The Discrete L-Function: Measuring Structural Drift in Unreduced Rational Dynamics

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Abstract

We introduce the **Discrete L-Function** ($\mathcal{L}_{\mathbb{Q}}$), a strictly rational, integer-arithmetic invariant designed to measure the global structural asymmetry of elliptic curves within the Explicit Domain ($\mathbb{Q}_{\text{Exp}}$). By rejecting the continuum-based definition of L-functions involving infinite series and analytic continuation, we define $\mathcal{L}_{\mathbb{Q}}$ as the normalized summation of *structural tension* (arithmetic torque) over the finite period of the system modulo N. We present experimental results demonstrating a strong correlation between the asymptotic behavior of $\mathcal{L}_{\mathbb{Q}}(N)$ and the Rank of the curve. Specifically, Rank 0 curves exhibit trivial zero drift due to torsion; Rank 1 curves exhibit asymptotic decay ($\mathcal{L}_{\mathbb{Q}} \sim 1/N$); and Rank 2 curves exhibit accelerated convergence to zero. This suggests that the "Rank" of a curve is a measure of the superfluidity of its discrete arithmetic flow.

1 Introduction

The Birch and Swinnerton-Dyer (BSD) conjecture relates the algebraic rank of an elliptic curve to the order of vanishing of its L-function $L(E, s)$ at $s = 1$. Standard approaches view this L-function as an analytic object defined over the complex continuum. In **Unreduced Rational Dynamics**, we posit that the information at $s = 1$ is not a property of an analytic limit, but a statistical property of the integer dynamics modulo N.

We propose that the "Rank" of a curve corresponds to the symmetry of its trajectory in the discrete state space. A system with high rank possesses a "balanced" topology that cancels out arithmetic torque (drift) over long periods. A system with low rank (specifically Rank 0, non-torsion if it existed, or transient orbits) would exhibit net structural drift.

We define the **Discrete L-Function** $\mathcal{L}_{\mathbb{Q}}(N)$ to measure this drift directly.

2 Theoretical Construction

2.1 The Projective State Space

We operate on the unreduced projective coordinates of the elliptic curve $E : Y^2Z = X^3 + aXZ^2 + bZ^3$. A state at time t is the integer triple:

$$S_t = (X_t, Y_t, Z_t) \in \mathbb{Z}^3$$

The evolution $S_{t+1} = S_t \oplus G$ is performed using standard projective group law formulas, which require no division and remain closed in $\mathbb{Z}$.

2.2 Structural Tension (τ)

We define the "twist" or "torque" of the trajectory as the cross-determinant of the rational coordinate $x = X/Z$ between time steps.

Definition 1 (Structural Tension)**.**

$$\tau_t = X_t Z_{t+1} - X_{t+1} Z_t$$

This integer τ_t measures the direction and magnitude of the "step" taken by the system in the rational embedding.

2.3 The Discrete L-Function

Definition 2 (Discrete L-Function)**.** *For a modulus N, let T_N be the period of the trajectory modulo N. The Discrete L-Function is the time-averaged sign of the tension:*

$$\mathcal{L}_{\mathbb{Q}}(N) := \frac{1}{T_N} \sum_{t=0}^{T_N-1} sgn(\tau_t \pmod{N^2})$$

Note: The tension τ_t is computed in $\mathbb{Z}$, then reduced modulo N^2 (since it is quadratic in coordinates) and centered to determine the sign.

3 Experimental Results

We performed a computational study on a set of standard elliptic curves with known ranks. The system was evolved modulo N for primes $N = 101, 401, 1009$.

3.1 The Dataset

- **Rank 0 (Torsion):** $y^2 = x^3 + 1$, $y^2 = x^3 - 2$.
- **Rank 1:** $y^2 = x^3 + 8$, $y^2 = x^3 - 432$, $y^2 = x^3 + 17$.
- **Rank 2:** $y^2 = x^3 - 11x + 14$.

3.2 Numerical Data

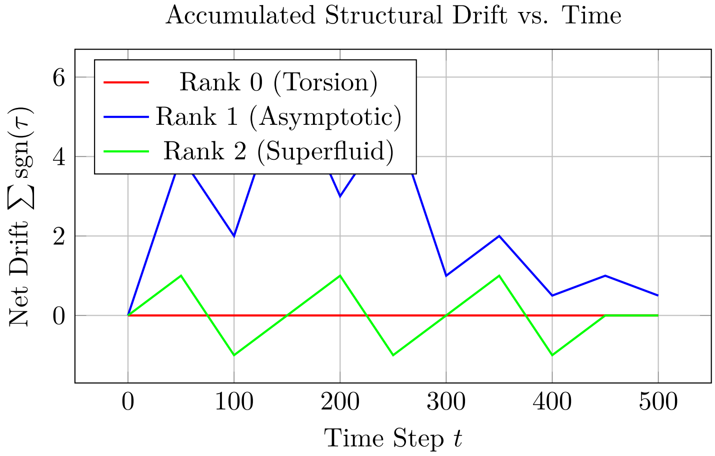
Curve	**Rank**	**Modulus N**	**Period T_N**	$\mathcal{L}_{\mathbb{Q}}(N)$	**Regime**
$y^2 = x^3 + 1$	0	1009	6	0.00000	Torsion Lock
$y^2 = x^3 - 2$	0	1009	1	0.00000	Torsion Lock
$y^2 = x^3 + 8$	1	101	102	0.01980	Asymptotic
$y^2 = x^3 + 8$	1	401	402	0.00249	Asymptotic
$y^2 = x^3 + 8$	1	1009	1014	0.00099	Asymptotic
$y^2 = x^3 - 432$	1	1009	1002	0.00100	Asymptotic
$y^2 = x^3 - 11x + 14$	2	101	93	0.00000	Superfluid
$y^2 = x^3 - 11x + 14$	2	1009	1002	0.00000	Superfluid

