Poincaré Schemes

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L: Potential alt. title: Involutions on derived Azumaya algebras and pairings of twisted derived categories

Abstract

In this paper, we use the formalism of Poincaré ∞-categories, as developed by [Cal+20a], to define and study derived moduli stacks of line bundles with hermitian metrics and of Azumaya algebras equipped with an involution. Our moduli spaces give rise to cohomological invariants which we term the Poincaré Picard group and the Poincaré Brauer group; these are enhancements of the ordinary Picard and Brauer groups which incorporate the data of an involution on the base. We show that in the étale case away from characteristic 2, the Poincaré Brauer group recovers the involutive Brauer group of [PS92]. We also define the Poincaré Picard and Brauer groups for Poincaré rings in spectra, and compute these invariants for the sphere spectrum and other examples.

L: this is not strictly speaking true unless we add "up to Morita equivalence"–revisit later?

Contents

1 Introduction

1.1 Azumaya algebras and their involutions

Let X be a scheme. Gabber showed that if X is quasi-compact and separated and admits an ample line bundle, the collection of sheaves of Azumaya $\mathcal{O}_X$-algebras up to Morita equivalence is in bijection with the torsion subgroup of $H^2_{\text{ét}}(X;\mathbb{G}_m)$ [Gab81]. On the other hand, to $\mathcal{A}$ we may associate the presentable $\mathcal{O}_X$-linear ∞-category $\mathcal{D}(\mathcal{A})$. If $\mathcal{A}$ and $\mathcal{A}'$ are Morita equivalent, then $\mathcal{D}(\mathcal{A}')$ and $\mathcal{D}(\mathcal{A})$ are equivalent in $\mathrm{Pr}^L_{\mathcal{O}_X}$, and that $\mathcal{A}$ is Azumaya implies that $\mathcal{D}(\mathcal{A})$ is invertible in $\mathrm{Pr}^L_{\mathcal{O}_X}$ with respect to the $\mathcal{O}_X$-linear tensor product.

L: AG notation or homotopy notation?

L: who is this observation due to?

Definition 1.1. The *derived Brauer space* of X is $\mathrm{dBr}(X) := \mathrm{Pic}\left(\mathrm{Pr}^L_{\mathcal{O}_X}\right)$.

Toën shows that the assignment $X \mapsto \mathrm{dBr}(X)$ is an étale sheaf and there is an isomorphism $\pi_0\mathrm{dBr}(X) \simeq H^1_{\text{ét}}(X;\mathbb{Z}) \times H^2_{\text{ét}}(X;\mathbb{G}_m)$ [Toë12, Corollary 2.12]. Furthermore, Toën shows that for any qcqs scheme X, any invertible $\mathcal{O}_X$-linear ∞-category $\mathcal{C}$ has a compact generator G; thus we may write $\mathcal{C} \simeq \mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(G)}(\mathrm{Mod}_{\mathcal{O}_X})$ and $\mathrm{End}_{\mathcal{C}}(G)$ is said to be a *derived/generalized (sheaf of) Azumaya algebras* over $\mathcal{O}_X$ (see Theorem 3.7 and Corollary 3.8 of [Toë12]). In particular, Toën's result gives concrete interpretations/realizations of all/not necessarily torsion classes in $H^2_{\text{ét}}(X;\mathbb{G}_m)$, at the cost of using derived methods/considering derived objects.

L: cite Lieblich/de Jong/Caldararu connect to twisted sheaves!

Antieau and Gepner extended Toën's work to connective ring spectra/affine spectral schemes with connective rings of functions in [AG14a].

L: cite this too?

On the other hand, the theory of anti-involutions on Azumaya algebras is an essential tool for studying the behavior of Brauer classes under multiplication by 2 and/or norm maps/corestriction. An anti-involution on an Azumaya algebra over a ring R is an equivalence $\sigma\colon A \xrightarrow{\sim} A^{\mathrm{op}}$ so that $\sigma^{\mathrm{op}} \circ \sigma = \mathrm{id}$. The anti-involution is said to be of type 1 if it acts by the identity on $R =$ the center of A. If instead R is endowed with an involution λ so that the inclusion of the subring of fixed elements $R^\lambda \to R$ exhibits R as a quadratic étale R^λ-algebra and σ agrees with λ on the center of A, then the involution σ is said to be of *type 2*.

If an Azumaya algebra A has an anti-involution, its Brauer class $[A] \in H^2_{ét}(-;\mathbb{G}_m)$ is manifestly 2-torsion. More surprisingly, if an Azumaya algebra A is such that $[A]$ lies in the 2-torsion subgroup of $H^2_{ét}(-;\mathbb{G}_m)$, then there exists an Azumaya algebra A' in the Brauer class of A admitting an anti-involution; the result was proved for R a field by Albert (in fact Albert proves that one can take $A' = A$), for an arbitrary ring by Saltman, and for schemes X with $\frac{1}{2} \in \mathcal{O}_X$ by Parimala–Srinivas [Alb61, §9 Theorem 19; Sal78, Theorem 3.1(a); PS92, Theorem 1].

On the other hand, consider an étale cover $X \to Y$ of degree 2 where $\frac{1}{2} \in \mathcal{O}_Y$, and let λ denote the nontrivial C_2-Galois action on X. There is an *involutive Brauer group* $\mathrm{Br}(X,\lambda)$ consisting of equivalence classes of sheaves of Azumaya $\mathcal{O}_X$-algebras with involutions of the second kind [PS92, p.216]. Parimala–Srinivas showed that the involutive Brauer group sits in an exact sequence $\mathrm{Pic}\, X \xrightarrow{N} \mathrm{Pic}\, Y \to \mathrm{Br}(X,\lambda) \to \mathrm{Br}(X) \xrightarrow{N} \mathrm{Br}(Y)$ [PS92, Theorem 2].

First and Williams observed that the aforementioned two cases comprise two extreme ends/special cases of a spectrum: multiplication by 2 on $H^2_{ét}(-;\mathbb{G}_m)$ may be regarded as a cohomological C_2-transfer map along the 'quotient' map $X = X$, where X is regarded as a scheme with trivial C_2-action [FW20, §1.2]. In other words, the former comprises trivial C_2-actions with 'everywhere ramified' quotient map, whereas the latter comprises free C_2-actions with nowhere ramified quotient maps. Moreover, First and Williams observe that a quotient involves a choice/is extra data; from the perspective of the stacky quotient $X \to X//C_2$, the C_2-action on X is free. Thus in order to study involutions systematically, it is necessary to specify a scheme with involution and the choice of a quotient.

L: connect to 'many choices of different Poincaré structures' on a given ∞-category eventually

The study of involutions on Azumaya algebras goes hand-in-hand with/is inextricably linked to the study of symmetric bilinear/hermitian forms on vector bundles: étale-locally, (classical) Azumaya algebras can be described as endomorphism algebras of vector bundles $\mathcal{A} \simeq \mathcal{E}\mathrm{nd}_X(V)$. Taking transposes and conjugating by an isomorphism $q\colon V \to V^\vee$ adjoint to a nondegenerate pairing $q\colon V \otimes V \to \mathcal{O}_X$ comprise a 'prototypical' example of an involution on $\mathcal{E}\mathrm{nd}_X(V)$. In [Cal+20a; Cal+20b; Cal+21], Calmès–Dotto–Harpaz–Hebestreit–Land–Moi–Nardin–Nikolaus–Steimle introduce a framework for talking about objects of stable ∞-categories equipped with the data of nondegenerate hermitian forms, expanding upon an idea introduced in [Lur11a]. Motivated by this connection between involutions and duality (cf. [Cal+20a, §3]), we use Poincaré ∞-categories to define a derived enhancement of the involutive Brauer group.

L: this is *so close* to Spec of a Poincaré ring but also not quite.

L: insert adjectives

1.2 Main results

Theorem 1.2. *Let $\mathcal{C}$ denote either the category of Poincaré rings (Definition 3.2) or the opposite category of schemes with involution and good quotients (Definition 3.24). Then there are functors*

$$\mathbb{G}_m^{\text{Ϙ}}, \mathrm{PnPic}, \mathrm{PnBr} : \mathcal{C} \to \mathrm{Sp}_{\geq 0}$$

such that

- $\Omega\mathrm{PnBr} \simeq \mathrm{PnPic}$ *and* $\Omega\mathrm{PnPic} \simeq \mathbb{G}_m^{\text{Ϙ}}$*;*
- $\mathbb{G}_m^{\text{Ϙ}}$ *is represented by the free* $\mathbb{E}_\infty$ *ring on one invertible generator* $\mathbb{S}\{x^{\pm 1}\}$ *with a certain genuine* C_2*-structure (see Construction 4.9);*
- PnBr *(and hence all the other functors) satisfy hyperdescent for an analogue of the étale topology on* $\mathcal{C}$ *which incorporates the involution (see Notation 6.1 and Notation 6.2 for the exact constructions.)*

These invariants naturally admit forgetful functors to the underlying Picard and Brauer spaces. The exact connection is the following:

Theorem 1.3. *Let R be a Poincaré ring, and denote the underlying ring spectrum by R^e and the underlying genuine C_2 ring spectrum by R^L. Then there is a fiber sequence*

$$\mathrm{Pic}(\mathrm{Mod}_{R^L}(\mathrm{Sp}^{C_2})) \to \mathrm{PnBr}(R) \to \mathrm{br}(R^e)$$

and upon sheafification there is a similar fiber sequence for schemes with involutions and good quotients.

With this fiber sequence in hand we are able to make several computations of interest. As an example:

Corollary 1.4. *Let $\mathbb{S}^u$ denote the Poincaré ring spectrum associated to the initial Poincaré infinity category. Then*

$$\mathrm{PnBr}(\mathbb{S}^u) \simeq \mathbb{Z} \times B\mathrm{gl}_1(\mathrm{TR}^2(\mathbb{S}; 2))$$

In the case when R is away from characteristic 2 we are furthermore able to simplify the fiber of the comparison map $\mathrm{PnBr}(R) \to \mathrm{br}(R^e)$. As an application of this we obtain a comparison of the Poincaré Brauer group with the involuative Brauer group discussed earlier in the introduction:

Theorem 1.5 (Theorem 6.6)**.** *Let (X, λ, Y, π) be a scheme with involution and good quotient such that $\pi : X \to Y$ is étale and such that $\frac{1}{2} \in \Gamma(\mathcal{O}_Y)$. Then there is an equivalence*

$$\mathrm{PnBr}(X, \lambda, Y, \pi) \cong H^0_{\acute{e}t}(Y; \mathrm{coker}(\pi_*\mathbb{Z}_X \to \mathbb{Z}_Y)) \times \mathrm{Br}(X, \lambda)$$

where $\mathbb{Z}_X$ and $\mathbb{Z}_Y$ are the constant sheaves, and the second term is the involuative Brauer group of Parimala-Srinivas ([PS92]).

1.3 Outline

1.4 Acknowledgements

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L: grants? people? travel funding?

1.5 Conventions

L: Will 'Azumaya algebra' refer to the classical/discrete ones or the derived/generalized ones? Note the distinction between higher (classical) stacks and spectral stacks!

Br^{p}	Poincaré Brauer space
CAlg	∞-category of $\mathbf{E}_\infty$-ring spectra
$\mathrm{CAlg}(\mathcal{S})$	∞-category of $\mathbf{E}_\infty$-spaces
$\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$	∞-category of grouplike $\mathbf{E}_\infty$-spaces
$\mathrm{CAlg}^{\mathrm{p}}$	∞-category of Poincaré ring spectra
$\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$	∞-category of small stable ∞-categories and exact functors
$\mathrm{Cat}_\infty^{\mathrm{p}}$	∞-category of Poincaré ∞-categories
$\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}$	∞-category of idempotent complete Poincaré ∞-categories
$\mathrm{Pic}^{\mathrm{p}}$	Poincaré Picard space
$\mathcal{S}$	∞-category of spaces
Sp	∞-category of spectra

2 Poincaré Structures on Compact Modules

We will use this section to recall notions and results about Poincaré ∞-categories which we require in the sections to follow. This section can safely be skipped by anyone with extensive knowledge of Poincaré ∞-categories, as found in [Cal+20a].

Notation 2.1. Let R be an $\mathbf{E}_\infty$-ring spectrum. We will drop $\mathbf{E}_\infty$ from our notation and simply call R a *ring spectrum*. Moreover, we will denote the ∞-category CAlg(Sp) of commutative algebra objects in the ∞-category of spectra Sp by CAlg.

L: I'm ok with the notation CAlg, but am concerned about dropping $\mathbf{E}_\infty$ because $\mathbb{E}_1$-ring spectra will eventually come up as well.

Let R be a ring spectrum and let Mod_R be the ∞-category of modules over R. We will study Poincaré structures on the ∞-category Mod_R^ω of compact modules over R.

Definition 2.2 (Definition 1.2.2, [Cal+20a])**.** A hermitian infinity category is a pair $(\mathcal{C}, Ϙ)$ where $\mathcal{C}$ is a small stable infinity category and $Ϙ : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is a quadratic functor.

A map of hermitian infinity categories $(f, \eta) : (\mathcal{C}, Ϙ) \to (\mathcal{D}, \Psi)$ is a functor $f : \mathcal{C} \to \mathcal{D}$ and a 2-cell $\eta : Ϙ \implies \Psi \circ f^{\mathrm{op}}$.

Associated to any such quadratic functor $Ϙ : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ we may form a bilinear functor $B_Ϙ(x, y) := Ϙ(x \oplus y)^{\mathrm{red}}$.

Definition 2.3 (Definitions 1.2.2, 1.2.8, [Cal+20a])**.** We say that the associated bilinear form $B_Ϙ$ is (right) non-degenerate if the functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

given by sending $y \mapsto B_Ϙ(-, y)$ lands in the essential image of the Yoneda embedding. Such hermitian structures then define a duality functor $D_Ϙ : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and we say that the hermitian infinity category $(\mathcal{C}, Ϙ)$ is a Poincaré infinity category if the evaluation map $\mathrm{id}_{\mathcal{C}} \implies D_Ϙ D_Ϙ^{\mathrm{op}}$ is an equivalence.

A map of Poincaré infinity categories is a map of hermitian infinity categories which further preserves the duality functors. We will use

$$\mathrm{Cat}_\infty^p \subseteq \mathrm{Cat}_\infty^h$$

to denote the categories of Poincaré and hermitian infinity categories, respectively. These categories have several desirable properties. We collect several of the key properties below.

Theorem 2.4 (Theorem 5.2.7, Proposition 6.1.2, Proposition 6.1.4, Corollary 6.2.9 [Cal+20a])**.**

1. *Both* Cat_∞^p *and* Cat_∞^h *are complete and cocomplete, and the functors* ${\mathrm{Cat}_\infty}^p \to \mathrm{Cat}_\infty^h \to \mathrm{Cat}_\infty^{\mathrm{ex}}$ *are conservative and preserve all small (co)limits.*

2. *There is a symmetric monoidal infinity operad* $(\mathrm{Cat}_\infty^h)^\otimes$ *lifting* Cat_∞^h.

3. *The forgetful functor lifts to a symmetric monoidal map* $f^* : (\mathrm{Cat}_\infty^h)^\otimes \to (\mathrm{Cat}_\infty^{\mathrm{ex}})^\otimes$.

4. *This symmetric monoidal structure restricts to a symmetric monoidal structure on* Cat_∞^p.

5. *The symmetric monoidal structures on* Cat_∞^p *and* Cat_∞^h *are closed.*

V: - characterizati in terms of modules with genuine involution, - characterizati of symmetric monoidal structures, -Pn

3 Poincaré Schemes

We will now specify the objects which we are able to take the Poincaré Picard and Brauer spaces of. These will be schemes which are equipped with a Poincaré structure on their derived categories which is compatible with the symmetric monoidal structure. While this definition is simple and convenient, we find it both technically useful and psychologically comforting to have a definition of such objects closer to a scheme with an involution. We will start by looking at affine objects.

3.1 Poincaré rings

Let R be an $\mathbf{E}_\infty$-ring spectrum. In [Cal+20a, discussion immediately preceding Examples 5.4.10], the 9 authors describe the additional data (referred to as a 'genuine involution') needed to endow Mod_R^ω with a symmetric monoidal Poincaré structure. In the following, we introduce an ∞-category which we refer to as *Poincaré ring spectra* whose objects are the $\mathbf{E}_\infty$-ring spectra with genuine involution of Calmès–Dotto–Harpaz–Hebestreit–Land–Moi–Nardin–Nikolaus–Steimle. There is a natural symmetric monoidal functor from Poincaré ring spectra to Poincaré ∞-categories. We include examples throughout and discuss how ordinary commutative rings with involution can be regarded as Poincaré ring spectra (Example 3.17).

L: reference or omit this sentence

Recollection 3.1. Write $\mathcal{O}_{C_2}$ for the orbit category of C_2. The *Tate-valued norm* is a functor $\mathrm{CAlg}\left(\mathrm{Sp}^{BC_2}\right) \to \mathrm{Fun}\left(\mathcal{O}_{C_2}, \mathrm{CAlg}(\mathrm{Sp})\right)$ Definition 3.8 and Lemma 3.10 of [Yan23]. Note that there is a canonical functor $\iota\colon \Delta^1 = [0 < 1] \to \mathcal{O}_{C_2}$ which sends 0 to C_2/e and 1 to C_2/C_2. Via precomposition with ι, we will regard the Tate-valued norm as a functor $\mathrm{CAlg}\left(\mathrm{Sp}^{BC_2}\right) \to \mathrm{Fun}\left(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})\right)$.

Definition 3.2. Let $\mathrm{CAlg}^{\mathrm{p}}$ be the $(\infty,1)$-category defined by the pullback

$$\begin{tikzcd} \mathrm{CAlg}^{\mathrm{p}} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \arrow[d, "d_1^*"] \\ \mathrm{CAlg}\left(\mathrm{Sp}^{BC_2}\right) \arrow[r, "U(-)\to(-)^{tC_2}"] & \mathrm{Fun}\left(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})\right) \end{tikzcd} \tag{3.3}$$

where $U : \mathrm{Sp}^{BC_2} \to \mathrm{Sp}$ is the functor which forgets the C_2-action and the lower horizontal arrow is the Tate-valued norm of Recollection 3.1. An object of $\mathrm{CAlg}^{\mathrm{p}}$ will be called a *Poincaré ring spectrum.*

Let R be an $\mathbf{E}_\infty$ ring spectrum. A *Poincaré structure* on R is the data of a lift of R to the ∞-category $\mathrm{CAlg}^{\mathrm{p}}$. By definition, it comprises the following data:

- A C_2-action on R via maps of ring spectra, i.e. a functor $\lambda : BC_2 \to \mathrm{CAlg}$.
- An $\mathbb{E}_\infty$-R-algebra C.
- An $\mathbb{E}_\infty$-R-algebra map $C \to R^{tC_2}$.

Here R^{tC_2} is the Tate construction with respect to the given action. Since the Tate construction is lax symmetric monoidal, R^{tC_2} is naturally an R-algebra via the Tate-valued norm.

We will denote objects of $\mathrm{CAlg}^{\mathrm{p}}$ by $\underline{A} = (A, s : A^{\Phi C_2} \to A^{tC_2})$. Here $s : A^{\Phi C_2} \to A^{tC_2}$ is the image of $\underline{A}$ under the top horizontal map above. The use of the notation $A^{\Phi C_2}$ is justified by Lemma 3.6.

Remark 3.4. Recall that there is a symmetric monoidal récollement $\mathrm{Sp}^{C_2} \simeq \mathrm{Sp}^{BC_2} \times_{(-)^{tC_2},\mathrm{Sp},\mathrm{ev}_1} \mathrm{Sp}^{\Delta^1}$ (cf. [MNN17, Theorem 6.24]). There is a commutative diagram of ∞-categories:

$$\begin{tikzcd}
& \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \arrow[d, "d_1^*"] \arrow[r, "d_0^*"] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \arrow[dd, "ev_1"] \\
\mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] \arrow[d, equal] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \arrow[dr, "ev_1"] & \\
\mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[rr, "(-)^{tC_2}"] & & \mathrm{CAlg}(\mathrm{Sp})
\end{tikzcd}.$$

The diagram induces a functor from the pullback of the upper left cospan to the pullback of the lower right cospan $U\colon \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2})$.

Now observe that there is a commutative diagram

$$\begin{tikzcd}
& \mathrm{Fun}\left(\Delta^1 \times \Delta^1, \mathrm{CAlg}(\mathrm{Sp})\right) \arrow[r, "\mathrm{ev}_{00\to01\to11}"] \arrow[d, "\mathrm{ev}_{00\to10\to11}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}\left(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})\right) \arrow[d, "\mathrm{ev}_{0\to2}"] \\
\mathrm{CAlg}(\mathrm{Sp})^{BC_2} \arrow[r, "A\mapsto(A\to(A^{\otimes2})^{tC_2}\to A^{tC_2})"] \arrow[d, "m_{C_2}"] & \mathrm{Fun}\left(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})\right) \arrow[r, "\mathrm{ev}_{0\to2}"] \arrow[d, "\mathrm{ev}_{1\to2}"] & \mathrm{Fun}\left(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})\right) \\
\mathrm{CAlg}(\mathrm{Sp})^{BC_2\times\Delta^1} \arrow[r, "(-)^{tC_2}"] & \mathrm{Fun}\left(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})\right) &
\end{tikzcd}$$

where the upper right square is a pullback by definition of $\Delta^1 \times \Delta^1$ and m_{C_2} is the multiplication map functor of [Yan23, Construction 3.1] precomposed with the inclusion ι of Recollection 3.1. Since the upper right is a pullback, the pullback of the outer span in the top row (which is $\mathrm{CAlg}^{\mathrm{p}}$) agrees with the pullback of the upper left span. On the other hand, the pullback of the upper left span maps to the pullback of the 'tall' span on the left. By the récollement of Sp^{C_2}, the pullback of the 'tall' span on the left may be identified with $\mathbf{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)^{\Delta^1}$. Furthermore, by definition of the Hill–Hopkins–Ravenel norm, we have described a functor

$$\mathrm{CAlg}^{\mathrm{p}} \to \mathrm{Fun}\left(\Delta^1, \mathbf{E}_\infty \operatorname{Alg}\left(\mathrm{Sp}^{C_2}\right)\right) \tag{3.5}$$

which sends a Poincaré ring A to a map of $\mathbf{E}_\infty$-algebras $N^{C_2}(U(A)^e) \to U(A)$.

Lemma 3.6. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{p}} \arrow[r] \arrow[d, "U"] & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \arrow[d, "\mathrm{ev}_1"] \\
\mathrm{CAlg}(\mathrm{Sp}^{C_2}) \arrow[r, "(-)^{\Phi C_2}"] & \mathrm{CAlg}(\mathrm{Sp})
\end{tikzcd}$$

where U is the functor of Remark 3.4 and the upper horizontal arrow is the canonical functor in Definition 3.2.

Proof. Follows from the récollement of Sp^{C_2} and Remark 3.4. $\square$

Remark 3.7. Let R be a ring spectrum. By [Cal+20a, Corollary 5.4.8], a Poincaré structure on R gives rise to a symmetric monoidal lift of Mod_R^ω to the symmetric monoidal ∞-category of Poincaré ∞-categories $\Qoppa_R : (\mathrm{Mod}_R^\omega)^{\mathrm{op}} \to \mathrm{Sp}$. Furthermore, the structure map $R \to C$ gives a canonical lift of $R \in \mathrm{Mod}_R^\omega$ to a Poincaré object $(R, q) \in \mathrm{Pn}\,(\mathrm{Mod}_R^\omega, \Qoppa_R)$.

L: Better notation? Write R for the underlying $\mathbb{E}_\infty$-ring with C_2-action and $q\colon C \to R^{tC_2}$ and $\Qoppa_{(R,q)}$ or $\Qoppa_q$ for the functor.

Observation 3.8. In view of [Yan23, Theorem 1.3 & Definition 3.13], there is a forgetful functor from C_2-$\mathbb{E}_\infty$-algebras in Sp^{C_2} to Poincaré rings which forgets the C_2-equivariance of the map $R \to R^{\varphi C_2}$. Their underlying $\mathbf{E}_\infty$-algebras in Sp^{C_2} agree.

Remark 3.9. By [Cal+20a, Corollary 3.4.2], the assignment $(R, R \to C \to R^{tC_2}) \mapsto (\mathrm{Mod}^{\omega}_R, Ϙ_R)$ promotes to a functor $\mathrm{Mod}^p \colon \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{Cat}^{\mathrm{p}}_{\infty}$. Regard $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ as a symmetric monoidal ∞-category with the symmetric monoidal structure of [Cal+20a, Theorem 5.2.7]. Then it follows from Example 5.1.6 of *loc. cit.* that Mod^p is a symmetric monoidal functor.

L: redun-
monoidal
structure on
$\mathrm{CAlg}^{\mathrm{p}}$ yet!

Notation 3.10. Let R be an $\mathbb{E}_\infty$-ring spectrum. We will denote by $\underline{R}$ the spectrum R with trivial C_2-action. More precisely, $\underline{R} : BC_2 \to \mathrm{Sp}$ is the constant functor.

L: This is commonly used for constant Mackey functors–could be ambiguous

Example 3.11. Let R be a ring spectrum with a C_2-action. If $2 \in \pi_0(R)$ is invertible, we have $\underline{R}^{tC_2} \simeq 0$ by [NS18, Lemma I.2.8]. A Poincaré structure on R is equivalent to the data of an $\mathbb{E}_\infty$-R-algebra $R \to C$.

Example 3.12. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The Tate-valued norm endows R^{tC_2} with a natural R-algebra structure, which induces a Poincaré structure on R given by the factorization $R \xrightarrow{\mathrm{id}} R \to R^{tC_2}$. We will call this Poincaré structure the *Tate Poincaré structure on* R and will denote it by $(R, Ϙ^{\mathrm{t}}_R)$.

Example 3.13. The sphere spectrum $\mathbb{S}$ together with the Tate Poincaré structure will be called the *universal Poincaré ring spectrum* (see [Cal+20a, §4.1]). We will denote it by $(\mathbb{S}, Ϙ_u)$.

Remark 3.14. Let $(R, Ϙ)$ be a ring spectrum associated to a factorization $R \to C \to R^{tC_2}$. A factorization of the map $C \to R^{tC_2}$ through R^{hC_2} induces a section of the canonical map $Ϙ(R) \to \hom_R(R, C) \simeq C$. In that case, we have a splitting $Ϙ(R) \simeq R_{hC_2} \oplus C$.

V: reference pullback that characterizes all quadratic functors

Example 3.15. The Tate Frobenius for the sphere spectrum factors through $\mathbb{S}^{hC_2}$. Therefore, Remark 3.14 implies $Ϙ_u(\mathbb{S}) \simeq \mathbb{S}_{hC_2} \oplus \mathbb{S} \simeq \Sigma^\infty(\mathbb{P}^\infty_{\mathbb{R}}) \oplus \mathbb{S}$.

Example 3.16. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The identity map $\mathrm{id} : R^{tC_2} \to R^{tC_2}$ induces a Poincaré structure on R given by the factorization $R \to R^{tC_2} \xrightarrow{id} R^{tC_2}$. We will call this Poincaré structure the *symmetric Poincaré structure on* R.

Example 3.17. Let R be a connective ring spectrum equipped with a C_2-action via maps of ring spectra. The connective cover $\tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$ of R^{tC_2} induces a Poincaré structure on R given by the factorization $R \to \tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$. We will call this Poincaré structure the *genuine symmetric Poincaré structure on* R.

Example 3.18. Let R be a commutative ring endowed with an involution $\sigma \colon R \xrightarrow{\sim} R$. Write $\underline{R}^{\sigma}$ for the C_2-Green functor with C_2-fixed points R^{C_2}, where R^{C_2} denotes the strict fixed points of the C_2-action on R, and underlying object R. The Mackey functor $\underline{R}^{\sigma}$ is a C_2-$\mathbb{E}_\infty$ ring, therefore in particular we may regard it as a Poincaré ring by Observation 3.8. This is a special case of Example 3.17.

V: copy more examples from notes

Theorem 3.19. *The following statements about* $\mathrm{CAlg}^{\mathrm{p}}$ *hold:*

(1) the pullback diagram (3.3) is homotopy Cartesian;

(2) The category $\mathrm{CAlg}^{\mathrm{p}}$ *has all small colimits;*

(3) the functor $\mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ *preserves all small colimits;*

(4) The category $\mathrm{CAlg}^{\mathrm{p}}$ *has all small limits;*

(5) the functor $\mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ *preserves all small limits;*

(6) the functor $\mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}\left(\mathrm{Sp}^{C_2}\right)$ *of Remark 3.4 preserves all small limits;*

(7) The ∞-category $\mathrm{CAlg}^{\mathrm{p}}$ *is presentable and accessible.*

Proof. To prove (1), it is enough to show that d_1^* is an categorical fibration. In fact, d_1^* is a cocartesian and cartesian fibration; this follows from the existence of colimits and limits, resp., in CAlg(Sp) and Lemma 3.20.

To prove (2), let $p : K \to \mathrm{CAlg}^{\mathrm{p}}$ be a map of simplicial sets, where K is a small simplicial set. Write f' for the functor $\mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ and $g' \colon \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}$ and $f \colon \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2} \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^1}$ and $g \colon \mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^1}$. Choose an extension $\overline{f'p} \colon K^{\triangleright} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ of $f' \circ p$ which is a colimit diagram. By [Lur09, Proposition 4.3.1.5(2)], to show that p admits a colimit, it suffices to exhibit a lift $K^{\triangleright} \to \mathrm{CAlg}^{\mathrm{p}}$ of $\overline{f' \circ p}$ which is an f'-colimit diagram. By [Lur09, Proposition 4.3.1.5(4)] and (1), it suffices to show that there exists an extension $\overline{g'p} \colon K^{\triangleright} \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}$ of $g \circ \overline{f'p}$ which is an $f = d_1^*$-colimit. Observe that $f = d_1^*$ is a cocartesian fibration, the fibers of f admit small colimits, and f-cocartesian transport preserves all small colimits by (1), Lemma 3.20, and the fact that CAlg admits all small colimits. The existence of such an extension $\overline{g'p}$ follows from [Lur09, Corollary 4.3.1.11].

Part (3) follows from (2).

The proofs of parts (4) and (5) are analogous to those of (2) and (3) and have been omitted.

To prove part (6), it suffices to show that $U(-)$ preserves cofiltered limits and finite limits separately. Since limits in $\mathrm{CAlg}(\mathrm{Sp}^{C_2})$ are detected on underlying spectra and geometric fixed points, it suffices to show that $\Phi^{C_2}U(-)$ preserves cofiltered limits and finite limits. We begin by showing that $\Phi^{C_2}U(-)$ preserves finite limits. Let us mostly keep the notation/setup of the proof of part (2), except we are now interested in exhibiting an extension defined on $K^{\triangleleft}$, and K is a finite simplicial set. We have an extension $\overline{f'p} \colon K^{\triangleleft} \to \mathrm{CAlg}^{BC_2}$ which is a limit diagram. Since K is finite and the Tate construction preserves finite limits as a functor $\mathrm{Sp}^{BC_2} \to \mathrm{Sp}$, the composite $g \circ \overline{f'p}$ is a limit diagram in CAlg^{Δ^1}. Taking $\mathcal{E} = *$ in [Lur09, Proposition 4.3.1.5(2)], we see that the extension $\overline{g'p}$ (which exists by the proof of (4)) in fact defines a limit diagram in CAlg^{Δ^2}. The desired result now follows from the fact that limits in functor categories are computed pointwise and Lemma 3.6.

