# Poincaré Schemes

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**Abstract**

We do stuff

N: Change this

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# 1 Introduction

**Theorem 1.1.** *Let $\underline{A}$ be an affine Poincaré scheme with underlying $\mathbb{E}_\infty$-ring spectrum with involution $A$. Then the natural maps*

$$\pi_i(\mathrm{PnPic}(\underline{A})) \to \pi_i(\mathrm{Pic}(A))$$

*are surjective on 2-torsion.*

N: I think there is some interaction with the homotopy fixed points, or maybe even the genuine fixed points

**Theorem 1.2.** *Let $A$ be an $\mathbb{E}_\infty$ ring with involution, and let $\underline{NA}$ be the associated Tate affine Poincaré scheme. Let $\mathrm{Br}_\nu(A)$ be the Brauer group of Azumaya algebras over $A$ with involution. Then the natural map*

$$\mathrm{PnBr}(\underline{NA}) \to \mathrm{Br}_\nu(A)$$

*is an equivalence*

N: I think we need to define this for ring spectra. For $A$ discrete this is done in [2].

**Theorem 1.3.** *The functors $\mathrm{PnPic}, \mathrm{PnBr} : \mathrm{APS} \to \mathcal{S}p$ are fppf sheaves.*

**Theorem 1.4.** *There is a Poincaré group scheme $\mathbb{G}_m^{\female}$ such that*

$$B\mathbb{G}_m^{\female} \simeq \mathrm{PnPic}$$

*as fppf stacks.*

# 2 Poincaré ring spectra

We begin by defining the ring theoretic building blocks of Poincaré schemes and the corresponding category they live in. Affine Poincaré Schemes will then be the dual objects, similar to how affine schemes are dual to commutative rings.

N: probably of $\mathbb{E}_\infty$ dodads

**Notation 2.1.** Let $R$ be an $\mathbf{E}_\infty$-ring spectrum. We will drop $\mathbf{E}_\infty$ from our notation and simply call $R$ a *ring spectrum.*

**Definition 2.2.** Let $R$ be a ring spectrum. A *Poincaré structure* on $R$ is a symmetric monoidal Poincaré $\infty$-category $\Qoppa : (\mathrm{Mod}_R^\omega)^{\mathrm{op}} \to \mathrm{Sp}$. We call such a symmetric monoidal Poincaré $\infty$-category a *Poincaré ring spectrum*. We will denote the full subcategory of $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{p}})$ spanned by Poincaré ring spectra by $\mathrm{CAlg}^{\mathrm{p}}$ and call it the $\infty$*-category of Poincaré ring spectra*.

**Remark 2.3.** Poincaré ring spectra, as defined in Definition 2.2, were studied in . Note that we chose a different notation. In Poincaré ring spectra are being referred to as $\mathbf{E}_\infty$*-ring spectra with genuine involution*.

V: cite 9

V: cite 9 authored paper

**Remark 2.4.** Let $R$ be a ring spectrum. By there is a natural equivalence between symmetric monoidal Poincaré structures on $\mathrm{Mod}_R^\omega$ and algebra objects over the genuine $C_2$-spectrum $NR$ . In particular, a Poincaré structure on $R$ can be identified with the following data:

V: cite 9-authors

V: reference

- A $C_2$-action on $R$ via maps of ring spectra, i.e. a functor $\lambda : BC_2 \to \mathrm{CAlg}$.
- An $R$-algebra $R \to C$.
- An $R$-algebra map $C \to R^{tC_2}$.

Here $R^{tC_2}$ is the Tate construction with respect to the above action. Since the Tate construction is symmetric monoidal, $R^{tC_2}$ is naturally an $R$-algebra. A ring spectrum equipped with a Poincaré structure will be called a *Poincaré ring spectrum*.

**Remark 2.5.** By Remark 2.4, a Poincaré structure on a ring spectrum $R$ with a $C_2$-action via maps of ring spectra is a factorization $R \to C \to R^{tC_2}$ in CAlg of the natural map $R \to R^{tC_2}$.

