Et cetera

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Abstract

Dumping ground for other stuff: Notes, one-off observations, stuff that we can collectively use when preparing talks, etc.

L: I make no promises re: organization but I will do my best to keep it reasonably readable

Contents

1 Talk prep

2 References

- Involutions of Azumaya algebras by First and Williams (2020 *Documenta*)
- Counterexamples in involutions of Azumaya algebras by First and Williams; much more readable than the 2020 Documenta paper

3 Questions and directions

Question 3.1 (Morita theory for $\mathrm{Cat}^{\mathrm{p}}_{\infty}$)**.** Let R be a Poincaré ring. Suppose given two R-algebras (suitably interpreted so their module categories are canonically endowed with R-linear Poincaré structures–perhaps $\mathbb{E}_\sigma$) A, B. Can we characterize

$$\hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty R}}\left((\mathrm{Mod}^{\omega}_{A}, \Qoppa_A), (\mathrm{Mod}^{\omega}_{B}, \Qoppa_B)\right)$$

in terms of something bimodule-like?

Question 3.2. On page 2 of the *Counterexamples* paper, First and Williams write that " existence of an extraordinary involution means classificaiton of Azumaya algebras with involution... *cannot* be reduced to questions about projective modules and hermitian forms on them."

What if we replaced projective modules by perfect complexes?

Question 3.3. First–Williams show (see discussion in §4 of the *Counterexamples* paper) that coarse type classify many (most?) Azumaya algebras up to (étale-local) *isomorphism*.

What is a suitable derived version of "coarse type"?

Question 3.4 (asked by Andrew Nov 2, 2024)**.** C. Schlichtkrull shows in this paper that a map $BGL_1(R) \to K(R) \to THH(R) \to R$ in terms of the Hopf map η.

Is there a "Poincaré" version of this result?

4 Thoughts & observations

Question 4.1. When R has the Tate Poincaré structure and $(\mathrm{Mod}_A^\omega, M_A, N_A, N_A \to M_A^{tC_2})$ is invertible, then by invertibility have an equivalence $\hom_R(A, R) \simeq N_A \otimes_R N_{A^{\mathrm{op}}}$ of $A \otimes_R A^{\mathrm{op}}$-modules. Restricting the left-hand side along the unit map $R \to A$ gives a map $N_A \otimes_R N_{A^{\mathrm{op}}} \to \hom_R(R, R) \simeq R$. Is this a perfect ($R$-linear) pairing?

I *think* using that $R^{\varphi C_2} \simeq R$ and combining the linear and bilinear part conditions, we get something like

$$M_A \otimes_R M_{A^{\mathrm{op}}} \simeq (N_A \otimes_R N_{A^{\mathrm{op}}})^{\otimes_R 2} \qquad \text{as } A \otimes_R A^{\mathrm{op}}\text{-bimodules.}$$

Is this useful?

Brauer-Severi schemes We know there is a correspondence between Azumaya algebras A over X and Brauer-Severi schemes. What does a Poincaré structure on Mod_A^ω mean 'geometrically' for D^b_{coh} of the corresponding Brauer-Severi scheme? (Lucy: I didn't get very far here, but just typing up what I had)

- Mod_A^ω corresponds to α-twisted sheaves on X (see Proposition 3.2.2.1 of Max Lieblich's thesis)
- The bounded derived category of α-twisted sheaves on X includes as one 'piece' of a semiorthogonal decomposition on D^b_{coh} of the corresponding Brauer-Severi scheme (see Theorem 5.1 here)

5 Desperate Flailing

This section is a cronical of my thoughts about $\mathbb{G}_m^{\text{♀}}$.

Goal The goal is to build a Poincaré ring $\mathbb{G}_m^{\text{♀}} := (\mathrm{Mod}_R, \text{♀}_R)$ such that $B\mathbb{G}_m^{\text{♀}}(\underline{S}) = \mathrm{Pic}^{\mathrm{p}}(\underline{S})$ for any Poincaré ring $\underline{S}$.

Lemma 5.1. *Let $\underline{S}$ be a Poincaré ring. Then* $\pi_0(\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}_S)}(S, u)) = \{s \in \pi_0(S)^\times | s = 1 \text{ in } \pi_0(S^{C_2})\}$.

Proof. Since the functor $\mathrm{Pn}(\mathrm{Mod}_S) \to \mathrm{Mod}_S$ is conservative it follows that an element of $\pi_0(\mathrm{Aut}_{\mathrm{Pn}(\mathrm{Mod}_S)}(S, u))$ must have underlying map an element of $\pi_0\mathrm{Aut}(S) = \pi_0(S)^\times$. Then in order for $s \in \pi_0(S)^\times$ to induce a map $(S, u) \to (S, u)$, the induced map $s^* : S^{C_2} \to S^{C_2}$ must satisfy $s^*(u) = u$. The pullback is given by multiplication by s, so this requirement translates into s being the unit, as desired. ☐

The problem I thought existed maybe doesn't. Here is a candidate construction:

Construction 5.2. Define R to be the $\mathbb{E}_\infty$ ring given by $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z\}} \mathbb{S}$ where the map $\mathbb{S}\{z\} \to \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by the map $z \mapsto xy$, and the map $\mathbb{S}\{z\} \to \mathbb{S}$ is induced by $z \mapsto 1$. We can give R an $\mathbb{E}_\infty$ ring structure in Sp^{BC_2} by taking the trivial action on $\mathbb{S}\{z\}$ and $\mathbb{S}$, and taking the action induced by $x \mapsto y$ and $y \mapsto x$ on $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$. Thus in $\mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ the ring R corepresents the functor $S \mapsto \{s \in \pi_0(S)^\times | s\sigma(s) = 1\}$.

Now take $\underline{R}$ to be the Poincaré ring with underlying Borel C_2 structure as described in the previous paragraph and geometric fixed points $R^{\varphi C_2} = \mathbb{S}$ and the map $R^{\varphi C_2} \to R^{tC_2}$ given by the unit map. Endowing $R^{\varphi C_2}$ with the R-module structre given by $x, y \mapsto 1$, it remains to show that the unit map $R^{\varphi C_2} \to R^{tC_2}$ factors the Tate valued Frobenius $R \to R^{tC_2}$ in order to promote $\underline{R}$ to a Poincaré ring. By construction of R it is then enough to show that on π_0 the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0(R^{tC_2})$. This map sends both x and y to $xy \in \pi_0(R^{tC_2})$. These areequal to 1 in $\pi_0(R^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so R^{tC_2} is a modules over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of xy equal to 1.

Now consider another Poincaré ring $\underline{S}$. We then have that maps $\pi_0(\mathrm{Maps}(\underline{R}, \underline{S}))$ is the data of a unit $s \in \pi_0(S)^\times$, a path $s\sigma(s) \to 1$ in $\Omega^\infty S$, and paths $x, y \to 1$ in $\Omega^\infty S^{\varphi C_2}$. This then agrees with $\mathbb{G}_m^{♀}$ by the following lemma.

Lemma 5.3. *Let $S \in \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ and $s \in \pi_0(S)^\times$. Then $s\sigma(s) = 1$ in $\pi_0(S)$ if and only if $(s \otimes s)^*$ acts by 1 on $\pi_0(S^{hC_2}) = \pi_0(\mathrm{Hom}_{S \otimes S}(S \otimes S, S)^{hC_2})$.*

Proof. The 'only if' direction follows from the fact that the map $S^{hC_2} \to S$ is an S-bimodule map. Now suppose that $s\sigma(s) = 1$ in S. Then before taking homotopy fixed points the induced map $s^* = id$ because S is $\mathbb{E}_\infty$.[1] □

6 Modules with genuine involution

Remark 6.1 (Lucy)**.** I'm just going to put drafts of stuff pertaining to hermitian modules here. Eventually when it gets to be more complete, I will hopefully move this entire section over to the main file.

L: or whatever we want to keep calling these

Meta-commentary There are (at least) three things we want to do:

(a) Define a category of 'bimodules with involution over algebras with anti-involution' equipped with a forgetful functor $\Theta \colon \mathrm{BMod}_{\mathrm{inv}}(-) \to \mathbb{E}_1 \mathrm{Alg}(-)^{hC_2}$.

