Poincaré Schemes

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Abstract

We do stuff

N: Change this

Contents

1 Introduction

1.1 Azumaya algebras and their involutions

Let X be a scheme. Gabber showed that if X is quasi-compact and separated and admits an ample line bundle, the collection of sheaves of Azumaya $\mathcal{O}_X$-algebras up to Morita equivalence is in bijection with the torsion subgroup of $H^2_{\text{ét}}(X;\mathbb{G}_m)$ [Gab81]. On the other hand, to $\mathcal{A}$ we may associate the presentable $\mathcal{O}_X$-linear ∞-category $\mathcal{D}(\mathcal{A})$. If $\mathcal{A}$ and $\mathcal{A}'$ are Morita equivalent, then $\mathcal{D}(\mathcal{A}')$ and $\mathcal{D}(\mathcal{A})$ are equivalent in $\mathrm{Pr}^L_{\mathcal{O}_X}$, and that $\mathcal{A}$ is Azumaya implies that $\mathcal{D}(\mathcal{A})$ is invertible in $\mathrm{Pr}^L_{\mathcal{O}_X}$ with respect to the $\mathcal{O}_X$-linear tensor product.

L: AG notation or homotopy notation?

L: who is this observation due to?

L: cite Lieblich/de Jong? connect to twisted sheaves!

Definition 1.1. The *derived Brauer space* of X is $\mathrm{dBr}(X) := \mathrm{Pic}\left(\mathrm{Pr}^L_{\mathcal{O}_X}\right)$.

Toën shows that the assignment $X \mapsto \mathrm{dBr}(X)$ is an étale sheaf and there is an isomorphism $\pi_0 \mathrm{dBr}(X) \simeq H^1_{\text{ét}}(X;\mathbb{Z}) \times H^2_{\text{ét}}(X;\mathbb{G}_m)$ [Toë12, Corollary 2.12]. Furthermore, Toën shows that for any qcqs scheme X, any invertible $\mathcal{O}_X$-linear ∞-category $\mathcal{C}$ has a compact generator G; thus we may write $\mathcal{C} \simeq \mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(G)}(\mathrm{Mod}_{\mathcal{O}_X})$ and $\mathrm{End}_{\mathcal{C}}(G)$ is said to be a *derived/generalized (sheaf of) Azumaya algebras* over $\mathcal{O}_X$ (see Theorem 3.7 and Corollary 3.8 of [Toë12]). In particular, Toën's result gives concrete interpretations/realizations of all/not necessarily torsion classes in $H^2_{\text{ét}}(X;\mathbb{G}_m)$, at the cost of using derived methods/considering derived objects.

Antieau and Gepner extended Toën's work to connective ring spectra/affine spectral schemes with connective rings of functions in [AG14].

L: cite this too?

On the other hand, the theory of anti-involutions on Azumaya algebras is an essential tool for studying the behavior of Brauer classes under multiplication by 2 and/or norm maps/corestriction. An anti-involution on an Azumaya algebra over a ring R is an equivalence $\sigma\colon A \xrightarrow{\sim} A^{\mathrm{op}}$ so that $\sigma^{\mathrm{op}} \circ \sigma = \mathrm{id}$. The anti-involution is said to be of type 1 if it acts by the identity on $R =$ the center of A. If instead R is endowed with an involution λ so that the inclusion of the subring of fixed elements $R^\lambda \to R$ exhibits R as a quadratic étale R^λ-algebra and σ agrees with λ on the center of A, then the involution σ is said to be of *type 2*.

If an Azumaya algebra A has an anti-involution, its Brauer class $[A] \in H^2_{ét}(-;\mathbb{G}_m)$ is manifestly 2-torsion. More surprisingly, if an Azumaya algebra A is such that $[A]$ lies in the 2-torsion subgroup of $H^2_{ét}(-;\mathbb{G}_m)$, then there exists an Azumaya algebra A' in the Brauer class of A admitting an anti-involution; the result was proved for R a field by Albert (in fact Albert proves that one can take $A' = A$), for an arbitrary ring by Saltman, and for schemes X with $\frac{1}{2} \in \mathcal{O}_X$ by Parimala–Srinivas [Alb61, §9 Theorem 19; Sal78, Theorem 3.1(a); PS92, Theorem 1].

On the other hand, consider an étale cover $X \to Y$ of degree 2 where $\frac{1}{2} \in \mathcal{O}_Y$, and let λ denote the nontrivial C_2-Galois action on X. There is an *involutive Brauer group* $\mathrm{Br}\,(X,\lambda)$ consisting of equivalence classes of sheaves of Azumaya $\mathcal{O}_X$-algebras with involutions of the second kind [PS92, p.216]. Parimala–Srinivas showed that the involutive Brauer group sits in an exact sequence $\mathrm{Pic}\,X \xrightarrow{N} \mathrm{Pic}\,Y \to \mathrm{Br}\,(X,\lambda) \to \mathrm{Br}\,(X) \xrightarrow{N} \mathrm{Br}\,(Y)$ [PS92, Theorem 2].

First and Williams observed that the aforementioned two cases comprise two extreme ends/special cases of a spectrum: multiplication by 2 on $H^2_{ét}(-;\mathbb{G}_m)$ may be regarded as a cohomological C_2-transfer map along the 'quotient' map $X = X$, where X is regarded as a scheme with trivial C_2-action [FW20, §1.2]. In other words, the former comprises trivial C_2-actions with 'everywhere ramified' quotient map, whereas the latter comprises free C_2-actions with nowhere ramified quotient maps. Moreover, First and William observe that a quotient involves a choice/is extra data; from the perspective of the stacky quotient $X \to X//C_2$, the C_2-action on X is free. Thus in order to study involutions systematically, it is necessary to specify a scheme with involution and the choice of a quotient.

L: connect to 'many choices of different Poincaré structures' on a given ∞-category eventually

The study of involutions on Azumaya algebras goes hand-in-hand with/is inextricably linked to the study of symmetric bilinear/hermitian forms on vector bundles: étale-locally, (classical) Azumaya algebras can be described as endomorphism algebras of vector bundles $\mathcal{A} \simeq \mathcal{E}\mathrm{nd}_X(V)$. Taking transposes and conjugating by an isomorphism $q\colon V \to V^\vee$ adjoint to a nondegenerate pairing $q\colon V \otimes V \to \mathcal{O}_X$ comprise a 'prototypical' example of an involution on $\mathcal{E}\mathrm{nd}_X(V)$. In [Cal+20a; Cal+20b; Cal+21], Calmès–Dotto–Harpaz–Hebestreit–Land–Moi–Nardin–Nikolaus–Steimle introduce a framework for talking about objects of stable ∞-categories equipped with the data of nondegenerate hermitian forms. Motivated by this connection between involutions and duality (cf. [Cal+20a, §3]), we use Poincaré ∞-categories to define a derived enhancement of the involutive Brauer group.

L: this is *so close* to Spec of a Poincaré ring but also not quite.

L: insert adjectives

L: also cite Jacob

1.2 Main results

Theorem 1.2. *Let $\underline{A}$ be an affine Poincaré scheme with underlying $\mathbb{E}_\infty$-ring spectrum with involution A. Then the natural maps*

$$\pi_i(\mathrm{PnPic}(\underline{A})) \to \pi_i(\mathrm{Pic}(A))$$

are surjective on 2-torsion.

N: I think there is some interaction with the homotopy fixed points, or maybe even the genuine

Theorem 1.3. *Let A be an $\mathbb{E}_\infty$ ring with involution, and let $\underline{NA}$ be the associated Tate affine Poincaré scheme. Let $\mathrm{Br}_\nu(A)$ be the Brauer group of Azumaya algebras over A with involution. Then the natural map*

$$\mathrm{PnBr}(\underline{NA}) \to \mathrm{Br}_\nu(A)$$

is an equivalence

N: probably of $\mathbb{E}_\infty$ do-dads

Theorem 1.4. *The functors* $\mathrm{PnPic}, \mathrm{PnBr} : \mathrm{APS} \to \mathrm{Sp}$ *are fppf sheaves.*

Theorem 1.5. *There is a Poincaré group scheme* $\mathbb{G}_m^{\text{♀}}$ *such that*

$$B\mathbb{G}_m^{\text{♀}} \simeq \mathrm{PnPic}$$

as fppf stacks.

1.3 Outline

1.4 Acknowledgements

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1.5 Conventions

Br^{p}	Poincaré Brauer space
CAlg	∞-categoriy of $\mathbf{E}_\infty$-ring spectra
$\mathrm{CAlg}(\mathcal{S})$	∞-categoriy of $\mathbf{E}_\infty$-spaces
$\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$	∞-categoriy of grouplike $\mathbf{E}_\infty$-spaces
$\mathrm{CAlg}^{\mathrm{p}}$	∞-categoriy of Poincaré ring spectra
$\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$	∞-category of small stable ∞-categories and exact functors
$\mathrm{Cat}_\infty^{\mathrm{p}}$	∞-category of Poincaré ∞-categories
$\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}$	∞-category of idempotent complete Poincaré ∞-categories
$\mathrm{Pic}^{\mathrm{p}}$	Poincaré Picard space
$\mathcal{S}$	∞-category of spaces
Sp	∞-category of spectra

2 Poincaré Structures on Compact Modules

We will use this section to recall notions and results about Poincaré ∞-categories which we require in the sections to follow. This section can safely be skipped by anyone with extensive knowledge of Poincaré ∞-categories, as found in [Cal+20a].

Notation 2.1. Let R be an $\mathbf{E}_\infty$-ring spectrum. We will drop $\mathbf{E}_\infty$ from our notation and simply call R a *ring spectrum.* Moreover, we will denote the ∞-category $\mathrm{CAlg}(\mathrm{Sp})$ of commutative algebra objects in the ∞-category of spectra Sp by CAlg.

Let R be a ring spectrum and let Mod_R be the ∞-category of modules over R. We will study Poincaré structures on the ∞-category Mod_R^ω of compact modules over R.

V: - characterizati in terms of modules with genuine involution, - characterizati of symmetric monoidal structures, -Pn

3 Poincaré Schemes

We will now specify the objects which we are able to take the Poincaré Picard and Brauer spaces of. These will be schemes which are equipped with a Poincaré structure on their derived categories which is compatible with the symmetric monoidal structure. While this definition is simple and convenient, we find it both technically useful and psychologically comforting to have a definition of such objects closer to a scheme with an involution. We will start by looking at affine objects.

3.1 Poincaré rings

Definition 3.1. Let R be a ring spectrum. A *Poincaré structure* on R is the following data:

- A C_2-action on R via maps of ring spectra, i.e. a functor $\lambda : BC_2 \to \mathrm{CAlg}$.
- An $\mathbb{E}_\infty$-R-algebra $R \to C$.
- An $\mathbb{E}_\infty$-R-algebra map $C \to R^{tC_2}$.

Here R^{tC_2} is the Tate construction with respect to the given action. Since the Tate construction is lax symmetric monoidal, R^{tC_2} is naturally an R-algebra via the Tate-valued norm. A ring spectrum equipped with a Poincaré structure will be called a *Poincaré ring spectrum*.

Remark 3.2. In [Cal+20a, discussion immediately preceding Examples 5.4.10], Poincaré ring spectra are referred to as $\mathbf{E}_\infty$*-ring spectra with genuine involution*.

Remark 3.3. Let R be a ring spectrum. By [Cal+20a, Corollary 5.4.8], a Poincaré structure on R gives rise to a symmetric monoidal lift of Mod_R^ω to the symmetric monoidal ∞-category of Poincaré ∞-categories $Ϙ_R : (\mathrm{Mod}_R^\omega)^{\mathrm{op}} \to \mathrm{Sp}$. Furthermore, the structure map $R \to C$ gives a canonical lift of $R \in \mathrm{Mod}_R^\omega$ to a Poincaré object $(R, q) \in \mathrm{Pn}\left(\mathrm{Mod}_R^\omega, Ϙ_R\right)$.

Remark 3.4. A Poincaré structure on a ring spectrum R with a C_2-action via maps of ring spectra is a factorization $R \to C \to R^{tC_2}$ in CAlg of the Tate Frobenius $R \to R^{tC_2}$.

Observation 3.5. There is a forgetful functor from C_2-$\mathbb{E}_\infty$-algebras to Poincaré rings which forgets the C_2-equivariance of the map $R \to R^{\varphi C_2}$.

Remark 3.6. By [Cal+20a, §5.1], the assignment $(R, R \to C \to R^{tC_2}) \mapsto (\mathrm{Mod}_R^\omega, Ϙ_R)$ promotes to a symmetric monoidal functor $\mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{p}})$.

Notation 3.7. Let R be an $\mathbb{E}_\infty$-ring spectrum. We will denote by $\underline{R}$ the spectrum R with trivial C_2-action. More precisely, $\underline{R} : BC_2 \to \mathrm{Sp}$ is the constant functor.

L: This is commonly used for constant Mackey functors–could be ambiguous

Example 3.8. Let R be a ring spectrum with a C_2-action. If $2 \in \pi_0(R)$ is invertible, we have $\underline{R}^{tC_2} \simeq 0$ by [NS18, Lemma I.2.8]. A Poincaré structure on R is equivalent to the data of an $\mathbb{E}_\infty$-R-algebra $R \to C$.

Example 3.9. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The Tate-valued norm endows R^{tC_2} with a natural R-algebra structure, which induces a Poincaré structure on R given by the factorization $R \xrightarrow{\mathrm{id}} R \to R^{tC_2}$. We will call this Poincaré structure the *Tate Poincaré structure on* R and will denote it by $(R, Ϙ_R^{\mathrm{t}})$.

Example 3.10. The sphere spectrum $\mathbb{S}$ together with the Tate Poincaré structure will be called the *universal Poincaré ring spectrum* (see [Cal+20a, §4.1]). We will denote it by $(\mathbb{S}, Ϙ_u)$.

Remark 3.11. Let $(R, Ϙ)$ be a ring spectrum associated to a factorization $R \to C \to R^{tC_2}$. A factorization of the map $C \to R^{tC_2}$ through R^{hC_2} induces a section of the canonical map $Ϙ(R) \to \hom_R(R, C) \simeq C$. In that case, we have a splitting $Ϙ(R) \simeq R_{hC_2} \oplus C$.

V: reference pullback that characterizes all quadratic functors

Example 3.12. The Tate Frobenius for the sphere spectrum factors through $\mathbb{S}^{hC_2}$. Therefore, Remark 3.11 implies $Ϙ_u(\mathbb{S}) \simeq \mathbb{S}_{hC_2} \oplus \mathbb{S} \simeq \Sigma^\infty(\mathbb{P}_{\mathbb{R}}^\infty) \oplus \mathbb{S}$.

Example 3.13. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The identity map $\mathrm{id} : R^{tC_2} \to R^{tC_2}$ induces a Poincaré structure on R given by the factorization $R \to R^{tC_2} \xrightarrow{id} R^{tC_2}$. We will call this Poincaré structure the *symmetric Poincaré structure on* R.

Example 3.14. Let R be a connective ring spectrum equipped with a C_2-action via maps of ring spectra. The connective cover $\tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$ of R^{tC_2} induces a Poincaré structure on R given by the factorization $R \to \tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$. We will call this Poincaré structure the *genuine symmetric Poincaré structure on* R.

Example 3.15. Let R be a commutative ring endowed with an involution $\sigma \colon R \xrightarrow{\sim} R$. Write $\underline{R}^\sigma$ for the C_2-Green functor with C_2-fixed points R^{C_2}, where R^{C_2} denotes the strict fixed points of the C_2-action on R, and underlying object R. The Mackey functor $\underline{R}^\sigma$ is a C_2-$\mathbb{E}_\infty$ ring, therefore in particular we may regard it as a Poincaré ring by Observation 3.5. This is a special case of Example 3.14.

V: copy more examples from notes

3.2 From schemes with involution to Poincaré structures on module categories

We will now turn our attention to the non-affine case. In this setting we will again want to work with schemes with some notion of a genuine C_2-structure as our model, and then show that this leads to the structure of a scheme together with a symmetric monoidal structure on its derived category.

Philosophically, a scheme with genuine C_2-action should, via a recollement, be given by a scheme X with an involution together with a choice of genuine C_2 quotient $X \to Y$ satisfying certain conditions. It turns out that such a notion has already appeared in the literature on Azumaya algebras with involution.

