Et cetera

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Abstract

Dumping ground for other stuff: Notes, one-off observations, stuff that we can collectively use when preparing talks, etc.

L: I make no promises re: organization but I will do my best to keep it reasonably readable

Contents

1 Talk prep

2 References

- Involutions of Azumaya algebras by First and Williams (2020 *Documenta*)
- Counterexamples in involutions of Azumaya algebras by First and Williams; much more readable than the 2020 Documenta paper

3 Questions and directions

Question 3.1 (Morita theory for $\mathrm{Cat}_\infty^{\mathrm{p}}$)**.** Let R be a Poincaré ring. Suppose given two R-algebras (suitably interpreted so their module categories are canonically endowed with R-linear Poincaré structures–perhaps $\mathbb{E}_\sigma$) A, B. Can we characterize

$$\hom_{\mathrm{Cat}_{\infty}^{\mathrm{p}}{}_R}\left(\left(\mathrm{Mod}_A^\omega, \Omega_A\right), \left(\mathrm{Mod}_B^\omega, \Omega_B\right)\right)$$

in terms of something bimodule-like?

Question 3.2. On page 2 of the *Counterexamples* paper, First and Williams write that " existence of an extraordinary involution means classificaiton of Azumaya algebras with involution…*cannot* be reduced to questions about projective modules and hermitian forms on them."

What if we replaced projective modules by perfect complexes?

Question 3.3. First–Williams show (see discussion in §4 of the *Counterexamples* paper) that coarse type classify many (most?) Azumaya algebras up to (étale-local) *isomorphism*.

What is a suitable derived version of "coarse type"?

Question 3.4 (asked by Andrew Nov 2, 2024)**.** C. Schlichtkrull shows in this paper that a map $BGL_1(R) \to K(R) \to THH(R) \to R$ in terms of the Hopf map η.

Is there a "Poincaré" version of this result?

4 Thoughts & observations

Question 4.1. When R has the Tate Poincaré structure and $(\mathrm{Mod}_A^\omega, M_A, N_A, N_A \to M_A^{tC_2})$ is invertible, then by invertibility have an equivalence $\hom_R(A, R) \simeq N_A \otimes_R N_{A^{\mathrm{op}}}$ of $A \otimes_R A^{\mathrm{op}}$-modules. Restricting the left-hand side along the unit map $R \to A$ gives a map $N_A \otimes_R N_{A^{\mathrm{op}}} \to \hom_R(R, R) \simeq R$. Is this a perfect ($R$-linear) pairing?

I *think* using that $R^{\varphi C_2} \simeq R$ and combining the linear and bilinear part conditions, we get something like

$$M_A \otimes_R M_{A^{\mathrm{op}}} \simeq (N_A \otimes_R N_{A^{\mathrm{op}}})^{\otimes_R 2} \qquad \text{as } A \otimes_R A^{\mathrm{op}}\text{-bimodules.}$$

Is this useful?

Brauer-Severi schemes We know there is a correspondence between Azumaya algebras A over X and Brauer-Severi schemes. What does a Poincaré structure on Mod_A^ω mean 'geometrically' for D^b_{coh} of the corresponding Brauer-Severi scheme? (Lucy: I didn't get very far here, but just typing up what I had)

- Mod_A^ω corresponds to α-twisted sheaves on X (see Proposition 3.2.2.1 of Max Lieblich's thesis)
- The bounded derived category of α-twisted sheaves on X includes as one 'piece' of a semiorthogonal decomposition on D^b_{coh} of the corresponding Brauer-Severi scheme (see Theorem 5.1 here)

5 Desperate Flailing

This section is a cronical of my thoughts about $\mathbb{G}_m^{\text{♀}}$.

Goal The goal is to build a Poincaré ring $\mathbb{G}_m^{\text{♀}} := (\mathrm{Mod}_R, \text{♀}_R)$ such that $B\mathbb{G}_m^{\text{♀}}(\underline{S}) = \mathrm{Pic}^{\mathrm{p}}(\underline{S})$ for any Poincaré ring $\underline{S}$.

Lemma 5.1. *Let $\underline{S}$ be a Poincaré ring. Then $\pi_0(\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}_S)}(S, u)) = \{s \in \pi_0(S)^\times | s = 1 \text{ in } \pi_0(S^{C_2})\}$.*

Proof. Since the functor $\mathrm{Pn}(\mathrm{Mod}_S) \to \mathrm{Mod}_S$ is conservative it follows that an element of $\pi_0(\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}_S)}(S, u))$ must have underlying map an element of $\pi_0\mathrm{Aut}(S) = \pi_0(S)^\times$. Then in order for $s \in \pi_0(S)^\times$ to induce a map $(S, u) \to (S, u)$, the induced map $s^* : S^{C_2} \to S^{C_2}$ must satisfy $s^*(u) = u$. The pullback is given by multiplication by s, so this requirement translates into s being the unit, as desired. ☐

The problem I thought existed maybe doesn't. Here is a candidate construction:

Construction 5.2. Define R to be the $\mathbb{E}_\infty$ ring given by $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z\}} \mathbb{S}$ where the map $\mathbb{S}\{z\} \to \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by the map $z \mapsto xy$, and the map $\mathbb{S}\{z\} \to \mathbb{S}$ is induced by $z \mapsto 1$. We can give R an $\mathbb{E}_\infty$ ring structure in Sp^{BC_2} by taking the trivial action on $\mathbb{S}\{z\}$ and $\mathbb{S}$, and taking the action induced by $x \mapsto y$ and $y \mapsto x$ on $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$. Thus in $\mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ the ring R corepresents the functor $S \mapsto \{s \in \pi_0(S)^\times | s\sigma(s) = 1\}$.

Now take $\underline{R}$ to be the Poincaré ring with underlying Borel C_2 structure as described in the previous paragraph and geometric fixed points $R^{\varphi C_2} = \mathbb{S}$ and the map $R^{\varphi C_2} \to R^{tC_2}$ given by the unit map. Endowing $R^{\varphi C_2}$ with the R-module structre given by $x, y \mapsto 1$, it remains to show that the unit map $R^{\varphi C_2} \to R^{tC_2}$ factors the Tate valued Frobenius $R \to R^{tC_2}$ in order to promote $\underline{R}$ to a Poincaré ring. By construction of R it is then enough to show that on π_0 the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0(R^{tC_2})$. This map sends both x and y to $xy \in \pi_0(R^{tC_2})$. These areequal to 1 in $\pi_0(R^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so R^{tC_2} is a modules over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of xy equal to 1.

Now consider another Poincaré ring $\underline{S}$. We then have that maps $\pi_0(\mathrm{Maps}(\underline{R}, \underline{S}))$ is the data of a unit $s \in \pi_0(S)^\times$, a path $s\sigma(s) \to 1$ in $\Omega^\infty S$, and paths $x, y \to 1$ in $\Omega^\infty S^{\varphi C_2}$. This then agrees with $\mathbb{G}_m^{\female}$ by the following lemma.

