Chapter 17

Problems 17.1

1. $f(x, y) = 2x^2 + 3xy + 4y^2 + 5x + 6y - 7$

$f_x(x, y) = 2(2x) + 3(1)y + 0 + 5(1) + 0 - 0$
$= 4x + 3y + 5$

$f_y(x, y) = 0 + 3x(1) + 4(2y) + 0 + 6(1) - 0$
$= 3x + 8y + 6$

2. $f(x, y) = 2x^2 + 3xy$

$f_x(x, y) = 2(2x) + 3(1)y = 4x + 3y$

$f_y(x, y) = 0 + 3x(1) = 3x$

3. $f(x, y) = 2y + 1$

$f_x(x, y) = 0 + 0 = 0$

$f_y(x, y) = 2(1) + 0 = 2$

4. $f(x, y) = \ln 2$

$f_x(x, y) = 0$

$f_y(x, y) = 0$

5. $g(x, y) = 3x^4y + 2xy^2 - 5xy + 8x - 9y$

$g_x(x, y) = 3(4)x^3y + 2(1)y^2 - 5(1)y + 8(1)$
$= 12x^3y + 2y^2 - 5y + 8$

$g_y(x, y) = 3x^4(1) + 2x(2)y - 5x(1) - 9(1)$
$= 3x^4 + 4xy - 5x - 9$

6. $g(x, y) = (x^2 + 1)^2 + (y^3 - 3)^3 + 5xy^3 - 2x^2y^2$

$g_x(x, y) = 2(x^2 + 1)(2x) + 0 + 5(1)y^3 - 2(2x)y^2$
$= 4x(x^2 + 1) + 5y^3 - 4xy^2$

$g_y(x, y)$
$= 0 + 3(y^3 - 3)^2(3y^2) + 5x(3y^2) - 2x^2(2y)$
$= 9y^2(y^3 - 3)^2 + 15xy^2 - 4x^2y$

7. $g(p, q) = \sqrt{pq} = (pq)^{\frac{1}{2}}$

$g_p(p, q) = \frac{1}{2}(pq)^{-\frac{1}{2}} \cdot q = \frac{q}{2\sqrt{pq}}$

$g_q(p, q) = \frac{1}{2}(pq)^{-\frac{1}{2}} \cdot p = \frac{p}{2\sqrt{pq}}$

8. $g(w, z) = \sqrt[3]{w^2 + z^2} = \left(w^2 + z^2\right)^{\frac{1}{3}}$

$g_w(w, z) = \frac{1}{3}\left(w^2 + z^2\right)^{-\frac{2}{3}}(2w) = \frac{2w}{3\left(w^2 + z^2\right)^{\frac{2}{3}}}$

$g_z(w, z) = \frac{1}{3}\left(w^2 + z^2\right)^{-\frac{2}{3}}(2z) = \frac{2z}{3\left(w^2 + z^2\right)^{\frac{2}{3}}}$

9. $h(s, t) = \frac{s^2 + 4}{t - 3}$

$h_s(s, t) = \frac{1}{t - 3}(2s) = \frac{2s}{t - 3}$

Rewriting $h(s, t)$ as $\left(s^2 + 4\right)(t - 3)^{-1}$, we have

$h_t(s, t) = \left(s^2 + 4\right)\left[(-1)(t - 3)^{-2}(1)\right] = -\frac{s^2 + 4}{(t - 3)^2}$

10. $h(u, v) = \frac{8uv^2}{u^2 + v^2}$

$h_u(u, v) = 8v^2 \frac{\left(u^2 + v^2\right)(1) - u(2u)}{\left(u^2 + v^2\right)^2}$

$= \frac{8v^2\left(v^2 - u^2\right)}{\left(u^2 + v^2\right)^2}$

$h_v(u, v) = 8u \frac{\left(u^2 + v^2\right)(2v) - v^2(2v)}{\left(u^2 + v^2\right)^2}$

$= \frac{16u^3v}{\left(u^2 + v^2\right)^2}$

11. $u\left(q_1, q_2\right) = \ln\sqrt{q_1 + 2} + \ln\sqrt[3]{q_2 + 5}$

$= \frac{1}{2}\ln(q_1 + 2) + \frac{1}{3}\ln(q_2 + 5)$

$u_{q_1}\left(q_1, q_2\right) = \frac{1}{2} \cdot \frac{1}{q_1 + 2} + 0 = \frac{1}{2(q_1 + 2)}$

$u_{q_2}\left(q_1, q_2\right) = 0 + \frac{1}{3} \cdot \frac{1}{q_2 + 5} = \frac{1}{3(q_2 + 5)}$

12. $Q(l,\ k)=2l^{0.38}k^{1.79}-3l^{1.03}+2k^{0.13}$

$Q_l(l,\ k)=2(0.38)l^{0.38-1}k^{1.79}-3(1.03)l^{1.03-1}+0=0.76l^{-0.62}k^{1.79}-3.09l^{0.03}$

$Q_k(l,\ k)=2l^{0.38}(1.79)k^{1.79-1}-0+2(0.13)k^{0.13-1}=3.58l^{0.38}k^{0.79}+0.26k^{-0.87}$

13. $h(x,\ y)=\dfrac{x^2+3xy+y^2}{\sqrt{x^2+y^2}}$

$$h_x(x,\ y)=\frac{\left(x^2+y^2\right)^{\frac{1}{2}}[2x+3y]-\left(x^2+3xy+y^2\right)\left[\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2x)\right]}{\left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]^2}$$

$$=\frac{\left(x^2+y^2\right)^{-\frac{1}{2}}\left[\left(x^2+y^2\right)(2x+3y)-\left(x^2+3xy+y^2\right)x\right]}{x^2+y^2}$$

$$=\frac{2x^3+3x^2y+2xy^2+3y^3-x^3-3x^2y-xy^2}{\left(x^2+y^2\right)^{\frac{3}{2}}}=\frac{x^3+xy^2+3y^3}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

$$h_y(x,\ y)=\frac{\left(x^2+y^2\right)^{\frac{1}{2}}[3x+2y]-\left(x^2+3xy+y^2\right)\left[\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2y)\right]}{\left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]^2}$$

$$=\frac{\left(x^2+y^2\right)^{-\frac{1}{2}}\left[\left(x^2+y^2\right)(3x+2y)-\left(x^2+3xy+y^2\right)y\right]}{x^2+y^2}$$

$$=\frac{3x^3+2x^2y+3xy^2+2y^3-x^2y-3xy^2-y^3}{\left(x^2+y^2\right)^{\frac{3}{2}}}=\frac{3x^3+x^2y+y^3}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

14. $h(x,\ y)=\dfrac{\sqrt{x+9}}{x^2y+y^2x}$

$$h_x(x,\ y)=\frac{\left(x^2y+y^2x\right)\frac{1}{2}(x+9)^{-\frac{1}{2}}-(x+9)^{\frac{1}{2}}\left(2xy+y^2\right)}{\left(x^2y+y^2x\right)^2}$$

$$=\frac{\frac{1}{2}(x+9)^{-\frac{1}{2}}\left[x^2y+y^2x-2(x+9)\left(2xy+y^2\right)\right]}{\left(x^2y+y^2x\right)^2}$$

$$=\frac{y\left[x^2+xy-2(x+9)(2x+y)\right]}{2(x+9)^{\frac{1}{2}}[xy(x+y)]^2}$$

$$=\frac{y\left[x^2+xy-4x^2-36x-2xy-18y\right]}{2(x+9)^{\frac{1}{2}}x^2y^2(x+y)^2}=\frac{-\left(3x^2+xy+36x+18y\right)}{2x^2y\sqrt{x+9}(x+y)^2}$$

Since $h(x,\ y)=\sqrt{x+9}\left(x^2y+y^2x\right)^{-1}$, then

$$h_y(x,\ y)=\sqrt{x+9}(-1)\left(x^2y+y^2x\right)^{-2}\left(x^2+2xy\right)$$

$$=\frac{-\sqrt{x+9}\left(x^2+2xy\right)}{\left(x^2y+y^2x\right)^2}=\frac{-x\sqrt{x+9}(x+2y)}{x^2y^2(x+y)^2}=\frac{-\sqrt{x+9}(x+2y)}{xy^2(x+y)^2}$$

15. $z=e^{5xy}$

$$\frac{\partial z}{\partial x}=e^{5xy}(5y)=5ye^{5xy};\ \frac{\partial z}{\partial y}=e^{5xy}(5x)=5xe^{5xy}$$

16. $z=(x^3+y^3)e^{xy+3x+3y}$

$$\frac{\partial z}{\partial x}=(x^3+y^3)[e^{xy+3x+3y}(y+3)]+e^{xy+3x+3y}[3x^2]$$

$$=[3x^2+(x^3+y^3)(y+3)]e^{xy+3x+3y}$$

$$\frac{\partial z}{\partial y}=(x^3+y^3)[e^{xy+3x+3y}(x+3)]+e^{xy+3x+3y}[3y^2]$$

$$=[3y^2+(x^3+y^3)(x+3)]e^{xy+3x+3y}$$

17. $z=5x\ln\left(x^2+y\right)$

$$\frac{\partial z}{\partial x}=5\left\{x\left[\frac{1}{x^2+y}(2x)\right]+\ln\left(x^2+y\right)[1]\right\}=5\left[\frac{2x^2}{x^2+y}+\ln\left(x^2+y\right)\right]$$

$$\frac{\partial z}{\partial y}=5x\left(\frac{1}{x^2+y}[1]\right)=\frac{5x}{x^2+y}$$

18. $z=\ln(5x^3y^2+2y^4)^4=4\ln(5x^3y^2+2y^4)$

$$\frac{\partial z}{\partial x}=4\cdot\frac{1}{5x^3y^2+2y^4}[5(3x^2)y^2+0]=\frac{60x^2y^2}{5x^3y^2+2y^4}=\frac{60x^2y^2}{y^2(5x^3+2y^2)}=\frac{60x^2}{5x^3+2y^2}$$

$$\frac{\partial z}{\partial y}=4\cdot\frac{1}{5x^3y^2+2y^4}[5x^3(2y)+2(4y^3)]=\frac{4(10x^3y+8y^3)}{5x^3y^2+2y^4}=\frac{8y(5x^3+4y^2)}{y(5x^3y+2y^3)}=\frac{8(5x^3+4y^2)}{5x^3y+2y^3}$$

19. $f(r,\ s)=(r+2s)^{\frac{1}{2}}\left(r^3-2rs+s^2\right)$

$$f_r(r,\ s)=(r+2s)^{\frac{1}{2}}\left[3r^2-2s\right]+\left(r^3-2rs+s^2\right)\left[\frac{1}{2}(r+2s)^{-\frac{1}{2}}(1)\right]$$

$$=\sqrt{r+2s}\left(3r^2-2s\right)+\frac{r^3-2rs+s^2}{2\sqrt{r+2s}}$$

$$f_s(r,\ s)=(r+2s)^{\frac{1}{2}}\left[-2r+2s\right]+\left(r^3-2rs+s^2\right)\left[\frac{1}{2}(r+2s)^{-\frac{1}{2}}(2)\right]$$

$$=2(s-r)\sqrt{r+2s}+\frac{r^3-2rs+s^2}{\sqrt{r+2s}}$$

20. $f(r,\ s) = (rs)^{\frac{1}{2}} e^{2+r}$

$f_r(r,\ s) = (rs)^{\frac{1}{2}}\left[e^{2+r}(1)\right] + e^{2+r}\left[\frac{1}{2}(rs)^{-\frac{1}{2}}(s)\right] = \left[\sqrt{rs} + \frac{s}{2\sqrt{rs}}\right]e^{2+r}$

$f_s(r,\ s) = e^{2+r}\left[\frac{1}{2}(rs)^{-\frac{1}{2}}(r)\right] = \frac{re^{2+r}}{2\sqrt{rs}}$

21. $f(r,\ s) = e^{3-r}\ln(7-s)$

$f_r(r,\ s) = \ln(7-s)\left[e^{3-r}(-1)\right] = -e^{3-r}\ln(7-s)$

$f_s(r,\ s) = e^{3-r}\left[\frac{1}{7-s}(-1)\right] = \frac{e^{3-r}}{s-7}$

22. $f(r,\ s) = \left(5r^2 + 3s^3\right)(2r - 5s)$

$f_r(r,\ s) = \left(5r^2 + 3s^3\right)[2] + (2r-5s)[10r] = 2\left(5r^2 + 3s^3\right) + 10r(2r-5s)$

$f_s(r,\ s) = \left(5r^2 + 3s^3\right)[-5] + (2r-5s)\left[9s^2\right] = -5\left(5r^2 + 3s^3\right) + 9s^2(2r-5s)$

23. $g(x,\ y,\ z) = 2x^3y^2 + 2xy^3z + 4z^2$

$g_x(x,\ y,\ z) = 2y^2(3x^2) + 2y^3z(1) + 0 = 6x^2y^2 + 2y^3z$

$g_y(x,\ y,\ z) = 2x^3(2y) + 2xz(3y^2) + 0 = 4x^3y + 6xy^2z$

$g_z(x,\ y,\ z) = 0 + 2xy^3(1) + 4(2z) = 2xy^3 + 8z$

24. $g(x,\ y,\ z) = 2xy^2z^6 - 4x^2y^3z^2 + 3xyz$

$$\begin{aligned} g_x(x,\ y,\ z) &= 2(1)y^2z^6 - 4(2x)y^3z^2 + 3(1)yz \\ &= 2y^2z^6 - 8xy^3z^2 + 3yz \end{aligned}$$

$$\begin{aligned} g_y(x,\ y,\ z) &= 2x(2y)z^6 - 4x^2(3y^2)z^2 + 3x(1)z \\ &= 4xyz^6 - 12x^2y^2z^2 + 3xz \end{aligned}$$

$$\begin{aligned} g_z(x,\ y,\ z) &= 2xy^2(6z^5) - 4x^2y^3(2z) + 3xy(1) \\ &= 12xy^2z^5 - 8x^2y^3z + 3xy \end{aligned}$$

25. $g(r,\ s,\ t) = e^{s+t}(r^2 + 7s^3)$

$g_r(r,\ s,\ t) = e^{s+t}[2r + 0] = 2re^{s+t}$

$$\begin{aligned} g_s(r,\ s,\ t) &= e^{s+t}\left[0 + 21s^2\right] + \left(r^2 + 7s^3\right)\left[e^{s+t}(1)\right] \\ &= \left(7s^3 + 21s^2 + r^2\right)e^{s+t} \end{aligned}$$

$g_t(r,\ s,\ t) = \left(r^2 + 7s^3\right)\left[e^{s+t}(1)\right] = e^{s+t}\left(r^2 + 7s^3\right)$

26. $g(r, s, t, u) = rs\ln(t)e^u$

$g_r(r, s, t, u) = s\ln(t)e^u$

$g_s(r, s, t, u) = r\ln(t)e^u$

$g_t(r, s, t, u) = rs\left(\frac{1}{t}\right)e^u = \frac{rse^u}{t}$

$g_u(r, s, t, u) = rs\ln(t)e^u$

27. $f(x, y) = x^3y + 7x^2y^2$

$f_x(x, y) = 3x^2y + 14xy^2$

$f_x(1, -2) = 3(1)^2(-2) + 14(1)(-2)^2 = 50$

28. $z = \sqrt{2x^3 + 5xy + 2y^2}$

$\frac{\partial z}{\partial x} = \frac{6x^2 + 5y}{2\sqrt{2x^3 + 5xy + 2y^2}}$

$\left.\frac{\partial z}{\partial x}\right|_{(0,\ 1)} = \frac{5}{2\sqrt{2}}$

29. $g(x, y, z) = e^x\sqrt{y + 2z}$

$g_z(x, y, z) = e^x\left[\frac{1}{2}(y + 2z)^{-\frac{1}{2}}(2)\right] = \frac{e^x}{\sqrt{y + 2z}}$

$g_z(0, 6, 4) = \frac{1}{\sqrt{6+8}} = \frac{1}{\sqrt{14}}$

30. $g(x, y, z) = \frac{3x^2y^2 + 2xy + x - y}{xy - yz + xz}$

$g_y(x, y, z) = \frac{(xy - yz + xz)(6x^2y + 2x - 1) - (3x^2y^2 + 2xy + x - y)(x - z)}{(xy - yz + xz)^2}$

$g_y(1, 1, 5) = \frac{(1-5+5)(6+2-1) - (3+2+1-1)(1-5)}{(1-5+5)^2} = 27$

31. $h(r, s, t, u) = (rst^2u)\ln(1 + rstu)$

$h_t(r, s, t, u) = [rs(2t)u]\ln(1 + rstu) + (rst^2u)\cdot\frac{rsu}{1 + rstu}$

$h_t(1, 1, 0, 1) = 0$

32. $h(r, s, t, u) = \frac{7r + 3s^2u^2}{s}$

$h_t(r, s, t, u) = 0$

$h_t(4, 3, 2, 1) = 0$

33. $f(r, s, t) = rst\left(r^2 + s^3 + t^4\right) = r^3st + rs^4t + rst^5$

$f_s(r, s, t) = r^3(1)t + r\left(4s^3\right)t + r(1)t^5 = r^3t + 4rs^3t + rt^5$

$f_s(1, -1, 2) = 2 + (-8) + 32 = 26$

34. $z = \dfrac{x^2 + y^2}{e^{x^2+y^2}} = (x^2 + y^2)e^{-(x^2+y^2)}$

$$\frac{\partial z}{\partial x} = (2x)e^{-(x^2+y^2)} + (x^2 + y^2)e^{-(x^2+y^2)}(-2x)$$

$$= 2xe^{-(x^2+y^2)}[1 - (x^2 + y^2)]$$

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=0}} = 2(0)e^0[1 - (0)] = 0$$

By symmetry, $\dfrac{\partial z}{\partial y} = 2ye^{-(x^2+y^2)}[1 - (x^2 + y^2)]$.

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=1}} = 2(1)e^{-2}[1 - (2)] = -\frac{2}{e^2}$$

35. $z = xe^{x-y} + ye^{y-x}$

$$\frac{\partial z}{\partial x} = \left[xe^{x-y} + e^{x-y}\right] + \left[ye^{y-x}(-1)\right]$$

$$\frac{\partial z}{\partial y} = \left[xe^{x-y}(-1)\right] + \left[ye^{y-x} + e^{y-x}\right]$$

Thus $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = e^{x-y} + e^{y-x}$, as was to be shown.

36. $u = f(t, r, z) = \dfrac{(1+r)^{1-z}\ln(1+r)}{(1+r)^{1-z} - t}$

$$\frac{\partial u}{\partial z} = \ln(1+r)\frac{\partial}{\partial z}\left[\frac{(1+r)^{1-z}}{(1+r)^{1-z} - t}\right]$$

$$= \ln(1+r)\frac{\left[(1+r)^{1-z} - t\right]\frac{\partial}{\partial z}\left[(1+r)^{1-z}\right] - (1+r)^{1-z}\left\{\frac{\partial}{\partial z}\left[(1+r)^{1-z}\right] - 0\right\}}{\left[(1+r)^{1-z} - t\right]^2}$$

$$= \ln(1+r)\frac{-t\frac{\partial}{\partial z}\left[(1+r)^{1-z}\right]}{\left[(1+r)^{1-z} - t\right]^2}$$

$$= \ln(1+r)\frac{-t\left\{(1+r)^{1-z}\ln(1+r)[-1]\right\}}{\left[(1+r)^{1-z} - t\right]^2}$$

$= \dfrac{t(1+r)^{1-z}\ln^2(1+r)}{\left[(1+r)^{1-z} - t\right]^2}$, as was to be shown.

