Poincaré Schemes

Ben Antieau, Viktor Burghardt, Noah Riggenbach, Lucy Yang

Abstract

We do stuff

N: Change this

Contents

1 Introduction

Theorem 1.1. *Let $\underline{A}$ be an affine Poincaré scheme with underlying $\mathbb{E}_\infty$-ring spectrum with involution A. Then the natural maps*

$$\pi_i(\mathrm{PnPic}(\underline{A})) \to \pi_i(\mathrm{Pic}(A))$$

are surjective on 2-torsion.

N: I think there is some interaction with the homotopy fixed points, or maybe even the genuine fixed points

Theorem 1.2. *Let A be an $\mathbb{E}_\infty$ ring with involution, and let $\underline{NA}$ be the associated Tate affine Poincaré scheme. Let $\mathrm{Br}_\nu(A)$ be the Brauer group of Azumaya algebras over A with involution. Then the natural map*

$$\mathrm{PnBr}(\underline{NA}) \to \mathrm{Br}_\nu(A)$$

is an equivalence

N: I think we need to define this for ring spectra. For A discrete this is done in [FW20].

Theorem 1.3. *The functors* $\mathrm{PnPic}, \mathrm{PnBr} : \mathrm{APS} \to \mathrm{Sp}$ *are fppf sheaves.*

Theorem 1.4. *There is a Poincaré group scheme $\mathbb{G}_m^{♀}$ such that*

$$B\mathbb{G}_m^{♀} \simeq \mathrm{PnPic}$$

as fppf stacks.

1.1 Conventions

Br^{p}	Poincaré Brauer space
CAlg	∞-categoriy of $\mathbf{E}_\infty$-ring spectra
CAlg($\mathcal{S}$)	∞-categoriy of $\mathbf{E}_\infty$-spaces

N: probably of $\mathbb{E}_\infty$ dodads

$\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$	∞-categoriy of grouplike $\mathbf{E}_\infty$-spaces
$\mathrm{CAlg}^{\mathrm{p}}$	∞-categoriy of Poincaré ring spectra
$\mathrm{Cat}_\infty^{\mathrm{ex}}$	∞-category of small stable ∞-categories and exact functors
$\mathrm{Cat}_\infty^{\mathrm{p}}$	∞-category of Poincaré ∞-categories
$\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}$	∞-category of idempotent complete Poincaré ∞-categories
$\mathrm{Pic}^{\mathrm{p}}$	Poincaré Picard space
$\mathcal{S}$	∞-category of spaces
Sp	∞-category of spectra

2 Poincaré Structures on Compact Modules

We will use this section to recall notions and results about Poincaré ∞-categories which we require in the sections to follow. This section can safely be skipped by anyone who posses extensive knowledge of Poincaré ∞-categories, as found in [Cal+20a].

Notation 2.1. Let R be an $\mathbf{E}_\infty$-ring spectrum. We will drop $\mathbf{E}_\infty$ from our notation and simply call R a *ring spectrum*. Moreover, we will denote the ∞-category $\mathrm{CAlg}(\mathrm{Sp})$ of commutative algebra objects in the ∞-category of spectra Sp by CAlg.

Let R be a ring spectrum and let Mod_R be the ∞-category of modules over R. We will study Poincaré structures on the ∞-category Mod_R^ω of compact modules over R.

V: -characterizati in terms of modules with genuine involution, -characterizati of symmetric monoidal structures, -Pn

3 Poincaré Ring Spectra

In this section we will define the ring theoretic building blocks of Poincaré schemes and the corresponding category they live in. Affine Poincaré Schemes will then be the dual objects, similar to how affine schemes are dual to commutative rings.

