Et cetera

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Abstract

Dumping ground for other stuff: Notes, one-off observations, stuff that we can collectively use when preparing talks, etc.

L: I make no promises re: organization but I will do my best to keep it reasonably readable

Contents

1 Talk prep

2 References

- Involutions of Azumaya algebras by First and Williams (2020 *Documenta*)
- Counterexamples in involutions of Azumaya algebras by First and Williams; much more readable than the 2020 Documenta paper
- Azumaya algebras without involution by Auel, First, and Williams: the introduction of this provides a very helpful historical overview of the connection between involutions on Azumaya algebras and 2-torsion/kernel of coRes

3 Questions and directions

Question 3.1 (Literature)**.**
- In [PS92] Parimala–Srinivas assume that 2 is invertible in the ring of functions. Has anyone been able to extend their results to the 2 not necessarily invertible case in the meantime?

Question 3.2 (Morita theory for $\mathrm{Cat}^{\mathrm{p}}_{\infty}$)**.** Let R be a Poincaré ring. Suppose given two R-algebras (suitably interpreted so their module categories are canonically endowed with R-linear Poincaré structures–perhaps $\mathbb{E}_{\sigma}$) A, B. Can we characterize

$$\hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty R}}\left(\left(\mathrm{Mod}^{\omega}_A, \Omega_A\right), \left(\mathrm{Mod}^{\omega}_B, \Omega_B\right)\right)$$

in terms of something bimodule-like?

Question 3.3. On page 2 of the *Counterexamples* paper, First and Williams write that " existence of an extraordinary involution means classificaiton of Azumaya algebras with involution... *cannot* be reduced to questions about projective modules and hermitian forms on them."

What if we replaced projective modules by perfect complexes?

Question 3.4. First–Williams show (see discussion in §4 of the *Counterexamples* paper) that coarse type classify many (most?) Azumaya algebras up to (étale-local) *isomorphism*.

What is a suitable derived version of "coarse type"?

Question 3.5 (asked by Andrew Nov 2, 2024)**.** C. Schlichtkrull shows in this paper that a map $BGL_1(R) \to K(R) \to THH(R) \to R$ in terms of the Hopf map η.

Is there a "Poincaré" version of this result?

Question 3.6. Are there some general conditions for a ring with involution R so that the inclusion $R^{C_2} \to R$ is 'nice'?

There's some stuff in section 4 here, idk. Also see M. Hochster and J. L. Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay.

Applications to Hodge theory?

- A Deligne pairing for Hermitian Azumaya modules
- Burt Totaro's paper on Hodge structures of type $(n, 0, \ldots, 0, n)$
- There's some stuff about endomorphisms of Hodge structures here

4 Thoughts & observations

Question 4.1. When R has the Tate Poincaré structure and $(\mathrm{Mod}^{\omega}_A, M_A, N_A, N_A \to M_A^{tC_2})$ is invertible, then by invertibility have an equivalence $\hom_R(A, R) \simeq N_A \otimes_R N_{A^{\mathrm{op}}}$ of $A \otimes_R A^{\mathrm{op}}$-modules. Restricting the left-hand side along the unit map $R \to A$ gives a map $N_A \otimes_R N_{A^{\mathrm{op}}} \to \hom_R(R, R) \simeq R$. Is this a perfect ($R$-linear) pairing?

I *think* using that $R^{\varphi C_2} \simeq R$ and combining the linear and bilinear part conditions, we get something like

$$M_A \otimes_R M_{A^{\mathrm{op}}} \simeq (N_A \otimes_R N_{A^{\mathrm{op}}})^{\otimes_R 2} \qquad \text{as } A \otimes_R A^{\mathrm{op}}\text{-bimodules.}$$

Is this useful?

Brauer-Severi schemes We know there is a correspondence between Azumaya algebras A over X and Brauer-Severi schemes. What does a Poincaré structure on Mod^{ω}_A mean 'geometrically' for D^b_{coh} of the corresponding Brauer-Severi scheme? (Lucy: I didn't get very far here, but just typing up what I had)

- Mod^{ω}_A corresponds to α-twisted sheaves on X (see Proposition 3.2.2.1 of Max Lieblich's thesis)
- The bounded derived category of α-twisted sheaves on X includes as one 'piece' of a semiorthogonal decomposition on D^b_{coh} of the corresponding Brauer-Severi scheme (see Theorem 5.1 here)

5 Desperate Flailing

This section is a cronical of my thoughts about $\mathbb{G}_m^{\text{♀}}$.

Goal The goal is to build a Poincaré ring $\mathbb{G}_m^{\text{♀}} := (\mathrm{Mod}_R, \text{♀}_R)$ such that $B\mathbb{G}_m^{\text{♀}}(\underline{S}) = \mathrm{Pic}^{\mathrm{p}}(\underline{S})$ for any Poincaré ring $\underline{S}$.

Lemma 5.1. *Let $\underline{S}$ be a Poincaré ring. Then $\pi_0(\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}_S)}(S,u)) = \{s \in \pi_0(S)^\times | s = 1$ in $\pi_0(S^{C_2})\}$.*

Proof. Since the functor $\mathrm{Pn}(\mathrm{Mod}_S) \to \mathrm{Mod}_S$ is conservative it follows that an element of $\pi_0(\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}_S)}(S,u))$ must have underlying map an element of $\pi_0\mathrm{Aut}(S) = \pi_0(S)^\times$. Then in order for $s \in \pi_0(S)^\times$ to induce a map $(S,u) \to (S,u)$, the induced map $s^* : S^{C_2} \to S^{C_2}$ must satisfy $s^*(u) = u$. The pullback is given by multiplication by s, so this requirement translates into s being the unit, as desired. □

The problem I thought existed maybe doesn't. Here is a candidate construction:

Construction 5.2. Define R to be the $\mathbb{E}_\infty$ ring given by $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z\}} \mathbb{S}$ where the map $\mathbb{S}\{z\} \to \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by the map $z \mapsto xy$, and the map $\mathbb{S}\{z\} \to \mathbb{S}$ is induced by $z \mapsto 1$. We can give R an $\mathbb{E}_\infty$ ring structure in Sp^{BC_2} by taking the trivial action on $\mathbb{S}\{z\}$ and $\mathbb{S}$, and taking the action induced by $x \mapsto y$ and $y \mapsto x$ on $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$. Thus in $\mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ the ring R corepresents the functor $S \mapsto \{s \in \pi_0(S)^\times | s\sigma(s) = 1\}$.

Now take $\underline{R}$ to be the Poincaré ring with underlying Borel C_2 structure as described in the previous paragraph and geometric fixed points $R^{\varphi C_2} = \mathbb{S}$ and the map $R^{\varphi C_2} \to R^{tC_2}$ given by the unit map. Endowing $R^{\varphi C_2}$ with the R-module structre given by $x, y \mapsto 1$, it remains to show that the unit map $R^{\varphi C_2} \to R^{tC_2}$ factors the Tate valued Frobenius $R \to R^{tC_2}$ in order to promote $\underline{R}$ to a Poincaré ring. By construction of R it is then enough to show that on π_0 the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0(R^{tC_2})$. This map sends both x and y to $xy \in \pi_0(R^{tC_2})$. These areequal to 1 in $\pi_0(R^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so R^{tC_2} is a modules over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of xy equal to 1.

Now consider another Poincaré ring $\underline{S}$. We then have that maps $\pi_0(\mathrm{Maps}(\underline{R}, \underline{S}))$ is the data of a unit $s \in \pi_0(S)^\times$, a path $s\sigma(s) \to 1$ in $\Omega^\infty S$, and paths $x, y \to 1$ in $\Omega^\infty S^{\varphi C_2}$. This then agrees with $\mathbb{G}_m^{\text{♀}}$ by the following lemma.

Lemma 5.3. *Let $S \in \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ and $s \in \pi_0(S)^\times$. Then $s\sigma(s) = 1$ in $\pi_0(S)$ if and only if $(s \otimes s)^*$ acts by 1 on $\pi_0(S^{hC_2}) = \pi_0(\mathrm{Hom}_{S\otimes S}(S \otimes S, S)^{hC_2})$.*

Proof. The 'only if' direction follows from the fact that the map $S^{hC_2} \to S$ is an S-bimodule map. Now suppose that $s\sigma(s) = 1$ in S. Then before taking homotopy fixed points the induced map $s^* = id$ because S is $\mathbb{E}_\infty$.[1] □

6 Modules with genuine involution

Remark 6.1 (Lucy)**.** I'm just going to put drafts of stuff pertaining to hermitian modules here. Eventually when it gets to be more complete, I will hopefully move this entire section over to the main file. L: or whatever we want to keep calling these

Meta-commentary There are (at least) three things we want to do:

(a) Define a category of 'bimodules with involution over algebras with anti-involution' equipped with a forgetful functor $\Theta\colon \mathrm{BMod}_{\mathrm{inv}}(-) \to \mathbb{E}_1\,\mathrm{Alg}(-)^{hC_2}$.

(b) Show that Θ is a coCartesian fibration. For this, it suffices to show that it is a *Cartesian* fibration and that it satisfies the hypotheses of [Lur09, Corollary 5.2.2.5]

- I used to think that we could obtain this by 'bootstrapping' a result from Higher Algebra, plus some facts about assembly. This doesn't seem to be working, so I'm just going to try to do this directly (imitating certain aspects of Chapter 4 of higher algebra.)

[1] Or just $\mathbb{E}_2$.

(c) Define a relative tensor product for hermitian bimodules

(d) Show that the formula for the cocartesian pushforward along a map $A \to B$ in $\mathbb{E}_1 \operatorname{Alg}(-)^{hC_2}$ is something like $- \otimes_{A \otimes A^{\mathrm{op}}} (B \otimes B^{\mathrm{op}}) \otimes_{B \otimes B^{\mathrm{op}}} B$.

- In Higher Algebra, the formula for the cocartesian pushforward is proven in [Lur17, §4.6]; in particular, this is in the section on duality. In particular, see Proposition 4.6.2.17 and the paragraph immediately preceding this.
- I don't know how to do this yet–while (a) and (b) are not useful if I can't show (c), I can't suss out the feasibility of (c) without (a) and (b) already in place.

(e) Towards an adjunction between $\mathbb{E}_\sigma$-algebras and categories with additional structure.

- Involutive version of statement that, for a monoidal ∞-category $\mathcal{C}$ and an $\mathbb{E}_1$-algebra A, $\operatorname{LMod}_A(\mathcal{C})$ is right-tensored over $\mathcal{C}$?
- Involutive version of endomorphism categories? [Lur17, §4.7.1]

I think that the equivalence of part (b) of the definition of an Azumaya algebra with genuine involution follows from the property of being Azumaya; see Lemma 1(b) (and p.216 for the 'type 2' case) of [PS92].

Lemma 6.2. *Let R be an $\mathbb{E}_\infty$-ring with an involution $\sigma\colon R \xrightarrow{\sim} R$ and suppose A is an $\mathbb{E}_1$-R-algebra with an anti-involution $\lambda\colon A \xrightarrow{\sim} \sigma^* A^{\mathrm{op}}$. Suppose A is further Azumaya in the sense of . Then the bilinear pairing*

L: reference

$$A \otimes_R \sigma^* A \xrightarrow{\mathrm{id} \otimes \sigma^* \lambda} A \otimes_R A^{\mathrm{op}} \simeq \operatorname{End}_R(A) \xrightarrow{\mathrm{tr}} R$$

is perfect, i.e. its adjoint $A \to (\sigma^* A)^\vee$ is an equivalence.

Question 6.3. Does the map in part (e) of the definition of an Azumaya algebra with genuine involution follow from property of being Azumaya?

6.1 Step (a)

Definition 6.4. Define a colored operad $\operatorname{Assoc}_\sigma$ as follows:

L: This is just an imitation of [Lur17, Definition 4.1.1.1], modified in accordance with ideas from §5.4.2.

(i) The colored operad has a single object, which we denote by $\mathfrak{a}$.

(ii) For every finite set I, the set of operations $\operatorname{Mul}_{\operatorname{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$, where $\mathcal{L}I$ is the set of linear orderings on I and an element of $\{\pm 1\}^I$ is a function $I \to \{\pm 1\}$.