Recollection 3.16 ([FW20, Remark 4.20])**.** Let X be a scheme with an involution $\lambda\colon X \to X$. A map $\pi\colon X \to Y$ is called a good quotient of X relative to λ if π is C_2-invariant and affine and $\pi_\#\colon \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ induces an isomorphism $\mathcal{O}_Y \simeq (\pi_*\mathcal{O}_X)^{C_2}$. A good quotient of X exists if and only if every C_2-orbit is contained in an affine open subscheme, in which case it is unique up to isomorphism.

Definition 3.17. Define a category qSch^{C_2} so that

- an object of qSch^{C_2} consists of the data of qcqs schemes X and Y, an involution $\lambda\colon X \to X$, and a morphism $p\colon X \to Y$ which exhibits Y as a *good quotient* of the involution on X in the sense of [FW20, Remark 4.20].
- a morphism from (X, λ, Y, p) to (Z, ν, W, q) consists of a C_2-equivariant morphism $X \to Z$ and a morphism $Y \to W$ so that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & W \end{tikzcd}$$

commutes.

Remark 3.18. Suppose (X, λ, Y, p) is an object of qSch^{C_2} and $j\colon U \to Y$ is a flat map. Then $(X_U, \lambda|_U, U, p|_U)$ is an object of qSch^{C_2}. Affineness and invariance under the C_2-action are stable under pullback, so it suffices to show that $j^*(\pi)$ satisfies $\mathcal{O}_U \simeq (j^*(\pi))_* (\mathcal{O}_{j^*U})^{C_2}$. This follows from the proof of [FW20, Theorem 4.35(i)].

Proposition 3.19. *Write $U\colon \mathrm{qSch}^{C_2} \to \mathrm{qSch}$ for the functor so that $U(X, \lambda, Y, p) = X$. The category qSch^{C_2} has a symmetric monoidal structure $\boxtimes$ so that U is symmetric monoidal, where qSch is endowed with the product symmetric monoidal structure.*

Proof. If X, Z are schemes with involutions λ_X, λ_Z, then $\lambda_X \times \lambda_Z$ endows $X \times Z$ with an involution. It suffices to show that $X \times Z$ admits a good quotient, as a good quotient is a categorical quotient and is therefore unique up to isomorphism. By [FW20, Remark 4.20], a good quotient exists if and only if every C_2-orbit is contained in an affine open subscheme. Consider a C_2-orbit in $X \times Z$. Its image under the projection $\pi_1\colon X \times Z \to X$ (resp. $\pi_2\colon X \times Z \to Z$) is contained in an affine open subscheme $U \subseteq X$ (resp. $V \subseteq Z$). Thus the orbit under consideration is contained in $U \times Z$, which is affine. □

Construction 3.20. Assume that X has a *good quotient* Y in the sense of [FW20, Remark 4.20]. We write $p\colon X \to Y$ for the quotient map. Let $j\colon \operatorname{Spec} A \simeq U \subseteq Y$ be an affine open subscheme of Y. Because p is an affine map, the fiber product $\operatorname{Spec} B := \operatorname{Spec} A \times_Y X$ is an affine open of X which is invariant under the C_2-action. In particular $\operatorname{Spec} B$ inherits a C_2-action from X (hence so does its ring of functions B). Now $A \to B$ acquires the structure of a C_2-Green functor $\underline{\mathcal{O}}(U)$. Regarding $\underline{\mathcal{O}}(U)$ as a C_2-spectrum, by the isotropy separation sequence, we have an equivalence of A-modules $\underline{\mathcal{O}}(U)^{\varphi C_2} \simeq \operatorname{cofib}(\mathrm{tr}\colon B_{hC_2} \to A)$.

Lemma 3.21. *Let X be a scheme with an involution. Assume that X has a* good quotient *Y in the sense of [FW20, Remark 4.20], and write $p\colon X \to Y$ for the quotient map.*

(i) The assignment of Construction 3.20 lifts to a contravariant functor from (the nerve of) the category of affine opens of Y to the ∞-category of Poincaré rings/C_2-$\mathbb{E}_\infty$-rings/Tambara functors.

(ii) The presheaf $\underline{\mathcal{O}}$ of (i) defines a Zariski sheaf.

(iii) Write $p_*\mathcal{O}_X$ for the sheaf of $\mathbb{E}_\infty$-$\mathcal{O}_Y$-algebras (all functors are derived). Then the pushforward p_* induces an equivalence $\mathcal{D}(X) \xrightarrow{\sim} \mathrm{Mod}_{p_*\mathcal{O}_X}$.

Proof. Part (i) follows from a similar argument to [Yan23, Theorem 5.1]; functoriality follows from noting that $\tau_{\geq 0}$ is a functor. Part (ii) follows from Lemma 3.22. To prove part (iii), consider a Zariski cover $\{j_i \colon U_i \to Y\}$ of Y by affine opens. By Zariski descent, $\mathcal{D}(X) \simeq \lim_{p^*(j_i)=p\times_Y j_i \colon U_i\times_Y X\to X} \mathrm{Mod}_{\mathcal{O}_X(U_i\times_Y X)}$ and $\mathrm{Mod}_{p_*(\mathcal{O}_X)} \simeq \lim_{j_i \colon U_i\to Y} \mathrm{Mod}_{p_*\mathcal{O}_X(U_i)}$, hence the result follows. □

Lemma 3.22. *Let K be a simplicial set, and let $f\colon K^{\triangleleft} \to C_2\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_2})$ be a diagram. Then f is a limit diagram if and only if $f^e\colon K^{\triangleleft} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ and $f^{C_2}\colon K^{\triangleleft} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ are both limit diagrams.*

Proof. The result follows from the observation that limits in $\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right)$ are computed in Sp^{C_2}. □

Construction 3.23. Let $p\colon X \to Y$ as before. Consider the composites

$$\begin{aligned} \mathrm{Mod}_{\underline{\mathcal{O}}}\colon \mathrm{Op}(Y)^{\mathrm{op}} &\xrightarrow{\underline{\mathcal{O}}} C_2\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{\mathrm{Mod}_{(-)}} \mathcal{C}\mathrm{at} \\ \mathrm{Mod}^{\otimes}_{\underline{\mathcal{O}}}\colon \mathrm{Op}(Y)^{\mathrm{op}} &\xrightarrow{\underline{\mathcal{O}}} C_2\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{\mathrm{Mod}^{\otimes}_{(-)}} C_2\otimes\mathcal{C}\mathrm{at}\,, \end{aligned} \tag{3.24}$$

where $C_2\otimes\mathcal{C}\mathrm{at}$ denotes the ∞-category of (small) C_2-symmetric monoidal C_2-∞-categories. In the notation of Construction 3.20, this functor sends the affine open $\operatorname{Spec} A \subseteq Y$ to the category of modules in C_2-spectra over the C_2-$\mathbb{E}_\infty$-algebra which has underlying C_2-Mackey functor $A \to B$. Define $\mathrm{Mod}_{\underline{\mathcal{O}}}$, $\mathrm{Mod}^{\otimes}_{\underline{\mathcal{O}}}$ to be the limits in $\mathcal{C}\mathrm{at}$, $C_2\otimes\mathcal{C}\mathrm{at}$, resp. of the functors in (3.24). In particular, if we write $s\colon \int \mathrm{Mod}_{\underline{\mathcal{O}}} \to \mathrm{Op}(Y)^{\mathrm{op}}$ for the cocartesian fibration obtained by taking the Grothendieck construction on (3.24), an object of $\mathrm{Mod}_{\underline{\mathcal{O}}}$ is a cocartesian section of s. In other words, it is a choice, for each affine open $\operatorname{Spec} A$ of Y (same notation as before), of a module over the C_2-$\mathbb{E}_\infty$-algebra which has underlying C_2-Mackey functor $A \to B$ which glue compatibly.

L: bleh...cardinals

L: invent better notation later

Observe that for each $A \to B$, there is a quadratic norm functor $N^{C_2}\colon \mathrm{Mod}_B(\mathrm{Sp}) \to \mathrm{Mod}_{N^{C_2}B}\left(\mathrm{Sp}^{C_2}\right)$ and a quadratic relative norm functor $N^{C_2}\colon \mathrm{Mod}_B(\mathrm{Sp}) \to \mathrm{Mod}_{A\to B}\left(\mathrm{Sp}^{C_2}\right)$.

Construction 3.25. Let X be a scheme with an involution, and let $p\colon X \to Y$ exhibit Y as a good quotient of X. Assume that p is affine. The norm functors (resp. relative norm functors) $N_e^{C_2}$ assemble under Construction 3.20 to a 'global' norm functor $N_Y^{C_2}\colon \pi_{\#}\mathcal{O}_X\,\mathrm{Mod} \to N^{C_2}\pi_{\#}\mathcal{O}_X\,\mathrm{Mod}$ (resp. relative norm functor $N_Y^{C_2}\colon \pi_{\#}\mathcal{O}_X\,\mathrm{Mod} \to \underline{\mathcal{O}}\,\mathrm{Mod}$). Moreover, these functors are quadratic.

For each affine open $j\colon \operatorname{Spec} A \subseteq Y$, write $B = \Gamma(\mathcal{O}_{\operatorname{Spec} A\times_Y X})$, consider the composite

$$\pi_{\#}\mathcal{O}_X\,\mathrm{Mod} \xrightarrow{j^*} \mathrm{Mod}_B(\mathrm{Sp}) \xrightarrow{N^{C_2}} \mathrm{Mod}_{N^{C_2}B}(\mathrm{Sp}^{C_2}) \xrightarrow{-\otimes_{N^{C_2}B}(A\to B)} \mathrm{Mod}_{A\to B}(\mathrm{Sp}^{C_2})\,,$$

where the last map is base change along the map $N^{C_2}B \to (A \to B)$ which is a structure map for the C_2-$\mathbb{E}_\infty$-algebra structure on $A \to B$. Now since quadratic functors are closed under limits [Lur17, Theorem 6.1.1.10] and $N_Y^{C_2}$ can be written as a limit of a diagram of quadratic functors, $N_Y^{C_2}$ is also quadratic.

L: todo: use effective descent/limit definition for $\underline{\mathcal{O}}$-modules.

Definition 3.26. Varying $X \to Y$, Constructions 3.23 and 3.25 define a functor

$$\begin{aligned} \left(\mathrm{qSch}^{C_2}\right)^{\mathrm{op}} &\to C_2\otimes\mathcal{C}\mathrm{at} \\ (X,\lambda,Y,p) &\mapsto \underline{\mathrm{Mod}}_{\underline{\mathcal{O}}}\left(\underline{\mathrm{Sp}}^{C_2}\right) \end{aligned}$$

L: want: codomain consists of C_2-stable C_2-presentable C_2-∞-categories

Definition 3.27. Suppose $\mathcal{C}$ is a C_2-stable C_2-symmetric monoidal C_2-∞-category. Define a functor

$$\begin{aligned} \mathrm{eInv}\colon C_2\otimes\mathcal{C}\mathrm{at}^{\mathrm{ex}} &\to \mathcal{S} \\ (\mathcal{C},\otimes) &\mapsto \left(\mathrm{C}^{C_2}\right)^{\simeq} \times_{(\mathrm{C}^e)^{\simeq,hC_2}} \mathrm{Pic}(\mathrm{C}^e)^{\simeq,hC_2}\,. \end{aligned}$$

In other words, eInv sends a C_2-symmetric monoidal C_2-∞-category to the full subgroupoid of $\mathcal{C}^{C_2}$ on those objects L so that L^e is an invertible object in $\mathcal{C}^e$.

Write $\widetilde{\text{eInv}}$ for the Grothendieck construction on eInv.

There is a functor

$$\begin{aligned} \widetilde{\text{eInv}} &\to \mathbb{E}_\infty\text{Alg}\left(\text{Cat}^h\right) \\ (\mathcal{C}, L) &\mapsto (\mathcal{C}^e, \mathcal{C}^e \xrightarrow{N_{\mathcal{C}}} \mathcal{C}^{C_2} \xrightarrow{\hom_{\mathcal{C}^{C_2}}(-,L)} \text{Sp}) \end{aligned} \tag{3.28}$$

Lemma 3.29. *The functor of (3.28) lifts to a functor* $\widetilde{\text{eInv}} \to \text{Cat}^{\text{p}}_\infty$.

Lemma 3.30. *Let X be a scheme with involution $\sigma\colon X \xrightarrow{\sim} X$ equipped with a good quotient $\pi\colon X \to Y$. Let L be a line bundle on Y. Then the canonical map*

$$L \to \pi_\# \pi^* L \tag{3.31}$$

promotes (3.31) to a sheaf of $\underline{\mathcal{O}}$-modules on Y. We will write $\underline{L}$ for (3.31).

L: Pretty sure distributive norm functors are 2-excisive (results here should generalize readily)...if not, can define the problem away.

Proof. Follows from naturality of the unit and the canonical identification $\pi^*\mathcal{O}_Y$ with $\mathcal{O}_X$. □

L: Example: Special case where X has trivial C_2 action and $X = Y$.

Definition 3.32. Let X be a scheme with involution $\sigma\colon X \xrightarrow{\sim} X$ equipped with a good quotient $\pi\colon X \to Y$. Let L be a line bundle on Y. Define $\Omega_{\sigma,L}$ to be the functor

$$\text{Perf}_X^{\text{op}} \xrightarrow{\pi_\#} \pi_\#\mathcal{O}_X \,\text{Mod}^{\omega,\text{op}} \xrightarrow{N^{C_2}} N^{C_2}\pi_\#\mathcal{O}_X \,\text{Mod}\left(\text{Sp}^{C_2}\right)^{\text{op}} \xrightarrow{\hom_{N^{C_2}\pi_\#\mathcal{O}_X}(-,\underline{L})} \text{Sp}\,,$$

where $\underline{L}$ is a $\underline{\mathcal{O}}$-module by Lemma 3.30 and $\underline{\mathcal{O}}$ is a $N^{C_2}\pi_\#\mathcal{O}_X$-algebra by Lemma 3.21. By Construction 3.25 and the fact that the composite of an exact (1-excisive) functor and an m-excisive functor is m-excisive (see [Bar+22, §2.2]), $\Omega_{\sigma,L}$ is quadratic.

Example 3.33. Suppose $L = \mathcal{O}_Y$. Then we drop L from notation and the quadratic functor Ω_σ of Definition 3.32 takes the form

$$\text{Perf}_X^{\text{op}} \xrightarrow{\pi_\#} \pi_\#\mathcal{O}_X \,\text{Mod}^{\omega,\text{op}} \xrightarrow{N^{C_2}} N^{C_2}\pi_\#\mathcal{O}_X \,\text{Mod}\left(\text{Sp}^{C_2}\right)^{\text{op}} \xrightarrow{\hom_{N^{C_2}\pi_\#\mathcal{O}_X}(-,\underline{\mathcal{O}})} \text{Sp}\,.$$

Lemma 3.34. *Let X be a scheme with involution $\sigma\colon X \xrightarrow{\sim} X$, and let Y be a good quotient of X. Let L be a line bundle on Y, and let $\Omega_{\sigma,L}$ be the quadratic functor on* Perf_X *of Definition 3.32. Then the bilinear part of $\Omega_{\sigma,L}$ agrees with that of Observation 5.4. In particular,* $(\text{Perf}_X, \Omega_{\sigma,L})$ *is a Poincaré ∞-category.*

Proof. By definition of the bilinear part of a quadratic functor, it suffices to show that there is an equivalence $\hom_{\pi_\#\mathcal{O}_X\,\text{Mod}}\left(\pi_\# E \otimes_{\pi_\#\mathcal{O}_X} \pi_\# E, \pi_\#\mathcal{O}_X\right) \simeq \hom_{\mathcal{O}_X\,\text{Mod}}\left(E \otimes_{\mathcal{O}_X} \sigma^* E, \mathcal{O}_X\right)$ for any perfect complex E on X. This follows from Lemma 3.21(iii). □

Remark 3.35. Compare the description of the space of bilinear forms in Lemma 3.34 with the description of a δ-hermitian form H in [PS92, p. 216].

4 The Poincaré Picard space

Recall that the Poincaré space functor $\text{Pn}\colon \text{Cat}^{\text{p}}_\infty \to \text{CAlg}(\mathcal{S})$ is lax symmetric monoidal with respect to tensor product of Poincaré ∞-categories and smash product of $\mathbf{E}_\infty$-spaces [Cal+20a, Corollary 5.2.8]. In particular, we can consider invertible objects in $\text{Pn}(A)$ for a Poincaré ring spectrum A.

