# Poincaré Schemes

Viktor Burghardt, Noah Riggenbach, Lucy Yang

**Abstract**

We do stuff

N: Change this

## Contents



# 1 Introduction

## 1.1 Azumaya algebras and their involutions

Let $X$ be a scheme. Gabber showed that if $X$ is quasi-compact and separated and admits an ample line bundle, the collection of sheaves of Azumaya $\mathcal{O}_X$-algebras up to Morita equivalence is in bijection with the torsion subgroup of $H^2_{\text{ét}}(X; \mathbb{G}_m)$ [Gab81]. On the other hand, to $\mathcal{A}$ we may associate the presentable $\mathcal{O}_X$-linear $\infty$-category $\mathcal{D}(\mathcal{A})$. If $\mathcal{A}$ and $\mathcal{A}'$ are Morita equivalent, then $\mathcal{D}(\mathcal{A}')$ and $\mathcal{D}(\mathcal{A})$ are equivalent in $\mathrm{Pr}^L_{\mathcal{O}_X}$, and that $\mathcal{A}$ is Azumaya implies that $\mathcal{D}(\mathcal{A})$ is invertible in $\mathrm{Pr}^L_{\mathcal{O}_X}$ with respect to the $\mathcal{O}_X$-linear tensor product.

L: AG notation or homotopy notation?

L: who is this observation due to?

L: cite Lieblich/de Jong? connect to twisted sheaves!

**Definition 1.1.** The *derived Brauer space* of $X$ is $\mathrm{dBr}(X) := \mathrm{Pic}\left(\mathrm{Pr}^L_{\mathcal{O}_X}\right)$.

Toën shows that the assignment $X \mapsto \mathrm{dBr}(X)$ is an étale sheaf and there is an isomorphism $\pi_0\mathrm{dBr}(X) \simeq H^1_{\text{ét}}(X;\mathbb{Z}) \times H^2_{\text{ét}}(X;\mathbb{G}_m)$ [Toë12, Corollary 2.12]. Furthermore, Toën shows that for any qcqs scheme $X$, any invertible $\mathcal{O}_X$-linear $\infty$-category $\mathcal{C}$ has a compact generator $G$; thus we may write $\mathcal{C} \simeq \mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(G)}(\mathrm{Mod}_{\mathcal{O}_X})$ and $\mathrm{End}_{\mathcal{C}}(G)$ is said to be a *derived/generalized (sheaf of) Azumaya algebras* over $\mathcal{O}_X$ (see Theorem 3.7 and Corollary 3.8 of [Toë12]). In particular, Toën's result gives concrete interpretations/realizations of all/not necessarily torsion classes in $H^2_{\text{ét}}(X;\mathbb{G}_m)$, at the cost of using derived methods/considering derived objects.

Antieau and Gepner extended Toën's work to connective ring spectra/affine spectral schemes with connective rings of functions in [AG14a].

L: cite this too?

On the other hand, the theory of anti-involutions on Azumaya algebras is an essential tool for studying the behavior of Brauer classes under multiplication by 2 and/or norm maps/corestriction. An anti-involution on an Azumaya algebra over a ring $R$ is an equivalence $\sigma\colon A \xrightarrow{\sim} A^{\mathrm{op}}$ so that $\sigma^{\mathrm{op}} \circ \sigma = \mathrm{id}$. The anti-involution is said to be of type 1 if it acts by the identity on $R =$ the center of $A$. If instead $R$ is endowed with an involution $\lambda$ so that the inclusion of the subring of fixed elements $R^\lambda \to R$ exhibits $R$ as a quadratic étale $R^\lambda$-algebra and $\sigma$ agrees with $\lambda$ on the center of $A$, then the involution $\sigma$ is said to be of *type 2*.

If an Azumaya algebra $A$ has an anti-involution, its Brauer class $[A] \in H^2_{ét}(-;\mathbb{G}_m)$ is manifestly 2-torsion. More surprisingly, if an Azumaya algebra $A$ is such that $[A]$ lies in the 2-torsion subgroup of $H^2_{ét}(-;\mathbb{G}_m)$, then there exists an Azumaya algebra $A'$ in the Brauer class of $A$ admitting an anti-involution; the result was proved for $R$ a field by Albert (in fact Albert proves that one can take $A' = A$), for an arbitrary ring by Saltman, and for schemes $X$ with $\frac{1}{2} \in \mathcal{O}_X$ by Parimala–Srinivas [Alb61, §9 Theorem 19; Sal78, Theorem 3.1(a); PS92, Theorem 1].

On the other hand, consider an étale cover $X \to Y$ of degree 2 where $\frac{1}{2} \in \mathcal{O}_Y$, and let $\lambda$ denote the nontrivial $C_2$-Galois action on $X$. There is an *involutive Brauer group* $\mathrm{Br}(X,\lambda)$ consisting of equivalence classes of sheaves of Azumaya $\mathcal{O}_X$-algebras with involutions of the second kind [PS92, p.216]. Parimala–Srinivas showed that the involutive Brauer group sits in an exact sequence $\mathrm{Pic}\, X \xrightarrow{N} \mathrm{Pic}\, Y \to \mathrm{Br}(X,\lambda) \to \mathrm{Br}(X) \xrightarrow{N} \mathrm{Br}(Y)$ [PS92, Theorem 2].

First and Williams observed that the aforementioned two cases comprise two extreme ends/special cases of a spectrum: multiplication by 2 on $H^2_{ét}(-;\mathbb{G}_m)$ may be regarded as a cohomological $C_2$-transfer map along the 'quotient' map $X = X$, where $X$ is regarded as a scheme with trivial $C_2$-action [FW20, §1.2]. In other words, the former comprises trivial $C_2$-actions with 'everywhere ramified' quotient map, whereas the latter comprises free $C_2$-actions with nowhere ramified quotient maps. Moreover, First and William observe that a quotient involves a choice/is extra data; from the perspective of the stacky quotient $X \to X//C_2$, the $C_2$-action on $X$ is free. Thus in order to study involutions systematically, it is necessary to specify a scheme with involution and the choice of a quotient.

L: connect to 'many choices of different Poincaré structures' on a given $\infty$-category eventually

The study of involutions on Azumaya algebras goes hand-in-hand with/is inextricably linked to the study of symmetric bilinear/hermitian forms on vector bundles: étale-locally, (classical) Azumaya algebras can be described as endomorphism algebras of vector bundles $\mathcal{A} \simeq \mathcal{E}\mathrm{nd}_X(V)$. Taking transposes and conjugating by an isomorphism $q\colon V \to V^\vee$ adjoint to a nondegenerate pairing $q\colon V \otimes V \to \mathcal{O}_X$ comprise a 'prototypical' example of an involution on $\mathcal{E}\mathrm{nd}_X(V)$. In [Cal+20a; Cal+20b; Cal+21], Calmès–Dotto–Harpaz–Hebestreit–Land–Moi–Nardin–Nikolaus–Steimle introduce a framework for talking about objects of stable $\infty$-categories equipped with the data of nondegenerate hermitian forms, expanding upon an idea introduced in [Lur11]. Motivated by this connection between involutions and duality (cf. [Cal+20a, §3]), we use Poincaré $\infty$-categories to define a derived enhancement of the involutive Brauer group.

L: this is *so close* to Spec of a Poincaré ring but also not quite.

## 1.2 Main results

**Theorem 1.2.** *Let $\underline{A}$ be an affine Poincaré scheme with underlying $\mathbb{E}_\infty$-ring spectrum with involution $A$. Then the natural maps*

$$\pi_i(\mathrm{PnPic}(\underline{A})) \to \pi_i(\mathrm{Pic}(A))$$

*are surjective on 2-torsion.*

L: insert adjectives

N: I think there is some interaction with the homotopy fixed points, or maybe even the genuine fixed points

**Theorem 1.3.** *Let $A$ be an $\mathbb{E}_\infty$ ring with involution, and let $\underline{NA}$ be the associated Tate affine Poincaré scheme. Let $\mathrm{Br}_\nu(A)$ be the Brauer group of Azumaya algebras over $A$ with involution. Then the natural map*

$$\mathrm{PnBr}(\underline{NA}) \to \mathrm{Br}_\nu(A)$$

N: I think

*is an equivalence*

N: probably of $\mathbb{E}_\infty$ dodads

**Theorem 1.4.** *The functors* $\mathrm{PnPic}, \mathrm{PnBr} : \mathrm{APS} \to \mathrm{Sp}$ *are fppf sheaves.*

**Theorem 1.5.** *There is a Poincaré group scheme* $\mathbb{G}_m^{\text{♀}}$ *such that*

$$B\mathbb{G}_m^{\text{♀}} \simeq \mathrm{PnPic}$$

*as fppf stacks.*

## 1.3 Outline

## 1.4 Acknowledgements

The authors wish to thank the Institute for Advanced Study and the organizers of the 2024 Park City Mathematics institute on motivic homotopy theory. Also thank: Columbia, Ben Antieau...??

## 1.5 Conventions

L: Will 'Azumaya algebra' refer to the classical/discrete ones or the derived/generalize ones?

| | |
|---|---|
| $\mathrm{Br}^{\mathrm{p}}$ | Poincaré Brauer space |
| $\mathrm{CAlg}$ | $\infty$-category of $\mathbf{E}_\infty$-ring spectra |
| $\mathrm{CAlg}(\mathcal{S})$ | $\infty$-category of $\mathbf{E}_\infty$-spaces |
| $\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$ | $\infty$-category of grouplike $\mathbf{E}_\infty$-spaces |
| $\mathrm{CAlg}^{\mathrm{p}}$ | $\infty$-category of Poincaré ring spectra |
| $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ | $\infty$-category of small stable $\infty$-categories and exact functors |
| $\mathrm{Cat}_\infty^{\mathrm{p}}$ | $\infty$-category of Poincaré $\infty$-categories |
| $\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}$ | $\infty$-category of idempotent complete Poincaré $\infty$-categories |
| $\mathrm{Pic}^{\mathrm{p}}$ | Poincaré Picard space |
| $\mathcal{S}$ | $\infty$-category of spaces |
| $\mathrm{Sp}$ | $\infty$-category of spectra |

# 2 Poincaré Structures on Compact Modules

We will use this section to recall notions and results about Poincaré $\infty$-categories which we require in the sections to follow. This section can safely be skipped by anyone with extensive knowledge of Poincaré $\infty$-categories, as found in [Cal+20a].

**Notation 2.1.** Let $R$ be an $\mathbf{E}_\infty$-ring spectrum. We will drop $\mathbf{E}_\infty$ from our notation and simply call $R$ a *ring spectrum*. Moreover, we will denote the $\infty$-category $\mathrm{CAlg}(\mathrm{Sp})$ of commutative algebra objects in the $\infty$-category of spectra $\mathrm{Sp}$ by $\mathrm{CAlg}$.

Let $R$ be a ring spectrum and let $\mathrm{Mod}_R$ be the $\infty$-category of modules over $R$. We will study Poincaré structures on the $\infty$-category $\mathrm{Mod}_R^\omega$ of compact modules over $R$.

V: -characterizati in terms of modules with genuine involution, -characterizati of symmetric monoidal structures, -Pn

# 3 Poincaré Schemes

We will now specify the objects which we are able to take the Poincaré Picard and Brauer spaces of. These will be schemes which are equipped with a Poincaré structure on their derived categories which is compatible with the symmetric monoidal structure. While this definition is simple and convenient, we find it both technically useful and psychologically comforting to have a definition of such objects closer to a scheme with an involution. We will start by looking at affine objects.

## 3.1 Poincaré rings

**Definition 3.1.** Let $R$ be a ring spectrum. A *Poincaré structure* on $R$ is the following data:

- A $C_2$-action on $R$ via maps of ring spectra, i.e. a functor $\lambda : BC_2 \to \mathrm{CAlg}$.
- An $\mathbb{E}_\infty$-$R$-algebra $R \to C$.
- An $\mathbb{E}_\infty$-$R$-algebra map $C \to R^{tC_2}$.

Here $R^{tC_2}$ is the Tate construction with respect to the given action. Since the Tate construction is lax symmetric monoidal, $R^{tC_2}$ is naturally an $R$-algebra via the Tate-valued norm. A ring spectrum equipped with a Poincaré structure will be called a *Poincaré ring spectrum*.

**Remark 3.2.** In [Cal+20a, discussion immediately preceding Examples 5.4.10], Poincaré ring spectra are referred to as $\mathbf{E}_\infty$-*ring spectra with genuine involution*.

**Remark 3.3.** Let $R$ be a ring spectrum. By [Cal+20a, Corollary 5.4.8], a Poincaré structure on $R$ gives rise to a symmetric monoidal lift of $\mathrm{Mod}_R^\omega$ to the symmetric monoidal $\infty$-category of Poincaré $\infty$-categories $\Qoppa_R : (\mathrm{Mod}_R^\omega)^{\mathrm{op}} \to \mathrm{Sp}$. Furthermore, the structure map $R \to C$ gives a canonical lift of $R \in \mathrm{Mod}_R^\omega$ to a Poincaré object $(R, q) \in \mathrm{Pn}\,(\mathrm{Mod}_R^\omega, \Qoppa_R)$.

**Remark 3.4.** A Poincaré structure on a ring spectrum $R$ with a $C_2$-action via maps of ring spectra is a factorization $R \to C \to R^{tC_2}$ in CAlg of the Tate Frobenius $R \to R^{tC_2}$.

**Observation 3.5.** There is a forgetful functor from $C_2$-$\mathbb{E}_\infty$-algebras to Poincaré rings which forgets the $C_2$-equivariance of the map $R \to R^{\varphi C_2}$.

**Remark 3.6.** By [Cal+20a, §5.1], the assignment $(R, R \to C \to R^{tC_2}) \mapsto (\mathrm{Mod}_R^\omega, \Qoppa_R)$ promotes to a symmetric monoidal functor $\mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{p}})$.

**Notation 3.7.** Let $R$ be an $\mathbb{E}_\infty$-ring spectrum. We will denote by $\underline{R}$ the spectrum $R$ with trivial $C_2$-action. More precisely, $\underline{R} : BC_2 \to \mathrm{Sp}$ is the constant functor.

L: This is commonly used for constant Mackey functors–could be ambiguous

**Example 3.8.** Let $R$ be a ring spectrum with a $C_2$-action. If $2 \in \pi_0(R)$ is invertible, we have $\underline{R}^{tC_2} \simeq 0$ by [NS18, Lemma I.2.8]. A Poincaré structure on $R$ is equivalent to the data of an $\mathbb{E}_\infty$-$R$-algebra $R \to C$.

**Example 3.9.** Let $R$ be a ring spectrum equipped with a $C_2$-action via maps of ring spectra. The Tate-valued norm endows $R^{tC_2}$ with a natural $R$-algebra structure, which induces a Poincaré structure on $R$ given by the factorization $R \xrightarrow{\mathrm{id}} R \to R^{tC_2}$. We will call this Poincaré structure the *Tate Poincaré structure on* $R$ and will denote it by $(R, \Qoppa_R^{\mathrm{t}})$.

**Example 3.10.** The sphere spectrum $\mathbb{S}$ together with the Tate Poincaré structure will be called the *universal Poincaré ring spectrum* (see [Cal+20a, §4.1]). We will denote it by $(\mathbb{S}, \Qoppa_u)$.

**Remark 3.11.** Let $(R, \Qoppa)$ be a ring spectrum associated to a factorization $R \to C \to R^{tC_2}$. A factorization of the map $C \to R^{tC_2}$ through $R^{hC_2}$ induces a section of the canonical map $\Qoppa(R) \to \hom_R(R, C) \simeq C$. In that case, we have a splitting $\Qoppa(R) \simeq R_{hC_2} \oplus C$.

V: reference pullback that characterizes all quadratic functors

**Example 3.12.** The Tate Frobenius for the sphere spectrum factors through $\mathbb{S}^{hC_2}$. Therefore, Remark 3.11 implies $\Qoppa_u(\mathbb{S}) \simeq \mathbb{S}_{hC_2} \oplus \mathbb{S} \simeq \Sigma^\infty(\mathbb{P}_\mathbb{R}^\infty) \oplus \mathbb{S}$.

**Example 3.13.** Let $R$ be a ring spectrum equipped with a $C_2$-action via maps of ring spectra. The identity map $\mathrm{id} : R^{tC_2} \to R^{tC_2}$ induces a Poincaré structure on $R$ given by the factorization $R \to R^{tC_2} \xrightarrow{id} R^{tC_2}$. We will call this Poincaré structure the *symmetric Poincaré structure on* $R$.

**Example 3.14.** Let $R$ be a connective ring spectrum equipped with a $C_2$-action via maps of ring spectra. The connective cover $\tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$ of $R^{tC_2}$ induces a Poincaré structure on $R$ given by the factorization $R \to \tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$. We will call this Poincaré structure the *genuine symmetric Poincaré structure on* $R$.

**Example 3.15.** Let $R$ be a commutative ring endowed with an involution $\sigma \colon R \xrightarrow{\sim} R$. Write $\underline{R}^\sigma$ for the $C_2$-Green functor with $C_2$-fixed points $R^{C_2}$, where $R^{C_2}$ denotes the strict fixed points of the $C_2$-action on $R$, and underlying object $R$. The Mackey functor $\underline{R}^\sigma$ is a $C_2$-$\mathbb{E}_\infty$ ring, therefore in particular we may regard it as a Poincaré ring by Observation 3.5. This is a special case of Example 3.14.

V: copy more examples from notes

## 3.2 From schemes with involution to Poincaré structures on module categories

We will now turn our attention to the non-affine case. In this setting we will again want to work with schemes with some notion of a genuine $C_2$-structure as our model, and then show that this leads to the structure of a scheme together with a symmetric monoidal structure on its derived category.

Philosophically, a scheme with genuine $C_2$-action should, via a recollement, be given by a scheme $X$ with an involution together with a choice of genuine $C_2$ quotient $X \to Y$ satisfying certain conditions. It turns out that such a notion has already appeared in the literature on Azumaya algebras with involution.

**Recollection 3.16** ([FW20, Remark 4.20])**.** Let $X$ be a scheme with an involution $\lambda\colon X \to X$. A map $\pi\colon X \to Y$ is called a good quotient of $X$ relative to $\lambda$ if $\pi$ is $C_2$-invariant and affine and $\pi_\#\colon \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ induces an isomorphism $\mathcal{O}_Y \simeq (\pi_*\mathcal{O}_X)^{C_2}$. A good quotient of $X$ exists if and only if every $C_2$-orbit is contained in an affine open subscheme, in which case it is unique up to isomorphism.

**Definition 3.17.** Define a category $\mathrm{qSch}^{C_2}$ so that

- an object of $\mathrm{qSch}^{C_2}$ consists of the data of qcqs schemes $X$ and $Y$, an involution $\lambda\colon X \to X$, and a morphism $p\colon X \to Y$ which exhibits $Y$ as a *good quotient* of the involution on $X$ in the sense of [FW20, Remark 4.20].
- a morphism from $(X, \lambda, Y, p)$ to $(Z, \nu, W, q)$ consists of a $C_2$-equivariant morphism $X \to Z$ and a morphism $Y \to W$ so that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & W \end{tikzcd}$$

commutes.

**Observation 3.18.** Suppose $R$ is a discrete commutative ring with a $C_2$-action, and regard $R$ as a $C_2$-Mackey functor via Example 3.15. Then $\operatorname{Spec} R \to \operatorname{Spec}(R^{C_2})$ may be regarded as an object of $\mathrm{qSch}^{C_2}$.

**Remark 3.19.** Suppose $(X, \lambda, Y, p)$ is an object of $\mathrm{qSch}^{C_2}$ and $j\colon U \to Y$ is a flat map. Then $(X_U, \lambda|_U, U, p|_U)$ is an object of $\mathrm{qSch}^{C_2}$. Affineness and invariance under the $C_2$-action are stable under pullback, so it suffices to show that $j^*(\pi)$ satisfies $\mathcal{O}_U \simeq (j^*(\pi))_* (\mathcal{O}_{j^*U})^{C_2}$. This follows from the proof of [FW20, Theorem 4.35(i)].