(iii) Suppose given a map of finite sets $\alpha\colon I \to J$, together with operations $(\preceq_j, f_j\colon I_j \to \{\pm 1\}) \in \operatorname{Mul}_{\operatorname{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{\alpha(i)=j}, \mathfrak{a}\right)$ and $(\preceq_J, g\colon J \to \{\pm 1\}) \in \operatorname{Mul}_{\operatorname{Assoc}_\sigma}\left(\{\mathfrak{a}_j\}_{j \in J}, \mathfrak{a}\right)$. Define a linear ordering on the set I as follows: $i \leq i'$ if $\alpha(i) \preceq_J \alpha(i')$ or $\alpha(i) = \alpha(i') = j$ and $i \preceq_j i'$ and $g(j) = +1$ or $\alpha(i) = \alpha(i') = j$ and $i \succeq_j i'$ and $g(j) = -1$. Finally, define a function

$$\begin{aligned} I &\to \{\pm 1\} \\ i &\mapsto f_{\alpha(i)}(i) \cdot g(\alpha(i)), \end{aligned}$$

where the multiplication on $\{\pm 1\}$ is the usual one.

Remark 6.5. There is a map of colored operads $\iota\colon \operatorname{Assoc} \to \operatorname{Assoc}_\sigma$ which is the identity on objects and on operations $\operatorname{Mul}_{\operatorname{Assoc}}\left(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \operatorname{Mul}_{\operatorname{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$ where c_1 is the constant function on I with value 1.

There is another map of colored operads $\iota^{\mathrm{rev}}\colon \operatorname{Assoc} \to \operatorname{Assoc}_\sigma$ which is the identity on objects and on operations $\operatorname{Mul}_{\operatorname{Assoc}}\left(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \operatorname{Mul}_{\operatorname{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ sends a linear ordering ℓ to $(\ell^{\mathrm{rev}}, c_{-1})$ where c_{-1} is the constant function on I with value 1.

Definition 6.6. Let $\operatorname{Assoc}_\sigma^\otimes$ denote the associated ∞-operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.7. Unwinding definitions

- Objects $\mathrm{Assoc}_\sigma^\otimes$ are finite pointed sets $\langle n\rangle \in \mathrm{Fin}_*$
- Morphisms $\langle m\rangle \to \langle n\rangle$ consist of
  - $\alpha\colon \langle m\rangle \to \langle n\rangle$ a map of finite pointed sets
  - for each $i \in \langle n\rangle^\circ$, a linear ordering $\preceq_i$ on the inverse image $\alpha^{-1}(\{i\})$
  - a map of sets $s\colon \alpha^{-1}(\langle m\rangle^\circ) \to \{\pm 1\}$
- For each pair of morphisms

$$(\beta\colon \langle \ell\rangle \to \langle m\rangle, \preceq_j, s) \qquad (\alpha\colon \langle m\rangle \to \langle n\rangle, \preceq_i, t)\ ,$$

the composite is the triple $\left(\alpha\circ\beta, \preceq_j'', u\right)$ where $\preceq_j''$ is the ordering on $(\alpha\circ\beta)^{-1}(\{i\})$ so that if $a, b \in \langle \ell\rangle$ so that $\alpha(\beta(a)) = \alpha(\beta(b))$, then $a \preceq_j'' b$ if $\beta(a) \preceq_i \beta(b)$ or $\beta(a) =_i \beta(b) = i$ and $a \preceq_i b$ if $s(i) = 1$ or $a \succeq_i b$ if $s(i) = -1$. Finally $u(l) = s(l)\cdot t(\beta(l))$.

L: Note that when s, t are identically one, the resulting order $\preceq_j''$ agrees with the lexicographic order defined in [Lur17, Remark 4.1.1.4].

Remark 6.8. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.5 induce maps of ∞-operads $\mathrm{Assoc}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$. There is a canonical identification $\iota^{\mathrm{rev}} = \sigma\circ\iota$, where σ is the automorphism of the associative operad considered in [Lur17, Remark 4.1.1.7].

Note that each object $\langle n\rangle \in \mathrm{Assoc}_\sigma^\otimes$ has a distinguished automorphism $\mathrm{rev}_{\langle n\rangle}$ of order two given by the identity map on $\langle n\rangle$ and the constant map $c_{-1}\colon \langle n\rangle^\circ \to \{\pm1\}$ at -1. There is a canonical natural equivalence $\iota \overset{\sim}{\Longrightarrow} \iota^{\mathrm{rev}}$ whose component at $\langle n\rangle$ is $\mathrm{rev}_{\langle n\rangle}$.

Definition 6.9. Let $\mathcal{C}^\otimes$ be a ∞-operad equipped with the data of a fibration $p\colon \mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$. Let $\mathrm{Alg}^\sigma(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{/\mathrm{Assoc}_\sigma}(\mathcal{C})$ of ∞-operad sections of p. We will refer to $\mathrm{Alg}^\sigma(\mathcal{C})$ as the ∞-category of *involutive algebra objects of* $\mathcal{C}$.

An *involutive monoidal ∞-category* is the data of a cocartesian fibration $\mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$.

L: do we need weaker than cocartesian fibration?

Remark 6.10. Suppose given a cocartesian fibration $f\colon \mathcal{D}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ of ∞-operads. Write $\mathcal{C}^\otimes := \mathcal{D}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes, \iota} \mathrm{Assoc}^\otimes$; $\mathcal{C}^\otimes$ is a monoidal ∞-category in the sense of [Lur17, Definition 4.1.1.10]. Furthermore, $\mathcal{C}^\otimes_{\mathrm{rev}} := \mathcal{D}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes, \iota^{\mathrm{rev}}} \mathrm{Assoc}^\otimes$ is a monoidal ∞-category. By Remark 6.8, this notation is consistent with that of [Lur17, Remark 4.1.1.7]. In particular, a Assoc_σ-monoidal ∞-category $\mathcal{D}^\otimes$ determines a monoidal ∞-category $\mathcal{C}^\otimes$ equipped with a monoidal equivalence $\sigma_{\mathcal{C}}\colon \mathcal{C}^\otimes \xrightarrow{\sim} \mathcal{C}^\otimes_{\mathrm{rev}}$. Pullback along the involution of $\mathrm{Assoc}^\otimes$ determines another monoidal equivalence $\sigma_{\mathcal{C}}^{\mathrm{rev}}\colon \mathcal{C}^\otimes_{\mathrm{rev}} \xrightarrow{\sim} \mathcal{C}^\otimes$, and our assumptions imply that $\sigma_{\mathcal{C}}^{\mathrm{rev}} \circ \sigma_{\mathcal{C}}$ is equivalent to the identity on $\mathcal{C}^\otimes$.

Now suppose that A is an involutive algebra object of $\mathcal{D}$. With the same notation as before, pullback along ι (resp. ι^{rev}) determines associative algebra objects $u(A)$, $u^{\mathrm{rev}}(A)$ of $\mathcal{C}$ and $\mathcal{C}_{\mathrm{rev}}$, respectively. Note that $\sigma_{\mathcal{C}}(u(A))$ is an algebra object of $\mathcal{C}_{\mathrm{rev}}$, which we may regard as an algebra object of $\mathcal{C}$ by precomposing with the autoequivalence $\sigma\colon \mathrm{Assoc}^\otimes \xrightarrow{\sim} \mathrm{Assoc}^\otimes$. It follows from Remark 6.8 that A determines an equivalence $\sigma_A\colon u(A) \xrightarrow{\sim} \sigma_{\mathcal{C}}(u(A))^{\mathrm{rev}}$ of algebra objects in $\mathcal{C}$.

Now suppose furthermore that $\mathcal{D}^\otimes$ is of the form $\mathcal{E}^\otimes \times_{\mathrm{Fin}_*} \mathrm{Assoc}_\sigma^\otimes$ for some symmetric monoidal ∞-category $\mathcal{E}$. Then the associated involution $\sigma_{\mathcal{C}}$ is the identity, and for any involutive algebra object A of $\mathcal{D}$, σ_A is an equivalence $u(A) \simeq u(A)^{\mathrm{rev}}$ satisfying $\sigma_A^{\mathrm{rev}} \circ \sigma_A \simeq \mathrm{id}_A$.

Definition 6.11. Define a category Δ_σ

- objects are pairs $([n], s\colon \{1,\dots,n\} \to \{\pm1\})$
- a morphism from $([n], s\colon \{1,\dots,n\} \to \{\pm1\})$ to $([m], t\colon \{0,1,\dots,m\} \to \{\pm1\})$ is an order-preserving map $[n] \to [m]$ in Δ.

L: maybe better to write s as a function defined on the set of *morphisms* $i < i+1$ in $[n]$.

Construction 6.12. Define a functor $\mathrm{Cut}\colon \Delta_\sigma^{\mathrm{op}} \to \mathrm{Assoc}_\sigma^\otimes$:

- For each $([n], s)$, we have $\mathrm{Cut}([n], s) = \langle n\rangle$.
- Given a morphism $\alpha\colon ([n], s) \to ([m], t)$, the associated morphism $\mathrm{Cut}([n], s) \to \mathrm{Cut}([m], t)$ consists of

– On underlying finite pointed sets $\langle m\rangle \to \langle n\rangle$, Cut agrees with that appearing in [Lur17, Construction 4.1.2.9]

– Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j-1 < j$, we may regard $s\colon \langle n\rangle^\circ \to \{\pm 1\}$ and likewise $t\colon \langle m\rangle^\circ \to \{\pm 1\}$. Define $u\colon \mathrm{Cut}(\alpha)^{-1}(\langle n\rangle^\circ) \to \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\mathrm{Cut}(\alpha)(j))$.

Lemma 6.13. *The functor* $\mathrm{Cut}\colon \Delta_\sigma^{\mathrm{op}} \to \mathrm{Assoc}_\sigma^\otimes$ *exhibits* $\Delta_\sigma^{\mathrm{op}}$ *as an approximation to the* ∞*-operad* $\mathrm{Assoc}_\sigma^\otimes$.

L: I think the proof of this lemma is not too different from the proof of Proposition 4.1.2.11 of [Lur17]; the point here is just to unravel the definitions of locally coCartesian and Cartesian; the morphisms in $\Delta_\sigma^{\mathrm{op}}$ *are a little more complicated than* Δ^{op}*, but not by much.*

Notation 6.14. Let $\mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ exhibit $\mathcal{C}$ as $\mathbb{E}_\sigma$-monoidal. Let $\mathcal{C}^\circledast$ denote the fiber product $\mathcal{C}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes} \Delta_\sigma^{\mathrm{op}}$.

Definition 6.15. Say that a morphism $([n], s) \to ([m], t)$ is *inert* if the induced map $\mathrm{Cut}([m], t) \to \mathrm{Cut}([n], s)$ is an inert morphism in $\mathrm{Assoc}_\sigma^\otimes$.

Definition 6.16. A $\mathbb{R}^\sigma$*-planar operad* is an ∞-category $\mathcal{O}^\circledast$ equipped with a functor $q\colon \mathcal{O}^\circledast \to \Delta_\sigma^{\mathrm{op}}$ so that

1. For every object $X \in \mathcal{O}^\circledast$ and every inert morphism $\alpha\colon ([n], s) \to q(X)$ in Δ_σ, there is a q-cocartesian morphism $\overline{\alpha}\colon X \to Y$ satisfying $q(\overline{\alpha}) = \alpha$

2. Let X be an object satisfying $q(X) = ([n], s)$, and choose q-cocartesian morphisms $\overline{\alpha}_i\colon X \to X_i$ corresponding to the morphism $([i-1 < i], s_i) \to ([n], s)$ which is the inclusion on underlying sets and satisfies $s_i(i) = s(i)$. Then the morphisms $\overline{\alpha}_i$ exhibit X as the q-product of the X_i.

