

# Et cetera

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## Abstract

Dumping ground for other stuff: Notes, one-off observations, stuff that we can collectively use when preparing talks, etc.

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## 1 Talk prep

## 2 References

- [Involutions of Azumaya algebras](#) by First and Williams (2020 *Documenta*)
- [Counterexamples in involutions of Azumaya algebras](#) by First and Williams; much more readable than the 2020 *Documenta* paper

## 3 Questions and directions

**Question 3.1** (Morita theory for  $\text{Cat}_\infty^{\text{P}}$ ). Let  $R$  be a Poincaré ring. Suppose given two  $R$ -algebras (suitably interpreted so their module categories are canonically endowed with  $R$ -linear Poincaré structures—perhaps  $\mathbb{E}_\sigma$ )  $A, B$ . Can we characterize

$$\text{hom}_{\text{Cat}_\infty^{\text{P}}_R}((\text{Mod}_A^\omega, \mathfrak{P}_A), (\text{Mod}_B^\omega, \mathfrak{P}_B))$$

in terms of something bimodule-like?

**Question 3.2.** On page 2 of the *Counterexamples* paper, First and Williams write that “existence of an extraordinary involution means classification of Azumaya algebras with involution...*cannot* be reduced to questions about projective modules and hermitian forms on them.”

What if we replaced projective modules by perfect complexes?

**Question 3.3.** First–Williams show (see discussion in §4 of the *Counterexamples* paper) that coarse type classify many (most?) Azumaya algebras up to (étale-local) *isomorphism*.

What is a suitable derived version of “coarse type”?

L: I make no promises re: organization but I will do my best to keep it reasonably readable

## 4 Thoughts & observations

**Question 4.1.** When  $R$  has the Tate Poincaré structure and  $(\text{Mod}_A^\omega, M_A, N_A, N_A \rightarrow M_A^{tC_2})$  is invertible, then by invertibility have an equivalence  $\text{hom}_R(A, R) \simeq N_A \otimes_R N_{A^{\text{op}}}$  of  $A \otimes_R A^{\text{op}}$ -modules. Restricting the left-hand side along the unit map  $R \rightarrow A$  gives a map  $N_A \otimes_R N_{A^{\text{op}}} \rightarrow \text{hom}_R(R, R) \simeq R$ . Is this a perfect ( $R$ -linear) pairing?

I *think* using that  $R^{\varphi C_2} \simeq R$  and combining the linear and bilinear part conditions, we get something like

$$M_A \otimes_R M_{A^{\text{op}}} \simeq (N_A \otimes_R N_{A^{\text{op}}})^{\otimes_R 2} \quad \text{as } A \otimes_R A^{\text{op}}\text{-bimodules.}$$

Is this useful?

**Brauer-Severi schemes** We know there is a correspondence between Azumaya algebras  $A$  over  $X$  and Brauer-Severi schemes. What does a Poincaré structure on  $\text{Mod}_A^\omega$  mean ‘geometrically’ for  $D_{\text{coh}}^b$  of the corresponding Brauer-Severi scheme? (Lucy: I didn’t get very far here, but just typing up what I had)

- $\text{Mod}_A^\omega$  corresponds to  $\alpha$ -twisted sheaves on  $X$  (see Proposition 3.2.2.1 of Max Lieblich’s thesis)
- The bounded derived category of  $\alpha$ -twisted sheaves on  $X$  includes as one ‘piece’ of a semiorthogonal decomposition on  $D_{\text{coh}}^b$  of the corresponding Brauer-Severi scheme (see Theorem 5.1 [here](#))

## 5 Desperate Flailing

This section is a cronical of my thoughts about  $\mathbb{G}_m^\omega$ .

**Goal** The goal is to build a Poincaré ring  $\mathbb{G}_m^\omega := (\text{Mod}_R, \Omega_R)$  such that  $B\mathbb{G}_m^\omega(\underline{S}) = \text{Pic}^P(\underline{S})$  for any Poincaré ring  $\underline{S}$ .

