

Et cetera

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Abstract

Dumping ground for other stuff: Notes, one-off observations, stuff that we can collectively use when preparing talks, etc.

L: I make no promises re: organization but I will do my best to keep it reasonably readable

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1 Talk prep

2 References

- [Involutions of Azumaya algebras](#) by First and Williams (2020 *Documenta*)
- [Counterexamples in involutions of Azumaya algebras](#) by First and Williams; much more readable than the 2020 *Documenta* paper
- [Azumaya algebras without involution](#) by Auel, First, and Williams: the introduction of this provides a very helpful historical overview of the connection between involutions on Azumaya algebras and 2-torsion/kernel of coRes

3 Questions and directions

Question 3.1 (Literature). • In [PS92] Parimala–Srinivas assume that 2 is invertible in the ring of functions. Has anyone been able to extend their results to the 2 not necessarily invertible case in the meantime?

Question 3.2 (Morita theory for $\text{Cat}_\infty^{\text{P}}$). Let R be a Poincaré ring. Suppose given two R -algebras (suitably interpreted so their module categories are canonically endowed with R -linear Poincaré structures—perhaps \mathbb{E}_σ) A, B . Can we characterize

$$\text{hom}_{\text{Cat}_\infty^{\text{P}} R}((\text{Mod}_A^\omega, \varphi_A), (\text{Mod}_B^\omega, \varphi_B))$$

in terms of something bimodule-like?

Question 3.3. On page 2 of the *Counterexamples* paper, First and Williams write that “existence of an extraordinary involution means classification of Azumaya algebras with involution...*cannot* be reduced to questions about projective modules and hermitian forms on them.”

What if we replaced projective modules by perfect complexes?

Question 3.4. First–Williams show (see discussion in §4 of the *Counterexamples* paper) that coarse type classify many (most?) Azumaya algebras up to (étale-local) *isomorphism*.

What is a suitable derived version of “coarse type”?

Question 3.5 (asked by Andrew Nov 2, 2024). C. Schlichtkrull shows in [this paper](#) that a map $BGL_1(R) \rightarrow K(R) \rightarrow THH(R) \rightarrow R$ in terms of the Hopf map η .

Is there a “Poincaré” version of this result?

4 Thoughts & observations

Question 4.1. When R has the Tate Poincaré structure and $(\text{Mod}_A^\omega, M_A, N_A, N_A \rightarrow M_A^{tC_2})$ is invertible, then by invertibility have an equivalence $\text{hom}_R(A, R) \simeq N_A \otimes_R N_{A^{\text{op}}}$ of $A \otimes_R A^{\text{op}}$ -modules. Restricting the left-hand side along the unit map $R \rightarrow A$ gives a map $N_A \otimes_R N_{A^{\text{op}}} \rightarrow \text{hom}_R(R, R) \simeq R$. Is this a perfect (R -linear) pairing?

I *think* using that $R^{\varphi^{C_2}} \simeq R$ and combining the linear and bilinear part conditions, we get something like

$$M_A \otimes_R M_{A^{\text{op}}} \simeq (N_A \otimes_R N_{A^{\text{op}}})^{\otimes R^2} \quad \text{as } A \otimes_R A^{\text{op}}\text{-bimodules.}$$

Is this useful?

Brauer-Severi schemes We know there is a correspondence between Azumaya algebras A over X and Brauer-Severi schemes. What does a Poincaré structure on Mod_A^ω mean ‘geometrically’ for D_{coh}^b of the corresponding Brauer-Severi scheme? (Lucy: I didn’t get very far here, but just typing up what I had)

- Mod_A^ω corresponds to α -twisted sheaves on X (see Proposition 3.2.2.1 of Max Lieblich’s thesis)
- The bounded derived category of α -twisted sheaves on X includes as one ‘piece’ of a semiorthogonal decomposition on D_{coh}^b of the corresponding Brauer-Severi scheme (see Theorem 5.1 [here](#))

5 Desperate Flailing

This section is a cronical of my thoughts about \mathbb{G}_m° .

Goal The goal is to build a Poincaré ring $\mathbb{G}_m^q := (\text{Mod}_R, \mathbb{Y}_R)$ such that $B\mathbb{G}_m^q(\underline{S}) = \text{Pic}^P(\underline{S})$ for any Poincaré ring \underline{S} .

Lemma 5.1. *Let \underline{S} be a Poincaré ring. Then $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u)) = \{s \in \pi_0(S)^\times \mid s = 1 \text{ in } \pi_0(S^{C_2})\}$.*

Proof. Since the functor $\text{Pn}(\text{Mod}_S) \rightarrow \text{Mod}_S$ is conservative it follows that an element of $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u))$ must have underlying map an element of $\pi_0 \text{Aut}(S) = \pi_0(S)^\times$. Then in order for $s \in \pi_0(S)^\times$ to induce a map $(S, u) \rightarrow (S, u)$, the induced map $s^* : S^{C_2} \rightarrow S^{C_2}$ must satisfy $s^*(u) = u$. The pullback is given by multiplication by s , so this requirement translates into s being the unit, as desired. \square

The problem I thought existed maybe doesn't. Here is a candidate construction:

Construction 5.2. Define R to be the \mathbb{E}_∞ ring given by $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z\}} \mathbb{S}$ where the map $\mathbb{S}\{z\} \rightarrow \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by the map $z \mapsto xy$, and the map $\mathbb{S}\{z\} \rightarrow \mathbb{S}$ is induced by $z \mapsto 1$. We can give R an \mathbb{E}_∞ ring structure in Sp^{BC_2} by taking the trivial action on $\mathbb{S}\{z\}$ and \mathbb{S} , and taking the action induced by $x \mapsto y$ and $y \mapsto x$ on $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$. Thus in $\text{CAlg}(\text{Sp}^{BC_2})$ the ring R corepresents the functor $S \mapsto \{s \in \pi_0(S)^\times \mid s\sigma(s) = 1\}$.

Now take \underline{R} to be the Poincaré ring with underlying Borel C_2 structure as described in the previous paragraph and geometric fixed points $R^{\varphi C_2} = \mathbb{S}$ and the map $R^{\varphi C_2} \rightarrow R^{tC_2}$ given by the unit map. Endowing $R^{\varphi C_2}$ with the R -module structure given by $x, y \mapsto 1$, it remains to show that the unit map $R^{\varphi C_2} \rightarrow R^{tC_2}$ factors the Tate valued Frobenius $R \rightarrow R^{tC_2}$ in order to promote \underline{R} to a Poincaré ring. By construction of R it is then enough to show that on π_0 the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0(R^{tC_2})$. This map sends both x and y to $xy \in \pi_0(R^{tC_2})$. These are equal to 1 in $\pi_0(R^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so R^{tC_2} is a module over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of xy equal to 1.

Now consider another Poincaré ring \underline{S} . We then have that maps $\pi_0(\text{Maps}(\underline{R}, \underline{S}))$ is the data of a unit $s \in \pi_0(S)^\times$, a path $s\sigma(s) \rightarrow 1$ in $\Omega^\infty S$, and paths $x, y \rightarrow 1$ in $\Omega^\infty S^{\varphi C_2}$. This then agrees with \mathbb{G}_m^q by the following lemma.

Lemma 5.3. *Let $S \in \text{CAlg}(\text{Sp}^{BC_2})$ and $s \in \pi_0(S)^\times$. Then $s\sigma(s) = 1$ in $\pi_0(S)$ if and only if $(s \otimes s)^*$ acts by 1 on $\pi_0(S^{hC_2}) = \pi_0(\text{Hom}_{S \otimes S}(S \otimes S, S)^{hC_2})$.*

Proof. The 'only if' direction follows from the fact that the map $S^{hC_2} \rightarrow S$ is an S -bimodule map. Now suppose that $s\sigma(s) = 1$ in S . Then before taking homotopy fixed points the induced map $s^* = id$ because S is \mathbb{E}_∞ .¹ \square

6 Modules with genuine involution

Remark 6.1 (Lucy). I'm just going to put drafts of stuff pertaining to hermitian modules here. Eventually when it gets to be more complete, I will hopefully move this entire section over to the main file.

L: or whatever we want to keep calling these

Meta-commentary There are (at least) three things we want to do:

- (a) Define a category of 'bimodules with involution over algebras with anti-involution' equipped with a forgetful functor $\Theta : \text{BMod}_{\text{inv}}(-) \rightarrow \mathbb{E}_1 \text{Alg}(-)^{hC_2}$.
- (b) Show that Θ is a coCartesian fibration. For this, it suffices to show that it is a *Cartesian* fibration and that it satisfies the hypotheses of [Lur09, Corollary 5.2.2.5]
 - I used to think that we could obtain this by 'bootstrapping' a result from Higher Algebra, plus some facts about assembly. This doesn't seem to be working, so I'm just going to try to do this directly (imitating certain aspects of Chapter 4 of higher algebra.)
- (c) Define a relative tensor product for hermitian bimodules
- (d) Show that the formula for the cocartesian pushforward along a map $A \rightarrow B$ in $\mathbb{E}_1 \text{Alg}(-)^{hC_2}$ is something like $- \otimes_{A \otimes A^{\text{op}}} (B \otimes B^{\text{op}}) \otimes_{B \otimes B^{\text{op}}} B$.

¹Or just \mathbb{E}_2 .

- In Higher Algebra, the formula for the cocartesian pushforward is proven in [Lur17, §4.6]; in particular, this is in the section on duality. In particular, see Proposition 4.6.2.17 and the paragraph immediately preceding this.
- I don't know how to do this yet—while (a) and (b) are not useful if I can't show (c), I can't suss out the feasibility of (c) without (a) and (b) already in place.

(e) Towards an adjunction between \mathbb{E}_σ -algebras and categories with additional structure.

- Involutive version of statement that, for a monoidal ∞ -category \mathcal{C} and an \mathbb{E}_1 -algebra A , $\mathrm{LMod}_A(\mathcal{C})$ is right-tensored over \mathcal{C} ?
- Involutive version of endomorphism categories? [Lur17, §4.7.1]

I think that the equivalence of part (b) of the definition of an Azumaya algebra with genuine involution follows from the property of being Azumaya; see Lemma 1(b) (and p.216 for the ‘type 2’ case) of [PS92].