3. For each $n \geq 0$, the construction $C \mapsto \{C_i\}_{1 \leq i \leq n}$ induces an equivalence of ∞-categories

$$\mathcal{O}^\circledast \times_{\Delta_\sigma^{\mathrm{op}}} \{([n], s)\} \xrightarrow{\sim} \left(\mathcal{O}^\circledast \times_{\Delta_\sigma^{\mathrm{op}}} \{([1], s|_{\{i\}})\}\right)^{\times n}$$

We say that a morphism α in $\mathbb{R}^\sigma$-planar operad is *inert* if it is q-cocartesian and $q(\alpha)$ is inert in $\Delta_\sigma^{\mathrm{op}}$ in the sense of Definition 6.15.

Definition 6.17. Let $q\colon \mathcal{O}^\circledast \to \Delta_\sigma^{\mathrm{op}}$ be a $\mathbb{R}^\sigma$-planar operad. An $\mathbb{A}_\infty^\sigma$-algebra object of $\mathcal{O}^\circledast$ is a section of q which carries inert morphisms to inert morphisms. Write $\mathrm{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O})$ for the full subcategory of $\mathrm{Fun}_{\Delta_\sigma^{\mathrm{op}}}(\Delta_\sigma^{\mathrm{op}}, \mathcal{O}^\circledast)$ on $\mathbb{A}_\infty^\sigma$-algebra objects.

Proposition 6.18. *Let* $\mathcal{O}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ *be a fibration of* ∞*-operads. Then precomposition with the functor* Cut *of Construction 6.12 induces an equivalence of* ∞*-categories*

$$\mathrm{Alg}_{\mathrm{Assoc}_\sigma}(\mathcal{O}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O}) .$$

Proof. Combine Lemma 6.13 with [Lur17, Theorem 2.3.3.23]. □

Definition 6.19. Define a colored operad $\mathbf{LM}_{\mathrm{inv}}$

(i) The set of objects of $\mathbf{LM}_{\mathrm{inv}}$ has two elements, which we denote by $\mathfrak{a}, \mathfrak{m}$.

(ii) Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{LM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{LM}_{\mathrm{inv}}$. If $Y = \mathfrak{a}$, then $\mathrm{Mul}_{\mathbf{LM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \to \{\pm 1\}$ if $X_i = \mathfrak{a}$ for all i, and empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{LM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying either

- $X_{i_n} = \mathfrak{m}$ and $c(i_n) = 1$ and $X_j = \mathfrak{a}$ otherwise
- $X_{i_1} = \mathfrak{m}$ and $c(i_n) = -1$ and $X_j = \mathfrak{a}$ otherwise

(iii) The composition law on $\mathbf{LM}_{\mathrm{inv}}$ is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4

Remark 6.20. There is a colored operad $\mathbf{RM}_{\mathrm{inv}}$ defined exactly in the same way as $\mathbf{LM}_{\mathrm{inv}}$ in Definition 6.19. In the interest of precision: $\mathbf{RM}_{\mathrm{inv}}$ has the same objects $\mathfrak{a}, \mathfrak{m}$. Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{RM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{RM}_{\mathrm{inv}}$. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{RM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying either

- $X_{i_n} = \mathfrak{m}$ and $c(i_n) = -1$ and $X_j = \mathfrak{a}$ otherwise
- $X_{i_1} = \mathfrak{m}$ and $c(i_n) = 1$ and $X_j = \mathfrak{a}$ otherwise

Remark 6.21. Restricting to the objects which are both called $\mathfrak{a}$, we see that both $\mathbf{LM}_{\mathrm{inv}}$ and $\mathbf{RM}_{\mathrm{inv}}$ have a sub-colored operad which is canonically identified with $\mathbf{Assoc}_{\mathrm{inv}}$ of Definition 6.4.

Remark 6.22. There is a map of colored operads $\iota\colon \mathrm{LM} \to \mathrm{LM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}$ to $\mathfrak{a}$. On $\mathrm{Mul}_{\mathrm{LM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{LM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.5. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{id}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is a map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{RM} \to \mathrm{LM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}$ to $\mathfrak{a}$. On $\mathrm{Mul}_{\mathrm{RM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{LM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{rev}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.5. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{rev}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

Definition 6.23. Define a colored operad $\mathbf{BM}_{\mathrm{inv}}$

(i) The set of objects of $\mathbf{BM}_{\mathrm{inv}}$ has three elements, which we denote by $\mathfrak{a}_\ell, \mathfrak{a}_r, \mathfrak{m}$.

(ii) Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{BM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{BM}_{\mathrm{inv}}$. If $Y = \mathfrak{a}_\ell$ (resp. $Y = \mathfrak{a}_r$), then $\mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \to \{\pm 1\}$ if $X_i = \mathfrak{a}_\ell$ (resp. $X_i = \mathfrak{a}_r$) for all i, and empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the subset of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying: if there is exactly one index i_k so that $X_{i_k} = \mathfrak{m}$, either

- $c(i_k) = 1$, $X_j = \mathfrak{a}_\ell$ for $j < i_k$ and $X_j = \mathfrak{a}_r$ for $j > i_k$; or
- $c(i_k) = -1$, $X_j = \mathfrak{a}_\ell$ for $j > i_k$ and $X_j = \mathfrak{a}_r$ for $j < i_k$

(iii) The composition law on $\mathbf{BM}_{\mathrm{inv}}$ is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4

Remark 6.24. The colored operad $\mathbf{BM}_{\mathrm{inv}}$ has a canonical involution σ which fixes $\mathfrak{m}$, exchanges $\mathfrak{a}_\ell$ and $\mathfrak{a}_r$, and sends a morphism (λ, c) to $(\lambda^{\mathrm{rev}}, I \xrightarrow{c} \{\pm 1\} \xrightarrow{\cdot(-1)} \{\pm 1\})$.

Remark 6.25. There is a map of colored operads $\iota\colon \mathrm{BM} \to \mathrm{BM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}_-$ to $\mathfrak{a}_\ell$ and $\mathfrak{a}_+$ to $\mathfrak{a}_r$. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}_\pm\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.5. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{id}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is *also* a map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{BM} \to \mathrm{BM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and and sends $\mathfrak{a}_-$ to $\mathfrak{a}_r$ and $\mathfrak{a}_+$ to $\mathfrak{a}_\ell$. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}_\pm\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.5. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{rev}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_{-1}\}$ where c_{-1} is the constant function on $I \sqcup \{j\}$ with value -1.

Definition 6.26. Let $\mathcal{LM}^{\otimes}_{\mathrm{inv}}$, $\mathcal{RM}^{\otimes}_{\mathrm{inv}}$, and $\mathcal{BM}^{\otimes}_{\mathrm{inv}}$ denote the associated ∞-operads (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.27. We can describe the category $\mathcal{LM}^{\otimes}_{\mathrm{inv}}$ as follows:

(1) An object of $\mathcal{LM}^{\otimes}_{\mathrm{inv}}$ is a pair $(\langle n\rangle, S)$ where S is a subset of $\langle n\rangle^{\circ}$.

(2) Morphisms $(\langle m\rangle, T) \to (\langle n\rangle, S)$ consist of a map $(\alpha\colon \langle m\rangle \to \langle n\rangle, \lambda\colon \langle m\rangle^{\circ} \to \{\pm 1\})$ in $\mathrm{Assoc}^{\otimes}_{\sigma}$ satisfying:

- The map α takes $T \cup \{*\}$ to $S \cup \{*\}$
- For each $s \in S$, then $\alpha^{-1}(\{s\})$ contains exactly one element t_s of T, and it is maximal (resp. minimal) with respect to the linear ordering on $\alpha^{-1}(\{s\})$ if $\lambda(t_s) = 1$ (resp. $\lambda(t_s) = -1$).

Remark 6.28. We can describe the category $\mathcal{BM}^{\otimes}_{\mathrm{inv}}$ as follows:

(1) An object of $\mathcal{BM}^{\otimes}_{\mathrm{inv}}$ is a triple $(\langle n\rangle, c_+, c_-)$ where $c_{\pm}$ are functions $\langle n\rangle^{\circ} \to \{0,1\}$ and $c_-(i) \le c_+(i)$ for all $i \in \langle n\rangle^{\circ}$.

(2) Morphisms $(\langle m\rangle, c_+, c_-) \to (\langle n\rangle, c'_+, c'_-)$ consist of a map $(\alpha\colon \langle m\rangle \to \langle n\rangle, \lambda\colon \langle m\rangle^{\circ} \to \{\pm 1\})$ in $\mathrm{Assoc}^{\otimes}_{\sigma}$ satisfying: if $j \in \langle n\rangle^{\circ}$ and $\alpha^{-1}(j) = \{i_1 < i_2 < \cdots < i_\ell\}$,

- If $c_-(j) = c_+(j)$, then
$$c'_-(j) = c_-(i_1) \le c_+(i_1) = c_-(i_2) \le c_+(i_2) \cdots\cdots c_-(i_{m-1}) \le c_+(i_m) = c'_+(j)$$
- If $c_-(j) < c_+(j)$, then there exists a unique k so that $c_-(i_k) < c_+(i_k)$ and
$$\begin{aligned}\lambda(i_k)\cdot c'_-(j) = \lambda(i_k)\cdot c_-(i_1) \le \lambda(i_k)\cdot c_+(i_1) = \lambda(i_k)\cdot c_-(i_2) \le \lambda(i_k)\cdot c_+(i_2)\cdots \\ \lambda(i_k)\cdot c_-(i_{m-1}) \le \lambda(i_k)\cdot c_+(i_m) = \lambda(i_k)\cdot c'_+(j)\end{aligned}$$

Remark 6.29. Each morphism $\varphi \in \mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ determines a linear ordering ℓ on the set I and a function $s\colon I \to \{\pm 1\}$. Passing from φ to the pair (ℓ, s) determines a map of colored operads $j\colon \mathbf{BM}_{\mathrm{inv}} \to \mathbf{Assoc}_{\mathrm{inv}}$. The map j induces a morphism of ∞-operads $\mathcal{BM}^{\otimes}_{\mathrm{inv}} \to \mathrm{Assoc}^{\otimes}_{\sigma}$ which we will also denote by j. For any $\mathbb{E}_{\sigma}$-monoidal ∞-category $\mathcal{C}$, restriction along j sends an $\mathbb{E}_{\sigma}$-algebra $A\colon \mathrm{Assoc}_{\sigma} \to \mathcal{C}^{\otimes}$ to the pair (A, A) where A is regarded as an involutive bimodule over itself. L: hermitian

Remark 6.30. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.22 induce maps of ∞-operads $\iota\colon \mathcal{LM}^{\otimes} \to \mathcal{LM}^{\otimes}_{\mathrm{inv}}$ and $\iota^{\mathrm{rev}}\colon \mathcal{RM}^{\otimes} \to \mathcal{LM}^{\otimes}_{\mathrm{inv}}$.

Remark 6.31. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.25 induce maps of ∞-operads $\iota, \iota^{\mathrm{rev}}\colon \mathcal{BM}^{\otimes} \to \mathrm{BM}^{\otimes}_{\sigma}$. There are canonical identifications $\iota \circ \mathrm{rev} \simeq \sigma \circ \iota^{\mathrm{rev}}$ where σ is the involution on $\mathrm{BM}^{\otimes}_{\sigma}$ induced by Remark 6.24 and rev is the involution on $\mathcal{BM}^{\otimes}$ of [Lur17, Construction 4.6.3.1].

Remark 6.32. There are canonical maps of operads $\mathcal{LM}^{\otimes}_{\mathrm{inv}} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ and $\mathcal{RM}^{\otimes}_{\mathrm{inv}} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ sending $\mathfrak{a}$ to $\mathfrak{a}_\ell$, resp. $\mathfrak{a}_r$ and making the diagram

$$\begin{tikzcd} \mathrm{Assoc}^{\otimes} \arrow[r] \arrow[d, "\sigma"] & \mathcal{LM}^{\otimes}_{\mathrm{inv}} \arrow[d, "\mathrm{rev}"] \arrow[dr] & \\ & & \mathcal{BM}^{\otimes}_{\mathrm{inv}} \\ \mathrm{Assoc}^{\otimes} \arrow[r] & \mathcal{RM}^{\otimes}_{\mathrm{inv}} \arrow[ur] & \end{tikzcd}$$

commute, where rev is (an involutive version of) the reversal involution of [Lur17, Remark 4.6.3.2].

