Bayesian Inference

Ricard Garcia Isern & Antonio Arcas

2023-11-09

1- Write the likelihood function. Estimate the mean life expectancy using a classical inference procedure.

Practical: Bayesian inference

1- $f(x|\mu) = \frac{1}{\mu} \cdot e^{-\frac{1}{\mu}x}$

$$L(x|\mu) = \left(\frac{1}{\mu}\right)^n \cdot e^{-1 \cdot \sum_{i=1}^{n} \frac{x_i}{\mu}} \rightarrow \boxed{\frac{1}{\mu^8} \cdot e^{\frac{-31'77}{\mu}}}$$

Classical mean of μ:

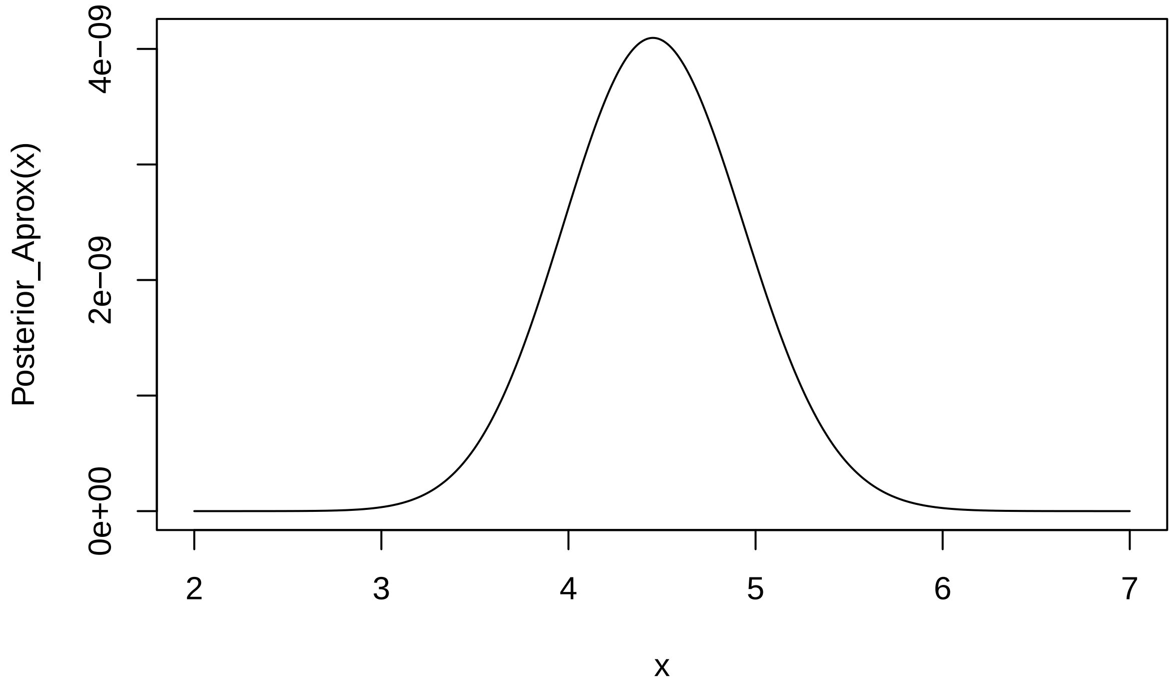
$$L(x|\mu)' = -\frac{8}{\mu^9} \cdot e^{\frac{-31'77}{\mu}} + \frac{1}{\mu^8} \cdot \frac{31'77 \cdot e^{-\frac{31'77}{\mu}}}{\mu^2} = \frac{-8}{\mu^9} \cdot e^{\frac{-31'77}{\mu}} + \frac{31'77 \cdot e^{\frac{-31'77}{\mu}}}{\mu^{10}} = \frac{e^{\frac{-31'77}{\mu}}}{\mu^9}\left[-8 + \frac{31'77}{\mu}\right]$$

$$\frac{e^{\frac{-31'77}{\mu}}}{\mu^9}\left[-8 + \frac{31'77}{\mu}\right] = 0; \quad -8 + \frac{31'77}{\mu} = 0; \quad \mu = \frac{31'77}{8} = \boxed{3'97}$$

