Mathematics for Business Analytics and Finance

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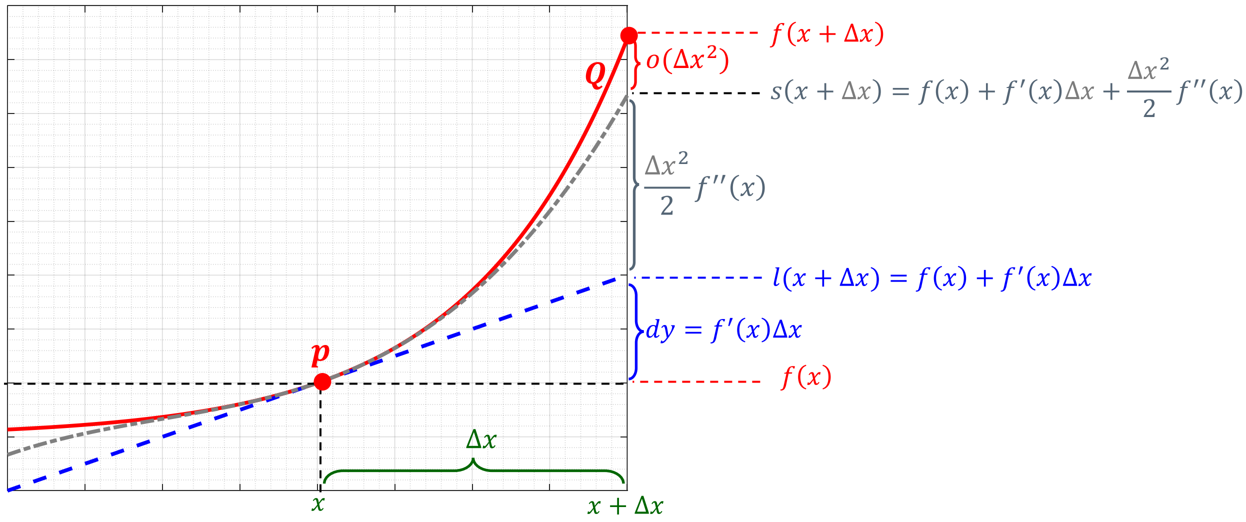
Module 2

Differentials

Differentials

The differential of y, denoted dy or $df(x)$, is given by:

$$dy = df(x, \Delta x) = f'(x)\Delta x$$



Differential

Example: Find the differential of $f(x) = x^3 - 2x^2 + 3x - 4$ and evaluate it when $x = 1$ and $\Delta x = 0.04$.

Solution: When $x = 1$ and $\Delta x = 0.04$,

$df(x, \Delta x) = f'(x)\Delta x = (3x^2 - 4x + 3)\Delta x$

$df(1,0.04) = [3(1)^2 - 4(1) + 3](0.04) = 0.08$

Remark: To find $f'(x)$ in R, type:

```
f=expression(x^3-2x^2+3x-4)
D(f,'x')
```

Differentials

If $f(x) = x$, then since $f'(x) = 1$, we have:

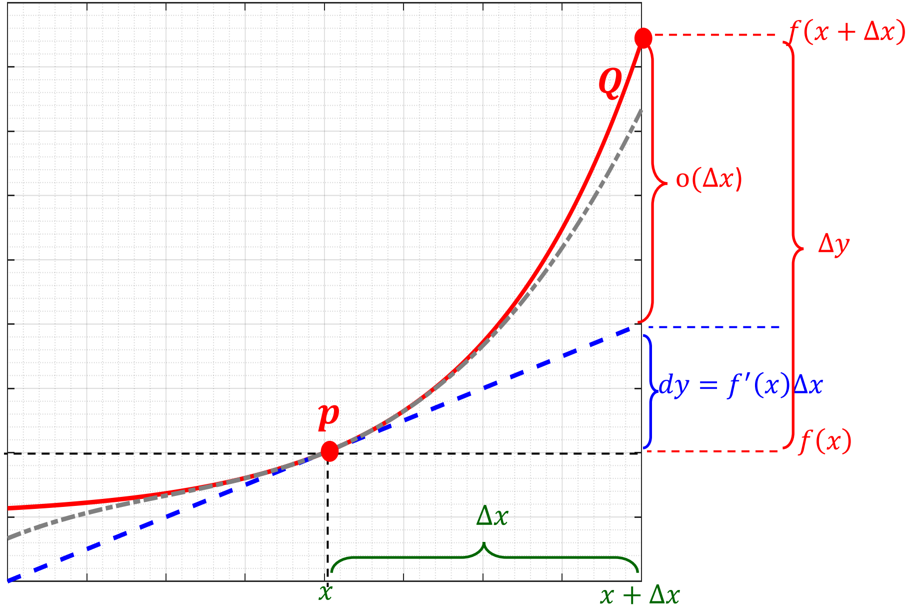
$$dx = dx(x, \Delta x) = f'(x)\Delta x \Rightarrow dx = \Delta x$$