Lemma 6.2. *Let R be an \mathbb{E}_∞ -ring with an involution $\sigma: R \xrightarrow{\sim} R$ and suppose A is an \mathbb{E}_1 - R -algebra with an anti-involution $\lambda: A \xrightarrow{\sim} \sigma^* A^{\mathrm{op}}$. Suppose A is further Azumaya in the sense of . Then the bilinear pairing*

$$A \otimes_R \sigma^* A \xrightarrow{\mathrm{id} \otimes \sigma^* \lambda} A \otimes_R A^{\mathrm{op}} \simeq \mathrm{End}_R(A) \xrightarrow{\mathrm{tr}} R$$

is perfect, i.e. its adjoint $A \rightarrow (\sigma^ A)^\vee$ is an equivalence.*

Question 6.3. Does the map in part (e) of the definition of an Azumaya algebra with genuine involution follow from property of being Azumaya?

6.1 Step (a)

Definition 6.4. Define a colored operad Assoc_σ as follows:

- The colored operad has a single object, which we denote by \mathbf{a} .
- For every finite set I , the set of operations $\mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$, where $\mathcal{L}I$ is the set of linear orderings on I and an element of $\{\pm 1\}^I$ is a function $I \rightarrow \{\pm 1\}$.
- Suppose given a map of finite sets $\alpha: I \rightarrow J$, together with operations $(\preceq_j, f_j: I_j \rightarrow \{\pm 1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathbf{a}_i\}_{\alpha(i)=j}, \mathbf{a})$ and $(\preceq_J, g: J \rightarrow \{\pm 1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathbf{a}_j\}_{j \in J}, \mathbf{a})$. Define a linear ordering on the set I as follows: $i \leq i'$ if $\alpha(i) \preceq_J \alpha(i')$ or $\alpha(i) = \alpha(i') = j$ and $i \preceq_j i'$ and $g(j) = +1$ or $\alpha(i) = \alpha(i') = j$ and $i \succeq_j i'$ and $g(j) = -1$. Finally, define a function

$$\begin{aligned} I &\rightarrow \{\pm 1\} \\ i &\mapsto f_{\alpha(i)}(i) \cdot g(\alpha(i)), \end{aligned}$$

where the multiplication on $\{\pm 1\}$ is the usual one.

Remark 6.5. There is a map of colored operads $\iota: \mathrm{Assoc} \rightarrow \mathrm{Assoc}_\sigma$ which is the identity on objects and on operations $\mathrm{Mul}_{\mathrm{Assoc}}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$ where c_1 is the constant function on I with value 1.

There is another map of colored operads $\iota^{\mathrm{rev}}: \mathrm{Assoc} \rightarrow \mathrm{Assoc}_\sigma$ which is the identity on objects and on operations $\mathrm{Mul}_{\mathrm{Assoc}}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ sends a linear ordering ℓ to $(\ell^{\mathrm{rev}}, c_{-1})$ where c_{-1} is the constant function on I with value 1.

Definition 6.6. Let $\mathrm{Assoc}_\sigma^\otimes$ denote the associated ∞ -operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.7. Unwinding definitions

- Objects $\mathrm{Assoc}_\sigma^\otimes$ are finite pointed sets $\langle n \rangle \in \mathrm{Fin}_*$

L: reference

L: This is just an imitation of [Lur17, Definition 4.1.1.1], modified in accordance with ideas from §5.4.2.

- Morphisms $\langle m \rangle \rightarrow \langle n \rangle$ consist of
 - $\alpha: \langle m \rangle \rightarrow \langle n \rangle$ a map of finite pointed sets
 - for each $i \in \langle n \rangle^\circ$, a linear ordering \preceq_i on the inverse image $\alpha^{-1}(\{i\})$
 - a map of sets $s: \alpha^{-1}(\langle m \rangle^\circ) \rightarrow \{\pm 1\}$
- For each pair of morphisms

$$(\beta: \langle \ell \rangle \rightarrow \langle m \rangle, \preceq_j, s) \quad (\alpha: \langle m \rangle \rightarrow \langle n \rangle, \preceq_i, t),$$

the composite is the triple $(\alpha \circ \beta, \preceq_j'', u)$ where \preceq_j'' is the ordering on $(\alpha \circ \beta)^{-1}(\{i\})$ so that if $a, b \in \langle \ell \rangle$ so that $\alpha(\beta(a)) = \alpha(\beta(b))$, then $a \preceq_j'' b$ if $\beta(a) \preceq_i \beta(b)$ or $\beta(a) =_i \beta(b) = i$ and $a \preceq_i b$ if $s(i) = 1$ or $a \succeq_i b$ if $s(i) = -1$. Finally $u(l) = s(l) \cdot t(\beta(l))$.

Remark 6.8. The maps $\iota, \iota^{\text{rev}}$ of Remark 6.5 induce maps of ∞ -operads $\text{Assoc}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$. There is a canonical identification $\iota^{\text{rev}} = \sigma \circ \iota$, where σ is the automorphism of the associative operad considered in [Lur17, Remark 4.1.1.7].

Note that each object $\langle n \rangle \in \text{Assoc}_\sigma^\otimes$ has a distinguished automorphism $\text{rev}_{\langle n \rangle}$ of order two given by the identity map on $\langle n \rangle$ and the constant map $c_{-1}: \langle n \rangle^\circ \rightarrow \{\pm 1\}$ at -1 . There is a canonical natural equivalence $\iota \xrightarrow{\sim} \iota^{\text{rev}}$ whose component at $\langle n \rangle$ is $\text{rev}_{\langle n \rangle}$.

Definition 6.9. Let \mathcal{C}^\otimes be a ∞ -operad equipped with the data of a fibration $p: \mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$. Let $\text{Alg}^\sigma(\mathcal{C})$ denote the ∞ -category $\text{Alg}/_{\text{Assoc}_\sigma^\otimes}(\mathcal{C})$ of ∞ -operad sections of p . We will refer to $\text{Alg}^\sigma(\mathcal{C})$ as the ∞ -category of *involutive algebra objects* of \mathcal{C} .

An *involutive monoidal ∞ -category* is the data of a cocartesian fibration $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$.

Remark 6.10. Suppose given a cocartesian fibration $f: \mathcal{D}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ of ∞ -operads. Write $\mathcal{C}^\otimes := \mathcal{D}^\otimes \times_{\text{Assoc}_\sigma^\otimes, \iota} \text{Assoc}^\otimes$; \mathcal{C}^\otimes is a monoidal ∞ -category in the sense of [Lur17, Definition 4.1.1.10]. Furthermore, $\mathcal{C}_{\text{rev}}^\otimes := \mathcal{D}^\otimes \times_{\text{Assoc}_\sigma^\otimes, \iota^{\text{rev}}} \text{Assoc}^\otimes$ is a monoidal ∞ -category. By Remark 6.8, this notation is consistent with that of [Lur17, Remark 4.1.1.7]. In particular, a Assoc_σ -monoidal ∞ -category \mathcal{D}^\otimes determines a monoidal ∞ -category \mathcal{C}^\otimes equipped with a monoidal equivalence $\sigma_{\mathcal{C}}: \mathcal{C}^\otimes \xrightarrow{\sim} \mathcal{C}_{\text{rev}}^\otimes$. Pullback along the involution of Assoc^\otimes determines another monoidal equivalence $\sigma_{\mathcal{C}}^{\text{rev}}: \mathcal{C}_{\text{rev}}^\otimes \xrightarrow{\sim} \mathcal{C}^\otimes$, and our assumptions imply that $\sigma_{\mathcal{C}}^{\text{rev}} \circ \sigma_{\mathcal{C}}$ is equivalent to the identity on \mathcal{C}^\otimes .

Now suppose that A is an involutive algebra object of \mathcal{D} . With the same notation as before, pullback along ι (resp. ι^{rev}) determines associative algebra objects $u(A)$, $u^{\text{rev}}(A)$ of \mathcal{C} and \mathcal{C}_{rev} , respectively. Note that $\sigma_{\mathcal{C}}(u(A))$ is an algebra object of \mathcal{C}_{rev} , which we may regard as an algebra object of \mathcal{C} by precomposing with the autoequivalence $\sigma: \text{Assoc}^\otimes \xrightarrow{\sim} \text{Assoc}^\otimes$. It follows from Remark 6.8 that A determines an equivalence $\sigma_A: u(A) \xrightarrow{\sim} \sigma_{\mathcal{C}}(u(A))^{\text{rev}}$ of algebra objects in \mathcal{C} .

Now suppose furthermore that \mathcal{D}^\otimes is of the form $\mathcal{E}^\otimes \times_{\text{Fin}_*} \text{Assoc}_\sigma^\otimes$ for some symmetric monoidal ∞ -category \mathcal{E} . Then the associated involution $\sigma_{\mathcal{C}}$ is the identity, and for any involutive algebra object A of \mathcal{D} , σ_A is an equivalence $u(A) \simeq u(A)^{\text{rev}}$ satisfying $\sigma_A^{\text{rev}} \circ \sigma_A \simeq \text{id}_A$.

Definition 6.11. Define a category Δ_σ

- objects are pairs $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$
- a morphism from $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$ to $([m], t: \{0, 1, \dots, m\} \rightarrow \{\pm 1\})$ is an order-preserving map $[n] \rightarrow [m]$ in Δ .

Construction 6.12. Define a functor $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$:

- For each $([n], s)$, we have $\text{Cut}([n], s) = \langle n \rangle$.
- Given a morphism $\alpha: ([n], s) \rightarrow ([m], t)$, the associated morphism $\text{Cut}([n], s) \rightarrow \text{Cut}([m], t)$ consists of
 - On underlying finite pointed sets $\langle m \rangle \rightarrow \langle n \rangle$, Cut agrees with that appearing in [Lur17, Construction 4.1.2.9]

L: Note that when s, t are identically one, the resulting order \preceq_j'' agrees with the lexicographic order defined in [Lur17, Remark 4.1.1.4].

L: do we need weaker than cocartesian fibration?

L: maybe better to write s as a function defined on the set of morphisms $i < i+1$ in $[n]$.

- Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j - 1 < j$, we may regard $s: \langle n \rangle^\circ \rightarrow \{\pm 1\}$ and likewise $t: \langle m \rangle^\circ \rightarrow \{\pm 1\}$. Define $u: \text{Cut}(\alpha)^{-1}(\langle n \rangle^\circ) \rightarrow \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\text{Cut}(\alpha)(j))$.