Definition 3.1. Let R be a ring spectrum. A *Poincaré structure* on R is a symmetric monoidal Poincaré ∞-category $Ϙ : (\mathrm{Mod}_R^\omega)^{\mathrm{op}} \to \mathrm{Sp}$. We call such a symmetric monoidal Poincaré ∞-category a *Poincaré ring spectrum*. We will denote the full subcategory of $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{p}})$ spanned by Poincaré ring spectra by $\mathrm{CAlg}^{\mathrm{p}}$ and call it the *∞-category of Poincaré ring spectra*.

Remark 3.2. Poincaré ring spectra, as defined in Definition 3.1, were studied in . Note that we chose a different notation. In Poincaré ring spectra are being referred to as *$\mathbf{E}_\infty$-ring spectra with genuine involution*.

V: cite 9 authored paper

Remark 3.3. Let R be a ring spectrum. By there is a natural equivalence between symmetric monoidal Poincaré structures on Mod_R^ω and algebra objects over the genuine C_2-spectrum NR . In particular, a Poincaré structure on R can be identified with the following data:

V: cite 9 authored paper

- A C_2-action on R via maps of ring spectra, i.e. a functor $\lambda : BC_2 \to \mathrm{CAlg}$.
- An R-algebra $R \to C$.
- An R-algebra map $C \to R^{tC_2}$.

V: cite 9-authors

V: reference

Here R^{tC_2} is the Tate construction with respect to the above action. Since the Tate construction is lax symmetric monoidal, R^{tC_2} is naturally an R-algebra via the Tate-valued norm. A ring spectrum equipped with a Poincaré structure will be called a *Poincaré ring spectrum*.

Remark 3.4. By Remark 3.3, a Poincaré structure on a ring spectrum R with a C_2-action via maps of ring spectra is a factorization $R \to C \to R^{tC_2}$ in CAlg of the natural map $R \to R^{tC_2}$.

Remark 3.5. Let $\mathcal{M}$ be the full subcategory of $\mathrm{Cat}_\infty^{\mathrm{p}}$ spanned by Poincaré ∞-categories with underlying ∞-category Mod_R^ω for some ring spectrum R. Then the symmetric monoidal structure of $\mathrm{Cat}_{\infty\infty}^{\mathrm{p}}$ restricts to a symmetric monoidal structure on $\mathcal{M}$ by Example 3.9 and . Then we have $\mathrm{CAlg}^{\mathrm{p}} \simeq \mathrm{CAlg}(\mathcal{M})$. In particular, the symmetric monoidal structure of $\mathrm{CAlg}(Catp)$ restricts to a symmetric monoidal structure on $\mathrm{CAlg}^{\mathrm{p}}$.

V: cite 9-authors I.5.1.5 and I.5.1.6

Notation 3.6. Let R be a ring spectrum. We will denote by $\underline{R}$ the spectrum R with trivial action. More precisely, $\underline{R} : BC_2 \to \mathrm{Sp}$ is the constant functor.

L: This is commonly used for constant Mackey functors–could be ambiguous

Example 3.7. Let R be a ring spectrum. If $2 \in \pi_0(R)$ is invertible, we have $\underline{R}^{tC_2} \simeq 0$. A Poincaré structure on R with the trivial action is then given by an R-algebra $R \to C$.

V: explain/referen

Example 3.8. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The Tate-valued norm endows R^{tC_2} with a natural R-algebra structure, which induces a Poincaré structure on R given by the factorization $R \xrightarrow{\mathrm{id}} R \to R^{tC_2}$. We will call this Poincaré structure the *Tate Poincaré structure on* R.

Example 3.9. The sphere spectrum $\mathbb{S}$ together with the Tate Poincaré structure will be called the *universal Poincaré ring spectrum*.

V: expain why/translat universality statement to poincare ring spectra

Example 3.10. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The identity map $\mathrm{id} : R^{tC_2} \to R^{tC_2}$ induces a Poincaré structure on R given by the factorization $R \to R^{tC_2} \xrightarrow{id} R^{tC_2}$. We will call this Poincaré structure the *symmetric Poincaré structure on* R.