Remark:

$$\Delta y = f(x + dx) - f(x) \approx f'(x)dx = dy$$

$$f(x + dx) \approx f(x) + dy$$

This formula gives us a way of estimating a function value $f(x + dx)$



Example

Using the Differential to Estimate a Change in a Quantity

A governmental health agency examined the records of a group of individuals who were hospitalized with a particular illness. It was found that the total proportion P that are discharged at the end of t days of hospitalization is given by

$$P = P(t) = 1 - \left(\frac{300}{300 + t}\right)^3$$

Use differentials to approximate the change in the proportion discharged if t changes from 300 to 305.

Solution

$$\Delta P \approx dP = P'(t)dt = -3\left(\frac{300}{300+t}\right)^2\left(\frac{-300}{(300+t)^2}\right)dt = 3\,\frac{300^3}{(300+t)^4}\,dt$$

$$t = 300, dt = 5 \Rightarrow \Delta P \approx 3\,\frac{300^3}{(300+300)^4}(5) = \frac{15}{16 \times 300} = \frac{1}{320} \approx 0.0031$$

$$P(305) - \mathrm{P}(300) = 0.87807 - 0.87500 = 0.00307$$

Example

Using the Differential to Estimate a Function Value

By using differentials, estimate the value of $\ln(1.06)$.

Solution:

$$\ln(x + dx) \approx \ln(x) + dy = \ln(x) + \frac{1}{x} dx$$

$$\ln(1.06) = \ln(1) + \frac{1}{1}(0.06) = 0.06$$

Differentials

The Relationship between $\frac{dy}{dx}$ From $\frac{dx}{dy}$

$$\boxed{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}$$

$$f^{-1}\big(f(x)\big) = 1 \Rightarrow \frac{df^{-1}(y)}{y} \times \frac{dy}{dx} = 1 \Rightarrow \frac{dx}{y} \times \frac{dy}{dx} = 1$$

Example: Find $\frac{dx}{dy}$ for the following: a) $y = x^3 + 4x + 5$ b) $y = \sqrt{2500 - x^2}$

Solution

a)

$$\frac{dy}{dx} = 3x^2 + 4 \Rightarrow \frac{dx}{dy} = \frac{1}{3x^2 + 4}$$

b)

$$\frac{dy}{dx} = -\frac{x}{\sqrt{2500 - x^2}} \Rightarrow \frac{dx}{dy} = -\frac{\sqrt{2500 - x^2}}{x}$$

Indefinite Integral

The Indefinite Integral

- An antiderivative of a function f is a function F such that $F'(x) = f(x)$.
- In differential notation $F'(x) = \frac{dF(x)}{dx} = f(x) \Rightarrow dF(x) = f(x)dx$.
- Any two antiderivatives of a function differ only by a constant. Indefinite Integral is expressed as

$$\boxed{\int f(x)dx = F(x) + C \Leftrightarrow F'(x) = \frac{d(\int f(x)dx)}{dx} = f(x)}$$

- $f(x)$: integrand, x: variable of integral, C: constant of integral

Basic Properties

- $\int k dx = kx + C$ k is a constant
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $n \neq -1$ (power rule)
- $\int x^{-1} dx = \int \frac{1}{x} dx = ln\,|x| + C$

- $\int e^x dx = e^x + C$
- $\int kf(x)dx = k\int f(x)dx + C$ k is a constant
- $\int \big(f(x) \pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx$

Example

Indefinite Integral of a Sum and Difference

Find $\int \left(2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1\right) dx$.

