Et cetera

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Abstract

Dumping ground for other stuff: Notes, one-off observations, stuff that we can collectively use when preparing talks, etc.

L: I make no promises re: organization but I will do my best to keep it reasonably readable

Contents

1 Talk prep

2 References

- Involutions of Azumaya algebras by First and Williams (2020 *Documenta*)
- Counterexamples in involutions of Azumaya algebras by First and Williams; much more readable than the 2020 Documenta paper
- Azumaya algebras without involution by Auel, First, and Williams: the introduction of this provides a very helpful historical overview of the connection between involutions on Azumaya algebras and 2-torsion/kernel of coRes

3 Questions and directions

Question 3.1 (Literature)**.**
- In [PS92] Parimala–Srinivas assume that 2 is invertible in the ring of functions. Has anyone been able to extend their results to the 2 not necessarily invertible case in the meantime?

Question 3.2 (Morita theory for $\mathrm{Cat}_\infty^{\mathrm{p}}$)**.** Let R be a Poincaré ring. Suppose given two R-algebras (suitably interpreted so their module categories are canonically endowed with R-linear Poincaré structures–perhaps $\mathbb{E}_\sigma$) A, B. Can we characterize

$$\hom_{\mathrm{Cat}_{\infty}^{\mathrm{p}}{}_R}\left((\mathrm{Mod}_A^\omega, \Omega_A), (\mathrm{Mod}_B^\omega, \Omega_B)\right)$$

in terms of something bimodule-like?

Question 3.3. On page 2 of the *Counterexamples* paper, First and Williams write that " existence of an extraordinary involution means classification of Azumaya algebras with involution…*cannot* be reduced to questions about projective modules and hermitian forms on them."

What if we replaced projective modules by perfect complexes?

Question 3.4. First–Williams show (see discussion in §4 of the *Counterexamples* paper) that coarse type classify many (most?) Azumaya algebras up to (étale-local) *isomorphism*.

What is a suitable derived version of "coarse type"?

Question 3.5 (asked by Andrew Nov 2, 2024)**.** C. Schlichtkrull shows in this paper that a map $BGL_1(R) \to K(R) \to THH(R) \to R$ in terms of the Hopf map η.

Is there a "Poincaré" version of this result?

Question 3.6. Are there some general conditions for a ring with involution R so that the inclusion $R^{C_2} \to R$ is 'nice'?

There's some stuff in section 4 here, idk. Also see M. Hochster and J. L. Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay.

Question 3.7. Derived category of non-generic cubic 4-fold has an application/use of pushforwards of Azumaya algebras along double covers. Kuznetsov has also written about this, see 'Derived categories of quadric fibrations and intersections of quadrics' here.

Applications to Hodge theory?

- A Deligne pairing for Hermitian Azumaya modules
- Burt Totaro's paper on Hodge structures of type $(n, 0, \dots, 0, n)$
- There's some stuff about endomorphisms of Hodge structures here

4 Thoughts & observations

Question 4.1. When R has the Tate Poincaré structure and $(\mathrm{Mod}_A^\omega, M_A, N_A, N_A \to M_A^{tC_2})$ is invertible, then by invertibility have an equivalence $\hom_R(A, R) \simeq N_A \otimes_R N_{A^{\mathrm{op}}}$ of $A \otimes_R A^{\mathrm{op}}$-modules. Restricting the left-hand side along the unit map $R \to A$ gives a map $N_A \otimes_R N_{A^{\mathrm{op}}} \to \hom_R(R, R) \simeq R$. Is this a perfect ($R$-linear) pairing?

I *think* using that $R^{\varphi C_2} \simeq R$ and combining the linear and bilinear part conditions, we get something like

$$M_A \otimes_R M_{A^{\mathrm{op}}} \simeq (N_A \otimes_R N_{A^{\mathrm{op}}})^{\otimes_R 2} \qquad \text{as } A \otimes_R A^{\mathrm{op}}\text{-bimodules.}$$

Is this useful?

Brauer-Severi schemes We know there is a correspondence between Azumaya algebras A over X and Brauer-Severi schemes. What does a Poincaré structure on Mod_A^ω mean 'geometrically' for D^b_{coh} of the corresponding Brauer-Severi scheme? (Lucy: I didn't get very far here, but just typing up what I had)

- Mod_A^ω corresponds to α-twisted sheaves on X (see Proposition 3.2.2.1 of Max Lieblich's thesis)
- The bounded derived category of α-twisted sheaves on X includes as one 'piece' of a semiorthogonal decomposition on D^b_{coh} of the corresponding Brauer-Severi scheme (see Theorem 5.1 here)

5 Desperate Flailing

This section is a cronical of my thoughts about $\mathbb{G}_m^{♀}$.

Goal The goal is to build a Poincaré ring $\mathbb{G}_m^{♀} := (\mathrm{Mod}_R, ♀_R)$ such that $B\mathbb{G}_m^{♀}(\underline{S}) = \mathrm{Pic}^{\mathrm{p}}(\underline{S})$ for any Poincaré ring $\underline{S}$.

Lemma 5.1. *Let $\underline{S}$ be a Poincaré ring. Then $\pi_0(\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}_S)}(S,u)) = \{s \in \pi_0(S)^\times | s = 1$ in $\pi_0(S^{C_2})\}$.*

Proof. Since the functor $\mathrm{Pn}(\mathrm{Mod}_S) \to \mathrm{Mod}_S$ is conservative it follows that an element of $\pi_0(\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}_S)}(S,u))$ must have underlying map an element of $\pi_0\mathrm{Aut}(S) = \pi_0(S)^\times$. Then in order for $s \in \pi_0(S)^\times$ to induce a map $(S,u) \to (S,u)$, the induced map $s^* : S^{C_2} \to S^{C_2}$ must satisfy $s^*(u) = u$. The pullback is given by multiplication by s, so this requirement translates into s being the unit, as desired. □

The problem I thought existed maybe doesn't. Here is a candidate construction:

Construction 5.2. Define R to be the $\mathbb{E}_\infty$ ring given by $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z\}} \mathbb{S}$ where the map $\mathbb{S}\{z\} \to \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by the map $z \mapsto xy$, and the map $\mathbb{S}\{z\} \to \mathbb{S}$ is induced by $z \mapsto 1$. We can give R an $\mathbb{E}_\infty$ ring structure in Sp^{BC_2} by taking the trivial action on $\mathbb{S}\{z\}$ and $\mathbb{S}$, and taking the action induced by $x \mapsto y$ and $y \mapsto x$ on $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$. Thus in $\mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ the ring R corepresents the functor $S \mapsto \{s \in \pi_0(S)^\times | s\sigma(s) = 1\}$.

Now take $\underline{R}$ to be the Poincaré ring with underlying Borel C_2 structure as described in the previous paragraph and geometric fixed points $R^{\varphi C_2} = \mathbb{S}$ and the map $R^{\varphi C_2} \to R^{tC_2}$ given by the unit map. Endowing $R^{\varphi C_2}$ with the R-module structre given by $x, y \mapsto 1$, it remains to show that the unit map $R^{\varphi C_2} \to R^{tC_2}$ factors the Tate valued Frobenius $R \to R^{tC_2}$ in order to promote $\underline{R}$ to a Poincaré ring. By construction of R it is then enough to show that on π_0 the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0(R^{tC_2})$. This map sends both x and y to $xy \in \pi_0(R^{tC_2})$. These areequal to 1 in $\pi_0(R^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so R^{tC_2} is a modules over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of xy equal to 1.

Now consider another Poincaré ring $\underline{S}$. We then have that maps $\pi_0(\mathrm{Maps}(\underline{R}, \underline{S}))$ is the data of a unit $s \in \pi_0(S)^\times$, a path $s\sigma(s) \to 1$ in $\Omega^\infty S$, and paths $x, y \to 1$ in $\Omega^\infty S^{\varphi C_2}$. This then agrees with $\mathbb{G}_m^{♀}$ by the following lemma.

Lemma 5.3. *Let $S \in \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ and $s \in \pi_0(S)^\times$. Then $s\sigma(s) = 1$ in $\pi_0(S)$ if and only if $(s \otimes s)^*$ acts by 1 on $\pi_0(S^{hC_2}) = \pi_0(\mathrm{Hom}_{S \otimes S}(S \otimes S, S)^{hC_2})$.*

Proof. The 'only if' direction follows from the fact that the map $S^{hC_2} \to S$ is an S-bimodule map. Now suppose that $s\sigma(s) = 1$ in S. Then before taking homotopy fixed points the induced map $s^* = id$ because S is $\mathbb{E}_\infty$.[1] □

6 Modules with genuine involution

Remark 6.1 (Lucy)**.** I'm just going to put drafts of stuff pertaining to hermitian modules here. Eventually when it gets to be more complete, I will hopefully move this entire section over to the main file.

L: or whatever we want to keep calling these

[1] Or just $\mathbb{E}_2$.

Meta-commentary There are (at least) three things we want to do:

(a) Define a category of 'bimodules with involution over algebras with anti-involution' equipped with a forgetful functor $\Theta\colon \mathrm{BMod}_{\mathrm{inv}}(-) \to \mathbb{E}_1 \operatorname{Alg}(-)^{hC_2}$.

(b) Show that Θ is a coCartesian fibration. For this, it suffices to show that it is a *Cartesian* fibration and that it satisfies the hypotheses of [Lur09, Corollary 5.2.2.5]

- I used to think that we could obtain this by 'bootstrapping' a result from Higher Algebra, plus some facts about assembly. This doesn't seem to be working, so I'm just going to try to do this directly (imitating certain aspects of Chapter 4 of higher algebra.)

(c) Define a relative tensor product for hermitian bimodules

(d) Show that the formula for the cocartesian pushforward along a map $A \to B$ in $\mathbb{E}_1 \operatorname{Alg}(-)^{hC_2}$ is something like $- \otimes_{A\otimes A^{\mathrm{op}}} (B \otimes B^{\mathrm{op}}) \otimes_{B\otimes B^{\mathrm{op}}} B$.

- In Higher Algebra, the formula for the cocartesian pushforward is proven in [Lur17, §4.6]; in particular, this is in the section on duality. In particular, see Proposition 4.6.2.17 and the paragraph immediately preceding this.
- I don't know how to do this yet–while (a) and (b) are not useful if I can't show (c), I can't suss out the feasibility of (c) without (a) and (b) already in place.

(e) Towards an adjunction between $\mathbb{E}_\sigma$-algebras and categories with additional structure.

- Involutive version of statement that, for a monoidal ∞-category $\mathcal{C}$ and an $\mathbb{E}_1$-algebra A, $\mathrm{LMod}_A(\mathcal{C})$ is right-tensored over $\mathcal{C}$?
- Involutive version of endomorphism categories? [Lur17, §4.7.1]

I think that the equivalence of part (b) of the definition of an Azumaya algebra with genuine involution follows from the property of being Azumaya; see Lemma 1(b) (and p.216 for the 'type 2' case) of [PS92].

Lemma 6.2. *Let R be an $\mathbb{E}_\infty$-ring with an involution $\sigma\colon R \xrightarrow{\sim} R$ and suppose A is an $\mathbb{E}_1$-R-algebra with an anti-involution $\lambda\colon A \xrightarrow{\sim} \sigma^* A^{\mathrm{op}}$. Suppose A is further Azumaya in the sense of . Then the bilinear pairing*

L: reference

$$A \otimes_R \sigma^* A \xrightarrow{\mathrm{id}\otimes\sigma^*\lambda} A \otimes_R A^{\mathrm{op}} \simeq \mathrm{End}_R(A) \xrightarrow{\mathrm{tr}} R$$

is perfect, i.e. its adjoint $A \to (\sigma^* A)^\vee$ is an equivalence.

Question 6.3. Does the map in part (e) of the definition of an Azumaya algebra with genuine involution follow from property of being Azumaya?

6.1 Step (a)

Definition 6.4. Define a colored operad Assoc_σ as follows:

L: This is just an imitation of [Lur17, Definition 4.1.1.1], modified in accordance with ideas from §5.4.2.

(i) The colored operad has a single object, which we denote by $\mathfrak{a}$.

(ii) For every finite set I, the set of operations $\mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$, where $\mathcal{L}I$ is the set of linear orderings on I and an element of $\{\pm1\}^I$ is a function $I \to \{\pm 1\}$.

(iii) Suppose given a map of finite sets $\alpha\colon I \to J$, together with operations $(\preceq_j, f_j\colon I_j \to \{\pm1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{\alpha(i)=j}, \mathfrak{a}\right)$ and $(\preceq_J, g\colon J \to \{\pm 1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_j\}_{j\in J}, \mathfrak{a}\right)$. Define a linear ordering on the set I as follows: $i \le i'$ if $\alpha(i) \preceq_J \alpha(i')$ or $\alpha(i) = \alpha(i') = j$ and $i \preceq_j i'$ and $g(j) = +1$ or $\alpha(i) = \alpha(i') = j$ and $i \succeq_j i'$ and $g(j) = -1$. Finally, define a function

$$\begin{aligned} I &\to \{\pm 1\} \\ i &\mapsto f_{\alpha(i)}(i) \cdot g(\alpha(i)), \end{aligned}$$

where the multiplication on $\{\pm1\}$ is the usual one.

Remark 6.5. There is a map of colored operads $\iota\colon \mathrm{Assoc} \to \mathrm{Assoc}_\sigma$ which is the identity on objects and on operations $\mathrm{Mul}_{\mathrm{Assoc}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$ where c_1 is the constant function on I with value 1.

There is another map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{Assoc} \to \mathrm{Assoc}_\sigma$ which is the identity on objects and on operations $\mathrm{Mul}_{\mathrm{Assoc}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ sends a linear ordering ℓ to $(\ell^{\mathrm{rev}}, c_{-1})$ where c_{-1} is the constant function on I with value 1.

Definition 6.6. Let $\mathrm{Assoc}_\sigma^\otimes$ denote the associated ∞-operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.7. Unwinding definitions

- Objects $\mathrm{Assoc}_\sigma^\otimes$ are finite pointed sets $\langle n\rangle \in \mathrm{Fin}_*$
- Morphisms $\langle m\rangle \to \langle n\rangle$ consist of
  - $\alpha\colon \langle m\rangle \to \langle n\rangle$ a map of finite pointed sets
  - for each $i \in \langle n\rangle^\circ$, a linear ordering $\preceq_i$ on the inverse image $\alpha^{-1}(\{i\})$
  - a map of sets $s\colon \alpha^{-1}\left(\langle m\rangle^\circ\right) \to \{\pm 1\}$
- For each pair of morphisms

$$(\beta\colon \langle \ell\rangle \to \langle m\rangle, \preceq_j, s) \qquad (\alpha\colon \langle m\rangle \to \langle n\rangle, \preceq_i, t)\ ,$$

  the composite is the triple $\left(\alpha\circ\beta, \preceq''_j, u\right)$ where $\preceq''_j$ is the ordering on $(\alpha\circ\beta)^{-1}(\{i\})$ so that if $a, b \in \langle \ell\rangle$ so that $\alpha(\beta(a)) = \alpha(\beta(b))$, then $a \preceq''_j b$ if $\beta(a) \precneq_i \beta(b)$ or $\beta(a) =_i \beta(b) = i$ and $a \preceq_i b$ if $s(i) = 1$ or $a \succeq_i b$ if $s(i) = -1$. Finally $u(l) = s(l)\cdot t(\beta(l))$.

L: Note that when s, t are identically one, the resulting order $\preceq''_j$ agrees with the lexicographic order defined in [Lur17, Remark 4.1.1.4].

Remark 6.8. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.5 induce maps of ∞-operads $\mathrm{Assoc}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$. There is a canonical identification $\iota^{\mathrm{rev}} = \sigma\circ\iota$, where σ is the automorphism of the associative operad considered in [Lur17, Remark 4.1.1.7].

Note that each object $\langle n\rangle \in \mathrm{Assoc}_\sigma^\otimes$ has a distinguished automorphism $\mathrm{rev}_{\langle n\rangle}$ of order two given by the identity map on $\langle n\rangle$ and the constant map $c_{-1}\colon \langle n\rangle^\circ \to \{\pm 1\}$ at -1. There is a canonical natural equivalence $\iota \overset{\sim}{\Longrightarrow} \iota^{\mathrm{rev}}$ whose component at $\langle n\rangle$ is $\mathrm{rev}_{\langle n\rangle}$.

Definition 6.9. Let $\mathcal{C}^\otimes$ be a ∞-operad equipped with the data of a fibration $p\colon \mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$. Let $\mathrm{Alg}^\sigma(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{/\mathrm{Assoc}_\sigma}(\mathcal{C})$ of ∞-operad sections of p. We will refer to $\mathrm{Alg}^\sigma(\mathcal{C})$ as the ∞-category of *involutive algebra objects of* $\mathcal{C}$.

An *involutive monoidal* ∞*-category* is the data of a cocartesian fibration $\mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$.

L: do we need weaker than cocartesian fibration?

Remark 6.10. Suppose given a cocartesian fibration $f\colon \mathcal{D}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ of ∞-operads. Write $\mathcal{C}^\otimes := \mathcal{D}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes, \iota} \mathrm{Assoc}^\otimes$; $\mathcal{C}^\otimes$ is a monoidal ∞-category in the sense of [Lur17, Definition 4.1.1.10]. Furthermore, $\mathcal{C}_{\mathrm{rev}}^\otimes := \mathcal{D}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes, \iota^{\mathrm{rev}}} \mathrm{Assoc}^\otimes$ is a monoidal ∞-category. By Remark 6.8, this notation is consistent with that of [Lur17, Remark 4.1.1.7]. In particular, a Assoc_σ-monoidal ∞-category $\mathcal{D}^\otimes$ determines a monoidal ∞-category $\mathcal{C}^\otimes$ equipped with a monoidal equivalence $\sigma_{\mathcal{C}}\colon \mathcal{C}^\otimes \xrightarrow{\sim} \mathcal{C}_{\mathrm{rev}}^\otimes$. Pullback along the involution of $\mathrm{Assoc}^\otimes$ determines another monoidal equivalence $\sigma_{\mathcal{C}}^{\mathrm{rev}}\colon \mathcal{C}_{\mathrm{rev}}^\otimes \xrightarrow{\sim} \mathcal{C}^\otimes$, and our assumptions imply that $\sigma_{\mathcal{C}}^{\mathrm{rev}} \circ \sigma_{\mathcal{C}}$ is equivalent to the identity on $\mathcal{C}^\otimes$.

Now suppose that A is an involutive algebra object of $\mathcal{D}$. With the same notation as before, pullback along ι (resp. ι^{rev}) determines associative algebra objects $u(A)$, $u^{\mathrm{rev}}(A)$ of $\mathcal{C}$ and $\mathcal{C}_{\mathrm{rev}}$, respectively. Note that $\sigma_{\mathcal{C}}(u(A))$ is an algebra object of $\mathcal{C}_{\mathrm{rev}}$, which we may regard as an algebra object of $\mathcal{C}$ by precomposing with the autoequivalence $\sigma\colon \mathrm{Assoc}^\otimes \xrightarrow{\sim} \mathrm{Assoc}^\otimes$. It follows from Remark 6.8 that A determines an equivalence $\sigma_A\colon u(A) \xrightarrow{\sim} \sigma_{\mathcal{C}}(u(A))^{\mathrm{rev}}$ of algebra objects in $\mathcal{C}$.

Now suppose furthermore that $\mathcal{D}^\otimes$ is of the form $\mathcal{E}^\otimes \times_{\mathrm{Fin}_*} \mathrm{Assoc}_\sigma^\otimes$ for some symmetric monoidal ∞-category $\mathcal{E}$. Then the associated involution $\sigma_{\mathcal{C}}$ is the identity, and for any involutive algebra object A of $\mathcal{D}$, σ_A is an equivalence $u(A) \simeq u(A)^{\mathrm{rev}}$ satisfying $\sigma_A^{\mathrm{rev}} \circ \sigma_A \simeq \mathrm{id}_A$.

Remark 6.11. Suppose given a cocartesian fibration $f\colon \mathcal{D}^{\otimes} \to \mathrm{Assoc}_{\sigma}^{\otimes}$ of ∞-operads, and write $\mathcal{C}$ for its underlying monoidal ∞-category with equivalence $\sigma_{\mathcal{C}}$ as in Remark 6.10. The functors of Remark 6.8 induce a functor

$$\mathrm{Alg}^{\sigma}(\mathcal{D}) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}_{\mathrm{rev}})\,;$$

moreover, the aforementioned functor factors through the fixed points of the C_2-action on the right-hand side given by $\mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}_{\mathrm{rev}}) \xrightarrow{\text{swap}} \mathrm{Alg}(\mathcal{C}_{\mathrm{rev}}) \times \mathrm{Alg}(\mathcal{C}) \xrightarrow{\sigma_{\mathcal{C}} \times \sigma_{\mathcal{C}}} \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}_{\mathrm{rev}})$.