Definition 4.1. Let A be a Poincaré ring spectrum. We define the *Picard space of A* to be

$$\mathrm{Pic}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Pn}(A)).$$

For $(X, \lambda, Y, p) \in \mathrm{qSch}^{C_2}$ we similarly define

$$\mathrm{Pic}^{\mathrm{p}}(X, \lambda, Y, p) := \mathrm{Pic}(\mathrm{Pn}(\mathrm{Perf}_X, \Qoppa_\sigma)).$$

Remark 4.2. Let $(\mathrm{Mod}_R^\omega, \Qoppa_R)$ be a Poincaré ring spectrum, where $(M_R = R, N_R = R^{\varphi C_2}, R^{\varphi C_2} \to R^{tC_2})$ is the module with genuine involution associated to $\Qoppa_R$. Then a point in the Poincaré Picard space is the data of a pair $(\mathcal{L}, q)$, where $\mathcal{L}$ is an invertible module in Mod_R^ω and q is a point in $\Omega^\infty \Qoppa_R(\mathcal{L})$. By [Cal+20a, Proposition 1.3.11], the data of q is equivalent to the data of points in the lower left and upper right corner of the square

$$\begin{tikzcd} \Qoppa(\mathcal{L}) \arrow[r] \arrow[d] & \hom_R(\mathcal{L}, R^{\varphi C_2}) \ni \ell(q) \arrow[d] \\ b(q) \in \hom_{R\otimes R}(\mathcal{L} \otimes \mathcal{L}, R)^{hC_2} \arrow[r] & \hom_R(\mathcal{L}, R^{tC_2}) \end{tikzcd} \tag{4.3}$$

and a path between their images in the lower right corner. In particular, the adjoint of $b(q)$ must define a nondegenerate hermitian form on $\mathcal{L}$, that is, an equivalence $\mathcal{L} \simeq \hom_R(\mathcal{L}, R^*)$ where R^* is considered as an R-module via the action of the generator of C_2.

L: add equivariance/symmetry data

Write $(\mathcal{L}^\vee, q^\vee)$ is for the inverse of $(\mathcal{L}, q)$. By definition of invertibility, there exists an R-linear map $\ell(q^\vee) \colon \mathcal{L}^\vee \to R^{\varphi C_2}$ so that the following diagram commutes

$$\begin{tikzcd} \mathcal{L} \otimes_R \mathcal{L}^\vee \arrow[r, "\ell(q)\otimes\ell(q^\vee)"] \arrow[d, "\mathrm{ev}", "\sim"'] & R^{\varphi C_2} \otimes_R R^{\varphi C_2} \arrow[d, "\text{multiplication}"] \\ R \arrow[r, "\text{given}"] & N_R \end{tikzcd} \tag{4.4}$$

Lemma 4.5. *Let $(R, \Qoppa)$ be a connective Poincaré ring spectrum. Then, for any integer n, the spectrum $\Qoppa(\Sigma^n R)$ is $(-2n)$-connective.*

V: define connectivity and make conditions here precise. As stated this works for R and C connective. More precisely, $conn(\Qoppa(\Sigma^n R)$ $\min(conn(\Sigma^-$

Proof. This follows from the fiber sequence

$$(\Sigma^{-2n} R)_{hC_2} \to \Qoppa(\Sigma^n R) \to \hom_R(\Sigma^n R, C) \simeq \Sigma^{-n} C.$$

$\square$

Remark 4.6. The functor $\mathrm{Pic}^{\mathrm{p}} : \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$ preserves certain structures. Let A be a Poincaré ring spectrum. Since A is a module over $(\mathbb{S}, \Qoppa_u)$, the space $\mathrm{Pic}^{\mathrm{p}}(A)$ is something over $\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Qoppa_u)$.

V: write out details

Since the forgetful functor $\mathrm{Pn}(A) \to \mathrm{Mod}_A^\omega$ is symmetric monoidal we get an induced map

$$U : \mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)$$

of spectra. For a point $(\mathcal{L}, q) \in \pi_0(\mathrm{Pic}^{\mathrm{p}}(A))$ we will refer to $\mathcal{L} := U(\mathcal{L}, q)$ as the *underlying invertible module.* Note that the A-module A^* is (nonequivariantly) isomorphic to A via the involution, and so the fact that $\mathcal{L} \simeq \hom_A(\mathcal{L}, A^*)$ forces $\mathcal{L}$ to be 2-torsion. In particular we get a refined map

$$U : \mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)[2]$$

which factors the underlying invertible module map.

Example 4.7. Let $(\mathbb{S}, \Qoppa_u)$ be the universal Poincaré ring spectrum from Example 3.10. The only 2-torsion element of $\mathrm{Pic}(\mathbb{S}) \simeq \mathbf{Z}$ is $\mathbb{S}$. Therefore, any element in $\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Qoppa_u)$ lies above $\mathbb{S}$ under U. With Remark 3.12, we conclude $\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Qoppa_u)) \simeq \pi_0(\mathbb{S}_{hC_2} \oplus \mathbb{S}^\times)^\times \simeq (\mathbf{Z} \times \mathbf{Z/2})^\times \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$.

V: there is no truth in here yet. Work in progress. Noah had an example using Witt vectors which showed that π_0 does not need to be 2-torsion

V:did we

Remark 4.8. One might hope that the map $\mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)[2]$ is close to an equivalence. This however is quite far from being true. Let k be a finite field of characteristic 2, and let $\mathbb{S}_{W(k)}$ be the spherical Witt vectors on k in the sense of [Lur18, Example 5.2.7]. Then by [Nik23, Example 3.4] we know that $\mathbb{S}_{W(k)}$ must satisfy that the map $\varphi_2 : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}^{tC_2}$ is an equivalence where the action is trivial.

Consider now the Poincaré ring $(\mathrm{Mod}^{\omega}_{\mathbb{S}_{W(k)}}, \Qoppa^{u}_{\mathbb{S}_{W(k)}})$ where $\Qoppa^{u}_{\mathbb{S}_{W(k)}}$ is the Tate Poincaré structure. We have that $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) \cong \mathbb{Z}$ and is generated by $\Sigma\mathbb{S}_{W(k)}$. To see this note that for $\mathcal{L}$ an invertible module over $\mathbb{S}_{W(k)}$, $\mathcal{L}$ must be bounded below since otherwise it would not be perfect. Then for $\pi_n(\mathcal{L})$ its bottom homotopy group, we have that $\pi_n(\mathcal{L}/2) \cong k$ since it must be an invertible k-module and k is a field. Thus we get a map $\mathbb{S}^n \to \mathcal{L}$ lifting a generator of k, and by adjunction an $\mathbb{S}_{W(k)}$-module map $\Sigma^n\mathbb{S}_{W(k)} \to \mathcal{L}$ which on $\pi_n((-)/2)$ gives an isomorphism $k \cong k$. Therefore

$$\mathbb{S}_{W(k)}[n] \otimes k \simeq k[n] \to k[n] \simeq \mathcal{L} \otimes k$$

is an equivalence, where the equivalence $k[n] \simeq \mathcal{L} \otimes k$ follows from the fact that base change preserves invertible objects. The map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is then a k-local, and therefore an $\mathbb{F}_p$-local, equivalence. Both sides are connective and p-complete so it follows that the map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is an equivalence.

N: There is probably a reference for this fact, I'll look around for one.

Thus $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) = 0$. On the other hand, we have that the unit map $\mathbb{S}_{W(k)} \to \Qoppa^{u}_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})$ is split by the map $\Qoppa^{u}_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)}) \to \mathbb{S}_{W(k)}^{\varphi C_2} = \mathbb{S}_{W(k)}$. Consequently $\pi_0(\Qoppa^{u}_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})) \cong \pi_0(\mathbb{S}_{W(k)} \oplus (\mathbb{S}_{W(k)})_{hC_2}) \cong W(k) \times W(k)$. As a ring this is $W_2(W(k))$, and in order for $q \in W_2(W(k))$ to induce a Poincaré structure we must have that $q \in W_2(W(k))^{\times} \cong W(k)^{\times} \times W(k)^{\times}$.

We then have that $\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong W(k)^{\times} \times W(k)^{\times}/H$ where H is the subgroup of Poincaré structures q on $\mathbb{S}_{W(k)}$ which are identified by some automorphism $f : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}$. By the defining property of spherical Witt vectors there is an equivalence $\mathrm{Maps}_{\mathrm{CAlg}}(\mathbb{S}_{W(k)}, \mathbb{S}_{W(k)}) \simeq \mathrm{Maps}_{Perf}(k, k) = \mathrm{Gal}(k/\mathbb{F}_2)$ and the action on $W(k)^{\times} \times W(k)^{\times}$ is given by $g \in \mathrm{Gal}(k/\mathbb{F}_2)$ acts via $W(g) \times W(g)$. Consequently

$$\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong (W(k)^{\times} \times W(k)^{\times})/\mathrm{Gal}(k/\mathbb{F}_2)$$

which even for $k = \mathbb{F}_2$ is not zero and in fact not even 2^{∞}-torsion.

In the usual Picard spectrum one has the relationship $\mathrm{Pic} = B\mathbb{G}_m$, where $\mathbb{G}_m$ is the spectral algebraic group scheme sending a ring spectrum E to the spectrum of E-linear equivalences of E $\mathrm{gl}_1 E := \mathrm{Aut}_E(E)$.[1] Equivalently, $\mathbb{G}_m$ is the affine group scheme given by $\mathbb{G}_m = \mathrm{Spét}(\mathbb{S}\{x^{\pm 1}\})$, where $\mathbb{S}\{x^{\pm 1}\}$ is the free $\mathbb{E}_\infty$ ring on the $\mathbb{E}_\infty$ space $\mathbb{Z}$. This relationship between Pic and $\mathbb{G}_m$ has many important applications, for example relating the higher homtopy groups of $\mathrm{Pic}(A)$ with those of A. We will spend the rest of this section on establishing such an equivalence in the Poincaré setting.

Construction 4.9. The underlying $\mathbb{E}_\infty$ ring of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ will again be $\mathbb{S}\{x^{\pm 1}\}$, but in order to promote this ring to a Poincare ring it will be helpful to write it as

$$\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z^{\pm 1}\}} \mathbb{S}$$

where the map $\mathbb{S}\{z^{\pm 1}\} \to \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by $z \mapsto xy$, This ring naturally lifts to a Borel C_2-ring given by C_2 swaps x and y and does nothing to z. Now take $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ to be the Poincaré ring with underlying Borel C_2 structure as described above and geometric fixed points $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} = \mathbb{S}$ and the map $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ given by the unit map. Endowing $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2}$ with the $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$-module structre given by $x, y \mapsto 1$, it remains to show that the unit map $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ factors the Tate valued Frobenius $\mathbb{G}_{\mathrm{m}}^{\Qoppa} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ in order to promote $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ to a Poincaré ring.

By construction of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ this amounts to showing that on π_0 the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$. This map sends both x and y to $xy \in \pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$. These are equal to 1 in $\pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ is a modules over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of xy equal to 1.

Theorem 4.10. *There is a natural equivalence of*

$$\Omega\,\mathrm{Pic}^{\mathrm{p}}(-) \simeq \mathbb{G}_{\mathrm{m}}^{\Qoppa}$$

of functors on Poincaré rings.

[1]Normally the automorphism space of an object is only $\mathbb{A}_\infty$, but as the unit in a symmetric monoidal category, the automorphisms of Einherit a canonical and in fact functorial $\mathbb{E}_\infty$ structure and this construction makes sense.

Proof. This amounts to identifying the space $\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod_A})}(A,u)$ functorially, where (A,u) is the Poincaré object A with bilinear form given by the unit map $\mathbb{S} \to Ϙ_A(A)$. Note that any automorphism of Hermetian objects will automatically be Poincaré and so we may instead describe the automorphisms as a Hermetian object. We then have that $\mathrm{He}(\mathrm{Mod}_A) \to \mathrm{Mod}_A$ is a cocartesian fibration by definition, and classified by the functor which takes a module M to the groupoid $\Omega^\infty Ϙ_A(M)$. Thus we get that $\mathrm{Aut}_{\mathrm{He}(\mathrm{Mod}_A)}((A,u))$ is exactly the fiber of the map

$$\mathrm{Aut}_{\mathrm{Mod}_A}(A) \to Ϙ_A(A)$$

or in other words an automorphism $(A,u) \to (A,u)$ is the data of an automorphism $a \in \mathrm{Aut}(A)$ together with a path $q : u \mapsto a^*u$ in $\Omega^{\infty+1}Ϙ_A(A)$.

There is a natural transformation $\mathbb{G}_{\mathrm{m}}^{Ϙ}(-) \to \Omega\,\mathrm{Pic}^{\mathrm{p}}(-)$ given as follows: we get a map $\mathbb{G}_{\mathrm{m}}^{Ϙ}((\mathrm{Mod}_A, Ϙ_A)) \to \mathrm{Aut}_A(A)$ given by forgetting the Poincaré structure everywhere, and so it is enough to see that on π_0 the automorphisms of A coming from $\mathbb{G}_{\mathrm{m}}^{Ϙ}$ preserve u. By using the linear and quadratic decomposition of $Ϙ_A$, for an element $a \in \pi_0(A)^\times$ send u to u is must be sent to $1 \in \pi_0(A^{\varphi C_2})$ and must act by 1 on A^{hC_2}. By the following Lemma this second condition is equivalent to $a\sigma(a) \in \pi_0(A)^\times$ being equal to 1, but then these two conditions are exactly describing a map out of $\mathbb{G}_{\mathrm{m}}^{Ϙ}$ as desired.

Consequently we have a comparison map $\mathbb{G}_{\mathrm{m}}^{Ϙ}(\mathrm{Mod}_A, Ϙ_A) \to \Omega\,\mathrm{Pic}^{\mathrm{p}}(\mathrm{Mod}_A, Ϙ_A)$, and the above argument in fact shows that this map is an equivalence on π_0. To finish the argument, note that the pushout description of $\mathbb{G}_{\mathrm{m}}^{Ϙ}$ induces a pullback of mapping spaces

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}}^{Ϙ}(\mathrm{Mod}_A, Ϙ_A) \arrow[r] \arrow[d] & \mathrm{Maps}_{\mathrm{CAlg}(\mathrm{Sp}^{C_2})}(\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}, A) \simeq \mathrm{gl}_1(A) \arrow[d] \\
* \arrow[r] & \mathrm{Maps}_{\mathrm{CAlg}(\mathrm{Sp}^{C_2})}(\mathbb{S}\{z\}, A) \simeq \Omega^\infty Ϙ_A(A)
\end{tikzcd}$$

which finishes the proof. ☐

Lemma 4.11. *Let $A \in \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ and $s \in \pi_0(A)^\times$. Then $a\sigma(a) = 1$ in $\pi_0(A)$ if and only if $(a \otimes a)^*$ acts by 1 on $\pi_0(A^{hC_2}) = \pi_0(\mathrm{Hom}_{A\otimes A}(A \otimes A, A)^{hC_2})$.*

Proof. The only if direction follows from the fact that the evaluation map $\mathrm{Hom}_{A\otimes A}(A \otimes A, A) \to A$ is an $A \otimes A$-module map. Now suppose that $a\sigma(a) = 1$ in A. Then before taking homotopy fixed points the induced map $a^* = id$ because A is $\mathbb{E}_\infty$.[2] ☐

4.1 prime factorization and the picard group of hearts

V: still in development

We establish a Poincaré analogue of Fausk's result which describes the picard group of the derived category of a scheme X in terms of connected components of X and the classical picard group of X.

Let X be a simplicial set. The set $\pi_0(X)$ is the set of connected compoents of X, i.e. simplicial subsets which are connected and form X via a coproduct. In other words, the functor π_0 records a unique and maximal decomposition of X into coproducts. To establish the the result mentioned above, we study the dual analogue of connected components in the sense of a maximal decomposition of X into products.

Definition 4.12. Let X be a simplicial set. We call a map of simplicial sets $X \to Y$ a factor of X, if there is an isomorphism $X \simeq Y_\times Z$, for some simplicial set Z, such that $Y \times Z \simeq X \to Y$ is a structure map of the given product.