(b) Show that Θ is a coCartesian fibration. For this, it suffices to show that it is a *Cartesian* fibration and that it satisfies the hypotheses of [Lur09, Corollary 5.2.2.5]

- I used to think that we could obtain this by 'bootstrapping' a result from Higher Algebra, plus some facts about assembly. This doesn't seem to be working, so I'm just going to try to do this directly (imitating certain aspects of Chapter 4 of higher algebra.)

(c) Define a relative tensor product for hermitian bimodules

(d) Show that the formula for the cocartesian pushforward along a map $A \to B$ in $\mathbb{E}_1 \mathrm{Alg}(-)^{hC_2}$ is something like $- \otimes_{A \otimes A^{\mathrm{op}}} (B \otimes B^{\mathrm{op}}) \otimes_{B \otimes B^{\mathrm{op}}} B$.

- In Higher Algebra, the formula for the cocartesian pushforward is proven in [Lur17, §4.6]; in particular, this is in the section on duality. In particular, see Proposition 4.6.2.17 and the paragraph immediately preceding this.
- I don't know how to do this yet–while (a) and (b) are not useful if I can't show (c), I can't suss out the feasibility of (c) without (a) and (b) already in place.

(e) Towards an adjunction between $\mathbb{E}_\sigma$-algebras and categories with additional structure.

- Involutive version of statement that, for a monoidal ∞-category $\mathcal{C}$ and an $\mathbb{E}_1$-algebra A, $\mathrm{LMod}_A(\mathcal{C})$ is right-tensored over $\mathcal{C}$?
- Involutive version of endomorphism categories? [Lur17, §4.7.1]

6.1 Step (a)

Definition 6.2. Define a colored operad Assoc_σ as follows:

L: This is just an imitation of [Lur17, Definition 4.1.1.1], modified in accordance with ideas from §5.4.2.

(i) The colored operad has a single object, which we denote by $\mathfrak{a}$.

(ii) For every finite set I, the set of operations $\mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$, where $\mathcal{L}I$ is the set of linear orderings on I and an element of $\{\pm 1\}^I$ is a function $I \to \{\pm 1\}$.

(iii) Suppose given a map of finite sets $\alpha\colon I \to J$, together with operations $(\preceq_j, f_j\colon I_j \to \{\pm 1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{\alpha(i)=j}, \mathfrak{a}\right)$ and $(\preceq_J, g\colon J \to \{\pm 1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_j\}_{j\in J}, \mathfrak{a}\right)$. Define a linear ordering on the set I as follows: $i \le i'$ if $\alpha(i) \preceq_J \alpha(i')$ or $\alpha(i) = \alpha(i') = j$ and $i \preceq_j i'$ and $g(j) = +1$ or $\alpha(i) = \alpha(i') = j$ and $i \succeq_j i'$ and $g(j) = -1$. Finally, define a function

$$\begin{aligned} I &\to \{\pm 1\} \\ i &\mapsto f_{\alpha(i)}(i) \cdot g(\alpha(i)), \end{aligned}$$

where the multiplication on $\{\pm 1\}$ is the usual one.

Remark 6.3. There is a map of colored operads $\iota\colon \mathrm{Assoc} \to \mathrm{Assoc}_\sigma$ which is the identity on objects and on operations $\mathrm{Mul}_{\mathrm{Assoc}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$ where c_1 is the constant function on I with value 1.

There is another map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{Assoc} \to \mathrm{Assoc}_\sigma$ which is the identity on objects and on operations $\mathrm{Mul}_{\mathrm{Assoc}}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{Assoc}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ sends a linear ordering ℓ to $(\ell^{\mathrm{rev}}, c_{-1})$ where c_{-1} is the constant function on I with value 1.

Definition 6.4. Let $\mathrm{Assoc}_\sigma^\otimes$ denote the associated ∞-operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.5. Unwinding definitions

- Objects $\mathrm{Assoc}_\sigma^\otimes$ are finite pointed sets $\langle n\rangle \in \mathrm{Fin}_*$
- Morphisms $\langle m\rangle \to \langle n\rangle$ consist of
  - $\alpha\colon \langle m\rangle \to \langle n\rangle$ a map of finite pointed sets
  - for each $i \in \langle n\rangle^\circ$, a linear ordering $\preceq_i$ on the inverse image $\alpha^{-1}(\{i\})$
  - a map of sets $s\colon \alpha^{-1}(\langle m\rangle^\circ) \to \{\pm 1\}$
- For each pair of morphisms

$$(\beta\colon \langle \ell\rangle \to \langle m\rangle, \preceq_j, s) \qquad (\alpha\colon \langle m\rangle \to \langle n\rangle, \preceq_i, t)\ ,$$

the composite is the triple $\left(\alpha\circ\beta, \preceq''_j, u\right)$ where $\preceq''_j$ is the ordering on $(\alpha\circ\beta)^{-1}(\{i\})$ so that if $a, b \in \langle \ell\rangle$ so that $\alpha(\beta(a)) = \alpha(\beta(b))$, then $a \preceq''_j b$ if $\beta(a) \preceq_i \beta(b)$ or $\beta(a) =_i \beta(b) = i$ and $a \preceq_i b$ if $s(i) = 1$ or $a \succeq_i b$ if $s(i) = -1$. Finally $u(l) = s(l)\cdot t(\beta(l))$.

L: Note that when s, t are identically one, the resulting order $\preceq''_j$ agrees with the lexicographic order defined in [Lur17, Remark 4.1.1.4].

Remark 6.6. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.3 induce maps of ∞-operads $\mathrm{Assoc}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$. There is a canonical identification $\iota^{\mathrm{rev}} = \sigma\circ\iota$, where σ is the automorphism of the associative operad considered in [Lur17, Remark 4.1.1.7].

Note that each object $\langle n\rangle \in \mathrm{Assoc}_\sigma^\otimes$ has a distinguished automorphism $\mathrm{rev}_{\langle n\rangle}$ of order two given by the identity map on $\langle n\rangle$ and the constant map $c_{-1}\colon \langle n\rangle^\circ \to \{\pm 1\}$ at -1. There is a canonical natural equivalence $\iota \overset{\sim}{\Longrightarrow} \iota^{\mathrm{rev}}$ whose component at $\langle n\rangle$ is $\mathrm{rev}_{\langle n\rangle}$.

Definition 6.7. Let $\mathcal{C}^\otimes$ be a ∞-operad equipped with the data of a fibration $p\colon \mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$. Let $\mathrm{Alg}^\sigma(\mathcal{C})$ denote the ∞-category $\mathrm{Alg}_{/\mathrm{Assoc}_\sigma}(\mathcal{C})$ of ∞-operad sections of p. We will refer to $\mathrm{Alg}^\sigma(\mathcal{C})$ as the ∞-category of *involutive algebra objects of* $\mathcal{C}$.

An *involutive monoidal ∞-category* is the data of a cocartesian fibration $\mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$.

L: do we need weaker than cocartesian fibration?

[1] Or just $\mathbb{E}_2$.

Remark 6.8. Suppose given a cocartesian fibration $f\colon \mathcal{D}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ of ∞-operads. Write $\mathcal{C}^\otimes := \mathcal{D}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes, \iota} \mathrm{Assoc}^\otimes$; $\mathcal{C}^\otimes$ is a monoidal ∞-category in the sense of [Lur17, Definition 4.1.1.10]. Furthermore, $\mathcal{C}^\otimes_{\mathrm{rev}} := \mathcal{D}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes, \iota^{\mathrm{rev}}} \mathrm{Assoc}^\otimes$ is a monoidal ∞-category. By Remark 6.6, this notation is consistent with that of [Lur17, Remark 4.1.1.7]. In particular, a Assoc_σ-monoidal ∞-category $\mathcal{D}^\otimes$ determines a monoidal ∞-category $\mathcal{C}^\otimes$ equipped with a monoidal equivalence $\sigma_{\mathcal{C}}\colon \mathcal{C}^\otimes \xrightarrow{\sim} \mathcal{C}^\otimes_{\mathrm{rev}}$. Pullback along the involution of $\mathrm{Assoc}^\otimes$ determines another monoidal equivalence $\sigma_{\mathcal{C}}^{\mathrm{rev}}\colon \mathcal{C}^\otimes_{\mathrm{rev}} \xrightarrow{\sim} \mathcal{C}^\otimes$, and our assumptions imply that $\sigma_{\mathcal{C}}^{\mathrm{rev}} \circ \sigma_{\mathcal{C}}$ is equivalent to the identity on $\mathcal{C}^\otimes$.