**Proposition 3.20.** *Write $U\colon \mathrm{qSch}^{C_2} \to \mathrm{qSch}$ for the functor so that $U(X, \lambda, Y, p) = X$. The category $\mathrm{qSch}^{C_2}$ has a symmetric monoidal structure $\boxtimes$ so that $U$ is symmetric monoidal, where $\mathrm{qSch}$ is endowed with the product symmetric monoidal structure.*

*Proof.* If $X$, $Z$ are schemes with involutions $\lambda_X$, $\lambda_Z$, then $\lambda_X \times \lambda_Z$ endows $X \times Z$ with an involution. It suffices to show that $X \times Z$ admits a good quotient, as a good quotient is a categorical quotient and is therefore unique up to isomorphism. By [FW20, Remark 4.20], a good quotient exists if and only if every $C_2$-orbit is contained in an affine open subscheme. Consider a $C_2$-orbit in $X \times Z$. Its image under the projection $\pi_1\colon X \times Z \to X$ (resp. $\pi_2\colon X \times Z \to Z$) is contained in an affine open subscheme $U \subseteq X$ (resp. $V \subseteq Z$). Thus the orbit under consideration is contained in $U \times Z$, which is affine. ☐

**Construction 3.21.** Assume that $X$ has a *good quotient* $Y$ in the sense of [FW20, Remark 4.20]. We write $p\colon X \to Y$ for the quotient map. Let $j\colon \operatorname{Spec} A \simeq U \subseteq Y$ be an affine open subscheme of $Y$. Because $p$ is an affine map, the fiber product $\operatorname{Spec} B := \operatorname{Spec} A \times_Y X$ is an affine open of $X$ which is invariant under the $C_2$-action. In particular $\operatorname{Spec} B$ inherits a $C_2$-action from $X$ (hence so does its ring of functions $B$). Now $A \to B$ acquires the structure of a $C_2$-Green functor $\underline{\mathcal{O}}(U)$. Regarding $\underline{\mathcal{O}}(U)$ as a $C_2$-spectrum, by the isotropy separation sequence, we have an equivalence of $A$-modules $\underline{\mathcal{O}}(U)^{\varphi C_2} \simeq \operatorname{cofib}(\mathrm{tr}\colon B_{hC_2} \to A)$.

**Lemma 3.22.** *Let $X$ be a scheme with an involution. Assume that $X$ has a* good quotient *$Y$ in the sense of [FW20, Remark 4.20], and write $p\colon X \to Y$ for the quotient map.*

*(i) The assignment of Construction 3.21 lifts to a contravariant functor from (the nerve of) the category of affine opens of $Y$ to the $\infty$-category of Poincaré rings/$C_2$-$\mathbb{E}_\infty$-rings/Tambara functors.*

*(ii)* *The presheaf $\underline{\mathcal{O}}$ of (i) defines a Zariski sheaf.*

*(iii)* *Write $p_*\mathcal{O}_X$ for the sheaf of $\mathbb{E}_\infty$-$\mathcal{O}_Y$-algebras (all functors are derived). Then the pushforward $p_*$ induces an equivalence $\mathcal{D}(X) \xrightarrow{\sim} \mathrm{Mod}_{p_*\mathcal{O}_X}$.*

*Proof.* Part (i) follows from a similar argument to [Yan23, Theorem 5.1]; functoriality follows from noting that $\tau_{\geq 0}$ is a functor. Part (ii) follows from Lemma 3.23. To prove part (iii), consider a Zariski cover $\{j_i : U_i \to Y\}$ of $Y$ by affine opens. By Zariski descent, $\mathcal{D}(X) \simeq \lim_{p^*(j_i)=p\times_Y j_i : U_i\times_Y X\to X} \mathrm{Mod}_{\mathcal{O}_X(U_i\times_Y X)}$ and $\mathrm{Mod}_{p_*(\mathcal{O}_X)} \simeq \lim_{j_i : U_i\to Y} \mathrm{Mod}_{p_*\mathcal{O}_X(U_i)}$, hence the result follows. □

**Lemma 3.23.** *Let $K$ be a simplicial set, and let $f : K^{\triangleleft} \to C_2\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_2})$ be a diagram. Then $f$ is a limit diagram if and only if $f^e : K^{\triangleleft} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ and $f^{C_2} : K^{\triangleleft} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ are both limit diagrams.*

*Proof.* The result follows from the observation that limits in $\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right)$ are computed in $\mathrm{Sp}^{C_2}$. □

**Construction 3.24.** Let $p : X \to Y$ as before. Consider the composites

$$\begin{aligned} \mathrm{Mod}_{\underline{\mathcal{O}}} &: \mathrm{Op}(Y)^{\mathrm{op}} \xrightarrow{\underline{\mathcal{O}}} C_2\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{\mathrm{Mod}_{(-)}} \mathcal{C}\mathrm{at} \\ \mathrm{Mod}^{\otimes}_{\underline{\mathcal{O}}} &: \mathrm{Op}(Y)^{\mathrm{op}} \xrightarrow{\underline{\mathcal{O}}} C_2\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{\mathrm{Mod}^{\otimes}_{(-)}} C_2 \otimes \mathcal{C}\mathrm{at}\,, \end{aligned} \tag{3.25}$$

where $C_2 \otimes \mathcal{C}\mathrm{at}$ denotes the $\infty$-category of (small) $C_2$-symmetric monoidal $C_2$-$\infty$-categories. In the notation of Construction 3.21, this functor sends the affine open $\mathrm{Spec}\, A \subseteq Y$ to the category of modules in $C_2$-spectra over the $C_2$-$\mathbb{E}_\infty$-algebra which has underlying $C_2$-Mackey functor $A \to B$. Define $\mathrm{Mod}_{\underline{\mathcal{O}}}$, $\mathrm{Mod}^{\otimes}_{\underline{\mathcal{O}}}$ to be the limits in $\mathcal{C}\mathrm{at}$, $C_2 \otimes \mathcal{C}\mathrm{at}$, resp. of the functors in (3.25). In particular, if we write $s : \int \mathrm{Mod}_{\underline{\mathcal{O}}} \to \mathrm{Op}(Y)^{\mathrm{op}}$ for the cocartesian fibration obtained by taking the Grothendieck construction on (3.25), an object of $\mathrm{Mod}_{\underline{\mathcal{O}}}$ is a cocartesian section of $s$. In other words, it is a choice, for each affine open $\mathrm{Spec}\, A$ of $Y$ (same notation as before), of a module over the $C_2$-$\mathbb{E}_\infty$-algebra which has underlying $C_2$-Mackey functor $A \to B$ which glue compatibly.

L: bleh...cardinals

L: invent better notation later

Observe that for each $A \to B$, there is a quadratic norm functor $N^{C_2} : \mathrm{Mod}_B(\mathrm{Sp}) \to \mathrm{Mod}_{N^{C_2}B}\left(\mathrm{Sp}^{C_2}\right)$ and a quadratic relative norm functor $N^{C_2} : \mathrm{Mod}_B(\mathrm{Sp}) \to \mathrm{Mod}_{A\to B}\left(\mathrm{Sp}^{C_2}\right)$.

**Construction 3.26.** Let $X$ be a scheme with an involution, and let $p : X \to Y$ exhibit $Y$ as a good quotient of $X$. Assume that $p$ is affine. The norm functors (resp. relative norm functors) $N_e^{C_2}$ assemble under Construction 3.21 to a 'global' norm functor $N_Y^{C_2} : \pi_\#\mathcal{O}_X\,\mathrm{Mod} \to N^{C_2}\pi_\#\mathcal{O}_X\,\mathrm{Mod}$ (resp. relative norm functor $N_Y^{C_2} : \pi_\#\mathcal{O}_X\,\mathrm{Mod} \to \underline{\mathcal{O}}\,\mathrm{Mod}$). Moreover, these functors are quadratic.

For each affine open $j : \mathrm{Spec}\, A \subseteq Y$, write $B = \Gamma(\mathcal{O}_{\mathrm{Spec}\, A\times_Y X})$, consider the composite

$$\pi_\#\mathcal{O}_X\,\mathrm{Mod} \xrightarrow{j^*} \mathrm{Mod}_B(\mathrm{Sp}) \xrightarrow{N^{C_2}} \mathrm{Mod}_{N^{C_2}B}(\mathrm{Sp}^{C_2}) \xrightarrow{-\otimes_{N^{C_2}B}(A\to B)} \mathrm{Mod}_{A\to B}(\mathrm{Sp}^{C_2})\,,$$

where the last map is base change along the map $N^{C_2}B \to (A \to B)$ which is a structure map for the $C_2$-$\mathbb{E}_\infty$-algebra structure on $A \to B$. Now since quadratic functors are closed under limits [Lur17, Theorem 6.1.1.10] and $N_Y^{C_2}$ can be written as a limit of a diagram of quadratic functors, $N_Y^{C_2}$ is also quadratic.

L: todo: use effective descent/limit definition for $\underline{\mathcal{O}}$-modules.

**Definition 3.27.** Varying $X \to Y$, Constructions 3.24 and 3.26 define a functor

$$\begin{aligned} \left(\mathrm{qSch}^{C_2}\right)^{\mathrm{op}} &\to C_2 \otimes \mathcal{C}\mathrm{at} \\ (X, \lambda, Y, p) &\mapsto \underline{\mathrm{Mod}}_{\underline{\mathcal{O}}}\left(\underline{\mathrm{Sp}}^{C_2}\right) \end{aligned}$$

L: want: codomain consists of $C_2$-stable $C_2$-presentable $C_2$-$\infty$-categories

**Definition 3.28.** Suppose $\mathcal{C}$ is a $C_2$-stable $C_2$-symmetric monoidal $C_2$-$\infty$-category. Define a functor

$$\begin{aligned}\mathrm{eInv}\colon C_2\otimes\mathcal{C}\mathrm{at}^{\mathrm{ex}} &\to \mathcal{S}\\ (\mathcal{C},\otimes) &\mapsto \left(\mathrm{C}^{C_2}\right)^{\simeq}\times_{(\mathrm{C}^e)^{\simeq,hC_2}}\mathrm{Pic}(\mathrm{C}^e)^{\simeq,hC_2}.\end{aligned}$$

In other words, eInv sends a $C_2$-symmetric monoidal $C_2$-$\infty$-category to the full subgroupoid of $\mathcal{C}^{C_2}$ on those objects $L$ so that $L^e$ is an invertible object in $\mathcal{C}^e$.

Write $\widetilde{\mathrm{eInv}}$ for the Grothendieck construction on eInv.

There is a functor

$$\begin{aligned}\widetilde{\mathrm{eInv}} &\to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Cat}^h\right)\\ (\mathcal{C},L) &\mapsto (\mathcal{C}^e,\mathcal{C}^e\xrightarrow{N_{\mathcal{C}}}\mathcal{C}^{C_2}\xrightarrow{\hom_{\mathcal{C}^{C_2}}(-,L)}\mathrm{Sp})\end{aligned} \tag{3.29}$$

**Lemma 3.30.** *The functor of (3.29) lifts to a functor* $\widetilde{\mathrm{eInv}}\to\mathrm{Cat}^{\mathrm{p}}_\infty$.

L: Pretty sure distributive norm functors are 2-excisive (results here should generalize readily)...if not, can define the problem away.

**Lemma 3.31.** *Let $X$ be a scheme with involution $\sigma\colon X\xrightarrow{\sim}X$ equipped with a good quotient $\pi\colon X\to Y$. Let $L$ be a line bundle on $Y$. Then the canonical map*

$$L\to\pi_\#\pi^*L \tag{3.32}$$

*promotes (3.32) to a sheaf of $\underline{\mathcal{O}}$-modules on $Y$. We will write $\underline{L}$ for (3.32).*

*Proof.* Follows from naturality of the unit and the canonical identification $\pi^*\mathcal{O}_Y$ with $\mathcal{O}_X$. □

L: Example: Special case where $X$ has trivial $C_2$ action and $X=Y$.

**Definition 3.33.** Let $X$ be a scheme with involution $\sigma\colon X\xrightarrow{\sim}X$ equipped with a good quotient $\pi\colon X\to Y$. Let $L$ be a line bundle on $Y$. Define $\Qoppa_{\sigma,L}$ to be the functor

$$\mathrm{Perf}_X^{\mathrm{op}}\xrightarrow{\pi_\#}\pi_\#\mathcal{O}_X\,\mathrm{Mod}^{\omega,\mathrm{op}}\xrightarrow{N^{C_2}}N^{C_2}\pi_\#\mathcal{O}_X\,\mathrm{Mod}\left(\mathrm{Sp}^{C_2}\right)^{\mathrm{op}}\xrightarrow{\hom_{N^{C_2}\pi_\#\mathcal{O}_X}(-,\underline{L})}\mathrm{Sp},$$

where $\underline{L}$ is a $\underline{\mathcal{O}}$-module by Lemma 3.31 and $\underline{\mathcal{O}}$ is a $N^{C_2}\pi_\#\mathcal{O}_X$-algebra by Lemma 3.22. By Construction 3.26 and the fact that the composite of an exact (1-excisive) functor and an $m$-excisive functor is $m$-excisive (see [Bar+22, §2.2]), $\Qoppa_{\sigma,L}$ is quadratic.

**Example 3.34.** Suppose $L=\mathcal{O}_Y$. Then we drop $L$ from notation and the quadratic functor $\Qoppa_\sigma$ of Definition 3.33 takes the form

$$\mathrm{Perf}_X^{\mathrm{op}}\xrightarrow{\pi_\#}\pi_\#\mathcal{O}_X\,\mathrm{Mod}^{\omega,\mathrm{op}}\xrightarrow{N^{C_2}}N^{C_2}\pi_\#\mathcal{O}_X\,\mathrm{Mod}\left(\mathrm{Sp}^{C_2}\right)^{\mathrm{op}}\xrightarrow{\hom_{N^{C_2}\pi_\#\mathcal{O}_X}(-,\underline{\mathcal{O}})}\mathrm{Sp}.$$

**Lemma 3.35.** *Let $X$ be a scheme with involution $\sigma\colon X\xrightarrow{\sim}X$, and let $Y$ be a good quotient of $X$. Let $L$ be a line bundle on $Y$, and let $\Qoppa_{\sigma,L}$ be the quadratic functor on* $\mathrm{Perf}_X$ *of Definition 3.33. Then the bilinear part of $\Qoppa_{\sigma,L}$ agrees with that of Observation* **??**. *In particular,* $(\mathrm{Perf}_X,\Qoppa_{\sigma,L})$ *is a Poincaré $\infty$-category.*

*Proof.* By definition of the bilinear part of a quadratic functor, it suffices to show that there is an equivalence $\hom_{\pi_\#\mathcal{O}_X\,\mathrm{Mod}}\left(\pi_\#E\otimes_{\pi_\#\mathcal{O}_X}\pi_\#E,\pi_\#\mathcal{O}_X\right)\simeq\hom_{\mathcal{O}_X\,\mathrm{Mod}}\left(E\otimes_{\mathcal{O}_X}\sigma^*E,\mathcal{O}_X\right)$ for any perfect complex $E$ on $X$. This follows from Lemma 3.22(iii). □

**Remark 3.36.** Compare the description of the space of bilinear forms in Lemma 3.35 with the description of a $\delta$-hermitian form $H$ in [PS92, p. 216].

# 4 The Poincaré Picard space

Recall that the Poincaré space functor $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{CAlg}(\mathcal{S})$ is lax symmetric monoidal with respect to tensor product of Poincaré $\infty$-categories and smash product of $\mathbf{E}_\infty$-spaces [Cal+20a, Corollary 5.2.8]. In particular, we can consider invertible objects in $\mathrm{Pn}(A)$ for a Poincaré ring spectrum $A$.

**Definition 4.1.** Let $A$ be a Poincaré ring spectrum. We define the *Picard space of $A$* to be

$$\mathrm{Pic}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Pn}(A)).$$

For $(X, \lambda, Y, p) \in \mathrm{qSch}^{C_2}$ we similarly define

$$\mathrm{Pic}^{\mathrm{p}}(X, \lambda, Y, p) := \mathrm{Pic}(\mathrm{Pn}(\mathrm{Perf}_X, \Qoppa_\sigma)).$$

**Remark 4.2.** Let $(\mathrm{Mod}^\omega_R, \Qoppa_R)$ be a Poincaré ring spectrum, where $(M_R = R, N_R = R^{\varphi C_2}, R^{\varphi C_2} \to R^{tC_2})$ is the module with genuine involution associated to $\Qoppa_R$. Then a point in the Poincaré Picard space is the data of a pair $(\mathcal{L}, q)$, where $\mathcal{L}$ is an invertible module in $\mathrm{Mod}^\omega_R$ and $q$ is a point in $\Omega^\infty \Qoppa_R(\mathcal{L})$. By [Cal+20a, Proposition 1.3.11], the data of $q$ is equivalent to the data of points in the lower left and upper right corner of the square

$$\begin{tikzcd}
\Qoppa(\mathcal{L}) \arrow[r] \arrow[d] & \hom_R(\mathcal{L}, R^{\varphi C_2}) \ni \ell(q) \arrow[d] \\
b(q) \in \hom_{R\otimes R}(\mathcal{L}\otimes\mathcal{L}, R)^{hC_2} \arrow[r] & \hom_R(\mathcal{L}, R^{tC_2})
\end{tikzcd} \tag{4.3}$$

and a path between their images in the lower right corner. In particular, the adjoint of $b(q)$ must define a nondegenerate hermitian form on $\mathcal{L}$, that is, an equivalence $\mathcal{L} \simeq \hom_R(\mathcal{L}, R^*)$ where $R^*$ is considered as an $R$-module via the action of the generator of $C_2$.

L: add equivari-ance/symmetry data

Write $(\mathcal{L}^\vee, q^\vee)$ is for the inverse of $(\mathcal{L}, q)$. By definition of invertibility, there exists an $R$-linear map $\ell(q^\vee)\colon \mathcal{L}^\vee \to R^{\varphi C_2}$ so that the following diagram commutes

$$\begin{tikzcd}
\mathcal{L}\otimes_R \mathcal{L}^\vee \arrow[r, "\ell(q)\otimes\ell(q^\vee)"] \arrow[d, "\mathrm{ev}", "\sim"'] & R^{\varphi C_2}\otimes_R R^{\varphi C_2} \arrow[d, "\text{multiplication}"] \\
R \arrow[r, "\text{given}"] & N_R
\end{tikzcd} \tag{4.4}$$

**Lemma 4.5.** *Let $(R, \Qoppa)$ be a connective Poincaré ring spectrum. Then, for any integer $n$, the spectrum $\Qoppa(\Sigma^n R)$ is $(-2n)$-connective.*

V: define connectivity and make conditions here precise. As stated this works for R and C connective. More precisely, $conn(\Qoppa(\Sigma^n R)$ $\min(conn(\Sigma^-$

*Proof.* This follows from the fiber sequence

$$(\Sigma^{-2n}R)_{hC_2} \to \Qoppa(\Sigma^n R) \to \hom_R(\Sigma^n R, C) \simeq \Sigma^{-n} C.$$

$\square$

**Remark 4.6.** The functor $\mathrm{Pic}^{\mathrm{p}} : \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$ preserves certain structures. Let $A$ be a Poincaré ring spectrum. Since $A$ is a module over $(\mathbb{S}, \Qoppa_u)$, the space $\mathrm{Pic}^{\mathrm{p}}(A)$ is something over $\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Qoppa_u)$.

V: write out details

Since the forgetful functor $\mathrm{Pn}(A) \to \mathrm{Mod}^\omega_A$ is symmetric monoidal we get an induced map

$$U : \mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)$$

of spectra. For a point $(\mathcal{L}, q) \in \pi_0(\mathrm{Pic}^{\mathrm{p}}(A))$ we will refer to $\mathcal{L} := U(\mathcal{L}, q)$ as the *underlying invertible module*. Note that the $A$-module $A^*$ is (nonequivariantly) isomorphic to $A$ via the involution, and so the fact that $\mathcal{L} \simeq \hom_A(\mathcal{L}, A^*)$ forces $\mathcal{L}$ to be 2-torsion. In particular we get a refined map

$$U : \mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)[2]$$

which factors the underlying invertible module map.

V: there is no truth in here yet. Work in progress. Noah had an example using Witt vectors which

**Example 4.7.** Let $(\mathbb{S}, \Qoppa_u)$ be the universal Poincaré ring spectrum from Example 3.10. The only 2-torsion element of $\mathrm{Pic}(\mathbb{S}) \simeq \mathbf{Z}$ is $\mathbb{S}$. Therefore, any element in $\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Qoppa_u)$ lies above $\mathbb{S}$ under $U$. With Remark 3.12, we conclude $\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Qoppa_u)) \simeq \pi_0(\mathbb{S}_{hC_2} \oplus \mathbb{S}^\times)^\times \simeq (\mathbf{Z} \times \mathbf{Z/2})^\times \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$.

**Remark 4.8.** One might hope that the map $\mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)[2]$ is close to an equivalence. This however is quite far from being true. Let $k$ be a finite field of characteristic 2, and let $\mathbb{S}_{W(k)}$ be the spherical Witt vectors on $k$ in the sense of [Lur18, Example 5.2.7]. Then by [Nik23, Example 3.4] we know that $\mathbb{S}_{W(k)}$ must satisfy that the map $\varphi_2 : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}^{tC_2}$ is an equivalence where the action is trivial.