Lemma 5.3. *Let $S \in \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ and $s \in \pi_0(S)^\times$. Then $s\sigma(s) = 1$ in $\pi_0(S)$ if and only if $(s \otimes s)^*$ acts by 1 on $\pi_0(S^{hC_2}) = \pi_0(\mathrm{Hom}_{S\otimes S}(S \otimes S, S)^{hC_2})$.*

Proof. The 'only if' direction follows from the fact that the map $S^{hC_2} \to S$ is an S-bimodule map. Now suppose that $s\sigma(s) = 1$ in S. Then before taking homotopy fixed points the induced map $s^* = id$ because S is $\mathbb{E}_\infty$.[1] □

6 Modules with genuine involution

Remark 6.1 (Lucy)**.** I'm just going to put drafts of stuff pertaining to hermitian modules here. Eventually when it gets to be more complete, I will hopefully move this entire section over to the main file.

L: or whatever we want to keep calling these

Meta-commentary There are (at least) three things we want to do:

(a) Define a category of 'bimodules with involution over algebras with anti-involution' equipped with a forgetful functor $\Theta\colon \mathrm{BMod}_{\mathrm{inv}}(-) \to \mathbb{E}_1\,\mathrm{Alg}(-)^{hC_2}$.

(b) Show that Θ is a coCartesian fibration. For this, it suffices to show that it is a *Cartesian* fibration and that it satisfies the hypotheses of [Lur09, Corollary 5.2.2.5]

- I used to think that we could obtain this by 'bootstrapping' a result from Higher Algebra, plus some facts about assembly. This doesn't seem to be working, so I'm just going to try to do this directly (imitating certain aspects of Chapter 4 of higher algebra.)

(c) Define a relative tensor product for hermitian bimodules

(d) Show that the formula for the cocartesian pushforward along a map $A \to B$ in $\mathbb{E}_1\,\mathrm{Alg}(-)^{hC_2}$ is something like $- \otimes_{A\otimes A^{\mathrm{op}}} (B \otimes B^{\mathrm{op}}) \otimes_{B\otimes B^{\mathrm{op}}} B$.

- In Higher Algebra, the formula for the cocartesian pushforward is proven in [Lur17, §4.6]; in particular, this is in the section on duality. In particular, see Proposition 4.6.2.17 and the paragraph immediately preceding this.
- I don't know how to do this yet–while (a) and (b) are not useful if I can't show (c), I can't suss out the feasibility of (c) without (a) and (b) already in place.

(e) Towards an adjunction between $\mathbb{E}_\sigma$-algebras and categories with additional structure.

- Involutive version of statement that, for a monoidal ∞-category $\mathcal{C}$ and an $\mathbb{E}_1$-algebra A, $\mathrm{LMod}_A(\mathcal{C})$ is right-tensored over $\mathcal{C}$?
- Involutive version of endomorphism categories? [Lur17, §4.7.1]

6.1 Step (a)

Definition 6.2. Define a colored operad Assoc_σ as follows:

(i) The colored operad has a single object, which we denote by $\mathfrak{a}$.

(ii) For every finite set I, the set of operations $\mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$, where $\mathcal{L}I$ is the set of linear orderings on I and an element of $\{\pm1\}^I$ is a function $I \to \{\pm 1\}$.

(iii) Suppose given a map of finite sets $\alpha\colon I \to J$, together with operations $(\preceq_j, f_j\colon I_j \to \{\pm1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{\alpha(i)=j}, \mathfrak{a}\right)$ and $(\preceq_J, g\colon J \to \{\pm1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_j\}_{j\in J}, \mathfrak{a}\right)$. Define a linear ordering on the set I as follows: $i \le i'$ if $\alpha(i) \preceq_J \alpha(i')$ or $\alpha(i) = \alpha(i') = j$ and $i \preceq_j i'$ and $g(j) = +1$ or $\alpha(i) = \alpha(i') = j$ and $i \succeq_j i'$ and $g(j) = -1$. Finally, define a function

$$\begin{aligned} I &\to \{\pm1\} \\ i &\mapsto f_{\alpha(i)}(i) \cdot g(\alpha(i)), \end{aligned}$$

where the multiplication on $\{\pm1\}$ is the usual one.

L: This is just an imitation of [Lur17, Definition 4.1.1.1], modified in accordance with ideas from §5.4.2.

Remark 6.3. There is a map of colored operads $\iota\colon \mathrm{Assoc} \to \mathrm{Assoc}_\sigma$ which is the identity on objects and on operations $\mathrm{Mul}_{\mathrm{Assoc}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$ where c_1 is the constant function on I with value 1.

There is another map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{Assoc} \to \mathrm{Assoc}_\sigma$ which is the identity on objects and on operations $\mathrm{Mul}_{\mathrm{Assoc}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm1\}^I$ sends a linear ordering ℓ to $(\ell^{\mathrm{rev}}, c_{-1})$ where c_{-1} is the constant function on I with value 1.

Definition 6.4. Let $\mathrm{Assoc}_\sigma^\otimes$ denote the associated ∞-operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.5. Unwinding definitions

- Objects $\mathrm{Assoc}_\sigma^\otimes$ are finite pointed sets $\langle n\rangle \in \mathrm{Fin}_*$
- Morphisms $\langle m\rangle \to \langle n\rangle$ consist of
  - $\alpha\colon \langle m\rangle \to \langle n\rangle$ a map of finite pointed sets
  - for each $i \in \langle n\rangle^\circ$, a linear ordering $\preceq_i$ on the inverse image $\alpha^{-1}(\{i\})$
  - a map of sets $s\colon \alpha^{-1}(\langle m\rangle^\circ) \to \{\pm1\}$
- For each pair of morphisms

$$(\beta\colon \langle \ell\rangle \to \langle m\rangle, \preceq_j, s) \qquad (\alpha\colon \langle m\rangle \to \langle n\rangle, \preceq_i, t),$$

  the composite is the triple $\left(\alpha\circ\beta, \preceq_j'', u\right)$ where $\preceq_j''$ is the ordering on $(\alpha\circ\beta)^{-1}(\{i\})$ so that if $a, b \in \langle \ell\rangle$ so that $\alpha(\beta(a)) = \alpha(\beta(b))$, then $a \preceq_j'' b$ if $\beta(a) \preceq_i \beta(b)$ or $\beta(a) =_i \beta(b) = i$ and $a \preceq_i b$ if $s(i) = 1$ or $a \succeq_i b$ if $s(i) = -1$. Finally $u(l) = s(l)\cdot t(\beta(l))$.

L: Note that when s, t are identically one, the resulting order $\preceq_j''$ agrees with the lexicographic order defined in [Lur17, Remark 4.1.1.4].

Remark 6.6. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.3 induce maps of ∞-operads $\mathrm{Assoc}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$. There is a canonical identification $\iota^{\mathrm{rev}} = \sigma\circ\iota$, where σ is the automorphism of the associative operad considered in [Lur17, Remark 4.1.1.7].

Note that each object $\langle n\rangle \in \mathrm{Assoc}_\sigma^\otimes$ has a distinguished automorphism $\mathrm{rev}_{\langle n\rangle}$ of order two given by the identity map on $\langle n\rangle$ and the constant map $c_{-1}\colon \langle n\rangle^\circ \to \{\pm1\}$ at -1. There is a canonical natural equivalence $\iota \overset{\sim}{\Longrightarrow} \iota^{\mathrm{rev}}$ whose component at $\langle n\rangle$ is $\mathrm{rev}_{\langle n\rangle}$.