Now suppose that A is an involutive algebra object of $\mathcal{D}$. With the same notation as before, pullback along ι (resp. ι^{rev}) determines associative algebra objects $u(A)$, $u^{\mathrm{rev}}(A)$ of $\mathcal{C}$ and $\mathcal{C}_{\mathrm{rev}}$, respectively. Note that $\sigma_{\mathcal{C}}(u(A))$ is an algebra object of $\mathcal{C}_{\mathrm{rev}}$, which we may regard as an algebra object of $\mathcal{C}$ by precomposing with the autoequivalence $\sigma\colon \mathrm{Assoc}^\otimes \xrightarrow{\sim} \mathrm{Assoc}^\otimes$. It follows from Remark 6.6 that A determines an equivalence $\sigma_A\colon u(A) \xrightarrow{\sim} \sigma_{\mathcal{C}}(u(A))^{\mathrm{rev}}$ of algebra objects in $\mathcal{C}$.

Now suppose furthermore that $\mathcal{D}^\otimes$ is of the form $\mathcal{E}^\otimes \times_{\mathrm{Fin}_*} \mathrm{Assoc}_\sigma^\otimes$ for some symmetric monoidal ∞-category $\mathcal{E}$. Then the associated involution $\sigma_{\mathcal{C}}$ is the identity, and for any involutive algebra object A of $\mathcal{D}$, σ_A is an equivalence $u(A) \simeq u(A)^{\mathrm{rev}}$ satisfying $\sigma_A^{\mathrm{rev}} \circ \sigma_A \simeq \mathrm{id}_A$.

Definition 6.9. Define a category Δ_σ

- objects are pairs $([n], s\colon \{1, \ldots, n\} \to \{\pm 1\})$
- a morphism from $([n], s\colon \{1, \ldots, n\} \to \{\pm 1\})$ to $([m], t\colon \{0, 1, \ldots, m\} \to \{\pm 1\})$ is an order-preserving map $[n] \to [m]$ in Δ.

L: maybe better to write s as a function defined on the set of *morphisms* $i < i+1$ in $[n]$.

Construction 6.10. Define a functor $\mathrm{Cut}\colon \Delta_\sigma^{\mathrm{op}} \to \mathrm{Assoc}_\sigma^\otimes$:

- For each $([n], s)$, we have $\mathrm{Cut}([n], s) = \langle n \rangle$.
- Given a morphism $\alpha\colon ([n], s) \to ([m], t)$, the associated morphism $\mathrm{Cut}([n], s) \to \mathrm{Cut}([m], t)$ consists of
  - On underlying finite pointed sets $\langle m \rangle \to \langle n \rangle$, Cut agrees with that appearing in [Lur17, Construction 4.1.2.9]
  - Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j - 1 < j$, we may regard $s\colon \langle n \rangle^\circ \to \{\pm 1\}$ and likewise $t\colon \langle m \rangle^\circ \to \{\pm 1\}$. Define $u\colon \mathrm{Cut}(\alpha)^{-1}(\langle n \rangle^\circ) \to \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\mathrm{Cut}(\alpha)(j))$.

Lemma 6.11. *The functor* $\mathrm{Cut}\colon \Delta_\sigma^{\mathrm{op}} \to \mathrm{Assoc}_\sigma^\otimes$ *exhibits* $\Delta_\sigma^{\mathrm{op}}$ *as an approximation to the* ∞*-operad* $\mathrm{Assoc}_\sigma^\otimes$.

L: I think the proof of this lemma is not too different from the proof of Proposition 4.1.2.11 of [Lur17]; the point here is just to unravel the definitions of locally coCartesian and Cartesian; the morphisms in $\Delta_\sigma^{\mathrm{op}}$ are a little more complicated than Δ^{op}, but not by much.

Notation 6.12. Let $\mathcal{C}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ exhibit $\mathcal{C}$ as $\mathbb{E}_\sigma$-monoidal. Let $\mathcal{C}^\circledast$ denote the fiber product $\mathcal{C}^\otimes \times_{\mathrm{Assoc}_\sigma^\otimes} \Delta_\sigma^{\mathrm{op}}$.

Definition 6.13. Say that a morphism $([n], s) \to ([m], t)$ is *inert* if the induced map $\mathrm{Cut}([m], t) \to \mathrm{Cut}([n], s)$ is an inert morphism in $\mathrm{Assoc}_\sigma^\otimes$.

Definition 6.14. A $\mathbb{R}^\sigma$*-planar operad* is an ∞-category $\mathcal{O}^\circledast$ equipped with a functor $q\colon \mathcal{O}^\circledast \to \Delta_\sigma^{\mathrm{op}}$ so that

1. For every object $X \in \mathcal{O}^\circledast$ and every inert morphism $\alpha\colon ([n], s) \to q(X)$ in Δ_σ, there is a q-cocartesian morphism $\overline{\alpha}\colon X \to Y$ satisfying $q(\overline{\alpha}) = \alpha$
2. Let X be an object satisfying $q(X) = ([n], s)$, and choose q-cocartesian morphisms $\overline{\alpha}_i\colon X \to X_i$ corresponding to the morphism $([i-1 < i], s_i) \to ([n], s)$ which is the inclusion on underlying sets and satisfies $s_i(i) = s(i)$. Then the morphisms $\overline{\alpha}_i$ exhibit X as the q-product of the X_i.
3. For each $n \geq 0$, the construction $C \mapsto \{C_i\}_{1 \leq i \leq n}$ induces an equivalence of ∞-categories

$$\mathcal{O}^\circledast \times_{\Delta_\sigma^{\mathrm{op}}} \{([n], s)\} \xrightarrow{\sim} \left(\mathcal{O}^\circledast \times_{\Delta_\sigma^{\mathrm{op}}} \{([1], s|_{\{i\}})\}\right)^{\times n}$$

We say that a morphism α in $\mathbb{R}^\sigma$-planar operad is *inert* if it is q-cocartesian and $q(\alpha)$ is inert in $\Delta^{\mathrm{op}}_\sigma$ in the sense of Definition 6.13.

Definition 6.15. Let $q\colon \mathcal{O}^{\circledast} \to \Delta^{\mathrm{op}}_\sigma$ be a $\mathbb{R}^\sigma$-planar operad. An $\mathbb{A}^\sigma_\infty$-algebra object of $\mathcal{O}^{\circledast}$ is a section of q which carries inert morphisms to inert morphisms. Write $\mathrm{Alg}_{\mathbb{A}^\sigma_\infty}(\mathcal{O})$ for the full subcategory of $\mathrm{Fun}_{\Delta^{\mathrm{op}}_\sigma}(\Delta^{\mathrm{op}}_\sigma, \mathcal{O}^{\circledast})$ on $\mathbb{A}^\sigma_\infty$-algebra objects.

Proposition 6.16. *Let $\mathcal{O}^{\otimes} \to \mathrm{Assoc}^{\otimes}_\sigma$ be a fibration of ∞-operads. Then precomposition with the functor* Cut *of Construction 6.10 induces an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathrm{Assoc}_\sigma}(\mathcal{O}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{A}^\sigma_\infty}(\mathcal{O})\ .$$

Proof. Combine Lemma 6.11 with [Lur17, Theorem 2.3.3.23]. □

Definition 6.17. Define a colored operad $\mathbf{LM}_{\mathrm{inv}}$

(i) The set of objects of $\mathbf{LM}_{\mathrm{inv}}$ has two elements, which we denote by $\mathfrak{a}, \mathfrak{m}$.