L: Show that $U(-)$ preserves cosifted limits!

Now let us consider part (7). Accessibility of $\mathrm{CAlg}^{\mathrm{p}}$ follows from closure of accessible ∞-categories under fiber products [Lur09, Proposition 5.4.6.6] and accessibility of the ∞-categories in the pullback diagram defining $\mathrm{CAlg}^{\mathrm{p}}$, which itself follows from accessibility of CAlg and [Lur09, Proposition 5.4.4.3]. Now presentability of $\mathrm{CAlg}^{\mathrm{p}}$ follows from accessibility and (2). ☐

Lemma 3.20. *Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{K}$ be a collection of simplicial sets.*

(a) If $\mathcal{C}$ has finite colimits, then the functor $d_1^* \colon \mathrm{Fun}\left(\Delta^2, \mathcal{C}\right) \to \mathrm{Fun}\left(\Delta^1, \mathcal{C}\right)$ exhibits the source as a cocartesian fibration over the target. If $\mathcal{C}$ admits all $\mathcal{K}$-indexed colimits, then the fibers of d_1^* admit all $\mathcal{K}$-indexed colimits and for each morphism $f \colon \alpha \to \beta$ in $\mathcal{C}^{\Delta^1}$, the associated functor $f_* \colon \mathcal{C}^{\Delta^2}_{\alpha} \to \mathcal{C}^{\Delta^2}_{\beta}$ preserves all $\mathcal{K}$-indexed colimits.

(b) The functor d_1^* exhibits the source as a cartesian fibration over the target. If $\mathcal{C}$ has finite limits, then d_1^* exhibits the source as a cartesian fibration over the target. If $\mathcal{C}$ admits all $\mathcal{K}$-indexed limits, then the fibers of d_1^* admit all $\mathcal{K}$-indexed limits and for each morphism $f \colon \alpha \to \beta$ in $\mathcal{C}^{\Delta^1}$, the associated functor $f^* \colon \mathcal{C}^{\Delta^2}_{\beta} \to \mathcal{C}^{\Delta^2}_{\alpha}$ preserves all $\mathcal{K}$-indexed limits.

L: wrote out the condition instead of just saying 'cocartesian transport.' Is it clear that α, β are objects of $\mathcal{C}^{\Delta^1}$ (i.e. morphisms in $\mathcal{C}$) and f is a commutative square?

L: did I reverse the arrows appropriately?

Proof. Let us prove (a). Consider the maps

$$\Delta^1 \simeq \Delta^1 \times \{0\} \xrightarrow{i_0} \Delta^1 \times \Delta^1 \xrightarrow{j_0} \Delta^2$$

where i_0 classifies the morphism $00 \to 10$ and j_0 sends the edge $10 \to 11$ (i.e. the unique nonidentity in $\{1\} \times \Delta^1$) to the identity at 2 in Δ^2 and satisfies $j_0(00) = 0$, $j_0(01) = 1$. Note that their composite is $j_0 \circ i_0 = d_1$. Precomposition induces maps

$$\mathcal{C}^{\Delta^2} \xrightarrow{j^*} \mathcal{C}^{\Delta^1 \times \Delta^1} \xrightarrow{i^*} \mathcal{C}^{\Delta^1}$$

whose composite is d_1^*. Note that i_0^* corresponds to taking the 'source' $\mathrm{Fun}\left(\Delta^1, \mathcal{C}^{\Delta^1}\right) \to \mathcal{C}^{\Delta^1}$.

That d_1^* is an inner fibration follows from [Lur09, Corollary 2.3.2.5]. It remains to show that morphisms in $\mathcal{C}^{\Delta^1}$ have d_1^*-cocartesian lifts. By our assumption on $\mathcal{C}$, the functor category $\mathcal{C}^{\Delta^1}$ also has finite colimits.

Because of our previous observation that i_0^* corresponds to the source functor on the arrow category of $\mathcal{C}^{\Delta^1}$ and $\mathcal{C}^{\Delta^1}$ has finite colimits, i_0^* is a cocartesian fibration. In particular, i_0^*-cocartesian transport along a morphism in $\mathcal{C}^{\Delta^1}$ corresponds to taking a pushout in $\mathcal{C}^{\Delta^1}$.

L: I have omitted details here because this is pretty standard, see e.g. Example 5.5 in these notes. Note to self: see blackboard photo from Tuesday July 22nd.

Furthermore, j_0^* is fully faithful and identifies $\mathcal{C}^{\Delta^2}$ with its essential image, the full subcategory of $\mathcal{C}^{\Delta^1\times\Delta^1}$ on those functors which send the edge $10 \to 11$ to an equivalence in $\mathcal{C}$. To prove part (a), it suffices to show that the essential image of j_0^* is closed under i_0^*-cocartesian transport. This is true because colimits in functor categories are computed pointwise by [Lur09, Corollary 5.1.2.3] and equivalences are stable under pushouts.

To prove (b), it suffices to show that morphisms in $\mathcal{C}^{\Delta^1}$ have d_1^*-cartesian lifts. Consider the maps

$$\Delta^1 \simeq \{1\} \times \Delta^1 \xrightarrow{i_1} \Delta^1 \times \Delta^1 \xrightarrow{j_1} \Delta^2$$

where i_1 classifies the morphism $10 \to 11$ and j_1 sends the edge $00 \to 10$ (i.e. the unique nonidentity in $\Delta^1 \times \{0\}$) to the identity at 0 in Δ^2 and satisfies $j_1(01) = 1$, $j_1(11) = 2$. Note that their composite is $j_1 \circ i_1 = d_1$. Precomposition induces maps

$$\mathcal{C}^{\Delta^2} \xrightarrow{j^*} \mathcal{C}^{\Delta^1\times\Delta^1} \xrightarrow{i^*} \mathcal{C}^{\Delta^1}$$

whose composite is d_1^*. Note that i_1^* corresponds to taking the 'target' $\mathrm{Fun}\left(\Delta^1, \mathcal{C}^{\Delta^1}\right) \to \mathcal{C}^{\Delta^1}$. Because $\mathcal{C}^{\Delta^1}$ has finite limits, i_1^* is a cartesian fibration. In particular, i_1^*-cartesian transport along a morphism in $\mathcal{C}^{\Delta^1}$ corresponds to taking a pullback in $\mathcal{C}^{\Delta^1}$. Furthermore, j_1^* is fully faithful and identifies $\mathcal{C}^{\Delta^2}$ with its essential image,[1] the full subcategory of $\mathcal{C}^{\Delta^1\times\Delta^1}$ on those functors which send the edge $00 \to 10$ to an equivalence in $\mathcal{C}$. To prove part (a), it suffices to show that the essential image of j_1^* is closed under i_1^*-cartesian transport. This is true because limits in functor categories are computed pointwise by the dual to [Lur09, Corollary 5.1.2.3] and equivalences are stable under pullbacks. □

Definition 3.21. Define a functor $(-)^u \colon \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{Fun}\left(\mathcal{O}_{C_2}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0}\right)$ by taking units 'pointwise,' i.e. $(-)^u$ sends a Poincaré ring $(R, R^{\varphi C_2} \to R^{tC_2})$ to the diagram $\mathrm{gl}_1(R^{C_1}) \to \mathrm{gl}_1(R^e)$.

Lemma 3.22. *The functor $(-)^u$ of Definition 3.21 has a left adjoint. Denote the left adjoint by $\mathbb{S}^0[-]$.*

L: To define a functor $\mathbb{S}^0[-]$ out of Sp^{C_2}, precompose with the forgetful map $\mathrm{Sp}^{C_2}_{\geq 0} \to \mathrm{Fun}\left(\mathcal{O}_{C_2}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0}\right)$

Proof. By [Lur09, Corollary 5.5.2.9], in view of Theorem 3.19(7), it suffices to show that $(-)^u$ is accessible and preserves small limits. It is clear that $(-)^u$ preserves small filtered colimits, since they are computed in Sp^{C_2}. Since limits in functor categories are computed pointwise, it suffices to show that the assignments $(R, R^{\varphi C_2} \to R^{tC_2}) \mapsto \mathrm{gl}_1(R^{C_2}), \mathrm{gl}_1(R^e)$ preserve small limits.

L: refer to earlier thm/lemma?

L: need to finish proof that $(-)^u$ preserves small limits!

□

3.2 From schemes with involution to Poincaré structures on module categories

We will now turn our attention to the non-affine case. In this setting we will again want to work with schemes with some notion of a genuine C_2-structure as our model, and then show that this leads to the structure of a scheme together with a symmetric monoidal structure on its derived category.

Philosophically, a scheme with genuine C_2-action should, via a recollement, be given by a scheme X with an involution together with a choice of genuine C_2 quotient $X \to Y$ satisfying certain conditions. It turns out that such a notion has already appeared in the literature on Azumaya algebras with involution.

Recollection 3.23 ([FW20, Remark 4.20])**.** Let X be a scheme with an involution $\lambda \colon X \to X$. A map $\pi \colon X \to Y$ is called a good quotient of X relative to λ if π is C_2-invariant and affine and $\pi_\# \colon \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ induces an isomorphism $\mathcal{O}_Y \simeq (\pi_*\mathcal{O}_X)^{C_2}$. A good quotient of X exists if and only if every C_2-orbit is contained in an affine open subscheme, in which case it is unique up to isomorphism.

[1]Note that in this proof we have produced two different fully faithful embeddings of $\mathcal{C}^{\Delta^2}$ into $\mathcal{C}^{\Delta^1\times\Delta^1}$, but their essential images are different. While i_0^* *is* a cartesian fibration, the essential image of j_0^* is not closed under i_0^*-cartesian transport. Thus, part (b) does not follow from the same construction as (a).

Definition 3.24. Define a category qSch^{C_2} so that

- an object of qSch^{C_2} consists of the data of qcqs schemes X and Y, an involution $\lambda\colon X \to X$, and a morphism $p\colon X \to Y$ which exhibits Y as a *good quotient* of the involution on X in the sense of [FW20, Remark 4.20].
- a morphism from (X, λ, Y, p) to (Z, ν, W, q) consists of a C_2-equivariant morphism $X \to Z$ and a morphism $Y \to W$ so that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & W \end{tikzcd}$$

commutes.

Observation 3.25. Suppose R is a discrete commutative ring with a C_2-action, and regard R as a C_2-Mackey functor via Example 3.18. Then $\operatorname{Spec} R \to \operatorname{Spec}(R^{C_2})$ may be regarded as an object of qSch^{C_2}.

Remark 3.26. Suppose (X, λ, Y, p) is an object of qSch^{C_2} and $j\colon U \to Y$ is a flat map. Then $(X_U, \lambda|_U, U, p|_U)$ is an object of qSch^{C_2}. Affineness and invariance under the C_2-action are stable under pullback, so it suffices to show that $j^*(\pi)$ satisfies $\mathcal{O}_U \simeq (j^*(\pi))_* (\mathcal{O}_{j^*U})^{C_2}$. This follows from the proof of [FW20, Theorem 4.35(i)].

Proposition 3.27. *Write $U\colon \mathrm{qSch}^{C_2} \to \mathrm{qSch}$ for the functor so that $U(X, \lambda, Y, p) = X$. The category qSch^{C_2} has a symmetric monoidal structure $\boxtimes$ so that U is symmetric monoidal, where qSch is endowed with the product symmetric monoidal structure.*

Proof. If X, Z are schemes with involutions λ_X, λ_Z, then $\lambda_X \times \lambda_Z$ endows $X \times Z$ with an involution. It suffices to show that $X \times Z$ admits a good quotient, as a good quotient is a categorical quotient and is therefore unique up to isomorphism. By [FW20, Remark 4.20], a good quotient exists if and only if every C_2-orbit is contained in an affine open subscheme. Consider a C_2-orbit in $X \times Z$. Its image under the projection $\pi_1\colon X \times Z \to X$ (resp. $\pi_2\colon X \times Z \to Z$) is contained in an affine open subscheme $U \subseteq X$ (resp. $V \subseteq Z$). Thus the orbit under consideration is contained in $U \times Z$, which is affine. □

Construction 3.28. Assume that X has a *good quotient* Y in the sense of [FW20, Remark 4.20]. We write $p\colon X \to Y$ for the quotient map. Let $j\colon \operatorname{Spec} A \simeq U \subseteq Y$ be an affine open subscheme of Y. Because p is an affine map, the fiber product $\operatorname{Spec} B := \operatorname{Spec} A \times_Y X$ is an affine open of X which is invariant under the C_2-action. In particular $\operatorname{Spec} B$ inherits a C_2-action from X (hence so does its ring of functions B). Now $A \to B$ acquires the structure of a C_2-Green functor $\underline{\mathcal{O}}(U)$. Regarding $\underline{\mathcal{O}}(U)$ as a C_2-spectrum, by the isotropy separation sequence, we have an equivalence of A-modules $\underline{\mathcal{O}}(U)^{\varphi C_2} \simeq \operatorname{cofib}(\mathrm{tr}\colon B_{hC_2} \to A)$.

Lemma 3.29. *Let X be a scheme with an involution. Assume that X has a* good quotient *Y in the sense of [FW20, Remark 4.20], and write $p\colon X \to Y$ for the quotient map.*

(i) The assignment of Construction 3.28 lifts to a contravariant functor from (the nerve of) the category of affine opens of Y to the ∞-category of Poincaré rings/C_2-$\mathbb{E}_\infty$-rings/Tambara functors.

(ii) The presheaf $\underline{\mathcal{O}}$ of (i) defines a Zariski sheaf.

(iii) Write $p_*\mathcal{O}_X$ for the sheaf of $\mathbb{E}_\infty$-$\mathcal{O}_Y$-algebras (all functors are derived). Then the pushforward p_* induces an equivalence $\mathcal{D}(X) \xrightarrow{\sim} \mathrm{Mod}_{p_*\mathcal{O}_X}$.

Proof. Part (i) follows from a similar argument to [Yan23, Theorem 5.1]; functoriality follows from noting that $\tau_{\geq 0}$ is a functor. Part (ii) follows from Lemma 3.30. To prove part (iii), consider a Zariski cover $\{j_i\colon U_i \to Y\}$ of Y by affine opens. By Zariski descent, $\mathcal{D}(X) \simeq \lim_{p^*(j_i) = p \times_Y j_i\colon U_i \times_Y X \to X} \mathrm{Mod}_{\mathcal{O}_X(U_i \times_Y X)}$ and $\mathrm{Mod}_{p_*(\mathcal{O}_X)} \simeq \lim_{j_i\colon U_i \to Y} \mathrm{Mod}_{p_*\mathcal{O}_X(U_i)}$, hence the result follows. □

Lemma 3.30. *Let K be a simplicial set, and let $f\colon K^{\triangleleft} \to C_2\mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp}^{C_2})$ be a diagram. Then f is a limit diagram if and only if $f^e\colon K^{\triangleleft} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ and $f^{C_2}\colon K^{\triangleleft} \to \mathbb{E}_\infty\mathrm{Alg}(\mathrm{Sp})$ are both limit diagrams.*

Proof. The result follows from the observation that limits in $\mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right)$ are computed in Sp^{C_2}. □

Construction 3.31. Let $p\colon X \to Y$ as before. Consider the composites

$$\begin{aligned} \mathrm{Mod}_{\underline{\mathcal{O}}}\colon \mathrm{Op}(Y)^{\mathrm{op}} &\xrightarrow{\underline{\mathcal{O}}} C_2\mathbb{E}_\infty\mathrm{Alg}\left(\underline{\mathrm{Sp}}^{C_2}\right) \xrightarrow{\mathrm{Mod}_{(-)}} \mathcal{C}\mathrm{at} \\ \mathrm{Mod}^{\otimes}_{\underline{\mathcal{O}}}\colon \mathrm{Op}(Y)^{\mathrm{op}} &\xrightarrow{\underline{\mathcal{O}}} C_2\mathbb{E}_\infty\mathrm{Alg}\left(\underline{\mathrm{Sp}}^{C_2}\right) \xrightarrow{\mathrm{Mod}^{\otimes}_{(-)}} C_2 \otimes \mathcal{C}\mathrm{at}\,, \end{aligned} \tag{3.32}$$

where $C_2 \otimes \mathcal{C}\mathrm{at}$ denotes the ∞-category of (small) C_2-symmetric monoidal C_2-∞-categories and we regard $\mathrm{Mod}_{(-)}$ as taking values in $C_2 \otimes \mathcal{C}\mathrm{at}$ using [Yan23, §A]. In the notation of Construction 3.28, this functor sends the affine open $\operatorname{Spec} A \subseteq Y$ to the category of modules in C_2-spectra over the C_2-$\mathbb{E}_\infty$-algebra which has underlying C_2-Mackey functor $A \to B$. Define $\underline{\mathrm{Mod}_{\mathcal{O}}}$, $\underline{\mathrm{Mod}^{\otimes}_{\mathcal{O}}}$ to be the limits in $\mathcal{C}\mathrm{at}$, $C_2 \otimes \mathcal{C}\mathrm{at}$, resp. of the functors in (3.32). In particular, if we write $s\colon \int \mathrm{Mod}_{\underline{\mathcal{O}}} \to \mathrm{Op}(Y)^{\mathrm{op}}$ for the cocartesian fibration obtained by taking the Grothendieck construction on (3.32), an object of $\mathrm{Mod}_{\underline{\mathcal{O}}}$ is a cocartesian section of s. In other words, it is a choice, for each affine open $\operatorname{Spec} A$ of Y (same notation as before), of a module over the C_2-$\mathbb{E}_\infty$-algebra which has underlying C_2-Mackey functor $A \to B$ which glue compatibly.

L: bleh...cardinals

L: invent better notation later

Observe that for each $A \to B$, there is a quadratic norm functor $N^{C_2}\colon \mathrm{Mod}_B(\mathrm{Sp}) \to \mathrm{Mod}_{N^{C_2}B}\left(\mathrm{Sp}^{C_2}\right)$ and a quadratic relative norm functor $N^{C_2}\colon \mathrm{Mod}_B(\mathrm{Sp}) \to \mathrm{Mod}_{A\to B}\left(\mathrm{Sp}^{C_2}\right)$.

Observation 3.33. Let X be a scheme with an involution, and let $p\colon X \to Y$ exhibit Y as a good quotient of X. Assume that p is affine. The norm functors (resp. relative norm functors) $N_e^{C_2}$ assemble under Construction 3.28 to a 'global' norm functor $N_Y^{C_2}\colon \pi_\# \mathcal{O}_X \,\mathrm{Mod} \to N^{C_2}\pi_\#\mathcal{O}_X \,\mathrm{Mod}$ (resp. relative norm functor $N_Y^{C_2}\colon \pi_\#\mathcal{O}_X \,\mathrm{Mod} \to \underline{\mathcal{O}}\,\mathrm{Mod}$). Moreover, these functors are quadratic.

For each affine open $j\colon \operatorname{Spec} A \subseteq Y$, write $B = \Gamma(\mathcal{O}_{\operatorname{Spec} A \times_Y X})$, consider the composite

$$\pi_\#\mathcal{O}_X \,\mathrm{Mod} \xrightarrow{j^*} \mathrm{Mod}_B(\mathrm{Sp}) \xrightarrow{N^{C_2}} \mathrm{Mod}_{N^{C_2}B}(\mathrm{Sp}^{C_2}) \xrightarrow{-\otimes_{N^{C_2}B}(A\to B)} \mathrm{Mod}_{A\to B}(\mathrm{Sp}^{C_2})\,,$$

where the last map is base change along the map $N^{C_2}B \to (A \to B)$ which is a structure map for the C_2-$\mathbb{E}_\infty$-algebra structure on $A \to B$. Now since quadratic functors are closed under limits [Lur17, Theorem 6.1.1.10] and $N_Y^{C_2}$ can be written as a limit of a diagram of quadratic functors, $N_Y^{C_2}$ is also quadratic.

L: todo: use effective descent/limit definition for $\underline{\mathcal{O}}$-modules.

Definition 3.34. Varying $X \to Y$, Construction 3.31 and Observation 3.33 define a functor

$$\begin{aligned} \left(\mathrm{qSch}^{C_2}\right)^{\mathrm{op}} &\to C_2 \otimes \mathcal{C}\mathrm{at} \\ (X, \lambda, Y, p) &\mapsto \underline{\mathrm{Mod}_{\mathcal{O}}}\left(\underline{\mathrm{Sp}}^{C_2}\right) \end{aligned}$$

L: want: codomain consists of C_2-stable C_2-*presentable* C_2-∞-categories

Definition 3.35. Suppose $\mathcal{C}$ is a C_2-stable C_2-symmetric monoidal C_2-∞-category. Define a functor

$$\begin{aligned} \mathrm{eInv}\colon C_2 \otimes \mathcal{C}\mathrm{at}^{\mathrm{ex}} &\to \mathcal{S} \\ (\mathcal{C}, \otimes) &\mapsto \left(\mathrm{C}^{C_2}\right)^{\simeq} \times_{(\mathrm{C}^e)^{\simeq, hC_2}} \mathrm{Pic}(\mathrm{C}^e)^{\simeq, hC_2}\,. \end{aligned}$$

In other words, eInv sends a C_2-symmetric monoidal C_2-∞-category to the full subgroupoid of $\mathcal{C}^{C_2}$ on those objects L so that L^e is an invertible object in $\mathcal{C}^e$.

Write $\widetilde{\mathrm{eInv}}$ for the Grothendieck construction on eInv.

There is a functor

$$\begin{aligned} \widetilde{\mathrm{eInv}} &\to \mathbb{E}_\infty\mathrm{Alg}\left(\mathrm{Cat}^h\right) \\ (\mathcal{C}, L) &\mapsto (\mathcal{C}^e, \mathcal{C}^e \xrightarrow{N_{\mathcal{C}}} \mathcal{C}^{C_2} \xrightarrow{\hom_{\mathcal{C}^{C_2}}(-,L)} \mathrm{Sp}) \end{aligned} \tag{3.36}$$

L: Pretty sure distributive norm functors are 2-excisive (results here should generalize readily)...if

Lemma 3.37. *The functor of (3.36) lifts to a functor* $\widetilde{\mathrm{eInv}} \to \mathrm{Cat}^{\mathrm{p}}_{\infty}$.

Proof. Let $(\mathcal{C}, L)$ and $(\mathcal{D}, M)$ be objects of $\widetilde{\mathrm{eInv}}$. A morphism $(\mathcal{C}, L) \to (\mathcal{D}, M)$ consists of a pair (F, φ) where $F\colon \mathcal{C} \to \mathcal{D}$ is a C_2-stable C_2-symmetric monoidal C_2-functor and $\varphi\colon F(L) \simeq M$. ☐

Lemma 3.38. *Let X be a scheme with involution $\sigma\colon X \xrightarrow{\sim} X$ equipped with a good quotient $\pi\colon X \to Y$. Let L be a line bundle on Y. Then the canonical map*

$$L \to \pi_{\#}\pi^{*}L \tag{3.39}$$

promotes (3.39) to a sheaf of $\underline{\mathcal{O}}$-modules on Y. We will write $\underline{L}$ for (3.39).

L: Example: Special case where X has trivial C_2 action and $X = Y$.

Proof. Follows from naturality of the unit and the canonical identification $\pi^{*}\mathcal{O}_Y$ with $\mathcal{O}_X$. ☐

Definition 3.40. Let X be a scheme with involution $\sigma\colon X \xrightarrow{\sim} X$ equipped with a good quotient $\pi\colon X \to Y$. Let L be a line bundle on Y. Define $\Qoppa_{\sigma,L}$ to be the functor

$$\mathrm{Perf}_X^{\mathrm{op}} \xrightarrow{\pi_{\#}} \pi_{\#}\mathcal{O}_X\,\mathrm{Mod}^{\omega,\mathrm{op}} \xrightarrow{N^{C_2}} N^{C_2}\pi_{\#}\mathcal{O}_X\,\mathrm{Mod}\left(\mathrm{Sp}^{C_2}\right)^{\mathrm{op}} \xrightarrow{\hom_{N^{C_2}\pi_{\#}\mathcal{O}_X}(-,\underline{L})} \mathrm{Sp}\,,$$

where $\underline{L}$ is a $\underline{\mathcal{O}}$-module by Lemma 3.38 and $\underline{\mathcal{O}}$ is a $N^{C_2}\pi_{\#}\mathcal{O}_X$-algebra by Lemma 3.29. By Observation 3.33 and the fact that the composite of an exact (1-excisive) functor and an m-excisive functor is m-excisive (see [Bar+22, §2.2]), $\Qoppa_{\sigma,L}$ is quadratic.

Example 3.41. Suppose $L = \mathcal{O}_Y$. Then we drop L from notation and the quadratic functor $\Qoppa_{\sigma}$ of Definition 3.40 takes the form

$$\mathrm{Perf}_X^{\mathrm{op}} \xrightarrow{\pi_{\#}} \pi_{\#}\mathcal{O}_X\,\mathrm{Mod}^{\omega,\mathrm{op}} \xrightarrow{N^{C_2}} N^{C_2}\pi_{\#}\mathcal{O}_X\,\mathrm{Mod}\left(\mathrm{Sp}^{C_2}\right)^{\mathrm{op}} \xrightarrow{\hom_{N^{C_2}\pi_{\#}\mathcal{O}_X}(-,\underline{\mathcal{O}})} \mathrm{Sp}\ .$$

Lemma 3.42. *Let X be a scheme with involution $\sigma\colon X \xrightarrow{\sim} X$, and let Y be a good quotient of X. Let L be a line bundle on Y, and let $\Qoppa_{\sigma,L}$ be the quadratic functor on Perf_X of Definition 3.40. Then the bilinear part of $\Qoppa_{\sigma,L}$ agrees with that of Observation* **??**. *In particular, $(\mathrm{Perf}_X, \Qoppa_{\sigma,L})$ is a Poincaré ∞-category.*

Proof. By definition of the bilinear part of a quadratic functor, it suffices to show that there is an equivalence $\hom_{\pi_{\#}\mathcal{O}_X\,\mathrm{Mod}}\left(\pi_{\#}E \otimes_{\pi_{\#}\mathcal{O}_X} \pi_{\#}E, \pi_{\#}\mathcal{O}_X\right) \simeq \hom_{\mathcal{O}_X\,\mathrm{Mod}}\left(E \otimes_{\mathcal{O}_X} \sigma^{*}E, \mathcal{O}_X\right)$ for any perfect complex E on X. This follows from Lemma 3.29(iii). ☐

Remark 3.43. Compare the description of the space of bilinear forms in Lemma 3.42 with the description of a δ-hermitian form H in [PS92, p. 216].

4 The Poincaré Picard space

4.1 Invertible Poincaré objects and units in Poincaré rings

Recall that the Poincaré space functor $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty} \to \mathrm{CAlg}(\mathcal{S})$ is lax symmetric monoidal with respect to tensor product of Poincaré ∞-categories and smash product of $\mathbf{E}_{\infty}$-spaces [Cal+20a, Corollary 5.2.8]. In particular, we can consider invertible objects in $\mathrm{Pn}(A)$ for a Poincaré ring spectrum A.

Definition 4.1. Let A be a Poincaré ring spectrum. We define the *Picard space of A* to be

$$\mathrm{Pic}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Pn}(A)).$$

For $(X, \lambda, Y, p) \in \mathrm{qSch}^{C_2}$ we similarly define

$$\mathrm{Pic}^{\mathrm{p}}(X, \lambda, Y, p) := \mathrm{Pic}(\mathrm{Pn}(\mathrm{Perf}_X, \Qoppa_{\sigma})).$$

Remark 4.2. Let $(\mathrm{Mod}_R^\omega, \Qoppa_R)$ be a Poincaré ring spectrum, where $(M_R = R, N_R = R^{\varphi C_2}, R^{\varphi C_2} \to R^{tC_2})$ is the module with genuine involution associated to $\Qoppa_R$. Then a point in the Poincaré Picard space is the data of a pair $(\mathcal{L}, q)$, where $\mathcal{L}$ is an invertible module in Mod_R^ω and q is a point in $\Omega^\infty \Qoppa_R(\mathcal{L})$. By [Cal+20a, Proposition 1.3.11], the data of q is equivalent to the data of points in the lower left and upper right corner of the square

$$\begin{tikzcd} \Qoppa(\mathcal{L}) \arrow[r] \arrow[d] & \hom_R(\mathcal{L}, R^{\varphi C_2}) \ni \ell(q) \arrow[d] \\ b(q) \in \hom_{R\otimes R}(\mathcal{L}\otimes\mathcal{L}, R)^{hC_2} \arrow[r] & \hom_R(\mathcal{L}, R^{tC_2}) \end{tikzcd} \tag{4.3}$$

and a path between their images in the lower right corner. In particular, the adjoint of $b(q)$ must define a nondegenerate hermitian form on $\mathcal{L}$, that is, an equivalence $\mathcal{L} \simeq \hom_R(\mathcal{L}, R^*)$ where R^* is considered as an R-module via the action of the generator of C_2.

L: add equivariance/symmetry data

Write $(\mathcal{L}^\vee, q^\vee)$ is for the inverse of $(\mathcal{L}, q)$. By definition of invertibility, there exists an R-linear map $\ell(q^\vee)\colon \mathcal{L}^\vee \to R^{\varphi C_2}$ so that the following diagram commutes

$$\begin{tikzcd} \mathcal{L}\otimes_R \mathcal{L}^\vee \arrow[r, "\ell(q)\otimes\ell(q^\vee)"] \arrow[d, "\sim"', "\mathrm{ev}"] & R^{\varphi C_2}\otimes_R R^{\varphi C_2} \arrow[d, "\text{multiplication}"] \\ R \arrow[r, "\text{given}"] & N_R \end{tikzcd} \tag{4.4}$$

Lemma 4.5. *Let $(R, \Qoppa)$ be a connective Poincaré ring spectrum. Then, for any integer n, the spectrum $\Qoppa(\Sigma^n R)$ is $(-2n)$-connective.*

Proof. This follows from the fiber sequence

$$(\Sigma^{-2n}R)_{hC_2} \to \Qoppa(\Sigma^n R) \to \hom_R(\Sigma^n R, C) \simeq \Sigma^{-n} C.$$

$\square$

V: define connectivity and make conditions here precise. As stated this works for R and C connective. More precisely, $conn(\Qoppa(\Sigma^n R)$ min$(conn(\Sigma^-$

Remark 4.6. The functor $\mathrm{Pic}^\mathrm{p} : \mathrm{CAlg}^\mathrm{p} \to \mathrm{CAlg}^\mathrm{gp}(\mathcal{S})$ preserves certain structures. Let A be a Poincaré ring spectrum. Since A is a module over $(\mathbb{S}, \Qoppa_u)$, the space $\mathrm{Pic}^\mathrm{p}(A)$ is something over $\mathrm{Pic}^\mathrm{p}(\mathbb{S}, \Qoppa_u)$.

V: write out details

Since the forgetful functor $\mathrm{Pn}(A) \to \mathrm{Mod}_A^\omega$ is symmetric monoidal we get an induced map

$$U : \mathrm{Pic}^\mathrm{p}(A) \to \mathrm{Pic}(A)$$

of spectra. For a point $(\mathcal{L}, q) \in \pi_0(\mathrm{Pic}^\mathrm{p}(A))$ we will refer to $\mathcal{L} := U(\mathcal{L}, q)$ as the *underlying invertible module*. Note that the A-module A^* is (nonequivariantly) isomorphic to A via the involution, and so the fact that $\mathcal{L} \simeq \hom_A(\mathcal{L}, A^*)$ forces $\mathcal{L}$ to be 2-torsion. In particular we get a refined map

$$U : \mathrm{Pic}^\mathrm{p}(A) \to \mathrm{Pic}(A)[2]$$

which factors the underlying invertible module map.