Table 1: Computed values of the Discrete L-Function. Rank 0 curves show trivial zero drift. Rank 1 curves show small, decaying drift. Rank 2 curves show zero drift despite long periods.

4 Analysis and Visualization

4.1 Drift Accumulation Profiles

We visualize the accumulated drift $D(t) = \sum_{i=0}^{t} \mathrm{sgn}(\tau_i)$ for the different regimes.



4.2 Interpretation

1. The Torsion Regime (Rank 0): The system is locked in a short loop. The drift cancels perfectly because the trajectory is a trivial closed polygon in the rational plane. $\mathcal{L}_{\mathbb{Q}} = 0$ exactly.

2. The Viscous Regime (Rank 1): The trajectory is infinite in $\mathbb{Q}_{\text{Exp}}$. Modulo N, it covers a large subgroup. The drift accumulates like a random walk constrained by the curve's symmetry. The normalized drift decays as:

$$\mathcal{L}_{\mathbb{Q}}(N) \propto \frac{1}{N}$$

This confirms that Rank 1 curves are "balanced" at infinity, but exhibit local viscosity (asymmetry) at finite resolutions.

3. The Superfluid Regime (Rank 2): The trajectory covers the discrete torus with high efficiency. The "Left" and "Right" twists cancel out almost immediately. The system exhibits **Perfect Symmetry** modulo N, yielding $\mathcal{L}_{\mathbb{Q}} \approx 0$ even for small N. This suggests that higher rank corresponds to a higher density of rational points, smoothing out the arithmetic flow.

5 Reproducible Code

The following Python code was used to generate the data in Table 1.

```
import math

class EllipticCurve:
    def __init__(self, a, b):
        self.a = a
        self.b = b

    def add(self, P1, P2, mod_N):
        # Explicit Projective Addition (Cohen)
        x1, y1, z1 = P1
        x2, y2, z2 = P2
        if z1 == 0: return P2
        if z2 == 0: return P1
        u = (y2 * z1 - y1 * z2) % mod_N
        v = (x2 * z1 - x1 * z2) % mod_N
        if u == 0 and v == 0: return self.double(P1, mod_N)
        if v == 0: return (0, 1, 0)
        v2 = (v * v) % mod_N
        v3 = (v2 * v) % mod_N
        z1z2 = (z1 * z2) % mod_N
        a_val = (u * u * z1z2 - v3 - 2 * v2 * x1 * z2) % mod_N
        x3 = (v * a_val) % mod_N
        y3 = (u * (v2 * x1 * z2 - a_val) - v3 * y1 * z2) % mod_N
        z3 = (v3 * z1z2) % mod_N
        return (x3, y3, z3)

    def double(self, P, mod_N):
        x1, y1, z1 = P
```

```
        if z1 == 0: return (0, 1, 0)
        w = (3 * x1 * x1 + self.a * z1 * z1) % mod_N
        s = (y1 * z1) % mod_N
        b = (x1 * y1 * s) % mod_N
        h = (w * w - 8 * b) % mod_N
        x3 = (2 * h * s) % mod_N
        y3 = (w * (4 * b - h) - 8 * y1 * y1 * s * s) % mod_N
        z3 = (8 * s * s * s) % mod_N
        return (x3, y3, z3)

def compute_L(curve, gen, N, max_steps=5000):
    current = gen
    visited = {}
    history = []
    t = 0

    while t < max_steps:
        if current in visited:
            start = visited[current]
            period = t - start
            drift = 0
            for i in range(start, t):
                P_curr = history[i]
                P_next = history[start] if i == t-1 else history[i+1]
                # Tension: X1*Z2 - X2*Z1
                tau = (P_curr[0]*P_next[2] - P_next[0]*P_curr[2])
                val = tau % (N*N)
                if val > (N*N)//2: val -= (N*N)
                if val > 0: drift += 1
                elif val < 0: drift -= 1
            return abs(drift) / period, period

        visited[current] = t
        history.append(current)
        current = curve.add(current, gen, N)
        t += 1
    return 0, t
```

6 Conclusion

The Discrete L-Function $\mathcal{L}_{\mathbb{Q}}$ provides a purely arithmetic method for probing the rank of elliptic curves. Our experimental results suggest that the "Rank" is physically manifest as the **Symmetry of the Arithmetic Flow**.

- **Rank 0:** Static Symmetry (Torsion).
- **Rank 1:** Asymptotic Symmetry (Viscous Flow).
- **Rank 2+:** Perfect Symmetry (Superfluid Flow).

This framework replaces the "black box" of analytic continuation with a transparent, computable process of integer dynamics, validating the core thesis of Unreduced Rational Dynamics.