $t$ hours. Write a function $N$ for the number of bacteria after $t$ hours.

**33. Recycling** Suppose the amount of plastic being recycled increases by 32% every year. Make a table of the factor by which recycling increases over the original amount for 0 to 5 years.

**34. Population Growth** Cities A and B presently have populations of 270,000 and 360,000, respectively. City A grows at the rate of 6% per year, and B grows at the rate of 4% per year. Determine the larger and by how much the populations differ at the end of five years. Give your answer to the nearest integer.

*Problems 35 and 36 involve a declining population. If a population declines at the rate of r per time period, then the population after t time periods is given by*

$$P = P_0(1 - r)^t$$

*where $P_0$ is the initial population (the population when $t = 0$).*

**35. Population** Because of an economic downturn, the population of a certain urban area declines at the rate of 1.5% per year. Initially, the population is 350,000. To the nearest person, what is the population after three years?

**36. Enrollment** After a careful demographic analysis, a university forecasts that student enrollments will drop by 3% per year for the the next 12 years. If the university currently has 14,000 students, how many students will it have 12 years from now?

*In Problems 37–40, use a calculator to find the value (rounded to four decimal places) of each expression.*

**37.** $e^{1.5}$ **38.** $e^{4.6}$ **39.** $e^{-0.8}$ **40.** $e^{-2/3}$

*In Problems 41 and 42, graph the functions.*

**41.** $y = -e^{-(x+1)}$ **42.** $y = 2e^x$

**43. Telephone Calls** The probability that a telephone operator will receive exactly $x$ calls during a certain period is given by

$$P = \frac{e^{-3}3^x}{x!}$$

Find the probability that the operator will receive exactly four calls. Round your answer to four decimal places.

**44. Normal Distribution** An important function used in economic and business decisions is the *normal distribution density function,* which, in standard form, is

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-(\frac{1}{2})x^2}$$

Evaluate $f(0)$, $f(1)$, and $f(2)$. Round your answers to three decimal places.

**45.** Express $e^{kt}$ in the form $b^t$. **46.** Express $\frac{1}{e^x}$ in the form $b^x$.

**47. Radioactive Decay** A radioactive element is such that $N$ grams remain after $t$ hours, where

$$N = 12e^{-0.031t}$$

**(a)** How many grams are initially present? To the nearest tenth of a gram, how many grams remain after **(b)** 10 hours? **(c)** 44 hours? **(d)** Based on your answer to part (c), what is your estimate of the half-life of this element?

**48. Radioactive Decay** At a certain time, there are 100 milligrams of a radioactive substance. The substance decays so that after $t$ years the number of milligrams present, $N$, is given by

$$N = 100e^{-0.045t}$$

How many milligrams, rounded to the nearest tenth of a milligram, are present after 20 years?

**49. Radioactive Decay** If a radioactive substance has a half-life of 9 years, how long does it take for 1 gram of the substance to decay to $\frac{1}{8}$ gram?

**50. Marketing** A mail-order company advertises in a national magazine. The company finds that, of all small towns, the percentage (given as a decimal) in which exactly $x$ people respond to an ad fits a Poisson distribution with $\mu = 0.5$. From what percentage of small towns can the company expect exactly two people to respond? Round your answer to four decimal places.

**51. Emergency-Room Admissions** Suppose the number of patients admitted into a hospital emergency room during a certain hour of the day has a Poisson distribution with mean 4. Find the probability that during that hour there will be exactly two emergency patients. Round your answer to four decimal places.


**52.** Graph $y = 17^x$ and $y = \left(\frac{1}{17}\right)^x$ on the same screen. Determine the intersection point.

**53.** Let $a > 0$ be a constant. Graph $y = 2^{-x}$ and $y = 2^a \cdot 2^{-x}$ on the same screen, for constant values $a = 2$ and $a = 3$. Observe that the graph of $y = 2^a \cdot 2^x$ *appears* to be the graph of $y = 2^{-x}$ shifted $a$ units to the *right*. Prove algebraically that, in this case, merely two observations predict what is true.

**54.** For $y = 5^x$, find $x$ if $y = 3$. Round your answer to two decimal places.

**55.** For $y = 2^x$, find $x$ if $y = 9$. Round your answer to two decimal places.

**56. Cell Growth** Cells are growing in a culture, and their number is increasing at the rate of 7% per hour. Initially, 1000 cells are present. After how many full hours will there be at least 3000 cells?

**57. Bacteria Growth** Refer to Example 1. How long will it take for 1000 bacteria to be present? Round your answer to the nearest tenth of a minute.

**58. Demand Equation** The demand equation for a new toy is

$$q = 100,000(0.95012)^p$$

Evaluate $q$ to the nearest integer when $p = 15$.

**59. Investment** If \$2000 is invested in a savings account that earns interest at the rate of 9.9% compounded annually, after how many full years will the amount at least double?

## Objective

To introduce logarithmic functions and their graphs. (Properties of logarithms will be discussed in Section 4.3.)

# 4.2 Logarithmic Functions

Since all exponential functions pass the horizontal line test, they are all one-to-one functions. It follows that each exponential function has an inverse. These functions, inverse to the exponential functions, are called the **logarithmic functions**.