V:did we mod out by isomorphisms here?

Consider now the Poincaré ring $(\mathrm{Mod}^{\omega}_{\mathbb{S}_{W(k)}}, \Qoppa^u_{\mathbb{S}_{W(k)}})$ where $\Qoppa^u_{\mathbb{S}_{W(k)}}$ is the Tate Poincaré structure. We have that $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) \cong \mathbb{Z}$ and is generated by $\Sigma\mathbb{S}_{W(k)}$. To see this note that for $\mathcal{L}$ an invertible module over $\mathbb{S}_{W(k)}$, $\mathcal{L}$ must be bounded below since otherwise it would not be perfect. Then for $\pi_n(\mathcal{L})$ its bottom homotopy group, we have that $\pi_n(\mathcal{L}/2) \cong k$ since it must be an invertible $k$-module and $k$ is a field. Thus we get a map $\mathbb{S}^n \to \mathcal{L}$ lifting a generator of $k$, and by adjunction an $\mathbb{S}_{W(k)}$-module map $\Sigma^n\mathbb{S}_{W(k)} \to \mathcal{L}$ which on $\pi_n((-)/2)$ gives an isomorphism $k \cong k$. Therefore

$$\mathbb{S}_{W(k)}[n] \otimes k \simeq k[n] \to k[n] \simeq \mathcal{L} \otimes k$$

is an equivalence, where the equivalence $k[n] \simeq \mathcal{L} \otimes k$ follows from the fact that base change preserves invertible objects. The map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is then a $k$-local, and therefore an $\mathbb{F}_p$-local, equivalence. Both sides are connective and $p$-complete so it follows that the map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is an equivalence.

N: There is probably a reference for this fact, I'll look around for one.

Thus $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) = 0$. On the other hand, we have that the unit map $\mathbb{S}_{W(k)} \to \Qoppa^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})$ is split by the map $\Qoppa^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)}) \to \mathbb{S}_{W(k)}^{\varphi C_2} = \mathbb{S}_{W(k)}$. Consequently $\pi_0(\Qoppa^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})) \cong \pi_0(\mathbb{S}_{W(k)} \oplus (\mathbb{S}_{W(k)})_{hC_2}) \cong W(k) \times W(k)$. As a ring this is $W_2(W(k))$, and in order for $q \in W_2(W(k))$ to induce a Poincaré structure we must have that $q \in W_2(W(k))^\times \cong W(k)^\times \times W(k)^\times$.

We then have that $\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong W(k)^\times \times W(k)^\times / H$ where $H$ is the subgroup of Poincaré structures $q$ on $\mathbb{S}_{W(k)}$ which are identified by some automorphism $f : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}$. By the defining property of spherical Witt vectors there is an equivalence $\mathrm{Maps}_{\mathrm{CAlg}}(\mathbb{S}_{W(k)}, \mathbb{S}_{W(k)}) \simeq \mathrm{Maps}_{Perf}(k, k) = \mathrm{Gal}(k/\mathbb{F}_2)$ and the action on $W(k)^\times \times W(k)^\times$ is given by $g \in \mathrm{Gal}(k/\mathbb{F}_2)$ acts via $W(g) \times W(g)$. Consequently

$$\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong (W(k)^\times \times W(k)^\times)/\mathrm{Gal}(k/\mathbb{F}_2)$$

which even for $k = \mathbb{F}_2$ is not zero and in fact not even $2^\infty$-torsion.

In the usual Picard spectrum one has the relationship $\mathrm{Pic} = B\mathbb{G}_m$, where $\mathbb{G}_m$ is the spectral algebraic group scheme sending a ring spectrum $E$ to the spectrum of $E$-linear equivalences of $E$ $\mathrm{gl}_1 E := \mathrm{Aut}_E(E)$.[1] Equivalently, $\mathbb{G}_m$ is the affine group scheme given by $\mathbb{G}_m = \mathrm{Spét}(\mathbb{S}\{x^{\pm 1}\})$, where $\mathbb{S}\{x^{\pm 1}\}$ is the free $\mathbb{E}_\infty$ ring on the $\mathbb{E}_\infty$ space $\mathbb{Z}$. This relationship between Pic and $\mathbb{G}_m$ has many important applications, for example relating the higher homotopy groups of $\mathrm{Pic}(A)$ with those of $A$. We will spend the rest of this section on establishing such an equivalence in the Poincaré setting.

**Construction 4.9.** The underlying $\mathbb{E}_\infty$ ring of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ will again be $\mathbb{S}\{x^{\pm 1}\}$, but in order to promote this ring to a Poincare ring it will be helpful to write it as

$$\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z^{\pm 1}\}} \mathbb{S}$$

where the map $\mathbb{S}\{z^{\pm 1}\} \to \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by $z \mapsto xy$, This ring naturally lifts to a Borel $C_2$-ring given by $C_2$ swaps $x$ and $y$ and does nothing to $z$. Now take $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ to be the Poincaré ring with underlying Borel $C_2$ structure as described above and geometric fixed points $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} = \mathbb{S}$ and the map $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ given by the unit map. Endowing $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2}$ with the $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$-module structre given by $x, y \mapsto 1$, it remains to show that the unit map $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ factors the Tate valued Frobenius $\mathbb{G}_{\mathrm{m}}^{\Qoppa} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ in order to promote $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ to a Poincaré ring.

By construction of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ this amounts to showing that on $\pi_0$ the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$. This map sends both $x$ and $y$ to $xy \in \pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$. These are equal to 1 in $\pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ is a modules over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of $xy$ equal to 1.

[1] Normally the automorphism space of an object is only $\mathbb{A}_\infty$, but as the unit in a symmetric monoidal category, the automorphisms of $E$inherit a canonical and in fact functorial $\mathbb{E}_\infty$ structure and this construction makes sense.

**Theorem 4.10.** *There is a natural equivalence of*

$$\Omega \operatorname{Pic}^{\mathrm{p}}(-) \simeq \mathbb{G}_{\mathrm{m}}^{\Qoppa}$$

*of functors on Poincaré rings.*

*Proof.* This amounts to identifying the space $\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}_A)}(A, u)$ functorially, where $(A, u)$ is the Poincaré object $A$ with bilinear form given by the unit map $\mathbb{S} \to \Qoppa_A(A)$. Note that any automorphism of Hermetian objects will automatically be Poincaré and so we may instead describe the automorphisms as a Hermetian object. We then have that $\mathrm{He}(\mathrm{Mod}_A) \to \mathrm{Mod}_A$ is a cocartesian fibration by definition, and classified by the functor which takes a module $M$ to the groupoid $\Omega^\infty \Qoppa_A(M)$. Thus we get that $\mathrm{Aut}_{\mathrm{He}(\mathrm{Mod}_A)}((A, u))$ is exactly the fiber of the map

$$\mathrm{Aut}_{\mathrm{Mod}_A}(A) \to \Qoppa_A(A)$$

or in other words an automorphism $(A, u) \to (A, u)$ is the data of an automorphism $a \in \mathrm{Aut}(A)$ together with a path $q : u \mapsto a^* u$ in $\Omega^{\infty+1} \Qoppa_A(A)$.

There is a natural transformation $\mathbb{G}_{\mathrm{m}}^{\Qoppa}(-) \to \Omega \operatorname{Pic}^{\mathrm{p}}(-)$ given as follows: we get a map $\mathbb{G}_{\mathrm{m}}^{\Qoppa}((\mathrm{Mod}_A, \Qoppa_A)) \to \mathrm{Aut}_A(A)$ given by forgetting the Poincaré structure everywhere, and so it is enough to see that on $\pi_0$ the automorphisms of $A$ coming from $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ preserve $u$. By using the linear and quadratic decomposition of $\Qoppa_A$, for an element $a \in \pi_0(A)^\times$ send $u$ to $u$ is must be sent to $1 \in \pi_0(A^{\varphi C_2})$ and must act by 1 on $A^{hC_2}$. By the following Lemma this second condition is equivalent to $a\sigma(a) \in \pi_0(A)^\times$ being equal to 1, but then these two conditions are exactly describing a map out of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ as desired.

Consequently we have a comparison map $\mathbb{G}_{\mathrm{m}}^{\Qoppa}(\mathrm{Mod}_A, \Qoppa_A) \to \Omega \operatorname{Pic}^{\mathrm{p}}(\mathrm{Mod}_A, \Qoppa_A)$, and the above argument in fact shows that this map is an equivalence on $\pi_0$. To finish the argument, note that the pushout description of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ induces a pullback of mapping spaces

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}}^{\Qoppa}(\mathrm{Mod}_A, \Qoppa_A) \arrow[r] \arrow[d] & \mathrm{Maps}_{\mathrm{CAlg}(\mathrm{Sp}^{C_2})}(\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}, A) \simeq \mathrm{gl}_1(A) \arrow[d] \\
* \arrow[r] & \mathrm{Maps}_{\mathrm{CAlg}(\mathrm{Sp}^{C_2})}(\mathbb{S}\{z\}, A) \simeq \Omega^\infty \Qoppa_A(A)
\end{tikzcd}$$

which finishes the proof. $\square$

**Lemma 4.11.** *Let $A \in \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ and $s \in \pi_0(A)^\times$. Then $a\sigma(a) = 1$ in $\pi_0(A)$ if and only if $(a \otimes a)^*$ acts by 1 on $\pi_0(A^{hC_2}) = \pi_0(\mathrm{Hom}_{A \otimes A}(A \otimes A, A)^{hC_2})$.*

*Proof.* The only if direction follows from the fact that the evaluation map $\mathrm{Hom}_{A \otimes A}(A \otimes A, A) \to A$ is an $A \otimes A$-module map. Now suppose that $a\sigma(a) = 1$ in $A$. Then before taking homotopy fixed points the induced map $a^* = id$ because $A$ is $\mathbb{E}_\infty$.[2] $\square$

## 4.1 prime factorization and the picard group of hearts

V: still in development

We establish a Poincaré analogue of Fausk's result which describes the picard group of the derived category of a scheme $X$ in terms of connected components of $X$ and the classical picard group of $X$.

Let $X$ be a simplicial set. The set $\pi_0(X)$ is the set of connected compoents of $X$, i.e. simplicial subsets which are connected and form $X$ via a coproduct. In other words, the functor $\pi_0$ records a unique and maximal decomposition of $X$ into coproducts. To establish the the result mentioned above, we study the dual analogue of connected components in the sense of a maximal decomposition of $X$ into products.

**Definition 4.12.** Let $X$ be a simplicial set. We call a map of simplicial sets $X \to Y$ a factor of $X$, if there is an isomorphism $X \simeq Y_\times Z$, for some simplicial set $Z$, such that $Y \times Z \simeq X \to Y$ is a structure map of the given product.

**Definition 4.13.** Let $X$ be a simplicial set. We say that $X$ is prime, or indecomposable, if it is nonempty and every factor of $X$ is isomorphic to either $\Delta^0$ or $X$. We let $\mathrm{prin}(X)$ denote the set of prime factors of $X$.

**Proposition 4.14.** *Let $X_\bullet$ be a simplicial set, then $X_\bullet$ is the product of its prime factors.*

*Proof.* □

V:

**Proposition 4.15.** *Let $X$ be a simplicial set. Then* $\mathrm{prim}(X) \simeq \mathrm{prim}(X^{\simeq})$.

*Proof.* Let $f : X \to Y$ be a weak equivalence of simplicial sets. Then $f^{\simeq} : X^{\simeq} \to Y^{\simeq}$ is a weak equivalence of spaces. □

V:

**Proposition 4.16.** *Let $X$ be a simplicial set. Then* $\mathrm{prim}(X) \simeq \pi_0(\mathrm{Spec}(X))$, *where* $\mathrm{Spec}(X^{\simeq})$ *is the Balmer spectrum of $X$ with respect to the symmetric monoidal structure given by cartesian product.*

*Proof.* □

V:

**Remark 4.17.** Let $R$ be a commutative ring. Then the scheme $\mathrm{Spec}(R)$ is isomorphic to the Balmer spectrum of $\mathrm{Mod}_R^{\heartsuit}$. When we view $R$ as a discrete simplicial set, we thus have

$$\mathrm{prim}(\mathrm{Mod}_R^{\heartsuit}) \simeq \pi_0(\mathrm{Spec}(R)).$$

V: does this need a proof?

**Definition 4.18.** Let $X$ be a prime simplicial set. A *c*-structure on $X$ is a map of simplicial sets $X \to \mathbf{Z}$ satisfying (todo). Let $Y$ be a simplicial set, then a *c*-structure on $Y$ is a product of *c*-structures on each of its prime components. We write $X_{\geq n}$ for the homotopy pullback of $\mathbf{Z}_{\geq n}$ along $c$, $X_{\leq n}$ for the homotopy pullback of $\mathbf{Z}_{\leq n}$ along $c$, and $X^{\heartsuit}$ for the pullback of $X_{\leq n}$ along $X_{\geq n} \to X$.

V: when X is a stable infinity category and prime, then a c-structure should be a t-structure on it

**Theorem 4.19.** *Let $X$ be a prime simplicial set and $c : X \to \mathbf{Z}$ a c-structure. Then we have a fiber sequence of monoids* >

$$X^{\heartsuit} \to X^{\simeq} \to \mathbf{Z}.$$

V: what kind exactly

*Proof.* □

V:

**Corollary 4.20** (Fausk)**.** *Let $R$ be a discrete ring. Then we have a short exact sequence:* >

$$0 \to \mathrm{Pic}(\mathrm{Mod}(R)^{\heartsuit}) \to \pi_0(\mathrm{Pic}(\mathrm{Mod}(R))) \to H^0(\mathrm{Spec}(R); \mathbf{Z}) \to 0.$$

V: apply pic to the previous sequence and take $\pi_0$

*Proof.* □

## 4.2 Hermitian line bundles

**Definition 4.21.** Let $R$ be a commutative discrete ring with a $C_2$-action $\sigma\colon R \to R$. Write $\sigma_* R$ for the $R$-module with underlying abelian group $R$ and action $r \cdot m = \sigma(r) \cdot m$. Let $M$ be an $R$-module. Define the *adjoint* of $M$ to be the $R$-module $M^{\dagger} := \hom_R(M, \sigma_* R)$.

L: From meeting May 29, 2025: Missing a $\sigma_*$ before hom? clarify the $R$-action!

L: see 3.8-3.11 in this paper

Also recall that there is a canonical $R$-linear isomorphism $\left(M^{\dagger}\right)^{\dagger} \simeq M$. Note that given two $R$-modules $M, N$, the adjoint satisfies $M^{\dagger} \otimes N^{\dagger} \simeq (M \otimes N)^{\dagger}$. Let $I$ be a projective $R$-module (in particular, there is a canonical identification $(I^{\dagger})^{\dagger} \simeq I$). A *$\sigma$-hermitian form on $I$* is an $R$-linear isomorphism $\varphi\colon I \xrightarrow{\sim} I^{\dagger}$ so that $\varphi^{\dagger} = \varphi$.

**Observation 4.22.** Let $R$ be a commutative discrete ring with a $C_2$-action $\sigma\colon R \to R$. Given two discrete $R$-modules $M, N$ equipped with $\sigma$-hermitian forms $\varphi, \psi$, respectively, $\varphi \otimes \psi$ defines a $\sigma$-hermitian form on $M \otimes_R N$. Using the canonical isomorphism mentioned above, if $\varphi$ is a $\sigma$-hermitian form on $M$, then $\varphi^{\dagger}$ induces a $\sigma$-hermitian form on $M^{\dagger}$. Finally, observe that $R$ has a canonical $\sigma$-hermitian form which is the adjoint of the map $R \otimes R \to R$, $r \otimes s \mapsto r\sigma(s)$.

**Definition 4.23.** Let $R$ be a commutative discrete ring with a $C_2$-action $\sigma\colon R \to R$. Define the *hermitian Picard group of $R$* to have underlying set consisting of pairs $(I, \varphi)$ where $I$ is an invertible $R$-module and $\varphi$ is a $\sigma$-hermitian form on $I$.

L: workshop the name later

By Observation 4.22, this set inherits a group structure. We write $\mathrm{hPic}(R)$ for the group of $\sigma$-hermitian line bundles on $\mathrm{Spec}\, R$.

[2]Or just $\mathbb{E}_2$.

**Theorem 4.24.** *Let $R$ be a discrete commutative ring with a $C_2$-action $\sigma\colon R \xrightarrow{\sim} R$ via ring maps. Regard $R$ as a Poincaré ring via Example 3.15. Then there is a split short exact sequence of abelian groups*

$$0 \to \mathrm{hPic}(R) \to \pi_0\mathrm{PnPic}(\underline{R}^{\sigma}) \to C_{C_2}(\mathrm{Spec}\, R, \mathbb{Z}^-) \to 0$$

*where $R$ is endowed with the genuine symmetric Poincaré structure and $\mathbb{Z}^-$ is endowed with the $C_2$-action given by multiplication by $-1$ and $C_{C_2}$ denotes continuous functions which are moreover $C_2$-equivariant. Moreover, forgetting the hermitian form (resp. forgetting the $C_2$-action) induces a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{hPic}(R) \arrow[r] \arrow[d] & \pi_0\mathrm{PnPic}(R) \arrow[r] \arrow[d] & C_{C_2}(\mathrm{Spec}\, R, \mathbb{Z}^-) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Pic}^{\mathrm{cl}}(R) \arrow[r] & \pi_0\mathrm{Pic}\left(\mathrm{Perf}_R\right) \arrow[r] & C(\mathrm{Spec}\, R, \mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

*where the bottom row is that of [Fau03, Theorem 3.5].*

*Proof.* An object of $\pi_0\mathrm{PnPic}(R)$ is a pair $(I, q)$ where $I$ is an invertible $R$-module and $q$ is a point in $\pi_0\Omega^\infty Ϙ_{R^{gs}}(I)$. By the proof of [Fau03, Theorem 3.5], $I$ induces a continuous map $\Psi(I)\colon \mathrm{Spec}\, R \to \mathbb{Z}$. Write $\sigma$ for the involution on $R$. Now $q$ in particular induces an equivalence $q\colon I \xrightarrow{\sim} I^\dagger \simeq (\sigma_* I)^\vee$. For each point $\mathfrak{p} \in \mathrm{Spec}\, R$, localizing $q$ gives an equivalence

$$q_{\mathfrak{p}}\colon I_{\mathfrak{p}} \xrightarrow{\sim} (\sigma_* I)^\vee_{\mathfrak{p}} \simeq \left(\sigma_*(I_{\sigma(\mathfrak{p})})\right)^\vee .$$

Since $I_{\mathfrak{p}}$ is an invertible module over a local ring, [Fau03, Proposition 3.2] implies that $q_{\mathfrak{p}}$ induces an equivalence

$$I_{\mathfrak{p}} \simeq R_{\mathfrak{p}}[\varphi(\mathfrak{p})] \xrightarrow{\sim} \left(\sigma_*(R_{\sigma(\mathfrak{p})}[\varphi(\sigma(\mathfrak{p}))])\right)^\vee \simeq (\sigma_*(R_{\sigma(\mathfrak{p})}))^\vee[-\varphi(\sigma(\mathfrak{p}))] .$$

Since $R$ is discrete, this implies in particular that $\Psi(I)(\sigma(\mathfrak{p})) = -\Psi(I)(\mathfrak{p})$, i.e. that $\Psi(I)$ is $C_2$-equivariant. It follows immediately from [Fau03, Theorem 3.5] that $\Psi$ is a homomorphism and that an element of the kernel of $\Psi$ lifts to $\mathrm{hPic}(R)$.

