Poincaré Schemes

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Abstract

We do stuff

N: Change this

Contents

1 Introduction

Theorem 1.1. *Let $\underline{A}$ be an affine Poincaré scheme with underlying $\mathbb{E}_\infty$-ring spectrum with involution A. Then the natural maps*

$$\pi_i(\mathrm{PnPic}(\underline{A})) \to \pi_i(\mathrm{Pic}(A))$$

are surjective on 2-torsion.

N: I think there is some interaction with the homotopy fixed points, or maybe even the genuine fixed points

Theorem 1.2. *Let A be an $\mathbb{E}_\infty$ ring with involution, and let $\underline{NA}$ be the associated Tate affine Poincaré scheme. Let $\mathrm{Br}_\nu(A)$ be the Brauer group of Azumaya algebras over A with involution. Then the natural map*

$$\mathrm{PnBr}(\underline{NA}) \to \mathrm{Br}_\nu(A)$$

is an equivalence

N: I think we need to define this for ring spectra. For A discrete this is done in [FW20].

Theorem 1.3. *The functors $\mathrm{PnPic}, \mathrm{PnBr} : \mathrm{APS} \to \mathrm{Sp}$ are fppf sheaves.*

Theorem 1.4. *There is a Poincaré group scheme $\mathbb{G}_m^{♀}$ such that*

$$B\mathbb{G}_m^{♀} \simeq \mathrm{PnPic}$$

as fppf stacks.

N: probably of $\mathbb{E}_\infty$ do

1.1 Acknowledgements

The authors wish to thank the Institute for Advanced Study and the organizers of the 2024 Park City Mathematics institute on motivic homotopy theory.

1.2 Conventions

Br^{p}	Poincaré Brauer space
CAlg	∞-categoriy of $\mathbf{E}_\infty$-ring spectra
$\mathrm{CAlg}(\mathcal{S})$	∞-categoriy of $\mathbf{E}_\infty$-spaces
$\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$	∞-categoriy of grouplike $\mathbf{E}_\infty$-spaces
$\mathrm{CAlg}^{\mathrm{p}}$	∞-categoriy of Poincaré ring spectra
$\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$	∞-category of small stable ∞-categories and exact functors
$\mathrm{Cat}_\infty^{\mathrm{p}}$	∞-category of Poincaré ∞-categories
$\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}$	∞-category of idempotent complete Poincaré ∞-categories
$\mathrm{Pic}^{\mathrm{p}}$	Poincaré Picard space
$\mathcal{S}$	∞-category of spaces
Sp	∞-category of spectra

2 Poincaré Structures on Compact Modules

We will use this section to recall notions and results about Poincaré ∞-categories which we require in the sections to follow. This section can safely be skipped by anyone who posses extensive knowledge of Poincaré ∞-categories, as found in [Cal+20a].

Notation 2.1. Let R be an $\mathbf{E}_\infty$-ring spectrum. We will drop $\mathbf{E}_\infty$ from our notation and simply call R a *ring spectrum*. Moreover, we will denote the ∞-category CAlg(Sp) of commutative algebra objects in the ∞-category of spectra Sp by CAlg.

Let R be a ring spectrum and let Mod_R be the ∞-category of modules over R. We will study Poincaré structures on the ∞-category Mod_R^ω of compact modules over R.

V: -
characterizati
in terms
of modules
with genuine
involution, -
characterizati
of symmetric
monoidal
structures,
-Pn

3 Poincaré Ring Spectra

In this section we will define the ring theoretic building blocks of Poincaré schemes and the corresponding category they live in. Affine Poincaré Schemes will then be the dual objects, similar to how affine schemes are dual to commutative rings.

Definition 3.1. Let R be a ring spectrum. A *Poincaré structure* on R is a symmetric monoidal Poincaré ∞-category $\Qoppa : (\mathrm{Mod}_R^\omega)^{\mathrm{op}} \to \mathrm{Sp}$. We call such a symmetric monoidal Poincaré ∞-category a *Poincaré ring spectrum*. We will denote the full subcategory of $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{p}})$ spanned by Poincaré ring spectra by $\mathrm{CAlg}^{\mathrm{p}}$ and call it the *∞-category of Poincaré ring spectra.*

Remark 3.2. Poincaré ring spectra, as defined in Definition 3.1, were studied in . Note that we chose a different notation. In [Cal+20a, discussion immediately preceding Examples 5.4.10], Poincaré ring spectra are referred to as $\mathbf{E}_\infty$-*ring spectra with genuine involution.*

V: cite 9
authored
paper

Remark 3.3. Let R be a ring spectrum. By there is a natural equivalence between symmetric monoidal Poincaré structures on Mod_R^ω and certain algebra objects over the genuine C_2-spectrum NR [Cal+20a, Corollary 5.4.8]. In particular, a Poincaré structure on R can be identified with the following data:

- A C_2-action on R via maps of ring spectra, i.e. a functor $\lambda : BC_2 \to \mathrm{CAlg}$.
- An R-algebra $R \to C$.
- An R-algebra map $C \to R^{tC_2}$.

Here R^{tC_2} is the Tate construction with respect to the above action. Since the Tate construction is lax symmetric monoidal, R^{tC_2} is naturally an R-algebra via the Tate-valued norm. A ring spectrum equipped with a Poincaré structure will be called a *Poincaré ring spectrum*.

Remark 3.4. By Remark 3.3, a Poincaré structure on a ring spectrum R with a C_2-action via maps of ring spectra is a factorization $R \to C \to R^{tC_2}$ in CAlg of the Tate Frobenius $R \to R^{tC_2}$.

Remark 3.5. Let $\mathcal{M}$ be the full subcategory of $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ spanned by Poincaré ∞-categories with underlying ∞-category $\mathrm{Mod}^{\omega}_{R}$ for some ring spectrum R. Then the symmetric monoidal structure of $\mathrm{Cat}^{\mathrm{p}}_{\infty\infty}$ restricts to a symmetric monoidal structure on $\mathcal{M}$ by Example 3.9 and . Then we have $\mathrm{CAlg}^{\mathrm{p}} \simeq \mathrm{CAlg}(\mathcal{M})$. In particular, the symmetric monoidal structure of $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{p}}_{\infty})$ restricts to a symmetric monoidal structure on $\mathrm{CAlg}^{\mathrm{p}}$.

V: cite 9-authors I.5.1.5 and I.5.1.6

Notation 3.6. Let R be a ring spectrum. We will denote by $\underline{R}$ the spectrum R with trivial action. More precisely, $\underline{R}: BC_2 \to \mathrm{Sp}$ is the constant functor.

L: This is commonly used for constant Mackey functors–could be ambiguous

Example 3.7. Let R be a ring spectrum. If $2 \in \pi_0(R)$ is invertible, we have $\underline{R}^{tC_2} \simeq 0$. A Poincaré structure on R with the trivial action is then given by an R-algebra $R \to C$.

V: explain/referen

Example 3.8. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The Tate-valued norm endows R^{tC_2} with a natural R-algebra structure, which induces a Poincaré structure on R given by the factorization $R \xrightarrow{\mathrm{id}} R \to R^{tC_2}$. We will call this Poincaré structure the *Tate Poincaré structure on* R and will denote it by $(R, \Qoppa^{\mathrm{t}}_{R})$.

Example 3.9. The sphere spectrum $\mathbb{S}$ together with the Tate Poincaré structure will be called the *universal Poincaré ring spectrum*. We will denote it by $(\mathbb{S}, \Qoppa_u)$.

V: expain why/translate universality statement to poincare ring spectra

Remark 3.10. Let $(R, \Qoppa)$ be a ring spectrum associated to a factorization $R \to C \to R^{tC_2}$. A factorization of the map $C \to R^{tC_2}$ through R^{hC_2} induces a section of the canonical map $\Qoppa(R) \to \hom_R(R, C) \simeq C$. In that case, we have a splitting $\Qoppa(R) \simeq R_{hC_2} \oplus C$.

V: reference pullback that characterizes all quadratic functors

Example 3.11. The Tate Frobenius for the sphere spectrum factors through $\mathbb{S}^{hC_2}$. Therefore, Remark 3.10 implies $\Qoppa_u(\mathbb{S}) \simeq \mathbb{S}_{hC_2} \oplus \mathbb{S} \simeq \Sigma^{\infty}(\mathbb{P}^{\infty}_{\mathbb{R}}) \oplus \mathbb{S}$.

Example 3.12. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The identity map $\mathrm{id}: R^{tC_2} \to R^{tC_2}$ induces a Poincaré structure on R given by the factorization $R \to R^{tC_2} \xrightarrow{id} R^{tC_2}$. We will call this Poincaré structure the *symmetric Poincaré structure on* R.

Example 3.13. Let R be a connective ring spectrum equipped with a C_2-action via maps of ring spectra. The connective cover $\tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$ of R^{tC_2} induces a Poincaré structure on R given by the factorization $R \to \tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$. We will call this Poincaré structure the *genuine symmetric Poincaré structure on* R.

V: copy more examples from notes

Definition 3.14. Let A and R be Poincaré ring spectra. A *map of Poincaré ring spectra* between A and R is a map of ring spectra $f : A \to R$ compatible with the corresponding Poincaré structures via the following additional data:

-

V: this should become a remark and go below the definition of calgp

3.1 Algebras with genuine involution

Recollection 3.15. Assume $\mathcal{C}$ is a presentable monoidal ∞-category such that the monoidal product $-\otimes -: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then there is an ∞-category $\mathrm{LMod}(\mathcal{C})$ [Lur17, Example 4.2.1.18] whose objects are pairs (A, M) where A is an associative algebra object of $\mathcal{C}$ and M is a left A-module. Write a, m respectively for the canonical forgetful functors $\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$, $\mathrm{LMod}(\mathcal{C}) \to \mathcal{C}$ which send (A, M) to A and M, resp. Then a is a cocartesian fibration [Lur17, Corollary 4.2.3.7], hence it is classified by a functor $\mathrm{mod}: \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_{\infty}$.

The functor s of [Lur17, Example 4.2.1.17] determines a natural transformation $\eta\colon * \to \mathrm{mod}$, where $*\colon \mathrm{Alg}(\mathcal{C}) \to \{*\} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ is the constant functor at the trivial category, or equivalently

$$\begin{tikzcd}
 & \mathcal{U} \arrow[d] \\
\mathrm{Alg}(\mathcal{C}) \arrow[ur, "\eta"] \arrow[r, "\mathrm{mod}"'] & \mathcal{C}\mathrm{at}_\infty
\end{tikzcd} \tag{3.16}$$

where $\mathcal{U}$ is the universal cocartesian fibration. Now consider the functor $o\colon \mathcal{U} \to \mathcal{C}\mathrm{at}_\infty$ which sends $(\mathcal{D}, d \in \mathcal{D})$ to the undercategory $\mathcal{D}_{d/-}$. Define $\mathrm{LMod}(\mathcal{C})_{*/-}$ to be the cocartesian fibration over $\mathrm{Alg}(\mathcal{C})$ classified by $o \circ \eta \circ \mathrm{mod}$.

Variant 3.17. Let C be as in Recollection 3.15. There is a similar construction where left modules is replaced by *bimodules* [Lur17, Definition 4.3.1.12].

Construction 3.18. Regard $\mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})$ as a category with C_2-action given by taking the opposite/reverse algebra. There are functors $b\colon \mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2} \to \mathrm{LMod}(\mathrm{Sp})$ and $b\colon \mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2} \to \mathrm{BiMod}(\mathrm{Sp})_{*/-}$ so that $(a, m) \circ b$ and $(a, m) \circ b_*$ are (canonically) equivalent to $(-^e) \otimes (-^e)^{\mathrm{op}}, (-)^e$. Informally, an $\mathbb{E}_1$-algebra with involution B can be regarded as a $B \otimes B^{\mathrm{op}}$-module in a canonical way, and there is a canonical $B \otimes B^{\mathrm{op}}$-module map $B \otimes B^{\mathrm{op}} \to B$.

Definition 3.19. The category of $\mathbb{E}_1$-*algebras with genuine involution* is defined to be the limit of the $\mathcal{C}\mathrm{at}_\infty$-valued diagram

$$\begin{tikzcd}
 & \mathrm{LMod}(\mathrm{Sp}) \arrow[dd, "{a,m}"] & \\
 & & \mathrm{LMod}\left(\mathrm{Sp}^{C_2}\right) \arrow[ul, "(-)^e"'] \arrow[dd, "{a,m}"] \\
 & \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp}) \times \mathrm{Sp} & \\
\mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \arrow[uuur, bend left, "b\circ(-^e)"] \arrow[ur, "{(-^e)\otimes(-^e)^{\mathrm{op}},(-)^e}"] \arrow[rr, "N^{C_2}(-^e)\times U"'] & & \mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \times \mathrm{Sp}^{C_2} \arrow[ul, "(-)^e\times(-)^e"]
\end{tikzcd} \tag{3.20}$$

where

- b is the functor/section of Construction 3.18
- U is the 'underlying' C_2-spectrum functor $\mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{BC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \to \mathrm{Sp}^{BC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \simeq \mathrm{Sp}^{C_2}$
- The upper right trapezoid commutes canonically by definition of LMod (and the fact that the functors a, m are given by restriction to subcategories of $LM^{\otimes}$).

Write $\mathbb{E}_1\mathrm{Alg}^{\mathrm{gi}}\left(\mathrm{Sp}^{C_2}\right)$ for the ∞-category of $\mathbb{E}_1$-algebras with genuine involution.