**Lemma 5.1.** *Let  $\underline{S}$  be a Poincaré ring. Then  $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u)) = \{s \in \pi_0(S)^\times \mid s = 1 \text{ in } \pi_0(S^{C_2})\}$ .*

*Proof.* Since the functor  $\text{Pn}(\text{Mod}_S) \rightarrow \text{Mod}_S$  is conservative it follows that an element of  $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u))$  must have underlying map an element of  $\pi_0 \text{Aut}(S) = \pi_0(S)^\times$ . Then in order for  $s \in \pi_0(S)^\times$  to induce a map  $(S, u) \rightarrow (S, u)$ , the induced map  $s^* : S^{C_2} \rightarrow S^{C_2}$  must satisfy  $s^*(u) = u$ . The pullback is given by multiplication by  $s$ , so this requirement translates into  $s$  being the unit, as desired.  $\square$

The problem I thought existed maybe doesn’t. Here is a candidate construction:

**Construction 5.2.** Define  $R$  to be the  $\mathbb{E}_\infty$  ring given by  $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z\}} \mathbb{S}$  where the map  $\mathbb{S}\{z\} \rightarrow \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$  is induced by the map  $z \mapsto xy$ , and the map  $\mathbb{S}\{z\} \rightarrow \mathbb{S}$  is induced by  $z \mapsto 1$ . We can give  $R$  an  $\mathbb{E}_\infty$  ring structure in  $\text{Sp}^{BC_2}$  by taking the trivial action on  $\mathbb{S}\{z\}$  and  $\mathbb{S}$ , and taking the action induced by  $x \mapsto y$  and  $y \mapsto x$  on  $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ . Thus in  $\text{CAlg}(\text{Sp}^{BC_2})$  the ring  $R$  corepresents the functor  $S \mapsto \{s \in \pi_0(S)^\times \mid s\sigma(s) = 1\}$ .

Now take  $\underline{R}$  to be the Poincaré ring with underlying Borel  $C_2$  structure as described in the previous paragraph and geometric fixed points  $R^{\varphi C_2} = \mathbb{S}$  and the map  $R^{\varphi C_2} \rightarrow R^{tC_2}$  given by the unit map. Endowing  $R^{\varphi C_2}$  with the  $R$ -module structure given by  $x, y \mapsto 1$ , it remains to show that the unit map  $R^{\varphi C_2} \rightarrow R^{tC_2}$  factors the Tate valued Frobenius  $R \rightarrow R^{tC_2}$  in order to promote  $\underline{R}$  to a Poincaré ring. By construction of  $R$  it is then enough to show that on  $\pi_0$  the Tate valued Frobenius sends  $x, y \mapsto 1$  in  $\pi_0(R^{tC_2})$ . This map sends both  $x$  and  $y$  to  $xy \in \pi_0(R^{tC_2})$ . These are equal to 1 in  $\pi_0(R^{tC_2})$  since the functor  $(-)^{tC_2}$  is lax-monoidal so  $R^{tC_2}$  is a modules over  $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$  which has the image of  $xy$  equal to 1.

Now consider another Poincaré ring  $\underline{S}$ . We then have that maps  $\pi_0(\text{Maps}(\underline{R}, \underline{S}))$  is the data of a unit  $s \in \pi_0(S)^\times$ , a path  $s\sigma(s) \rightarrow 1$  in  $\Omega^\infty S$ , and paths  $x, y \rightarrow 1$  in  $\Omega^\infty S^{\varphi C_2}$ . This then agrees with  $\mathbb{G}_m^\omega$  by the following lemma.

**Lemma 5.3.** *Let  $S \in \text{CAlg}(\text{Sp}^{BC_2})$  and  $s \in \pi_0(S)^\times$ . Then  $s\sigma(s) = 1$  in  $\pi_0(S)$  if and only if  $(s \otimes s)^*$  acts by 1 on  $\pi_0(S^{hC_2}) = \pi_0(\text{Hom}_{S \otimes S}(S \otimes S, S)^{hC_2})$ .*

*Proof.* The ‘only if’ direction follows from the fact that the map  $S^{hC_2} \rightarrow S$  is an  $S$ -bimodule map. Now suppose that  $s\sigma(s) = 1$  in  $S$ . Then before taking homotopy fixed points the induced map  $s^* = id$  because  $S$  is  $\mathbb{E}_\infty$ .<sup>1</sup>  $\square$

## 6 Modules with genuine involution

**Remark 6.1** (Lucy). I’m just going to put drafts of stuff pertaining to hermitian modules here. Eventually when it gets to be more complete, I will hopefully move this entire section over to the main file.