(ii) Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{LM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{LM}_{\mathrm{inv}}$. If $Y = \mathfrak{a}$, then $\mathrm{Mul}_{\mathbf{LM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \to \{\pm 1\}$ if $X_i = \mathfrak{a}$ for all i, and empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{LM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying either

- $X_{i_n} = \mathfrak{m}$ and $c(i_n) = 1$ and $X_j = \mathfrak{a}$ otherwise
- $X_{i_1} = \mathfrak{m}$ and $c(i_n) = -1$ and $X_j = \mathfrak{a}$ otherwise

(iii) The composition law on $\mathbf{LM}_{\mathrm{inv}}$ is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.2

Remark 6.18. There is a colored operad $\mathbf{RM}_{\mathrm{inv}}$ defined exactly in the same way as $\mathbf{LM}_{\mathrm{inv}}$ in Definition 6.17. In the interest of precision: $\mathbf{RM}_{\mathrm{inv}}$ has the same objects $\mathfrak{a}, \mathfrak{m}$. Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{RM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{RM}_{\mathrm{inv}}$. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{RM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is a subset of the set of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying either

- $X_{i_n} = \mathfrak{m}$ and $c(i_n) = -1$ and $X_j = \mathfrak{a}$ otherwise
- $X_{i_1} = \mathfrak{m}$ and $c(i_n) = 1$ and $X_j = \mathfrak{a}$ otherwise

Remark 6.19. Restricting to the objects which are both called $\mathfrak{a}$, we see that both $\mathbf{LM}_{\mathrm{inv}}$ and $\mathbf{RM}_{\mathrm{inv}}$ have a sub-colored operad which is canonically identified with $\mathbf{Assoc}_{\mathrm{inv}}$ of Definition 6.2.

Remark 6.20. There is a map of colored operads $\iota\colon \mathrm{LM} \to \mathrm{LM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}$ to $\mathfrak{a}$. On $\mathrm{Mul}_{\mathrm{LM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{LM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.3. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{id}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is a map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{RM} \to \mathrm{LM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}$ to $\mathfrak{a}$. On $\mathrm{Mul}_{\mathrm{RM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{LM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{rev}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.3. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{rev}_{\mathcal{L}(I\sqcup\{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

Definition 6.21. Define a colored operad $\mathbf{BM}_{\mathrm{inv}}$

(i) The set of objects of $\mathbf{BM}_{\mathrm{inv}}$ has three elements, which we denote by $\mathfrak{a}_\ell, \mathfrak{a}_r, \mathfrak{m}$.

(ii) Let $\{X_i\}_{i\in I}$ be a finite collection of objects of $\mathbf{BM}_{\mathrm{inv}}$ and let Y be another object of $\mathbf{BM}_{\mathrm{inv}}$. If $Y = \mathfrak{a}_\ell$ (resp. $Y = \mathfrak{a}_r$), then $\mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \to \{\pm 1\}$ if $X_i = \mathfrak{a}_\ell$ (resp. $X_i = \mathfrak{a}_r$) for all i, and empty otherwise. If $Y = \mathfrak{m}$, then $\mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ is the subset of pairs (λ, c) consisting of a linear ordering $\lambda = \{i_1 < i_2 < \cdots < i_n\}$ on I and a function $c\colon I \to \{\pm 1\}$ satisfying: if there is exactly one index i_k so that $X_{i_k} = \mathfrak{m}$, either

- $c(i_k) = 1$, $X_j = \mathfrak{a}_\ell$ for $j < i_k$ and $X_j = \mathfrak{a}_r$ for $j > i_k$; or
- $c(i_k) = -1$, $X_j = \mathfrak{a}_\ell$ for $j > i_k$ and $X_j = \mathfrak{a}_r$ for $j < i_k$

(iii) The composition law on $\mathbf{BM}_{\mathrm{inv}}$ is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.2

Remark 6.22. The colored operad $\mathbf{BM}_{\mathrm{inv}}$ has a canonical involution σ which fixes $\mathfrak{m}$, exchanges $\mathfrak{a}_\ell$ and $\mathfrak{a}_r$, and sends a morphism (λ, c) to $(\lambda^{\mathrm{rev}}, I \xrightarrow{c} \{\pm 1\} \xrightarrow{\cdot(-1)} \{\pm 1\})$.

Remark 6.23. There is a map of colored operads $\iota\colon \mathrm{BM} \to \mathrm{BM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends $\mathfrak{a}_-$ to $\mathfrak{a}_\ell$ and $\mathfrak{a}_+$ to $\mathfrak{a}_r$. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}_\pm\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι of Remark 6.3. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{id}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$ where c_1 is the constant function on $I \sqcup \{j\}$ with value 1.

There is *also* a map of colored operads $\iota^{\mathrm{rev}}\colon \mathrm{BM} \to \mathrm{BM}_\sigma$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and and sends $\mathfrak{a}_-$ to $\mathfrak{a}_r$ and $\mathfrak{a}_+$ to $\mathfrak{a}_\ell$. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I}, \mathfrak{a}_\pm\right) \simeq \mathcal{L}I \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I}, \mathfrak{a}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$, this map agrees with ι^{rev} of Remark 6.3. On $\mathrm{Mul}_{\mathrm{BM}}\left(\{(\mathfrak{a}_\pm)_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \subseteq \mathcal{L}(I \sqcup \{j\}) \to \mathrm{Mul}_{\mathrm{BM}_\sigma}\left(\{\mathfrak{a}_i\}_{i\in I} \sqcup \{\mathfrak{m}\}, \mathfrak{m}\right) \simeq \mathcal{L}I \times \{\pm 1\}^I$ is the restriction of the map $\mathrm{rev}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_{-1}\}$ where c_{-1} is the constant function on $I \sqcup \{j\}$ with value -1.

Definition 6.24. Let $\mathcal{LM}^\otimes_{\mathrm{inv}}$, $\mathcal{RM}^\otimes_{\mathrm{inv}}$, and $\mathcal{BM}^\otimes_{\mathrm{inv}}$ denote the associated ∞-operads (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.25. We can describe the category $\mathcal{LM}^\otimes_{\mathrm{inv}}$ as follows:

(1) An object of $\mathcal{LM}^\otimes_{\mathrm{inv}}$ is a pair $(\langle n\rangle, S)$ where S is a subset of $\langle n\rangle^\circ$.

(2) Morphisms $(\langle m\rangle, T) \to (\langle n\rangle, S)$ consist of a map $(\alpha\colon \langle m\rangle \to \langle n\rangle, \lambda\colon \langle m\rangle^\circ \to \{\pm 1\})$ in $\mathrm{Assoc}^\otimes_\sigma$ satisfying:

- The map α takes $T \cup \{*\}$ to $S \cup \{*\}$
- For each $s \in S$, then $\alpha^{-1}(\{s\})$ contains exactly one element t_s of T, and it is maximal (resp. minimal) with respect to the linear ordering on $\alpha^{-1}(\{s\})$ if $\lambda(t_s) = 1$ (resp. $\lambda(t_s) = -1$).

Remark 6.26. We can describe the category $\mathcal{BM}^\otimes_{\mathrm{inv}}$ as follows:

(1) An object of $\mathcal{BM}^\otimes_{\mathrm{inv}}$ is a triple $(\langle n\rangle, c_+, c_-)$ where $c_\pm$ are functions $\langle n\rangle^\circ \to \{0,1\}$ and $c_-(i) \le c_+(i)$ for all $i \in \langle n\rangle^\circ$.

(2) Morphisms $(\langle m\rangle, c_+, c_-) \to (\langle n\rangle, c'_+, c'_-)$ consist of a map $(\alpha\colon \langle m\rangle \to \langle n\rangle, \lambda\colon \langle m\rangle^\circ \to \{\pm 1\})$ in $\mathrm{Assoc}^\otimes_\sigma$ satisfying: if $j \in \langle n\rangle^\circ$ and $\alpha^{-1}(j) = \{i_1 < i_2 < \cdots < i_\ell\}$,

- If $c_-(j) = c_+(j)$, then
$$c'_-(j) = c_-(i_1) \le c_+(i_1) = c_-(i_2) \le c_+(i_2) \cdot \cdots \cdot c_-(i_{m-1}) \le c_+(i_m) = c'_+(j)$$
- If $c_-(j) < c_+(j)$, then there exists a unique k so that $c_-(i_k) < c_+(i_k)$ and
$$\begin{aligned}\lambda(i_k)\cdot c'_-(j) = \lambda(i_k)\cdot c_-(i_1) \le \lambda(i_k)\cdot c_+(i_1) = \lambda(i_k)\cdot c_-(i_2) \le \lambda(i_k)\cdot c_+(i_2)\cdots \\ \lambda(i_k)\cdot c_-(i_{m-1}) \le \lambda(i_k)\cdot c_+(i_m) = \lambda(i_k)\cdot c'_+(j)\end{aligned}$$

Remark 6.27. Each morphism $\varphi \in \mathrm{Mul}_{\mathbf{BM}_{\mathrm{inv}}}(\{X_i\}_{i\in I}, Y)$ determines a linear ordering ℓ on the set I and a function $s\colon I \to \{\pm 1\}$. Passing from φ to the pair (ℓ, s) determines a map of colored operads $j\colon \mathbf{BM}_{\mathrm{inv}} \to \mathbf{Assoc}_{\mathrm{inv}}$. The map j induces a morphism of ∞-operads $\mathcal{BM}^\otimes_{\mathrm{inv}} \to \mathrm{Assoc}^\otimes_\sigma$ which we will also denote by j. For any $\mathbb{E}_\sigma$-monoidal ∞-category $\mathcal{C}$, restriction along j sends an $\mathbb{E}_\sigma$-algebra $A\colon \mathrm{Assoc}_\sigma \to \mathcal{C}^\otimes$ to the pair (A, A) where A is regarded as an involutive bimodule over itself. L: hermitian

Remark 6.28. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.20 induce maps of ∞-operads $\iota\colon \mathcal{LM}^\otimes \to \mathcal{LM}^\otimes_{\mathrm{inv}}$ and $\iota^{\mathrm{rev}}\colon \mathcal{RM}^\otimes \to \mathcal{LM}^\otimes_{\mathrm{inv}}$.