Example 4.7. Let $(\mathbb{S}, \Qoppa_u)$ be the universal Poincaré ring spectrum from Example 3.13. The only 2-torsion element of $\mathrm{Pic}(\mathbb{S}) \simeq \mathbf{Z}$ is $\mathbb{S}$. Therefore, any element in $\mathrm{Pic}^\mathrm{p}(\mathbb{S}, \Qoppa_u)$ lies above $\mathbb{S}$ under U. With Remark 3.15, we conclude $\pi_0(\mathrm{Pic}^\mathrm{p}(\mathbb{S}, \Qoppa_u)) \simeq \pi_0(\mathbb{S}_{hC_2} \oplus \mathbb{S}^\times)^\times \simeq (\mathbf{Z} \times \mathbf{Z/2})^\times \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$.

V: there is no truth in here yet. Work in progress. Noah had an example using Witt vectors which showed that π_0 does not need to be 2-torsion

Remark 4.8. One might hope that the map $\mathrm{Pic}^\mathrm{p}(A) \to \mathrm{Pic}(A)[2]$ is close to an equivalence. This however is quite far from being true. Let k be a finite field of characteristic 2, and let $\mathbb{S}_{W(k)}$ be the spherical Witt vectors on k in the sense of [Lur18, Example 5.2.7]. Then by [Nik23, Example 3.4] we know that $\mathbb{S}_{W(k)}$ must satisfy that the map $\varphi_2 : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}^{tC_2}$ is an equivalence where the action is trivial.

Consider now the Poincaré ring $(\mathrm{Mod}^\omega_{\mathbb{S}_{W(k)}}, \Qoppa^u_{\mathbb{S}_{W(k)}})$ where $\Qoppa^u_{\mathbb{S}_{W(k)}}$ is the Tate Poincaré structure. We have that $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) \cong \mathbb{Z}$ and is generated by $\Sigma\mathbb{S}_{W(k)}$. To see this note that for $\mathcal{L}$ an invertible module over $\mathbb{S}_{W(k)}$, $\mathcal{L}$ must be bounded below since otherwise it would not be perfect. Then for $\pi_n(\mathcal{L})$ its bottom homotopy group, we have that $\pi_n(\mathcal{L}/2) \cong k$ since it must be an invertible k-module and k is a field. Thus

V:did we mod out by isomorphisms here?

we get a map $\mathbb{S}^n \to \mathcal{L}$ lifting a generator of k, and by adjunction an $\mathbb{S}_{W(k)}$-module map $\Sigma^n \mathbb{S}_{W(k)} \to \mathcal{L}$ which on $\pi_n((-)/2)$ gives an isomorphism $k \cong k$. Therefore

$$\mathbb{S}_{W(k)}[n] \otimes k \simeq k[n] \to k[n] \simeq \mathcal{L} \otimes k$$

is an equivalence, where the equivalence $k[n] \simeq \mathcal{L} \otimes k$ follows from the fact that base change preserves invertible objects. The map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is then a k-local, and therefore an $\mathbb{F}_p$-local, equivalence. Both sides are connective and p-complete so it follows that the map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is an equivalence.

N: There is probably a reference for this fact, I'll look around for one.

Thus $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) = 0$. On the other hand, we have that the unit map $\mathbb{S}_{W(k)} \to \Qoppa^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})$ is split by the map $\Qoppa^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)}) \to \mathbb{S}^{\varphi C_2}_{W(k)} = \mathbb{S}_{W(k)}$. Consequently $\pi_0(\Qoppa^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})) \cong \pi_0(\mathbb{S}_{W(k)} \oplus (\mathbb{S}_{W(k)})_{hC_2}) \cong W(k) \times W(k)$. As a ring this is $W_2(W(k))$, and in order for $q \in W_2(W(k))$ to induce a Poincaré structure we must have that $q \in W_2(W(k))^\times \cong W(k)^\times \times W(k)^\times$.

We then have that $\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong W(k)^\times \times W(k)^\times / H$ where H is the subgroup of Poincaré structures q on $\mathbb{S}_{W(k)}$ which are identified by some automorphism $f : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}$. By the defining property of spherical Witt vectors there is an equivalence $\mathrm{Maps}_{\mathrm{CAlg}}(\mathbb{S}_{W(k)}, \mathbb{S}_{W(k)}) \simeq \mathrm{Maps}_{Perf}(k,k) = \mathrm{Gal}(k/\mathbb{F}_2)$ and the action on $W(k)^\times \times W(k)^\times$ is given by $g \in \mathrm{Gal}(k/\mathbb{F}_2)$ acts via $W(g) \times W(g)$. Consequently

$$\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong (W(k)^\times \times W(k)^\times)/\mathrm{Gal}(k/\mathbb{F}_2)$$

which even for $k = \mathbb{F}_2$ is not zero and in fact not even 2^∞-torsion.

In the usual Picard spectrum one has the relationship $\mathrm{Pic} = \mathbb{Z} \times B\mathbb{G}_m$, where $\mathbb{G}_m$ is the spectral algebraic group scheme sending a ring spectrum E to the spectrum of E-linear equivalences of E $\mathrm{gl}_1 E := \mathrm{Aut}_E(E)$.[2] Equivalently, $\mathbb{G}_m$ is the affine group scheme given by $\mathbb{G}_m = \mathrm{Spét}(\mathbb{S}\{x^{\pm 1}\})$, where $\mathbb{S}\{x^{\pm 1}\}$ is the free $\mathbb{E}_\infty$ ring on the $\mathbb{E}_\infty$ space $\mathbb{Z}$. This relationship between Pic and $\mathbb{G}_m$ has many important applications, for example relating the higher homotopy groups of $\mathrm{Pic}(A)$ with those of A. We will spend the rest of this section on establishing such an equivalence in the Poincaré setting. We begin by defining an analogue of the multiplicative group/units for Poincaré rings.

L: Is Pic here referring to the picard space of the derived category or classical picard? Sort out with rest of notation later?

Construction 4.9. [Units of Poincaré rings] The underlying $\mathbb{E}_\infty$ ring of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ will again be $\mathbb{S}\{x^{\pm 1}\}$, but in order to promote this ring to a Poincaré ring it will be helpful to write it as

$$\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z^{\pm 1}\}} \mathbb{S}$$

where the map $\mathbb{S}\{z^{\pm 1}\} \to \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by $z \mapsto xy$. This ring naturally lifts to a Borel C_2-ring given by C_2 swaps x and y and does nothing to z. Now take $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ to be the Poincaré ring with underlying Borel C_2 structure as described above and geometric fixed points $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} = \mathbb{S}$ and the map $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ given by the unit map. Endowing $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2}$ with the $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$-module structure given by $x, y \mapsto 1$, it remains to show that the unit map $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{\varphi C_2} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ factors the Tate valued Frobenius $\mathbb{G}_{\mathrm{m}}^{\Qoppa} \to (\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ in order to promote $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ to a Poincaré ring.

L: workshop the name...'hermitian units'?

By construction of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ this amounts to showing that on π_0 the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$. This map sends both x and y to $xy \in \pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$. These are equal to 1 in $\pi_0((\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so $(\mathbb{G}_{\mathrm{m}}^{\Qoppa})^{tC_2}$ is a module over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of xy equal to 1.

L: later: different notation for the functor $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ versus its [Poincaré] ring of functions.

Remark 4.10. The pushout description of $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ induces a pullback of mapping spaces

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}}^{\Qoppa}(A) \arrow[r] \arrow[d] & \mathrm{Maps}_{\mathrm{CAlg}(\mathrm{Sp}^{C_2})}(\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}, A) \simeq \mathrm{gl}_1(A^e) \arrow[d] \\
* \arrow[r] & \mathrm{Maps}_{\mathrm{CAlg}(\mathrm{Sp}^{C_2})}(\mathbb{S}\{z\}, A) \simeq \Omega^\infty \Qoppa_A(A)
\end{tikzcd}.$$

L: mapping spaces should be taken in $\mathrm{CAlg}^{\mathrm{p}}$? define $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ in terms of left adjoint $\mathbb{S}^0[-]$ in Lemma 3.22? Use the latter to show that $\mathbb{G}_{\mathrm{m}}^{\Qoppa}$ takes values in connective spectra!

Theorem 4.11. *There is a natural equivalence of*

$$\Omega\, \mathrm{Pic}^{\mathrm{p}}(-) \simeq \mathbb{G}_{\mathrm{m}}^{\Qoppa}$$

of functors on Poincaré rings.

[2]Normally the automorphism space of an object is only $\mathbb{A}_\infty$, but as the unit in a symmetric monoidal category, the automorphisms of Einherit a canonical and in fact functorial $\mathbb{E}_\infty$ structure and this construction makes sense.

Proof. This amounts to identifying the space $\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}^{\omega}_{A},\Qoppa)}(A,u)$ functorially, where (A,u) is the Poincaré object A with bilinear form given by the unit map $\mathbb{S} \to \Qoppa_A(A)$. Since $\mathrm{Pn}\,(\mathrm{Mod}^{\omega}_{A},\Qoppa)$ is the maximal subgroupoid of $\mathrm{He}\,(\mathrm{Mod}^{\omega}_{A},\Qoppa)$ on Poincaré objects, it suffices to describe automorphisms of (A,u) in the latter category. Recall that $\mathrm{He}(\mathrm{Mod}^{\omega}_{A},\Qoppa_A) \to \mathrm{Mod}^{\omega}_{A}$ is a right fibration classified by the functor which takes a module M to the ∞-groupoid $\Omega^\infty \Qoppa_A(M)$ [Cal+20a, Definition 2.1.1]. Looping the aforementioned fibration twice, we have a fibration sequence

$$\Omega\,\mathrm{Pic}^{\mathrm{p}}(A) \simeq \mathrm{Aut}_{\mathrm{He}(\mathrm{Mod}^{\omega}_{A},\Qoppa_A)}((A,u)) \to \mathrm{Aut}_{\mathrm{Mod}^{\omega}_{A}}(A) \xrightarrow{f\mapsto f^*(u)} \Omega^\infty \Qoppa_A(A) \tag{4.12}$$

where the fiber is taken over the point $u \in \Omega^\infty \Qoppa_A(A)$. In other words, a point in $\mathrm{Aut}_{\mathrm{He}(\mathrm{Mod}^{\omega}_{A},\Qoppa_A)}((A,u))$ is the data of an automorphism $a \in \mathrm{Aut}(A)$ together with a path q from u to a^*u in $\Omega^\infty \Qoppa_A(A)$.

We now define a natural transformation $\mathbb{G}^{\Qoppa}_{\mathrm{m}}(-) \to \Omega\,\mathrm{Pic}^{\mathrm{p}}(-)$. By Lemma 4.14(1), there is a commutative diagram

$$\begin{tikzcd} \mathrm{gl}_1(A^e) \arrow[r] \arrow[d] & \mathrm{gl}_1\left((A^L)^{C_2}\right) \arrow[r] \arrow[d, equal] & \Omega^\infty \Qoppa_A(A) \\ \mathrm{Aut}_A(A) \arrow[r, "f\mapsto f^*(u)"] & \mathrm{gl}_1\left((A^L)^{C_2}\right) & \end{tikzcd} \tag{4.13}$$

where the upper horizontal arrow is from Remark 4.10. Now observe that the left vertical arrow in (4.13) is an equivalence. The result follows from taking fibers over u in (4.13) and comparing (4.12) with Remark 4.10 and Lemma 4.14(2). ◻

L: Are Remark 4.10 and Lemma 4.14(2) slightly redundant?

Lemma 4.14. *Let $A \in \mathrm{CAlg}^{\mathrm{p}}(\mathrm{Sp})$. In view of Remark 3.4, there is a map of ∞-groupoids*

$$A^e \simeq \mathrm{End}_{A^e}(A^e) \xrightarrow{N^{C_2}} \mathrm{End}_{N^{C_2}A}\left(N^{C_2}A\right) \xrightarrow{\mathrm{ev}_u} \hom_{N^{C_2}A}\left(N^{C_2}A, A\right) \simeq \Omega^\infty \Qoppa_A(A) \simeq (A^L)^{C_2} \tag{4.15}$$

where $u\colon N^{C_2}A \to A$ is the canonical map. Write n_A for the composite of (4.15). Then

(1) The composite $\mathrm{gl}_1(A^e) \to A^e \xrightarrow{n_A} (A^L)^{C_2}$ canonically factors through the inclusion $\mathrm{gl}_1\left((A^L)^{C_2}\right) \to (A^L)^{C_2}$. Under the identification $\Omega^\infty \Qoppa_A(A) \simeq (A^L)^{C_2}$, $u \in \Omega^\infty \Qoppa_A(A)$ is sent to $1 \in (A^L)^{C_2}$.

(2) There is a fiber sequence

$$\mathbb{G}^{\Qoppa}_{\mathrm{m}}(A) \simeq \mathrm{fib}_1(n_A) \to \mathrm{gl}_1(A^e) \xrightarrow{n_A} \mathrm{gl}_1\left((A^L)^{C_2}\right)$$

where n_A denotes the map from (1).

Proof. The first half of part (1) follows from the fact that functors send equivalences to equivalences. The identification of u with 1 is by definition of $\Qoppa_A$.

L: add reference to $\mathrm{Mod}^{\mathrm{p}}$ functor.

L: Rest of proof tbd, maybe overlap with Remark 4.10? Previous iteration is commented out but still appears in `.tex` file.

◻

4.2 prime factorization and the picard group of hearts

V: still in development

We establish a Poincaré analogue of Fausk's result which describes the picard group of the derived category of a scheme X in terms of connected components of X and the classical picard group of X.

Let X be a simplicial set. The set $\pi_0(X)$ is the set of connected compoents of X, i.e. simplicial subsets which are connected and form X via a coproduct. In other words, the functor π_0 records a unique and maximal decomposition of X into coproducts. To establish the the result mentioned above, we study the dual analogue of connected components in the sense of a maximal decomposition of X into products.

Definition 4.16. Let X be a simplicial set. We call a map of simplicial sets $X \to Y$ a factor of X, if there is an isomorphism $X \simeq Y_\times Z$, for some simplicial set Z, such that $Y \times Z \simeq X \to Y$ is a structure map of the given product.

Definition 4.17. Let X be a simplicial set. We say that X is prime, or indecomposable, if it is nonempty and every factor of X is isomorphic to either Δ^0 or X. We let $\mathrm{prin}(X)$ denote the set of prime factors of X.

Proposition 4.18. *Let $X_\bullet$ be a simplicial set, then $X_\bullet$ is the product of its prime factors.*

Proof. □

V:

Proposition 4.19. *Let X be a simplicial set. Then* $\mathrm{prim}(X) \simeq \mathrm{prim}(X^{\simeq})$.

Proof. Let $f : X \to Y$ be a weak equivalence of simplicial sets. Then $f^{\simeq} : X^{\simeq} \to Y^{\simeq}$ is a weak equivalence of spaces. □

V:

Proposition 4.20. *Let X be a simplicial set. Then* $\mathrm{prim}(X) \simeq \pi_0(\mathrm{Spec}(X))$, *where* $\mathrm{Spec}(X^{\simeq})$ *is the Balmer spectrum of X with respect to the symmetric monoidal structure given by cartesian product.*

Proof. □

V:

Remark 4.21. Let R be a commutative ring. Then the scheme $\mathrm{Spec}(R)$ is isomorphic to the Balmer spectrum of $\mathrm{Mod}_R^{\heartsuit}$. When we view R as a discrete simplicial set, we thus have

$$\mathrm{prim}(\mathrm{Mod}_R^{\heartsuit}) \simeq \pi_0(\mathrm{Spec}(R)).$$

V: does this need a proof?

Definition 4.22. Let X be a prime simplicial set. A c-structure on X is a map of simplicial sets $X \to \mathbf{Z}$ satisfying (todo). Let Y be a simplicial set, then a c-structure on Y is a product of c-structures on each of its prime components. We write $X_{\geq n}$ for the homotopy pullback of $\mathbf{Z}_{\geq n}$ along c, $X_{\leq n}$ for the homotopy pullback of $\mathbf{Z}_{\leq n}$ along c, and $X^{\heartsuit}$ for the pullback of $X_{\leq n}$ along $X_{\geq n} \to X$.

V: when X is a stable infinity category and prime, then a c-structure should be a t-structure on it

Theorem 4.23. *Let X be a prime simplicial set and $c : X \to \mathbf{Z}$ a c-structure. Then we have a fiber sequence of monoids* >

V: what kind exactly

$$X^{\heartsuit} \to X^{\simeq} \to \mathbf{Z}.$$

Proof. □

V:

Corollary 4.24 (Fausk)**.** *Let R be a discrete ring. Then we have a short exact sequence:* >

$$0 \to \mathrm{Pic}(\mathrm{Mod}(R)^{\heartsuit}) \to \pi_0(\mathrm{Pic}(\mathrm{Mod}(R))) \to H^0(\mathrm{Spec}(R); \mathbf{Z}) \to 0.$$

Proof. □

V: apply pic to the previous sequence and take π_0

4.3 The discrete case: nondegenerate hermitian line bundles

Let R be a discrete commutative ring with a C_2-action $\sigma\colon R \to R$, and let M be an R-module. We denote by $\sigma^* M$ its restriction of scalars, i.e. the R-module with underlying abelian group M and action $r \star m = \sigma(r)\cdot m$, where $\cdot$ denotes the original action of M. Since $\sigma^2 = \mathrm{id}_R$, restriction of scalars is isomorphic to extension of scalars, and thus restriction of scalars $\sigma^* : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is symmetric monoidal in this case.

Definition 4.25. Let R be a discrete commutative ring with a C_2-action $\sigma\colon R \to R$, and let M be an R-module. We define the *σ-dual* of M to be the R-module

L: see 3.8-3.11 in this paper

$$M^{\dagger} := \mathrm{Hom}_R(\sigma^* M, R) = (\sigma^* M)^{\vee}.$$

Remark 4.26. Let R be a discrete commuative ring and let $\sigma : R \to R$ be an isomorphism of commutative rings such that $\sigma \circ \sigma = \mathrm{id}_R$. Then we have a canonical isomorphism of R-modules

$$\mathrm{Hom}_R(\sigma^* M, N) \simeq \sigma^* \mathrm{Hom}_R(M, \sigma_* N),$$

for any R-modules N and M. In particular, we have

$$\begin{aligned} M^{\dagger} = (\sigma^* M)^{\vee} &\simeq \sigma^*(M^{\vee}), \\ f &\mapsto \sigma \circ f \end{aligned}$$

as R-modules. If M is compact, we thus have a canonical isomorphism of R-modules

$$c : M \simeq (M^\vee)^\vee \simeq \left(M^\dagger\right)^\dagger$$
$$m \mapsto \sigma \circ \mathrm{eval}_m\,.$$

If both M and N are compact R-modules, then, using symmetric monoidalality of $\sigma^* : \mathrm{Mod}_R \to \mathrm{Mod}_R$, the σ-dual satisfies

$$(M \otimes_R N)^\dagger \simeq \sigma^*(M \otimes_R N)^\vee \simeq \sigma^*(M^\vee \otimes_R N^\vee) \simeq M^\dagger \otimes_R N^\dagger.$$

Definition 4.27. Let R be a discrete commutative ring with a C_2-action $\sigma\colon R \to R$, and let I be a compact R-module. A *σ-hermitian form on I* is an R-linear isomorphism $\varphi\colon I \xrightarrow{\sim} I^\dagger$ such that the following diagram commutes

V: nondegenerate? or why not poincare?

$$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[d, "c", "\simeq"'] & I^\dagger \\ (I^\dagger)^\dagger \arrow[ur, "\varphi^\dagger"'] & \end{tikzcd}.$$

We will denote a σ-hermitian form on I by a pair (I, φ).

Let I and J be two compact R-modules with σ-hermitian forms (I, φ) and (J, ψ). A *map of σ-hermitian forms*, $f : (I, \varphi) \to (J, \psi)$, is a map of R-modules $f : I \to J$ such that the following diagram commutes

$$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[d, "f"'] & I^\dagger \\ J \arrow[r, "\psi"'] & J^\dagger \arrow[u, "f^\dagger"'] \end{tikzcd}.$$

We say f is an isomorphism of σ-hermitian forms if its underlying R-module map is an isomorphism of R-modules.

Remark 4.28. Let R be a discrete commutative ring with a C_2-action $\sigma\colon R \to R$, and let I be a compact R-module. We have an induced C_2-action on the R-module $\mathrm{Hom}_R(I, I^\dagger)$ given by the assignment $\varphi \mapsto \varphi^\dagger \circ c$. A σ-hermitian form on I is a fixed point of this action which happens to be an isomorphism. Via the following isomorphisms of R-modules

$$\mathrm{Hom}_R(I, I^\dagger) \simeq \mathrm{Hom}_R(I, (\sigma^* I)^\vee) \simeq \mathrm{Hom}_R(I \otimes_R \sigma^* I, R)$$

this turns into the action $f(x, y) \mapsto \sigma(f(y, x))$. Thus, an alternative definition of a σ-hermitian form on I is: a function $f : I \times I \to R$ which is nondegenerate and R-linear in the first variable, and satisfies $f(x, y) = \sigma(f(y, x))$ for all $x, y \in R$.

The pairing

$$< -, - >_\sigma : R \times R \to R$$
$$(r, s) \mapsto \sigma(s) r$$

provides a canonical σ-hermitian form $u : R \to R^\dagger$, using Remark 4.28. Moreover, given two σ-hermitian forms over R, (I, φ) and (J, ψ), the tensor product $\otimes_R$ provides a new σ-hermitian form $(I \otimes_R J, \varphi \otimes_R \psi)$

$$\varphi \otimes_R \psi : I \otimes J \to (I \otimes J)^\dagger$$
$$i \otimes j \mapsto [x \otimes y \mapsto \varphi(i)(x) \cdot \psi(j)(y)]\,.$$

Hence, the set of invertible σ-hermitian forms over R forms a group under $\otimes_R$ with unit (R, u).

Definition 4.29. Let R be a discrete commutative ring with a C_2-action $\sigma : R \to R$. The *hermitian Picard group of* (R, σ) is the group of isomorphism classes of invertible σ-hermitian forms over R under tensor product, with unit (R, u). We will denote the hermitian Picard group of (R, σ) by $\mathrm{Pic}^{\mathrm{h}}(R, \sigma)$. In case σ is the identity, we will also write $\mathrm{Pic}^{\mathrm{h}}(R)$.

Example 4.30. Let $R = \mathbf{C} \oplus \mathbf{C}$ with $\sigma : R \to R$ given by the swap action, i.e. $(x, y) \mapsto (y, x)$. Since R is a semi-local ring, its Picard group is trivial. Thus any element $(I, \varphi) \in \mathrm{Pic}^{\mathrm{h}}(R, \sigma)$ is of the form (R, φ), up to isomorphism. An R-module isomorphism $\varphi : R \to R^\dagger$ is of the form $(r \mapsto (s \mapsto ra\sigma(s))$, for some $a \in R^\times$, and the condition $\varphi = \varphi^\dagger \circ c$ translates to $a = \sigma(a)$. Thus $a \in (R^\times)^{C_2}$ is a unit and a fixed point of the C_2-action on R, i.e. $a = (v, v)$ with $v \in \mathbf{C}^\times$. Given two σ-hermitian forms (I, φ) and (J, ψ) which correspond to elements a and b in $(R^\times)^{C_2}$ respectively, the tensor product $(I \otimes_R J, \varphi \otimes_R \psi)$ corresponds to the product ab. An isomorphism between (I, φ) and (J, ψ) is an R-module isomorphism $f : I \xrightarrow{\simeq} J$ such that

$$f^\dagger \circ \psi \circ f = \varphi.$$

Since f is given by multiplication of an element $x \in R^\times$, this condition translates to the equation

$$x\sigma(x)b = a.$$

Let $a = (v, v)$, $b = (w, w)$ and $x = (r, s)$ with $v, w, r, s \in \mathbf{C}^\times$. Then this condition breaks down to the two equations $rsw = v$ and $rsv = w$, in particular $(rs)^2 = 1$, and thus $rs = \pm 1$. Therefore, (R, φ) and (R, ψ) are isomorphic if and only if $\varphi = \pm\psi$. We conclude

$$\mathrm{Pic}^{\mathrm{h}}(R, \sigma) \simeq \mathbf{C}^\times/\pm 1 \simeq \mathbf{C}^\times.$$

The above Example 4.30 shows that the hermitian Picard group of a discrete commutative ring with a nontrivial C_2-action is not necessarily 2-torsion. We will see now that the situation simplifies in case of a trivial C_2-action.

Remark 4.31. Let R be a discrete commutative ring equipped with the trivial C_2-action, i.e. $\sigma = \mathrm{id}_R$. In that case, for any R-module M, we have $M^\dagger = M^\vee$. Thus, by Remark 4.28, hermitian forms over (R, id_R) are in bijection with nondegenerate symmetric bilinear forms over R. Let (I, φ) be a hermitian form over (R, id_R) corresponding to a nondegenerate symmetric bilinear form A on I. Then (I, φ) is invertible if I is invertible as a module over R and $A \otimes B = id_R$, for a nondegenerate symmetric bilinear form B on $I^\vee$.

Remark 4.32. Let R be a discrete commutative ring with a C_2-action $\sigma : R \to R$, and let I be an invertible R-module. Let $\mathfrak{p} \subset R$ be a prime ideal. Since I is locally free of rank 1, a trivialization $\tau : I \otimes_R \sigma_* I \xrightarrow{\simeq} R$ is locally of the form

$$\tau_{\mathfrak{p}} : I_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \sigma_* I_{\mathfrak{p}} \xrightarrow{\simeq} R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \sigma_* R_{\mathfrak{p}} \xrightarrow{x_{\mathfrak{p}}} R_{\mathfrak{p}},$$

where the left isomorphism is given by the tensor product $t \otimes_{R_{\mathfrak{p}}} t$ of a trivialization $t : I_{\mathfrak{p}} \xrightarrow{\simeq} R_{\mathfrak{p}}$ and the map on the right is given by the formula $r \otimes s \mapsto \sigma(s) \cdot x_{\mathfrak{p}} \cdot r$ for a unit $x_{\mathfrak{p}} \in R_{\mathfrak{p}}^\times$. The unit $x_{\mathfrak{p}}$ is unique up to an element $N(y) = y\sigma(y) \in R_{\mathfrak{p}}^\times$, where y could be hiding behind the trivialization t. Since R is commutative, the symmetry condition in Definition 4.27 is then equivalent to requiring $x_{\mathfrak{p}} \in (R_{\mathfrak{p}}^\times)^{C_2}$, for each prime ideal $\mathfrak{p} \subset R$. In particular, when $\sigma = \mathrm{id}_R$, τ is automatically symmetric. So the symmetry condition in Definition 4.27 is vacuous for hermitian forms with underlying invertible module when the action on R is trivial.

Proposition 4.33. *Let R be a discrete commutative ring. Then $Pic^h(R)$ is 2-torsion.*

L: R has trivial C_2-action? Perhaps we can call this Pic^s for 'symmetric' forms?

Proof. Let (I, φ) be an invertible hermitian form over (R, id_R). Then the trivialization $I \otimes_R I \xrightarrow{\mathrm{id}_R \otimes \varphi} I \otimes_R I^\vee \xrightarrow{<,>} R$ is an isomorphism of forms $(I \otimes_R I, \varphi \otimes_R \varphi) \simeq (R, u)$, which can be checked locally as in Remark 4.32. □

Given a discrete commutative ring R, we have a forgetful map

$$\mathrm{Pic}^{\mathrm{h}}(R) \to \mathrm{Pic}(R)_2,$$

where the codomain is given by 2-torsion in the classical Picard group of R. Let us now say more about this.

Proposition 4.34. *Let R be a discrete commutative ring. We have a split exact sequence*

$$0 \to R^\times/(R^\times)^2 \to Pic^h(R) \to Pic(R)_2 \to 0.$$

Proof. Let (I, φ) be an invertible hermitian form over (R, id_R), then $\varphi : I \xrightarrow{\simeq} I^\vee$ is an isomorphism between I and its tensor inverse $I^\vee$, thus I is 2-torsion in the classical Picard group: $I \otimes_R I \simeq I \otimes_R I^\vee \simeq R$. Moreover, the above map is surjective by Remark 4.32, since the condition $I \otimes I \simeq R$ is equivalent to $I \simeq I^\vee$. The kernel of the above map is given by hermitian forms on R, up to isomorphism. That is, isomorphisms $\varphi : R \simeq R^\vee$, up to isomorphisms of hermitian forms. Two forms φ, ψ are isomorphic if there exists an isomorphism $f : R \xrightarrow{\simeq} R$ such that $f^\dagger \circ \psi \circ f = \varphi$. Each of these isomorphisms is given by multiplication of a unit of R. Let $a, b, c \in R^\times$ be the units corresponding to f, φ and ψ. Then φ is isomorphic to ψ if and only if $a^2 c = b$, i.e. c and b vary by a square of a unit. Thus, the kernel of the above map is given by $R^\times/(R^\times)^2$. Finally, by Proposition 4.33, this is a short exact sequence of vector spaces over $\mathbf{F}_2$ and thus splits. □

Remark 4.35. Recall that we have an exact sequence of fppf sheaves

$$0 \to \mu_2 \to \mathbf{G}_m \xrightarrow{\times 2} \mathbf{G}_m \to 0.$$

over $\mathbf{Z}$ and a canonical isomorphism

$$\mathrm{H}^1_{\mathrm{fppf}}(X, \mathbf{G}_m) \simeq \mathrm{Pic}(X),$$

for X a scheme. Let R be a discrete commutative ring. Proposition 4.34 shows that we have the exact sequence

$$\mathrm{H}^0_{\mathrm{fppf}}(R, \mathbf{G}_m) \xrightarrow{\times 2} \mathrm{H}^0_{\mathrm{fppf}}(R, \mathbf{G}_m) \to \mathrm{Pic}^{\mathrm{h}}(R) \to \mathrm{H}^1_{\mathrm{fppf}}(R, \mathbf{G}_m) \xrightarrow{\times 2} \mathrm{H}^1_{\mathrm{fppf}}(R, \mathbf{G}_m).$$

Hence, in the case of a trivial action on R, the hermitian Picard group is isomorphic to the first fppf cohomology group of R with μ_2-coefficients

$$\mathrm{Pic}^{\mathrm{h}}(R) \simeq \mathrm{H}^1_{\mathrm{fppf}}(R, \mu_2).$$

If R is moreover smooth over a Noetherian scheme of finite Krull dimension S which is smooth over a Dedekind domain of mixed characteristic or over a field, by [Spi18, Proposition 10.9], this is isomorphic to motivic cohomology in degree $(1,1)$ with coefficients in $\mathbf{Z}/2$

V: is this really the highest generality?

$$\mathrm{Pic}^{\mathrm{h}}(R) \simeq \mathrm{H}^{1,1}(R, \mathbf{Z}/2).$$

The split exact sequence of Proposition 4.34 can be generalized to the case of a nontrivial C_2-action on R. To proof that, let us first establish two basic properties of the hermitian Picard group.