L: or whatever we want to keep calling these

**Meta-commentary** There are (at least) three things we want to do:

- (a) Define a category of ‘bimodules with involution over algebras with anti-involution’ equipped with a forgetful functor  $\Theta: \text{BMod}_{\text{inv}}(-) \rightarrow \mathbb{E}_1 \text{Alg}(-)^{hC_2}$ .
- (b) Show that  $\Theta$  is a coCartesian fibration. For this, it suffices to show that it is a *Cartesian* fibration and that it satisfies the hypotheses of [Lur09, Corollary 5.2.2.5]
  - I used to think that we could obtain this by ‘bootstrapping’ a result from Higher Algebra, plus some facts about assembly. This doesn’t seem to be working, so I’m just going to try to do this directly (imitating certain aspects of Chapter 4 of higher algebra.)
- (c) Define a relative tensor product for hermitian bimodules
- (d) Show that the formula for the cocartesian pushforward along a map  $A \rightarrow B$  in  $\mathbb{E}_1 \text{Alg}(-)^{hC_2}$  is something like  $- \otimes_{A \otimes A^{\text{op}}} (B \otimes B^{\text{op}}) \otimes_{B \otimes B^{\text{op}}} B$ .
  - In Higher Algebra, the formula for the cocartesian pushforward is proven in [Lur17, §4.6]; in particular, this is in the section on duality. In particular, see Proposition 4.6.2.17 and the paragraph immediately preceding this.
  - I don’t know how to do this yet—while (a) and (b) are not useful if I can’t show (c), I can’t suss out the feasibility of (c) without (a) and (b) already in place.

**Definition 6.2.** Define a colored operad  $\text{Assoc}_\sigma$  as follows:

- (i) The colored operad has a single object, which we denote by  $\mathbf{a}$ .
- (ii) For every finite set  $I$ , the set of operations  $\text{Mul}_{\text{Assoc}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ , where  $\mathcal{L}I$  is the set of linear orderings on  $I$  and an element of  $\{\pm 1\}^I$  is a function  $I \rightarrow \{\pm 1\}$ .
- (iii) Suppose given a map of finite sets  $\alpha: I \rightarrow J$ , together with operations  $(\preceq_j, f_j: I_j \rightarrow \{\pm 1\}) \in \text{Mul}_{\text{Assoc}_\sigma}(\{\mathbf{a}_i\}_{\alpha(i)=j}, \mathbf{a})$  and  $(\preceq_J, g: J \rightarrow \{\pm 1\}) \in \text{Mul}_{\text{Assoc}_\sigma}(\{\mathbf{a}_j\}_{j \in J}, \mathbf{a})$ . Define a linear ordering on the set  $I$  as follows:  $i \leq i'$  if  $\alpha(i) \preceq_J \alpha(i')$  or  $\alpha(i) = \alpha(i') = j$  and  $i \preceq_j i'$  and  $g(j) = +1$  or  $\alpha(i) = \alpha(i') = j$  and  $i \succeq_j i'$  and  $g(j) = -1$ . Finally, define a function

$$\begin{aligned} I &\rightarrow \{\pm 1\} \\ i &\mapsto f_{\alpha(i)}(i) \cdot g(\alpha(i)), \end{aligned}$$

where the multiplication on  $\{\pm 1\}$  is the usual one.

**Remark 6.3.** There is a map of colored operads  $\iota: \text{Assoc} \rightarrow \text{Assoc}_\sigma$  which is the identity on objects and on operations  $\text{Mul}_{\text{Assoc}}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\text{Assoc}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{id}_{\mathcal{L}I} \times \{c_1\}$  where  $c_1$  is the constant function on  $I$  with value 1.

There is another map of colored operads  $\iota^{\text{rev}}: \text{Assoc} \rightarrow \text{Assoc}_\sigma$  which is the identity on objects and on operations  $\text{Mul}_{\text{Assoc}}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\text{Assoc}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  sends a linear ordering  $\ell$  to  $(\ell^{\text{rev}}, c_{-1})$  where  $c_{-1}$  is the constant function on  $I$  with value 1.