**Remark 2.6.** Let $\mathcal{M}$ be the full subcategory of $\mathrm{Cat}_\infty^{\mathrm{p}}$ spanned by Poincaré $\infty$-categories with underlying $\infty$-category $\mathrm{Mod}_R^\omega$ for some ring spectrum $R$. Then the symmetric monoidal structure of $\mathrm{Cat}_\infty^{\mathrm{p}}$ restricts to a symmetric monoidal structure on $\mathcal{M}$ by Example 2.10 and . Then we have $\mathrm{CAlg}^{\mathrm{p}} \simeq \mathrm{CAlg}(\mathcal{M})$. In particular, the symmetric monoidal structure of $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{p}})$ restricts to a symmetric monoidal structure on $\mathrm{CAlg}^{\mathrm{p}}$.

V: cite 9-authors I.5.1.5 and I.5.1.6

**Notation 2.7.** Let $R$ be a ring spectrum. We will denote by $\underline{R}$ the spectrum $R$ with trivial action. More precisely, $\underline{R} : BC_2 \to \mathrm{Sp}$ is the constant functor.

**Example 2.8.** Let $R$ be a ring spectrum. If $2 \in \pi_0(R)$ is invertible, we have $\underline{R}^{tC_2} \simeq 0$. A Poincaré structure on $R$ with the trivial action is then given by an $R$-algebra $R \to C$.

V: ex-plain/referen

**Example 2.9.** Let $R$ be a ring spectrum equipped with a $C_2$-action via maps of ring spectra. The natural $R$-algebra structure on $R^{tC_2}$ induces a Poincaré structure on $R$ given by the factorization $R \xrightarrow{\mathrm{id}} R \to R^{tC_2}$. We will call this Poincaré structure the *Tate Poincaré structure on* $R$.

**Example 2.10.** The sphere spectrum $\mathbb{S}$ together with the Tate Poincaré structure will be called the *universal Poincaré ring spectrum*.

V: expain why/translat universality statement to poincare ring spectra

**Example 2.11.** Let $R$ be a ring spectrum equipped with a $C_2$-action via maps of ring spectra. The identity map $\mathrm{id} : R^{tC_2} \to R^{tC_2}$ induces a Poincaré structure on $R$ given by the factorization $R \to R^{tC_2} \xrightarrow{\mathrm{id}} R^{tC_2}$. We will call this Poincaré structure the *symmetric Poincaré structure on* $R$.

**Example 2.12.** Let $R$ be a connective ring spectrum equipped with a $C_2$-action via maps of ring spectra. The connective cover $\tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$ of $R^{tC_2}$ induces a Poincaré structure on $R$ given by the factorization $R \to \tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$. We will call this Poincaré structure the *genuine symmetric Poincaré structure on* $R$.

V: copy more examples from notes

**Definition 2.13.** Let $A$ and $R$ be Poincaré ring spectra. A *map of Poincaré ring spectra* between $A$ and $R$ is a map of ring spectra $f : A \to R$ compatible with the corresponding Poincaré structures via the following additional data:

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V: this should become a remark and go below the definition of calgp

# 3 Modules over Poincaré ring spectra

Let $A$ be a Poincaré ring spectrum. Then $A$ is an algebra object in the $\infty$-category of Poincaré $\infty$-categories $\mathrm{Cat}^{\mathrm{p}}_{\infty}$. We may thus consider the $\infty$-category of modules over it $\mathrm{Mod}_A(\mathrm{Cat}^{\mathrm{p}}_{\infty})$, which we will simply denote by $\mathrm{Mod}_A$. In this section we will use modules over Poincaré ring spectra to define analogues of the Brauer and Picard groups for Poincaré ring spectra.

V: this is the beginning of an attempt to define brauer groups for poincare ring spectra

**Notation 3.1.** We will abbreviate the $\infty$-category of modules over a Poincaré ring $A$ by $\mathrm{Mod}_A$.

**Definition 3.2.** Let $A$ be a Poincaré ring spectrum. We define the *Picard space of $A$* to be

$$\mathrm{Pic}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Pn}(A)).$$

**Definition 3.3.** Let $A$ be a Poincaré ring spectrum. We define the *Brauer space of $A$* as

$$\mathrm{Br}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Mod}_A(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})).$$

Recall that a Poincaré $\infty$-category is called idempotent complete if the underlying stable $\infty$-category is idempotent complete. The full subcategory of $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ spanned by idempotent complete Poincaré $\infty$-categories is denoted by $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$.