More precisely, if $f(x) = b^x$, the exponential function base $b$ (where $0 < b < 1$ or $1 < b$), then the inverse function $f^{-1}(x)$ is called the *logarithm function base b* and is denoted $\log_b x$. It follows from our general remarks about inverse functions in Section 2.4 that

To review inverse functions, refer to Section 2.4.

$$y = \log_b x \quad \text{if and only if} \quad b^y = x$$

and we have the following fundamental equations:

$$\log_b b^x = x \tag{1}$$

and

$$b^{\log_b x} = x \tag{2}$$

where Equation (1) holds for all $x$ in $(-\infty, \infty)$ and Equation (2) holds for all $x$ in $(0, \infty)$. We recall that $(-\infty, \infty)$ is the domain of the exponential function base $b$ and $(0, \infty)$ is the range of the exponential function base $b$. It follows that $(0, \infty)$ is the domain of the logarithm function base $b$ and $(-\infty, \infty)$ is the range of the logarithm function base $b$.

Stated otherwise, given positive $x$, $\log_b x$ is the unique number with the property that $b^{\log_b x} = x$. The generalities about inverse functions also enable us to see immediately what the graph of a logarithmic function looks like.

In Figure 4.15 we have shown the graph of the particular exponential function $y = f(x) = 2^x$, whose general shape is typical of exponential functions $y = b^x$ for which the base $b$ satisfies $1 < b$. We have added a (dashed) copy of the line $y = x$. The graph of $y = f^{-1}(x) = \log_2 x$ is obtained as the mirror image of $y = f(x) = 2^x$ in the line $y = x$.

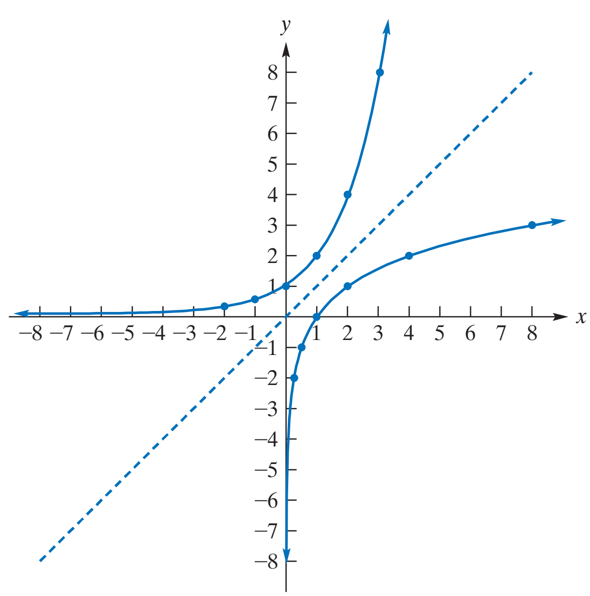


**FIGURE 4.15** Graphs of $y = 2^x$ and $y = \log_2 x$.

**Solution:** Here the decay constant $\lambda$ is 0.00501. By Equation (4), the half-life is given by

$$T = \frac{\ln 2}{\lambda} = \frac{\ln 2}{0.00501} \approx 138.4 \text{ days}$$

Now Work Problem 63 ◁

## PROBLEMS 4.2

*In Problems 1–8, express each logarithmic form exponentially and each exponential form logarithmically.*

**1.** $10^4 = 10{,}000$
**2.** $2 = \log_{12} 144$
**3.** $\log_2 1024 = 10$
**4.** $32^{3/5} = 3$
**5.** $e^3 \approx 20.0855$
**6.** $e^{0.33647} \approx 1.4$
**7.** $\ln 3 \approx 1.09861$
**8.** $\log 7 \approx 0.84509$

*In Problems 9–16, graph the functions.*

**9.** $y = f(x) = \log_5 x$
**10.** $y = f(x) = \log_4 2x$
**11.** $y = f(x) = \log_{1/4} x$
**12.** $y = f(x) = \log_{1/5} x$
**13.** $y = f(x) = \log_2(x + 4)$
**14.** $y = f(x) = \ln(-x)$
**15.** $y = f(x) = -2 \ln x$
**16.** $y = f(x) = \ln(x + 2)$

*In Problems 17–28, evaluate the expression.*

**17.** $\log_6 36$
**18.** $\log_2 512$
**19.** $\log_5 625$
**20.** $\log_{16} 4$
**21.** $\log_7 7$
**22.** $\log 10{,}000$
**23.** $\log 0.0001$
**24.** $\log_2 \sqrt[5]{8}$
**25.** $\log_5 1$
**26.** $\log_5 \frac{1}{25}$
**27.** $\log_2 \frac{1}{8}$
**28.** $\log_3 \sqrt[7]{3}$

*In Problems 29–48, find x.*

**29.** $\log_7 x = 3$
**30.** $\log_2 x = 8$
**31.** $\log_5 x = 3$
**32.** $\log_4 x = 0$
**33.** $\log x = -3$
**34.** $\log_\pi x = 1$
**35.** $\ln x = -3$
**36.** $\log_x 25 = 2$
**37.** $\log_x 8 = 3$
**38.** $\log_x 4 = \frac{1}{3}$
**39.** $\log_x \frac{1}{4} = -1$
**40.** $\log_x y = 1$
**41.** $\log_3 x = -3$
**42.** $\log_x(2x - 3) = 1$
**43.** $\log_x(12 - x) = 2$
**44.** $\log_6 36 = x - 1$
**45.** $2 + \log_2 4 = 3x - 1$
**46.** $\log_3(x + 2) = -2$
**47.** $\log_x(2x + 8) = 2$
**48.** $\log_x(16 - 4x - x^2) = 2$

*In Problems 49–52, find x and express your answer in terms of natural logarithms.*

**49.** $e^{5x} = 7$
**50.** $0.1e^{0.1x} = 0.5$
**51.** $e^{2x-5} + 1 = 4$
**52.** $6e^{2x} - 1 = \frac{1}{2}$

*In Problems 53–56, use your calculator to find the approximate value of each expression, correct to five decimal places.*

**53.** $\ln 11$
**54.** $\ln 3.19$
**55.** $\ln 7.39$
**56.** $\ln 9.98$

**57. Appreciation** Suppose an antique gains 10% in value every year. Graph the number of years it is owned as a function of the multiplicative increase in its original value. Label the graph with the name of the function.