Example 3.11. Let R be a connective ring spectrum equipped with a C_2-action via maps of ring spectra. The connective cover $\tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$ of R^{tC_2} induces a Poincaré structure on R given by the factorization $R \to \tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$. We will call this Poincaré structure the *genuine symmetric Poincaré structure on* R.

V: copy more examples from notes

Definition 3.12. Let A and R be Poincaré ring spectra. A *map of Poincaré ring spectra* between A and R is a map of ring spectra $f : A \to R$ compatible with the corresponding Poincaré structures via the following additional data:

-

V: this should become a remark and go below the definition of calgp

4 Modules over Poincaré Ring Spectra

Let A be a Poincaré ring spectrum. Then A is a commutative algebra object in the ∞-category of Poincaré ∞-categories $\mathrm{Cat}^{\mathrm{p}}_\infty$. We may thus consider modules over it. In this section we will use modules over Poincaré ring spectra to define analogues of the Brauer and Picard groups for Poincaré ring spectra.

V: ref

4.1 The Poincaré Picard Group

Recall that the Poincaré space functor $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{CAlg}(\mathcal{S})$ is lax symmetric monoidal with respect to tensor product of Poincaré ∞-categories and smash product of $\mathbb{E}_\infty$-spaces [Cal+20a, Corollary 5.2.8]. In particular, we can consider invertible objects in $\mathrm{Pn}(A)$ for a Poincaré ring spectrum A.

Definition 4.1. Let A be a Poincaré ring spectrum. We define the *Picard space of* A to be

$$\mathrm{Pic}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Pn}(A)).$$

Remark 4.2. Let $(\mathrm{Mod}^\omega_R, \Qoppa_R)$ be a Poincaré ring spectrum, where $(M_R = R, N_R = R^{\varphi C_2}, R^{\varphi C_2} \to R^{tC_2})$ is the module with genuine involution associated to $\Qoppa_R$. Then a point in the Poincaré Picard space is the data of a pair $(\mathcal{L}, q)$, where $\mathcal{L}$ is an invertible module in Mod^ω_R and q is a point in $\Omega^\infty \Qoppa_R(\mathcal{L})$. By [Cal+20a, Proposition 1.3.11], the data of q is equivalent to the data of points in the lower left and upper right corner of the square

$$\begin{tikzcd}
\Qoppa(\mathcal{L}) \arrow[r] \arrow[d] & \hom_R(\mathcal{L}, R^{\varphi C_2}) \ni \ell(q) \arrow[d] \\
b(q) \in \hom_{R\otimes R}(\mathcal{L} \otimes \mathcal{L}, R)^{hC_2} \arrow[r] & \hom_R(\mathcal{L}, R^{tC_2})
\end{tikzcd} \tag{4.1}$$

and a path between their images in the lower right corner. In particular, the adjoint of $b(q)$ must define a nondegenerate hermitian form on $\mathcal{L}$, that is, an equivalence $\mathcal{L} \simeq \hom_R(\mathcal{L}, R^*)$ where R^* is considered as an R-module via the action of the generator of C_2.

Write $(\mathcal{L}^\vee, q^\vee)$ is for the inverse of $(\mathcal{L}, q)$. By definition of invertibility, there exists an R-linear map $\ell(q^\vee)\colon \mathcal{L}^\vee \to R^{\varphi C_2}$ so that the following diagram commutes

L: add equivariance/symmet data

$$\begin{tikzcd}
\mathcal{L} \otimes_R \mathcal{L}^\vee \arrow[r, "\ell(q)\otimes\ell(q^\vee)"] \arrow[d, "\sim"', "\mathrm{ev}"] & R^{\varphi C_2} \otimes_R R^{\varphi C_2} \arrow[d, "\text{multiplication}"] \\
R \arrow[r, "\text{given}"] & N_R
\end{tikzcd} \tag{4.2}$$

4.2 The Poincaré Brauer Group

Recall that a Poincaré ∞-category is called idempotent complete if the underlying stable ∞-category is idempotent complete. The full subcategory of $\mathrm{Cat}_\infty^{\mathrm{p}}$ spanned by idempotent complete Poincaré ∞-categories is denoted by $\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}$ [Cal+20b, Definition 1.3.2].