Solution

$$\int \left(2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1\right) dx = (2)\frac{x^{9/5}}{9/5} - (7)\frac{x^4}{4} + (10)e^x - x + C$$

$$= \frac{10}{9}x^{9/5} - \frac{7}{4}x^4 + 10e^x - x + C$$

Example

Using Algebraic Manipulation to Find an Indefinite Integral

Find: a) $\int \frac{(2x-1)(x+3)}{6} dx$ b) $\int \frac{x^3-1}{x^2} dx$

Solution

a) $\int \frac{(2x-1)(x+3)}{6} dx = \frac{1}{6}\left((2)\frac{x^3}{3} + (5)\frac{x^2}{2} - 3x\right) + C = \frac{x^3}{9} + \frac{5x^2}{12} - \frac{x}{2} + C$

b) $\int \frac{x^3 - 1}{x^2} dx = \int \left(x - x^{-2}\right) d = \frac{x^2}{2} + \frac{1}{x} + C$

Integration with Initial Conditions

Using Initial Conditions To Find The Constant Value

If y is a function of x such that $y' = (8x - 4)$ and $y(2) = 5$, find y.

Solution

We find the integral,

$$y = \int (8x - 4)\, dx = (8)\frac{x^2}{2} - 4x + C = 4x^2 - 4x + C$$

Using the condition,

$$5 = 4(2)^2 - 4(2) + C \Rightarrow C = -3$$

The equation is

$$y = 4x^2 - 4x - 3$$

Example

Finding the Demand Function from Marginal Revenue

If the marginal-revenue function for a manufacturer's product is

$$\frac{dr}{dq} = 2000 - 20q - 3q^2$$

find the price as a function of quantity.

Solution

$$\frac{dr}{dq} = 2000 - 20q - 3q^2 \Rightarrow r = \int (2000 - 20q - 3q^2)dq$$

$$\Rightarrow r = 2000q - 10q^2 - q^3 + C$$

If no units are sold, i.e., $q = 0$, the revenue will be zero, i.e., $r = 0$, which gives $C = 0.$

The price function will become:

$$p = \frac{r}{q} = \frac{2000q - 10q^2 - q^3}{q} = 2000 - 10q - q^2$$

Example

Finding Cost from Marginal Cost

In the manufacture of a product, fixed costs per week are \$4000. (Fixed costs are costs, such as rent and insurance, that remain constant at all levels of production during a given time period.) If the marginal-cost function is

$$\frac{dc}{dq} = 0.000001\left(0.002q^2 - 25q\right) + 0.2$$

where c is the total cost (in dollars) of producing q pounds of product per week, find the total cost of producing 10,000 lbs. in 1 week.

Solution

The total cost c is

$$c(q) = \int \left[0.000001\left(0.002q^2 - 25q\right) + 0.2\right] dq$$

$$= 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + C$$

When $q = 0$, $c = 4000$. Cost of 10,000 lbs. in one week,

$$c(q) = 0.000001\left(\frac{0.002q^3}{3} - \frac{25q^2}{2}\right) + 0.2q + 4000$$

$$c(10000) = \$5416.67$$

Integration by Substitution

Example

Find the integrals of $\int (x+1)^{20}dx$ and $\int 3x^2\left(x^3+7\right)^3 dx$.

Solution

a)

$u = x + 1 \Rightarrow du = dx$

$$\int (x+1)^{20} dx = \int u^{20} du = \frac{u^{21}}{21} + C = \frac{(x+1)^{21}}{21} + C$$

b)

$u = x^3 + 7 \Rightarrow du = 3x^2 dx$

$$\int 3x^2 \left(x^3 + 7\right)^3 dx = \int u^3\, du = \frac{u^4}{4} + C = \frac{\left(x^3 + 7\right)^4}{4} + C$$

Example

Integrals Involving Logarithmic Functions

Find $\int \frac{(2x^3+3x)}{x^4+3x^2+7}\,dx$.

Solution

Let $u = x^4 + 3x^2 + 7 \Rightarrow du = (4x^3 + 6x)dx$

$$\int \frac{(2x^3 + 3x)}{x^4 + 3x^2 + 7} dx = \frac{1}{2} ln|u| + C = \frac{1}{2} ln|x^4 + 3x^2 + 7| + C = \frac{1}{2} ln(x^4 + 3x^2 + 7) + C$$

Example

Preliminary Division before Integration

Find $\int \frac{2x^3+3x^2+x+1}{2x+1}\,dx$.