Definition 4.13. Let X be a simplicial set. We say that X is prime, or indecomposable, if it is nonempty and every factor of X is isomorphic to either Δ^0 or X. We let $\mathrm{prin}(X)$ denote the set of prime factors of X.

Proposition 4.14. *Let $X_\bullet$ be a simplicial set, then $X_\bullet$ is the product of its prime factors.*

Proof. ☐

V:

Proposition 4.15. *Let X be a simplicial set. Then* $\mathrm{prim}(X) \simeq \mathrm{prim}(X^{\simeq})$.

Proof. Let $f : X \to Y$ be a weak equivalence of simplicial sets. Then $f^{\simeq} : X^{\simeq} \to Y^{\simeq}$ is a weak equivalence of spaces. □

V:

Proposition 4.16. *Let X be a simplicial set. Then* $\mathrm{prim}(X) \simeq \pi_0(\mathrm{Spec}(X))$, *where* $\mathrm{Spec}(X^{\simeq})$ *is the Balmer spectrum of X with respect to the symmetric monoidal structure given by cartesian product.*

Proof. □

V:

Remark 4.17. Let R be a commutative ring. Then the scheme $\mathrm{Spec}(R)$ is isomorphic to the Balmer spectrum of $\mathrm{Mod}_R^{\heartsuit}$. When we view R as a discrete simplicial set, we thus have

$$\mathrm{prim}(\mathrm{Mod}_R^{\heartsuit}) \simeq \pi_0(\mathrm{Spec}(R)).$$

V: does this need a proof?

Definition 4.18. Let X be a prime simplicial set. A c-structure on X is a map of simplicial sets $X \to \mathbf{Z}$ satisfying (todo). Let Y be a simplicial set, then a c-structure on Y is a product of c-structures on each of its prime components. We write $X_{\geq n}$ for the homotopy pullback of $\mathbf{Z}_{\geq n}$ along c, $X_{\leq n}$ for the homotopy pullback of $\mathbf{Z}_{\leq n}$ along c, and $X^{\heartsuit}$ for the pullback of $X_{\leq n}$ along $X_{\geq n} \to X$.

V: when X is a stable infinity category and prime, then a c-structure should be a t-structure on it

Theorem 4.19. *Let X be a prime simplicial set and $c : X \to \mathbf{Z}$ a c-structure. Then we have a fiber sequence of monoids >*

$$X^{\heartsuit} \to X^{\simeq} \to \mathbf{Z}.$$

V: what kind exactly

Proof. □

V:

Corollary 4.20 (Fausk). *Let R be a discrete ring. Then we have a short exact sequence: >*

$$0 \to \mathrm{Pic}(\mathrm{Mod}(R)^{\heartsuit}) \to \pi_0(\mathrm{Pic}(\mathrm{Mod}(R))) \to H^0(\mathrm{Spec}(R); \mathbf{Z}) \to 0.$$

V: apply pic to the previous sequence and take π_0

Proof. □

4.2 Hermitian line bundles

Definition 4.21. Let R be a commutative discrete ring with a C_2-action $\sigma\colon R \to R$. Write $\sigma_* R$ for the R-module with underlying abelian group R and action $r \cdot m = \sigma(r) \cdot m$. Let M be an R-module. Define the *adjoint* of M to be the R-module $M^{\dagger} := \hom_R(M, \sigma_* R)$. Also recall that there is a canonical R-linear isomorphism $(M^{\dagger})^{\dagger} \simeq M$. Note that given two R-modules M, N, the adjoint satisfies $M^{\dagger} \otimes N^{\dagger} \simeq (M \otimes N)^{\dagger}$. Let I be a projective R-module (in particular, there is a canonical identification $(I^{\dagger})^{\dagger} \simeq I$). A *$\sigma$-hermitian form on I* is an R-linear isomorphism $\varphi\colon I \xrightarrow{\sim} I^{\dagger}$ so that $\varphi^{\dagger} = \varphi$.

L: see 3.8-3.11 in this paper

Observation 4.22. Let R be a commutative discrete ring with a C_2-action $\sigma\colon R \to R$. Given two discrete R-modules M, N equipped with σ-hermitian forms φ, ψ, respectively, $\varphi \otimes \psi$ defines a σ-hermitian form on $M \otimes_R N$. Using the canonical isomorphism mentioned above, if φ is a σ-hermitian form on M, then $\varphi^{\dagger}$ induces a σ-hermitian form on $M^{\dagger}$. Finally, observe that R has a canonical σ-hermitian form which is the adjoint of the map $R \otimes R \to R$, $r \otimes s \mapsto r\sigma(s)$.

Definition 4.23. Let R be a commutative discrete ring with a C_2-action $\sigma\colon R \to R$. Define the *hermitian Picard group of R* to have underlying set consisting of pairs (I, φ) where I is an invertible R-module and φ is a σ-hermitian form on I.

L: workshop the name later

By Observation 4.22, this set inherits a group structure. We write $\mathrm{hPic}(R)$ for the group of σ-hermitian line bundles on $\mathrm{Spec}\, R$.

[2] Or just $\mathbb{E}_2$.

Theorem 4.24. *Let R be a discrete commutative ring with a C_2-action $\sigma\colon R \xrightarrow{\sim} R$ via ring maps. Regard R as a Poincaré ring via Example 3.15. Then there is a split short exact sequence of abelian groups*

$$0 \to \mathrm{hPic}(R) \to \pi_0\mathrm{PnPic}(\underline{R}^\sigma) \to C_{C_2}(\operatorname{Spec} R, \mathbb{Z}^-) \to 0$$

where R is endowed with the genuine symmetric Poincaré structure and $\mathbb{Z}^-$ is endowed with the C_2-action given by multiplication by -1 and C_{C_2} denotes continuous functions which are moreover C_2-equivariant. Moreover, forgetting the hermitian form (resp. forgetting the C_2-action) induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{hPic}(R) \arrow[r] \arrow[d] & \pi_0\mathrm{PnPic}(R) \arrow[r] \arrow[d] & C_{C_2}(\operatorname{Spec} R, \mathbb{Z}^-) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Pic}^{\mathrm{cl}}(R) \arrow[r] & \pi_0\mathrm{Pic}\left(\mathrm{Perf}_R\right) \arrow[r] & C(\operatorname{Spec} R, \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is that of [Fau03, Theorem 3.5].

Proof. An object of $\pi_0\mathrm{PnPic}(R)$ is a pair (I, q) where I is an invertible R-module and q is a point in $\pi_0\Omega^\infty Ϙ_{R^{gs}}(I)$. By the proof of [Fau03, Theorem 3.5], I induces a continuous map $\Psi(I)\colon \operatorname{Spec} R \to \mathbb{Z}$. Write σ for the involution on R. Now q in particular induces an equivalence $q\colon I \xrightarrow{\sim} I^\dagger \simeq (\sigma_* I)^\vee$. For each point $\mathfrak{p} \in \operatorname{Spec} R$, localizing q gives an equivalence

$$q_\mathfrak{p}\colon I_\mathfrak{p} \xrightarrow{\sim} (\sigma_* I)^\vee_\mathfrak{p} \simeq \left(\sigma_*(I_{\sigma(\mathfrak{p})})\right)^\vee .$$

Since $I_\mathfrak{p}$ is an invertible module over a local ring, [Fau03, Proposition 3.2] implies that $q_\mathfrak{p}$ induces an equivalence

$$I_\mathfrak{p} \simeq R_\mathfrak{p}[\varphi(\mathfrak{p})] \xrightarrow{\sim} \left(\sigma_*(R_{\sigma(\mathfrak{p})}[\varphi(\sigma(\mathfrak{p}))])\right)^\vee \simeq (\sigma_*(R_{\sigma(\mathfrak{p})}))^\vee[-\varphi(\sigma(\mathfrak{p}))] .$$

Since R is discrete, this implies in particular that $\Psi(I)(\sigma(\mathfrak{p})) = -\Psi(I)(\mathfrak{p})$, i.e. that $\Psi(I)$ is C_2-equivariant. It follows immediately from [Fau03, Theorem 3.5] that Ψ is a homomorphism and that an element of the kernel of Ψ lifts to $\mathrm{hPic}(R)$.

Now consider a C_2-equivariant map $g\colon \operatorname{Spec} R \to \mathbb{Z}$. As in *loc. cit.*, the image of g is finite and C_2-invariant, say $\{n_1, -n_1, \dots, n_m, -n_m\}$ or $\{0, n_1, -n_1, \dots, n_m, -n_m\}$ for some $n_i \neq 0$. As in *loc. cit.*, the disjoint subsets $U_{\pm n_i} := g^{-1}(\pm n_i)$ correspond to an orthogonal basis of idempotents $e_{U_{\pm n_i}}$ in R. Since g is C_2-equivariant with respect to the sign action on $\mathbb{Z}$, we have $\sigma(U_{n_i}) = U_{-n_i}$. Moreover, it follows from Lemma 3.4 *ibid.* that $\sigma(e_{U_{n_i}}) = e_{U_{-n_i}}$. Consider the R-module $\Phi(g) := \bigoplus_{n\in \mathrm{Im}(g)} e_{g^{-1}(\{n\})}R[n]$. In other words, $\Phi(g) := \bigoplus_{i=1}^m \left(e_{U_{n_i}}R[n_i] \oplus e_{U_{-n_i}}R[-n_i]\right)$ if 0 is not in the image of g and $\Phi(g) := e_{U_0} \oplus \bigoplus_{i=1}^m \left(e_{U_{n_i}}R[n_i] \oplus e_{U_{-n_i}}R[-n_i]\right)$ otherwise. Observe that $\left(e_{U_{-n_i}}R[-n_i]\right)^\dagger = \hom_R(e_{U_{-n_i}}R[-n_i], \sigma_* R) = \hom_R\left(\sigma_*(e_{U_{-n_i}}R), R\right)[n_i] = \hom_R\left(e_{U_{n_i}}R, R\right)[n_i]$. Finally, we claim that there is a canonical σ-hermitian form $q_g \in \Omega^\infty Ϙ_{R^{\mathrm{gs}}}(\Phi(g))$ whose adjoint $q_g^\dagger\colon \Phi(g) \xrightarrow{\sim} \Phi(g)^\dagger$ corresponds to the identity. That q_g defines a point of $\hom_{R^{\otimes 2}}(\Phi(g)^{\otimes 2}, R)^{hC_2}$ is evident. Observe that to give a lift of q_g to $Ϙ_{R^{\mathrm{gs}}}(\Phi(g)) = \hom_{N^{C_2}R}\left(N^{C_2}\Phi(g), R\right)$ is equivalent to giving a commutative diagram

$$\begin{tikzcd}
\Phi(g) \otimes_R R^{\varphi C_2} \arrow[r, dashed, "\exists ?"] \arrow[d] & R^{\varphi C_2} \arrow[d] \\
\left(\Phi(g)^{\otimes 2}\right)^{tC_2} \arrow[r, "q_g^{tC_2}"] & R^{tC_2}
\end{tikzcd} . \tag{4.25}$$

Let us write $\eta\colon R \to \pi_0 R^{\varphi C_2}$ for the ring map induced by the structure map. Since R is a C_2-$\mathbb{E}_\infty$-ring, η is invariant with respect to the given action on R and the trivial action on $\pi_0 R^{\varphi C_2}$. Consider $e_{U_{n_i}}$ an idempotent corresponding to an element of the image of g so that $n_i \neq 0$. Then

$$\begin{aligned}
\eta(e_{U_{n_i}}) &= \eta(e_{U_{n_i}})^2 && \text{ring maps preserve idempotents} \\
&= \eta(e_{U_{n_i}}) \cdot \eta(e_{U_{-n_i}}) && C_2\text{-invariance of } \eta \\
&= \eta(e_{U_{n_i}} e_{U_{-n_i}}) && \eta \text{ is a ring map} \\
&= 0 && \text{orthogonality and } n_i \neq 0 .
\end{aligned}$$

In particular, if 0 is not in the image of g, $\Phi(g) \otimes_R R^{\varphi C_2} \simeq 0$ and (4.25) commutes vacuously. If 0 is in the image of g, then $e_{U_0}R$ is a discrete/projective $e_{U_0}R$-module and q_g evidently defines a genuine hermitian form on $e_{U_0}R$ (compare [Cal+20a, Remark 4.2.21]).

Thus, $g \mapsto (\Phi(g), q_g)$ defines a splitting of Ψ which agrees with the splitting constructed in [Fau03, Theorem 3.5] on underlying objects. $\square$

5 Poincaré structures on schemes with involution

Let X be a scheme with an involution $\sigma\colon X \xrightarrow{\sim} X$. We want to introduce a Poincaré structure ♀ on $\mathrm{Perf}(X)$ so that the duality is given by $E \mapsto E^\vee \otimes \sigma_*(\mathcal{O}_X)$ (contrast with §3 of this paper).

5.1 Poincaré structures associated to rigid symmetric monoidal ∞-categories with involution

Let $\mathcal{C}$ be a stably symmetric monoidal ∞-category with an involution, i.e. an exact autoequivalence $\sigma\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ and a functor $BC_2 \to \mathbb{E}_\infty \mathrm{Alg}\,\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ sending $* \mapsto \mathcal{C}$ and a generator of $\mathrm{End}_{BC_2}(*) \simeq C_2$ to σ. Then σ induces a C_2-action on the ∞-groupoid of $\otimes$-invertible objects $\mathrm{Pic}(\mathcal{C})$. Let $L \in \mathrm{Pic}(\mathcal{C})^{hC_2}$ be a homotopy fixed point of this action. In other words, L is endowed with the choice of an equivalence $\varphi\colon L \simeq \sigma(L)$, a homotopy from $\sigma(\varphi) \circ \varphi$ to the identity on L, and higher coherences.

Consider a functor $f\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ which is C_2-equivariant with respect to the σ-action on $\mathcal{C}$ and the trivial action on Sp. In particular, the "C_2-equivariance" of f is additional data: for each $x \in \mathcal{C}$, an equivalence of spectra $c_x\colon f(x) \simeq f(\sigma(x))$ which is natural in x, a homotopy from $c_{\sigma(x)} \circ c_x$ to id_x, and higher coherences.

Lemma 5.1. *Let $\mathcal{C}$ be a stably symmetric monoidal ∞-category with an involution, and suppose given $L \in \mathrm{Pic}(\mathcal{C})^{hC_2}$ a homotopy fixed point of this action. Then the functor $\hom_{\mathcal{C}}(-, L)$ promotes canonically to a C_2-equivariant functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ in the sense of the previous paragraph.*

Proof. Note that the Yoneda embedding $y\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is equivariant with respect to the given action on $\mathcal{C}$ and the action of C_2 on the functor category via $F \mapsto \sigma^* F = F \circ \sigma$. Now since $\mathrm{Pic}(\mathcal{C}) \subseteq \mathcal{C}$ induces $\mathrm{Pic}(\mathcal{C})^{hC_2} \to \mathcal{C}^{hC_2}$, we may take the image of L under the Yoneda embedding: $y(L) \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})^{hC_2} \simeq \mathrm{Fun}_{C_2}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$. $\square$

L: This may be 'overkill,' but this is true because $\mathcal{C}$ can be regarded as a $\mathcal{O}_{C_2}^{\mathrm{op}}$-parametrized ∞-category (with empty fiber over C_2/C_2). Then there is a parametrized Yoneda embedding.

Now fix a presentably symmetric monoidal ∞-category $\mathcal{D}$, and regard it as having the trivial C_2-action. Recall the Fin_*-cartesian fibration $\left(\mathcal{C}\mathrm{at}^{BC_2}_{\mathrm{op}//q^*\mathcal{D}}\right)^\otimes \simeq \left(\left(\mathcal{C}\mathrm{at}_{\mathrm{op}//\mathcal{D}}\right)^{BC_2}\right)^\otimes \to (\mathcal{C}\mathrm{at}^{BC_2})^\times$ from [CHN24, p. 13]. Set

$$\mathcal{W}_{\mathcal{D}}^\otimes := \mathbb{E}_\infty \mathrm{Alg}\left(\mathcal{C}\mathrm{at}^{BC_2}\right)^\times \times_{(\mathcal{C}\mathrm{at}^{BC_2})^\times} \left(\mathcal{C}\mathrm{at}^{BC_2}_{\mathrm{op}//q^*\mathcal{D}}\right)^\otimes$$

This is a Fin_*-cartesian fibration classified by the lax symmetric monoidal functor

$$\begin{aligned} \mathbb{E}_\infty \mathrm{Alg}(\mathcal{C}\mathrm{at})^{BC_2} &\to \mathcal{C}\mathrm{at} \\ \mathcal{C}^\otimes &\mapsto \mathrm{Fun}_{C_2}(\mathcal{C}^{\mathrm{op}}, q^*\mathcal{D})\,. \end{aligned} \tag{5.2}$$

In particular, an object of the underlying ∞-category of $\mathcal{W}_{\mathcal{D}}^\otimes$ is a pair $(\mathcal{C}, f)$ where $\mathcal{C}$ is a symmetric monoidal ∞-category with a C_2-action (via a symmetric monoidal functor) and $f\colon \mathcal{C}^{\mathrm{op}} \to q^*\mathcal{D}$ is a C_2-equivariant functor.