Definition 6.12. Define a category Δ_{σ}

- objects are pairs $([n], s\colon \{1, \ldots, n\} \to \{\pm 1\})$

L: maybe better to write s as a function defined on the set of *morphisms* $i < i+1$ in $[n]$.

- a morphism from $([n], s\colon \{1, \ldots, n\} \to \{\pm 1\})$ to $([m], t\colon \{0, 1, \ldots, m\} \to \{\pm 1\})$ is an order-preserving map $[n] \to [m]$ in Δ.

Construction 6.13. Define a functor $\mathrm{Cut}\colon \Delta_{\sigma}^{\mathrm{op}} \to \mathrm{Assoc}_{\sigma}^{\otimes}$:

- For each $([n], s)$, we have $\mathrm{Cut}([n], s) = \langle n \rangle$.
- Given a morphism $\alpha\colon ([n], s) \to ([m], t)$, the associated morphism $\mathrm{Cut}([n], s) \to \mathrm{Cut}([m], t)$ consists of
  - On underlying finite pointed sets $\langle m \rangle \to \langle n \rangle$, Cut agrees with that appearing in [Lur17, Construction 4.1.2.9]
  - Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j - 1 < j$, we may regard $s\colon \langle n \rangle^{\circ} \to \{\pm 1\}$ and likewise $t\colon \langle m \rangle^{\circ} \to \{\pm 1\}$. Define $u\colon \mathrm{Cut}(\alpha)^{-1}(\langle n \rangle^{\circ}) \to \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\mathrm{Cut}(\alpha)(j))$.

Lemma 6.14. *The functor* $\mathrm{Cut}\colon \Delta_{\sigma}^{\mathrm{op}} \to \mathrm{Assoc}_{\sigma}^{\otimes}$ *exhibits* $\Delta_{\sigma}^{\mathrm{op}}$ *as an approximation to the* ∞*-operad* $\mathrm{Assoc}_{\sigma}^{\otimes}$*.*

L: I think the proof of this lemma is not too different from the proof of Proposition 4.1.2.11 of [Lur17]; the point here is just to unravel the definitions of locally coCartesian and Cartesian; the morphisms in $\Delta_{\sigma}^{\mathrm{op}}$ are a little more complicated than Δ^{op}, but not by much.

Notation 6.15. Let $\mathcal{C}^{\otimes} \to \mathrm{Assoc}_{\sigma}^{\otimes}$ exhibit $\mathcal{C}$ as $\mathbb{E}_{\sigma}$-monoidal. Let $\mathcal{C}^{\circledast}$ denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathrm{Assoc}_{\sigma}^{\otimes}} \Delta_{\sigma}^{\mathrm{op}}$.

Definition 6.16. Say that a morphism $([n], s) \to ([m], t)$ is *inert* if the induced map $\mathrm{Cut}([m], t) \to \mathrm{Cut}([n], s)$ is an inert morphism in $\mathrm{Assoc}_{\sigma}^{\otimes}$.

Definition 6.17. A $\mathbb{R}^{\sigma}$*-planar operad* is an ∞-category $\mathcal{O}^{\circledast}$ equipped with a functor $q\colon \mathcal{O}^{\circledast} \to \Delta_{\sigma}^{\mathrm{op}}$ so that

1. For every object $X \in \mathcal{O}^{\circledast}$ and every inert morphism $\alpha\colon ([n], s) \to q(X)$ in Δ_{σ}, there is a q-cocartesian morphism $\overline{\alpha}\colon X \to Y$ satisfying $q(\overline{\alpha}) = \alpha$
2. Let X be an object satisfying $q(X) = ([n], s)$, and choose q-cocartesian morphisms $\overline{\alpha}_i\colon X \to X_i$ corresponding to the morphism $([i-1 < i], s_i) \to ([n], s)$ which is the inclusion on underlying sets and satisfies $s_i(i) = s(i)$. Then the morphisms $\overline{\alpha}_i$ exhibit X as the q-product of the X_i.
3. For each $n \geq 0$, the construction $C \mapsto \{C_i\}_{1 \leq i \leq n}$ induces an equivalence of ∞-categories

$$\mathcal{O}^{\circledast} \times_{\Delta_{\sigma}^{\mathrm{op}}} \{([n], s)\} \xrightarrow{\sim} \left(\mathcal{O}^{\circledast} \times_{\Delta_{\sigma}^{\mathrm{op}}} \{([1], s|_{\{i\}})\}\right)^{\times n}$$

We say that a morphism α in $\mathbb{R}^{\sigma}$-planar operad is *inert* if it is q-cocartesian and $q(\alpha)$ is inert in $\Delta_{\sigma}^{\mathrm{op}}$ in the sense of Definition 6.16.

Definition 6.18. Let $q\colon \mathcal{O}^{\circledast} \to \Delta_{\sigma}^{\mathrm{op}}$ be a $\mathbb{R}^{\sigma}$-planar operad. An $\mathbb{A}_{\infty}^{\sigma}$-algebra object of $\mathcal{O}^{\circledast}$ is a section of q which carries inert morphisms to inert morphisms. Write $\mathrm{Alg}_{\mathbb{A}_{\infty}^{\sigma}}(\mathcal{O})$ for the full subcategory of $\mathrm{Fun}_{\Delta_{\sigma}^{\mathrm{op}}}(\Delta_{\sigma}^{\mathrm{op}}, \mathcal{O}^{\circledast})$ on $\mathbb{A}_{\infty}^{\sigma}$-algebra objects.

Proposition 6.19. *Let $\mathcal{O}^{\otimes} \to \mathrm{Assoc}_{\sigma}^{\otimes}$ be a fibration of ∞-operads. Then precomposition with the functor* Cut *of Construction 6.13 induces an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{O}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{A}_{\infty}^{\sigma}}(\mathcal{O})\ .$$

Proof. Combine Lemma 6.14 with [Lur17, Theorem 2.3.3.23]. □

Definition 6.20. Define a colored operad $\mathbf{LM}_{\mathrm{inv}}$

(i) The set of objects of $\mathbf{LM}_{\mathrm{inv}}$ has two elements, which we denote by $\mathfrak{a}, \mathfrak{m}$.

(ii) Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{LM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{LM}_{\mathrm{inv}}$. If $Y = \mathfrak{a}$, then $\mathrm{Mul}_{\mathbf{LM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \to \{\pm 1\}$ if $X_i = \mathfrak{a}$ for all i, and empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{LM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying either

- $X_{i_n} = \mathfrak{m}$ and $c(i_n) = 1$ and $X_j = \mathfrak{a}$ otherwise
- $X_{i_1} = \mathfrak{m}$ and $c(i_n) = -1$ and $X_j = \mathfrak{a}$ otherwise

(iii) The composition law on $\mathbf{LM}_{\mathrm{inv}}$ is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4

Remark 6.21. There is a colored operad $\mathbf{RM}_{\mathrm{inv}}$ defined exactly in the same way as $\mathbf{LM}_{\mathrm{inv}}$ in Definition 6.20. In the interest of precision: $\mathbf{RM}_{\mathrm{inv}}$ has the same objects $\mathfrak{a}, \mathfrak{m}$. Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{RM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{RM}_{\mathrm{inv}}$. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{RM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying either

- $X_{i_n} = \mathfrak{m}$ and $c(i_n) = -1$ and $X_j = \mathfrak{a}$ otherwise
- $X_{i_1} = \mathfrak{m}$ and $c(i_n) = 1$ and $X_j = \mathfrak{a}$ otherwise

Remark 6.22. Restricting to the objects which are both called $\mathfrak{a}$, we see that both $\mathbf{LM}_{\mathrm{inv}}$ and $\mathbf{RM}_{\mathrm{inv}}$ have a sub-colored operad which is canonically identified with $\mathbf{Assoc}_{\mathrm{inv}}$ of Definition 6.4.

Remark 6.23. There is a map of colored operads $\iota\colon \mathrm{LM} \to \mathrm{LM}_{\sigma}$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}$ to $\mathfrak{a}$. On $\mathrm{Mul}_{\mathrm{LM}}\left(\{(\mathfrak{a}_{\pm})_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{LM}_{\sigma}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.5. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_{\pm})_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_{\sigma}}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{id}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is a map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{RM} \to \mathrm{LM}_{\sigma}$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}$ to $\mathfrak{a}$. On $\mathrm{Mul}_{\mathrm{RM}}\left(\{(\mathfrak{a}_{\pm})_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{LM}_{\sigma}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{rev}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.5. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_{\pm})_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_{\sigma}}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{rev}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

Definition 6.24. Define colored operads $\mathbf{BM}_{\mathrm{inv}}$ and $\mathbf{BM}_{\sigma}$

(i) The set of objects of $\mathbf{BM}_{\mathrm{inv}}$ has three elements, which we denote by $\mathfrak{a}_{\ell}, \mathfrak{a}_r, \mathfrak{m}$. $\mathbf{BM}_{\sigma}$ has the same objects

(ii) Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{BM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{BM}_{\mathrm{inv}}$. If $Y = \mathfrak{a}_{\ell}$ (resp. $Y = \mathfrak{a}_r$), then $\mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \to \{\pm 1\}$ if $X_i = \mathfrak{a}_{\ell}$ (resp. $X_i = \mathfrak{a}_r$) for all i, and empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the subset of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying: if there is exactly one index i_k so that $X_{i_k} = \mathfrak{m}$, either

- $c(i_k) = 1$, $X_j = \mathfrak{a}_{\ell}$ for $j < i_k$ and $X_j = \mathfrak{a}_r$ for $j > i_k$; or
- $c(i_k) = -1$, $X_j = \mathfrak{a}_{\ell}$ for $j > i_k$ and $X_j = \mathfrak{a}_r$ for $j < i_k$

$\mathrm{Mul}_{\mathbf{BM}_\sigma}(\{X_i\}_{i\in I}, Y)$ is the subset of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying: if there is exactly one index i_k so that $X_{i_k} = \mathfrak{m}$, $c(i_k) = 1$, $X_j = \mathfrak{a}_\ell$ for $j < i_k$ and $X_j = \mathfrak{a}_r$ for $j > i_k$.

(iii) The composition law on $\mathbf{BM}_{\mathrm{inv}}$ is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4. The composition law on $\mathbf{BM}_\sigma$ is determined by the composition of linear orderings

Remark 6.25. The colored operad $\mathbf{BM}_{\mathrm{inv}}$ has a canonical involution σ which fixes $\mathfrak{m}$, exchanges $\mathfrak{a}_\ell$ and $\mathfrak{a}_r$, and sends a morphism (λ, c) to $(\lambda^{\mathrm{rev}}, I \xrightarrow{c} \{\pm 1\} \xrightarrow{\cdot(-1)} \{\pm 1\})$.

Remark 6.26. There is a map of colored operads $\iota\colon \mathbf{BM} \to \mathbf{BM}_{\mathrm{inv}}$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}_-$ to $\mathfrak{a}_\ell$ and $\mathfrak{a}_+$ to $\mathfrak{a}_r$. On $\mathrm{Mul}_{\mathbf{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}_\pm\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.5. On $\mathrm{Mul}_{\mathbf{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{id}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is *also* a map of colored operads $\iota^{\mathrm{rev}}\colon \mathbf{BM} \to \mathbf{BM}_{\mathrm{inv}}$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and and sends $\mathfrak{a}_-$ to $\mathfrak{a}_r$ and $\mathfrak{a}_+$ to $\mathfrak{a}_\ell$. On $\mathrm{Mul}_{\mathbf{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}_\pm\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.5. On $\mathrm{Mul}_{\mathbf{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathbf{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{rev}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_{-1}\}$ where c_{-1} is the constant function on $I \sqcup \{j\}$ with value -1.

Definition 6.27. Let $\mathcal{LM}^\otimes_{\mathrm{inv}}$, $\mathcal{RM}^\otimes_{\mathrm{inv}}$, $\mathcal{BM}^\otimes_{\mathrm{inv}}$, and $\mathcal{BM}^\otimes_\sigma$ denote the ∞-operads associated to the colored operads of Definitions 6.20 and 6.24 (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.28. We can describe the category $\mathcal{LM}^\otimes_{\mathrm{inv}}$ as follows:

(1) An object of $\mathcal{LM}^\otimes_{\mathrm{inv}}$ is a pair $(\langle n\rangle, S)$ where S is a subset of $\langle n\rangle^\circ$.

(2) Morphisms $(\langle m\rangle, T) \to (\langle n\rangle, S)$ consist of a map $(\alpha\colon \langle m\rangle \to \langle n\rangle, \lambda\colon \langle m\rangle^\circ \to \{\pm 1\})$ in $\mathrm{Assoc}^\otimes_\sigma$ satisfying:

- The map α takes $T \cup \{*\}$ to $S \cup \{*\}$
- For each $s \in S$, then $\alpha^{-1}(\{s\})$ contains exactly one element t_s of T, and it is maximal (resp. minimal) with respect to the linear ordering on $\alpha^{-1}(\{s\})$ if $\lambda(t_s) = 1$ (resp. $\lambda(t_s) = -1$).

Remark 6.29. We can describe the category $\mathcal{BM}^\otimes_{\mathrm{inv}}$ as follows:

(1) An object of $\mathcal{BM}^\otimes_{\mathrm{inv}}$ is a triple $(\langle n\rangle, c_+, c_-)$ where $c_\pm$ are functions $\langle n\rangle^\circ \to \{0,1\}$ and $c_-(i) \le c_+(i)$ for all $i \in \langle n\rangle^\circ$.

(2) Morphisms $(\langle m\rangle, c_+, c_-) \to (\langle n\rangle, c'_+, c'_-)$ consist of a map $(\alpha\colon \langle m\rangle \to \langle n\rangle, \lambda\colon \langle m\rangle^\circ \to \{\pm 1\})$ in $\mathrm{Assoc}^\otimes_\sigma$ satisfying: if $j \in \langle n\rangle^\circ$ and $\alpha^{-1}(j) = \{i_1 < i_2 < \cdots < i_\ell\}$,

- If $c_-(j) = c_+(j)$, then
$$c'_-(j) = c_-(i_1) \le c_+(i_1) = c_-(i_2) \le c_+(i_2) \cdots\cdots c_-(i_{m-1}) \le c_+(i_m) = c'_+(j)$$
- If $c_-(j) < c_+(j)$, then there exists a unique k so that $c_-(i_k) < c_+(i_k)$ and
$$\begin{aligned}\lambda(i_k)\cdot c'_-(j) = \lambda(i_k)\cdot c_-(i_1) \le \lambda(i_k)\cdot c_+(i_1) = \lambda(i_k)\cdot c_-(i_2) \le \lambda(i_k)\cdot c_+(i_2)\cdots\\ \lambda(i_k)\cdot c_-(i_{m-1}) \le \lambda(i_k)\cdot c_+(i_m) = \lambda(i_k)\cdot c'_+(j)\end{aligned}$$

Remark 6.30. Each morphism $\varphi \in \mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ determines a linear ordering ℓ on the set I and a function $s\colon I \to \{\pm 1\}$. Passing from φ to the pair (ℓ, s) determines a map of colored operads $j\colon \mathbf{BM}_{\mathrm{inv}} \to \mathbf{Assoc}_{\mathrm{inv}}$. The map j induces a morphism of ∞-operads $\mathcal{BM}^\otimes_{\mathrm{inv}} \to \mathrm{Assoc}^\otimes_\sigma$ which we will also denote by j.

Remark 6.31. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.23 induce maps of ∞-operads $\iota\colon \mathcal{LM}^\otimes \to \mathcal{LM}^\otimes_{\mathrm{inv}}$ and $\iota^{\mathrm{rev}}\colon \mathcal{RM}^\otimes \to \mathcal{LM}^\otimes_{\mathrm{inv}}$.

Remark 6.32. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.26 induce maps of ∞-operads $\iota, \iota^{\mathrm{rev}} \colon \mathcal{BM}^{\otimes} \to \mathrm{BM}_{\sigma}^{\otimes}$. There are canonical identifications $\iota \circ \mathrm{rev} \simeq \sigma \circ \iota^{\mathrm{rev}}$ where σ is the involution on $\mathrm{BM}_{\sigma}^{\otimes}$ induced by Remark 6.25 and rev is the involution on $\mathcal{BM}^{\otimes}$ of [Lur17, Construction 4.6.3.1].

Remark 6.33. There are canonical maps of ∞-operads $\mathcal{LM}_{\mathrm{inv}}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ and $\mathcal{RM}_{\mathrm{inv}}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ sending $\mathfrak{a}$ to $\mathfrak{a}_\ell$, resp. $\mathfrak{a}_r$ and making the diagram

$$\begin{tikzcd} \mathrm{Assoc}^{\otimes} \arrow[r] \arrow[d, "\sigma"] & \mathcal{LM}_{\mathrm{inv}}^{\otimes} \arrow[d, "\mathrm{rev}"] \arrow[dr] & \\ & & \mathcal{BM}_{\mathrm{inv}}^{\otimes} \\ \mathrm{Assoc}^{\otimes} \arrow[r] & \mathcal{RM}_{\mathrm{inv}}^{\otimes} \arrow[ur] & \end{tikzcd}$$

commute, where rev is (an involutive version of) the reversal involution of [Lur17, Remark 4.6.3.2].

Definition 6.34. Let $\mathcal{C}^{\otimes} \to \mathrm{Assoc}_{\sigma}^{\otimes}$ and $\mathcal{D}^{\otimes} \to \mathrm{Assoc}_{\sigma}^{\otimes}$ be fibrations of ∞-operads and let $\mathcal{M}$ be an ∞-category. Suppose given a fibration of ∞-operads $q \colon \mathcal{O}^{\otimes} \to \mathcal{LM}_{\mathrm{inv}}^{\otimes}$ together with equivalences $\mathcal{O}_{\mathfrak{a}}^{\otimes} \simeq \mathcal{C}^{\otimes}$ and $\mathcal{O}_{\mathfrak{m}}^{\otimes} \simeq \mathcal{M}$. Let $L^{\sigma}\mathrm{Mod}(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{LM}_{\mathrm{inv}}}(\mathcal{O})$. We will refer to $L^{\sigma}\mathrm{Mod}(\mathcal{M})$ as the *∞-category of left hermitian module objects of $\mathcal{M}$.*

Suppose given a fibration of ∞-operads $q \colon \mathcal{O}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ together with equivalences $\mathcal{O}_{\mathfrak{a}_\ell}^{\otimes} \simeq \mathcal{C}^{\otimes}$, $\mathcal{O}_{\mathfrak{a}_R}^{\otimes} \simeq \mathcal{D}^{\otimes}$ and $\mathcal{O}_{\mathfrak{m}}^{\otimes} \simeq \mathcal{M}$. Let ${}^{\sigma}\mathrm{Mod}(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{BM}_{\mathrm{inv}}}(\mathcal{O})$. We will refer to ${}^{\sigma}\mathrm{Mod}(\mathcal{M})$ as the *∞-category of hermitian bimodule objects of $\mathcal{M}$.* Composition with the inclusions $\mathrm{Assoc}_{\sigma}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ induces a categorical fibration

$$ {}^{\sigma}\mathrm{Mod}(\mathcal{M}) = \mathrm{Alg}_{/\mathcal{BM}_{\mathrm{inv}}}(\mathcal{O}) \to \mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{C}) \times \mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{D}). $$

L: Lurie gives this a name (Definition 4.2.1.12 *weakly enriched*)– not sure what to call this. something *bi-enriched*?

If A is an Assoc_{σ}-algebra object of $\mathcal{C}$, we let ${}^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ denote the fiber ${}^{\sigma}\mathrm{Mod}(\mathcal{M}) \times_{\mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{C})} \{A\}$. We will refer to ${}^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ as the *∞-category of hermitian A-bimodule objects of $\mathcal{M}$.*

Definition 6.35. Let $q \colon \mathcal{O}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ be a fibration of ∞-operads. We say that q exhibits $\mathcal{O}_{\mathfrak{m}}$ as $\mathbb{E}_{\sigma}$-*bitensored over $\mathcal{O}_{\mathfrak{a}_\ell}$ and $\mathcal{O}_{\mathfrak{a}_r}$* if q is a cocartesian fibration.

Remark 6.36. Let $q \colon \mathcal{O}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ be a cocartesian fibration of ∞-operads. Then q is classified by a map $\chi \colon \mathcal{BM}_{\mathrm{inv}}^{\otimes} \to \mathrm{Cat}_{\infty}$. By Remark 6.32, we can think of q as giving two $\mathbb{E}_{\sigma}$ algebras $\mathcal{C}$, $\mathcal{D}$ in Cat_{∞} with an ∞-category $\mathcal{M}$ equipped with both the structure of a $\mathcal{C}$-$\mathcal{D}$-bimodule (equivalently, the structure of a left $\mathcal{C} \times \mathcal{D}_{\mathrm{rev}}$-module) and of a $\mathcal{D}$-$\mathcal{C}$-bimodule, and an autoequivalence $\sigma_{\mathcal{M}} \colon \mathcal{M} \simeq \mathcal{M}$ of order two which is linear with respect to the autoequivalence $\mathcal{C} \times \mathcal{D}_{\mathrm{rev}} \xrightarrow{\mathrm{flip}} \mathcal{D}_{\mathrm{rev}} \times \mathcal{C} \xrightarrow{\sigma_{\mathcal{D}}^{-1} \times \sigma_{\mathcal{C}}} \mathcal{D} \times \mathcal{C}_{\mathrm{rev}}$.