Definition 3.21. The category of $\mathbb{E}_\sigma$-*algebras* is defined to be the limit of the $\mathcal{C}\mathrm{at}_\infty$-valued diagram

$$\begin{tikzcd}
 & \mathrm{BiMod}(\mathrm{Sp})_{*/-} \arrow[dd, "{a,m}"] & \\
 & & \mathrm{BiMod}\left(\mathrm{Sp}^{C_2}\right)_{*/-} \arrow[ul, "(-)^e"'] \arrow[dd, "{a,m}"] \\
 & \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp}) \times \mathrm{Sp} & \\
\mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \arrow[uuur, bend left, "b_*\circ(-^e)"] \arrow[ur, "{(-^e)\otimes(-^e)^{\mathrm{op}},(-)^e}"] \arrow[rr, "N^{C_2}(-^e)\times U"'] & & \mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \times \mathrm{Sp}^{C_2} \arrow[ul, "(-)^e\times(-)^e"]
\end{tikzcd} \quad . \tag{3.22}$$

Write $\mathbb{E}_\sigma\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right)$ for the ∞-category of $\mathbb{E}_\sigma$-algebras.

Variant 3.23. Let the base be R an $\mathbb{E}_\infty$-algebra or Poincaré ring instead of $\mathbb{S}^0$.

Remarks 3.24.
1. Compare [AGH21, Corollary 3.10].
2. There are canonical forgetful functors $\mathbb{E}_\sigma\mathrm{Alg} \to \mathbb{E}_1\mathrm{Alg}^{\mathrm{gi}} \to \mathbb{E}_1\mathrm{Alg}^{hC_2} \to \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})$.

Construction 3.25. Let $R, R^{\varphi C_2} \to R^{tC_2}$ be a Poincaré ring. There is a functor $\left(\mathrm{Mod}^\omega_{(-)}, ϙ_{(-)}\right) : \mathbb{E}_\sigma\mathrm{Alg}_R \to \left(\mathcal{C}\mathrm{at}^h_R\right)_{\left(\mathrm{Mod}^\omega_R, ϙ_R\right)/-}$.

Lemma 3.26. *Let $R, R^{\varphi C_2} \to R^{tC_2}$ be a Poincaré ring. The functor of Construction 3.25 is fully faithful.*

Proof. □

Now we observe that given a hermitian object (x, q) of $(\mathcal{C}, ϙ_\mathcal{C})$, its endomorphism algebra admits a canonical lift to a $\mathbb{E}_\sigma$-algebra.

Construction 3.27. There is a functor $\mathrm{End}(-)\colon \left(\mathcal{C}\mathrm{at}^h_R\right)_{\left(\mathrm{Mod}^\omega_R, ϙ_R\right)} \to \mathbb{E}_\sigma\mathrm{Alg}$ lifting the functor $\left(\mathcal{C}\mathrm{at}^h_R\right)_{\left(\mathrm{Mod}^\omega_R, ϙ_R\right)} \to \mathbb{E}_1\mathrm{Alg}^{hC_2}$ of [Cal+20a, Proposition 3.1.16].

Theorem 3.28. *The functors of Construction 3.25 and 3.27 form an adjoint pair.*

4 The Poincaré Picard space

Let A be a Poincaré ring spectrum. Then A is a commutative algebra object in the ∞-category of Poincaré ∞-categories $\mathrm{Cat}^{\mathrm{p}}_\infty$. We may thus consider modules over it. In this section we will use modules over Poincaré ring spectra to define analogues of the Picard group for Poincaré ring spectra. V: ref

Recall that the Poincaré space functor $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{CAlg}(\mathcal{S})$ is lax symmetric monoidal with respect to tensor product of Poincaré ∞-categories and smash product of $\mathbf{E}_\infty$-spaces [Cal+20a, Corollary 5.2.8]. In particular, we can consider invertible objects in $\mathrm{Pn}(A)$ for a Poincaré ring spectrum A.

Definition 4.1. Let A be a Poincaré ring spectrum. We define the *Picard space of A* to be

$$\mathrm{Pic}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Pn}(A)).$$

Remark 4.2. Let $(\mathrm{Mod}^\omega_R, ϙ_R)$ be a Poincaré ring spectrum, where $(M_R = R, N_R = R^{\varphi C_2}, R^{\varphi C_2} \to R^{tC_2})$ is the module with genuine involution associated to $ϙ_R$. Then a point in the Poincaré Picard space is the data of a pair $(\mathcal{L}, q)$, where $\mathcal{L}$ is an invertible module in Mod^ω_R and q is a point in $\Omega^\infty ϙ_R(\mathcal{L})$. By [Cal+20a, Proposition 1.3.11], the data of q is equivalent to the data of points in the lower left and upper right corner of the square

$$\begin{tikzcd} ϙ(\mathcal{L}) \arrow[r] \arrow[d] & \hom_R(\mathcal{L}, R^{\varphi C_2}) \ni \ell(q) \arrow[d] \\ b(q) \in \hom_{R\otimes R}(\mathcal{L}\otimes\mathcal{L}, R)^{hC_2} \arrow[r] & \hom_R(\mathcal{L}, R^{tC_2}) \end{tikzcd} \tag{4.3}$$

and a path between their images in the lower right corner. In particular, the adjoint of $b(q)$ must define a nondegenerate hermitian form on $\mathcal{L}$, that is, an equivalence $\mathcal{L} \simeq \hom_R(\mathcal{L}, R^*)$ where R^* is considered as an R-module via the action of the generator of C_2. L: add equivariance/symmetry data

Write $(\mathcal{L}^\vee, q^\vee)$ is for the inverse of $(\mathcal{L}, q)$. By definition of invertibility, there exists an R-linear map $\ell(q^\vee)\colon \mathcal{L}^\vee \to R^{\varphi C_2}$ so that the following diagram commutes

$$\begin{tikzcd} \mathcal{L}\otimes_R \mathcal{L}^\vee \arrow[r, "\ell(q)\otimes\ell(q^\vee)"] \arrow[d, "\sim"', "\mathrm{ev}"] & R^{\varphi C_2}\otimes_R R^{\varphi C_2} \arrow[d, "\text{multiplication}"] \\ R \arrow[r, "\text{given}"] & N_R \end{tikzcd} \tag{4.4}$$

Lemma 4.5. *Let $(R, \Qoppa)$ be a connective Poincaré ring spectrum. Then, for any integer n, the spectrum $\Qoppa(\Sigma^n R)$ is $(-2n)$-connective.*

V: define connectivity and make conditions here precise. As stated … connective.

Proof. This follows from the fiber sequence

$$(\Sigma^{-2n} R)_{hC_2} \to \Qoppa(\Sigma^n R) \to \hom_R(\Sigma^n R, C) \simeq \Sigma^{-n} C.$$

$\square$

V: write out details

Remark 4.6. The functor $\mathrm{Pic}^{\mathrm{p}} : \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$ preserves certain structures. Let A be a Poincaré ring spectrum. Since A is a module over $(\mathbb{S}, \Qoppa_u)$, the space $\mathrm{Pic}^{\mathrm{p}}(A)$ is something over $\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Qoppa_u)$.

V: there is no truth in here yet. Work in progress. Noah had an example using Witt vectors which showed that π_0 does not need to be 2-torsion

Since the forgetful functor $\mathrm{Pn}(A) \to \mathrm{Mod}_A^{\omega}$ is symmetric monoidal we get an induced map

$$U : \mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)$$

of spectra. For a point $(\mathcal{L}, q) \in \pi_0(\mathrm{Pic}^{\mathrm{p}}(A))$ we will refer to $\mathcal{L} := U(\mathcal{L}, q)$ as the *underlying invertible module*. Note that the A-module A^* is (nonequivariantly) isomorphic to A via the involution, and so the fact that $\mathcal{L} \simeq \hom_A(\mathcal{L}, A^*)$ forces $\mathcal{L}$ to be 2-torsion. In particular we get a refined map

$$\mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)[2]$$

which factors the underlying invertible module map.

Example 4.7. Let $(\mathbb{S}, \Qoppa_u)$ be the universal Poincaré ring spectrum from Example 3.9. The only 2-torsion element of $\mathrm{Pic}(\mathbb{S}) \simeq \mathbf{Z}$ is $\mathbb{S}$. Therefore, any element in $\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Qoppa_u)$ lies above $\mathbb{S}$. With Remark 3.11, we conclude $\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Qoppa_u)) \simeq \pi_0(\mathbb{S}_{hC_2} \oplus \mathbb{S}^{\times})^{\times} \simeq (\mathbf{Z} \times \mathbf{Z/2})^{\times} \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$.

Remark 4.8. One might hope that the map $\mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)[2]$ is close to an equivalence. This however is quite fare from being true. Let k be a finite field of characteristic 2, and let $\mathbb{S}_{W(k)}$ be the spherical Witt vectors on k in the sense of [Lur18, Example 5.2.7]. Then by [Nik23, Example 3.4] we know that $\mathbb{S}_{W(k)}$ must satisfy that the map $\varphi_2 : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}^{tC_2}$ is an equivalence where the action is trivial.

Consider now the Poincaré ring $(\mathrm{Mod}^{\omega}_{\mathbb{S}_{W(k)}}, \Qoppa^u_{\mathbb{S}_{W(k)}})$ where $\Qoppa^u_{\mathbb{S}_{W(k)}}$ is the Tate Poincaré structure. We have that $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) \cong \mathbb{Z}$ and is generated by $\Sigma\mathbb{S}_{W(k)}$. To see this note that for $\mathcal{L}$ an invertible module over $\mathbb{S}_{W(k)}$, $\mathcal{L}$ must be bounded below since otherwise it would not be perfect. Then for $\pi_n(\mathcal{L})$ its bottom homotopy group, we have that $\pi_n(\mathcal{L}/2) \cong k$ since it must be an invertible k-module and k is a field. Thus we get a map $\mathbb{S}^n \to \mathcal{L}$ lifting a generator of k, and by adjunction an $\mathbb{S}_{W(k)}$-module map $\Sigma^n \mathbb{S}_{W(k)} \to \mathcal{L}$ which on $\pi_n((-)/2)$ gives an isomorphism $k \cong k$. Therefore

$$\mathbb{S}_{W(k)}[n] \otimes k \simeq k[n] \to k[n] \simeq \mathcal{L} \otimes k$$

is an equivalence, where the equivalence $k[n] \simeq \mathcal{L} \otimes k$ follows from the fact that base change preserves invertible objects. The map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is then a k-local, and therefore an $\mathbb{F}_p$-local, equivalence. Both sides are connective and p-complete so it follows that the map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is an equivalence.

N: There is probably a reference for this fact, I'll look around for one.

Thus $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) = 0$. On the other hand, we have that the unit map $\mathbb{S}_{W(k)} \to \Qoppa^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})$ is split by the map $\Qoppa^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)}) \to \mathbb{S}_{W(k)}^{\varphi C_2} = \mathbb{S}_{W(k)}$. Consequently $\pi_0(\Qoppa^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})) \cong \pi_0(\mathbb{S}_{W(k)} \oplus (\mathbb{S}_{W(k)})_{hC_2}) \cong W(k) \times W(k)$. As a ring this is $W_2(W(k))$, and in order for $q \in W_2(W(k))$ to induce a Poincaré structure we must have that $q \in W_2(W(k))^{\times} \cong W(k)^{\times} \times W(k)^{\times}$.

We then have that $\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong W(k)^{\times} \times W(k)^{\times}/H$ where H is the subgroup of Poincaré structures q on $\mathbb{S}_{W(k)}$ which are identified by some automorphism $f : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}$. By the defining property of spherical Witt vectors there is an equivalence $\mathrm{Maps}_{\mathrm{CAlg}}(\mathbb{S}_{W(k)}, \mathbb{S}_{W(k)}) \simeq \mathrm{Maps}_{Perf}(k, k) = \mathrm{Gal}(k/\mathbb{F}_2)$ and the action on $W(k)^{\times} \times W(k)^{\times}$ is given by $g \in \mathrm{Gal}(k/\mathbb{F}_2)$ acts via $W(g) \times W(g)$. Consequently

$$\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong (W(k)^{\times} \times W(k)^{\times})/\mathrm{Gal}(k/\mathbb{F}_2)$$

which even for $k = \mathbb{F}_2$ is not zero and in fact not even 2^{∞}-torsion.

In the usual Picard spectrum one has the relationship $\mathrm{Pic} = B\mathbb{G}_m$, where $\mathbb{G}_m$ is the spectral algebraic group scheme sending a ring spectrum E to the spectrum of E-linear equivalences of E $\mathrm{gl}_1 E := \mathrm{Aut}_E(E)$.[1] Equivalently $\mathbb{G}_m$ is the affine groupscheme given by $\mathbb{G}_m = \mathrm{Sp\acute{e}t}(\mathbb{S}\{x^{\pm 1}\})$, where $\mathbb{S}\{x^{\pm 1}\}$ is the free $\mathbb{E}_\infty$ ring on the $\mathbb{E}_\infty$ space $\mathbb{Z}$. This relationship between Pic and $\mathbb{G}_m$ has many important applications, for example relating the higher homtopy groups of $\mathrm{Pic}(A)$ with those of A. We will spend the rest of this section on establishing such an equivalence in the Poincare setting.

Construction 4.9. The underlying $\mathbb{E}_\infty$ ring of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ will again be $\mathbb{S}\{x^{\pm 1}\}$, but in order to promote this ring to a Poincare ring it will be helpful to write it as

$$\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z^{\pm 1}\}} \mathbb{S}$$

where the map $\mathbb{S}\{z^{\pm 1}\} \to \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by $z \mapsto xy$, This ring naturally lifts to a Borel C_2-ring given by C_2 swaps x and y and does nothing to z. Now take $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ to be the Poincaré ring with underlying Borel C_2 structure as described above and geometric fixed points $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} = \mathbb{S}$ and the map $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ given by the unit map. Endowing $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2}$ with the $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$-module structre given by $x, y \mapsto 1$, it remains to show that the unit map $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ factors the Tate valued Frobenius $\mathbb{G}_{\mathrm{m}}^{\Qoppa} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ in order to promote $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ to a Poincaré ring.