37. $F(b, C, T, i) = \dfrac{bT}{C} + \dfrac{iC}{2}$

$$\frac{\partial F}{\partial C} = \frac{\partial}{\partial C}\left[\frac{bT}{C}\right] + \frac{\partial}{\partial C}\left[\frac{iC}{2}\right] = -\frac{bT}{C^2} + \frac{i}{2}$$

38. From $\eta = \dfrac{\frac{r}{D}}{\frac{\partial r}{\partial D}}$, we have $\dfrac{\partial r}{\partial D} = \dfrac{r}{D\eta}$. Substituting into Equation (3) gives

$$r_L = r + D \cdot \frac{r}{D\eta} + \frac{dC}{dD}$$

$$r_L = r + \frac{r}{\eta} + \frac{dC}{dD}$$

$$r_L = r\left[1 + \frac{1}{\eta}\right] + \frac{dC}{dD}$$

$$r_L = r\left[\frac{\eta + 1}{\eta}\right] + \frac{dC}{dD}$$

$$r_L = r\left[\frac{1 + \eta}{\eta}\right] + \frac{dC}{dD}$$

which is Equation (4).

39. $R = f(r, a, n) = \dfrac{r}{1 + a\left(\frac{n-1}{2}\right)} = r\left[1 + a\left(\dfrac{n-1}{2}\right)\right]^{-1}$

$$\frac{\partial R}{\partial n} = r(-1)\left[1 + a\left(\frac{n-1}{2}\right)\right]^{-2} \cdot \frac{a}{2}$$

$$= -\frac{ra}{2\left[1 + a\left(\frac{n-1}{2}\right)\right]^2}$$

Problems 17.2

1. $c = 7x + 0.3y^2 + 2y + 900$

$$\frac{\partial c}{\partial y} = 0.6y + 2$$

When $x = 20$ and $y = 30$, then

$$\frac{\partial c}{\partial y} = 0.6(30) + 2 = 20.$$

2. $c = 2x\sqrt{x + y} + 6000$

$$\frac{\partial c}{\partial x} = \frac{x}{\sqrt{x + y}} + 2\sqrt{x + y}$$

When $x = 70$ and $y = 74$, then

$$\frac{\partial c}{\partial x} = \frac{70}{\sqrt{70 + 74}} + 2\sqrt{70 + 74} = \frac{179}{6}.$$

3. $c = 0.03(x + y)^3 - 0.6(x + y)^2 + 9.5(x + y) + 7700$

$$\frac{\partial c}{\partial x} = 0.09(x + y)^2 - 1.2(x + y) + 9.5$$

When $x = 50$ and $y = 80$, then

$$\frac{\partial c}{\partial x} = 0.09(130)^2 - 1.2(130) + 9.5 = 1374.5.$$

4. $P = 15lk - 3l^2 + 5k^2 + 500$

$$\frac{\partial P}{\partial k} = 15l + 10k$$

$$\frac{\partial P}{\partial l} = 15k - 6l$$

5. $P = 2.314l^{0.357}k^{0.643}$

$$\frac{\partial P}{\partial l} = 2.314(0.357)l^{-0.643}k^{0.643}$$

$$= 0.826098\left(\frac{k}{l}\right)^{0.643}$$

$$\frac{\partial P}{\partial k} = 2.314(0.643)l^{0.357}k^{-0.357}$$

$$= 1.487902\left(\frac{l}{k}\right)^{0.357}$$

6. $P = Al^{\alpha}k^{\beta}$

a. $\dfrac{\partial P}{\partial l} = A\alpha l^{\alpha-1}k^{\beta} = \left(\dfrac{\alpha}{l}\right)Al^{\alpha}k^{\beta} = \dfrac{\alpha P}{l}$

b. $\dfrac{\partial P}{\partial k} = A\beta l^{\alpha}k^{\beta-1} = \left(\dfrac{\beta}{k}\right)Al^{\alpha}k^{\beta} = \dfrac{\beta P}{k}$

c. From parts (a) and (b),

$$l\frac{\partial P}{\partial l} + k\frac{\partial P}{\partial k} = l\left(\frac{\alpha P}{l}\right) + k\left(\frac{\beta P}{k}\right)$$

$$= \alpha P + \beta P = P(\alpha + \beta) = P(1) = P$$

7. $\dfrac{\partial q_{\mathrm{A}}}{\partial p_{\mathrm{A}}} = -40$, $\dfrac{\partial q_{\mathrm{A}}}{\partial p_{\mathrm{B}}} = 3$, $\dfrac{\partial q_{\mathrm{B}}}{\partial p_{\mathrm{A}}} = 5$, $\dfrac{\partial q_{\mathrm{B}}}{\partial p_{\mathrm{B}}} = -20$

Since $\dfrac{\partial q_{\mathrm{A}}}{\partial p_{\mathrm{B}}} > 0$ and $\dfrac{\partial q_{\mathrm{B}}}{\partial p_{\mathrm{A}}} > 0$ the products are competitive.

8. $\frac{\partial q_A}{\partial p_A}=-1$, $\frac{\partial q_A}{\partial p_B}=-2$, $\frac{\partial q_B}{\partial p_A}=-2$, $\frac{\partial q_B}{\partial p_B}=-3$

Since $\frac{\partial q_A}{\partial p_B}<0$ and $\frac{\partial q_B}{\partial p_A}<0$ the products are complementary.

9. $q_A=100p_A^{-1}p_B^{-\frac{1}{2}}$

$q_B=500p_B^{-1}p_A^{-\frac{1}{3}}$

$$\frac{\partial q_A}{\partial p_A}=100(-1)p_A^{-2}p_B^{-\frac{1}{2}}=\frac{-100}{p_A^2p_B^{\frac{1}{2}}}$$

$$\frac{\partial q_A}{\partial p_B}=100\left(-\frac{1}{2}\right)p_A^{-1}p_B^{-\frac{3}{2}}=\frac{-50}{p_Ap_B^{\frac{3}{2}}}$$

$$\frac{\partial q_B}{\partial p_A}=500\left(-\frac{1}{3}\right)p_B^{-1}p_A^{-\frac{4}{3}}=\frac{-500}{3p_Bp_A^{\frac{4}{3}}}$$

$$\frac{\partial q_B}{\partial p_B}=500(-1)p_B^{-2}p_A^{-\frac{1}{3}}=-\frac{500}{p_B^2p_A^{\frac{1}{3}}}$$

Since $\frac{\partial q_A}{\partial p_B}<0$ and $\frac{\partial q_B}{\partial p_A}<0$, the products are complementary.

10. $\frac{\partial P}{\partial l}=15.18l^{-0.54}k^{0.52}$

$\frac{\partial P}{\partial k}=17.16l^{0.46}k^{-0.48}$

If l = 1 and k = 1, then $\frac{\partial P}{\partial l}=15.18$ and

$\frac{\partial P}{\partial k}=17.16$

11. $\frac{\partial P}{\partial B}=0.01A^{0.27}B^{-0.99}C^{0.01}D^{0.23}E^{0.09}F^{0.27}$

$\frac{\partial P}{\partial C}=0.01A^{0.27}B^{0.01}C^{-0.99}D^{0.23}E^{0.09}F^{0.27}$

12. $P=\frac{kl}{3k+5l}$

a. $$\frac{\partial P}{\partial k}=\frac{l(3k+5l)-kl(3)}{(3k+5l)^2}=\frac{5l^2}{(3k+5l)^2}$$

$$\frac{\partial P}{\partial l}=\frac{k(3k+5l)-kl(5)}{(3k+5l)^2}=\frac{3k^2}{(3k+5l)^2}$$

b. When $k = l$, then

$$\frac{\partial P}{\partial k}+\frac{\partial P}{\partial l}=\frac{5l^2}{(3l+5l)^2}+\frac{3l^2}{(3l+5l)^2}=\frac{8l^2}{64l^2}=\frac{1}{8}$$

13. $\frac{\partial z}{\partial x}=4480$. If a staff manager with an M.B.A. degree had an extra year of work experience before the degree, the manager would receive \$4480 more per year in extra compensation.

14. $S_g=7S_e^{\frac{1}{3}}S_i^{\frac{1}{2}}$

$$\frac{\partial S_g}{\partial S_e}=7\left(\frac{1}{3}\right)S_e^{-\frac{2}{3}}S_i^{\frac{1}{2}}=\left(\frac{7}{3}\right)\frac{\sqrt{S_i}}{\sqrt[3]{S_e^2}}$$

$$\frac{\partial S_g}{\partial S_i}=7\left(\frac{1}{2}\right)S_e^{\frac{1}{3}}S_i^{-\frac{1}{2}}=\left(\frac{7}{2}\right)\frac{\sqrt[3]{S_e}}{\sqrt{S_i}}$$

If $S_e=125$ and $S_i=100$, then

$\frac{\partial S_g}{\partial S_e}=\left(\frac{7}{3}\right)\frac{10}{5^2}=\frac{14}{15}$ and $\frac{\partial S_g}{\partial S_i}=\left(\frac{7}{2}\right)\frac{5}{10}=\frac{7}{4}$.

Thus if S_e increases to 126 and S_i remains at 100, then S_g increases by approximately $\frac{14}{15}$; if S_i increases to 101 and S_e remains at 125, then S_g increases by approximately $\frac{7}{4}$.

15. a. $\frac{\partial R}{\partial w}=-1.015$; $\frac{\partial R}{\partial s}=-0.846$

b. One for which $w=w_0$ and $s=s_0$ since increasing w by 1 while holding s fixed decreases the reading ease score.

16. $\omega=b^{-1}L^{-1}\sqrt{\frac{\tau}{\pi\rho}}=\frac{1}{bL}\sqrt{\frac{\tau}{\pi}}\rho^{-\frac{1}{2}}=\frac{1}{bL}\sqrt{\frac{1}{\pi\rho}}\tau^{\frac{1}{2}}$

$$\frac{\partial\omega}{\partial b}=(-1)b^{-2}L^{-1}\sqrt{\frac{\tau}{\pi\rho}}=-\frac{1}{b^2L}\sqrt{\frac{\tau}{\pi\rho}}$$

$$\frac{\partial\omega}{\partial L}=b^{-1}(-1)L^{-2}\sqrt{\frac{\tau}{\pi\rho}}=-\frac{1}{bL^2}\sqrt{\frac{\tau}{\pi\rho}}$$

$$\frac{\partial \omega}{\partial \rho} = \frac{1}{bL}\sqrt{\frac{\tau}{\pi}}\left(-\frac{1}{2}\right)\rho^{-\frac{3}{2}} = -\frac{1}{2bL\rho^{\frac{3}{2}}}\sqrt{\frac{\tau}{\pi}}$$

$$\frac{\partial \omega}{\partial \tau} = \frac{1}{bL}\sqrt{\frac{1}{\pi\rho}}\left(\frac{1}{2}\right)\tau^{-\frac{1}{2}} = \frac{1}{2bL}\sqrt{\frac{1}{\pi\rho\tau}}$$

17. $\frac{\partial g}{\partial x} = \frac{1}{V_F} > 0$ for $V_F > 0$. Thus if x increases and V_F and V_S are fixed, then g increases.

18. $q_A = e^{-(p_A + p_B)}$ and $q_B = \frac{16}{p_A^2 p_B^2} = 16 p_A^{-2} p_B^{-2}$

a. $\frac{\partial q_A}{\partial p_B} = -e^{-(p_A + p_B)} < 0$

$\frac{\partial q_B}{\partial p_A} = -32 p_A^{-3} p_B^{-2} < 0$

Since both are < 0, A and B are complementary.

b. Note that p_A and p_B are in units of thousands of dollars. When $p_A = 1$ and $p_A = 2$, then $\frac{\partial q_A}{\partial p_B} = -e^{-3} = -\frac{1}{e^3}$.

A decrease in the price of B of \$20 is a decrease in p_B of $\frac{20}{2000} = 0.01$. Thus the change in q_B is approximately $-\frac{1}{e^3}(-0.01) = \frac{0.01}{e^3}$. So demand increases by approximately $\frac{0.01}{e^3}$ unit.

19. a. $\frac{\partial q_A}{\partial p_A} = 10\sqrt{p_B}\left(-\frac{1}{2}p_A^{-\frac{3}{2}}\right)$

$\frac{\partial q_A}{\partial p_B} = \frac{10}{\sqrt{p_A}}\left(\frac{1}{2}p_B^{-\frac{1}{2}}\right)$

When $p_A = 9$ and $p_B = 16$, then $\frac{\partial q_A}{\partial p_A} = 10(4)\left(-\frac{1}{2}\cdot\frac{1}{27}\right) = -\frac{20}{27}$ and $\frac{\partial q_A}{\partial p_B} = \frac{10}{3}\left(\frac{1}{2}\cdot\frac{1}{4}\right) = \frac{5}{12}$.

b. From (a), when $p_A = 9$ and $p_B = 16$, then $\frac{\partial q_A}{\partial p_B} = \frac{5}{12}$. Hence each \$1 reduction in p_B decreases q_A by approximately $\frac{5}{12}$ unit. Thus a \$2 reduction in p_B (from \$16 to \$14) decreases the demand for A by approximately $\frac{5}{12}(2) = \frac{5}{6}$ unit.

20. $c = \dfrac{q_A^2\left(q_B^3 + q_A\right)^{\frac{1}{2}}}{17} + q_A q_B^{\frac{1}{3}} + 600$

a. $\dfrac{\partial c}{\partial q_A} = \dfrac{1}{17}\left[q_A^2 \cdot \dfrac{1}{2}\left(q_B^3 + q_A\right)^{-\frac{1}{2}} + \left(q_B^3 + q_A\right)^{\frac{1}{2}}(2q_A)\right] + q_B^{\frac{1}{3}}$

$= \dfrac{1}{17}\left[\dfrac{1}{2}q_A^2\left(q_B^3 + q_A\right)^{-\frac{1}{2}} + 2q_A\left(q_B^3 + q_A\right)^{\frac{1}{2}}\right] + q_B^{\frac{1}{3}}$

$\dfrac{\partial c}{\partial q_B} = \dfrac{1}{17}\left[q_A^2 \cdot \dfrac{1}{2}\left(q_B^3 + q_A\right)^{-\frac{1}{2}}\left(3q_B^2\right)\right] + q_A \cdot \dfrac{1}{3}q_B^{-\frac{2}{3}}$

$= \dfrac{1}{17}\left[\dfrac{3}{2}q_A^2 q_B^2\left(q_B^3 + q_A\right)^{-\frac{1}{2}}\right] + \dfrac{1}{3}q_A q_B^{-\frac{2}{3}}$

b. When $q_A = 17$ and $q_B = 8$, then

$\dfrac{\partial c}{\partial q_A} = \dfrac{1}{17}\left[\dfrac{1}{2}(17)^2\left(\dfrac{1}{23}\right) + 2(17)(23)\right] + 2 = \left[\dfrac{1}{2}(17)\dfrac{1}{23} + 2(23)\right] + 2 \approx 48.37$.

c. From (b), if q_A is reduced by one unit (from 17 to 16) while q_B remains at 8, then the cost will decrease by approximately \$48.37.

21. a. $\dfrac{\partial R}{\partial E_r} = 2.5945 - 0.1608E_r - 0.0277I_r$

If $E_r = 18.8$ and $I_r = 10$, then $\dfrac{\partial R}{\partial E_r} = -0.70564$. Since $\dfrac{\partial R}{\partial E_r} < 0$, such a candidate should not be so advised.

b. $\dfrac{\partial R}{\partial N} = 0.8579 - 0.0122N$

If $\dfrac{\partial R}{\partial N} < 0$, then $N > 70.3 \approx 70\%$

22. $S = \dfrac{AT + 450}{\sqrt{A + T^2}}$. Note: A is expressed in hundreds of dollars.

a. $\dfrac{\partial S}{\partial T} = \dfrac{\left(A + T^2\right)^{\frac{1}{2}}(A) - (AT + 450)\left[\frac{1}{2}\left(A + T^2\right)^{-\frac{1}{2}}(2T)\right]}{\left(\sqrt{A + T^2}\right)^2}$

$= \dfrac{\left(A + T^2\right)^{-\frac{1}{2}}\left[\left(A + T^2\right)A - (AT + 450)T\right]}{A + T^2} = \dfrac{A^2 - 450T}{\left(A + T^2\right)^{\frac{3}{2}}}$

as was to be shown.

b. We want to find when $\frac{\partial S}{\partial T} < 0$ and $A = \frac{9000}{100} = 90$. First we find when $\frac{\partial S}{\partial T} = 0$ and $A = 90$:

$$\frac{90^2 - 450T}{\left(90 + T^2\right)^{\frac{3}{2}}} = 0 \Rightarrow 90^2 - 450T = 0$$

$$\Rightarrow T = \frac{90^2}{450} = 18.$$

$\frac{\partial S}{\partial T} > 0$ for $T < 18$, and $\frac{\partial S}{\partial T} < 0$ for $T > 18$.
Thus 18 months elapse before the sales volume begins to decrease.

23. $q_A = 1000 - 50p_A + 2p_B$

$$\eta_{p_A} = \left(\frac{p_A}{q_A}\right)\frac{\partial q_A}{\partial p_A} = \left(\frac{p_A}{q_A}\right)(-50)$$

$$\eta_{p_B} = \left(\frac{p_B}{q_A}\right)\frac{\partial q_A}{\partial p_B} = \left(\frac{p_B}{q_A}\right)(2)$$

When $p_A = 2$ and $p_B = 10$, then $q_A = 920$, from which $\eta_{p_A} = -\frac{5}{46}$ and $\eta_{p_B} = \frac{1}{46}$

24. $q_A = 60 - 3p_A - 2p_B$

$$\eta_{p_A} = \left(\frac{p_A}{q_A}\right)\frac{\partial q_A}{\partial p_A} = \left(\frac{p_A}{q_A}\right)(-3)$$

$$\eta_{p_B} = \left(\frac{p_B}{q_A}\right)\frac{\partial q_A}{\partial p_B} = \left(\frac{p_B}{q_A}\right)(-2)$$

When $p_A = 5$ and $p_B = 3$, then $q_A = 39$, from which $\eta_{p_A} = -\frac{5}{13}$ and $\eta_{p_B} = -\frac{2}{13}$.

25. $q_A = \frac{100}{p_A\sqrt{p_B}}$

$$\eta_{p_A} = \left(\frac{p_A}{q_A}\right)\frac{\partial q_A}{\partial p_A} = \left(\frac{p_A}{q_A}\right)\left(\frac{-100}{p_A^2\sqrt{p_B}}\right)$$

$$\eta_{p_B} = \left(\frac{p_B}{q_A}\right)\frac{\partial q_A}{\partial p_B} = \left(\frac{p_B}{q_A}\right)\left(\frac{-50}{p_A\sqrt{p_B^3}}\right)$$

When $p_A = 1$ and $p_B = 4$, then $q_A = 50$. This gives $\eta_{p_A} = -1$ and $\eta_{p_B} = -\frac{1}{2}$.