Now consider a $C_2$-equivariant map $g\colon \mathrm{Spec}\, R \to \mathbb{Z}$. As in *loc. cit.*, the image of $g$ is finite and $C_2$-invariant, say $\{n_1, -n_1, \ldots, n_m, -n_m\}$ or $\{0, n_1, -n_1, \ldots, n_m, -n_m\}$ for some $n_i \neq 0$. As in *loc. cit.*, the disjoint subsets $U_{\pm n_i} := g^{-1}(\pm n_i)$ correspond to an orthogonal basis of idempotents $e_{U_{\pm n_i}}$ in $R$. Since $g$ is $C_2$-equivariant with respect to the sign action on $\mathbb{Z}$, we have $\sigma(U_{n_i}) = U_{-n_i}$. Moreover, it follows from Lemma 3.4 *ibid.* that $\sigma(e_{U_{n_i}}) = e_{U_{-n_i}}$. Consider the $R$-module $\Phi(g) := \bigoplus_{n\in \mathrm{Im}(g)} e_{g^{-1}(\{n\})}R[n]$. In other words, $\Phi(g) := \bigoplus_{i=1}^m \left(e_{U_{n_i}}R[n_i] \oplus e_{U_{-n_i}}R[-n_i]\right)$ if 0 is not in the image of $g$ and $\Phi(g) := e_{U_0} \oplus \bigoplus_{i=1}^m \left(e_{U_{n_i}}R[n_i] \oplus e_{U_{-n_i}}R[-n_i]\right)$ otherwise. Observe that $\left(e_{U_{-n_i}}R[-n_i]\right)^\dagger = \hom_R(e_{U_{-n_i}}R[-n_i], \sigma_* R) = \hom_R\left(\sigma_*(e_{U_{-n_i}}R), R\right)[n_i] = \hom_R\left(e_{U_{n_i}}R, R\right)[n_i]$. Finally, we claim that there is a canonical $\sigma$-hermitian form $q_g \in \Omega^\infty Ϙ_{R^{\mathrm{gs}}}(\Phi(g))$ whose adjoint $q_g^\dagger\colon \Phi(g) \xrightarrow{\sim} \Phi(g)^\dagger$ corresponds to the identity. That $q_g$ defines a point of $\hom_{R^{\otimes 2}}(\Phi(g)^{\otimes 2}, R)^{hC_2}$ is evident. Observe that to give a lift of $q_g$ to $Ϙ_{R^{\mathrm{gs}}}(\Phi(g)) = \hom_{N^{C_2}R}\left(N^{C_2}\Phi(g), R\right)$ is equivalent to giving a commutative diagram

$$\begin{tikzcd} \Phi(g) \otimes_R R^{\varphi C_2} \arrow[r, dashed, "\exists ?"] \arrow[d] & R^{\varphi C_2} \arrow[d] \\ \left(\Phi(g)^{\otimes 2}\right)^{tC_2} \arrow[r, "q_g^{tC_2}"] & R^{tC_2} \end{tikzcd} \tag{4.25}$$

of $R^{\varphi C_2}$-modules. Let us write $\eta\colon R \to \pi_0 R^{\varphi C_2}$ for the ring map induced by the structure map. Since $R$ is a $C_2$-$\mathbb{E}_\infty$-ring, $\eta$ is invariant with respect to the given action on $R$ and the trivial action on $\pi_0 R^{\varphi C_2}$. Consider $e_{U_{n_i}}$ an idempotent corresponding to an element of the image of $g$ so that $n_i \neq 0$. Then

$$\begin{aligned} \eta(e_{U_{n_i}}) &= \eta(e_{U_{n_i}})^2 && \text{ring maps preserve idempotents} \\ &= \eta(e_{U_{n_i}}) \cdot \eta(e_{U_{-n_i}}) && C_2\text{-invariance of } \eta \\ &= \eta(e_{U_{n_i}} e_{U_{-n_i}}) && \eta \text{ is a ring map} \\ &= 0 && \text{orthogonality and } n_i \neq 0 . \end{aligned}$$

In particular, if 0 is not in the image of $g$, $\Phi(g) \otimes_R R^{\varphi C_2} \simeq 0$ and (4.25) commutes vacuously. If 0 is in the image of $g$, then $e_{U_0}R$ is a discrete/projective $e_{U_0}R$-module and $q_g$ evidently defines a genuine hermitian form on $e_{U_0}R$ (compare [Cal+20a, Remark 4.2.21]).

Thus, $g \mapsto (\Phi(g), q_g)$ defines a splitting of $\Psi$ which agrees with the splitting constructed in [Fau03, Theorem 3.5] on underlying objects. □

N: I'm commenting out section 5 since we do not yet have an application for it. Feel free to put it back in if you want.

# 5 The Poincaré Brauer Group

Let $A$ be a Poincaré ring spectrum. By Remark 3.3, $\mathrm{Mod}_A^\omega$ promotes to a commutative algebra object in the $\infty$-category of Poincaré $\infty$-categories $\mathrm{Cat}_\infty^\mathrm{p}$ , and we may thus consider modules over it. In this section, we will use modules over Poincaré ring spectra to define derived analogues of the involutive Brauer group for Poincaré ring spectra.

Recall that a Poincaré $\infty$-category is called idempotent complete if the underlying stable $\infty$-category is idempotent complete. The full subcategory of $\mathrm{Cat}_\infty^\mathrm{p}$ spanned by idempotent complete Poincaré $\infty$-categories is denoted by $\mathrm{Cat}_{\infty,\mathrm{idem}}^\mathrm{p}$ [Cal+20b, Definition 1.3.2].

**Definition 5.1.** Let $A$ be a Poincaré ring spectrum. We define the *Poincaré Brauer space of $A$* as

$$\mathrm{Br}^\mathrm{p}(A) := \mathrm{Pic}(\mathrm{Mod}_A(\mathrm{Cat}_{\infty,\mathrm{idem}}^\mathrm{p})).$$

The assignment $A \mapsto \mathrm{Br}^\mathrm{p}(A)$ defines a functor

$$\mathrm{Br}^\mathrm{p} \colon \mathrm{CAlg}^\mathrm{p} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$$

valued in grouplike $\mathbf{E}_\infty$-spaces.

**Remark 5.2.** The symmetric monoidal forgetful functor $\mathrm{Mod}_A(\mathrm{Cat}_{\infty,\mathrm{idem}}^\mathrm{p}) \to \mathrm{Mod}_A(\mathcal{C}\mathrm{at}_\infty^\mathrm{ex})$ induces a map $\mathrm{Br}^\mathrm{p}(A) \to \mathrm{Br}(A)$ of grouplike $\mathbf{E}_\infty$-spaces, where $\mathrm{Br}(A)$ is the Brauer space $\mathrm{br}_{\mathrm{alg}}(A)$ of [AG14a, pp. 1154–1155].

**Proposition 5.3.** *Let $A$ be a Poincaré ring spectrum. Then we have a canonical equivalence*

$$\Omega\,\mathrm{Br}^\mathrm{p}(A) \simeq \mathrm{Pic}^\mathrm{p}(A).$$

*Proof.* Since $\Omega\,\mathrm{Br}^\mathrm{p}(R)$ is given by the space of automorphisms of any object in $\mathrm{Br}^\mathrm{p}(R)$, it suffices to determine the space of autoequivalences of $(\mathrm{Mod}_R^\omega, \Qoppa_R)$. An autoequivalence is the data of a pair $(f, \eta)$ where $f \colon \mathrm{Mod}_R^\omega \xrightarrow{\simeq} \mathrm{Mod}_R^\omega$ is an exact $R$-linear autoequivalence and $\eta \colon \Qoppa_R \xrightarrow{\sim} \Qoppa_R \circ f^{\mathrm{op}}$ is a natural equivalence. Since $\mathrm{Cat}_{\infty R}^\mathrm{p} \to \mathcal{C}\mathrm{at}_{\infty R}^\mathrm{ex}$ is symmetric monoidal (and hence $f$ will be $\mathrm{Mod}_R^\omega$-linear), $f$ is of the form $- \otimes_R \mathcal{L}$ where $\mathcal{L}$ is an invertible $R$-module. Since taking bilinear and linear parts is functorial by [Cal+20a, Proposition 1.3.11], $\eta$ is equivalently the data of a pair of equivalences

$$\begin{aligned} b(\eta) &\colon \hom_{R\otimes R}((- \otimes \mathcal{L}) \otimes (- \otimes \mathcal{L}), R)^{hC_2} \simeq \hom_{R\otimes R}(- \otimes -, R)^{hC_2} \\ \ell(\eta) &\colon \hom_R(- \otimes \mathcal{L}, R^{\varphi C_2}) \simeq \hom_R(-, R^{\varphi C_2}) \end{aligned}$$

plus a path between their images in $\hom_R(\mathcal{L}, R^{tC_2})$. The transformation $b(\eta)$ is equivalent to the data of an $R$-bilinear equivalence $R \simeq \mathcal{L}^\vee \otimes \mathcal{L}^\vee$ , and the transformation $\ell(\eta)$ is equivalent to the data of an $R^{\varphi C_2}$-linear equivalence $\ell(\eta) \colon R^{\varphi C_2} \otimes_R \mathcal{L}^\vee \xrightarrow{\sim} R^{\varphi C_2}$.

L: maybe one of these should be conjugate dual here?

L: is the $R^{\varphi C_2}$-linearity of this $\simeq$ correct?

Now consider the composites

$$R \otimes_R \mathcal{L}^\vee \xrightarrow{\mathrm{unit}\,\otimes \mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L}^\vee \xrightarrow{\ell(\eta)} R^{\varphi C_2}$$

$$R \otimes_R \mathcal{L} \xrightarrow{\mathrm{unit}\,\otimes \mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L} \xrightarrow{\ell(\eta)^{-1} \otimes \mathrm{id}_\mathcal{L}} R^{\varphi C_2}\,.$$

These correspond to the $\ell(q^\vee), \ell(q)$ of Remark 4.2, respectively. In particular, the condition that $\ell(q^\vee), \ell(q)$ make the diagram (4.4) commute is equivalent to the condition that $\ell(\eta)$ is an equivalence by an adjunction argument. The data of the path in $\hom_R(\mathcal{L}, R^{tC_2})$ is exactly the data needed to show that the maps $\ell(q)$ and $b(q)$ glue together to give a form on $\mathcal{L}$.

The above thus produces a natural transformation $\Omega\,\mathrm{Br}^{\mathrm{p}}(-) \to \mathrm{Pic}^{\mathrm{p}}(-)$. In the other direction, to any $(\mathcal{L}, q) \in \mathrm{Pn}(\mathrm{Mod}_A^\omega)$ invertible we may define an autoequivalence $(\mathrm{Mod}_A^\omega, \Qoppa) \to (\mathrm{Mod}_A^\omega, \Qoppa)$ via tensoring with $(\mathcal{L}, q)$, which will be an autoequivalence by the assumption that $(\mathcal{L}, q)$ is invertible. We have that these two natural trnasformations are inverse to each other, hence the result. $\square$

## 5.1 Generalities on $R$-linear Poincaré $\infty$-categories

**Proposition 5.4.** *Let $(\mathrm{Mod}_R^\omega, \Qoppa_R)$ be a Poincaré ring spectrum.*

*(1) The $\infty$-category $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ admits all small limits and colimits, and it inherits a canonical symmetric monoidal structure, and for every morphism $\left(R, R^{\varphi C_2} \to R^{tC_2}\right) \to (S, S^{\varphi C_2} \to S^{tC_2})$, the functor $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_{(\mathrm{Mod}_S^\omega, \Qoppa_S)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ is a symmetric monoidal left adjoint.*

*(2) Let $A$ be an $\mathbb{E}_1$-$R$-algebra in spectra, and regard the category of compact right $A$-modules $\mathrm{Mod}_A^\omega$ as left-tensored over $\mathrm{Mod}_R^\omega$ in the canonical way. Then the pullback*

$$\begin{tikzcd} & \mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}\left(\mathrm{Cat}^{\mathrm{h}}_\infty\right) \arrow[d] \\ \{\mathrm{Mod}_A^\omega\} \arrow[r] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R} \end{tikzcd} \tag{5.5}$$

*is canonically equivalent to $\mathrm{Mod}_{N_R A \otimes_{N_R R} R^L}\left(\mathrm{Sp}^{C_2}\right)$ where $R^L$ is the $\mathbb{E}_\infty$-$N_R R$-algebra with $(R^L)^e \simeq R$ and $(R^L)^{\varphi C_2} \simeq C$.*

*A $N_R A \otimes_{N_R R} R^L$-module classifies a $(\mathrm{Mod}_R^\omega, \Qoppa_R)$-module in Poincaré $\infty$-categories if its underlying $A$-module is invertible in the sense of [Cal+20a, Definition 3.1.4].*

*(3) Let $A, B$ be $R$-algebras with associated ($R$-linear) modules with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and $(M_B, N_B, N_B \to M_B^{tC_2})$, respectively so that (under item (2)) $(\mathrm{Mod}_A^\omega, \Qoppa_A)$ and $(\mathrm{Mod}_B^\omega, \Qoppa_B)$ are objects of $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$. Then the symmetric monoidal structure of item (1) is so that the underlying $R$-linear $\infty$-category with perfect duality $(\mathrm{Mod}_A^\omega, \Qoppa_A) \otimes_{(\mathrm{Mod}_R^\omega, \Qoppa_R)} (\mathrm{Mod}_B^\omega, \Qoppa_B)$ is $\mathrm{Mod}_A^\omega \otimes_{\mathrm{Mod}_R^\omega} \mathrm{Mod}_B^\omega \simeq \mathrm{Mod}_{A \otimes_R B}^\omega$, and the associated module with genuine involution is given by $M_A \otimes_R M_B$, $N_A \otimes_{R^{\varphi C_2}} N_B$, and the structure map is $N_A \otimes_{R^{\varphi C_2}} N_B \to M_A^{tC_2} \otimes_{R^{tC_2}} M_B^{tC_2} \to (M_A \otimes_R M_B)^{tC_2}$ where the latter map arises canonically from lax monoidality of the Tate construction.*

*(4) Let $(\mathcal{C}, \Qoppa_{\mathcal{C}}), (\mathcal{D}, \Qoppa_{\mathcal{D}})$ be objects of $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{h}}_\infty)$. Then the forgetful functor induces $\hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}((\mathcal{C}, \Qoppa_{\mathcal{C}}), (\mathcal{D}, \Qoppa_{\mathcal{D}})) \to \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}(\mathcal{C}, \mathcal{D})$ on mapping spaces so that the fiber over an $R$-linear functor $F\colon \mathcal{C} \to \mathcal{D}$ is the mapping space $\mathrm{map}_{\Qoppa_R}(F_! \Qoppa_{\mathcal{C}}, \Qoppa_{\mathcal{D}}) \simeq \mathrm{map}_{\Qoppa_R}(\Qoppa_{\mathcal{C}}, \Qoppa_{\mathcal{D}} \circ F^{\mathrm{op}})$, where the mapping space is taken in $\mathrm{Fun}^q_{\Qoppa_R}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp})$ and $\mathrm{Fun}^q_{\Qoppa_R}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$, respectively.*[3]

*(5) The symmetric monoidal forgetful functor $\theta\colon \mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{h}}_\infty) \to \mathrm{Mod}_{\mathrm{Mod}_R^\omega}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)$ is a (co)cartesian fibration.*

L: what is it classified by?

**Remark 5.6.** A special case of part (2) is [Cal+20a, Example 5.4.13].

*Proof.* (1) The first part of the statement follows from [Cal+20a, §6.1] and [Lur17, §4.2.3].

(2) Let $\mathcal{LM}^\otimes$ denote the $\infty$-operad of [Lur17, Definition 4.2.1.7]. Our strategy of proof will be similar to that of [Cal+20a, §5.3]: First, we show that an $\mathcal{LM}^\otimes$-algebra object in $\mathrm{Cat}^{\mathrm{h}}_\infty$ is equivalent to an $\mathcal{LM}^\otimes$-algebra object in an operad of functor categories. Then, we use a (suitably coherent version

[3]The proof of (2) in particular shows that $\mathrm{Fun}^q(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is left-tensored over $\mathrm{Fun}^q(\mathrm{Mod}_R^{\omega,\mathrm{op}}, \mathrm{Sp})$ in the sense of [Lur17, Definition 4.2.1.19], so this makese sense.

of) the classification of hermitian structures on module categories as categories of modules over the Hill–Hopkins–Ravenel norm [Cal+20a, Theorem 3.3.1] to conclude. Recall that the action of $\mathrm{Mod}_R^\omega$ on $\mathrm{Mod}_A^\omega$ is given by a functor $\mathcal{LM}^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$, and define $\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes$ via the following pullback square of $\infty$-operads:

$$\begin{tikzcd}
\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \arrow[r, "p"] \arrow[d] & \mathcal{LM}^\otimes \arrow[d, "{\mathrm{Mod}_R^\omega, \mathrm{Mod}_A^\omega}"] \\
(\mathcal{C}\mathrm{at}_\infty)^\otimes_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}_\infty^\times
\end{tikzcd} . \tag{5.7}$$

Informally, an object $F \in \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes_{\mathfrak{a}}$ is a functor $F\colon \mathrm{Mod}_R^{\omega,\mathrm{op}} \to \mathrm{Sp}$ and an object $G$ over the fiber of $\mathfrak{m}$ is a functor $G\colon \mathrm{Mod}_A^{\omega,\mathrm{op}} \to \mathrm{Sp}$. The $p$-cocartesian edge over the canonical map $(\mathfrak{a}, \mathfrak{m}) \to \mathfrak{m}$ in $\mathcal{LM}^\otimes$ sends $(F, G)$ to the lower arrow in the diagram

$$\begin{tikzcd}
\mathrm{Mod}_R^{\omega,\mathrm{op}} \times \mathrm{Mod}_A^{\omega,\mathrm{op}} \arrow[r, "F\times G"] \arrow[d, "-\otimes_R -"'] & \mathrm{Sp}\times\mathrm{Sp} \arrow[d, "\otimes_{\mathrm{Sp}}"] \\
\mathrm{Mod}_A^{\omega,\mathrm{op}} \arrow[r, "F\otimes G := \mathrm{LKE}_{\otimes_R}(\otimes_{\mathrm{Sp}} \circ (F\times G))"] & \mathrm{Sp}
\end{tikzcd} .$$

Now define $\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes$ to consist of the full subcategory of $\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes$ consisting of those tuples of functors which are all quadratic. The inclusion $\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \to \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes$ exhibits the former as an $\infty$-operad, and moreover the localization is compatible with the $\mathcal{LM}^\otimes$-monoidal structure in the sense of [Lur17, Definition 2.2.1.6]. We can extend the previous diagram to

$$\begin{tikzcd}
\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \arrow[r, "p"] \arrow[d] & \mathcal{LM}^\otimes \arrow[d, "{\mathrm{Mod}_R^\omega, \mathrm{Mod}_A^\omega}"] \\
\mathrm{Cat}_\infty^{\mathrm{h}\ \otimes} \arrow[r] & (\mathcal{C}\mathrm{at}_\infty)^\otimes_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}_\infty^\otimes
\end{tikzcd} . \tag{5.8}$$

Modifying [Cal+20a, Construction 5.3.15 & Lemma 5.3.15] slightly (note that Corollary 5.1.4 did not assume the tensor factors to be equivalent), we obtain an analogous commutative diagram of $\infty$-operads

$$\begin{tikzcd}
\mathrm{Fun}^p_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \arrow[r] \arrow[d] & \mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \arrow[r, "p"] \arrow[d] & \mathcal{LM}^\otimes \arrow[d, "{\mathrm{Mod}_R^\omega, \mathrm{Mod}_A^\omega}"] \\
\mathrm{Cat}_\infty^{\mathrm{p}\ \otimes} \arrow[r] & \mathrm{Cat}_\infty^{\mathrm{h}\ \otimes} \arrow[r] & (\mathcal{C}\mathrm{at}_\infty)^\otimes_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}_\infty^\otimes
\end{tikzcd} \tag{5.9}$$

in which all squares are pullbacks. Now suppose $A$ is given a module with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and call the associated Poincaré $\infty$-category $\overline{\mathrm{Mod}}_A$. Then to lift $\overline{\mathrm{Mod}}_A$ to a module over $(\mathrm{Mod}_R^\omega, \Qoppa_R)$ compatibly with the $\mathrm{Mod}_R^\omega$-module structure on $\mathrm{Mod}_A^\omega$ is to give a map of $\infty$-operads $\mathcal{LM}^\otimes \to \mathrm{Cat}_\infty^{\mathrm{h}\ \otimes}$ so that the restriction along the canonical inclusion $\mathrm{Assoc}^\otimes \to \mathcal{LM}^\otimes$ gives the algebra object $(\mathrm{Mod}_R^\omega, \Qoppa_R)$ and postcomposing with the canonical projection to $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}\,\times}$ recovers the given $\mathrm{Mod}_R^\omega$-module structure on $\mathrm{Mod}_A^\omega$. By the pullback square (5.8), this is equivalent to giving an object of $\mathrm{Alg}_{\mathcal{LM}/\mathcal{LM}}\left(\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes\right)$. Now let us identify the bilinear functor $\mathrm{Mod}_R^\omega \times \mathrm{Mod}_A^\omega \xrightarrow{-\otimes_R -} \mathrm{Mod}_A^\omega$ with the exact functor $\mathrm{Mod}_R^\omega \otimes \mathrm{Mod}_A^\omega \simeq \mathrm{Mod}_{R\otimes A}^\omega \to \mathrm{Mod}_A^\omega$ which is induction along the action map $R\otimes A \to A$. Using [Cal+20a, Corollary 3.4.1] and unravelling definitions gives the claim for $R$-linear hermitian structures. The proof for $R$-linear Poincaré structures considers (5.9) instead but otherwise proceeds in an identical fashion.