Definition 6.7. Let $\mathcal{C}^\otimes$ be a ∞-operad equipped with the data of a fibration $p\colon \mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$. Let $\mathrm{Alg}^\sigma(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{/\mathrm{Assoc}_\sigma}(\mathcal{C})$ of ∞-operad sections of p. We will refer to $\mathrm{Alg}^\sigma(\mathcal{C})$ as the ∞-category of *involutive algebra objects of* $\mathcal{C}$.

L: do we need weaker than cocartesian fibration?

An *involutive monoidal ∞-category* is the data of a cocartesian fibration $\mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$.

[1]Or just $\mathbb{E}_2$.

Remark 6.8. Suppose given a cocartesian fibration $f\colon \mathcal{D}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ of ∞-operads. Write $\mathcal{C}^\otimes := \mathcal{D}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes, \iota} \mathrm{Assoc}^\otimes$; $\mathcal{C}^\otimes$ is a monoidal ∞-category in the sense of [Lur17, Definition 4.1.1.10]. Furthermore, $\mathcal{C}_{\mathrm{rev}}^\otimes := \mathcal{D}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes, \iota^{\mathrm{rev}}} \mathrm{Assoc}^\otimes$ is a monoidal ∞-category. By Remark 6.6, this notation is consistent with that of [Lur17, Remark 4.1.1.7]. In particular, a Assoc_σ-monoidal ∞-category $\mathcal{D}^\otimes$ determines a monoidal ∞-category $\mathcal{C}^\otimes$ equipped with a monoidal equivalence $\sigma_{\mathcal{C}}\colon \mathcal{C}^\otimes \xrightarrow{\sim} \mathcal{C}_{\mathrm{rev}}^\otimes$. Pullback along the involution of $\mathrm{Assoc}^\otimes$ determines another monoidal equivalence $\sigma_{\mathcal{C}}^{\mathrm{rev}}\colon \mathcal{C}_{\mathrm{rev}}^\otimes \xrightarrow{\sim} \mathcal{C}^\otimes$, and our assumptions imply that $\sigma_{\mathcal{C}}^{\mathrm{rev}} \circ \sigma_{\mathcal{C}}$ is equivalent to the identity on $\mathcal{C}^\otimes$.

Now suppose that A is an involutive algebra object of $\mathcal{D}$. With the same notation as before, pullback along ι (resp. ι^{rev}) determines associative algebra objects $u(A)$, $u^{\mathrm{rev}}(A)$ of $\mathcal{C}$ and $\mathcal{C}_{\mathrm{rev}}$, respectively. Note that $\sigma_{\mathcal{C}}(u(A))$ is an algebra object of $\mathcal{C}_{\mathrm{rev}}$, which we may regard as an algebra object of $\mathcal{C}$ by precomposing with the autoequivalence $\sigma\colon \mathrm{Assoc}^\otimes \xrightarrow{\sim} \mathrm{Assoc}^\otimes$. It follows from Remark 6.6 that A determines an equivalence $\sigma_A\colon u(A) \xrightarrow{\sim} \sigma_{\mathcal{C}}(u(A))^{\mathrm{rev}}$ of algebra objects in $\mathcal{C}$.

Now suppose furthermore that $\mathcal{D}^\otimes$ is of the form $\mathcal{E}^\otimes \times_{\mathrm{Fin}_*} \mathrm{Assoc}_\sigma^\otimes$ for some symmetric monoidal ∞-category $\mathcal{E}$. Then the associated involution $\sigma_{\mathcal{C}}$ is the identity, and for any involutive algebra object A of $\mathcal{D}$, σ_A is an equivalence $u(A) \simeq u(A)^{\mathrm{rev}}$ satisfying $\sigma_A^{\mathrm{rev}} \circ \sigma_A \simeq \mathrm{id}_A$.

Definition 6.9. Define a category Δ_σ

- objects are pairs $([n], s\colon \{1, \ldots, n\} \to \{\pm 1\})$
- a morphism from $([n], s\colon \{1, \ldots, n\} \to \{\pm 1\})$ to $([m], t\colon \{0, 1, \ldots, m\} \to \{\pm 1\})$ is an order-preserving map $[n] \to [m]$ in Δ.

L: maybe better to write s as a function defined on the set of *morphisms* $i < i+1$ in $[n]$.

Construction 6.10. Define a functor $\mathrm{Cut}\colon \Delta_\sigma^{\mathrm{op}} \to \mathrm{Assoc}_\sigma^\otimes$:

- For each $([n], s)$, we have $\mathrm{Cut}([n], s) = \langle n \rangle$.
- Given a morphism $\alpha\colon ([n], s) \to ([m], t)$, the associated morphism $\mathrm{Cut}([n], s) \to \mathrm{Cut}([m], t)$ consists of
  - On underlying finite pointed sets $\langle m \rangle \to \langle n \rangle$, Cut agrees with that appearing in [Lur17, Construction 4.1.2.9]
  - Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j - 1 < j$, we may regard $s\colon \langle n \rangle^\circ \to \{\pm 1\}$ and likewise $t\colon \langle m \rangle^\circ \to \{\pm 1\}$. Define $u\colon \mathrm{Cut}(\alpha)^{-1}(\langle n \rangle^\circ) \to \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\mathrm{Cut}(\alpha)(j))$.

Lemma 6.11. *The functor* $\mathrm{Cut}\colon \Delta_\sigma^{\mathrm{op}} \to \mathrm{Assoc}_\sigma^\otimes$ *exhibits* $\Delta_\sigma^{\mathrm{op}}$ *as an approximation to the* ∞*-operad* $\mathrm{Assoc}_\sigma^\otimes$.

L: I think the proof of this lemma is not too different from the proof of Proposition 4.1.2.11 of [Lur17]; the point here is just to unravel the definitions of locally coCartesian and Cartesian; the morphisms in $\Delta_\sigma^{\mathrm{op}}$ *are a little more complicated than* Δ^{op}*, but not by much.*

Notation 6.12. Let $\mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ exhibit $\mathcal{C}$ as $\mathbb{E}_\sigma$-monoidal. Let $\mathcal{C}^\circledast$ denote the fiber product $\mathcal{C}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes} \Delta_\sigma^{\mathrm{op}}$.

Definition 6.13. Say that a morphism $([n], s) \to ([m], t)$ is *inert* if the induced map $\mathrm{Cut}([m], t) \to \mathrm{Cut}([n], s)$ is an inert morphism in $\mathrm{Assoc}_\sigma^\otimes$.

Definition 6.14. A $\mathbb{R}^\sigma$*-planar operad* is an ∞-category $\mathcal{O}^\circledast$ equipped with a functor $q\colon \mathcal{O}^\circledast \to \Delta_\sigma^{\mathrm{op}}$ so that

1. For every object $X \in \mathcal{O}^\circledast$ and every inert morphism $\alpha\colon ([n], s) \to q(X)$ in Δ_σ, there is a q-cocartesian morphism $\overline{\alpha}\colon X \to Y$ satisfying $q(\overline{\alpha}) = \alpha$
2. Let X be an object satisfying $q(X) = ([n], s)$, and choose q-cocartesian morphisms $\overline{\alpha}_i\colon X \to X_i$ corresponding to the morphism $([i-1 < i], s_i) \to ([n], s)$ which is the inclusion on underlying sets and satisfies $s_i(i) = s(i)$. Then the morphisms $\overline{\alpha}_i$ exhibit X as the q-product of the X_i.
3. For each $n \geq 0$, the construction $C \mapsto \{C_i\}_{1 \leq i \leq n}$ induces an equivalence of ∞-categories

$$\mathcal{O}^\circledast \times_{\Delta_\sigma^{\mathrm{op}}} \{([n], s)\} \xrightarrow{\sim} \left(\mathcal{O}^\circledast \times_{\Delta_\sigma^{\mathrm{op}}} \{([1], s|_{\{i\}})\}\right)^{\times n}$$

We say that a morphism α in $\mathbb{R}^\sigma$-planar operad is *inert* if it is q-cocartesian and $q(\alpha)$ is inert in $\Delta_\sigma^{\mathrm{op}}$ in the sense of Definition 6.13.