Problems 17.3

1. $4x + 0 + 10z\frac{\partial z}{\partial x} = 0$

$$\frac{\partial z}{\partial x} = -\frac{4x}{10z} = -\frac{2x}{5z}$$

2. $2z\frac{\partial z}{\partial x} - 10x + 0 = 0$

$$\frac{\partial z}{\partial x} = \frac{10x}{2z} = \frac{5x}{z}$$

3. $6z\frac{\partial z}{\partial y} - 0 - 14y = 0$

$$\frac{\partial z}{\partial y} = \frac{14y}{6z} = \frac{7y}{3z}$$

4. $0 + 2y + 6z^2\frac{\partial z}{\partial y} = 0$

$$\frac{\partial z}{\partial y} = \frac{-2y}{6z^2} = -\frac{y}{3z^2}$$

5. $x^2 - 2y - z^2 + y\left(x^2z^2\right) = 20$

$$2x - 0 - 2z\frac{\partial z}{\partial x} + y\left[x^2 \cdot 2z\frac{\partial z}{\partial x} + z^2 \cdot 2x\right] = 0$$

$$\left(2x^2yz - 2z\right)\frac{\partial z}{\partial x} = -2x - 2xyz^2$$

$$\frac{\partial z}{\partial x} = \frac{-2x\left(1 + yz^2\right)}{2z\left(x^2y - 1\right)} = \frac{x\left(yz^2 + 1\right)}{z\left(1 - x^2y\right)}$$

6. $3z^2\frac{\partial z}{\partial x} + 2x^2\left(2z\frac{\partial z}{\partial x}\right) + 2z^2(2x) - y = 0$

$$(3z^2 + 4x^2z)\frac{\partial z}{\partial x} = y - 4xz^2$$

$$\frac{\partial z}{\partial x} = \frac{y - 4xz^2}{3z^2 + 4x^2z}$$

7. $0 + e^y + e^z\frac{\partial z}{\partial y} = 0$

$$\frac{\partial z}{\partial y} = -\frac{e^y}{e^z} = -e^{y-z}$$

8. $xyz + xy^2z^3 - \ln z^4 = 0$ so

$xyz + xy^2z^3 - 4\ln z = 0.$

$$xz + xy\frac{dz}{dy} + 2xyz^3 + 3xy^2z^2\frac{dz}{dy} - \frac{4}{z}\cdot\frac{dz}{dy} = 0$$

$$\left(xy + 3xy^2z^2 - \frac{4}{z}\right)\frac{dz}{dy} = -xz - 2xyz^3$$

$$\left(\frac{xyz + 3xy^2z^3 - 4}{z}\right)\frac{dz}{dy} = -(xz + 2xyz^3)$$

$$\frac{dz}{dy} = -\frac{(xz + 2xyz^3)z}{xyz + 3xy^2z^3 - 4}$$

9. $\frac{1}{z}\frac{\partial z}{\partial x} + 9\frac{\partial z}{\partial x} - y = 0$

$$\left(\frac{1}{z} + 9\right)\frac{\partial z}{\partial x} = y$$

$$\left(\frac{1+9z}{z}\right)\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial x} = \frac{yz}{9+z}$$

10. $\frac{1}{x} + 0 - \frac{1}{z}\frac{\partial z}{\partial x} = 0$

$$-\frac{1}{z}\frac{\partial z}{\partial x} = -\frac{1}{x}$$

$$\frac{\partial z}{\partial x} = \frac{z}{x}$$

11. $\left(2z\frac{\partial z}{\partial y} + 6x\right)\sqrt{x^3+5} = 0$

$$2z\frac{\partial z}{\partial y} + 6x = 0$$

$$\frac{\partial z}{\partial y} = \frac{-6x}{2z} = -\frac{3x}{z}$$

12. $xz(1 + y) - 5 = 0$

$$\left[x\frac{\partial z}{\partial x} + z\cdot 1\right](1+y) - 0 = 0$$

$$x\frac{\partial z}{\partial x} + z = 0$$

$$\frac{\partial z}{\partial x} = -\frac{z}{x}$$

If $x = 1$, $y = 4$, $z = 1$, then $\frac{\partial z}{\partial x} = -\frac{1}{1} = -1$.

13. $z^2 + 2xz\frac{\partial z}{\partial x} + 2yz\frac{\partial z}{\partial x} - 2xy = 0$

$$(2xz + 2yz)\frac{\partial z}{\partial x} = 2xy - z^2$$

$$\frac{\partial z}{\partial x} = \frac{2xy - z^2}{2z(x+y)}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,\,0,\,1)} = \frac{2(1)(0) - 1^2}{2(1)(1+0)} = -\frac{1}{2}$$

14. $e^{zx}\cdot x\frac{\partial z}{\partial y} = x\left[y\frac{\partial z}{\partial y} + z\cdot 1\right]$

$$\left(xe^{zx} - xy\right)\frac{\partial z}{\partial y} = xz$$

$$\frac{\partial z}{\partial y} = \frac{xz}{x\left(e^{zx} - y\right)}$$

$$\frac{\partial z}{\partial y} = \frac{z}{e^{zx} - y}$$

If $x = 1$, $y = -e^{-1}$, $z = -1$, then

$$\frac{\partial z}{\partial y} = \frac{-1}{e^{-1} - \left(-e^{-1}\right)} = -\frac{e}{2}.$$

15. $e^{yz}\cdot y\frac{\partial z}{\partial x} = -y\left[x\frac{\partial z}{\partial x} + z\cdot 1\right].$

$$\left(ye^{yz} + xy\right)\frac{\partial z}{\partial x} = -yz$$

$$\frac{\partial z}{\partial x} = -\frac{yz}{y\left(e^{yz} + x\right)}$$

$$\frac{\partial z}{\partial x} = -\frac{z}{e^{yz} + x}$$

If $x = -\frac{e^2}{2}$, $y = 1$, $z = 2$, then

$$\frac{\partial z}{\partial x} = -\frac{2}{e^2 + \frac{-e^2}{2}} = -\frac{2}{\frac{e^2}{2}} = -\frac{4}{e^2}.$$

16. $\frac{1}{2}\left(xz+y^2\right)^{-\frac{1}{2}}\left[x\frac{\partial z}{\partial y}+2y\right]-x=0$

$$\frac{x}{2\sqrt{xz+y^2}}\frac{\partial z}{\partial y}=x-\frac{y}{\sqrt{xz+y^2}}=\frac{x\sqrt{xz+y^2}-y}{\sqrt{xz+y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{2\left(x\sqrt{xz+y^2}-y\right)}{x}$$

If $x = 2$, $y = 2$, $z = 6$, then $\frac{\partial z}{\partial y}=\frac{2(2\cdot 4-2)}{2}=6$.

17. $\frac{1}{z}\frac{\partial z}{\partial x}=4+0$

$$\frac{\partial z}{\partial x}=4z$$

If $x = 5$, $y = -20$, $z = 1$, then $\frac{\partial z}{\partial x}=4$.

18. $\frac{(s^2+t^2)\left[2rs^2\frac{\partial r}{\partial t}\right]-r^2s^2[2t]}{(s^2+t^2)^2}=t$

$$2rs^2(s^2+t^2)\frac{\partial r}{\partial t}-2r^2s^2t=t(s^2+t^2)^2$$

$$2rs^2(s^2+t^2)\frac{\partial r}{\partial t}=t(s^2+t^2)^2+2r^2s^2t$$

$$\frac{\partial r}{\partial t}=\frac{t[(s^2+t^2)^2+2r^2s^2]}{2rs^2(s^2+t^2)}$$

If $r = 1$, $s = 1$, $t = 1$, then $\frac{\partial r}{\partial t}=\frac{1[(1^2+1^2)^2+2(1)^2(1)^2]}{2(1)(1)^2(1^2+1^2)}=\frac{6}{4}=\frac{3}{2}$.

19. $\frac{(rs)\left[2t\frac{\partial t}{\partial r}\right]-\left(s^2+t^2\right)[s]}{(rs)^2}=0 \quad 2rst\frac{\partial t}{\partial r}-s\left(s^2+t^2\right)=0$

$$2rst\frac{\partial t}{\partial r}=s\left(s^2+t^2\right)$$

$$\frac{\partial t}{\partial r}=\frac{s\left(s^2+t^2\right)}{2rst}=\frac{s^2+t^2}{2rt}$$

If $r = 1$, $s = 2$, $t = 4$, then $\frac{\partial t}{\partial r}=\frac{4+16}{2\cdot 1\cdot 4}=\frac{20}{8}=\frac{5}{2}$.

20. $\frac{1}{x+y+z}\left(1+\frac{\partial z}{\partial x}\right)+yz+xy\frac{\partial z}{\partial x}$
$$=\frac{\partial z}{\partial x}e^{x+y+z}+ze^{x+y+z}\left(1+\frac{\partial z}{\partial x}\right)$$

When $x = 0$, $y = 1$, and $z = 0$, then $\frac{1}{1}\left(1+\frac{\partial z}{\partial x}\right)+(1)(0)+(0)(1)\frac{\partial z}{\partial x}$
$$=\frac{\partial z}{\partial x}e^{1}+0(e^{1})\left(1+\frac{\partial z}{\partial x}\right)$$

$1+\frac{\partial z}{\partial x}=e\frac{\partial z}{\partial x},\ 1=\frac{\partial z}{\partial x}(e-1),\ \frac{\partial z}{\partial x}=\frac{1}{e-1}$

21. $c+\sqrt{c}=12+q_{\text{A}}\sqrt{9+q_{\text{B}}^2}$

a. If $q_{\text{A}}=6$ and $q_{\text{B}}=4$, then $c+\sqrt{c}=12+6(5)=42$, $\sqrt{c}=42-c$, $c=(42-c)^2=42^2-84c+c^2$,
$c^2-85c+1764=0$, $c=\frac{85\pm\sqrt{(-85)^2-4(1)(1764)}}{2}=\frac{85\pm\sqrt{169}}{2}=\frac{85\pm 13}{2}$.
Thus $c = 49$ or $c = 36$. However $c = 49$ is extraneous but $c = 36$ is not. Thus $c = 36$.

b. Differentiating with respect to q_{A}:
$\frac{\partial c}{\partial q_{\text{A}}}+\frac{1}{2\sqrt{c}}\cdot\frac{\partial c}{\partial q_{\text{A}}}=\sqrt{9+q_{\text{B}}^2}$. $\left(1+\frac{1}{2\sqrt{c}}\right)\frac{\partial c}{\partial q_{\text{A}}}=\sqrt{9+q_{\text{B}}^2}$.
When $q_{\text{A}}=6$ and $q_{\text{B}}=4$, then $c = 36$ and $\left(1+\frac{1}{12}\right)\frac{\partial c}{\partial q_{\text{A}}}=5$, $\frac{13}{12}\cdot\frac{\partial c}{\partial q_{\text{A}}}=5$, or $\frac{\partial c}{\partial q_{\text{A}}}=\frac{60}{13}$.

Differentiating with respect to q_{B}:
$$\frac{\partial c}{\partial q_{\text{B}}}+\frac{1}{2\sqrt{c}}\cdot\frac{\partial c}{\partial q_{\text{B}}}=q_{\text{A}}\cdot\frac{q_{\text{B}}}{\sqrt{9+q_{\text{B}}^2}}$$
$$\left(1+\frac{1}{2\sqrt{c}}\right)\frac{\partial c}{\partial q_{\text{B}}}=\frac{q_{\text{A}}q_{\text{B}}}{\sqrt{9+q_{\text{B}}^2}}$$
When $q_{\text{A}}=6$ and $q_{\text{B}}=4$, then $c = 36$ and
$\left(1+\frac{1}{12}\right)\frac{\partial c}{\partial q_{\text{B}}}=\frac{24}{5}$, $\frac{13}{12}\cdot\frac{\partial c}{\partial q_{\text{B}}}=\frac{24}{5}$, or $\frac{\partial c}{\partial q_{\text{B}}}=\frac{288}{65}$.

Problems 17.4

1. $f_x(x,\ y)=6(1)y^2=6y^2$
$f_{xy}(x,\ y)=6(2y)=12y$
$f_y(x,\ y)=6x(2y)=12xy$
$f_{yx}(x,\ y)=12(1)y=12y$

2. $f_x(x,\ y)=6x^2y^2+12xy^3-3y$
$f_{xx}(x,\ y)=12xy^2+12y^3$

3. $f_y(x,\ y) = 3$

$f_{yy}(x,\ y) = 0$

$f_{yyx}(x,\ y) = 0$

4. $f_x(x,\ y) = \left(x^2 + xy + y^2\right)[y+1] + (xy + x + y)[2x + y]$

$= 3x^2y + 3x^2 + 2xy^2 + 4xy + y^3 + 2y^2$

$f_{xy}(x,\ y) = 3x^2 + 0 + 2x(2y) + 4x(1) + 3y^2 + 4y$

$= 3x^2 + 4xy + 4x + 3y^2 + 4y$

5. $f_y(x,\ y) = 9\left[e^{2xy}(2x)\right] = 18xe^{2xy}$

$f_{yx}(x,\ y) = 18\left[x\left(e^{2xy} \cdot 2y\right) + e^{2xy}(1)\right] = 18e^{2xy}(2xy+1)$

$f_{yxy}(x,\ y) = 18\left[e^{2xy}(2x) + (2xy+1)\left(e^{2xy} \cdot 2x\right)\right]$

$= 18e^{2xy}(2x)[1 + (2xy+1)] = 18e^{2xy}(2x)[2 + 2xy]$

$= 72x(1+xy)e^{2xy}$

6. $f_x(x,\ y) = \dfrac{1}{x^2 + y^2} \cdot 2x = \dfrac{2x}{x^2 + y^2}$

$f_{xx}(x,\ y) = \dfrac{\left(x^2 + y^2\right)[2] - (2x)[2x]}{\left(x^2 + y^2\right)^2} = \dfrac{2\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2}$

$f_{xy}(x,\ y) = (2x)(-1)\left(x^2 + y^2\right)^{-2}[2y] = -\dfrac{4xy}{\left(x^2 + y^2\right)^2}$

7. $f(x,\ y) = (x+y)^2(xy) = \left(x^2 + 2xy + y^2\right)(xy) = x^3y + 2x^2y^2 + xy^3$

$f_x(x,\ y) = 3x^2y + 4xy^2 + y^3$

$f_y(x,\ y) = x^3 + 4x^2y + 3xy^2$

$f_{xx}(x,\ y) = 6xy + 4y^2$

$f_{yy}(x,\ y) = 4x^2 + 6xy$

8. $f_x(x,\ y,\ z) = 2xy^3z^4$

$f_{xz}(x,\ y,\ z) = 8xy^3z^3$

$f_z(x,\ y,\ z) = 4x^2y^3z^3$

$f_{zx}(x,\ y,\ z) = 8xy^3z^3$

9. $z = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$

$\frac{\partial z}{\partial y} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot(2y) = \frac{y}{x^2+y^2}$

$\frac{\partial^2 z}{\partial y^2} = \frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

10. $\frac{\partial z}{\partial x} = \frac{1}{y}\cdot\frac{1}{x^2+5}(2x) = \frac{2x}{y\left(x^2+5\right)}$

$\frac{\partial^2 z}{\partial y\partial x} = \frac{2x}{x^2+5}\left(-\frac{1}{y^2}\right) = -\frac{2x}{y^2\left(x^2+5\right)}$

11. $f_y(x, y, z) = 0$

$f_{yx}(x, y, z) = 0$

$f_{yxx}(x, y, z) = 0$

Thus $f_{yxx}(4, 3, -2) = 0$.

12. $f_x(x, y, z) = z^2\left(6x-4y^3\right)$

$f_{xy}(x, y, z) = z^2\left(-12y^2\right) = -12y^2z^2$

$f_{xyz}(x, y, z) = -24y^2z$. Thus

$f_{xyz}(1, 2, 3) = -24(4)(3) = -288$.

13. $f_k(l, k) = 18l^3k^5 - 14l^2k^6$

$f_{kl}(l, k) = 54l^2k^5 - 28lk^6$

$f_{klk}(l, k) = 270l^2k^4 - 168lk^5$

Thus $f_{klk}(2, 1) = 270(4)(1) - 168(2)(1) = 744.$

14. $f_x(x, y) = 3x^2y^2 + 2xy - 2xy^2$

$f_{xx}(x, y) = 6xy^2 + 2y - 2y^2$

$f_{xxy}(x, y) = 12xy + 2 - 4y$

$f_{xy}(x, y) = 6x^2y + 2x - 4xy$

$f_{xyx}(x, y) = 12xy + 2 - 4y$

Thus

$f_{xxy}(2, 3) = f_{xyx}(2, 3) = 12(2)(3) + 2 - 4(3) = 62.$

15. $f_x(x, y) = y^2e^x + \frac{1}{x}$

$f_{xy}(x, y) = 2ye^x$

$f_{xyy}(x, y) = 2e^x$

Thus $f_{xyy}(1, 1) = 2e$.

16. $f_x(x, y) = 3x^2 - 6y^2 + 2x$

$f_{xy}(x, y) = -12y$

Thus $f_{xy}(1, -1) = 12$.

17. $\frac{\partial c}{\partial q_{\text{B}}} = \frac{1}{3}\left(3q_{\text{A}}^2 + q_{\text{B}}^3 + 4\right)^{-\frac{2}{3}}\left(3q_{\text{B}}^2\right)$

$= q_{\text{B}}^2\left(3q_{\text{A}}^2 + q_{\text{B}}^3 + 4\right)^{-\frac{2}{3}}$

$\frac{\partial^2 c}{\partial q_{\text{A}}\partial q_{\text{B}}} = -\frac{2}{3}q_{\text{B}}^2\left(3q_{\text{A}}^2 + q_{\text{B}}^3 + 4\right)^{-\frac{5}{3}}(6q_{\text{A}})$

$= -4q_{\text{A}}q_{\text{B}}^2\left(3q_{\text{A}}^2 + q_{\text{B}}^3 + 4\right)^{-\frac{5}{3}}$

When $p_{\text{A}} = 25$ and $p_{\text{B}} = 4$, then

$q_{\text{A}} = 10 - 25 + 16 = 1$ and $q_{\text{B}} = 20 + 25 - 44 = 1$,

and $\frac{\partial^2 c}{\partial q_{\text{A}}\partial q_{\text{B}}} = -4(8)^{-\frac{5}{3}} = -\frac{4}{32} = -\frac{1}{8}$.

18. $f_x(x, y) = 4x^3y^4 + 9x^2y^2 - 7$

$f_{xy}(x, y) = 16x^3y^3 + 18x^2y$

$f_{xx}(x, y) = 12x^2y^4 + 18xy^2$

$f_{xyx}(x, y) = 48x^2y^3 + 36xy$

$f_{xxy}(x, y) = 48x^2y^3 + 36xy$

Thus $f_{xyx}(x, y) = f_{xxy}(xy)$.