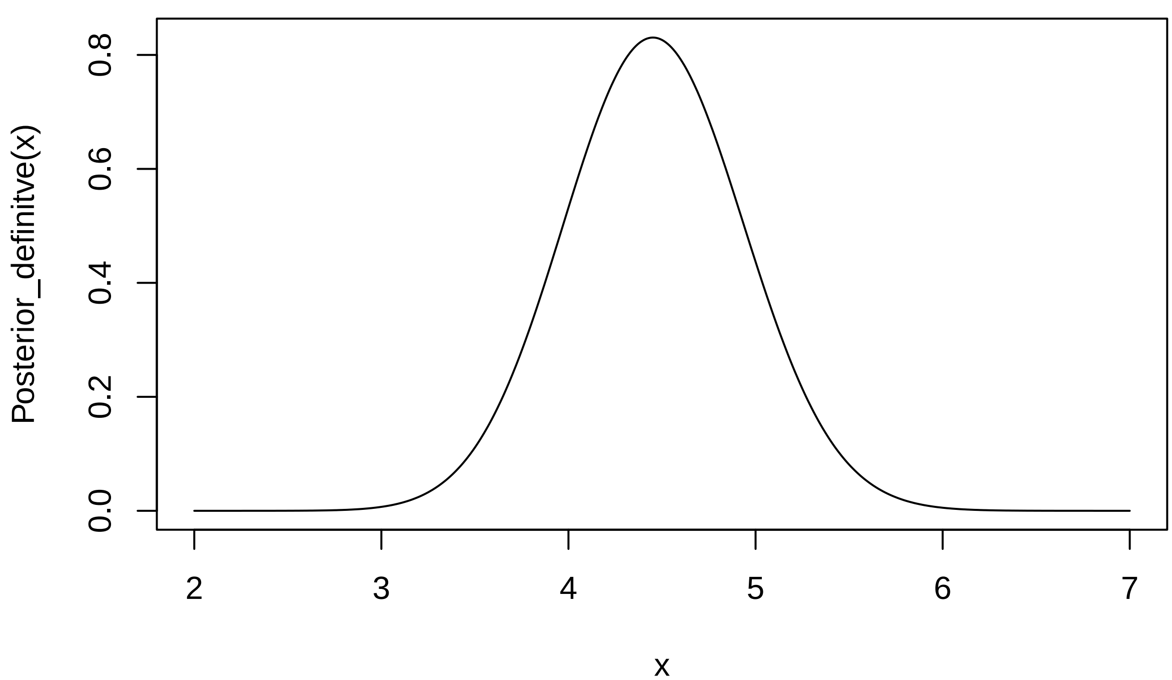
Since $L(x|3'97)'' < 0$, $\mu = 3'97$ is a maximum. Our MLE is 3'97.

2- The authors of a paper claim that the mean life expectancy of this bacterium is 4.5±0.5h (reading the paper we see that 0.5h is just the standard deviation). Using a Gaussian density for implementing the prior information for parameter , compute the posterior density, the expectation and a 95% credibility interval. You can follow the following steps: Posterior distribution = (Prior Distribution) * (Likelihood Function)