**58. Cost Equation** The cost for a firm producing $q$ units of a product is given by the cost equation

$$c = (5q \ln q) + 15$$

Evaluate the cost when $q = 12$. (Round your answer to two decimal places.)

**59. Supply Equation** A manufacturer's supply equation is

$$p = \log\left(15 + \frac{5q}{8}\right)$$

where $q$ is the number of units supplied at a price $p$ per unit. At what price will the manufacturer supply 1576 units?

**60. Earthquake** The magnitude, $M$, of an earthquake and its energy, $E$, are related by the equation[3]

$$1.5M = \log\left(\frac{E}{2.5 \times 10^{11}}\right)$$

where $M$ is given in terms of Richter's preferred scale of 1958 and $E$ is in ergs. Solve the equation for $E$.

**61. Biology** For a certain population of cells, the number of cells at time $t$ is given by $N = N_0(2^{t/k})$, where $N_0$ is the number of cells at $t = 0$ and $k$ is a positive constant. **(a)** Find $N$ when $t = k$. **(b)** What is the significance of $k$? **(c)** Show that the time it takes to have population $N_1$ can be written

$$t = k \log_2 \frac{N_1}{N_0}$$

**62. Inferior Good** In a discussion of an inferior good, Persky[4] solves an equation of the form

$$u_0 = A \ln(x_1) + \frac{x_2^2}{2}$$

for $x_1$, where $x_1$ and $x_2$ are quantities of two products, $u_0$ is a measure of utility, and $A$ is a positive constant. Determine $x_1$.

**63. Radioactive Decay** A 1-gram sample of radioactive lead 211 ($^{211}$Pb) decays according to the equation $N = e^{-0.01920t}$, where $N$ is the number of grams present after $t$ minutes. How long will it take until only 0.25 grams remain? Express the answer to the nearest tenth of a minute.

**64. Radioactive Decay** The half-life of radioactive actinium 227 ($^{227}$Ac) is approximately 21.70514 years. If a lab currently has a 100-milligram sample, how many milligrams will it have one year from now?

[3] K. E. Bullen, *An Introduction to the Theory of Seismology* (Cambridge, U.K.: Cambridge at the University Press, 1963).

[4] A. L. Persky, "An Inferior Good and a Novel Indifference Map," *The American Economist*, XXIX, no. 1 (Spring 1985).

**65.** If $\log_y x = 3$ and $\log_z x = 2$, find a formula for $z$ as an explicit function of $y$ only.

**66.** Solve for $y$ as an explicit function of $x$ if

$$x + 3e^{2y} - 8 = 0$$

**67.** Suppose $y = f(x) = x \ln x$. **(a)** For what values of $x$ is $y < 0$? (*Hint:* Determine when the graph is below the $x$-axis.) **(b)** Determine the range of $f$.

**68.** Find the $x$-intercept of $y = x^3 \ln x$.

**69.** Use the graph of $y = e^x$ to estimate $\ln 2$ to two decimal places

**70.** Use the graph of $y = \ln x$ to estimate $e^2$ to two decimal places.

**71.** Determine the $x$-values of points of intersection of the graphs of $y = (x-2)^2$ and $y = \ln x$. Round your answers to two decimal places.

## Objective

To study basic properties of logarithmic functions.

## 4.3 Properties of Logarithms

The logarithmic function has many important properties. For example,

**1.** $\log_b(mn) = \log_b m + \log_b n$

which says that the logarithm of a product of two numbers is the sum of the logarithms of the numbers. We can prove this property by deriving the exponential form of the equation:

$$b^{\log_b m + \log_b n} = mn$$

Using first a familiar rule for exponents, we have

$$\begin{aligned} b^{\log_b m + \log_b n} &= b^{\log_b m} b^{\log_b n} \\ &= mn \end{aligned}$$

where the second equality uses two instances of the fundamental equation (2) of Section 4.2. We will not prove the next two properties, since their proofs are similar to that of Property 1.

**2.** $\log_b \dfrac{m}{n} = \log_b m - \log_b n$

That is, the logarithm of a quotient is the difference of the logarithm of the numerator and the logarithm of the denominator.

**3.** $\log_b m^r = r \log_b m$

That is, the logarithm of a power of a number is the exponent times the logarithm of the number.

**Table 4.4 Common Logarithms**

| $x$ | $\log x$ | $x$ | $\log x$ |
|---|---|---|---|
| 2 | 0.3010 | 7 | 0.8451 |
| 3 | 0.4771 | 8 | 0.9031 |
| 4 | 0.6021 | 9 | 0.9542 |
| 5 | 0.6990 | 10 | 1.0000 |
| 6 | 0.7782 | $e$ | 0.4343 |

Table 4.4 gives the values of a few common logarithms. Most entries are approximate. For example, $\log 4 \approx 0.6021$, which means $10^{0.6021} \approx 4$. To illustrate the use of properties of logarithms, we will use this table in some of the examples that follow.