(3) By [Lur17, Theorem 4.4.2.8], the relative tensor product $(\mathrm{Mod}_A^\omega, \Qoppa_A) \otimes_{(\mathrm{Mod}_R^\omega, \Qoppa_R)} (\mathrm{Mod}_B^\omega, \Qoppa_B)$ is computed as the geometric realization of the bar construction

$$\begin{aligned} p\colon \Delta^{\mathrm{op}} &\to \mathrm{Cat}_\infty^{\mathrm{h}} \\ [n] &\mapsto (\mathrm{Mod}_A^\omega, \Qoppa_A) \otimes (\mathrm{Mod}_R^\omega, \Qoppa_R)^{\otimes n} \otimes (\mathrm{Mod}_B^\omega, \Qoppa_B) \end{aligned}$$

Write $f\colon \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ for the forgetful functor. Then $f \circ p$ has a colimit with value $\mathrm{Mod}_A^\omega \otimes_{\mathrm{Mod}_R^\omega} \mathrm{Mod}_B^\omega \simeq \mathrm{Mod}_{A\otimes_R B}^\omega$. Writing $g\colon \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \to \{*\}$, by Example 4.3.1.3 of [Lur09] we see that $f \circ p$ is a $g$-colimit. By Proposition 4.3.1.5(2) and Example 4.3.1.3 of [Lur09], $p$ admits a colimit in $\mathrm{Cat}_\infty^{\mathrm{h}}$ if and only if it admits an $f$-colimit. Now recall that $f$ is a cocartesian fibration with pushforward given by left Kan extension [Cal+20a, Corollary 1.4.2]. We show that $f$ satisfies the conditions of [Lur09, Corollary 4.3.1.11].

- Condition (1) follows from Theorem 6.1.1.10 of [Lur17] applied to $\mathrm{Sp}^{\mathrm{op}}$ (see the end of [Cal+20a, Construction 1.1.26]).
- Condition (2) follows from [Cal+20a, Corollary 1.4.2], the adjoint functor theorem, and presentability of $\mathrm{Fun}^q(\mathcal{C})$, which is discussed in the proof of [Cal+20a, Lemma 5.3.3] (also see [Lur17, Remark 6.1.1.11]).

Thus the preceding discussion shows that there exists a map of simplicial sets $p'$ making the diagram commute

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}_\infty^{\mathrm{h}} \arrow[d, "f"] \\ (\Delta^{\mathrm{op}})^{\triangleright} \arrow[r] \arrow[ur, "p'"] & \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \end{tikzcd}.$$

Since $\{0\} \to \Delta^1$ is left anodyne, by [Lur09, Corollary 2.1.2.7] the inclusions

$$\begin{aligned} \{0\} \times \Delta^{\mathrm{op}} &\to \Delta^1 \times \Delta^{\mathrm{op}} \\ \iota\colon\ (\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) &\to \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright} \end{aligned}$$

are left anodyne. The former implies that there exists a map $p''$ making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}_\infty^{\mathrm{h}} \arrow[d, "f"] \\ \Delta^1 \times \Delta^{\mathrm{op}} \arrow[r] \arrow[ur, "p''"] & \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \end{tikzcd}$$

commute. The maps $p'$ and $p''$ assemble to give a map $p''' := p' \sqcup_p p''$ making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[rr, "p"] \arrow[d] & & \mathrm{Cat}_\infty^{\mathrm{h}} \arrow[d, "f"] \\ (\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) \arrow[r, "\iota"] \arrow[urr, "p'''"] & \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright} \arrow[r] \arrow[ur, bend left, "\overline{p}"'] & \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \end{tikzcd}$$

commute, and likewise $\overline{p}$ exists making the diagram commute since $\iota$ is left anodyne. Now we show that $\overline{p}$ satisfies the conditions of [Lur09, Proposition 4.3.1.9]. By (the opposite/dual/cocartesian version of) [Lur09, Remark 3.1.1.10] and Proposition 3.1.1.5(2") *ibid.* and the fact that $f$ is a cocartesian fibration, we can choose $\overline{p}$ so that for all $k \in (\Delta^{\mathrm{op}})^{\triangleright}$, $\overline{p}|_{\Delta^1\times\{k\}}$ is $f$-cocartesian. Furthermore, since we can choose $\Delta^{\mathrm{op}}$, $(\Delta^{\mathrm{op}})^{\triangleright}$ to have the markings $(-)^\flat$ in [Lur09, Remark 3.1.1.10], $f \circ \overline{p}|_{\Delta^1\times\{\infty\}}$ is a degenerate edge in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$.

Now [Lur09, Proposition 4.3.1.9] implies that $\overline{p}_0$ is an $f$-colimit diagram if and only if $\overline{p}_1$ is an $f$-colimit diagram. Now notice that $\overline{p}|_{\{1\}\times(\Delta^{\mathrm{op}})^{\triangleright}}$ has image contained in the fiber of $f$ over $\mathrm{Mod}_{A\otimes_R B}^\omega$.

By [Lur09, Proposition 4.3.1.10], it suffices to show that $\overline{p}_1$ is a colimit diagram in $\mathrm{Fun}^q\left(\mathrm{Mod}^{\omega}_{A\otimes_R B}\right)$. Write $\overline{M}_A \in \mathrm{Mod}_{N^{C_2}A}$ and $\overline{M}_B \in \mathrm{Mod}_{N^{C_2}B}$ for the corresponding modules (see introduction to §3.3 of [Cal+20a]). Unraveling definitions and using [Cal+20a, Theorem 3.3.1 & Corollary 3.4.1 & Lemma 5.4.6], it follows that the diagram $\overline{p}_1|_{\{1\}\times\Delta^{\mathrm{op}}}$ is the bar construction

$$[n] \mapsto \overline{M}_A \otimes_{N^{C_2}R} R^{\otimes_{N^{C_2}R} n} \otimes_{N^{C_2}R} \overline{M}_B\,.$$

This proves the result.

(4) Let $(\mathcal{C}, \Qoppa_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R, \Qoppa_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$ and let $F\colon \mathcal{C} = \theta\left(\mathcal{C}, \Qoppa_{\mathcal{C}}\right) \to \mathcal{D}$ be an $R$-linear functor. Now define $\Qoppa_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp}$ to be the left Kan extension of $\Qoppa_{\mathcal{C}}$ along $F^{\mathrm{op}}$. Now $(\mathcal{D}, \Qoppa_{\mathcal{D}}) \in \mathrm{Cat}^{\mathrm{h}}_{\infty}$ and there is a canonical map $(f, \eta)\colon (\mathcal{C}, \Qoppa_{\mathcal{C}}) \to (\mathcal{D}, \Qoppa_{\mathcal{D}})$. Now $F$ is classified by a functor $\Delta^1 \times \mathcal{LM}^{\otimes} \to \mathcal{C}at^{\mathrm{ex}\otimes}_{\infty}$, and we may form the pullback

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \Delta^1 \times \mathcal{LM}^{\otimes} \arrow[d] \\ \mathrm{Cat}^{\mathrm{h}\,\otimes}_{\infty} \arrow[r, "p"] & \mathcal{C}at^{\mathrm{ex}\otimes}_{\infty} \end{tikzcd}\,. \tag{5.10}$$

Since $p$ is a cocartesian fibration [Cal+20a, Theorem 5.2.7], $\mathcal{N} \to \Delta^1 \times \mathcal{LM}^{\otimes}$ is a cocartesian fibration, and the nontrivial morphism in $\Delta^1$ classifies a map $F_!\colon \mathrm{Fun}^q_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})^{\otimes} \to \mathrm{Fun}^q_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp})^{\otimes}$ of $\infty$-operads over $\mathcal{LM}^{\otimes}$. Passing to algebra objects, we obtain the desired result on mapping spaces.

(5) By [Lur09, Proposition 2.4.2.8], it suffices to show that $\theta$ is a locally (co)cartesian fibration, and that locally (co)cartesian edges are closed under composition. We give the proof that $\theta$ is a cocartesian fibration; the proof that $\theta$ is a cartesian fibration is formally dual and will be left to the reader.

Let $(\mathcal{C}, \Qoppa_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R, \Qoppa_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$ and let $F\colon \mathcal{C} = \theta\left(\mathcal{C}, \Qoppa_{\mathcal{C}}\right) \to \mathcal{D}$ be an $R$-linear functor. Now define $\Qoppa_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp}$ to be the left Kan extension of $\Qoppa_{\mathcal{C}}$ along $F^{\mathrm{op}}$. By the proof of (4), we see that the image of $\Qoppa_{\mathcal{C}}$ under $F_!$ is a lift of $(\mathcal{D}, \Qoppa_{\mathcal{D}})$ to an object of $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R, \Qoppa_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$ and $(f, \eta)$ to a morphism in $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R, \Qoppa_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$.

Now by Lemma 2.4.4.1 and the locally cocartesian version of Proposition 2.4.1.10 of [Lur09], we must show that for all choices $\Qoppa'_{\mathcal{D}}$ of an $R$-linear Hermitian structure on $\mathcal{D}$, precomposition with $F_!$ induces a pullback square

$$\begin{tikzcd} \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty\,R}}\left(\left(\mathcal{D}, \Qoppa_{\mathcal{D}}\right), \left(\mathcal{D}, \Qoppa'_{\mathcal{D}}\right)\right) \arrow[r] \arrow[d] & \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty\,R}}\left(\left(\mathcal{C}, \Qoppa_{\mathcal{C}}\right), \left(\mathcal{D}, \Qoppa'_{\mathcal{D}}\right)\right) \arrow[d] \\ \hom_{\mathcal{C}at^{\mathrm{ex}}_{\infty\,R}}\left(\mathcal{D}, \mathcal{D}\right) \arrow[r] & \hom_{\mathcal{C}at^{\mathrm{ex}}_{\infty\,R}}\left(\mathcal{C}, \mathcal{D}\right) \end{tikzcd}\,. \tag{5.11}$$

By (4), $F_!$ induces equivalences on the fibers of the vertical maps, hence $(f, \eta)$ is locally $\theta$-cocartesian. The locally $\theta$-cocartesian maps are manifestly closed under composition, hence we are done. □

**Corollary 5.12.** *Let $R$ be a Poincaré ring, and let $A, B$ be $\mathbb{E}_1$-$R$-algebras with genuine involution. Then there is an equivalence* $\hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}}\left(\left(\mathrm{Mod}^{\omega}_A, \Qoppa_A\right), \left(\mathrm{Mod}^{\omega}_B, \Qoppa_B\right)\right) \simeq \left(\mathrm{BiMod}_{A\otimes_R B^{\mathrm{op}}}\right)_{A^{\varphi C_2}\otimes_R B^{\varphi C_2}/-}$.

*Proof.* □

L: todo– probably need to fix the statement with *duals* when the proof is written

**Definition 5.13.** Let $R$ be a Poincaré ring, and let $A$ be an $\mathbb{E}_1$-$R$-algebra with an anti-involution. We will refer to the data of $(M_A, N_A, N_A \to M_A^{tC_2})$ of Proposition 5.4(2) as an *$R$-linear $A$-module with genuine involution.*

**Remark 5.14.** When $R = \mathbb{S}^0$ is the initial Poincaré ring, then a $\mathbb{S}^0$-linear $A$-module with genuine involution is simply an $A$-mmodule with genuine involution in the sense of [Cal+20a, Defintiion 3.2.3].

**Proposition 5.15.** *Let $R$ be a Poincaré ring, and let $A, B$ be $\mathbb{E}_1$-$R$-algebras with anti-involutions and let $(M_A, N_A, \alpha\colon N_A \to M_A^{tC_2})$, $(M_B, N_B, \beta\colon N_B \to M_B^{tC_2})$ be $R$-linear modules with genuine involution over $A$ and $B$, respectively. Suppose given a map $f\colon A \to B$ of $\mathbb{E}_1$-$R$-algebras with anti-involution, and write $f_*$ for the functor $B \otimes_A -\colon \operatorname{LMod}_A^\omega \to \operatorname{LMod}_B^\omega$. Then*

L: 'geometrically' this should be $f^*$

1. *the data of a $R$-linear hermitian functor $\left(\operatorname{LMod}_A^\omega, \Qoppa_{M_A}^\alpha\right) \to \left(\operatorname{LMod}_B^\omega, \Qoppa_{M_B}^\beta\right)$ covering the base change functor $f_*$ can be encoded by a triple $(\delta, \gamma, \sigma)$ where $\delta\colon M_A \to M_B$ is a morphism in $\operatorname{LMod}_{A\otimes_R A}^{hC_2}$, $\gamma\colon N_A \to N_B$ is a morphism in $\operatorname{LMod}_{A\otimes_R R^{\varphi C_2}}$, and $\sigma$ is a homotopy making the square*

$$\begin{tikzcd} N_A \arrow[r, "\gamma"] \arrow[d, "\alpha"] & N_B \arrow[d, "\beta"] \\ M_A^{tC_2} \arrow[r, "\delta^{tC_2}"] & M_B^{tC_2} \end{tikzcd}$$

*commute.*

L: This is an $R$-linear version of [Cal+20a, Corollary 3.4.2]

2. *$(\delta, \gamma, \sigma)$ defines an $R$-linear Poincaré functor if the maps*

$$B \otimes_A M_A \to (B \otimes_R B) \otimes_{A\otimes_R A} M_A \to M_B$$
$$B \otimes_A N_A \to N_B$$

*are equivalences.*

As in the Picard group case, the symmetric monoidal forgetful functor $\theta\colon \operatorname{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}$ induces a map of spectra $\theta\colon \operatorname{Br}^{\mathrm{p}}(A) \to \operatorname{Br}(A^e)$. When $A^e$ is endowed with the trivial action, $\theta$ will factor through the 2-torsion on $\pi_0$. As a consequence of Proposition 5.4(2) we can identify the fiber of this map.

**Corollary 5.16.** *Let $(\operatorname{Mod}_A^\omega, \Qoppa_A)$ be a Poincaré ring with underlying genuine $C_2$ spectrum $A^L$ as in Proposition 5.4(2). Write $\sigma\colon A^e \simeq A^e$ for the $C_2$-action on the underlying $\mathbb{E}_\infty$-ring associated to $A$. Then the fiber of the map*

$$\theta\colon \operatorname{Br}^{\mathrm{p}}(A) \to \operatorname{Br}(A^e)$$

*can be naturally identified with $\operatorname{Pic}\left(\operatorname{Mod}_{A^L}\left(\mathrm{Sp}^{C_2}\right)\right)$. Moreover, the connecting map $\Omega \operatorname{Br}(A^e) \simeq \operatorname{Pic}(A^e) \to \operatorname{fib}(\theta)$ is induced by the norm $\operatorname{Mod}_{A^e}^{\mathrm{op}} \to \operatorname{Mod}_{A^L}\left(\mathrm{Sp}^{C_2}\right)$, $X \mapsto N^{C_2}(X^\vee) \otimes_{N^{C_2}A^e} A^L$, which on underlying spectra is given by $X \mapsto X^\vee \otimes_A \sigma^* X^\vee$.*

*Proof of Corollary 5.16.* Since $\theta : \operatorname{Mod}_{(\operatorname{Mod}_A^\omega, \Qoppa_A)}(\operatorname{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \operatorname{Mod}_{\operatorname{Mod}_{A^e}^\omega}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)$ is symmetric monoidal and conservative, it induces a map $\theta^\simeq\colon \operatorname{PnBr}(A) \to \operatorname{br}(A^e)$ on the groupoid core of invertible objects. Now observe that $\theta$ is an isofibration; it follows that $\theta^\simeq$ is a Kan fibration by [Lur24, Proposition 01EZ]. Consequently, to identify the homotopy fiber of $\theta$, it suffices to identify the fiber of $\theta$ over a single point. Consider $(\operatorname{Mod}_{A^e}^\omega, \Qoppa)$ a point in the fiber of $\theta$ over $\operatorname{Mod}_{A^e}^\omega$. By Proposition 5.4(2), $\Qoppa$ is associated to an $A$-linear invertible module with involution $\left(M, N, N \to M^{tC_2}\right)$. By Proposition 5.4(3), invertibility of $(\operatorname{Mod}_{A^e}^\omega, \Qoppa)$ implies that $\left(M, N, N \to M^{tC_2}\right)$ is invertible as a module over $A^L$.

Now we give the description of the connecting map. Write $(\operatorname{Mod}_{A^e}^\omega, \Qoppa_A)$ for the identity element in the fiber of $\theta$ over $\operatorname{Mod}_{A^e}^\omega$, and let $\gamma\colon S^1 \to \operatorname{Br}(A^e)$. Write $\mathcal{L}_\gamma$ for $\gamma$ regarded as a point in $\operatorname{Pic}(A^e) \simeq \Omega \operatorname{Br}(A^e)$. Lift $\gamma$ to a path $\widetilde{\gamma}$ in $\operatorname{PnBr}(A)$ starting at $(\operatorname{Mod}_{A^e}^\omega, \Qoppa_A)$, and write $(\operatorname{Mod}_{A^e}^\omega, \Phi)$ for the other endpoint of $\widetilde{\gamma}$. By Proposition 5.4(2), $\Qoppa_A$ is associated to the invertible $A^L$-module with involution $A^L$ and $\Phi$ is associated to some invertible $A^L$-module with involution $(M, N, N \to M^{tC_2})$. We may regard $\widetilde{\gamma}$ as an $A$-linear hermitian equivalence from $(\operatorname{Mod}_{A^e}^\omega, \Qoppa_A)$ to $(\operatorname{Mod}_{A^e}^\omega, \Phi)$, which by Proposition 5.4(4) consists of an $A$-linear hermitian functor $(F, \eta\colon \Qoppa_A \to \Phi \circ F)$ so that $F, \eta$ are both equivalences. Since $\widetilde{\gamma}$ projects to $\gamma$, we must have that $F = - \otimes \mathcal{L}_\gamma$. Now the natural equivalence $\eta$ classifies an equivalence $A^L \simeq \hom_{A^L}(N_A^{C_2}(L), (M, N, N \to M^{tC_2}))$ of $A^L$-modules (Proposition 5.15), hence the result. $\square$

**Example 5.17.** Let $\mathbb{S}^u$ denote the universal Poincaré structure on the sphere spectrum, or equivalently $\mathbb{S}^u$ is the Poincaré ring associated to the genuine equivariant sphere spectrum. By Corollary 5.16 we have a fiber sequence

$$\operatorname{Pic}(\mathrm{Sp}^{C_2}) \to \operatorname{PnBr}(\mathbb{S}^u) \to \operatorname{br}(\mathbb{S})$$

and by [AG14b, Corollary 7.17] we have that $\pi_0(\mathrm{br}(\mathbb{S})) = 0$. Therefore we get a long exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & \pi_1(\mathrm{PnBr}(\mathbb{S}^u)) \arrow[r] \arrow[d, "\cong"] & \pi_1(\mathrm{br}(\mathbb{S})) \arrow[r] \arrow[d, "\cong"] & \pi_0(\mathrm{Pic}(\mathrm{Sp}^{C_2})) \arrow[r] \arrow[d, "\cong"] & \pi_0(\mathrm{PnBr}(\mathbb{S}^u)) \arrow[r] & 0 \\
& \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \times \mathbb{Z} & &
\end{tikzcd}$$

where the third term is identified via [Kra25, Section 8.1]. From this and the fact that $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is the identity on the first component since they are both suspension of the underlying spectrum we see that $\pi_0(\mathrm{PnBr}(\mathbb{S}^u)) \simeq \mathbb{Z}$. In fact this allows us to identify the space $\mathrm{PnBr}(\mathbb{S}^u) \simeq \mathbb{Z} \times B\mathrm{TR}^2(\mathbb{S}; 2)^\times$.

**Example 5.18.** Let $k$ be an algebraically closed field, and regard $k$ as a Poincaré ring $\underline{k}$ via Example 3.15 with the trivial involution. Then

$$\pi_0 \mathrm{PnBr}(\underline{k}) \simeq \begin{cases} \mathbb{Z}/2 & \text{if char } k \neq 2 \\ \mathbb{Z} & \text{if char } k = 2\,. \end{cases}$$

To see this, note that by [Toë12, Proposition 1.9], $\pi_0 \mathrm{br}(k^e) \simeq 0$. Thus by Corollary 5.16, it suffices to understand the fiber sequence

$$\mathrm{Pic}(k) \to \mathrm{Pic}(\mathrm{Mod}_{\underline{k}}(\mathrm{Sp}^{C_2})) \to \mathrm{PnBr}(\underline{k})$$

.