Definition 4.3. Let A be a Poincaré ring spectrum. We define the *Poincaré Brauer space of A* as

$$\mathrm{Br}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Mod}_A(\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}})).$$

The assignment $A \mapsto \mathrm{Br}^{\mathrm{p}}(A)$ defines a functor

$$\mathrm{Br}^{\mathrm{p}}\colon \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$$

valued in grouplike $\mathbf{E}_\infty$-spaces.

Remark 4.4. The symmetric monoidal forgetful functor $\mathrm{Mod}_A(\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}) \to \mathrm{Mod}_A(\mathrm{Cat}_\infty^{\mathrm{ex}})$ induces a map $\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A)$ of grouplike $\mathbf{E}_\infty$-spaces, where $\mathrm{Br}(A)$ is the Brauer space $\mathrm{br}_{\mathrm{alg}}(A)$ of [AG14, pp. 1154–1155].

Proposition 4.5. *Let A be a Poincaré ring spectrum. Then we have a canonical equivalence*

$$\Omega\,\mathrm{Br}^{\mathrm{p}}(A) \simeq \mathrm{Pic}^{\mathrm{p}}(A).$$

Proof. Since $\Omega\,\mathrm{Br}^{\mathrm{p}}(R)$ is given by the space of automorphisms of any object in $\mathrm{Br}^{\mathrm{p}}(R)$, it suffices to determine the space of autoequivalences of $(\mathrm{Mod}_R^\omega, Ϙ_R)$. An autoequivalence is the data of a pair (f, η) where $f\colon \mathrm{Mod}_R^\omega \xrightarrow{\simeq} \mathrm{Mod}_R^\omega$ is an exact R-linear autoequivalence and $\eta\colon Ϙ_R \xrightarrow{\sim} Ϙ_R \circ f^{\mathrm{op}}$ is a natural equivalence. Since $\mathrm{Cat}_{\infty R}^{\mathrm{p}} \to \mathrm{Cat}_{\infty R}^{\mathrm{ex}}$ is symmetric monoidal, f is of the form $- \otimes_R \mathcal{L}$ where $\mathcal{L}$ is an invertible R-module. Since taking bilinear and linear parts is functorial/by [Cal+20a, Proposition 1.3.11], η is equivalently the data of a pair of equivalences

L: What else do we need to do to show that we have an equivalence of *functors*?

V: todo

$$\begin{aligned}
b(\eta)\colon\ & \hom_{R\otimes R}((- \otimes \mathcal{L}) \otimes (- \otimes \mathcal{L}), R)^{hC_2} \simeq \hom_{R\otimes R}(- \otimes -, R)^{hC_2} \\
\ell(\eta)\colon\ & \hom_R(- \otimes \mathcal{L}, R^{\varphi C_2}) \simeq \hom_R(-, R^{\varphi C_2})
\end{aligned}$$

plus a path between their images in $\hom_R(\mathcal{L}, R^{tC_2})$. The transformation $b(\eta)$ is equivalent to the data of an R-bilinear equivalence $R \simeq \mathcal{L}^\vee \otimes \mathcal{L}^\vee$, and the transformation $\ell(\eta)$ is equivalent to the data of an $R^{\varphi C_2}$-linear equivalence $\ell(\eta)\colon R^{\varphi C_2} \otimes_R \mathcal{L}^\vee \xrightarrow{\sim} R^{\varphi C_2}$.

L: maybe one of these should be conjugate dual here?