By construction of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ this amounts to showing that on π_0 the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$. This map sends both x and y to $xy \in \pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$. These are equal to 1 in $\pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ is a modules over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of xy equal to 1.

Theorem 4.10. *There is a natural equivalence of*

$$\Omega\, \mathrm{Pic}^{\mathrm{p}}(-) \simeq \mathbb{G}_{\mathrm{m}}^{\Qoppa}$$

of functors on Poincaré rings.

Proof. This amounts to identifying the space $\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod_A})}(A, u)$ functorially, where (A, u) is the Poincaré object A with bilinear form given by the unit map $\mathbb{S} \to \Qoppa_A(A)$. Note that any automorphism of Hermetian objects will automatically be Poincaré and so we may instead describe the automorphisms as a Hermetian object. We then have that $\mathrm{He}(\mathrm{Mod}_A) \to \mathrm{Mod}_A$ is a cocartesian fibration by definition, and classified by the functor which takes a module M to the groupoid $\Omega^\infty \Qoppa_A(M)$. Thus we get that $\mathrm{Aut}_{\mathrm{He}(\mathrm{Mod}_A)}((A, u))$ is exactly the fiber of the map

$$\mathrm{Aut}_{\mathrm{Mod}_A}(A) \to \Qoppa_A(A)$$

or in other words an automorphism $(A, u) \to (A, u)$ is the data of an automorphism $a \in \mathrm{Aut}(A)$ together with a path $q : u \mapsto a^* u$ in $\Omega^{\infty+1} \Qoppa_A(A)$.

There is a natural transformation $\mathbb{G}_{\mathrm{m}}^{\Qoppa}(-) \to \Omega\, \mathrm{Pic}^{\mathrm{p}}(-)$ given as follows: we get a map $\mathbb{G}_{\mathrm{m}}^{\Qoppa}((\mathrm{Mod}_A, \Qoppa_A)) \to \mathrm{Aut}_A(A)$ given by forgetting the Poincaré structure everywhere, and so it is enough to see that on π_0 the automorphisms of A coming from $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ preserve u. By using the linear and quadratic decomposition of $\Qoppa_A$, for an element $a \in \pi_0(A)^\times$ send u to u is must be sent to $1 \in \pi_0(A^{\varphi C_2})$ and must act by 1 on A^{hC_2}. By the following Lemma this second condition is equivalent to $a\sigma(a) \in \pi_0(A)^\times$ being equal to 1, but then these two conditions are exactly describing a map out of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ as desired.

Consequently we have a comparison map $\mathbb{G}_{\mathrm{m}}^{\Qoppa}(\mathrm{Mod}_A, \Qoppa_A) \to \Omega\, \mathrm{Pic}^{\mathrm{p}}(\mathrm{Mod}_A, \Qoppa_A)$, and the above argument in fact shows that this map is an equivalence on π_0. To finish the argument, note that the pushout description of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ induces a pullback of mapping spaces

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}}^{\Qoppa}(\mathrm{Mod}_A, \Qoppa_A) \arrow[r] \arrow[d] & \mathrm{Maps}_{\mathrm{CAlg}(\mathrm{Sp}^{C_2})}(\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}, A) \simeq \mathrm{gl}_1(A) \arrow[d] \\
* \arrow[r] & \mathrm{Maps}_{\mathrm{CAlg}(\mathrm{Sp}^{C_2})}(\mathbb{S}\{z\}, A) \simeq \Omega^\infty \Qoppa_A(A)
\end{tikzcd}$$

which finishes the proof. $\square$

[1]Normally the automorphism space of an object is only $\mathbb{A}_\infty$, but as the unit in a symmetric monoidal category, the automorphisms of Einherit a canonical and in fact functorial $\mathbb{E}_\infty$ structure and this construction makes sense.

Lemma 4.11. *Let $A \in \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ and $s \in \pi_0(A)^\times$. Then $a\sigma(a) = 1$ in $\pi_0(A)$ if and only if $(a \otimes a)^*$ acts by 1 on $\pi_0(A^{hC_2}) = \pi_0(\mathrm{Hom}_{A\otimes A}(A \otimes A, A)^{hC_2})$.*

Proof. The only if direction follows from the fact that the evaluation map $\mathrm{Hom}_{A\otimes A}(A \otimes A, A) \to A$ is an $A \otimes A$-module map. Now suppose that $a\sigma(a) = 1$ in A. Then before taking homotopy fixed points the induced map $a^* = id$ because A is $\mathbb{E}_\infty$.[2] □

4.1 Hermitian line bundles

Definition 4.12. Let R be a commutative discrete ring with a C_2-action $\sigma\colon R \to R$. Write $\sigma_* R$ for the R-module with underlying abelian group R and action $r \cdot m = \sigma(r) \cdot m$. Let M be an R-module. Define the *adjoint* of M to be the R-module $M^\dagger := \hom_R(M, \sigma_* R)$. Also recall that there is a canonical R-linear isomorphism $\left(M^\dagger\right)^\dagger \simeq M$. Note that given two R-modules M, N, the adjoint satisfies $M^\dagger \otimes N^\dagger \simeq (M \otimes N)^\dagger$. A *$\sigma$-hermitian form on I* is an R-linear isomorphism $\varphi\colon I \xrightarrow{\sim} I^\dagger$.

L: see 3.8-3.11 in this paper

Observation 4.13. Let R be a commutative discrete ring with a C_2-action $\sigma\colon R \to R$. Given two R-modules M, N equipped with σ-hermitian forms φ, ψ, respectively, $\varphi \otimes \psi$ defines a σ-hermitian form on $M \otimes_R N$. Using the canonical isomorphism mentioned above, if φ is a σ-hermitian form on M, then $\varphi^\dagger$ induces a σ-hermitian form on $M^\dagger$. Finally, observe that R has a canonical σ-hermitian form which is the adjoint of the map $R \otimes R \to R$, $r \otimes s \mapsto r\sigma(s)$.

Definition 4.14. Let R be a commutative discrete ring with a C_2-action $\sigma\colon R \to R$. Define the *hermitian Picard group of R* to have underlying set consisting of pairs (I, φ) where I is an invertible R-module and φ is a σ-hermitian form on I.

L: workshop the name later

By Observation 4.13, this set inherits a group structure. We write $\mathrm{hPic}(R)$ for the group of σ-hermitian line bundles on $\mathrm{Spec}\, R$.

Theorem 4.15. *Let R be a discrete commutative ring with a C_2-action via ring maps. Then there is a short exact sequence of abelian groups*

$$0 \to \mathrm{hPic}(R) \to \pi_0 \mathrm{PnPic}(R) \to C_{C_2}(\mathrm{Spec}\, R, \mathbb{Z}^-) \to 0$$

where R is endowed with the genuine symmetric Poincaré structure and $\mathbb{Z}^-$ is endowed with the C_2-action given by multiplication by -1 and C_{C_2} denotes continuous functions which are moreover C_2-equivariant.

Proof. An object of $\mathrm{PnPic}(R)$ is a pair (I, q) where I is an invertible R-module and q is a point in $\Omega^\infty Ϙ_{R^{gs}}(I)$. By the proof of [Fau03, Theorem 3.5], I induces a continuous map $\varphi\colon\ \mathrm{Spec}\, R \to \mathbb{Z}$. Write σ for the involution on R. Now q in particular induces an equivalence $q\colon I \xrightarrow{\sim} I^\dagger \simeq (\sigma_* I)^\vee$. For each point $\mathfrak{p} \in \mathrm{Spec}\, R$, localizing q gives an equivalence

$$q_\mathfrak{p}\colon I_\mathfrak{p} \xrightarrow{\sim} (\sigma_* I)^\vee_\mathfrak{p} \simeq \left(\sigma_*(I_{\sigma(\mathfrak{p})})\right)^\vee .$$

Since $I_\mathfrak{p}$ is an invertible module over a local ring, [Fau03] implies that $q_\mathfrak{p}$ induces an equivalence

$$I_\mathfrak{p} \simeq R_\mathfrak{p}[\varphi(\mathfrak{p})] \xrightarrow{\sim} \left(\sigma_*(R_{\sigma(\mathfrak{p})}[\varphi(\sigma(\mathfrak{p}))])\right)^\vee \simeq (\sigma_*(R_{\sigma(\mathfrak{p})}))^\vee[-\varphi(\sigma(\mathfrak{p}))] .$$

Since R is discrete, this implies in particular that $\varphi(\sigma(\mathfrak{p})) = -\varphi(\mathfrak{p})$, i.e. that φ is C_2-equivariant. □

L: todo: show that the sequence is exact.

4.2 Poincaré structures on schemes with involution

Let X be a scheme with an involution $\sigma\colon X \xrightarrow{\sim} X$. We want to introduce a Poincaré structure $Ϙ$ on $\mathrm{Perf}(X)$ so that the duality is given by $E \mapsto E^\vee \otimes \sigma_*(\mathcal{O}_X)$ (contrast with §3 of this paper).

Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category with an involution, i.e. an exact autoequivalence $\sigma\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ and a functor $BC_2 \to \mathbb{E}_\infty \mathrm{Alg}\, \mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty$ sending $* \mapsto \mathcal{C}$ and a generator of $\mathrm{End}_{BC_2}(*) \simeq C_2$ to σ. Then σ induces a C_2-action on the ∞-groupoid of $\otimes$-invertible objects $\mathrm{Pic}(\mathcal{C})$. Let $L \in \mathrm{Pic}(\mathcal{C})^{hC_2}$ be a homotopy fixed point of this action. In other words, L is endowed with the choice of an equivalence $\varphi\colon L \simeq \sigma(L)$, a homotopy from $\sigma(\varphi) \circ \varphi$ to the identity on L, and higher coherences.

[2] Or just $\mathbb{E}_2$.

Consider a functor $f\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ which is C_2-equivariant with respect to the σ-action on $\mathcal{C}$ and the trivial action on Sp. In particular, the "C_2-equivariance" of f is additional data: for each $x \in \mathcal{C}$, an equivalence of spectra $c_x\colon f(x) \simeq f(\sigma(x))$ which is natural in x, a homotopy from $c_{\sigma(x)}\colon c_x$ to id_x, and higher coherences.

Lemma 4.16. *Given $L \in \mathrm{Pic}(\mathcal{C})^{hC_2}$ be a homotopy fixed point of this action, the functor $\hom_{\mathcal{C}}(-, L)$ promotes canonically to a C_2-equivariant functor in the sense of the previous paragraph.*

Proof. Note that the Yoneda embedding $y\colon \mathcal{C} \to \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is equivariant with respect to the given action on $\mathcal{C}$ and the action of C_2 on the functor category via $F \mapsto \sigma^* F = F \circ \sigma$. Now since $\mathrm{Pic}(\mathcal{C}) \subseteq \mathcal{C}$ induces $\mathrm{Pic}(\mathcal{C})^{hC_2} \to \mathcal{C}^{hC_2}$, we may take the image of L under the Yoneda embedding: $y(L) \in \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})^{hC_2} \simeq \mathrm{Fun}_{C_2}\,(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$. □

L: This may be 'overkill,' but this is true because $\mathcal{C}$ can be regarded as a $\mathcal{O}_{C_2}^{\mathrm{op}}$-parametrized ∞-category (with empty fiber over C_2/C_2). Then there is a parametrized Yoneda embedding.

Now fix a presentably symmetric monoidal ∞-category $\mathcal{D}$, and regard it as having the trivial C_2-action. Recall the Fin_*-cartesian fibration $\left(\mathcal{C}\mathrm{at}^{BC_2}_{\mathrm{op}//q^*\mathcal{D}}\right)^{\otimes} \simeq \left(\left(\mathcal{C}\mathrm{at}_{\mathrm{op}//\mathcal{D}}\right)^{BC_2}\right)^{\otimes} \to (\mathcal{C}\mathrm{at}^{BC_2})^{\times}$ from [CHN24, p. 13]. Set

$$\mathcal{W}_{\mathcal{D}}^{\otimes} := \mathbb{E}_\infty\mathrm{Alg}\left(\mathcal{C}\mathrm{at}^{BC_2}\right)^{\times} \times_{(\mathcal{C}\mathrm{at}^{BC_2})^{\times}} \left(\mathcal{C}\mathrm{at}^{BC_2}_{\mathrm{op}//q^*\mathcal{D}}\right)^{\otimes}$$

This is a Fin_*-cartesian fibration classified by the lax symmetric monoidal functor

$$\begin{aligned} \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at})^{BC_2} &\to \mathcal{C}\mathrm{at} \\ \mathcal{C}^{\otimes} &\mapsto \mathrm{Fun}_{C_2}(\mathcal{C}^{\mathrm{op}}, q^*\mathcal{D})\,. \end{aligned} \tag{4.17}$$

In particular, an object of the underlying ∞-category of $\mathcal{W}_{\mathcal{D}}^{\otimes}$ is a pair $(\mathcal{C}, f)$ where $\mathcal{C}$ is a symmetric monoidal ∞-category with a C_2-action (via a symmetric monoidal functor) and $f\colon \mathcal{C}^{\mathrm{op}} \to q^*\mathcal{D}$ is a C_2-equivariant functor.