Solution

Divide the numerator by the denominator:

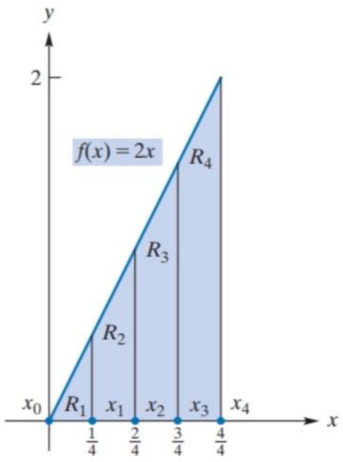
$$\int \frac{2x^3 + 3x^2 + x + 1}{2x + 1} dx = \int \left(\frac{(x^2 + x)(2x + 1) + 1}{2x + 1} \right) dx = \int \left(x^2 + x + \frac{1}{2x + 1} \right) dx = \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{2} \ln|2x + 1| + C$$

Definite Integrals: The Concept

The Definite Integral

Motivation

- Consider the region R, bounded by the lines $y = 2x, y = 0$ and $x = 1$. The region is a right triangle, so its area is 1.
- Let us divide the interval into four subintervals of equal length. Each subinterval has length $\Delta x = \frac{1}{4}$.



Four subregions of R.

$$x_k = \frac{k}{4}, k = 1,2,3,4$$

$$\Delta x = \frac{1}{4}$$

The Definite Integral

Motivation

We can approximate the area by using rectangular subregions in two ways:

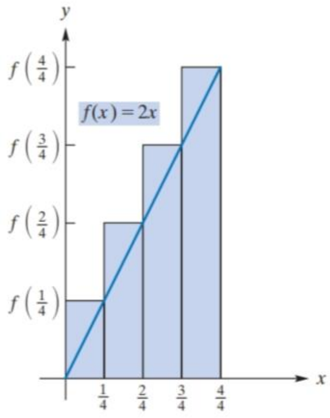
- By using circumscribed (the outside) rectangles:

$$\bar{S}_4 = \sum_{k=1}^{4} f(x_k)\Delta x = \sum_{k=1}^{4} 2\left(\frac{k}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{8}\sum_{k=1}^{4} k = \frac{10}{8} = \frac{5}{4}$$

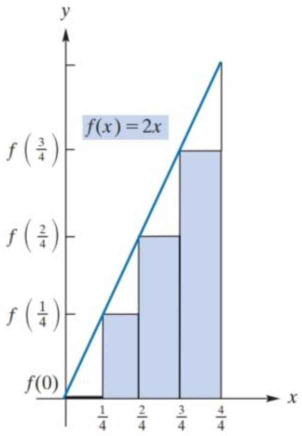
- By using inscribed (the inside) rectangles:

$$\underline{S}_4 = \sum_{k=1}^{3} f(x_k)\Delta x = \sum_{k=1}^{3} 2\left(\frac{k}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{8}\sum_{k=1}^{3} k = \frac{6}{8} = \frac{3}{4}$$

$$\frac{3}{4} = \underline{S}_4 \le Area \le \bar{S}_4 = \frac{5}{4}$$



Four circumscribed rectangles.



Four inscribed rectangles.

The Definite Integral

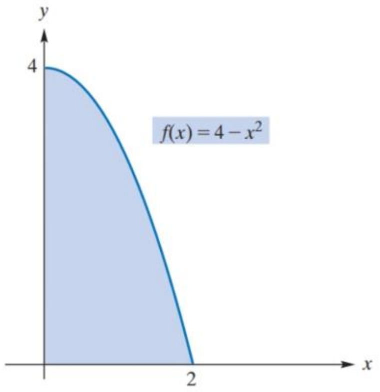
For area under the graph from $x = a$ to $x = b$:

$$S = \lim_{n\to\infty} \bar{S}_n = \lim_{n\to\infty} \underline{S}_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)dx$$

x: the variable of integration, $f(x)$: the integrand, a: the lower bound, b: the upper bound

Coding in R:

```
f=function(x){4-x^2}
integrate(f,0,2)
5.333333 with absolute error < 5.9e-14
```



Creating Loops in R

Creating Loops

```
for (variable in vector) {
expressions
}

while (condition) {

}
```

```
s=0
for (i in 1:3){
s=s+(1/8)*i
}
print(s)

s=0
i=1
while(i<=3){
s=s+(1/8)*i
i=i+1}
print(s)
```

The Definite Integral

Recap: Definite Integral as a Limit of a Sum

$$S = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

$$\Delta x = \frac{b-a}{n}, \qquad x_k = a + k\Delta x, \qquad f(x_k) = f(k\Delta x)$$

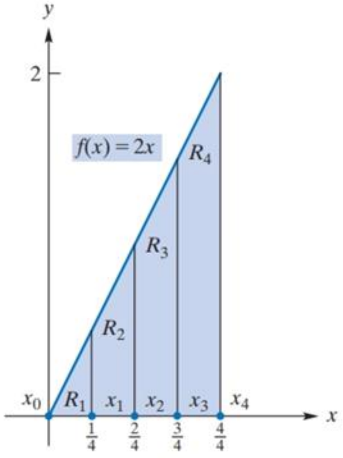
Example: Find $S = \int_0^1 2x dx$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}, \qquad x_k = k\Delta x = \frac{k}{n}, \qquad f(x_k) = 2x_k = \frac{2k}{n}$$

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n} \left(\frac{2k}{n}\right)\left(\frac{1}{n}\right)$$

Now, to find the approximate value of $S = \lim_{n\to\infty} S_n$, use R:

```
s=0
n=1000
for (k in 1:n){
s=s+2*k/n^2}
print(s)
[1] 1.001
```



Four subregions of R.

