Ricard Garcia Isern

NIA: 107742

4. Let $f(x,y) = 2x^2 + y^2 - xy + 2x + y + 4$.

(a) Write $f(x,y)$ as a quadratic function

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + c.$$

(b) Explain why $f(x,y)$ has a global minimum, and find it by the Newton method with initial point $(-1,-1)$.

(c) Find the first iteration of the steepest descent method with initial point $(-1,-1)$.

a)

$$= \begin{pmatrix} q_1 x + q_3 y, & q_2 x + q_4 y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (b_1 x + b_2 y) + C$$

$$= x(q_1 x + q_3 y) + y(q_2 x + q_4 y) + (b_1 x + b_2 y) + C \quad ; \ \text{let } q_2 = q_3$$

$$= \boxed{q_1 x^2 + 2 q_2 xy + q_4 y^2 + b_1 x + b_2 y + C}$$

* We compare it with $f(x,y)$ & we get that:

* $q_1 = 2$
* $q_2 = q_3 = -1/2$
* $q_4 = 1$
* $b_1 = 2$
* $b_2 = 1$
* $C = 4$

$$\Rightarrow \boxed{f(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & -1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 4}$$

b) $f(x,y)$ to have a global minimum $\rightarrow$ must be CONVEX.

$$H f(x,y) = 2 \cdot \begin{vmatrix} q_1 & q_2 \\ q_2 & q_4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & -1/2 \\ -1/2 & 1 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix}$$

$$|H_{f(x,y)}| = (4 \cdot 2) - ((-1)\cdot(-1)) = 8 - 1 > 0$$

&

$4 > 0$

$\Rightarrow$ $\boxed{H f(x,y) \text{ is POSITIVE DEFINITE} \rightarrow \text{CONVEX}}$, by Silvestrian criteria

Newton METHOD

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} x_{k-1} \\ y_{k-1} \end{pmatrix} - H_{f(x,y)}(x_{k-1}, y_{k-1})^{-1} \cdot \nabla f(x_{k-1}, y_{k-1})$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M^{-1} = \frac{1}{|M|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{cases} f_x = 4x - y + 2 \\ f_y = 2y - x + 1 \end{cases} \Rightarrow \nabla f = \begin{pmatrix} 4x - y + 2 \\ 2y - x + 1 \end{pmatrix}$$

$$\begin{cases} f_{xx} = 4 \\ f_{yy} = 2 \\ f_{xy} = f_{yx} = -1 \end{cases} \Rightarrow H = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad H(-1,-1) = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} \qquad H(-1,-1)^{-1} = \begin{pmatrix} 2/7 & 1/7 \\ 1/7 & 4/7 \end{pmatrix}$$

$$\nabla f\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 + 1 + 2 \\ -2 + 1 + 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2/7 & 1/7 \\ 1/7 & 4/7 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} -2/7 \\ -1/7 \end{pmatrix} = \boxed{\begin{pmatrix} -5/7 \\ -6/7 \end{pmatrix}}$$

$(5/7, -6/7)$ is a a crit. point because $\nabla f(-5/7, -6/7) = (0,0)$

$\Downarrow$

minimum

c) $\left|\begin{array}{l} f_x = 4x - y + 2 \\ f_y = 2y - x + 1 \end{array}\right. \Rightarrow \nabla f = (4x - y + 2, 2y - x + 1)$

$\nabla f(-1,-1) = (-1, 0)$

$\phi_{x_0}(t) = f((-1,-1) - t(-1,0))$

$\phi'_{x_0}(t) = \nabla f(-1+1t, -1) \cdot (-1,0) = \begin{pmatrix} 4(-1+t)+1+2 \\ -2+1-t+1 \end{pmatrix} \cdot (-1, 0) = (4t-1, -t) \cdot (-1,0) = -4t+1 \rightarrow \boxed{t_0 = 1/4}$

- $\mathbf{x}_{k+1} = \mathbf{x}_k - t_k \nabla f(\mathbf{x}_k)$

$x_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - 1/4 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} -1/4 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -3/4 \\ -1 \end{pmatrix}}$