Construction 4.18. Given a C_2-equivariant functor $f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, we may regard the data of the C_2-equivariance of f as a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}^{\mathrm{op}}} \arrow[r, "\widetilde{f}"] \arrow[d] & \mathcal{D} \times BC_2 \arrow[d] \\ BC_2 \arrow[r, equal] & BC_2 \end{tikzcd}$$

where the vertical maps are cocartesian fibrations and the restriction of $\widetilde{f}$ to the fiber over the point $* \in BC_2$ recovers f. The diagram induces a map on cocartesian sections

$$\overline{f}\colon\ \mathrm{Fun}^{\mathrm{cocart}}_{BC_2}(BC_2, \widetilde{\mathcal{C}^{\mathrm{op}}}) \to \mathcal{D}^{BC_2}\,.$$

Now if C is a symmetric monoidal ∞-category, we can associate to f the composite

$$T_f\colon \mathcal{C}^{\mathrm{op}} \xrightarrow{x \mapsto x \otimes \sigma(x)} \mathrm{Fun}^{\mathrm{cocart}}_{BC_2}(BC_2, \widetilde{\mathcal{C}^{\mathrm{op}}}) \xrightarrow{\overline{f}} \mathcal{D}^{BC_2}\,.$$

Finally, if $\mathcal{D}$ admits BC_2-indexed limits, we can take homotopy fixed points of C_2-objects in which case we define the functor $\Qoppa^s_f\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ as the composite

$$\Qoppa^s_f\colon \mathcal{C}^{\mathrm{op}} \xrightarrow{T_f} \mathcal{D}^{BC_2} \xrightarrow{(-)^{hC_2}} \mathcal{D}\,.$$

L: I'm just using the same notation as Harpaz-Nardin-Shah here, but $(-)^s$ is maybe a little weird because it should be 'hermitian,' not 'symmetric.'

Observation 4.19. Let $\mathcal{C}$, L, be as before. Then the cross effect B_L of $\Qoppa^s_L$ is given by $B_L(x, y) = \hom_{\mathcal{C}}(x \otimes \sigma(y), L)$.

L: The 'untwisted' version of this appears in [CHN24, proof

Proposition 4.20. *Let $\mathcal{D}$ be a symmetric monoidal ∞-category which admits BC_2-indexed limits. Then the assignment $(\mathcal{C}, f) \mapsto (\mathcal{C}, \Qoppa^s_f)$ of Construction 4.18 assembles to form a lax symmetric monoidal functor*

$$\mathcal{W}_{\mathcal{D}}^{\otimes} \to \left(\mathcal{C}\mathrm{at}^{BC_2}\right)^{\otimes}_{\mathrm{op}//\mathcal{D}}$$

sitting in a commutative diagram

$$\begin{tikzcd} \mathcal{W}_{\mathcal{D}}^{\otimes} \arrow[r] \arrow[d] & (\mathcal{C}\mathrm{at})^{\otimes}_{\mathrm{op}//\mathcal{D}^{BC_2}} \arrow[d] \\ \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{BC_2})^{\times} \arrow[r, "\mathcal{C}^{\otimes}\mapsto\mathcal{C}"] & \mathcal{C}\mathrm{at}^{\times} \end{tikzcd} \tag{4.21}$$

in which both vertical arrows are both Fin_**-cartesian fibrations and cocartesian fibrations of* ∞*-operads.*

Construction 4.22. Consider the functors $r\colon \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{BC_2}) \xrightarrow{\mathcal{C}^{\otimes}\mapsto\mathcal{C}} \mathcal{C}\mathrm{at}^{BC_2}$ and $p\colon \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{BC_2}) \xrightarrow{\text{forget}} \mathcal{C}\mathrm{at} \xrightarrow{q^*} \mathcal{C}\mathrm{at}^{BC_2}$ where $q\colon BC_2 \to *$. We construct a symmetric monoidal natural transformation $\tau\colon r^{\times} \implies p^{\times}$ whose component at a given category $\mathcal{C}$ with C_2-action $\sigma\colon \mathcal{C}\simeq\mathcal{C}$ is the C_2-equivariant functor $\mathcal{C} \xrightarrow{x\mapsto x\otimes\sigma(x)} p(\mathcal{C})$.

L: This is basically Construction 3.1.4 of [CHN24] with minor edits; we could possibly include less detail later.

Let $\mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right)$ be the span ∞-category of finite sets with free C_2-action. For an ∞-category with finite products $\mathcal{E}$, there is a natural equivalence $\mathrm{Fun}^{\times}\left(\mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right),\mathcal{E}\right) \simeq \mathrm{CMon}(\mathcal{E})^{BC_2}$ between product-preserving functors $\mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right) \to \mathcal{E}$ and commutative monoids in $\mathcal{E}$ with C_2-action. Taking $\mathcal{E}\simeq\mathcal{C}\mathrm{at}$, we may identify the functor r as restriction along the inclusion $i\colon BC_2 \to \mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right)$ of the maximal subgroupoid in the full subcategory on a finite C_2-set with a single orbit. On the other hand, we can identify the functor p as restriction along the map $j\colon BC_2 \to \{*\} \xrightarrow{*\mapsto C_2} \mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right)$. Now the span $C_2 \xleftarrow{\pi_1} C_2\times C_2 \xrightarrow{\pi_2} C_2$ determines a morphism in $\mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right)$ which is equivariant with respect to the given action on the source C_2 and the *trivial action* on the target C_2. This morphism determines a functor $\Delta^1\times BC_2 \to \mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right)$ whose restriction to $\{0\}\times BC_2$ agrees with i and whose restriction to $\{1\}\times BC_2$ agrees with j. This determines a natural transformation $i^* \implies j^*$ of functors $\mathrm{Fun}\left(\mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right),\mathcal{C}\mathrm{at}\right) \to \mathrm{Fun}(BC_2,\mathcal{C}\mathrm{at})$, and precomposing with the inclusion $\mathrm{Fun}^{\times}\left(\mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right),\mathcal{C}\mathrm{at}\right) \subset \mathrm{Fun}\left(\mathrm{Span}\left(\mathrm{Fin}_{C_2}^{\mathrm{free}}\right),\mathcal{C}\mathrm{at}\right)$ gives the desired natural transformation $\tau\colon r \implies p$. Since r and p preserve products, we may lift them to symmetric monoidal functors $r^{\times}, p^{\times}\colon \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}^{BC_2})^{\times} \to (\mathcal{C}\mathrm{at}^{BC_2})^{\times}$, and τ refines to a symmetric monoidal natural transformation $\tau^{\times}\colon r^{\times} \implies p^{\times}$.

Proof of Proposition 4.20. Horizontally composing the natural transformation of Construction 4.22 with the functor 4.18 and unstraightening induces a commutative diagram

L: Pretty similar to proof of [CHN24, Proposition 3.1.3], the main thing is the fact (stated after proof of Lemma 3.1.1 of *op. cit.*) that Fin_*-cartesian fibrations are classified by lax symmetric monoidal functors.

$$\begin{tikzcd} \mathcal{W}_{\mathcal{D}}^{\otimes} \arrow[r] \arrow[d] & \left(\mathcal{C}\mathrm{at}_{\mathrm{op}//\mathcal{D}^{BC_2}}\right)^{\otimes} \arrow[d] \\ \left(\mathcal{C}\mathrm{at}^{BC_2}\right)^{\otimes}_{\mathrm{op}//\mathcal{D}} \arrow[r, "\mathcal{C}^{\otimes}\mapsto\mathcal{C}"] & \mathcal{C}\mathrm{at}^{\times} \end{tikzcd}, \tag{4.23}$$

where we have used that $(q^*\mathcal{D})^{hC_2} \simeq \mathcal{D}^{BC_2}$. ☐

Definition 4.24. Define

$$\mathcal{W}_{\mathrm{ex}}^{\otimes} \subseteq \mathcal{W}_{\mathrm{Sp}}^{\otimes} \times_{\mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at})} \mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}})$$

to be the full sub-operad on those colors $(\mathcal{C}, f)$ so that f is exact.

Observation 4.25. The commutative square (4.21) restricts to a commutative square of ∞-operads

$$\begin{tikzcd} \mathcal{W}_{\mathrm{ex}}^{\otimes} \arrow[r] \arrow[d] & {\mathrm{Cat}_\infty^{\mathrm{h}}}^{\otimes} \arrow[d] \\ \left(\mathbb{E}_\infty\mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}})^{BC_2}\right)^{\otimes} \arrow[r] & \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}\otimes} \end{tikzcd} \tag{4.26}$$

Lemma 4.27. *Both vertical maps in 4.26 are cocartesian fibrations of ∞-operads. In particular, $\mathcal{W}_{\mathrm{ex}}^{\otimes}$ is a symmetric monoidal ∞-category.*

Proof. $\square$

As in [CHN24, p. 15], we can identify objects of the underlying ∞-category of $\mathcal{W}_{\mathrm{ex}}^{\otimes}$ with pairs $(\mathcal{C}^{\otimes}, \sigma_{\mathcal{C}}, L, \lambda)$ where $\mathcal{C}$ is a symmetric monoidal stable ∞-category with involution $\sigma_{\mathcal{C}}$, and $(L, \lambda) \in \mathrm{Ind}(\mathcal{C})^{hC_2}$ is a fixed point with respect to the induced action on $\mathrm{Ind}(\mathcal{C})$.

Proposition 4.28. *Let $(\mathcal{C}^{\otimes}, \sigma_{\mathcal{C}}, L, \lambda) \in \mathcal{W}_{\mathrm{ex}}$ be as above. Then the hermitian structure $\Qoppa^s_L$ is non-degenerate if and only if (L, λ) belongs to $\mathcal{C}^{hC_2}$, and it is furthermore Poincaré if and only if the underlying object L is tensor-invertible in $\mathcal{C}$. In addition, if $g\colon \mathcal{C} \to \mathcal{C}'$ is a C_2-equivariant exact functor, $L \in \mathcal{C}^{BC_2}$ and $L' \in (\mathcal{C}')^{BC_2}$ are tensor-invertible, and $g(L) \simeq L'$ an equivalence, then the induced hermitian functor $(\mathcal{C}, \Qoppa^s_L)\,(\mathcal{C}', \Qoppa^s_{L'})$ is Poincaré.*

4.3 Poincaré Picard groups of schemes

Let X be a scheme with an involution. Assume that X has a *good quotient* Y in the sense of [FW20, Remark 4.20]. We write $p\colon X \to Y$ for the quotient map.

Construction 4.29. Let $j\colon \operatorname{Spec} A \simeq U \subseteq Y$ be an affine open subscheme of Y. Because p is an affine map, the fiber product $\operatorname{Spec} B := \operatorname{Spec} A \times_Y X$ is an affine open of X which is invariant under the C_2-action. In particular $\operatorname{Spec} B$ inherits a C_2-action from X (hence so does its ring of functions B). Now $A \to B$ acquires the structure of a C_2-Green functor $\underline{\mathcal{O}}(U)$. Regarding $\underline{\mathcal{O}}(U)$ as a C_2-spectrum, by the isotropy separation sequence, we have an equivalence of A-modules $\underline{\mathcal{O}}(U)^{\varphi C_2} \simeq \operatorname{cofib}(\mathrm{tr}\colon B_{hC_2} \to A)$.

L: by assumption!

Lemma 4.30. *Let X be a scheme with an involution. Assume that X has a* good quotient *Y in the sense of [FW20, Remark 4.20].*

(i) The assignment of Construction 4.29 lifts to a contravariant functor from (the nerve of) the category of affine opens of Y to the ∞-category of Poincaré rings/C_2-$\mathbb{E}_\infty$-rings/Tambara functors.

(ii) The presheaf $\underline{\mathcal{O}}$ of (i) defines a Zariski sheaf.

Proof. Part (i) follows from a similar argument to [Yan23, Theorem 5.1]; functoriality follows from noting that $\tau_{\geq 0}$ is a functor. Part (ii) follows from Lemma 4.31. $\square$

Lemma 4.31. *Let K be a simplicial set, and let $f\colon K^{\triangleleft} \to C_2\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_2})$ be a diagram. Then f is a limit diagram if and only if $f^e\colon K^{\triangleleft} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ and $f^{C_2}\colon K^{\triangleleft} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ are both limit diagrams.*

Proof. The result follows from the observation that limits in $\mathbb{E}_\infty\left(\mathrm{Sp}^{C_2}\right)$ are computed in Sp^{C_2}. $\square$

Construction 4.32. Let $p\colon X \to Y$ as before. Consider the composites

$$\begin{aligned} \mathrm{Mod}_{\underline{\mathcal{O}}}\colon \mathrm{Op}(Y)^{\mathrm{op}} &\xrightarrow{\underline{\mathcal{O}}} C_2\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{\mathrm{Mod}_{(-)}} \mathcal{C}\mathrm{at} \\ \mathrm{Mod}^{\otimes}_{\underline{\mathcal{O}}}\colon \mathrm{Op}(Y)^{\mathrm{op}} &\xrightarrow{\underline{\mathcal{O}}} C_2\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{\mathrm{Mod}^{\otimes}_{(-)}} C_2 \otimes \mathcal{C}\mathrm{at}\,, \end{aligned} \tag{4.33}$$

where $C_2 \otimes \mathcal{C}\mathrm{at}$ denotes the ∞-category of (small) C_2-symmetric monoidal C_2-∞-categories. In the notation of Construction 4.29, this functor sends the affine open $\operatorname{Spec} A \subseteq Y$ to the category of modules in C_2-spectra over the C_2-$\mathbb{E}_\infty$-algebra which has underlying C_2-Mackey functor $A \to B$. Define $\mathrm{Mod}_{\underline{\mathcal{O}}}$, $\mathrm{Mod}^{\otimes}_{\underline{\mathcal{O}}}$ to be the limits in $\mathcal{C}\mathrm{at}$, $C_2 \otimes \mathcal{C}\mathrm{at}$, resp. of the functors in (4.33). In particular, if we write $s\colon \int \mathrm{Mod}_{\underline{\mathcal{O}}} \to \mathrm{Op}(Y)^{\mathrm{op}}$ for the cocartesian fibration obtained by taking the Grothendieck construction on (4.33), an object of $\mathrm{Mod}_{\underline{\mathcal{O}}}$ is a cocartesian section of s. In other words, it is a choice, for each affine open $\operatorname{Spec} A$ of Y (same notation as before), of a module over the C_2-$\mathbb{E}_\infty$-algebra which has underlying C_2-Mackey functor $A \to B$ which glue compatibly.

