The Picard Stack of Poincaré Stacks

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Abstract

We do stuff

N: Change this

Contents

1 Introduction

Theorem 1.1. *Let $\underline{A}$ be an affine Poincaré scheme with underlying $\mathbb{E}_\infty$-ring spectrum with involution A. Then the natural maps*

$$\pi_i(\mathrm{PnPic}(\underline{A})) \to \pi_i(\mathrm{Pic}(A))$$

are surjective on 2-torsion.

N: I think there is some interaction with the homotopy fixed points, or maybe even the genuine fixed points

Theorem 1.2. *Let A be an $\mathbb{E}_\infty$ ring with involution, and let $\underline{NA}$ be the associated Tate affine Poincaré scheme. Let $\mathrm{Br}_\nu(A)$ be the Brauer group of Azumaya algebras over A with involution. Then the natural map*

$$\mathrm{PnBr}(\underline{NA}) \to \mathrm{Br}_\nu(A)$$

is an equivalence

N: I think we need to define this for ring spectra. For A discrete this is done in [2].

N: probably of $\mathbb{E}_\infty$ dodads

Theorem 1.3. *The functors $\mathrm{PnPic}, \mathrm{PnBr} : \mathrm{APS} \to \mathcal{S}p$ are fppf sheaves.*

Theorem 1.4. *There is a Poincaré group scheme $\mathbb{G}_m^{♀}$ such that*

$$B\mathbb{G}_m^{♀} \simeq \mathrm{PnPic}$$

as fppf stacks.

2 Poincaré schemes

Definition 2.1. *Let APS be the $(\infty,1)$-category defined by the pullback*

$$\begin{tikzcd}
\mathrm{APS} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \arrow[d, "d_1^*"] \\
\mathrm{CAlg}(\mathcal{S}p^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p))
\end{tikzcd}$$

where $U : \mathcal{S}p^{BC_2} \to \mathcal{S}p$ is the functor which forgets the C_2-action.

We record here a few structural results about this category.

Theorem 2.2. *The following statements about APS hold:*

1. *The category* APS *is a cocomplete and symmetric monoidal infinite category;*
2. *the pullback diagram above is homotopy Cartesian;*
3. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{BC_2})$ *is symmetric monoidal and (co)continuous;*
4. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p)^{\Delta^2}$ *is lax symmetric monoidal;*
5. *and the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p)^{\Delta^2} \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathcal{S}p)$ *is symmetric monoidal.*

Proof. For (2) it is enough to show that d_1^* is a cartesian fibration which follows from [3, Corollary 2.4.6.5]. There is a (pseudo-)functor

L: Is this reference correct? The conclusion asserts that some map of simplicial sets is a *categorical* fibration. The following argument is 'sketchy'– depending on how precise we want to be about quasicategories, we may want to argue with left/right anodyne maps instead.

$$\begin{aligned} F\colon \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p)) &\to \mathcal{C}\mathrm{at}_\infty \\ (\varphi\colon A \to B) &\mapsto \big(\big(\mathrm{CAlg}(\mathcal{S}p)_{A/-/B}\big)_{/\varphi}\big) \end{aligned}$$

which sends a square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\psi"] & D \end{tikzcd} \tag{2.1}$$

regarded as a morphism from φ to ψ, to the functor

$$\begin{aligned} \big(\mathrm{CAlg}(\mathcal{S}p)_{A/-/B}\big)_{/\varphi} &\to \big(\mathrm{CAlg}(\mathcal{S}p)_{C/-/D}\big)_{/\psi} \\ (A \to R \to B) &\mapsto C \simeq A \otimes_A C \xrightarrow{\varphi \otimes \mathrm{id}_C} B \otimes_A C \to D \end{aligned} \tag{2.2}$$

where $B \otimes_A C \to D$ is the canonical map induced by the commuting square (2.1). The functor F classifies the cocartesian fibration d_1^*.

For (3), let $p : K \to \mathrm{APS}$ be a map of simplicial sets, K a small simplicial set. Suppose the $K^{\triangleright} \to \mathrm{APS}$ be an extension such that $K^{\triangleright} \to \mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{BC_2})$ is a colimit diagram. By [3, Proposition 2.4.3.2] the diagram

$$\begin{tikzcd} \mathrm{APS}_{p/} \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathcal{S}p)^{\Delta^2}_{p/-} \arrow[d] \\ \mathrm{CAlg}(\mathcal{S}p^{BC_2})_{p/} \arrow[r] & \mathrm{CAlg}(\mathcal{S}p)^{\Delta^1}_{p/-} \end{tikzcd}$$

is again homotopy cartesian. Then

$$\begin{aligned} \hom_{\mathrm{APS}}(p(\infty), -) &\simeq \hom_{\mathrm{CAlg}(\mathcal{S}p^{BC_2})}(p(\infty), -) \times_{\hom_{\mathrm{CAlg}(\mathcal{S}p)^{\Delta^1}}(p(\infty),-)} \hom_{\mathrm{CAlg}(\mathcal{S}p)^{\Delta^2}}(p(\infty)) \\ &\simeq \end{aligned}$$

□

We will denote elements of APS by $\underline{A} = (A, s : A^{\Phi C_2} \to A^{tC_2})$. Here $s : A^{\Phi C_2} \to A^{tC_2}$ is the image of $\underline{A}$ under the top horizontal map above. The use of the notation $A^{\Phi C_2}$ is justified by the following.