Remark 6.37. Let $q \colon \mathcal{O}^{\otimes} \to \mathcal{LM}_{\mathrm{inv}}^{\otimes}$ be a cocartesian fibration of ∞-operads. Consider a left hermitian module object $F \colon \mathcal{LM}_{\mathrm{inv}}^{\otimes} \to \mathcal{O}^{\otimes}$. By Remark 6.33, F determines an associative algebra A of $\mathcal{C}$ with an equivalence of algebras $\sigma_A \colon A \simeq \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$, an object $M \in \mathcal{M}$ so that M (resp. $\sigma_{\mathcal{M}}(M)$) is equipped with the structure of a left A-module (resp. right $\sigma_{\mathcal{C}}(A)$-module). Furthermore, we have an equivalence $\sigma_M \colon M \simeq \sigma_{\mathcal{M}}(M)$ which is linear with respect to the equivalence $A \xrightarrow{\sigma_A} \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$.

L: is this related to "modules with involution" from [Cal+20, §3.1]?

Remark 6.38. Let $q \colon \mathcal{O}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ be a cocartesian fibration of ∞-operads. Consider a hermitian module object $F \colon \mathcal{BM}_{\mathrm{inv}}^{\otimes} \to \mathcal{O}^{\otimes}$ and keep the notation of Remark 6.36. By Remark 6.33, F determines an associative algebra A of $\mathcal{C}$ with an equivalence of algebras $\sigma_A \colon A \simeq \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$ and an associative algebra B of $\mathcal{D}$ with an equivalence of algebras $\sigma_B \colon B \simeq \sigma_{\mathcal{D}}(B)^{\mathrm{rev}}$, an object $M \in \mathcal{M}$ so that M (resp. $\sigma_{\mathcal{M}}(M)$) is equipped with the structure of a A-B-bimodule (resp. $\sigma_{\mathcal{D}}(B)$-$\sigma_{\mathcal{C}}(A)$-bimodule). Furthermore, we have an equivalence $\sigma_M \colon M \simeq \sigma_{\mathcal{M}}(M)$ which is linear with respect to the equivalence $A \otimes B \xrightarrow{\mathrm{flip}} B \otimes A \xrightarrow{\sigma_B^{-1} \otimes \sigma_A} \sigma_{\mathcal{D}}(B)^{\mathrm{rev}} \otimes \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$.

L: when $\mathcal{C} = \mathcal{D}$ and $\sigma_{\mathcal{M}}$ and $\sigma_{\mathcal{C}}$ are both the identity and $A = B$, I think this recovers the "module with involution" from [Cal+20, §3.1].

Remark 6.39. Let $q\colon \mathcal{O}^{\otimes} \to \mathrm{Assoc}_{\sigma}$ be a cocartesian fibration of ∞-operads exhibiting $\mathcal{C}$ as a monoidal ∞-category with an involution $\sigma_{\mathcal{C}}$. Let us write $\mathcal{C}_{\pm}^{\otimes} \to \mathcal{BM}^{\otimes}$ for the pullback of q the functors of Remark 6.26 and 6.30. Restriction along the functors of Remark 6.26 and 6.30 defines functors $\mathrm{Alg}^{\sigma}(\mathcal{D}) \to {}^{\sigma}\mathrm{Mod}(\mathcal{D}) \to \mathrm{BMod}(\mathcal{C}_{\pm}^{\otimes})$ which is equivariant with respect to the action of $\sigma_{\mathcal{C}}$ on the target. Moreover, there is a commutative diagram

L: explain the action on $\mathcal{C}_{\pm}$ or delete the last half of the sentence.

$$\begin{tikzcd}
\mathrm{Alg}^{\sigma}(\mathcal{D}) \arrow[r] \arrow[d, "\Delta_{\tau}"] & {}^{\sigma}\mathrm{Mod}(\mathcal{D}) \arrow[r] \arrow[dl] & \mathrm{BMod}\left(\mathcal{C}_{\pm}^{\otimes}\right) \arrow[d] \\
\mathrm{Alg}^{\sigma}(\mathcal{D}) \times \mathrm{Alg}^{\sigma}(\mathcal{D}) \arrow[r, "(A,B)\mapsto A"'] & \mathrm{Alg}^{\sigma}(\mathcal{D}) \arrow[r] & \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{C}_{\mathrm{rev}})
\end{tikzcd}$$

where the right vertical arrow is the categorical fibration of [Lur17, Definition 4.3.1.12], the central diagonal arrow is the categorical fibration of Definition 6.34, and the left vertical arrow Δ_{τ} is the identity on one component and precomposition with the involution of Remark 6.25. The total composite from the upper left to the lower right agrees with the forgetful functor of Remark 6.11. If $A\colon \mathrm{Assoc}_{\sigma} \to \mathcal{O}^{\otimes}$ is an involutive algebra object (Definition 6.9), informally the functor sends A to A, regarded as an A-$\sigma_{\mathcal{C}}(A)$-bimodule in the notation of Remark 6.38 (compare [Lur17, Remark 4.2.1.17]).

Construction 6.40. Define a functor $\mathrm{MCut}\colon \Delta_{\sigma}^{\mathrm{op}} \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$:

L: maybe this overloaded notation is not good. I'm running out of ideas.

- For each $([n], s)$, we have $\mathrm{MCut}([n], s) = \langle n{+}1\rangle \simeq \mathrm{RCut}_0([n])$ where RCut is from [Lur17, Construction 4.8.4.4].
- Given a morphism $\alpha\colon ([n], s) \to ([m], t)$, the associated morphism $\mathrm{MCut}([m], t) \to \mathrm{MCut}([n], s)$ consists of
  - On underlying finite pointed sets $\langle m+1\rangle \to \langle n+1\rangle$, MCut agrees with (the reverse of) that appearing in [Lur17, Construction 4.2.2.6]
  - Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j-1 < j$, we may regard $s\colon \langle n+1\rangle^{\circ} \to \{\pm 1\}$ and likewise $t\colon \langle m+1\rangle^{\circ} \to \{\pm 1\}$. Define $u\colon \mathrm{MCut}(\alpha)^{-1}\left(\langle n+1\rangle^{\circ}\right) \to \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\mathrm{MCut}(\alpha)(j))$.

L: check later

Remark 6.41. We can identify $\mathrm{Assoc}_{\sigma}^{\otimes}$ with the full subcategory of $\mathcal{RM}_{\mathrm{inv}}^{\otimes}$ spanned by objects of the form $(\langle n\rangle, \langle n\rangle^{\circ})$. We can regard Construction 6.13 as defining a functor $\Delta_{\sigma}^{\mathrm{op}} \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$. For each $([n], s) \in \Delta_{\sigma}^{\mathrm{op}}$, there is a map of sets $\theta\colon \mathrm{MCut}([n], s) \to \mathrm{Cut}([n], s)$ defined as in [Lur17, Remark 4.2.2.8]. Concretely, on underlying pointed sets, θ takes the form

L: check that the signs s work out!

$$\theta\colon \langle n+1\rangle \to \langle n\rangle$$
$$k \mapsto \begin{cases} k-1 & \text{if } k > 0 \\ * & \text{if } k = 0, *. \end{cases}$$

This construction determines a morphism γ in the ∞-category $\mathrm{Fun}\left(\Delta_{\sigma}^{\mathrm{op}}, \mathcal{RM}_{\mathrm{inv}}^{\otimes}\right)$, or equivalently a map $\gamma\colon \Delta_{\sigma}^{\mathrm{op}} \times \Delta^1 \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$.

Lemma 6.42. *The morphism $\gamma\colon \Delta_{\sigma}^{\mathrm{op}} \times \Delta^1 \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$ defined in Remark 6.41 exhibits $\Delta_{\sigma}^{\mathrm{op}} \times \Delta^1$ as an approximation to the ∞-operad $\mathcal{RM}_{\mathrm{inv}}^{\otimes}$.*

Definition 6.43. Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^{\otimes}$ as weakly bi-enriched over $\mathcal{O}_{\mathfrak{a}}^{\otimes}$. Let γ be as in Remark 6.41. Let $R\,{}^{\sigma}\mathrm{Mod}^{\mathbb{A}_{\infty}^{\sigma}}(\mathcal{M})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{RM}_{\mathrm{inv}}^{\otimes}}\left(\Delta_{\sigma}^{\mathrm{op}} \times \Delta^1, \mathcal{O}^{\otimes}\right)$ spanned by those maps $f\colon \Delta_{\sigma}^{\mathrm{op}} \times \Delta^1 \to \mathcal{O}^{\otimes}$ satisfying

1. The restriction of f to $\Delta_{\sigma}^{\mathrm{op}} \times \{1\}$ belongs to $\mathrm{Alg}_{\mathbb{A}_{\infty}^{\sigma}}(\mathcal{O})$ of Definition 6.18
2. If $\alpha\colon ([m], s) \to ([n], t)$ so that $\alpha(0) = 0$, then the induced map $f([m], s, 0) \to f([n], t, 0)$ is an inert map in $\mathcal{O}^{\otimes}$
3. for each object $([n], s)$ in $\Delta_{\sigma}^{\mathrm{op}}$, the induced map $f([n], s, 0) \to f([n], s, 1)$ is an inert map in $\mathcal{O}^{\otimes}$

Example 6.44. Let $\mathcal{C}^{\otimes} \to \mathcal{RM}^{\otimes}$ be a fibration of ∞-operads. Restriction along the map of ∞-operads $\mathcal{RM}^{\otimes}_{\mathrm{inv}} \to \mathrm{Assoc}^{\otimes}_{\sigma}$ induced by Remark 6.30 induces a map $\mathbb{E}_{\sigma}\,\mathrm{Alg}\,(\mathcal{C}) \to R^{\sigma}\mathrm{Mod}\,(\mathcal{C})$ which is a section of the projection map $R^{\sigma}\mathrm{Mod}\,(\mathcal{C}) \to \mathbb{E}_{\sigma}\,\mathrm{Alg}\,(\mathcal{C})$.

L: see Example 4.2.1.17 of higher algebra

Notation 6.45. Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{RM}^{\otimes}_{\mathrm{inv}}$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}^{\otimes}_{\mathfrak{m}}$ as weakly bi-enriched over $\mathcal{O}^{\otimes}_{\mathfrak{a}}$. Define a new simplicial set $\overline{\mathcal{M}}^{\circledast}$ by the following universal property

L: fibration?

$$\hom_{\mathrm{sSet}_{/\Delta^{\mathrm{op}}_{\sigma}}}\left(K, \overline{\mathcal{M}}^{\circledast}\right) \simeq \hom_{\mathrm{sSet}_{/\mathcal{RM}^{\otimes}_{\mathrm{inv}}}}\left(K \times \Delta^1, \mathcal{O}^{\otimes}\right).$$

Here we regard $K \times \Delta^1$ as a simplicial set over $\mathcal{RM}^{\otimes}_{\mathrm{inv}}$ via the composite $K \times \Delta^1 \to \Delta^{\mathrm{op}}_{\sigma} \times \Delta^1 \xrightarrow{\gamma} \mathcal{RM}^{\otimes}_{\mathrm{inv}}$ where γ is from Remark 6.41.

Unwinding definitions, we see that a vertex in $\overline{\mathcal{M}}^{\circledast}$ lying over an object $([n], s\colon \{1,\dots,n\} \to \{\pm 1\}) \in \Delta^{\mathrm{op}}_{\sigma}$ corresponds to a morphism α in $\mathcal{O}^{\otimes}$ whose image in $\mathcal{RM}^{\otimes}_{\mathrm{inv}}$ is the map $(\langle n+1\rangle, \{0\}) \to (\langle n\rangle, \varnothing)$. Now let $\mathcal{M}^{\circledast}$ denote the full simplicial subset of $\overline{\mathcal{M}}^{\circledast}$ spanned by those vertices for which α is inert.

L: this might be off–revisit later!

Remark 6.46. Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{RM}^{\otimes}_{\mathrm{inv}}$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}^{\otimes}_{\mathfrak{m}}$ as weakly enriched over $\mathcal{O}^{\otimes}_{\mathfrak{a}}$. By [Lur09, Example 4.3.1.4 & Proposition 4.3.2.15], composition with the inclusion $\{0\} \to \Delta^1$ induces a trivial Kan fibration $\mathcal{M}^{\circledast} \xrightarrow{\sim} \mathcal{O}^{\otimes} \times_{\mathcal{RM}^{\otimes}_{\mathrm{inv}}} \Delta^{\mathrm{op}}_{\sigma}$. In particular, the fiber of $\mathcal{M}^{\circledast}$ over an object $([n], s) \in \Delta^{\mathrm{op}}_{\sigma}$ is canonically equivalent to $\mathcal{M} \times \mathcal{C}^{\times n}$.

Finally, since q is a categorical fibration and categorical fibrations are closed under pullback and composition with trivial fibrations, q induces categorical fibrations $\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast} \to \Delta^{\mathrm{op}}_{\sigma}$.

L: Jacob explains this in a really terse way–just by citing Prop 4.3.2.15 of HTT. It does just follow from definitions/observatio but there are many (for instance, definition of inert edge).

Lemma 6.47. *Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{RM}^{\otimes}_{\mathrm{inv}}$ be a cocartesian fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}^{\otimes}_{\mathfrak{m}}$ as tensored over $\mathcal{O}^{\otimes}_{\mathfrak{a}}$. Then the associated functor $\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ (Notation 6.15) is a locally coCartesian fibration.*

Proposition 6.48. *Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{RM}^{\otimes}_{\mathrm{inv}}$ be a cocartesian fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}^{\otimes}_{\mathfrak{m}}$ as tensored over $\mathcal{O}^{\otimes}_{\mathfrak{a}}$. Then precomposition with the functor* MCut *of Construction 6.40 induces an equivalence of ∞-categories*

$$R^{\sigma}\mathrm{Mod}(\mathcal{M}) \simeq \mathrm{Alg}_{/\mathcal{RM}_{\mathrm{inv}}}(\mathcal{O}) \xrightarrow{\sim} R^{\sigma}\mathrm{Mod}^{\mathbb{A}^{\sigma}_{\infty}}(\mathcal{M}).$$

Proof. Combine Lemma 6.42 with [Lur17, Theorem 2.3.3.23]. ☐

Definition 6.49. Introduction to section 3.4 here gives a recipe for defining $\mathbb{E}^{\otimes}_{\sigma}$ as a genuine/parametrized ∞-operad. Relate the 'underlying' ∞-operad with C_2-action to Assoc_{σ}?

L: Unravel what it means to be $\mathbb{E}_{\sigma}$-monoidal in a genuine way!

Lemma 6.50. *Let R be a Poincaré ring. Then $\mathrm{He}(\mathrm{Mod}^{\omega}_R, Ϙ_R) \to \mathrm{He}\,(\mathrm{Hyp}(\mathrm{Mod}^{\omega}_R))$ is canonically endowed with the structure of an $\mathbb{E}_{\sigma}$-monoidal C_2-∞-category. Recall that $\mathrm{He}\,(\mathrm{Hyp}(\mathrm{Mod}^{\omega}_R)) \simeq \mathrm{TwAr}(\mathrm{Mod}^{\omega}_R)$ by [Cal+20, §2.2].*

L: What does it mean for a C_2-∞-category to be tensored over a $\mathbb{E}_{\sigma}$-monoidal C_2-∞-category? Show that if $(\mathcal{C}, Ϙ)$ is an R-linear Poincaré ∞-category, then $\mathrm{He}(\mathcal{C}, Ϙ) \to \mathrm{He}(\mathcal{C}^e)$ is an $\mathbb{E}_{\sigma}$-linear/-tensored over $\underline{He}\left(\mathrm{Mod}^{\omega}_R\right)$?

6.2 Part (b)

Proposition 6.51. *Let $\mathcal{C}$ be an involutive monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is bitensored over $\mathcal{C}$. Let K be a simplicial set so that $\mathcal{M}$ admits K-indexed limits, and let $\theta\colon R^{\sigma}\mathrm{Mod}(\mathcal{M}) \to \mathrm{Alg}^{\sigma}(\mathcal{C})$ be the forgetful functor. Then*

L: This statement is [Lur17, Proposition 4.2.3.1] with some words changed; no claim of originality here.

(1) For every commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & R^{\sigma}\mathrm{Mod}(\mathcal{M}) \arrow[d, "\theta"] \\ K^{\triangleleft} \arrow[r] \arrow[ur, dashed] & \mathrm{Alg}^{\sigma}(\mathcal{C}), \end{tikzcd}$$

there exists a dashed arrow which is a θ-limit diagram.

(2) An arbitrary map $\overline{g}\colon K^{\triangleleft} \to R^{\sigma}\mathrm{Mod}(\mathcal{M})$ is a θ-limit diagram if and only if the induced map $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.

Proof. □ L: todo

Corollary 6.52. *θ is a cartesian fibration, and a morphism $f\colon \Delta^1 \to R^{\sigma}\mathrm{Mod}(\mathcal{M})$ is θ-cartesian if and only if the image of f in $\mathcal{M}$ is an equivalence.*

Corollary 6.53. *Let $\mathcal{C}$ be an involutive monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is bitensored over $\mathcal{C}$. Let K be a simplicial set so that $\mathcal{M}$ admits K-indexed limits, and let $\theta\colon R^{\sigma}\mathrm{Mod}(\mathcal{M}) \to \mathrm{Alg}^{\sigma}(\mathcal{C})$ be the forgetful functor. Let A be an involutive algebra object of $\mathcal{C}$. Then*

(1) $R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ admits K-indexed limits.

(2) A diagram $K^{\triangleleft} \to R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ is a limit diagram if and only if the induced diagram $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.

(3) Given a morphism $A \to B$ of involutive algebra objects of $\mathcal{C}$, the induced functor $R^{\sigma}\mathrm{Mod}_B(\mathcal{M}) \to R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ preserves K-indexed limits.

6.3 Towards (e)

Construction 6.54. Define a functor $\mathrm{Pr}\colon \mathbf{LM}^{\otimes}_{\mathrm{inv}} \times \mathbf{RM}^{\otimes}_{\mathrm{inv}} \to \mathbf{BM}^{\otimes}_{\mathrm{inv}}$.

(1) Let $(\langle m\rangle, S)$ be an object of $\mathbf{LM}^{\otimes}_{\mathrm{inv}}$ and let $(\langle n\rangle, T)$ be an object of $\mathbf{RM}^{\otimes}_{\mathrm{inv}}$. Let $\mathrm{Pr}((\langle m\rangle, S), (\langle n\rangle, T)) = (X_*, c_-, c_+)$ where X_*, c_-, c_+ are described in [Lur17, Construction 4.3.2.1(1)].

(2) Let $(\alpha, \lambda)\colon (\langle m\rangle, S) \to (\langle m'\rangle, S')$ be a morphism in $\mathbf{LM}^{\otimes}_{\mathrm{inv}}$ and let $(\beta, \mu)\colon (\langle n\rangle, T) \to (\langle n'\rangle, T')$ be a morphism in $\mathbf{RM}^{\otimes}_{\mathrm{inv}}$. Write $\mathrm{Pr}((\langle m'\rangle, S'), (\langle n'\rangle, T')) = (X'_*, c'_-, c'_+)$. Then $\mathrm{Pr}\left((\alpha, \lambda), (\beta, \mu)\right)$ is the unique morphism in $\mathbf{BM}^{\otimes}_{\mathrm{inv}}$ lying over the map $\gamma\colon X_* \to X'_*$ described by

   (i) $\gamma(i,j) = \begin{cases} (\alpha(i), \beta(j)) & \text{if } \alpha(i) \in \langle m'\rangle^{\circ}, \beta(j) \in \langle n'\rangle^{\circ} \\ * & \text{otherwise.} \end{cases}$

   (ii) Let $i' \in \langle m'\rangle^{\circ} \setminus S'$ and $j' \in T'$ so $j' = \beta(j)$ for a unique $j \in T$. Then the linear ordering on $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\}$ is (a) determined by the map α if $\mu(j) = 1$, and (b) it is the reverse of the linear ordering determined by α if $\mu(j) = -1$. The map $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\} \to \{\pm 1\}$ is determined by λ if $\mu(j) = 1$ and it is $-\lambda$ if $\mu(j) = -1$.