L: bleh...cardinals

L: invent better notation later

Observe that for each $A \to B$, there is a quadratic norm functor $N^{C_2}\colon \operatorname{Mod}_B(\mathrm{Sp}) \to \operatorname{Mod}_{N^{C_2}B}\left(\mathrm{Sp}^{C_2}\right)$ and a quadratic relative norm functor $N^{C_2}\colon \operatorname{Mod}_B(\mathrm{Sp}) \to \operatorname{Mod}_{A\to B}\left(\mathrm{Sp}^{C_2}\right)$.

Lemma 4.34. *The norm functors (resp. relative norm functors) $N_e^{C_2}$ assemble under Construction 4.29 to a 'global' norm functor $N_Y^{C_2}\colon \pi_\#\mathcal{O}_X \operatorname{Mod} \to N^{C_2}\pi_\#\mathcal{O}_X \operatorname{Mod}$ (resp. relative norm functor $N_Y^{C_2}\colon \pi_\#\mathcal{O}_X \operatorname{Mod} \to \underline{\mathcal{O}} \operatorname{Mod}$). Moreover, these functors are quadratic.*

Proof. For each affine open $\operatorname{Spec} A \subseteq Y$, write $B = \Gamma\mathcal{O}_{\operatorname{Spec} A\times_Y X}$. Then the functor $N_Y^{C_2}$ is the limit over all $\operatorname{Spec} A \subseteq Y$ of the functors $\pi_\#\mathcal{O}_X \operatorname{Mod} \to B \operatorname{Mod}(\mathrm{Sp}) \xrightarrow{N^{C_2}} \operatorname{Mod}_{N^{C_2}B}(\mathrm{Sp}^{C_2}) \xrightarrow{-\otimes_{N^{C_2}B}(A\to B)} \operatorname{Mod}_{A\to B}(\mathrm{Sp}^{C_2})$, where the last map is base change along the map $N^{C_2}B \to (A \to B)$ which is a structure map for the C_2-$\mathbb{E}_\infty$-algebra structure on $A \to B$. Now since quadratic functors are closed under limits [Lur17, Theorem 6.1.1.10] and $N_Y^{C_2}$ can be written as a limit of a diagram of quadratic functors, $N_Y^{C_2}$ is also quadratic. ◻

L: todo: rewrite this as a diagram in a functor category with fixed domain and target.

Lemma 4.35. *Let X be a scheme with involution $\sigma\colon X \xrightarrow{\sim} X$ equipped with a good quotient $\pi\colon X \to Y$. Let L be a line bundle on Y. Then the canonical map*

$$L \to \pi_\#\pi^* L \tag{4.36}$$

promotes (4.36) to a sheaf of $\underline{\mathcal{O}}$-modules on Y. We will write $\underline{L}$ for (4.36).

Definition 4.37. Let X be a scheme with involution $\sigma\colon X \xrightarrow{\sim} X$ equipped with a good quotient $\pi\colon X \to Y$. Let L be a line bundle on Y. Define $Ϙ_{\sigma,L}$ to be the functor

$$\mathrm{Perf}_X^{\mathrm{op}} \xrightarrow{\pi_\#} \pi_\#\mathcal{O}_X \operatorname{Mod}^{\omega,\mathrm{op}} \xrightarrow{N^{C_2}} N^{C_2}\pi_\#\mathcal{O}_X \operatorname{Mod}\left(\mathrm{Sp}^{C_2}\right)^{\mathrm{op}} \xrightarrow{\hom_{N^{C_2}\pi_\#\mathcal{O}_X}(-,\underline{L})} \mathrm{Sp}\,,$$

where $\underline{L}$ is a $\underline{\mathcal{O}}$-module by Lemma 4.35 and $\underline{\mathcal{O}}$ is a $N^{C_2}\pi_\#\mathcal{O}_X$-algebra by Lemma 4.30. By Lemma 4.34 and the fact that the composite of an exact (1-excisive) functor and an m-excisive functor is m-excisive (see [Bar+22, §2.2]), $Ϙ_\sigma$ is quadratic.

L: want to show that the assignment $(\sigma : X \to X, \pi : X \to Y, L) \mapsto (\mathrm{Perf}_X, Ϙ_{\sigma,L})$ defines a functor from some category of schemes with involution (+line bundle) to the ∞-category of Poincaré ∞-categories.

Example 4.38. Suppose $L = \mathcal{O}_Y$. Then we drop L from notation and the quadratic functor $Ϙ_\sigma$ of Definition 4.37 takes the form

$$\mathrm{Perf}_X^{\mathrm{op}} \xrightarrow{\pi_\#} \pi_\#\mathcal{O}_X \operatorname{Mod}^{\omega,\mathrm{op}} \xrightarrow{N^{C_2}} N^{C_2}\pi_\#\mathcal{O}_X \operatorname{Mod}\left(\mathrm{Sp}^{C_2}\right)^{\mathrm{op}} \xrightarrow{\hom_{N^{C_2}\pi_\#\mathcal{O}_X}(-,\underline{\mathcal{O}})} \mathrm{Sp}\ .$$

L: maybe for the purposes of this example, take Y to be affine?

Lemma 4.39. *Let $\pi\colon X \to Y$ be a finite étale map. Then the canonical functor $\operatorname{Mod}_{\mathcal{O}_X} \to \operatorname{Mod}_{\pi_\#\mathcal{O}_X}$ is fully faithful.*

L: asked JH Nov 22nd: James says that this should definitely be true, and moreover in greater gen-

Proof. Affine-locally on Y, I think this is an *equivalence*. Conclude by descent. ◻

Lemma 4.40. *Let X be a scheme with involution $\sigma\colon X \xrightarrow{\sim} X$, and let Y be a good quotient of X. Assume that the quotient map $q\colon X \to Y$ is étale. Let $Ϙ_\sigma$ be the Poincaré structure on Perf_X of Definition 4.37. Then the bilinear part of $Ϙ_\sigma$ agrees with that of Observation 4.19.*

Proof. By definition of the bilinear part of a quadratic functor, it suffices to show that there is an equivalence $\hom_{\pi_\#\mathcal{O}_X \operatorname{Mod}}\left(\pi_\# E \otimes_{\pi_\#\mathcal{O}_X} \pi_\# E, \pi_\#\mathcal{O}_X\right) \simeq \hom_{\mathcal{O}_X \operatorname{Mod}}\left(E \otimes_{\mathcal{O}_X} \sigma^* E, \mathcal{O}_X\right)$ for any perfect complex E on X. This follows from Lemma 4.39. ◻

Remark 4.41. Compare the description of the space of bilinear forms in Lemma 4.40 with the description of a δ-hermitian form H in [PS92, p. 216].

5 The Poincaré Brauer Group

Recall that a Poincaré ∞-category is called idempotent complete if the underlying stable ∞-category is idempotent complete. The full subcategory of $\mathrm{Cat}_\infty^{\mathrm{p}}$ spanned by idempotent complete Poincaré ∞-categories is denoted by $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$ [Cal+20b, Definition 1.3.2].

Definition 5.1. Let A be a Poincaré ring spectrum. We define the *Poincaré Brauer space of A* as

$$\mathrm{Br}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Mod}_A(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})).$$

The assignment $A \mapsto \mathrm{Br}^{\mathrm{p}}(A)$ defines a functor

$$\mathrm{Br}^{\mathrm{p}} \colon \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$$

valued in grouplike $\mathbf{E}_\infty$-spaces.

Remark 5.2. The symmetric monoidal forgetful functor $\mathrm{Mod}_A(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_A(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)$ induces a map $\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A)$ of grouplike $\mathbf{E}_\infty$-spaces, where $\mathrm{Br}(A)$ is the Brauer space $\mathrm{br}_{\mathrm{alg}}(A)$ of [AG14, pp. 1154–1155].

Proposition 5.3. *Let A be a Poincaré ring spectrum. Then we have a canonical equivalence*

$$\Omega\,\mathrm{Br}^{\mathrm{p}}(A) \simeq \mathrm{Pic}^{\mathrm{p}}(A).$$

L: What else do we need to do to show that we have an equivalence of *functors*?

V: todo

Proof. Since $\Omega\,\mathrm{Br}^{\mathrm{p}}(R)$ is given by the space of automorphisms of any object in $\mathrm{Br}^{\mathrm{p}}(R)$, it suffices to determine the space of autoequivalences of $(\mathrm{Mod}_R^\omega, Ϙ_R)$. An autoequivalence is the data of a pair (f,η) where $f\colon \mathrm{Mod}_R^\omega \xrightarrow{\simeq} \mathrm{Mod}_R^\omega$ is an exact R-linear autoequivalence and $\eta\colon Ϙ_R \xrightarrow{\sim} Ϙ_R \circ f^{\mathrm{op}}$ is a natural equivalence. Since $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}$ is symmetric monoidal, f is of the form $-\otimes_R \mathcal{L}$ where $\mathcal{L}$ is an invertible R-module. Since taking bilinear and linear parts is functorial/by [Cal+20a, Proposition 1.3.11], η is equivalently the data of a pair of equivalences

$$\begin{aligned}
b(\eta)\colon\ & \hom_{R\otimes R}((-\otimes\mathcal{L})\otimes(-\otimes\mathcal{L}), R)^{hC_2} \simeq \hom_{R\otimes R}(-\otimes -, R)^{hC_2}\\
\ell(\eta)\colon\ & \hom_R(-\otimes\mathcal{L}, R^{\varphi C_2}) \simeq \hom_R(-, R^{\varphi C_2})
\end{aligned}$$

plus a path between their images in $\hom_R(\mathcal{L}, R^{tC_2})$. The transformation $b(\eta)$ is equivalent to the data of an R-bilinear equivalence $R \simeq \mathcal{L}^\vee \otimes \mathcal{L}^\vee$, and the transformation $\ell(\eta)$ is equivalent to the data of an $R^{\varphi C_2}$-linear equivalence $\ell(\eta)\colon R^{\varphi C_2} \otimes_R \mathcal{L}^\vee \xrightarrow{\sim} R^{\varphi C_2}$.

L: maybe one of these should be conjugate dual here?

Now consider the composites

$$R \otimes_R \mathcal{L}^\vee \xrightarrow{\text{unit}\otimes\mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L}^\vee \xrightarrow{\ell(\eta)} R^{\varphi C_2}$$

$$R \otimes_R \mathcal{L} \xrightarrow{\text{unit}\otimes\mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L} \xrightarrow{\ell(\eta)^{-1}\otimes\mathrm{id}_{\mathcal{L}}} R^{\varphi C_2}\,.$$

L: is the $R^{\varphi C_2}$-linearity of this $\simeq$ correct?

These correspond to the $\ell(q^\vee), \ell(q)$ of Remark **??**, respectively. In particular, the condition that $\ell(q^\vee), \ell(q)$ make the diagram (4.4) commute is equivalent to the condition that $\ell(\eta)$ is an equivalence by an adjunction argument. $\square$

L: under construction– not sure what to say about the $(-)^{tC_2}$ part yet.

Proposition 5.4. *Let $(\mathrm{Mod}_R^\omega, Ϙ_R)$ be a Poincaré ring spectrum.*

1. *The ∞-category $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, Ϙ_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ admits all small limits and colimits, and it inherits a canonical symmetric monoidal structure, and for every morphism $\big(R, R^{\varphi C_2} \to R^{tC_2}\big) \to (S, S^{\varphi C_2} \to S^{tC_2})$, the functor $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, Ϙ_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_{(\mathrm{Mod}_S^\omega, Ϙ_S)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ is a symmetric monoidal left adjoint.*

2. *Let A be an $\mathbb{E}_1$-R-algebra in spectra, and regard the category of compact right A-modules Mod_A^ω as left-tensored over Mod_R^ω in the canonical way. Then the pullback*

$$\begin{tikzcd} & \mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Omega_R)}\left(\mathrm{Cat}_\infty^{\mathrm{h}}\right) \arrow[d] \\ \{\mathrm{Mod}_A^\omega\} \arrow[r] & \mathcal{C}\mathrm{at}_{\infty R}^{\mathrm{ex}} \end{tikzcd} \tag{5.5}$$

is canonically equivalent to $\mathrm{Mod}_{N_R A \otimes_{N_R R} R^L}\left(\mathrm{Sp}^{C_2}\right)$ where R^L is the $\mathbb{E}_\infty$-$N_R R$-algebra with $(R^L)^e \simeq R$ and $(R^L)^{\varphi C_2} \simeq C$.

A $N_R A \otimes_{N_R R} R^L$-module classifies a $(\mathrm{Mod}_R^\omega, \Omega_R)$-module in Poincaré ∞-categories if its underlying R-module is invertible in the sense of [Cal+20a, Definition 3.1.4].