Lemma 2.3. *Let* $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p)$ *be the composition of the functors*

$$\mathrm{APS} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathcal{S}p).$$

Then this functor factors as a composition $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{C_2}) \xrightarrow{(-)^{\Phi C_2}} \mathrm{CAlg}(\mathcal{S}p)$.

Proof. The commutativity of the diagram

$$\begin{tikzcd}
 & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \arrow[d, "d_1^*"] \arrow[dr, "d_0^*"] & \\
 & & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p)) \arrow[dd, "ev_{[1]}"] \\
\mathrm{CAlg}(\mathcal{S}p^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] \arrow[dr, "id"'] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p)) \arrow[dr, "ev_{[1]}"] & \\
 & \mathrm{CAlg}(\mathcal{S}p^{BC_2}) \arrow[r, "(-)^{tc_2}"] & \mathrm{CAlg}(\mathcal{S}p)
\end{tikzcd}$$

induces a functor on the pullback infinity categories $\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{C_2})$ which makes the corresponding cube commute. The functor $ev_{[1]} : \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \to \mathrm{CAlg}(\mathcal{S}p)$ factors through d_0^* and so $\mathrm{APS} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathcal{S}p)) \to \mathrm{CAlg}(\mathcal{S}p)$ is equivalent to the composition

$$\mathrm{APS} \to \mathrm{CAlg}(\mathcal{S}p^{C_2}) \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathcal{S}p)) \to \mathrm{CAlg}(\mathcal{S}p)$$

and the composition of the last two maps is the geometric fixed point functor as desired. $\square$

The following Lemma gives the justification of the name Poincaré scheme.

Construction 2.4. We shall construct a functor

$$\mathrm{Perf}^{\mathrm{Pn}} : \mathrm{APS} \to \mathrm{Cat}_\infty^{\mathrm{Pn}}$$

to the category of Poincaré infinity categories.

Recall that $\mathcal{C}\mathrm{at}_\infty^h \to (\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}})^{\mathrm{op}}$ is a cocartesian fibration [1, §1.4.] We will first construct a map of cocartesian fibrations

L: For symmetric monoidal structure–maybe want to swap out Mod_{NA} for CAlg_{NA}?

$$\begin{tikzcd}
\mathrm{APS} \arrow[r, dotted] \arrow[d] & \mathcal{C}\mathrm{at}_\infty^h \arrow[d] \\
\mathrm{CAlg}\left(\mathcal{S}p^{BC_2}\right) \arrow[r] & (\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}})^{\mathrm{op}}
\end{tikzcd}, \tag{2.3}$$

then show that the dotted arrow factors through the subcategory $\mathcal{C}\mathrm{at}_\infty^p \subseteq \mathcal{C}\mathrm{at}_\infty^h$. To construct a map of cartesian fibrations, it suffices to exhibit a natural transformation of classifying functors. Unraveling the definitions, by Theorem 3.2.13 of [1] we must exhibit for each $A \in \mathrm{CAlg}(\mathcal{S}p)^{BC_2}$, a functor

$$\left(\mathrm{CAlg}(\mathcal{S}p)_{A/-/A^{tC_2}}\right)_{/\varphi} \to \mathrm{Mod}_{N^{C_2}(A^e)}\left(\mathcal{S}p^{C_2}\right) \tag{2.4}$$

(where $\varphi\colon A \to A^{tC_2}$ is the Tate-valued Frobenius and N^{C_2} is the Hill–Hopkins–Ravenel norm) which is natural in A.

That the resulting functor factors through the subcategory $\mathcal{C}\mathrm{at}_\infty^p$ follows from Proposition 3.1.3 and Lemma 3.3.3 of *loc. cit.*

Lemma 2.5. *The functor of Construction 2.4 is symmetric monoidal and has essential image the subcategory spanned by objects* $(\mathrm{Perf}(R), \Omega)$ *which are* $\mathbb{E}_\infty$*-algebras.*

Definition 2.6. *A map* $f : \underline{A} \to \underline{B} \in \mathrm{APS}$ *is faithfully flat if the underlying map* $f : A \to B$ *is faithfully flat and the map* $f^{\Phi C_2} : A^{\Phi C_2} \to B^{\Phi C_2}$ *is also faithfully flat.*

Lemma 2.7. *The fpqc covers on* APS *form a Grothendieck site.*

References

[1] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable ∞-categories I: Foundations*. 2020. DOI: 10.48550/ARXIV.2009.07223. URL: https://arxiv.org/abs/2009.07223.

[2] Uriya A. First and Ben Williams. "Involutions of Azumaya Algebras". en. In: *DOCUMENTA MATHEMATICA* Vol 25 (2020 2020), p. 527–633. DOI: 10.25537/DM.2020V25.527-633. URL: https://www.elibm.org/article/10012038.

[3] Jacob Lurie. *Higher topos theory.* Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.