### EXAMPLE 1 Finding Logarithms by Using Table 4.4

**a.** Find log 56.

**Solution:** Log 56 is not in the table. But we can write 56 as the product $8 \cdot 7$. Thus, by Property 1,

$$\log 56 = \log(8 \cdot 7) = \log 8 + \log 7 \approx 0.9031 + 0.8451 = 1.7482$$

## PROBLEMS 4.3

*In Problems 1–10, let* $\log 2 = a$, $\log 3 = b$, *and* $\log 5 = c$. *Express the indicated logarithm in terms of a, b, and c.*

**1.** $\log 30$ **2.** $\log 1024$ **3.** $\log \frac{2}{3}$ **4.** $\log \frac{5}{2}$

**5.** $\log \frac{27}{5}$ **6.** $\log \frac{6}{25}$ **7.** $\log 100$

**8.** $\log 0.00003$ **9.** $\log_2 3$ **10.** $\log_2 3$

*In Problems 11–20, determine the value of the expression without the use of a calculator.*

**11.** $\log_7 7^{48}$ **12.** $\log_{11}(11\sqrt[3]{11})^7$ **13.** $\log 0.0000001$

**14.** $10^{\log 3.4}$ **15.** $\ln e^{2.77}$ **16.** $\ln e$ **17.** $\ln \frac{1}{\sqrt{e}}$

**18.** $\log_3 81$ **19.** $\log \frac{1}{10} + \ln e^3$ **20.** $e^{\ln e}$

*In Problems 21–32, write the expression in terms of* $\ln x$, $\ln(x+1)$, *and* $\ln(x+2)$.

**21.** $\ln(x(x+1)^2)$ **22.** $\ln \frac{\sqrt[5]{x}}{(x+1)^3}$

**23.** $\ln \frac{x^2}{(x+1)^3}$ **24.** $\ln(x(x+1))^3$

**25.** $\ln\left(\frac{x+1}{x^2(x+2)}\right)^{-3}$ **26.** $\ln \sqrt{x(x+1)(x+2)}$

**27.** $\ln \frac{x(x+1)}{x+2}$ **28.** $\ln \frac{x^2(x+1)}{x+2}$

**29.** $\ln \frac{\sqrt{x}}{(x+1)^2(x+2)^3}$ **30.** $\ln \frac{x^5}{(x+1)^2(x+2)^3}$

**31.** $\ln\left(\frac{1}{x+2}\sqrt[5]{\frac{x^2}{x+1}}\right)$ **32.** $\ln \sqrt[4]{\frac{x^2(x+2)^3}{(x+1)^5}}$

*In Problems 33–40, express each of the given forms as a single logarithm.*

**33.** $\log 6 + \log 4$ **34.** $\log_3 10 - \log_3 5$

**35.** $3\log_2(2x) - 5\log_2(x+2)$ **36.** $2\log x - \frac{1}{2}\log(x-2)$

**37.** $7\log_3 5 + 4\log_3 17$ **38.** $5(2\log x + 3\log y - 2\log z)$

**39.** $2 + 10\log 1.05$

**40.** $\frac{1}{3}(2\log 13 + 7\log 5 - 3\log 2)$

*In Problems 41–44, determine the values of the expressions without using a calculator.*

**41.** $e^{4\ln 3 - 3\ln 4}$

**42.** $\log_3(\ln(\sqrt{7+e^3} + \sqrt{7}) + \ln(\sqrt{7+e^3} - \sqrt{7}))$

**43.** $\log_6 54 - \log_6 9$

**44.** $\log_3 \sqrt{3} - \log_2 \sqrt[3]{2} - \log_5 \sqrt[4]{5}$

*In Problems 45–48, find x.*

**45.** $e^{\ln(x^2-5x)} = -15$ **46.** $4^{\log_4 x + \log_4 2} = 3$

**47.** $10^{\log(x^2+2x)} = 3$ **48.** $e^{3\ln x} = 8$

*In Problems 49–53, write each expression in terms of natural logarithms.*

**49.** $\log_2(2x+1)$ **50.** $\log_2(3x^2+3x+3)$

**51.** $\log_3(x^2+1)$ **52.** $\log_7(x^2+1)$

**53.** If $e^{\ln z} = 7e^y$, solve for $y$ in terms of $z$.

**54. Statistics** In statistics, the sample regression equation $y = ab^x$ is reduced to a linear form by taking logarithms of both sides. Express $\log y$ in terms of $x$, $\log a$, and $\log b$ and explain what is meant by saying that the resulting expression is linear.

**55. Logarithm of a Sum?** In a study of military enlistments, Brown[5] considers total military compensation $C$ as the sum of basic military compensation $B$ (which includes the value of allowances, tax advantages, and base pay) and educational benefits $E$. Thus, $C = B + E$. Brown states that

$$\ln C = \ln(B+E) = \ln B + \ln\left(1 + \frac{E}{B}\right)$$

Verify this but explain why it is not really a "formula" for the logarithm of a sum.

**56. Earthquake** According to Richter,[6] the magnitude $M$ of an earthquake occurring 100 km from a certain type of seismometer is given by $M = \log(A) + 3$, where $A$ is the recorded trace amplitude (in millimeters) of the quake. **(a)** Find the magnitude of an earthquake that records a trace amplitude of 10 mm. **(b)** If a particular earthquake has amplitude $A_1$ and magnitude $M_1$, determine the magnitude of a quake with amplitude $10A_1$ in terms of $M_1$.

**57.** Display the graph of $y = \log_4 x$.

**58.** Display the graph of $y = \log_4(x+2)$.

**59.** Display the graphs of $y = \log x$ and $y = \frac{\ln x}{\ln 10}$ on the same screen. The graphs appear to be identical. Why?

**60.** On the same screen, display the graphs of $y = \ln x$ and $y = \ln(2x)$. It appears that the graph of $y = \ln(2x)$ is the graph of $y = \ln x$ shifted upward. Determine algebraically the value of this shift.