Remark 6.29. The maps $\iota, \iota^{\mathrm{rev}}$ of Remark 6.23 induce maps of ∞-operads $\iota, \iota^{\mathrm{rev}} \colon \mathcal{BM}^{\otimes} \to \mathrm{BM}^{\otimes}_{\sigma}$. There are canonical identifications $\iota \circ \mathrm{rev} \simeq \sigma \circ \iota^{\mathrm{rev}}$ where σ is the involution on $\mathrm{BM}^{\otimes}_{\sigma}$ induced by Remark 6.22 and rev is the involution on $\mathcal{BM}^{\otimes}$ of [Lur17, Construction 4.6.3.1].

Remark 6.30. There are canonical maps of operads $\mathcal{LM}^{\otimes}_{\mathrm{inv}} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ and $\mathcal{RM}^{\otimes}_{\mathrm{inv}} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ sending $\mathfrak{a}$ to $\mathfrak{a}_{\ell}$, resp. $\mathfrak{a}_r$ and making the diagram

$$\begin{tikzcd}
\mathrm{Assoc}^{\otimes} \arrow[r] \arrow[d, "\sigma"] & \mathcal{LM}^{\otimes}_{\mathrm{inv}} \arrow[d, "\mathrm{rev}"] \arrow[dr] & \\
& & \mathcal{BM}^{\otimes}_{\mathrm{inv}} \\
\mathrm{Assoc}^{\otimes} \arrow[r] & \mathcal{RM}^{\otimes}_{\mathrm{inv}} \arrow[ur] &
\end{tikzcd}$$

commute, where rev is (an involutive version of) the reversal involution of [Lur17, Remark 4.6.3.2].

Definition 6.31. Let $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}_{\sigma}$ and $\mathcal{D}^{\otimes} \to \mathrm{Assoc}^{\otimes}_{\sigma}$ be fibrations of ∞-operads and let $\mathcal{M}$ be an ∞-category. Suppose given a fibration of ∞-operads $q \colon \mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}_{\mathrm{inv}}$ together with equivalences $\mathcal{O}^{\otimes}_{\mathfrak{a}} \simeq \mathcal{C}^{\otimes}$ and $\mathcal{O}^{\otimes}_{\mathfrak{m}} \simeq \mathcal{M}$. Let $L\,{}^{\sigma}\mathrm{Mod}\,(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{LM}_{\mathrm{inv}}}(\mathcal{O})$. We will refer to $L\,{}^{\sigma}\mathrm{Mod}\,(\mathcal{M})$ as the *∞-category of left hermitian module objects of $\mathcal{M}$*.

Suppose given a fibration of ∞-operads $q \colon \mathcal{O}^{\otimes} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ together with equivalences $\mathcal{O}^{\otimes}_{\mathfrak{a}_{\ell}} \simeq \mathcal{C}^{\otimes}$, $\mathcal{O}^{\otimes}_{\mathfrak{a}_R} \simeq \mathcal{D}^{\otimes}$ and $\mathcal{O}^{\otimes}_{\mathfrak{m}} \simeq \mathcal{M}$. Let ${}^{\sigma}\mathrm{Mod}\,(\mathcal{M})$ denote the ∞-category $\mathrm{Alg}_{/\mathcal{BM}_{\mathrm{inv}}}(\mathcal{O})$. We will refer to ${}^{\sigma}\mathrm{Mod}\,(\mathcal{M})$ as the *∞-category of hermitian bimodule objects of $\mathcal{M}$*. Composition with the inclusions $\mathrm{Assoc}^{\otimes}_{\sigma} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ induces a categorical fibration

L: Lurie gives this a name (Definition 4.2.1.12 *weakly enriched*)– not sure what to call this. something *bi-enriched*?

$$ {}^{\sigma}\mathrm{Mod}\,(\mathcal{M}) = \mathrm{Alg}_{/\mathcal{BM}_{\mathrm{inv}}}(\mathcal{O}) \to \mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{C}) \times \mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{D}) \,. $$

If A is an Assoc_{σ}-algebra object of $\mathcal{C}$, we let ${}^{\sigma}\mathrm{Mod}_A\,(\mathcal{M})$ denote the fiber ${}^{\sigma}\mathrm{Mod}\,(\mathcal{M}) \times_{\mathrm{Alg}_{\mathrm{Assoc}_{\sigma}}(\mathcal{C})} \{A\}$. We will refer to ${}^{\sigma}\mathrm{Mod}_A\,(\mathcal{M})$ as the *∞-category of hermitian A-bimodule objects of $\mathcal{M}$*.

Definition 6.32. Let $q \colon \mathcal{O}^{\otimes} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ be a fibration of ∞-operads. We say that q exhibits $\mathcal{O}_{\mathfrak{m}}$ as $\mathbb{E}_{\sigma}$-bitensored over $\mathcal{O}_{\mathfrak{a}_{\ell}}$ and $\mathcal{O}_{\mathfrak{a}_r}$ if q is a cocartesian fibration.

Remark 6.33. Let $q \colon \mathcal{O}^{\otimes} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ be a cocartesian fibration of ∞-operads. Then q is classified by a map $\chi \colon \mathcal{BM}^{\otimes}_{\mathrm{inv}} \to \mathrm{Cat}_{\infty}$. By Remark 6.29, we can think of q as giving two $\mathbb{E}_{\sigma}$ algebras $\mathcal{C}$, $\mathcal{D}$ in Cat_{∞} with an ∞-category $\mathcal{M}$ equipped with both the structure of a $\mathcal{C}$-$\mathcal{D}$-bimodule (equivalently, the structure of a left $\mathcal{C} \times \mathcal{D}_{\mathrm{rev}}$-module) and of a $\mathcal{D}$-$\mathcal{C}$-bimodule, and an autoequivalence $\sigma_{\mathcal{M}} \colon \mathcal{M} \simeq \mathcal{M}$ of order two which is linear with respect to the autoequivalence $\mathcal{C} \times \mathcal{D}_{\mathrm{rev}} \xrightarrow{\mathrm{flip}} \mathcal{D}_{\mathrm{rev}} \times \mathcal{C} \xrightarrow{\sigma_{\mathcal{D}}^{-1} \times \sigma_{\mathcal{C}}} \mathcal{D} \times \mathcal{C}_{\mathrm{rev}}$.

Remark 6.34. Let $q \colon \mathcal{O}^{\otimes} \to \mathcal{LM}^{\otimes}_{\mathrm{inv}}$ be a cocartesian fibration of ∞-operads. Consider a left hermitian module object $F \colon \mathcal{LM}^{\otimes}_{\mathrm{inv}} \to \mathcal{O}^{\otimes}$. By Remark 6.30, F determines an associative algebra A of $\mathcal{C}$ with an equivalence of algebras $\sigma_A \colon A \simeq \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$, an object $M \in \mathcal{M}$ so that M (resp. $\sigma_{\mathcal{M}}(M)$) is equipped with the structure of a left A-module (resp. right $\sigma_{\mathcal{C}}(A)$-module). Furthermore, we have an equivalence $\sigma_M \colon M \simeq \sigma_{\mathcal{M}}(M)$ which is linear with respect to the equivalence $A \xrightarrow{\sigma_A} \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$.

L: is this related to "modules with involution" from [Cal+20, §3.1]?