19. $f_x(x, y) = (2x+y)e^{x^2+xy+y^2}$

$f_y(x, y) = (x+2y)e^{x^2+xy+y^2}$

$f_{xy}(x, y)$

$= (2x+y)(x+2y)e^{x^2+xy+y^2} + e^{x^2+xy+y^2}$

$f_{yx}(x, y)$

$= (x+2y)(2x+y)e^{x^2+xy+y^2} + e^{x^2+xy+y^2}$

Thus $f_{xy}\left(x, y\right) = f_{yx}(x, y)$.

20. $f_x(x, y) = ye^{xy}$

$f_{xx}(x, y) = y^2e^{xy}$

$f_{xy}(x, y) = y(xe^{xy}) + e^{xy}(1) = e^{xy}(xy+1)$

$f_y(x, y) = xe^{xy}$

$f_{yy}(x, y) = x^2e^{xy}$

$f_{yx}(x, y) = x(ye^{xy}) + e^{xy}(1) = e^{xy}(xy+1)$

Thus, $f_{xx}(x, y) + f_{xy}(x, y) + f_{yx}(x, y) + f_{yy}(x, y)$

$= y^2e^{xy} + e^{xy}(xy+1) + e^{xy}(xy+1) + x^2e^{xy}$

$= e^{xy}(x^2 + 2xy + y^2 + 2)$

$= f(x, y)((x+y)^2 + 2)$

21. $\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\left(x^2+y^2\right)(2) - (2x)(2x)}{\left(x^2+y^2\right)^2} = \frac{2\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$$

$\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\left(x^2+y^2\right)(2) - (2y)(2y)}{\left(x^2+y^2\right)^2} = \frac{2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} + \frac{2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} = 0$$

22. $6z\frac{\partial z}{\partial x} - 6x^2 = 0$

$\frac{\partial z}{\partial x} = \frac{6x^2}{6z} = \frac{x^2}{z}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{z(2x) - x^2\,\frac{\partial z}{\partial x}}{z^2} = \frac{2xz - x^2\left(\frac{x^2}{z}\right)}{z^2} = \frac{2xz^2 - x^4}{z^3}$$

23. $2z\frac{\partial z}{\partial y} + 2y = 0$

$\frac{\partial z}{\partial y} = -\frac{2y}{2z} = -\frac{y}{z}$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{z(1) - y\cdot\frac{\partial z}{\partial y}}{z^2} = -\frac{z - y\left(-\frac{y}{z}\right)}{z^2} = -\frac{z^2+y^2}{z^3}$$

From the original equation, $z^2 + y^2 = 3x^2$. Thus

$\frac{\partial^2 z}{\partial y^2} = -\frac{3x^2}{z^3}$.

24. $2z^2 = x^2 + 2xy + xz$ (Eq. 1).

Differentiating both sides of Eq. 1 with respect to y:

$4z\frac{\partial z}{\partial y} = 0 + 2x + x\frac{\partial z}{\partial y}$, $(4z-x)\frac{\partial z}{\partial y} = 2x$,

$\frac{\partial z}{\partial y} = \frac{2x}{4z-x}$.

Differentiating both sides of Eq. 1 with respect to x:

$4z\frac{\partial z}{\partial x} = 2x + 2y + x\frac{\partial z}{\partial x} + z(1)$,

$(4z-x)\frac{\partial z}{\partial x} = 2x + 2y + z$, $\frac{\partial z}{\partial x} = \frac{2x+2y+z}{4z-x}$.

Differentiating $\frac{\partial z}{\partial y}$ with respect to x:

$$\frac{\partial^2 z}{\partial x \partial y} = 2\cdot\frac{(4z-x)[1] - x\left[4\frac{\partial z}{\partial x} - 1\right]}{(4z-x)^2}$$

$$= 2\cdot\frac{(4z-x) - x\left[\frac{4(2x+2y+z)}{4z-x} - 1\right]}{(4z-x)^2}$$

$$= 2\cdot\frac{(4z-x)^2 - x[4(2x+2y+z) - (4z-x)]}{(4z-x)^3}$$

$$= 2\cdot\frac{16z^2 - 8xz - 8x^2 - 8xy}{(4z-x)^3}$$

$$= 16\cdot\frac{2z^2 - xz - x^2 - xy}{(4z-x)^3}$$

$$= 16\cdot\frac{\left(x^2 + 2xy + xz\right) - xz - x^2 - xy}{(4z-x)^3}$$

$$= \frac{16xy}{(4z-x)^3}.$$

Problems 17.5

1. $z = 5x + 3y$, $x = 2r + 3s$, $y = r - 2s$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (5)(2) + (3)(1) = 13$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (5)(3) + (3)(-2) = 9$$

2. $z = 2x^2 + 3xy + 2y^2$, $x = r^2 - s^2$, $y = r^2 + s^2$

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\\ &= (4x+3y)(2r) + (3x+4y)(2r)\\ &= 14r(x+y)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ &= (4x+3y)(-2s) + (3x+4y)(2s)\\ &= -2s(x-y)\end{aligned}$$

3. $z = e^{x+y}$, $x = t^2 + 3$, $y = \sqrt{t^3}$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial t}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= e^{x+y}(2t) + e^{x+y}\left(\frac{3}{2}t^{1/2}\right)\\ &= e^{x+y}\left(2t + \frac{3}{2}\sqrt{t}\right)\end{aligned}$$

4. $z = \sqrt{8x+y}$, $x = t^2 + 3t + 4$, $y = t^3 + 4$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{4}{\sqrt{8x+y}}(2t+3) + \frac{1}{2\sqrt{8x+y}}\left(3t^2\right)$$

$$= \frac{3t^2 + 16t + 24}{2\sqrt{8x+y}}$$

5. $w = x^2yz + xy^2z + xyz^2$, $x = e^t$, $y = te^t$, $z = t^2e^t$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}\\ &= (2xyz + y^2z + yz^2)e^t + (x^2z + 2xyz + xz^2)e^t(1+t) + (x^2y + xy^2 + 2xyz)e^t(2t+t^2)\end{aligned}$$

6. $w = \ln\left(x^2 + y^2 + z^2\right)$, $x = 2 - 3t$, $y = t^2 + 3$, $z = 4 - t$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}\\ &= \frac{2x}{x^2+y^2+z^2}(-3) + \frac{2y}{x^2+y^2+z^2}(2t) + \frac{2z}{x^2+y^2+z^2}(-1)\\ &= \frac{-2(3x - 2yt + z)}{x^2+y^2+z^2}\end{aligned}$$

7. $z = \left(x^2 + xy^2\right)^3$, $x = r + s + t$, $y = 2r - 3s + 8t$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= 3\left(x^2 + xy^2\right)^2\left(2x + y^2\right)[1] + 3\left(x^2 + xy^2\right)^2(2xy)[8]$$

$$= 3\left(x^2 + xy^2\right)^2\left(2x + y^2 + 16xy\right)$$

8. $z = \sqrt{x^2 + y^2}$, $x = r^2 + s - t$, $y = r - s + t$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{x}{\sqrt{x^2 + y^2}}(2r) + \frac{y}{\sqrt{x^2 + y^2}}(1) = \frac{2xr + y}{\sqrt{x^2 + y^2}}$$

9. $w = x^2 + xyz + z^2$, $x = r^2 - s^2$, $y = rs$, $z = r^2 + s^2$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (2x + yz)(-2s) + (xz)(r) + \left(xy + 2z\right)(2s)$$

$$= -2s(2x + yz) + r(xz) + 2s(xy + 2z)$$

10. $w = \ln(xyz)$, $x = r^2 s$, $y = rs$, $z = rs^2$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= \frac{yz}{xyz}(2rs) + \frac{xz}{xyz}(s) + \frac{xy}{xyz}(s^2)$$

$$= \frac{2rs}{x} + \frac{s}{y} + \frac{s^2}{z}$$

11. $y = x^2 - 7x + 5$, $x = 19rs + 2s^2t^2$

$$\frac{\partial y}{\partial r} = \frac{dy}{dx}\frac{\partial x}{\partial r} = (2x - 7)(19s) = 19s(2x - 7)$$

12. $y = 4 - x^2$, $x = 2r + 3s - 4t$

$$\frac{\partial y}{\partial t} = \frac{dy}{dx}\frac{\partial x}{\partial t} = (-2x)(-4) = 8x$$

13. $z = (4x + 3y)^3$, $x = r^2 s$, $y = r - 2s$; $r = 0$, $s = 1$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= 12(4x + 3y)^2(2rs) + 9(4x + 3y)^2(1)$$

$$= 3(4x + 3y)^2(8rs + 3)$$

When $r = 0$, $s = 1$, then $x = 0$, $y = -2$, and $\frac{\partial z}{\partial r} = 324$.

14. $z = \sqrt{2x+3y}$, $x = 3t + 5$, $y = t^2 + 2t + 1$; $t = 1$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= \frac{2}{2\sqrt{2x+3y}}(3) + \frac{3}{2\sqrt{2x+3y}}(2t+2)$$
$$= \frac{3(t+2)}{\sqrt{2x+3y}}$$

When $t = 1$, then $x = 8$, $y = 4$ and $\frac{dz}{dt} = \frac{9}{\sqrt{28}} = \frac{9}{2\sqrt{7}}$.

15. $w = e^{x+y+z}(x^2 + y^2 + z^2)$, $x = (r-s)^2$, $y = (r+s)^2$, $z = (s-r)^2$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$
$$= [e^{x+y+z}(x^2 + y^2 + z^2) + e^{x+y+z}(2x)][2(r-s)(-1)] + [e^{x+y+z}(x^2 + y^2 + z^2) + e^{x+y+z}(2y)][2(r+s)]$$
$$+ [e^{x+y+z}(x^2 + y^2 + z^2) + e^{x+y+z}(2z)][2(s-r)]$$

When $r = 1$, $s = 1$, then $x = 0$, $y = 4$, $z = 0$.

$$\frac{\partial w}{\partial s} = e^4(16+8)(4) = 96e^4$$

16. $y = \frac{x}{x-5}$, $x = 2t^2 - 3rs - r^2 t$; $r = 0$, $s = 2$, $t = -1$

$$\frac{\partial y}{\partial t} = \frac{dy}{dx}\frac{\partial x}{\partial t} = \frac{-5}{(x-5)^2}\left(4t - r^2\right)$$

When $r = 0$, $s = 2$, and $t = -1$, then $x = 2$ and $\frac{\partial y}{\partial t} = \frac{20}{9}$

17. $$\frac{\partial c}{\partial p_A} = \frac{\partial c}{\partial q_A}\frac{\partial q_A}{\partial p_A} + \frac{\partial c}{\partial q_B}\frac{\partial q_B}{\partial p_A}$$
$$= \left[\frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}(6q_A)\right](-1) + \left[\frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(3q_B^2\right)\right](1)$$
$$= \left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(-2q_A + q_B^2\right)$$
$$\frac{\partial c}{\partial p_B} = \frac{\partial c}{\partial q_A}\frac{\partial q_A}{\partial p_B} + \frac{\partial c}{\partial q_B}\frac{\partial q_B}{\partial p_B}$$
$$= \left[\frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}(6q_A)\right](2p_B) + \left[\frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(3q_B^2\right)\right](-11)$$
$$= \left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(4q_A p_B - 11q_B^2\right)$$

When $p_A = 25$ and $p_B = 4$, then $q_A = 10 - 25 + 16 = 1$, $q_B = 20 + 25 - 44 = 1$,

and $\frac{\partial c}{\partial p_A} = (8)^{-\frac{2}{3}}(-1) = -\frac{1}{4}$ and $\frac{\partial c}{\partial p_B} = (8)^{-\frac{2}{3}}(5) = \frac{5}{4}$.

18. a. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

b. Since $\dfrac{dy}{dt} = 1$, from (a), $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}$

19. a. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$

b. $w = 2x^2 \ln|3x - 5y|$, $x = s\sqrt{t^2+2}$ and $y = t - 3e^{2-s}$.

$$\frac{\partial w}{\partial t} = \left[4x\ln|3x-5y| + \frac{2x^2(3)}{3x-5y}\right]\frac{s(2t)}{2\sqrt{t^2+2}} + \left[\frac{2x^2}{3x-5y}(-5)\right](1)$$

When $s = 1$ and $t = 0$, then $x = \sqrt{2}$ and $y = -3e$.

$$\frac{\partial w}{\partial t} = \left[4\sqrt{2}\ln\left|3\sqrt{2} - 5(-3e)\right| + \frac{2(2)(3)}{3\sqrt{2} - 5(-3e)}\right](0) + \left[\frac{2(2)}{3\sqrt{2} - 5(-3e)}(-5)\right]$$

$$= -\frac{20}{3\sqrt{2} + 15e}$$

20. $p = aP - whL$, where $P = f(l, k)$ and $l = Lg(h)$.

$$\frac{\partial p}{\partial L} = a\frac{\partial P}{\partial L} - wh = a\left[\frac{\partial P}{\partial l}\frac{\partial l}{\partial L} + \frac{\partial P}{\partial k}\frac{\partial k}{\partial L}\right] - wh$$

$$= a\left[\frac{\partial P}{\partial l}g(h) + \frac{\partial P}{\partial k}\cdot 0\right] - wh = a\frac{\partial P}{\partial l}g(h) - wh$$

$$\frac{\partial p}{\partial h} = a\frac{\partial P}{\partial h} - wL = a\left[\frac{\partial P}{\partial l}\frac{\partial l}{\partial h} + \frac{\partial P}{\partial k}\frac{\partial k}{\partial h}\right] - wL$$

$$= a\left[\frac{\partial P}{\partial l}Lg'(h) + \frac{\partial P}{\partial k}\cdot 0\right] - wL$$

$$= a\frac{\partial P}{\partial l}Lg'(h) - wL$$

Problems 17.6

1. $f(x, y) = x^2 - 3y^2 - 8x + 9y + 3xy$

$$\begin{cases} f_x(x, y) = 2x - 8 + 3y = 0 \\ f_y(x, y) = -6y + 9 + 3x = 0 \end{cases}$$

Solving the system gives the critical point (1, 2).

2. $f(x, y) = x^2 + 4y^2 - 6x + 16y$

$$\begin{cases} f_x(x, y) = 2x - 6 = 0 \\ f_y(x, y) = 8y + 16 = 0 \end{cases}$$

Critical point: (3, –2)

3. $f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7$

$$\begin{cases} f_x(x, y) = 5x^2 - 15x = 0 \\ f_y(x, y) = 2y^2 + 2y - 4 = 0 \end{cases}$$

Both equations are easily solved by factoring.
Critical points: (0, –2), (0, 1), (3, –2), (3, 1)

4. $f(x, y) = xy - x + y$

$f_x(x, y) = y - 1$

$f_y(x, y) = x + 1$

Critical point: (–1, 1)

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

$$\begin{cases} f_x(x, y, z) = 4x + y - z = 0 \\ f_y(x, y, z) = x + 2y - z = 0 \\ f_z(x, y, z) = -x - y + 200 = 0 \end{cases}$$

Solving the system gives the critical point (50, 150, 350).

6. $f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3)$

$$\begin{cases} f_x(x, y, z, w) = 2x + w = 0 \\ f_y(x, y, z, w) = 2y + w = 0 \\ f_z(x, y, z, w) = 2z + w = 0 \\ f_w(x, y, z, w) = x + y + z - 3 = 0 \end{cases}$$

Solving the system gives the critical point (1, 1, 1, –2).

7. $f(x, y) = x^2 + 3y^2 + 4x - 9y + 3$

$$\begin{cases} f_x(x, y) = 2x + 4 = 0 \\ f_y(x, y) = 6y - 9 = 0 \end{cases}$$

Critical point $\left(-2, \frac{3}{2}\right)$

Second-Derivative Test

$f_{xx}(x, y) = 2$, $f_{yy}(x, y) = 6$, $f_{xy}(x, y) = 0$. At $\left(-2, \frac{3}{2}\right)$, $D = (2)(6) - 0^2 = 12 > 0$ and $f_{xx}(x, y) = 2 > 0$. Thus at $\left(-2, \frac{3}{2}\right)$ there is a relative minimum.

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

$$\begin{cases} f_x(x, y) = -4x + 8 = 0 \\ f_y(x, y) = -6y + 24 = 0 \end{cases}$$

Critical point: (2, 4)

Second-Derivative Test

$f_{xx}(x, y) = -4$, $f_{yy}(x, y) = -6$, $f_{xy}(x, y) = 0$. At (2, 4), $D = (-4)(-6) - 0^2 = 24 > 0$ and $f_{xx}(x, y) = -4 < 0$; thus there is a relative maximum at (2, 4).

9. $f(x, y) = y - y^2 - 3x - 6x^2$

$$\begin{cases} f_x(x, y) = -3 - 12x = 0 \\ f_y(x, y) = 1 - 2y = 0 \end{cases}$$

Critical point $\left(-\frac{1}{4}, \frac{1}{2}\right)$

Second-Derivative Test

$f_{xx}(x, y) = -12$, $f_{yy}(x, y) = -2$, $f_{xy}(x, y) = 0$

At $\left(-\frac{1}{4}, \frac{1}{2}\right)$, $D = (-12)(-2) - 0^2 = 24 > 0$ and $f_{xx}(x, y) = -12 < 0$. Thus at $\left(-\frac{1}{4}, \frac{1}{2}\right)$ there is a relative maximum.

10. $f(x, y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$

$$\begin{cases} f_x(x, y) = 4x + 3y - 10 = 0 \\ f_y(x, y) = 3y + 3x - 9 = 0 \end{cases}$$

Critical point: (1, 2)

Second-Derivative Test

$f_{xx}(x, y) = 4,\ f_{yy}(x, y) = 3,\ f_{xy}(x, y) = 3.$

At (1, 2), $D = (4)(3) - 3^2 = 3 > 0$ and $f_{xx}(x, y) = 4 > 0$; thus there is a relative minimum at (1, 2).

11. $f(x, y) = x^2 + 3xy + y^2 - 9x - 11y + 3$

$$\begin{cases} f_x(x, y) = 2x + 3y - 9 = 0 \\ f_y(x, y) = 3x + 2y - 11 = 0 \end{cases}$$

Critical point: (3, 1)

Second-Derivative Test

$f_{xx}(x, y) = 2,\ f_{yy} = 2,\ f_{xy} = 3.$ At (3, 1), $D = (2)(2) - (3)^2 = -5 < 0,$ so there is no relative extremum at (3, 1).

12. $f(x, y) = \frac{x^3}{3} + y^2 - 2x + 2y - 2xy$

$$\begin{cases} f_x(x, y) = x^2 - 2 - 2y = 0 \\ f_y(x, y) = 2y + 2 - 2x = 0 \end{cases}$$

Critical points: (2, 1), (0, –1)
Second-Derivative Test
$f_{xx}(x, y) = 2x$, $f_{yy}(x, y) = 2$, $f_{xy}(x, y) = -2$.
At (2, 1), $D = (4)(2) - (-2)^2 = 4 > 0$ and $f_{xx}(x, y) = 4 > 0$, so a relative minimum at (2, 1). At (0, –1), $D = (0)(2) - (-2)^2 = -4 < 0$; thus neither at (0, –1).