3. *Let A, B be R-algebras with associated (R-linear) modules with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and $(M_B, N_B, N_B \to M_B^{tC_2})$, respectively so that (under item 2) $(\mathrm{Mod}_A^\omega, \Omega_A)$ and $(\mathrm{Mod}_B^\omega, \Omega_B)$ are objects of $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Omega_R)}(\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}})$. Then the symmetric monoidal structure of item 1 is so that the underlying R-linear ∞-category with perfect duality $(\mathrm{Mod}_A^\omega, \Omega_A) \otimes_{(\mathrm{Mod}_R^\omega, \Omega_R)} (\mathrm{Mod}_B^\omega, \Omega_B)$ is $\mathrm{Mod}_A^\omega \otimes_{\mathrm{Mod}_R^\omega} \mathrm{Mod}_B^\omega \simeq \mathrm{Mod}_{A \otimes_R B}^\omega$, and the associated module with genuine involution is given by $M_A \otimes_R M_B$, $N_A \otimes_{R^{\varphi C_2}} N_B$, and the structure map is $N_A \otimes_{R^{\varphi C_2}} N_B \to M_A^{tC_2} \otimes_{R^{tC_2}} M_B^{tC_2} \to (M_A \otimes_R M_B)^{tC_2}$ where the latter map arises canonically from lax monoidality of the Tate construction.*

4. *Let $(\mathcal{C}, \Omega_\mathcal{C}), (\mathcal{D}, \Omega_\mathcal{D})$ be objects of $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Omega_R)}(\mathrm{Cat}_\infty^{\mathrm{h}})$. Then the forgetful functor induces $\hom_{\mathrm{Cat}_{\infty R}^{\mathrm{h}}}((\mathcal{C}, \Omega_\mathcal{C}), (\mathcal{D}, \Omega_\mathcal{D})) \to \hom_{\mathcal{C}\mathrm{at}_{\infty R}^{\mathrm{ex}}}(\mathcal{C}, \mathcal{D})$ on mapping spaces so that the fiber over an R-linear functor $F\colon \mathcal{C} \to \mathcal{D}$ is the mapping space $\mathrm{map}_{\Omega_R}(F_! \Omega_\mathcal{C}, \Omega_\mathcal{D}) \simeq \mathrm{map}_{\Omega_R}(\Omega_\mathcal{C}, \Omega_\mathcal{D} \circ F^{\mathrm{op}})$, where the mapping space is taken in $\mathrm{Fun}_{\Omega_R}^q(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp})$ and $\mathrm{Fun}_{\Omega_R}^q(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$, respectively.*[3]

5. *The symmetric monoidal forgetful functor $\theta\colon \mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Omega_R)}(\mathrm{Cat}_\infty^{\mathrm{h}}) \to \mathrm{Mod}_{\mathrm{Mod}_R^\omega}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}})$ is a (co)cartesian fibration.*

L: what is it classified by?

Remark 5.6. A special case of part 2 is [Cal+20a, Example 5.4.13].

Proof. 1. The first part of the statement follows from [Cal+20a, §6.1] and [Lur17, §4.2.3].

2. Let $\mathcal{LM}^\otimes$ denote the ∞-operad of [Lur17, Definition 4.2.1.7]. Our strategy of proof will be similar to that of [Cal+20a, §5.3]: First, we show that an $\mathcal{LM}^\otimes$-algebra object in $\mathrm{Cat}_\infty^{\mathrm{h}}$ is equivalent to an $\mathcal{LM}^\otimes$-algebra object in an operad of functor categories. Then, we use a (suitably coherent version of) the classification of hermitian structures on module categories as categories of modules over the Hill–Hopkins–Ravenel norm [Cal+20a, Theorem 3.3.1] to conclude. Recall that the action of Mod_R^ω on Mod_A^ω is given by a functor $\mathcal{LM}^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$, and define $\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes$ via the following pullback square of ∞-operads:

$$\begin{tikzcd} \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \arrow[r, "p"] \arrow[d] & \mathcal{LM}^\otimes \arrow[d, "\mathrm{Mod}_R^\omega, \mathrm{Mod}_A^\omega"] \\ (\mathcal{C}\mathrm{at}_\infty)^\otimes_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}_\infty^\times \end{tikzcd} . \tag{5.7}$$

Informally, an object $F \in \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})_{\mathfrak{a}}^\otimes$ is a functor $F\colon \mathrm{Mod}_R^{\omega,\mathrm{op}} \to \mathrm{Sp}$ and an object G over the fiber of $\mathfrak{m}$ is a functor $G\colon \mathrm{Mod}_A^{\omega,\mathrm{op}} \to \mathrm{Sp}$. The p-cocartesian edge over the canonical map

[3]The proof of 2 in particular shows that $\mathrm{Fun}^q(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is left-tensored over $\mathrm{Fun}^q(\mathrm{Mod}_R^{\omega,\mathrm{op}}, \mathrm{Sp})$ in the sense of [Lur17, Definition 4.2.1.19], so this makese sense.

$(\mathfrak{a},\mathfrak{m}) \to \mathfrak{m}$ in $\mathcal{LM}^{\otimes}$ sends (F,G) to the lower arrow in the diagram

$$\begin{tikzcd}
\mathrm{Mod}_R^{\omega,\mathrm{op}} \times \mathrm{Mod}_A^{\omega,\mathrm{op}} \arrow[r, "F\times G"] \arrow[d, "-\otimes_R -"'] & \mathrm{Sp}\times\mathrm{Sp} \arrow[d, "\otimes_{\mathrm{Sp}}"] \\
\mathrm{Mod}_A^{\omega,\mathrm{op}} \arrow[r, "F\otimes G := \mathrm{LKE}_{\otimes_R}(\otimes_{\mathrm{Sp}}\circ(F\times G))"] & \mathrm{Sp}
\end{tikzcd}.$$

Now define $\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes}$ to consist of the full subcategory of $\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes}$ consisting of those tuples of functors which are all quadratic. The inclusion $\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes} \to \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes}$ exhibits the former as an ∞-operad, and moreover the localization is compatible with the $\mathcal{LM}^{\otimes}$-monoidal structure in the sense of [Lur17, Definition 2.2.1.6]. We can extend the previous diagram to

$$\begin{tikzcd}
\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes} \arrow[r, "p"] \arrow[d] & \mathcal{LM}^{\otimes} \arrow[d, "\mathrm{Mod}_R^{\omega},\mathrm{Mod}_A^{\omega}"] \\
{\mathrm{Cat}^{\mathrm{h}}_{\infty}}^{\otimes} \arrow[r] & (\mathcal{C}\mathrm{at}_{\infty})^{\otimes}_{\mathrm{op}/-/\,\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}_{\infty}^{\otimes}
\end{tikzcd}. \tag{5.8}$$

Modifying [Cal+20a, Construction 5.3.15 & Lemma 5.3.15] slightly (note that Corollary 5.1.4 did not assume the tensor factors to be equivalent), we obtain an analogous commutative diagram of ∞-operads

$$\begin{tikzcd}
\mathrm{Fun}^p_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes} \arrow[r, "p"] \arrow[d] & \mathcal{LM}^{\otimes} \arrow[d, "\mathrm{Mod}_R^{\omega},\mathrm{Mod}_A^{\omega}"] \\
{\mathrm{Cat}^{\mathrm{p}}_{\infty}}^{\otimes} \arrow[r] & {\mathrm{Cat}^{\mathrm{h}}_{\infty}}^{\otimes} \arrow[r] & (\mathcal{C}\mathrm{at}_{\infty})^{\otimes}_{\mathrm{op}/-/\,\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}_{\infty}^{\otimes}
\end{tikzcd} \tag{5.9}$$

in which all squares are pullbacks. Now suppose A is given a module with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and call the associated Poincaré ∞-category $\overline{\mathrm{Mod}}_A$. Then to lift $\overline{\mathrm{Mod}}_A$ to a module over $(\mathrm{Mod}_R^{\omega}, Ϙ_R)$ compatibly with the Mod_R^{ω}-module structure on Mod_A^{ω} is to give a map of ∞-operads $\mathcal{LM}^{\otimes} \to {\mathrm{Cat}^{\mathrm{h}}_{\infty}}^{\otimes}$ so that the restriction along the canonical inclusion $\mathrm{Assoc}^{\otimes} \to \mathcal{LM}^{\otimes}$ gives the algebra object $(\mathrm{Mod}_R^{\omega}, Ϙ_R)$ and postcomposing with the canonical projection to $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}\times}$ recovers the given Mod_R^{ω}-module structure on Mod_A^{ω}. By the pullback square (5.8), this is equivalent to giving an object of $\mathrm{Alg}_{\mathcal{LM}/\mathcal{LM}}\left(\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}},\mathrm{Sp})^{\otimes}\right)$. Now let us identify the bilinear functor $\mathrm{Mod}_R^{\omega}\times\mathrm{Mod}_A^{\omega} \xrightarrow{-\otimes_R -} \mathrm{Mod}_A^{\omega}$ with the exact functor $\mathrm{Mod}_R^{\omega}\otimes\mathrm{Mod}_A^{\omega} \simeq \mathrm{Mod}_{R\otimes A}^{\omega} \to \mathrm{Mod}_A^{\omega}$ which is induction along the action map $R\otimes A \to A$. Using [Cal+20a, Corollary 3.4.1] and unravelling definitions gives the claim for R-linear hermitian structures. The proof for R-linear Poincaré structures considers (5.9) instead but otherwise proceeds in an identical fashion.

3. By [Lur17, Theorem 4.4.2.8], the relative tensor product $(\mathrm{Mod}_A^{\omega}, Ϙ_A) \otimes_{(\mathrm{Mod}_R^{\omega}, Ϙ_R)} (\mathrm{Mod}_B^{\omega}, Ϙ_B)$ is computed as the geometric realization of the bar construction

$$\begin{aligned} p\colon \Delta^{\mathrm{op}} &\to \mathrm{Cat}^{\mathrm{h}}_{\infty} \\ [n] &\mapsto (\mathrm{Mod}_A^{\omega}, Ϙ_A)\otimes(\mathrm{Mod}_R^{\omega}, Ϙ_R)^{\otimes n}\otimes(\mathrm{Mod}_B^{\omega}, Ϙ_B) \end{aligned}$$

Write $f\colon \mathrm{Cat}^{\mathrm{h}}_{\infty} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty}$ for the forgetful functor. Then $f\circ p$ has a colimit with value $\mathrm{Mod}_A^{\omega}\otimes_{\mathrm{Mod}_R^{\omega}}\mathrm{Mod}_B^{\omega} \simeq \mathrm{Mod}^{\omega}_{A\otimes_R B}$. Writing $g\colon \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \to \{*\}$, by Example 4.3.1.3 of [Lur09] we see that $f\circ p$ is a g-colimit. By Proposition 4.3.1.5(2) and Example 4.3.1.3 of [Lur09], p admits a colimit in $\mathrm{Cat}^{\mathrm{h}}_{\infty}$ if and only if it admits an f-colimit. Now recall that f is a cocartesian fibration with pushforward given by left Kan extension [Cal+20a, Corollary 1.4.2]. We show that f satisfies the conditions of [Lur09, Corollary 4.3.1.11].

- Condition (1) follows from Theorem 6.1.1.10 of [Lur17] applied to $\mathrm{Sp}^{\mathrm{op}}$ (see the end of [Cal+20a, Construction 1.1.26]).
- Condition (2) follows from [Cal+20a, Corollary 1.4.2], the adjoint functor theorem, and presentability of $\mathrm{Fun}^q(\mathcal{C})$, which is discussed in the proof of [Cal+20a, Lemma 5.3.3] (also see [Lur17, Remark 6.1.1.11]).

Thus the preceding discussion shows that there exists a map of simplicial sets p' making the diagram commute

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}^{\mathrm{h}}_{\infty} \arrow[d, "f"] \\ (\Delta^{\mathrm{op}})^{\triangleright} \arrow[r] \arrow[ur, "p'"] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \end{tikzcd} .$$

Since $\{0\} \to \Delta^1$ is left anodyne, by [Lur09, Corollary 2.1.2.7] the inclusions

$$\{0\} \times \Delta^{\mathrm{op}} \to \Delta^1 \times \Delta^{\mathrm{op}}$$
$$\iota\colon \left(\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}\right) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) \to \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright}$$

are left anodyne. The former implies that there exists a map p'' making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}^{\mathrm{h}}_{\infty} \arrow[d, "f"] \\ \Delta^1 \times \Delta^{\mathrm{op}} \arrow[r] \arrow[ur, "p''"] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \end{tikzcd}$$

commute. The maps p' and p'' assemble to give a map $p''' := p' \sqcup_p p''$ making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[rrr, "p"] \arrow[d] & & & \mathrm{Cat}^{\mathrm{h}}_{\infty} \arrow[d, "f"] \\ \left(\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}\right) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) \arrow[r, "\iota"] \arrow[urrr, "p'''"] & \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright} \arrow[rr] \arrow[urr, bend left, "\overline{p}"'] & & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \end{tikzcd}$$

commute, and likewise $\overline{p}$ exists making the diagram commute since ι is left anodyne. Now we show that $\overline{p}$ satisfies the conditions of [Lur09, Proposition 4.3.1.9]. By (the opposite/dual/cocartesian version of) [Lur09, Remark 3.1.1.10] and Proposition 3.1.1.5(2") *ibid.* and the fact that f is a cocartesian fibration, we can choose $\overline{p}$ so that for all $k \in (\Delta^{\mathrm{op}})^{\triangleright}$, $\overline{p}|_{\Delta^1\times\{k\}}$ is f-cocartesian. Furthermore, since we can choose Δ^{op}, $(\Delta^{\mathrm{op}})^{\triangleright}$ to have the markings $(-)^{\flat}$ in [Lur09, Remark 3.1.1.10], $f \circ \overline{p}|_{\Delta^1\times\{\infty\}}$ is a degenerate edge in $\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty}$.

