Chain Rule assignment

1. Given $f(z) = \log_e(1+z)$ where $z = x^T x$, $x \in \mathbb{R}^d$

$\Rightarrow$ if $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$ then, $x^T = [x_1, x_2 \cdots\cdots x_d]$

$$x^T x = [x_1^2 + x_2^2 + \cdots\cdots + x_d^2]$$

Applying chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}(\log(1+z)) \cdot \frac{d}{dx} \cdot (x^T \cdot x)$$

$$= \frac{1}{1+z} \cdot \frac{d}{dz}(z) \cdot \frac{d}{dx}(x_1^2 + x_2^2 + \cdots + x_d^2)$$

$$= \frac{1}{1+z}(2x_1 + 2x_2 + \cdots + 2x_d)$$

$$= \frac{1}{1+z} \cdot 2(x_1 + x_2 + \cdots + x_d)$$

$$= \frac{2}{1+z} \sum_{i=1}^{d} x_i$$ (Ans.)

2. $f(z) = e^{-z/2}$; where $z = g(z)$, $g(y) = y^T \cdot S^{-1} y$,
$y = h(x)$, $h(x) = x - u$

$\Rightarrow$ using chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

here,

$$\frac{df}{dz} = \frac{d}{dz}\left(e^{-z/2}\right) = -\frac{e^{-z/2}}{2}$$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right)$$

$$= \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1}}{h}$$

$$= \lim_{h \to 0} \frac{h(y^T S^{-1} + S^{-1} y + h S^{-1})}{h}$$

$=$

$$= \lim_{h \to 0} \left(y^T s^{-1} + s^{-1} y + h s^{-1} \right)$$

$$= y^T s^{-1} + s^{-1} y$$

$$\frac{dy}{dx} = \frac{d(x-u)}{dx} = 1$$

$$\therefore \frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= -\frac{e^{-z/2}}{2} \left(y^T s^{-1} + s^{-1} y \right) \cdot 1$$

$$= -\frac{e^{-z/2}}{2} \cdot \frac{1}{s} \left(y^T + y \right)$$

Ans.