13. $f(x, y) = \frac{1}{3}\left(x^3 + 8y^3\right) - 2\left(x^2 + y^2\right) + 1$

$$\begin{cases} f_x(x, y) = x^2 - 4x = 0 \\ f_y(x, y) = 8y^2 - 4y = 0 \end{cases}$$

Critical points: (0, 0), $\left(4, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}\right)$, (4, 0)
Second-Derivative Test
$f_{xx}(x, y) = 2x - 4$, $f_{yy}(x, y) = 16y - 4$, $f_{xy}(x, y) = 0$. At (0, 0), $D = (-4)(-4) - 0^2 = 16 > 0$ and $f_{xx}(x, y) = -4 < 0$; thus a relative maximum.
At $\left(4, \frac{1}{2}\right)$, $D = (4)(4) - 0^2 = 16 > 0$ and $f_{xx}(x, y) = 4 > 0$; thus a relative minimum.
At $\left(0, \frac{1}{2}\right)$, $D = (-4)(4) - 0^2 = -16 < 0$; thus neither.
At (4, 0), $D = (4)(-4) - 0^2 = -16 < 0$, thus neither.

14. $f(x, y) = x^2 + y^2 - xy + x^3$

$$\begin{cases} f_x(x, y) = 2x - y + 3x^2 = 0 \\ f_y(x, y) = 2y - x = 0 \end{cases}$$

Critical points: (0, 0), $\left(-\frac{1}{2}, -\frac{1}{4}\right)$
Second-Derivative Test
$f_{xx}(x, y) = 2 + 6x$, $f_{yy}(x, y) = 2$, $f_{xy}(x, y) = -1$. At (0, 0), $D = (2)(2) - (-1)^2 = 3 > 0$ and $f_{xx}(x, y) = 2 > 0$; thus relative minimum.
At $\left(-\frac{1}{2}, -\frac{1}{4}\right)$, $D = (-1)(2) - (-1)^2 = -3 < 0$; thus neither.

15. $f(l, k) = \frac{l^2}{2} + 2lk + 3k^2 - 69l - 164k + 17$

$$\begin{cases} f_l(l, k) = l + 2k - 69 = 0 \\ f_k(l, k) = 2l + 6k - 164 = 0 \end{cases}$$

Critical point: (43, 13)
Second-Derivative Test
$f_{ll}(l, k) = 1$, $f_{kk}(l, k) = 6$, $f_{lk}(l, k) = 2$
At (43, 13), $D = (1)(6) - 2^2 = 2 > 0$ and $f_{ll}(l, k) = 1 > 0$; thus there is a relative minimum at (43, 13).

16. $f(l, k) = l^2 + 4k^2 - 4lk$

$$\begin{cases} f_l(l, k) = 2l - 4k \\ f_k(l, k) = 8k - 4l \end{cases}$$

Critical points: (2*r*, *r*) where *r* is any real number.
Second Derivative Test
$f_{ll}(l, k) = 2,$ $f_{kk}(l, k) = 8,$ and $f_{lk}(l, k) = -4.$
At (2*r*, *r*), $D = (2)(8) - (-4)^2 = 0,$ thus we cannot make a conclusion.

17. $f(p, q) = pq - \frac{1}{p} - \frac{1}{q}$

$$\begin{cases} f_p(p, q) = q + \frac{1}{p^2} = 0 \\ f_q(p, q) = p + \frac{1}{q^2} = 0 \end{cases}$$

Critical point: (–1, –1)
Second-Derivative Test
$f_{pp}(p, q) = -\frac{2}{p^3}$, $f_{qq}(p, q) = -\frac{2}{q^3}$, $f_{pq}(p, q) = 1$. At (–1, –1), $D = (2)(2) - 1^2 = 3 > 0$ and $f_{pp}(p, q) = 2 > 0$; thus there is a relative minimum at (–1, –1).

18.
$$\begin{aligned} f(x, y) &= (x-3)(y-3)(x+y-3) \\ &= (y-3)(x^2 + xy - 6x - 3y + 9) \\ &= (x-3)(xy - 3x + y^2 - 6y + 9) \end{aligned}$$

$$\begin{cases} f_x(x, y) = (y-3)(2x + y - 6) = 0 \\ f_y(x, y) = (x-3)(x + 2y - 6) = 0 \end{cases}$$

Critical points: (2, 2), (3, 3), (3, 0), (0, 3)
Second-Derivative Test
$f_{xx}(x, y) = 2(y-3)$, $f_{yy}(x, y) = 2(x-3)$, $f_{xy}(x, y) = 2x + 2y - 9$. At (2, 2),

$D = (-2)(-2) - (-1)^2 = 3 > 0$ and

$f_{xx}(x, y) = -2 < 0$; thus relative maximum.

At (3, 3), $D = (0)(0) - 3^2 = -9 < 0$; thus neither.

At (3, 0), $D = (-6)(0) - (-3)^2 = -9 < 0$; thus neither. At (0, 3), $D = (0)(-6) - (-3)^2 = -9 < 0$; thus neither.

19. $f(x, y) = \left(y^2 - 4\right)\left(e^x - 1\right)$

$$\begin{cases} f_x(x, y) = e^x\left(y^2 - 4\right) = 0 & (1) \\ f_y(x, y) = 2y\left(e^x - 1\right) = 0 & (2) \end{cases}$$

Critical points: (0, –2), (0, 2)
[Note that $y = 0$ does not give rise to a common solution of (1) and (2).]
Second-Derivative Test

$f_{xx}(x, y) = e^x\left(y^2 - 4\right)$, $f_{yy}(x, y) = 2\left(e^x - 1\right)$,

$f_{xy}(x, y) = 2ye^x$. At (0, –2),

$D = (0)(0) - (-4)^2 = -16 < 0$; thus neither. At (0, 2), $D = (0)(0) - (4)^2 = -16 < 0$; thus neither.

20. $f(x, y) = \ln(xy) + 2x^2 - xy - 6x$

$$\begin{cases} f_x(x, y) = \dfrac{1}{x} + 4x - y - 6 = 0 \\ f_y(x, y) = \dfrac{1}{y} - x = 0 \end{cases}$$

The only critical point is $\left(\frac{3}{2}, \frac{2}{3}\right)$.

$f_{xx}(x, y) = -\dfrac{1}{x^2} + 4$, $f_{yy}(x, y) = -\dfrac{1}{y^2}$,

$f_{xy}(x, y) = -1$. At $\left(\frac{3}{2}, \frac{2}{3}\right)$,

$D = \left(\frac{32}{9}\right)\left(\frac{-9}{4}\right) - (-1)^2 = -9 < 0$; thus neither.

21. $P = f(l, k) = 2.18l^2 - 0.02l^3 + 1.97k^2 - 0.03k^3$

$$\begin{cases} P_l = 4.36l - 0.06l^2 = 0 \\ P_k = 3.94k - 0.09k^2 = 0 \end{cases}$$

Critical points: (0, 0), $\left(0, \frac{394}{9}\right)$, $\left(\frac{218}{3}, 0\right)$, $\left(\frac{218}{3}, \frac{394}{9}\right)$

Second-Derivative Test

$P_{ll} = 4.36 - 0.12l$, $P_{kk} = 3.94 - 0.18k$, $P_{lk} = 0$.

At (0, 0), $D = (4.36)(3.94) - 0^2 > 0$ and $P_{ll} = 4.36 > 0$; thus relative minimum.

At $\left(0, \frac{394}{9}\right)$, $D = (4.36)(-3.94) - 0^2 < 0$; thus no extremum.

At $\left(\frac{218}{3}, 0\right)$, $D = (-4.36)(3.94) - 0^2 < 0$; thus no extremum.

At $\left(\frac{218}{3}, \frac{394}{9}\right)$, $D = (-4.36)(-3.94) - 0^2 > 0$ and $P_{ll} = -4.36 < 0$; thus $l = \frac{218}{3} \approx 72.67$, $k = \frac{394}{9} \approx 43.78$ gives a relative maximum.

22. $Q = 18c + 20d - 2c^2 - 4d^2 - cd$

$$\begin{cases} Q_c = 18 - 4c - d = 0 \\ Q_d = 20 - 8d - c = 0 \end{cases}$$

Critical point: $c = 4$, $d = 2$

$Q_{cc} = -4$, $Q_{dd} = -8$, $Q_{cd} = -1$

When $c = 4$ and $d = 2$, then

$D = (-4)(-8) - (-1)^2 > 0$ and $Q_{cc} = -4 < 0$; thus relative maximum at $c = 4$, $d = 2$.

23. Profit per lb for A $= p_A - 60$.

Profit per lb for B $= p_B - 70$.

Total Profit $= P = (p_A - 60)q_A + (p_B - 70)q_B$

$P = (p_A - 60)\left[5(p_B - p_A)\right]$
$+ (p_B - 70)\left[500 + 5(p_A - 2p_B)\right]$

Thus

$$\begin{cases} \dfrac{\partial P}{\partial p_A} = -10(p_A - p_B + 5) = 0 \\ \dfrac{\partial P}{\partial p_B} = 10(p_A - 2p_B + 90) = 0 \end{cases}$$

Critical point: $p_A = 80$, $p_B = 85$

$\dfrac{\partial^2 P}{\partial p_A^2} = -10$, $\dfrac{\partial^2 P}{\partial p_B^2} = -20$, $\dfrac{\partial^2 P}{\partial p_B \partial p_A} = 10$.

When $p_A = 80$ and $p_B = 85$, then

$D = (-10)(-20) - (10)^2 = 100 > 0$ and

$\dfrac{\partial^2 P}{\partial p_A^2} = -10 < 0$; thus relative maximum at $p_A = 80$, $p_B = 85$.

24. Profit per lb for $A = p_A - a$.
Profit per lb for $B = p_B - b$.
Total Profit $= P = (p_A - a)q_A + (p_B - b)q_B$
$P = (p_A - a)[5(p_B - p_A)] + (p_B - b)[500 + 5(p_A - 2p_B)]$

$$\begin{cases} \frac{\partial P}{\partial p_A} = -5(2p_A - 2p_B + b - a) = 0 \\ \frac{\partial P}{\partial p_B} = 5(2p_A - 4p_B + 2b - a + 100) = 0 \end{cases}$$

Critical point: $p_A = 50 + \frac{a}{2}$, $p_B = 50 + \frac{b}{2}$

$$\frac{\partial^2 P}{\partial p_A^2} = -10, \quad \frac{\partial^2 P}{\partial p_B^2} = -20, \quad \frac{\partial^2 P}{\partial p_B \partial p_A} = 10$$

When $p_A = 50 + \frac{a}{2}$ and $p_B = 50 + \frac{b}{2}$, then $D = (-10)(-20) - (10)^2 = 100 > 0$ and $\frac{\partial^2 P}{\partial p_A^2} = -10 < 0$; thus a relative maximum at $p_A = 50 + \frac{a}{2}$, $p_B = 50 + \frac{b}{2}$.

25. $p_A = 100 - q_A$, $p_B = 84 - q_B$, $c = 600 + 4(q_A + q_B)$.
Revenue from market A $= r_A = p_A q_A = (100 - q_A)q_A$. Revenue from market B $= r_B = p_B q_B = (84 - q_B)q_B$.
Total Profit = Total Revenue – Total Cost
$P = (100 - q_A)q_A + (84 - q_B)q_B - [600 + 4(q_A + q_B)]$

$$\begin{cases} \frac{\partial P}{\partial q_A} = 96 - 2q_A = 0 \\ \frac{\partial P}{\partial q_B} = 80 - 2q_B = 0 \end{cases}$$

Critical point: $q_A = 48$, $q_B = 40$

$$\frac{\partial^2 P}{\partial q_A^2} = -2, \quad \frac{\partial^2 P}{\partial q_B^2} = -2, \quad \frac{\partial^2 P}{\partial q_B \partial q_A} = 0.$$

At $q_A = 48$ and $q_B = 40$, then $D = (-2)(-2) - 0^2 = 4 > 0$ and $\frac{\partial^2 P}{\partial q_A^2} = -2 < 0$; thus relative maximum at $q_A = 48$, $q_B = 40$. When $q_A = 48$ and $q_B = 40$, then selling prices are $p_A = 52$, $p_B = 44$, and profit = 3304.

26. $q_A = 16 - p_A + p_B$, $q_B = 24 + 2p_A - 4p_B$
Revenue from A $= p_A q_A$. Revenue from B $= p_B q_B$.
Total cost of producing q_A units of A and q_B units of B is $2q_A + 4q_B$.
Total Profit = Total Revenue – Total Cost
$P = p_A q_A + p_B q_B - (2q_A + 4q_B)$

$$\begin{aligned} P &= 16p_A - p_A^2 + p_A p_B + 24p_B + 2p_A p_B - 4p_B^2 - 32 + 2p_A - 2p_B - 96 - 8p_A + 16p_B \\ &= -p_A^2 - 4p_B^2 + 3p_A p_B + 10p_A + 38p_B - 128 \end{aligned}$$

$$\begin{cases} \dfrac{\partial P}{\partial p_A} = -2p_A + 3p_B + 10 \\ \dfrac{\partial P}{\partial p_B} = 3p_A - 8p_B + 38 \end{cases}$$

Critical point: $p_A = \dfrac{194}{7}$, $p_B = \dfrac{106}{7}$

$\dfrac{\partial^2 P}{\partial p_A^2} = -2$, $\dfrac{\partial^2 P}{\partial p_B^2} = -8$, $\dfrac{\partial^2 P}{\partial p_B \partial p_A} = 3$

At $p_A = \dfrac{194}{7}$, $p_B = \dfrac{106}{7}$, then $D = (-2)(-8) - 3^2 = 7 > 0$ and $\dfrac{\partial^2 P}{\partial p_A^2} = -2 < 0$; thus relative maximum at

$p_A = \dfrac{194}{7}$, $p_B = \dfrac{106}{7}$.

$q_A = 16 - \dfrac{194}{7} + \dfrac{106}{7} = \dfrac{24}{7}$

$q_B = 24 + 2\left(\dfrac{194}{7}\right) - 4\left(\dfrac{106}{7}\right) = \dfrac{132}{7}$

So, $\dfrac{24}{7} \approx 3$ of A and $\dfrac{132}{7} \approx 19$ of B should be sold.

27. $c = \dfrac{3}{2}q_A^2 + 3q_B^2$, $p_A = 60 - q_A^2$, $p_B = 72 - 2q_B^2$

Total Profit = Total Revenue – Total Cost

$P = (p_A q_A + p_B q_B) - c$

$P = 60q_A - q_A^3 + 72q_B - 2q_B^3 - \left(\dfrac{3}{2}q_A^2 + 3q_B^2\right)$

$$\begin{cases} \dfrac{\partial P}{\partial q_A} = 60 - 3q_A - 3q_A^2 = 3(5 + q_A)(4 - q_A) \\ \dfrac{\partial P}{\partial q_B} = 72 - 6q_B - 6q_B^2 = 6(4 + q_B)(3 - q_B) \end{cases}$$

Since we want $q_A \geq 0$ and $q_B \geq 0$, the critical point occurs when $q_A = 4$ and $q_B = 3$.

$\dfrac{\partial^2 P}{\partial q_A^2} = -3 - 6q_A$, $\dfrac{\partial^2 P}{\partial q_B^2} = -6 - 12q_B$, $\dfrac{\partial^2 P}{\partial q_B \partial q_A} = 0$. When $q_A = 4$ and $q_B = 3$, then $D = (-27)(-42) - 0^2 > 0$

and $\dfrac{\partial^2 P}{\partial q_A^2} = -27 < 0$; thus relative maximum at $q_A = 4$, $q_B = 3$.

28. $c = 2(q_A + q_B + q_A q_B)$,

Total Profit = Total Revenue – Total Cost

$$\begin{aligned} P &= (p_A q_A + p_B q_B) - c \\ &= p_A(20 - 2p_A) + p_B(10 - p_B) - [20 - 2p_A + 10 - p_B + (20 - 2p_A)(10 - p_B)] \\ &= -2p_A^2 - p_B^2 - 2p_A p_B + 42p_A + 31p_B + 230 \end{aligned}$$

$$\begin{cases} \dfrac{\partial P}{\partial p_A} = -4p_A - 2p_B + 42 \\ \dfrac{\partial P}{\partial p_B} = -2p_A - 2p_B + 31 \end{cases}$$

Critical point: $p_A = \frac{11}{2}$, $p_B = 10$

$$\frac{\partial^2 P}{\partial p_A^2} = -4, \frac{\partial^2 P}{\partial p_B^2} = -2, \frac{\partial^2 P}{\partial p_B \partial p_A} = -2$$

When $p_A = \frac{11}{2}$, $p_B = 10$, then $D = (-4)(-2) - (-2)^2 = 4 > 0$, and $\frac{\partial^2 p}{\partial p_A^2} = -4 < 0$, so the maximum profit occurs when $p_A = 5.5$ and $p_B = 10$. At these prices, $q_A = 9$, $q_B = 0$, and the total profit is 40.5.

29. Refer to the diagram in the text.
$xyz = 6$
$C = 3xy + 2[1(xz)] + 2[0.5(yz)]$
Note that $z = \frac{6}{xy}$. Thus

$$C = 3xy + 2xz + yz = 3xy + 2x\left(\frac{6}{xy}\right) + y\left(\frac{6}{xy}\right) = 3xy + \frac{12}{y} + \frac{6}{x}$$

$$\begin{cases} \dfrac{\partial C}{\partial x} = 3y - \dfrac{6}{x^2} = 0 \\ \dfrac{dC}{dy} = 3x - \dfrac{12}{y^2} = 0 \end{cases}$$

A critical point occurs at $x = 1$ and $y = 2$. Thus $z = 3$.

$$\frac{\partial^2 C}{\partial x^2} = \frac{12}{x^3}, \frac{\partial^2 C}{\partial y^2} = \frac{24}{y^3}, \frac{\partial^2 C}{\partial x \partial y} = 3.$$

When $x = 1$ and $y = 2$, then $d = (12)(3) - (3)^2 = 27 > 0$ and $\frac{\partial^2 C}{\partial x^2} = 12 > 0$. Thus we have a minimum. The dimensions should be 1 ft by 2 ft by 3 ft.

30. $p = 92 - q_A - q_B$, $c_A = 10q_A$, $c_B = 0.5q_B^2$
Since Profit = Total Revenue – Total Cost, then
Profit of A $= pq_A - c_A$ and
Profit of B $= pq_B - c_B$.
Thus profit P of monopoly is

$$\begin{aligned} P &= pq_A - c_A + pq_B - c_B \\ &= p(q_A + q_B) - c_A - c_B \\ &= (92 - q_A - q_B)(q_A + q_B) - 10q_A - 0.5q_B^2 \\ &= 82q_A + 92q_B - q_A^2 - 2q_Aq_B - 1.5q_B^2 \end{aligned}$$

$$\begin{cases} \dfrac{\partial P}{\partial q_A} = 82 - 2q_A - 2q_B = 0 \\ \dfrac{\partial P}{\partial q_B} = 92 - 2q_A - 3q_B = 0 \end{cases}$$

Critical point: $q_A = 31,\ q_B = 10$

$\dfrac{\partial^2 P}{\partial q_A^2} = -2,\ \dfrac{\partial^2 P}{\partial q_B^2} = -3,\ \dfrac{\partial^2 P}{\partial q_B \partial q_A} = -2$

When $q_A = 31$ and $q_B = 10$, then

$D = (-2)(-3) - (-2)^2 = 2 > 0$ and

$\dfrac{\partial^2 P}{\partial q_A^2} = -2 < 0$; thus relative maximum at

$q_A = 31,\ q_B = 10$.