Lemma 6.13. *The functor $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$ exhibits $\Delta_\sigma^{\text{op}}$ as an approximation to the ∞ -operad $\text{Assoc}_\sigma^\otimes$.*

L: I think the proof of this lemma is not too different from the proof of Proposition 4.1.2.11 of [Lur17]; the point here is just to unravel the definitions of locally coCartesian and Cartesian; the morphisms in $\Delta_\sigma^{\text{op}}$ are a little more complicated than Δ^{op} , but not by much.

Notation 6.14. Let $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ exhibit \mathcal{C} as \mathbb{E}_σ -monoidal. Let \mathcal{C}^\otimes denote the fiber product $\mathcal{C}^\otimes \times_{\text{Assoc}_\sigma^\otimes} \Delta_\sigma^{\text{op}}$.

Definition 6.15. Say that a morphism $([n], s) \rightarrow ([m], t)$ is *inert* if the induced map $\text{Cut}([m], t) \rightarrow \text{Cut}([n], s)$ is an inert morphism in $\text{Assoc}_\sigma^\otimes$.

Definition 6.16. A \mathbb{R}^σ -planar operad is an ∞ -category \mathcal{O}^\otimes equipped with a functor $q: \mathcal{O}^\otimes \rightarrow \Delta_\sigma^{\text{op}}$ so that

1. For every object $X \in \mathcal{O}^\otimes$ and every inert morphism $\alpha: ([n], s) \rightarrow q(X)$ in Δ_σ , there is a q -cocartesian morphism $\bar{\alpha}: X \rightarrow Y$ satisfying $q(\bar{\alpha}) = \alpha$
2. Let X be an object satisfying $q(X) = ([n], s)$, and choose q -cocartesian morphisms $\bar{\alpha}_i: X \rightarrow X_i$ corresponding to the morphism $([i-1 < i], s_i) \rightarrow ([n], s)$ which is the inclusion on underlying sets and satisfies $s_i(i) = s(i)$. Then the morphisms $\bar{\alpha}_i$ exhibit X as the q -product of the X_i .
3. For each $n \geq 0$, the construction $C \mapsto \{C_i\}_{1 \leq i \leq n}$ induces an equivalence of ∞ -categories

$$\mathcal{O}^\otimes \times_{\Delta_\sigma^{\text{op}}} \{([n], s)\} \xrightarrow{\sim} (\mathcal{O}^\otimes \times_{\Delta_\sigma^{\text{op}}} \{([1], s|_{\{i\}})\})^{\times n}$$

We say that a morphism α in \mathbb{R}^σ -planar operad is *inert* if it is q -cocartesian and $q(\alpha)$ is inert in $\Delta_\sigma^{\text{op}}$ in the sense of Definition 6.15.

Definition 6.17. Let $q: \mathcal{O}^\otimes \rightarrow \Delta_\sigma^{\text{op}}$ be a \mathbb{R}^σ -planar operad. An \mathbb{A}_∞^σ -algebra object of \mathcal{O}^\otimes is a section of q which carries inert morphisms to inert morphisms. Write $\text{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O})$ for the full subcategory of $\text{Fun}_{\Delta_\sigma^{\text{op}}}(\Delta_\sigma^{\text{op}}, \mathcal{O}^\otimes)$ on \mathbb{A}_∞^σ -algebra objects.

Proposition 6.18. *Let $\mathcal{O}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ be a fibration of ∞ -operads. Then precomposition with the functor Cut of Construction 6.12 induces an equivalence of ∞ -categories*

$$\text{Alg}_{\text{Assoc}_\sigma^\otimes}(\mathcal{O}) \xrightarrow{\sim} \text{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O}).$$

Proof. Combine Lemma 6.13 with [Lur17, Theorem 2.3.3.23]. □

Definition 6.19. Define a colored operad \mathbf{LM}_{inv}

- (i) The set of objects of \mathbf{LM}_{inv} has two elements, which we denote by \mathbf{a}, \mathbf{m} .
- (ii) Let $\{X_i\}_{i \in I}$ be a finite collection of objects of \mathbf{LM}_{inv} and let Y be another object of \mathbf{LM}_{inv} . If $Y = \mathbf{a}$, then $\text{Mul}_{\mathbf{LM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \rightarrow \{\pm 1\}$ if $X_i = \mathbf{a}$ for all i , and empty otherwise. If $Y = \mathbf{m}$, then $\text{Mul}_{\mathbf{LM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \dots < i_n\}$ on I and a function $c: I \rightarrow \{\pm 1\}$ satisfying either
 - $X_{i_n} = \mathbf{m}$ and $c(i_n) = 1$ and $X_j = \mathbf{a}$ otherwise
 - $X_{i_1} = \mathbf{m}$ and $c(i_1) = -1$ and $X_j = \mathbf{a}$ otherwise
- (iii) The composition law on \mathbf{LM}_{inv} is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4

Remark 6.20. There is a colored operad \mathbf{RM}_{inv} defined exactly in the same way as \mathbf{LM}_{inv} in Definition 6.19. In the interest of precision: \mathbf{RM}_{inv} has the same objects \mathbf{a}, \mathbf{m} . Let $\{X_i\}_{i \in I}$ be a finite collection of objects of \mathbf{RM}_{inv} and let Y be another object of \mathbf{RM}_{inv} . If $Y = \mathbf{m}$, then $\text{Mul}_{\mathbf{RM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \dots < i_n\}$ on I and a function $c: I \rightarrow \{\pm 1\}$ satisfying either

- $X_{i_n} = \mathbf{m}$ and $c(i_n) = -1$ and $X_j = \mathbf{a}$ otherwise
- $X_{i_1} = \mathbf{m}$ and $c(i_1) = 1$ and $X_j = \mathbf{a}$ otherwise

Remark 6.21. Restricting to the objects which are both called \mathbf{a} , we see that both \mathbf{LM}_{inv} and \mathbf{RM}_{inv} have a sub-colored operad which is canonically identified with $\mathbf{Assoc}_{\text{inv}}$ of Definition 6.4.

Remark 6.22. There is a map of colored operads $\iota: \mathbf{LM} \rightarrow \mathbf{LM}_\sigma$ which sends \mathbf{m} to \mathbf{m} and sends \mathbf{a} to \mathbf{a} . On $\text{Mul}_{\mathbf{LM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{LM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\text{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.5. On $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\text{id}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is a map of colored operads $\iota^{\text{rev}}: \mathbf{RM} \rightarrow \mathbf{LM}_\sigma$ which sends \mathbf{m} to \mathbf{m} and sends \mathbf{a} to \mathbf{a} . On $\text{Mul}_{\mathbf{RM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{LM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\text{rev}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.5. On $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\text{rev}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

Definition 6.23. Define a colored operad \mathbf{BM}_{inv}

- (i) The set of objects of \mathbf{BM}_{inv} has three elements, which we denote by $\mathbf{a}_\ell, \mathbf{a}_r, \mathbf{m}$.
- (ii) Let $\{X_i\}_{i \in I}$ be a finite collection of objects of \mathbf{BM}_{inv} and let Y be another object of \mathbf{BM}_{inv} . If $Y = \mathbf{a}_\ell$ (resp. $Y = \mathbf{a}_r$), then $\text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \rightarrow \{\pm 1\}$ if $X_i = \mathbf{a}_\ell$ (resp. $X_i = \mathbf{a}_r$) for all i , and empty otherwise. If $Y = \mathbf{m}$, then $\text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$ is the subset of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \dots < i_n\}$ on I and a function $c: I \rightarrow \{\pm 1\}$ satisfying: if there is exactly one index i_k so that $X_{i_k} = \mathbf{m}$, either
 - $c(i_k) = 1$, $X_j = \mathbf{a}_\ell$ for $j < i_k$ and $X_j = \mathbf{a}_r$ for $j > i_k$; or
 - $c(i_k) = -1$, $X_j = \mathbf{a}_\ell$ for $j > i_k$ and $X_j = \mathbf{a}_r$ for $j < i_k$
- (iii) The composition law on \mathbf{BM}_{inv} is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4

Remark 6.24. The colored operad \mathbf{BM}_{inv} has a canonical involution σ which fixes \mathbf{m} , exchanges \mathbf{a}_ℓ and \mathbf{a}_r , and sends a morphism (λ, c) to $(\lambda^{\text{rev}}, I \xrightarrow{c} \{\pm 1\} \xrightarrow{\cdot(-1)} \{\pm 1\})$.

Remark 6.25. There is a map of colored operads $\iota: \mathbf{BM} \rightarrow \mathbf{BM}_\sigma$ which sends \mathbf{m} to \mathbf{m} and sends \mathbf{a}_- to \mathbf{a}_ℓ and \mathbf{a}_+ to \mathbf{a}_r . On $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}_\pm) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\text{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.5. On $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\text{id}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is *also* a map of colored operads $\iota^{\text{rev}}: \mathbf{BM} \rightarrow \mathbf{BM}_\sigma$ which sends \mathbf{m} to \mathbf{m} and sends \mathbf{a}_- to \mathbf{a}_r and \mathbf{a}_+ to \mathbf{a}_ℓ . On $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}_\pm) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\text{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.5. On $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\text{rev}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_{-1}\}$ where c_{-1} is the constant function on $I \sqcup \{j\}$ with value -1 .

Definition 6.26. Let $\mathcal{LM}_{\text{inv}}^\otimes$, $\mathcal{RM}_{\text{inv}}^\otimes$, and $\mathcal{BM}_{\text{inv}}^\otimes$ denote the associated ∞ -operads (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.27. We can describe the category $\mathcal{LM}_{\text{inv}}^\otimes$ as follows:

- (1) An object of $\mathcal{LM}_{\text{inv}}^\otimes$ is a pair $(\langle n \rangle, S)$ where S is a subset of $\langle n \rangle^\circ$.

(2) Morphisms $(\langle m \rangle, T) \rightarrow (\langle n \rangle, S)$ consist of a map $(\alpha: \langle m \rangle \rightarrow \langle n \rangle, \lambda: \langle m \rangle^\circ \rightarrow \{\pm 1\})$ in $\text{Assoc}_\sigma^\otimes$ satisfying:

- The map α takes $T \cup \{*\}$ to $S \cup \{*\}$
- For each $s \in S$, then $\alpha^{-1}(\{s\})$ contains exactly one element t_s of T , and it is maximal (resp. minimal) with respect to the linear ordering on $\alpha^{-1}(\{s\})$ if $\lambda(t_s) = 1$ (resp. $\lambda(t_s) = -1$).