Definition 6.15. Let $q\colon \mathcal{O}^\circledast \to \Delta_\sigma^{\mathrm{op}}$ be a $\mathbb{R}^\sigma$-planar operad. An $\mathbb{A}_\infty^\sigma$-algebra object of $\mathcal{O}^\circledast$ is a section of q which carries inert morphisms to inert morphisms. Write $\mathrm{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O})$ for the full subcategory of $\mathrm{Fun}_{\Delta_\sigma^{\mathrm{op}}}(\Delta_\sigma^{\mathrm{op}}, \mathcal{O}^\circledast)$ on $\mathbb{A}_\infty^\sigma$-algebra objects.

Proposition 6.16. *Let $\mathcal{O}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ be a fibration of ∞-operads. Then precomposition with the functor* Cut *of Construction 6.10 induces an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathrm{Assoc}_\sigma}(\mathcal{O}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O}) \ .$$

Proof. Combine Lemma 6.11 with [Lur17, Theorem 2.3.3.23]. □

Definition 6.17. Define a colored operad $\mathbf{LM}_{\mathrm{inv}}$

(i) The set of objects of $\mathbf{LM}_{\mathrm{inv}}$ has two elements, which we denote by $\mathfrak{a}, \mathfrak{m}$.

(ii) Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{LM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{LM}_{\mathrm{inv}}$. If $Y = \mathfrak{a}$, then $\mathrm{Mul}_{\mathbf{LM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \to \{\pm 1\}$ if $X_i = \mathfrak{a}$ for all i, and empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{LM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying either

- $X_{i_n} = \mathfrak{m}$ and $c(i_n) = 1$ and $X_j = \mathfrak{a}$ otherwise
- $X_{i_1} = \mathfrak{m}$ and $c(i_n) = -1$ and $X_j = \mathfrak{a}$ otherwise

(iii) The composition law on $\mathbf{LM}_{\mathrm{inv}}$ is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.2

Remark 6.18. There is a colored operad $\mathbf{RM}_{\mathrm{inv}}$ defined exactly in the same way as $\mathbf{LM}_{\mathrm{inv}}$ in Definition 6.17. In the interest of precision: $\mathbf{RM}_{\mathrm{inv}}$ has the same objects $\mathfrak{a}, \mathfrak{m}$. Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{RM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{RM}_{\mathrm{inv}}$. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{RM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying either

- $X_{i_n} = \mathfrak{m}$ and $c(i_n) = -1$ and $X_j = \mathfrak{a}$ otherwise
- $X_{i_1} = \mathfrak{m}$ and $c(i_n) = 1$ and $X_j = \mathfrak{a}$ otherwise

Remark 6.19. Restricting to the objects which are both called $\mathfrak{a}$, we see that both $\mathbf{LM}_{\mathrm{inv}}$ and $\mathbf{RM}_{\mathrm{inv}}$ have a sub-colored operad which is canonically identified with $\mathbf{Assoc}_{\mathrm{inv}}$ of Definition 6.2.

Remark 6.20. There is a map of colored operads $\iota\colon \mathrm{LM} \to \mathrm{LM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}$ to $\mathfrak{a}$. On $\mathrm{Mul}_{\mathrm{LM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{LM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.3. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{id}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is a map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{RM} \to \mathrm{LM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}$ to $\mathfrak{a}$. On $\mathrm{Mul}_{\mathrm{RM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{LM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{rev}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.3. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{rev}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

Definition 6.21. Define a colored operad $\mathbf{BM}_{\mathrm{inv}}$

(i) The set of objects of $\mathbf{BM}_{\mathrm{inv}}$ has three elements, which we denote by $\mathfrak{a}_\ell, \mathfrak{a}_r, \mathfrak{m}$.

(ii) Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{BM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{BM}_{\mathrm{inv}}$. If $Y = \mathfrak{a}_\ell$ (resp. $Y = \mathfrak{a}_r$), then $\mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \to \{\pm 1\}$ if $X_i = \mathfrak{a}_\ell$ (resp. $X_i = \mathfrak{a}_r$) for all i, and empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the subset of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying: if there is exactly one index i_k so that $X_{i_k} = \mathfrak{m}$, either

- $c(i_k) = 1$, $X_j = \mathfrak{a}_\ell$ for $j < i_k$ and $X_j = \mathfrak{a}_r$ for $j > i_k$; or
- $c(i_k) = -1$, $X_j = \mathfrak{a}_\ell$ for $j > i_k$ and $X_j = \mathfrak{a}_r$ for $j < i_k$

(iii) The composition law on $\mathbf{BM}_{\mathrm{inv}}$ is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.2

Remark 6.22. The colored operad $\mathbf{BM}_{\mathrm{inv}}$ has a canonical involution σ which fixes $\mathfrak{m}$, exchanges $\mathfrak{a}_\ell$ and $\mathfrak{a}_r$, and sends a morphism (λ, c) to $(\lambda^{\mathrm{rev}}, I \xrightarrow{c} \{\pm 1\} \xrightarrow{\cdot(-1)} \{\pm 1\})$.

Remark 6.23. There is a map of colored operads $\iota\colon \mathrm{BM} \to \mathrm{BM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}_-$ to $\mathfrak{a}_\ell$ and $\mathfrak{a}_+$ to $\mathfrak{a}_r$. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}_\pm\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.3. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{id}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is *also* a map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{BM} \to \mathrm{BM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and and sends $\mathfrak{a}_-$ to $\mathfrak{a}_r$ and $\mathfrak{a}_+$ to $\mathfrak{a}_\ell$. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}_\pm\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.3. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{rev}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_{-1}\}$ where c_{-1} is the constant function on $I \sqcup \{j\}$ with value -1.

Definition 6.24. Let $\mathcal{LM}^{\otimes}_{\mathrm{inv}}$, $\mathcal{RM}^{\otimes}_{\mathrm{inv}}$, and $\mathcal{BM}^{\otimes}_{\mathrm{inv}}$ denote the associated ∞-operads (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.25. We can describe the category $\mathcal{LM}^{\otimes}_{\mathrm{inv}}$ as follows:

(1) An object of $\mathcal{LM}^{\otimes}_{\mathrm{inv}}$ is a pair $(\langle n \rangle, S)$ where S is a subset of $\langle n \rangle^\circ$.