Example

Definite Integral as a Limit of a Sum

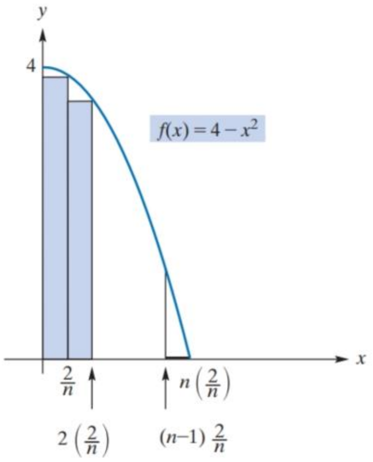
Find the area of the region bounded by $f(x) = 4 - x^2$ between $x = 0$ and $x = 2$.

Since the length of $[0,2]$ is 2, $\Delta x = \frac{2}{n}$. Considering inscribed rectangles, $x_k = \frac{2}{n}k$. Summing the areas, we get

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n}\left(4 - \left(\frac{2k}{n}\right)^2\right)\left(\frac{2}{n}\right)$$

Now, to find the approximate value of $S = \lim_{n\to\infty} S_n$, use R:

```
s=0
n=1000
for (k in 1:n){
s=s+(4-(2*k/n)^2)*(2/n)
}
print(s)
[1] 5.33
```

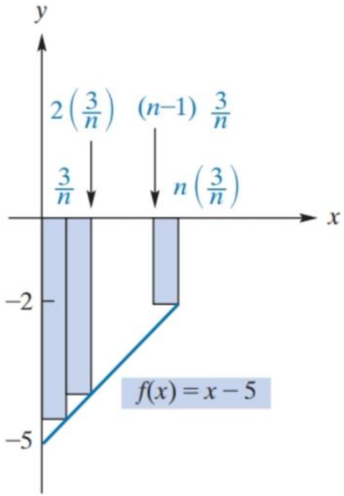


n subintervals and corresponding rectangles

Example

Definite Integral as a Limit of a Sum

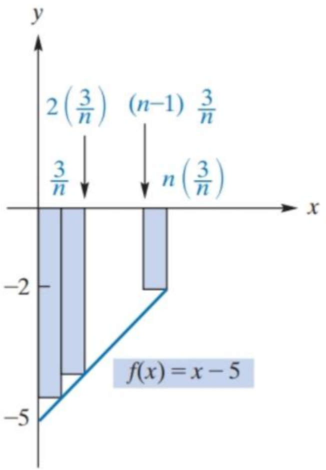
Integrate $f(x) = x - 5$ from $x = 0$ to $x = 3$



Solution

Since the length of $[0,3]$ is 3, $\Delta x = \frac{3}{n}$. Considering inscribed rectangles, $x_k = \frac{3}{n}k$. Summing the areas, we get

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n} \left(\frac{3k}{n} - 5\right)\left(\frac{3}{n}\right)$$



Now, to find the approximate value of $S = \lim_{n\to\infty} S_n$, use R:

```
s=0
n=1000
for (k in 1:n){
s=s+(3*k/n-5)*(3/n)
}
print(s)
[1] -10.50
```

Remark: Since $f(x) \leq 0$ for all x, the definite integral has become a negative number.

The Fundamental Theorem of Integral Calculus

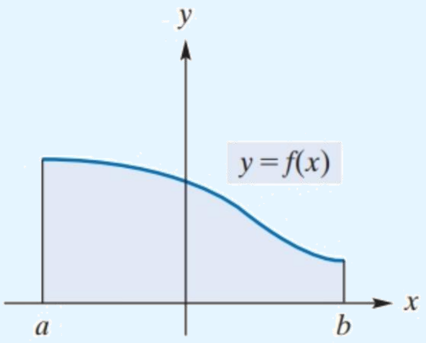
Fundamental Theorem of Integral Calculus

If f is continuous on the interval $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

a) $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$

b) $\int_a^b f(x)dx = F(b) - F(a)$

If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x)dx$ represents the area under the curve.