Remark 6.28. We can describe the category $\mathcal{BM}_{\text{inv}}^\otimes$ as follows:

- (1) An object of $\mathcal{BM}_{\text{inv}}^\otimes$ is a triple $(\langle n \rangle, c_+, c_-)$ where c_\pm are functions $\langle n \rangle^\circ \rightarrow \{0, 1\}$ and $c_-(i) \leq c_+(i)$ for all $i \in \langle n \rangle^\circ$.
- (2) Morphisms $(\langle m \rangle, c_+, c_-) \rightarrow (\langle n \rangle, c'_+, c'_-)$ consist of a map $(\alpha: \langle m \rangle \rightarrow \langle n \rangle, \lambda: \langle m \rangle^\circ \rightarrow \{\pm 1\})$ in $\text{Assoc}_\sigma^\otimes$ satisfying: if $j \in \langle n \rangle^\circ$ and $\alpha^{-1}(j) = \{i_1 < i_2 < \dots < i_\ell\}$,

- If $c_-(j) = c_+(j)$, then

$$c'_-(j) = c_-(i_1) \leq c_+(i_1) = c_-(i_2) \leq c_+(i_2) \cdots \cdots c_-(i_{m-1}) \leq c_+(i_m) = c'_+(j)$$

- If $c_-(j) < c_+(j)$, then there exists a unique k so that $c_-(i_k) < c_+(i_k)$ and

$$\begin{aligned} \lambda(i_k) \cdot c'_-(j) &= \lambda(i_k) \cdot c_-(i_1) \leq \lambda(i_k) \cdot c_+(i_1) = \lambda(i_k) \cdot c_-(i_2) \leq \lambda(i_k) \cdot c_+(i_2) \cdots \\ &\quad \lambda(i_k) \cdot c_-(i_{m-1}) \leq \lambda(i_k) \cdot c_+(i_m) = \lambda(i_k) \cdot c'_+(j) \end{aligned}$$

Remark 6.29. Each morphism $\varphi \in \text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$ determines a linear ordering ℓ on the set I and a function $s: I \rightarrow \{\pm 1\}$. Passing from φ to the pair (ℓ, s) determines a map of colored operads $j: \mathbf{BM}_{\text{inv}} \rightarrow \mathbf{Assoc}_{\text{inv}}$. The map j induces a morphism of ∞ -operads $\mathcal{BM}_{\text{inv}}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ which we will also denote by j . For any \mathbb{E}_σ -monoidal ∞ -category \mathcal{C} , restriction along j sends an \mathbb{E}_σ -algebra $A: \text{Assoc}_\sigma \rightarrow \mathcal{C}^\otimes$ to the pair (A, A) where A is regarded as an involutive bimodule over itself.

L: hermitian

Remark 6.30. The maps $\iota, \iota^{\text{rev}}$ of Remark 6.22 induce maps of ∞ -operads $\iota: \mathcal{LM}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$ and $\iota^{\text{rev}}: \mathcal{RM}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$.

Remark 6.31. The maps $\iota, \iota^{\text{rev}}$ of Remark 6.25 induce maps of ∞ -operads $\iota, \iota^{\text{rev}}: \mathcal{BM}^\otimes \rightarrow \mathcal{BM}_\sigma^\otimes$. There are canonical identifications $\iota \circ \text{rev} \simeq \sigma \circ \iota^{\text{rev}}$ where σ is the involution on $\mathcal{BM}_\sigma^\otimes$ induced by Remark 6.24 and rev is the involution on \mathcal{BM}^\otimes of [Lur17, Construction 4.6.3.1].

Remark 6.32. There are canonical maps of operads $\mathcal{LM}_{\text{inv}}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$ and $\mathcal{RM}_{\text{inv}}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$ sending \mathbf{a} to \mathbf{a}_ℓ , resp. \mathbf{a}_r and making the diagram

$$\begin{array}{ccc} \text{Assoc}^\otimes & \longrightarrow & \mathcal{LM}_{\text{inv}}^\otimes \\ \downarrow \sigma & & \downarrow \text{rev} \\ \text{Assoc}^\otimes & \longrightarrow & \mathcal{RM}_{\text{inv}}^\otimes \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad \mathcal{BM}_{\text{inv}}^\otimes$$

commute, where rev is (an involutive version of) the reversal involution of [Lur17, Remark 4.6.3.2].

Definition 6.33. Let $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ and $\mathcal{D}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ be fibrations of ∞ -operads and let \mathcal{M} be an ∞ -category. Suppose given a fibration of ∞ -operads $q: \mathcal{O}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$ together with equivalences $\mathcal{O}_{\mathbf{a}}^\otimes \simeq \mathcal{C}^\otimes$ and $\mathcal{O}_{\mathbf{m}}^\otimes \simeq \mathcal{M}$. Let $L^\sigma \text{Mod}(\mathcal{M})$ denote the ∞ -category $\text{Alg}_{/\mathcal{LM}_{\text{inv}}}(\mathcal{O})$. We will refer to $L^\sigma \text{Mod}(\mathcal{M})$ as the ∞ -category of left hermitian module objects of \mathcal{M} .

Suppose given a fibration of ∞ -operads $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$ together with equivalences $\mathcal{O}_{\mathbf{a}_\ell}^\otimes \simeq \mathcal{C}^\otimes$, $\mathcal{O}_{\mathbf{a}_R}^\otimes \simeq \mathcal{D}^\otimes$ and $\mathcal{O}_{\mathbf{m}}^\otimes \simeq \mathcal{M}$. Let ${}^\sigma \text{Mod}(\mathcal{M})$ denote the ∞ -category $\text{Alg}_{/\mathcal{BM}_{\text{inv}}}(\mathcal{O})$. We will refer to ${}^\sigma \text{Mod}(\mathcal{M})$ as the ∞ -category of hermitian bimodule objects of \mathcal{M} . Composition with the inclusions $\text{Assoc}_\sigma^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$ induces a categorical fibration

$${}^\sigma \text{Mod}(\mathcal{M}) = \text{Alg}_{/\mathcal{BM}_{\text{inv}}}(\mathcal{O}) \rightarrow \text{Alg}_{\text{Assoc}_\sigma}(\mathcal{C}) \times \text{Alg}_{\text{Assoc}_\sigma}(\mathcal{D}).$$

If A is an Assoc_σ -algebra object of \mathcal{C} , we let ${}^\sigma \text{Mod}_A(\mathcal{M})$ denote the fiber ${}^\sigma \text{Mod}(\mathcal{M}) \times_{\text{Alg}_{\text{Assoc}_\sigma}(\mathcal{C})} \{A\}$. We will refer to ${}^\sigma \text{Mod}_A(\mathcal{M})$ as the ∞ -category of hermitian A -bimodule objects of \mathcal{M} .

L: Lurie gives this a name (Definition 4.2.1.12 weakly enriched)–not sure what to call this. something bi-enriched?

Definition 6.34. Let $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$ be a fibration of ∞ -operads. We say that q exhibits \mathcal{O}_m as \mathbb{E}_σ -bitensored over \mathcal{O}_{a_ℓ} and \mathcal{O}_{a_r} if q is a cocartesian fibration.

Remark 6.35. Let $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$ be a cocartesian fibration of ∞ -operads. Then q is classified by a map $\chi: \mathcal{BM}_{\text{inv}}^\otimes \rightarrow \text{Cat}_\infty$. By Remark 6.31, we can think of q as giving two \mathbb{E}_σ algebras \mathcal{C}, \mathcal{D} in Cat_∞ with an ∞ -category \mathcal{M} equipped with both the structure of a \mathcal{C} - \mathcal{D} -bimodule (equivalently, the structure of a left $\mathcal{C} \times \mathcal{D}_{\text{rev}}$ -module) and of a \mathcal{D} - \mathcal{C} -bimodule, and an autoequivalence $\sigma_{\mathcal{M}}: \mathcal{M} \simeq \mathcal{M}$ of order two which is linear with respect to the autoequivalence $\mathcal{C} \times \mathcal{D}_{\text{rev}} \xrightarrow{\text{flip}} \mathcal{D}_{\text{rev}} \times \mathcal{C} \xrightarrow{\sigma_{\mathcal{D}}^{-1} \times \sigma_{\mathcal{C}}} \mathcal{D} \times \mathcal{C}_{\text{rev}}$.

Remark 6.36. Let $q: \mathcal{O}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$ be a cocartesian fibration of ∞ -operads. Consider a left hermitian module object $F: \mathcal{LM}_{\text{inv}}^\otimes \rightarrow \mathcal{O}^\otimes$. By Remark 6.32, F determines an associative algebra A of \mathcal{C} with an equivalence of algebras $\sigma_A: A \simeq \sigma_{\mathcal{C}}(A)^{\text{rev}}$, an object $M \in \mathcal{M}$ so that M (resp. $\sigma_{\mathcal{M}}(M)$) is equipped with the structure of a left A -module (resp. right $\sigma_{\mathcal{C}}(A)$ -module). Furthermore, we have an equivalence $\sigma_M: M \simeq \sigma_{\mathcal{M}}(M)$ which is linear with respect to the equivalence $A \xrightarrow{\sigma_A} \sigma_{\mathcal{C}}(A)^{\text{rev}}$.

L: is this related to “modules with involution” from [Cal+20, §3.1]?

Remark 6.37. Let $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$ be a cocartesian fibration of ∞ -operads. Consider a hermitian module object $F: \mathcal{BM}_{\text{inv}}^\otimes \rightarrow \mathcal{O}^\otimes$. By Remark 6.32, F determines an associative algebra A of \mathcal{C} with an equivalence of algebras $\sigma_A: A \simeq \sigma_{\mathcal{C}}(A)^{\text{rev}}$ and an associative algebra B of \mathcal{D} with an equivalence of algebras $\sigma_B: B \simeq \sigma_{\mathcal{D}}(B)^{\text{rev}}$, an object $M \in \mathcal{M}$ so that M (resp. $\sigma_{\mathcal{M}}(M)$) is equipped with the structure of a A - B -bimodule (resp. $\sigma_{\mathcal{D}}(B)$ - $\sigma_{\mathcal{C}}(A)$ -bimodule). Furthermore, we have an equivalence $\sigma_M: M \simeq \sigma_{\mathcal{M}}(M)$ which is linear with respect to the equivalence $A \otimes B \xrightarrow{\text{flip}} B \otimes A \xrightarrow{\sigma_B^{-1} \otimes \sigma_A} \sigma_{\mathcal{D}}(B)^{\text{rev}} \otimes \sigma_{\mathcal{C}}(A)^{\text{rev}}$.