**61.** On the same screen, display the graphs of $y = \ln(2x)$ and $y = \ln(6x)$. It appears that the graph of $y = \ln(6x)$ is the graph of $y = \ln(2x)$ shifted upward. Determine algebraically the value of this shift.

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[5] C. Brown, "Military Enlistments: What Can We Learn from Geographic Variation?" *The American Economic Review,* 75, no. 1 (1985), 228–34.

[6] C. F. Richter, *Elementary Seismology* (San Francisco: W. H. Freeman and Company, 1958).

while replacing $x$ by 4 in $5 - \log_2(x+4)$ gives $5 - \log_2(4+4) = 5 - \log_2(8) = 5 - \log_2 2^3 = 5 - 3 = 2$. Since the results are the same, 4 is a solution of the equation.

In solving a logarithmic equation, it is a good idea to check for extraneous solutions.

**Now Work Problem 5** ◁

## PROBLEMS 4.4

*In Problems 1–36, find x rounded to three decimal places.*

**1.** $\log(7x+2) = \log(5x+3)$ **2.** $\log x - \log 5 = \log 7$

**3.** $\log 7 - \log(x-1) = \log 4$ **4.** $\log_2 x + 3\log_2 2 = \log_2 \frac{2}{x}$

**5.** $\ln(-x) = \ln(x^2 - 6)$ **6.** $\ln(x+7.5) + \ln 2 = 2\ln x$

**7.** $e^{2x} \cdot e^{5x} = e^{14}$ **8.** $(e^{3x-2})^3 = e^3$ **9.** $(81)^{4x} = 9$

**10.** $(27)^{2x+1} = \dfrac{1}{3}$ **11.** $e^{3x} = 11$ **12.** $e^{4x} = \frac{3}{4}$

**13.** $2e^{5x+2} = 17$ **14.** $5e^{2x-1} - 2 = 23$ **15.** $10^{4/x} = 6$

**16.** $\frac{7(10)^{0.2x}}{5} = 3$ **17.** $\dfrac{5}{10^{2x}} = 7$

**18.** $2(10)^x + (10)^{x+1} = 4$ **19.** $2^x = 5$

**20.** $7^{2x+3} = 9$ **21.** $3^{5x+7} = 11$

**22.** $4^{x/2} = 20$ **23.** $2^{-2x/3} = \dfrac{4}{5}$

**24.** $5(3^x - 6) = 10$ **25.** $(4)5^{3-x} - 7 = 2$

**26.** $\dfrac{127}{3^x} = 67$ **27.** $\log(x-3) = 3$

**28.** $\log_2(x+1) = 4$ **29.** $\log_4(9x-4) = 2$

**30.** $\log_4(2x+4) - 3 = \log_4 3$ **31.** $\ln(x+3) + \ln(x+5) = 1$

**32.** $\log(x-3) + \log(x-5) = 1$

**33.** $\log_2(5x+1) = 4 - \log_2(3x-2)$

**34.** $\log(x+2)^2 = 2$, where $x > 0$

**35.** $\log_2\left(\dfrac{2}{x}\right) = 3 + \log_2 x$ **36.** $\log(x+1) = \log(x+2) + 1$

**37. Rooted Plants** In a study of rooted plants in a certain geographic region,[9] it was determined that on plots of size $A$ (in square meters), the average number of species that occurred was $S$. When $\log S$ was graphed as a function of $\log A$, the result was a straight line given by

$$\log S = \log 12.4 + 0.26 \log A$$

Solve for $S$.

**38. Gross National Product** In an article, Taagepera and Hayes[10] refer to an equation of the form

$$\log T = 1.7 + 0.2068 \log P - 0.1334(\log P)^2$$

Here $T$ is the percentage of a country's gross national product (GNP) that corresponds to foreign trade (exports plus imports), and $P$ is the country's population (in units of 100,000). Verify the claim that

$$T = 50P^{(0.2068 - 0.1334 \log P)}$$

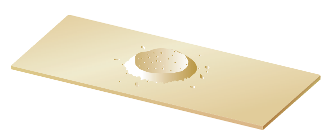
You may assume that $\log 50 = 1.7$. Also verify that, for any base $b$, $(\log_b x)^2 = \log_b(x^{\log_b x})$.

**39. Radioactivity** The number of milligrams of a radioactive substance present after $t$ years is given by

$$Q = 100e^{-0.035t}$$

**(a)** How many milligrams are present after 0 years?
**(b)** After how many years will there be 20 milligrams present? Give your answer to the nearest year.

**40. Blood Sample** On the surface of a glass slide is a grid that divides the surface into 225 equal squares. Suppose a blood sample containing $N$ red cells is spread on the slide and the cells are randomly distributed. Then the number of squares containing no cells is (approximately) given by $225e^{-N/225}$. If 100 of the squares contain no cells, estimate the number of cells the blood sample contained.



**41. Population** In Springfield the population $P$ grows at the rate of 2% per year. The equation $P = 1{,}500{,}000(1.02)^t$ gives the population $t$ years after 2015. Find the value of $t$ for which the population will be 1,900,000. Give your answer to the nearest tenth of a year.

**42. Market Penetration** In a discussion of market penetration by new products, Hurter and Rubenstein[11] refer to the function

$$F(t) = \frac{q - pe^{-(t+C)(p+q)}}{q[1 + e^{(t+C)(p+q)}]}$$

[9] R. W. Poole, *An Introduction to Quantitative Ecology* (New York: McGraw-Hill, 1974).