Construction 5.3. Given a C_2-equivariant functor $f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, we may regard the data of the C_2-equivariance of f as a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}^{\mathrm{op}}} \arrow[r, "\widetilde{f}"] \arrow[d] & \mathcal{D} \times BC_2 \arrow[d] \\ BC_2 \arrow[r, equal] & BC_2 \end{tikzcd}$$

where the vertical maps are cocartesian fibrations and the restriction of $\widetilde{f}$ to the fiber over the point $* \in BC_2$ recovers f. The diagram induces a map on cocartesian sections

$$\overline{f}\colon \operatorname{Fun}^{\mathrm{cocart}}_{BC_2}(BC_2, \widetilde{\mathcal{C}^{\mathrm{op}}}) \to \mathcal{D}^{BC_2}\,.$$

Now if C is a symmetric monoidal ∞-category, we can associate to f the composite

$$T_f\colon \mathcal{C}^{\mathrm{op}} \xrightarrow{x \mapsto x\otimes\sigma(x)} \operatorname{Fun}^{\mathrm{cocart}}_{BC_2}(BC_2, \widetilde{\mathcal{C}^{\mathrm{op}}}) \xrightarrow{\overline{f}} \mathcal{D}^{BC_2}\,.$$

Finally, if $\mathcal{D}$ admits BC_2-indexed limits, we can take homotopy fixed points of C_2-objects in which case we define the functor $♀^s_f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ as the composite

L: I'm just using the same notation as Harpaz-Nardin-Shah here, but $(-)^s$ is maybe a little weird because it should be 'hermitian,' not 'symmetric.'

$$♀^s_f\colon \mathcal{C}^{\mathrm{op}} \xrightarrow{T_f} \mathcal{D}^{BC_2} \xrightarrow{(-)^{hC_2}} \mathcal{D}\,.$$

Observation 5.4. Let $\mathcal{C}$, L, be as before. Then the cross effect B_L of $♀^s_L$ is given by $B_L(x,y) = \hom_{\mathcal{C}}(x \otimes \sigma(y), L)$.

Proposition 5.5. *Let $\mathcal{D}$ be a symmetric monoidal ∞-category which admits BC_2-indexed limits; endow $\mathcal{D}$ with the trivial C_2-action. Then the assignment $(\mathcal{C}, f) \mapsto (\mathcal{C}, ♀^s_f)$ of Construction 5.3 assembles to form a lax symmetric monoidal functor*

$$\mathcal{W}^{\otimes}_{\mathcal{D}} \to \left(\mathcal{C}\mathrm{at}^{BC_2}\right)^{\otimes}_{\mathrm{op}//\mathcal{D}}$$

sitting in a commutative diagram

$$\begin{tikzcd}
\mathcal{W}^{\otimes}_{\mathcal{D}} \arrow[r] \arrow[d] & \left(\mathcal{C}\mathrm{at}^{BC_2}\right)^{\otimes}_{\mathrm{op}//\mathcal{D}} \arrow[d] \\
\mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{BC_2})^{\times} \arrow[r, "\mathcal{C}^{\otimes}\mapsto\mathcal{C}"] & \left(\mathcal{C}\mathrm{at}^{BC_2}\right)^{\times}
\end{tikzcd} \tag{5.6}$$

in which both vertical arrows are both Fin_*-cartesian fibrations and cocartesian fibrations of ∞-operads.

Construction 5.7. Consider the functors $r\colon \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{BC_2}) \xrightarrow{\mathcal{C}^{\otimes}\mapsto\mathcal{C}} \mathcal{C}\mathrm{at}^{BC_2}$ and $p\colon \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{BC_2}) \xrightarrow{\text{forget}} \mathcal{C}\mathrm{at} \xrightarrow{q^*} \mathcal{C}\mathrm{at}^{BC_2}$ where $q\colon BC_2 \to *$. We construct a symmetric monoidal natural transformation $\tau\colon r^{\times} \implies p^{\times}$ whose component at a given category $\mathcal{C}$ with C_2-action $\sigma\colon \mathcal{C} \simeq \mathcal{C}$ is the C_2-equivariant functor $\mathcal{C} \xrightarrow{x\mapsto x\otimes\sigma(x)} p(\mathcal{C})$.

L: This is basically Construction 3.1.4 of [CHN24] with minor edits.

Let $\operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right)$ be the span ∞-category of finite sets with free C_2-action. For an ∞-category with finite products $\mathcal{E}$, there is a natural equivalence $\operatorname{Fun}^{\times}\left(\operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right), \mathcal{E}\right) \simeq \mathrm{CMon}(\mathcal{E})^{BC_2}$ between product-preserving functors $\operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right) \to \mathcal{E}$ and commutative monoids in $\mathcal{E}$ with C_2-action. Taking $\mathcal{E} \simeq \mathcal{C}\mathrm{at}$, we may identify the functor r as restriction along the inclusion $i\colon BC_2 \to \operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right)$ of the maximal subgroupoid in the full subcategory on a finite C_2-set with a single orbit. On the other hand, we can identify the functor p as restriction along the map $j\colon BC_2 \to \{*\} \xrightarrow{*\mapsto C_2} \operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right)$. Now the span $C_2 \xleftarrow{\pi_1} C_2 \times C_2 \xrightarrow{\pi_2} C_2$ determines a morphism in $\operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right)$ which is equivariant with respect to the given action on the source C_2 and the *trivial action* on the target C_2. This morphism determines a functor $\Delta^1 \times BC_2 \to \operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right)$ whose restriction to $\{0\} \times BC_2$ agrees with i and whose restriction to $\{1\} \times BC_2$ agrees with j. This determines a natural transformation $i^* \implies j^*$ of functors $\operatorname{Fun}\left(\operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right), \mathcal{C}\mathrm{at}\right) \to \operatorname{Fun}(BC_2, \mathcal{C}\mathrm{at})$, and precomposing with the inclusion $\operatorname{Fun}^{\times}\left(\operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right), \mathcal{C}\mathrm{at}\right) \subset \operatorname{Fun}\left(\operatorname{Span}\left(\mathrm{Fin}^{\mathrm{free}}_{C_2}\right), \mathcal{C}\mathrm{at}\right)$ gives the desired natural transformation $\tau\colon r \implies p$. Since r and p preserve products, we may lift them to symmetric monoidal functors $r^{\times}, p^{\times}\colon \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{BC_2})^{\times} \to (\mathcal{C}\mathrm{at}^{BC_2})^{\times}$, and τ refines to a symmetric monoidal natural transformation $\tau^{\times}\colon r^{\times} \implies p^{\times}$.

Proof of Proposition 5.5. Horizontally composing the natural transformation of Construction 5.7 with the functor 5.3 and unstraightening induces a commutative diagram

L: Pretty similar to proof of [CHN24, Proposition 3.1.3], the main thing is the fact (stated after proof of Lemma 3.1.1 of *op. cit.*) that Fin_*-cartesian fibrations are classified by lax symmetric monoidal functors.

$$\begin{tikzcd} \mathcal{W}^{\otimes}_{\mathcal{D}} \arrow[r] \arrow[d] & \left(\mathcal{C}\mathrm{at}^{BC_2}_{\mathrm{op}//\mathcal{D}}\right)^{\otimes} \arrow[d] \\ \left(\mathcal{C}\mathrm{at}^{BC_2}\right)^{\otimes}_{\mathrm{op}//\mathcal{D}} \arrow[r, "\mathcal{C}^{\otimes}\mapsto\mathcal{C}"] & \left(\mathcal{C}\mathrm{at}^{BC_2}\right)^{\times} \end{tikzcd}, \tag{5.8}$$

where we have used that $(q^*\mathcal{D})^{hC_2} \simeq \mathcal{D}^{BC_2}$. □

Definition 5.9. Define

$$\mathcal{W}^{\otimes}_{\mathrm{ex}} \subseteq \mathcal{W}^{\otimes}_{\mathrm{Sp}} \times_{\mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at})} \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)$$

to be the full sub-operad on those colors $(\mathcal{C}, f)$ so that f is exact.

Observation 5.10. The commutative square (5.6) restricts to a commutative square of ∞-operads

$$\begin{tikzcd} \mathcal{W}^{\otimes}_{\mathrm{ex}} \arrow[r, "(\mathcal{C},f)\mapsto(\mathcal{C},\Qoppa^s_f)"] \arrow[d] & {\mathrm{Cat}^{\mathrm{h}}_\infty}^{\otimes} \arrow[d] \\ \left(\mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)^{BC_2}\right)^{\otimes} \arrow[r] & \mathcal{C}\mathrm{at}^{\mathrm{ex}\otimes}_\infty \end{tikzcd} \tag{5.11}$$

where $\Qoppa^s_f$ was defined in Construction 5.3.

Lemma 5.12. *Both vertical maps in 5.11 are cocartesian fibrations of ∞-operads. In particular, $\mathcal{W}^{\otimes}_{\mathrm{ex}}$ is a symmetric monoidal ∞-category.*

Proof. That ${\mathrm{Cat}^{\mathrm{h}}_\infty}^{\otimes}$ is symmetric monoidal and the right-hand projection is a symmetric monoidal functor is [Cal+20a, Theorem 5.2.7]. Furthermore, by Proposition 5.5 and base change, we have a cocartesian fibration of ∞-operads:

$$\pi\colon \left(\mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)^{BC_2}\right)^{\otimes} \times_{\left(\mathcal{C}\mathrm{at}^{BC_2}\right)^{\times}} \left(\mathcal{C}\mathrm{at}^{BC_2}_{\mathrm{op}//\mathcal{D}}\right)^{\otimes} \to \left(\mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)^{BC_2}\right)^{\otimes}$$

so that $\mathcal{W}^{\otimes}_{\mathrm{ex}}$ includes as a full sub-operad on the fiber product on the left on those colors $(\mathcal{C}, f\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp})$ so that f is exact. It suffices to show that if $\alpha\colon (\mathcal{C}_i, f_i\colon \mathcal{C}_i^{\mathrm{op}} \to \mathrm{Sp})_{i\in I} \to (\mathcal{D}, f\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp})$ is a π-cocartesian arrow so that $(\mathcal{C}_i, f_i\colon \mathcal{C}_i^{\mathrm{op}} \to \mathrm{Sp})_{i\in I}$ is in $\mathcal{W}^{\otimes}_{\mathrm{ex}}$, then $(\mathcal{D}, f\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp})$ is also in $\mathcal{W}^{\otimes}_{\mathrm{ex}}$. This holds because the left Kan extension of the multi-exact functor $\Pi_i f_i\colon \Pi_i\mathcal{C}_i^{\mathrm{op}} \to \mathrm{Sp}$ along $\Pi_i\mathcal{C}_i^{\mathrm{op}} \to \bigotimes_i \mathcal{C}_i^{\mathrm{op}} \xrightarrow{\alpha} \mathcal{D}^{\mathrm{op}}$ is exact. □

As in [CHN24, p. 15], we can identify objects of the underlying ∞-category of $\mathcal{W}^{\otimes}_{\mathrm{ex}}$ with pairs $(\mathcal{C}^{\otimes}, \sigma_{\mathcal{C}}, L, \lambda)$ where $\mathcal{C}$ is a symmetric monoidal stable ∞-category with involution $\sigma_{\mathcal{C}}$, and $(L, \lambda) \in \mathrm{Ind}(\mathcal{C})^{hC_2}$ is a fixed point with respect to the induced action on $\mathrm{Ind}(\mathcal{C})$.

Recollection 5.13. A symmetric monoidal ∞-category $\mathcal{C}$ is said to be *rigid* if every object in $\mathcal{C}$ is dualizable.

L: reference for when derived categories of connective objects over more general objects are compactly generated

Proposition 5.14. *Suppose $\mathcal{C}$ is a rigid stably symmetric monoidal ∞-category with a C_2-action $\sigma_{\mathcal{C}}$ via symmetric monoidal functors, and let $(L, \lambda) \in \mathrm{Ind}(\mathcal{C})^{BC_2}$. Then the hermitian structure $\Qoppa^s_L$ is non-degenerate if and only if L belongs to $\mathcal{C}$, and it is furthermore Poincaré if and only if the underlying object L is tensor-invertible in $\mathcal{C}$. In addition, if $g\colon \mathcal{C} \to \mathcal{C}'$ is a symmetric monoidal C_2-equivariant exact functor, $(L, \lambda) \in \mathcal{C}^{BC_2}$ and $(L', \lambda') \in (\mathcal{C}')^{BC_2}$ are tensor-invertible, and $g(L, \lambda) \simeq (L', \lambda')$ an equivalence in $(\mathcal{C}')^{BC_2}$, then the induced hermitian functor $(\mathcal{C}, \Qoppa^s_L) \to (\mathcal{C}', \Qoppa^s_{L'})$ is Poincaré.*

Proof. If the bilinear part B_L is represented by $D\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, then $D(1_{\mathcal{C}}) = \sigma_{\mathcal{C}}(L)$; in particular, $\sigma_{\mathcal{C}}(L) \simeq L$ is an object of $\mathcal{C}$. On the other hand, if L belongs to $\mathcal{C}$, then for $x, y \in \mathcal{C}$, the bilinear part B_L is $B_L(x, y) = \hom_{\mathrm{Ind}(\mathcal{C})}(x \otimes \sigma_{\mathcal{C}}(y), L) \simeq \hom_{\mathcal{C}}(x \otimes \sigma_{\mathcal{C}}(y), L) \simeq \hom_{\mathcal{C}}(x, L \otimes \sigma_{\mathcal{C}}(y)^\vee)$. The natural transformation $\mathrm{id}_{\mathcal{C}} \to D^{\mathrm{op}} \circ D$ is given by $y \to L \otimes \sigma_{\mathcal{C}}\left(L \otimes \sigma_{\mathcal{C}}(y)^\vee\right)^\vee$ induced by the adjoint of the equivalence $L \simeq \sigma_{\mathcal{C}}(L)$, which is an equivalence if and only if L is invertible.

Finally, the natural transformation $g \circ D_L \Rightarrow D_{L'} \circ g$ is $g\left(L \otimes \sigma_{\mathcal{C}}(y)^\vee\right) \simeq g(L) \otimes g\left(\sigma_{\mathcal{C}}(y)^\vee\right) \to L' \otimes \sigma_{\mathcal{C}'}(g(y))^\vee$ which is an equivalence if and only if the map $g(L) \to L'$ is an equivalence. □

L: how much equivariance on $g(L) \to L'$ is necessary?

6 The Poincaré Brauer Group

Let A be a Poincaré ring spectrum. By Remark 3.3, Mod_A^ω promotes to a commutative algebra object in the ∞-category of Poincaré ∞-categories $\mathrm{Cat}_\infty^{\mathrm{p}}$, and we may thus consider modules over it. In this section, we will use modules over Poincaré ring spectra to define derived analogues of the involutive Brauer group for Poincaré ring spectra.

Recall that a Poincaré ∞-category is called idempotent complete if the underlying stable ∞-category is idempotent complete. The full subcategory of $\mathrm{Cat}_\infty^{\mathrm{p}}$ spanned by idempotent complete Poincaré ∞-categories is denoted by $\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}$ [Cal+20b, Definition 1.3.2].

Definition 6.1. Let A be a Poincaré ring spectrum. We define the *Poincaré Brauer space of A* as

$$\mathrm{Br}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Mod}_A(\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}})).$$

The assignment $A \mapsto \mathrm{Br}^{\mathrm{p}}(A)$ defines a functor

$$\mathrm{Br}^{\mathrm{p}} \colon \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$$

valued in grouplike $\mathbf{E}_\infty$-spaces.