Remark 6.35. Let $q \colon \mathcal{O}^{\otimes} \to \mathcal{BM}^{\otimes}_{\mathrm{inv}}$ be a cocartesian fibration of ∞-operads. Consider a hermitian module object $F \colon \mathcal{BM}^{\otimes}_{\mathrm{inv}} \to \mathcal{O}^{\otimes}$. By Remark 6.30, F determines an associative algebra A of $\mathcal{C}$ with an equivalence of algebras $\sigma_A \colon A \simeq \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$ and an associative algebra B of $\mathcal{D}$ with an equivalence of algebras $\sigma_B \colon B \simeq \sigma_{\mathcal{D}}(B)^{\mathrm{rev}}$, an object $M \in \mathcal{M}$ so that M (resp. $\sigma_{\mathcal{M}}(M)$) is equipped with the structure of a A-B-bimodule (resp. $\sigma_{\mathcal{D}}(B)$-$\sigma_{\mathcal{C}}(A)$-bimodule). Furthermore, we have an equivalence $\sigma_M \colon M \simeq \sigma_{\mathcal{M}}(M)$ which is linear with respect to the equivalence $A \otimes B \xrightarrow{\mathrm{flip}} B \otimes A \xrightarrow{\sigma_B^{-1} \otimes \sigma_A} \sigma_{\mathcal{D}}(B)^{\mathrm{rev}} \otimes \sigma_{\mathcal{C}}(A)^{\mathrm{rev}}$.

L: when $\mathcal{C} = \mathcal{D}$ and $\sigma_{\mathcal{M}}$ and $\sigma_{\mathcal{C}}$ are both the identity and $A = B$, I think this recovers the "module with involution" from [Cal+20, §3.1].

Construction 6.36. Define a functor $\mathrm{MCut} \colon \Delta^{\mathrm{op}}_{\sigma} \to \mathcal{RM}^{\otimes}_{\mathrm{inv}}$:

L: maybe this overloaded notation is not good. I'm running out of ideas.

- For each $([n], s)$, we have $\mathrm{MCut}([n], s) = \langle n{+}1 \rangle \simeq \mathrm{RCut}_0([n])$ where RCut is from [Lur17, Construction 4.8.4.4].
- Given a morphism $\alpha\colon ([n], s) \to ([m], t)$, the associated morphism $\mathrm{MCut}([m], t) \to \mathrm{MCut}([n], s)$ consists of
  - On underlying finite pointed sets $\langle m+1 \rangle \to \langle n+1 \rangle$, MCut agrees with (the reverse of) that appearing in [Lur17, Construction 4.2.2.6]
  - Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j-1 < j$, we may regard $s\colon \langle n+1 \rangle^\circ \to \{\pm 1\}$ and likewise $t\colon \langle m+1 \rangle^\circ \to \{\pm 1\}$. Define $u\colon \mathrm{MCut}(\alpha)^{-1}(\langle n+1 \rangle^\circ) \to \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\mathrm{MCut}(\alpha)(j))$.

L: check later

Remark 6.37. We can identify $\mathrm{Assoc}_\sigma^\otimes$ with the full subcategory of $\mathcal{RM}_{\mathrm{inv}}^\otimes$ spanned by objects of the form $(\langle n \rangle, \langle n \rangle^\circ)$. We can regard Construction 6.10 as defining a functor $\Delta_\sigma^{\mathrm{op}} \to \mathcal{RM}_{\mathrm{inv}}^\otimes$. For each $([n], s) \in \Delta_\sigma^{\mathrm{op}}$, there is a map of sets $\theta\colon \mathrm{MCut}([n], s) \to \mathrm{Cut}([n], s)$ defined as in [Lur17, Remark 4.2.2.8]. Concretely, on underlying pointed sets, θ takes the form

L: check that the signs s work out!

$$\theta\colon \langle n+1 \rangle \to \langle n \rangle$$
$$k \mapsto \begin{cases} k-1 & \text{if } k > 0 \\ * & \text{if } k = 0, *. \end{cases}$$

This construction determines a morphism γ in the ∞-category $\mathrm{Fun}\left(\Delta_\sigma^{\mathrm{op}}, \mathcal{RM}_{\mathrm{inv}}^\otimes\right)$, or equivalently a map $\gamma\colon \Delta_\sigma^{\mathrm{op}} \times \Delta^1 \to \mathcal{RM}_{\mathrm{inv}}^\otimes$.

Lemma 6.38. *The morphism $\gamma\colon \Delta_\sigma^{\mathrm{op}} \times \Delta^1 \to \mathcal{RM}_{\mathrm{inv}}^\otimes$ defined in Remark 6.37 exhibits $\Delta_\sigma^{\mathrm{op}} \times \Delta^1$ as an approximation to the ∞-operad $\mathcal{RM}_{\mathrm{inv}}^\otimes$.*

Definition 6.39. Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}_{\mathrm{inv}}^\otimes$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^\otimes$ as weakly bi-enriched over $\mathcal{O}_{\mathfrak{a}}^\otimes$. Let γ be as in Remark 6.37. Let $R^\sigma\mathrm{Mod}^{\mathbb{A}_\infty^\sigma}(\mathcal{M})$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{RM}_{\mathrm{inv}}^\otimes}\left(\Delta_\sigma^{\mathrm{op}} \times \Delta^1, \mathcal{O}^\otimes\right)$ spanned by those maps $f\colon \Delta_\sigma^{\mathrm{op}} \times \Delta^1 \to \mathcal{O}^\otimes$ satisfying

1. The restriction of f to $\Delta_\sigma^{\mathrm{op}} \times \{1\}$ belongs to $\mathrm{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O})$ of Definition 6.15
2. If $\alpha\colon ([m], s) \to ([n], t)$ so that $\alpha(0) = 0$, then the induced map $f([m], s, 0) \to f([n], t, 0)$ is an inert map in $\mathcal{O}^\otimes$
3. for each object $([n], s)$ in $\Delta_\sigma^{\mathrm{op}}$, the induced map $f([n], s, 0) \to f([n], s, 1)$ is an inert map in $\mathcal{O}^\otimes$

Example 6.40. Let $\mathcal{C}^\otimes \to \mathcal{RM}^\otimes$ be a fibration of ∞-operads. Restriction along the map of ∞-operads $\mathcal{RM}_{\mathrm{inv}}^\otimes \to \mathrm{Assoc}_\sigma^\otimes$ induced by Remark 6.27 induces a map $\mathbb{E}_\sigma \mathrm{Alg}(\mathcal{C}) \to R^\sigma\mathrm{Mod}(\mathcal{C})$ which is a section of the projection map $R^\sigma\mathrm{Mod}(\mathcal{C}) \to \mathbb{E}_\sigma \mathrm{Alg}(\mathcal{C})$.

L: see Example 4.2.1.17 of higher algebra

L: fibration?

Notation 6.41. Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}_{\mathrm{inv}}^\otimes$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^\otimes$ as weakly bi-enriched over $\mathcal{O}_{\mathfrak{a}}^\otimes$. Define a new simplicial set $\overline{\mathcal{M}}^\circledast$ by the following universal property

$$\hom_{\mathrm{sSet}_{/\Delta_\sigma^{\mathrm{op}}}}\left(K, \overline{\mathcal{M}}^\circledast\right) \simeq \hom_{\mathrm{sSet}_{/\mathcal{RM}_{\mathrm{inv}}^\otimes}}\left(K \times \Delta^1, \mathcal{O}^\otimes\right).$$

Here we regard $K \times \Delta^1$ as a simplicial set over $\mathcal{RM}_{\mathrm{inv}}^\otimes$ via the composite $K \times \Delta^1 \to \Delta_\sigma^{\mathrm{op}} \times \Delta^1 \xrightarrow{\gamma} \mathcal{RM}_{\mathrm{inv}}^\otimes$ where γ is from Remark 6.37.

Unwinding definitions, we see that a vertex in $\overline{\mathcal{M}}^\circledast$ lying over an object $([n], s\colon \{1, \ldots, n\} \to \{\pm 1\}) \in \Delta_\sigma^{\mathrm{op}}$ corresponds to a morphism α in $\mathcal{O}^\otimes$ whose image in $\mathcal{RM}_{\mathrm{inv}}^\otimes$ is the map $(\langle n+1 \rangle, \{0\}) \to (\langle n \rangle, \varnothing)$. Now let $\mathcal{M}^\circledast$ denote the full simplicial subset of $\overline{\mathcal{M}}^\circledast$ spanned by those vertices for which α is inert.