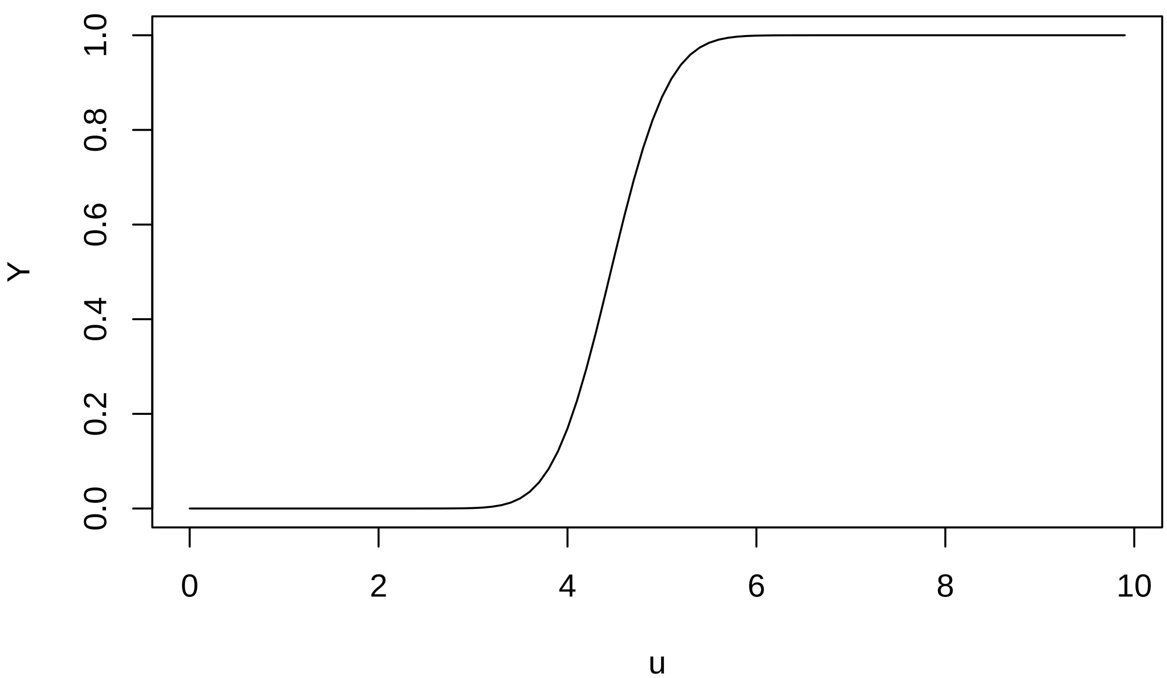
```
Prior_Function <-function(mu){dnorm(mu,4.5,0.5)}
Likelihood_Function <- function(mu){(1/(mu^8))*exp(-31.77/mu)}

Posterior_Aprox <- function(mu){Prior_Function(mu)*Likelihood_Function(mu)}

# Plot the approximated posterior:
x=seq(2,7,0.01)
plot(x,Posterior_Aprox(x),type='l')
```




```r
#Construct a proper density dividing the approximated posterior by its integral:
c <- integrate(Posterior_Aprox,0,Inf,rel.tol = 1e-15)$value

#Plot the posterior density:
Posterior_definitve <- function(mu){Posterior_Aprox(mu)/c}
x=seq(2,7,0.01)
plot(x,Posterior_definitve(x),type='l')
```



```r
#Plot the posterior distribution function:

Distribution_function_posterior<- function(u){integrate(Posterior_definitve,0,u)$value}

u=seq(0.0000001,10,0.1)
Y=numeric(length(u))

for(i in 1:length(u)){
  Y[i]=Distribution_function_posterior(u[i])
}

plot(u,Y,type = 'l')
```




```r
#Expectation and credibility intervals:
expectation_function <- function(mu){mu*Posterior_definitve(mu)}
expectation <- integrate(expectation_function,0,Inf)$value
expectation
```

```
## [1] 4.46012
```

```r
#F(u) = 0.025
lower_bound_result <- uniroot(function(u)
  Distribution_function_posterior(u) - 0.025, interval = c(2, 6))

#F(u) = 0.975
upper_bound_result <- uniroot(function(u)
  Distribution_function_posterior(u) - 0.975, interval = c(4, 8))

#Credibility interval: [3.53, 5.41]
```

```r
#With non-informative prior probabilities for H0 and H1:
#P(H0|X):

probabH0 <- (Likelihood_Function(4.5))/(Likelihood_Function(4.5)+
                                          integrate(Posterior_Aprox, 0, Inf)$value)
probabH1 <- 1 - probabH0
#Bayesian factor (in favor of H0):
bayesfactor <- probabH0/probabH1
bayesfactor
```

3- Test if your results agree with those of the authors of the paper. In other words, consider H0: = 4.5 and H1: 4.5. Be non informative for the prior probabilities of H0 and H1 , and consider for H1 the same Gaussian prior than in part 2. Compute P(H0|X), P(H1|X), and the Bayes factor. Which are the conclusions?

```
## [1] 1.259583
```

```r
#Bayesian factor (in favor of H1):
1/bayesfactor
```

```
## [1] 0.7939133
```

```r
#The bayesian factor in favour of H0 is not worth more than a bare mention but
#it's still higher than the bayesian factor in favor of H1.s
```