Now consider the composites

$$R \otimes_R \mathcal{L}^\vee \xrightarrow{\text{unit}\otimes\mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L}^\vee \xrightarrow{\ell(\eta)} R^{\varphi C_2}$$

$$R \otimes_R \mathcal{L} \xrightarrow{\text{unit}\otimes\mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L} \xrightarrow{\ell(\eta)^{-1}\otimes\mathrm{id}_{\mathcal{L}}} R^{\varphi C_2}\,.$$

L: is the $R^{\varphi C_2}$-linearity of this $\simeq$ correct?

These correspond to the $\ell(q^\vee), \ell(q)$ of Remark **??**, respectively. In particular, the condition that $\ell(q^\vee), \ell(q)$ make the diagram (4.2) commute is equivalent to the condition that $\ell(\eta)$ is an equivalence by an adjunction argument. ∎

L: under construction– not sure what to say about the $(-)^{tC_2}$

Proposition 4.6. *Let* $(\mathrm{Mod}_R^\omega, \Omega_R)$ *be a Poincaré ring spectrum. Let* A *be an* $\mathbb{E}_1$*-*R*-algebra in spectra. Then the pullback*

$$\begin{tikzcd} & \mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Omega_R)}\left(\mathrm{Cat}_\infty^{\mathrm{h}}\right) \arrow[d] \\ \{\mathrm{Mod}_A^\omega\} \arrow[r] & \mathrm{Cat}_R \end{tikzcd} \tag{4.3}$$

is canonically equivalent to $\mathrm{Mod}_{N_R A \otimes_{N_R R} R^L}\left(\mathrm{Sp}^{C_2}\right)$ *where* R^L *is the* $\mathbb{E}_\infty$*-*$N_R R$*-algebra with* $(R^L)^e \simeq R$ *and* $(R^L)^{\varphi C_2} \simeq C$*.*

A $N_R A \otimes_{N_R R} R^L$*-module classifies a* $(\mathrm{Mod}_R^\omega, \Omega_R)$*-module in Poincaré* ∞*-categories if its underlying* R*-module is invertible in the sense of [Cal+20a, Definition 3.1.4].*

Proof. □

L: Todo–this should be similar to [Cal+20a, Example 5.4.13].

5 Poincaré schemes

Definition 5.1. Let APS be the $(\infty, 1)$-category defined by the pullback

$$\begin{tikzcd} \mathrm{APS} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \arrow[d, "d_1^*"] \\ \mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \end{tikzcd}$$

where $U : \mathrm{Sp}^{BC_2} \to \mathrm{Sp}$ is the functor which forgets the C_2-action.

N: I keep trying to make this work but the technical details are actively killing me. Better, I think, to use the following definition instead.

Definition 5.2. Define the catgeory of affine Hermetian schemes, denoted AHS, to be the infinity category given by the Grothendieck construction applied to the functor

$$\mathrm{CAlg}(\mathrm{Sp})^{op} \to \mathrm{Cat}_\infty$$

given by sending a ring R to the category $\mathrm{CAlg}(\mathrm{Mod}_{NR})$ of $\mathbb{E}_\infty$ algebras in modules with genuine involutions over R. Then define the category of affine Poincare schemes, denoted by APS, to be the full subcategory of AHS spanned by the pairs (R, M) where $M \in \mathrm{CAlg}(\mathrm{Mod}_{NR})$ is invertible.

We record here a few structural results about this category.

Theorem 5.3. *The following statements about* APS *hold:*

1. *The category* APS *is a cocomplete and symmetric monoidal infinite category;*
2. *the pullback diagram above is homotopy Cartesian;*
3. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ *is symmetric monoidal and (co)continuous;*
4. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}$ *is lax symmetric monoidal;*
5. *and the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2} \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathrm{Sp})$ *is symmetric monoidal.*

Proof. For (2) it is enough to show that d_1^* is an categorical fibration which follows from [Lur09, Corollary 2.3.2.5] and [Lur09, Corollary 2.4.6.5]. In fact $d_!^*$ is a left fibration by [(]taking $p = id$, $i = d_1 : \Delta^1 \to \Delta^2$) There is a (pseudo-)functor