[10] R. Taagepera and J. P. Hayes, "How Trade/GNP Ratio Decreases with Country Size," *Social Science Research,* 6 (1977), 108–32.

[11] A. P. Hurter, Jr., A. H. Rubenstein et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change,* 11 (1978), 197–221.

where $p$, $q$, and $C$ are constants. They claim that if $F(0) = 0$, then

$$C = -\frac{1}{p+q}\ln\frac{q}{p}$$

Show that their claim is true.

**43. Demand Equation** The demand equation for a consumer product is $q = 80 - 2^p$. Solve for $p$ and express your answer in terms of common logarithms, as in Example 4. Evaluate $p$ to two decimal places when $q = 60$.

**44. Investment** The equation $A = P(1.105)^t$ gives the value $A$ at the end of $t$ years of an investment of $P$ dollars compounded annually at an annual interest rate of 10.5%. How many years will it take for an investment to double? Give your answer to the nearest year.

**45. Sales** After $t$ years the number of units of a product sold per year is given by $q = 1000\left(\frac{1}{2}\right)^{0.8^t}$. Such an equation is called a *Gompertz equation* and describes natural growth in many areas of study. Solve this equation for $t$ in the same manner as in Example 4, and show that

$$t = \frac{\log\left(\dfrac{3-\log q}{\log 2}\right)}{\log 0.8}$$

Also, for any $A$ and suitable $b$ and $a$, solve $y = Ab^{a^x}$ for $x$ and explain how the previous solution is a special case.

**46. Learning Equation** Suppose that the daily output of units of a new product on the $t$th day of a production run is given by

$$q = q(t) = 100(1 - e^{-0.1t})$$

Such an equation is called a *learning equation* and indicates that as time increases, output per day will increase. This may be due to a gain in a worker's proficiency at his or her job. Determine, to the nearest complete unit, the output **(a)** initially (that is when $t = 0$), **(b)** on the first day, and **(c)** on the second day. **(d)** After how many days, correct to the nearest whole day, will a daily production run of 90 units be reached? **(e)** Will production increase indefinitely?

**47.** Verify that 4 is the only solution to the logarithmic equation in Example 6 by graphing the function

$$y = 5 - \log_2(x+4) - \log_2 x$$

and observing when $y = 0$.

**48.** Solve $2^{3x+0.5} = 17$. Round your answer to two decimal places.

**49.** Solve $\ln(x+2) = 5 - x$. Round your answer to two decimal places.

**50.** Graph the equation $(3)2^y - 4x = 5$. (*Hint:* Solve for $y$ as a function of $x$.)

# Chapter 4 Review

## Important Terms and Symbols

| | | Examples |
|---|---|---|
| **Section 4.1** | **Exponential Functions** | |
| | exponential function, $b^x$, for $b > 1$ and for $0 < b < 1$ | Ex. 2,3, p. 177,178 |
| | compound interest principal compound amount | Ex. 6, p. 180 |
| | interest period periodic rate nominal rate | |
| | $e$ natural exponential function, $e^x$ | Ex. 8, p. 183 |
| | exponential law of decay initial amount decay constant half-life | Ex. 11, p. 186 |
| **Section 4.2** | **Logarithmic Functions** | |
| | logarithmic function, $\log_b x$ common logarithm, $\log x$ | Ex. 5, p. 191 |
| | natural logarithm, $\ln x$ | Ex. 5, p. 191 |
| **Section 4.3** | **Properties of Logarithms** | |
| | change-of-base formula | Ex. 8, p. 198 |
| **Section 4.4** | **Logarithmic and Exponential Equations** | |
| | logarithmic equation exponential equation | Ex. 1, p. 200 |

## Summary

An exponential function has the form $f(x) = b^x$. The graph of $y = b^x$ has one of two general shapes, depending on the value of the base $b$. (See Figure 4.3.) The compound interest formula

$$\boxed{S = P(1+r)^n}$$

expresses the compounded future amount $S$ of a principal $P$ at periodic rate $r$, as an exponential function of the number of interest periods $n$.

The irrational number $e \approx 2.71828$ provides the most important base for an exponential function. This base occurs in economic analysis and many situations involving growth of populations or decay of radioactive elements. Radioactive elements follow the exponential law of decay,

$$\boxed{N = N_0 e^{-\lambda t}}$$

where $N$ is the amount of an element present at time $t$, $N_0$ is the initial amount, and $\lambda$ is the decay constant. The time required for half of the amount of the element to decay is called the half-life and denoted by $T$.

The logarithmic function is the inverse function of the exponential function, and vice versa. The logarithmic function with base $b$ is denoted $\log_b$, and $y = \log_b x$ if and only if $b^y = x$. The graph of $y = \log_b x$ has one of two general shapes, depending on the value of the base $b$. (See Figure 4.18.) Logarithms with base $e$ are called natural logarithms and are denoted ln; those with base 10 are called common logarithms and are denoted log. The half-life $T$ of a radioactive element can be given in terms of a natural logarithm and the decay constant: $T = (\ln 2)/\lambda$.

Some important properties of logarithms are the following:

$$\log_b(mn) = \log_b m + \log_b n$$
$$\log_b \frac{m}{n} = \log_b m - \log_b n$$
$$\log_b m^r = r\log_b m$$
$$\log_b \frac{1}{m} = -\log_b m$$
$$\log_b 1 = 0$$
$$\log_b b = 1$$
$$\log_b b^r = r$$
$$b^{\log_b m} = m$$
$$\log_b m = \frac{\log_a m}{\log_a b}$$

Moreover, if $\log_b m = \log_b n$, then $m = n$. Similarly, if $b^m = b^n$, then $m = n$. Many of these properties are used in solving logarithmic and exponential equations.