(2) Morphisms $(\langle m \rangle, T) \to (\langle n \rangle, S)$ consist of a map $(\alpha\colon \langle m \rangle \to \langle n \rangle, \lambda\colon \langle m \rangle^\circ \to \{\pm 1\})$ in $\mathrm{Assoc}^{\otimes}_\sigma$ satisfying:

  - The map α takes $T \cup \{*\}$ to $S \cup \{*\}$
  - For each $s \in S$, then $\alpha^{-1}(\{s\})$ contains exactly one element t_s of T, and it is maximal (resp. minimal) with respect to the linear ordering on $\alpha^{-1}(\{s\})$ if $\lambda(t_s) = 1$ (resp. $\lambda(t_s) = -1$).

Remark 6.26. We can describe the category $\mathcal{BM}^{\otimes}_{\mathrm{inv}}$ as follows:

(1) An object of $\mathcal{BM}^{\otimes}_{\mathrm{inv}}$ is a triple $(\langle n \rangle, c_+, c_-)$ where $c_\pm$ are functions $\langle n \rangle^\circ \to \{0, 1\}$ and $c_-(i) \le c_+(i)$ for all $i \in \langle n \rangle^\circ$.

(2) Morphisms $(\langle m \rangle, c_+, c_-) \to (\langle n \rangle, c'_+, c'_-)$ consist of a map $(\alpha\colon \langle m \rangle \to \langle n \rangle, \lambda\colon \langle m \rangle^\circ \to \{\pm 1\})$ in $\mathrm{Assoc}^{\otimes}_\sigma$ satisfying: if $j \in \langle n \rangle^\circ$ and $\alpha^{-1}(j) = \{i_1 < i_2 < \cdots < i_\ell\}$,

  - If $c_-(j) = c_+(j)$, then
$$c'_-(j) = c_-(i_1) \le c_+(i_1) = c_-(i_2) \le c_+(i_2) \cdot \cdots \cdot c_-(i_{m-1}) \le c_+(i_m) = c'_+(j)$$
  - If $c_-(j) < c_+(j)$, then there exists a unique k so that $c_-(i_k) < c_+(i_k)$ and
$$\begin{aligned}\lambda(i_k) \cdot c'_-(j) = \lambda(i_k) \cdot c_-(i_1) \le \lambda(i_k) \cdot c_+(i_1) = \lambda(i_k) \cdot c_-(i_2) \le \lambda(i_k) \cdot c_+(i_2) \cdots \\ \lambda(i_k) \cdot c_-(i_{m-1}) \le \lambda(i_k) \cdot c_+(i_m) = \lambda(i_k) \cdot c'_+(j)\end{aligned}$$

Remark 6.27. Each morphism $\varphi \in \mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ determines a linear ordering ℓ on the set I and a function $s\colon I \to \{\pm 1\}$. Passing from φ to the pair (ℓ, s) determines a map of colored operads $j\colon \mathbf{BM}_{\mathrm{inv}} \to \mathbf{Assoc}_{\mathrm{inv}}$. The map j induces a morphism of ∞-operads $\mathcal{BM}^{\otimes}_{\mathrm{inv}} \to \mathrm{Assoc}^{\otimes}_\sigma$ which we will also denote by j. For any $\mathbb{E}_\sigma$-monoidal ∞-category $\mathcal{C}$, restriction along j sends an $\mathbb{E}_\sigma$-algebra $A\colon \mathrm{Assoc}_\sigma \to \mathcal{C}^{\otimes}$ to the pair (A, A) where A is regarded as an involutive bimodule over itself.

L: hermitian

Remark 6.28. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.20 induce maps of ∞-operads $\iota\colon \mathcal{LM}^{\otimes} \to \mathcal{LM}^{\otimes}_{\mathrm{inv}}$ and $\iota^{\mathrm{rev}}\colon \mathcal{RM}^{\otimes} \to \mathcal{LM}^{\otimes}_{\mathrm{inv}}$.

Remark 6.29. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.23 induce maps of ∞-operads $\iota, \iota^{\mathrm{rev}} \colon \mathcal{BM}^\otimes \to \mathrm{BM}^\otimes_\sigma$. There are canonical identifications $\iota \circ \mathrm{rev} \simeq \sigma \circ \iota^{\mathrm{rev}}$ where σ is the involution on $\mathrm{BM}^\otimes_\sigma$ induced by Remark 6.22 and rev is the involution on $\mathcal{BM}^\otimes$ of [Lur17, Construction 4.6.3.1].

Remark 6.30. There are canonical maps of operads $\mathcal{LM}^\otimes_{\mathrm{inv}} \to \mathcal{BM}^\otimes_{\mathrm{inv}}$ and $\mathcal{RM}^\otimes_{\mathrm{inv}} \to \mathcal{BM}^\otimes_{\mathrm{inv}}$ sending $\mathfrak{a}$ to $\mathfrak{a}_\ell$, resp. $\mathfrak{a}_r$ and making the diagram

$$\begin{tikzcd} \mathrm{Assoc}^\otimes \arrow[r] \arrow[d, "\sigma"] & \mathcal{LM}^\otimes_{\mathrm{inv}} \arrow[d, "\mathrm{rev}"] \arrow[dr] & \\ & & \mathcal{BM}^\otimes_{\mathrm{inv}} \\ \mathrm{Assoc}^\otimes \arrow[r] & \mathcal{RM}^\otimes_{\mathrm{inv}} \arrow[ur] & \end{tikzcd}$$

commute, where rev is (an involutive version of) the reversal involution of [Lur17, Remark 4.6.3.2].

Definition 6.31. Let $\mathcal{C}^\otimes \to \mathrm{Assoc}^\otimes_\sigma$ and $\mathcal{D}^\otimes \to \mathrm{Assoc}^\otimes_\sigma$ be fibrations of ∞-operads and let $\mathcal{M}$ be an ∞-category. Suppose given a fibration of ∞-operads $q \colon \mathcal{O}^\otimes \to \mathcal{LM}^\otimes_{\mathrm{inv}}$ together with equivalences $\mathcal{O}^\otimes_{\mathfrak{a}} \simeq \mathcal{C}^\otimes$ and $\mathcal{O}^\otimes_{\mathfrak{m}} \simeq \mathcal{M}$. Let $L\,{}^\sigma\mathrm{Mod}\,(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{LM}_{\mathrm{inv}}}(\mathcal{O})$. We will refer to $L\,{}^\sigma\mathrm{Mod}\,(\mathcal{M})$ as the *∞-category of left hermitian module objects of $\mathcal{M}$.*

Suppose given a fibration of ∞-operads $q \colon \mathcal{O}^\otimes \to \mathcal{BM}^\otimes_{\mathrm{inv}}$ together with equivalences $\mathcal{O}^\otimes_{\mathfrak{a}_\ell} \simeq \mathcal{C}^\otimes$, $\mathcal{O}^\otimes_{\mathfrak{a}_R} \simeq \mathcal{D}^\otimes$ and $\mathcal{O}^\otimes_{\mathfrak{m}} \simeq \mathcal{M}$. Let ${}^\sigma\mathrm{Mod}\,(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{BM}_{\mathrm{inv}}}(\mathcal{O})$. We will refer to ${}^\sigma\mathrm{Mod}\,(\mathcal{M})$ as the *∞-category of hermitian bimodule objects of $\mathcal{M}$*. Composition with the inclusions $\mathrm{Assoc}^\otimes_\sigma \to \mathcal{BM}^\otimes_{\mathrm{inv}}$ induces a categorical fibration

$$ {}^\sigma\mathrm{Mod}\,(\mathcal{M}) = \mathrm{Alg}_{/\mathcal{BM}_{\mathrm{inv}}}(\mathcal{O}) \to \mathrm{Alg}_{\mathrm{Assoc}_\sigma}(\mathcal{C}) \times \mathrm{Alg}_{\mathrm{Assoc}_\sigma}(\mathcal{D}). $$

L: Lurie gives this a name (Definition 4.2.1.12 *weakly enriched*)– not sure what to call this. something *bi-enriched*?