Now if char $k \neq 2$, then $\mathrm{Pic}(\underline{k}) \simeq \mathrm{Pic}(k)^{hC_2} \simeq (\mathbb{Z} \times Bk^\times)^{hC_2}$ where the generator of $C_2$ acts on $\pi_1 Bk^\times$ by $u \mapsto u^{-1}$. We then have that $H^1(C_2, k^\times) = \{x \in k^\times | x \cdot x^{-1} = 1\}/\{y^{-1}/y | y \in k^\times\} = k^\times/(k^\times)^2 = 0$ since $k$ is algebraically closed.

L: what is the $C_2$-action on $k^\times$? Come to think of it it might just be trivial.

We can then deduce that

$$\pi_0 \mathrm{Pic}(\mathrm{Mod}_{\underline{k}}(\mathrm{Sp}^{C_2})) \simeq \begin{cases} \mathbb{Z} \times \mathbb{Z} & \text{if char } k = 2 \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

where the char$(k) = 2$ case is handled by an argument similar to [Kra25, Section 8.1]. If char $k \neq 2$ then the map $\pi_0 \mathrm{Pic}(k) \to \pi_0 \mathrm{Pic}(\underline{k})$ is $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$. On the other hand, if char $k = 2$, then the map $\pi_0 \mathrm{Pic}(k) \to \pi_0 \mathrm{Pic}(\underline{k})$ is $\mathbb{Z} \xrightarrow{n \mapsto (2n, n)} \mathbb{Z} \times \mathbb{Z}$.

L: This and Example 5.19 are a pair; if we move one, also move the other.

N: Why is the characteristic away from 2 case zero? It looks to me like that map is the identity?

**Example 5.19.** Let $k$ be an algebraically closed field, and consider the Poincaré ring associated to $\prod_{C_2} k$ (i.e. $k \times k$ with the swap action) via Example 3.15. Similarly to Example 5.18, by [Toë12, Proposition 1.9] we have $\pi_0 \mathrm{br}(\mathrm{Spec}\, k \sqcup \mathrm{Spec}\, k) \simeq \pi_0 \mathrm{br}(\mathrm{Spec}\, k)^{\times 2} = 0$. Thus by Corollary 5.16, it suffices to understand the cokernel of the connecting homomorphism $\pi_1 \mathrm{br}(k \times k) \to \pi_0\, \mathrm{Pic}(\prod_{C_2} k)$. Now since $\prod_{C_2} k$ is Borel and $(\prod_{C_2} k)^{tC_2} = 0$,

$$\mathrm{Pic}\left(\mathrm{Mod}_{\prod_{C_2} k}\left(\mathrm{Sp}^{C_2}\right)\right) \simeq \mathrm{Pic}\, \mathrm{Mod}_{k \times k}(\mathrm{Sp})^{hC_2} \simeq \left(\prod_{C_2} (\mathbb{Z} \oplus k^\times[1])\right)^{hC_2} \simeq \mathbb{Z} \oplus k^\times[1]$$

thus $\pi_0 \mathrm{Pic}\left(\mathrm{Mod}_{\prod_{C_2} k}\left(\mathrm{Sp}^{C_2}\right)\right) \simeq \mathbb{Z}$. On the other hand, $\pi_1 \mathrm{br}(k \times k) \simeq \pi_0\, \mathrm{Pic}(k \times k) \simeq \mathbb{Z}^{\times 2}$ and the connecting homomorphism is $(n, m) \mapsto n + m$, whence $\pi_0 \mathrm{PnBr}\left(\prod_{C_2} k\right) = 0$.

Write Fm, resp. Pn for the composite $\mathrm{Cat}^{\mathrm{h}}_{\infty R} \xrightarrow{U} \mathrm{Cat}^{\mathrm{p}}_\infty \xrightarrow{\mathrm{Fm}} \mathcal{S}$, resp. $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \xrightarrow{U} \mathrm{Cat}^{\mathrm{p}}_\infty \xrightarrow{\mathrm{Pn}} \mathcal{S}$.

**Proposition 5.20.** *Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Then $(\mathrm{Mod}^\omega_R, \Omega_R)$ corepresents the functors* $\mathrm{Fm}\colon \mathrm{Cat}^{\mathrm{h}}_{\infty R} \to \mathcal{S}$ *and* $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{S}$.

*Proof.* We prove the statement for Pn; the proof for Fm is similar and is left to the reader. Recall that Proposition 5.4.(1) furnishes an adjoint pair $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \rightleftarrows \mathrm{Cat}^{\mathrm{p}}_{\infty}$ of functors. Write $\overline{\mathcal{C}} = (\mathcal{C}, \Qoppa_{\mathcal{C}}) \in \mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}$. Then

$$\mathrm{Pn}(\mathcal{C}) = \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty}}\left((\mathrm{Sp}^f, \Qoppa^u), U(\overline{\mathcal{C}})\right) \simeq \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty R}}\left((\mathrm{Mod}^{\omega}_R, \Qoppa_R) \otimes (\mathrm{Sp}^f, \Qoppa^u), \overline{\mathcal{C}}\right),$$

where the first equivalence is [Cal+20a, Proposition 4.1.3]. $\square$

## 5.2 Azumaya algebras with genuine involution

In this section, we introduce the notion of a derived/generalized Azumaya algebra with involution. Recall that a classical/discrete Azumaya algebra $\mathcal{A}$ is étale-locally given by the endomorphism algebra of a vector bundle $V$, and a derived/generalized Azumaya algebra over $X$ is étale-locally given by the endomorphism algebra of a perfect complex. Observe that if $\mathcal{A} \simeq \mathrm{End}_X(V)$, then $\sigma^*\mathcal{A}^{\mathrm{op}} \simeq \mathrm{End}_X(\sigma^*(V^\vee))$. The prototypical anti-involution on a classical/discrete Azumaya algebra $\mathcal{A}$ is given by taking transposes and conjugating by an isomorphism $V \simeq \sigma^*(V^\vee) \otimes \mathcal{L}$, where $\mathcal{L}$ is a line bundle on $X$ [PS92, §1.1(ii), p.209, §1.2, p.216]. If we now consider a derived/generalized Azumaya algebra $\mathcal{A} \simeq \mathrm{End}_X(P)$ where $P$ is a perfect complex on $X$, then the prototypical anti-involution on $\mathcal{A}$ arises from transposition and conjugation by an equivalence $P \simeq \sigma^*(P^\vee) \otimes \mathcal{L}[n]$.

L: Example? $R\Gamma$ of an oriented family of smooth proper schemes over $X$? relate to Serre duality!

Let $R$ be an $\mathbb{E}_\infty$-ring spectrum.

**Recollection 5.21.** Recall [BRS12; AG14a] that an $\mathbb{E}_1$-$R$-algebra $A$ is said to be *Azumaya* if it is a compact generator of $\mathrm{Mod}_R$ and if the natural $R$-algebra map giving the bimodule structure on $A$

$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$$

is an equivalence of $R$-algebras.

**Definition 5.22.** Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring spectrum, and write $\sigma\colon R \xrightarrow{\sim} R$ for the involution on $R$. An *Azumaya algebra with genuine (anti-)involution* over $R$ is the data of

(a) An $\mathbb{E}_1$-$R$-algebra $A$ equipped with an anti-involution $\tau\colon A \to \sigma^*A^{\mathrm{op}}$ so that the underlying $\mathbb{E}_1$-$R$-algebra $A$ is an Azumaya $R$-algebra in the sense of Recollection 5.21

(b) an $(A \otimes_R \sigma^*A)^{\otimes_R 2}$-linear equivalence $\hom_{R\otimes R}(A \otimes_R A, R) \simeq A \otimes_R \sigma^*A^{\mathrm{op}}$

(c) A left $A \otimes_R R^{\varphi C_2}$-module $P$ and an $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-module $\overline{P}$

(d) An $A \otimes_R R^{\varphi C_2}$-linear map $P \to A^{tC_2}$ and an $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-linear map $\overline{P} \to A^{tC_2}$. Here we regard $A^{tC_2}$, which is canonically a $(A \otimes_R \sigma^*A)^{tC_2}$-module, as an $A \otimes_R R^{\varphi C_2}$-module (resp. $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-module) via the twisted Tate-valued diagonal $A \to (A \otimes \sigma^*A)^{tC_2}$ (resp. $A^{\mathrm{op}} \to (\sigma^*A^{\mathrm{op}} \otimes A^{\mathrm{op}})^{tC_2}$).

(e) An equivalence of $(A \otimes_R \sigma^*A^{\mathrm{op}}) \otimes_R R^{\varphi C_2}$-modules

$$\hom_R(A, R^{\varphi C_2}) \simeq P \otimes_{R^{\varphi C_2}} \overline{P}$$

and a homotopy making the diagram

$$\begin{tikzcd}
\hom_R(A, R^{\varphi C_2}) \arrow[r] \arrow[d] & P \otimes_{R^{\varphi C_2}} \overline{P} \arrow[d] \\
\hom_R(A, R^{tC_2}) \simeq \hom_R(A \otimes A^{\mathrm{op}}, R)^{tC_2} \arrow[r] & \hom_{(A\otimes A^{\mathrm{op}})^{\otimes 2}}\left((A \otimes A^{\mathrm{op}})^{\otimes 2}, A \otimes A^{\mathrm{op}}\right)^{tC_2} \simeq (A \otimes A^{\mathrm{op}})^{tC_2}
\end{tikzcd}$$

commute, where the lower horizontal arrow is induced by (b) and the right vertical arrow is induced by (d).

**Observation 5.23.** Let $A$ be an Azumaya algebra with genuine involution over $R$, and suppose given a map $R \to S$ of Poincaré rings. Then $(A \otimes_R S, P \otimes_{R^{\varphi C_2}} S^{\varphi C_2})$ is an Azumaya algebra with genuine involution over $S$.

**Remark 5.24.** If $A$ is an Azumaya algebra with genuine involution over $R$, then in particular $M_A = A$, $N_A = P$ is a module with genuine involution over $A$ in the sense of [Cal+20a, Definition 3.2.3].

**Definition 5.25.** Let $(X, \sigma)$ be a scheme with an involution and let $\pi\colon X \to Y$ exhibit $Y$ as a good quotient of $X$. Recall that there is a sheaf of $C_2$-$\mathbb{E}_\infty$-algebras $\underline{\mathcal{O}}$ (Construction 3.21), and write $\sigma$ for the involution $\pi_*\mathcal{O}_X \xrightarrow{\sim} \pi_*\mathcal{O}_X$. An *Azumaya algebra with genuine (anti-)involution* over $X$ is the data of

(a) An $\mathbb{E}_1$-$\underline{\mathcal{O}}^e = \pi_*\mathcal{O}_X$-algebra $A$ equipped with an anti-involution $\tau\colon A \to \sigma^*A^{\mathrm{op}}$ (i.e. $\sigma^*(\tau^{\mathrm{op}}) \circ \tau \simeq \mathrm{id}_A$, and higher coherences) so that the underlying $\mathbb{E}_1$-$\underline{\mathcal{O}}^e$-algebra $A$ is a [generalized] Azumaya $\underline{\mathcal{O}}^e$-algebra in the sense of [Toë12, Definition 2.11].

(b) an $(A \otimes_{\underline{\mathcal{O}}^e} \sigma^*A)^{\otimes_{\underline{\mathcal{O}}^e} 2}$-linear equivalence $\hom_{\underline{\mathcal{O}}^e}(A \otimes_{\underline{\mathcal{O}}^e} A, R) \simeq A \otimes_{\underline{\mathcal{O}}^e} \sigma^*A^{\mathrm{op}}$

(c) A left $A \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-module $P$ and an $A^{\mathrm{op}} \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-module $\overline{P}$

(d) An $A \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-linear map $P \to A^{tC_2}$ and an $A^{\mathrm{op}} \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-linear map $\overline{P} \to A^{tC_2}$. Here we regard $A^{tC_2}$, which is canonically a $(A \otimes_{\underline{\mathcal{O}}^e} \sigma^*A)^{tC_2}$-module, as an $A \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-module (resp. $A^{\mathrm{op}} \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-module) via the twisted Tate-valued diagonal $A \to (A \otimes \sigma^*A)^{tC_2}$ (resp. $A^{\mathrm{op}} \to (\sigma^*A^{\mathrm{op}} \otimes A^{\mathrm{op}})^{tC_2}$).

(e) An equivalence of $(A \otimes \sigma^*A^{\mathrm{op}}) \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-modules

$$\hom_{\underline{\mathcal{O}}^e}(A, \underline{\mathcal{O}}^{\varphi C_2}) \simeq P \otimes_{\underline{\mathcal{O}}^{\varphi C_2}} \overline{P}$$

and a homotopy making the diagram

$$\begin{tikzcd}
\hom_{\underline{\mathcal{O}}^e}(A, \underline{\mathcal{O}}^{\varphi C_2}) \simeq P \otimes_{\underline{\mathcal{O}}^{\varphi C_2}} \overline{P} \arrow[r] \arrow[d] & P \otimes_{\underline{\mathcal{O}}^{\varphi C_2}} \overline{P} \arrow[d] \\
\hom_{\underline{\mathcal{O}}^e}(A, \underline{\mathcal{O}}^{etC_2}) \simeq \hom_{\underline{\mathcal{O}}^e}(A \otimes A^{\mathrm{op}}, \underline{\mathcal{O}}^e)^{tC_2} \arrow[r] & \hom_{(A \otimes A^{\mathrm{op}})^{\otimes 2}}\left((A \otimes A^{\mathrm{op}})^{\otimes 2}, A \otimes A^{\mathrm{op}}\right)^{tC_2} \simeq (A \otimes A^{\mathrm{op}})^{tC_2}
\end{tikzcd}$$

commute, where the lower horizontal arrow is induced by (b) and the right vertical arrow is induced by (d).

**Example 5.26.** Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(P, q) \in \mathrm{Pn}\,(\mathrm{Mod}^\omega_R, Ϙ_R)$.

L: What if I replaced $Ϙ_R$ by a shift $Ϙ^{[n]}_R$? Or any $R$-linear Poincaré structure $Ϙ'$ so that $(\mathrm{Mod}^\omega_R, Ϙ')$ is in $\mathrm{PnBr}(R)$? see [Cal+20a, §3.5], and furthermore see [PS92, p.216].

Then $A := \mathrm{End}_R(P)$ admits a canonical lift to an $\mathbb{E}_1$ algebra with genuine involution over $R$ with $A^{\varphi C_2} := \hom_R(P, R^{\varphi C_2})$. If $P$ is a generator of $\mathrm{Mod}^\omega_R$, then $A$ is furthermore Azumaya.

By [Cal+20a, Proposition 3.1.16], $A$ inherits a canonical anti-involution. To exhibit (b), observe that $q^\dagger$ induces a canonical $A \otimes A^{\mathrm{op}}$-linear equivalence $A = \mathrm{End}(P) \simeq \mathrm{End}(P^\vee) \simeq A^\vee$. If $P$ is a generator, $\hom_R(P, -)$ induces an equivalence $\mathrm{Mod}^\omega_R \simeq \mathrm{Mod}^\omega_A$, thus we can regard $\mathrm{Mod}^\omega_A$ as equipped with a Poincaré structure. By the classification of $R$-linear Poincaré structures of Proposition 5.4(2), the Poincaré structure on $\mathrm{Mod}^\omega_A$ is associated to an $A$-module with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$. We claim that $M_A \simeq A$ with the canonical $A$-$A$-bimodule structure: By [Cal+20a, Proposition 3.1.6], as an $A^{\mathrm{op}}$-module $M_A$ is the image of $A$ under the composite

L: todo: an $R$-linear enhancement of Proposition 3.1.16?

$$\mathrm{Mod}^\omega_A \xrightarrow{\hom_R(P,-)^{-1}} \mathrm{Mod}^\omega_R \xrightarrow{D_R = \hom_R(-,R)} \mathrm{Mod}^{\omega,\mathrm{op}}_R \xrightarrow{\hom_R(P,-)} \mathrm{Mod}^{\omega,\mathrm{op}}_A .$$

Observe that the image of $A$ in $\mathrm{Mod}^{\omega,\mathrm{op}}_R$ is $D_R(P)$ and $q^\dagger$ induces an equivalence $D_R(P) \simeq P$, hence $M_A \simeq A$ as $A^{\mathrm{op}}$-modules.

A similar argument with the linear part of $Ϙ$ shows that we have an equivalence $N_A \simeq \hom_R(P, R^{\varphi C_2})$.

L: continue. discuss consequences of being Morita trivial (distinguished 'point' in $N_A$)? can we identify $\overline{P}$?

**Proposition 5.27.** *Let $R$ be a discrete ring with a given $C_2$-action $\lambda$, and suppose that the branch locus in $\mathrm{Spec}(R)/C_2 = \mathrm{Spec}(R^{C_2})$ is empty. Let $A$ be a classical Azumaya algebra over $R$ with an involution of type 2. Regard $R$ as a Poincaré ring spectrum $\underline{R}^\lambda$ via Example 3.15.*

*Then there is a canonical Azumaya algebra with genuine involution over $\underline{R}^\lambda$ whose underlying Azumaya algebra is $A$.*

L: In what other cases does a lift exist triv-

**Remark 5.28.** If $\frac{1}{2} \in R$, then $\mathrm{Br}(\mathrm{Spec}\, R, \lambda)$ is defined [PS92, p. 216]. In view of Propositions 5.27 and 5.29, there is a homomorphism $\mathrm{Br}(\mathrm{Spec}\, R, \lambda) \to \pi_0 \mathrm{PnBr}(\underline{R}^{\lambda})$.

L: rewrite for scheme with involution

*Proof of Proposition 5.27.* Since $R^{\varphi C_2} = 0$, conditions (c)-(e) of Definition 5.22 are vacuous. ☐

**Proposition 5.29.** *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(A, A^{\varphi C_2} \to A^{tC_2})$ be an Azumaya algebra with genuine involution over $R$. Then*

1. $(\mathrm{Mod}^{\omega}_A, \Omega_A)$ *defines an $R$-linear Poincaré $\infty$-category.*
2. $(\mathrm{Mod}^{\omega}_A, \Omega_A)$ *is an invertible object in* $\mathrm{Mod}_{(\mathrm{Mod}^{\omega}_R, \Omega_R)}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}\right)$.

*Proof of Proposition 5.29.* The first statement follows from Proposition 5.4(2); we prove the second statement. First, by [Cal+20a, Example 3.2.9], we see that $(\mathrm{Mod}^{\omega}_A, \Omega_A)$ is indeed an $R$-linear Poincaré $\infty$-category (and not merely hermitian). To show that the associated Poincaré $\infty$-category is invertible, we must identify a dual $(\mathrm{Mod}^{\omega}_A, \Omega_A)^{\vee}$ and exhibit an equivalence $(\mathrm{Mod}^{\omega}_A, \Omega_A) \otimes (\mathrm{Mod}^{\omega}_A, \Omega_A)^{\vee} \simeq (\mathrm{Mod}^{\omega}_R, \Omega_R)$. Since $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty\,R}$ is symmetric monoidal, we see that the underlying $R$-linear $\infty$-category associated to the dual must be $\mathrm{Mod}^{\omega}_{A^{\mathrm{op}}}$. Moreover, the canonical evaluation map $\mathrm{ev}\colon \mathrm{Mod}^{\omega}_A \otimes \mathrm{Mod}^{\omega}_{A^{\mathrm{op}}} \xrightarrow{\simeq} \mathrm{Mod}^{\omega}_R$ sends $A \otimes A^{\mathrm{op}}$ to $A$. Endow $\mathrm{Mod}^{\omega}_{\underline{A}^{\mathrm{op}}}$ with a Poincaré structure corresponding to the module with genuine involution $M_{A^{\mathrm{op}}} := A^{\mathrm{op}}$, $N_{A^{\mathrm{op}}} := \overline{P}$. It remains to exhibit a natural equivalence

$$\eta\colon\ (\Omega_A \otimes \Omega_{A^{\mathrm{op}}}) \xrightarrow{\simeq} \mathrm{ev}^{*}\Omega_R \tag{5.30}$$

of [quadratic] functors $\mathrm{Mod}^{\omega}_A \otimes \mathrm{Mod}^{\omega}_{A^{\mathrm{op}}} \to \mathrm{Sp}$. By [Cal+20a, Theorem 3.3.1], it suffices to exhibit equivalences on the bilinear and linear parts of (5.30) which glue compatibly. By Proposition 5.4(3), on linear parts, it suffices to exhibit an $A \otimes_R A^{\mathrm{op}}$-linear equivalence

$$\hom_R(A, R^{\varphi C_2}) \simeq N_A \otimes_{R^{\varphi C_2}} N_{A^{\mathrm{op}}}$$

and on bilinear parts, it suffices to exhibit an $(A \otimes_R A^{\mathrm{op}})^{\otimes_R 2}$-linear equivalence

$$\hom_{R \otimes R}(A \otimes_R A, R) \simeq M_A \otimes_R M_{A^{\mathrm{op}}}$$

which glue compatibly. This follows from the definitions, concluding the proof. ☐

**Observation 5.31.** Let $k$ be an algebraically closed field, and regard $\underline{k}$ as a Poincaré ring spectrum with the trivial involution via Example 3.15. Let $(A, A^{\varphi C_2}, A^{\varphi C_2} \to A^{tC_2})$ be an Azumaya algebra with genuine involution over $\underline{k}$. By Proposition 5.29, $(\mathrm{Mod}^{\omega}_A, \Omega_A) \in \mathrm{PnBr}(\underline{k})$. By [Toë12, Corollary 1.15], $A$ is equivalent to $\mathrm{End}_k(P)$ for some compact $k$-module $P$. Observe that there is a canonical identification $A^{\mathrm{op}} \simeq \mathrm{End}_k(P^{\vee})$. By the derived Skolem–Noether theorem [Lie09, Theorem 5.1.5], there exists a unique $n \in \mathbb{Z}$ so that the involution $A \simeq A^{\mathrm{op}}$ is induced by an equivalence $P \simeq P^{\vee}[n]$ (which is unique up to multiplication by a unit in $k$).