   (iii) Likewise if $i' \in S'$ and $j' \in \langle n'\rangle^{\circ} \setminus T'$

   (iv) Let $i' \in S'$ and $j' \in T'$ so $i' = \alpha(i)$ for a unique $i \in S$ and $j' = \beta(j)$ for a unique $j \in T$. Then $\gamma^{-1}\{(i', j')\} = \{i\} \times \beta^{-1}\{(j')\} \sqcup_{\{(i,j)\}} \alpha^{-1}\{(i')\} \times \{j\}$. Define $\gamma^{-1}\{(i', j')\} \to \{\pm 1\}$ by $\lambda \times \mu$. Endow $\gamma^{-1}\{(i', j')\}$ with the linear ordering from [Lur17, Construction 4.3.2.1(2)(iv)] if $\lambda(i) = \mu(j)$ and endow $\gamma^{-1}\{(i', j')\}$ with the opposite ordering if $\lambda(i) \neq \mu(j)$ (or equivalently, if $\lambda(i) = -\mu(j)$).

Write Pr for the induced map $\mathcal{LM}^{\otimes}_{\sigma} \times \mathcal{RM}^{\otimes}_{\sigma} \to \mathcal{BM}^{\otimes}_{\sigma}$ of ∞-categories.

Construction 6.55. Let $q\colon \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}_{\sigma}$ be a fibration of ∞-operads. We define a map of simplicial sets $\overline{L^{\sigma}\mathrm{Mod}}(\mathcal{C}_{\mathfrak{m}})^{\otimes} \to \mathcal{RM}^{\otimes}_{\sigma}$ by the universal property: For any simplicial set $K \to \mathcal{RM}^{\otimes}_{\sigma}$, there is a bijection

$$\mathrm{Hom}_{\mathrm{sSet}_{/\mathcal{RM}^{\otimes}_{\sigma}}}\left(K, \overline{L^{\sigma}\mathrm{Mod}}(\mathcal{C}_{\mathfrak{m}})^{\otimes}\right) \simeq \mathrm{Hom}_{\mathrm{sSet}_{/\mathcal{BM}^{\otimes}_{\sigma}}}\left(\mathcal{LM}^{\otimes}_{\sigma} \times K, \mathcal{C}^{\otimes}\right).$$

Let $L^{\sigma}\mathrm{Mod}(\mathcal{C}_{\mathfrak{m}})^{\otimes}$ denote the full simplicial subset of $\overline{L^{\sigma}\mathrm{Mod}}(\mathcal{C}_{\mathfrak{m}})^{\otimes}$ spanned by those vertices which correspond to a vertex $X \in \mathcal{RM}^{\otimes}_{\sigma}$ and a functor $F\colon \mathcal{LM}^{\otimes}_{\sigma}\{X\} \to \mathcal{BM}^{\otimes}_{\sigma}$ which takes inert morphisms in $\mathcal{LM}^{\otimes}_{\sigma}$ to inert morphisms in $\mathcal{BM}^{\otimes}_{\sigma}$.

Remark 6.56. The composite $\mathcal{LM}_\sigma^\otimes \times \{\mathfrak{m}\} \hookrightarrow \mathcal{LM}_\sigma^\otimes \times \mathcal{RM}_\sigma^\otimes \xrightarrow{\mathrm{Pr}} \mathcal{BM}_\sigma^\otimes$ agrees with the inclusion of Remark 6.33. Taking $K \to \mathcal{RM}_\sigma^\otimes$ to be the inclusion $\{\mathfrak{m}\} \hookrightarrow \mathcal{RM}_\sigma^\otimes$, we have an isomorphism of simplicial sets $L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \times_{\mathcal{RM}_\sigma^\otimes} \{\mathfrak{m}\} \simeq L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})$ where $L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})$ is the ∞-category of left modules associated to the fibration of ∞-operads $\mathcal{C}^\otimes \times_{\mathcal{BM}_\sigma^\otimes} \mathcal{LM}_\sigma^\otimes \to \mathcal{LM}_\sigma^\otimes$.

Proposition 6.57. *Let $q\colon \mathcal{C}^\otimes \to \mathcal{BM}_\sigma^\otimes$ be a fibration of ∞-operads. Then*

(1) the induced map $p\colon L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \to \mathcal{RM}_\sigma^\otimes$ is a fibration of ∞-operads

(2) a morphism α in $L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$ is inert if and only if $p(\alpha)$ is inert in $\mathcal{RM}_\sigma^\otimes$ and for all $X \in \mathcal{LM}_\sigma$, $\alpha(X)$ is an inert morphism in $\mathcal{C}^\otimes$.

(3) if q is a cocartesian fibration of ∞-operads, then so is p

(4) if q is a cocartesian fibration of ∞-operads, a morphism α in $L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$ is p-cocartesian if and only if, for all $X \in \mathcal{LM}_\sigma^\otimes$, $\alpha(X)$ is q-cocartesian in $\mathcal{C}^\otimes$.

Proof. Similar to [Lur17, Proposition 4.3.2.5]. □

Theorem 6.58. *Let $\mathcal{C}$ be an $\mathbb{E}_\sigma$-monoidal ∞-category, and let A be an $\mathbb{E}_\sigma$-algebra in $\mathcal{C}$. Then $L^\sigma\mathrm{Mod}_A(\mathcal{C})$ is right $\mathbb{E}_\sigma$-tensored over $\mathcal{C}$.*

6.4 Endomorphisms

Let $\mathcal{C}$ be an $\mathbb{E}_\sigma$-monoidal ∞-category, and write $\sigma_\mathcal{C}\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ for its involution. Suppose $M \in \mathcal{C}$ is an object equipped with an equivalence $\sigma_M\colon M \simeq \sigma_\mathcal{C}(M)$. By [Lur17, §4.7.1], endomorphisms of M can be regarded as an $\mathbb{E}_1$-algebra in $u(\mathcal{C})^\otimes$, where u is from Remark 6.10. Now σ_M induces an equivalence $\mathrm{End}_\mathcal{C}(M) \simeq \mathrm{End}_\mathcal{C}(\sigma_\mathcal{C}(M))$ On the other hand, $\sigma_\mathcal{C}$ induces an equivalence $\mathrm{End}_\mathcal{C}(\sigma_\mathcal{C}(M)) \simeq \mathrm{End}_\mathcal{C}(M)^{\mathrm{rev}}$. In particular, for any ∞-category $\mathcal{M}$ left $\mathbb{E}_\sigma$-tensored over $\mathcal{C}$ and any object $M \in \mathcal{M}$ which is fixed by the involution on $\mathcal{M}$, we expect the endomorphisms of M to admit the structure of an $\mathbb{E}_\sigma$-algebra in $\mathcal{C}$.

To this end, we will define an ∞-category of objects acting on M, show that it has an $\mathbb{E}_\sigma$-monoidal structure, and locate endomorphisms of M as the final object in this ∞-category. Informally, we may define a category $\mathcal{C}[M]$ whose objects consist of either

- pairs (C, η) where $C \in \mathcal{C}$ and $\eta\colon C \otimes M \to M$ is a morphism in $\mathcal{M}$; or
- pairs (C', η') where $C' \in \mathcal{C}$ and $\eta'\colon \sigma_\mathcal{M}(M) \otimes C' \to \sigma_\mathcal{M}(M)$.
- pairs (D, ξ) where $D \in \mathcal{C}$ and $\xi\colon M \otimes D \to \sigma_\mathcal{M}(M)$ L: added July 26th
- pairs (D, ξ') where $D \in \mathcal{C}$ and $\xi'\colon D \otimes \sigma_\mathcal{M}(M) \to M$

The monoidal structure is as described in [Lur17, §4.7.1]. Note that given an object (C, η), the involution $\sigma_\mathcal{M}$ on $\mathcal{M}$ sends η to the map $\sigma_\mathcal{M}(C \otimes M) \simeq \sigma_\mathcal{M}(M) \otimes \sigma_\mathcal{C}(C) \to \sigma_\mathcal{M}(M)$. Given an object (D, ξ), the involution $\sigma_\mathcal{M}$ on $\mathcal{M}$ we may consider the map $\xi'\colon \sigma_\mathcal{C}(D) \otimes M \xrightarrow{\mathrm{id}\otimes\sigma_M} \sigma_\mathcal{C}(D) \otimes \sigma_\mathcal{M}^2(M) \simeq \sigma_\mathcal{M}(M \otimes D) \xrightarrow{\sigma_\mathcal{M}(\xi)} \sigma_\mathcal{M}(M)$. The assignment $(D, \xi) \mapsto (\sigma_\mathcal{C}(D), \xi')$... This is the involution on $\mathcal{C}[M]$. L: is an involution?

Definition 6.59. Let $p\colon \mathcal{M}^\circledast \to \Delta^1 \times \Delta_\sigma^{\mathrm{op}}$ exhibit $\mathcal{M}^\circledast$ as weakly enriched over $\mathcal{C}^\circledast$. An *enriched morphism* of $\mathcal{M}$ is a diagram

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

satisfying either

- $p(\alpha)$ is the morphism $(0, [1], c_1) \to (0, [0])$ in $\Delta_\sigma^{\mathrm{op}}$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_1\colon \{1\} \to \{\pm 1\}$ is the constant function at $+1$, and
- the map β is inert, and $p(\beta)$ is the morphism $(0, [1], c_1) \to (0, [0])$ in $\Delta^1 \times \Delta_\sigma^{\mathrm{op}}$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

or

- $p(\alpha)$ is the morphism $(0, [1], c_{-1}) \to (0, [0])$ in $\Delta^{\mathrm{op}}_{\sigma}$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_{-1} \colon \{1\} \to \{\pm 1\}$ is the constant function at -1.

- the map β is inert, and $p(\beta)$ is the morphism $(0, [1], c_{-1}) \to (0, [0])$ in $\Delta^1 \times \Delta^{\mathrm{op}}_{\sigma}$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

Let $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]}$ denote the full subcategory of $\mathrm{Fun}_{\Delta^1 \times \Delta^{\mathrm{op}}_{\sigma}}\left(\Lambda^2_0, \mathcal{M}^{\circledast}\right)$ spanned by the enriched morphisms of $\mathcal{M}$.

Note that there are two evaluation functors $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]} \to \mathcal{M}$. Given $M \in \mathcal{M}$, write $\mathcal{C}[M] := \{M\} \times_{\mathcal{M}} \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]} \times_{\mathcal{M}} \{M\}$ and refer to it as the endomorphism ∞-category of M.

Definition 6.60. *enriched n-string*

Proposition 6.61 (Segal condition)**.**

In the course of thinking about the 'involutive' generalization of the statement that given an $\mathbb{E}_1$-algebra, its category of modules is $\mathbb{E}_0$ (and conversely, that given an object in a stable ∞-category, that its endomorphism spectrum is an $\mathbb{E}_1$-algebra), I have run up against some questions.

Question 6.62.

- Can we sidestep an involutive version of the construction of endomorphism categories of [Lur17, §4.7.1]?

- Suppose $\mathcal{C}$ is a monoidal ∞-category and $\mathcal{M}$ is an ∞-category which is enriched over $\mathcal{C}$ in the sense of [Lur17, §4.2.1]. The opposite category $\mathcal{M}^{\mathrm{op}}$ is enriched over $\mathcal{C}$ by [Hei23, §10].

7 From Poincaré rings to Poincaré ∞-categories

Use genuine operadic nerve here to go from simplicial genuine operads to parametrized operads.

Notation 7.1. Let S, T be finite C_2-sets, and let $f \colon S \to T$ be a C_2-equivariant map. A *t-ordering* on f is the data of, for all free orbits $V \subset S$, $V \simeq C_2$ so that $f(V) = \{*\}$, a choice of an ordering $\leq$ on V. A *t-ordered map* $S \to T$ is the data of f and a t-ordering on f.

Note that if $f_i \colon S_i \to T_i$ are all t-ordered, there is a canonical t-ordering on $\coprod_i f_i$. If $f \colon S \to T$ is t-ordered and $g \colon T \to U$ is t-ordered, there is a canonical t-ordering on $g \circ f$. Let us spell this out: Suppose given an orbit $V \subset S$ on which C_2 acts freely so that $g \circ f(V)$ is trivial. There are two cases

- Suppose f restricts to an isomorphism $f|_V \colon V \xrightarrow{\sim} f(V)$ and g sends $f(V)$ to $*$. Then the t-ordering on g means we have an ordering on $f(V)$, which we transport to an ordering on V via $f|_V$.

- Suppose $f(V) = *$. Then the t-ordering on f endows V with a canonical ordering.

Finally, we define a t-ordering on pullbacks as follows: Suppose given a pullback square of C_2-sets.

$$\begin{tikzcd}
Z \times_X W \arrow[r, "g^* f"] \arrow[d, "\pi_1"] & W \arrow[d, "g"] \\
Z \arrow[r, "f"] & X
\end{tikzcd}$$

and a t-ordering on f. Suppose given a free orbit $C_2 \simeq U \subseteq Z \times_X W$ so that $g^*(f)(U)$ is a singleton (with trivial C_2-action). Because the square above is a pullback, $\pi_1(U) \subset Z$ is acted on by C_2 freely, and $\pi_1|_U$ defines an isomorphism $U \simeq \pi_1(U)$. By C_2-equivariance of g, $g(g^*(f)(U)) = f(\pi_1(U))$ is a singleton. The given t-ordering on f means $\pi_1(U)$ has a given ordering, which we use to endow U with an ordering using the isomorphism $\pi_1|_U$.

Remark 7.2. In the following pullback diagram in Fin_{C_2}, both t-orderings on f induce the same t-ordering on $g^*(f)$

$$\begin{tikzcd} C_2 \times C_2 \arrow[r] \arrow[d] & C_2 \arrow[d, "g"] \\ C_2 \arrow[r, "f"] & C_2/C_2 \end{tikzcd}$$

(in fact the set of t-orderings on $g^*(f)$ is a singleton.)

In what follows, we will refer to the following diagram

$$\begin{tikzcd} U \arrow[d] & Z \arrow[l, "g"'] \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ V & Y \arrow[l] \arrow[r, equal] & Y \end{tikzcd} \tag{7.3}$$

in $\underline{\mathrm{Fin}}_{C_2}$ or $\underline{\mathrm{Fin}}_{C_2,*}$ repeatedly (see [NS22, §2.1]). In particular, recall that in $\underline{\mathrm{Fin}}_{C_2,*}$, the induced map $Z \to U \times_V Y$ is always assumed to be a summand inclusion.

Definition 7.4. Define a C_2-∞-operad $\mathbb{E}_{\mathrm{p}}^{\otimes} \to \underline{\mathrm{Fin}}_{C_2,*}$ as having

- objects consist of those arrows $U \to V$ so that C_2 acts freely on U and transitively on V
- morphisms consist of spans (7.3) in $\underline{\mathrm{Fin}}_{C_2,*}$, plus the data of a t-ordering on f in the sense of Notation 7.1.
- composition is defined to agree with that in $\underline{\mathrm{Fin}}_{C_2,*}$.

There is a canonical map to $\underline{\mathrm{Fin}}_{C_2,*}$ which is an inclusion on objects and on morphisms forgets the t-ordering.

L: Say something about how (1) these are all C_2-∞-operads and (2) the underlying ∞-categories are actually the nerve of $(2,1)$-categories (see [Yan23, Remark 4.8]).

Definition 7.5. Define a C_2-∞-operad $\underline{\mathrm{Assoc}}^{\otimes}_{\mathrm{naive}} \to \underline{\mathrm{Fin}}_{C_2,*}$ as having

- objects consist of those arrows $U \to V$ so that C_2 acts freely on U and transitively on V
- morphisms consist of spans (7.3) in $\underline{\mathrm{Fin}}_{C_2,*}$ plus the data of, for all orbits $T \subset X$ an ordering on the orbits in the preimage $f^{-1}(T) \subseteq Z$.
- composition is defined to agree with that in $\underline{\mathrm{Assoc}}^{\otimes}$.

there is a canonical map to $\underline{\mathrm{Fin}}_{C_2,*}$ which is an inclusion on objects and on morphisms forgets the ordering.

Remark 7.6. Let $\underline{\mathrm{Free}}_{C_2,*}$ be the full subcategory of $\underline{\mathrm{Fin}}_{C_2,*}$ on those arrows $U \to V$ where U is a free C_2-set. It is evident that $\underline{\mathrm{Free}}_{C_2,*}$ is a sub C_2-∞-operad of $\underline{\mathrm{Fin}}_{C_2,*}$ in the sense of [NS22, Definition 2.2.7]. By [BHS22, Lemma 4.1.13], there is an equivalence of C_2-∞-operads $\underline{\mathrm{Free}}_{C_2,*} \times_{\underline{\mathrm{Fin}}_{C_2,*}} \underline{\mathrm{Assoc}}^{\otimes} \simeq \underline{\mathrm{Assoc}}^{\otimes}_{\mathrm{naive}}$. Now for any C_2-∞-operad $\mathcal{C}^{\otimes}$, there is an equivalence

$$\hom_{\underline{\mathrm{Fin}}_{C_2,*}} \left(\underline{\mathrm{Assoc}}^{\otimes}_{\mathrm{naive}}, \mathcal{C}^{\otimes}\right) \simeq \hom_{\underline{\mathrm{Free}}_{C_2,*}} \left(\underline{\mathrm{Assoc}}^{\otimes}_{\mathrm{naive}}, \mathcal{C}^{\otimes} \times_{\underline{\mathrm{Fin}}_{C_2,*}} \underline{\mathrm{Free}}_{C_2,*}\right).$$

The functor $\mathrm{Free}_{C_2,*} \to \mathrm{Fin}_*$ which sends a (pointed, finite) free C_2-set to its set of orbits exhibits $\mathrm{Free}_{C_2,*}$ as a generalized ∞-operad in the sense of [Lur17, Definition 2.3.2.1]. Now choose a free C_2-set with transitive action; this determines a functor $BC_2 \to \mathrm{Free}_{C_2,*}$. This induces a functor $p\colon BC_2 \times \mathrm{Fin}_* \to \mathrm{Free}_{C_2,*}$. By [Lur17, Corollary 2.3.2.13], there is an equivalence

$$\hom_{\mathrm{Free}_{C_2,*}} \left(\mathrm{Assoc}^{\otimes}_{\mathrm{naive}}, \mathcal{C}^{\otimes}_{\mathrm{naive}}\right) \simeq \hom_{BC_2 \times \underline{\mathrm{Fin}}_*} \left(p^*(\mathrm{Assoc}^{\otimes}_{\mathrm{naive}}), p^*(\mathcal{C}^{\otimes}_{\mathrm{naive}})\right)$$

where the left hand side denotes morphisms in generalized ∞-operads over $\mathrm{Free}_{C_2,*}$ and the right hand side denotes morphisms of BC_2-families of ∞-operads in the sense of [Lur17, Definition 2.3.2.10]. Now the latter is equivalent to $\mathbb{E}_1$-algebra objects in $(\mathcal{C}^e)^{hC_2}$.

L: Now pass to spans of finite free C_2-sets (not spans of arrows!), and write $\mathcal{C}^{\otimes}_{\mathrm{naive}} \to \mathrm{Free}_{C_2,*}$ and $\mathrm{Assoc}^{\otimes}_{\mathrm{naive}}$. Use Higher algebra Remark 2.3.3.4 and (Thm 2.3.3.23 or Cor 2.3.2.13).

Remark 7.7. Note that in the definition of $\mathrm{Assoc}_{\mathrm{naive}}$ (7.3), f is equivalent to a fold map because X is acted upon freely by C_2.

Definition 7.8. Define a C_2-∞-operad $\mathcal{HM}^{\otimes} \to \underline{\mathrm{Fin}}_{C_2,*}$ as having

- objects consist of pairs $(h : U \to V, \ell\colon U \to \{a, m\})$ where $h : U \to V$ is an object of $\underline{\mathrm{Fin}}_{C_2,*}$ and ℓ is a function which is constant on orbits (think 'label') satisfying the condition that C_2 acts freely on $\ell^{-1}(\{a\})$.
- a morphism from $(h : U \to V, \ell_U\colon U \to \{a, m\})$ to $(h' : X \to Y, \ell_X\colon X \to \{a, m\})$ consists of
  - a span (7.3) in $\underline{\mathrm{Fin}}_{C_2,*}$
  - Z is equipped with a labeling ℓ_Z satisfying $\ell_U = g \circ \ell_Z$
  - a t-ordering on f in the sense of Notation 7.1
  - for all orbits $T \subset X$, an ordering $\leq$ on the orbits in the preimage $f^{-1}(T) \subseteq Z$.

  These are required to satisfy the conditions

  - if $x \in X$ is so that $\ell_X(x) = m$, there exists at exactly one orbit, call it Z_x in $f^{-1}(x)$ which ℓ_Z sends to m, which is maximal with respect to the ordering.
  - let $x \in X$ be so that $\ell_X(x) = m$, and write Z_x for the orbit in $f^{-1}(x)$ which ℓ_Z sends to m of the previous bullet point. Then f restricts to an isomorphism $f|_{Z_x}$.

there is a canonical map to $\underline{\mathrm{Fin}}_{C_2,*}$ which is an inclusion on objects and on morphisms forgets the ordering.