<sup>1</sup>Or just  $\mathbb{E}_2$ .

L: This is just an imitation of [Lur17, Definition 4.1.1.1], modified in accordance with ideas from §5.4.2.

**Definition 6.4.** Let  $\text{Assoc}_\sigma^\otimes$  denote the associated  $\infty$ -operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

**Remark 6.5.** Unwinding definitions

- Objects  $\text{Assoc}_\sigma^\otimes$  are finite pointed sets  $\langle n \rangle \in \text{Fin}_*$
- Morphisms  $\langle m \rangle \rightarrow \langle n \rangle$  consist of
  - $\alpha: \langle m \rangle \rightarrow \langle n \rangle$  a map of finite pointed sets
  - for each  $i \in \langle n \rangle^\circ$ , a linear ordering  $\preceq_i$  on the inverse image  $\alpha^{-1}(\{i\})$
  - a map of sets  $s: \alpha^{-1}(\langle m \rangle^\circ) \rightarrow \{\pm 1\}$
- For each pair of morphisms

$$(\beta: \langle \ell \rangle \rightarrow \langle m \rangle, \preceq_j, s) \quad (\alpha: \langle m \rangle \rightarrow \langle n \rangle, \preceq_i, t),$$

the composite is the triple  $(\alpha \circ \beta, \preceq_j'', u)$  where  $\preceq_j''$  is the ordering on  $(\alpha \circ \beta)^{-1}(\{i\})$  so that if  $a, b \in \langle \ell \rangle$  so that  $\alpha(\beta(a)) = \alpha(\beta(b))$ , then  $a \preceq_j'' b$  if  $\beta(a) \preceq_i \beta(b)$  or  $\beta(a) =_i \beta(b) = i$  and  $a \preceq_i b$  if  $s(i) = 1$  or  $a \succeq_i b$  if  $s(i) = -1$ . Finally  $u(l) = s(l) \cdot t(\beta(l))$ .

**Definition 6.6.** Define a category  $\Delta_\sigma$

- objects are pairs  $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$
- a morphism from  $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$  to  $([m], t: \{0, 1, \dots, m\} \rightarrow \{\pm 1\})$  is an order-preserving map  $[n] \rightarrow [m]$  in  $\Delta$ .

**Construction 6.7.** Define a functor  $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$ :

- For each  $([n], s)$ , we have  $\text{Cut}([n], s) = \langle n \rangle$ .
- Given a morphism  $\alpha: ([n], s) \rightarrow ([m], t)$ , the associated morphism  $\text{Cut}([n], s) \rightarrow \text{Cut}([m], t)$  consists of
  - On underlying finite pointed sets  $\langle m \rangle \rightarrow \langle n \rangle$ , Cut agrees with that appearing in [Lur17, Construction 4.1.2.9]
  - Identifying the cut  $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$  with the morphism  $j - 1 < j$ , we may regard  $s: \langle n \rangle^\circ \rightarrow \{\pm 1\}$  and likewise  $t: \langle m \rangle^\circ \rightarrow \{\pm 1\}$ . Define  $u: \text{Cut}(\alpha)^{-1}(\langle n \rangle^\circ) \rightarrow \{\pm 1\}$  to be the unique function so that  $u(j)t(j) = s(\text{Cut}(\alpha)(j))$ .

**Lemma 6.8.** The functor  $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$  exhibits  $\Delta_\sigma^{\text{op}}$  as an approximation to the  $\infty$ -operad  $\text{Assoc}_\sigma^\otimes$ .

*L: I think the proof of this lemma is not too different from the proof of Proposition 4.1.2.11 of [Lur17]; the point here is just to unravel the definitions of locally coCartesian and Cartesian; the morphisms in  $\Delta_\sigma^{\text{op}}$  are a little more complicated than  $\Delta^{\text{op}}$ , but not by much.*

**Notation 6.9.** Let  $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  exhibit  $\mathcal{C}$  as  $\mathbb{E}_\sigma$ -monoidal. Let  $\mathcal{C}^\otimes$  denote the fiber product  $\mathcal{C}^\otimes \times_{\text{Assoc}_\sigma^\otimes} \Delta_\sigma^{\text{op}}$ .