Definition 6.33. Let $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}_{\sigma}$ and $\mathcal{D}^{\otimes} \to \mathrm{Assoc}^{\otimes}_{\sigma}$ be fibrations of ∞-operads and let $\mathcal{M}$ be an ∞-category. Suppose given a fibration of ∞-operads $q\colon \mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}_{\mathrm{inv}}$ together with equivalences $\mathcal{O}^{\otimes}_{\mathfrak{a}} \simeq \mathcal{C}^{\otimes}$ and $\mathcal{O}^{\otimes}_{\mathfrak{m}} \simeq \mathcal{M}$. Let $L\,{}^{\sigma}\mathrm{Mod}(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{LM}_{\mathrm{inv}}}(\mathcal{O})$. We will refer to $L\,{}^{\sigma}\mathrm{Mod}(\mathcal{M})$ as the *∞-category of left hermitian module objects of $\mathcal{M}$*.

Suppose given a fibration of ∞-operads $q\colon \mathcal{O}^{\otimes} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ together with equivalences $\mathcal{O}^{\otimes}_{\mathfrak{a}_\ell} \simeq \mathcal{C}^{\otimes}$, $\mathcal{O}^{\otimes}_{\mathfrak{a}_R} \simeq \mathcal{D}^{\otimes}$ and $\mathcal{O}^{\otimes}_{\mathfrak{m}} \simeq \mathcal{M}$. Let ${}^{\sigma}\mathrm{Mod}(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{BM}_{\mathrm{inv}}}(\mathcal{O})$. We will refer to ${}^{\sigma}\mathrm{Mod}(\mathcal{M})$ as the *∞-category of hermitian bimodule objects of $\mathcal{M}$*. Composition with the inclusions $\mathrm{Assoc}^{\otimes}_{\sigma} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ induces a categorical fibration

L: Lurie gives this a name (Definition 4.2.1.12 *weakly enriched*)– not sure what to call this. something *bi-enriched*?

$${}^{\sigma}\mathrm{Mod}(\mathcal{M}) = \mathrm{Alg}_{/\mathcal{BM}_{\mathrm{inv}}}(\mathcal{O}) \to \mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{C}) \times \mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{D}).$$

If A is an Assoc_{σ}-algebra object of $\mathcal{C}$, we let ${}^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ denote the fiber ${}^{\sigma}\mathrm{Mod}(\mathcal{M}) \times_{\mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{C})} \{A\}$. We will refer to ${}^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ as the *∞-category of hermitian A-bimodule objects of $\mathcal{M}$*.

Definition 6.34. Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ be a fibration of ∞-operads. We say that q exhibits $\mathcal{O}_{\mathfrak{m}}$ as $\mathbb{E}_{\sigma}$-bitensored over $\mathcal{O}_{\mathfrak{a}_\ell}$ and $\mathcal{O}_{\mathfrak{a}_r}$ if q is a cocartesian fibration.

Remark 6.35. Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ be a cocartesian fibration of ∞-operads. Then q is classified by a map $\chi\colon \mathcal{BM}_{\mathrm{inv}}^{\otimes} \to \mathrm{Cat}_{\infty}$. By Remark 6.31, we can think of q as giving two $\mathbb{E}_{\sigma}$ algebras $\mathcal{C}$, $\mathcal{D}$ in Cat_{∞} with an ∞-category $\mathcal{M}$ equipped with both the structure of a $\mathcal{C}$-$\mathcal{D}$-bimodule (equivalently, the structure of a left $\mathcal{C} \times \mathcal{D}_{\mathrm{rev}}$-module) and of a $\mathcal{D}$-$\mathcal{C}$-bimodule, and an autoequivalence $\sigma_{\mathcal{M}}\colon \mathcal{M} \simeq \mathcal{M}$ of order two which is linear with respect to the autoequivalence $\mathcal{C} \times \mathcal{D}_{\mathrm{rev}} \xrightarrow{\mathrm{flip}} \mathcal{D}_{\mathrm{rev}} \times \mathcal{C} \xrightarrow{\sigma_{\mathcal{D}}^{-1} \times \sigma_{\mathcal{C}}} \mathcal{D} \times \mathcal{C}_{\mathrm{rev}}$.

Remark 6.36. Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{LM}_{\mathrm{inv}}^{\otimes}$ be a cocartesian fibration of ∞-operads. Consider a left hermitian module object $F\colon \mathcal{LM}_{\mathrm{inv}}^{\otimes} \to \mathcal{O}^{\otimes}$. By Remark 6.32, F determines an associative algebra A of $\mathcal{C}$ with an equivalence of algebras $\sigma_A\colon A \simeq \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$, an object $M \in \mathcal{M}$ so that M (resp. $\sigma_{\mathcal{M}}(M)$) is equipped with the structure of a left A-module (resp. right $\sigma_{\mathcal{C}}(A)$-module). Furthermore, we have an equivalence $\sigma_M\colon M \simeq \sigma_{\mathcal{M}}(M)$ which is linear with respect to the equivalence $A \xrightarrow{\sigma_A} \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$.

L: is this related to "modules with involution" from [Cal+20, §3.1]?

Remark 6.37. Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{BM}_{\mathrm{inv}}^{\otimes}$ be a cocartesian fibration of ∞-operads. Consider a hermitian module object $F\colon \mathcal{BM}_{\mathrm{inv}}^{\otimes} \to \mathcal{O}^{\otimes}$. By Remark 6.32, F determines an associative algebra A of $\mathcal{C}$ with an equivalence of algebras $\sigma_A\colon A \simeq \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$ and an associative algebra B of $\mathcal{D}$ with an equivalence of algebras $\sigma_B\colon B \simeq \sigma_{\mathcal{D}}(B)^{\mathrm{rev}}$, an object $M \in \mathcal{M}$ so that M (resp. $\sigma_{\mathcal{M}}(M)$) is equipped with the structure of a A-B-bimodule (resp. $\sigma_{\mathcal{D}}(B)$-$\sigma_{\mathcal{C}}(A)$-bimodule). Furthermore, we have an equivalence $\sigma_M\colon M \simeq \sigma_{\mathcal{M}}(M)$ which is linear with respect to the equivalence $A \otimes B \xrightarrow{\mathrm{flip}} B \otimes A \xrightarrow{\sigma_B^{-1} \otimes \sigma_A} \sigma_{\mathcal{D}}(B)^{\mathrm{rev}} \otimes \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$.

L: when $\mathcal{C} = \mathcal{D}$ and $\sigma_{\mathcal{M}}$ and $\sigma_{\mathcal{C}}$ are both the identity and $A = B$, I think this recovers the "module with involution" from [Cal+20, §3.1].

Construction 6.38. Define a functor $\mathrm{MCut}\colon \Delta_{\sigma}^{\mathrm{op}} \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$:

L: maybe this overloaded notation is not good. I'm running out of ideas.

- For each $([n], s)$, we have $\mathrm{MCut}([n], s) = \langle n{+}1 \rangle \simeq \mathrm{RCut}_0([n])$ where RCut is from [Lur17, Construction 4.8.4.4].
- Given a morphism $\alpha\colon ([n], s) \to ([m], t)$, the associated morphism $\mathrm{MCut}([m], t) \to \mathrm{MCut}([n], s)$ consists of
  - On underlying finite pointed sets $\langle m+1 \rangle \to \langle n+1 \rangle$, MCut agrees with (the reverse of) that appearing in [Lur17, Construction 4.2.2.6]
  - Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j-1 < j$, we may regard $s\colon \langle n+1 \rangle^{\circ} \to \{\pm 1\}$ and likewise $t\colon \langle m+1 \rangle^{\circ} \to \{\pm 1\}$. Define $u\colon \mathrm{MCut}(\alpha)^{-1}(\langle n+1 \rangle^{\circ}) \to \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\mathrm{MCut}(\alpha)(j))$.

L: check later

Remark 6.39. We can identify $\mathrm{Assoc}_{\sigma}^{\otimes}$ with the full subcategory of $\mathcal{RM}_{\mathrm{inv}}^{\otimes}$ spanned by objects of the form $(\langle n \rangle, \langle n \rangle^{\circ})$. We can regard Construction 6.12 as defining a functor $\Delta_{\sigma}^{\mathrm{op}} \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$. For each $([n], s) \in \Delta_{\sigma}^{\mathrm{op}}$, there is a map of sets $\theta\colon \mathrm{MCut}([n], s) \to \mathrm{Cut}([n], s)$ defined as in [Lur17, Remark 4.2.2.8]. Concretely, on underlying pointed sets, θ takes the form

L: check that the signs s work out!

$$\theta\colon \langle n+1 \rangle \to \langle n \rangle$$
$$k \mapsto \begin{cases} k-1 & \text{if } k > 0 \\ * & \text{if } k = 0, *. \end{cases}$$

This construction determines a morphism γ in the ∞-category $\mathrm{Fun}\left(\Delta_{\sigma}^{\mathrm{op}}, \mathcal{RM}_{\mathrm{inv}}^{\otimes}\right)$, or equivalently a map $\gamma\colon \Delta_{\sigma}^{\mathrm{op}} \times \Delta^1 \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$.

Lemma 6.40. *The morphism $\gamma\colon \Delta_{\sigma}^{\mathrm{op}} \times \Delta^1 \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$ defined in Remark 6.39 exhibits $\Delta_{\sigma}^{\mathrm{op}} \times \Delta^1$ as an approximation to the ∞-operad $\mathcal{RM}_{\mathrm{inv}}^{\otimes}$.*

Definition 6.41. Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}^\otimes_{\mathrm{inv}}$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}^\otimes_{\mathfrak{m}}$ as weakly bi-enriched over $\mathcal{O}^\otimes_{\mathfrak{a}}$. Let γ be as in Remark 6.39. Let $R^\sigma\mathrm{Mod}^{\mathbb{A}^\sigma_\infty}(\mathcal{M})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{RM}^\otimes_{\mathrm{inv}}}\left(\Delta^{\mathrm{op}}_\sigma \times \Delta^1, \mathcal{O}^\otimes\right)$ spanned by those maps $f\colon \Delta^{\mathrm{op}}_\sigma \times \Delta^1 \to \mathcal{O}^\otimes$ satisfying

1. The restriction of f to $\Delta^{\mathrm{op}}_\sigma \times \{1\}$ belongs to $\mathrm{Alg}_{\mathbb{A}^\sigma_\infty}(\mathcal{O})$ of Definition 6.17
2. If $\alpha\colon ([m], s) \to ([n], t)$ so that $\alpha(0) = 0$, then the induced map $f([m], s, 0) \to f([n], t, 0)$ is an inert map in $\mathcal{O}^\otimes$
3. for each object $([n], s)$ in $\Delta^{\mathrm{op}}_\sigma$, the induced map $f([n], s, 0) \to f([n], s, 1)$ is an inert map in $\mathcal{O}^\otimes$

Example 6.42. Let $\mathcal{C}^\otimes \to \mathcal{RM}^\otimes$ be a fibration of ∞-operads. Restriction along the map of ∞-operads $\mathcal{RM}^\otimes_{\mathrm{inv}} \to \mathrm{Assoc}^\otimes_\sigma$ induced by Remark 6.29 induces a map $\mathbb{E}_\sigma\,\mathrm{Alg}\,(\mathcal{C}) \to R^\sigma\mathrm{Mod}\,(\mathcal{C})$ which is a section of the projection map $R^\sigma\mathrm{Mod}\,(\mathcal{C}) \to \mathbb{E}_\sigma\,\mathrm{Alg}\,(\mathcal{C})$.

L: see Example 4.2.1.17 of higher algebra

L: fibration?