L: when $\mathcal{C} = \mathcal{D}$ and $\sigma_{\mathcal{M}}$ and $\sigma_{\mathcal{C}}$ are both the identity and $A = B$, I think this recovers the “module with involution” from [Cal+20, §3.1].

Construction 6.38. Define a functor $\text{MCut}: \Delta_\sigma^{\text{op}} \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$:

- For each $([n], s)$, we have $\text{MCut}([n], s) = \langle n+1 \rangle \simeq \text{RCut}_0([n])$ where RCut is from [Lur17, Construction 4.8.4.4].
- Given a morphism $\alpha: ([n], s) \rightarrow ([m], t)$, the associated morphism $\text{MCut}([m], t) \rightarrow \text{MCut}([n], s)$ consists of
 - On underlying finite pointed sets $\langle m+1 \rangle \rightarrow \langle n+1 \rangle$, MCut agrees with (the reverse of) that appearing in [Lur17, Construction 4.2.2.6]
 - Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j-1 < j$, we may regard $s: \langle n+1 \rangle^\circ \rightarrow \{\pm 1\}$ and likewise $t: \langle m+1 \rangle^\circ \rightarrow \{\pm 1\}$. Define $u: \text{MCut}(\alpha)^{-1}(\langle n+1 \rangle^\circ) \rightarrow \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\text{MCut}(\alpha)(j))$.

L: maybe this overloaded notation is not good. I’m running out of ideas.

L: check later

Remark 6.39. We can identify $\text{Assoc}_\sigma^\otimes$ with the full subcategory of $\mathcal{RM}_{\text{inv}}^\otimes$ spanned by objects of the form $(\langle n \rangle, \langle n \rangle^\circ)$. We can regard Construction 6.12 as defining a functor $\Delta_\sigma^{\text{op}} \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$. For each $([n], s) \in \Delta_\sigma^{\text{op}}$, there is a map of sets $\theta: \text{MCut}([n], s) \rightarrow \text{Cut}([n], s)$ defined as in [Lur17, Remark 4.2.2.8]. Concretely, on underlying pointed sets, θ takes the form

$$\theta: \langle n+1 \rangle \rightarrow \langle n \rangle$$

$$k \mapsto \begin{cases} k-1 & \text{if } k > 0 \\ * & \text{if } k = 0, * \end{cases}$$

L: check that the signs s work out!

This construction determines a morphism γ in the ∞ -category $\text{Fun}(\Delta_\sigma^{\text{op}}, \mathcal{RM}_{\text{inv}}^\otimes)$, or equivalently a map $\gamma: \Delta_\sigma^{\text{op}} \times \Delta^1 \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$.

Lemma 6.40. The morphism $\gamma: \Delta_\sigma^{\text{op}} \times \Delta^1 \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ defined in Remark 6.39 exhibits $\Delta_\sigma^{\text{op}} \times \Delta^1$ as an approximation to the ∞ -operad $\mathcal{RM}_{\text{inv}}^\otimes$.

Definition 6.41. Let $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ be a fibration of ∞ -operads, so q exhibits $\mathcal{M} := \mathcal{O}_m^\otimes$ as weakly bi-enriched over \mathcal{O}_a^\otimes . Let γ be as in Remark 6.39. Let $R^\sigma \text{Mod}^{\mathbb{A}^\sigma_\infty}(\mathcal{M})$ denote the full subcategory of $\text{Fun}_{\mathcal{RM}_{\text{inv}}^\otimes}(\Delta_\sigma^{\text{op}} \times \Delta^1, \mathcal{O}^\otimes)$ spanned by those maps $f: \Delta_\sigma^{\text{op}} \times \Delta^1 \rightarrow \mathcal{O}^\otimes$ satisfying

1. The restriction of f to $\Delta_\sigma^{\text{op}} \times \{1\}$ belongs to $\text{Alg}_{\mathbb{A}^\sigma_\infty}(\mathcal{O})$ of Definition 6.17
2. If $\alpha: ([m], s) \rightarrow ([n], t)$ so that $\alpha(0) = 0$, then the induced map $f([m], s, 0) \rightarrow f([n], t, 0)$ is an inert map in \mathcal{O}^\otimes
3. for each object $([n], s)$ in $\Delta_\sigma^{\text{op}}$, the induced map $f([n], s, 0) \rightarrow f([n], s, 1)$ is an inert map in \mathcal{O}^\otimes

Example 6.42. Let $\mathcal{C}^\otimes \rightarrow \mathcal{RM}^\otimes$ be a fibration of ∞ -operads. Restriction along the map of ∞ -operads $\mathcal{RM}_{\text{inv}}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ induced by Remark 6.29 induces a map $\mathbb{E}_\sigma \text{Alg}(\mathcal{C}) \rightarrow R^\sigma \text{Mod}(\mathcal{C})$ which is a section of the projection map $R^\sigma \text{Mod}(\mathcal{C}) \rightarrow \mathbb{E}_\sigma \text{Alg}(\mathcal{C})$.

Notation 6.43. Let $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ be a fibration of ∞ -operads, so q exhibits $\mathcal{M} := \mathcal{O}_m^\otimes$ as weakly bi-enriched over \mathcal{O}_a^\otimes . Define a new simplicial set $\overline{\mathcal{M}}^\otimes$ by the following universal property

$$\text{hom}_{\text{sSet}/\Delta_\sigma^{\text{op}}} (K, \overline{\mathcal{M}}^\otimes) \simeq \text{hom}_{\text{sSet}/\mathcal{RM}_{\text{inv}}^\otimes} (K \times \Delta^1, \mathcal{O}^\otimes) .$$

Here we regard $K \times \Delta^1$ as a simplicial set over $\mathcal{RM}_{\text{inv}}^\otimes$ via the composite $K \times \Delta^1 \rightarrow \Delta_\sigma^{\text{op}} \times \Delta^1 \xrightarrow{\gamma} \mathcal{RM}_{\text{inv}}^\otimes$ where γ is from Remark 6.39.

Unwinding definitions, we see that a vertex in $\overline{\mathcal{M}}^\otimes$ lying over an object $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\}) \in \Delta_\sigma^{\text{op}}$ corresponds to a morphism α in \mathcal{O}^\otimes whose image in $\mathcal{RM}_{\text{inv}}^\otimes$ is the map $(\langle n+1 \rangle, \{0\}) \rightarrow (\langle n \rangle, \emptyset)$. Now let \mathcal{M}^\otimes denote the full simplicial subset of $\overline{\mathcal{M}}^\otimes$ spanned by those vertices for which α is inert.

Remark 6.44. Let $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ be a fibration of ∞ -operads, so q exhibits $\mathcal{M} := \mathcal{O}_m^\otimes$ as weakly enriched over \mathcal{O}_a^\otimes . By [Lur09, Example 4.3.1.4 & Proposition 4.3.2.15], composition with the inclusion $\{0\} \rightarrow \Delta^1$ induces a trivial Kan fibration $\mathcal{M}^\otimes \xrightarrow{\sim} \mathcal{O}^\otimes \times_{\mathcal{RM}_{\text{inv}}^\otimes} \Delta_\sigma^{\text{op}}$. In particular, the fiber of \mathcal{M}^\otimes over an object $([n], s) \in \Delta_\sigma^{\text{op}}$ is canonically equivalent to $\mathcal{M} \times \mathcal{C}^{\times n}$.

Finally, since q is a categorical fibration and categorical fibrations are closed under pullback and composition with trivial fibrations, q induces categorical fibrations $\mathcal{M}^\otimes \rightarrow \mathcal{C}^\otimes \rightarrow \Delta_\sigma^{\text{op}}$.

Lemma 6.45. Let $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ be a cocartesian fibration of ∞ -operads, so q exhibits $\mathcal{M} := \mathcal{O}_m^\otimes$ as tensored over \mathcal{O}_a^\otimes . Then the associated functor $\mathcal{M}^\otimes \rightarrow \mathcal{C}^\otimes$ (Notation 6.14) is a locally coCartesian fibration.

Proposition 6.46. Let $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ be a cocartesian fibration of ∞ -operads, so q exhibits $\mathcal{M} := \mathcal{O}_m^\otimes$ as tensored over \mathcal{O}_a^\otimes . Then precomposition with the functor MCut of Construction 6.38 induces an equivalence of ∞ -categories

$$R^\sigma \text{Mod}(\mathcal{M}) \simeq \text{Alg}_{/\mathcal{RM}_{\text{inv}}^\otimes}(\mathcal{O}) \xrightarrow{\sim} R^\sigma \text{Mod}^{\mathbb{A}^\sigma_\infty}(\mathcal{M}) .$$

Proof. Combine Lemma 6.40 with [Lur17, Theorem 2.3.3.23]. □

6.2 Part (b)

Proposition 6.47. Let \mathcal{C} be an involutive monoidal ∞ -category and let \mathcal{M} be an ∞ -category which is bitensored over \mathcal{C} . Let K be a simplicial set so that \mathcal{M} admits K -indexed limits, and let $\theta: R^\sigma \text{Mod}(\mathcal{M}) \rightarrow \text{Alg}^\sigma(\mathcal{C})$ be the forgetful functor. Then

- (1) For every commutative square

$$\begin{array}{ccc} K & \longrightarrow & R^\sigma \text{Mod}(\mathcal{M}) \\ \downarrow & \nearrow & \downarrow \theta \\ K^\triangleleft & \longrightarrow & \text{Alg}^\sigma(\mathcal{C}) , \end{array}$$

there exists a dashed arrow which is a θ -limit diagram.

L: see Example 4.2.1.17 of higher algebra

L: fibration?

L: this might be off-revisit later!

L: Jacob explains this in a really terse way—just by citing Prop 4.3.2.15 of HTT. It does just follow from definitions/observations but there are many (for instance, definition of inert edge).