31. $y = 2 - x$

$f(x, y) = x^2 + 3(2 - x)^2 + 9$

Setting the derivative equal to 0 gives

$2x + 6(2 - x)(-1) = 0$.

$2x - 12 + 6x = 0$, $8x - 12 = 0$,

$8x = 12$, or $x = \dfrac{3}{2}$. The second-derivative is

$8 > 0$, so we have a relative minimum. If $x = \dfrac{3}{2}$,

then $y = \dfrac{1}{2}$. Thus there is a relative minimum at

$\left(\dfrac{3}{2}, \dfrac{1}{2}\right)$.

32. $y = \dfrac{x-10}{4}$

$f(x, y) = x^2 + 4\left(\dfrac{x-10}{4}\right)^2 + 6$

Setting the derivative equal to 0 gives

$2x + 4(2)\left(\dfrac{x-10}{4}\right)\left(\dfrac{1}{4}\right) = 0$, from which $x = 2$.

The second-derivative is $\dfrac{5}{2} > 0$,so we have a

relative minimum If $x = 2$, then $y = -2$. Thus at

$(2, -2)$ there is a relative minimum

33. $c = q_A^2 + 3q_B^2 + 2q_A q_B + aq_A + bq_B + d$

We are given that $(q_A, q_B) = (3, 1)$ is a critical point.

$$\begin{cases} \dfrac{\partial c}{\partial q_A} = 2q_A + 2q_B + a = 0 \\ \dfrac{\partial c}{\partial q_B} = 6q_B + 2q_A + b = 0 \end{cases}$$

Substituting the given values for q_A and q_B into both equations gives $a = -8$ and $b = -12$. Since

$c = 15$ when $q_A = 3$ and $q_B = 1$, from the joint-cost function we have

$15 = 3^2 + 3\left(1^2\right) + 2(3)(1) + (-8)(3) + (-12) + d$,

$15 = -18 + d$, $33 = d$. Thus $a = -8$, $b = -12$, $d = 33$.

34. $D(a, b) > 0$

$f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2 > 0$

$f_{xx}(a, b) f_{yy}(a, b) > (f_{xy}(a, b))^2 \ge 0$

a. Since the product $f_{xx}(a, b) f_{yy}(a, b)$ is positive, $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. That is $f_{xx}(a, b) < 0$ if and only if $f_{yy}(a, b) < 0$.

b. Since the product $f_{xx}(a, b) f_{yy}(a, b)$ is positive, $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. That is $f_{xx}(a, b) > 0$ if and only if $f_{yy}(a, b) > 0$.

35. a. Profit = Total Revenue – Total Cost

$P = p_A q_A + p_B q_B -$ total cost

$= \left(35 - 2q_A^2 + q_B\right)q_A + (20 - q_B + q_A)q_B - \left(-8 - 2q_A^3 + 3q_A q_B + 30q_A + 12q_B + \frac{1}{2}q_A^2\right)$

$P = 5q_A - \frac{1}{2}q_A^2 - q_A q_B + 8q_B - q_B^2 + 8$

$$\begin{cases} \frac{\partial P}{\partial q_A} = 5 - q_A - q_B = 0 \\ \frac{\partial P}{\partial q_B} = -q_A + 8 - 2q_B = 0 \end{cases}$$

Critical point: $q_A = 2$, $q_B = 3$

$\frac{\partial^2 P}{\partial q_A^2} = -1$, $\frac{\partial^2 P}{\partial q_B^2} = -2$, $\frac{\partial^2 P}{\partial q_B \partial q_A} = -1$

At $q_A = 2$ and $q_B = 3$, then $D = (-1)(-2) - (-1)^2 = 1 > 0$ and $\frac{\partial^2 P}{\partial q_A^2} = -1 < 0$; thus there is a relative maximum profit for 2 units of A and 3 units of B.

b. Substituting $q_A = 2$ and $q_B = 3$ into the formulas for p_A, p_B, and P gives a selling price for A of 30, a selling price for B of 19, and a relative maximum profit of 25.

36. $P = 300\left[\frac{7x}{2+x} + \frac{4y}{5+y}\right] - x - y$

$$\begin{cases} \frac{\partial P}{\partial x} = 300 \cdot \frac{(2+x)(7) - 7x}{(2+x)^2} - 1 = \frac{4200}{(2+x)^2} - 1 = 0 \\ \frac{\partial P}{\partial y} = 300 \cdot \frac{(5+y)(4) - 4y}{(5+y)^2} - 1 = \frac{6000}{(5+y)^2} - 1 = 0 \end{cases}$$

$4200 = (2+x)^2$

$\pm\sqrt{4200} = 2 + x$

$x = -2 \pm \sqrt{4200} = -2 \pm 10\sqrt{42}$

$6000 = (5+y)^2$

$\pm\sqrt{6000} = 5 + y$

$y = -5 \pm \sqrt{6000} = -5 \pm 20\sqrt{15}$

The values of *x* and *y* must be nonnegative.

Critical point: $x = 10\sqrt{42} - 2$, $y = 20\sqrt{15} - 5$

$\frac{\partial^2 P}{\partial x^2} = -\frac{8400}{(2+x)^3}$, $\frac{\partial^2 P}{\partial y^2} = -\frac{12{,}000}{(5+y)^3}$, $\frac{\partial^2 P}{\partial y \partial x} = 0$

At $x = 10\sqrt{42} - 2$ and $y = 20\sqrt{15} - 5$, then $D \approx (0.031)(0.026) - 0^2 > 0$ and $\frac{\partial^2 P}{\partial x^2} \approx -0.031 < 0$.

Thus relative maximum profit at $x = 10\sqrt{42} - 2 \approx 62.81$, $y = 20\sqrt{15} - 5 \approx 72.46$.

37. a. $P = 5T\left(1 - e^{-x}\right) - 20x - 0.1T^2$

b. $\frac{\partial P}{\partial T} = 5\left(1 - e^{-x}\right) - 0.2T$

$\frac{\partial P}{\partial x} = 5Te^{-x} - 20$

At the point $(T, x) = (20, \ln 5)$,

$$\frac{\partial P}{\partial T} = 5\left(1-e^{-\ln 5}\right) - 0.2(20) = 5\left(1-\frac{1}{5}\right) - 4 = 0$$

$$\frac{\partial P}{\partial x} = 5(20)e^{-\ln 5} - 20 = 100\left(\frac{1}{5}\right) - 20 = 0$$

Thus (20, ln 5) is a critical point. In a similar fashion we verify that $\left(5, \ln\frac{5}{4}\right)$ is a critical point.

c. $\dfrac{\partial^2 P}{\partial T^2} = -0.2$, $\dfrac{\partial^2 P}{\partial x^2} = -5Te^{-x}$, $\dfrac{\partial^2 P}{\partial T \partial x} = 5e^{-x}$

At (20, ln 5),

$$D = (-0.2)\left[-5(20)e^{-\ln 5}\right] - \left(5e^{-\ln 5}\right)^2 = 20\left(\frac{1}{5}\right) - \left[5\left(\frac{1}{5}\right)\right]^2 = 3 > 0,$$

and $\dfrac{\partial^2 P}{\partial T^2} = -0.2 < 0$. Thus we get a relative maximum at (20, ln 5).

At $\left(5, \ln\frac{5}{4}\right)$,

$D = (-0.2)\left[-5(5)e^{-\ln\left(\frac{5}{4}\right)}\right] - \left[5e^{-\ln\left(\frac{5}{4}\right)}\right]^2 = 5\left(\frac{4}{5}\right) - \left[5\left(\frac{4}{5}\right)\right]^2 = -12 < 0$, so there is no relative extremum at $\left(5, \ln\frac{5}{4}\right)$.

Problems 17.7

1. $f(x, y) = x^2 + 4y^2 + 6,\ 2x - 8y = 20$

$F(x, y, \lambda) = x^2 + 4y^2 + 6 - \lambda(2x - 8y - 20)$

$$\begin{cases} F_x = 2x - 2\lambda = 0 & (1) \\ F_y = 8y + 8\lambda = 0 & (2) \\ F_\lambda = -2x + 8y + 20 = 0 & (3) \end{cases}$$

From (1), $x = \lambda$; from (2), $y = -\lambda$. Substituting $x = \lambda$ and $y = -\lambda$ into (3) gives $-2\lambda - 8\lambda + 20 = 0$, $-10\lambda = -20$, so $\lambda = 2$. Thus $x = 2$ and $y = -2$. Critical point of F: (2, –2, 2). Critical point of f: (2, –2).

2. $f(x, y) = 3x^2 - 2y^2 + 9,\ x + y = 1$

$F(x, y, \lambda) = 3x^2 - 2y^2 + 9 - \lambda(x + y - 1)$

$$\begin{cases} F_x = 6x - \lambda = 0 & (1) \\ F_y = -4y - \lambda = 0 & (2) \\ F_\lambda = -x - y + 1 = 0 & (3) \end{cases}$$

From (1), $x = \dfrac{\lambda}{6}$; from (2), $y = -\dfrac{\lambda}{4}$.

Substituting $x = \dfrac{\lambda}{6}$ and $y = -\dfrac{\lambda}{4}$ into (3) gives $-\dfrac{\lambda}{6} + \dfrac{\lambda}{4} + 1 = 0$, from which $\lambda = -12$. Thus $x = -2$ and $y = 3$.

Critical point of F: (–2, 3, –12). Critical point of f: (–2, 3).

3. $f(x, y, z) = x^2 + y^2 + z^2,\ 2x + y - z = 9$

$F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(2x + y - z - 9)$

$$\begin{cases} F_x = 2x - 2\lambda = 0 & (1) \\ F_y = 2y - \lambda = 0 & (2) \\ F_z = 2z + \lambda = 0 & (3) \\ F_\lambda = -2x - y + z + 9 = 0 & (4) \end{cases}$$

From (1), $x = \lambda$; from (2), $y = \frac{\lambda}{2}$; from (3), $z = -\frac{\lambda}{2}$. Substituting into (4) gives $-2\lambda - \frac{\lambda}{2} + \left(\frac{-\lambda}{2}\right) + 9 = 0$, $-6\lambda + 18 = 0$, so $\lambda = 3$. Thus $x = 3$, $y = \frac{3}{2}$, $z = -\frac{3}{2}$. Critical point of F: $\left(3, \frac{3}{2}, -\frac{3}{2}, 3\right)$. Critical point of f: $\left(3, \frac{3}{2}, -\frac{3}{2}\right)$.

4. $f(x, y, z) = x + y + z,\ xyz = 8$

$F(x, y, z, \lambda) = x + y + z - \lambda(xyz - 8)$

$$\begin{cases} F_x = 1 - \lambda yz = 0 & (1) \\ F_y = 1 - \lambda xz = 0 & (2) \\ F_z = 1 - \lambda xy = 0 & (3) \\ F_\lambda = -xzy + 8 = 0 & (4) \end{cases}$$

From (1) and (2), $\lambda yz = \lambda xz$, so $y = x$. From (2) and (3), $\lambda xz = \lambda xy$, so $y = z$. Therefore $x = y = z$, so from (4), $x = y = z = 2$. Hence, Critical point of f is (2, 2, 2). Note that it is not necessary to determine λ.

5. $f(x, y, z) = 2x^2 + xy + y^2 + z,\ x + 2y + 4z = 3$

$F(x, y, z, \lambda)$

$= 2x^2 + xy + y^2 + z - \lambda(x + 2y + 4z - 3)$

$$\begin{cases} F_x = 4x + y - \lambda = 0 \\ F_y = x + 2y - 2\lambda = 0 \\ F_z = 1 - 4\lambda = 0 \\ F_\lambda = -x - 2y - 4z - 3 = 0 \end{cases}$$

From the third equation we have $\lambda = \frac{1}{4}$. Substituting this value into the first two equations and then eliminating y gives $x = 0$ and $y = \frac{1}{4}$. Finally, solving for z in the last equation gives $z = -\frac{7}{8}$.

Critical point of F: $\left(0, \frac{1}{4}, -\frac{7}{8}, \frac{1}{4}\right)$

Critical point of f: $\left(0, \frac{1}{4}, -\frac{7}{8}\right)$

6. $f(x, y, z) = xyz^2$, $x - y + z = 20$ $\left(xyz^2 \neq 0\right)$

$f(x, y, z, \lambda) = xyz^2 - \lambda(x - y + z - 20)$

$$\begin{cases} F_x = yz^2 - \lambda = 0 & (1) \\ F_y = xz^2 + \lambda = 0 & (2) \\ F_z = 2xyz - \lambda = 0 & (3) \\ F_\lambda = -x + y - z + 20 = 0 & (4) \end{cases}$$

From (1) and (2), $y = -x$, From (1) and (3), $z = 2x$. Hence from (4), $x = 5$, so $y = -5$ and $z = 10$. Critical point of f is (5, –5, 10). Note that it is not necessary to determine λ.

7. $f(x, y, z) = xyz$, $x + y + z = 1$ $(xyz \neq 0)$

$F(x, y, z, \lambda) = xyz - \lambda(x + y + z - 1)$

$$\begin{cases} F_x = yz - \lambda = 0 & (1) \\ F_y = xz - \lambda = 0 & (2) \\ F_z = xy - \lambda = 0 & (3) \\ F_\lambda = -x - y - z + 1 = 0 & (4) \end{cases}$$

From (1) and (2), $y = x$. From (1) and (3), $x = z$. Hence from (4) $x = y = z = \frac{1}{3}$. Critical point of f is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Note that it is not necessary to determine λ.

8. $f(x, y, z) = x^2 + y^2 + z^2$, $x + y + z = 3$

$F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x + y + z - 3)$

$$\begin{cases} F_x = 2x - \lambda = 0 & (1) \\ F_y = 2y - \lambda = 0 & (2) \\ F_z = 2z - \lambda = 0 & (3) \\ F_\lambda = -x - y - z + 3 = 0 & (4) \end{cases}$$

From (1)–(3), $x = y = z = \frac{\lambda}{2}$. Substituting into (4), $-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} + 3 = 0$, so $\lambda = 2$. Thus $x = 1,\ y = 1,\ z = 1$. Critical point of F: $(1, 1, 1, 2)$. Critical point of f: $(1, 1, 1)$.

9. $f(x, y, z) = x^2 + 2y - z^2$, $2x - y = 0$, $y + z = 0$

Since there are two constraints, two Lagrange multipliers are used.

$F(x, y, z, \lambda_1, \lambda_2) = x^2 + 2y - z^2 - \lambda_1(2x - y) - \lambda_2(y + z)$

$$\begin{cases} F_x = 2x - 2\lambda_1 = 0 & (1) \\ F_y = 2 + \lambda_1 - \lambda_2 = 0 & (2) \\ F_z = -2z - \lambda_2 = 0 & (3) \\ F_{\lambda_1} = -2x + y = 0 & (4) \\ F_{\lambda_2} = -y - z = 0 & (5) \end{cases}$$

From (1), $x = \lambda_1$. From (3), $z = -\frac{\lambda_2}{2}$. From (4) and (5), $2x = -z$, so $\lambda_1 = \frac{\lambda_2}{4}$. Substituting $\lambda_1 = \frac{\lambda_2}{4}$ into (2) yields $\lambda_2 = \frac{8}{3}$. Thus $\lambda_1 = \frac{2}{3}$, $x = \frac{2}{3}$, and $z = -\frac{4}{3}$. From (5), $y = -z$ and hence $y = \frac{4}{3}$. Critical point of *f*: $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$

10. $f(x, y, z) = x^2 + y^2 + z^2$, $x + y + z = 4$, $x - y + z = 4$

Since there are two constraints, two Lagrange multipliers are used.

$F(x, y, z, \lambda_1, \lambda_2) = x^2 + y^2 + z^2 - \lambda_1(x + y + z - 4) - \lambda_2(x - y + z - 4)$

$$\begin{cases} F_x = 2x - \lambda_1 - \lambda_2 = 0 & (1) \\ F_y = 2y - \lambda_1 + \lambda_2 = 0 & (2) \\ F_z = 2z - \lambda_1 - \lambda_2 = 0 & (3) \\ F_{\lambda_1} = -x - y - z + 4 = 0 & (4) \\ F_{\lambda_2} = -x + y - z + 4 = 0 & (5) \end{cases}$$

From (4) and (5), $y = 0$. From (1) and (3), $z = x$. Substituting into (5) gives $x = 2$. Thus $z = 2$.
Critical point of *f*: $(2, 0, 2)$

11. $f(x, y, z) = xy^2z$, $x + y + z = 1$, $x - y + z = 0$ $(xyz \neq 0)$

Since there are two constraints, two Lagrange multipliers are used.

$F(x, y, z, \lambda_1, \lambda_2) = xy^2z - \lambda_1(x + y + z - 1) - \lambda_2(x - y + z)$

$$\begin{cases} F_x = y^2z - \lambda_1 - \lambda_2 = 0 & (1) \\ F_y = 2xyz - \lambda_1 + \lambda_2 = 0 & (2) \\ F_z = xy^2 - \lambda_1 - \lambda_2 = 0 & (3) \\ F_{\lambda_1} = -x - y - z + 1 = 0 & (4) \\ F_{\lambda_2} = -x + y - z = 0 & (5) \end{cases}$$

Subtracting (3) from (1) gives $y^2z - xy^2 = 0$, so $x = z$ (since $xy^2z \neq 0$). Subtracting (5) from (4) gives $-2y + 1 = 0$, so $y = \frac{1}{2}$. Substituting $z = x$ and $y = \frac{1}{2}$ in (5) gives $-2x + \frac{1}{2} = 0$, so $x = \frac{1}{4}$. Thus, $z = \frac{1}{4}$. Critical point of *f*: $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

12. $f(x, y, z, w) = x^2 + 2y^2 + 3z^2 - w^2$, $4x + 3y + 2z + w = 10$

$F(x, y, z, w, \lambda) = x^2 + 2y^2 + 3z^2 - w^2 - \lambda(4x + 3y + 2z + w - 10)$

$$\begin{cases} F_x = 2x - 4\lambda = 0 \\ F_y = 4y - 3\lambda = 0 \\ F_z = 6z - 2\lambda = 0 \\ F_w = -2w - \lambda = 0 \\ F_\lambda = -4x - 3y - 2z - w + 10 = 0 \end{cases}$$

Solving the first four equations for x, y, z, and w in terms of λ gives $x = 2\lambda$, $y = \frac{3\lambda}{4}$, $z = \frac{\lambda}{3}$, and $w = -\frac{\lambda}{2}$.