Remark 6.2. The symmetric monoidal forgetful functor $\mathrm{Mod}_A(\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}) \to \mathrm{Mod}_A(\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}})$ induces a map $\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A)$ of grouplike $\mathbf{E}_\infty$-spaces, where $\mathrm{Br}(A)$ is the Brauer space $\mathrm{br}_{\mathrm{alg}}(A)$ of [AG14, pp. 1154–1155].

Proposition 6.3. *Let A be a Poincaré ring spectrum. Then we have a canonical equivalence*

$$\Omega\,\mathrm{Br}^{\mathrm{p}}(A) \simeq \mathrm{Pic}^{\mathrm{p}}(A).$$

L: What else do we need to do to show that we have an equivalence of *functors*?

Proof. Since $\Omega\,\mathrm{Br}^{\mathrm{p}}(R)$ is given by the space of automorphisms of any object in $\mathrm{Br}^{\mathrm{p}}(R)$, it suffices to determine the space of autoequivalences of $(\mathrm{Mod}_R^\omega, \Omega_R)$. An autoequivalence is the data of a pair (f, η) where $f\colon \mathrm{Mod}_R^\omega \xrightarrow{\simeq} \mathrm{Mod}_R^\omega$ is an exact R-linear autoequivalence and $\eta\colon \Omega_R \xrightarrow{\sim} \Omega_R \circ f^{\mathrm{op}}$ is a natural equivalence. Since $\mathrm{Cat}_{\infty R}^{\mathrm{p}} \to \mathcal{C}\mathrm{at}_{\infty R}^{\mathrm{ex}}$ is symmetric monoidal, f is of the form $-\otimes_R \mathcal{L}$ where $\mathcal{L}$ is an invertible R-module. Since taking bilinear and linear parts is functorial/by [Cal+20a, Proposition 1.3.11], η is equivalently the data of a pair of equivalences

$$\begin{aligned} b(\eta)\colon\ &\hom_{R\otimes R}((-\otimes\mathcal{L})\otimes(-\otimes\mathcal{L}), R)^{hC_2} \simeq \hom_{R\otimes R}(-\otimes -, R)^{hC_2} \\ \ell(\eta)\colon\ &\hom_R(-\otimes\mathcal{L}, R^{\varphi C_2}) \simeq \hom_R(-, R^{\varphi C_2}) \end{aligned}$$

plus a path between their images in $\hom_R(\mathcal{L}, R^{tC_2})$. The transformation $b(\eta)$ is equivalent to the data of an R-bilinear equivalence $R \simeq \mathcal{L}^\vee \otimes \mathcal{L}^\vee$, and the transformation $\ell(\eta)$ is equivalent to the data of an $R^{\varphi C_2}$-linear equivalence $\ell(\eta)\colon R^{\varphi C_2} \otimes_R \mathcal{L}^\vee \xrightarrow{\sim} R^{\varphi C_2}$.

L: maybe one of these should be conjugate dual here?

L: is the $R^{\varphi C_2}$-linearity of this $\simeq$ correct?

Now consider the composites

$$R \otimes_R \mathcal{L}^\vee \xrightarrow{\mathrm{unit}\otimes\mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L}^\vee \xrightarrow{\ell(\eta)} R^{\varphi C_2}$$

$$R \otimes_R \mathcal{L} \xrightarrow{\mathrm{unit}\otimes\mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L} \xrightarrow{\ell(\eta)^{-1}\otimes\mathrm{id}_{\mathcal{L}}} R^{\varphi C_2}\,.$$

These correspond to the $\ell(q^\vee), \ell(q)$ of Remark 4.2, respectively. In particular, the condition that $\ell(q^\vee), \ell(q)$ make the diagram (4.4) commute is equivalent to the condition that $\ell(\eta)$ is an equivalence by an adjunction argument. $\square$

L: under construction–not sure what to say about the $(-)^{tC_2}$ part yet.

6.1 Generalities on R-linear Poincaré ∞-categories

Proposition 6.4. *Let $(\mathrm{Mod}_R^\omega, \Omega_R)$ be a Poincaré ring spectrum.*

(1) *The ∞-category $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ admits all small limits and colimits, and it inherits a canonical symmetric monoidal structure, and for every morphism $\left(R, R^{\varphi C_2} \to R^{tC_2}\right) \to (S, S^{\varphi C_2} \to S^{tC_2})$, the functor $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_{(\mathrm{Mod}_S^\omega, \Qoppa_S)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ is a symmetric monoidal left adjoint.*

(2) *Let A be an $\mathbb{E}_1$-R-algebra in spectra, and regard the category of compact right A-modules Mod_A^ω as left-tensored over Mod_R^ω in the canonical way. Then the pullback*

$$
\begin{tikzcd}
 & \mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}\left(\mathrm{Cat}^{\mathrm{h}}_\infty\right) \arrow[d] \\
\{\mathrm{Mod}_A^\omega\} \arrow[r] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}
\end{tikzcd}
\tag{6.5}
$$

is canonically equivalent to $\mathrm{Mod}_{N_R A \otimes_{N_R R} R^L}\left(\mathrm{Sp}^{C_2}\right)$ where R^L is the $\mathbb{E}_\infty$-$N_R R$-algebra with $(R^L)^e \simeq R$ and $(R^L)^{\varphi C_2} \simeq C$.

A $N_R A \otimes_{N_R R} R^L$-module classifies a $(\mathrm{Mod}_R^\omega, \Qoppa_R)$-module in Poincaré ∞-categories if its underlying A-module is invertible in the sense of [Cal+20a, Definition 3.1.4].

(3) *Let A, B be R-algebras with associated (R-linear) modules with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and $(M_B, N_B, N_B \to M_B^{tC_2})$, respectively so that (under item (2)) $(\mathrm{Mod}_A^\omega, \Qoppa_A)$ and $(\mathrm{Mod}_B^\omega, \Qoppa_B)$ are objects of $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$. Then the symmetric monoidal structure of item (1) is so that the underlying R-linear ∞-category with perfect duality $(\mathrm{Mod}_A^\omega, \Qoppa_A) \otimes_{(\mathrm{Mod}_R^\omega, \Qoppa_R)} (\mathrm{Mod}_B^\omega, \Qoppa_B)$ is $\mathrm{Mod}_A^\omega \otimes_{\mathrm{Mod}_R^\omega} \mathrm{Mod}_B^\omega \simeq \mathrm{Mod}_{A \otimes_R B}^\omega$, and the associated module with genuine involution is given by $M_A \otimes_R M_B$, $N_A \otimes_{R^{\varphi C_2}} N_B$, and the structure map is $N_A \otimes_{R^{\varphi C_2}} N_B \to M_A^{tC_2} \otimes_{R^{tC_2}} M_B^{tC_2} \to (M_A \otimes_R M_B)^{tC_2}$ where the latter map arises canonically from lax monoidality of the Tate construction.*

(4) *Let $(\mathcal{C}, \Qoppa_{\mathcal{C}}), (\mathcal{D}, \Qoppa_{\mathcal{D}})$ be objects of $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{h}}_\infty)$. Then the forgetful functor induces $\hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}((\mathcal{C}, \Qoppa_{\mathcal{C}}), (\mathcal{D}, \Qoppa_{\mathcal{D}})) \to \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}(\mathcal{C}, \mathcal{D})$ on mapping spaces so that the fiber over an R-linear functor $F\colon \mathcal{C} \to \mathcal{D}$ is the mapping space $\mathrm{map}_{\Qoppa_R}(F_! \Qoppa_{\mathcal{C}}, \Qoppa_{\mathcal{D}}) \simeq \mathrm{map}_{\Qoppa_R}(\Qoppa_{\mathcal{C}}, \Qoppa_{\mathcal{D}} \circ F^{\mathrm{op}})$, where the mapping space is taken in $\mathrm{Fun}^q_{\Qoppa_R}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp})$ and $\mathrm{Fun}^q_{\Qoppa_R}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$, respectively.*[3]

(5) *The symmetric monoidal forgetful functor $\theta\colon \mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{h}}_\infty) \to \mathrm{Mod}_{\mathrm{Mod}_R^\omega}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)$ is a (co)cartesian fibration.*

L: what is it classified by?

Remark 6.6. A special case of part (2) is [Cal+20a, Example 5.4.13].

Proof. (1) The first part of the statement follows from [Cal+20a, §6.1] and [Lur17, §4.2.3].

(2) Let $\mathcal{LM}^\otimes$ denote the ∞-operad of [Lur17, Definition 4.2.1.7]. Our strategy of proof will be similar to that of [Cal+20a, §5.3]: First, we show that an $\mathcal{LM}^\otimes$-algebra object in $\mathrm{Cat}^{\mathrm{h}}_\infty$ is equivalent to an $\mathcal{LM}^\otimes$-algebra object in an operad of functor categories. Then, we use a (suitably coherent version of) the classification of hermitian structures on module categories as categories of modules over the Hill–Hopkins–Ravenel norm [Cal+20a, Theorem 3.3.1] to conclude. Recall that the action of Mod_R^ω on Mod_A^ω is given by a functor $\mathcal{LM}^\otimes \to \mathcal{C}\mathrm{at}^\times_\infty$, and define $\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes$ via the following pullback square of ∞-operads:

$$
\begin{tikzcd}
\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \arrow[r, "p"] \arrow[d] & \mathcal{LM}^\otimes \arrow[d, "{\mathrm{Mod}_R^\omega, \mathrm{Mod}_A^\omega}"] \\
(\mathcal{C}\mathrm{at}_\infty)^\otimes_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}^\times_\infty
\end{tikzcd}
.
\tag{6.7}
$$

[3] The proof of (2) in particular shows that $\mathrm{Fun}^q(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is left-tensored over $\mathrm{Fun}^q(\mathrm{Mod}_R^{\omega,\mathrm{op}}, \mathrm{Sp})$ in the sense of [Lur17, Definition 4.2.1.19], so this makese sense.

Informally, an object $F \in \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes}_{\mathfrak{a}}$ is a functor $F\colon \mathrm{Mod}_R^{\omega,\mathrm{op}} \to \mathrm{Sp}$ and an object G over the fiber of $\mathfrak{m}$ is a functor $G\colon \mathrm{Mod}_A^{\omega,\mathrm{op}} \to \mathrm{Sp}$. The p-cocartesian edge over the canonical map $(\mathfrak{a}, \mathfrak{m}) \to \mathfrak{m}$ in $\mathcal{LM}^{\otimes}$ sends (F, G) to the lower arrow in the diagram

$$\begin{tikzcd}
\mathrm{Mod}_R^{\omega,\mathrm{op}} \times \mathrm{Mod}_A^{\omega,\mathrm{op}} \arrow[r, "F\times G"] \arrow[d, "-\otimes_R -"'] & \mathrm{Sp}\times\mathrm{Sp} \arrow[d, "\otimes_{\mathrm{Sp}}"] \\
\mathrm{Mod}_A^{\omega,\mathrm{op}} \arrow[r, "F\otimes G := \mathrm{LKE}_{\otimes_R}(\otimes_{\mathrm{Sp}}\circ(F\times G))"] & \mathrm{Sp}
\end{tikzcd}.$$

Now define $\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes}$ to consist of the full subcategory of $\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes}$ consisting of those tuples of functors which are all quadratic. The inclusion $\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \to \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes}$ exhibits the former as an ∞-operad, and moreover the localization is compatible with the $\mathcal{LM}^{\otimes}$-monoidal structure in the sense of [Lur17, Definition 2.2.1.6]. We can extend the previous diagram to

$$\begin{tikzcd}
\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r, "p"] \arrow[d] & \mathcal{LM}^{\otimes} \arrow[d, "\mathrm{Mod}_R^{\omega}{,}\mathrm{Mod}_A^{\omega}"] \\
\mathrm{Cat}^{\mathrm{h}\,\otimes}_{\infty} \arrow[r] & (\mathcal{C}\mathrm{at}_{\infty})^{\otimes}_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}^{\otimes}_{\infty}
\end{tikzcd}. \tag{6.8}$$

Modifying [Cal+20a, Construction 5.3.15 & Lemma 5.3.15] slightly (note that Corollary 5.1.4 did not assume the tensor factors to be equivalent), we obtain an analogous commutative diagram of ∞-operads

$$\begin{tikzcd}
\mathrm{Fun}^p_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r, "p"] \arrow[d] & \mathcal{LM}^{\otimes} \arrow[d, "\mathrm{Mod}_R^{\omega}{,}\mathrm{Mod}_A^{\omega}"] \\
\mathrm{Cat}^{\mathrm{p}\,\otimes}_{\infty} \arrow[r] & \mathrm{Cat}^{\mathrm{h}\,\otimes}_{\infty} \arrow[r] & (\mathcal{C}\mathrm{at}_{\infty})^{\otimes}_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}^{\otimes}_{\infty}
\end{tikzcd} \tag{6.9}$$

in which all squares are pullbacks. Now suppose A is given a module with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and call the associated Poincaré ∞-category $\overline{\mathrm{Mod}}_A$. Then to lift $\overline{\mathrm{Mod}}_A$ to a module over $(\mathrm{Mod}_R^{\omega}, \Qoppa_R)$ compatibly with the Mod_R^{ω}-module structure on Mod_A^{ω} is to give a map of ∞-operads $\mathcal{LM}^{\otimes} \to \mathrm{Cat}^{\mathrm{h}\,\otimes}_{\infty}$ so that the restriction along the canonical inclusion $\mathrm{Assoc}^{\otimes} \to \mathcal{LM}^{\otimes}$ gives the algebra object $(\mathrm{Mod}_R^{\omega}, \Qoppa_R)$ and postcomposing with the canonical projection to $\mathcal{C}\mathrm{at}^{\mathrm{ex}\,\times}_{\infty}$ recovers the given Mod_R^{ω}-module structure on Mod_A^{ω}. By the pullback square (6.8), this is equivalent to giving an object of $\mathrm{Alg}_{\mathcal{LM}/\mathcal{LM}}\left(\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes}\right)$. Now let us identify the bilinear functor $\mathrm{Mod}_R^{\omega} \times \mathrm{Mod}_A^{\omega} \xrightarrow{-\otimes_R -} \mathrm{Mod}_A^{\omega}$ with the exact functor $\mathrm{Mod}_R^{\omega} \otimes \mathrm{Mod}_A^{\omega} \simeq \mathrm{Mod}_{R\otimes A}^{\omega} \to \mathrm{Mod}_A^{\omega}$ which is induction along the action map $R \otimes A \to A$. Using [Cal+20a, Corollary 3.4.1] and unravelling definitions gives the claim for R-linear hermitian structures. The proof for R-linear Poincaré structures considers (6.9) instead but otherwise proceeds in an identical fashion.

(3) By [Lur17, Theorem 4.4.2.8], the relative tensor product $(\mathrm{Mod}_A^{\omega}, \Qoppa_A) \otimes_{(\mathrm{Mod}_R^{\omega}, \Qoppa_R)} (\mathrm{Mod}_B^{\omega}, \Qoppa_B)$ is computed as the geometric realization of the bar construction

$$\begin{aligned} p\colon \Delta^{\mathrm{op}} &\to \mathrm{Cat}^{\mathrm{h}}_{\infty} \\ [n] &\mapsto (\mathrm{Mod}_A^{\omega}, \Qoppa_A) \otimes (\mathrm{Mod}_R^{\omega}, \Qoppa_R)^{\otimes n} \otimes (\mathrm{Mod}_B^{\omega}, \Qoppa_B) \end{aligned}$$

Write $f\colon \mathrm{Cat}^{\mathrm{h}}_{\infty} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty}$ for the forgetful functor. Then $f \circ p$ has a colimit with value $\mathrm{Mod}_A^{\omega} \otimes_{\mathrm{Mod}_R^{\omega}} \mathrm{Mod}_B^{\omega} \simeq \mathrm{Mod}_{A\otimes_R B}^{\omega}$. Writing $g\colon \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \to \{*\}$, by Example 4.3.1.3 of [Lur09] we see that $f \circ p$ is a g-colimit. By Proposition 4.3.1.5(2) and Example 4.3.1.3 of [Lur09], p admits a colimit in $\mathrm{Cat}^{\mathrm{h}}_{\infty}$ if and only if it admits an f-colimit. Now recall that f is a cocartesian fibration with pushforward given by left Kan extension [Cal+20a, Corollary 1.4.2]. We show that f satisfies the conditions of [Lur09, Corollary 4.3.1.11].