L: This statement is [Lur17, Proposition 4.2.3.1] with some words changed; no claim of originality here.

- (2) An arbitrary map $\bar{g}: K^\triangleleft \rightarrow R^\sigma \text{Mod}(\mathcal{M})$ is a θ -limit diagram if and only if the induced map $K^\triangleleft \rightarrow \mathcal{M}$ is a limit diagram.

Proof.

□

L: todo

Corollary 6.48. θ is a cartesian fibration, and a morphism $f: \Delta^1 \rightarrow R^\sigma \text{Mod}(\mathcal{M})$ is θ -cartesian if and only if the image of f in \mathcal{M} is an equivalence.

Corollary 6.49. Let \mathcal{C} be an involutive monoidal ∞ -category and let \mathcal{M} be an ∞ -category which is bitensored over \mathcal{C} . Let K be a simplicial set so that \mathcal{M} admits K -indexed limits, and let $\theta: R^\sigma \text{Mod}(\mathcal{M}) \rightarrow \text{Alg}^\sigma(\mathcal{C})$ be the forgetful functor. Let A be an involutive algebra object of \mathcal{C} . Then

- (1) $R^\sigma \text{Mod}_A(\mathcal{M})$ admits K -indexed limits.
- (2) A diagram $K^\triangleleft \rightarrow R^\sigma \text{Mod}_A(\mathcal{M})$ is a limit diagram if and only if the induced diagram $K^\triangleleft \rightarrow \mathcal{M}$ is a limit diagram.
- (3) Given a morphism $A \rightarrow B$ of involutive algebra objects of \mathcal{C} , the induced functor $R^\sigma \text{Mod}_B(\mathcal{M}) \rightarrow R^\sigma \text{Mod}_A(\mathcal{M})$ preserves K -indexed limits.

6.3 Towards (e)

Construction 6.50. Define a functor $\text{Pr}: \mathbf{LM}_{\text{inv}}^\otimes \times \mathbf{RM}_{\text{inv}}^\otimes \rightarrow \mathbf{BM}_{\text{inv}}^\otimes$.

- (1) Let $(\langle m \rangle, S)$ be an object of $\mathbf{LM}_{\text{inv}}^\otimes$ and let $(\langle n \rangle, T)$ be an object of $\mathbf{RM}_{\text{inv}}^\otimes$. Let $\text{Pr}((\langle m \rangle, S), (\langle n \rangle, T)) = (X_*, c_-, c_+)$ where X_*, c_-, c_+ are described in [Lur17, Construction 4.3.2.1(1)].
- (2) Let $(\alpha, \lambda): (\langle m \rangle, S) \rightarrow (\langle m' \rangle, S')$ be a morphism in $\mathbf{LM}_{\text{inv}}^\otimes$ and let $(\beta, \mu): (\langle n \rangle, T) \rightarrow (\langle n' \rangle, T')$ be a morphism in $\mathbf{RM}_{\text{inv}}^\otimes$. Write $\text{Pr}((\langle m' \rangle, S'), (\langle n' \rangle, T')) = (X'_*, c'_-, c'_+)$. Then $\text{Pr}((\alpha, \lambda), (\beta, \mu))$ is the unique morphism in $\mathbf{BM}_{\text{inv}}^\otimes$ lying over the map $\gamma: X_* \rightarrow X'_*$ described by

$$(i) \quad \gamma(i, j) = \begin{cases} (\alpha(i), \beta(j)) & \text{if } \alpha(i) \in \langle m' \rangle^\circ, \beta(j) \in \langle n' \rangle^\circ \\ * & \text{otherwise.} \end{cases}$$

- (ii) Let $i' \in \langle m' \rangle^\circ \setminus S'$ and $j' \in T'$ so $j' = \beta(j)$ for a unique $j \in T$. Then the linear ordering on $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\}$ is (a) determined by the map α if $\mu(j) = 1$, and (b) it is the reverse of the linear ordering determined by α if $\mu(j) = -1$. The map $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\} \rightarrow \{\pm 1\}$ is determined by λ if $\mu(j) = 1$ and it is $-\lambda$ if $\mu(j) = -1$.
- (iii) Likewise if $i' \in S'$ and $j' \in \langle n' \rangle^\circ \setminus T'$
- (iv) Let $i' \in S'$ and $j' \in T'$ so $i' = \alpha(i)$ for a unique $i \in S$ and $j' = \beta(j)$ for a unique $j \in T$. Then $\gamma^{-1}\{(i', j')\} = \{i\} \times \beta^{-1}\{j'\} \sqcup_{\{(i, j)\}} \alpha^{-1}\{(i')\} \times \{j\}$. Define $\gamma^{-1}\{(i', j')\} \rightarrow \{\pm 1\}$ by $\lambda \times \mu$. Endow $\gamma^{-1}\{(i', j')\}$ with the linear ordering from [Lur17, Construction 4.3.2.1(2)(iv)] if $\lambda(i) = \mu(j)$ and endow $\gamma^{-1}\{(i', j')\}$ with the opposite ordering if $\lambda(i) \neq \mu(j)$ (or equivalently, if $\lambda(i) = -\mu(j)$).

Write Pr for the induced map $\mathcal{LM}_\sigma^\otimes \times \mathcal{RM}_\sigma^\otimes \rightarrow \mathcal{BM}_\sigma^\otimes$ of ∞ -categories.

Construction 6.51. Let $q: \mathcal{C}^\otimes \rightarrow \mathcal{BM}_\sigma^\otimes$ be a fibration of ∞ -operads. We define a map of simplicial sets $\overline{L^\sigma \text{Mod}(\mathcal{C}_\mathbf{m})}^\otimes \rightarrow \mathcal{RM}_\sigma^\otimes$ by the universal property: For any simplicial set $K \rightarrow \mathcal{RM}_\sigma^\otimes$, there is a bijection

$$\text{Hom}_{\text{sSet}}_{/\mathcal{RM}_\sigma^\otimes} (K, \overline{L^\sigma \text{Mod}(\mathcal{C}_\mathbf{m})}^\otimes) \simeq \text{Hom}_{\text{sSet}}_{/\mathcal{BM}_\sigma^\otimes} (\mathcal{LM}_\sigma^\otimes \times K, \mathcal{C}^\otimes).$$

Let $L^\sigma \text{Mod}(\mathcal{C}_\mathbf{m})^\otimes$ denote the full simplicial subset of $\overline{L^\sigma \text{Mod}(\mathcal{C}_\mathbf{m})}^\otimes$ spanned by those vertices which correspond to a vertex $X \in \mathcal{RM}_\sigma^\otimes$ and a functor $F: \mathcal{LM}_\sigma^\otimes \{X\} \rightarrow \mathcal{BM}_\sigma^\otimes$ which takes inert morphisms in $\mathcal{LM}_\sigma^\otimes$ to inert morphisms in $\mathcal{BM}_\sigma^\otimes$.

Remark 6.52. The composite $\mathcal{LM}_\sigma^\otimes \times \{\mathfrak{m}\} \hookrightarrow \mathcal{LM}_\sigma^\otimes \times \mathcal{RM}_\sigma^\otimes \xrightarrow{\text{Pr}} \mathcal{BM}_\sigma^\otimes$ agrees with the inclusion of Remark 6.32. Taking $K \rightarrow \mathcal{RM}_\sigma^\otimes$ to be the inclusion $\{\mathfrak{m}\} \hookrightarrow \mathcal{RM}_\sigma^\otimes$, we have an isomorphism of simplicial sets $L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \times_{\mathcal{RM}_\sigma^\otimes} \{\mathfrak{m}\} \simeq L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})$ where $L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})$ is the ∞ -category of left modules associated to the fibration of ∞ -operads $\mathcal{C}^\otimes \times_{\mathcal{BM}_\sigma^\otimes} \mathcal{LM}_\sigma^\otimes \rightarrow \mathcal{LM}_\sigma^\otimes$.

Proposition 6.53. *Let $q: \mathcal{C}^\otimes \rightarrow \mathcal{BM}_\sigma^\otimes$ be a fibration of ∞ -operads. Then*

- (1) *the induced map $p: L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \rightarrow \mathcal{RM}_\sigma^\otimes$ is a fibration of ∞ -operads*
- (2) *a morphism α in $L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$ is inert if and only if $p(\alpha)$ is inert in $\mathcal{RM}_\sigma^\otimes$ and for all $X \in \mathcal{LM}_\sigma$, $\alpha(X)$ is an inert morphism in \mathcal{C}^\otimes .*
- (3) *if q is a cocartesian fibration of ∞ -operads, then so is p*
- (4) *if q is a cocartesian fibration of ∞ -operads, a morphism α in $L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$ is p -cocartesian if and only if, for all $X \in \mathcal{LM}_\sigma$, $\alpha(X)$ is q -cocartesian in \mathcal{C}^\otimes .*

Proof. Similar to [Lur17, Proposition 4.3.2.5]. □

Theorem 6.54. *Let \mathcal{C} be an \mathbb{E}_σ -monoidal ∞ -category, and let A be an \mathbb{E}_σ -algebra in \mathcal{C} . Then $L^\sigma \text{Mod}_A(\mathcal{C})$ is right \mathbb{E}_σ -tensored over \mathcal{C} .*

6.4 Endomorphisms

Let \mathcal{C} be an \mathbb{E}_σ -monoidal ∞ -category, and write $\sigma_\mathcal{C}: \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ for its involution. Suppose $M \in \mathcal{C}$ is an object equipped with an equivalence $\sigma_M: M \simeq \sigma_\mathcal{C}(M)$. By [Lur17, §4.7.1], endomorphisms of M can be regarded as an \mathbb{E}_1 -algebra in $u(\mathcal{C})^\otimes$, where u is from Remark 6.10. Now σ_M induces an equivalence $\text{End}_\mathcal{C}(M) \simeq \text{End}_\mathcal{C}(\sigma_\mathcal{C}(M))$. On the other hand, $\sigma_\mathcal{C}$ induces an equivalence $\text{End}_\mathcal{C}(\sigma_\mathcal{C}(M)) \simeq \text{End}_\mathcal{C}(M)^{\text{rev}}$. In particular, for any ∞ -category \mathcal{M} left \mathbb{E}_σ -tensored over \mathcal{C} and any object $M \in \mathcal{M}$ which is fixed by the involution on \mathcal{M} , we expect the endomorphisms of M to admit the structure of an \mathbb{E}_σ -algebra in \mathcal{C} .