Notation 6.43. Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}^\otimes_{\mathrm{inv}}$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}^\otimes_{\mathfrak{m}}$ as weakly bi-enriched over $\mathcal{O}^\otimes_{\mathfrak{a}}$. Define a new simplicial set $\overline{\mathcal{M}}^\circledast$ by the following universal property

$$\hom_{\mathrm{sSet}_{/\Delta^{\mathrm{op}}_\sigma}}\left(K, \overline{\mathcal{M}}^\circledast\right) \simeq \hom_{\mathrm{sSet}_{/\mathcal{RM}^\otimes_{\mathrm{inv}}}}\left(K \times \Delta^1, \mathcal{O}^\otimes\right) .$$

Here we regard $K \times \Delta^1$ as a simplicial set over $\mathcal{RM}^\otimes_{\mathrm{inv}}$ via the composite $K \times \Delta^1 \to \Delta^{\mathrm{op}}_\sigma \times \Delta^1 \xrightarrow{\gamma} \mathcal{RM}^\otimes_{\mathrm{inv}}$ where γ is from Remark 6.39.

Unwinding definitions, we see that a vertex in $\overline{\mathcal{M}}^\circledast$ lying over an object $([n], s\colon \{1, \dots, n\} \to \{\pm 1\}) \in \Delta^{\mathrm{op}}_\sigma$ corresponds to a morphism α in $\mathcal{O}^\otimes$ whose image in $\mathcal{RM}^\otimes_{\mathrm{inv}}$ is the map $(\langle n+1\rangle, \{0\}) \to (\langle n\rangle, \varnothing)$. Now let $\mathcal{M}^\circledast$ denote the full simplicial subset of $\overline{\mathcal{M}}^\circledast$ spanned by those vertices for which α is inert.

L: this might be off–revisit later!

Remark 6.44. Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}^\otimes_{\mathrm{inv}}$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}^\otimes_{\mathfrak{m}}$ as weakly enriched over $\mathcal{O}^\otimes_{\mathfrak{a}}$. By [Lur09, Example 4.3.1.4 & Proposition 4.3.2.15], composition with the inclusion $\{0\} \to \Delta^1$ induces a trivial Kan fibration $\mathcal{M}^\circledast \xrightarrow{\sim} \mathcal{O}^\otimes \times_{\mathcal{RM}^\otimes_{\mathrm{inv}}} \Delta^{\mathrm{op}}_\sigma$. In particular, the fiber of $\mathcal{M}^\circledast$ over an object $([n], s) \in \Delta^{\mathrm{op}}_\sigma$ is canonically equivalent to $\mathcal{M} \times \mathcal{C}^{\times n}$.

Finally, since q is a categorical fibration and categorical fibrations are closed under pullback and composition with trivial fibrations, q induces categorical fibrations $\mathcal{M}^\circledast \to \mathcal{C}^\circledast \to \Delta^{\mathrm{op}}_\sigma$.

L: Jacob explains this in a really terse way–just by citing Prop 4.3.2.15 of HTT. It does just follow from definitions/observatio but there are many (for instance, definition of inert edge).

Lemma 6.45. *Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}^\otimes_{\mathrm{inv}}$ be a cocartesian fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}^\otimes_{\mathfrak{m}}$ as tensored over $\mathcal{O}^\otimes_{\mathfrak{a}}$. Then the associated functor $\mathcal{M}^\circledast \to \mathcal{C}^\circledast$ (Notation 6.14) is a locally coCartesian fibration.*

Proposition 6.46. *Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}^\otimes_{\mathrm{inv}}$ be a cocartesian fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}^\otimes_{\mathfrak{m}}$ as tensored over $\mathcal{O}^\otimes_{\mathfrak{a}}$. Then precomposition with the functor* MCut *of Construction 6.38 induces an equivalence of ∞-categories*

$$R^\sigma\mathrm{Mod}(\mathcal{M}) \simeq \mathrm{Alg}_{/\mathcal{RM}_{\mathrm{inv}}}(\mathcal{O}) \xrightarrow{\sim} R^\sigma\mathrm{Mod}^{\mathbb{A}^\sigma_\infty}(\mathcal{M}) .$$

Proof. Combine Lemma 6.40 with [Lur17, Theorem 2.3.3.23]. ☐

6.2 Part (b)

Proposition 6.47. *Let $\mathcal{C}$ be an involutive monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is bitensored over $\mathcal{C}$. Let K be a simplicial set so that $\mathcal{M}$ admits K-indexed limits, and let $\theta\colon R^\sigma\mathrm{Mod}(\mathcal{M}) \to \mathrm{Alg}^\sigma(\mathcal{C})$ be the forgetful functor. Then*

L: This statement is [Lur17, Proposition 4.2.3.1] with some words changed; no claim of originality here.

(1) For every commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & R^\sigma\mathrm{Mod}(\mathcal{M}) \arrow[d, "\theta"] \\ K^\triangleleft \arrow[r] \arrow[ur, dashed] & \mathrm{Alg}^\sigma(\mathcal{C}) , \end{tikzcd}$$

there exists a dashed arrow which is a θ-limit diagram.

(2) *An arbitrary map $\overline{g}\colon K^{\triangleleft} \to R^{\sigma}\mathrm{Mod}(\mathcal{M})$ is a θ-limit diagram if and only if the induced map $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.*

Proof. $\square$ L: todo

Corollary 6.48. *θ is a cartesian fibration, and a morphism $f\colon \Delta^1 \to R^{\sigma}\mathrm{Mod}(\mathcal{M})$ is θ-cartesian if and only if the image of f in $\mathcal{M}$ is an equivalence.*

Corollary 6.49. *Let $\mathcal{C}$ be an involutive monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is bitensored over $\mathcal{C}$. Let K be a simplicial set so that $\mathcal{M}$ admits K-indexed limits, and let $\theta\colon R^{\sigma}\mathrm{Mod}(\mathcal{M}) \to \mathrm{Alg}^{\sigma}(\mathcal{C})$ be the forgetful functor. Let A be an involutive algebra object of $\mathcal{C}$. Then*

(1) *$R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ admits K-indexed limits.*

(2) *A diagram $K^{\triangleleft} \to R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ is a limit diagram if and only if the induced diagram $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.*

(3) *Given a morphism $A \to B$ of involutive algebra objects of $\mathcal{C}$, the induced functor $R^{\sigma}\mathrm{Mod}_B(\mathcal{M}) \to R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ preserves K-indexed limits.*

6.3 Towards (e)

Construction 6.50. Define a functor $\mathrm{Pr}\colon \mathbf{LM}^{\otimes}_{\mathrm{inv}} \times \mathbf{RM}^{\otimes}_{\mathrm{inv}} \to \mathbf{BM}^{\otimes}_{\mathrm{inv}}$.

(1) Let $(\langle m\rangle, S)$ be an object of $\mathbf{LM}^{\otimes}_{\mathrm{inv}}$ and let $(\langle n\rangle, T)$ be an object of $\mathbf{RM}^{\otimes}_{\mathrm{inv}}$. Let $\mathrm{Pr}((\langle m\rangle, S),(\langle n\rangle, T)) = (X_*, c_-, c_+)$ where X_*, c_-, c_+ are described in [Lur17, Construction 4.3.2.1(1)].

(2) Let $(\alpha,\lambda)\colon (\langle m\rangle, S) \to (\langle m'\rangle, S')$ be a morphism in $\mathbf{LM}^{\otimes}_{\mathrm{inv}}$ and let $(\beta,\mu)\colon (\langle n\rangle, T) \to (\langle n'\rangle, T')$ be a morphism in $\mathbf{RM}^{\otimes}_{\mathrm{inv}}$. Write $\mathrm{Pr}((\langle m'\rangle, S'),(\langle n'\rangle, T')) = (X'_*, c'_-, c'_+)$. Then $\mathrm{Pr}\left((\alpha,\lambda),(\beta,\mu)\right)$ is the unique morphism in $\mathbf{BM}^{\otimes}_{\mathrm{inv}}$ lying over the map $\gamma\colon X_* \to X'_*$ described by

   (i) $\gamma(i,j) = \begin{cases} (\alpha(i),\beta(j)) & \text{if } \alpha(i) \in \langle m'\rangle^{\circ}, \beta(j) \in \langle n'\rangle^{\circ} \\ * & \text{otherwise.} \end{cases}$

   (ii) Let $i' \in \langle m'\rangle^{\circ} \setminus S'$ and $j' \in T'$ so $j' = \beta(j)$ for a unique $j \in T$. Then the linear ordering on $\gamma^{-1}(i',j') = \alpha^{-1}(i') \times \{j\}$ is (a) determined by the map α if $\mu(j) = 1$, and (b) it is the reverse of the linear ordering determined by α if $\mu(j) = -1$. The map $\gamma^{-1}(i',j') = \alpha^{-1}(i') \times \{j\} \to \{\pm 1\}$ is determined by λ if $\mu(j) = 1$ and it is $-\lambda$ if $\mu(j) = -1$.

   (iii) Likewise if $i' \in S'$ and $j' \in \langle n'\rangle^{\circ} \setminus T'$

   (iv) Let $i' \in S'$ and $j' \in T'$ so $i' = \alpha(i)$ for a unique $i \in S$ and $j' = \beta(j)$ for a unique $j \in T$. Then $\gamma^{-1}\{(i',j')\} = \{i\} \times \beta^{-1}\{(j')\} \sqcup_{\{(i,j)\}} \alpha^{-1}\{(i')\} \times \{j\}$. Define $\gamma^{-1}\{(i',j')\} \to \{\pm 1\}$ by $\lambda \times \mu$. Endow $\gamma^{-1}\{(i',j')\}$ with the linear ordering from [Lur17, Construction 4.3.2.1(2)(iv)] if $\lambda(i) = \mu(j)$ and endow $\gamma^{-1}\{(i',j')\}$ with the opposite ordering if $\lambda(i) \neq \mu(j)$ (or equivalently, if $\lambda(i) = -\mu(j)$).

Write Pr for the induced map $\mathcal{LM}^{\otimes}_{\sigma} \times \mathcal{RM}^{\otimes}_{\sigma} \to \mathcal{BM}^{\otimes}_{\sigma}$ of ∞-categories.

Construction 6.51. Let $q\colon \mathcal{C}^{\otimes} \to \mathcal{BM}^{\otimes}_{\sigma}$ be a fibration of ∞-operads. We define a map of simplicial sets $\overline{L^{\sigma}\mathrm{Mod}}(\mathcal{C}_{\mathfrak{m}})^{\otimes} \to \mathcal{RM}^{\otimes}_{\sigma}$ by the universal property: For any simplicial set $K \to \mathcal{RM}^{\otimes}_{\sigma}$, there is a bijection

$$\mathrm{Hom}_{\mathrm{sSet}_{/\mathcal{RM}^{\otimes}_{\sigma}}}\left(K, \overline{L^{\sigma}\mathrm{Mod}}(\mathcal{C}_{\mathfrak{m}})^{\otimes}\right) \simeq \mathrm{Hom}_{\mathrm{sSet}_{/\mathcal{BM}^{\otimes}_{\sigma}}}\left(\mathcal{LM}^{\otimes}_{\sigma} \times K, \mathcal{C}^{\otimes}\right).$$

Let $L^{\sigma}\mathrm{Mod}(\mathcal{C}_{\mathfrak{m}})^{\otimes}$ denote the full simplicial subset of $\overline{L^{\sigma}\mathrm{Mod}}(\mathcal{C}_{\mathfrak{m}})^{\otimes}$ spanned by those vertices which correspond to a vertex $X \in \mathcal{RM}^{\otimes}_{\sigma}$ and a functor $F\colon \mathcal{LM}^{\otimes}_{\sigma}\{X\} \to \mathcal{BM}^{\otimes}_{\sigma}$ which takes inert morphisms in $\mathcal{LM}^{\otimes}_{\sigma}$ to inert morphisms in $\mathcal{BM}^{\otimes}_{\sigma}$.

Remark 6.52. The composite $\mathcal{LM}_\sigma^\otimes \times \{\mathfrak{m}\} \hookrightarrow \mathcal{LM}_\sigma^\otimes \times \mathcal{RM}_\sigma^\otimes \xrightarrow{\mathrm{Pr}} \mathcal{BM}_\sigma^\otimes$ agrees with the inclusion of Remark 6.32. Taking $K \to \mathcal{RM}_\sigma^\otimes$ to be the inclusion $\{\mathfrak{m}\} \hookrightarrow \mathcal{RM}_\sigma^\otimes$, we have an isomorphism of simplicial sets $L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \times_{\mathcal{RM}_\sigma^\otimes} \{\mathfrak{m}\} \simeq L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})$ where $L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})$ is the ∞-category of left modules associated to the fibration of ∞-operads $\mathcal{C}^\otimes \times_{\mathcal{BM}_\sigma^\otimes} \mathcal{LM}_\sigma^\otimes \to \mathcal{LM}_\sigma^\otimes$.