L: this might be off–revisit later!

Remark 6.42. Let $q\colon \mathcal{O}^\otimes \to \mathcal{RM}_{\mathrm{inv}}^\otimes$ be a fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^\otimes$ as weakly enriched over $\mathcal{O}_{\mathfrak{a}}^\otimes$. By [Lur09, Example 4.3.1.4 & Proposition 4.3.2.15], composition with the inclusion $\{0\} \to \Delta^1$ induces a trivial Kan fibration $\mathcal{M}^\circledast \xrightarrow{\sim} \mathcal{O}^\otimes \times_{\mathcal{RM}_{\mathrm{inv}}^\otimes} \Delta_\sigma^{\mathrm{op}}$. In particular, the fiber of $\mathcal{M}^\circledast$ over an object $([n], s) \in \Delta_\sigma^{\mathrm{op}}$ is canonically equivalent to $\mathcal{M} \times \mathcal{C}^{\times n}$.

L: Jacob explains this in a really terse way–just by citing Prop 4.3.2.15 of HTT. It does just follow from definitions/observatio

Finally, since q is a categorical fibration and categorical fibrations are closed under pullback and composition with trivial fibrations, q induces categorical fibrations $\mathcal{M}^\circledast \to \mathcal{C}^\circledast \to \Delta_\sigma^{\mathrm{op}}$.

Lemma 6.43. *Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$ be a cocartesian fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^{\otimes}$ as tensored over $\mathcal{O}_{\mathfrak{a}}^{\otimes}$. Then the associated functor $\mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ (Notation 6.12) is a locally coCartesian fibration.*

Proposition 6.44. *Let $q\colon \mathcal{O}^{\otimes} \to \mathcal{RM}_{\mathrm{inv}}^{\otimes}$ be a cocartesian fibration of ∞-operads, so q exhibits $\mathcal{M} := \mathcal{O}_{\mathfrak{m}}^{\otimes}$ as tensored over $\mathcal{O}_{\mathfrak{a}}^{\otimes}$. Then precomposition with the functor* MCut *of Construction 6.36 induces an equivalence of ∞-categories*

$$R^{\sigma}\mathrm{Mod}(\mathcal{M}) \simeq \mathrm{Alg}_{/\mathcal{RM}_{\mathrm{inv}}}(\mathcal{O}) \xrightarrow{\sim} R^{\sigma}\mathrm{Mod}^{\mathbb{A}_{\infty}^{\sigma}}(\mathcal{M}) \ .$$

Proof. Combine Lemma 6.38 with [Lur17, Theorem 2.3.3.23]. □

6.2 Part (b)

Proposition 6.45. *Let $\mathcal{C}$ be an involutive monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is bitensored over $\mathcal{C}$. Let K be a simplicial set so that $\mathcal{M}$ admits K-indexed limits, and let $\theta\colon R^{\sigma}\mathrm{Mod}(\mathcal{M}) \to \mathrm{Alg}^{\sigma}(\mathcal{C})$ be the forgetful functor. Then*

L: This statement is [Lur17, Proposition 4.2.3.1] with some words changed; no claim of originality here.

(1) For every commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & R^{\sigma}\mathrm{Mod}(\mathcal{M}) \arrow[d, "\theta"] \\ K^{\triangleleft} \arrow[r] \arrow[ur, dashed] & \mathrm{Alg}^{\sigma}(\mathcal{C}) \, , \end{tikzcd}$$

there exists a dashed arrow which is a θ-limit diagram.

(2) An arbitrary map $\overline{g}\colon K^{\triangleleft} \to R^{\sigma}\mathrm{Mod}(\mathcal{M})$ is a θ-limit diagram if and only if the induced map $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.

Proof. □

L: todo

Corollary 6.46. *θ is a cartesian fibration, and a morphism $f\colon \Delta^1 \to R^{\sigma}\mathrm{Mod}(\mathcal{M})$ is θ-cartesian if and only if the image of f in $\mathcal{M}$ is an equivalence.*

Corollary 6.47. *Let $\mathcal{C}$ be an involutive monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is bitensored over $\mathcal{C}$. Let K be a simplicial set so that $\mathcal{M}$ admits K-indexed limits, and let $\theta\colon R^{\sigma}\mathrm{Mod}(\mathcal{M}) \to \mathrm{Alg}^{\sigma}(\mathcal{C})$ be the forgetful functor. Let A be an involutive algebra object of $\mathcal{C}$. Then*

(1) $R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ admits K-indexed limits.

(2) A diagram $K^{\triangleleft} \to R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ is a limit diagram if and only if the induced diagram $K^{\triangleleft} \to \mathcal{M}$ is a limit diagram.

(3) Given a morphism $A \to B$ of involutive algebra objects of $\mathcal{C}$, the induced functor $R^{\sigma}\mathrm{Mod}_B(\mathcal{M}) \to R^{\sigma}\mathrm{Mod}_A(\mathcal{M})$ preserves K-indexed limits.

6.3 Towards (e)

Construction 6.48. Define a functor $\mathrm{Pr}\colon \mathbf{LM}_{\mathrm{inv}}^{\otimes} \times \mathbf{RM}_{\mathrm{inv}}^{\otimes} \to \mathbf{BM}_{\mathrm{inv}}^{\otimes}$.

(1) Let $(\langle m\rangle, S)$ be an object of $\mathbf{LM}_{\mathrm{inv}}^{\otimes}$ and let $(\langle n\rangle, T)$ be an object of $\mathbf{RM}_{\mathrm{inv}}^{\otimes}$. Let $\mathrm{Pr}((\langle m\rangle, S), (\langle n\rangle, T)) = (X_*, c_-, c_+)$ where X_*, c_-, c_+ are described in [Lur17, Construction 4.3.2.1(1)].

(2) Let $(\alpha, \lambda)\colon (\langle m\rangle, S) \to (\langle m'\rangle, S')$ be a morphism in $\mathbf{LM}_{\mathrm{inv}}^{\otimes}$ and let $(\beta, \mu)\colon (\langle n\rangle, T) \to (\langle n'\rangle, T')$ be a morphism in $\mathbf{RM}_{\mathrm{inv}}^{\otimes}$. Write $\mathrm{Pr}((\langle m'\rangle, S'), (\langle n'\rangle, T')) = (X'_*, c'_-, c'_+)$. Then $\mathrm{Pr}\left((\alpha, \lambda), (\beta, \mu)\right)$ is the unique morphism in $\mathbf{BM}_{\mathrm{inv}}^{\otimes}$ lying over the map $\gamma\colon X_* \to X'_*$ described by

(i) $\gamma(i,j) = \begin{cases} (\alpha(i), \beta(j)) & \text{if } \alpha(i) \in \langle m'\rangle^{\circ}, \beta(j) \in \langle n'\rangle^{\circ} \\ * & \text{otherwise.} \end{cases}$

(ii) Let $i' \in \langle m' \rangle^\circ \setminus S'$ and $j' \in T'$ so $j' = \beta(j)$ for a unique $j \in T$. Then the linear ordering on $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\}$ is (a) determined by the map α if $\mu(j) = 1$, and (b) it is the reverse of the linear ordering determined by α if $\mu(j) = -1$. The map $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\} \to \{\pm 1\}$ is determined by λ if $\mu(j) = 1$ and it is $-\lambda$ if $\mu(j) = -1$.

(iii) Likewise if $i' \in S'$ and $j' \in \langle n' \rangle^\circ \setminus T'$

(iv) Let $i' \in S'$ and $j' \in T'$ so $i' = \alpha(i)$ for a unique $i \in S$ and $j' = \beta(j)$ for a unique $j \in T$. Then $\gamma^{-1}\{(i', j')\} = \{i\} \times \beta^{-1}\{(j')\} \sqcup_{\{(i,j)\}} \alpha^{-1}\{(i')\} \times \{j\}$. Define $\gamma^{-1}\{(i', j')\} \to \{\pm 1\}$ by $\lambda \times \mu$. Endow $\gamma^{-1}\{(i', j')\}$ with the linear ordering from [Lur17, Construction 4.3.2.1(2)(iv)] if $\lambda(i) = \mu(j)$ and endow $\gamma^{-1}\{(i', j')\}$ with the opposite ordering if $\lambda(i) \neq \mu(j)$ (or equivalently, if $\lambda(i) = -\mu(j)$).