Now [Lur09, Proposition 4.3.1.9] implies that $\overline{p}_0$ is an f-colimit diagram if and only if $\overline{p}_1$ is an f-colimit diagram. Now notice that $\overline{p}|_{\{1\}\times(\Delta^{\mathrm{op}})^{\triangleright}}$ has image contained in the fiber of f over $\mathrm{Mod}^{\omega}_{A\otimes_R B}$. By [Lur09, Proposition 4.3.1.10], it suffices to show that $\overline{p}_1$ is a colimit diagram in $\mathrm{Fun}^q\left(\mathrm{Mod}^{\omega}_{A\otimes_R B}\right)$. Write $\overline{M}_A \in \mathrm{Mod}_{N^{C_2}A}$ and $\overline{M}_B \in \mathrm{Mod}_{N^{C_2}B}$ for the corresponding modules (see introduction to §3.3 of [Cal+20a]). Unraveling definitions and using [Cal+20a, Theorem 3.3.1 & Corollary 3.4.1 & Lemma 5.4.6], it follows that the diagram $\overline{p}_1|_{\{1\}\times\Delta^{\mathrm{op}}}$ is the bar construction

$$[n] \mapsto \overline{M}_A \otimes_{N^{C_2}R} R^{\otimes_{N^{C_2}R} n} \otimes_{N^{C_2}R} \overline{M}_B \,.$$

This proves the result.

4. Let $(\mathcal{C}, \Qoppa_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R, \Qoppa_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$ and let $F\colon \mathcal{C} = \theta\,(\mathcal{C}, \Qoppa_{\mathcal{C}}) \to \mathcal{D}$ be an R-linear functor. Now define $\Qoppa_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp}$ to be the left Kan extension of $\Qoppa_{\mathcal{C}}$ along F^{op}. Now $(\mathcal{D}, \Qoppa_{\mathcal{D}}) \in \mathrm{Cat}^{\mathrm{h}}_{\infty}$ and there

is a canonical map $(f,\eta)\colon (\mathcal{C},\Qoppa_{\mathcal{C}}) \to (\mathcal{D},\Qoppa_{\mathcal{D}})$. Now F is classified by a functor $\Delta^1 \times \mathcal{LM}^{\otimes} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}\otimes}$, and we may form the pullback

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \Delta^1 \times \mathcal{LM}^{\otimes} \arrow[d] \\ {\mathrm{Cat}_\infty^{\mathrm{h}}}^{\otimes} \arrow[r, "p"] & \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}\otimes} \end{tikzcd} . \tag{5.10}$$

Since p is a cocartesian fibration [Cal+20a, Theorem 5.2.7], $\mathcal{N} \to \Delta^1 \times \mathcal{LM}^{\otimes}$ is a cocartesian fibration, and the nontrivial morphism in Δ^1 classifies a map $F_!\colon \mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})^{\otimes} \to \mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathcal{D}^{\mathrm{op}},\mathrm{Sp})^{\otimes}$ of ∞-operads over $\mathcal{LM}^{\otimes}$. Passing to algebra objects, we obtain the desired result on mapping spaces.

5. By [Lur09, Proposition 2.4.2.8], it suffices to show that θ is a locally (co)cartesian fibration, and that locally (co)cartesian edges are closed under composition. We give the proof that θ is a cocartesian fibration; the proof that θ is a cartesian fibration is formally dual and will be left to the reader.

   Let $(\mathcal{C},\Qoppa_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{(\mathrm{Mod}_R^{\omega},\Qoppa_R)}(\mathrm{Cat}_\infty^{\mathrm{h}})$ and let $F\colon \mathcal{C} = \theta(\mathcal{C},\Qoppa_{\mathcal{C}}) \to \mathcal{D}$ be an R-linear functor. Now define $\Qoppa_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp}$ to be the left Kan extension of $\Qoppa_{\mathcal{C}}$ along F^{op}. By the proof of 4, we see that the image of $\Qoppa_{\mathcal{C}}$ under $F_!$ is a lift of $(\mathcal{D},\Qoppa_{\mathcal{D}})$ to an object of $\mathrm{Mod}_{(\mathrm{Mod}_R^{\omega},\Qoppa_R)}(\mathrm{Cat}_\infty^{\mathrm{h}})$ and (f,η) to a morphism in $\mathrm{Mod}_{(\mathrm{Mod}_R^{\omega},\Qoppa_R)}(\mathrm{Cat}_\infty^{\mathrm{h}})$.

   Now by Lemma 2.4.4.1 and the locally cocartesian version of Proposition 2.4.1.10 of [Lur09], we must show that for all choices $\Qoppa'_{\mathcal{D}}$ of an R-linear Hermitian structure on $\mathcal{D}$, precomposition with $F_!$ induces a pullback square

$$\begin{tikzcd} \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}\left((\mathcal{D},\Qoppa_{\mathcal{D}}),(\mathcal{D},\Qoppa'_{\mathcal{D}})\right) \arrow[r] \arrow[d] & \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}\left((\mathcal{C},\Qoppa_{\mathcal{C}}),(\mathcal{D},\Qoppa'_{\mathcal{D}})\right) \arrow[d] \\ \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}(\mathcal{D},\mathcal{D}) \arrow[r] & \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}(\mathcal{C},\mathcal{D}) \end{tikzcd} . \tag{5.11}$$

   By 4, $F_!$ induces equivalences on the fibers of the vertical maps, hence (f,η) is locally θ-cocartesian. The locally θ-cocartesian maps are manifestly closed under composition, hence we are done. □

Corollary 5.12. *Let R be a Poincaré ring, and let A,B be $\mathbb{E}_1$-R-algebras with genuine involution. Then there is an equivalence* $\hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}}\left((\mathrm{Mod}_A^{\omega},\Qoppa_A),(\mathrm{Mod}_B^{\omega},\Qoppa_B)\right) \simeq (\mathrm{BiMod}_{A\otimes_R B^{\mathrm{op}}})_{A^{\varphi C_2}\otimes_R B^{\varphi C_2}/-}$.

Proof. □

L: todo– probably need to fix the statement with *duals* when the proof is written

As in the Picard group case, the forgetful functor θ induces a map of spectra $\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A)$ which will again factor through the 2-trosion on π_0. As a consequence of Proposition 2 we can identify the fiber of this map.

Corollary 5.13. *Let $(\mathrm{Mod}_A^{\omega},\Qoppa_A)$ be a Poincaré ring with underlying genuine C_2 spectrum A^L as in Proposition 2. Then the fiber of the map*

$$\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A)$$

can be naturally identified with $\mathrm{Pic}(\mathrm{Mod}_{A^L}(\mathrm{Sp}^{C_2}))$.

Proof. Since $\theta : \mathrm{Mod}_{(\mathrm{Mod}_A^{\omega},\Qoppa_A)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_{\mathrm{Mod}_A}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}})$ is (co)cartesian, symmetric monoidal, and conservative, it follows that the induced functor on the groupoid core of invertible objects is a Kan fibration by [Lur24, Proposition 01EZ]. Consequently we need only identify the fiber instead of the homotopy fiber, which follows from the identifiaction of the fiber in Proposition 2. □

N: Needs a lot more detail I know, but I think the skeleton is here.

Write Pn for the composite $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \xrightarrow{U} \mathrm{Cat}^{\mathrm{p}}_{\infty} \xrightarrow{\mathrm{Pn}} \mathcal{S}$.

Proposition 5.14. *Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Then $(\mathrm{Mod}_R^{\omega},\Qoppa_R)$ corepresents the functor* $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{S}$.

Proof. Recall that Proposition 5.4.1 furnishes an adjoint pair $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \rightleftarrows \mathrm{Cat}^{\mathrm{p}}_{\infty}$ of functors. Write $\overline{\mathcal{C}} = (\mathcal{C}, \Qoppa_{\mathcal{C}}) \in \mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}$. Then

$$\mathrm{Pn}(\mathcal{C}) = \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty}}\left((\mathrm{Sp}^f, \Qoppa^u), U(\overline{\mathcal{C}})\right) \simeq \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty R}}\left((\mathrm{Mod}^{\omega}_R, \Qoppa_R) \otimes (\mathrm{Sp}^f, \Qoppa^u), \overline{\mathcal{C}}\right),$$

where the first equivalence is [Cal+20a, Proposition 4.1.3]. □

5.1 Azumaya algebras with genuine involution

Let R be an $\mathbb{E}_\infty$-ring spectrum.

Recollection 5.15. Recall [BRS12; AG14] that an $\mathbb{E}_1$-R-algebra is said to be *Azumaya* if it is a compact generator of Mod_R and if the natural R-algebra map giving the bimodule structure on A

$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$$

is an equivalence of R-algebras.

Definition 5.16. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring spectrum. An *Azumaya algebra with genuine type 1 (anti-)involution* over R is the data of

(a) An $\mathbb{E}_1$-R-algebra A equipped with an anti-involution $\tau\colon A \to A^{\mathrm{op}}$ so that A is an Azumaya R-algebra in the sense of Recollection 5.15

(b) A left $A \otimes_R R^{\varphi C_2}$-module $A^{\varphi C_2}$

(c) An equivalence of $A \otimes_R A^{\mathrm{op}}$-modules

$$\hom_R(A, R^{\varphi C_2}) \simeq A^{\varphi C_2} \otimes_{R^{\varphi C_2}} \hom_R(A^{\varphi C_2}, R^{\varphi C_2})$$

(d) An A-linear map $A^{\varphi C_2} \to A^{tC_2}$ where A^{tC_2} is regarded as an A-module via the Tate-valued Frobenius.

Similarly, an *Azumaya algebra with genuine type 2 (anti-)involution* over R is obtained by replacing τ in item (a) with $\tau\colon A \to \sigma_R^* A^{\mathrm{op}}$, where $\sigma_R\colon R \xrightarrow{\sim} R$ is the given involution on R.

L: For context, compare Examples 3.1.9 and 3.2.9 of the first 9-author paper.

L: compatibility with Tate?

L: Tate-valued norm?

L: define the category!

Remark 5.17. If A is an Azumaya algebra with genuine involution over R, then in particular $M_A = A$, $N_A = A^{\varphi C_2}$ is a module with genuine involution over A in the sense of [Cal+20a, Definition 3.2.3].

With ordinary Azumaya algebras, the prototypical Azumaya algebra with anti-involution arises from endomorphism rings of perfect modules. Choosing a (nondegenerate symmetric bilinear) form on a perfect module P endows its endomorphism algebra with additional structure.

Example 5.18. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(P, q) \in \mathrm{Pn}\,(\mathrm{Mod}^{\omega}_R, \Qoppa_R)$.

Then $A := \mathrm{End}_R(P)$ admits a canonical lift to an Azumaya algebra with genuine involution over R with $A^{\varphi C_2} := \hom_R(P, R^{\varphi C_2})$.

By [Cal+20a, Proposition 3.1.16], A inherits a canonical anti-involution. Since P is compact, it is dualizable with respect to the symmetric monoidal structure on Mod^{ω}_R [Elm+07, Theorem III.7.9]. Since $\otimes_R R^{\varphi C_2}$ is symmetric monoidal, in particular it takes P to a dualizable object–call it $\overline{P}$. Now there is a canonical choice of equivalence (c) since both sides are canonically equivalent to $\overline{P} \otimes_{R^{\varphi C_2}} \overline{P}^{\vee}$.

L: to-do: explain! Also $A^{\varphi C_2}$ could also be $P \otimes_R R^{\varphi C_2}$?

Proposition 5.19. *Let R be the Eilenberg–Mac Lane spectrum associated to a discrete ring, and suppose R has a given C_2-action. Let A be a classical Azumaya algebra over R with an involution of type 2. Regard R as a Poincaré ring spectrum with the genuine symmetric Poincaré structure of Example 3.13.*

Then there is a canonical Azumaya algebra with genuine involution over R^{gs} so that $A^{\varphi C_2} := \tau_{\geq 0} A^{tC_2}$.

L: todo: Make some noises about an R-linear enhancement of Proposition 3.1.16?

Proof. □

Proposition 5.20. *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(A, A^{\varphi C_2} \to A^{tC_2})$ be an Azumaya algebra with genuine involution over R. Then $(\mathrm{Mod}^{\omega}_A, \Qoppa_A)$ is an invertible object in $\mathrm{Mod}_{(\mathrm{Mod}^{\omega}_R, \Qoppa_R)}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}\right)$.*

L: todo

Remark 5.21. Contrast Proposition 5.20 with [AG14, Theorem 3.15], where it is shown that an R-linear stable ∞-category is invertible *if and only if* it is equivalent to modules over an Azumaya R-algebra. The difference lies in the fact that not every R-linear (Morita anti-)equivalence $\mathrm{Mod}^\omega_A \simeq \mathrm{Mod}^\omega_{A^{\mathrm{op}}}$ is induced by a map of $\mathbb{E}_1$-rings $A \to A^{\mathrm{op}}$.