**Definition 6.10.** Define a colored operad  $\mathbf{BM}_{\text{inv}}$

- The set of objects of  $\mathbf{BM}_{\text{inv}}$  has two elements, which we denote by  $\mathbf{a}, \mathbf{m}$ .
- Let  $\{X_i\}_{i \in I}$  be a finite collection of objects of  $\mathbf{BM}_{\text{inv}}$  and let  $Y$  be another object of  $\mathbf{BM}_{\text{inv}}$ . If  $Y = \mathbf{a}$ , then  $\text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is the set of pairs consisting of a linear ordering on  $I$  and a function  $I \rightarrow \{\pm 1\}$  if  $X_i = \mathbf{a}$  for all  $i$ , and empty otherwise. If  $Y = \mathbf{m}$ , then  $\text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is the set of pairs consisting of a linear ordering  $\{i_1 < i_2 < \dots < i_n\}$  on  $I$  and a function  $I \rightarrow \{\pm 1\}$  IF  $X_{i_1} = \mathbf{m}$  and  $X_j = \mathbf{a}$  for all  $j \neq i_1$ , and  $\text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is empty otherwise.
- The composition law on  $\mathbf{BM}_{\text{inv}}$  is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.2

L: Note that when  $s, t$  are identically one, the resulting order  $\preceq_j''$  agrees with the lexicographic order defined in [Lur17, Remark 4.1.1.4].

L: maybe better to write  $s$  as a function defined on the set of morphisms  $i < i + 1$  in  $[n]$ .

L: weakly enriched?? maybe need to add a new defn..

**Remark 6.11.** Restricting to the object  $\mathbf{a} \in \mathbf{BM}_{\text{inv}}$ , we see that  $\mathbf{BM}_{\text{inv}}$  has a sub-colored operad which is canonically identified with  $\mathbf{Assoc}_{\text{inv}}$  of Definition 6.2.

**Definition 6.12.** Let  $\mathcal{BM}_{\text{inv}}^{\otimes}$  denote the associated  $\infty$ -operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

**Remark 6.13.** We can describe the category  $\mathcal{BM}_{\text{inv}}^{\otimes}$  as follows:

- (1) An object of  $\mathcal{BM}_{\text{inv}}^{\otimes}$  is a pair  $(\langle n \rangle, S)$  where  $S$  is a subset of  $\langle n \rangle^{\circ}$ .
- (2) Morphisms  $(\langle m \rangle, T) \rightarrow (\langle n \rangle, S)$  consist of a map  $\alpha: \langle m \rangle \rightarrow \langle n \rangle$  in  $\mathbf{Assoc}_{\sigma}^{\otimes}$  satisfying:
  - The map  $\alpha$  takes  $T \cup \{*\}$  to  $S \cup \{*\}$
  - For each  $s \in S$ , then  $\alpha^{-1}(\{s\})$  contains exactly one element of  $t$ , and that element is minimal with respect to the linear ordering on  $\alpha^{-1}(\{s\})$ .

L: I've changed things a little so that  $S$  (in the notation of Higher Algebra) has been replaced by  $S^c$ —this way, we can regard  $[n]$  as representing the ordered set  $\{-n < -n+1 < \dots < -1 < 0 < 1 < \dots < n-1 < n\}$  where  $C_2$  acts by  $\cdot(-1)$  (or something along these lines). This is really a generalization of Notation 4.2.1.7 but for  $RM^{\otimes}$ .

**Remark 6.14.** Each morphism  $\varphi \in \mathbf{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  determines a linear ordering  $\ell$  on the set  $I$  and a function  $s: I \rightarrow \{\pm 1\}$ . Passing from  $\varphi$  to the pair  $(\ell, s)$  determines a map of colored operads  $j: \mathbf{BM}_{\text{inv}} \rightarrow \mathbf{Assoc}_{\text{inv}}$ . For any monoidal  $\infty$ -category  $\mathcal{C}$ , restriction along  $j$  sends an  $\mathbb{E}_{\sigma}$ -algebra  $A: \mathbf{Assoc}_{\text{inv}} \rightarrow \mathcal{C}^{\otimes}$  to the pair  $(A, A)$  where  $A$  is regarded as an involutive bimodule over itself.