## Review Problems

*In Problems 1–6, write each exponential form logarithmically and each logarithmic form exponentially.*

**1.** $3^5 = 243$ **2.** $\log_7 343 = 3$ **3.** $\log_{81} 3 = \frac{1}{4}$

**4.** $10^5 = 100{,}000$ **5.** $e^7 \approx 1096.63$ **6.** $\log_9 9 = 1$

*In Problems 7–12, find the value of the expression without using a calculator.*

**7.** $\log_5 3125$ **8.** $\log_4 16$ **9.** $\log_3 \frac{1}{81}$

**10.** $\log_{1/5} \frac{1}{625}$ **11.** $\log_{1/3} 9$ **12.** $\log_{81} 3$

*In Problems 13–18, find x without using a calculator.*

**13.** $\log_5 625 = x$ **14.** $\log_x \frac{1}{81} = -4$ **15.** $\log_2 x = -10$

**16.** $\ln \frac{1}{e} = x$ **17.** $\ln(5x + 7) = 0$ **18.** $e^{\ln(x+4)} = 7$

*In Problems 19 and 20, let* $\log 2 = a$ *and* $\log 3 = b$*. Express the given logarithm in terms of a and b.*

**19.** $\log 8000$ **20.** $\log \dfrac{1024}{\sqrt[5]{3}}$

*In Problems 21–26, write each expression as a single logarithm.*

**21.** $3\log 7 - 2\log 5$ **22.** $3\log_2 x + 5\log_2 y + 7\log z$

**23.** $2\ln x + \ln y - 3\ln z$ **24.** $\log_6 2 - \log_6 4 - 9\log_6 3$

**25.** $\frac{1}{3}\ln x + 3\ln(x^2) - 2\ln(x - 1) - 3\ln(x - 2)$

**26.** $4\log x + 2\log y - 3(\log z + \log w)$

*In Problems 27–32, write the expression in terms of* $\ln x$, $\ln y$, *and* $\ln z$.

**27.** $\ln \dfrac{x^7y^5}{z^{-3}}$ **28.** $\ln \dfrac{\sqrt{x}}{(yz)^2}$ **29.** $\ln \sqrt[3]{xyz}$

**30.** $\ln\left(\dfrac{x^4y^3}{z^2}\right)^5$ **31.** $\ln\left[\dfrac{1}{x}\sqrt{\dfrac{y}{z}}\right]$ **32.** $\ln\left(\left(\dfrac{x}{y}\right)^3\left(\dfrac{y}{z}\right)^5\right)$

**33.** Write $\log_3(x + 5)$ in terms of natural logarithms.

**34.** Write $\log_2(7x^3 + 5)$ in terms of common logarithms.

**35.** We have $\log_2 37 \approx 5.20945$ and $\log_2 7 \approx 2.80735$. Find $\log_7 37$.

**36.** Use natural logarithms to determine the value of $\log_4 5$.

**37.** If $\ln 2 = x$ and $\ln 3 = y$, express $\ln\left(\dfrac{\sqrt[5]{2}}{81}\right)$ in terms of $x$ and $y$.

**38.** Express $\log \dfrac{x^2\sqrt[3]{x+1}}{\sqrt[5]{x^2+2}}$ in terms of $\log x$, $\log(x + 1)$, and $\log(x^2 + 2)$.

**39.** Simplify $10^{\log x} + \log 10^x + \log 10$.

**40.** Simplify $\log \dfrac{1}{1000} + \log 1000$.

**41.** If $\ln y = x^2 + 2$, find $y$.

**42.** Sketch the graphs of $y = (1/3)^x$ and $y = \log_{1/3} x$.

**43.** Sketch the graph of $y = 2^{x+3}$.

**44.** Sketch the graph of $y = -2\log_2 x$.

*In Problems 45–52, find x.*

**45.** $\log(6x - 2) = \log(8x - 10)$ **46.** $\log 3x + \log 3 = 2$

**47.** $2^{3x} = 16^{x+1}$ **48.** $4^{3-x} = \frac{1}{16}$

**49.** $\log x + \log(10x) = 3$ **50.** $\ln\left(\frac{x-5}{x-1}\right) = \ln 6$

**51.** $\ln(\log_x 3) = 2$ **52.** $\log_3 x + \log_9 x = 5$

*In Problems 53–58, find x correct to three decimal places.*

**53.** $e^{3x} = 14$ **54.** $10^{3x/2} = 5$ **55.** $5(e^{x+2} - 6) = 10$

**56.** $7e^{3x-1} - 2 = 1$ **57.** $5^{x+1} = 11$ **58.** $3^{5/x} = 2$

**59. Investment** If \$2600 is invested for $6\frac{1}{2}$ years at 6% compounded quarterly, find (**a**) the compound amount and (**b**) the compound interest.

**60. Investment** Find the compound amount of an investment of \$2000 for five years and four months at the rate of 12% compounded monthly.

**61.** Find the nominal rate that corresponds to a periodic rate of $1\frac{1}{6}\%$ per month.

**62. Bacteria Growth** Bacteria are growing in a culture, and their number is increasing at the rate of 7% per hour. Initially, 500 bacteria are present. **(a)** Determine an equation that gives the number, $N$, of bacteria present after $t$ hours. **(b)** How many bacteria are present after one hour? **(c)** After five hours, correct to the nearest integer.

**63. Population Growth** The population of a small town *grows* at the rate of $-0.5\%$ per year because the outflow of people to nearby cities in search of jobs exceeds the birth rate. In 2006 the population was 6000. **(a)** Determine an equation that gives the population, $P$, $t$ years from 2006. **(b)** Find what the population will be in 2016 (be careful to express your answer as an integer).