L: the next part is contingent on the computation in Example 5.18; omit if the computation in the example works out to be zero. If the computation is nonzero, consider this a numerical invariant of derived Azumaya algebras w involution and put it in a definition?

By Example 5.18, we may associate to $(\mathrm{Mod}^{\omega}_A, \Omega_A) \in \pi_0\mathrm{PnBr}(\underline{k})$ an integer $n$.

**Proposition 5.32.** *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Let $(\mathcal{C}, \Omega)$ be an invertible idempotent-complete $R$-linear Poincaré $\infty$-category. Suppose given a Poincaré object $(P, q)$ of $(\mathcal{C}, \Omega)$ so that $P$ is a generator for $\mathcal{C}$. Then $(\mathcal{C}, \Omega)$ is of the form $(\mathrm{Mod}^{\omega}_A, \Omega_A)$ for some Azumaya algebra over $R$ with genuine involution.*

L: later: add case of schemes with involution

**Corollary 5.33.** *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Let $(\mathcal{C}, \Omega)$ be an invertible idempotent-complete $R$-linear Poincaré $\infty$-category. Then $(\mathcal{C}, \Omega)$ is of the form $(\mathrm{Mod}^{\omega}_A, \Omega_A)$ for some Azumaya algebra over $R$ with genuine involution.*

*Proof.* By Proposition 5.32, it suffices to exhibit a Poincaré object $(P, q)$ whose underlying object is a generator of $\mathcal{C}$. Since the underlying $R$-linear stable infinity category $\mathcal{C}$ is dualizable it follows that it is an $R$-linear category which staisfies étale hyperdescent by [AG14b, Example 4.4]. By [AG14a, Theorem 6.1], $\mathcal{C}$ has a generator $G$. Now $G \oplus D_{\Omega} G$ promotes canonically to a Poincaré object of $(\mathcal{C}, \Omega)$ by [Cal+20a, Proposition 2.2.5]. $\square$

*Proof of Proposition 5.32.* Example 5.26 produces from this data an Azumaya algebra with genuine involution $A$. Since $P$ generates $\mathcal{C}$ we then get an equivalence $\mathcal{C} \simeq \mathrm{Mod}_A^\omega$ which by the construction of the genuine involution on $A$ will in fact be an equivalence of $R$-linear Poincaré infinity categories. $\square$

## 5.3 Stacks associated to Poincaré-$\infty$-categories

**Notation 5.34.** Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. There is a functor

$$\begin{aligned} \mathbb{E}_\infty \operatorname{Alg}_{R/}^{BC_2} &\to \mathrm{CAlg^p}_{R/-} \\ S &\mapsto (S, S \to R^{\varphi C_2} \otimes_R S \to S^{tC_2}) =: (R, R \to R^{\varphi C_2} \to R^{tC_2}) \otimes S\,, \end{aligned}$$

where the map $R^{\varphi C_2} \otimes_R S \to R^{tC_2} \otimes_{R^{tC_2}} S^{tC_2} \simeq S^{tC_2}$ is given by base change along the Tate-valued norm composed with the structure map $R^{\varphi C_2} \to R^{tC_2}$. Composing the aforementioned functor with the functor that sends a Poincaré ring to its category of compact modules equipped with the canonical Poincaré structure defines a functor

$$\mathrm{Mod}^p \colon \mathbb{E}_\infty \operatorname{Alg}_{R/-}^{BC_2} \to \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}\right).$$

**Notation 5.35.** Let $X$ be a scheme with an involution $\sigma$ and let $\pi \colon X \to Y$ exhibit $Y$ as a good quotient of $X$. If $j \colon U \to Y$ is flat, let us write $\pi^* U$ for the tuple $(X \times_Y U, j^*(\sigma), U, j^*(\pi))$ of Remark 3.19. Then the assignment $(j \colon U \to Y) \mapsto (\mathrm{Mod}^\omega_{j^*X}, \Omega_{j^*\underline{\mathcal{O}}})$ defines a functor

$$\mathrm{Mod}^p \colon \text{Ét}_Y^{\mathrm{op}} \to \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Mod}_{(\mathrm{Mod}^\omega_X, \Omega_{\underline{\mathcal{O}}})}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})\right).$$

**Observation 5.36.** Let $R$ be a discrete commutative ring with a $C_2$-action, and recall that $X := \operatorname{Spec} R \to \operatorname{Spec}(R^{C_2}) = Y$ may be regarded as a $C_2$-scheme (Observation 3.18). Then base change along $R^{C_2} \to R$ defines a map from étale covers of $Y$ of Notation 5.35 to $C_2$-equivariant étale covers of $X$ of Notation 5.34. However, not all $C_2$-equivariant étale covers of $X$ arise from this construction: consider the étale cover associated to the map of rings $R \xrightarrow{r \mapsto (r, \sigma(r))} R \times R$, where $R \times R$ is endowed with the flip action $(r, s) \mapsto (s, r)$.

L: verify later

**Proposition 5.37.** *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and assume that $R^{\varphi C_2}$ and $R$ are connective. Then the assignment of Notation 5.34 is a hypersheaf on the small étale site of $R$.*

L: I think Propositions 5.37 and 5.42 have essentially the same proof–present them together?

**Corollary 5.38.** *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and assume that $R^{\varphi C_2}$ and $R$ are connective. Write* PnBr *for the composite functor $\text{Ét}_R \xrightarrow{\mathrm{Mod}^p} \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}\right) \xrightarrow{\mathrm{PnBr}} \mathrm{Sp}_{\geq 0}$, where $\mathrm{Mod}^p$ is from Notation 5.34 and* PnBr *is Definition 5.1. Then* PnBr *is an étale sheaf.*

*Proof.* Follows from Proposition 5.37 and the fact that PnBr preserves all limits.

N: Does PnBr preserve limits? This isn't clear to me...

Here is an alternative proof. By the above we have that $\mathrm{Pic}((-)^e) : \text{Ét}_Y^{\mathrm{op}} \to \mathrm{Sp}_{\geq 0}$ is an étale hypersheaf, and similarly $\mathrm{Pic}(\mathrm{Mod}_{(-)^L}(\mathrm{Sp}^{C_2})) : \text{Ét}_Y^{\mathrm{op}} \to \mathrm{Sp}_{\geq 0}$ is an étale hypersheaf by the same argument as in Proposition 5.37. Therefore $\mathrm{PnBr} : \text{Ét}_Y^{\mathrm{op}} \to \mathrm{Sp}_{\geq 0}$ also is a hypersheaf by Corollary 5.16 since hypersheaves are closed under extensions. $\square$

L: recall étale hypersheaf/hypercove

**Notation 5.39** (Moduli (pre)sheaf of hermitian objects)**.** Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and let $(\mathcal{C}, \Omega_{\mathcal{C}})$ be an $R$-linear Poincaré $\infty$-category. Define a presheaf $\mathbf{M}^h_{(\mathcal{C}, \Omega_{\mathcal{C}})} \colon \left(\mathbb{E}_\infty \operatorname{Alg}_R^{BC_2}\right) \to \mathcal{S}$ whose value at $R \to S$ is the $\infty$-groupoid of $R$-linear hermitian functors from $(\mathcal{C}, \Omega_{\mathcal{C}})$ to $(\mathrm{Mod}^\omega_R, \Omega_R) \otimes_R S$.

L: Antieau–Gepner write *stack* for 'sheaf of categories' and *sheaf* for 'sheaf of spaces.' Do we want to use this convention?

L: Antieau–Gepner consider the $\infty$-*category* of functors, then pass to the maximal sub-groupoid [AG14a, §5.2]. Do we need to consider an enhancement of hermitian functors to $\infty$-categories?

Now suppose $A$ is an $\mathbb{E}_1$-$R$-algebra with genuine involution. Then we will write $\mathbf{M}^h_A$ for $\mathbf{M}^h_{(\mathrm{Mod}^\omega_A, \Omega_A)}$.

**Observation 5.40.** Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and suppose $A$ is an $\mathbb{E}_1$-$R$-algebra with genuine involution. A point of $\mathbf{M}^h_A(S)$ is classified by pair $(F, \eta)$ where $F$ is an $R$-linear functor $F\colon \mathrm{Mod}^\omega_A \to S$ and $\eta$ is a natural transformation $\eta\colon Ϙ_A \to Ϙ_S \circ F^{\mathrm{op}}$ of $R$-linear hermitian structures. Now $F$ is classified by an $A \otimes_R S$-module $P$ which is compact as an $S$-module (cf. [AG14a, Proposition 5.10]), and $\eta$ is classified by a map of $N^{C_2}_S(A \otimes_R S)$-modules $A \otimes_R S \to N^{C_2}_S(P^\vee)$ (Proposition 5.4(2)). Thus $\mathbf{M}^h_A(S)$ may be identified with the full sub-$\infty$-groupoid of the pullback of the diagram

$$\begin{tikzcd} & \mathrm{Mod}_{A\otimes_R S}(\mathrm{Sp}) \arrow[d, "N^{C_2}_S"] \\ \mathrm{Mod}_{N^{C_2}_S(A\otimes_R S)}\left(\mathrm{Sp}^{C_2}\right)_{A\otimes_R S/} \arrow[r] & \mathrm{Mod}_{N^{C_2}_S(A\otimes_R S)}\left(\mathrm{Sp}^{C_2}\right). \end{tikzcd}$$

**Lemma 5.41.** *Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and let $(\mathcal{C}, Ϙ_{\mathcal{C}})$ be an $R$-linear Poincaré $\infty$-category. Suppose that $R$ is connective. Then the assignment $\mathbf{M}^h_{(\mathcal{C},Ϙ_{\mathcal{C}})}$ of Notation 5.39 satisfies étale hyperdescent.*

L: maybe want $R^{\varphi C_2}$ to be connective too?

**Proposition 5.42.** *Let $(X, \lambda, Y, \pi)$ be a scheme with involution $X$ and a good quotient $Y$. Then the assignment of Notation 5.35 is a hypersheaf on the small étale site of $Y$.*

*Proof of Proposition 5.37.* For now, use the notational shorthand $R^p = (R, R \to R^{\varphi C_2} \to R^{tC_2})$. Since limits in categories of algebras and modules are computed at the level of underlying objects, it suffices to show that the functor sends an étale hypercovering $j_\bullet\colon S \to T^\bullet$ to a limit diagram in $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$. By Proposition 6.1.4 of [Cal+20a], it suffices to show that the relevant diagram is a limit diagram in $\mathrm{Cat}^{\mathrm{h}}_{\infty,\mathrm{idem}}$. The proof of Lemma 5.4 in [AG14a] implies that the diagram defines a limit diagram on underlying $\infty$-categories. Thus by Remark 6.1.3 of [Cal+20a], it suffices to show that $j^*_\bullet\colon \mathrm{Mod}^\omega_{R^e\otimes_{R^{C_2}}S} \to \mathrm{Mod}^\omega_{R^e\otimes_{R^{C_2}}T^\bullet}$ induces an equivalence $Ϙ_{R^p\otimes S} \xrightarrow{\sim} \lim_\Delta Ϙ_{R^p\otimes T^\bullet} \circ (j^*_\bullet)^{\mathrm{op}}$ of quadratic functors $\mathrm{Mod}^{\omega,\mathrm{op}}_{R^e\otimes_{R^{C_2}}S} \to \mathrm{Sp}$. This follows from our assumption on $S \to T^\bullet$ and [Cal+20a, Theorem 3.3.1]. ☐

**Notation 5.43** (Moduli (pre)sheaf of hermitian objects)**.** Let $(X, \sigma, Y, \pi)$ be a scheme with involution and good quotient and let $(\mathcal{C}, Ϙ_{\mathcal{C}})$ be an $\left(\mathrm{Mod}^\omega_X, Ϙ_{\underline{O}}\right)$-linear Poincaré $\infty$-category. Define a presheaf $\mathbf{M}^h_{(\mathcal{C},Ϙ_{\mathcal{C}})}\colon \mathrm{\acute{E}t}^{\mathrm{op}}_Y \to \mathcal{S}$ whose value at $\mathrm{Spec}\, S \to Y$ is the $\infty$-groupoid of $\left(\mathrm{Mod}^\omega_X, Ϙ_{\underline{O}}\right)$-linear hermitian functors from $(\mathcal{C}, Ϙ_{\mathcal{C}})$ to $\left(\mathrm{Mod}^\omega_{X_S}, Ϙ_{\underline{O}|_S}\right)$.

L: same comment about $\infty$-*category* of functors vs $\infty$-groupoid.

Now suppose $A$ is an Azumaya algebra over $X$ with genuine $\sigma$-linear (anti-)involution. Then we will write $\mathbf{M}^h_A$ for $\mathbf{M}^h_{\left(\mathrm{Mod}^\omega_A, Ϙ_A\right)}$.

**Lemma 5.44.** *Let $(X, \sigma, Y, \pi)$ be a scheme with involution and good quotient and let $(\mathcal{C}, Ϙ_{\mathcal{C}})$ be an $\left(\mathrm{Mod}^\omega_X, Ϙ_{\underline{O}}\right)$-linear Poincaré $\infty$-category. Then the functor $\mathbf{M}^h_{(\mathcal{C},Ϙ_{\mathcal{C}})}$ defines a hypersheaf on the small étale site of $Y$.*

**Observation 5.45.** Let $(X, \sigma, Y, \pi)$ be a scheme with involution and good quotient and let $A$ be an Azumaya algebra over $X$ with genuine $\sigma$-linear (anti-)involution. Recall that there is a sheaf $\mathbf{M}_{A^e}$ on the small étale site of $X$ [AG14a, §5.2]. The forgetful functor $\mathrm{Cat}^{\mathrm{h}}_{\infty R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}$ induces a morphism of (pre)stacks $\mathbf{M}^h_A \to \pi_*\mathbf{M}_{A^e}$ on the small étale site of $Y$, where $\mathbf{M}^h_A$ is from Notation 5.43. Given a point $\mathrm{Spec}\, S \to Y$, the map

L: cite Toën? §2.2?

$$\begin{tikzcd} \mathbf{M}^h_A(\mathrm{Spec}\, S) \arrow[r] \arrow[d, phantom, "\wr|"] & \pi_*\mathbf{M}^h_{A^e}(\mathrm{Spec}\, S) \arrow[d, phantom, "\wr|"] \\ \hom_{\left(\mathrm{Mod}^\omega_X, Ϙ_{\underline{O}}\right)-\mathrm{Mod}(\mathrm{Cat}^{\mathrm{h}}_\infty)}\left(\left(\mathrm{Mod}^\omega_A, Ϙ_A\right), \left(\mathrm{Mod}^\omega_{X_S}, Ϙ_{\underline{O}|_S}\right)\right) & \hom_{\left(\mathrm{Mod}^\omega_X\right)-\mathrm{Mod}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)}\left(\mathrm{Mod}^\omega_A, \mathrm{Mod}^\omega_{X_S}\right) \end{tikzcd}$$

sends a hermitian functor $(F, \eta)$ to its underlying $\mathcal{O}_X$-linear functor $F$.

L: I think we can combine Lemma 5.47, Lemma 5.48, and [AG14a, Lemma 4.25] to show that $\mathbf{M}^h_A$ is locally geometric.

**Recollection 5.46.** An affine morphism of schemes is always quasi-compact [Stacks, Tag 01S5]. In particular, if $(X, \sigma, Y, \pi)$ is a scheme with involution and good quotient, then $\pi$ is quasi-compact.

**Lemma 5.47.** *Let $(X, \sigma, Y, \pi)$ be a scheme with involution and good quotient and let $A$ be an Azumaya algebra over $X$ with genuine $\sigma$-linear (anti-)involution. Assume that $Y$ is quasi-compact. Then the sheaf $\pi_*\mathbf{M}_{A^e}$ is locally geometric.*

*L: Other desired properties: locally of finite presentation, smoothness. Possibly need to impose different assumptions on $\pi$ for different properties to be preserved.*

*Proof.* Recall [AG14a, Theorem 5.8] that the sheaf $\mathbf{M}_{A^e}$ is locally geometric, i.e. there exists a filtered system $\mathbf{M}_{A^e} = \operatorname{colim}_{[a,b]} \mathbf{M}_{A^e}^{[a,b]}$ where each $\mathbf{M}_{A^e}^{[a,b]} \to \mathbf{M}_{A^e}$ is a monomorphism and each $\mathbf{M}_{A^e}^{[a,b]}$ is $n_i$-geometric for some $n_i$.

While $\pi_*$ does not preserve filtered colimits in general, quasi-compactness of $\pi$ (Recollection 5.46) and of $Y$ implies that $\operatorname{colim}_{[a,b]} \pi_*\mathbf{M}_{A^e}^{[a,b]} \to \pi_*\mathbf{M}_{A^e}$ is an equivalence [TT90, §3.1.2].

Since pushforward is left exact (and assuming that $\pi_*$ preserves this particular filtered colimit), it suffices to show that the pushforward of an $n$-geometric sheaf on $X$ is $m$-geometric for some $m$. □

**Lemma 5.48.** *Let $(X, \sigma, Y, \pi)$ be a scheme with involution and good quotient and let $A$ be an Azumaya algebra over $X$ with genuine $\sigma$-linear (anti-)involution. Then the morphism $\mathbf{M}_A^h \to \pi_*\mathbf{M}_{A^e}$ of Observation 5.45 is locally geometric in the sense of [AG14a, §4.3].*

*Proof.* It suffices to show that for all $p\colon \operatorname{Spec} S \to \pi_*\mathbf{M}_{A^e}$, the fiber product $\operatorname{Spec} S \times_{\pi_*\mathbf{M}_{A^e}} \mathbf{M}_A^h$ is locally geometric over $\operatorname{Spec} S$. Write $\overline{p}$ for the composite $\operatorname{Spec} S \to \pi_*\mathbf{M}_{A^e} \to Y$; there is a pullback diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "\widetilde{p}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\pi"] \\ \operatorname{Spec} S \arrow[r, "\overline{p}"] & Y \end{tikzcd}$$

where the pullback $\operatorname{Spec} R$ is affine by assumption. Recall that the map $p$ classifies an $\widetilde{p}^*A$-module $M$ which is compact as an $R$-module. Note that the fiber product $\operatorname{Spec} S \times_{\pi_*\mathbf{M}_{A^e}} \mathbf{M}_A^h$classifies lifts of the $R$-linear functor $\otimes M\colon \operatorname{Mod}_{p^*A}^{\omega} \to \operatorname{Mod}_R^{\omega}$ to an $R$-linear hermitian functor. A lift of $\otimes M$ to an $R$-linear hermitian functor is equivalent to the data of a map $\widetilde{p}^*A \to N_R^{C_2}(M^\vee)$ of $N_R^{C_2}(\widetilde{p}^*A)$-modules in $\mathrm{Sp}^{C_2}$, where we write $M^\vee$ for the $R$-linear dual of $M$. Thus the fiber product $\operatorname{Spec} S \times_{\pi_*\mathbf{M}_{A^e}} \mathbf{M}_A^h$ is given by the moduli sheaf $\operatorname{Spec} \operatorname{Sym}_S \tau_{\geq 0} \hom_{N_R^{C_2}(\widetilde{p}^*A)}\left(\widetilde{p}^*A, N_R^{C_2}(M^\vee)\right)$.