To this end, we will define an ∞ -category of objects acting on M , show that it has an \mathbb{E}_σ -monoidal structure, and locate endomorphisms of M as the final object in this ∞ -category. Informally, we may define a category $\mathcal{C}[M]$ whose objects consist of either

- pairs (C, η) where $C \in \mathcal{C}$ and $\eta: C \otimes M \rightarrow M$ is a morphism in \mathcal{M} ; or
- pairs (C', ξ) where $C' \in \mathcal{C}$ and $\xi: \sigma_\mathcal{M}(M) \otimes C' \rightarrow \sigma_\mathcal{M}(M)$.

The monoidal structure is as described in [Lur17, §4.7.1]. Note that given an object (C, η) , the involution $\sigma_\mathcal{M}$ on \mathcal{M} sends η to the map $\sigma_\mathcal{M}(C \otimes M) \simeq \sigma_\mathcal{M}(M) \otimes \sigma_\mathcal{C}(C) \rightarrow \sigma_\mathcal{M}(M)$. This is the involution on $\mathcal{C}[M]$.

Definition 6.55. Let $p: \mathcal{M}^\otimes \rightarrow \Delta^1 \times \Delta_\sigma^{\text{op}}$ exhibit \mathcal{M}^\otimes as weakly enriched over \mathcal{C}^\otimes . An *enriched morphism* of \mathcal{M} is a diagram

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

satisfying either

- $p(\alpha)$ is the morphism $(0, [1], c_1) \rightarrow (0, [0])$ in $\Delta_\sigma^{\text{op}}$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_1: \{1\} \rightarrow \{\pm 1\}$ is the constant function at $+1$, and
- the map β is inert, and $p(\beta)$ is the morphism $(0, [1], c_1) \rightarrow (0, [0])$ in $\Delta^1 \times \Delta_\sigma^{\text{op}}$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

or

- $p(\alpha)$ is the morphism $(0, [1], c_{-1}) \rightarrow (0, [0])$ in $\Delta_\sigma^{\text{op}}$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_{-1}: \{1\} \rightarrow \{\pm 1\}$ is the constant function at -1 .
- the map β is inert, and $p(\beta)$ is the morphism $(0, [1], c_{-1}) \rightarrow (0, [0])$ in $\Delta^1 \times \Delta_\sigma^{\text{op}}$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

Let $\text{Str } \mathcal{M}_{[1]}^{\text{en}}$ denote the full subcategory of $\text{Fun}_{\Delta^1 \times \Delta_{\sigma}^{\text{op}}}(\Lambda_0^2, \mathcal{M}^{\otimes})$ spanned by the enriched morphisms of \mathcal{M} .

Note that there are two evaluation functors $\text{Str } \mathcal{M}_{[1]}^{\text{en}} \rightarrow \mathcal{M}$. Given $M \in \mathcal{M}$, write $\mathcal{C}[M] := \{M\} \times_{\mathcal{M}} \text{Str } \mathcal{M}_{[1]}^{\text{en}} \times_{\mathcal{M}} \{M\}$ and refer to it as the endomorphism ∞ -category of M .

Definition 6.56. *enriched n -string*

Proposition 6.57 (Segal condition).

7 Categorification and structure

In the course of thinking about the ‘involutive’ generalization of the statement that given an \mathbb{E}_1 -algebra, its category of modules is \mathbb{E}_0 (and conversely, that given an object in a stable ∞ -category, that its endomorphism spectrum is an \mathbb{E}_1 -algebra), I have run up against some questions.

Question 7.1. • Can we sidestep an involutive version of the construction of endomorphism categories of [Lur17, §4.7.1]?

- Suppose \mathcal{C} is a monoidal ∞ -category and \mathcal{M} is an ∞ -category which is enriched over \mathcal{C} in the sense of [Lur17, §4.2.1]. The opposite category \mathcal{M}^{op} is enriched over \mathcal{C} by [Hei23, §10].

8 Comparing involutive classical Brauer and involutive higher Brauer

Question 8.1. • If, for a Poincaré ∞ -category $(\mathcal{C}, \mathfrak{Q})$, there exists a Poincaré object (E, q) so that E is a compact generator, can we rewrite both the category and its Poincaré structure in terms of $\text{End}_{\mathcal{C}}(E)$?

- Can the *property* of an existence of a Poincaré object (E, q) in $(\text{Perf}_X, \mathfrak{Q}_L)$ so that E is a compact generator be checked Zariski-locally? See Toën’s paper §3.

9 Other

Proposition 9.1. *Assume that X has a good quotient Y in the sense of [FW20, Remark 4.20], and write $p: X \rightarrow Y$ for the quotient map. Let $i: U \subseteq Y$ be the largest open subscheme on which $\pi|_{X_U}$ is étale [FW20, Proposition 4.45]. Write $\text{RamLoc}(\pi)$ for the closed complement of U regarded as a topological space, and let $j: \text{RamLoc}(\pi) \rightarrow Y$ denote the inclusion. Then $\mathcal{Q}^{\varphi C_2}$ is in the essential image of $j_*: \text{Shv}_{\text{Zar}}(\text{RamLoc}(\pi)) \rightarrow \text{Shv}_{\text{Zar}}(Y)$. In other words, there exists a sheaf \mathcal{Q} of \mathbb{E}_{∞} -rings on $\text{RamLoc}(\pi)$ so that $j_* \mathcal{Q} \simeq \mathcal{Q}^{\varphi C_2}$.*

Proof. Recall that the open-closed decomposition of Y induces a symmetric monoidal *récollement*

$$\text{Shv}_{\text{Zar}}(U) \xleftarrow{i^*} \text{Shv}_{\text{Zar}}(Y) \xrightarrow{j^*} \text{Shv}_{\text{Zar}}(\text{RamLoc}(\pi)).$$

Therefore, to show that $\mathcal{Q}^{\varphi C_2}$ is in the essential image of j_* , it suffices to show that $i^*(\mathcal{Q}^{\varphi C_2}) \simeq 0$ as a sheaf on U .

By [FW20, Proposition 4.45], it suffices to show that if y is a point in U , then $\mathcal{Q}_y^{\varphi C_2} = 0$. Since $\mathcal{Q}_y^{\varphi C_2} = \tau_{\geq 0} \left(\Gamma \mathcal{O}_{X \times_Y \{y\}}^{tC_2} \right)$ where $A = \Gamma \mathcal{O}_{Y, y} \rightarrow B = \Gamma \mathcal{O}_{X \times_Y \{y\}}$ is a quadratic étale map so that B has an involution λ and $A = B^{\lambda}$ is a local ring with maximal ideal \mathfrak{m}_A (therefore B is semilocal by [FW20, Proposition 3.15]), it suffices to show that $\pi_0 B^{tC_2} = 0$. By [NS18, Lemma I.2.9], we may without loss of generality replace A and B by their 2-completions. By the *récollement* of A -modules in terms of \mathfrak{m}_A -complete and $A[\mathfrak{m}_A^{-1}]$ -modules, it suffices to show that $(B_{\mathfrak{m}_A}^{\wedge})^{tC_2} = 0$ and $(B[\mathfrak{m}_A^{-1}])^{tC_2} = 0$.

By [FW20, Propositions 3.4 & 3.15], $B\mathfrak{m}_A = J \subseteq B$, where J denotes the Jacobson radical of B . We claim that $B \simeq \lim_i B/J^i$ induces an equivalence $B^{tC_2} \rightarrow \lim_i (B/J^i)^{tC_2}$. Granting the claim, it suffices to show that $(B[\mathfrak{m}_A^{-1}])^{tC_2} = 0$ and $(B/J^i)^{tC_2}$ is zero for each i . Since $(-)^{tC_2}$ is exact and lax symmetric monoidal and each B/J^i can be written as an extension of finitely many B/J -modules, it suffices to show that $(B[\mathfrak{m}_A^{-1}])^{tC_2}$ and $(B/J)^{tC_2}$ are zero.

L: hypothesis?
move to main
text?

L: Need hypercompleteness to reduce to checking on points?
Appeal to Clausen–Mathew and add finite Krull dim hypothesis?

L: this is unnecessary to the proof—but shows that U is contained in the open subscheme $Y \setminus [\frac{1}{\lambda}]$

Now observe that A/\mathfrak{m}_A (resp. $A[\mathfrak{m}_A^{-1}]$ -algebra) is a field and B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is a quadratic étale A/\mathfrak{m}_A -algebra (resp. $A[\mathfrak{m}_A^{-1}]$ -algebra). By [FW20, Proposition 3.4(ii)], B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is either a separable quadratic field extension of A/\mathfrak{m}_A -algebra (resp. $A[\mathfrak{m}_A^{-1}]$ -algebra), or it is isomorphic to $\prod_{C_2} A/\mathfrak{m}_A$ (resp. $\prod_{C_2} A[\mathfrak{m}_A^{-1}]$). In the latter case, the action of C_2 on B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is manifestly free, hence $(B/J)^{tC_2} = 0$ (resp. $B[\mathfrak{m}_A^{-1}]^{tC_2} = 0$). Suppose instead that B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is a separable quadratic field extension of A/\mathfrak{m}_A -algebra (resp. $A[\mathfrak{m}_A^{-1}]$ -algebra). By [FW20, Proposition 3.4(ii)], $\lambda \otimes_B B/J$ (resp. $\lambda \otimes_B B[\mathfrak{m}_A^{-1}]$) is nontrivial, hence by [Stacks, Lemma 9.21.2, Tag 09DU] the extension $A/\mathfrak{m}_A \rightarrow B/J$ (resp. $A[\mathfrak{m}_A^{-1}] \rightarrow B[\mathfrak{m}_A^{-1}]$) is Galois. Since C_2 acts freely on B/J as an A/\mathfrak{m}_A -module by the normal basis theorem, $(B/J)^{tC_2} = 0$ (resp. $B[\mathfrak{m}_A^{-1}]^{tC_2} = 0$).