- Condition (1) follows from Theorem 6.1.1.10 of [Lur17] applied to $\mathrm{Sp}^{\mathrm{op}}$ (see the end of [Cal+20a, Construction 1.1.26]).
- Condition (2) follows from [Cal+20a, Corollary 1.4.2], the adjoint functor theorem, and presentability of $\mathrm{Fun}^q(\mathcal{C})$, which is discussed in the proof of [Cal+20a, Lemma 5.3.3] (also see [Lur17, Remark 6.1.1.11]).

Thus the preceding discussion shows that there exists a map of simplicial sets p' making the diagram commute

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}^{\mathrm{h}}_{\infty} \arrow[d, "f"] \\ (\Delta^{\mathrm{op}})^{\triangleright} \arrow[r] \arrow[ur, "p'"] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \end{tikzcd} .$$

Since $\{0\} \to \Delta^1$ is left anodyne, by [Lur09, Corollary 2.1.2.7] the inclusions

$$\{0\} \times \Delta^{\mathrm{op}} \to \Delta^1 \times \Delta^{\mathrm{op}}$$
$$\iota \colon (\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) \to \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright}$$

are left anodyne. The former implies that there exists a map p'' making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}^{\mathrm{h}}_{\infty} \arrow[d, "f"] \\ \Delta^1 \times \Delta^{\mathrm{op}} \arrow[r] \arrow[ur, "p''"] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \end{tikzcd}$$

commute. The maps p' and p'' assemble to give a map $p''' := p' \sqcup_p p''$ making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[rr, "p"] \arrow[d] & & \mathrm{Cat}^{\mathrm{h}}_{\infty} \arrow[d, "f"] \\ (\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) \arrow[r, "\iota"] \arrow[urr, "p'''"] & \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright} \arrow[r] \arrow[ur, bend left, "\overline{p}"'] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \end{tikzcd}$$

commute, and likewise $\overline{p}$ exists making the diagram commute since ι is left anodyne. Now we show that $\overline{p}$ satisfies the conditions of [Lur09, Proposition 4.3.1.9]. By (the opposite/dual/cocartesian version of) [Lur09, Remark 3.1.1.10] and Proposition 3.1.1.5(2") *ibid.* and the fact that f is a cocartesian fibration, we can choose $\overline{p}$ so that for all $k \in (\Delta^{\mathrm{op}})^{\triangleright}$, $\overline{p}|_{\Delta^1\times\{k\}}$ is f-cocartesian. Furthermore, since we can choose Δ^{op}, $(\Delta^{\mathrm{op}})^{\triangleright}$ to have the markings $(-)^{\flat}$ in [Lur09, Remark 3.1.1.10], $f \circ \overline{p}|_{\Delta^1\times\{\infty\}}$ is a degenerate edge in $\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty}$.

Now [Lur09, Proposition 4.3.1.9] implies that $\overline{p}_0$ is an f-colimit diagram if and only if $\overline{p}_1$ is an f-colimit diagram. Now notice that $\overline{p}|_{\{1\}\times(\Delta^{\mathrm{op}})^{\triangleright}}$ has image contained in the fiber of f over $\mathrm{Mod}^{\omega}_{A\otimes_R B}$. By [Lur09, Proposition 4.3.1.10], it suffices to show that $\overline{p}_1$ is a colimit diagram in $\mathrm{Fun}^q\left(\mathrm{Mod}^{\omega}_{A\otimes_R B}\right)$. Write $\overline{M}_A \in \mathrm{Mod}_{N^{C_2}A}$ and $\overline{M}_B \in \mathrm{Mod}_{N^{C_2}B}$ for the corresponding modules (see introduction to §3.3 of [Cal+20a]). Unraveling definitions and using [Cal+20a, Theorem 3.3.1 & Corollary 3.4.1 & Lemma 5.4.6], it follows that the diagram $\overline{p}_1|_{\{1\}\times\Delta^{\mathrm{op}}}$ is the bar construction

$$[n] \mapsto \overline{M}_A \otimes_{N^{C_2}R} R^{\otimes_{N^{C_2}R} n} \otimes_{N^{C_2}R} \overline{M}_B\,.$$

This proves the result.

(4) Let $(\mathcal{C}, Ϙ_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R, Ϙ_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$ and let $F\colon \mathcal{C} = \theta\,(\mathcal{C}, Ϙ_{\mathcal{C}}) \to \mathcal{D}$ be an R-linear functor. Now define $Ϙ_{\mathcal{D}} \colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp}$ to be the left Kan extension of $Ϙ_{\mathcal{C}}$ along F^{op}. Now $(\mathcal{D}, Ϙ_{\mathcal{D}}) \in \mathrm{Cat}^{\mathrm{h}}_{\infty}$ and there

is a canonical map $(f,\eta)\colon (\mathcal{C},\Qoppa_{\mathcal{C}}) \to (\mathcal{D},\Qoppa_{\mathcal{D}})$. Now F is classified by a functor $\Delta^1 \times \mathcal{LM}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}\otimes}$, and we may form the pullback

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \Delta^1 \times \mathcal{LM}^{\otimes} \arrow[d] \\ \mathrm{Cat}_{\infty}^{\mathrm{h}\ \otimes} \arrow[r, "p"] & \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}\otimes} \end{tikzcd}. \tag{6.10}$$

Since p is a cocartesian fibration [Cal+20a, Theorem 5.2.7], $\mathcal{N} \to \Delta^1 \times \mathcal{LM}^{\otimes}$ is a cocartesian fibration, and the nontrivial morphism in Δ^1 classifies a map $F_!\colon \mathrm{Fun}^{q}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})^{\otimes} \to \mathrm{Fun}^{q}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathcal{D}^{\mathrm{op}},\mathrm{Sp})^{\otimes}$ of ∞-operads over $\mathcal{LM}^{\otimes}$. Passing to algebra objects, we obtain the desired result on mapping spaces.

(5) By [Lur09, Proposition 2.4.2.8], it suffices to show that θ is a locally (co)cartesian fibration, and that locally (co)cartesian edges are closed under composition. We give the proof that θ is a cocartesian fibration; the proof that θ is a cartesian fibration is formally dual and will be left to the reader.

Let $(\mathcal{C},\Qoppa_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{(\mathrm{Mod}_R^{\omega},\Qoppa_R)}(\mathrm{Cat}_{\infty}^{\mathrm{h}})$ and let $F\colon \mathcal{C} = \theta(\mathcal{C},\Qoppa_{\mathcal{C}}) \to \mathcal{D}$ be an R-linear functor. Now define $\Qoppa_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp}$ to be the left Kan extension of $\Qoppa_{\mathcal{C}}$ along F^{op}. By the proof of (4), we see that the image of $\Qoppa_{\mathcal{C}}$ under $F_!$ is a lift of $(\mathcal{D},\Qoppa_{\mathcal{D}})$ to an object of $\mathrm{Mod}_{(\mathrm{Mod}_R^{\omega},\Qoppa_R)}(\mathrm{Cat}_{\infty}^{\mathrm{h}})$ and (f,η) to a morphism in $\mathrm{Mod}_{(\mathrm{Mod}_R^{\omega},\Qoppa_R)}(\mathrm{Cat}_{\infty}^{\mathrm{h}})$.

Now by Lemma 2.4.4.1 and the locally cocartesian version of Proposition 2.4.1.10 of [Lur09], we must show that for all choices $\Qoppa'_{\mathcal{D}}$ of an R-linear Hermitian structure on $\mathcal{D}$, precomposition with $F_!$ induces a pullback square

$$\begin{tikzcd} \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}\left((\mathcal{D},\Qoppa_{\mathcal{D}}),(\mathcal{D},\Qoppa'_{\mathcal{D}})\right) \arrow[r] \arrow[d] & \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}\left((\mathcal{C},\Qoppa_{\mathcal{C}}),(\mathcal{D},\Qoppa'_{\mathcal{D}})\right) \arrow[d] \\ \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}(\mathcal{D},\mathcal{D}) \arrow[r] & \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}(\mathcal{C},\mathcal{D}) \end{tikzcd}. \tag{6.11}$$

By (4), $F_!$ induces equivalences on the fibers of the vertical maps, hence (f,η) is locally θ-cocartesian. The locally θ-cocartesian maps are manifestly closed under composition, hence we are done. ☐

Corollary 6.12. *Let R be a Poincaré ring, and let A, B be $\mathbb{E}_1$-R-algebras with genuine involution. Then there is an equivalence* $\hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}}\left((\mathrm{Mod}_A^{\omega},\Qoppa_A),(\mathrm{Mod}_B^{\omega},\Qoppa_B)\right) \simeq (\mathrm{BiMod}_{A\otimes_R B^{\mathrm{op}}})_{A^{\varphi C_2}\otimes_R B^{\varphi C_2}/-}$.

Proof. ☐

L: todo– probably need to fix the statement with *duals* when the proof is written

As in the Picard group case, the symmetric monoidal forgetful functor $\theta\colon \mathrm{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}$ induces a map of spectra $\theta\colon \mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A^e)$. When A^e is endowed with the trivial action, θ will factor through the 2-torsion on π_0. As a consequence of Proposition 6.4(2) we can identify the fiber of this map.

Corollary 6.13. *Let $(\mathrm{Mod}_A^{\omega},\Qoppa_A)$ be a Poincaré ring with underlying genuine C_2 spectrum A^L as in Proposition 6.4(2). Then the fiber of the map*

$$\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A^e)$$

can be naturally identified with the cofiber $\mathrm{cofib}\left(\mathrm{Pic}\left(\mathrm{Mod}_{A^e}(\mathrm{Sp})^{hC_2}\right) \to \mathrm{Pic}\left(\mathrm{Mod}_{A^L}\left(\mathrm{Sp}^{C_2}\right)\right)\right)$*, where the latter cofiber is taken in connective spectra.*

Proof of Corollary 6.13. Since $\theta : \mathrm{Mod}_{(\mathrm{Mod}_A^{\omega},\Qoppa_A)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_{\mathrm{Mod}^{\omega}_{A^e}}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty})$ is symmetric monoidal and conservative, it induces a map $\theta^{\simeq}\colon \mathrm{PnBr}(A) \to \mathrm{br}(A^e)$ on the groupoid core of invertible objects. Now observe that θ is an isofibration; it follows that $\theta^{\simeq}$ is a Kan fibration by [Lur24, Proposition 01EZ]. Consequently we need only identify the fiber over a single point instead of the homotopy fiber. Consider $(\mathrm{Mod}^{\omega}_{A^e},\Qoppa)$ a point in the fiber of θ over $\mathrm{Mod}^{\omega}_{A^e}$. By Proposition 6.4(2), $\Qoppa$ is associated to an A-linear invertible module with involution $\left(M,N,N \to M^{tC_2}\right)$. By Proposition 6.4(3), invertibility of $(\mathrm{Mod}^{\omega}_{A^e},\Qoppa)$ implies that $\left(M,N,N \to M^{tC_2}\right)$ is invertible as a module over A^L. Now suppose $(\mathrm{Mod}^{\omega}_{A^e},\Qoppa)$ and $(\mathrm{Mod}^{\omega}_{A^e},\Phi)$ are two points in the fiber of θ over $\mathrm{Mod}^{\omega}_{A^e}$. We know that $\Qoppa$ and Φ are associated to invertible A^L-modules

$(M, N, N \to M^{tC_2})$ and $(M', N', N' \to (M')^{tC_2})$, resp. An equivalence from $(\mathrm{Mod}^{\omega}_{A^e}, \Qoppa)$ to $(\mathrm{Mod}^{\omega}_{A^e}, \Phi)$ consists of an A^e-linear equivalence $F\colon \mathrm{Mod}^{\omega}_{A^e} \xrightarrow{\sim} \mathrm{Mod}^{\omega}_{A^e}$ and a natural equivalence $\eta\colon \Phi \circ F^{\mathrm{op}} \xrightarrow{\sim} \Qoppa$. Now F is classified by an invertible A^e-bimodule β and the natural equivalence η is classified by an equivalence $\beta \otimes_{A^L} (M, N, N \to M^{tC_2}) \simeq (M', N', N' \to (M')^{tC_2})$ of A^L-modules [Cal+20a, Lemma 3.4.3]. $\square$

N: Needs a lot more detail I know, but I think the skeleton is here.

Example 6.14. Let k be an algebraically closed field, and regard k as a Poincaré ring $\underline{k}$ via Example 3.15 with the trivial involution. Then

$$\pi_0 \mathrm{PnBr}(\underline{k}) = \begin{cases} 0 & \mathrm{char}\, k \neq 2 \\ \mathbb{Z} & \mathrm{char}\, k = 2\,. \end{cases}$$

Write Pn for the composite $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \xrightarrow{U} \mathrm{Cat}^{\mathrm{p}}_{\infty} \xrightarrow{\mathrm{Pn}} \mathcal{S}$.

Proposition 6.15. *Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Then $(\mathrm{Mod}^{\omega}_R, \Qoppa_R)$ corepresents the functor* $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{S}$.

L: Pretty sure this follows from Corollary 6.13; will return to this later. Also, maybe put this example somewhere else.

Proof. Recall that Proposition 6.4.(1) furnishes an adjoint pair $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \rightleftarrows \mathrm{Cat}^{\mathrm{p}}_{\infty}$ of functors. Write $\overline{\mathcal{C}} = (\mathcal{C}, \Qoppa_{\mathcal{C}}) \in \mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}$. Then

$$\mathrm{Pn}(\mathcal{C}) = \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty}}\left((\mathrm{Sp}^f, \Qoppa^u), U(\overline{\mathcal{C}})\right) \simeq \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty R}}\left((\mathrm{Mod}^{\omega}_R, \Qoppa_R) \otimes (\mathrm{Sp}^f, \Qoppa^u), \overline{\mathcal{C}}\right)\,,$$

where the first equivalence is [Cal+20a, Proposition 4.1.3]. $\square$

6.2 Azumaya algebras with genuine involution

Let R be an $\mathbb{E}_\infty$-ring spectrum.

Recollection 6.16. Recall [BRS12; AG14] that an $\mathbb{E}_1$-R-algebra A is said to be *Azumaya* if it is a compact generator of Mod_R and if the natural R-algebra map giving the bimodule structure on A

$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$$

is an equivalence of R-algebras.

Definition 6.17. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring spectrum. An *Azumaya algebra with genuine type 1 (anti-)involution* over R is the data of

(a) An $\mathbb{E}_1$-R-algebra A equipped with an anti-involution $\tau\colon A \to A^{\mathrm{op}}$ so that A is an Azumaya R-algebra in the sense of Recollection 6.16

(b) an $(A \otimes_R A^{\mathrm{op}})^{\otimes_R 2}$-linear equivalence $\hom_{R\otimes R}(A \otimes_R A, R) \simeq A \otimes_R A^{\mathrm{op}}$

(c) A left $A \otimes_R R^{\varphi C_2}$-module P and an $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-module $\overline{P}$

(d) An $A \otimes_R R^{\varphi C_2}$-linear map $P \to A^{tC_2}$ and an $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-linear map $\overline{P} \to A^{tC_2}$. Here we regard A^{tC_2}, which is canonically a $(A \otimes A)^{tC_2}$-module, as an $A \otimes_R R^{\varphi C_2}$-module (resp. $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-module) via the twisted Tate-valued diagonal $A \to (A \otimes A^{\mathrm{op}})^{tC_2}$ (resp. $A^{\mathrm{op}} \to (A \otimes A^{\mathrm{op}})^{tC_2}$).

(e) An equivalence of $(A \otimes_R A^{\mathrm{op}}) \otimes_R R^{\varphi C_2}$-modules

$$\hom_R(A, R^{\varphi C_2}) \simeq P \otimes_{R^{\varphi C_2}} \overline{P}$$

and a homotopy making the diagram

$$\begin{tikzcd} \hom_R(A, R^{\varphi C_2}) \arrow[r] \arrow[d] & P \otimes_{R^{\varphi C_2}} \overline{P} \arrow[d] \\ \hom_R(A, R^{tC_2}) \simeq \hom_R(A \otimes A^{\mathrm{op}}, R)^{tC_2} \arrow[r] & \hom_{(A\otimes A^{\mathrm{op}})^{\otimes 2}}\left((A \otimes A^{\mathrm{op}})^{\otimes 2}, A \otimes A^{\mathrm{op}}\right)^{tC_2} \simeq (A \otimes A^{\mathrm{op}})^{tC_2} \end{tikzcd}$$

commute, where the lower horizontal arrow is induced by (b) and the right vertical arrow is induced by (d).