Properties of The Definite Integrals:

- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_b^c f(x)dx$

Example

Fundamental Theorem of Calculus

Find the following integrals:

a) $\int_1^2 \left[4t^{1/3} + t\left(t^2 + 1\right)^3\right] dt$

b) $\int_0^1 e^{3t} dt$

Solution

a) $\int_1^2 \left[4t^{1/3} + t\left(t^2+1\right)^3\right] dx = \left[(4)\frac{t^{4/3}}{\frac{4}{3}} + \left(\frac{1}{2}\right)\frac{\left(t^2+1\right)^4}{4}\right]_1^2$

$$= 3\left(2^{4/3} - 1\right) + \frac{1}{8}\left(5^4 - 2^4\right) = 6\sqrt[3]{2} + \frac{585}{8}$$

b) $\int_0^1 e^{3t} dt = \left(\frac{1}{3}\right)\left[e^{3t}\right]_0^1 = \frac{1}{3}\left(e^3 - e^0\right) = \frac{1}{3}\left(e^3 - 1\right)$

Example

Finding a Change in Function Values by Definite Integration

A manufacturer's marginal-cost function is $\frac{dc}{dq} = 0.6q + 2$. If production is presently set at $q = 80$ units per week, how much more would it cost to increase production to $q = 100$ units per week?

Solution

$$\frac{dc}{dq} = 0.6q + 2 \Rightarrow dc = (0.6q + 2)dq \Rightarrow c(q) = \int_0^q (0.6q + 2)dq = 0.3q^2 + 2q$$

$$\Rightarrow c(100) - c(80) = 3200 - 2080 = 1120$$

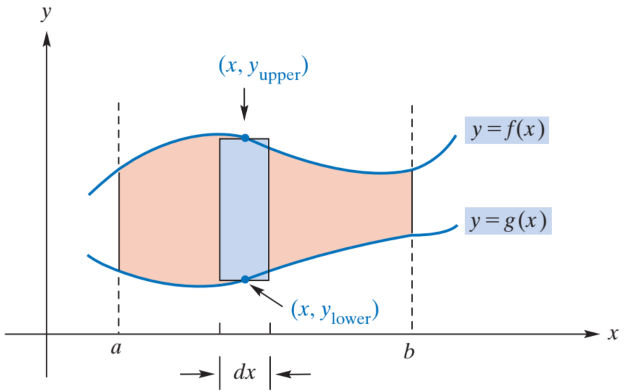
```
Coding in R:
f=function(q){0.6*q+2}
integrate(f,80,100)
```

Finding an Area between Two Curves

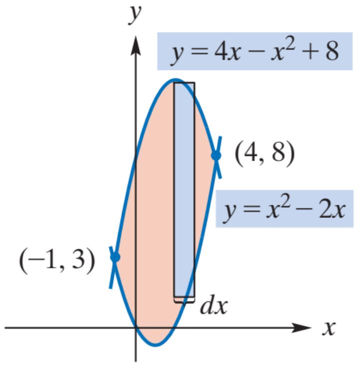
Vertical Elements

The area of the vertical strip is

$$Area = \int_a^b (f(x) - g(x))dx$$



Example: Find the area of the region bounded by the graphs of $y = 4x - x^2 + 8$ and $y = x^2 - 2x$.



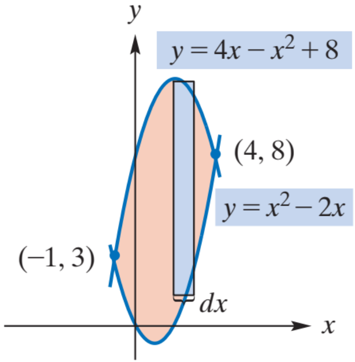
Solution

Vertical Elements

$4x - x^2 + 8 = x^2 - 2x \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x - 4)(x + 1) = 0 \Rightarrow x = -1{,}4$

Area $= 4x - x^2 + 8 - \left(x^2 - 2x\right) = -2x^2 + 6x + 8$

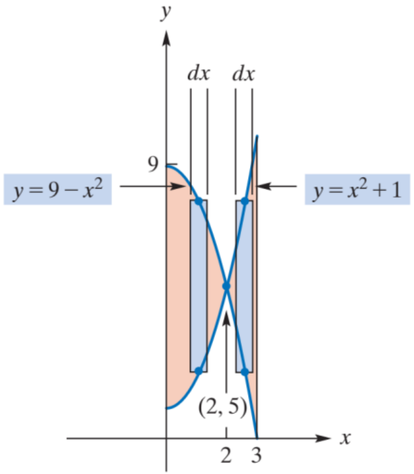
Area $= \int_{-1}^{2}(-2x^2 + 6x + 8)dx = 41\frac{2}{3}$



Finding an Area between Two Curves

Area of a Region Having Two Different Upper Curves

Find the area of the region bounded by the graphs of $y = 9 - x^2$ and $y = x^2 + 1$ from $x = 0$ to $x = 3$.