We conclude the proof by proving the claim. Since homotopy fixed points commute with arbitrary limits, it suffices to show that $B \simeq \lim_i B/\mathfrak{m}_B^i$ induces an equivalence $B_{hC_2} \rightarrow \lim_i (B/\mathfrak{m}_B^i)_{hC_2}$. This is true because the B/\mathfrak{m}_B^i are uniformly bounded below. \square

Example 9.2. If $\lambda = \text{id}_X$ and $Y = X$, then $\pi_0 \underline{\mathcal{O}}^{\varphi C_2} = \mathcal{O}_Y/2$. On π_0 , the norm map $\underline{\mathcal{O}}^e \simeq \mathcal{O}_X \rightarrow \underline{\mathcal{O}}^{\varphi C_2}$ takes $f \mapsto f^2$.

L: compare [CMM21, Remark 2.8].

pushforwards along quotient maps An attempt to show the pushforward preserves filtered colimits.

L: As of June 3, I am suspicious of this argument/think something has gone wrong—(9.2) is not supposed to hold in this level of generality. I'm not sure what the problem is yet—maybe that π_* does not in fact define a morphism of recollements? Will revisit later.

Lemma 9.3. Let (X, σ, Y, π) be a scheme with involution and good quotient. Let $\tilde{U} \subseteq X$ be the largest open subscheme of X on which π is quadratic étale and let $W \subseteq Y$ and $Z = \pi^{-1}(W) \subseteq X$ be the branch and ramification loci of π in the sense of [FW20, Proposition 4.45-4.47] (in particular, W and Z are endowed with the reduced subscheme structure). Assume that $2 \in \mathcal{O}_Y^\times$. Then the pushforward $\pi_*: \text{Shv}_{\text{ét}}(X; \mathcal{S}) \rightarrow \text{Shv}_{\text{ét}}(Y; \mathcal{S})$ preserves filtered colimits.

Proof. Since $W \subseteq Y$, $Z \subseteq X$ are closed immersions, there exist recollements

$$\begin{array}{ccccc} \text{Shv}_{\text{ét}}(\tilde{U}; \mathcal{S}) & \xleftarrow{j_U^*} & \text{Shv}_{\text{ét}}(X; \mathcal{S}) & \xleftarrow{i_Z^*} & \text{Shv}_{\text{ét}}(Z; \mathcal{S}) \\ \text{Shv}_{\text{ét}}(U; \mathcal{S}) & \xleftarrow{j_U^*} & \text{Shv}_{\text{ét}}(Y; \mathcal{S}) & \xleftarrow{i_W^*} & \text{Shv}_{\text{ét}}(W; \mathcal{S}) \end{array} \quad \begin{array}{c} \xrightarrow{j_U^*} \\ \xrightarrow{j_U^*} \end{array}$$

L: this is automatic, but [FW20, Lemma 5.36] asserts that $W \rightarrow Y$ is a closed embedding without proof

Moreover, the pushforward functor π_* is a morphism of recollements in the sense of [Sha21, Definition 2.3]. In particular, the ‘components’ of π_* (see Observation 2.4 of *loc. cit.*) are

$$\begin{aligned} \pi_*|_{\text{Shv}_{\text{ét}}(\tilde{U}; \mathcal{S})} &= j_U^* \circ \pi_* \circ j_{\tilde{U}*} = j_U^* \circ (\pi \circ j_{\tilde{U}})_* = j_U^* \circ (j_U \circ \pi|_{\tilde{U}})_* = j_U^* \circ j_{U*} \circ (\pi|_{\tilde{U}})_* \\ \pi_*|_{\text{Shv}_{\text{ét}}(Z; \mathcal{S})} &= i_W^* \circ \pi_* \circ i_{Z*} = i_W^* \circ (\pi \circ i_Z)_* = i_W^* \circ (i_W \circ \pi|_Z)_* = i_W^* \circ i_{W*} \circ (\pi|_Z)_* \end{aligned}$$

Now $j_U^* \circ j_{U*} \circ (\pi|_{\tilde{U}})_* \simeq (\pi|_{\tilde{U}})_*$ (resp. $i_W^* \circ i_{W*} \circ (\pi|_Z)_* \rightarrow (\pi|_Z)_*$) induced by the counit of the adjunction (j_U^*, j_{U*}) (resp. (i_W^*, i_{W*})) is an equivalence because j_{U*} (resp. i_{W*}) is fully faithful, thus

$$\pi_*|_{\text{Shv}_{\text{ét}}(\tilde{U}; \mathcal{S})} \simeq (\pi|_{\tilde{U}})_* \quad \pi_*|_{\text{Shv}_{\text{ét}}(Z; \mathcal{S})} \simeq (\pi|_Z)_* \quad (9.1)$$

We note for further reference the equivalences

$$\pi_*|_{\text{Shv}_{\text{ét}}(\tilde{U}; \mathcal{S})} \circ j_{\tilde{U}}^* \simeq j_U^* \circ \pi_* \quad \pi_*|_{\text{Shv}_{\text{ét}}(Z; \mathcal{S})} \circ i_W^* \simeq i_Z^* \circ \pi_* \quad (9.2)$$

L: diagram instead?

Suppose given a filtered diagram \mathcal{F}_\bullet in $\text{Shv}_{\text{ét}}(X; \mathcal{S})$ and write \mathcal{F} for its colimit. We would like to show that the canonical map $\text{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \rightarrow \pi_*(\mathcal{F})$ is an equivalence. Since j_U^*, i_W^* are jointly conservative (by definition of a recollement, see [Lur17, Definition A.8.1(e)]), it suffices to show that the canonical maps

$$\begin{aligned} j_U^* \left(\text{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \right) &\rightarrow j_U^* \pi_*(\mathcal{F}) \\ i_W^* \left(\text{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \right) &\rightarrow i_W^* \pi_*(\mathcal{F}) \end{aligned} \quad (9.3)$$

are equivalences. Since j_U^* and i_W^* preserve all colimits, the morphisms of (9.3) can be identified with the canonical maps

$$\begin{aligned} \operatorname{colim}_{\bullet} j_U^* \pi_* (\mathcal{F}_{\bullet}) &\simeq \operatorname{colim}_{\bullet} (\pi|_{\tilde{U}})_* (j_U^* \mathcal{F}_{\bullet}) \rightarrow (\pi|_{\tilde{U}})_* (j_U^* \mathcal{F}_{\bullet}) \\ \operatorname{colim}_{\bullet} i_W^* \pi_* (\mathcal{F}_{\bullet}) &\simeq \operatorname{colim}_{\bullet} (\pi|_W)_* (i_W^* \mathcal{F}_{\bullet}) \rightarrow (\pi|_Z)_* i_Z^* (\mathcal{F}), \end{aligned}$$

respectively, where we have used (9.2). Now $\pi|_W$ is an equivalence and $\pi|_{\tilde{U}}$ is finite étale by [FW20, Proposition 4.47], hence $(\pi|_{\tilde{U}})_*$ and $(\pi|_W)_*$ preserve filtered colimits. \square

10 Speculative norm for Brauer group at the level of infinity categories

We would want a functor from A^e -linear stable idempotent complete infinity categories to A^L -linear stable idempotent complete infinity categories in order to get an extension of our exact sequence to the right. Here is what I think might do it:

Construction 10.1. Let $\lambda : A^e \rightarrow A^e$ denote the involution. Consider the functor

$$\operatorname{Mod}_{\operatorname{Mod}_{A^e}^{\omega}} (\operatorname{Cat}_{\infty, \operatorname{idem}}^{st}) \xrightarrow{\left(- \otimes_{\operatorname{Mod}_{A^e}^{\omega}} \lambda^* - \right)^{hC_2}} \operatorname{Mod}_{(\operatorname{Mod}_{A^e}^{\omega})^{hC_2}} (\operatorname{Cat}_{\infty, \operatorname{idem}}^{st})$$

which we will denote by $N_{A^{hC_2}/A}$. This functor is symmetric monoidal and we have that the composite with the base change functor gives

$$\left(\mathcal{C} \otimes_{(\operatorname{Mod}_{A^e}^{\omega})^{hC_2}} \operatorname{Mod}_{A^e}^{\omega} \otimes_{\operatorname{Mod}_{A^e}^{\omega}} \lambda^* (\mathcal{C} \otimes_{(\operatorname{Mod}_{A^e}^{\omega})^{hC_2}} \operatorname{Mod}_{A^e}^{\omega}) \right)^{hC_2} \simeq \mathcal{C}^{\otimes_{(\operatorname{Mod}_{A^e}^{\omega})^{hC_2}} 2}$$

For $\mathcal{C} \in \operatorname{Mod}_{A^e}$, we have that

$$(\mathcal{C} \otimes_{\operatorname{Mod}_{A^e}} \lambda^* \mathcal{C})^{hC_2} \otimes_{\operatorname{Mod}_{A^e}^{hC_2}} \operatorname{Mod}_{A^e} \simeq \mathcal{C}^{\otimes_{\operatorname{Mod}_{A^e}} 2}$$

via the functor which forgets the C_2 -action.

Lemma 10.2. *The composite $\operatorname{PnBr}(A) \rightarrow \operatorname{br}(A^e) \rightarrow \operatorname{Pic}(\operatorname{Mod}_{\operatorname{Mod}_{A^e}^{hC_2}})$ is nullhomotopic.*

Proof. The underlying category of a Poincaré invertible category is self-dual, and so its square will vanish. Since the functor is naturally nullhomotopic so too is the composite after applying the functor $\operatorname{Pic}(-)$. \square

There is thus a map $\operatorname{PnBr}(-) \rightarrow \mathcal{F}(-)$, where $\mathcal{F}(-)$ is the fiber. Delooping both fiber sequences we see that we get a map of fiber sequences

$$\begin{array}{ccccc} \operatorname{Pic}(\operatorname{Mod}_{A^L}(\operatorname{Sp}^{C_2})) & \longrightarrow & \operatorname{PnBr}(A) & \longrightarrow & \operatorname{br}(A^e) \\ \downarrow & & \downarrow & & \downarrow = \\ \operatorname{Pic}(\operatorname{Mod}_{A^e}^{hC_2}) & \longrightarrow & \mathcal{F}(A) & \longrightarrow & \operatorname{br}(A^e) \end{array}$$

from which we see that the middle horizontal map must be an equivalence whenever $\frac{1}{2} \in A^e$ and $A^{\varphi C_2} = 0$.

References

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N: Maybe...
We would want this to be true but I don't see a proof immediately...

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