Proposition 6.53. *Let $q\colon \mathcal{C}^\otimes \to \mathcal{BM}_\sigma^\otimes$ be a fibration of ∞-operads. Then*

(1) the induced map $p\colon L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \to \mathcal{RM}_\sigma^\otimes$ is a fibration of ∞-operads

(2) a morphism α in $L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$ is inert if and only if $p(\alpha)$ is inert in $\mathcal{RM}_\sigma^\otimes$ and for all $X \in \mathcal{LM}_\sigma$, $\alpha(X)$ is an inert morphism in $\mathcal{C}^\otimes$.

(3) if q is a cocartesian fibration of ∞-operads, then so is p

(4) if q is a cocartesian fibration of ∞-operads, a morphism α in $L^\sigma\mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$ is p-cocartesian if and only if, for all $X \in \mathcal{LM}_\sigma^\otimes$, $\alpha(X)$ is q-cocartesian in $\mathcal{C}^\otimes$.

Proof. Similar to [Lur17, Proposition 4.3.2.5]. □

Theorem 6.54. *Let $\mathcal{C}$ be an $\mathbb{E}_\sigma$-monoidal ∞-category, and let A be an $\mathbb{E}_\sigma$-algebra in $\mathcal{C}$. Then $L^\sigma\mathrm{Mod}_A(\mathcal{C})$ is right $\mathbb{E}_\sigma$-tensored over $\mathcal{C}$.*

6.4 Endomorphisms

Let $\mathcal{C}$ be an $\mathbb{E}_\sigma$-monoidal ∞-category, and write $\sigma_\mathcal{C}\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ for its involution. Suppose $M \in \mathcal{C}$ is an object equipped with an equivalence $\sigma_M\colon M \simeq \sigma_\mathcal{C}(M)$. By [Lur17, §4.7.1], endomorphisms of M can be regarded as an $\mathbb{E}_1$-algebra in $u(\mathcal{C})^\otimes$, where u is from Remark 6.10. Now σ_M induces an equivalence $\mathrm{End}_\mathcal{C}(M) \simeq \mathrm{End}_\mathcal{C}(\sigma_\mathcal{C}(M))$ On the other hand, $\sigma_\mathcal{C}$ induces an equivalence $\mathrm{End}_\mathcal{C}(\sigma_\mathcal{C}(M)) \simeq \mathrm{End}_\mathcal{C}(M)^{\mathrm{rev}}$. In particular, for any ∞-category $\mathcal{M}$ left $\mathbb{E}_\sigma$-tensored over $\mathcal{C}$ and any object $M \in \mathcal{M}$ which is fixed by the involution on $\mathcal{M}$, we expect the endomorphisms of M to admit the structure of an $\mathbb{E}_\sigma$-algebra in $\mathcal{C}$.

To this end, we will define an ∞-category of objects acting on M, show that it has an $\mathbb{E}_\sigma$-monoidal structure, and locate endomorphisms of M as the final object in this ∞-category. Informally, we may define a category $\mathcal{C}[M]$ whose objects consist of either

- pairs (C, η) where $C \in \mathcal{C}$ and $\eta\colon C \otimes M \to M$ is a morphism in $\mathcal{M}$; or
- pairs (C', ξ) where $C' \in \mathcal{C}$ and $\xi\colon \sigma_\mathcal{M}(M) \otimes C' \to \sigma_\mathcal{M}(M)$.

The monoidal structure is as described in [Lur17, §4.7.1]. Note that given an object (C, η), the involution $\sigma_\mathcal{M}$ on $\mathcal{M}$ sends η to the map $\sigma_\mathcal{M}(C \otimes M) \simeq \sigma_\mathcal{M}(M) \otimes \sigma_\mathcal{C}(C) \to \sigma_\mathcal{M}(M)$. This is the involution on $\mathcal{C}[M]$.

Definition 6.55. Let $p\colon \mathcal{M}^\circledast \to \Delta^1 \times \Delta_\sigma^{\mathrm{op}}$ exhibit $\mathcal{M}^\circledast$ as weakly enriched over $\mathcal{C}^\circledast$. An *enriched morphism* of $\mathcal{M}$ is a diagram

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

satisfying either

- $p(\alpha)$ is the morphism $(0, [1], c_1) \to (0, [0])$ in $\Delta_\sigma^{\mathrm{op}}$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_1\colon \{1\} \to \{\pm 1\}$ is the constant function at $+1$, and
- the map β is inert, and $p(\beta)$ is the morphism $(0, [1], c_1) \to (0, [0])$ in $\Delta^1 \times \Delta_\sigma^{\mathrm{op}}$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

or

- $p(\alpha)$ is the morphism $(0, [1], c_{-1}) \to (0, [0])$ in $\Delta_\sigma^{\mathrm{op}}$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_{-1}\colon \{1\} \to \{\pm 1\}$ is the constant function at -1.
- the map β is inert, and $p(\beta)$ is the morphism $(0, [1], c_{-1}) \to (0, [0])$ in $\Delta^1 \times \Delta_\sigma^{\mathrm{op}}$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

Let $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]}$ denote the full subcategory of $\mathrm{Fun}_{\Delta^1\times\Delta^{\mathrm{op}}_{\sigma}}\left(\Lambda^2_0,\mathcal{M}^{\circledast}\right)$ spanned by the enriched morphisms of $\mathcal{M}$.

Note that there are two evaluation functors $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]}\to\mathcal{M}$. Given $M\in\mathcal{M}$, write $\mathcal{C}[M]:=\{M\}\times_{\mathcal{M}}\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]}\times_{\mathcal{M}}\{M\}$ and refer to it as the endomorphism ∞-category of M.

Definition 6.56. *enriched n-string*

Proposition 6.57 (Segal condition)**.**

7 Categorification and structure

In the course of thinking about the 'involutive' generalization of the statement that given an $\mathbb{E}_1$-algebra, its category of modules is $\mathbb{E}_0$ (and conversely, that given an object in a stable ∞-category, that its endomorphism spectrum is an $\mathbb{E}_1$-algebra), I have run up against some questions.

Question 7.1.

- Can we sidestep an involutive version of the construction of endomorphism categories of [Lur17, §4.7.1]?
- Suppose $\mathcal{C}$ is a monoidal ∞-category and $\mathcal{M}$ is an ∞-category which is enriched over $\mathcal{C}$ in the sense of [Lur17, §4.2.1]. The opposite category $\mathcal{M}^{\mathrm{op}}$ is enriched over $\mathcal{C}$ by [Hei23, §10].

8 Comparing involutive classical Brauer and involutive higher Brauer

Question 8.1.

- If, for a Poincaré ∞-category $(\mathcal{C},\Qoppa)$, there exists a Poincaré object (E,q) so that E is a compact generator, can we rewrite both the category and its Poincaré structure in terms of $\mathrm{End}_{\mathcal{C}}(E)$?
- Can the *property* of an existence of a Poincaré object (E,q) in $(\mathrm{Perf}_X,\Qoppa_L)$ so that E is a compact generator be checked Zariski-locally? See Toën's paper §3.

9 Other

Proposition 9.1. *Assume that X has a* good quotient *Y in the sense of [FW20, Remark 4.20], and write $p\colon X\to Y$ for the quotient map. Let $i\colon U\subseteq Y$ be the largest open subscheme on which $\pi|_{X_U}$ is étale [FW20, Proposition 4.45]. Write $Z(\pi)$ for the closed complement of U regarded as a topological space, and let $j\colon Z(\pi)\to Y$ denote the inclusion[2]. Then $\underline{\mathcal{O}}^{\varphi C_2}$ is in the essential image of $j_*\colon \mathrm{Shv}_{\mathrm{Zar}}(Z(\pi))\to\mathrm{Shv}_{\mathrm{Zar}}(Y)$. In other words, there exists a sheaf $\mathcal{Q}$ of $\mathbb{E}_\infty$-rings on $Z(\pi)$ so that $j_*\mathcal{Q}\simeq\underline{\mathcal{O}}^{\varphi C_2}$.*

L: hypothesis? move to main text?

L: How does $\mathcal{Q}$ relate to the structure sheaf on the branch locus (as reduced subscheme of Y) used in First–Williams?

Proof. Recall that the open-closed decomposition of Y induces a symmetric monoidal récollement

$$\mathrm{Shv}_{\mathrm{Zar}}(U)\xleftarrow{i^*}\mathrm{Shv}_{\mathrm{Zar}}(Y)\xrightarrow{j^*}\mathrm{Shv}_{\mathrm{Zar}}(Z(\pi))\,.$$

Therefore, to show that $\underline{\mathcal{O}}^{\varphi C_2}$ is in the essential image of j_*, it suffices to show that $i^*\left(\underline{\mathcal{O}}^{\varphi C_2}\right)\simeq 0$ as a sheaf on U.

By [FW20, Proposition 4.45], it suffices to show that if y is a point in U, then $\underline{\mathcal{O}}^{\varphi C_2}_y=0$. Since $\underline{\mathcal{O}}^{\varphi C_2}_y=\tau_{\geq 0}\left(\Gamma\mathcal{O}^{tC_2}_{X\times_Y\{y\}}\right)$ where $A=\Gamma\mathcal{O}_{Y,y}\to B=\Gamma\mathcal{O}_{X\times_Y\{y\}}$ is a quadratic étale map so that B has an involution λ and $A=B^\lambda$ is a local ring with maximal ideal $\mathfrak{m}_A$ (therefore B is semilocal by [FW20, Proposition 3.15]), it suffices to show that $\pi_0B^{tC_2}=0$. By [NS18, Lemma I.2.9], we may without loss of generality replace A and B by their 2-completions.

L: Need hypercompleteness to reduce to checking on points? Appeal to Clausen–Mathew and add finite Krull dim hypothesis?

L: this is unnecessary to the proof–but shows that the support of $\mathcal{Q}$ intersects trivially with the open subscheme $Y\left[\frac{1}{2}\right]$.

[2] $Z(\pi)$ is referred to as the *branch locus* in [FW20]

By the recollement of A-modules in terms of $\mathfrak{m}_A$-complete and $A[\mathfrak{m}_A^{-1}]$-modules, it suffices to show that $\left(B^{\wedge}_{\mathfrak{m}_A}\right)^{tC_2} = 0$ and $\left(B[\mathfrak{m}_A^{-1}]\right)^{tC_2} = 0$.

By [FW20, Propositions 3.4 & 3.15], $B\mathfrak{m}_A = J \subseteq B$, where J denotes the Jacobson radical of B. We claim that $B \simeq \lim_i B/J^i$ induces an equivalence $B^{tC_2} \to \lim_i \left(B/J^i\right)^{tC_2}$. Granting the claim, it suffices to show that $\left(B[\mathfrak{m}_A^{-1}]\right)^{tC_2} = 0$ and $\left(B/J^i\right)^{tC_2}$ is zero for each i. Since $(-)^{tC_2}$ is exact and lax symmetric monoidal and each B/J^i can be written as an extension of finitely many B/J-modules, it suffices to show that $\left(B[\mathfrak{m}_A^{-1}]\right)^{tC_2}$ and $(B/J)^{tC_2}$ are zero.