L: Similar to [AG14a, Theorem 5.6, Proposition 5.7]. Later: show that the map is locally of finite presentation?

□

**Notation 5.49.** Let $(X, \sigma, Y, \pi)$ be a scheme with involution and good quotient and let $A$ be an Azumaya algebra over $X$ with genuine $\sigma$-linear (anti-)involution; write $\lambda\colon A \to \sigma^*A^{\mathrm{op}}$ for the involution on $A$. Write $\mathbf{Mor}_A^p \to \mathbf{M}_A^h$ for the subsheaf of *Poincaré Morita equivalences*, i.e. $\mathbf{Mor}_A^h(\operatorname{Spec} S \to Y)$ is the full sub-groupoid of $\mathbf{M}_A^h(\operatorname{Spec} S \to Y)$ on those $\underline{\mathcal{O}}$-linear hermitian functors $(F, \eta)\colon (\operatorname{Mod}_A^{\omega}, \Qoppa_A) \to \left(\operatorname{Mod}^{\omega}_{\underline{\mathcal{O}}|_{\operatorname{Spec} S}}, \Qoppa_{\underline{\mathcal{O}}}|_{\operatorname{Spec} S}\right)$ so that $F$ is an equivalence and $(F, \eta)$ is duality-preserving.

**Lemma 5.50.** *Suppose given the setup of Notation 5.49. Supppose that the quotient map $\pi\colon X \to Y$ is* quasi-perfect *in the sense of [CHN24, Definition A.6.3], i.e. the pushforward $\pi_*$ of quasicoherent sheaves preserves perfect complexes. Suppose further that $\pi$ is flat. Then the inclusion $\mathbf{Mor}_A^p \to \mathbf{M}_A^h$ exhibits $\mathbf{Mor}_A^h$ as a quasicompact open subsheaf of $\mathbf{M}_A^h$.*

**Remark 5.51.** If either $\pi\colon X \to Y$ is an isomorphism or quadratic étale on each connected component, then $\pi_*$ satisfies the assumptions of Lemma 5.50.

Here is an example which doesn't fall under either of the preceding cases: If $X = \operatorname{Spec} \mathbb{C}[x]$ with the action $x \mapsto -x$, then $Y = \operatorname{Spec} \mathbb{C}[x^2]$ and the quotient map $X \to Y$ is quasi-perfect because $\mathbb{C}[x]$ is a free module of finite rank over $\mathbb{C}[x^2]$. In general, one may check quasi-perfection Zariski-locally on the quotient

[CHN24, Corollary A.6.5]. However, not all schemes with involution and good quotient are quasi-perfect; for a counterexample, see [FW20, Example 3.9].

L: general conditions for this to hold? References: IV.Proposition 2.2.3 and §4.3 here.

*Proof of Lemma 5.50.* Use the notation of Lemma 5.48. By the proof of Lemma 5.48, we know that $(F, \eta)$ is given by $(- \otimes_R M, \widetilde{p}^* A \to N_R^{C_2}(M^\vee))$ where $M$ is a compact $R$-module. By quasiperfection of $\pi_*$ and flatness of $\pi$, $\operatorname{Spec} R \to \operatorname{Spec} S$ is quasiperfect, i.e. $R$ is a compact $S$-module. It follows that $M$ is a compact $S$-module.

L: quasiperfection is not preserved under arbitrary base change, only tor-independent base change (which is guaranteed under flatness assumption, see p.5 here or [CHN24, Lemma A.6.4] which is a special case of the former). OTOH, if we only work with the small étale site of $Y$, then all $\operatorname{Spec} S \to Y$ would be flat and we can remove flatness assumption on $\pi$ (but still need quasi-perfection).

By the proof of [AG14a, Proposition 5.10], the subsheaf of points of $\operatorname{Spec} S$ on which the functor $F$ is an equivalence is a quasicompact Zariski open $U \subseteq \operatorname{Spec} S$. The hermitian functor $(F, \eta)$ is duality-preserving if and only if the canonical map $\tau_\eta \colon M \otimes_{\widetilde{p}^* A} \lambda^* \widetilde{p}^* A \to \hom_R(M, \sigma^* R)$ is an equivalence (cf. [Cal+20a, Lemma 3.4.3]). By [AG14a, Proposition 2.14] (compare the proof of Proposition 5.10 of *loc.cit.*), the subsheaf of points of $\operatorname{Spec} S$ on which the map $\tau_\eta$ is an equivalence is a quasicompact Zariski open $V \subseteq \operatorname{Spec} S$.

Finally $(F, \eta)$ defines a Poincaré Morita equivalence if, in addition to $F$ being an equivalence and $\eta$ being duality-preserving, $\eta$ is an equivalence of $N_R^{C_2}(\widetilde{p}^* A)$-modules. Since categorical fixed points are jointly conservative, $\eta$ is an equivalence if and only if $\eta^{C_2}$ and $\eta^e$ are equivalences. By Lemma 5.52, $(\widetilde{p}^* A)^{C_2}$ and $(N_R^{C_2}(M^\vee))^{C_2}$ are compact $S$-modules, hence by a similar argument to the above, there exists a quasicompact Zariski open $W \subseteq \operatorname{Spec} S$ on whicn $\eta$ is an equivalence. Taking the intersection $U \cap V \cap W$ gives the quasicompact Zariski open subsheaf of $\operatorname{Spec} S$ on which $(F, \eta)$ defines a Poincaré Morita equivalence. ☐

**Lemma 5.52.** *Let $R = (R^{C_2} \to R^e)$ be a $C_2$-$\mathbb{E}_\infty$-ring in $C_2$-spectra and suppose that $R$ is connective (i.e. $R^e$ and $R^{C_2}$ are both connective). Then the composite $\operatorname{Mod}_{R^e}(\mathrm{Sp}) \xrightarrow{N_R^{C_2}} \operatorname{Mod}_R\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{(-)^{C_2}} \operatorname{Mod}_{R^{C_2}}(\mathrm{Sp})$ sends compact $R^e$-modules to compact $R^{C_2}$-modules if and only if the restriction map $R^{C_2} \to R^e$ exhibits $R^e$ as a perfect $R^{C_2}$-module. Here $N_R^{C_2}$ denotes the relative norm.*

*Proof.* Observe that the relative norm satisfies $N_R^{C_2}(P \oplus Q) \simeq N_R^{C_2}(P) \oplus N_R^{C_2}(Q) \oplus C_2 \otimes (P \oplus Q)$ for all $P, Q \in \operatorname{Mod}_{R^e}$, where $C_2 \otimes - \colon \operatorname{Mod}_{R^e} \to \operatorname{Mod}_R$ is the left adjoint to the restriction functor.

L: This is essentially implied by [Nar17, Example 3.17 & Corollary 3.28] but we should find an earlier reference if possible. Same for the other references in this proof.

To prove the 'only if' direction, observe that

$$\left(N_R^{C_2}(R^e \oplus R^e)\right)^{C_2} = \left(N_R^{C_2}(R^e) \oplus N_R^{C_2}(R^e) \oplus C_2 \otimes (R^e \oplus R^e)\right)^{C_2} \simeq R^{C_2} \oplus R^{C_2} \oplus R^e \oplus R^e .$$

Since perfect $R^{C_2}$-modules contain $R^{C_2}$ and are closed under cofibers and taking summands, this implies that $R^e$ must be a perfect $R^{C_2}$-module.

Note that the above argument also implies that if $R^e$ is a perfect $R^{C_2}$-module, then the exact functor $(-)^{C_2} \colon \operatorname{Mod}_R\left(\mathrm{Sp}^{C_2}\right) \to \operatorname{Mod}_{R^{C_2}}(\mathrm{Sp})$ sends perfect $R$-modules (which are generated as a thick subcategory by $R$ and $C_2 \otimes R^e$) to perfect $R^{C_2}$-modules. To prove the 'only if' direction, it suffices to show that if the restriction map $R^{C_2} \to R^e$ exhibits $R^e$ as a perfect $R^{C_2}$-module, then $N_R^{C_2}$ sends perfect $R^e$-modules to perfect $R$-modules in $\mathrm{Sp}^{C_2}$. Recall that any perfect $R^e$-module $P$ can be written as a finite extension of $\{\Sigma^n R^e\}_{n \in \mathbb{Z}}$; write $\ell(P)$ for the minimum $\ell$ so that $P$ can be written as an extension of $\Sigma^{n_i} R^e$ for some $n_1, \ldots, n_\ell \in \mathbb{Z}$. The result follows from induction on $\ell(P)$ and the following observations:

- $N_R^{C_2}(\Sigma^a R^e) = \Sigma^{\varrho a} N_R^{C_2}(R^e) = \Sigma^{\varrho a} R$ is a perfect $R$-module
- If $Q, S$ are perfect $R^e$-modules so that $N_R^{C_2}(Q)$, $N_R^{C_2}(S)$ are perfect $R$-modules, then $N_R^{C_2}(Q \oplus S)$ is a perfect $R$-module.
- If $\ell(P) > 1$, then there exists a perfect $R^e$-module $Q$ with $\ell(Q) < \ell(P)$ and an exact sequence of $R^e$-modules $P \to Q \to \Sigma^a R^e$ for some $a \in \mathbb{Z}$.

- $N_R^{C_2}(-)$ is a quadratic functor and the following cube

$$\begin{tikzcd}
& Q \arrow[rr] \arrow[dd] & & Q \oplus \Sigma^a R^e \arrow[dd] \\
P \arrow[ur] \arrow[rr] \arrow[dd] & & Q \arrow[ur] \arrow[dd] & \\
& \Sigma^a R^e \arrow[rr] & & Q \oplus \Sigma^a R^e \oplus \Sigma^a R^e \\
0 \arrow[ur] \arrow[rr] & & \Sigma^a R^e \arrow[ur] &
\end{tikzcd}$$

is strongly cartesian [Lur17, Corollary 6.1.1.16]. Therefore, $N_R^{C_2}$ sends the cube to a cartesian cube in $\mathrm{Mod}_R\left(\mathrm{Sp}^{C_2}\right)$. By the inductive hypothesis and the second point, $N^{C_2}$ sends the vertices of the cube excluding $P$ to perfect $R$-modules. Thus we have shown that $N_R^{C_2}(P)$ can be written as a finite limit of perfect $R$-modules. $\square$

## 5.4 The Poincaré Brauer space spectral sequence

L: next bit is sketchy; working towards a spectral sequence like the one in [AG14a, §7]

**Notation 5.53.** Let $(X, \lambda, Y, \pi)$ be a scheme with involution $X$ and a good quotient $Y$. Let $\mathbb{Z}_X$ be the étale sheafification of the constant presheaf. Write $\mathbb{Z}_X^\sigma$ for the equalizer

$$\mathbb{Z}_X^\sigma := \mathrm{Eq}\left(\mathbb{Z}_X \overset{\lambda, \cdot(-1)}{\rightrightarrows} \mathbb{Z}_X\right).$$

Write disc for the cokernel

N: How is $\lambda$ inducing a map on $\mathbb{Z}_X$?

$$\mathrm{disc} := \mathrm{coKer}\left(\mathcal{O}_X^\times \xrightarrow{f \mapsto f \cdot \lambda(f)} \mathcal{O}_X^\times\right); \tag{5.54}$$

here the cokernel is taken in the category of sheaves of abelian groups on the small étale site of $X$. Consider the assignment

N: Why the name disc? In any event I think as a sheaf this vanishes because for strict henselian rings there is no contribution here.

L: inspired by 'discriminant.' To the second point: I think if $X = \mathrm{Spec}\, k \times k$ with the flip action, then the cokernel of $(u, v) \mapsto (u, v) \cdot (v, u) = (uv, uv)$ is nontrivial.

$$\begin{aligned} U_1 \colon \acute{\mathrm{E}}\mathrm{t}_Y &\to \mathbb{E}_\infty \mathrm{Mon}(\mathcal{S}) \\ (V = \mathrm{Spec}\, A \to Y) &\mapsto \mathrm{Ker}\left(R\Gamma(\mathcal{O}_{X_V}(X_V))^\times \xrightarrow{f \mapsto f \cdot \lambda(f)} R\Gamma(\mathcal{O}_{X_V}(X_V))^\times\right). \end{aligned}$$

Then $U_1$ is a sheaf of groups on the small étale site of $Y$. By Theorem 4.10, there is a natural map of sheaves $BU_1 \to \mathrm{PnPic}$ given by inclusion of the identity component.

L: check later

**Example 5.55.** If $X$ has the trivial involution $\lambda = \mathrm{id}$, then $\mathbb{Z}_X^\sigma$ is the trivial sheaf and $\mathrm{disc}_X$ is the units in $\mathcal{O}_X$ mod 2.

If $\pi$ is quadratic étale, then $\pi_* \mathbb{Z}_X^\sigma$ is a $\mathbb{Z}$-torsor.

**Corollary 5.56.** *Let $(X, \lambda, Y, \pi)$ be a scheme with involution $X$ and a good quotient $Y$. The homotopy sheaves of* PnBr *are*

$$\pi_* \mathrm{PnBr} = \begin{cases} ? & * = 0 \\ \pi_* \mathbb{Z}_X^\sigma \times \pi_* \mathrm{disc} & * = 1 \\ U_1 & * = 2 \\ 0 & \textit{else.} \end{cases}$$

*L: If we believe what's in Example 5.18, then $\pi_0 \mathrm{PnBr}$ should be supported on the branch locus (terminology from [FW20]); maybe to simplify presentation we could state it as: Assume $X$ is over a field $k$. Then the $\pi_0 \mathrm{PnBr}$ is the pushforward of either (sheafification of constant sheaves) $\mathbb{Z}$ or $\mathbb{Z}/2$ from the branch locus (depending on char $k$).*

## References


[AG14a] Benjamin Antieau and David Gepner. "Brauer groups and étale cohomology in derived algebraic geometry". In: *Geom. Topol.* 18.2 (2014), pp. 1149–1244. ISSN: 1465-3060,1364-0380. DOI: 10.2140/gt.2014.18.1149. URL: https://doi.org/10.2140/gt.2014.18.1149.

[AG14b] Benjamin Antieau and David Gepner. "Brauer groups and étale cohomology in derived algebraic geometry". In: *Geom. Topol.* 18.2 (2014), pp. 1149–1244. ISSN: 1465-3060,1364-0380. DOI: 10.2140/gt.2014.18.1149. URL: https://doi.org/10.2140/gt.2014.18.1149.

[Alb61] A. Adrian Albert. *Structure of algebras*. Vol. Vol. XXIV. American Mathematical Society Colloquium Publications. Revised printing. American Mathematical Society, Providence, RI, 1961, pp. xi+210.

[Bar+22] Clark Barwick, Saul Glasman, Akhil Mathew, and Thomas Nikolaus. *K-theory and polynomial functors*. 2022. arXiv: 2102.00936 [math.KT]. URL: https://arxiv.org/abs/2102.00936.

[BRS12] Andrew Baker, Birgit Richter, and Markus Szymik. "Brauer groups for commutative $S$-algebras". In: *J. Pure Appl. Algebra* 216.11 (2012), pp. 2361–2376. ISSN: 0022-4049,1873-1376. DOI: 10.1016/j.jpaa.2012.03.001. URL: https://doi.org/10.1016/j.jpaa.2012.03.001.

[Cal+20a] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable $\infty$-categories I: Foundations*. 2020. DOI: 10.48550/ARXIV.2009.07223. URL: https://arxiv.org/abs/2009.07223.

[Cal+20b] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable $\infty$-categories II: Cobordism categories and additivity*. 2020. DOI: 10.48550/ARXIV.2009.07224. URL: https://arxiv.org/abs/2009.07224.

[Cal+21] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable $\infty$-categories III: Grothendieck-Witt groups of rings*. 2021. arXiv: 2009.07225 [math.KT].

[CHN24] Baptiste Calmès, Yonatan Harpaz, and Denis Nardin. *A motivic spectrum representing hermitian K-theory*. 2024. arXiv: 2402.15136 [math.KT]. URL: https://arxiv.org/abs/2402.15136.

[Fau03] H. Fausk. "Picard groups of derived categories". In: *J. Pure Appl. Algebra* 180.3 (2003), pp. 251–261. ISSN: 0022-4049,1873-1376. DOI: 10.1016/S0022-4049(02)00145-7. URL: https://doi.org/10.1016/S0022-4049(02)00145-7.

[FW20] Uriya A. First and Ben Williams. "Involutions of Azumaya Algebras". en. In: *DOCUMENTA MATHEMATICA* Vol 25 (2020 2020), p. 527–633. DOI: 10.25537/DM.2020V25.527-633. URL: https://www.elibm.org/article/10012038.

[Gab81] Ofer Gabber. "Some theorems on Azumaya algebras". In: *The Brauer group (Sem., Les Plans-sur-Bex, 1980)*. Vol. 844. Lecture Notes in Math. Springer, Berlin, 1981, pp. 129–209. ISBN: 3-540-10562-X.

[Kra25] Achim Krause. "The Picard group in equivariant homotopy theory via stable module categories". In: *Journal of Topology* 18.2 (2025), e70020. DOI: https://doi.org/10.1112/topo.70020. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/topo.70020. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/topo.70020.

[Lie09] Max Lieblich. "Compactified moduli of projective bundles". In: *Algebra Number Theory* 3.6 (2009), pp. 653–695. ISSN: 1937-0652,1944-7833. DOI: 10.2140/ant.2009.3.653. URL: https://doi.org/10.2140/ant.2009.3.653.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[Lur11] Jacob Lurie. "Algebraic L-theory and Surgery". Course notes. 2011. URL: https://www.math.ias.edu/~lurie/287x.html.

[Lur17] Jacob Lurie. *Higher algebra*. Manuscript available at https://www.math.ias.edu/~lurie/papers/HA.pdf. 2017.

[Lur18] Jacob Lurie. “Elliptic cohomology II: orientations”. In: *available at https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf* (version dated 26 April 2018).

[Lur24] Jacob Lurie. *Kerodon*. https://kerodon.net. 2024.

[Nar17] Denis Nardin. *Stability and distributivity over orbital $\infty$-categories*. Thesis (Ph.D.)–Massachusetts Institute of Technology. ProQuest LLC, Ann Arbor, MI, 2017, (no paging). URL: http://gateway.proquest.com.ezp-prod1.hul.harvard.edu/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:10797490.

[Nik23] Thomas Nikolaus. “Frobenius homomorphisms in higher algebra”. In: *ICM—International Congress of Mathematicians. Vol. 4. Sections 5–8*. EMS Press, Berlin, [2023] ©2023, pp. 2826–2854. ISBN: 978-3-98547-062-4; 978-3-98547-562-9; 978-3-98547-058-7.

[NS18] Thomas Nikolaus and Peter Scholze. “On topological cyclic homology”. In: *Acta Math.* 221.2 (2018), pp. 203–409. ISSN: 0001-5962. DOI: 10.4310/ACTA.2018.v221.n2.a1. URL: https://doi-org.proxyiub.uits.iu.edu/10.4310/ACTA.2018.v221.n2.a1.

[PS92] R. Parimala and V. Srinivas. “Analogues of the Brauer group for algebras with involution”. In: *Duke Math. J.* 66.2 (1992), pp. 207–237. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-92-06606-3. URL: https://doi.org/10.1215/S0012-7094-92-06606-3.

[Sal78] David J. Saltman. “Azumaya algebras with involution”. In: *J. Algebra* 52.2 (1978), pp. 526–539. ISSN: 0021-8693. DOI: 10.1016/0021-8693(78)90253-3. URL: https://doi.org/10.1016/0021-8693(78)90253-3.

[Stacks] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2024.

[Toë12] Bertrand Toën. “Derived Azumaya algebras and generators for twisted derived categories”. In: *Invent. Math.* 189.3 (2012), pp. 581–652. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-011-0372-1. URL: https://doi.org/10.1007/s00222-011-0372-1.

[TT90] R. W. Thomason and Thomas Trobaugh. “Higher algebraic $K$-theory of schemes and of derived categories”. In: Progr. Math. 88 (1990), pp. 247–435. DOI: 10.1007/978-0-8176-4576-2_10. URL: https://doi.org/10.1007/978-0-8176-4576-2_10.

[Yan23] Lucy Yang. *On normed $\mathbb{E}_\infty$-rings in genuine equivariant $C_p$-spectra*. 2023. arXiv: 2308.16107 [math.AT].