Similarly, an *Azumaya algebra with genuine type 2 (anti-)involution* over R is obtained by replacing τ in item (a) with $\tau\colon A \to \sigma_R^* A^{\mathrm{op}}$, where $\sigma_R\colon R \xrightarrow{\simeq} R$ is the given involution on R.

L: define the category!

Remark 6.18. If A is an Azumaya algebra with genuine involution over R, then in particular $M_A = A$, $N_A = P$ is a module with genuine involution over A in the sense of [Cal+20a, Definition 3.2.3].

With ordinary Azumaya algebras, the prototypical Azumaya algebra with anti-involution arises from endomorphism rings of perfect modules. Choosing a (nondegenerate symmetric bilinear) form on a perfect module P endows its endomorphism algebra with additional structure.

Example 6.19. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(P, q) \in \mathrm{Pn}\,(\mathrm{Mod}_R^\omega, \Qoppa_R)$.

Then $A := \mathrm{End}_R(P)$ admits a canonical lift to an $\mathbb{E}_1$ with genuine involution over R with $A^{\varphi C_2} := \hom_R(P, R^{\varphi C_2})$. If P is a generator of Mod_R^ω, then A is furthermore Azumaya.

By [Cal+20a, Proposition 3.1.16], A inherits a canonical anti-involution. To exhibit (b), observe that $q^\dagger$ induces a canonical $A \otimes A^{\mathrm{op}}$-linear equivalence $A = \mathrm{End}(P) \simeq \mathrm{End}(P^\vee) \simeq A^\vee$. If P is a generator, $\hom_R(P, -)$ induces an equivalence $\mathrm{Mod}_R^\omega \simeq \mathrm{Mod}_A^\omega$, thus we can regard Mod_A^ω as equipped with a Poincaré structure. By the classification of R-linear Poincaré structures of Proposition 6.4(2), the Poincaré structure on Mod_A^ω is associated to an A-module with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$. We claim that $M_A \simeq A$ with the canonical A-A-bimodule structure: By [Cal+20a, Proposition 3.1.6], as an A^{op}-module M_A is the image of A under the composite

L: todo: an R-linear enhancement of Proposition 3.1.16?

$$\mathrm{Mod}_A^\omega \xrightarrow{\hom_R(P,-)^{-1}} \mathrm{Mod}_R^\omega \xrightarrow{D_R = \hom_R(-,R)} \mathrm{Mod}_R^{\omega,\mathrm{op}} \xrightarrow{\hom_R(P,-)} \mathrm{Mod}_A^{\omega,\mathrm{op}}\ .$$

Observe that the image of A in $\mathrm{Mod}_R^{\omega,\mathrm{op}}$ is $D_R(P)$ and $q^\dagger$ induces an equivalence $D_R(P) \simeq P$, hence $M_A \simeq A$ as A^{op}-modules.

A similar argument with the linear part of $\Qoppa$ shows that we have an equivalence $N_A \simeq \hom_R(P, R^{\varphi C_2})$.

L: continue. discuss consequences of being Morita trivial (distinguished 'point' in N_A)? can we identify $\overline{P}$?

Proposition 6.20. *Let R be a discrete ring with a given C_2-action λ, and suppose that the ramification locus in $\mathrm{Spec}(R)/C_2 = \mathrm{Spec}(R^{C_2})$ is empty. Let A be a classical Azumaya algebra over R with an involution of type 2. Regard R as a Poincaré ring spectrum $\underline{R}^\lambda$ via Example 3.15.*

Then there is a canonical Azumaya algebra with genuine involution over $\underline{R}^\lambda$ whose underlying Azumaya algebra is A.

Remark 6.21. If $\frac{1}{2} \in R$, then $\mathrm{Br}(\mathrm{Spec}\, R, \lambda)$ is defined [PS92, p. 216]. In view of Propositions 6.20 and 6.22, there is a homomorphism $\mathrm{Br}(\mathrm{Spec}\, R, \lambda) \to \pi_0 \mathrm{PnBr}(\underline{R}^\lambda)$.

L: In what other cases does a lift exist trivially? wrt Tate Poincaré structure, suffices to take a trivialization of A?

Proof of Proposition 6.20. Since $R^{\varphi C_2} = 0$, conditions (c)-(e) of Definition 6.17 are vacuous. $\square$

Proposition 6.22. *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(A, A^{\varphi C_2} \to A^{tC_2})$ be an Azumaya algebra with genuine involution over R. Then*

1. *$(\mathrm{Mod}_A^\omega, \Qoppa_A)$ defines an R-linear Poincaré ∞-category.*
2. *$(\mathrm{Mod}_A^\omega, \Qoppa_A)$ is an invertible object in $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}\left(\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}\right)$.*

Proof of Proposition 6.22. The first statement follows from Proposition 6.4(2); we prove the second statement. First, by [Cal+20a, Example 3.2.9], we see that $(\mathrm{Mod}_A^\omega, \Qoppa_A)$ is indeed an R-linear Poincaré ∞-category (and not merely hermitian). To show that the associated Poincaré ∞-category is invertible, we must identify a dual $(\mathrm{Mod}_A^\omega, \Qoppa_A)^\vee$ and exhibit an equivalence $(\mathrm{Mod}_A^\omega, \Qoppa_A) \otimes (\mathrm{Mod}_A^\omega, \Qoppa_A)^\vee \simeq (\mathrm{Mod}_R^\omega, \Qoppa_R)$. Since $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty\,R}$ is symmetric monoidal, we see that the underlying R-linear ∞-category associated to the dual must be $\mathrm{Mod}_{A^{\mathrm{op}}}^\omega$. Moreover, the canonical evaluation map $\mathrm{ev}\colon \mathrm{Mod}_A^\omega \otimes \mathrm{Mod}_{A^{\mathrm{op}}}^\omega \xrightarrow{\simeq} \mathrm{Mod}_R^\omega$ sends $A \otimes A^{\mathrm{op}}$ to A. Endow $\mathrm{Mod}_{\underline{A}^{\mathrm{op}}}^\omega$ with a Poincaré structure corresponding to the module with genuine involution $M_{A^{\mathrm{op}}} := A^{\mathrm{op}}$, $N_{A^{\mathrm{op}}} := \overline{P}$. It remains to exhibit a natural equivalence

L: rewrite for scheme with involution

$$\eta\colon\ (\Qoppa_A \otimes \Qoppa_{A^{\mathrm{op}}}) \xrightarrow{\simeq} \mathrm{ev}^* \Qoppa_R \tag{6.23}$$

of [quadratic] functors $\mathrm{Mod}_A^\omega \otimes \mathrm{Mod}_{A^{\mathrm{op}}}^\omega \to \mathrm{Sp}$. By [Cal+20a, Theorem 3.3.1], it suffices to exhibit equivalences on the bilinear and linear parts of (6.23) which glue compatibly. By Proposition 6.4(3), on linear parts, it suffices to exhibit an $A \otimes_R A^{\mathrm{op}}$-linear equivalence

$$\hom_R(A, R^{\varphi C_2}) \simeq N_A \otimes_{R^{\varphi C_2}} N_{A^{\mathrm{op}}}$$

and on bilinear parts, it suffices to exhibit an $(A \otimes_R A^{\mathrm{op}})^{\otimes_R 2}$-linear equivalence

$$\hom_{R\otimes R}(A \otimes_R A, R) \simeq M_A \otimes_R M_{A^{\mathrm{op}}}$$

which glue compatibly. This follows from the definitions, concluding the proof. □

Proposition 6.24. *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Let $(\mathcal{C}, Ϙ)$ be an invertible idempotent-complete R-linear Poincaré ∞-category. Suppose given a Poincaré object (P, q) of $(\mathcal{C}, Ϙ)$ so that P is a generator for $\mathcal{C}$. Then $(\mathcal{C}, Ϙ)$ is of the form $(\mathrm{Mod}_A^\omega, Ϙ_A)$ for some Azumaya algebra over R with genuine involution.*

L: later: add case of schemes with involution

Corollary 6.25. *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Let $(\mathcal{C}, Ϙ)$ be an invertible idempotent-complete R-linear Poincaré ∞-category. Then $(\mathcal{C}, Ϙ)$ is of the form $(\mathrm{Mod}_A^\omega, Ϙ_A)$ for some Azumaya algebra over R with genuine involution.*

L: connectivity hypothesis on R?

Proof. By Proposition 6.24, it suffices to exhibit a Poincaré object (P, q) whose underlying object is a generator of $\mathcal{C}$. By [AG14, Theorem 6.1], $\mathcal{C}$ has a generator G. Now $G \oplus D_Ϙ G$ promotes canonically to a Poincaré object of $(\mathcal{C}, Ϙ)$ by [Cal+20a, Proposition 2.2.5]. □

L: Need to show $\mathcal{C}$ (or $\mathrm{Ind}(\mathcal{C})$) defines a R-linear category with descent?

Proof of Proposition 6.24. Similar to Example 6.19 □

L: to-do

6.3 Stacks associated to Poincaré-∞-categories

Notation 6.26. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. There is a functor

$$\begin{aligned} \mathbb{E}_\infty \mathrm{Alg}_{R/}^{BC_2} &\to \mathrm{CAlg}^{\mathrm{p}}{}_{R/-} \\ S &\mapsto (S, S \to R^{\varphi C_2} \otimes_R S \to S^{tC_2}) =: (R, R \to R^{\varphi C_2} \to R^{tC_2}) \otimes S\,, \end{aligned}$$

where the map $R^{\varphi C_2} \otimes_R S \to R^{\varphi C_2} \otimes_{R^{tC_2}} S^{tC_2} \to R^{tC_2} \otimes_{R^{tC_2}} S^{tC_2} \simeq S^{tC_2}$ is given by base change along the Tate-valued norm composed with the structure map $R^{\varphi C_2} \to R^{tC_2}$. Composing the aforementioned functor with the functor that sends a Poincaré ring to its category of compact modules equipped with the canonical Poincaré structure defines a functor

$$\mathrm{Mod}^p \colon \mathbb{E}_\infty \mathrm{Alg}_{R/-}^{BC_2} \to \mathbb{E}_\infty \mathrm{Alg}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}{}_R\right).$$

Notation 6.27. Let X be a scheme with an involution σ and let $\pi \colon X \to Y$ exhibit Y as a good quotient of X. If $j \colon U \to Y$ is flat, let us write $\pi^* U$ for the tuple $(X \times_Y U, j^*(\sigma), U, j^*(\pi))$ of Remark 3.18. Then the assignment $(j \colon U \to Y) \mapsto \left(\mathrm{Mod}_{j^*X}^\omega, Ϙ_{j^*\underline{\mathcal{O}}}\right)$ defines a functor

$$\mathrm{Mod}^p \colon \text{Ét}_Y^{\mathrm{op}} \to \mathbb{E}_\infty \mathrm{Alg}\left(\mathrm{Mod}_{(\mathrm{Mod}_X^\omega, Ϙ_{\underline{\mathcal{O}}})}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})\right).$$

Proposition 6.28. *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and assume that $R^{\varphi C_2}$ and R are connective. Then the assignment of Notation 6.26 is a hypersheaf on the small étale site of R.*

L: I think Propositions ?? and 6.30 have essentially the same proof–present them together?

Corollary 6.29. *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and assume that $R^{\varphi C_2}$ and R are connective. Write PnBr for the composite functor $\text{Ét}_R \xrightarrow{\mathrm{Mod}^p} \mathbb{E}_\infty \mathrm{Alg}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}{}_R\right) \xrightarrow{\mathrm{PnBr}} \mathrm{Sp}_{\geq 0}$, where Mod^p is from Notation 6.26 and PnBr is Definition 6.1. Then PnBr is an étale sheaf.*

Proof. Follows from Proposition 6.28 and the fact that PnBr preserves all limits. □

L: recall étale hypersheaf/hypercove

L: Antieau–Gepner write *stack* for 'sheaf of categories' and *sheaf*

Proposition 6.30. *Let (X, λ, Y, π) be a scheme with involution X and a good quotient Y. Then the assignment of Notation 6.27 is a hypersheaf on the small étale site of Y.*

Proof of Proposition 6.28. For now, use the notational shorthand $R^p = (R, R \to R^{\varphi C_2} \to R^{tC_2})$. Since limits in categories of algebras and modules are computed at the level of underlying objects, it suffices to show that the functor sends an étale hypercovering $j_\bullet \colon S \to T^\bullet$ to a limit diagram in $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$. By Proposition 6.1.4 of [Cal+20a], it suffices to show that the relevant diagram is a limit diagram in $\mathrm{Cat}^{\mathrm{h}}_{\infty,\mathrm{idem}}$. The proof of Lemma 5.4 in [AG14] implies that the diagram defines a limit diagram on underlying ∞-categories. Thus by Remark 6.1.3 of [Cal+20a], it suffices to show that $j_\bullet^* \colon \mathrm{Mod}^\omega_{R^e \otimes_{R^{C_2}} S} \to \mathrm{Mod}^\omega_{R^e \otimes_{R^{C_2}} T^\bullet}$ induces an equivalence $Ϙ_{R^p \otimes S} \xrightarrow{\sim} \lim_\Delta Ϙ_{R^p \otimes T^\bullet} \circ (j_\bullet^*)^{\mathrm{op}}$ of quadratic functors $\mathrm{Mod}^{\omega,\mathrm{op}}_{R^e \otimes_{R^{C_2}} S} \to \mathrm{Sp}$. This follows from our assumption on $S \to T^\bullet$ and [Cal+20a, Theorem 3.3.1]. □

L: next bit is sketchy; working towards a spectral sequence like the one in [AG14, §7]

Notation 6.31. Let (X, λ, Y, π) be a scheme with involution X and a good quotient Y. Let $\mathbb{Z}_X$ be the étale sheafification of the constant presheaf. Write $\mathbb{Z}_X^\sigma$ for the equalizer

$$\mathbb{Z}_X^\sigma := \mathrm{Eq}\left(\mathbb{Z}_X \overset{\lambda, \cdot(-1)}{\rightrightarrows} \mathbb{Z}_X\right).$$

Write disc for the cokernel

$$\mathrm{disc} := \mathrm{coKer}\left(\mathcal{O}_X^\times \xrightarrow{f \mapsto f \cdot \lambda(f)} \mathcal{O}_X^\times\right); \tag{6.32}$$

here the cokernel is taken in the category of sheaves of abelian groups on the small étale site of X. Consider the assignment

$$\begin{aligned} U_1 \colon \acute{\mathrm{E}}\mathrm{t}_Y &\to \mathbb{E}_\infty\mathrm{Mon}(\mathcal{S}) \\ (V = \mathrm{Spec}\, A \to Y) &\mapsto \mathrm{Ker}\left(R\Gamma(\mathcal{O}_{X_V}(X_V))^\times \xrightarrow{f \mapsto f \cdot \lambda(f)} R\Gamma(\mathcal{O}_{X_V}(X_V))^\times\right). \end{aligned}$$

Then U_1 is a sheaf of groups on the small étale site of Y. By Theorem 4.10, there is a natural map of sheaves $BU_1 \to \mathrm{PnPic}$ given by inclusion of the identity component. L: check later

Example 6.33. If X has the trivial involution $\lambda = \mathrm{id}$, then $\mathbb{Z}_X^\sigma$ is the trivial sheaf and disc_X is the units in $\mathcal{O}_X$ mod 2.

If π is quadratic étale, then $\pi_* \mathbb{Z}_X^\sigma$ is a $\mathbb{Z}$-torsor.

Corollary 6.34. *Let (X, λ, Y, π) be a scheme with involution X and a good quotient Y. The homotopy sheaves of* PnBr *are*

$$\pi_* \mathrm{PnBr} = \begin{cases} ? & * = 0 \\ \pi_* \mathbb{Z}_X^\sigma \times \pi_* \mathrm{disc} & * = 1 \\ U_1 & * = 2 \\ 0 & \textit{else.} \end{cases}$$

L: If we believe what's in Example 6.14, then $\pi_0 \mathrm{PnBr}$ should be the pushforward of $\mathbb{Z}$ from the mod 2 reduction of the branch locus (terminology from [FW20])

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