Example

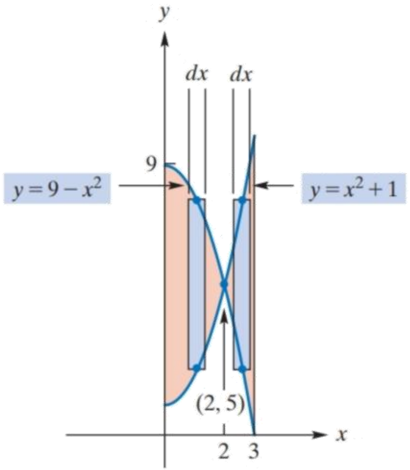
Area of a Region Having Two Different Upper Curves

$9 - x^2 = x^2 + 1 \Rightarrow x^2 = 4 \Rightarrow x = -2{,}2$

For $x \in [0,2]$: Area between two curves $= \int_0^2 \left(9 - x^2 - (x^2 + 1)\right) dx = \int_0^2 (8 - 2x^2) dx$

For $x \in [2,3]$: Area between two curves $= \int_2^3 \left(x^2 + 1 - (9 - x^2)\right) dx = \int_2^3 (-8 + 2x^2) dx$

Total Area $= \int_0^2 (8 - 2x^2) dx + \int_2^3 (-8 + 2x^2) dx = \frac{46}{3}$



To find the value in R, type:

```
y1 = function(x) (8-2*x^2)
Area1 = integrate(y1, 0, 2)
y2 = function(x) (2*x^2-8)
Area2 = integrate(y2, 2, 3)
print(Area1$value + Area2$value)

[1] 15.33333
```

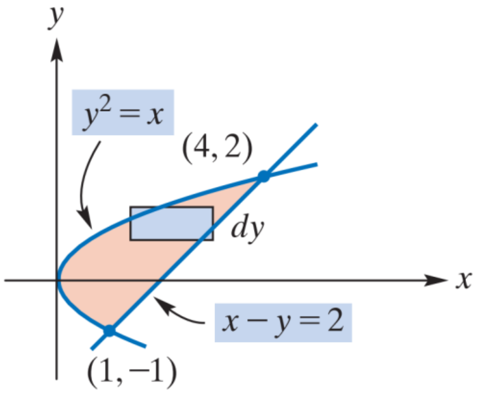
To plot these functions in R, type:

```
f=function(x) 9-x^2
g=function(x) x^2+1
plot(f,0,3,col="red")
curve(g,0,3,add = TRUE,col="blue")
```

Example

Horizontal Elements

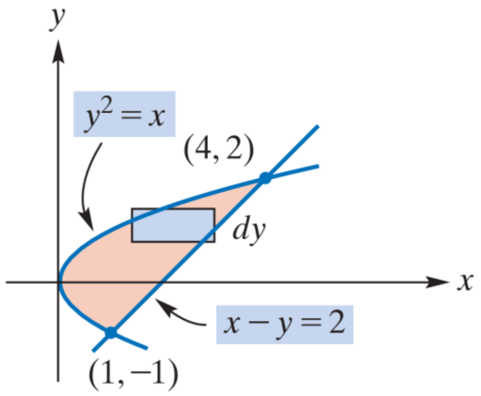
Find the area of the region bounded by the graphs of $y^2 = x$ and $x - y = 2$.