**Construction 6.15.** Define a functor  $\text{MCut}: \Delta_{\sigma}^{\text{op}} \rightarrow \mathcal{BM}_{\text{inv}}^{\otimes}$ :

- For each  $([n], s)$ , we have  $\text{MCut}([n], s) = \langle n+1 \rangle \simeq \text{RCut}_0([n])$  where  $\text{RCut}$  is from [Lur17, Construction 4.8.4.4].
- Given a morphism  $\alpha: ([n], s) \rightarrow ([m], t)$ , the associated morphism  $\text{MCut}([m], t) \rightarrow \text{MCut}([n], s)$  consists of
  - On underlying finite pointed sets  $\langle m+1 \rangle \rightarrow \langle n+1 \rangle$ ,  $\text{MCut}$  agrees with (the reverse of) that appearing in [Lur17, Construction 4.2.2.6]
  - Identifying the cut  $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$  with the morphism  $j-1 < j$ , we may regard  $s: \langle n+1 \rangle^{\circ} \rightarrow \{\pm 1\}$  and likewise  $t: \langle m+1 \rangle^{\circ} \rightarrow \{\pm 1\}$ . Define  $u: \text{MCut}(\alpha)^{-1}(\langle n+1 \rangle^{\circ}) \rightarrow \{\pm 1\}$  to be the unique function so that  $u(j)t(j) = s(\text{MCut}(\alpha)(j))$ .

**Remark 6.16.** We can identify  $\mathbf{Assoc}_{\sigma}^{\otimes}$  with the full subcategory of  $\mathcal{BM}_{\text{inv}}^{\otimes}$  spanned by objects of the form  $(\langle n \rangle, \langle n \rangle^{\circ})$ . We can regard Construction 6.7 as defining a functor  $\Delta_{\sigma}^{\text{op}} \rightarrow \mathcal{BM}_{\text{inv}}^{\otimes}$ . For each  $([n], s) \in \Delta_{\sigma}^{\text{op}}$ , there is a map of sets  $\theta: \text{MCut}([n], s) \rightarrow \text{Cut}([n], s)$  defined as in [Lur17, Remark 4.2.2.8]. Concretely, on underlying pointed sets,  $\theta$  takes the form

$$\theta: \langle n+1 \rangle \rightarrow \langle n \rangle$$

$$k \mapsto \begin{cases} k-1 & \text{if } k > 0 \\ * & \text{if } k = 0, * \end{cases}$$

This construction determines a morphism  $\gamma$  in the  $\infty$ -category  $\mathbf{Fun}(\Delta_{\sigma}^{\text{op}}, \mathcal{BM}_{\text{inv}}^{\otimes})$ , or equivalently a map  $\Delta_{\sigma}^{\text{op}} \times \Delta^1 \rightarrow \mathcal{BM}_{\text{inv}}^{\otimes}$ .

**Lemma 6.17.** The morphism  $\gamma: \Delta_{\sigma}^{\text{op}} \times \Delta^1 \rightarrow \mathcal{BM}_{\text{inv}}^{\otimes}$  defined in Remark 6.16 exhibits  $\Delta_{\sigma}^{\text{op}} \times \Delta^1$  as an approximation to the  $\infty$ -operad  $\mathcal{BM}_{\text{inv}}^{\otimes}$ .

L: compare Higher Algebra Notation 4.2.1.6

L: more general?

L: hermitian

L: maybe this over-loaded notation is not good. I'm running out of ideas.

L: check later

L: check that the signs  $s$  work out!