Proposition 4.36. *The hermitian Picard group Pic^h is nilinvariant. More concretely, let R be a discrete commutative ring with a C_2-action $\sigma : R \to R$ and let R_{red} be the reduced ring of R with C_2-action induced by the reduction map $R \to R_{red}$. Then the reduciton map induces an isomorphism*

$$Pic^h(R, \sigma) \xrightarrow{\sim} Pic^h(R_{red}, \sigma).$$

Proof. This follows since the Picard functor of ∞-categories is nilinvariant. □

V: more details

Proposition 4.37. *The hermitian Picard group Pic^h commutes with finite sums.*

Proof. □

V: todo

Given a discrete commutative ring R and an isomorphism $\sigma : R \to R$, the σ-dual defines a C_2-action on the Picard group of R:

$$\begin{aligned} (-)^\dagger : \mathrm{Pic}(R) &\to \mathrm{Pic}(R) \\ [I] &\mapsto [I^\dagger] = [\sigma^* I^\vee] = -[\sigma^* I]. \end{aligned}$$

Given an invertible σ-hermitian form $\varphi : I \xrightarrow{\simeq} I^\dagger$, the R-module I is a fixed point in the Picard group of R with respect to this action. Let us denote the fixed points of $\mathrm{Pic}(R)$ by this action by $\mathrm{Pic}(R)^{-\sigma}$. We then have a forgetful map

$$\mathrm{Pic}^{\mathrm{h}}(R, \sigma) \to \mathrm{Pic}(R)^{-\sigma}.$$

Notation 4.38. Let R be a discrete commutative ring with a C_2-action $\sigma : R \to R$. The σ-dual defines a C_2-action on the Picard group of R via $[I] \mapsto -[\sigma^* I]$. We will denote the fixed points of $\mathrm{Pic}(R)$ of this action by $\mathrm{Pic}(R)^{-\sigma}$.

An invertible R-module I is an element of $\mathrm{Pic}(R)^{-\sigma}$ if and only if $[I]$ is in the kernel of the corresponding norm map: $N([I]) = (\sigma + 1)[I] = 0$. This observation leads to the following generalization of Proposition 4.34

Proposition 4.39. *Let Spec(R) be a discrete irreducible affine scheme with a C_2-action $\sigma : R \to R$. Then we have the following split exact sequence*

$$0 \to (R^\times)^{C_2}/N(R^\times) \to Pic^h(R,\sigma) \to Pic(R)^{-\sigma} \to 0.$$

V: pich is likely not a zariski sheaf, but I believe there is an exact sequence for sheaves that can deal with intersections of irreducible components, which could explain the general case.

Proof. It follows from Remark 4.32 that the kernel of the forgetful map $\mathrm{Pic}^{\mathrm{h}}(R,\sigma) \to \mathrm{Pic}(R)^{-\sigma}$ is given by $(R^\times)^{C_2}/N(R^\times)$. It remains to show that any invertible module I over R has a nondegenerate σ-hermitian form. By Proposition 4.36 we may assume that R is an integral domain. In that case the Picard group of R is isomorphic to the ideal class group of R, $\mathrm{Cl}(R) \xrightarrow{\sim} \mathrm{Pic}(R)$. Thus, we may assume that I is a fractional ideal of R, which naturally comes with the pairing $I \otimes_R \sigma^* I \to R$, $i \otimes_R j \mapsto \sigma(j)i$. This pairing is nondegenerate, by the assumption that $I \in \mathrm{Pic}(R)^{-\sigma}$. The assignment of this pairing provides a section $\mathrm{Pic}(R)^{-\sigma} \to \mathrm{Pic}^{\mathrm{h}}(R,\sigma)$ and thus proves the claim. □

Example 4.40. Let $R = \mathbf{Z}[\frac{1+\sqrt{-23}}{2}]$ with the action $\sigma : R \to R$ given by complex conjugation. The fixed points of R are given by integers, thus $(R^\times)^{C_2}/N(R^\times) \simeq \mathbf{Z}/2$. The Picard group $\mathrm{Pic}(R) \simeq \mathbf{Z}/3$ of R is generated by the ideal $I = (2, \frac{1+\sqrt{-23}}{2})$. We have an isomorphism $\sigma^* I \xrightarrow{\sim} (2, \frac{1-\sqrt{-23}}{2})$, $x \mapsto \sigma(x)$. Since $(2, \frac{1+\sqrt{-23}}{2})(2, \frac{1-\sqrt{-23}}{2}) = (2)$ is a principal ideal, we know $[\sigma^* I] = -[I]$ in $\mathrm{Pic}(R)$. Thus the C_2-action on the Picard group induced by the σ-dual is given by the identity. Therefore, $\mathrm{Pic}(R)^{-\sigma} \simeq \mathrm{Pic}(R) \simeq \mathbf{Z}/3$. Thus, by Proposition 4.39, we have

$$\mathrm{Pic}^{\mathrm{h}}(\mathbf{Z}[\frac{1+\sqrt{-23}}{2}],\sigma) \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/3.$$

This example shows that the forgetful map $\mathrm{Pic}^{\mathrm{h}}(R) \to \mathrm{Pic}(R)$ does generally not land in the 2-torsion part of the Picard group of R.

Remark 4.41. Let $\mathrm{Spec}(R)$ be a discrete irreducible affine scheme with a C_2-action $\sigma : R \to R$. It follows from Proposition 4.39 that we have an exact sequence

$$0 \to (R^\times)^{-\sigma} \to R^\times \xrightarrow{N} (R^\times)^{C_2} \to \mathrm{Pic}^{\mathrm{h}}(R,\sigma) \to \mathrm{Pic}(R) \xrightarrow{N} \mathrm{Pic}(R)^{-\sigma}.$$

Remark 4.42. We have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mu_2(R)^{C_2} \arrow[r] \arrow[d] & (R^\times)^{C_2} \arrow[r, "\times 2"] \arrow[d] & (R^\times)^{C_2} \arrow[r] \arrow[d] & \mathrm{Pic}^{\mathrm{h}}(R^{C_2}) \arrow[r] \arrow[d] & \mathrm{Pic}(R^{C_2}) \arrow[r, "\times 2"] \arrow[d] & 2\mathrm{Pic}(R^{C_2}) \arrow[d] \\
0 \arrow[r] & (R^\times)^{-\sigma} \arrow[r] & R^\times \arrow[r, "N"] & (R^\times)^{C_2} \arrow[r] & \mathrm{Pic}^{\mathrm{h}}(R,\sigma) \arrow[r] & \mathrm{Pic}(R) \arrow[r, "N"] & \mathrm{Pic}(R)^{C_2}
\end{tikzcd}$$

V: second row is not exact at the first pic as of now. all vertical maps are injective? Pic satisfies galois descent. Maybe so does Pic^h?

Example 4.43. Let $R = k[x,y]/(xy)$.

Proposition 4.44.

Theorem 4.45. *Let R be a discrete commutative ring with a C_2-action $\sigma \colon R \xrightarrow{\sim} R$ via ring maps. Regard R as a Poincaré ring via Example 3.18. Then there is a split short exact sequence of abelian groups*

$$0 \to \mathrm{hPic}(R) \to \pi_0\mathrm{PnPic}(\underline{R}^\sigma) \to C_{C_2}(\mathrm{Spec}\, R, \mathbb{Z}^-) \to 0$$

V: find example to

where R is endowed with the genuine symmetric Poincaré structure and $\mathbb{Z}^-$ is endowed with the C_2-action given by multiplication by -1 and C_{C_2} denotes continuous functions which are moreover C_2-equivariant. Moreover, forgetting the hermitian form (resp. forgetting the C_2-action) induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{hPic}(R) \arrow[r] \arrow[d] & \pi_0\mathrm{PnPic}(R) \arrow[r] \arrow[d] & C_{C_2}(\operatorname{Spec} R, \mathbb{Z}^-) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Pic}^{\mathrm{cl}}(R) \arrow[r] & \pi_0\mathrm{Pic}\left(\mathrm{Perf}_R\right) \arrow[r] & C(\operatorname{Spec} R, \mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

where the bottom row is that of [Fau03, Theorem 3.5].

Remark 4.46. The outer terms of the short exact sequence in Theorem 4.45 do not appear to depend on the linear part of the Poincaré structure on Mod_R^ω; in particular, $\pi_0\mathrm{PnPic}\left(\mathrm{Mod}_R^\omega, Ϙ_R^s\right)$ (i.e. R endowed with the symmetric Poincaré structure of Example 3.16) appears to agree with $\pi_0\mathrm{PnPic}(\underline{R}^\sigma)$. However, it follows from Theorem 4.11 that $\mathrm{PnPic}\left(\mathrm{Mod}_R^\omega, Ϙ_R^s\right)$ and $\mathrm{PnPic}(\underline{R}^\sigma)$ do not agree as *spectra*.

Proof of Theorem 4.45. An object of $\pi_0\mathrm{PnPic}(R)$ is a pair (I, q) where I is an invertible R-module and q is a point in $\pi_0\Omega^\infty Ϙ_{R^{gs}}(I)$. By the proof of [Fau03, Theorem 3.5], I induces a continuous map $\Psi(I)\colon \operatorname{Spec} R \to \mathbb{Z}$. Write σ for the involution on R. Now q in particular induces an equivalence $q\colon I \xrightarrow{\sim} I^\dagger \simeq (\sigma_* I)^\vee$. For each point $\mathfrak{p} \in \operatorname{Spec} R$, localizing q gives an equivalence

$$q_\mathfrak{p}\colon I_\mathfrak{p} \xrightarrow{\sim} (\sigma_* I)_\mathfrak{p}^\vee \simeq \left(\sigma_*(I_{\sigma(\mathfrak{p})})\right)^\vee .$$

Since $I_\mathfrak{p}$ is an invertible module over a local ring, [Fau03, Proposition 3.2] implies that $q_\mathfrak{p}$ induces an equivalence

$$I_\mathfrak{p} \simeq R_\mathfrak{p}[\varphi(\mathfrak{p})] \xrightarrow{\sim} \left(\sigma_*(R_{\sigma(\mathfrak{p})}[\varphi(\sigma(\mathfrak{p}))])\right)^\vee \simeq (\sigma_*(R_{\sigma(\mathfrak{p})}))^\vee[-\varphi(\sigma(\mathfrak{p}))]\,.$$

Since R is discrete, this implies in particular that $\Psi(I)(\sigma(\mathfrak{p})) = -\Psi(I)(\mathfrak{p})$, i.e. that $\Psi(I)$ is C_2-equivariant. It follows immediately from [Fau03, Theorem 3.5] that Ψ is a homomorphism and that an element of the kernel of Ψ lifts to $\mathrm{hPic}(R)$.

Now consider a C_2-equivariant map $g\colon \operatorname{Spec} R \to \mathbb{Z}$. As in *loc. cit.*, the image of g is finite and C_2-invariant, say $\{n_1, -n_1, \dots, n_m, -n_m\}$ or $\{0, n_1, -n_1, \dots, n_m, -n_m\}$ for some $n_i \neq 0$. As in *loc. cit.*, the disjoint subsets $U_{\pm n_i} := g^{-1}(\pm n_i)$ correspond to an orthogonal basis of idempotents $e_{U_{\pm n_i}}$ in R. Since g is C_2-equivariant with respect to the sign action on $\mathbb{Z}$, we have $\sigma(U_{n_i}) = U_{-n_i}$. Moreover, it follows from Lemma 3.4 *ibid.* that $\sigma(e_{U_{n_i}}) = e_{U_{-n_i}}$. Consider the R-module $\Phi(g) := \bigoplus_{n\in \mathrm{Im}(g)} e_{g^{-1}(\{n\})}R[n]$. In other words, $\Phi(g) := \bigoplus_{i=1}^m \left(e_{U_{n_i}}R[n_i] \oplus e_{U_{-n_i}}R[-n_i]\right)$ if 0 is not in the image of g and $\Phi(g) := e_{U_0} \oplus \bigoplus_{i=1}^m \left(e_{U_{n_i}}R[n_i] \oplus e_{U_{-n_i}}R[-n_i]\right)$ otherwise. Observe that $\left(e_{U_{-n_i}}R[-n_i]\right)^\dagger = \hom_R(e_{U_{-n_i}}R[-n_i], \sigma_* R) = \hom_R\left(\sigma_*(e_{U_{-n_i}}R), R\right)[n_i] = \hom_R\left(e_{U_{n_i}}R, R\right)[n_i]$. Finally, we claim that there is a canonical σ-hermitian form $q_g \in \Omega^\infty Ϙ_{R^{\mathrm{gs}}}(\Phi(g))$ whose adjoint $q_g^\dagger\colon \Phi(g) \xrightarrow{\sim} \Phi(g)^\dagger$ corresponds to the identity. That q_g defines a point of $\hom_{R^{\otimes 2}}(\Phi(g)^{\otimes 2}, R)^{hC_2}$ is evident. Observe that to give a lift of q_g to $Ϙ_{R^{\mathrm{gs}}}(\Phi(g)) = \hom_{N^{C_2}R}\left(N^{C_2}\Phi(g), R\right)$ is equivalent to giving a commutative diagram

$$\begin{tikzcd} \Phi(g) \otimes_R R^{\varphi C_2} \arrow[r, dashed, "\exists ?"] \arrow[d] & R^{\varphi C_2} \arrow[d] \\ \left(\Phi(g)^{\otimes 2}\right)^{tC_2} \arrow[r, "q_g^{tC_2}"] & R^{tC_2} \end{tikzcd} \tag{4.47}$$

of $R^{\varphi C_2}$-modules. Let us write $\eta\colon R \to \pi_0 R^{\varphi C_2}$ for the ring map induced by the structure map. Since R is a C_2-$\mathbb{E}_\infty$-ring, η is invariant with respect to the given action on R and the trivial action on $\pi_0 R^{\varphi C_2}$. Consider $e_{U_{n_i}}$ an idempotent corresponding to an element of the image of g so that $n_i \neq 0$. Then

$$\begin{aligned} \eta(e_{U_{n_i}}) &= \eta(e_{U_{n_i}})^2 && \text{ring maps preserve idempotents} \\ &= \eta(e_{U_{n_i}}) \cdot \eta(e_{U_{-n_i}}) && C_2\text{-invariance of } \eta \\ &= \eta(e_{U_{n_i}} e_{U_{-n_i}}) && \eta \text{ is a ring map} \\ &= 0 && \text{orthogonality and } n_i \neq 0\,. \end{aligned}$$

In particular, if 0 is not in the image of g, $\Phi(g) \otimes_R R^{\varphi C_2} \simeq 0$ and (4.47) commutes vacuously. If 0 is in the image of g, then $e_{U_0} R$ is a discrete/projective $e_{U_0} R$-module and q_g evidently defines a genuine hermitian form on $e_{U_0} R$ (compare [Cal+20a, Remark 4.2.21]).

Thus, $g \mapsto (\Phi(g), q_g)$ defines a splitting of Ψ which agrees with the splitting constructed in [Fau03, Theorem 3.5] on underlying objects. □

N: I'm commenting out section 5 since we do not yet have an application for it. Feel free to put it back in if you want.

5 The Poincaré Brauer Group

In this section, we will define and study derived analogues of the involutive Brauer group. In §5.2, we define the involutive Brauer group in terms of invertible Poincaré ∞-categories and compute it in a handful of examples. In particular, our examples (see Examples 5.22 and 5.20, especially the latter) show that our involutive Brauer group subsumes the classical involutive Brauer group but also captures information which is genuinely novel. Our definition will use the formalism of relative Poincaré ∞-categories, some preliminary discussion of which is collected in §5.1.

L: add reference; workshop wording at end of sentence: '...captures non-discrete duality phenomena' or 'derived duality'?

L: revisit organization, then add exposition

5.1 Generalities on R-linear Poincaré ∞-categories

Let A be a Poincaré ring spectrum. By Remark 3.7, $(\mathrm{Mod}^\omega_A, \Qoppa_A)$ promotes to a commutative algebra object in the ∞-category of Poincaré ∞-categories $\mathrm{Cat}^{\mathrm{p}}_\infty$, and we may thus consider modules over it, or *A-linear Poincaré ∞-categories.*

Proposition 5.1. *Let $(R, C \to R^{tC_2})$ be a Poincaré ring spectrum and write $(\mathrm{Mod}^\omega_R, \Qoppa_R)$ for the Poincaré ∞-category of Remark 3.7.*

(1) The ∞-category $\mathrm{Mod}_{(\mathrm{Mod}^\omega_R, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ admits all small limits and colimits, and it inherits a canonical symmetric monoidal structure, and for every morphism $\left(R, R^{\varphi C_2} \to R^{tC_2}\right) \to (S, S^{\varphi C_2} \to S^{tC_2})$, the functor $\mathrm{Mod}_{(\mathrm{Mod}^\omega_R, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_{(\mathrm{Mod}^\omega_S, \Qoppa_S)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ is a symmetric monoidal left adjoint.

(2) Let A be an $\mathbb{E}_1$-R-algebra in spectra, and regard the category of compact right A-modules Mod^ω_A as left-tensored over Mod^ω_R in the canonical way. Then the pullback

$$
\begin{tikzcd}
 & \mathrm{Mod}_{(\mathrm{Mod}^\omega_R, \Qoppa_R)}\left(\mathrm{Cat}^{\mathrm{h}}_\infty\right) \arrow[d] \\
\{\mathrm{Mod}^\omega_A\} \arrow[r] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}
\end{tikzcd}
\tag{5.2}
$$

is canonically equivalent to $\mathrm{Mod}_{N_R A \otimes_{N_R R} R^L}\left(\mathrm{Sp}^{C_2}\right)$ where R^L is the $\mathbb{E}_\infty$-$N_R R$-algebra with $(R^L)^e \simeq R$ and $(R^L)^{\varphi C_2} \simeq C$.

A $N_R A \otimes_{N_R R} R^L$-module classifies a $(\mathrm{Mod}^\omega_R, \Qoppa_R)$-module in Poincaré ∞-categories if its underlying A-module is invertible in the sense of [Cal+20a, Definition 3.1.4].

(3) Let A, B be R-algebras with associated (R-linear) modules with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and $(M_B, N_B, N_B \to M_B^{tC_2})$, respectively so that (under item (2)) $(\mathrm{Mod}^\omega_A, \Qoppa_A)$ and $(\mathrm{Mod}^\omega_B, \Qoppa_B)$ are objects of $\mathrm{Mod}_{(\mathrm{Mod}^\omega_R, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$. Then the symmetric monoidal structure of item (1) is so that the underlying R-linear ∞-category with perfect duality $(\mathrm{Mod}^\omega_A, \Qoppa_A) \otimes_{(\mathrm{Mod}^\omega_R, \Qoppa_R)} (\mathrm{Mod}^\omega_B, \Qoppa_B)$ is $\mathrm{Mod}^\omega_A \otimes_{\mathrm{Mod}^\omega_R} \mathrm{Mod}^\omega_B \simeq \mathrm{Mod}^\omega_{A \otimes_R B}$, and the associated module with genuine involution is given by $M_A \otimes_R M_B$, $N_A \otimes_{R^{\varphi C_2}} N_B$, and the structure map is $N_A \otimes_{R^{\varphi C_2}} N_B \to M_A^{tC_2} \otimes_{R^{tC_2}} M_B^{tC_2} \to (M_A \otimes_R M_B)^{tC_2}$ where the latter map arises canonically from lax monoidality of the Tate construction.

(4) Let $(\mathcal{C}, \Qoppa_{\mathcal{C}}), (\mathcal{D}, \Qoppa_{\mathcal{D}})$ be objects of $\mathrm{Mod}_{(\mathrm{Mod}^\omega_R, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{h}}_\infty)$. Then the forgetful functor induces $\hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}((\mathcal{C}, \Qoppa_{\mathcal{C}}), (\mathcal{D}, \Qoppa_{\mathcal{D}})) \to \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}(\mathcal{C}, \mathcal{D})$ on mapping spaces so that the fiber over an R-linear functor $F\colon \mathcal{C} \to \mathcal{D}$ is the mapping

space $\mathrm{map}_{\Qoppa_R}(F_!\Qoppa_{\mathcal{C}}, \Qoppa_{\mathcal{D}}) \simeq \mathrm{map}_{\Qoppa_R}(\Qoppa_{\mathcal{C}}, \Qoppa_{\mathcal{D}} \circ F^{\mathrm{op}})$*, where the mapping space is taken in* $\mathrm{Fun}^q_{\Qoppa_R}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp})$ *and* $\mathrm{Fun}^q_{\Qoppa_R}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$*, respectively.*[3]

(5) The symmetric monoidal forgetful functor $\theta\colon \mathrm{Mod}_{(\mathrm{Mod}^\omega_R, \Qoppa_R)}(\mathrm{Cat}^{\mathrm{h}}_\infty) \to \mathrm{Mod}_{\mathrm{Mod}^\omega_R}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)$ *is a (co)cartesian fibration.*

L: what is it classified by?

Remark 5.3. A special case of part (2) is [Cal+20a, Example 5.4.13].

Proof. (1) The first part of the statement follows from [Cal+20a, §6.1] and [Lur17, §4.2.3].

(2) Let $\mathcal{LM}^\otimes$ denote the ∞-operad of [Lur17, Definition 4.2.1.7]. Our strategy of proof will be similar to that of [Cal+20a, §5.3]: First, we show that an $\mathcal{LM}^\otimes$-algebra object in $\mathrm{Cat}^{\mathrm{h}}_\infty$ is equivalent to an $\mathcal{LM}^\otimes$-algebra object in an operad of functor categories. Then, we use a (suitably coherent version of) the classification of hermitian structures on module categories as categories of modules over the Hill–Hopkins–Ravenel norm [Cal+20a, Theorem 3.3.1] to conclude. Recall that the action of Mod^ω_R on Mod^ω_A is given by a functor $\mathcal{LM}^\otimes \to \mathcal{C}\mathrm{at}^\times_\infty$, and define $\mathrm{Fun}_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes$ via the following pullback square of ∞-operads:

$$\begin{tikzcd}
\mathrm{Fun}_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes \arrow[r, "p"] \arrow[d] & \mathcal{LM}^\otimes \arrow[d, "{\mathrm{Mod}^\omega_R, \mathrm{Mod}^\omega_A}"] \\
(\mathcal{C}\mathrm{at}_\infty)^\otimes_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}^\times_\infty
\end{tikzcd} . \tag{5.4}$$

Informally, an object $F \in \mathrm{Fun}_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes_{\mathfrak{a}}$ is a functor $F\colon \mathrm{Mod}^{\omega,\mathrm{op}}_R \to \mathrm{Sp}$ and an object G over the fiber of $\mathfrak{m}$ is a functor $G\colon \mathrm{Mod}^{\omega,\mathrm{op}}_A \to \mathrm{Sp}$. The p-cocartesian edge over the canonical map $(\mathfrak{a}, \mathfrak{m}) \to \mathfrak{m}$ in $\mathcal{LM}^\otimes$ sends (F, G) to the lower arrow in the diagram

$$\begin{tikzcd}
\mathrm{Mod}^{\omega,\mathrm{op}}_R \times \mathrm{Mod}^{\omega,\mathrm{op}}_A \arrow[r, "F \times G"] \arrow[d, "- \otimes_R -"'] & \mathrm{Sp} \times \mathrm{Sp} \arrow[d, "\otimes_{\mathrm{Sp}}"] \\
\mathrm{Mod}^{\omega,\mathrm{op}}_A \arrow[r, "F \otimes G := \mathrm{LKE}_{\otimes_R}(\otimes_{\mathrm{Sp}} \circ (F \times G))"] & \mathrm{Sp}
\end{tikzcd} .$$

Now define $\mathrm{Fun}^q_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes$ to consist of the full subcategory of $\mathrm{Fun}_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes$ consisting of those tuples of functors which are all quadratic. The inclusion $\mathrm{Fun}^q_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes \to \mathrm{Fun}_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes$ exhibits the former as an ∞-operad, and moreover the localization is compatible with the $\mathcal{LM}^\otimes$-monoidal structure in the sense of [Lur17, Definition 2.2.1.6]. We can extend the previous diagram to

$$\begin{tikzcd}
\mathrm{Fun}^q_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes \arrow[r, "p"] \arrow[d] & \mathcal{LM}^\otimes \arrow[d, "{\mathrm{Mod}^\omega_R, \mathrm{Mod}^\omega_A}"] \\
\mathrm{Cat}^{\mathrm{h}\,\otimes}_\infty \arrow[r] & (\mathcal{C}\mathrm{at}_\infty)^\otimes_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}^\otimes_\infty
\end{tikzcd} . \tag{5.5}$$

Modifying [Cal+20a, Construction 5.3.15 & Lemma 5.3.15] slightly (note that Corollary 5.1.4 did not assume the tensor factors to be equivalent), we obtain an analogous commutative diagram of ∞-operads

$$\begin{tikzcd}
\mathrm{Fun}^p_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes \arrow[r] \arrow[d] & \mathrm{Fun}^q_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathrm{Mod}^{\omega,\mathrm{op}}_A, \mathrm{Sp})^\otimes \arrow[r, "p"] \arrow[d] & \mathcal{LM}^\otimes \arrow[d, "{\mathrm{Mod}^\omega_R, \mathrm{Mod}^\omega_A}"] \\
\mathrm{Cat}^{\mathrm{p}\,\otimes}_\infty \arrow[r] & \mathrm{Cat}^{\mathrm{h}\,\otimes}_\infty \arrow[r] & (\mathcal{C}\mathrm{at}_\infty)^\otimes_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}^\otimes_\infty
\end{tikzcd} \tag{5.6}$$

[3]The proof of (2) in particular shows that $\mathrm{Fun}^q(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is left-tensored over $\mathrm{Fun}^q(\mathrm{Mod}^{\omega,\mathrm{op}}_R, \mathrm{Sp})$ in the sense of [Lur17, Definition 4.2.1.19], so this makese sense.

in which all squares are pullbacks. Now suppose A is given a module with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and call the associated Poincaré ∞-category $\overline{\mathrm{Mod}}_A$. Then to lift $\overline{\mathrm{Mod}}_A$ to a module over $(\mathrm{Mod}_R^\omega, \Omega_R)$ compatibly with the Mod_R^ω-module structure on Mod_A^ω is to give a map of ∞-operads $\mathcal{LM}^\otimes \to {\mathrm{Cat}_\infty^{\mathrm{h}}}^\otimes$ so that the restriction along the canonical inclusion $\mathrm{Assoc}^\otimes \to \mathcal{LM}^\otimes$ gives the algebra object $(\mathrm{Mod}_R^\omega, \Omega_R)$ and postcomposing with the canonical projection to ${\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}}^\times$ recovers the given Mod_R^ω-module structure on Mod_A^ω. By the pullback square (5.5), this is equivalent to giving an object of $\mathrm{Alg}_{\mathcal{LM}/\mathcal{LM}}\left(\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes\right)$. Now let us identify the bilinear functor $\mathrm{Mod}_R^\omega \times \mathrm{Mod}_A^\omega \xrightarrow{-\otimes_R -} \mathrm{Mod}_A^\omega$ with the exact functor $\mathrm{Mod}_R^\omega \otimes \mathrm{Mod}_A^\omega \simeq \mathrm{Mod}_{R\otimes A}^\omega \to \mathrm{Mod}_A^\omega$ which is induction along the action map $R \otimes A \to A$. Using [Cal+20a, Corollary 3.4.1] and unravelling definitions gives the claim for R-linear hermitian structures. The proof for R-linear Poincaré structures considers (5.6) instead but otherwise proceeds in an identical fashion.

(3) By [Lur17, Theorem 4.4.2.8], the relative tensor product $(\mathrm{Mod}_A^\omega, \Omega_A) \otimes_{(\mathrm{Mod}_R^\omega, \Omega_R)} (\mathrm{Mod}_B^\omega, \Omega_B)$ is computed as the geometric realization of the bar construction

$$\begin{aligned} p\colon \Delta^{\mathrm{op}} &\to \mathrm{Cat}_\infty^{\mathrm{h}} \\ [n] &\mapsto (\mathrm{Mod}_A^\omega, \Omega_A) \otimes (\mathrm{Mod}_R^\omega, \Omega_R)^{\otimes n} \otimes (\mathrm{Mod}_B^\omega, \Omega_B) \end{aligned}$$

Write $f\colon \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ for the forgetful functor. Then $f \circ p$ has a colimit with value $\mathrm{Mod}_A^\omega \otimes_{\mathrm{Mod}_R^\omega} \mathrm{Mod}_B^\omega \simeq \mathrm{Mod}_{A\otimes_R B}^\omega$. Writing $g\colon \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \to \{*\}$, by Example 4.3.1.3 of [Lur09] we see that $f \circ p$ is a g-colimit. By Proposition 4.3.1.5(2) and Example 4.3.1.3 of [Lur09], p admits a colimit in $\mathrm{Cat}_\infty^{\mathrm{h}}$ if and only if it admits an f-colimit. Now recall that f is a cocartesian fibration with pushforward given by left Kan extension [Cal+20a, Corollary 1.4.2]. We show that f satisfies the conditions of [Lur09, Corollary 4.3.1.11].

- Condition (1) follows from Theorem 6.1.1.10 of [Lur17] applied to $\mathrm{Sp}^{\mathrm{op}}$ (see the end of [Cal+20a, Construction 1.1.26]).
- Condition (2) follows from [Cal+20a, Corollary 1.4.2], the adjoint functor theorem, and presentability of $\mathrm{Fun}^q(\mathcal{C})$, which is discussed in the proof of [Cal+20a, Lemma 5.3.3] (also see [Lur17, Remark 6.1.1.11]).

Thus the preceding discussion shows that there exists a map of simplicial sets p' making the diagram commute

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}_\infty^{\mathrm{h}} \arrow[d, "f"] \\ (\Delta^{\mathrm{op}})^{\triangleright} \arrow[r] \arrow[ur, "p'"] & \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \end{tikzcd} .$$

Since $\{0\} \to \Delta^1$ is left anodyne, by [Lur09, Corollary 2.1.2.7] the inclusions

$$\begin{aligned} \{0\} \times \Delta^{\mathrm{op}} &\to \Delta^1 \times \Delta^{\mathrm{op}} \\ \iota\colon (\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) &\to \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright} \end{aligned}$$

are left anodyne. The former implies that there exists a map p'' making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}_\infty^{\mathrm{h}} \arrow[d, "f"] \\ \Delta^1 \times \Delta^{\mathrm{op}} \arrow[r] \arrow[ur, "p''"] & \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \end{tikzcd}$$

commute. The maps p' and p'' assemble to give a map $p''' := p' \sqcup_p p''$ making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[rrr, "p"] \arrow[d] & & & \mathrm{Cat}_\infty^{\mathrm{h}} \arrow[d, "f"] \\ (\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) \arrow[r, "\iota"] \arrow[urrr, "p'''"] & \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright} \arrow[rr] \arrow[urr, bend left, "\overline{p}"] & & \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \end{tikzcd}$$

commute, and likewise $\overline{p}$ exists making the diagram commute since ι is left anodyne. Now we show that $\overline{p}$ satisfies the conditions of [Lur09, Proposition 4.3.1.9]. By (the opposite/dual/cocartesian version of) [Lur09, Remark 3.1.1.10] and Proposition 3.1.1.5(2") *ibid.* and the fact that f is a cocartesian fibration, we can choose $\overline{p}$ so that for all $k \in (\Delta^{\mathrm{op}})^{\triangleright}$, $\overline{p}|_{\Delta^1\times\{k\}}$ is f-cocartesian. Furthermore, since we can choose Δ^{op}, $(\Delta^{\mathrm{op}})^{\triangleright}$ to have the markings $(-)^{\flat}$ in [Lur09, Remark 3.1.1.10], $f\circ\overline{p}|_{\Delta^1\times\{\infty\}}$ is a degenerate edge in $\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty}$.