If A is an Assoc_σ-algebra object of $\mathcal{C}$, we let ${}^\sigma\mathrm{Mod}_A\,(\mathcal{M})$ denote the fiber ${}^\sigma\mathrm{Mod}\,(\mathcal{M}) \times_{\mathrm{Alg}_{\mathrm{Assoc}_\sigma}(\mathcal{C})} \{A\}$. We will refer to ${}^\sigma\mathrm{Mod}_A\,(\mathcal{M})$ as the *∞-category of hermitian A-bimodule objects of $\mathcal{M}$.*

Definition 6.32. Let $q \colon \mathcal{O}^\otimes \to \mathcal{BM}^\otimes_{\mathrm{inv}}$ be a fibration of ∞-operads. We say that q exhibits $\mathcal{O}_{\mathfrak{m}}$ as $\mathbb{E}_\sigma$-bitensored over $\mathcal{O}_{\mathfrak{a}_\ell}$ and $\mathcal{O}_{\mathfrak{a}_r}$ if q is a cocartesian fibration.

Remark 6.33. Let $q \colon \mathcal{O}^\otimes \to \mathcal{BM}^\otimes_{\mathrm{inv}}$ be a cocartesian fibration of ∞-operads. Then q is classified by a map $\chi \colon \mathcal{BM}^\otimes_{\mathrm{inv}} \to \mathrm{Cat}_\infty$. By Remark 6.29, we can think of q as giving two $\mathbb{E}_\sigma$ algebras $\mathcal{C}$, $\mathcal{D}$ in Cat_∞ with an ∞-category $\mathcal{M}$ equipped with both the structure of a $\mathcal{C}$-$\mathcal{D}$-bimodule (equivalently, the structure of a left $\mathcal{C} \times \mathcal{D}_{\mathrm{rev}}$-module) and of a $\mathcal{D}$-$\mathcal{C}$-bimodule, and an autoequivalence $\sigma_{\mathcal{M}} \colon \mathcal{M} \simeq \mathcal{M}$ of order two which is linear with respect to the autoequivalence $\mathcal{C} \times \mathcal{D}_{\mathrm{rev}} \xrightarrow{\mathrm{flip}} \mathcal{D}_{\mathrm{rev}} \times \mathcal{C} \xrightarrow{\sigma_{\mathcal{D}}^{-1} \times \sigma_{\mathcal{C}}} \mathcal{D} \times \mathcal{C}_{\mathrm{rev}}$.

Remark 6.34. Let $q \colon \mathcal{O}^\otimes \to \mathcal{BM}^\otimes_{\mathrm{inv}}$ be a cocartesian fibration of ∞-operads. Consider a hermitian module object $F \colon \mathcal{BM}^\otimes_{\mathrm{inv}} \to \mathcal{O}^\otimes$. By Remark 6.30, F determines an associative algebra A of $\mathcal{C}$ with an equivalence of algebras $\sigma_A \colon A \simeq \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$ and an associative algebra B of $\mathcal{D}$ with an equivalence of algebras $\sigma_B \colon B \simeq \sigma_{\mathcal{D}}(B)^{\mathrm{rev}}$, an object $M \in \mathcal{M}$ so that M (resp. $\sigma_{\mathcal{M}}(M)$) is equipped with the structure of a A-B-bimodule (resp. $\sigma_{\mathcal{D}}(B)$-$\sigma_{\mathcal{C}}(A)$-bimodule). Furthermore, we have an equivalence $\sigma_M \colon M \simeq \sigma_{\mathcal{M}}(M)$ which is linear with respect to the equivalence $A \otimes B \xrightarrow{\mathrm{flip}} B \otimes A \xrightarrow{\sigma_B^{-1} \otimes \sigma_A} \sigma_{\mathcal{D}}(B)^{\mathrm{rev}} \otimes \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$.

L: when $\mathcal{C} = \mathcal{D}$ and $\sigma_{\mathcal{M}}$ and $\sigma_{\mathcal{C}}$ are both the identity and $A = B$, I think this recovers the "module with involution" from [Cal+20, §3.1].

Construction 6.35. Define a functor $\mathrm{MCut} \colon \Delta^{\mathrm{op}}_\sigma \to \mathcal{RM}^\otimes_{\mathrm{inv}}$:

L: maybe this overloaded notation is not good. I'm running out of ideas.

- For each $([n], s)$, we have $\mathrm{MCut}([n], s) = \langle n{+}1 \rangle \simeq \mathrm{RCut}_0([n])$ where RCut is from [Lur17, Construction 4.8.4.4].
- Given a morphism $\alpha \colon ([n], s) \to ([m], t)$, the associated morphism $\mathrm{MCut}([m], t) \to \mathrm{MCut}([n], s)$ consists of
  - On underlying finite pointed sets $\langle m+1 \rangle \to \langle n+1 \rangle$, MCut agrees with (the reverse of) that appearing in [Lur17, Construction 4.2.2.6]

– Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j-1 < j$, we may regard $s\colon \langle n+1\rangle^\circ \to \{\pm 1\}$ and likewise $t\colon \langle m+1\rangle^\circ \to \{\pm 1\}$. Define $u\colon \mathrm{MCut}(\alpha)^{-1}\left(\langle n+1\rangle^\circ\right) \to \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\mathrm{MCut}(\alpha)(j))$.

L: check later

Remark 6.36. We can identify $\mathrm{Assoc}_\sigma^\otimes$ with the full subcategory of $\mathcal{RM}_{\mathrm{inv}}^\otimes$ spanned by objects of the form $(\langle n\rangle, \langle n\rangle^\circ)$. We can regard Construction 6.10 as defining a functor $\Delta_\sigma^{\mathrm{op}} \to \mathcal{RM}_{\mathrm{inv}}^\otimes$. For each $([n], s) \in \Delta_\sigma^{\mathrm{op}}$, there is a map of sets $\theta\colon \mathrm{MCut}([n], s) \to \mathrm{Cut}([n], s)$ defined as in [Lur17, Remark 4.2.2.8]. Concretely, on underlying pointed sets, θ takes the form

L: check that the signs s work out!

$$\begin{aligned}\theta\colon \langle n+1\rangle &\to \langle n\rangle \\ k &\mapsto \begin{cases} k-1 & \text{if } k > 0 \\ * & \text{if } k = 0, *. \end{cases}\end{aligned}$$

This construction determines a morphism γ in the ∞-category $\mathrm{Fun}\left(\Delta_\sigma^{\mathrm{op}}, \mathcal{RM}_{\mathrm{inv}}^\otimes\right)$, or equivalently a map $\gamma\colon \Delta_\sigma^{\mathrm{op}} \times \Delta^1 \to \mathcal{RM}_{\mathrm{inv}}^\otimes$.