**Proposition 3.4.** *Let $A$ be a Poincaré ring spectrum. Then we have a canonical equivalence*

$$\Omega\,\mathrm{Br}^{\mathrm{p}}(A) \simeq \mathrm{Pic}^{\mathrm{p}}(A).$$

*Proof.* □

V: todo

# 4 Poincaré schemes

**Definition 4.1.** Let APS be the $(\infty,1)$-category defined by the pullback

$$\begin{tikzcd}
\mathrm{APS} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \arrow[d, "d_1^*"] \\
\mathrm{CAlg}(\mathcal{S}p^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p))
\end{tikzcd}$$

where $U : \mathcal{S}p^{BC_2} \to \mathcal{S}p$ is the functor which forgets the $C_2$-action.

We record here a few structural results about this category.

**Theorem 4.2.** *The following statements about* APS *hold:*

1. *The category* APS *is a cocomplete and symmetric monoidal infinite category;*
2. *the pullback diagram above is homotopy Cartesian;*
3. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{BC_2})$ *is symmetric monoidal and (co)continuous;*
4. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p)^{\Delta^2}$ *is lax symmetric monoidal;*
5. *and the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p)^{\Delta^2} \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathcal{S}p)$ *is symmetric monoidal.*

*Proof.* For (2) it is enough to show that $d_1^*$ is a cartesian fibration which follows from [3, Corollary 2.4.6.5]. There is a (pseudo-)functor

L: Is this reference correct? The conclusion asserts that some map of simplicial sets is a *categorical* fibration. The following ar-

$$\begin{aligned} F\colon \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p)) &\to \mathcal{C}\mathrm{at}_{\infty} \\ (\varphi\colon A \to B) &\mapsto \left(\left(\mathrm{CAlg}(\mathcal{S}p)_{A/-/B}\right)_{/\varphi}\right) \end{aligned}$$

which sends a square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\psi"] & D \end{tikzcd} \tag{4.1}$$

regarded as a morphism from $\varphi$ to $\psi$, to the functor

$$\begin{aligned} \big(\mathrm{CAlg}(\mathcal{S}p)_{A/-/B}\big)_{/\varphi} &\to \big(\mathrm{CAlg}(\mathcal{S}p)_{C/-/D}\big)_{/\psi} \\ (A \to R \to B) &\mapsto C \simeq A \otimes_A C \xrightarrow{\varphi \otimes \mathrm{id}_C} B \otimes_A C \to D \end{aligned} \tag{4.2}$$

where $B \otimes_A C \to D$ is the canonical map induced by the commuting square (4.1). The functor $F$ classifies the cocartesian fibration $d_1^*$.

For (3), let $p : K \to \mathrm{APS}$ be a map of simplicial sets, $K$ a small simplicial set. Suppose the $K^{\triangleright} \to \mathrm{APS}$ be an extension such that $K^{\triangleright} \to \mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{BC_2})$ is a colimit diagram. By [3, Proposition 2.4.3.2] the diagram

$$\begin{tikzcd} \mathrm{APS}_{p/} \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathcal{S}p)^{\Delta^2}_{p/-} \arrow[d] \\ \mathrm{CAlg}(\mathcal{S}p^{BC_2})_{p/} \arrow[r] & \mathrm{CAlg}(\mathcal{S}p)^{\Delta^1}_{p/-} \end{tikzcd}$$

is again homotopy cartesian. Then

$$\begin{aligned} \hom_{\mathrm{APS}}(p(\infty), -) &\simeq \hom_{\mathrm{CAlg}(\mathcal{S}p^{BC_2})}(p(\infty), -) \times_{\hom_{\mathrm{CAlg}(\mathcal{S}p)^{\Delta^1}}(p(\infty),-)} \hom_{\mathrm{CAlg}(\mathcal{S}p)^{\Delta^2}}(p(\infty)) \\ &\simeq \end{aligned}$$

□

We will denote elements of APS by $\underline{A} = (A, s : A^{\Phi C_2} \to A^{tC_2})$. Here $s : A^{\Phi C_2} \to A^{tC_2}$ is the image of $\underline{A}$ under the top horizontal map above. The use of the notation $A^{\Phi C_2}$ is justified by the following.

**Lemma 4.3.** *Let* $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p)$ *be the composition of the functors*

$$\mathrm{APS} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathcal{S}p).$$

*Then this functor factors as a composition* $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{C_2}) \xrightarrow{(-)^{\Phi C_2}} \mathrm{CAlg}(\mathcal{S}p)$.