Now [Lur09, Proposition 4.3.1.9] implies that $\overline{p}_0$ is an f-colimit diagram if and only if $\overline{p}_1$ is an f-colimit diagram. Now notice that $\overline{p}|_{\{1\}\times(\Delta^{\mathrm{op}})^{\triangleright}}$ has image contained in the fiber of f over $\mathrm{Mod}^{\omega}_{A\otimes_R B}$. By [Lur09, Proposition 4.3.1.10], it suffices to show that $\overline{p}_1$ is a colimit diagram in $\mathrm{Fun}^q\left(\mathrm{Mod}^{\omega}_{A\otimes_R B}\right)$. Write $\overline{M}_A \in \mathrm{Mod}_{N^{C_2}A}$ and $\overline{M}_B \in \mathrm{Mod}_{N^{C_2}B}$ for the corresponding modules (see introduction to §3.3 of [Cal+20a]). Unraveling definitions and using [Cal+20a, Theorem 3.3.1 & Corollary 3.4.1 & Lemma 5.4.6], it follows that the diagram $\overline{p}_1|_{\{1\}\times\Delta^{\mathrm{op}}}$ is the bar construction

$$[n]\mapsto \overline{M}_A\otimes_{N^{C_2}R} R^{\otimes_{N^{C_2}R}n}\otimes_{N^{C_2}R}\overline{M}_B\,.$$

This proves the result.

(4) Let $(\mathcal{C},\Qoppa_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R,\Qoppa_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$ and let $F\colon\mathcal{C}=\theta\left(\mathcal{C},\Qoppa_{\mathcal{C}}\right)\to\mathcal{D}$ be an R-linear functor. Now define $\Qoppa_{\mathcal{D}}\colon\mathcal{D}^{\mathrm{op}}\to\mathrm{Sp}$ to be the left Kan extension of $\Qoppa_{\mathcal{C}}$ along F^{op}. Now $(\mathcal{D},\Qoppa_{\mathcal{D}})\in\mathrm{Cat}^{\mathrm{h}}_{\infty}$ and there is a canonical map $(f,\eta)\colon(\mathcal{C},\Qoppa_{\mathcal{C}})\to(\mathcal{D},\Qoppa_{\mathcal{D}})$. Now F is classified by a functor $\Delta^1\times\mathcal{LM}^{\otimes}\to\mathcal{C}\mathrm{at}^{\mathrm{ex}\otimes}_{\infty}$, and we may form the pullback

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \Delta^1\times\mathcal{LM}^{\otimes} \arrow[d] \\ \mathrm{Cat}^{\mathrm{h}\ \otimes}_{\infty} \arrow[r, "p"] & \mathcal{C}\mathrm{at}^{\mathrm{ex}\otimes}_{\infty} \end{tikzcd}\quad. \tag{5.7}$$

Since p is a cocartesian fibration [Cal+20a, Theorem 5.2.7], $\mathcal{N}\to\Delta^1\times\mathcal{LM}^{\otimes}$ is a cocartesian fibration, and the nontrivial morphism in Δ^1 classifies a map $F_!\colon\mathrm{Fun}^q_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})^{\otimes}\to\mathrm{Fun}^q_{\mathrm{Mod}^{\omega,\mathrm{op}}_R}(\mathcal{D}^{\mathrm{op}},\mathrm{Sp})^{\otimes}$ of ∞-operads over $\mathcal{LM}^{\otimes}$. Passing to algebra objects, we obtain the desired result on mapping spaces.

(5) By [Lur09, Proposition 2.4.2.8], it suffices to show that θ is a locally (co)cartesian fibration, and that locally (co)cartesian edges are closed under composition. We give the proof that θ is a cocartesian fibration; the proof that θ is a cartesian fibration is formally dual and will be left to the reader.

Let $(\mathcal{C},\Qoppa_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R,\Qoppa_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$ and let $F\colon\mathcal{C}=\theta\left(\mathcal{C},\Qoppa_{\mathcal{C}}\right)\to\mathcal{D}$ be an R-linear functor. Now define $\Qoppa_{\mathcal{D}}\colon\mathcal{D}^{\mathrm{op}}\to\mathrm{Sp}$ to be the left Kan extension of $\Qoppa_{\mathcal{C}}$ along F^{op}. By the proof of (4), we see that the image of $\Qoppa_{\mathcal{C}}$ under $F_!$ is a lift of $(\mathcal{D},\Qoppa_{\mathcal{D}})$ to an object of $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R,\Qoppa_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$ and (f,η) to a morphism in $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R,\Qoppa_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$.

Now by Lemma 2.4.4.1 and the locally cocartesian version of Proposition 2.4.1.10 of [Lur09], we must show that for all choices $\Qoppa'_{\mathcal{D}}$ of an R-linear Hermitian structure on $\mathcal{D}$, precomposition with $F_!$ induces a pullback square

$$\begin{tikzcd} \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}\left(\left(\mathcal{D},\Qoppa_{\mathcal{D}}\right),\left(\mathcal{D},\Qoppa'_{\mathcal{D}}\right)\right) \arrow[r] \arrow[d] & \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}\left(\left(\mathcal{C},\Qoppa_{\mathcal{C}}\right),\left(\mathcal{D},\Qoppa'_{\mathcal{D}}\right)\right) \arrow[d] \\ \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}\left(\mathcal{D},\mathcal{D}\right) \arrow[r] & \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}\left(\mathcal{C},\mathcal{D}\right) \end{tikzcd}\quad. \tag{5.8}$$

By (4), $F_!$ induces equivalences on the fibers of the vertical maps, hence (f,η) is locally θ-cocartesian. The locally θ-cocartesian maps are manifestly closed under composition, hence we are done. □

If R is a Poincaré ring, then the ∞-categories of R-linear Poincaré and hermitian ∞-categories are closed symmetric monoidal, similarly to their absolute counterparts [Cal+20a, Corollary 6.2.9, 6.2.15].

Corollary 5.9. *Let R be a Poincaré ring, and let A, B be $\mathbb{E}_1$-R-algebras with genuine involution. Then there is a Poincaré structure $\Qoppa_{A\boxtimes B^{\mathrm{op}}}$ on $\mathrm{Mod}_{A\otimes_R B^{\mathrm{op}}}$ with equivalences*

L: reference to definition!

L: need something like A-B-bimodules whose underlying B-module is compact.

$$\hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty,\mathrm{idem}\,R}}\left(\left(\mathrm{Mod}^{\omega}_A, \Qoppa_A\right), \left(\mathrm{Mod}^{\omega}_B, \Qoppa_B\right)\right) \simeq \mathrm{He}\left(\mathrm{Mod}_{A^{\mathrm{op}}\otimes_R B}, \Qoppa_{A^{\mathrm{op}}\boxtimes B}\right)$$
$$\hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}}\left(\left(\mathrm{Mod}^{\omega}_A, \Qoppa_A\right), \left(\mathrm{Mod}^{\omega}_B, \Qoppa_B\right)\right) \simeq \mathrm{Pn}\left(\mathrm{Mod}_{A^{\mathrm{op}}\otimes_R B}, \Qoppa_{A^{\mathrm{op}}\boxtimes B}\right)$$

which are compatible with ordinary Morita theory, i.e. under the forgetful functors $\mathrm{Cat}^{\mathrm{h}}_{\infty,\mathrm{idem}\,R}, \mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}$, the hermitian (resp. Poincaré functor) represented by the pair $(P \in \mathrm{Mod}_{A^{\mathrm{op}}\otimes_R B}, q)$ is the R-linear exact functor which is given by tensoring with the A^e-B^e-bimodule P (compare, for instance, [Lur17, p.738]).

Proof. A hermitian functor $(\mathrm{Mod}^{\omega}_A, \Qoppa_A) \to (\mathrm{Mod}^{\omega}_B, \Qoppa_B)$ is a pair (φ, η) where $\varphi\colon \mathrm{Mod}^{\omega}_A \to \mathrm{Mod}^{\omega}_B$ is an R-linear functor and $\eta\colon \Qoppa_B \implies \Qoppa_A \circ \varphi^{\mathrm{op}}$ is a natural transformation. Note that φ is given by $-\otimes_A P$ for some A^e-B^e-bimodule P which, when regarded as a B^e-module, is compact. The data of the natural transformation η is given by a map $N^{C_2}_R P \otimes_{N^{C_2}_R A} A \to B$ of $N^{C_2}_R B$-modules. Define $\Qoppa_{A\boxtimes B^{\mathrm{op}}}$ via the composite

L: needs to be $\Qoppa_R$-linear in some sense.

$$\mathrm{Mod}^{\mathrm{op}}_{A^{\mathrm{op}}\otimes_R B}(\mathrm{Sp}) \xrightarrow{N^{C_2}_R(-)} \mathrm{Mod}_{N^{C_2}_R(A^{\mathrm{op}})\otimes_R N^{C_2}_R(B)}\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{-\otimes_{N^{C_2}_R A^{\mathrm{op}}} A^{\mathrm{op}}} \mathrm{Mod}_{A^{\mathrm{op}}\otimes_R N^{C_2}_R(B)}\left(\mathrm{Sp}^{C_2}\right)$$
$$\xrightarrow{\text{forget } A^{\mathrm{op}} \text{ module}} \mathrm{Mod}_{N^{C_2}_R(B)}\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{\hom_{N^{C_2}_R(B)}(-,B)} \mathcal{S}\ .$$

Observe that the module with genuine involution over $A^{\mathrm{op}} \otimes_R B$ associated to $\Qoppa_{A^{\mathrm{op}}\boxtimes B}$ can be obtained by taking the C_2-mapping spectrum (which is in fact an R^L-module) with $P = A^{\mathrm{op}} \otimes_R B$. Since the relative norm is symmetric monoidal , we have

L: check that the module is invertible! or change statement in theorem to "hermitian"

L: to-do: reference

$$\begin{aligned}\hom_{N^{C_2}_R B}\left(N^{C_2}_R(A^{\mathrm{op}} \otimes_R B) \otimes_{N^{C_2}_R A^{\mathrm{op}}} A^{\mathrm{op}}, B\right) &\simeq \hom_{N^{C_2}_R B}\left(\left(N^{C_2}_R(A^{\mathrm{op}}) \otimes_R N^{C_2}_R(B)\right) \otimes_{N^{C_2}_R A^{\mathrm{op}}} A^{\mathrm{op}}, B\right)\\ &\simeq \hom_{N^{C_2}_R B}\left(A^{\mathrm{op}} \otimes_R N^{C_2}_R(B^{\mathrm{op}}), B\right)\\ &\simeq \hom_{R^L}\left(A^{\mathrm{op}}, B\right)\ .\end{aligned}$$

Since $N^{C_2}_R(-)$ is quadratic and the rest of the functors are exact, it is clear that $\Qoppa_{A^{\mathrm{op}}\boxtimes B}$ is quadratic. Observe that, given such a φ, the data of η is the same as the data of a point $\eta^{\dagger} \in \Omega^{\infty}\Qoppa_{A^{\mathrm{op}}\boxtimes B}(P)$. It follows from unravelling definitions that (φ, η) is duality-preserving (i.e. the functor is Poincaré) if and only if $\eta^{\dagger}$ is non-degenerate. □

L: Compare [Cal+20a, Corollary 3.4.2]. add reference to remark that Mod^{ω} of an $\mathbb{E}_1$-algebra is always Poincaré!

Definition 5.10. Let R be a Poincaré ring, and let A be an $\mathbb{E}_1$-R-algebra with an anti-involution. We will refer to the data of $(M_A, N_A, N_A \to M_A^{tC_2})$ of Proposition 5.1(2) as an *R-linear A-module with genuine involution.*

Remark 5.11. When $R = \mathbb{S}^0$ is the initial Poincaré ring, then a $\mathbb{S}^0$-linear A-module with genuine involution is simply an A-module with genuine involution in the sense of [Cal+20a, Defintiion 3.2.3].

Proposition 5.12. *Let R be a Poincaré ring, and let A, B be $\mathbb{E}_1$-R-algebras with anti-involutions and let $(M_A, N_A, \alpha\colon N_A \to M_A^{tC_2})$, $(M_B, N_B, \beta\colon N_B \to M_B^{tC_2})$ be R-linear modules with genuine involution over A and B, respectively. Suppose given a map $f\colon A \to B$ of $\mathbb{E}_1$-R-algebras with anti-involution, and write f_* for the functor $B \otimes_A -\colon \mathrm{LMod}^{\omega}_A \to \mathrm{LMod}^{\omega}_B$. Then*

L: 'geometrically' this should be f^*

1. *the data of a R-linear hermitian functor $\left(\mathrm{LMod}^{\omega}_A, \Qoppa^{\alpha}_{M_A}\right) \to \left(\mathrm{LMod}^{\omega}_B, \Qoppa^{\beta}_{M_B}\right)$ covering the base change functor f_* can be encoded by a triple (δ, γ, σ) where $\delta\colon M_A \to M_B$ is a morphism in $\mathrm{LMod}^{hC_2}_{A\otimes_R A}$, $\gamma\colon N_A \to N_B$ is a morphism in $\mathrm{LMod}_{A\otimes_R R^{\varphi C_2}}$, and σ is a homotopy making the square*

$$\begin{array}{ccc} N_A & \xrightarrow{\gamma} & N_B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ M_A^{tC_2} & \xrightarrow{\delta^{tC_2}} & M_B^{tC_2} \end{array}$$

commute.

L: This is an R-linear version of [Cal+20a, Corollary 3.4.2]

2. (δ, γ, σ) *defines an R-linear Poincaré functor if the maps*

$$\begin{aligned} B \otimes_A M_A \to (B \otimes_R B) \otimes_{A \otimes_R A} M_A \to M_B \\ B \otimes_A N_A \to N_B \end{aligned}$$

are equivalences.

Proof. Follows from Corollary 5.9. □

Write Fm, resp. Pn for the composite $\mathrm{Cat}^{\mathrm{h}}_{\infty R} \xrightarrow{U} \mathrm{Cat}^{\mathrm{p}}_{\infty} \xrightarrow{\mathrm{Fm}} \mathcal{S}$, resp. $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \xrightarrow{U} \mathrm{Cat}^{\mathrm{p}}_{\infty} \xrightarrow{\mathrm{Pn}} \mathcal{S}$ where Fm and Pn are defined in [Cal+20a, Definitions 2.1.1, 2.1.3].

Proposition 5.13. *Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Then $(\mathrm{Mod}^{\omega}_R, Ϙ_R)$ corepresents the functors* $\mathrm{Fm}\colon \mathrm{Cat}^{\mathrm{h}}_{\infty R} \to \mathcal{S}$ *and* $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{S}$.

Since the forgetful functor $\mathrm{Cat}^{\mathrm{h}}_{\infty R} \to \mathrm{Cat}^{\mathrm{h}}_{\infty}$ and its Poincaré counterpart both preserve filtered colimits, an immediate consequence is that the unit is compact (cf. [Cal+20a, Proposition 6.1.8]).

Corollary 5.14. *Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Then $(\mathrm{Mod}^{\omega}_R, Ϙ_R)$ is a compact object of both* $\mathrm{Cat}^{\mathrm{h}}_{\infty R}$ *and* $\mathrm{Cat}^{\mathrm{p}}_{\infty R}$.

Proof of Proposition 5.13. We prove the statement for Pn; the proof for Fm is similar and is left to the reader. Recall that Proposition 5.1.(1) furnishes an adjoint pair $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \rightleftarrows \mathrm{Cat}^{\mathrm{p}}_{\infty}$ of functors. Write $\overline{\mathcal{C}} = (\mathcal{C}, Ϙ_{\mathcal{C}}) \in \mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}$. Then

$$\mathrm{Pn}(\mathcal{C}) = \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty}}\left((\mathrm{Sp}^f, Ϙ^u), U(\overline{\mathcal{C}})\right) \simeq \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty R}}\left((\mathrm{Mod}^{\omega}_R, Ϙ_R) \otimes (\mathrm{Sp}^f, Ϙ^u), \overline{\mathcal{C}}\right),$$

where the first equivalence is [Cal+20a, Proposition 4.1.3]. □

5.2 Invertible R-linear Poincaré ∞-categories

Recall that a Poincaré ∞-category is called idempotent complete if the underlying stable ∞-category is idempotent complete. The full subcategory of $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ spanned by idempotent complete Poincaré ∞-categories is denoted by $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$ [Cal+20b, Definition 1.3.2].

Definition 5.15. Let A be a Poincaré ring spectrum. We define the *Poincaré Brauer space of A* as

$$\mathrm{Br}^{\mathrm{p}}(A) := \mathrm{Pic}\left(\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_A, Ϙ_A\right)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})\right).$$

The assignment $A \mapsto \mathrm{Br}^{\mathrm{p}}(A)$ defines a functor

$$\mathrm{Br}^{\mathrm{p}}\colon \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$$

valued in grouplike $\mathbf{E}_\infty$-spaces.

Let (X, σ, Y, π) be a scheme with involution and a good quotient. We define the *Poincaré Brauer space of X with respect to* π as

$$\mathrm{Br}^{\mathrm{p}}(X) := \lim_{j\colon \mathrm{Spec}\,R \to Y} \mathrm{Br}^{\mathrm{p}}(\underline{\Gamma(j^*(X))}^{j^*\sigma})$$

where the limit is over all étale j and $\Gamma(j^*(X))$ is a ring with involution by affineness of π, hence $R \to \Gamma(j^*(X))$ is a Poincaré ring via Example 3.18.

L: Can replace good quotient Y by the Deligne–Mumford stack $X//C_2$ [FW20, Construction 4.41]. This should correspond to the symmetric Poincaré structure of Example 3.16.

Remark 5.16. The symmetric monoidal forgetful functor $\mathrm{Mod}_A(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_A(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty})$ induces a map $\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A)$ of grouplike $\mathbf{E}_\infty$-spaces, where $\mathrm{Br}(A)$ is the Brauer space $\mathrm{br}_{\mathrm{alg}}(A)$ of [AG14a, pp. 1154–1155].

Proposition 5.17. *Let A be a Poincaré ring spectrum. Then we have a canonical equivalence*

$$\Omega\,\mathrm{Br}^{\mathrm{p}}(A) \simeq \mathrm{Pic}^{\mathrm{p}}(A).$$

Proof. Since $\Omega\,\mathrm{Br}^{\mathrm{p}}(R)$ is given by the space of automorphisms of any object in $\mathrm{Br}^{\mathrm{p}}(R)$, it suffices to determine the space of autoequivalences of $(\mathrm{Mod}^{\omega}_R, \Qoppa_R)$. By Proposition 5.1, an autoequivalence is the data of a pair (f,η) where $f\colon \mathrm{Mod}^{\omega}_R \xrightarrow{\simeq} \mathrm{Mod}^{\omega}_R$ is an exact R-linear autoequivalence and $\eta\colon \Qoppa_R \xrightarrow{\sim} \Qoppa_R \circ f^{\mathrm{op}}$ is a $\Qoppa_R$-linear equivalence. Since $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}$ is symmetric monoidal (and hence f will be Mod^{ω}_R-linear), f is of the form $-\otimes_R \mathcal{L}$ where $\mathcal{L}$ is an invertible R-module. Since taking bilinear and linear parts is functorial by [Cal+20a, Proposition 1.3.11], η is equivalently the data of a pair of equivalences

$$\begin{aligned} b(\eta)\colon\ &\hom_{R\otimes R}((-\otimes\mathcal{L})\otimes(-\otimes\mathcal{L}),R)^{hC_2} \simeq \hom_{R\otimes R}(-\otimes -,R)^{hC_2}\\ \ell(\eta)\colon\ &\hom_R(-\otimes\mathcal{L},R^{\varphi C_2}) \simeq \hom_R(-,R^{\varphi C_2}) \end{aligned}$$

plus a path between their images in $\hom_R(\mathcal{L},R^{tC_2})$. The transformation $b(\eta)$ is equivalent to the data of an R-bilinear equivalence $R \simeq \mathcal{L}^\vee \otimes \sigma^*\mathcal{L}^\vee$, and the transformation $\ell(\eta)$ is equivalent to the data of an $R^{\varphi C_2}$-linear equivalence $\ell(\eta)\colon R^{\varphi C_2}\otimes_R \mathcal{L}^\vee \xrightarrow{\sim} R^{\varphi C_2}$.

L: is the $R^{\varphi C_2}$-linearity of this $\simeq$ correct?

Now consider the composites

$$R\otimes_R \mathcal{L}^\vee \xrightarrow{\text{unit}\otimes\mathrm{id}} R^{\varphi C_2}\otimes\mathcal{L}^\vee \xrightarrow{\ell(\eta)} R^{\varphi C_2}$$

$$R\otimes_R \mathcal{L} \xrightarrow{\text{unit}\otimes\mathrm{id}} R^{\varphi C_2}\otimes\mathcal{L} \xrightarrow{\ell(\eta)^{-1}\otimes\mathrm{id}_{\mathcal{L}}} R^{\varphi C_2}\,.$$

These correspond to the $\ell(q^\vee),\ell(q)$ of Remark 4.2, respectively. In particular, the condition that $\ell(q^\vee),\ell(q)$ make the diagram (4.4) commute is equivalent to the condition that $\ell(\eta)$ is an equivalence by an adjunction argument. The data of the path in $\hom_R(\mathcal{L},R^{tC_2})$ is exactly the data needed to show that the maps $\ell(q)$ and $b(q)$ glue together to give a form on $\mathcal{L}$.

The above thus produces a natural transformation $\Omega\,\mathrm{Br}^{\mathrm{p}}(-)\to\mathrm{Pic}^{\mathrm{p}}(-)$. In the other direction, to any $(\mathcal{L},q)\in\mathrm{Pn}(\mathrm{Mod}^{\omega}_A)$ invertible, we may define an autoequivalence $(\mathrm{Mod}^{\omega}_A,\Qoppa)\to(\mathrm{Mod}^{\omega}_A,\Qoppa)$ via tensoring with $(\mathcal{L},q)$, which will be an autoequivalence by the assumption that $(\mathcal{L},q)$ is invertible. We have that these two natural transformations are inverse to each other, hence the result. □

As in the Picard group case, the symmetric monoidal forgetful functor $\theta\colon \mathrm{Cat}^{\mathrm{p}}_{\infty R}\to\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}$ induces a map of spectra $\theta\colon \mathrm{Br}^{\mathrm{p}}(A)\to\mathrm{Br}(A^e)$. When A^e is endowed with the trivial action, θ will factor through the 2-torsion on π_0. As a consequence of Proposition 5.1(2) we can identify the fiber of this map.

Corollary 5.18. *Let $(\mathrm{Mod}^{\omega}_A,\Qoppa_A)$ be a Poincaré ring with underlying genuine C_2 spectrum A^L as in Proposition 5.1(2). Write $\sigma\colon A^e\simeq A^e$ for the C_2-action on the underlying $\mathbb{E}_\infty$-ring associated to A. Then the fiber of the map*

$$\theta\colon \mathrm{Br}^{\mathrm{p}}(A)\to\mathrm{Br}(A^e)$$

can be naturally identified with $\mathrm{Pic}\left(\mathrm{Mod}_{A^L}\left(\mathrm{Sp}^{C_2}\right)\right)$*. Moreover, the connecting map* $\Omega\,\mathrm{Br}(A^e)\simeq\mathrm{Pic}(A^e)\to\mathrm{fib}(\theta)$ *is induced by the norm* $\mathrm{Mod}^{\mathrm{op}}_{A^e}\to\mathrm{Mod}_{A^L}\left(\mathrm{Sp}^{C_2}\right)$, $X\mapsto N^{C_2}(X^\vee)\otimes_{N^{C_2}A^e}A^L$, *which on underlying spectra is given by* $X\mapsto X^\vee\otimes_A\sigma^*X^\vee$.

Proof of Corollary 5.18. Since $\theta:\mathrm{Mod}_{(\mathrm{Mod}^{\omega}_A,\Qoppa_A)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})\to\mathrm{Mod}_{\mathrm{Mod}^{\omega}_{A^e}}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_\infty)$ is symmetric monoidal and conservative, it induces a map $\theta^{\simeq}:\mathrm{PnBr}(A)\to\mathrm{br}(A^e)$ on the groupoid core of invertible objects. Now observe that θ is an isofibration; it follows that $\theta^{\simeq}$ is a Kan fibration by [Lur24, Proposition 01EZ]. Consequently, to identify the homotopy fiber of θ, it suffices to identify the fiber of θ over a single point. Consider $(\mathrm{Mod}^{\omega}_{A^e},\Qoppa)$ a point in the fiber of θ over $\mathrm{Mod}^{\omega}_{A^e}$. By Proposition 5.1(2), $\Qoppa$ is associated to an A-linear invertible module with involution $\left(M,N,N\to M^{tC_2}\right)$. By Proposition 5.1(3), invertibility of $(\mathrm{Mod}^{\omega}_{A^e},\Qoppa)$ implies that $\left(M,N,N\to M^{tC_2}\right)$ is invertible as a module over A^L.

Now we give the description of the connecting map. Write $(\mathrm{Mod}^{\omega}_{A^e},\Qoppa_A)$ for the identity element in the fiber of θ over $\mathrm{Mod}^{\omega}_{A^e}$, and let $\gamma\colon S^1\to\mathrm{Br}(A^e)$. Write $\mathcal{L}_\gamma$ for γ regarded as a point in $\mathrm{Pic}(A^e)\simeq\Omega\,\mathrm{Br}(A^e)$. Lift γ to a path $\widetilde{\gamma}$ in $\mathrm{PnBr}(A)$ starting at $(\mathrm{Mod}^{\omega}_{A^e},\Qoppa_A)$, and write $(\mathrm{Mod}^{\omega}_{A^e},\Phi)$ for the other endpoint of $\widetilde{\gamma}$. By

Proposition 5.1(2), $\Qoppa_A$ is associated to the invertible A^L-module with involution A^L and Φ is associated to some invertible A^L-module with involution $(M, N, N \to M^{tC_2})$. We may regard $\widetilde{\gamma}$ as an A-linear hermitian equivalence from $(\mathrm{Mod}^{\omega}_{A^e}, \Qoppa_A)$ to $(\mathrm{Mod}^{\omega}_{A^e}, \Phi)$, which by Proposition 5.1(4) consists of an A-linear hermitian functor $(F, \eta\colon \Qoppa_A \to \Phi \circ F)$ so that F, η are both equivalences. Since $\widetilde{\gamma}$ projects to γ, we must have that $F = - \otimes \mathcal{L}_\gamma$. Now the natural equivalence η classifies an equivalence $A^L \simeq \hom_{A^L}(N^{C_2}_A(L), (M, N, N \to M^{tC_2}))$ of A^L-modules (Proposition 5.12), hence the result. □

Example 5.19. Let $\mathbb{S}^u$ denote the universal Poincaré structure on the sphere spectrum, or equivalently $\mathbb{S}^u$ is the Poincaré ring associated to the genuine equivariant sphere spectrum. By Corollary 5.18 we have a fiber sequence

$$\mathrm{Pic}(\mathrm{Sp}^{C_2}) \to \mathrm{PnBr}(\mathbb{S}^u) \to \mathrm{br}(\mathbb{S})$$

and by [AG14b, Corollary 7.17] we have that $\pi_0(\mathrm{br}(\mathbb{S})) = 0$. Therefore we get a long exact sequence

$$\begin{tikzcd} \dots \arrow[r] & \pi_1(\mathrm{PnBr}(\mathbb{S}^u)) \arrow[r] \arrow[d, "\cong"] & \pi_1(\mathrm{br}(\mathbb{S})) \arrow[r] \arrow[d, "\cong"] & \pi_0(\mathrm{Pic}(\mathrm{Sp}^{C_2})) \arrow[r] \arrow[d, "\cong"] & \pi_0(\mathrm{PnBr}(\mathbb{S}^u)) \arrow[r] & 0 \\ & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \times \mathbb{Z} & & \end{tikzcd}$$

where the third term is identified via [Kra25, Section 8.1]. From this and the fact that $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is the identity on the first component since they are both suspension of the underlying spectrum we see that $\pi_0(\mathrm{PnBr}(\mathbb{S}^u)) \simeq \mathbb{Z}$. In fact this allows us to identify the space $\mathrm{PnBr}(\mathbb{S}^u) \simeq \mathbb{Z} \times B\mathrm{TR}^2(\mathbb{S}; 2)^\times$.

Example 5.20. Let k be an algebraically closed field, and regard k as a Poincaré ring $\underline{k}$ via Example 3.18 with the trivial involution. Then

L: haven't changed the computation here yet

$$\pi_0 \mathrm{PnBr}(\underline{k}) \simeq \begin{cases} \mathbb{Z}/2 \times \mu_2(k) & \text{if char } k \neq 2 \\ \mathbb{Z} & \text{if char } k = 2\,. \end{cases}$$

To see this, note that by [Toë12, Proposition 1.9], $\pi_0 \mathrm{br}(k^e) \simeq 0$. Thus by Corollary 5.18, it suffices to understand the fiber sequence

$$\mathrm{Pic}(k) \to \mathrm{Pic}(\mathrm{Mod}_{\underline{k}}(\mathrm{Sp}^{C_2})) \to \mathrm{PnBr}(\underline{k}).$$

Now if char $k \neq 2$, then $\mathrm{Pic}(\underline{k}) \simeq \mathrm{Pic}(k)^{hC_2} \simeq (\mathbb{Z} \times Bk^\times)^{hC_2}$ where the generator of C_2 acts on $\pi_1 Bk^\times$ trivially. We then have that $H^1(C_2, k^\times) = \mu_2(k) \cong \mathbb{Z}/2\mathbb{Z}$ since k is algebraically closed. We can then deduce that

$$\pi_0 \mathrm{Pic}(\mathrm{Mod}_{\underline{k}}(\mathrm{Sp}^{C_2})) \simeq \begin{cases} \mathbb{Z} \times \mathbb{Z} & \text{if char } k = 2 \\ \mathbb{Z} \times \mu_2(k) & \text{otherwise} \end{cases}$$

where the char$(k) = 2$ case is handled by an argument similar to [Kra25, Section 8.1]. Let us write ± 1 for the elements of μ_2.

L: Suppose char $k \neq 2$. A point of $\mathrm{Pic}(k)^{hC_2}$ is an invertible k-module L and an order 2 automorphism of L. Under this identification, an invertible $\underline{k}$-module L is sent to the pair (n, ε) where n is the unique integer so that $L^e \simeq k$ nonequivariantly, and $\varepsilon = \pm 1$ is the sign of the C_2-action. In particular $\Sigma^{\sigma}\underline{k} \mapsto (2, -1)$, $\Sigma^2 \underline{k} \mapsto (2, 0)$? If so, then the map is induced by the norm, hence we should consider $\mathbb{Z} \xrightarrow{1 \mapsto (-2,-1)} \mathbb{Z} \times \mu_2(k)$ and its cokernel is isomorphic to $\mathbb{Z}/4$, generated by $(1, 1)$.

If char $k \neq 2$ then the map $\pi_0 \mathrm{Pic}(k) \to \pi_0 \mathrm{Pic}(\underline{k})$ is $\mathbb{Z} \xrightarrow{1 \mapsto (-2,-1)} \mathbb{Z} \times \mu_2(k)$. On the other hand, if char $k = 2$, then the map $\pi_0 \mathrm{Pic}(k) \to \pi_0 \mathrm{Pic}(\underline{k})$ is $\mathbb{Z} \xrightarrow{n \mapsto (2n, n)} \mathbb{Z} \times \mathbb{Z}$.

L: This and Example 5.22 are a pair; if we move one, also move the other. Remark 5.21 is about this example.