Remark 7.9. There is a map of C_2-∞-operads $s\colon \mathcal{HM}^{\otimes} \to \mathbb{E}_p^{\otimes}$ which forgets the data of the labelings and the orderings on the preimages of the orbits (but remembers the t-orderings!). Furthermore, there is a map of C_2-∞-operads $\iota_a\colon \mathrm{Assoc}^{\otimes}_{\mathrm{naive}} \to \mathcal{HM}^{\otimes}$ which sends $U \to V$ to the pair $(U \to V, U \to \{a\} \hookrightarrow \{a, m\})$.

Definition 7.10. Let $\mathcal{C}^{\otimes}$ be a C_2-symmetric monoidal C_2-∞-category. Write $\mathrm{Mod}^h(\mathcal{C})$ denote the C_2-∞-category $\underline{\mathrm{Alg}}_{\mathcal{HM}}(\mathcal{C})$. Note that precomposition with the map ι_a of Remark 7.9 defines a map of C_2-∞-categories $\theta\colon\ \mathrm{Mod}^h(\mathcal{C}) \to \underline{\mathrm{Assoc}_{\mathrm{naive}}}(\mathcal{C})$.

Remark 7.11. Let $\mathcal{C}^{\otimes}$ be the C_2-symmetric monoidal C_2-∞-category of C_2-spectra equipped with the Hill–Hopkins–Ravenel norm. In view of Remark 7.6, we may regard an $\mathcal{O}_{C_2}^{\mathrm{op}}$-object of $\mathrm{Mod}^h\left(\underline{\mathrm{Sp}}^{C_2}\right)$ as consisting of a pair (A, M) where A is an $\mathbb{E}_1$-algebra in Sp^{BC_2} and M is an $N^{C_2}A$-module in Sp^{BC_2}. Precomposition with the maps of Remark 7.9 induces a commutative diagram

$$\begin{tikzcd} & \mathrm{Mod}^h \arrow[d] \\ \underline{\mathrm{Alg}}_{\mathbb{E}_p}\left(\underline{\mathrm{Sp}}^{C_2}\right) \arrow[ur] \arrow[r] & \underline{\mathrm{Alg}}_{\underline{\mathrm{Assoc}}_{\mathrm{naive}}}(\mathcal{C}) \end{tikzcd}$$

Construction 7.12. Recall that by [NS22, Proposition 2.8.7(1)], there is an equivalence $\underline{\mathrm{Alg}}_{\underline{\mathrm{Assoc}}_{\mathrm{naive}}}\left(\underline{\mathrm{Sp}}^{C_2}\right) \simeq \mathrm{Fun}^{\otimes}_{C_2}\left(\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}), \left(\underline{\mathrm{Sp}}^{C_2}\right)^{\otimes}\right)$. The assignment $S_+ \mapsto [S \times C_2 \sqcup \{*\} \to C_2/C_2]$ sends a finite pointed set to a finite pointed set with free C_2-action (away from the basepoint), and for any tuple $\left(f\colon S_+ \to T_+, \left(\leq_t \text{ ordering on } f^{-1}(\{t\})\right)_{t\in T}\right)$ (which defines a morphism in $\mathrm{Assoc}^{\otimes}$, see [Lur17, Remark 4.1.1.4]), we have a span

$$\begin{tikzcd} S \times C_2 \sqcup \{*\} \arrow[d] & f^{-1}(T) \times C_2 \sqcup \{*\} \arrow[l] \arrow[r, "f|_{f^{-1}(T)} \times \mathrm{id}_{C_2}"] \arrow[d] & T \times C_2 \sqcup \{*\} \arrow[d] \\ C_2/C_2 \arrow[r, equal] & C_2/C_2 \arrow[r, equal] & C_2/C_2 \end{tikzcd}$$

and for each C_2-orbit $Z \simeq \{t\} \times C_2 \subset T \times C_2$, an ordering on the orbits in $\left(f|_{f^{-1}(T)} \times \mathrm{id}_{C_2}\right)^{-1}(Z)$. In other words, we have a functor

$$q_{-1} \colon \mathrm{Assoc}^{\otimes} \to (\underline{\mathrm{Assoc}}_{\mathrm{naive}})_{C_2/C_2} \tag{7.13}$$

where $(-)_{C_2/C_2}$ denotes the non-parametrized fiber. Now recall that $\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}) \simeq \underline{\mathrm{Assoc}}_{\mathrm{naive}} \times_{\underline{\mathrm{Fin}}_{C_2,*}} Ar^{\mathrm{act}}(\underline{\mathrm{Fin}}_{C_2,*})$ [NS22, Definition 2.8.4]. Since the identity map on any object in $\underline{\mathrm{Fin}}_{C_2,*}$ is an active arrow, (7.13) induces a map

$$q_0 \colon \mathrm{Assoc}^{\otimes} \to \mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}) \tag{7.14}$$

Observe that $\mathrm{Assoc}^{\otimes} \times \{0\} \to \mathrm{Assoc}^{\otimes} \times \Delta^1$ is left anodyne by (the dual to) [Lur09, Corollary 2.1.2.7]. It follows from (the dual to) [Lur09, Corollary 2.4.2.5; Lur24, Tag 01VF Theorem 5.2.1.1(1')] that there exists an essentially unique extension

$$q \colon \Delta^1 \times \mathrm{Assoc}^{\otimes} \to \mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}) \tag{7.15}$$

of (7.14) which sends $(0 < 1, \mathrm{id}_A)$ to a morphism which is cocartesian with respect to the structure map $\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}) \to \underline{\mathrm{Fin}}_{C_2,*}$. Write p for the composite $\Delta^1 \times \mathrm{Assoc}^{\otimes} \to \mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}) \to \underline{\mathrm{Fin}}_{C_2,*}$. Restriction along the functor of (7.15) defines

$$\mathrm{Fun}^{\otimes}_{C_2}\left(\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}), \left(\underline{\mathrm{Sp}}^{C_2}\right)^{\otimes}\right) \to \mathrm{Fun}_{\Delta^1 \times \mathrm{Assoc}^{\otimes}}\left(\Delta^1 \times \mathrm{Assoc}^{\otimes}, p^*\left(\underline{\mathrm{Sp}}^{C_2}\right)\right). \tag{7.16}$$

Observe that $p^*\left(\underline{\mathrm{Sp}}^{C_2}\right) \to \Delta^1 \times \mathrm{Assoc}^{\otimes}$ is a Δ^1-cocartesian family of monoidal ∞-categories in the sense of [Lur17, Definition 4.8.3.1], and the image of the aforementioned restriction functor consists of Δ^1-cocartesian families of associative algebra objects in $p^*\left(\underline{\mathrm{Sp}}^{C_2}\right)$. In particular, by [Lur17, Remark 4.8.3.7], the aforementioned restriction functor refines to

$$\mathrm{Fun}^{\otimes}_{C_2}\left(\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}), \left(\underline{\mathrm{Sp}}^{C_2}\right)^{\otimes}\right) \to \mathrm{Fun}\left(\Delta^1, \mathrm{Cat}^{\mathrm{Alg}}_{\infty}\right).$$

By construction, its image in $\mathrm{Fun}\left(\Delta^1, \mathrm{Alg}\left(\mathrm{Cat}_{\infty}\right)\right)$ consists of the single arrow $N^{C_2} \colon \mathrm{Sp} \to \mathrm{Sp}^{C_2}$.

Construction 7.17. Recall that by [NS22, Proposition 2.8.7(1)], there is an equivalence ${}^{\sigma}\mathrm{Mod}\left(\underline{\mathrm{Sp}}^{C_2}\right) \simeq \mathrm{Fun}^{\otimes}_{C_2}\left(\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\mathcal{HM}^{\otimes}), \left(\underline{\mathrm{Sp}}^{C_2}\right)^{\otimes}\right)$.

Choose a C_2-set U on which C_2 acts freely and transitively, and fix an ordering $\leq$ on U. Define a functor

$$\begin{aligned}
\iota_m \colon \mathcal{LM}^{\otimes} &\to \left(\mathcal{HM}^{\otimes}\right)_{C_2/C_2} \\
(\langle n \rangle, S) &\mapsto \left((\langle n \rangle^{\circ} \setminus S) \times U \sqcup S_+ \to C_2/C_2, \ell(i) = \begin{cases} a & i \in (\langle n \rangle^{\circ} \setminus S) \times U \\ m & i \in S \end{cases}\right) \\
(\alpha \colon (\langle n \rangle, S) \to (\langle m \rangle, T)) &\mapsto \left(\alpha|_{(\langle n \rangle^{\circ} \setminus S)} \times \mathrm{id}_U\right) \sqcup \alpha_S
\end{aligned}$$

where $\left(\mathcal{HM}^{\otimes}\right)_{C_2/C_2}$ denotes the fiber of $\mathcal{HM}^{\otimes}$ over C_2/C_2. To define ι_m, we must give a t-ordering on $\iota_m(\alpha)$ and, for all orbits $Z \subseteq (\langle m \rangle^{\circ} \setminus T) \times U \sqcup T$, an ordering on the orbits in the preimage $\iota_m(\alpha)^{-1}(Z)$. If $Z \simeq \{j\} \times U \subseteq (\langle m \rangle^{\circ} \setminus T) \times U$ or $Z = \{j\} \subseteq T$, then there is a canonical bijection from the orbits in $\iota_m(\alpha)^{-1}(Z)$ and $\alpha^{-1}(j)$; the given ordering on $\alpha^{-1}(j)$ induces an ordering on the orbits of $\iota_m(\alpha)^{-1}(Z)$. If $Z = \{t\} \subseteq T$, then we use the given ordering on U to endow $\iota_m(\alpha)$ with a t-ordering.

Now recall that $\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\mathcal{HM}^{\otimes}) \simeq \mathcal{HM}^{\otimes} \times_{\underline{\mathrm{Fin}}_{C_2,*}} Ar^{\mathrm{act}}(\underline{\mathrm{Fin}}_{C_2,*})$ [NS22, Definition 2.8.4]. Notice that the fiber of $\underline{\mathrm{Fin}}_{C_2,*}$ over C_2/C_2 has a final object given by the identity arrow at $[C_2/C_2 = C_2/C_2]$ and for every object $[U \to C_2/C_2]$ in $\underline{\mathrm{Fin}}_{C_2,*}$, the unique arrow from $[U \to C_2/C_2]$ to $[C_2/C_2 \to C_2/C_2]$, hence the composite $\mathcal{LM}^{\otimes} \to \mathcal{HM}^{\otimes}_{C_2/C_2} \to \left(\underline{\mathrm{Fin}}_{C_2,*}\right)_{C_2/C_2}$ lifts to $\mathcal{LM}^{\otimes} \to \mathcal{HM}^{\otimes}_{C_2/C_2} \to \left(\underline{\mathrm{Fin}}_{C_2,*}\right)_{C_2/C_2/[\mathrm{id}_{C_2/C_2}]} \to Ar^{\mathrm{act}}(\underline{\mathrm{Fin}}_{C_2,*})$. This defines a functor

L: observe that r sends inert maps to inert maps? What does it mean for a morphism

$$r\colon \mathcal{LM}^{\otimes} \to \mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\mathcal{HM}^{\otimes}) \simeq \mathcal{HM}^{\otimes} \times_{\underline{\mathrm{Fin}}_{C_2,*}} Ar^{\mathrm{act}}(\underline{\mathrm{Fin}}_{C_2,*})\,. \tag{7.18}$$

Writing ι_a for the canonical inclusion $\mathrm{Assoc}^{\otimes} \to \mathcal{LM}^{\otimes}$, there is an equivalence[2]

$$r \circ \iota_a = p \circ (\iota_1 \times \mathrm{id}_{\mathrm{Assoc}^{\otimes}}) =: p_1\,. \tag{7.19}$$

Restriction along the functor (7.18) sends maps of C_2-∞-operads to maps of ordinary ∞-operads, hence it defines

$$\mathrm{Fun}^{\otimes}_{C_2}\left(\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\mathcal{HM}^{\otimes}), \left(\underline{\mathrm{Sp}}^{C_2}\right)^{\otimes}\right) \to \mathrm{Alg}_{\mathcal{LM}/\mathrm{Assoc}}\left(p_1^*\left(\underline{\mathrm{Sp}}^{C_2,\otimes}\right)\right) \to \mathrm{Alg}_{/\mathrm{Assoc}}\left(p_1^*\left(\underline{\mathrm{Sp}}^{C_2,\otimes}\right)\right) \tag{7.20}$$

in the notation of [Lur17, Definition 2.1.3.1]. Now observe that $p_1^*\left(\underline{\mathrm{Sp}}^{C_2,\otimes}\right)$ is the ordinary ∞-category of C_2-spectra, regarded as a monoidal ∞-category via smash product, and there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\otimes}_{C_2}\left(\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\mathcal{HM}^{\otimes}), \left(\underline{\mathrm{Sp}}^{C_2}\right)^{\otimes}\right) \arrow[r] \arrow[d] & \mathrm{LMod}\left(\mathrm{Sp}^{C_2,\otimes}\right) \arrow[r] \arrow[d] & \widetilde{\mathrm{Cat}}^{\mathrm{ex}}_{\infty} \arrow[d] \\
\mathrm{Fun}^{\otimes}_{C_2}\left(\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}), \left(\underline{\mathrm{Sp}}^{C_2}\right)^{\otimes}\right) \arrow[r] & \mathrm{Alg}_{\mathbb{E}_1}\left(\mathrm{Sp}^{C_2}\right) \arrow[r] & \mathrm{Cat}^{\mathrm{ex}}_{\infty}
\end{tikzcd} \tag{7.21}$$

where $\widetilde{\mathrm{Cat}}^{\mathrm{ex}}_{\infty} \to \mathrm{Cat}^{\mathrm{ex}}_{\infty}$ is the restriction of the universal cocartesian fibration to $\mathrm{Cat}^{\mathrm{ex}}_{\infty} \subseteq \mathrm{Cat}_{\infty}$, the left vertical arrow is restriction along the map $\underline{\mathrm{Assoc}}_{\mathrm{naive}} \to \mathcal{HM}^{\otimes}$ of Remark 7.9, and the left horizontal arrow is (7.16) composed with restriction along $\mathrm{Assoc} \times \{1\} \hookrightarrow \mathrm{Assoc} \times \Delta^1$. The right hand square is a pullback by [Lur17, Corollary 4.2.3.7(3)] and the straightening-unstraightening equivalence.

There is a canonical forgetful functor $U\colon \mathrm{Cat}^{\mathrm{Mod}}_{\infty} \to \mathrm{Cat}_{\infty}$. Combining Constructions 7.12 and 7.17 and composing with the forgetful functor U, we obtain a commutative diagram

$$\begin{tikzcd}
{}^{\sigma}\mathrm{Mod} \arrow[rrr] \arrow[d] & & & \widetilde{\mathrm{Cat}}^{\mathrm{ex}}_{\infty} \arrow[d] \\
\underline{\mathrm{Alg}}_{\underline{\mathrm{Assoc}}_{\mathrm{naive}}}\left(\underline{\mathrm{Sp}}^{C_2}\right) \arrow[r] & \mathrm{Fun}\left(\Delta^1, \mathrm{Cat}_{\infty}\right) \arrow[r, "\mathrm{ev}_1"] & \mathrm{Cat}_{\infty} & \mathrm{Cat}^{\mathrm{ex}}_{\infty} \arrow[l]
\end{tikzcd} \tag{7.22}$$

where the lower arrow is Construction 7.12 and Θ is from [Lur17, Construction 4.8.3.24].

Lemma 7.23. *Write* $\widetilde{\mathrm{Cat}}^{\mathrm{ex}}_{\infty} \to \mathrm{Cat}^{\mathrm{ex}}_{\infty}$ *for the restriction of the universal cocartesian fibration to* $\mathrm{Cat}^{\mathrm{ex}}_{\infty} \subseteq \mathrm{Cat}_{\infty}$*. Consider the cocartesian fibration* $\mathrm{Cat}^{\mathrm{ex}}_{\infty,(-)^{\mathrm{op}}/\mathrm{Sp}} \to \mathrm{Cat}_{\infty}$ *classified by the functor* $\mathrm{Fun}^{\mathrm{ex}}\left((-)^{\mathrm{op}}, \mathrm{Sp}\right)$*, with cocartesian transport given by left Kan extension. Then there is a functor* $\widetilde{\mathrm{Cat}}^{\mathrm{ex}}_{\infty} \to \mathrm{Cat}^{\mathrm{ex}}_{\infty,(-)^{\mathrm{op}}/\mathrm{Sp}}$ *over* $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$ *which preserves cocartesian edges.*

L: replace $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$ by 'large' stable ∞-categories

Proof.

L: inspired by Yonatan's answer here

Write $\mathrm{RFib} \subset \mathrm{Ar}\,(\mathrm{Cat}) \times_{\mathrm{Cat}} \mathrm{Cat}^{\mathrm{ex}}_{\infty}$ be the full subcategory of the arrow category spanned by the right fibrations. Write $\mathrm{RFib}_* \subseteq \mathrm{RFib}$ for the full subcategory on those arrows $\mathcal{C} \to \mathcal{D}$ so that $\mathcal{C}$ has a final object. The maps $\mathrm{RFib}_* \to \mathrm{Cat}^{\mathrm{ex}}_{\infty}$, $\mathrm{RFib} \to \mathrm{Cat}^{\mathrm{ex}}_{\infty}$ which send an arrow to its target are cocartesian fibrations; in particular by [CH22, Corollary 6.5], the former is equivalent to the *universal* cocartesian fibration $\mathrm{RFib}_* \simeq \widetilde{\mathrm{Cat}}^{\mathrm{ex}}_{\infty}$. For $\mathcal{C} \in \mathrm{Cat}^{\mathrm{ex}}_{\infty}$, write $\mathrm{RFib}_*(\mathcal{C})$ for the ∞-category of right fibrations over $\mathcal{C}$ with a terminal object. Corollary A.32 of [GHN17] implies that there is a functor $\mathrm{Cat} \to \mathrm{Fun}\left(\Delta^1, \widehat{\mathrm{Cat}}\right)$ which

L: further discussion here

[2]Actually, need to postcompose right hand side with the canonical map $\mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}(\underline{\mathrm{Assoc}}_{\mathrm{naive}}) \to \mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}}\left(\mathcal{HM}^{\otimes}\right)$ induced by Remark 7.9.

sends an ∞-category $\mathcal{C}$ to the equivalence $\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathrm{Gpd}_\infty) \simeq \mathrm{RFib}_*(\mathcal{C})$, where Fun' denotes functors which preserve all limits.

L: In progress: Now use $\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ to construct a map of cocartesian fibrations.