Now observe that $A/\mathfrak{m}_A$ (resp. $A[\mathfrak{m}_A^{-1}]$-algebra) is a field and B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is a quadratic étale $A/\mathfrak{m}_A$-algebra (resp. $A[\mathfrak{m}_A^{-1}]$-algebra). By [FW20, Proposition 3.4(ii)], B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is either a separable quadratic field extension of $A/\mathfrak{m}_A$-algebra (resp. $A[\mathfrak{m}_A^{-1}]$-algebra), or it is isomorphic to $\prod_{C_2} A/\mathfrak{m}_A$ (resp. $\prod_{C_2} A[\mathfrak{m}_A^{-1}]$). In the latter case, the action of C_2 on B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is manifestly free, hence $(B/J)^{tC_2} = 0$ (resp. $B[\mathfrak{m}_A^{-1}]^{tC_2} = 0$). Suppose instead that B/J (resp. $B[\mathfrak{m}_A^{-1}]$) is a separable quadratic field extension of $A/\mathfrak{m}_A$-algebra (resp. $A[\mathfrak{m}_A^{-1}]$-algebra). By [FW20, Proposition 3.4(ii)], $\lambda \otimes_B B/J$ (resp. $\lambda \otimes_B B[\mathfrak{m}_A^{-1}]$) is nontrivial, hence by [Stacks, Lemma 9.21.2, Tag 09DU] the extension $A/\mathfrak{m}_A \to B/J$ (resp. $A[\mathfrak{m}_A^{-1}] \to B[\mathfrak{m}_A^{-1}]$) is Galois. Since C_2 acts freely on B/J as an $A/\mathfrak{m}_A$-module by the normal basis theorem, $(B/J)^{tC_2} = 0$ (resp. $B[\mathfrak{m}_A^{-1}]^{tC_2}$).

We conclude the proof by proving the claim. Since homotopy fixed points commute with arbitrary limits, it suffices to show that $B \simeq \lim_i B/\mathfrak{m}_B^i$ induces an equivalence $B_{hC_2} \to \lim_i \left(B/\mathfrak{m}_B^i\right)_{hC_2}$. This is true because the $B/\mathfrak{m}_B^i$ are uniformly bounded below. □

L: compare [CMM21, Remark 2.8].

Example 9.2. If $\lambda = \mathrm{id}_X$ and $Y = X$, then $\pi_0 \underline{\mathcal{O}}^{\varphi C_2} = \mathcal{O}_Y/2$. On π_0, the norm map $\underline{\mathcal{O}}^e \simeq \mathcal{O}_X \to \underline{\mathcal{O}}^{\varphi C_2}$ takes $f \mapsto f^2$.

pushforwards along quotient maps Two attempts to show the pushforward preserves filtered colimits.

L: DEPRECATED JUNE 3RD: Here are some thoughts towards showing that the canonical map $\mathrm{colim}_{[a,b]} \pi_* \mathbf{M}_{A^e}^{[a,b]} \to \pi_* \mathbf{M}_{A^e}$ is an equivalence. Later: see if Example 3.1.2 here could be useful?

1. Each $\mathbf{M}_{A^e}^{[a,b]}$ and $\mathbf{M}_{A^e}$ is a hypersheaf. For $\mathbf{M}_{A^e}$ this follows from [AG14, Lemma 5.4]; need to prove for $\mathbf{M}_{A^e}^{[a,b]}$.

2. π_* sends hypersheaves on the small étale site of X to hypersheaves on the small étale site of Y; this follows from the proof of [Lur09, Proposition 6.5.2.13].

   L: which I learned from Recollection 1.14 of this paper.

3. Is $\mathrm{colim}_{[a,b]} \pi_* \mathbf{M}_{A^e}^{[a,b]}$ still a hypersheaf? Hypersheaves are not in general closed under colimits, but maybe we can argue using an explicit model for this sheaf?

4. The hypercompletion of the étale ∞-topos of Y has enough points; this follows from [Lur18, Theorem A.4.0.5] and Proposition 3.7.3 of Exodromy.

5. It suffices to show that the canonical map $\mathrm{colim}_{[a,b]} \pi_* \mathbf{M}_{A^e}^{[a,b]} \to \pi_* \mathbf{M}_{A^e}$ is an equivalence on points; use the explicit model from [FW20, Theorem 3.16]?

L: As of June 3, I am suspicious of the next argument/think something has gone wrong–(9.2) is not supposed to hold in this level of generality. I'm not sure what the problem is yet–maybe that π_* does not in fact define a morphism of recollements? Will revisit later.

Lemma 9.3. *Let (X, σ, Y, π) be a scheme with involution and good quotient. Let $\widetilde{U} \subseteq X$ be the largest open subscheme of X on which π is quadratic étale and let $W \subseteq Y$ and $Z = \pi^{-1}(W) \subseteq X$ be the branch and ramification loci of π in the sense of [FW20, Proposition 4.45-4.47] (in particular, W and Z are endowed with the reduced subscheme structure). Assume that $2 \in \mathcal{O}_Y^\times$. Then the pushforward $\pi_* \colon \mathrm{Shv}_{\text{ét}}(X; \mathcal{S}) \to \mathrm{Shv}_{\text{ét}}(Y; \mathcal{S})$ preserves filtered colimits.*

Proof. Since $W \subseteq Y$, $Z \subseteq X$ are closed immersions, there exist récollements

L: this is automatic, but [FW20, Lemma 5.36] asserts that $W \to Y$ is a closed embedding without proof

$$\begin{tikzcd}
\mathrm{Shv}_{\text{ét}}(\widetilde{U};\mathcal{S}) \arrow[r, shift right, "j_{\widetilde{U}*}"'] & \mathrm{Shv}_{\text{ét}}(X;\mathcal{S}) \arrow[l, shift right, "j^*_{\widetilde{U}}"'] \arrow[r, shift right, "i^*_Z"'] & \mathrm{Shv}_{\text{ét}}(Z;\mathcal{S}) \arrow[l, shift right, "i_{Z*}"'] \\
\mathrm{Shv}_{\text{ét}}(U;\mathcal{S}) \arrow[r, shift right, "j_{U*}"'] & \mathrm{Shv}_{\text{ét}}(Y;\mathcal{S}) \arrow[l, shift right, "j^*_U"'] \arrow[r, shift right, "i^*_W"'] & \mathrm{Shv}_{\text{ét}}(W;\mathcal{S}) \arrow[l, shift right, "i_{W*}"']
\end{tikzcd} .$$

Moreover, the pushforward functor π_* is a morphism of récollements in the sense of [Sha21, Definition 2.3]. In particular, the 'components' of π_* (see Observation 2.4 of *loc. cit.*) are

$$\pi_*|_{\mathrm{Shv}_{\text{ét}}(\widetilde{U};\mathcal{S})} = j^*_U \circ \pi_* \circ j_{\widetilde{U}*} = j^*_U \circ (\pi \circ j_{\widetilde{U}})_* = j^*_U \circ (j_U \circ \pi|_{\widetilde{U}})_* = j^*_U \circ j_{U*} \circ (\pi|_{\widetilde{U}})_*$$
$$\pi_*|_{\mathrm{Shv}_{\text{ét}}(Z;\mathcal{S})} = i^*_W \circ \pi_* \circ i_{Z*} = i^*_W \circ (\pi \circ i_Z)_* = i^*_W \circ (i_W \circ \pi|_Z)_* = i^*_W \circ i_{W*} \circ (\pi|_Z)_* .$$

Now $j^*_U \circ j_{U*} \circ (\pi|_{\widetilde{U}})_* \simeq (\pi|_{\widetilde{U}})_*$ (resp. $i^*_W \circ i_{W*} \circ (\pi|_Z)_* \to (\pi|_Z)_*$) induced by the counit of the adjunction (j^*_U, j_{U*}) (resp. (i^*_W, i_{W*})) is an equivalence because j_{U*} (resp. i_{W*}) is fully faithful, thus

$$\pi_*|_{\mathrm{Shv}_{\text{ét}}(\widetilde{U};\mathcal{S})} \simeq (\pi|_{\widetilde{U}})_* \qquad\qquad \pi_*|_{\mathrm{Shv}_{\text{ét}}(Z;\mathcal{S})} \simeq (\pi|_Z)_* . \tag{9.1}$$

We note for further reference the equivalences

L: diagram instead?

$$\pi_*|_{\mathrm{Shv}_{\text{ét}}(\widetilde{U};\mathcal{S})} \circ j^*_{\widetilde{U}} \simeq j^*_U \circ \pi_* \qquad\qquad \pi_*|_{\mathrm{Shv}_{\text{ét}}(Z;\mathcal{S})} \circ i^*_W \simeq i^*_Z \circ \pi_* . \tag{9.2}$$

Suppose given a filtered diagram $\mathcal{F}_\bullet$ in $\mathrm{Shv}_{\text{ét}}(X;\mathcal{S})$ and write $\mathcal{F}$ for its colimit. We would like to show that the canonical map $\operatorname{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \to \pi_*(\mathcal{F})$ is an equivalence. Since j^*_U, i^*_W are jointly conservative (by definition of a récollement, see [Lur17, Definition A.8.1(e)]), it suffices to show that the canonical maps

$$\begin{aligned} j^*_U\left(\operatorname*{colim}_\bullet \pi_*(\mathcal{F}_\bullet)\right) &\to j^*_U\pi_*(\mathcal{F}) \\ i^*_W\left(\operatorname*{colim}_\bullet \pi_*(\mathcal{F}_\bullet)\right) &\to i^*_W\pi_*(\mathcal{F}) \end{aligned} \tag{9.3}$$

are equivalences. Since j^*_U and i^*_W preserve all colimits, the morphisms of (9.3) can be identified with the canonical maps

$$\operatorname*{colim}_\bullet j^*_U\pi_*(\mathcal{F}_\bullet) \simeq \operatorname*{colim}_\bullet(\pi|_{\widetilde{U}})_*(j^*_{\widetilde{U}}\mathcal{F}_\bullet) \to (\pi|_{\widetilde{U}})_*(j^*_{\widetilde{U}}\mathcal{F}_\bullet)$$
$$\operatorname*{colim}_\bullet i^*_W\pi_*(\mathcal{F}_\bullet) \simeq \operatorname*{colim}_\bullet(\pi|_W)_*(i^*_W\mathcal{F}_\bullet) \to (\pi|_Z)_* i^*_Z(\mathcal{F}) ,$$

respectively, where we have used (9.2). Now $\pi|_W$ is an equivalence and $\pi|_{\widetilde{U}}$ is finite étale by [FW20, Proposition 4.47], hence $(\pi|_{\widetilde{U}})_*$ and $(\pi|_W)_*$ preserve filtered colimits. $\square$

9.1 Algebras with genuine involution

Recollection 9.4. Assume $\mathcal{C}$ is a presentable monoidal ∞-category such that the monoidal product $- \otimes -: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then there is an ∞-category $\mathrm{LMod}(\mathcal{C})$ [Lur17, Example 4.2.1.18] whose objects are pairs (A, M) where A is an associative algebra object of $\mathcal{C}$ and M is a left A-module. Write a, m respectively for the canonical forgetful functors $\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$, $\mathrm{LMod}(\mathcal{C}) \to \mathcal{C}$ which send (A, M) to A and M, resp. Then a is a cocartesian fibration [Lur17, Corollary 4.2.3.7], hence it is classified by a functor $\mathrm{mod}\colon \mathrm{Alg}(\mathcal{C}) \to \mathrm{Cat}_\infty$.

The functor s of [Lur17, Example 4.2.1.17] determines a natural transformation $\eta\colon * \to \mathrm{mod}$, where $*\colon \mathrm{Alg}(\mathcal{C}) \to \{*\} \hookrightarrow \mathrm{Cat}_\infty$ is the constant functor at the trivial category, or equivalently

$$\begin{tikzcd}
& \mathcal{U} \arrow[d] \\
\mathrm{Alg}(\mathcal{C}) \arrow[ur, "\eta"] \arrow[r, "\mathrm{mod}"'] & \mathrm{Cat}_\infty
\end{tikzcd} \tag{9.5}$$

where $\mathcal{U}$ is the universal cocartesian fibration. Now consider the functor $o\colon \mathcal{U} \to \mathrm{Cat}_\infty$ which sends $(\mathcal{D}, d \in \mathcal{D})$ to the undercategory $\mathcal{D}_{d/-}$. Define $\mathrm{LMod}(\mathcal{C})_{*/-}$ to be the cocartesian fibration over $\mathrm{Alg}(\mathcal{C})$ classified by $o \circ \eta \circ \mathrm{mod}$.

Variant 9.6. Let C be as in Recollection 9.4. There is a similar construction where left modules is replaced by *bimodules* [Lur17, Definition 4.3.1.12].