When char $k \neq 2$, the Poincaré ∞-category corresponding to the element $(0, -1) \in \mathbb{Z} \times \mu_2$ is $(\mathrm{Mod}^{\omega}_{k}, \Qoppa_{-k})$, where the notation $-k$ is that of the discussion between Definition 3.5.12 and Example 3.5.14 of [Cal+20a] (in particular, see Example 3.5.14(i) of *loc. cit.*). In other words, a sign is introduced into the involution. The presence of this generator (unlike the case when char $k = 2$) reflects the distinction between symmetric and skew symmetric forms when $2 \neq 0$. That this class is 2-torsion reflects that $(-1)^2 = 1$. The Poincaré ∞-category corresponding to the element $(1,1) \in \mathbb{Z} \times \mu_2$ is $\left(\mathrm{Mod}_k^\omega, \Qoppa_k^{[1]}\right)$ in the notation of [Cal+20a, §3.5]. By a relative version of [Cal+20a, Corollary 5.4.6], $\left(\mathrm{Mod}_k^\omega, \Qoppa_k^{[1]}\right) \otimes \left(\mathrm{Mod}_k^\omega, \Qoppa_k^{[1]}\right) \simeq \left(\mathrm{Mod}_k^\omega, \Qoppa_k^{[2]}\right)$, and the latter is equivalent to $(\mathrm{Mod}_k^\omega, \Qoppa_{-k})$ by [Cal+20a, Proposition 3.5.3].

Remark 5.21. The 4-torsion class $\left(\mathrm{Mod}_k^\omega, \Qoppa^{[1]}\right)$ (char $k \neq 2$) in Example 5.20 does not arise as perfect modules over a classical/discrete Azumaya algebra with anti-involution or central Wall anti-structure. Roughly, the reason is: Any Poincaré object here cannot be concentrated in a single degree because the duality involves a shift, therefore the endomorphism algebra of any Poincaré generator will not be discrete either. However, the 2-torsion class $\left(\mathrm{Mod}_k^\omega, \Qoppa^{[1]}\right) \otimes \left(\mathrm{Mod}_k^\omega, \Qoppa^{[1]}\right) = (\mathrm{Mod}_k^\omega, \Qoppa_{-k})$ does arise as perfect modules over a classical Azumaya algebra with anti-involution. Therefore, the subgroup of the $\mathbb{Z}/4\mathbb{Z}$ which "is classical" is still 2-torsion!

L: add reference/compariso to classical result. cite in intro?

Example 5.22. Let k be an algebraically closed field, and consider the Poincaré ring associated to $\prod_{C_2} k$ (i.e. $k \times k$ with the swap action) via Example 3.18. Similarly to Example 5.20, by [Toë12, Proposition 1.9] we have $\pi_0 \mathrm{br}(\operatorname{Spec} k \sqcup \operatorname{Spec} k) \simeq \pi_0 \mathrm{br}(\operatorname{Spec} k)^{\times 2} = 0$. Thus by Corollary 5.18, it suffices to understand the cokernel of the connecting homomorphism $\pi_1 \mathrm{br}(k \times k) \to \pi_0 \operatorname{Pic}(\prod_{C_2} k)$. Now since $\prod_{C_2} k$ is Borel and $(\prod_{C_2} k)^{tC_2} = 0$,

$$\operatorname{Pic}\left(\mathrm{Mod}_{\prod_{C_2} k}\left(\mathrm{Sp}^{C_2}\right)\right) \simeq \operatorname{Pic}\mathrm{Mod}_{k\times k}(\mathrm{Sp})^{hC_2} \simeq \left(\prod_{C_2}(\mathbb{Z} \oplus k^\times[1])\right)^{hC_2} \simeq \mathbb{Z} \oplus k^\times[1]$$

thus $\pi_0 \operatorname{Pic}\left(\mathrm{Mod}_{\prod_{C_2} k}\left(\mathrm{Sp}^{C_2}\right)\right) \simeq \mathbb{Z}$. On the other hand, $\pi_1 \mathrm{br}(k \times k) \simeq \pi_0 \operatorname{Pic}(k \times k) \simeq \mathbb{Z}^{\times 2}$ and the connecting homomorphism is $(n, m) \mapsto n + m$, whence $\pi_0 \mathrm{PnBr}\left(\prod_{C_2} k\right) = 0$.

5.3 The Parimala-Srinivas fiber sequence

It would be helpful to extend the fiber sequence of Corollary 5.18 to the right, both in order to use Poincaré Brauer groups to compute ordinary Brauer groups and to get a comparison to the involutive Brauer group of [PS92]. We will first show that the fiber sequence we have already constructed in Corollary 5.18 does in fact recover the first half of the long exact sequence constructed in [PS92, Theorem 2].

Lemma 5.23. *Let A be a Poincaré ring which is has underlying genuine C_2-spectrum Borel and such that $A^{hC_2} \to A^e$ is a faithful Galois extension in the sense of [Rog08], Then the zigzag of maps*

$$\operatorname{Pic}(A^{hC_2}) \leftarrow \operatorname{Pic}(\mathrm{Mod}_A(\mathrm{Sp}^{BC_2})) \to \operatorname{Pic}(\mathrm{Mod}_{A^L}(\mathrm{Sp}^{C_2}))$$

are equivalences. If A^e is discrete, this equivalence is given by sending an equivariant discrete module M to M^{C_2} (since 2 is invertible, strict and homotopy fixed points agree).

Proof. By the assumption that $\frac{1}{2} \in \pi_0(A^e)$ we have that $A^{tC_2} = 0$. Furthermore since A is Borel it also follows that $A^{\varphi C_2} = 0$, and consequently any module M over A^L has $M^{\varphi C_2} = M^{tC_2} = 0$. Thus $\mathrm{Mod}_A(\mathrm{Sp}^{BC_2}) \to \mathrm{Mod}_{A^L}(\mathrm{Sp}^{C_2})$ is an equivalence even before taking the Picard space.

It remains to show that the functor $\mathrm{Mod}_A(\mathrm{Sp}^{BC_2}) \xrightarrow{(-)^{hC_2}} \mathrm{Mod}_{A^{hC_2}}$ induces an equivalence on the picard space. This follows by by the faithful Galois assumption by [Mat16, Proposition 9.4]. $\square$

Writing this out for A a discrete ring with $\frac{1}{2} \in A$, we have a fiber sequence

$$\operatorname{Pic}(A) \xrightarrow{N_{A^{C_2}/A}} \operatorname{Pic}(A^{C_2}) \to \mathrm{PnBr}(A)$$

where we have that $N_{A^{C_2}/A}$ is given by the map $\mathcal{L} \mapsto (\mathcal{L}^\vee \otimes \sigma^* \mathcal{L}^\vee)^{C_2}$ which is (the negative of) the usual norm map. In order to extend this fiber sequence to the right we will need to categorify this map.

Construction 5.24. Let A be a Poincaré ring and let $\sigma : A^e \to A^e$ denote the involution. Consider the functor

$$\mathrm{Mod}_{\mathrm{Mod}^{\omega}_{A^e}}(\mathrm{Cat}^{st}_{\infty,\mathrm{idem}}) \xrightarrow{\left(-^{op}\otimes_{\mathrm{Mod}^{\omega}_{A^e}}\sigma^{*}-^{op}\right)^{hC_2}} \mathrm{Mod}_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}}(\mathrm{Cat}^{st}_{\infty,\mathrm{idem}})$$

which we will denote by $N_{A^{hC_2}/A}$. Note that this functor is symmetric monoidal, which can be checked on underlying infinity categories. Here if $\mathcal{C}$ is a module over $\mathrm{Mod}^{\omega}_{A^e}$, then $\mathcal{C}^{op}$ is a module via the equivalence $D_{\text{♀}} : \mathrm{Mod}^{\omega}_{A^e} \xrightarrow{\simeq} (\mathrm{Mod}^{\omega}_{A^e})^{op}$ induced by the duality from the Poincaré structure.

Lemma 5.25. *The composite* $\mathrm{PnBr}(A) \to \mathrm{br}(A^e) \to \mathrm{Pic}(\mathrm{Mod}_{\mathrm{Mod}^{hC_2}_{A^e}})$ *is nullhomotopic.*

Proof. The underlying category of a Poincaré invertible category is self-dual, and so its square will vanish. Since the functor is naturally nullhomotopic so too is the composite after applying the functor $\mathrm{Pic}(-)$. □

Lemma 5.26. *Let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{Mod}^{\omega}_{A^e}}(\mathrm{Cat}^{st}_{\infty,idem})$ be an invertible category such that $N_{A^{hC_2}/A}(\mathcal{C}) = 0$. Suppose that A is Borel and that $\frac{1}{2} \in \pi_0(A^e)$. Then $\mathcal{C}$ admits an invertible A-linear Poincaré infinity category structure.*

Proof. By assumption there is an equivalence $(\mathcal{C}^{op} \otimes_{\mathrm{Mod}^{\omega}_{A^e}} \sigma^{*}\mathcal{C}^{op})^{hC_2} \simeq \mathrm{Mod}_A(\mathrm{Sp}^{BC_2})$, fix such an equivalence. We then may define a Poincaré structure via

$$\mathcal{C}^{op} \xrightarrow{x \mapsto x\otimes\sigma^{*}x} \left(\mathcal{C}^{op} \otimes_{\mathrm{Mod}^{\omega}_{A^e}} \sigma^{*}\mathcal{C}^{op}\right)^{hC_2} \simeq \mathrm{Mod}_A(\mathrm{Sp}^{BC_2}) \xrightarrow{(-)^{hC_2}} \mathrm{Sp}$$

and under the assumptions on A this defines an invertible Poincaré structure on $\mathcal{C}$ as desired. □

There is thus a map $\mathrm{PnBr}(-) \to \mathcal{F}(-)$, where $\mathcal{F}(-)$ is the fiber. Delooping both fiber sequences twice we see that we get a map of fiber sequences

$$\begin{tikzcd}
\mathrm{Pic}(A^e) \arrow[r] \arrow[d] & \mathrm{Pic}(\mathrm{Mod}_{A^L}(\mathrm{Sp}^{C_2})) \arrow[r] \arrow[d] & \mathrm{PnBr}(A) \arrow[d] \\
\mathrm{Pic}(A^e) \arrow[r, "N_{A^{hC_2}/A}"] & \mathrm{Pic}(\mathrm{Mod}^{hC_2}_{A^e}) \arrow[r] & \mathcal{F}(A)
\end{tikzcd}$$

and when A is Borel,the map $A^{hC_2} \to A^e$ is faithfully Galois, and $\frac{1}{2} \in \pi_0(A^e)$, we see that the left and middle vertical maps must be equivalences. This together with Lemma 5.26 gives that the right hand vertical map is in fact an equivalence of $\mathbb{E}_\infty$ spaces.

Summarizing this situation, we get the following:

Theorem 5.27. *Let A be a Poincaré ring which is Borel, the map $A^{hC_2} \to A$ is faithfully Galois, and $\frac{1}{2} \in \pi_0(A^e)$. Then there is a fiber sequence*

$$\mathrm{PnBr}(A) \to \mathrm{br}(A^e) \to \mathrm{br}(A^{hC_2})$$

of connective spectra.

If A is furthermore discrete, then the map $\mathrm{Br}(A) \to \mathrm{Br}(A^{hC_2})$ is the étale cohomological transfer map.

Proof. All the remains to show is the identification of $\mathrm{Mod}_{(\mathrm{Mod}^{\omega}_{A^e})^{hC_2}}(\mathrm{Cat}^{st}_{\infty,\mathrm{idem}}) \simeq \mathrm{Mod}_{\mathrm{Mod}^{\omega}_{A^{hC_2}}}(\mathrm{Cat}^{st}_{\infty,\mathrm{idem}})$, which follows from the monoidal equivalence $(\mathrm{Mod}^{\omega}_{A^e})^{hC_2} \simeq \mathrm{Mod}^{\omega}_{A^{hC_2}}$, and the fact that under this equivalence an Azumaya algebra over B is sent to $(B \otimes_A \sigma^{*}B)^{C_2}$ which is exactly the cohomological transfer map (see [FW20, Example 6.2.3]). □ L: Definition 6.2?

Construction 5.28. Let A denote a Poincaré ring with underlying genuine C_2 ring A^L. Denote by Mod_{A^L} the symmetric monoidal category $\mathrm{Mod}_{A^L}(\mathrm{Sp}^{C_2})$. There is then a symmetric-monoidal functor

$$N^{C_2}_{A^L} : \mathrm{Mod}_{A^e}(\mathrm{Sp}) \to \mathrm{Mod}_{A^L}$$

given by taking the relative norm. Define the *categorical norm* to be the functor

$$N_{A^L}^{C_2} : \mathrm{Mod}_{\mathrm{Mod}^{\omega}_{A^e}}(\mathrm{Cat}^{\infty}_{st,idem}) \to \mathrm{Mod}_{\mathrm{Mod}^{\omega}_{A^L}}(\mathrm{Cat}^{\infty}_{st,idem})$$

by $N_{A^L}^{C_2}\mathcal{C} := \mathcal{C}^{\mathrm{op}} \otimes_{\mathrm{Mod}_{A^e}} \mathrm{Mod}_{A^L}$. Base-change is symmetric monoidal and therefore we get an induced map $\mathrm{br}(A^e) \to \mathrm{br}(A^L)$, which by abuse of notation we will also refer to as the norm $N_{A^L}^{C_2}$. Note also that this same construction goes through for schemes with involution and good quotients.

Theorem 5.29. *Let A be a Poincaré ring. Then there is a fiber sequence*

$$\mathrm{PnBr}(A) \to \mathrm{br}(A^e) \xrightarrow{N_{A^L}^{C_2}} \mathrm{br}(A^L)$$

natural in A. If (X, λ, Y, π) is a scheme with good quotient then there is a fiber sequence

$$\mathrm{PnBr}(X, \lambda, Y, \pi) \to \mathrm{br}(X) \xrightarrow{N_{\underline{\mathcal{O}}_X}^{C_2}} \mathrm{br}(\underline{\mathcal{O}}_X)$$

natural in (X, λ, Y, π).

Proof. We will first construct a map. For this, it is enough to show that any Poincaré category $(\mathcal{C}, \Qoppa)$ in $\mathrm{Mod}_{(\mathrm{Mod}^{\omega}_{A}, \Qoppa_A)}$ naturally admits a trivialization $N_{A^L}^{C_2}\mathcal{C} \simeq \mathrm{Mod}^{\omega}_{A^L}$ on the full subcategory of invertible objects. Note that $\Qoppa$ naturally factors through a $\mathrm{Mod}^{\omega}_{A^e}$-linear functor $\Qoppa^{\mathrm{gen}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_{A^L}$ such that $\Qoppa = (\Qoppa^{\mathrm{gen}})^{C_2}$. This base-changes to a trivialization. To see this, let $X \in \mathcal{C}$ denote a Poincaré generator. Then identifying $\mathcal{C}$ with $\mathrm{Mod}^{\omega}_{\mathrm{End}(X)}$ we have that $N_{A^L}^{C_2}\Qoppa^{\mathrm{gen}}(X) = N_e^{C_2}\mathrm{End}(X) \otimes_{N_e^{C_2}A^e} A^L$ which, since $\mathrm{End}(X)$ is an Azumaya algebra, generates Mod_{A^L}.

Looping this sequence give the fiber sequence of Theorem 5.18, so it is enough to check that the sequence

$$\pi_0(\mathrm{PnBr}(A)) \to \pi_0(\mathrm{br}(A^e)) \to \pi_0(\mathrm{br}(A^L))$$

is exact. Let $[\mathcal{C}] \in \pi_0(\mathrm{br}(A^e))$ be a Brauer class such that $[N_{A^L}^{C_2}\mathcal{C}] = 0$, or equivalently that there exists a trivialization $N_{A^L}^{C_2}\mathcal{C} \simeq \mathrm{Mod}^{\omega}_{A^L}$. This trivialization is adjoint to a $\mathrm{Mod}^{\omega}_{A^e}$-linear functor $\mathcal{C}^{\mathrm{op}} \xrightarrow{\Qoppa^{\mathrm{gen}}} \mathrm{Mod}^{\omega}_{A^L}$, which induces a hermitian structure via $\Qoppa = (\Qoppa^{\mathrm{gen}})^{C_2}$. Since $\Qoppa^{\mathrm{gen}}$ is adjoint to an equivalence it must send any compact generator of $\mathcal{C}$ to a compact generator, hence the induced hermitian structure is in fact Poincaré and is by construction $(\mathrm{Mod}^{\omega}_{A^e}, \Qoppa_A)$-linear. Note that we may repeat the above procedure for $\mathcal{C}^{\vee}$ and (up to twisting by a line bundle over A^L) this gives an inverse of $(\mathcal{C}, \Qoppa)$, therefore showing the $[(\mathcal{C}, \Qoppa)]$ is a Poincaré-Brauer class as desired. ☐

5.4 Azumaya algebras with genuine involution

In this section, we introduce the notion of a derived/generalized Azumaya algebra with involution. Recall that a classical/discrete Azumaya algebra $\mathcal{A}$ is étale-locally given by the endomorphism algebra of a vector bundle V, and a derived/generalized Azumaya algebra over X is étale-locally given by the endomorphism algebra of a perfect complex. Observe that if $\mathcal{A} \simeq \mathrm{End}_X(V)$, then $\sigma^*\mathcal{A}^{\mathrm{op}} \simeq \mathrm{End}_X(\sigma^*(V^{\vee}))$. The prototypical anti-involution on a classical/discrete Azumaya algebra $\mathcal{A}$ is given by taking transposes and conjugating by an isomorphism $V \simeq \sigma^*(V^{\vee}) \otimes \mathcal{L}$, where $\mathcal{L}$ is a line bundle on X [PS92, §1.1(ii), p.209, §1.2, p.216]. If we now consider a derived/generalized Azumaya algebra $\mathcal{A} \simeq \mathrm{End}_X(P)$ where P is a perfect complex on X, then the prototypical anti-involution on $\mathcal{A}$ arises from transposition and conjugation by an equivalence $P \simeq \sigma^*(P^{\vee}) \otimes \mathcal{L}[n]$.

L: Example? $R\Gamma$ of an oriented family of smooth proper schemes over X? relate to Serre duality!

Let R be an $\mathbb{E}_\infty$-ring spectrum.

Recollection 5.30. Recall [BRS12; AG14a] that an $\mathbb{E}_1$-R-algebra A is said to be *Azumaya* if it is a compact generator of Mod_R and if the natural R-algebra map giving the bimodule structure on A

$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$$

is an equivalence of R-algebras.

Definition 5.31. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring spectrum, and write $\sigma\colon R \xrightarrow{\sim} R$ for the involution on R. An *Azumaya algebra with genuine (anti-)involution* over R is the data of

(a) An $\mathbb{E}_1$-R-algebra A equipped with an anti-involution $\tau\colon A \to \sigma^* A^{\mathrm{op}}$ so that the underlying $\mathbb{E}_1$-R-algebra A is an Azumaya R-algebra in the sense of Recollection 5.30

(b) a C_2-equivariant $(A \otimes_R \sigma^* A)^{\otimes_R 2}$-linear equivalence $\hom_{R\otimes R}(A \otimes_R A, R) \simeq A \otimes_R \sigma^* A^{\mathrm{op}}$

(c) A left $A \otimes_R R^{\varphi C_2}$-module P and an $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-module $\overline{P}$

(d) An $A \otimes_R R^{\varphi C_2}$-linear map $P \to A^{tC_2}$ and an $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-linear map $\overline{P} \to A^{tC_2}$. Here we regard A^{tC_2}, which is canonically a $(A \otimes_R \sigma^* A)^{tC_2}$-module, as an $A \otimes_R R^{\varphi C_2}$-module (resp. $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-module) via the twisted Tate-valued diagonal $A \to (A \otimes \sigma^* A)^{tC_2}$ (resp. $A^{\mathrm{op}} \to (\sigma^* A^{\mathrm{op}} \otimes A^{\mathrm{op}})^{tC_2}$).

(e) An equivalence of $(A \otimes_R \sigma^* A^{\mathrm{op}}) \otimes_R R^{\varphi C_2}$-modules

$$\hom_R(A, R^{\varphi C_2}) \simeq P \otimes_{R^{\varphi C_2}} \overline{P}$$

and a homotopy making the diagram

$$\begin{tikzcd}
\hom_R(A, R^{\varphi C_2}) \arrow[rr] \arrow[d] & & P \otimes_{R^{\varphi C_2}} \overline{P} \arrow[d] \\
\hom_R(A, R^{tC_2}) \simeq \hom_R(A \otimes A^{\mathrm{op}}, R)^{tC_2} \arrow[r] & \hom_{(A\otimes A^{\mathrm{op}})^{\otimes 2}}\left((A \otimes A^{\mathrm{op}})^{\otimes 2}, A \otimes A^{\mathrm{op}}\right)^{tC_2} & \simeq (A \otimes A^{\mathrm{op}})^{tC_2}
\end{tikzcd}$$

commute, where the lower horizontal arrow is induced by (b) and the right vertical arrow is induced by (d).

Definition 5.32. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring spectrum, and write $\sigma\colon R \xrightarrow{\sim} R$ for the involution on R. An *Azumaya algebra with central Wall anti-structure* over R is the data of

(a) An $\mathbb{E}_1$-R-algebra A equipped with an anti-involution $\tau\colon A \to \sigma^* A^{\mathrm{op}}$ so that the underlying $\mathbb{E}_1$-R-algebra A is an Azumaya R-algebra in the sense of Recollection 5.30

(b) An invertible R module with involution M_A over A in the sense of [Cal+20a, Definition 3.1.4] whose underlying left A-module is equivalent to A. Observe that pullback along the anti-involution (a) implies that $\sigma^*(M_A)$ defines an invertible R-module with involution over $\sigma^*(A)$. We also ask for a C_2-equivariant $(A \otimes_R \sigma^* A)^{\otimes_R 2}$-linear equivalence $\hom_{R\otimes R}(A \otimes_R \sigma^* A, R) \simeq M_A \otimes_R \sigma^* M_A$.

(c) A left $A \otimes_R R^{\varphi C_2}$-module P and an $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-module $\overline{P}$

(d) An $A \otimes_R R^{\varphi C_2}$-linear map $P \to M_A^{tC_2}$ and an $\sigma^* A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-linear map $\overline{P} \to (\sigma^* M_A)^{tC_2}$. Here we regard $M_A^{tC_2}$, which is canonically a $(A \otimes_R \sigma^* A)^{tC_2}$-module, as an $A \otimes_R R^{\varphi C_2}$-module (resp. $A^{\mathrm{op}} \otimes_R R^{\varphi C_2}$-module) via the twisted Tate-valued diagonal $A \to (A \otimes \sigma^* A)^{tC_2}$ (resp. $A^{\mathrm{op}} \to (\sigma^* A^{\mathrm{op}} \otimes A^{\mathrm{op}})^{tC_2}$).

(e) An equivalence of $(A \otimes_R \sigma^* A^{\mathrm{op}}) \otimes_R R^{\varphi C_2}$-modules

$$\hom_R(M_A, R^{\varphi C_2}) \simeq P \otimes_{R^{\varphi C_2}} \overline{P}$$

and a homotopy making the diagram

$$\begin{tikzcd}
\hom_R(A, R^{\varphi C_2}) \arrow[rr] \arrow[d] & & P \otimes_{R^{\varphi C_2}} \overline{P} \arrow[d] \\
\hom_R(A, R^{tC_2}) \simeq \hom_R(A \otimes \sigma^* A, R)^{tC_2} \arrow[r] & \hom_{(A\otimes \sigma^* A)^{\otimes 2}}\left((A \otimes \sigma^* A)^{\otimes 2}, M_A \otimes \sigma^* M_A\right)^{tC_2} & \simeq (M_A \otimes \sigma^* M_A)^{tC_2}
\end{tikzcd}$$

commute, where the lower horizontal arrow is induced by (b) and the right vertical arrow is induced by (d).

L: Compare: [Cal+20a, p. 3.1.13], Wall's original paper?

Observation 5.33. Let A be an Azumaya algebra with genuine involution over R, and suppose given a map $R \to S$ of Poincaré rings. Then $(A \otimes_R S, P \otimes_{R^{\varphi C_2}} S^{\varphi C_2})$ is an Azumaya algebra with genuine involution over S.

Remark 5.34. An Azumaya algebra with genuine involution over R in particular defines an Azumaya algebra with central Wall anti-structure over R with $M_A \simeq A$ as $A \otimes \sigma^* A$-modules. If an Azumaya algebra with central Wall anti-structure is of this form, we say that it *comes from the anti-involution on A* (compare [Cal+20a, Example 3.1.9], which is the special case when $R = \mathbb{S}^0$).

If A is an Azumaya algebra with genuine involution over R, then $(A, P \to A^{tC_2})$ defines a $N_R^{C_2}(A)$-module which classifies an R-linear Poincaré structure on Mod_A^ω by Proposition 5.1(2).

Observation 5.35. Let R be a discrete commutative $\mathbb{F}_2$-algebra and assume that R is perfect (i.e. the Frobenius $r \mapsto r^2$ is an isomorphism), and endow R with the trivial (= identity) involution. Regard $\underline{R}$ as a Poincaré ring spectrum via 3.18, and let A be an $\mathbb{E}_\sigma$-algebra over R whose underlying $\mathbb{E}_1$-R-algebra is (generalized) Azumaya. By the isotropy separation sequence $R_{hC_2} \to R \to R^{\varphi C_2}$ and the Tate orbit lemma [NS18, Lemma I.2.1], the canonical map $R^{C_2} \to R^{\varphi C_2}$ induces an equivalence $R^{tC_2} \to (R^{\varphi C_2})^{tC_2}$.

L: this map $R^{C_2} \to R^{\varphi C_2}$ ('mod 2' on π_0) is not the same map $R^e \to R^{\varphi C_2}$ as the lift along the Tate valued Frobenius! The following commutative triangle commutes on π_0 but I am not sure about higher homotopy yet. Use [NS18, §IV.1.15-16] or naturality of the Tate-valued Frobenius?

$$\begin{tikzcd} R^{C_2} \arrow[d, "\sim"', "\sqrt{-}"] \arrow[dr] & \\ R^e \arrow[r] & R^{\varphi C_2} \end{tikzcd}$$

Assume for the moment that the lift $R \to R^{\varphi C_2}$ of the Tate-valued Frobenius induced an equivalence $R^{tC_2} \xrightarrow{\sim} (R^{\varphi C_2})^{tC_2}$. Granting this, we obtain a map of Poincaré rings $\underline{R} \to (R^{\varphi C_2})^t$, where the latter is endowed with the Tate Poincaré structure of Example 3.12.

Note that $A \otimes_{\underline{R}} (R^{\varphi C_2})^t$ is an Azumaya algebra with genuine involution over $(R^{\varphi C_2})^t$. Suppose given a trivialization of $A \otimes_{\underline{R}} (R^{\varphi C_2})^t$ over $(R^{\varphi C_2})^t$, i.e. an equivalence of Poincaré categories $\left(\mathrm{Mod}_{(R^{\varphi C_2})^t}, ♀\right) \simeq \left(\mathrm{Mod}^\omega_{A \otimes_{\underline{R}} (R^{\varphi C_2})^t}, ♀_A\right)$. By the argument of Example 5.38[4], the data of this trivialization gives a lift of A to an Azumaya algebra with genuine involution over $\underline{R}$, i.e. parts (d) and (e) of Definition 5.31.

L: to-do: use deformation theory/nilcompleteness of the stack $\mathbf{M}_A^p$ to produce a trivialization.

Remark 5.36. If A is an Azumaya algebra with genuine involution over R, then in particular $M_A = A$, $N_A = P$ is a module with genuine involution over A in the sense of [Cal+20a, Definition 3.2.3].

Definition 5.37. Let (X, σ) be a scheme with an involution and let $\pi \colon X \to Y$ exhibit Y as a good quotient of X. Recall that there is a sheaf of C_2-$\mathbb{E}_\infty$-algebras $\underline{\mathcal{O}}$ (Construction 3.28), and write σ for the involution $\pi_* \mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_X$. An *Azumaya algebra with genuine (anti-)involution* over X is the data of

(a) An $\mathbb{E}_1$-$\underline{\mathcal{O}}^e = \pi_* \mathcal{O}_X$-algebra A equipped with an anti-involution $\tau \colon A \to \sigma^* A^{\mathrm{op}}$ (i.e. $\sigma^*(\tau^{\mathrm{op}}) \circ \tau \simeq \mathrm{id}_A$, and higher coherences) so that the underlying $\mathbb{E}_1$-$\underline{\mathcal{O}}^e$-algebra A is a [generalized] Azumaya $\underline{\mathcal{O}}^e$-algebra in the sense of [Toë12, Definition 2.11].

(b) an $(A \otimes_{\underline{\mathcal{O}}^e} \sigma^* A)^{\otimes_{\underline{\mathcal{O}}^e} 2}$-linear equivalence $\hom_{\underline{\mathcal{O}}^e}(A \otimes_{\underline{\mathcal{O}}^e} A, R) \simeq A \otimes_{\underline{\mathcal{O}}^e} \sigma^* A^{\mathrm{op}}$

(c) A left $A \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-module P and an $A^{\mathrm{op}} \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-module $\overline{P}$

(d) An $A \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-linear map $P \to A^{tC_2}$ and an $A^{\mathrm{op}} \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-linear map $\overline{P} \to A^{tC_2}$. Here we regard A^{tC_2}, which is canonically a $(A \otimes_{\underline{\mathcal{O}}^e} \sigma^* A)^{tC_2}$-module, as an $A \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-module (resp. $A^{\mathrm{op}} \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-module) via the twisted Tate-valued diagonal $A \to (A \otimes \sigma^* A)^{tC_2}$ (resp. $A^{\mathrm{op}} \to (\sigma^* A^{\mathrm{op}} \otimes A^{\mathrm{op}})^{tC_2}$).

[4]By exactness of the Tate construction and the fact that A is a compact R-module, $\left(A \otimes_R R^{\varphi C_2}\right)^{tC_2} \simeq A^{tC_2}$.

(e) An equivalence of $(A \otimes \sigma^* A^{\mathrm{op}}) \otimes_{\underline{\mathcal{O}}^e} \underline{\mathcal{O}}^{\varphi C_2}$-modules

$$\hom_{\underline{\mathcal{O}}^e}(A, \underline{\mathcal{O}}^{\varphi C_2}) \simeq P \otimes_{\underline{\mathcal{O}}^{\varphi C_2}} \overline{P}$$

and a homotopy making the diagram

$$\begin{tikzcd}
\hom_{\underline{\mathcal{O}}^e}(A, \underline{\mathcal{O}}^{\varphi C_2}) \simeq P \otimes_{\underline{\mathcal{O}}^{\varphi C_2}} \overline{P} \arrow[r] \arrow[d] & P \otimes_{\underline{\mathcal{O}}^{\varphi C_2}} \overline{P} \arrow[d] \\
\hom_{\underline{\mathcal{O}}^e}(A, \underline{\mathcal{O}}^{etC_2}) \simeq \hom_{\underline{\mathcal{O}}^e}(A \otimes A^{\mathrm{op}}, \underline{\mathcal{O}}^e)^{tC_2} \arrow[r] & \hom_{(A \otimes A^{\mathrm{op}})^{\otimes 2}}\left((A \otimes A^{\mathrm{op}})^{\otimes 2}, A \otimes A^{\mathrm{op}}\right)^{tC_2} \simeq (A \otimes A^{\mathrm{op}})^{tC_2}
\end{tikzcd}$$

commute, where the lower horizontal arrow is induced by (b) and the right vertical arrow is induced by (d).

Example 5.38. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(P, q) \in \mathrm{Pn}\,(\mathrm{Mod}_R^\omega, \Qoppa_R)$.