□

Construction 7.24. Combining (7.22) with Lemma 7.23 and Remark 7.11, we obtain a commutative diagram

L: replace $\mathrm{Cat}_\infty^{\mathrm{ex}}$ by *'large'* stable ∞-categories, add remark about $\mathrm{Cat}^{\mathrm{epoly}}$ earlier

$$\begin{tikzcd} & & \int \mathrm{Fun}^{\mathrm{ex}}((-)^{\mathrm{op}}, \mathrm{Sp}) \arrow[d] \\ \mathrm{CAlg}^{\mathrm{p}} \arrow[r] \arrow[urr, dashed, bend left] & \left(\mathrm{Cat}^{\mathrm{epoly}}\right)^{\Delta^1} \times_{\mathrm{Cat}^{\mathrm{epoly}}} \mathrm{Cat}_\infty^{\mathrm{ex}} \arrow[r] & \mathrm{Cat}_\infty^{\mathrm{ex}} \end{tikzcd} \tag{7.25}$$

Define $\mathrm{Cat}^{\mathrm{epoly}}$ to be a large ∞-category of large stable ∞-categories and reduced excisively polynomial functors between them. There is a functor $\mathrm{Fun}^{\mathrm{epoly}}((-)^{\mathrm{op}}, \mathrm{Sp})\colon \mathrm{Cat}^{\mathrm{epoly}} \to \mathrm{Cat}$. Under the non-full subcategory inclusion $\mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Cat}^{\mathrm{epoly}}$, for any $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{ex}}$, there is an inclusion $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \subseteq \mathrm{Fun}^{\mathrm{epoly}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$, where $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ denotes the full subcategory on exact functors. In particular, there is a map of cartesian fibrations

$$\begin{tikzcd} \int \mathrm{Fun}^{\mathrm{ex}}((-)^{\mathrm{op}}, \mathrm{Sp}) \arrow[r] \arrow[d] & \int \mathrm{Fun}^{\mathrm{epoly}}((-)^{\mathrm{op}}, \mathrm{Sp}) \arrow[d] \\ \mathrm{Cat}_\infty^{\mathrm{ex}} \arrow[r] & \mathrm{Cat}^{\mathrm{epoly}} . \end{tikzcd} \tag{7.26}$$

Composing (7.25) with (7.26) and noting that the map $\left(\mathrm{Cat}^{\mathrm{epoly}}\right)^{\Delta^1} \to \mathrm{Cat}^{\mathrm{epoly}}$ defining the middle term in (7.25) is given by evaluation at the target, we obtain a diagram

$$\begin{tikzcd} & & \int \mathrm{Fun}^{\mathrm{epoly}}((-)^{\mathrm{op}}, \mathrm{Sp}) \arrow[dd] \\ \mathrm{CAlg}^{\mathrm{p}} \times \{1\} \arrow[r, "M"] \arrow[d] \arrow[urr, bend left, "M_*"] & \left(\mathrm{Cat}^{\mathrm{epoly}}\right)^{\Delta^1} \times \{1\} \arrow[d] & \\ \mathrm{CAlg}^{\mathrm{p}} \times \Delta^1 \arrow[r, "M \times \mathrm{id}_{\Delta^1}"] & \left(\mathrm{Cat}^{\mathrm{epoly}}\right)^{\Delta^1} \times \Delta^1 \arrow[r, "\mathrm{ev}"] & \mathrm{Cat}^{\mathrm{epoly}} . \end{tikzcd} \tag{7.27}$$

Observe that in the bottom row of (7.27), given a Poincaré ring A, the morphism $(\mathrm{id}_A, 0 \to 1)$ from $(A, 0)$ to $(A, 1)$ on the lower left is sent to the morphism $N^{C_2}\colon \mathrm{Mod}_{A^e}(\mathrm{Sp}) \to \mathrm{Mod}_{N^{C_2}A}\left(\mathrm{Sp}^{C_2}\right)$ in the lower right. Now observe that $\{1\} \to \Delta^1$ and $\mathrm{CAlg}^{\mathrm{p}} \times \{1\} \to \mathrm{CAlg}^{\mathrm{p}} \times \Delta^1$ are right anodyne by [Lur09, Corollary 2.1.2.7]. It follows from [Lur09, Corollary 2.4.2.5; Lur24, Tag 01VF Theorem 5.2.1.1(1)] that there exists a functor $\overline{M}_*\colon \mathrm{CAlg}^{\mathrm{p}} \times \Delta^1 \to \int \mathrm{Fun}^{\mathrm{epoly}}((-)^{\mathrm{op}}, \mathrm{Sp})$ in (7.27) extending M_*. Moreover, by [Lur24, Tag 01VF Theorem 5.2.1.1(2)], we see that $\overline{M}(A, 0)$ is the cartesian transport of $M(A, 1) = \hom_{N^{C_2}A}(-, A)$ along the functor

$$\mathrm{Fun}^{\mathrm{epoly}}\left(\mathrm{Mod}_{N^{C_2}A}\left(\mathrm{Sp}^{C_2}\right)^{\mathrm{op}}, \mathrm{Sp}\right) \to \mathrm{Fun}^{\mathrm{epoly}}\left(\mathrm{Mod}_{A^e}(\mathrm{Sp}), \mathrm{Sp}\right)$$

classified by $\mathrm{ev} \circ (M \times \mathrm{id}_{\Delta^1})$. Consider the restriction of $\overline{M}_*$ to $\mathrm{CAlg}^{\mathrm{p}} \times \{0\}$. Since the restriction of $M \times \mathrm{id}_{\Delta^1}$ to $\mathrm{CAlg}^{\mathrm{p}} \times \{0\}$ agrees with $\mathrm{Mod}_{(-)^e}$ by Construction 7.12, we have that the image of $(M \times \mathrm{id}_{\Delta^1})|_{\mathrm{CAlg}^{\mathrm{p}} \times \{0\}}$ factors through the inclusion $\mathrm{Cat}_\infty^{\mathrm{ex}} \subseteq \mathrm{Cat}^{\mathrm{epoly}}$. It follows that we may regard $\overline{M}_*|_{\mathrm{CAlg}^{\mathrm{p}} \times \{0\}}$ as having codomain the total space of the cartesian fibration $\int \mathrm{Fun}^{\mathrm{epoly}}((-)^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Cat}_\infty^{\mathrm{ex}}$. L: todo: size issue

We claim that $\overline{M}_*|_{\mathrm{CAlg}^{\mathrm{p}} \times \{0\}}$ admits an essentially unique factorization through the cartesian fibration $\int \mathrm{Fun}^q(-) \to \mathrm{Cat}_\infty^{\mathrm{ex}}$. Since the natural transformation $\mathrm{Fun}^q(-) \subseteq \mathrm{Fun}^{\mathrm{epoly}}((-)^{\mathrm{op}}, \mathrm{Sp})$ of functors $\mathrm{Cat}_\infty^{\mathrm{ex}} \to$

Cat is given pointwise by full inclusions, the induced map $\int \mathrm{Fun}^q(-) \to \int \mathrm{Fun}^{\mathrm{epoly}}\left((-)^{\mathrm{op}}, \mathrm{Sp}\right)$ of cartesian fibrations over $\mathrm{Cat}_\infty^{\mathrm{ex}}$ is a full inclusion by [Lur09, Proposition 2.4.4.2]. Therefore, to show that $\overline{M}_*|_{\mathrm{CAlg}^{\mathrm{p}} \times \{0\}}$ factors through $\int \mathrm{Fun}^q(-)$, it suffices to check that $\overline{M}_*|_{\mathrm{CAlg}^{\mathrm{p}} \times \{0\}}$ sends objects of $\mathrm{CAlg}^{\mathrm{p}}$ to objects of $\int \mathrm{Fun}^q(-)$. This is true because $\overline{M}_*|_{\mathrm{CAlg}^{\mathrm{p}} \times \{0\}}(A, 0) = \left(\mathrm{Mod}_{A^e}, \hom_{N^{C_2} A}\left(N^{C_2}(-), A\right)\right)$, $N^{C_2} \colon \mathrm{Mod}_{A^e} \to \mathrm{Mod}_{N^{C_2} A}$ is quadratic, and a composite of a reduced quadratic functor and an exact functor is reduced and quadratic.

L: reference for last fact?

In sum, we have produced a commutative diagram like so:

$$\begin{tikzcd}
 & & \int \mathrm{Fun}^q(-) \arrow[d] \arrow[dr] & \\
\mathrm{CAlg}^{\mathrm{p}} \times \{0\} \arrow[urr, bend left, "\overline{M}_*|_{\mathrm{CAlg}^{\mathrm{p}} \times \{0\}}"] \arrow[rr, "\mathrm{ev}_0 \circ M \simeq \mathrm{Mod}_{(-)^e}"] \arrow[d] & & \mathrm{Cat}_\infty^{\mathrm{ex}} \arrow[dr] & \int \mathrm{Fun}^{\mathrm{epoly}}\left((-)^{\mathrm{op}}, \mathrm{Sp}\right) \arrow[d] \\
\mathrm{CAlg}^{\mathrm{p}} \times \Delta^1 \arrow[r, "M \times \mathrm{id}_{\Delta^1}"] & \left(\mathrm{Cat}^{\mathrm{epoly}}\right)^{\Delta^1} \times \Delta^1 \arrow[rr, "\mathrm{ev}"] & & \mathrm{Cat}^{\mathrm{epoly}} .
\end{tikzcd} \tag{7.28}$$

L: Todo: pass to compact/dualizable modules.

Observation 7.29. Consider the cocartesian fibration $p \colon \mathrm{LMod}\,(\mathrm{Sp}) \to \mathbb{E}_1\,\mathrm{Alg}\,(\mathrm{Sp})$ [Lur17, Corollary 4.2.3.7(3)].

Write $\mathrm{LMod}^\omega\,(\mathrm{Sp})$ for the full subcategory of $\mathrm{LMod}\,(\mathrm{Sp})$ on those pairs (A, M) so that M is ω-compact as an A-module. Observe that given any $(A, M) \in \mathrm{LMod}^\omega$ and any map $f \colon A \to B$ in $\mathbb{E}_1\,\mathrm{Alg}\,(\mathrm{Sp})$, then p-cocartesian pushforward of (A, M) along f is equivalent to $(B, B \otimes_A M) \in \mathrm{LMod}^\omega$. In other words, p-cocartesian pushforward preserves the subcategory $\mathrm{LMod}^\omega\,(\mathrm{Sp})$ because the functor $B \otimes_A (-)$ admits a right adjoint which preserves ω-filtered colimits. It follows that the restriction $\overline{p}$ of p to $\mathrm{LMod}^\omega\,(\mathrm{Sp})$ exhibits $\mathrm{LMod}^\omega\,(\mathrm{Sp}) \to \mathbb{E}_1\,\mathrm{Alg}\,(\mathrm{Sp})$ as a cocartesian fibration, and the inclusion $\mathrm{LMod}^\omega\,(\mathrm{Sp}) \to \mathrm{LMod}\,(\mathrm{Sp})$ takes $\overline{p}$-cocartesian edges to p-cocartesian edges. Unstraightening the inclusion $\mathrm{LMod}^\omega\,(\mathrm{Sp}) \to \mathrm{LMod}\,(\mathrm{Sp})$, we obtain a functor

$$\mathbb{E}_1\,\mathrm{Alg}\,(\mathrm{Sp}) \to \mathrm{Fun}\left(\Delta^1, \mathrm{Cat}_\infty^{\mathrm{st}}\right)$$

which sends an $\mathbb{E}_1$-algebra A to the inclusion $\mathrm{LMod}_A^\omega \to \mathrm{LMod}_A$. Here $\mathrm{Cat}_\infty^{\mathrm{st}}$ is the large ∞-category of stable ∞-categories and exact functors between them.

7.1 An operadic description of Poincaré rings

We show that algebras over $\mathbb{E}_p$ are equivalent to Poincaré rings defined diagrammatically.

Let K denote the ∞-categorical nerve of the 1-category

$$\begin{tikzcd}
 & 3 \arrow[d] & \\
 & & 2 \arrow[ul] \arrow[dd] \\
 & 5 & \\
1 \arrow[uuur, bend left] \arrow[ur] \arrow[rr] & & 4 \arrow[ul]
\end{tikzcd}$$

in which all triangles and squares commute.

Definition 7.30. Consider the diagram $\mathcal{P}\colon K \to \mathrm{Cat}_\infty$

$$\begin{tikzcd}
& \mathrm{Fun}\left(\Delta^1, \mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp})^{BC_2}\right) \arrow[dd, "\mathrm{ev}_0{,}\mathrm{ev}_1"] & \\
& & \mathrm{Fun}\left(\Delta^1, \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)\right) \arrow[ul, "(-)^e"'] \arrow[dd, "\mathrm{ev}_0{,}\mathrm{ev}_1"] \\
& \mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp})^{BC_2} \times \mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp})^{BC_2} & \\
\mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right) \arrow[uuur, bend left, "m\circ(-^e)"] \arrow[ur] \arrow[rr, "N^{C_2}(-^e)\times\mathrm{id}"'] & & \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right) \times \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right) \arrow[ul, "(-)^e\times(-)^e"]
\end{tikzcd} \tag{7.31}$$

where m is the functor of [Yan23, Construction 3.1] precomposed with the canonical map $\Delta^1 \to \mathcal{O}_{C_2}$. Observe that the right-hand trapezoid commutes essentially by definition, and the leftmost triangle commutes because $\left(N^{C_2}A\right)^e \simeq (A^e)^{\otimes 2}$. Consider the limit of the diagram

$$\mathrm{CAlg}^{\mathrm{p}\prime}\left(\mathrm{Sp}^{C_2}\right) := \lim_K \mathcal{P}. \tag{7.32}$$

There is a canonical forgetful functor $G'\colon \mathrm{CAlg}^{\mathrm{p}}\left(\mathrm{Sp}^{C_2}\right) \to \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)$ given by the canonical projection to the lower left corner of the diagram.

Observation 7.33. Let $F\colon K \to \mathcal{C}$ be any diagram and let $\mathcal{C}$ be any ∞-category with finite limits. Then there is a canonical equivalence $\lim_K F \simeq F(1) \times_{\left(F(3)\times_{F(5)}F(4)\right)} F(2)$.

Observation 7.34. By the recollement decomposition of Sp^{C_2}, $\lim_K \mathcal{P}$ is equivalent to the limit of the diagram[3]

$$\begin{tikzcd}
& \mathrm{Fun}\left(\Delta^1, \mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp})\right) \arrow[dd, "\mathrm{ev}_0{,}\mathrm{ev}_1"] & \\
& & \mathrm{Fun}\left(\Delta^1\times\Delta^1, \mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp})\right) \arrow[ul, "\mathrm{res}_{\{1\}\times\Delta^1}"'] \arrow[dd, "(\mathrm{res}_{\Delta^1\times\{0\}}{,}\mathrm{res}_{\Delta^1\times\{1\}})"] \\
& \mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp}) \times \mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp}) & \\
\mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right) \arrow[uuur, bend left, "m^{tC_2}\circ(-^e)"] \arrow[ur] \arrow[rr, "N^{C_2}(-^e)\times\mathrm{id}"'] & & \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)^{\Delta^1} \times \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)^{\Delta^1} \arrow[ul, "\mathrm{ev}_1\times\mathrm{ev}_1"]
\end{tikzcd} \tag{7.35}$$

By Observation 7.33 and the fact that all commutative trapezoids below *except* the upper left are pullbacks:

$$\begin{tikzcd}
\mathbb{E}\operatorname{Alg}(\mathrm{Sp})^{\Delta^1\times\Delta^1} \arrow[r, "\mathrm{ev}_{00\to10\to11}"] \arrow[d, "(\mathrm{res}_{\Delta^1\times\{0\}}{,}\mathrm{res}_{\Delta^1\times\{1\}})"'] & \mathbb{E}\operatorname{Alg}(\mathrm{Sp})^{\Delta^2} \arrow[rr, "d_0"] \arrow[d, "(d_2{,}\mathrm{ev}_2)"] & & \mathbb{E}\operatorname{Alg}(\mathrm{Sp})^{\Delta^1} \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] & \\
\left(\mathbb{E}\operatorname{Alg}(\mathrm{Sp})^{\Delta^1}\right)^{\times 2} \arrow[r, "(\mathrm{id}{,}\mathrm{ev}_1)"] \arrow[d, "\pi_2"] & \mathbb{E}\operatorname{Alg}(\mathrm{Sp})^{\Delta^1} \times \mathbb{E}\operatorname{Alg}(\mathrm{Sp}) \arrow[rr, "(\mathrm{ev}_1{,}\mathrm{id})"] \arrow[d, "\pi_2"] \arrow[dr, "\pi_1"] & & \mathbb{E}\operatorname{Alg}(\mathrm{Sp})^{\times 2} \arrow[dr, "\pi_1"] & \\
\mathbb{E}\operatorname{Alg}(\mathrm{Sp})^{\Delta^1} \arrow[r, "\mathrm{ev}_1"] & \mathbb{E}\operatorname{Alg}(\mathrm{Sp}) & \mathbb{E}\operatorname{Alg}(\mathrm{Sp})^{\Delta^1} \arrow[rr, "\mathrm{ev}_1"] & & \mathbb{E}\operatorname{Alg}(\mathrm{Sp}),
\end{tikzcd} \tag{7.36}$$

[3] 'res' means restriction.

(Here π_1, π_2 denote projections onto the first and second factors, resp.) it follows that

$$\lim_K \mathcal{P} \simeq \left(\mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)\right) \times_{\left(\mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp})^{\Delta^2} \times_{\mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp})} \mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp})^{\Delta^1}\right)} \mathbb{E}_\infty \operatorname{Alg}(\mathrm{Sp})^{\Delta^1 \times \Delta^1}.$$

Thus $\lim_K \mathcal{P}$ is equivalent to $\mathrm{CAlg}^{\mathrm{p}}$ as defined in main.tex (see the limit of diagram `diagram:Poincare_ring_alternate_diag` in Remark 3.4).

Remark 7.37. There is a map of C_2-∞-operads $\mathrm{Com}_{\mathcal{O}^{\simeq}_{C_2}} \to \mathbb{E}_p$. By [Yan23, Lemma 4.24], this induces a functor $G' \colon \mathbb{E}_p \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right) \to \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)$, which by an argument similar to Proposition 4.23 *loc.cit.* is monadic.

Construction 7.38. Define a functor $\gamma \colon \mathbb{E}_p \operatorname{Alg}(\mathrm{Sp}^{C_2}) \to \mathrm{CAlg}^{\mathrm{p}}$ (similar to [Yan23, §4.1]).

Theorem 7.39. *The functor of Construction 7.38 is an equivalence.*

Proof. The proof is nearly identical to that of [Yan23, Theorem 4.21], but we include it here for the reader's convenience. Consider the commutative diagram of forgetful functors

$$\begin{tikzcd} \mathbb{E}_p \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right) \arrow[rr, "\gamma"] \arrow[dr, "G"'] & & \mathrm{CAlg}^{\mathrm{p}}\left(\mathrm{Sp}^{C_2}\right) \arrow[dl, "G'"] \\ & \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right) & \end{tikzcd} \tag{7.40}$$

where G is from Remark 7.37 and G' is from Definition 7.30.

The functor G is monadic by Remark 7.37. The functor G' is monadic by Proposition 7.41. Now for any $A \in \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)$, the unit $A \to \gamma F(A)$ of Theorem 7.42 induces an equivalence $F'(A) \simeq \gamma F(A)$ by Corollary 7.47. The result follows from [Lur17, Proposition 4.7.3.16]. ☐

Proposition 7.41. *The forgetful functor $G' \colon \mathrm{CAlg}^{\mathrm{p}}\left(\mathrm{Sp}^{C_2}\right) \to \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)$ of Definition 7.30 is monadic.*

Proof.

L: Similar to [Yan23, Proposition 3.23], except there I forgot to show that G' *has* a left adjoint (oops!). That's ok; it does, and we showed that the relevant functor for $\mathrm{CAlg}^{\mathrm{p}}$ does in `main.tex`.

☐

Theorem 7.42. *Let $A \in \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)$ and consider the adjunction $F \dashv G$ between $\mathbb{E}_p$ and $\mathbb{E}_\infty$-algebras in Sp^{C_2}.*

(1) The underlying C_2-spectrum of the free $\mathbb{E}_p$ algebra $F(A)$ on A is given by

$$F(A) \simeq \begin{tikzcd} & A^{\varphi C_2} \otimes A^e \arrow[d, "s_A \otimes \nu_A"] \\ A^e \arrow[r, mapsto] & A^{tC_2} \end{tikzcd} \tag{7.43}$$

where u is the unit, $s_A : A^{\varphi C_2} \to A^{tC_2}$ is the structure map, and ν_A is the twisted Tate-valued Frobenius.

(2) There is a canonical $\mathbb{E}_\infty$ ring map $\eta_A \colon A \to GF(A)$ given by $\mathrm{id}_{A^{\varphi C_2}} \otimes (\eta_{A^e} : \mathbb{S}^0 \to A^e)$ on geometric fixed points and the identity on underlying.

Proof. Similar to proof of [Yan23, Theorem 4.15], substituting $\mathrm{Env}_{\mathbb{E}_p, \underline{\mathrm{Fin}}_{C_2,*}}\left(\mathrm{Com}_{\mathcal{O}^{\simeq,\mathrm{op}}_{C_2}}\right)$ for $\mathrm{Com}^{\otimes}_{\mathcal{T}^{\simeq},act} = \mathrm{Env}_{\underline{\mathrm{Fin}}_{C_2,*}, \underline{\mathrm{Fin}}_{C_2,*}}\left(\mathrm{Com}_{\mathcal{O}^{\simeq,\mathrm{op}}_{C_2}}\right)$. Since the indexing diagram is different, one uses Lemma 7.44 instead of Lemma 4.19 *loc. cit.* ☐

Lemma 7.44. *Consider the (non-parametrized) fiber I of $\mathrm{Env}_{\mathbb{E}_p,\underline{\mathrm{Fin}}_{C_2,*}}\left(\mathrm{Com}_{\mathcal{O}^{\simeq,\mathrm{op}}_{C_2}}\right)$ over id_{C_2/C_2}. An object of I can be thought of as a pair $\big([U \to V], \alpha\colon i([U \to V]) \to [\mathrm{id}_{C_2/C_2}]\big)$ where $[U \to V]$ is an object in the fiber of $\mathrm{Com}_{\mathcal{O}^{\simeq,\mathrm{op}}_{C_2}}$ over C_2/C_2, i denotes the inclusion $\mathrm{Com}_{\mathcal{O}^{\simeq,\mathrm{op}}_{C_2}} \to \mathbb{E}_p$, and α is an active arrow in $\mathbb{E}_p$.*

Choose a C_2-set U with free and transitive C_2-action, and fix an ordering $\leq$ on U. This determines a t-ordering on the collapse map $C_2/C_2 \sqcup U \to C_2/C_2$, hence an active arrow β in $\mathbb{E}_p^{\otimes}$. The inclusion

$$\{*\} \xrightarrow{* \mapsto \beta} I$$

is cofinal.