L: Is this reference correct? The conclusion asserts that some map of simplicial sets is a *categorical* fibration. The following argument is

$$\begin{aligned} F\colon\ \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) &\to \mathrm{Cat}_\infty \\ (\varphi\colon A \to B) &\mapsto \left(\left(\mathrm{CAlg}(\mathrm{Sp})_{A/-/B}\right)_{/\varphi}\right) \end{aligned}$$

which sends a square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\psi"] & D \end{tikzcd} \tag{5.1}$$

regarded as a morphism from φ to ψ, to the functor

$$\begin{aligned} \big(\mathrm{CAlg}(\mathrm{Sp})_{A/-/B}\big)_{/\varphi} &\to \big(\mathrm{CAlg}(\mathrm{Sp})_{C/-/D}\big)_{/\psi} \\ (A \to R \to B) &\mapsto C \simeq A \otimes_A C \xrightarrow{\varphi\otimes \mathrm{id}_C} B \otimes_A C \to D \end{aligned} \tag{5.2}$$

where $B \otimes_A C \to D$ is the canonical map induced by the commuting square (5.1). The functor F classifies the cocartesian fibration d_1^*.

For (3), let $p : K \to \mathrm{APS}$ be a map of simplicial sets, K a small simplicial set. Suppose the $K^{\triangleright} \to \mathrm{APS}$ be an extension such that $K^{\triangleright} \to \mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ is a colimit diagram. By [Lur09, Proposition 2.4.3.2] the diagram

$$\begin{tikzcd} \mathrm{APS}_{p/} \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}_{p/-} \arrow[d] \\ \mathrm{CAlg}(\mathrm{Sp}^{BC_2})_{p/} \arrow[r] & \mathrm{CAlg}(\mathrm{Sp})^{\Delta^1}_{p/-} \end{tikzcd}$$

is again homotopy cartesian. Then

$$\begin{aligned} \hom_{\mathrm{APS}}(p(\infty), -) &\simeq \hom_{\mathrm{CAlg}(\mathrm{Sp}^{BC_2})}(p(\infty), -) \times_{\hom_{\mathrm{CAlg}(\mathrm{Sp})^{\Delta^1}}(p(\infty),-)} \hom_{\mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}}(p(\infty)) \\ &\simeq \end{aligned}$$

□

We will denote elements of APS by $\underline{A} = (A, s : A^{\Phi C_2} \to A^{tC_2})$. Here $s : A^{\Phi C_2} \to A^{tC_2}$ is the image of $\underline{A}$ under the top horizontal map above. The use of the notation $A^{\Phi C_2}$ is justified by the following.

Lemma 5.4. *Let* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp})$ *be the composition of the functors*

$$\mathrm{APS} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathrm{Sp}).$$

Then this functor factors as a composition $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2}) \xrightarrow{(-)^{\Phi C_2}} \mathrm{CAlg}(\mathrm{Sp})$.

Proof. The commutativity of the diagram

$$\begin{tikzcd} & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \arrow[dd, "d_1^*"] \arrow[dr, "d_0^*"] & \\ & & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \arrow[dd, "ev_{[1]}"] \\ \mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] \arrow[dr, "id"'] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \arrow[dr, "ev_{[1]}"] & \\ & \mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "(-)^{tc_2}"] & \mathrm{CAlg}(\mathrm{Sp}) \end{tikzcd}$$

induces a functor on the pullback infinity categories $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2})$ which makes the corresponding cube commute. The functor $ev_{[1]} : \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \to \mathrm{CAlg}(\mathrm{Sp})$ factors through d_0^* and so $\mathrm{APS} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \to \mathrm{CAlg}(\mathrm{Sp})$ is equivalent to the composition

$$\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2}) \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \to \mathrm{CAlg}(\mathrm{Sp})$$

and the composition of the last two maps is the geometric fixed point functor as desired. □

The following Lemma gives the justification of the name Poincaré scheme.