*Proof.* The commutativity of the diagram

$$\begin{tikzcd} & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \arrow[d, "d_1^*"] \arrow[dr, "d_0^*"] & \\ & & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p)) \arrow[dd, "ev_{[1]}"] \\ \mathrm{CAlg}(\mathcal{S}p^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] \arrow[dr, "id"] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p)) \arrow[dr, "ev_{[1]}"] & \\ & \mathrm{CAlg}(\mathcal{S}p^{BC_2}) \arrow[r, "(-)^{tc_2}"] & \mathrm{CAlg}(\mathcal{S}p) \end{tikzcd}$$

induces a functor on the pullback infinity categories $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{C_2})$ which makes the corresponding cube commute. The functor $ev_{[1]} : \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \to \mathrm{CAlg}(\mathcal{S}p)$ factors through $d_0^*$ and so $\mathrm{APS} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \to \mathrm{CAlg}(\mathcal{S}p)$ is equivalent to the composition

$$\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{C_2}) \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p)) \to \mathrm{CAlg}(\mathcal{S}p)$$

and the composition of the last two maps is the geometric fixed point functor as desired. □

The following Lemma gives the justification of the name Poincaré scheme.

**Construction 4.4.** We shall construct a functor

$$\mathrm{Perf}^{\mathrm{Pn}} : \mathrm{APS} \to \mathrm{Cat}_{\infty}^{\mathrm{Pn}}$$

to the category of Poincaré infinity categories.

Recall that $\mathcal{C}\mathrm{at}_{\infty}^{h} \to (\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}})^{\mathrm{op}}$ is a cocartesian fibration [1, §1.4.] We will first construct a map of cocartesian fibrations

L: For symmetric monoidal structure–maybe want to swap out $\mathrm{Mod}_{NA}$ for $\mathrm{CAlg}_{NA}$?

$$\begin{tikzcd}
\mathrm{APS} \arrow[r, dotted] \arrow[d] & \mathcal{C}\mathrm{at}_{\infty}^{h} \arrow[d] \\
\mathrm{CAlg}\left(\mathcal{S}p^{BC_2}\right) \arrow[r] & (\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}})^{\mathrm{op}}
\end{tikzcd}, \tag{4.3}$$

then show that the dotted arrow factors through the subcategory $\mathcal{C}\mathrm{at}_{\infty}^{p} \subseteq \mathcal{C}\mathrm{at}_{\infty}^{h}$. To construct a map of cartesian fibrations, it suffices to exhibit a natural transformation of classifying functors. Unraveling the definitions, by Theorem 3.2.13 of [1] we must exhibit for each $A \in \mathrm{CAlg}(\mathcal{S}p)^{BC_2}$, a functor

$$\left(\mathrm{CAlg}(\mathcal{S}p)_{A/-/A^{tC_2}}\right)_{/\varphi} \to \mathrm{Mod}_{N^{C_2}(A^e)}\left(\mathcal{S}p^{C_2}\right) \tag{4.4}$$

(where $\varphi\colon A \to A^{tC_2}$ is the Tate-valued Frobenius and $N^{C_2}$ is the Hill–Hopkins–Ravenel norm) which is natural in $A$.

That the resulting functor factors through the subcategory $\mathcal{C}\mathrm{at}_{\infty}^{p}$ follows from Proposition 3.1.3 and Lemma 3.3.3 of *loc. cit.*

**Lemma 4.5.** *The functor of Construction 4.4 is symmetric monoidal and has essential image the subcategory spanned by objects* $(\mathrm{Perf}(R), \text{♀})$ *which are* $\mathbb{E}_{\infty}$*-algebras.*

**Definition 4.6.** A map $f : \underline{A} \to \underline{B} \in \mathrm{APS}$ is faithfully flat if the underlying map $f : A \to B$ is faithfully flat and the map $f^{\Phi C_2} : A^{\Phi C_2} \to B^{\Phi C_2}$ is also faithfully flat.

**Lemma 4.7.** *The fpqc covers on* APS *form a Grothendieck site.*

# References


[1] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories I: Foundations.* 2020. DOI: 10.48550/ARXIV.2009.07223. URL: https://arxiv.org/abs/2009.07223.

[2] Uriya A. First and Ben Williams. "Involutions of Azumaya Algebras". en. In: *DOCUMENTA MATHEMATICA* Vol 25 (2020 2020), p. 527–633. DOI: 10.25537/DM.2020V25.527-633. URL: https://www.elibm.org/article/10012038.

[3] Jacob Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.