Lemma 6.37. *The morphism $\gamma\colon \Delta_\sigma^{\mathrm{op}} \times \Delta^1 \to \mathcal{RM}_{\mathrm{inv}}^\otimes$ defined in Remark 6.36 exhibits $\Delta_\sigma^{\mathrm{op}} \times \Delta^1$ as an approximation to the ∞-operad $\mathcal{RM}_{\mathrm{inv}}^\otimes$.*

Definition 6.38. Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}_{\mathrm{inv}}^\otimes$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^\otimes$ as weakly bi-enriched over $\mathcal{O}_{\mathfrak{a}}^\otimes$. Let γ be as in Remark 6.36. Let $R^\sigma\mathrm{Mod}^{\mathbb{A}_\infty^\sigma}(\mathcal{M})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{RM}_{\mathrm{inv}}^\otimes}\left(\Delta_\sigma^{\mathrm{op}} \times \Delta^1, \mathcal{O}^\otimes\right)$ spanned by those maps $f\colon \Delta_\sigma^{\mathrm{op}} \times \Delta^1 \to \mathcal{O}^\otimes$ satisfying

1. The restriction of f to $\Delta_\sigma^{\mathrm{op}} \times \{1\}$ belongs to $\mathrm{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O})$ of Definition 6.15
2. If $\alpha\colon ([m], s) \to ([n], t)$ so that $\alpha(0) = 0$, then the induced map $f([m], s, 0) \to f([n], t, 0)$ is an inert map in $\mathcal{O}^\otimes$
3. for each object $([n], s)$ in $\Delta_\sigma^{\mathrm{op}}$, the induced map $f([n], s, 0) \to f([n], s, 1)$ is an inert map in $\mathcal{O}^\otimes$

Example 6.39. Let $\mathcal{C}^\otimes \to \mathcal{RM}^\otimes$ be a fibration of ∞-operads. Restriction along the map of ∞-operads $\mathcal{RM}_{\mathrm{inv}}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ induced by Remark 6.27 induces a map $\mathbb{E}_\sigma\,\mathrm{Alg}\,(\mathcal{C}) \to R^\sigma\mathrm{Mod}\,(\mathcal{C})$ which is a section of the projection map $R^\sigma\mathrm{Mod}\,(\mathcal{C}) \to \mathbb{E}_\sigma\,\mathrm{Alg}\,(\mathcal{C})$.

L: see Example 4.2.1.17 of higher algebra

Notation 6.40. Let $q\colon \mathcal{O}^\otimes \to \mathcal{BM}_{\mathrm{inv}}^\otimes$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^\otimes$ as weakly bi-enriched over $\mathcal{O}_{\mathfrak{a}}^\otimes$. Define a new simplicial set $\overline{\mathcal{M}}^\circledast$ by the following universal property

L: fibration?

$$\hom_{\mathrm{sSet}_{/\Delta_\sigma^{\mathrm{op}}}}\left(K, \overline{\mathcal{M}}^\circledast\right) \simeq \hom_{\mathrm{sSet}_{/\mathcal{BM}_{\mathrm{inv}}^\otimes}}\left(K \times \Delta^1, \mathcal{O}^\otimes\right).$$

Here we regard $K \times \Delta^1$ as a simplicial set over $\mathcal{BM}_{\mathrm{inv}}^\otimes$ via the composite $K \times \Delta^1 \to \Delta_\sigma^{\mathrm{op}} \times \Delta^1 \xrightarrow{\gamma} \mathcal{BM}_{\mathrm{inv}}^\otimes$ where γ is from Remark 6.36.

Unwinding definitions, we see that a vertex in $\overline{\mathcal{M}}^\circledast$ lying over an object $([n], s\colon \{1, \ldots, n\} \to \{\pm 1\}) \in \Delta_\sigma^{\mathrm{op}}$ corresponds to a morphism α in $\mathcal{O}^\otimes$ whose image in $\mathcal{RM}_{\mathrm{inv}}^\otimes$ is the map $(\langle n+1\rangle, \{0\}) \to (\langle n\rangle, \varnothing)$. Now let $\mathcal{M}^\circledast$ denote the full simplicial subset of $\overline{\mathcal{M}}^\circledast$ spanned by those vertices for which α is inert.

L: this might be off–revisit later!

Remark 6.41. Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}_{\mathrm{inv}}^\otimes$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^\otimes$ as weakly enriched over $\mathcal{O}_{\mathfrak{a}}^\otimes$. By [Lur09, Example 4.3.1.4 & Proposition 4.3.2.15], composition with the inclusion $\{0\} \to \Delta^1$ induces a trivial Kan fibration $\mathcal{M}^\circledast \xrightarrow{\sim} \mathcal{O}^\otimes \times_{\mathcal{RM}_{\mathrm{inv}}^\otimes} \Delta_\sigma^{\mathrm{op}}$. In particular, the fiber of $\mathcal{M}^\circledast$ over an object $([n], s) \in \Delta_\sigma^{\mathrm{op}}$ is canonically equivalent to $\mathcal{M} \times \mathcal{C}^{\times n}$.

Finally, since q is a categorical fibration and categorical fibrations are closed under pullback and composition with trivial fibrations, q induces categorical fibrations $\mathcal{M}^\circledast \to \mathcal{C}^\circledast \to \Delta_\sigma^{\mathrm{op}}$.

L: Jacob explains this in a really terse way–just by citing Prop 4.3.2.15 of HTT. It does just follow from definitions/observatio but there are many (for instance, definition of inert edge).

Lemma 6.42. *Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}_{\mathrm{inv}}^\otimes$ be a cocartesian fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^\otimes$ as tensored over $\mathcal{O}_{\mathfrak{a}}^\otimes$. Then the associated functor $\mathcal{M}^\circledast \to \mathcal{C}^\circledast$ (Notation 6.12) is a locally coCartesian fibration.*

Proposition 6.43. *Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}_{\mathrm{inv}}^\otimes$ be a cocartesian fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^\otimes$ as tensored over $\mathcal{O}_{\mathfrak{a}}^\otimes$. Then precomposition with the functor* MCut *of Construction 6.35 induces an equivalence of ∞-categories*

$$R^\sigma\mathrm{Mod}(\mathcal{M}) \simeq \mathrm{Alg}_{/\mathcal{RM}_{\mathrm{inv}}}(\mathcal{O}) \xrightarrow{\sim} R^\sigma\mathrm{Mod}^{\mathbb{A}_\infty^\sigma}(\mathcal{M}).$$

Proof. Combine Lemma 6.37 with [Lur17, Theorem 2.3.3.23]. ☐

6.2 Part (b)

Proposition 6.44. *Let $\mathcal{C}$ be an involutive monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is bitensored over $\mathcal{C}$. Let K be a simplicial set so that $\mathcal{M}$ admits K-indexed limits, and let $\theta\colon R^{\sigma}\mathrm{Mod}(\mathcal{M}) \to \mathrm{Alg}^{\sigma}(\mathcal{C})$ be the forgetful functor. Then*

L: This statement is [Lur17, Proposition 4.2.3.1] with some words changed; no claim of originality here.

(1) For every commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & R^{\sigma}\mathrm{Mod}(\mathcal{M}) \arrow[d, "\theta"] \\ K^{\triangleleft} \arrow[r] \arrow[ur, dashed] & \mathrm{Alg}^{\sigma}(\mathcal{C})\,, \end{tikzcd}$$

there exists a dashed arrow which is a θ-limit diagram.