Construction 9.4. Regard $\mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})$ as a category with C_2-action given by taking the opposite/reverse algebra. There are functors $b\colon \mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2} \to \mathrm{LMod}(\mathrm{Sp})$ and $b\colon \mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2} \to \mathrm{BiMod}(\mathrm{Sp})_{*/-}$ so that $(a,m)\circ b$ and $(a,m)\circ b_*$ are (canonically) equivalent to $(-^e)\otimes(-^e)^{\mathrm{op}}, (-)^e$. Informally, an $\mathbb{E}_1$-algebra with involution B can be regarded as a $B\otimes B^{\mathrm{op}}$-module in a canonical way, and there is a canonical $B\otimes B^{\mathrm{op}}$-module map $B\otimes B^{\mathrm{op}} \to B$.

Definition 9.5. The category of $\mathbb{E}_1$-*algebras with genuine involution* is defined to be the limit of the Cat_∞-valued diagram

$$\begin{tikzcd}
 & \mathrm{LMod}(\mathrm{Sp}) \arrow[d, "{a,m}"] & \mathrm{LMod}\left(\mathrm{Sp}^{C_2}\right) \arrow[l, "(-)^e"'] \arrow[dd, "{a,m}"] \\
 & \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})\times\mathrm{Sp} & \\
\mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2}\times_{\mathrm{Sp}}\mathrm{Sp}^{\Delta^1} \arrow[uur, bend left, "b\circ(-^e)"] \arrow[ur, "{(-^e)\otimes(-^e)^{\mathrm{op}},(-)^e}"] \arrow[rr, "N^{C_2}(-^e)\times U"'] & & \mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right)\times\mathrm{Sp}^{C_2} \arrow[ul, "(-)^e\times(-)^e"]
\end{tikzcd} \tag{9.7}$$

where

- b is the functor/section of Construction 9.4
- U is the 'underlying' C_2-spectrum functor $\mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{BC_2}\times_{\mathrm{Sp}}\mathrm{Sp}^{\Delta^1} \to \mathrm{Sp}^{BC_2}\times_{\mathrm{Sp}}\mathrm{Sp}^{\Delta^1} \simeq \mathrm{Sp}^{C_2}$
- The upper right trapezoid commutes canonically by definition of LMod (and the fact that the functors a, m are given by restriction to subcategories of $LM^{\otimes}$).

Write $\mathbb{E}_1\mathrm{Alg}^{\mathrm{gi}}\left(\mathrm{Sp}^{C_2}\right)$ for the ∞-category of $\mathbb{E}_1$-algebras with genuine involution.

Definition 9.6. The category of $\mathbb{E}_\sigma$-*algebras* is defined to be the limit of the Cat_∞-valued diagram

$$\begin{tikzcd}
 & \mathrm{BiMod}(\mathrm{Sp})_{*/-} \arrow[d, "{a,m}"] & \mathrm{BiMod}\left(\mathrm{Sp}^{C_2}\right)_{*/-} \arrow[l, "(-)^e"'] \arrow[dd, "{a,m}"] \\
 & \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})\times\mathrm{Sp} & \\
\mathbb{E}_1\mathrm{Alg}\,(\mathrm{Sp})^{hC_2}\times_{\mathrm{Sp}}\mathrm{Sp}^{\Delta^1} \arrow[uur, bend left, "b_*\circ(-^e)"] \arrow[ur, "{(-^e)\otimes(-^e)^{\mathrm{op}},(-)^e}"] \arrow[rr, "N^{C_2}(-^e)\times U"'] & & \mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right)\times\mathrm{Sp}^{C_2} \arrow[ul, "(-)^e\times(-)^e"]
\end{tikzcd} \quad . \tag{9.8}$$

Write $\mathbb{E}_\sigma\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right)$ for the ∞-category of $\mathbb{E}_\sigma$-algebras.

Variant 9.9. Let the base be R an $\mathbb{E}_\infty$-algebra or Poincaré ring instead of $\mathbb{S}^0$.

Remarks 9.7. 1. Compare [AGH21, Corollary 3.10].

2. There are canonical forgetful functors $\mathbb{E}_\sigma\mathrm{Alg} \to \mathbb{E}_1\mathrm{Alg}^{\mathrm{gi}} \to \mathbb{E}_1\mathrm{Alg}^{hC_2} \to \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})$.

Construction 9.8. Let $R, R^{\varphi C_2} \to R^{tC_2}$ be a Poincaré ring. There is a functor $\left(\mathrm{Mod}^\omega_{(-)}, Ϙ_{(-)}\right) : \mathbb{E}_\sigma\mathrm{Alg}_R \to \left(\mathrm{Cat}^h_R\right)_{(\mathrm{Mod}^\omega_R, Ϙ_R)/-}$.

Lemma 9.9. *Let $R, R^{\varphi C_2} \to R^{tC_2}$ be a Poincaré ring.*

1. *The functor of Construction 9.8 factors through the subcategory $(\mathrm{Cat}^p_R)_{(\mathrm{Mod}^\omega_R, Ϙ_R)/-}$. In other words, a map of $\mathbb{E}_\sigma$-R-algebras $A \to B$ induces a duality-preserving map of R-linear Poincaré ∞-categories.*
2. *Write $\mathrm{Mod}^p \colon \mathbb{E}_1^{\mathrm{gi}}\mathrm{Alg}_R \to (\mathrm{Cat}^p_R)_{(\mathrm{Mod}^\omega_R, Ϙ_R)/-}$ for the canonical factorization from part 1. Then Mod^p is fully faithful.*

Proof.

L: Proof of the first point should be quite similar to/a relative variant on Corollary 3.4.2, Lemma 3.4.3 of [Cal+20].

Let A, B be $\mathbb{E}_1$-R-algebras with genuine involution. Then there is a fiber sequence

$$\begin{tikzcd} \hom_{\mathrm{Mod}^p_R/-}(\mathrm{Mod}^p_A, \mathrm{Mod}^p_B) \arrow[r] & \hom_{\mathrm{Mod}^p_R\text{-linear}}(\mathrm{Mod}^p_A, \mathrm{Mod}^p_B) \arrow[r] \arrow[d, "\sim"] & \hom_{\mathrm{Mod}^p_R\text{-linear}}(\mathrm{Mod}^p_R, \mathrm{Mod}^p_B) \arrow[d, "\sim"] \\ & \mathrm{Pn}(\mathrm{Mod}_{A^{\mathrm{op}}\otimes_R B}, Ϙ_{A^{\mathrm{op}}\boxtimes B}) \arrow[r] & \mathrm{Pn}(\mathrm{Mod}_B, Ϙ_B) \end{tikzcd}$$

where we have used Corollary ??. The fiber of the horizontal map over the point $(B, q_B) \in \mathrm{Pn}(\mathrm{Mod}_B, Ϙ_B)$ (q_B is the canonical nondegenerate form on B) is

L: this refers to main text; will be fixed when we move it.

L: IN PROGRESS: should be some sort of $\mathbb{E}_\sigma$ version of [AG14, Proposition 3.1].

□

Now we observe that given a Poincaré object (x, q) of $(\mathcal{C}, Ϙ_\mathcal{C})$, its endomorphism algebra admits a canonical lift to a $\mathbb{E}_\sigma$-algebra.

Construction 9.10. There is a functor $\mathrm{End}(-) \colon \left(\mathrm{Cat}^h_R\right)_{(\mathrm{Mod}^\omega_R, Ϙ_R)} \to \mathbb{E}_\sigma\mathrm{Alg}$ lifting the functor $\left(\mathrm{Cat}^h_R\right)_{(\mathrm{Mod}^\omega_R, Ϙ_R)} \to \mathbb{E}_1\mathrm{Alg}^{hC_2}$ of [Cal+20, Proposition 3.1.16].

Theorem 9.11. *The functors of Construction 9.8 and 9.10 form an adjoint pair.*

L: make everything Poincaré

Lemma 9.12. *The right adjoint of Construction 9.10 preserves filtered colimits.*

L: similar to [AG14, Lemma 3.4].

Proof. □

Proposition 9.13. *Let A be an $\mathbb{E}_1$-R-algebra with genuine involution. Then A is compact in $\mathbb{E}_1^{\mathrm{gi}}\mathrm{Alg}_R$ if and only if Mod^p_A is compact in $\mathrm{Cat}^{\mathrm{p}}_{\infty R}$.*

L: similar to [AG14, Proposition 3.5].

Proof. The only if part of the statement follows from observing that Mod^p admits a right adjoint which preserves filtered colimits (Lemma 9.12) and [Lur09, Lemma 5.5.1.4]. □

L: other half of the statement

Proposition 9.14. *Let A be an $\mathbb{E}_1$-R-algebra with genuine involution. If Mod^p_A is dualizable in $\mathrm{Cat}^{\mathrm{p}}_{\infty R}$, then A is compact in $\mathbb{E}_1^{\mathrm{gi}}\mathrm{Alg}_R$.*

L: similar to [AG14, Proposition 3.11].

Proof. □

10 Speculative norm for Brauer group at the level of infinity categories

We would want a functor from A^e-linear stable idempotent complete infinity categories to A^L-linear stable idempotent complete infinity categories in order to get an extension of our exact sequence to the right. Here is what I think might do it:

Construction 10.1. Let $\lambda : A^e \to A^e$ denote the involution. Consider the functor

$$\mathrm{Mod}_{\mathrm{Mod}^{\omega}_{A^e}}(\mathrm{Cat}^{st}_{\infty,\mathrm{idem}}) \xrightarrow{\left(-\otimes_{\mathrm{Mod}^{\omega}_{A^e}} \lambda^* -\right)^{hC_2}} \mathrm{Mod}_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}}(\mathrm{Cat}^{st}_{\infty,\mathrm{idem}})$$

which we will denote by $N_{A^{hC_2}/A}$. This functor is symmetric monoidal and we have that the composite with the base change functor gives

$$\left(\mathcal{C} \otimes_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}} \mathrm{Mod}^{\omega}_{A^e} \otimes_{\mathrm{Mod}^{\omega}_{A^e}} \lambda^*(\mathcal{C} \otimes_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}} \mathrm{Mod}^{\omega}_{A^e})\right)^{hC_2} \simeq \mathcal{C}^{\otimes_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}} 2}$$

For $\mathcal{C} \in \mathrm{Mod}_{A^e}$, we have that

N: Maybe... We would want this to be true but I don't see a proof immediately...

$$(\mathcal{C} \otimes_{\mathrm{Mod}_{A^e}} \lambda^*\mathcal{C})^{hC_2} \otimes_{\mathrm{Mod}^{hC_2}_{A^e}} \mathrm{Mod}_{A^e} \simeq \mathcal{C}^{\otimes_{\mathrm{Mod}_{A^e}} 2}$$

via the functor which forgets the C_2-action.

Lemma 10.2. *The composite* $\mathrm{PnBr}(A) \to \mathrm{br}(A^e) \to \mathrm{Pic}(\mathrm{Mod}_{\mathrm{Mod}^{hC_2}_{A^e}})$ *is nullhomotopic.*

Proof. The underlying category of a Poincaré invertible category is self-dual, and so its square will vanish. Since the functor is naturally nullhomotopic so too is the composite after applying the functor $\mathrm{Pic}(-)$. $\square$

There is thus a map $\mathrm{PnBr}(-) \to \mathcal{F}(-)$, where $\mathcal{F}(-)$ is the fiber. Delooping both fiber sequences we see that we get a map of fiber sequences

$$\begin{tikzcd} \mathrm{Pic}(\mathrm{Mod}_{A^L}(\mathrm{Sp}^{C_2})) \arrow[r] \arrow[d] & \mathrm{PnBr}(A) \arrow[r] \arrow[d] & \mathrm{br}(A^e) \arrow[d, "="] \\ \mathrm{Pic}(\mathrm{Mod}^{hC_2}_{A^e}) \arrow[r] & \mathcal{F}(A) \arrow[r] & \mathrm{br}(A^e) \end{tikzcd}$$

from which we see that the middle horizontal map must be an equivalence whenever $\frac{1}{2} \in A^e$ and $A^{\varphi C_2} = 0$.

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