L: this is not quite the same (in a literal sense), but I think similar in spirit to the "counterexamples" paper by First–Williams

Proof. First, by [Cal+20a, Example 3.2.9], we see that $(\mathrm{Mod}^\omega_A, Ϙ_A)$ is indeed an R-linear Poincaré ∞-category (and not merely hermitian). To show that the associated Poincaré ∞-category is invertible, we must identify a dual $(\mathrm{Mod}^\omega_A, Ϙ_A)^\vee$ and exhibit an equivalence $(\mathrm{Mod}^\omega_A, Ϙ_A)\otimes(\mathrm{Mod}^\omega_A, Ϙ_A)^\vee \simeq (\mathrm{Mod}^\omega_R, Ϙ_R)$. Since $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R} \to \mathcal{C}at^{\mathrm{ex}}_{\infty R}$ is symmetric monoidal, we see that the underlying R-linear ∞-category associated to the dual must be $\mathrm{Mod}^\omega_{A^{\mathrm{op}}}$. Moreover, the canonical evaluation map $\mathrm{ev}\colon \mathrm{Mod}^\omega_A \otimes \mathrm{Mod}^\omega_{A^{\mathrm{op}}} \xrightarrow{\simeq} \mathrm{Mod}^\omega_R$ sends $A \otimes A^{\mathrm{op}}$ to A. Endow $\mathrm{Mod}_{A^{\mathrm{op}}}$ with a Poincaré structure corresponding to the module with genuine involution $M_{A^{\mathrm{op}}} := A^{\mathrm{op}}$, $N_{A^{\mathrm{op}}} := \hom_R(A^{\varphi C_2}, R^{\varphi C_2})$. It remains to exhibit a natural equivalence

$$\eta\colon\ (Ϙ_A \otimes Ϙ_{A^{\mathrm{op}}}) \xrightarrow{\simeq} \mathrm{ev}^* Ϙ_R \tag{5.22}$$

of [quadratic] functors $\mathrm{Mod}^\omega_A \otimes \mathrm{Mod}^\omega_{A^{\mathrm{op}}} \to \mathrm{Sp}$. By [Cal+20a, Theorem 3.3.1], it suffices to exhibit equivalences on the bilinear and linear parts of (5.22) which glue compatibly. □

L: finish

6 Poincaré schemes

Definition 6.1. Let APS be the $(\infty, 1)$-category defined by the pullback

$$\begin{tikzcd} \mathrm{APS} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \arrow[d, "d_1^*"] \\ \mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \end{tikzcd}$$

where $U : \mathrm{Sp}^{BC_2} \to \mathrm{Sp}$ is the functor which forgets the C_2-action.

N: I keep trying to make this work but the technical details are actively killing me. Better, I think, to use the following definition instead.

Definition 6.2. Define the catgeory of affine Hermetian schemes, denoted AHS, to be the infinity category given by the Grothendieck construction applied to the functor

$$\mathrm{CAlg}(\mathrm{Sp})^{op} \to \mathrm{Cat}_\infty$$

given by sending a ring R to the category $\mathrm{CAlg}(\mathrm{Mod}_{NR})$ of $\mathbb{E}_\infty$ algebras in modules with genuine involutions over R. Then define the category of affine Poincare schemes, denoted by APS, to be the full subcategory of AHS spanned by the pairs (R, M) where $M \in \mathrm{CAlg}(\mathrm{Mod}_{NR})$ is invertible.

We record here a few structural results about this category.

Theorem 6.3. *The following statements about* APS *hold:*

1. *The category* APS *is a cocomplete and symmetric monoidal infinite category;*
2. *the pullback diagram above is homotopy Cartesian;*
3. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ *is symmetric monoidal and (co)continuous;*
4. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}$ *is lax symmetric monoidal;*
5. *and the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2} \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathrm{Sp})$ *is symmetric monoidal.*

Proof. For (2) it is enough to show that d_1^* is an categorical fibration which follows from [Lur09, Corollary 2.3.2.5] and [Lur09, Corollary 2.4.6.5]. In fact $d_!^*$ is a left fibration by [(]taking $p = id$, $i = d_1 : \Delta^1 \to \Delta^2$) There is a (pseudo-)functor

L: Is this reference correct? The conclusion asserts that some map of simplicial sets is a *categorical* fibration. The follow-

$$\begin{aligned} F\colon \operatorname{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) &\to \mathcal{C}\mathrm{at}_\infty \\ (\varphi\colon A \to B) &\mapsto (\big(\mathrm{CAlg}(\mathrm{Sp})_{A/-/B}\big)_{/\varphi}) \end{aligned}$$

which sends a square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\psi"] & D \end{tikzcd} \tag{6.4}$$

regarded as a morphism from φ to ψ, to the functor

$$\begin{aligned} \big(\mathrm{CAlg}(\mathrm{Sp})_{A/-/B}\big)_{/\varphi} &\to \big(\mathrm{CAlg}(\mathrm{Sp})_{C/-/D}\big)_{/\psi} \\ (A \to R \to B) &\mapsto C \simeq A \otimes_A C \xrightarrow{\varphi \otimes \mathrm{id}_C} B \otimes_A C \to D \end{aligned} \tag{6.5}$$

where $B \otimes_A C \to D$ is the canonical map induced by the commuting square (6.4). The functor F classifies the cocartesian fibration d_1^*.

For (3), let $p : K \to \mathrm{APS}$ be a map of simplicial sets, K a small simplicial set. Suppose the $K^\triangleright \to \mathrm{APS}$ be an extension such that $K^\triangleright \to \mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ is a colimit diagram. By [Lur09, Proposition 2.4.3.2] the diagram

$$\begin{tikzcd} \mathrm{APS}_{p/} \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}_{p/-} \arrow[d] \\ \mathrm{CAlg}(\mathrm{Sp}^{BC_2})_{p/} \arrow[r] & \mathrm{CAlg}(\mathrm{Sp})^{\Delta^1}_{p/-} \end{tikzcd}$$

is again homotopy cartesian. Then

$$\begin{aligned} \hom_{\mathrm{APS}}(p(\infty), -) &\simeq \hom_{\mathrm{CAlg}(\mathrm{Sp}^{BC_2})}(p(\infty), -) \times_{\hom_{\mathrm{CAlg}(\mathrm{Sp})^{\Delta^1}}(p(\infty), -)} \hom_{\mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}}(p(\infty)) \\ &\simeq \end{aligned}$$

$\square$

We will denote elements of APS by $\underline{A} = (A, s : A^{\Phi C_2} \to A^{tC_2})$. Here $s : A^{\Phi C_2} \to A^{tC_2}$ is the image of $\underline{A}$ under the top horizontal map above. The use of the notation $A^{\Phi C_2}$ is justified by the following.

Lemma 6.6. *Let* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp})$ *be the composition of the functors*

$$\mathrm{APS} \to \operatorname{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathrm{Sp}).$$

Then this functor factors as a composition $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2}) \xrightarrow{(-)^{\Phi C_2}} \mathrm{CAlg}(\mathrm{Sp})$.

Proof. The commutativity of the diagram

$$\begin{tikzcd} & \operatorname{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \arrow[d, "d_1^*"] \arrow[dr, "d_0^*"] & \\ & & \operatorname{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \arrow[dd, "ev_{[1]}"] \\ \mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] \arrow[dr, "id"] & \operatorname{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \arrow[dr, "ev_{[1]}"] & \\ & \mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "(-)^{tc_2}"] & \mathrm{CAlg}(\mathrm{Sp}) \end{tikzcd}$$

induces a functor on the pullback infinity categories $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2})$ which makes the corresponding cube commute. The functor $ev_{[1]} : \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \to \mathrm{CAlg}(\mathrm{Sp})$ factors through d_0^* and so $\mathrm{APS} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \to \mathrm{CAlg}(\mathrm{Sp})$ is equivalent to the composition

$$\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2}) \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \to \mathrm{CAlg}(\mathrm{Sp})$$

and the composition of the last two maps is the geometric fixed point functor as desired. □

The following Lemma gives the justification of the name Poincaré scheme.

Construction 6.7. We shall construct a functor

$$\mathrm{Perf}^{\mathrm{Pn}} : \mathrm{APS} \to \mathrm{Cat}_{\infty}^{\mathrm{Pn}}$$

to the category of Poincaré infinity categories.

L: For symmetric monoidal structure–maybe want to swap out Mod_{NA} for CAlg_{NA}?

Recall that $\mathrm{Cat}^{\mathrm{h}}_{\infty\infty} \to (\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty\infty})^{op}$ is a cocartesian fibration [Cal+20a, §1.4.] We will first construct a map of cocartesian fibrations

$$\begin{tikzcd} \mathrm{APS} \arrow[r, dotted] \arrow[d] & \mathrm{Cat}^{\mathrm{h}}_{\infty\infty} \arrow[d] \\ \mathrm{CAlg}\left(\mathrm{Sp}^{BC_2}\right) \arrow[r] & (\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty\infty})^{op} \end{tikzcd} , \tag{6.8}$$

then show that the dotted arrow factors through the subcategory $\mathcal{C}\mathrm{at}^p_\infty \subseteq \mathcal{C}\mathrm{at}^h_\infty$. To construct a map of cartesian fibrations, it suffices to exhibit a natural transformation of classifying functors. Unraveling the definitions, by Theorem 3.2.13 of [Cal+20a] we must exhibit for each $A \in \mathrm{CAlg}(\mathrm{Sp})^{BC_2}$, a functor

$$\left(\mathrm{CAlg}(\mathrm{Sp})_{A/-/A^{tC_2}}\right)_{/\varphi} \to \mathrm{Mod}_{N^{C_2}(A^e)}\left(\mathrm{Sp}^{C_2}\right) \tag{6.9}$$

(where $\varphi\colon A \to A^{tC_2}$ is the Tate-valued Frobenius and N^{C_2} is the Hill–Hopkins–Ravenel norm) which is natural in A.

That the resulting functor factors through the subcategory $\mathcal{C}\mathrm{at}^p_\infty$ follows from Proposition 3.1.3 and Lemma 3.3.3 of *loc. cit.*

Lemma 6.10. *The functor of Construction 6.7 is symmetric monoidal and has essential image the subcategory spanned by objects* $(\mathrm{Perf}(R), Ϙ)$ *which are* $\mathbb{E}_\infty$*-algebras.*

Definition 6.11. A map $f : \underline{A} \to \underline{B} \in \mathrm{APS}$ is faithfully flat if the underlying map $f : A \to B$ is faithfully flat and the map $f^{\Phi C_2} : A^{\Phi C_2} \to B^{\Phi C_2}$ is also faithfully flat.

Lemma 6.12. *The fpqc covers on* APS *form a Grothendieck site.*

References

[AG14] Benjamin Antieau and David Gepner. "Brauer groups and étale cohomology in derived algebraic geometry". In: *Geom. Topol.* 18.2 (2014), pp. 1149–1244. ISSN: 1465-3060,1364-0380. DOI: 10.2140/gt.2014.18.1149. URL: https://doi.org/10.2140/gt.2014.18.1149.

[AGH21] Gabriel Angelini-Knoll, Teena Gerhardt, and Michael Hill. *Real topological Hochschild homology via the norm and Real Witt vectors.* 2021. arXiv: 2111.06970 [math.AT]. URL: https://arxiv.org/abs/2111.06970.

[Bar+22] Clark Barwick, Saul Glasman, Akhil Mathew, and Thomas Nikolaus. *K-theory and polynomial functors.* 2022. arXiv: 2102.00936 [math.KT]. URL: https://arxiv.org/abs/2102.00936.

[BRS12] Andrew Baker, Birgit Richter, and Markus Szymik. "Brauer groups for commutative S-algebras". In: *J. Pure Appl. Algebra* 216.11 (2012), pp. 2361–2376. ISSN: 0022-4049,1873-1376. DOI: 10.1016/j.jpaa.2012.03.001. URL: https://doi.org/10.1016/j.jpaa.2012.03.001.

[Cal+20a] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories I: Foundations.* 2020. DOI: 10.48550/ARXIV.2009.07223. URL: https://arxiv.org/abs/2009.07223.

[Cal+20b] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity.* 2020. DOI: 10.48550/ARXIV.2009.07224. URL: https://arxiv.org/abs/2009.07224.

[CHN24] Baptiste Calmès, Yonatan Harpaz, and Denis Nardin. *A motivic spectrum representing hermitian K-theory.* 2024. arXiv: 2402.15136 [math.KT]. URL: https://arxiv.org/abs/2402.15136.

[Elm+07] Anthony D Elmendorf, Igor Kriz, Michael A Mandell, and J Peter May. *Rings, modules, and algebras in stable homotopy theory.* 47. American Mathematical Soc., 2007.

[Fau03] H. Fausk. "Picard groups of derived categories". In: *J. Pure Appl. Algebra* 180.3 (2003), pp. 251–261. ISSN: 0022-4049,1873-1376. DOI: 10.1016/S0022-4049(02)00145-7. URL: https://doi.org/10.1016/S0022-4049(02)00145-7.

[FW20] Uriya A. First and Ben Williams. "Involutions of Azumaya Algebras". en. In: *DOCUMENTA MATHEMATICA* Vol 25 (2020 2020), p. 527–633. DOI: 10.25537/DM.2020V25.527-633. URL: https://www.elibm.org/article/10012038.

[Lur09] Jacob Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[Lur17] Jacob Lurie. *Higher algebra.* Manuscript available at https://www.math.ias.edu/~lurie/papers/HA.pdf. 2017.

[Lur18] Jacob Lurie. "Elliptic cohomology II: orientations". In: *available at https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf* (version dated 26 April 2018).

[Lur24] Jacob Lurie. *Kerodon.* https://kerodon.net. 2024.

[Nik23] Thomas Nikolaus. "Frobenius homomorphisms in higher algebra". In: *ICM—International Congress of Mathematicians. Vol. 4. Sections 5–8.* EMS Press, Berlin, [2023] ©2023, pp. 2826–2854. ISBN: 978-3-98547-062-4; 978-3-98547-562-9; 978-3-98547-058-7.

[PS92] R. Parimala and V. Srinivas. "Analogues of the Brauer group for algebras with involution". In: *Duke Math. J.* 66.2 (1992), pp. 207–237. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-92-06606-3. URL: https://doi.org/10.1215/S0012-7094-92-06606-3.

[Yan23] Lucy Yang. *On normed $\mathbb{E}_\infty$-rings in genuine equivariant C_p-spectra.* 2023. arXiv: 2308.16107 [math.AT].