(2) An arbitrary map $\overline{g}\colon K^{\triangleleft} \to R^{\sigma}\mathrm{Mod}(\mathcal{M})$ is a θ-limit diagram if and only if the induced map $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.

Proof. □

L: todo

Corollary 6.45. *θ is a cartesian fibration, and a morphism $f\colon \Delta^1 \to R^{\sigma}\mathrm{Mod}(\mathcal{M})$ is θ-cartesian if and only if the image of f in $\mathcal{M}$ is an equivalence.*

Corollary 6.46. *Let $\mathcal{C}$ be an involutive monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is bitensored over $\mathcal{C}$. Let K be a simplicial set so that $\mathcal{M}$ admits K-indexed limits, and let $\theta\colon R^{\sigma}\mathrm{Mod}(\mathcal{M}) \to \mathrm{Alg}^{\sigma}(\mathcal{C})$ be the forgetful functor. Let A be an involutive algebra object of $\mathcal{C}$. Then*

(1) $R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ admits K-indexed limits.

(2) A diagram $K^{\triangleleft} \to R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ is a limit diagram if and only if the induced diagram $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.

(3) Given a morphism $A \to B$ of involutive algebra objects of $\mathcal{C}$, the induced functor $R^{\sigma}\mathrm{Mod}_B(\mathcal{M}) \to R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ preserves K-indexed limits.

6.3 Towards (e)

Construction 6.47. Define a functor $\mathrm{Pr}\colon \mathbf{LM}^{\otimes}_{\mathrm{inv}} \times \mathbf{RM}^{\otimes}_{\mathrm{inv}} \to \mathbf{BM}^{\otimes}_{\mathrm{inv}}$.

Theorem 6.48. *Let $\mathcal{C}$ be an $\mathbb{E}_{\sigma}$-monoidal ∞-category, and let A be an $\mathbb{E}_{\sigma}$-algebra in $\mathcal{C}$. Then $L^{\sigma}\mathrm{Mod}_A(\mathcal{C})$ is right $\mathbb{E}_{\sigma}$-tensored over $\mathcal{C}$.*

6.4 Endomorphisms

Let $\mathcal{C}$ be an $\mathbb{E}_{\sigma}$-monoidal ∞-category, and write $\sigma_{\mathcal{C}}\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ for its involution. Suppose $M \in \mathcal{C}$ is an object equipped with an equivalence $\sigma_M\colon M \simeq \sigma_{\mathcal{C}}(M)$. By [Lur17, §4.7.1], endomorphisms of M can be regarded as an $\mathbb{E}_1$-algebra in $u(\mathcal{C})^{\otimes}$, where u is from Remark 6.8. Now σ_M induces an equivalence $\mathrm{End}_{\mathcal{C}}(M) \simeq \mathrm{End}_{\mathcal{C}}(\sigma_{\mathcal{C}}(M))$ On the other hand, $\sigma_{\mathcal{C}}$ induces an equivalence $\mathrm{End}_{\mathcal{C}}(\sigma_{\mathcal{C}}(M)) \simeq \mathrm{End}_{\mathcal{C}}(M)^{\mathrm{rev}}$. In particular, for any ∞-category $\mathcal{M}$ left $\mathbb{E}_{\sigma}$-tensored over $\mathcal{C}$ and any object $M \in \mathcal{M}$ which is fixed by the involution on $\mathcal{M}$, we expect the endomorphisms of $\mathcal{M}$ to admit the structure of an $\mathbb{E}_{\sigma}$-algebra in $\mathcal{C}$.

To this end, we will define an ∞-category of objects acting on M, show that it has an $\mathbb{E}_{\sigma}$-monoidal structure, and locate endomorphisms of M as the final object in this ∞-category. Informally, we may define a category $\mathcal{C}[M]$ whose objects consist of either

- pairs (C, η) where $C \in \mathcal{C}$ and $\eta\colon C \otimes M \to M$ is a morphism in $\mathcal{M}$; or
- pairs (C', ξ) where $C' \in \mathcal{C}$ and $\xi\colon \sigma_{\mathcal{M}}(M) \otimes C' \to \sigma_{\mathcal{M}}(M)$.

The monoidal structure is as described in [Lur17, §4.7.1]. Note that given an object (C, η), the involution $\sigma_{\mathcal{M}}$ on $\mathcal{M}$ sends η to the map $\sigma_{\mathcal{M}}(C \otimes M) \simeq \sigma_{\mathcal{M}}(M) \otimes \sigma_{\mathcal{C}}(C) \to \sigma_{\mathcal{M}}(M)$. This is the involution on $\mathcal{C}[M]$.

Definition 6.49. Let $p\colon \mathcal{M}^\circledast \to \Delta^1 \times \Delta^{\mathrm{op}}_\sigma$ exhibit $\mathcal{M}^\circledast$ as weakly enriched over $\mathcal{C}^\circledast$. An *enriched morphism* of $\mathcal{M}$ is a diagram

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

satisfying either

- $p(\alpha)$ is the morphism $(0,[1],c_1) \to (0,[0])$ in $\Delta^{\mathrm{op}}_\sigma$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_1\colon \{1\} \to \{\pm 1\}$ is the constant function at $+1$, and
- the map β is inert, and $p(\beta)$ is the morphism $(0,[1],c_1) \to (0,[0])$ in $\Delta^1 \times \Delta^{\mathrm{op}}_\sigma$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

or

- $p(\alpha)$ is the morphism $(0,[1],c_{-1}) \to (0,[0])$ in $\Delta^{\mathrm{op}}_\sigma$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_{-1}\colon \{1\} \to \{\pm 1\}$ is the constant function at -1.
- the map β is inert, and $p(\beta)$ is the morphism $(0,[1],c_{-1}) \to (0,[0])$ in $\Delta^1 \times \Delta^{\mathrm{op}}_\sigma$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

Let $\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[1]}$ denote the full subcategory of $\operatorname{Fun}_{\Delta^1 \times \Delta^{\mathrm{op}}_\sigma}\left(\Lambda^2_0, \mathcal{M}^\circledast\right)$ spanned by the enriched morphisms of $\mathcal{M}$.

Note that there are two evaluation functors $\operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[1]} \to \mathcal{M}$. Given $M \in \mathcal{M}$, write $\mathcal{C}[M] := \{M\} \times_{\mathcal{M}} \operatorname{Str}\mathcal{M}^{\mathrm{en}}_{[1]} \times_{\mathcal{M}} \{M\}$ and refer to it as the endomorphism ∞-category of M.

Definition 6.50. *enriched n-string*

Proposition 6.51 (Segal condition)**.**

7 Categorification and structure

In the course of thinking about the 'involutive' generalization of the statement that given an $\mathbb{E}_1$-algebra, its category of modules is $\mathbb{E}_0$ (and conversely, that given an object in a stable ∞-category, that its endomorphism spectrum is an $\mathbb{E}_1$-algebra), I have run up against some questions.

Question 7.1.

- Can we sidestep an involutive version of the construction of endomorphism categories of [Lur17, §4.7.1]?
- Suppose $\mathcal{C}$ is a monoidal ∞-category and $\mathcal{M}$ is an ∞-category which is enriched over $\mathcal{C}$ in the sense of [Lur17, §4.2.1]. The opposite category $\mathcal{M}^{\mathrm{op}}$ is enriched over $\mathcal{C}$ by [Hei23, §10].

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