Proof. By [Lur09, Theorem 4.1.3.1], it suffices to show that for all $X \in I$, the category $\{*\} \times_I I_{X/}$ has an initial object. Informally, an object X of I is given by a tuple $(S, \alpha\colon S \to C_2/C_2)$ where S is a finite C_2-set and α is an active arrow in $\left(\mathbb{E}_p^{\otimes}\right)_{C_2/C_2}$. We observe that α is given by a C_2-equivariant map f (in fact, f is unique) and a t-ordering on f in the sense of Notation 7.1. Morphisms are given by tuples; note that a morphism $\psi\colon (S,\alpha) \to (T,\gamma)$ determines a map $S \to T$ in $\left(\mathrm{Com}_{\mathcal{O}^{\simeq}_{C_2}}\right)_{C_2/C_2}$; in particular, the map $S \to T$ must be equivalent to a disjoint union of fold maps. Now the t-ordering on f is equivalent to giving, for each free orbit $Z \subseteq S$, $Z \simeq C_2$, an ordering $\leq_Z$ on Z. For each such Z, there is a unique order-preserving, C_2-equivariant isomorphism $j_Z\colon Z \simeq U$. Write $S \simeq S^f \sqcup S^t$ for the canonical decomposition where C_2 acts freely on S^f and trivially on S^t. Then $S \simeq S^t \sqcup \left(\coprod_{Z \in \mathrm{Orbit}(S^f)} Z\right) \xrightarrow{\nabla \sqcup (\coprod j_Z)} C_2/C_2 \sqcup U$ defines a morphism in $\left(\mathrm{Com}_{\mathcal{O}^{\simeq,\mathrm{op}}_{C_2}}\right)_{C_2/C_2}$ so that the diagram

$$\begin{tikzcd} S \arrow[rr, "\nabla \sqcup (\coprod j_Z)"] \arrow[dr, "\alpha"'] & & C_2/C_2 \sqcup U \arrow[dl, "\beta"] \\ & C_2/C_2 & \end{tikzcd}$$

commutes in $\mathbb{E}_p$; this is the desired initial object of $\{*\} \times_I I_{X/}$. ☐

Proposition 7.45. *Let A be an $\mathbb{E}_\infty$-ring in Sp^{C_2} and let $(B, n_B\colon N^{C_2}B \to B)$ be a Poincaré ring in Sp^{C_2}. Then precomposition with the $\mathbb{E}_\infty$-map $\eta_A : A \to GF(A)$ of Theorem 7.42 induces an equivalence of morphism spaces*

$$\begin{tikzcd} \hom_{\mathrm{CAlg}^{\mathrm{p}}}\big((\gamma F(A), n_{F(A)}\colon N^{C_2}\gamma F(A) \to \gamma F(A)), (B, n_B\colon N^{C_2}B \to B)\big) \arrow[d, "G'"] \arrow[dd, bend left, "\sim"] \\ \hom_{\mathbb{E}_\infty}(G'\gamma F(A), G'(B)) \arrow[d, "\eta^*"] \\ \hom_{\mathbb{E}_\infty}(A, G'(B)). \end{tikzcd} \tag{7.46}$$

where G' is the forgetful functor and γ is the functor of Construction 7.38. That is, $\eta_{(-)}$ is a unit for the functors $(\gamma \circ F, G')$ in the sense of [Lur09, Definition 5.2.2.7].

Corollary 7.47. *The natural transformation $\eta_{(-)}$ exhibits $\gamma \circ F$ as a left adjoint to G'.*

Proof of Proposition 7.45. Notice that

$$\hom_{\mathbb{E}_\infty \mathrm{Alg}(\mathrm{Sp}^{C_2})}(F(A), B) \simeq \hom_{\mathbb{E}_\infty \mathrm{Alg}(\mathrm{Sp}^{C_2})}(A, B) \underset{\hom(A^{tC_2}, B^{tC_2})}{\times} \hom\left(\begin{matrix} A \\ \downarrow \nu_A \\ A^{tC_2} \end{matrix}, \begin{matrix} B^{\varphi C_2} \\ \downarrow \Delta \\ B^{tC_2} \end{matrix}\right)$$

and moreover the composite

$$\hom_{\mathrm{CAlg}^{\mathrm{p}}}(F(A), B) \xrightarrow{G'} \hom_{\mathbb{E}_\infty \mathrm{Alg}(\mathrm{Sp}^{C_2})}(F(A), B) \xrightarrow{\pi_1} \hom_{\mathbb{E}_\infty \mathrm{Alg}(\mathrm{Sp}^{C_2})}(A, B)$$

is equivalent to $\eta^* \circ G'$. Unravelling definitions, we see that given a point $f \in \hom_{\mathbb{E}_\infty \mathrm{Alg}(\mathrm{Sp}^{C_2})}(A, B)$, the fiber of $\eta^* \circ G'$ over f is given by the space of fillings of the below diagram to a commutative diagram $(\Delta^1)^{\times 3} \to \mathbb{E}_\infty \mathrm{Alg}(\mathrm{Sp})$:

$$\begin{tikzcd}
A^e \arrow[rr, "f^e"] \arrow[dr, equal] \arrow[dd] & & B^e \arrow[dr, "n_B"] \arrow[dd] & \\
& A^e \arrow[rr, dotted] \arrow[dd, "\nu_{A^e}"] & & B^{\varphi C_2} \arrow[dd] \\
(A^{\otimes 2})^{tC_2} \arrow[rr, "(f^{\otimes 2})^{tC_2}"] \arrow[dr] & & (B^{\otimes 2})^{tC_2} \arrow[dr] & \\
& A^{tC_2} \arrow[rr, "f^{tC_2}"] & & B^{tC_2}
\end{tikzcd}$$

wherein all but the top and front face are given. This space is contractible. $\square$

8 Categorification and structure–new leaf

Definition 8.1. Define a C_2-∞-operad $\underline{\mathrm{Assoc}}_\sigma^\otimes \to \underline{\mathrm{Fin}}_{C_2,*}$ as having

- objects consist of pairs $(h : U \to V)$ where $h : U \to V$ is an object of $\underline{\mathrm{Fin}}_{C_2,*}$.
- a morphism from $(h : U \to V)$ to $(h' : X \to Y)$ consists of
  - a span (7.3) in $\underline{\mathrm{Fin}}_{C_2,*}$
  - for each $x \in X$, an ordering $\leq_x$ on $f^{-1}(\{x\})$.

  These are required to satisfy the conditions (write $\sigma \in C_2$ for the generator)

  - under the isomorphism $f^{-1}(\{x\}) \simeq f^{-1}(\{\sigma(x)\})$ induced by the C_2-action, the ordering $\leq_x$ is sent to the reverse of the ordering $\leq_{\sigma(x)}$.

there is a canonical map to $\underline{\mathrm{Fin}}_{C_2,*}$ which is an inclusion on objects and on morphisms forgets the data of the orderings.

Remark 8.2. Definition 8.1 is equivalent to the C_2-genuine nerve of the little disks operad $\mathbb{E}_\sigma$ associated to the sign C_2-representation on $\mathbb{R}$ [Hor19, Example 3.9.4].

Observation 8.3. Suppose given a finite C_2-set Z and an ordering $\leq_Z$ so that the C_2-action on Z sends $\leq_Z$ to its reverse. We may associate to this a canonical t-ordering on the map $f \colon Z \to C_2/C_2$: For each free orbit $U \subseteq Z$, take the restriction of $\leq_Z$ to U. This assignment induces a canonical forgetful map of C_2-∞-operads $\underline{\mathrm{Assoc}}_\sigma^\otimes \to \mathbb{E}_p^\otimes$.

Construction 8.4. We will construct a map $\mathrm{Assoc}^\otimes \to \underline{\mathrm{Assoc}}^\otimes_{\sigma, C_2/e}$. Here the subscript C_2/e means we consider the (non-parametrized) fiber over C_2/e.

Observe that if $\nabla \colon (C_2/e)^{\sqcup n} \simeq C_2/e \times \langle n \rangle \to C_2/e$ is the fold map, then a lift of ∇ to a morphism in $\underline{\mathrm{Assoc}}^\otimes_{\sigma, C_2/e}$ is equivalent to the choice of an ordering on $\{1, 2, \ldots, n\} \simeq \nabla^{-1}(\{e\})$; the ordering on $\nabla^{-1}(\{\sigma\})$ is uniquely determined by the ordering on $\nabla^{-1}(\{e\})$. There are functors

$$\begin{aligned}
\iota_e, \iota_\sigma \colon \mathrm{Assoc}^\otimes &\to \underline{\mathrm{Assoc}}^\otimes_{\sigma, C_2/e} \\
\langle n \rangle &\mapsto C_2/e \times \langle n \rangle
\end{aligned}$$

$$\iota_e \colon \big(f \colon \langle n \rangle \to \langle m \rangle, (\leq_i)_{i \in \langle m \rangle^\circ}\big) \mapsto \big(\mathrm{id}_{C_2} \times f \colon \langle n \rangle \to \langle m \rangle, (\leq_i \text{ ordering on } f^{-1}(i) \times \{e\})_{i \in \langle m \rangle^\circ}\big)$$

$$\iota_\sigma \colon \big(f \colon \langle n \rangle \to \langle m \rangle, (\leq_i)_{i \in \langle m \rangle^\circ}\big) \mapsto \big(\mathrm{id}_{C_2} \times f \colon \langle n \rangle \to \langle m \rangle, (\leq_i \text{ ordering on } f^{-1}(i) \times \{\sigma\})_{i \in \langle m \rangle^\circ}\big) .$$

Furthermore, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Assoc}^{\otimes} \arrow[r, "\iota_e"] \arrow[d, "\mathrm{rev}"] & \underline{\mathrm{Assoc}}^{\otimes}_{\sigma, C_2/e} \arrow[d, "\sigma^*"] \\ \mathrm{Assoc}^{\otimes} \arrow[r, "\iota_e"] \arrow[ur, "\iota_\sigma"] & \underline{\mathrm{Assoc}}^{\otimes}_{\sigma, C_2/e} \end{tikzcd} \tag{8.5}$$

where rev is the automorphism considered in [Lur17, Remark 4.1.1.7] and σ^* is the cocartesian pushforward along the nontrivial automorphism of C_2/e in $\mathcal{O}_{C_2}^{\mathrm{op}}$. Now if $\mathcal{C}^{\otimes}$ is any C_2-symmetric monoidal C_2-∞-category, let us write $\mathcal{C}^{e,\otimes}$ for its underlying symmetric monoidal ∞-category with naive C_2-action. Then the aforementioned diagram induces a commutative diagram

$$\begin{tikzcd} & \mathbb{E}_1\,\mathrm{Alg}\left(\mathcal{C}^{e,\otimes}\right) \arrow[d, "\mathrm{rev}", "\sim"'] \\ \mathrm{Alg}_{\underline{\mathrm{Assoc}}^{\otimes}_{\sigma}}\left(\mathcal{C}^{\otimes}\right) \arrow[ur] \arrow[r] & \mathbb{E}_1\,\mathrm{Alg}\left(\mathcal{C}^{e,\otimes}\right)\,. \end{tikzcd}$$

L: Define Azumaya algebras with involution in terms of algebras over $\underline{\mathrm{Assoc}}^{\otimes}_{\sigma}$ instead? I think it is 'better' to use $\underline{\mathrm{Assoc}}^{\otimes}_{\sigma}$ instead of $\mathcal{HM}^{\otimes}$. To define a functor from algebras over $\underline{\mathrm{Assoc}}^{\otimes}_{\sigma}$ to Hermitian ∞-categories, we can define a functor from $\mathcal{LM}^{\otimes}$ to $\underline{\mathrm{Assoc}}^{\otimes}_{\sigma,C_2/C_2}$, then use the same argument as in Construction 7.17.

Choose a C_2-set $U = \{u, v\}$ on which C_2 acts freely and transitively, and fix an ordering $u \leq v$ on U. Define a functor

$$\begin{aligned} \iota_m \colon \mathcal{LM}^{\otimes} &\to \left(\underline{\mathrm{Assoc}}^{\otimes}_{\sigma}\right)_{C_2/C_2} \\ (\langle n\rangle, S) &\mapsto \left((\langle n\rangle^{\circ} \setminus S) \times U \sqcup S_+ \to C_2/C_2, \ell(i) = \begin{cases} a & i \in (\langle n\rangle^{\circ} \setminus S) \times U \\ m & i \in S \end{cases} \right) \\ (\alpha\colon (\langle n\rangle, S) \to (\langle m\rangle, T)) &\mapsto \left(\alpha|_{(\langle n\rangle^{\circ} \setminus S)} \times \mathrm{id}_U \right) \sqcup \alpha_S\,. \end{aligned}$$

To define ι_m, we must give, for each $t \in (\langle m\rangle^{\circ} \setminus T) \times U \sqcup T$, an ordering $\leq_t$ on $\iota_m(\alpha)^{-1}(t)$ so that the C_2-action sends $\leq_t$ to the reverse of $\leq_{\sigma(t)}$. There are three cases

- If $t = (j, u)$, endow $\iota_m(\alpha)^{-1}(t)$ with the ordering induced by the canonical isomorphism $\iota_m(\alpha)^{-1}(t) \simeq \alpha^{-1}(t)$
- If $t = (j, v)$, endow $\iota_m(\alpha)^{-1}(t)$ with the reverse of the ordering induced by the canonical isomorphism $\iota_m(\alpha)^{-1}(t) \simeq \alpha^{-1}(t)$.
- If $t \in T$, endow $\iota_m(\alpha)^{-1}(t)$ with the ordering so that if $i, i' \notin S$, $s \in S$ and $\alpha(i) = \alpha(i') = \alpha(s) = t$ and $i \leq i'$, then
$$(i, u) \leq s \leq (i, v) \qquad (i, u) \leq (i', u) \qquad (i', v) \leq (i, v)\,.$$

9 Comparing involutive classical Brauer and involutive higher Brauer

Question 9.1.

- If, for a Poincaré ∞-category $(\mathcal{C}, \Omega)$, there exists a Poincaré object (E, q) so that E is a compact generator, can we rewrite both the category and its Poincaré structure in terms of $\mathrm{End}_{\mathcal{C}}(E)$?
- Can the *property* of an existence of a Poincaré object (E, q) in $(\mathrm{Perf}_X, \Omega_L)$ so that E is a compact generator be checked Zariski-locally? See Toën's paper §3.

10 Other

Proposition 10.1. *Assume that X has a* good quotient *Y in the sense of [FW20, Remark 4.20], and write $p\colon X \to Y$ for the quotient map. Let $i\colon U \subseteq Y$ be the largest open subscheme on which $\pi|_{X_U}$ is étale [FW20, Proposition 4.45]. Write $Z(\pi)$ for the closed complement of U regarded as a topological space, and let $j\colon Z(\pi) \to Y$ denote the inclusion[4]. Then $\underline{\mathcal{O}}^{\varphi C_2}$ is in the essential image of $j_*\colon \mathrm{Shv}_{\mathrm{Zar}}(Z(\pi)) \to \mathrm{Shv}_{\mathrm{Zar}}(Y)$. In other words, there exists a sheaf $\mathcal{Q}$ of $\mathbb{E}_\infty$-rings on $Z(\pi)$ so that $j_*\mathcal{Q} \simeq \underline{\mathcal{O}}^{\varphi C_2}$.*

L: hypothesis? move to main text?

L: How does $\mathcal{Q}$ relate to the structure sheaf on the branch locus (as reduced subscheme of Y) used in First–Williams?

Proof. Recall that the open-closed decomposition of Y induces a symmetric monoidal récollement

$$\mathrm{Shv}_{\mathrm{Zar}}(U) \xleftarrow{i^*} \mathrm{Shv}_{\mathrm{Zar}}(Y) \xrightarrow{j^*} \mathrm{Shv}_{\mathrm{Zar}}(Z(\pi))\,.$$

Therefore, to show that $\underline{\mathcal{O}}^{\varphi C_2}$ is in the essential image of j_*, it suffices to show that $i^*\left(\underline{\mathcal{O}}^{\varphi C_2}\right) \simeq 0$ as a sheaf on U.

By [FW20, Proposition 4.45], it suffices to show that if y is a point in U, then $\underline{\mathcal{O}}^{\varphi C_2}_y = 0$. Since $\underline{\mathcal{O}}^{\varphi C_2}_y = \tau_{\geq 0}\left(\Gamma\mathcal{O}^{tC_2}_{X\times_Y\{y\}}\right)$ where $A = \Gamma\mathcal{O}_{Y,y} \to B = \Gamma\mathcal{O}_{X\times_Y\{y\}}$ is a quadratic étale map so that B has an involution λ and $A = B^\lambda$ is a local ring with maximal ideal $\mathfrak{m}_A$ (therefore B is semilocal by [FW20, Proposition 3.15]), it suffices to show that $\pi_0 B^{tC_2} = 0$. By [NS18, Lemma I.2.9], we may without loss of generality replace A and B by their 2-completions.

L: Need hypercompleteness to reduce to checking on points? Appeal to Clausen–Mathew and add finite Krull dim hypothesis?

L: this is unnecessary to the proof–but shows that the support of $\mathcal{Q}$ intersects trivially with the open subscheme $Y\left[\frac{1}{2}\right]$.

By the recollement of A-modules in terms of $\mathfrak{m}_A$-complete and $A[\mathfrak{m}_A^{-1}]$-modules, it suffices to show that $\left(B^\wedge_{\mathfrak{m}_A}\right)^{tC_2} = 0$ and $\left(B[\mathfrak{m}_A^{-1}]\right)^{tC_2} = 0$.

By [FW20, Propositions 3.4 & 3.15], $B\mathfrak{m}_A = J \subseteq B$, where J denotes the Jacobson radical of B. We claim that $B \simeq \lim_i B/J^i$ induces an equivalence $B^{tC_2} \to \lim_i \left(B/J^i\right)^{tC_2}$. Granting the claim, it suffices to show that $\left(B[\mathfrak{m}_A^{-1}]\right)^{tC_2} = 0$ and $\left(B/J^i\right)^{tC_2}$ is zero for each i. Since $(-)^{tC_2}$ is exact and lax symmetric monoidal and each B/J^i can be written as an extension of finitely many B/J-modules, it suffices to show that $\left(B[\mathfrak{m}_A^{-1}]\right)^{tC_2}$ and $(B/J)^{tC_2}$ are zero.

Now observe that $A/\mathfrak{m}_A$ (resp. $A[\mathfrak{m}_A^{-1}]$-algebra) is a field and B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is a quadratic étale $A/\mathfrak{m}_A$-algebra (resp. $A[\mathfrak{m}_A^{-1}]$-algebra). By [FW20, Proposition 3.4(ii)], B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is either a separable quadratic field extension of $A/\mathfrak{m}_A$-algebra (resp. $A[\mathfrak{m}_A^{-1}]$-algebra), or it is isomorphic to $\prod_{C_2} A/\mathfrak{m}_A$ (resp. $\prod_{C_2} A[\mathfrak{m}_A^{-1}]$). In the latter case, the action of C_2 on B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is manifestly free, hence $(B/J)^{tC_2} = 0$ (resp. $B[\mathfrak{m}_A^{-1}]^{tC_2} = 0$). Suppose instead that B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is a separable quadratic field extension of $A/\mathfrak{m}_A$-algebra (resp. $A[\mathfrak{m}_A^{-1}]$-algebra). By [FW20, Proposition 3.4(ii)], $\lambda \otimes_B B/J$ (resp. $\lambda \otimes_B B[\mathfrak{m}_A^{-1}]$) is nontrivial, hence by [Stacks, Lemma 9.21.2, Tag 09DU] the extension $A/\mathfrak{m}_A \to B/J$ (resp. $A[\mathfrak{m}_A^{-1}] \to B[\mathfrak{m}_A^{-1}]$) is Galois. Since C_2 acts freely on B/J as an $A/\mathfrak{m}_A$-module by the normal basis theorem, $(B/J)^{tC_2} = 0$ (resp. $B[\mathfrak{m}_A^{-1}]^{tC_2}$).

We conclude the proof by proving the claim. Since homotopy fixed points commute with arbitrary limits, it suffices to show that $B \simeq \lim_i B/\mathfrak{m}_B^i$ induces an equivalence $B_{hC_2} \to \lim_i \left(B/\mathfrak{m}_B^i\right)_{hC_2}$. This is true because the $B/\mathfrak{m}_B^i$ are uniformly bounded below. ∎

L: compare [CMM21, Remark 2.8].