Solution

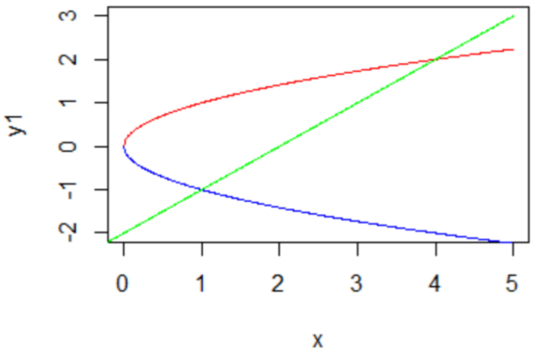
The intersection points are $(1,-1)$ and $(4,2)$. The total area is

$$\text{Area} = \int_{-1}^{2} (y + 2 - y^2)\,dy = \frac{9}{2}$$



To see the region between the graphs in R, you can type and compare the resulting graph with the book's figure:

```
x=seq(-2,5,0.01)
y1=x^0.5
y2=-x^0.5
y3=x-2
plot(x,y1,type="l",col="red",xlim=c(0,5),ylim=c(-2,3))
lines(x,y2,col="blue")
lines(x,y3,col="green")
```



Integration: Further Solution Techniques

Integration by Parts

Integration by Parts Formulation

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: Find $\int \frac{\ln x}{\sqrt{x}}\,dx$ by integration by parts.

Solution

Let $u = \ln x$ and $dv = \frac{1}{\sqrt{x}}\ dx$. We have:

$$du = \frac{1}{x}dx, \quad v = \int x^{-1/2}\ dx = 2x^{1/2}$$

Thus,

$$\int \frac{\ln x}{\sqrt{x}}\ dx = (\ln x)\left(2\sqrt{x}\right) - \int \left(2x^{1/2}\right)\left(\frac{1}{x}\ dx\right) = 2\sqrt{x}[\ln(x) - 2] + C$$

Integration by Parts

The LATE Rule for choosing u and dv

Between the two functions, u is often the one that comes upper on the list. The other will be dv :

- L: Logarithmic functions, e.g., $ln(x), log_b x, etc$
- A: Algebraic functions, e.g., $x^2, 3x^{50}, etc$
- T: Trigonometric functions, e.g., $sin(\mathrm{x}), etc$
- E: Exponential functions, e.g., $e^x, 19^x, etc$

Example: Find $\int \ln(y)\ dy$.

Solution

$u = \ln y$

$dv = dy$

$du = \left(\frac{1}{y}\right) dy$

$v = y$

$$\int \ln y \ dy = (\ln y)(y) - \int y\left(\frac{1}{y}\ dy\right) = y \ln y - y + C = y[\ln y - 1] + C$$

Integration by Partial Fractions

Remark: A polynomial is defined as $P(x) = \sum_{m=0}^{k} a_m x^m$ in which the highest exponent k is known as the degree of the polynomial. Suppose in $\int \frac{P(x)}{Q(x)} dx$ both $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is less than the degree of $Q(x)$. Then, we typically can solve it by partial fractions. To make this done we factor $Q(x)$ as completely as possible. Then for each factor in $Q(x)$ we can use the following table to determine the term(s) we pick up in the partial fraction decomposition.

Factor in $Q(x)$	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$\left(ax^2+bx+c\right)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Integration by Partial Fractions

Express The Integrand as Partial Fractions

Example: Determine $\int \frac{2x+1}{3x^2-27}\, dx$

Solution

$$\frac{2x+1}{x^2-9}=\frac{2x+1}{(x+3)(x-3)}=\frac{A}{(x+3)}+\frac{B}{(x-3)}\Rightarrow 2x+1=A(x-3)+B(x+3)$$

To solve for A and B, equate the like powers:

$$\Rightarrow\begin{cases} A+B=2 \\ 3(B-A)=1 \end{cases}\Rightarrow A=\frac{5}{6},\ B=\frac{7}{6}$$

$$\Rightarrow\int\frac{2x+1}{3x^2-27}\,dx=\frac{1}{3}\int\left(\frac{(5/6)}{x+3}+\frac{(7/6)}{x-3}\right)dx=\frac{1}{3}\left(\frac{5}{6}\ln|x+3|+\frac{7}{6}\ln|x-3|\right)+C$$

Example

An Integral with a Distinct Irreducible Quadratic Factor

Determine $\int \frac{-2x-4}{x^3+x^2+x}\, dx$

Solution

$$\frac{-2x-4}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$$-2x-4 = A(x^2+x+1) + (Bx+C)x$$

To solve for A and B, equate the like powers:

$$\Rightarrow \begin{cases} A+B=0 \\ A+C=-2 \\ A=-4 \end{cases} \Rightarrow A=-4,\ B=4, C=2$$

$$\Rightarrow \int \frac{-2x-4}{x(x^2+x+1)}dx = \int \left(\frac{-4}{x} + \frac{4x+2}{x^2+x+1}\right)dx = -4\ln|x| + 2\ln\left|x^2+x+1\right| + C$$