Substituting into the last equation gives $\lambda = \frac{24}{25}$. Thus $x = \frac{48}{25}$, $y = \frac{18}{25}$, $z = \frac{8}{25}$, and $w = -\frac{12}{25}$.

Critical point of F: $\left(\frac{48}{25}, \frac{18}{25}, \frac{8}{25}, -\frac{12}{25}, \frac{24}{25}\right)$

Critical point of f: $\left(\frac{48}{25}, \frac{18}{25}, \frac{8}{25}, -\frac{12}{25}\right)$

13. We minimize $c = f(q_1, q_2) = 0.1q_1^2 + 7q_1 + 15q_2 + 1000$ subject to the constraint $q_1 + q_2 = 100$.

$F(q_1, q_2, \lambda) = 0.1q_1^2 + 7q_1 + 15q_2 + 1000 - \lambda(q_1 + q_2 - 100)$

$$\begin{cases} F_{q_1} = 0.2q_1 + 7 - \lambda = 0 & (1) \\ F_{q_2} = 15 - \lambda = 0 & (2) \\ F_\lambda = -q_1 - q_2 + 100 = 0 & (3) \end{cases}$$

From (2), $\lambda = 15$. Substituting $\lambda = 15$ into (1) gives $0.2q_1 + 7 - 15 = 0$, so $q_1 = 40$. Substituting $q_1 = 40$ into (3) gives $-40 - q_2 + 100 = 0$, so $q_2 = 60$. Thus $\lambda = 15$, $q_1 = 40$, and $q = 60$. Thus plant 1 should produce 40 units and plant 2 should produce 60 units.

14. We minimize $c = 3q_1^2 + q_1q_2 + 2q_2^2$ subject to the constraint $q_1 + q_2 = 200$.

$F(q_1, q_2, \lambda) = 3q_1^2 + q_1q_2 + 2q_2^2 - \lambda(q_1 + q_2 - 200)$

$$\begin{cases} F_{q_1} = 6q_1 + q_2 - \lambda = 0 & (1) \\ F_{q_2} = q_1 + 4q_2 - \lambda = 0 & (2) \\ F_\lambda = -q_1 - q_2 + 200 = 0 & (3) \end{cases}$$

Eliminating λ from (1) and (2) yields $q_1 = \frac{3}{5}q_2$. Substituting $q_1 = \frac{3}{5}q_2$ into (3) yields $q_2 = 125$ and thus $q_1 = 75$. Thus plant 1 should produce 75 units and plant 2 should produce 125 units.

15. We maximize $f(l, k) = 12l + 20k - l^2 - 2k^2$ subject to the constraint $4l + 8k = 88$.

$F(l, k, \lambda) = 12l + 20k - l^2 - 2k^2 - \lambda(4l + 8k - 88)$

$$\begin{cases} F_l = 12 - 2l - 4\lambda = 0 & (1) \\ F_k = 20 - 4k - 8\lambda = 0 & (2) \\ F_\lambda = -4l - 8k + 88 = 0 & (3) \end{cases}$$

Eliminating λ from (1) and (2) yields $k = l - 1$. Substituting $k = l - 1$ into (3) yields $l = 8$, so $k = 7$. Therefore the greatest output is $f(8, 7) = 74$ units (when $l = 8$, $k = 7$).

16. We maximize $f(l,k) = 20l + 25k - l^2 - 3k^2$ subject to the constraint $2l + 4k = 50$.

$F(l, k, \lambda) = 20l + 25k - l^2 - 3k^2 - \lambda(2l + 4k - 50)$

$$\begin{cases} F_l = 20 - 2l - 2\lambda = 0 & (1) \\ F_k = 25 - 6k - 4\lambda = 0 & (2) \\ F_\lambda = -2l - 4k + 50 = 0 & (3) \end{cases}$$

From (1), $l = 10 - \lambda$ and from (2), $k = \frac{25}{6} - \frac{2}{3}\lambda$. Substituting these expressions for l and k into (3) yields $\lambda = -\frac{20}{7}$. Thus $l = \frac{90}{7}$ and $k = \frac{85}{14}$. Therefore the greatest output is $f\left(\frac{90}{7}, \frac{85}{14}\right) = \frac{3725}{28} \approx 133$ units (when $l = \frac{90}{7}, k = \frac{85}{14}\Big)$.

17. We maximize $P(x, y) = 8x^{1/4}y^{3/4} - x - y$ subject to the constraint $x + y = 20{,}000$.

$F(x, y, \lambda) = 8x^{1/4}y^{3/4} - x - y - \lambda(x + y - 20{,}000)$

$$\begin{cases} F_x = 2x^{-3/4}y^{3/4} - 1 - \lambda = 0 & (1) \\ F_y = 6x^{1/4}y^{-1/4} - 1 - \lambda = 0 & (2) \\ F_\lambda = -x - y + 20{,}000 = 0 & (3) \end{cases}$$

Solving (2) for λ and substituting in (1) gives $2x^{-3/4}y^{3/4} - 6x^{1/4}y^{-1/4} = 0$, $2x^{-3/4}y^{3/4} = 6x^{1/4}y^{-1/4}$, $y = 3x$. Substituting for y in (3) gives $-4x + 20{,}000 = 0$, so $x = 5{,}000$, from which $y = 15{,}000$. Thus each month \$5000 should be spent on newspaper advertising and \$15,000 on TV advertising.

18. We maximize $f(l, k) = 6l^{\frac{2}{5}}k^{\frac{3}{5}}$ subject to the constraint $25l + 69k = 25{,}875$.

$F(l, k, \lambda) = 6l^{\frac{2}{5}}k^{\frac{3}{5}} - \lambda(25l + 69k - 25{,}875)$

$$\begin{cases} F_l = \frac{12}{5}l^{-\frac{3}{5}}k^{\frac{3}{5}} - 25\lambda = 0 \\ F_k = \frac{18}{5}l^{\frac{2}{5}}k^{-\frac{2}{5}} - 69\lambda = 0 \\ F_\lambda = -25l - 69k + 25{,}875 = 0 \end{cases}$$

From the first two equations, $\frac{12}{5}l^{-\frac{3}{5}}k^{\frac{3}{5}} = 25\lambda$ and $\frac{18}{5}l^{\frac{2}{5}}k^{-\frac{2}{5}} = 69\lambda$. Thus, $\dfrac{\frac{12}{5}l^{-\frac{3}{5}}k^{\frac{3}{5}}}{\frac{18}{5}l^{\frac{2}{5}}k^{-\frac{2}{5}}} = \frac{25\lambda}{69\lambda} = \frac{25}{29}$, from which $k = \frac{25}{46}l$. Substituting this for k in the third equation and solving for l gives $l = 414$ so $k = 225$.
414 units of labor and 225 units of capital should be invested.

19. We minimize $B(x, y, z) = x^2 + y^2 + 2z^2$ subject to $x + y = 20$ and $y + z = 20$.
Since there are two constraints, two Lagrange multipliers are used.
$F(x, y, z, \lambda_1, \lambda_2) = x^2 + y^2 + 2z^2 - \lambda_1(x + y - 20) - \lambda_2(y + z - 20)$

$$\begin{cases} F_x = 2x - \lambda_1 = 0 & (1) \\ F_y = 2y - \lambda_1 - \lambda_2 = 0 & (2) \\ F_z = 4z - \lambda_2 = 0 & (3) \\ F_{\lambda_1} = -x - y + 20 = 0 & (4) \\ F_{\lambda_2} = -y - z + 20 = 0 & (5) \end{cases}$$

Eliminating y from (4) and (5) gives $x = z$. From (1) and (3), $\lambda_1 = 2x$ and $\lambda_2 = 4z$. Substituting in (2) we have $2y - 2x - 4z = 0$, $2y - 2x - 4x = 0$, $2y - 6x = 0$, $y = 3x$. Substituting in (5) gives $-(3x) - x + 20 = 0$, so $x = 5$. Thus $z = 5$ and $y = 15$. Therefore, $x = 5$, $y = 15$, $z = 5$.

20. **a.** $P = TR - TC = 64q - (8l + 16k)$

$$= 64\left[\frac{65 - 4(l-4)^2 - 2(k-5)^2}{16}\right] - 8l - 16k$$

$$P = -196 - 16l^2 + 120l - 8k^2 + 64k$$

b. $P_l = -32l + 120 = 0 \Rightarrow l = \frac{15}{4}$

$P_k = -16k + 64 = 0 \Rightarrow k = 4$

Thus there is one critical point: $(l, k) = \left(\frac{15}{4}, 4\right)$

Second-Derivative Test: $P_{ll} = -32$, $P_{kk} = -16$, $P_{lk} = 0$.

Thus $D(l, k) = P_{ll}P_{kk} - [P_{lk}]^2 = (-32)(-16) - 0^2 = 512$. At $\left(\frac{15}{4}, 4\right)$, $D\left(\frac{15}{4}, 4\right) = 512 > 0$ and $P_{ll} = -32 < 0$.

Thus there is a relative maximum at $l = \frac{15}{4}$, $k = 4$. Substituting these values into the profit function gives a relative maximum profit of \$157.00.

c. $F(l, k, q, \lambda) = 64q - 8l - 16k - \lambda\left[16q - 65 + 4(l-4)^2 + 2(k-5)^2\right]$

$$\begin{cases} F_l = -8 - 8\lambda(l-4) = 0 & (1) \\ F_k = -16 - 4\lambda(k-5) = 0 & (2) \\ F_q = 64 - 16\lambda = 0 & (3) \\ F_\lambda = -16q + 65 - 4(l-4)^2 - 2(k-5)^2 = 0 & (4) \end{cases}$$

From (3), $\lambda = 4$. Substituting $\lambda = 4$ into (1) gives $-8 - 32(l - 4) = 0$, so $l = \frac{15}{4}$. Similarly, from (2) we get $k = 4$. Substituting for l and k in (4) gives $q = \frac{251}{64}$. Thus $(l, k, q) = \left(\frac{15}{4}, 4, \frac{251}{64}\right)$.

21. $U = x^3y^3$, $p_X = 2$, $p_Y = 3$, $I = 48$ $\left(x^3y^3 \neq 0\right)$

We want to maximize $U = x^3y^3$ subject to $2x + 3y = 48$.

$F(x, y, \lambda) = x^3y^3 - \lambda(2x + 3y - 48)$

$$\begin{cases} F_x = 3x^2y^3 - 2\lambda = 0 & (1) \\ F_y = 3x^3y^2 - 3\lambda = 0 & (2) \\ F_\lambda = -2x - 3y + 48 = 0 & (3) \end{cases}$$

From (1), $\lambda = \frac{3}{2}x^2y^3$ and from (2), $\lambda = x^3y^2$. Thus $\frac{3}{2}x^2y^3 = x^3y^2$, so $x = \frac{3}{2}y$.

Substituting this expression for x into (3) yields $y = 8$. Hence $x = \left(\frac{3}{2}\right)8 = 12$.

22. $U = 40x - 8x^2 + 2y - y^2$, $p_X = 4$, $p_Y = 6$, $I = 100$

We want to maximize $U = 40x - 8x^2 + 2y - y^2$ subject to $4x + 6y = 100$.

$$F(x, y, \lambda) = 40x - 8x^2 + 2y - y^2 - \lambda(4x + 6y - 100) \begin{cases} F_x = 40 - 16x - 4\lambda = 0 \\ F_y = 2 - 2y - 6\lambda = 0 \\ F_\lambda = -4x - 6y + 100 = 0 \end{cases}$$

From the first equation, $x = \frac{5}{2} - \frac{\lambda}{4}$ and from the second equation $y = 1 - 3\lambda$.

Substituting these values into the third equation gives $\lambda = -\frac{84}{19}$. Thus $x = \frac{137}{38}$ and $y = \frac{271}{19}$.

23. $U = f(x, y, z) = xyz$
$p_X = p_Y = p_Z = 1$, $I = 100$
$(xyz \neq 0)$
We want to maximize $U = xyz$ subject to
$x + y + z = 100$.
$F(x, y, z, \lambda) = xyz - \lambda(x + y + z - 100)$

$$\begin{cases} F_x = yz - \lambda = 0 & (1) \\ F_y = xz - \lambda = 0 & (2) \\ F_z = xy - \lambda = 0 & (3) \\ F_\lambda = -x - y - z + 100 = 0 & (4) \end{cases}$$

From (1) and (2), $yz = xz$, so $y = x$. Similarly, from (1) and (3), $z = x$. Substituting $y = x$ and $z = x$ into (4) yields $x = \frac{100}{3}$. Thus $y = \frac{100}{3}$ and $z = \frac{100}{3}$.

24. To maximize $U = f(x, y)$ subject to the constraint $xp_X + yp_Y = I$, we consider
$F(x, y, \lambda) = f(x, y) - \lambda(xp_X + yp_Y - I)$.
For maximum satisfaction,
$F_x = f_x(x, y) - \lambda p_X = 0$ (1)
and
$F_y = f_y(x, y) - \lambda p_Y = 0$ (2)

From (1), $\lambda = \frac{f_x(x, y)}{p_X}$ and from (2), $\lambda = \frac{f_y(x, y)}{p_Y}$. Thus $\lambda = \frac{f_x(x, y)}{p_X} = \frac{f_y(x, y)}{p_Y}$

Since $f_x(x, y)$ represents change in total utility from a one unit change in X (which costs p_X), then $\frac{f_x(x, y)}{p_X}$ is the marginal utility of a dollar's worth of X. Likewise $\frac{f_y(x, y)}{p_Y}$ is the marginal utility of a dollar's worth of Y.

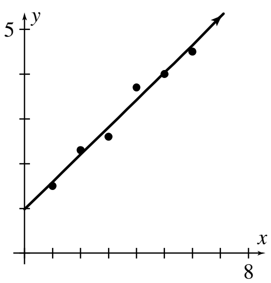
Thus maximum satisfaction is obtained when the consumer allocates the budget so that the marginal utility of a dollar's worth of X is equal to the marginal utility of a dollars worth of Y. Similarly, for $U = f(x, y, z, w)$ subject to the constraint $xp_X + yp_Y + zp_Z + wp_W = I$, U is maximized when

$$\lambda = \frac{f_x(x, y, z, w)}{p_X} = \frac{f_y(x, y, z, w)}{p_Y}$$
$$= \frac{f_z(x, y, z, w)}{p_Z} = \frac{f_w(x, y, z, w)}{p_W}.$$

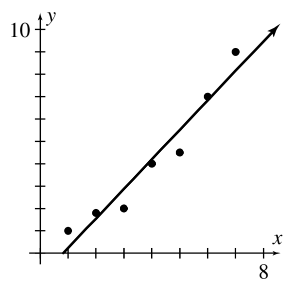
That is, U is maximized when the marginal utility of a dollar's worth of each of the products is the same.

Problems 17.8

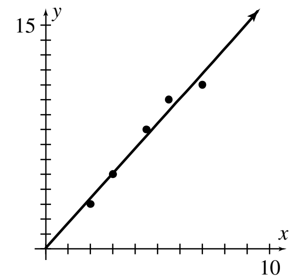
1. $n = 6$, $\Sigma x_i = 21$, $\Sigma y_i = 18.6$, $\Sigma x_i y_i = 75.7$, $\Sigma x_i^2 = 91$.
$a = 0.98$
$b = 0.61$
Thus $\hat{y} = 0.98 + 0.61x$. When $x = 3.5$, then $\hat{y} = 3.12$.



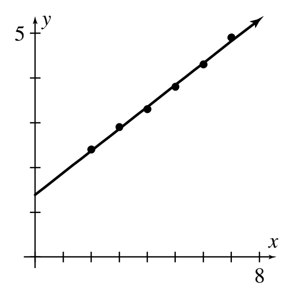
2. $n = 7$, $\Sigma x_i = 28$, $\Sigma y_i = 29.3$, $\Sigma x_i y_i = 154.1$, $\Sigma x_i^2 = 140$. $a = -1.09$, $b = 1.32$. Thus $\hat{y} = -1.09 + 1.32x$.
When $x = 3.5$, then $\hat{y} = 3.53$.



3. $n = 5$, $\Sigma x_i = 22$, $\Sigma y_i = 37$, $\Sigma x_i y_i = 189$, $\Sigma x_i^2 = 112.5$. $a = 0.057$, $b = 1.67$. Thus $\hat{y} = 0.057 + 1.67x$.
When $x = 3.5$, then $\hat{y} = 5.90$.



4. $n = 6$, $\Sigma x_i = 27$, $\Sigma y_i = 21.6$, $\Sigma x_i y_i = 105.8$, $\Sigma x_i^2 = 139$. $a = 1.39$, $b = 0.49$. Thus $\hat{y} = 1.39 + 0.49x$. When $x = 3.5$, then $\hat{y} = 3.12$.



5. $n = 6$, $\Sigma p_i = 250$, $\Sigma q_i = 322$, $\Sigma p_i q_i = 11,690$, $\Sigma p_i^2 = 13,100$.
$a = 80.5$
$b = -0.643$
Thus $\hat{q} = 80.5 - 0.643p$.

6. $n = 4$, $\sum x_i = 80$, $\sum y_i = 23.9$, $\sum x_i y_i = 498.4$, $\sum x_i^2 = 1920$, $a = 4.7$, $b = 0.06$.
Thus $\hat{y} = 4.7 + 0.06x$. When $x = 20$, then $\hat{y} = 5.9$.

7. $n = 4$, $\Sigma x_i = 160$, $\Sigma y_i = 420.8$, $\Sigma x_i y_i = 16,915.2$, $\Sigma x_i^2 = 7040$. $a = 100$, $b = 0.13$. Thus $\hat{y} = 100 + 0.13x$. When $x = 40$, then $\hat{y} = 105.2$.

8. $n = 4$, $\sum x_i = 539$, $\sum y_i = 569$, $\sum x_i y_i = 76,736$, $\sum x_i^2 = 72,691$, $a = 1.95$, $b = 1.04$.
Thus $\hat{y} = 1.95 + 1.04x$.

9.

Year (x)	1	2	3	4	5
Production (y)	10	15	16	18	21

$n = 5,\ \Sigma x_i = 15,\ \Sigma y_i = 80,\ \Sigma x_i y_i = 265,\ \Sigma x_i^2 = 55$.
$a = 8.5$
$b = 2.5$
Thus $\hat{y} = 8.5 + 2.5x$

10.

Year (x)	1	3	5	7
Index (y)	77	100	126	134

$n = 4,\ \Sigma x_i = 16,\ \Sigma y_i = 437,\ \Sigma x_i y_i = 1945$,

$\Sigma x_i^2 = 84$. $a = 69.85$, $b = 9.85$. Thus

$\hat{y} = 69.85 + 9.85x$.

11. a.

Year (x)	1	2	3	4	5
Quantity (y)	35	31	26	24	26

$n = 5,\ \Sigma x_i = 15,\ \Sigma y_i = 142,\ \Sigma x_i y_i = 401$,

$\Sigma x_i^2 = 55$. $a = 35.9$, $b = -2.5$. Thus

$\hat{y} = 35.9 - 2.5x$.

b.