Example 10.2. If $\lambda = \mathrm{id}_X$ and $Y = X$, then $\pi_0\underline{\mathcal{O}}^{\varphi C_2} = \mathcal{O}_Y/2$. On π_0, the norm map $\underline{\mathcal{O}}^e \simeq \mathcal{O}_X \to \underline{\mathcal{O}}^{\varphi C_2}$ takes $f \mapsto f^2$.

pushforwards along quotient maps Two attempts to show the pushforward preserves filtered colimits.

L: DEPRECATED JUNE 3RD: Here are some thoughts towards showing that the canonical map $\mathrm{colim}_{[a,b]}\, \pi_*\mathbf{M}^{[a,b]}_{A^e} \to \pi_*\mathbf{M}_{A^e}$ is an equivalence. Later: see if Example 3.1.2 here could be useful?

[4] $Z(\pi)$ is referred to as the *branch locus* in [FW20]

1. Each $\mathbf{M}^{[a,b]}_{A^e}$ and $\mathbf{M}_{A^e}$ is a hypersheaf. For $\mathbf{M}_{A^e}$ this follows from [AG14, Lemma 5.4]; need to prove for $\mathbf{M}^{[a,b]}_{A^e}$.

2. π_* sends hypersheaves on the small étale site of X to hypersheaves on the small étale site of Y; this follows from the proof of [Lur09, Proposition 6.5.2.13].

L: which I learned from Recollection 1.14 of this paper.

3. Is $\operatorname{colim}_{[a,b]} \pi_* \mathbf{M}^{[a,b]}_{A^e}$ still a hypersheaf? Hypersheaves are not in general closed under colimits, but maybe we can argue using an explicit model for this sheaf?

4. The hypercompletion of the étale ∞-topos of Y has enough points; this follows from [Lur18, Theorem A.4.0.5] and Proposition 3.7.3 of Exodromy.

5. It suffices to show that the canonical map $\operatorname{colim}_{[a,b]} \pi_* \mathbf{M}^{[a,b]}_{A^e} \to \pi_* \mathbf{M}_{A^e}$ is an equivalence on points; use the explicit model from [FW20, Theorem 3.16]?

L: As of June 3, I am suspicious of the next argument/think something has gone wrong–(10.5) is not supposed to hold in this level of generality. I'm not sure what the problem is yet–maybe that π_* does not in fact define a morphism of recollements? Will revisit later.

Lemma 10.3. *Let (X, σ, Y, π) be a scheme with involution and good quotient. Let $\widetilde{U} \subseteq X$ be the largest open subscheme of X on which π is quadratic étale and let $W \subseteq Y$ and $Z = \pi^{-1}(W) \subseteq X$ be the branch and ramification loci of π in the sense of [FW20, Proposition 4.45-4.47] (in particular, W and Z are endowed with the reduced subscheme structure). Assume that $2 \in \mathcal{O}_Y^\times$. Then the pushforward $\pi_* \colon \mathrm{Shv}_{\text{ét}}(X; \mathcal{S}) \to \mathrm{Shv}_{\text{ét}}(Y; \mathcal{S})$ preserves filtered colimits.*

Proof. Since $W \subseteq Y$, $Z \subseteq X$ are closed immersions, there exist récollements

L: this is automatic, but [FW20, Lemma 5.36] asserts that $W \to Y$ is a closed embedding without proof

$$\begin{tikzcd}
\mathrm{Shv}_{\text{ét}}(\widetilde{U}; \mathcal{S}) \arrow[r, shift right, "j_{\widetilde{U}*}"'] & \mathrm{Shv}_{\text{ét}}(X; \mathcal{S}) \arrow[l, shift right, "j^*_{\widetilde{U}}"'] \arrow[r, shift right, "i^*_Z"'] & \mathrm{Shv}_{\text{ét}}(Z; \mathcal{S}) \arrow[l, shift right, "i_{Z*}"'] \\
\mathrm{Shv}_{\text{ét}}(U; \mathcal{S}) \arrow[r, shift right, "j_{U*}"'] & \mathrm{Shv}_{\text{ét}}(Y; \mathcal{S}) \arrow[l, shift right, "j^*_U"'] \arrow[r, shift right, "i^*_W"'] & \mathrm{Shv}_{\text{ét}}(W; \mathcal{S}) \arrow[l, shift right, "i_{W*}"']
\end{tikzcd}.$$

Moreover, the pushforward functor π_* is a morphism of récollements in the sense of [Sha21, Definition 2.3]. In particular, the 'components' of π_* (see Observation 2.4 of *loc. cit.*) are

$$\pi_*|_{\mathrm{Shv}_{\text{ét}}(\widetilde{U};\mathcal{S})} = j^*_U \circ \pi_* \circ j_{\widetilde{U}*} = j^*_U \circ (\pi \circ j_{\widetilde{U}})_* = j^*_U \circ (j_U \circ \pi|_{\widetilde{U}})_* = j^*_U \circ j_{U*} \circ (\pi|_{\widetilde{U}})_*$$
$$\pi_*|_{\mathrm{Shv}_{\text{ét}}(Z;\mathcal{S})} = i^*_W \circ \pi_* \circ i_{Z*} = i^*_W \circ (\pi \circ i_Z)_* = i^*_W \circ (i_W \circ \pi|_Z)_* = i^*_W \circ i_{W*} \circ (\pi|_Z)_* .$$

Now $j^*_U \circ j_{U*} \circ (\pi|_{\widetilde{U}})_* \simeq (\pi|_{\widetilde{U}})_*$ (resp. $i^*_W \circ i_{W*} \circ (\pi|_Z)_* \to (\pi|_Z)_*$) induced by the counit of the adjunction (j^*_U, j_{U*}) (resp. (i^*_W, i_{W*})) is an equivalence because j_{U*} (resp. i_{W*}) is fully faithful, thus

$$\pi_*|_{\mathrm{Shv}_{\text{ét}}(\widetilde{U};\mathcal{S})} \simeq (\pi|_{\widetilde{U}})_* \qquad \pi_*|_{\mathrm{Shv}_{\text{ét}}(Z;\mathcal{S})} \simeq (\pi|_Z)_* . \tag{10.4}$$

We note for further reference the equivalences

L: diagram instead?

$$\pi_*|_{\mathrm{Shv}_{\text{ét}}(\widetilde{U};\mathcal{S})} \circ j^*_{\widetilde{U}} \simeq j^*_U \circ \pi_* \qquad \pi_*|_{\mathrm{Shv}_{\text{ét}}(Z;\mathcal{S})} \circ i^*_W \simeq i^*_Z \circ \pi_* . \tag{10.5}$$

Suppose given a filtered diagram $\mathcal{F}_\bullet$ in $\mathrm{Shv}_{\text{ét}}(X; \mathcal{S})$ and write $\mathcal{F}$ for its colimit. We would like to show that the canonical map $\operatorname{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \to \pi_*(\mathcal{F})$ is an equivalence. Since j^*_U, i^*_W are jointly conservative (by definition of a récollement, see [Lur17, Definition A.8.1(e)]), it suffices to show that the canonical maps

$$\begin{aligned} j^*_U \left(\operatorname*{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \right) &\to j^*_U \pi_*(\mathcal{F}) \\ i^*_W \left(\operatorname*{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \right) &\to i^*_W \pi_*(\mathcal{F}) \end{aligned} \tag{10.6}$$

are equivalences. Since j_U^* and i_W^* preserve all colimits, the morphisms of (10.6) can be identified with the canonical maps

$$\begin{aligned}\operatorname*{colim}_{\bullet} j_U^* \pi_*(\mathcal{F}_\bullet) &\simeq \operatorname*{colim}_{\bullet} (\pi|_{\widetilde{U}})_*(j_{\widetilde{U}}^* \mathcal{F}_\bullet) \to (\pi|_{\widetilde{U}})_*(j_{\widetilde{U}}^* \mathcal{F}_\bullet)\\ \operatorname*{colim}_{\bullet} i_W^* \pi_*(\mathcal{F}_\bullet) &\simeq \operatorname*{colim}_{\bullet} (\pi|_W)_*(i_W^* \mathcal{F}_\bullet) \to (\pi|_Z)_* i_Z^*(\mathcal{F}),\end{aligned}$$

respectively, where we have used (10.5). Now $\pi|_W$ is an equivalence and $\pi|_{\widetilde{U}}$ is finite étale by [FW20, Proposition 4.47], hence $(\pi|_{\widetilde{U}})_*$ and $(\pi|_W)_*$ preserve filtered colimits. □

10.1 Algebras with genuine involution

Construction 10.7. Assume $\mathcal{C}$ is a presentable monoidal ∞-category such that the monoidal product $-\otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then there is an ∞-category $\mathrm{LMod}(\mathcal{C})$ [Lur17, Example 4.2.1.18] whose objects are pairs (A, M) where A is an associative algebra object of $\mathcal{C}$ and M is a left A-module. Write a, m respectively for the canonical forgetful functors $\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$, $\mathrm{LMod}(\mathcal{C}) \to \mathcal{C}$ which send (A, M) to A and M, resp. Then a is a cocartesian fibration [Lur17, Corollary 4.2.3.7], hence it is classified by a functor $\mathrm{mod}\colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Cat}_\infty$.

The functor s of [Lur17, Example 4.2.1.17] determines a commutative diagram

$$\begin{tikzcd} & \mathcal{U} \arrow[d] \\ \mathrm{Alg}(\mathcal{C}) \arrow[ur, "\eta"] \arrow[r, "\mathrm{mod}"'] & \mathrm{Cat}_\infty \end{tikzcd} \tag{10.8}$$

where $\mathcal{U}$ is the universal cocartesian fibration. Now consider the functor $o\colon \mathcal{U} \to \mathrm{Cat}_\infty$ which sends $(\mathcal{D}, d \in \mathcal{D})$ to the undercategory $\mathcal{D}_{d/-}$. Define $\mathrm{LMod}(\mathcal{C})_{*/-}$ to be the cocartesian fibration over $\mathrm{Alg}(\mathcal{C})$ classified by $o \circ \eta$. Informally, an object of $\mathrm{LMod}(\mathcal{C})_{*/-}$ lying over $A \in \mathrm{Alg}(\mathcal{C})$ is the data of a left A-module M and a map of left A-modules $A \to M$.

Variant 10.9. Let $\mathcal{C}$ be an involutive monoidal ∞-category in the sense of Definition 6.9. There is a variation on Construction 10.7 where left modules is replaced by *bimodules* [Lur17, Definition 4.3.1.12]. L: involutive bimodules?

Construction 10.10. Regard $\mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})$ as a category with C_2-action given by taking the opposite/reverse algebra. There are functors $b\colon \mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2} \to \mathrm{LMod}(\mathrm{Sp})$ and $b\colon \mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2} \to \mathrm{BiMod}(\mathrm{Sp})_{*/-}$ so that $(a, m) \circ b$ and $(a, m) \circ b_*$ are (canonically) equivalent to $(-^e) \otimes (-^e)^{\mathrm{op}}, (-)^e$. Informally, an $\mathbb{E}_1$-algebra with involution B can be regarded as a $B \otimes B^{\mathrm{op}}$-module in a canonical way, and there is a canonical $B \otimes B^{\mathrm{op}}$-module map $B \otimes B^{\mathrm{op}} \to B$.

Definition 10.11. The category of $\mathbb{E}_1$-*algebras with genuine involution* is defined to be the limit of the Cat_∞-valued diagram

$$\begin{tikzcd} & \mathrm{LMod}(\mathrm{Sp}) \arrow[dd, "a{,}m"] & \\ & & \mathrm{LMod}\left(\mathrm{Sp}^{C_2}\right) \arrow[ul, "(-)^e"'] \arrow[dd, "a{,}m"] \\ & \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp}) \times \mathrm{Sp} & \\ \mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \arrow[uuur, bend left, "b\circ(-^e)"] \arrow[ur, "(-^e)\otimes(-^e)^{\mathrm{op}}{,}(-)^e"] \arrow[rr, "N^{C_2}(-^e)\times U"'] & & \mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \times \mathrm{Sp}^{C_2} \arrow[ul, "(-)^e\times(-)^e"] \end{tikzcd} \tag{10.12}$$

where

- b is the functor/section of Construction 10.10

- U is the 'underlying' C_2-spectrum functor $\mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{BC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \to \mathrm{Sp}^{BC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \simeq \mathrm{Sp}^{C_2}$
- The upper right trapezoid commutes canonically by definition of LMod (and the fact that the functors a, m are given by restriction to subcategories of $LM^{\otimes}$).

Write $\mathbb{E}_1\mathrm{Alg}^{\mathrm{gi}}\left(\mathrm{Sp}^{C_2}\right)$ for the ∞-category of $\mathbb{E}_1$-algebras with genuine involution.

Variant 10.13. Let the base be R an $\mathbb{E}_\infty$-algebra or Poincaré ring instead of $\mathbb{S}^0$.

Remarks 10.14.
1. Compare [AGH21, Corollary 3.10].
2. There are canonical forgetful functors $\mathbb{E}_\sigma\mathrm{Alg} \to \mathbb{E}_1\mathrm{Alg}^{\mathrm{gi}} \to \mathbb{E}_1\mathrm{Alg}^{hC_2} \to \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})$.

Construction 10.15. Let $R, R^{\varphi C_2} \to R^{tC_2}$ be a Poincaré ring. There is a functor $\left(\mathrm{Mod}^\omega_{(-)}, Ϙ_{(-)}\right) : \mathbb{E}_\sigma\mathrm{Alg}_R \to \left(\mathrm{Cat}^h_R\right)_{\left(\mathrm{Mod}^\omega_R, Ϙ_R\right)/-}$.

Lemma 10.16. *Let $R, R^{\varphi C_2} \to R^{tC_2}$ be a Poincaré ring.*

1. *The functor of Construction 10.15 factors through the subcategory $(\mathrm{Cat}^p_R)_{(\mathrm{Mod}^\omega_R, Ϙ_R)/-}$. In other words, a map of $\mathbb{E}_\sigma$-R-algebras $A \to B$ induces a duality-preserving map of R-linear Poincaré ∞-categories.*
2. *Write $\mathrm{Mod}^p \colon \mathbb{E}_1^{\mathrm{gi}}\,\mathrm{Alg}_R \to (\mathrm{Cat}^p_R)_{(\mathrm{Mod}^\omega_R, Ϙ_R)/-}$ for the canonical factorization from part 1. Then Mod^p is fully faithful.*

Proof.

L: Proof of the first point should be quite similar to/a relative variant on Corollary 3.4.2, Lemma 3.4.3 of [Cal+20].

Let A, B be $\mathbb{E}_1$-R-algebras with genuine involution. Then there is a fiber sequence

$$\begin{tikzcd} \hom_{\mathrm{Mod}^p_R/-}\left(\mathrm{Mod}^p_A, \mathrm{Mod}^p_B\right) \arrow[r] & \hom_{\mathrm{Mod}^p_R\text{-linear}}\left(\mathrm{Mod}^p_A, \mathrm{Mod}^p_B\right) \arrow[r] \arrow[d, "\sim"] & \hom_{\mathrm{Mod}^p_R\text{-linear}}\left(\mathrm{Mod}^p_R, \mathrm{Mod}^p_B\right) \arrow[d, "\sim"] \\ & \mathrm{Pn}\left(\mathrm{Mod}_{A^{\mathrm{op}}\otimes_R B}, Ϙ_{A^{\mathrm{op}}\boxtimes B}\right) \arrow[r] & \mathrm{Pn}\left(\mathrm{Mod}_B, Ϙ_B\right) \end{tikzcd}$$

where we have used Corollary **??**. The fiber of the horizontal map over the point $(B, q_B) \in \mathrm{Pn}\left(\mathrm{Mod}_B, Ϙ_B\right)$ (q_B is the canonical nondegenerate form on B) is

L: this refers to main text; will be fixed when we move it.

L: IN PROGRESS: should be some sort of $\mathbb{E}_\sigma$ version of [AG14, Proposition 3.1].

□

Now we observe that given a Poincaré object (x, q) of $(\mathcal{C}, Ϙ_{\mathcal{C}})$, its endomorphism algebra admits a canonical lift to a $\mathbb{E}_\sigma$-algebra.

Construction 10.17. There is a functor $\mathrm{End}(-)\colon \left(\mathrm{Cat}^h_R\right)_{\left(\mathrm{Mod}^\omega_R, Ϙ_R\right)} \to \mathbb{E}_\sigma\mathrm{Alg}$ lifting the functor $\left(\mathrm{Cat}^h_R\right)_{\left(\mathrm{Mod}^\omega_R, Ϙ_R\right)} \to \mathbb{E}_1\mathrm{Alg}^{hC_2}$ of [Cal+20, Proposition 3.1.16].

Theorem 10.18. *The functors of Construction 10.15 and 10.17 form an adjoint pair.*

L: make everything Poincaré

Lemma 10.19. *The right adjoint of Construction 10.17 preserves filtered colimits.*

L: similar to [AG14, Lemma 3.4].

Proof. □

Proposition 10.20. *Let A be an $\mathbb{E}_1$-R-algebra with genuine involution. Then A is compact in $\mathbb{E}_1^{\mathrm{gi}}\,\mathrm{Alg}_R$ if and only if Mod^p_A is compact in $\mathrm{Cat}^{\mathrm{p}}_{\infty R}$.*

L: similar to [AG14, Proposition 3.5].

Proof. The only if part of the statement follows from observing that Mod^p admits a right adjoint which preserves filtered colimits (Lemma 10.19) and [Lur09, Lemma 5.5.1.4]. $\square$

L: other half of the statement

Proposition 10.21. *Let A be an $\mathbb{E}_1$-R-algebra with genuine involution. If Mod^p_A is dualizable in $\mathrm{Cat}^{\mathrm{p}}_{\infty R}$, then A is compact in $\mathbb{E}_1^{\mathrm{gi}}\mathrm{Alg}_R$.*

L: similar to [AG14, Proposition 3.11].

Proof. $\square$

11 Speculative norm for Brauer group at the level of infinity categories

We would want a functor from A^e-linear stable idempotent complete infinity categories to A^L-linear stable idempotent complete infinity categories in order to get an extension of our exact sequence to the right. Here is what I think might do it:

Construction 11.1. Let $\lambda : A^e \to A^e$ denote the involution. Consider the functor

$$\mathrm{Mod}_{\mathrm{Mod}^{\omega}_{A^e}}(\mathrm{Cat}^{st}_{\infty,\mathrm{idem}}) \xrightarrow{\left(-\otimes_{\mathrm{Mod}^{\omega}_{A^e}}\lambda^*-\right)^{hC_2}} \mathrm{Mod}_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}}(\mathrm{Cat}^{st}_{\infty,\mathrm{idem}})$$

which we will denote by $N_{A^{hC_2}/A}$. This functor is symmetric monoidal and we have that the composite with the base change functor gives

$$\left(\mathcal{C}\otimes_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}}\mathrm{Mod}^{\omega}_{A^e}\otimes_{\mathrm{Mod}^{\omega}_{A^e}}\lambda^*(\mathcal{C}\otimes_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}}\mathrm{Mod}^{\omega}_{A^e})\right)^{hC_2}\simeq\mathcal{C}^{\otimes_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}}2}$$

For $\mathcal{C}\in\mathrm{Mod}_{A^e}$, we have that

N: Maybe... We would want this to be true but I don't see a proof immediately...

$$(\mathcal{C}\otimes_{\mathrm{Mod}_{A^e}}\lambda^*\mathcal{C})^{hC_2}\otimes_{\mathrm{Mod}^{hC_2}_{A^e}}\mathrm{Mod}_{A^e}\simeq\mathcal{C}^{\otimes_{\mathrm{Mod}_{A^e}}2}$$

via the functor which forgets the C_2-action.

Lemma 11.2. *The composite $\mathrm{PnBr}(A)\to\mathrm{br}(A^e)\to\mathrm{Pic}(\mathrm{Mod}_{\mathrm{Mod}^{hC_2}_{A^e}})$ is nullhomotopic.*

Proof. The underlying category of a Poincaré invertible category is self-dual, and so its square will vanish. Since the functor is naturally nullhomotopic so too is the composite after applying the functor $\mathrm{Pic}(-)$. $\square$

There is thus a map $\mathrm{PnBr}(-)\to\mathcal{F}(-)$, where $\mathcal{F}(-)$ is the fiber. Delooping both fiber sequences we see that we get a map of fiber sequences

$$\begin{tikzcd}
\mathrm{Pic}(\mathrm{Mod}_{A^L}(\mathrm{Sp}^{C_2})) \arrow[r] \arrow[d] & \mathrm{PnBr}(A) \arrow[r] \arrow[d] & \mathrm{br}(A^e) \arrow[d, "="] \\
\mathrm{Pic}(\mathrm{Mod}^{hC_2}_{A^e}) \arrow[r] & \mathcal{F}(A) \arrow[r] & \mathrm{br}(A^e)
\end{tikzcd}$$

from which we see that the middle horizontal map must be an equivalence whenever $\frac{1}{2}\in A^e$ and $A^{\varphi C_2}=0$.

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