**64. Revenue** Due to ineffective advertising, the Kleer-Kut Razor Company finds that its annual revenues have been cut sharply. Moreover, the annual revenue, $R$, at the end of $t$ years of business satisfies the equation $R = 200{,}000e^{-0.2t}$. Find the annual revenue at the end of two years and at the end of three years.

**65. Radioactivity** A radioactive substance decays according to the formula

$$N = 10e^{-0.41t}$$

where $N$ is the number of milligrams present after $t$ hours. **(a)** Determine the initial amount of the substance present. **(b)** To the nearest tenth of a milligram, determine the amount present after 1 hour and **(c)** after 5 hours. **(d)** To the nearest tenth of an hour, determine the half-life of the substance, and **(e)** determine the number of hours for 0.1 milligram to remain.

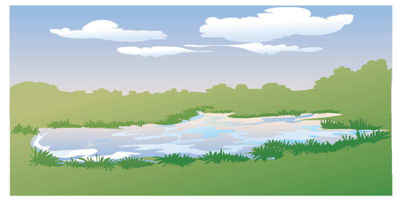
**66. Radioactivity** If a radioactive substance has a half-life of 10 days, in how many days will $\frac{1}{8}$ of the initial amount be present?

**67. Marketing** A marketing-research company needs to determine how people adapt to the taste of a new toothpaste. In one experiment, a person was given a pasted toothbrush and was asked periodically to assign a number, on a scale from 0 to 10, to the perceived taste. This number was called the *response magnitude*. The number 10 was assigned to the initial taste. After conducting the experiment several times, the company estimated that the response magnitude is given by

$$R = 10e^{-t/50}$$

where $t$ is the number of seconds after the person is given the toothpaste. **(a)** Find the response magnitude after 20 seconds, correct to the nearest integer. **(b)** After how many seconds, correct to the nearest second, does a person have a response magnitude of 3?

**68. Sediment in Water** The water in a Midwestern lake contains sediment, and the presence of the sediment reduces the transmission of light through the water. Experiments indicate that the intensity of light is reduced by 10% by passage through 20 cm of water. Suppose that the lake is uniform with respect to the amount of sediment contained by the water. A measuring instrument can detect light at the intensity of 0.17% of full sunlight. This measuring instrument is lowered into the lake. At what depth will it first cease to record the presence of light? Give your answer to the nearest 10 cm.



**69. Body Cooling** In a discussion of the rate of cooling of isolated portions of the body when they are exposed to low temperatures, there occurs the equation[12]

$$T_t - T_e = (T_t - T_e)_o e^{-at}$$

where $T_t$ is the temperature of the portion at time $t$, $T_e$ is the environmental temperature, the subscript $o$ refers to the initial temperature difference, and $a$ is a constant. Show that

$$a = \frac{1}{t} \ln \frac{(T_t - T_e)_o}{T_t - T_e}$$

**70. Depreciation** An alternative to straight-line depreciation is *declining-balance* depreciation. This method assumes that an item loses value more steeply at the beginning of its life than later on. A fixed percentage of the value is subtracted each month. Suppose an item's initial cost is $C$ and its useful life is $N$ months. Then the value, $V$ (in dollars), of the item at the end of $n$ months is given by

$$V = C\left(1 - \frac{1}{N}\right)^n$$

so that each month brings a depreciation of $\frac{100}{N}$ percent. (This is called *single declining-balance depreciation;* if the annual depreciation were $\frac{200}{N}$ percent, then we would speak of *double-declining-balance depreciation*.) A notebook computer is purchased for \$1500 and has a useful life of 36 months. It undergoes double-declining-balance depreciation. After how many months, to the nearest integer, does its value drop below \$500?

**71.** If $y = f(x) = \dfrac{\ln x}{x}$, determine the range of $f$. Round values to two decimal places.

**72.** Determine the points of intersection of the graphs of $y = \ln x$ and $y = x - 2$ with coordinates rounded to two decimal places.

**73.** Solve $\ln x = 6 - 2x$. Round your answer to two decimal places.

**74.** Solve $6^{3-4x} = 15$. Round your answer to two decimal places.

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[12] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill, 1955).

**75. Bases** We have seen that there are two kinds of bases, $b$, for exponential and logarithmic functions: those $b$ in $(0, 1)$ and those $b$ in $(1, \infty)$. It might be supposed that there are *more* of the second kind but this is not the case. Consider the function $f$: $(0,1) \longrightarrow (1, \infty)$ given by $f(x) = 1/x$.

**(a)** Show that the domain of $f$ can be taken to be $(0, 1)$.

**(b)** Show that with domain $(0, 1)$ the range of $f$ is $(1, \infty)$.

**(c)** Show that $f$ has an inverse $g$ and determine a formula for $g(x)$.

The exercise shows that the numbers in $(0, 1)$ are in one-to-one correspondence with the numbers in $(1, \infty)$ so that every *base* of either kind corresponds to exactly one of the other kind. Who would have thought it? "$(1, \infty)$—so many numbers; $(0, 1)$—so little space."

**76.** Display the graph of the equation $(6)5^y + x = 2$. (*Hint:* Solve for $y$ as an explicit function of $x$.)

**77.** Graph $y = 2^x$ and $y = \dfrac{2^x}{8}$ on the same screen. It appears that the graph of $y = \dfrac{2^x}{8}$ is the graph of $y = 2^x$ shifted three units to the right. Prove algebraically that this is true.