Year (x)	−2	−1	0	1	2
Quantity (y)	35	31	26	24	26

$n = 5,\ \Sigma x_i = 0,\ \Sigma y_i = 142,\ \Sigma x_i y_i = -25$,

$\Sigma x_i^2 = 10$. $a = \dfrac{\Sigma y_i}{n} = 28.4$ and

$b = \dfrac{\Sigma x_i y_i}{\Sigma x_i^2} = -2.5$. Thus $\hat{y} = 28.4 - 2.5x$.

12.

Year (x)	−2	−1	0	1	2
Index (y)	357	380	403	434	462

$n = 5,\ \Sigma x_i = 0,\ \Sigma y_i = 2036,\ \Sigma x_i y_i = 264$,

$\Sigma x_i^2 = 10$. $a = \dfrac{\Sigma y_i}{n} = 407.2$ and

$b = \dfrac{\Sigma x_i y_i}{\Sigma x_i^2} = 26.4$. Thus $\hat{y} = 407.2 + 26.4x$.

Problems 17.9

1. $\displaystyle\int_0^3\int_0^4 x\,dy\,dx = \int_0^3 xy\Big|_0^4 dx = \int_0^3 4x\,dx = 2x^2\Big|_0^3 = 18$

2. $\displaystyle\int_1^4\int_0^3 y\,dy\,dx = \int_1^4 \frac{y^2}{2}\Big|_0^3 dx = \int_1^4 \frac{9}{2}dx = \frac{9x}{2}\Big|_1^4 = \frac{27}{2}$

3. $\displaystyle\int_0^1\int_0^1 xy\,dx\,dy = \int_0^1 \frac{x^2y}{2}\Big|_0^1 dy = \int_0^1 \frac{y}{2}dy = \frac{y^2}{4}\Big|_0^1 = \frac{1}{4}$

4. $\displaystyle\int_0^1\int_0^1 x^2y^2\,dy\,dx = \int_0^1 x^2\cdot\frac{y^3}{3}\Big|_0^1 dx$

$\displaystyle = \frac{1}{3}\int_0^1 x^2dx$

$\displaystyle = \frac{1}{3}\cdot\frac{x^3}{3}\Big|_0^1$

$\displaystyle = \frac{1}{9}$

5. $\displaystyle\int_1^3\int_1^2 \left(x^2 - y\right)dx\,dy = \int_1^3 \left(\frac{x^3}{3} - xy\right)\Bigg|_1^2 dy$

$\displaystyle = \int_1^3\left[\left(\frac{8}{3} - 2y\right) - \left(\frac{1}{3} - y\right)\right]dy = \int_1^3\left(\frac{7}{3} - y\right)dy$

$\displaystyle = \left(\frac{7}{3}y - \frac{y^2}{2}\right)\Bigg|_1^3 = \left(7 - \frac{9}{2}\right) - \left(\frac{7}{3} - \frac{1}{2}\right) = \frac{2}{3}$

6. $\displaystyle\int_{-2}^3\int_0^2 (y^2 - 2xy)dy\,dx = \int_{-2}^3\left(\frac{y^3}{3} - xy^2\right)\Bigg|_0^2 dx$

$\displaystyle = \int_{-2}^3\left[\left(\frac{8}{3} - 4x\right) - 0\right]dx = \int_{-2}^3\left(\frac{8}{3} - 4x\right)dx$

$\displaystyle = \left(\frac{8}{3}x - 2x^2\right)\Bigg|_{-2}^3 = (8 - 18) - \left(-\frac{16}{3} - 8\right)$

$\displaystyle = \frac{10}{3}$

7. $\displaystyle\int_0^1\int_0^2 (x + y)dy\,dx = \int_0^1\left(xy + \frac{y^2}{2}\right)\Bigg|_0^2 dx$

$\displaystyle = \int_0^1 (2x + 2)dx = \left(x^2 + 2x\right)\Big|_0^1 = 3$

8. $\displaystyle\int_0^3\int_0^x \left(x^2 + y^2\right)dy\,dx = \int_0^3\left(x^2y + \frac{y^3}{3}\right)\Bigg|_0^x dx$

$\displaystyle = \int_0^3\left(x^3 + \frac{x^3}{3}\right)dx = \int_0^3 \frac{4x^3}{3}dx = \frac{x^4}{3}\Big|_0^3 = 27$

9. $\int_2^3\int_0^{2x} y\,dy\,dx = \int_2^3 \frac{y^2}{2}\Big|_0^{2x} dx$
$= \int_2^3 2x^2 dx$
$= \frac{2}{3}x^3\Big|_2^3$
$= \frac{2}{3}(27-8)$
$= \frac{38}{3}$

10. $\int_1^2\int_0^{x-1} 2y\,dy\,dx = \int_1^2 y^2\Big|_0^{x-1} dx$
$= \int_1^2 (x-1)^2\,dx = \frac{(x-1)^3}{3}\Big|_1^2 = \frac{1}{3}$

11. $\int_0^1\int_{3x}^{x^2} 14x^2 y\,dy\,dx = \int_0^1\left(7x^2y^2\right)\Big|_{3x}^{x^2} dx$
$= \int_0^1\left(7x^6 - 63x^4\right)dx = \left(x^7 - \frac{63x^5}{5}\right)\Bigg|_0^1 = -\frac{58}{5}$

12. $\int_0^2\int_0^{x^2} xy\,dy\,dx = \int_0^2 \frac{xy^2}{2}\Big|_0^{x^2} dx$
$= \int_0^2 \frac{x^5}{2}\,dx = \frac{x^6}{12}\Big|_0^2 = \frac{16}{3}$

13. $\int_0^3\int_0^{\sqrt{9-x^2}} y\,dy\,dx = \int_0^3 \frac{y^2}{2}\Big|_0^{\sqrt{9-x^2}} dx$
$= \int_0^3\left(\frac{9-x^2}{2} - 0\right)dx = \frac{1}{2}\int_0^3(9-x^2)dx$
$= \frac{1}{2}\left(9x - \frac{x^3}{3}\right)\Bigg|_0^3 = \frac{1}{2}(27-9) - 0 = 9$

14. $\int_0^1\int_{y^2}^{y} x\,dx\,dy = \int_0^1 \frac{x^2}{2}\Big|_{y^2}^{y} dy$
$= \frac{1}{2}\int_0^1 (y^2 - y^4)dy$
$= \frac{1}{2}\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Bigg|_0^1$
$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$
$= \frac{1}{15}$

15. $\int_{-1}^1\int_x^{1-x} 3(x+y)dy\,dx = \int_{-1}^1 3\left(xy + \frac{y^2}{2}\right)\Bigg|_x^{1-x} dx$
$= \int_{-1}^1 3\left[x(1-x) + \frac{(1-x)^2}{2} - \left(x^2 + \frac{x^2}{2}\right)\right]dx$
$= \int_{-1}^1 3\left[x - \frac{5x^2}{2} + \frac{(1-x)^2}{2}\right]dx$
$= 3\left[\frac{x^2}{2} - \frac{5x^3}{6} - \frac{(1-x)^3}{6}\right]\Bigg|_{-1}^1$
$= 3\left[\frac{1}{2} - \frac{5}{6} - 0\right] - 3\left[\frac{1}{2} + \frac{5}{6} - \frac{4}{3}\right] = -1$

16. $\int_0^3\int_{y^2}^{3y} 5x\,dx\,dy = \int_0^3 \frac{5x^2}{2}\Big|_{y^2}^{3y} dy$
$= \int_0^3\left(\frac{45y^2}{2} - \frac{5y^4}{2}\right)dy$
$= \left(\frac{15y^3}{2} - \frac{y^5}{2}\right)\Bigg|_0^3 = \frac{405}{2} - \frac{243}{2} = 81$

17. $\int_0^1\int_0^y e^{x+y}dx\,dy = \int_0^1 e^{x+y}\Big|_0^y dy = \int_0^1\left(e^{2y} - e^y\right)dy$
$= \left[\frac{e^{2y}}{2} - e^y\right]\Bigg|_0^1 = \frac{e^2}{2} - e - \left(\frac{1}{2} - 1\right) = \frac{e^2}{2} - e + \frac{1}{2}$

18. $\int_0^1\int_0^1 e^{y-x}dx\,dy = \int_0^1 -e^{y-x}\Big|_0^1 dy$
$= \int_0^1 (-e^{y-1}+e^y)dy = (-e^{y-1}+e^y)\Big|_0^1$
$= (-e^0+e^1)-(-e^{-1}+e^0) = -1+e+e^{-1}-1$
$= -2+e+e^{-1}$

19. $\int_0^1\int_0^2\int_0^3 xy^2z^3dx\,dy\,dz = \int_0^1\int_0^2 \frac{1}{2}x^2y^2z^3\Big|_0^3 dy\,dz$
$= \frac{9}{2}\int_0^1\int_0^2 y^2z^3dy\,dz$
$= \frac{9}{2}\int_0^1 \frac{1}{3}y^3z^3\Big|_0^2 dz$
$= \frac{9}{2}\cdot\frac{8}{3}\int_0^1 z^3dz$
$= 12\cdot\frac{z^4}{4}\Big|_0^1$
$= 3$

20. $\int_0^1\int_0^x\int_0^{x+y} x^2dz\,dy\,dx = \int_0^1\int_0^x x^2z\Big|_0^{x+y} dy\,dx$
$= \int_0^1\int_0^x [x^2(x+y)-0]dy\,dx = \int_0^1\int_0^x (x^3+x^2y)dy\,dx$
$= \int_0^1 \left(x^3y+\frac{x^2y^2}{2}\right)\Bigg|_0^x dx = \int_0^1\left[\left(x^4+\frac{x^4}{2}\right)-0\right]dx$
$= \int_0^1 \frac{3x^4}{2}dx = \frac{3x^5}{10}\Big|_0^1 = \frac{3(1)^5}{10}-0 = \frac{3}{10}$

21. $\int_0^1\int_{x^2}^x\int_0^{xy} dz\,dy\,dx = \int_0^1\int_{x^2}^x z\Big|_0^{xy} dy\,dx$
$= \int_0^1\int_{x^2}^x xy\,dy\,dx = \int_0^1 \frac{xy^2}{2}\Big|_{x^2}^x dx$
$= \int_0^1\left[\frac{x^3}{2}-\frac{x^5}{2}\right]dx = \left[\frac{x^4}{8}-\frac{x^6}{12}\right]\Bigg|_0^1 = \frac{1}{24}$

22. $\int_1^e\int_{\ln x}^x\int_0^y dz\,dy\,dx = \int_1^e\int_{\ln x}^x z\Big|_0^y dy\,dx$
$= \int_1^e\int_{\ln x}^x y\,dy\,dx = \int_1^e \frac{y^2}{2}\Big|_{\ln x}^x dx = \int_1^e \frac{x^2}{2}-\frac{(\ln x)^2}{2}dx$
$= \left[\frac{x^3}{6}-\frac{1}{2}(x\ln^2 x-2x\ln x+2x)\right]\Bigg|_1^e$
$= \frac{e^3}{6}-\frac{e}{2}-\left(\frac{1}{6}-1\right) = \frac{e^3}{6}-\frac{e}{2}+\frac{5}{6}$

23. $P(0\le x\le 2,\ 1\le y\le 2) = \int_1^2\int_0^2 e^{-(x+y)}dx\,dy$
$= \int_1^2 -e^{-(x+y)}\Big|_0^2 dy = \int_1^2\left[-e^{-(2+y)}+e^{-y}\right]dy$
$= \left[e^{-(2+y)}-e^{-y}\right]\Big|_1^2 = e^{-4}-e^{-2}-e^{-3}+e^{-1}$

24. $P(1\le x\le 3,\ 2\le y\le 4)$
$= \int_2^4\int_1^3 6e^{-(2x+3y)}dx\,dy$
$= \int_2^4 (-3e^{-2x-3y})\Big|_1^3 dy$
$= \int_2^4 (-3e^{-6-3y}+3e^{-2-3y})dy$
$= (e^{-6-3y}-e^{-2-3y})\Big|_2^4$
$= e^{-18}-e^{-14}-e^{-12}+e^{-8}$

25. $P\left(x\ge\frac{1}{2},\ y\ge\frac{1}{3}\right) = \int_{1/3}^1\int_{1/2}^1 1\,dx\,dy$
$= \int_{1/3}^1 x\Big|_{1/2}^1 dy = \int_{1/3}^1\left(1-\frac{1}{2}\right)dy$
$= \int_{1/3}^1 \frac{1}{2}dy = \frac{1}{2}y\Big|_{1/3}^1 = \frac{1}{2}\left(1-\frac{1}{3}\right) = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$

26. $\int_0^1\int_0^1 \frac{1}{8}dx\,dy = \int_0^1 \frac{x}{8}\Big|_0^1 dy = \int_0^1 \frac{1}{8}dy = \frac{y}{8}\Big|_0^1 = \frac{1}{8}$

Chapter 17 Review Problems

1. $f_x(x,\ y) = \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2}$
$f_y(x,\ y) = \frac{1}{x^2+y^2}(2y) = \frac{2y}{x^2+y^2}$

2. $\frac{\partial P}{\partial l} = 3l^2 + 0 - (1)k = 3l^2 - k$

$\frac{\partial P}{\partial k} = 0 + 3k^2 - l(1) = 3k^2 - l$

3. $\frac{\partial z}{\partial x} = \frac{(x+y)(1) - x(1)}{(x+y)^2} = \frac{y}{(x+y)^2}$

Because $z = x(x+y)^{-1}$,

$\frac{\partial z}{\partial y} = x\left[(-1)(x+y)^{-2}(1)\right] = -\frac{x}{(x+y)^2}$.

4. $f_{p_B}(p_A, p_B) = 0 + 5(1-0) = 5$

5. $f(x, y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln\left(x^2+y^2\right)$

$\frac{\partial}{\partial y}[f(x, y)] = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$

6. $w = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$

$\frac{\partial w}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}}$

7. $w_x(x, y, z) = 2xyze^{x^2yz}$

$w_{xy}(x, y, z) = 2xz\left[y\left(e^{x^2yz}\cdot x^2z\right) + e^{x^2yz}\cdot 1\right]$

$= 2xze^{x^2yz}\left(x^2yz+1\right)$

8.

$f_x(x, y) = y\left[x\left(\frac{1}{xy}\cdot y\right) + \ln(xy)\cdot 1\right]$

$= y[1+\ln(xy)]$

$f_{xy}(x, y) = y\left[\frac{1}{xy}\cdot x\right] + [1+\ln(xy)]\cdot 1$

$= 1 + 1 + \ln(xy) = 2 + \ln(xy)$

9. $\frac{\partial}{\partial z}[f(x, y, z)]$

$= (x+y+z)(2z) + (x^2+y^2+z^2)(1)$

$= 3z^2 + 2z(x+y) + x^2 + y^2$

$\frac{\partial^2}{\partial z^2}[f(x, y, z)] = 6z + 2(x+y) = 2x + 2y + 6z$

10. $z = \left(x^2 - y\right)\left(y^2 - 2xy\right) = x^2y^2 - 2x^3y - y^3 + 2xy^2$

$\frac{\partial z}{\partial y} = 2x^2y - 2x^3 - 3y^2 + 4xy$

$\frac{\partial^2 z}{\partial y^2} = 2x^2 - 6y + 4x$

11. $w = e^{x+y+z}\ln(xyz) = e^{x+y+z}(\ln x + \ln y + \ln z)$

$\frac{\partial w}{\partial x} = e^{x+y+z}(\ln x + \ln y + \ln z) + e^{x+y+z}\left(\frac{1}{x}\right)$

$= e^{x+y+z}\left[\ln(xyz) + \frac{1}{x}\right]$

$\frac{\partial^2 w}{\partial y\partial x} = e^{x+y+z}\left[\ln(xyz) + \frac{1}{x}\right] + e^{x+y+z}\left[\frac{1}{y}\right]$

$= e^{x+y+z}\left[\ln(xyz) + \frac{1}{x} + \frac{1}{y}\right]$.

$\frac{\partial^3 w}{\partial z\partial y\partial x} = e^{x+y+z}\left[\ln(xyz) + \frac{1}{x} + \frac{1}{y}\right] + e^{x+y+z}\left[\frac{1}{z}\right]$

$= e^{x+y+z}\left[\ln(xyz) + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right]$

12. $\frac{\partial P}{\partial l} = 100\left[(0.11)l^{0.11-1}\right]k^{0.89} = 11l^{-0.89}k^{0.89}$

$\frac{\partial^2 P}{\partial k\partial l} = 11l^{-0.89}\left[(0.89)k^{0.89-1}\right] = 9.79l^{-0.89}k^{-0.11}$

13. $f(x, y, z) = \frac{x+y}{xz} = \frac{1}{z} + \frac{y}{xz}$

$f_x(x, y, z) = -\frac{y}{x^2z}$

$f_{xy}(x, y, z) = -\frac{1}{x^2z}$

$f_{xyz}(x, y, z) = \frac{1}{x^2z^2}$

$f_{xyz}(2, 7, 4) = \frac{1}{2^2\cdot 4^2} = \frac{1}{64}$

14. $f_x(x,\ y,\ z) = 6e^{y^2 \ln(z+1)}$

$f_{xy}(x,\ y,\ z) = 12y\ln(z+1)e^{y^2 \ln(z+1)}$

$f_{xyz}(x,\ y,\ z) = 12y\left[\ln(z+1)\left\{e^{y^2 \ln(z+1)} \cdot \frac{y^2}{z+1}\right\} + e^{y^2 \ln(z+1)} \cdot \frac{1}{z+1}\right]$

$f_{xyz}(0,\ 1,\ 0) = 12[0+1] = 12$

15. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = (2x+2y)\left(e^r\right) + (2x+6y)\left(\frac{1}{r+s}\right)$

$= 2(x+y)e^r + \frac{2(x+3y)}{r+s}$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = (2x+2y)(0) + (2x+6y)\left(\frac{1}{r+s}\right)$

$= \frac{2(x+3y)}{r+s}$

16.
$$\begin{aligned}\frac{\partial z}{\partial r} - \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} - \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} - \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ &= \frac{\partial z}{\partial x}\left[\frac{\partial x}{\partial r} - \frac{\partial x}{\partial s}\right] + \frac{\partial z}{\partial y}\left[\frac{\partial y}{\partial r} - \frac{\partial y}{\partial s}\right]\\ &= \frac{1}{\frac{x}{y}}\left(\frac{1}{y}\right)[2r-2s] + \frac{1}{\frac{x}{y}}\left(-\frac{x}{y^2}\right)[2(r+s)-2(r+s)]\\ &= \frac{1}{x}(2r-2s)\\ &= \frac{2}{x}(r-s)\\ &= \frac{2(r-s)}{r^2+s^2}\end{aligned}$$

17. $2x + 2y - 4z\frac{\partial z}{\partial x} + \left[x\frac{\partial z}{\partial x} + z(1)\right] + 0 = 0$

$(-4z+x)\frac{\partial z}{\partial x} = -(2x+2y+z)$

$\frac{\partial z}{\partial x} = \frac{-(2x+2y+z)}{-4z+x} = \frac{2x+2y+z}{4z-x}$