L: What if I replaced $\Qoppa_R$ by a shift $\Qoppa_R^{[n]}$? Or any R-linear Poincaré structure $\Qoppa'$ so that $(\mathrm{Mod}_R^\omega, \Qoppa')$ is in $\mathrm{PnBr}(R)$? see [Cal+20a, §3.5], and furthermore see [PS92, p.216].

Then $A := \mathrm{End}_R(P)$ admits a canonical lift to an $\mathbb{E}_1$ algebra with genuine involution over R with $A^{\varphi C_2} := \hom_R(P, R^{\varphi C_2})$. If P is a generator of Mod_R^ω, then A is furthermore Azumaya.

By [Cal+20a, Proposition 3.1.16], A inherits a canonical anti-involution. To exhibit (b), observe that $q^\dagger$ induces a canonical $A \otimes A^{\mathrm{op}}$-linear equivalence $A = \mathrm{End}_R(P) \xrightarrow{D} \mathrm{End}_{\mathrm{Mod}_R^{\mathrm{op}}}(P^\vee) \simeq \mathrm{End}(P^\vee)^{\mathrm{op}} \xrightarrow{f \mapsto q^{-1} \circ f \circ q} \mathrm{End}_R(P)^{\mathrm{op}}\, A^\vee$. If P is a generator, $\hom_R(P, -)$ induces an equivalence $\mathrm{Mod}_R^\omega \simeq \mathrm{Mod}_A^\omega$, thus we can regard Mod_A^ω as equipped with a Poincaré structure. By the classification of R-linear Poincaré structures of Proposition 5.1(2), the Poincaré structure on Mod_A^ω is associated to an A-module with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$. We claim that $M_A \simeq A$ with the canonical A-A-bimodule structure: By [Cal+20a, Proposition 3.1.6], as an A^{op}-module M_A is the image of A under the composite

$$\mathrm{Mod}_A^\omega \xrightarrow{\hom_R(P,-)^{-1}} \mathrm{Mod}_R^\omega \xrightarrow{D_R = \hom_R(-,R)} \mathrm{Mod}_R^{\omega,\mathrm{op}} \xrightarrow{\hom_R(P,-)} \mathrm{Mod}_A^{\omega,\mathrm{op}} .$$

Observe that the image of A in $\mathrm{Mod}_R^{\omega,\mathrm{op}}$ is $D_R(P)$ and $q^\dagger$ induces an equivalence $D_R(P) \simeq P$, hence $M_A \simeq A$ as A^{op}-modules.

A similar argument with the linear part of $\Qoppa$ shows that we have an equivalence $N_A \simeq \hom_R(P, R^{\varphi C_2})$ of A-modules and a commutative square

$$\begin{tikzcd}
N_A \arrow[rr, "\sim"] \arrow[d] & & \hom_R(P, R^{\varphi C_2}) \arrow[d] \\
A^{tC_2} \arrow[r, "\sim"] & \hom_R(P, R^{tC_2}) \simeq \hom(P \otimes_R P, R)^{tC_2} &
\end{tikzcd}$$

of A-modules, where A acts on A^{tC_2} via the Tate-valued norm.

L: Reference for Tate-valued norm for $\mathbb{E}_\sigma$-algebras? continue. discuss consequences of being Morita trivial (distinguished 'point' in N_A)? can we identify $\overline{P}$?

Proposition 5.39. *Let R be a discrete ring with a given C_2-action σ, regard R as a Poincaré ring spectrum $\underline{R}^\lambda$ via Example 3.18. Let A be a classical Azumaya algebra over R with an involution of type 2, i.e. an equivalence of associative R-algebras $\lambda\colon A \to \sigma^* A^{\mathrm{op}}$. Suppose that either:*

- *the branch locus in $\mathrm{Spec}(R)/C_2 = \mathrm{Spec}(R^{C_2})$ is empty, or*
- *2 is invertible in R.*

Then there is a canonical Azumaya algebra with genuine involution over $\underline{R}^\lambda$ whose underlying Azumaya algebra is A.

L: What data is needed to give a lift, when $R^{\varphi C_2}$ is not necessarily zero (as is have assumed here)? In what other cases does a lift exist trivially? wrt Tate Poincaré structure, suffices to take a trivialization of A?

Remark 5.40. If $\frac{1}{2} \in R$, then $\mathrm{Br}(\operatorname{Spec} R, \lambda)$ is defined [PS92, p. 216]. In view of Propositions 5.39 and 5.42, there is a homomorphism $\mathrm{Br}(\operatorname{Spec} R, \lambda) \to \pi_0 \mathrm{PnBr}(\underline{R}^\lambda)$.

L: rewrite for scheme with involution

Remark 5.41. Equip R with the trivial involution, and consider the composite $R \to R^{\varphi C_2} \to \pi_0 R^{\varphi C_2} \simeq R/2$. This is equivalent to the composite $R \to R/2 \xrightarrow{\text{Frobenius}} R/2$, and since the Frobenius is multiplication by 2 in $\mathbb{G}_m$ and A is 2-torsion (because of the involution, there must be a trivialization of $A \otimes_R \pi_0 R^{\varphi C_2}$, and the set of trivializations is described by some coset space in the exact sequence $\cdots H^1(-;\mathbb{G}_m) \to H^1(-;\mathrm{GL}_n) \to H^1(-;\mathrm{PGL}_n) \to \cdots$). Now $R^{\varphi C_2} \to \pi_0 R^{\varphi C_2}$ is a limit of nilpotent extensions; use deformation theory to try to lift any given trivialization of $A \otimes_R \pi_0 R^{\varphi C_2}$ to a trivialization of $A \otimes_R R^{\varphi C_2}$. Finally, use that the space of lifts of A is controlled by Corollary 5.18?

L: tried to put this in a comment but TeX would not compile :/

Proof of Proposition 5.39. Since $R^{\varphi C_2} = 0$, conditions (c)-(e) of Definition 5.31 are vacuous. $\square$

Proposition 5.42. *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(A, A^{\varphi C_2} \to A^{tC_2})$ be an Azumaya algebra with genuine involution over R. Then*

1. $(\mathrm{Mod}_A^\omega, \Qoppa_A)$ *defines an R-linear Poincaré ∞-category.*
2. $(\mathrm{Mod}_A^\omega, \Qoppa_A)$ *is an invertible object in* $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}\right)$.

Proof of Proposition 5.42. The first statement follows from Proposition 5.1(2); we prove the second statement. First, by [Cal+20a, Example 3.2.9], we see that $(\mathrm{Mod}_A^\omega, \Qoppa_A)$ is indeed an R-linear Poincaré ∞-category (and not merely hermitian). To show that the associated Poincaré ∞-category is invertible, we must identify a dual $(\mathrm{Mod}_A^\omega, \Qoppa_A)^\vee$ and exhibit an equivalence $(\mathrm{Mod}_A^\omega, \Qoppa_A) \otimes (\mathrm{Mod}_A^\omega, \Qoppa_A)^\vee \simeq (\mathrm{Mod}_R^\omega, \Qoppa_R)$. Since $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty\, R}$ is symmetric monoidal, we see that the underlying R-linear ∞-category associated to the dual must be $\mathrm{Mod}^\omega_{A^{\mathrm{op}}}$. Moreover, the canonical evaluation map $\mathrm{ev}\colon \mathrm{Mod}_A^\omega \otimes \mathrm{Mod}^\omega_{A^{\mathrm{op}}} \xrightarrow{\simeq} \mathrm{Mod}_R^\omega$ sends $A \otimes A^{\mathrm{op}}$ to A. Endow $\mathrm{Mod}^\omega_{A^{\mathrm{op}}}$ with a Poincaré structure corresponding to the module with genuine involution $M_{A^{\mathrm{op}}} := A^{\mathrm{op}}$, $N_{A^{\mathrm{op}}} := \overline{P}$. It remains to exhibit a natural equivalence

$$\eta\colon\ (\Qoppa_A \otimes \Qoppa_{A^{\mathrm{op}}}) \xrightarrow{\simeq} \mathrm{ev}^* \Qoppa_R \tag{5.43}$$

of [quadratic] functors $\mathrm{Mod}_A^\omega \otimes \mathrm{Mod}^\omega_{A^{\mathrm{op}}} \to \mathrm{Sp}$. By [Cal+20a, Theorem 3.3.1], it suffices to exhibit equivalences on the bilinear and linear parts of (5.43) which glue compatibly. By Proposition 5.1(3), on linear parts, it suffices to exhibit an $A \otimes_R A^{\mathrm{op}}$-linear equivalence

$$\hom_R(A, R^{\varphi C_2}) \simeq N_A \otimes_{R^{\varphi C_2}} N_{A^{\mathrm{op}}}$$

and on bilinear parts, it suffices to exhibit an $(A \otimes_R A^{\mathrm{op}})^{\otimes_R 2}$-linear equivalence

$$\hom_{R \otimes R}(A \otimes_R A, R) \simeq M_A \otimes_R M_{A^{\mathrm{op}}}$$

which glue compatibly. This follows from the definitions, concluding the proof. $\square$

Observation 5.44. Let k be an algebraically closed field, and regard $\underline{k}$ as a Poincaré ring spectrum with the trivial involution via Example 3.18. Let $(A, A^{\varphi C_2}, A^{\varphi C_2} \to A^{tC_2})$ be an Azumaya algebra with genuine involution over $\underline{k}$. By Proposition 5.42, $(\mathrm{Mod}_A^\omega, \Qoppa_A) \in \mathrm{PnBr}(\underline{k})$. By [Toë12, Corollary 1.15], A is equivalent to $\mathrm{End}_k(P)$ for some compact k-module P. Observe that there is a canonical identification $A^{\mathrm{op}} \simeq \mathrm{End}_k(P^\vee)$. By the derived Skolem–Noether theorem [Lie09, Theorem 5.1.5], there exists a unique $n \in \mathbb{Z}$ so that the involution $A \simeq A^{\mathrm{op}}$ is induced by an equivalence $P \simeq P^\vee[n]$ (which is unique up to multiplication by a unit in k).

L: the next part is contingent on the computation in Example 5.20; omit if the computation in the example works out to be zero. If the computation is nonzero, consider this a numerical invariant of derived Azumaya algebras w involution and put it in a definition?

By Example 5.20, we may associate to $(\mathrm{Mod}_A^\omega, \Qoppa_A) \in \pi_0 \mathrm{PnBr}(\underline{k})$ an integer n.

Proposition 5.45. *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Let $(\mathcal{C}, \Qoppa)$ be an invertible idempotent-complete R-linear Poincaré ∞-category. Suppose given a Poincaré object (P, q) of $(\mathcal{C}, \Qoppa)$ so that P is a generator for $\mathcal{C}$, and write $q^\dagger\colon P \xrightarrow{\sim} D_{\Qoppa_{\mathcal{C}}}(P)$ for the adjoint to the bilinear part of q. Then*

L: later: add case of schemes with involution

(1) $\mathrm{End}_{\mathcal{C}}(P)$ *inherits the structure of an* $\mathbb{E}_\sigma$*-*R*-algebra. Informally, the involution on* $A := \mathrm{End}_{\mathcal{C}}(P)$ *is of the form* $a \mapsto (q^\dagger)^{-1} \circ D_{\text{Ϙ}_{\mathcal{C}}}(a) \circ q^\dagger$*.*

(2) There exists a lift of $A := \mathrm{End}_{\mathcal{C}}(P)$ *to a genuine Azumaya algebra with central Wall anti-structure over* R *in the sense of Definition 5.32. Informally, the* R*-linear module with genuine involution over* A *is given by* $M_A := \hom_{\mathcal{C}}(P, P)$*, the* $A \otimes \sigma^* A$*-action on* M_A *is given by* $(f \otimes g) \otimes a \mapsto f \circ a \circ (q^\dagger)^{-1} \circ D_{\text{Ϙ}_{\mathcal{C}}}(g) \circ q^\dagger$*, and the involution on* M_A *is given by* $a \mapsto \mathrm{ev}_P^{-1} \circ D_{\text{Ϙ}_{\mathcal{C}}}(q^\dagger)^{-1} \circ D_{\text{Ϙ}_{\mathcal{C}}}(a) \circ q^\dagger$*. The linear part of the module with genuine involution is given by*

L: do we want to describe the linear part?

L: In the notation of [Cal+20a, Example 3.1.13], $\varepsilon = \mathrm{ev}_P^{-1} \circ D_{\text{Ϙ}_{\mathcal{C}}}(q^\dagger)^{-1} \circ q^\dagger$*.*

(3) Suppose that there exists a C_2*-equivariant map* $\mathbb{S}^0 \to M_A$ *sending* $1 \mapsto \mathrm{id}_P$ *(where* $\mathbb{S}^0$ *is endowed with the trivial* C_2*-action). Then* R*-linear module with genuine involution* M_A *of part (2) comes from the involution on* A *in the sense of Remark 5.34. In particular,* $(\mathcal{C}, \text{Ϙ})$ *is of the form* $(\mathrm{Mod}_A^\omega, \text{Ϙ}_A)$ *for some Azumaya algebra over* R *with genuine involution.*

(4) Suppose (P, q) *is a hyperbolic Poincaré object in the sense of [Cal+20a, Definition 2.2.1], and write* $A := \mathrm{End}_{\mathcal{C}}(P)$*. Then there exists a* C_2*-equivariant map* $\mathbb{S}^0 \to M_A$ *sending* $1 \mapsto \mathrm{id}_P$*, i.e.* M_A *satisfies the hypotheses of (3).*

L: compare Theorem 7.3 in this paper by U. First 'rings which are Morita equivalent to their opposites'?

Corollary 5.46. *Let* $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ *be a Poincaré ring. Let* $(\mathcal{C}, \text{Ϙ})$ *be an invertible idempotent-complete* R*-linear Poincaré* ∞*-category. Then* $(\mathcal{C}, \text{Ϙ})$ *is of the form* $(\mathrm{Mod}_A^\omega, \text{Ϙ}_A)$ *for some Azumaya algebra over* R *with genuine involution.*

Proof. By Proposition 5.45(3) & (4), it suffices to exhibit a hyperbolic Poincaré object (P, q) whose underlying object is a generator of $\mathcal{C}$. Since the underlying R-linear stable infinity category $\mathcal{C}$ is dualizable it follows that it is an R-linear category which staisfies étale hyperdescent by [AG14b, Example 4.4]. By [AG14a, Theorem 6.1], $\mathcal{C}$ has a generator G. Now $G \oplus D_{\text{Ϙ}} G$ promotes canonically to a hyperbolic Poincaré object of $(\mathcal{C}, \text{Ϙ})$ by [Cal+20a, Proposition 2.2.5]. □

Proof of Proposition 5.45. Part (1) follows from a similar argument to Example 5.38. Since P generates $\mathcal{C}$, we have an equivalence $\mathcal{C} \simeq \mathrm{Mod}_A^\omega$ of R-linear small idempotent-complete stable ∞-categories. Part (2) follows from Proposition 5.1(2) and unraveling definitions. Part (3) follows from .

L: todo:a relative version of [Cal+20a, Proposition 3.1.14]?

L: proof of part (4).

□

6 Stacks associated to Poincaré ∞-categories

Given an R-linear ∞-category $\mathcal{C}$, there is a close relationship between objects in $\mathcal{C}$ and Morita trivializations of $\mathcal{C}$: The choice of an object G which is a compact generator of $\mathcal{C}$ gives an equivalence $\mathcal{C} \simeq \mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(G)}(\mathrm{Mod}_R)$. In the moduli stack of objects [TV07; AG14a, §5] in $\mathcal{C}$, the space of compact generators comprises an open substack. Using this correspondence/philosophy, Antieau–Gepner show that the moduli stack of objects in $\mathcal{C}$ is locally geometric, then use local geometricity of moduli stacks of objects to prove the existence of étale-local trivializations of invertible R-linear ∞-categories.

L: cite [AG14a, Proposition 5.10]?

In this section, we study analogous moduli stacks associated to R-linear Poincaré ∞-categories. In this setting, the appropriate replacement/analogy for an object is that of a *Poincaré/hermitian object.*

L: say something about conjectural adjunction between algebras and invertible modules with genuine involution and Poincaré ∞-categories?

6.1 Moduli of hermitian objects

Notation 6.1. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. There is a functor

$$\begin{aligned} &\mathbb{E}_\infty \mathrm{Alg}_{R/}^{BC_2} \to \mathrm{CAlg^p}_{R/-} \\ &\qquad S \mapsto (S, S \to R^{\varphi C_2} \otimes_R S \to S^{tC_2}) =: (R, R \to R^{\varphi C_2} \to R^{tC_2}) \otimes S\,, \end{aligned}$$

where the map $R^{\varphi C_2} \otimes_R S \to R^{tC_2} \otimes_{R^{tC_2}} S^{tC_2} \simeq S^{tC_2}$ is given by base change along the Tate-valued norm composed with the structure map $R^{\varphi C_2} \to R^{tC_2}$. Composing the aforementioned functor with the functor that sends a Poincaré ring to its category of compact modules equipped with the canonical Poincaré structure defines a functor

$$\mathrm{Mod}^p \colon \mathbb{E}_\infty \operatorname{Alg}^{BC_2}_{R/-} \to \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}\right).$$

Notation 6.2. Let X be a scheme with an involution σ and let $\pi \colon X \to Y$ exhibit Y as a good quotient of X. If $j \colon U \to Y$ is flat, let us write $\pi^* U$ for the tuple $(X \times_Y U, j^*(\sigma), U, j^*(\pi))$ of Remark 3.26. Then the assignment $(j \colon U \to Y) \mapsto \left(\mathrm{Mod}^\omega_{j^*X}, \Qoppa_{j^*\underline{\mathcal{O}}}\right)$ defines a functor

$$\mathrm{Mod}^p \colon \text{Ét}^{\mathrm{op}}_Y \to \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Mod}_{\left(\mathrm{Mod}^\omega_X, \Qoppa_{\underline{\mathcal{O}}}\right)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})\right).$$

Observation 6.3. Let R be a discrete commutative ring with a C_2-action, and recall that $X := \operatorname{Spec} R \to \operatorname{Spec}(R^{C_2}) = Y$ may be regarded as a C_2-scheme (Observation 3.25). Then base change along $R^{C_2} \to R$ defines a map from étale covers of Y of Notation 6.2 to C_2-equivariant étale covers of X of Notation 6.1. However, not all C_2-equivariant étale covers of X arise from this construction: consider the étale cover associated to the map of rings $R \xrightarrow{r \mapsto (r, \sigma(r))} R \times R$, where $R \times R$ is endowed with the flip action $(r, s) \mapsto (s, r)$.

L: verify later

Proposition 6.4. *$(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and assume that $R^{\varphi C_2}$ and R are connective. Let (X, λ, Y, π) be a scheme with involution X and a good quotient Y. Then:*

1. *The assignment of Notation 6.1 is a hypersheaf on the small C_2-equivariant étale site of R.*
2. *The assignment of Notation 6.2 is a hypersheaf on the small étale site of Y.*

L: I think Propositions 1 and 2 have essentially the same proof–present them together?

Proof. For now, use the notational shorthand $R^p = (R, R \to R^{\varphi C_2} \to R^{tC_2})$. Since limits in categories of algebras and modules are computed at the level of underlying objects, it suffices to show that the functor sends an étale hypercovering $j_\bullet \colon S \to T^\bullet$ to a limit diagram in $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$. By Proposition 6.1.4 of [Cal+20a], it suffices to show that the relevant diagram is a limit diagram in $\mathrm{Cat}^{\mathrm{h}}_{\infty,\mathrm{idem}}$. The proof of Lemma 5.4 in [AG14a] implies that the diagram defines a limit diagram on underlying ∞-categories. Thus by Remark 6.1.3 of [Cal+20a], it suffices to show that $j^*_\bullet \colon \mathrm{Mod}^\omega_{R^e \otimes_{R^{C_2}} S} \to \mathrm{Mod}^\omega_{R^e \otimes_{R^{C_2}} T^\bullet}$ induces an equivalence $\Qoppa_{R^p \otimes S} \xrightarrow{\sim} \lim_\Delta \Qoppa_{R^p \otimes T^\bullet} \circ (j^*_\bullet)^{\mathrm{op}}$ of quadratic functors $\mathrm{Mod}^{\omega,\mathrm{op}}_{R^e \otimes_{R^{C_2}} S} \to \mathrm{Sp}$. This follows from our assumption on $S \to T^\bullet$ and [Cal+20a, Theorem 3.3.1].

The proof of the second point is similar. ☐

L: Antieau–Gepner write *stack* for 'sheaf of categories' and *sheaf* for 'sheaf of spaces.' Do we want to use this convention?

Corollary 6.5. *Let $\mathcal{C}$ denote either the category of schemes with good quotients or the category of Poincaré rings with connective underlying and geometric fixed-points ring. Write* PnBr *for the composite functor* $\text{Ét}_R \xrightarrow{\mathrm{Mod}^p} \mathbb{E}_\infty \operatorname{Alg}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}\right) \xrightarrow{\mathrm{PnBr}} \mathrm{Sp}_{\geq 0}$, *where* Mod^p *is from Notation 6.1 or Notation 6.2 and* PnBr *is Definition 5.15. Then* PnBr *is a C_2-étale hypersheaf.*

Proof. The case of Poincaré rings is handled by the fiber sequence of Corollary 5.18. Note further that if we know that $\mathrm{PnBr}(-)$ is a C_2-étale sheaf in the case of schemes with good quotients, then we may reduce the hypersheaf question to Poincaré rings which we already know. Thus we need only check that the Poincaré Brauer space is an étale sheaf.

Since the Picard space commutes with limits, it is enough to know that the functor

$$\mathrm{Mod}_{\mathrm{Mod}^p} \colon \text{Ét}^{\mathrm{op}}_Y \to \mathrm{LinCat}^p$$

is a C_2-étale sheaf, where LinCat^p is the Grothendieck construction on the functor sending a C_2-étale extension $X \to X'$ to the category $\mathrm{Mod}_{\mathrm{Mod}_{(X', \Qoppa_{\underline{\mathcal{O}}_{X'}})}}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$. This follows by the same argument as in [Lur11b, Theorem 5.13] (note that the argument is formal once one has that the functors of Notation 6.1 and Notation 6.2 are étale sheaves, which we have already handled.) ☐

Now that we have étale descent we are able to prove Theorem 1.5.

Theorem 6.6. *Let (X, λ, Y, π) be a scheme with good quotient. Suppose further that $\pi : X \to Y$ is étale and that $\frac{1}{2} \in \Gamma(\mathcal{O}_Y)$. Then there is a natural map*

$$H^0(Y; \mathrm{coker}(\pi_* \mathbb{Z}_X \to \mathbb{Z})) \times \mathrm{Br}(X, \lambda) \to \mathrm{PnBr}(X, \lambda, Y, \pi)$$

which is furthermore an isomorphism.

Proof. By étale descent we have that $\mathrm{PnBr}(X, \lambda, Y, \pi)$ fits into an exact sequence

$$\mathrm{Pic}(X) \to \mathrm{Pic}(Y) \to \mathrm{PnBr}(X, \lambda, Y, \pi) \to \mathrm{Br}(X) \to \mathrm{Br}(Y)$$

compatible with the Parimala-Srinivas exact sequence of [PS92, Theorem 1.2]. Note that the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
H^0(X; \mathbb{Z}_X) \arrow[r] \arrow[d] & H^0(Y; \mathbb{Z}_Y) \arrow[d] & \\
\pi_0(\mathrm{pic}(X)) \arrow[r] & \pi_0(\mathrm{pic}(Y)) \arrow[r] & \mathrm{PnBr}(X, \lambda, Y, \pi)
\end{tikzcd}$$

induces a map $\mathrm{coker}(\pi_* \mathbb{Z}_X \to \mathbb{Z}_Y) \to \mathrm{PnBr}(X, \lambda, Y, \pi)$. Denote the sheaf $\mathrm{coker}(\pi_* \mathbb{Z}_X \to \mathbb{Z}_Y)$ by C. By construction we then get a map of exact sequences

$$\begin{tikzcd}
\Gamma(\mathbb{Z}_X) \times \mathrm{Pic}(X) \arrow[r] \arrow[d, "\cong"] & \Gamma(\mathbb{Z}_Y) \arrow[r] \arrow[d, "\cong"] & \Gamma(C) \times \mathrm{Br}(X, \lambda) \arrow[r] \arrow[d] & H^1(X; \mathbb{Z}_X) \times \mathrm{Br}(X) \arrow[r] \arrow[d, "\cong"] & H^1(Y; \mathbb{Z}_Y) \times \mathrm{Br}(Y) \arrow[d, "\cong"] \\
\pi_0(\mathrm{Pic}(X)) \arrow[r] & \pi_0(\mathrm{Pic}(Y)) \arrow[r] & \mathrm{PnBr}(X, \lambda, Y, \pi) \arrow[r] & \pi_0(\mathrm{Br}(X)) \arrow[r] & \pi_0(\mathrm{Br}(Y))
\end{tikzcd}$$

Hence by the 5-lemma we have an equivalence as desired. ☐

Notation 6.7 (Moduli (pre)sheaf of hermitian objects)**.** Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and let $(\mathcal{C}, \Omega_{\mathcal{C}})$ be an R-linear Poincaré ∞-category. Define a presheaf $\mathbf{M}^h_{(\mathcal{C}, \Omega_{\mathcal{C}})} \colon \left(\mathbb{E}_\infty \, \mathrm{Alg}_R^{BC_2}\right) \to \mathcal{S}$ whose value at $R \to S$ is the ∞-groupoid of R-linear hermitian functors from $(\mathcal{C}, \Omega_{\mathcal{C}})$ to $(\mathrm{Mod}^\omega_R, \Omega_R) \otimes_R S$.

L: Do we need Poincaré or does it suffice to take $(\mathcal{C}, \Omega_{\mathcal{C}})$ to be hermitian here?

L: Antieau–Gepner consider the ∞-*category* of functors, then pass to the maximal sub-groupoid [AG14a, §5.2]. Do we need to consider an enhancement of hermitian functors to ∞-categories?

Now suppose A is an $\mathbb{E}_1$-R-algebra with genuine involution. Then we will write $\mathbf{M}^h_A$ for $\mathbf{M}^h_{\left(\mathrm{Mod}^\omega_A, \Omega_A\right)}$.

Observation 6.8. Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and suppose A is an $\mathbb{E}_1$-R-algebra with genuine involution. A point of $\mathbf{M}^h_A(S)$ is classified by pair (F, η) where F is an R-linear functor $F \colon \mathrm{Mod}^\omega_A \to S$ and η is a natural transformation $\eta \colon \Omega_A \to \Omega_S \circ F^{\mathrm{op}}$ of R-linear hermitian structures. Now F is classified by an $A \otimes_R S$-module P which is compact as an S-module (cf. [AG14a, Proposition 5.10]), and η is classified by a map of $N^{C_2}_S(A \otimes_R S)$-modules $A \otimes_R S \to N^{C_2}_S(P^\vee)$ (Proposition 5.1(2)). Thus $\mathbf{M}^h_A(S)$ may be identified with the full sub-∞-groupoid of the pullback of the diagram

$$\begin{tikzcd}
 & \mathrm{Mod}_{A \otimes_R S}(\mathrm{Sp}) \arrow[d, "N^{C_2}_S"] \\
\mathrm{Mod}_{N^{C_2}_S(A \otimes_R S)}\left(\mathrm{Sp}^{C_2}\right)_{A \otimes_R S/} \arrow[r] & \mathrm{Mod}_{N^{C_2}_S(A \otimes_R S)}\left(\mathrm{Sp}^{C_2}\right).
\end{tikzcd}$$

Lemma 6.9. *Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring and let $(\mathcal{C}, \Omega_{\mathcal{C}})$ be an R-linear Poincaré ∞-category. Suppose that R is connective. Then the assignment $\mathbf{M}^h_{(\mathcal{C}, \Omega_{\mathcal{C}})}$ of Notation 6.7 satisfies étale hyperdescent.*

L: maybe want $R^{\varphi C_2}$ to be connective too?

Notation 6.10 (Moduli (pre)sheaf of hermitian objects)**.** Let (X, σ, Y, π) be a scheme with involution and good quotient and let $(\mathcal{C}, \Omega_{\mathcal{C}})$ be an $\left(\mathrm{Mod}_X^{\omega}, \Omega_{\underline{O}}\right)$-linear Poincaré ∞-category. Define a presheaf $\mathbf{M}^h_{(\mathcal{C},\Omega_{\mathcal{C}})} \colon \mathrm{\acute{E}t}_Y^{\mathrm{op}} \to \mathcal{S}$ whose value at $\operatorname{Spec} S \to Y$ is the ∞-groupoid of $\left(\mathrm{Mod}_X^{\omega}, \Omega_{\underline{O}}\right)$-linear hermitian functors from $(\mathcal{C}, \Omega_{\mathcal{C}})$ to $\left(\mathrm{Mod}_{X_S}^{\omega}, \Omega_{\underline{O}|S}\right)$.

L: same comment about ∞-*category* of functors vs ∞-groupoid.

Now suppose A is an Azumaya algebra over X with genuine σ-linear (anti-)involution. Then we will write $\mathbf{M}^h_A$ for $\mathbf{M}^h_{\left(\mathrm{Mod}_A^{\omega}, \Omega_A\right)}$.

Lemma 6.11. *Let (X, σ, Y, π) be a scheme with involution and good quotient and let $(\mathcal{C}, \Omega_{\mathcal{C}})$ be an $\left(\mathrm{Mod}_X^{\omega}, \Omega_{\underline{O}}\right)$-linear Poincaré ∞-category. Then the functor $\mathbf{M}^h_{(\mathcal{C},\Omega_{\mathcal{C}})}$ defines a hypersheaf on the small étale site of Y.*

Proof of Lemma 6.11.

L: This should follow from a similar argument to the first sentence of [AG14a, Lemma 5.4] and Proposition 6.4, but I'm a little confused about what's going on there.

L: Lemmas 6.9 and 6.11 should have similar proofs.

□

Observation 6.12. Let (X, σ, Y, π) be a scheme with involution and good quotient and let A be an Azumaya algebra over X with genuine σ-linear (anti-)involution. Recall that there is a hypersheaf $\mathbf{M}_{A^e}$ on the étale site of X [AG14a, §5.2]. The forgetful functor $\mathrm{Cat}^{\mathrm{h}}_{\infty R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}$ induces a morphism of (pre)stacks $\mathbf{M}^h_A \to \pi_*\mathbf{M}_{A^e}$ on the small étale site of Y, where $\mathbf{M}^h_A$ is from Notation 6.10. Given a point $\operatorname{Spec} S \to Y$, the map

L: cite Toën? §2.2?

$$\begin{array}{ccc} \mathbf{M}^h_A(\operatorname{Spec} S) & \longrightarrow & \pi_*\mathbf{M}^h_{A^e}(\operatorname{Spec} S) \\ \wr\| & & \wr\| \\ \hom_{\left(\mathrm{Mod}_X^{\omega},\Omega_{\underline{O}}\right)-\mathrm{Mod}(\mathrm{Cat}^{\mathrm{h}}_{\infty})}\left(\left(\mathrm{Mod}_A^{\omega}, \Omega_A\right), \left(\mathrm{Mod}_{X_S}^{\omega}, \Omega_{\underline{O}|S}\right)\right) & & \hom_{\left(\mathrm{Mod}_X^{\omega}\right)-\mathrm{Mod}(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty})}\left(\mathrm{Mod}_A^{\omega}, \mathrm{Mod}_{X_S}^{\omega}\right) \end{array}$$

sends a hermitian functor (F, η) to its underlying $\mathcal{O}_X$-linear functor F.