**Definition 6.18.** Let  $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  be a fibration of  $\infty$ -operads and let  $\mathcal{M}$  be an  $\infty$ -category. Suppose given a fibration of  $\infty$ -operads  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  together with equivalences  $\mathcal{O}_a^\otimes \simeq \mathcal{C}^\otimes$  and  $\mathcal{O}_m^\otimes \simeq \mathcal{M}$ . Let  ${}^\sigma\text{Mod}(\mathcal{M})$  denote the  $\infty$ -category  $\text{Alg}_{/\mathcal{BM}}(\mathcal{O})$ . We will refer to  ${}^\sigma\text{Mod}(\mathcal{M})$  as the  *$\infty$ -category of hermitian module objects of  $\mathcal{M}$* . Composition with the inclusion  $\text{Assoc}_\sigma^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  induces a categorical fibration

$${}^\sigma\text{Mod}(\mathcal{M}) = \text{Alg}_{/\mathcal{BM}}(\mathcal{O}) \rightarrow \text{Alg}_{\text{Assoc}_\sigma}(\mathcal{C}).$$

If  $A$  is an  $\text{Assoc}_\sigma$ -algebra object of  $\mathcal{C}$ , we let  ${}^\sigma\text{Mod}_A(\mathcal{M})$  denote the fiber  ${}^\sigma\text{Mod}(\mathcal{M}) \times_{\text{Alg}_{\text{Assoc}_\sigma}(\mathcal{C})} \{A\}$ . We will refer to  ${}^\sigma\text{Mod}_A(\mathcal{M})$  as the  *$\infty$ -category of hermitian  $A$ -module objects of  $\mathcal{M}$* .

**Example 6.19.** Let  $\mathcal{C}^\otimes \rightarrow \mathcal{BM}^\otimes$  be a fibration of  $\infty$ -operads. Restriction along the map of  $\infty$ -operads  $\mathcal{BM}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  induced by Remark 6.14 induces a map  $\mathbb{E}_\sigma \text{Alg}(\mathcal{C}) \rightarrow {}^\sigma\text{Mod}(\mathcal{C})$  which is a section of the projection map  ${}^\sigma\text{Mod}(\mathcal{C}) \rightarrow \mathbb{E}_\sigma \text{Alg}(\mathcal{C})$ .

**Notation 6.20.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as weakly bi-enriched over  $\mathcal{O}_a^\otimes$ . Define a new simplicial set  $\overline{\mathcal{M}}^\otimes$  by the following universal property

$$\text{hom}_{\text{sSet}/\Delta_\sigma^{\text{op}}}(K, \overline{\mathcal{M}}^\otimes) \simeq \text{hom}_{\text{sSet}/\mathcal{BM}_{\text{inv}}^\otimes}(K \times \Delta^1, \mathcal{O}^\otimes).$$

Here we regard  $K \times \Delta^1$  as a simplicial set over  $\mathcal{BM}_{\text{inv}}^\otimes$  via the composite  $K \times \Delta^1 \rightarrow \Delta_\sigma^{\text{op}} \times \Delta^1 \xrightarrow{\gamma} \mathcal{BM}_{\text{inv}}^\otimes$  where  $\gamma$  is from Remark 6.16.

Unwinding definitions, we see that a vertex in  $\overline{\mathcal{M}}^\otimes$  lying over an object  $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\}) \in \Delta_\sigma^{\text{op}}$  corresponds to a morphism  $\alpha$  in  $\mathcal{O}^\otimes$  whose image in  $\mathcal{BM}_{\text{inv}}^\otimes$  is the map  $(\langle n+1 \rangle, \{0\}) \rightarrow (\langle n \rangle, \emptyset)$ . Now let  $\mathcal{M}^\otimes$  denote the full simplicial subset of  $\overline{\mathcal{M}}^\otimes$  spanned by those vertices for which  $\alpha$  is inert.

**Lemma 6.21.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as weakly bi-enriched over  $\mathcal{O}_a^\otimes$ . Then the associated functor  $\mathcal{M}^\otimes \rightarrow \mathcal{C}^\otimes$  (Notation 6.9) is a locally coCartesian fibration.

## References

- [Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: [10.1515/9781400830558](https://doi.org/10.1515/9781400830558). URL: <https://doi.org/10.1515/9781400830558>.
- [Lur17] Jacob Lurie. *Higher algebra*. Manuscript available at <https://www.math.ias.edu/~lurie/papers/HA.pdf>. 2017.

L: Lurie gives this a name (Definition 4.2.1.12 weakly enriched)—not sure what to call this  
L: see Example 4.2.1.17 of higher algebra  
L: fibration?

L: this might be off-revisit later!