Construction 5.5. We shall construct a functor

$$\mathrm{Perf}^{\mathrm{Pn}} : \mathrm{APS} \to \mathrm{Cat}_\infty^{\mathrm{Pn}}$$

to the category of Poincaré infinity categories.

L: For symmetric monoidal structure–maybe want to swap out Mod_{NA} for CAlg_{NA}?

Recall that $\mathrm{Cat}^{\mathrm{h}}_{\infty\infty} \to (\mathrm{Cat}^{\mathrm{ex}}_{\infty\infty})^{op}$ is a cocartesian fibration [Cal+20a, §1.4.] We will first construct a map of cocartesian fibrations

$$\begin{tikzcd} \mathrm{APS} \arrow[r, dotted] \arrow[d] & \mathrm{Cat}^{\mathrm{h}}_{\infty\infty} \arrow[d] \\ \mathrm{CAlg}\left(\mathrm{Sp}^{BC_2}\right) \arrow[r] & (\mathrm{Cat}^{\mathrm{ex}}_{\infty\infty})^{op} \end{tikzcd}, \tag{5.3}$$

then show that the dotted arrow factors through the subcategory $\mathrm{Cat}^p_\infty \subseteq \mathrm{Cat}^h_\infty$. To construct a map of cartesian fibrations, it suffices to exhibit a natural transformation of classifying functors. Unraveling the definitions, by Theorem 3.2.13 of [Cal+20a] we must exhibit for each $A \in \mathrm{CAlg}(\mathrm{Sp})^{BC_2}$, a functor

$$\left(\mathrm{CAlg}(\mathrm{Sp})_{A/-/A^{tC_2}}\right)_{/\varphi} \to \mathrm{Mod}_{N^{C_2}(A^e)}\left(\mathrm{Sp}^{C_2}\right) \tag{5.4}$$

(where $\varphi \colon A \to A^{tC_2}$ is the Tate-valued Frobenius and N^{C_2} is the Hill–Hopkins–Ravenel norm) which is natural in A.

That the resulting functor factors through the subcategory Cat^p_∞ follows from Proposition 3.1.3 and Lemma 3.3.3 of *loc. cit.*

Lemma 5.6. *The functor of Construction 5.5 is symmetric monoidal and has essential image the subcategory spanned by objects* $(\mathrm{Perf}(R), ♀)$ *which are* $\mathbb{E}_\infty$*-algebras.*

Definition 5.7. A map $f : \underline{A} \to \underline{B} \in \mathrm{APS}$ is faithfully flat if the underlying map $f : A \to B$ is faithfully flat and the map $f^{\Phi C_2} : A^{\Phi C_2} \to B^{\Phi C_2}$ is also faithfully flat.

Lemma 5.8. *The fpqc covers on* APS *form a Grothendieck site.*

References

[AG14] Benjamin Antieau and David Gepner. "Brauer groups and étale cohomology in derived algebraic geometry". In: *Geom. Topol.* 18.2 (2014), pp. 1149–1244. ISSN: 1465-3060,1364-0380. DOI: 10.2140/gt.2014.18.1149. URL: https://doi.org/10.2140/gt.2014.18.1149.

[Cal+20a] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories I: Foundations.* 2020. DOI: 10.48550/ARXIV.2009.07223. URL: https://arxiv.org/abs/2009.07223.

[Cal+20b] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity.* 2020. DOI: 10.48550/ARXIV.2009.07224. URL: https://arxiv.org/abs/2009.07224.

[FW20] Uriya A. First and Ben Williams. "Involutions of Azumaya Algebras". en. In: *DOCUMENTA MATHEMATICA* Vol 25 (2020 2020), p. 527–633. DOI: 10.25537/DM.2020V25.527-633. URL: https://www.elibm.org/article/10012038.

[Lur09] Jacob Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.