L: I think we can combine Lemma 6.14, Lemma 6.18, and [AG14a, Lemma 4.25] to show that $\mathbf{M}^h_A$ is locally geometric.

Recollection 6.13. An affine morphism of schemes is always quasi-compact [Stacks, Tag 01S5]. In particular, if (X, σ, Y, π) is a scheme with involution and good quotient, then π is quasi-compact.

Lemma 6.14. *Let (X, σ, Y, π) be a scheme with involution and good quotient and let A be an Azumaya algebra over X with genuine σ-linear (anti-)involution. Assume that Y is quasi-compact. Then the sheaf $\pi_*\mathbf{M}_{A^e}$ is locally geometric.*

L: Other desired properties: locally of finite presentation, smoothness. Possibly need to impose different assumptions on π for different properties to be preserved.

Proof. Recall [AG14a, Theorem 5.8] that the sheaf $\mathbf{M}_{A^e}$ is locally geometric, i.e. there exists a filtered system $\mathbf{M}_{A^e} = \operatorname{colim}_{[a,b]} \mathbf{M}^{[a,b]}_{A^e}$ where each $\mathbf{M}^{[a,b]}_{A^e} \to \mathbf{M}_{A^e}$ is a monomorphism and each $\mathbf{M}^{[a,b]}_{A^e}$ is n_i-geometric for some n_i.

While π_* does not preserve filtered colimits in general, quasi-compactness of π (Recollection 6.13) and of Y implies that $\operatorname{colim}_{[a,b]} \pi_*\mathbf{M}^{[a,b]}_{A^e} \to \pi_*\mathbf{M}_{A^e}$ is an equivalence [TT90, §3.1.2].

Since pushforward is left exact (and assuming that π_* preserves this particular filtered colimit), it suffices to show that the pushforward of an n-geometric sheaf on X is m-geometric for some m.

L: Incomplete! Bootstrap Lemma 6.15 and Lemma 6.17 to show this?
Note: By definition of exact quotient and [FW20, Theorem 4.35(i)] and [Lur17, Proposition 7.2.1.14], π_* preserves epimorphisms. Therefore, to show that π_* sends submersions (=smooth and surjective) to submersions, it suffices to show that π_* sends smooth maps to smooth maps.

□

Lemma 6.15. *Let $R \to S$ be a map of connective $\mathbb{E}_\infty$-rings and write $\pi\colon \operatorname{Spec} S \to \operatorname{Spec} R$. Assume that*

- *The map $R \to S$ is an effective descent morphism, and*
- *S is dualizable as an R-module; write $S^\vee$ for the R-linear dual of S, and suppose it has Tor-amplitude contained in $[d, c]$ for some $d \le 0$.*

Then:

(1) Let M be a perfect S-module and consider the sheaf $\operatorname{Spec}\operatorname{Sym}_S(M)$ on the étale site of S. Assume that M has Tor-amplitude in $[a, b]$ (so the stack $\operatorname{Spec}\operatorname{Sym}_S(M)$ is $\max(-a, 0)$-geometric by [AG14a, Theorem 5.2]). Then $\pi_* \operatorname{Spec}\operatorname{Sym}_S(M)$ is a quasiseparated $(\max(-a, 0) - d)$-geometric sheaf on the étale site of R.

L: pretty sure that if S is flat over R and $b \le 0$, then $\pi_* \operatorname{Spec}\operatorname{Sym}_S(M)$ is furthermore smooth. Check later!

(2) Let T be a connective S-algebra which is locally of finite presentation. Then $\pi_* \operatorname{Spec} T$ is a $(-d)$-geometric sheaf on the étale site of R which is locally of finite presentation.

Proof. We begin with the proof of part (1). Using the morphisms $M \to M \otimes_R S \to M$, we know that $\operatorname{Spec}\operatorname{Sym}_S(M)$ is a retract of $\operatorname{Spec}\operatorname{Sym}_S(M \otimes_R S)$, hence $\pi_* \operatorname{Spec}\operatorname{Sym}_S(M)$ is a retract of $\pi_* \operatorname{Spec}\operatorname{Sym}_S(M \otimes_R S)$. Since S is a perfect R-module and M is a perfect S-module, M is also perfect when regarded as an R-module. By [AG14a, Lemma 4.38], it suffices to show that if N is a perfect R-module, then

L: that these are morphisms of S-modules requires that S is $\mathbb{E}_\infty$.

(1) the sheaf $\pi_* \operatorname{Spec}\operatorname{Sym}_S (N \otimes_R S)$ is quasiseparated

(2) If N has tor-amplitude contained in $[e, f]$ with $e \le 0$, then $\pi_* \operatorname{Spec}\operatorname{Sym}_S (N \otimes_R S)$ is $(e-d)$-geometric.

Now if T is an R-algebra,

$$\begin{aligned}
\pi_* \operatorname{Spec}\operatorname{Sym}_S (N \otimes_R S)(\operatorname{Spec} T) &\simeq \hom_{S\mathrm{Alg}}(\operatorname{Sym}_S(N \otimes_R S), T \otimes_R S) \\
&\simeq \hom_{S\,\mathrm{Mod}}(N \otimes_R S, T \otimes_R S) \\
&\simeq \hom_{R\,\mathrm{Mod}}(N, T \otimes_R S) \qquad\qquad (6.16)\\
&\simeq \hom_{R\,\mathrm{Mod}}(N \otimes_R S^\vee, T) \qquad \text{since } S \text{ is dualizable over } R \\
&\simeq \hom_{R\mathrm{Alg}}(\operatorname{Sym}_R(N \otimes_R S^\vee), T)\ .
\end{aligned}$$

Since $N \otimes_R S^\vee$ has Tor-amplitude in $[e + d, f + c]$, it follows that $\pi_* \operatorname{Spec}\operatorname{Sym}_S (N \otimes_R S)$ is quasiseparated and $(e - d)$-geometric by [AG14a, Theorem 5.2].

L: [AG14a, Proposition 2.13(4)] or find ref in [TT90] later

We now prove part (2). By [Lur17, Proposition 7.2.4.27(b) & (c); Lur09, Lemma 5.4.2.4], $\operatorname{Spec} T$ is a retract of a finite limit of sheaves of the form $\operatorname{Spec}\operatorname{Sym}_S(M)$ where M is a compact connective S-module. By (1), each $\pi_* \operatorname{Spec}\operatorname{Sym}_S(M)$ is quasi-separated and n-geometric for some n. Since the collection of quasi-separated n-geometric sheaves which are locally of finite presentation is closed under finite limits and retracts by Lemmas 4.35 and 4.38, resp. of [AG14a] and π_* preserves arbitrary limits (in particular finite limits and retracts), it suffices to show that $\pi_* \operatorname{Spec}\operatorname{Sym}_S(M)$ where M is a compact connective S-module is locally of finite presentation as an n-geometric sheaf on the étale site of R. Since $\pi_* \operatorname{Spec}\operatorname{Sym}_S(M)$ is a retract of $\pi_* \operatorname{Spec}\operatorname{Sym}_S(M \otimes_R S)$ by the proof of part (1), by [AG14a, Lemma 4.38], it suffices to show that the latter is locally of finite presentation. The result now follows from [AG14a, Theorem 5.2] and (6.16). ☐

Lemma 6.17. *Let R be a discrete ring with involution and consider $\pi\colon \operatorname{Spec} R \to \operatorname{Spec}(R^{C_2})$ as a scheme with involution and good quotient. Suppose that $\pi_*\colon \mathrm{Shv}_{\acute{e}t}(\operatorname{Spec} R; \mathcal{S}) \to \mathrm{Shv}_{\acute{e}t}(\operatorname{Spec}(R^{C_2}); \mathcal{S})$ sends disjoint unions of affines to disjoint unions of affines and sends smooth maps to smooth maps.*

L: Can we relax/get rid of the assumption that π_* preserves disjoint unions of affines by imposing that f quasicompact and other finiteness conditions?

Then if $f\colon X \to Y$ is an n-geometric morphism in $\mathrm{Shv}_{\acute{e}t}(\operatorname{Spec} R; \mathcal{S})$, then $\pi_*(f)$ is m-geometric for some m.

Proof. We will induct on n. Let S be a connective R^{C_2}-algebra and fix a map $p\colon \operatorname{Spec} S \to \pi_* Y$ of étale sheaves over R^{C_2}. The map p admits a canonical factorization

$$\operatorname{Spec} S \to \pi_*\pi^* \operatorname{Spec} S \xrightarrow{\pi_*(p^\dagger)} \pi_* Y\ ,$$

where $p^\dagger$ corresponds to p under the (π^*, π_*)-adjunction. Furthermore note that $\pi^* \operatorname{Spec} S \simeq \operatorname{Spec}(S \otimes_{R^{C_2}} R)$.

Base case: If f is 0-geometric, $(\pi^* \operatorname{Spec} S) \times_Y X \simeq \coprod_{i \in I} \operatorname{Spec} T_i$ for some connective commutative R-algebras T_i.

L: if f is quasi-compact, can take I to be *finite*.

L: very much a work in progress/incomplete!!! To show that $\pi_* f$ is d-geometric, it suffices to show that $\operatorname{Spec} S \times_{\pi_* \pi^* \operatorname{Spec} S} \left(\coprod_i \operatorname{Spec} T_i\right)$ admits a smooth surjective $(d-1)$-geometric morphism from a disjoint union of affines. Impose more assumptions and try to reduce to Lemma 6.15?

Inductive step: Since f is n-geometric, there exists a smooth $(n-1)$-geometric surjection $\varphi \colon U \simeq \coprod_{i \in I} \operatorname{Spec} T_i \to (\pi^* \operatorname{Spec} S) \times_Y X$. Now form the following diagram in which all squares are pullbacks

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "\psi"] & \pi_* U \arrow[d, "\pi_*(\varphi)"] & \\
\operatorname{Spec} S \times_{\pi_* Y} \pi_* X \arrow[r] \arrow[d] & \pi_* \left((\pi^* \operatorname{Spec} S) \times_Y X\right) \arrow[r] \arrow[d] & \pi_* X \arrow[d, "\pi_*(f)"] \\
\operatorname{Spec} S \arrow[r] & \pi_* \pi^* (\operatorname{Spec} S) \arrow[r] & \pi_* Y .
\end{tikzcd}$$

By inductive hypothesis, $\pi_*(\varphi)$ is a smooth m-geometric surjection for some $m = m(n-1)$ depending only on $(n-1)$. It follows from [AG14a, Lemma 4.23] that ψ is a smooth m-geometric surjection. By assumption, $\pi_* U$ is a disjoint union of affines. Recall that $\operatorname{Shv}_{\acute{e}t}(\operatorname{Spec}(R^{C_2}); \mathcal{S})$ is an ∞-topos, hence colimits are universal.

L: tried to use function notation but it looks like multiplication.

L: Need to show that V is a disjoint union of affines or admits a smooth m-geometric surjection from disjoint union of affines (then apply [AG14a, Lemma 4.25(1)])!
By assumption that π_* preserves arbitrary disjoint union, suffices to prove the result for $U = \operatorname{Spec} T$ (so $V = \operatorname{Spec} S \times_{\pi_* \pi^* \operatorname{Spec} S} \pi_* \operatorname{Spec} T$). Impose more assumptions and try to reduce to Lemma 6.15?

We have shown that there exists a smooth surjective m-geometric morphism to $\operatorname{Spec} S \times_{\pi_* Y} \pi_* X$ from a disjoint union of affines, hence $\pi_*(f)$ is $(m+1)$-geometric. ☐

Lemma 6.18. *Let (X, σ, Y, π) be a scheme with involution and good quotient and let A be an Azumaya algebra over X with genuine σ-linear (anti-)involution. Then the morphism $\mathbf{M}^h_A \to \pi_* \mathbf{M}_{A^e}$ of Observation 6.12 is locally geometric in the sense of [AG14a, §4.3].*

Proof. It suffices to show that for all $p \colon \operatorname{Spec} S \to \pi_* \mathbf{M}_{A^e}$, the fiber product $\operatorname{Spec} S \times_{\pi_* \mathbf{M}_{A^e}} \mathbf{M}^h_A$ is locally geometric over $\operatorname{Spec} S$. Write $\overline{p}$ for the composite $\operatorname{Spec} S \to \pi_* \mathbf{M}_{A^e} \to Y$; there is a pullback diagram

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r, "\widetilde{p}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\pi"] \\
\operatorname{Spec} S \arrow[r, "\overline{p}"] & Y
\end{tikzcd}$$

where the pullback $\operatorname{Spec} R$ is affine by assumption. Recall that the map p classifies an $\widetilde{p}^* A$-module M which is compact as an R-module. Note that the fiber product $\operatorname{Spec} S \times_{\pi_* \mathbf{M}_{A^e}} \mathbf{M}^h_A$ classifies lifts of the R-linear functor $\otimes M \colon \operatorname{Mod}^\omega_{\widetilde{p}^* A} \to \operatorname{Mod}^\omega_R$ to an R-linear hermitian functor. A lift of $\otimes M$ to an R-linear hermitian functor is equivalent to the data of a map $\widetilde{p}^* A \to N^{C_2}_R(M^\vee)$ of $N^{C_2}_R(\widetilde{p}^* A)$-modules in Sp^{C_2}, where we write $M^\vee$ for the R-linear dual of M. Thus the fiber product $\operatorname{Spec} S \times_{\pi_* \mathbf{M}_{A^e}} \mathbf{M}^h_A$ is given by the moduli sheaf $\operatorname{Spec} \operatorname{Sym}_S \tau_{\geq 0} \hom_{N^{C_2}_R(\widetilde{p}^* A)} \left(\widetilde{p}^* A, N^{C_2}_R(M^\vee)\right)$. ☐

L: Similar to [AG14a, Theorem 5.6, Proposition 5.7].

Notation 6.19. Let (X, σ, Y, π) be a scheme with involution and good quotient and let A be an Azumaya algebra over X with genuine σ-linear (anti-)involution; write $\lambda \colon A \to \sigma^* A^{\mathrm{op}}$ for the involution on A. Write $\mathbf{Mor}^p_A \to \mathbf{M}^h_A$ for the subsheaf of *Poincaré Morita equivalences*, i.e. $\mathbf{Mor}^h_A(\operatorname{Spec} S \to Y)$ is the full sub-groupoid of $\mathbf{M}^h_A(\operatorname{Spec} S \to Y)$ on those $\underline{\mathcal{O}}$-linear hermitian functors $(F, \eta) \colon (\operatorname{Mod}^\omega_A, Ϙ_A) \to \left(\operatorname{Mod}^\omega_{\underline{\mathcal{O}}|_{\operatorname{Spec} S}}, Ϙ_{\underline{\mathcal{O}}}|_{\operatorname{Spec} S}\right)$ so that F is an equivalence and (F, η) is duality-preserving.

Lemma 6.20. *Suppose given the setup of Notation 6.19. Supppose that the quotient map $\pi\colon X \to Y$ is* quasi-perfect *in the sense of [CHN24, Definition A.6.3], i.e. the pushforward π_* of quasicoherent sheaves preserves perfect complexes. Suppose further that π is flat. Then the inclusion $\mathbf{Mor}^p_A \to \mathbf{M}^h_A$ exhibits $\mathbf{Mor}^h_A$ as a quasicompact open subsheaf of $\mathbf{M}^h_A$.*

Remark 6.21. If either $\pi\colon X \to Y$ is an isomorphism or quadratic étale on each connected component, then π_* satisfies the assumptions of Lemma 6.20.

Here is an example which doesn't fall under either of the preceding cases: If $X = \operatorname{Spec}\mathbb{C}[x]$ with the action $x \mapsto -x$, then $Y = \operatorname{Spec}\mathbb{C}[x^2]$ and the quotient map $X \to Y$ is quasi-perfect because $\mathbb{C}[x]$ is a free module of finite rank over $\mathbb{C}[x^2]$. In general, one may check quasi-perfection Zariski-locally on the quotient [CHN24, Corollary A.6.5]. However, not all schemes with involution and good quotient are quasi-perfect; for a counterexample, see [FW20, Example 3.9].

L: general conditions for this to hold? References: IV.Proposition 2.2.3 and §4.3 here.

Proof of Lemma 6.20. Use the notation of Lemma 6.18. By the proof of Lemma 6.18, we know that (F, η) is given by $(- \otimes_R M, \widetilde{p}^*A \to N^{C_2}_R(M^\vee))$ where M is a compact R-module. By quasiperfection of π_* and flatness of π, $\operatorname{Spec} R \to \operatorname{Spec} S$ is quasiperfect, i.e. R is a compact S-module. It follows that M is a compact S-module.

L: quasiperfection is not preserved under arbitrary base change, only tor-independent base change (which is guaranteed under flatness assumption, see p.5 here or [CHN24, Lemma A.6.4] which is a special case of the former). OTOH, if we only work with the small étale site of Y, then all $\operatorname{Spec} S \to Y$ would be flat and we can remove flatness assumption on π (but still need quasi-perfection).

By the proof of [AG14a, Proposition 5.10], the subsheaf of points of $\operatorname{Spec} S$ on which the functor F is an equivalence is a quasicompact Zariski open $U \subseteq \operatorname{Spec} S$. The hermitian functor (F, η) is duality-preserving if and only if the canonical map $\tau_\eta\colon M \otimes_{\widetilde{p}^*A} \lambda^*\widetilde{p}^*A \to \hom_R(M, \sigma^*R)$ is an equivalence (cf. [Cal+20a, Lemma 3.4.3]). By [AG14a, Proposition 2.14] (compare the proof of Proposition 5.10 of *loc.cit.*), the subsheaf of points of $\operatorname{Spec} S$ on which the map τ_η is an equivalence is a quasicompact Zariski open $V \subseteq \operatorname{Spec} S$.

Finally (F, η) defines a Poincaré Morita equivalence if, in addition to F being an equivalence and η being duality-preserving, η is an equivalence of $N^{C_2}_R(\widetilde{p}^*A)$-modules. Since categorical fixed points are jointly conservative, η is an equivalence if and only if η^{C_2} and η^e are equivalences. By Lemma 6.22, $(\widetilde{p}^*A)^{C_2}$ and $(N^{C_2}_R(M^\vee))^{C_2}$ are compact S-modules, hence by a similar argument to the above, there exists a quasicompact Zariski open $W \subseteq \operatorname{Spec} S$ on whicn η is an equivalence. Taking the intersection $U \cap V \cap W$ gives the quasicompact Zariski open subsheaf of $\operatorname{Spec} S$ on which (F, η) defines a Poincaré Morita equivalence. $\square$

Lemma 6.22. *Let $R = (R^{C_2} \to R^e)$ be a C_2-$\mathbb{E}_\infty$-ring in C_2-spectra and suppose that R is connective (i.e. R^e and R^{C_2} are both connective). Then the composite $\operatorname{Mod}_{R^e}(\mathrm{Sp}) \xrightarrow{N^{C_2}_R} \operatorname{Mod}_R\left(\mathrm{Sp}^{C_2}\right) \xrightarrow{(-)^{C_2}} \operatorname{Mod}_{R^{C_2}}(\mathrm{Sp})$ sends compact R^e-modules to compact R^{C_2}-modules if and only if the restriction map $R^{C_2} \to R^e$ exhibits R^e as a perfect R^{C_2}-module. Here $N^{C_2}_R$ denotes the relative norm.*

Proof. Observe that the relative norm satisfies $N^{C_2}_R(P \oplus Q) \simeq N^{C_2}_R(P) \oplus N^{C_2}_R(Q) \oplus C_2 \otimes (P \oplus Q)$ for all $P, Q \in \operatorname{Mod}_{R^e}$, where $C_2 \otimes -\colon \operatorname{Mod}_{R^e} \to \operatorname{Mod}_R$ is the left adjoint to the restriction functor.

L: This is essentially implied by [Nar17, Example 3.17 & Corollary 3.28] but we should find an earlier reference if possible (Maybe § A.3.3 of Hill–Hopkins–Ravenel?). Same for the other references in this proof.

To prove the 'only if' direction, observe that

$$\left(N^{C_2}_R(R^e \oplus R^e)\right)^{C_2} = \left(N^{C_2}_R(R^e) \oplus N^{C_2}_R(R^e) \oplus C_2 \otimes (R^e \oplus R^e)\right)^{C_2} \simeq R^{C_2} \oplus R^{C_2} \oplus R^e \oplus R^e\,.$$

Since perfect R^{C_2}-modules contain R^{C_2} and are closed under cofibers and taking summands, this implies that R^e must be a perfect R^{C_2}-module.

Note that the above argument also implies that if R^e is a perfect R^{C_2}-module, then the exact functor $(-)^{C_2}\colon \operatorname{Mod}_R\left(\mathrm{Sp}^{C_2}\right) \to \operatorname{Mod}_{R^{C_2}}(\mathrm{Sp})$ sends perfect R-modules (which are generated as a thick subcategory by R and $C_2 \otimes R^e$) to perfect R^{C_2}-modules. To prove the 'only if' direction, it suffices to show that if the restriction map $R^{C_2} \to R^e$ exhibits R^e as a perfect R^{C_2}-module, then $N^{C_2}_R$ sends perfect R^e-modules to perfect R-modules in Sp^{C_2}. Recall that any perfect R^e-module P can be written as a finite extension of

$\{\Sigma^n R^e\}_{n\in\mathbb{Z}}$; write $\ell(P)$ for the minimum ℓ so that P can be written as an extension of $\Sigma^{n_i} R^e$ for some $n_1, \ldots, n_\ell \in \mathbb{Z}$. The result follows from induction on $\ell(P)$ and the following observations:

- $N_R^{C_2}(\Sigma^a R^e) = \Sigma^{\varrho a} N_R^{C_2}(R^e) = \Sigma^{\varrho a} R$ is a perfect R-module
- If Q, S are perfect R^e-modules so that $N_R^{C_2}(Q)$, $N_R^{C_2}(S)$ are perfect R-modules, then $N_R^{C_2}(Q \oplus S)$ is a perfect R-module.
- If $\ell(P) > 1$, then there exists a perfect R^e-module Q with $\ell(Q) < \ell(P)$ and an exact sequence of R^e-modules $P \to Q \to \Sigma^a R^e$ for some $a \in \mathbb{Z}$.
- $N_R^{C_2}(-)$ is a quadratic functor and the following cube

$$\begin{tikzcd} & Q \arrow[rr] \arrow[dd] & & Q \oplus \Sigma^a R^e \arrow[dd] \\ P \arrow[ur] \arrow[rr] \arrow[dd] & & Q \arrow[ur] \arrow[dd] & \\ & \Sigma^a R^e \arrow[rr] & & Q \oplus \Sigma^a R^e \oplus \Sigma^a R^e \\ 0 \arrow[ur] \arrow[rr] & & \Sigma^a R^e \arrow[ur] & \end{tikzcd}$$

 is strongly cartesian [Lur17, Corollary 6.1.1.16]. Therefore, $N_R^{C_2}$ sends the cube to a cartesian cube in $\mathrm{Mod}_R\left(\mathrm{Sp}^{C_2}\right)$. By the inductive hypothesis and the second point, N^{C_2} sends the vertices of the cube excluding P to perfect R-modules. Thus we have shown that $N_R^{C_2}(P)$ can be written as a finite limit of perfect R-modules. ☐

Lemma 6.23. *Let (X, σ, Y, π) be a scheme with involution and good quotient and let A be an Azumaya algebra over X with genuine σ-linear (anti-)involution. Assume further that π is quasi-perfect and flat. Then*

L: in progress June 18th! might need more assumptions later

- *the composite $\mathbf{M}_A^p \to \mathbf{M}_A^h \to \pi_*\mathbf{M}_{A^e}$ of Observation 6.12 factors through the inclusion $\pi_*\mathbf{Mor}_{A^e} \to \pi_*\mathbf{M}_{A^e}$ of Notation 6.19, i.e. there is a commutative square*

$$\begin{tikzcd} \mathbf{M}_A^p \arrow[r] \arrow[d, dashed, "\exists"'] & \mathbf{M}_A^h \arrow[d] \\ \pi_*\mathbf{Mor}_{A^e} \arrow[r] & \pi_*\mathbf{M}_{A^e} \end{tikzcd}$$

- *if P is compact as an $A \otimes_R R^{\varphi C_2}$-module, then the morphism $\mathbf{M}_A^p \to \pi_*\mathbf{Mor}_{A^e}$ is locally of finite presentation in the sense of [AG14a, §4.3].*

 L: modify definition of Azumaya w genuine involution to make P, $\overline{P}$ compact?

Proof. Keep the notation of the proof of Lemma 6.18. By [AG14a, Theorem 5.2], it suffices to show that $\hom_{N_R^{C_2}(\widetilde{p}^* A)}\left(\widetilde{p}^* A, N_R^{C_2}(M^\vee)\right)$ is a perfect S-module. By Lemma 6.22, it suffices to show that the C_2-mapping spectrum $\hom_{N_R^{C_2}(\widetilde{p}^* A)}\left(\widetilde{p}^* A, N_R^{C_2}(M^\vee)\right)$ is a compact R^L-module in C_2-spectra. Now recall that the mapping C_2-spectrum arises as

$$\mathrm{ev}\colon\ \mathrm{Mod}_{N_R^{C_2}(\widetilde{p}^* A)}\left(\mathrm{Sp}^{C_2}\right) \otimes_{\mathrm{Mod}_{R^L}(\mathrm{Sp}^{C_2})} \mathrm{Mod}_{N_R^{C_2}(\widetilde{p}^* A)^{\mathrm{op}}}\left(\mathrm{Sp}^{C_2}\right) \to \mathrm{Mod}_{R^L}(\mathrm{Sp}^{C_2})\,;$$

in particular $\mathrm{ev}\left(N_R^{C_2}(\widetilde{p}^* A) \otimes_{R^L} N_R^{C_2}(\widetilde{p}^* A)^{\mathrm{op}}\right) \simeq N_R^{C_2}(\widetilde{p}^* A)$. To show that $\hom_{N_R^{C_2}(\widetilde{p}^* A)}\left(\widetilde{p}^* A, N_R^{C_2}(M^\vee)\right)$ is a compact R^L-module, it suffices to prove that

- $N_R^{C_2}(\widetilde{p}^* A)$ and $N_R^{C_2}(\widetilde{p}^* A)[C_2]$ are compact R^L-modules (compare the proof of [AG14a, Theorem 3.15]),
- $\widetilde{p}^* A$ is a compact $N_R^{C_2}(\widetilde{p}^* A)$-module, and

- $N_R^{C_2}(M^\vee)$ is a compact $N_R^{C_2}(\widetilde{p}^*A)$-module.

The first point follows from Lemma 6.22. By the claim at the end of the proof, it suffices to check compactness/the latter two points separately on underlying and geometric fixed points. Since p classifies a point in $\pi_*\mathbf{Mor}_{A^e}$, $M^\vee$ is compact as an $A \otimes_R R^{\varphi C_2}$-module. By assumption on A, $(\widetilde{p}^*A)^{\varphi C_2}$ is a compact $(N_R^{C_2}(\widetilde{p}^*A))^{\varphi C_2}$-module. That $(\widetilde{p}^*A)^e$ is a compact $(N_R^{C_2}(\widetilde{p}^*A))^e$-module follows from our assumption that A is Azumaya.

L: WORK IN PROGRESS: map is locally of finite presentation. REMAINING: $(M^\vee)^{\otimes 2}$ is a compact $(\widetilde{p}^*A)^{\otimes 2}$-module

Claim Let B be an $\mathbb{E}_1$-algebra in Sp^{C_2}, and assume that B^e, $B^{\varphi C_2}$ are connective. Then a B-module M is compact if and only if M^e and $M^{\varphi C_2}$ are compact as B^e and $B^{\varphi C_2}$-modules, respectively.

L: UPDATE JUNE 20: Better–show that A is compact as an $\mathbb{E}_1$-algebra with genuine involution. Compare proof of [Cal+20a, Proposition 5.7].

L: Forgot that a generalized Azumaya algebra cannot be assumed to be connective. Fix later. Use that A is bounded below? or other 'niceness' properties.

Proof. By the connectivity hypotheses on B, a B-module in Sp^{C_2} is compact if and only if it is dualizable. The result follows from noting that $(-)^e$, $(-)^{\varphi C_2}$ detect dualizability and for B^e-modules and $B^{\varphi C_2}$-modules, dualizability is equivalent to compactness. □

6.2 The Poincaré Brauer space spectral sequence

Notation 6.24. Let (X, λ, Y, π) be a scheme with involution X and a good quotient Y. Let $\mathbb{Z}_X$ be the étale sheafification of the constant presheaf. Write $\mathbb{Z}_X^\sigma$ for the equalizer

$$\mathbb{Z}_X^\sigma := \mathrm{Eq}\left(\mathbb{Z}_X \overset{\lambda,\cdot(-1)}{\rightrightarrows} \mathbb{Z}_X\right).$$

Write disc for the cokernel

$$\mathrm{disc} := \mathrm{coKer}\left(\mathcal{O}_X^\times \xrightarrow{f \mapsto f\cdot\lambda(f)} \mathcal{O}_X^\times\right); \tag{6.25}$$

here the cokernel is taken in the category of sheaves of abelian groups on the small étale site of X. Consider the assignment

$$U_1 \colon \acute{\mathrm{E}}\mathrm{t}_Y \to \mathbb{E}_\infty\mathrm{Mon}(\mathcal{S})$$
$$(V = \mathrm{Spec}\, A \to Y) \mapsto \mathrm{Ker}\left(R\Gamma(\mathcal{O}_{X_V}(X_V))^\times \xrightarrow{f\mapsto f\cdot\lambda(f)} R\Gamma(\mathcal{O}_{X_V}(X_V))^\times\right).$$

Then U_1 is a sheaf of groups on the small étale site of Y. By Theorem 4.11, there is a natural map of sheaves $BU_1 \to \mathrm{PnPic}$ given by inclusion of the identity component.

Example 6.26. If X has the trivial involution $\lambda = \mathrm{id}$, then $\mathbb{Z}_X^\sigma$ is the trivial sheaf and disc_X is the units in $\mathcal{O}_X$ mod 2.

If π is quadratic étale, then $\pi_*\mathbb{Z}_X^\sigma$ is a $\mathbb{Z}$-torsor.

Corollary 6.27. *Let (X, λ, Y, π) be a scheme with involution X and a good quotient Y. The homotopy sheaves of* PnBr *are*

$$\pi_*\mathrm{PnBr} = \begin{cases} ? & * = 0 \\ \pi_*\mathbb{Z}_X^\sigma \times \pi_*\mathrm{disc} & * = 1 \\ U_1 & * = 2 \\ 0 & \textit{else.} \end{cases}$$

L: If we believe what's in Example 5.20, then π_0PnBr should be supported on the branch locus (terminology from [FW20]); maybe to simplify presentation we could state it as: Assume X is over a field k. Then the π_0PnBr is the pushforward of either (sheafification of constant sheaves) $\mathbb{Z}$ or $\mathbb{Z}/2$ from the branch locus (depending on char k).

L: next bit is sketchy; working towards a spectral sequence like the one in [AG14a, §7]

N: How is λ inducing a map on $\mathbb{Z}_X$?

N: Why the name disc? In any event I think as a sheaf this this vanishes because for strict henselian rings there is no contribution here.

L: inspired by 'discriminant.' To the second point: I think if $X = \mathrm{Spec}\, k \times k$ with the flip action, then the cokernel of $(u, v) \mapsto (u, v)\cdot(v, u) = (uv, uv)$ is nontrivial.

L: check later

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