Write Pr for the induced map $\mathcal{LM}_\sigma^\otimes \times \mathcal{RM}_\sigma^\otimes \to \mathcal{BM}_\sigma^\otimes$ of ∞-categories.

Construction 6.49. Let $q\colon \mathcal{C}^\otimes \to \mathcal{BM}_\sigma^\otimes$ be a fibration of ∞-operads. We define a map of simplicial sets $\overline{L^\sigma \mathrm{Mod}}(\mathcal{C}_\mathfrak{m})^\otimes \to \mathcal{RM}_\sigma^\otimes$ by the universal property: For any simplicial set $K \to \mathcal{RM}_\sigma^\otimes$, there is a bijection

$$\mathrm{Hom}_{\mathrm{sSet}_{/\mathcal{RM}_\sigma^\otimes}}\left(K, \overline{L^\sigma \mathrm{Mod}}(\mathcal{C}_\mathfrak{m})^\otimes\right) \simeq \mathrm{Hom}_{\mathrm{sSet}_{/\mathcal{BM}_\sigma^\otimes}}\left(\mathcal{LM}_\sigma^\otimes \times K, \mathcal{C}^\otimes\right).$$

Let $L^\sigma \mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$ denote the full simplicial subset of $\overline{L^\sigma \mathrm{Mod}}(\mathcal{C}_\mathfrak{m})^\otimes$ spanned by those vertices which correspond to a vertex $X \in \mathcal{RM}_\sigma^\otimes$ and a functor $F\colon \mathcal{LM}_\sigma^\otimes\{X\} \to \mathcal{BM}_\sigma^\otimes$ which takes inert morphisms in $\mathcal{LM}_\sigma^\otimes$ to inert morphisms in $\mathcal{BM}_\sigma^\otimes$.

Remark 6.50. The composite $\mathcal{LM}_\sigma^\otimes \times \{\mathfrak{m}\} \hookrightarrow \mathcal{LM}_\sigma^\otimes \times \mathcal{RM}_\sigma^\otimes \xrightarrow{\mathrm{Pr}} \mathcal{BM}_\sigma^\otimes$ agrees with the inclusion of Remark 6.30. Taking $K \to \mathcal{RM}_\sigma^\otimes$ to be the inclusion $\{\mathfrak{m}\} \hookrightarrow \mathcal{RM}_\sigma^\otimes$, we have an isomorphism of simplicial sets $L^\sigma \mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \times_{\mathcal{RM}_\sigma^\otimes} \{\mathfrak{m}\} \simeq L^\sigma \mathrm{Mod}(\mathcal{C}_\mathfrak{m})$ where $L^\sigma \mathrm{Mod}(\mathcal{C}_\mathfrak{m})$ is the ∞-category of left modules associated to the fibration of ∞-operads $\mathcal{C}^\otimes \times_{\mathcal{BM}_\sigma^\otimes} \mathcal{LM}_\sigma^\otimes \to \mathcal{LM}_\sigma^\otimes$.

Proposition 6.51. *Let $q\colon \mathcal{C}^\otimes \to \mathcal{BM}_\sigma^\otimes$ be a fibration of ∞-operads. Then*

(1) the induced map $p\colon L^\sigma \mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \to \mathcal{RM}_\sigma^\otimes$ is a fibration of ∞-operads

(2) a morphism α in $L^\sigma \mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$ is inert if and only if $p(\alpha)$ is inert in $\mathcal{RM}_\sigma^\otimes$ and for all $X \in \mathcal{LM}_\sigma$, $\alpha(X)$ is an inert morphism in $\mathcal{C}^\otimes$.

(3) if q is a cocartesian fibration of ∞-operads, then so is p

(4) if q is a cocartesian fibration of ∞-operads, a morphism α in $L^\sigma \mathrm{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$ is p-cocartesian if and only if, for all $X \in \mathcal{LM}_\sigma^\otimes$, $\alpha(X)$ is q-cocartesian in $\mathcal{C}^\otimes$.

Proof. Similar to [Lur17, Proposition 4.3.2.5]. □

Theorem 6.52. *Let $\mathcal{C}$ be an $\mathbb{E}_\sigma$-monoidal ∞-category, and let A be an $\mathbb{E}_\sigma$-algebra in $\mathcal{C}$. Then $L^\sigma \mathrm{Mod}_A(\mathcal{C})$ is right $\mathbb{E}_\sigma$-tensored over $\mathcal{C}$.*

6.4 Endomorphisms

Let $\mathcal{C}$ be an $\mathbb{E}_\sigma$-monoidal ∞-category, and write $\sigma_\mathcal{C}\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ for its involution. Suppose $M \in \mathcal{C}$ is an object equipped with an equivalence $\sigma_M\colon M \simeq \sigma_\mathcal{C}(M)$. By [Lur17, §4.7.1], endomorphisms of M can be regarded as an $\mathbb{E}_1$-algebra in $u(\mathcal{C})^\otimes$, where u is from Remark 6.8. Now σ_M induces an equivalence $\mathrm{End}_\mathcal{C}(M) \simeq \mathrm{End}_\mathcal{C}(\sigma_\mathcal{C}(M))$ On the other hand, $\sigma_\mathcal{C}$ induces an equivalence $\mathrm{End}_\mathcal{C}(\sigma_\mathcal{C}(M)) \simeq \mathrm{End}_\mathcal{C}(M)^{\mathrm{rev}}$. In particular, for any ∞-category $\mathcal{M}$ left $\mathbb{E}_\sigma$-tensored over $\mathcal{C}$ and any object $M \in \mathcal{M}$ which is fixed by the involution on $\mathcal{M}$, we expect the endomorphisms of M to admit the structure of an $\mathbb{E}_\sigma$-algebra in $\mathcal{C}$.

To this end, we will define an ∞-category of objects acting on M, show that it has an $\mathbb{E}_\sigma$-monoidal structure, and locate endomorphisms of M as the final object in this ∞-category. Informally, we may define a category $\mathcal{C}[M]$ whose objects consist of either

- pairs (C, η) where $C \in \mathcal{C}$ and $\eta\colon C \otimes M \to M$ is a morphism in $\mathcal{M}$; or

- pairs (C', ξ) where $C' \in \mathcal{C}$ and $\xi\colon \sigma_{\mathcal{M}}(M) \otimes C' \to \sigma_{\mathcal{M}}(M)$.

The monoidal structure is as described in [Lur17, §4.7.1]. Note that given an object (C, η), the involution $\sigma_{\mathcal{M}}$ on $\mathcal{M}$ sends η to the map $\sigma_{\mathcal{M}}(C \otimes M) \simeq \sigma_{\mathcal{M}}(M) \otimes \sigma_{\mathcal{C}}(C) \to \sigma_{\mathcal{M}}(M)$. This is the involution on $\mathcal{C}[M]$.

Definition 6.53. Let $p\colon \mathcal{M}^{\circledast} \to \Delta^1 \times \Delta^{\mathrm{op}}_{\sigma}$ exhibit $\mathcal{M}^{\circledast}$ as weakly enriched over $\mathcal{C}^{\circledast}$. An *enriched morphism* of $\mathcal{M}$ is a diagram

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

satisfying either

- $p(\alpha)$ is the morphism $(0, [1], c_1) \to (0, [0])$ in $\Delta^{\mathrm{op}}_{\sigma}$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_1\colon \{1\} \to \{\pm 1\}$ is the constant function at $+1$, and
- the map β is inert, and $p(\beta)$ is the morphism $(0, [1], c_1) \to (0, [0])$ in $\Delta^1 \times \Delta^{\mathrm{op}}_{\sigma}$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

or

- $p(\alpha)$ is the morphism $(0, [1], c_{-1}) \to (0, [0])$ in $\Delta^{\mathrm{op}}_{\sigma}$ determined by the embedding $[0] \simeq \{0\} \hookrightarrow [1]$ and $c_{-1}\colon \{1\} \to \{\pm 1\}$ is the constant function at -1.
- the map β is inert, and $p(\beta)$ is the morphism $(0, [1], c_{-1}) \to (0, [0])$ in $\Delta^1 \times \Delta^{\mathrm{op}}_{\sigma}$ determined by the embedding $[0] \simeq \{1\} \hookrightarrow [1]$

Let $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]}$ denote the full subcategory of $\mathrm{Fun}_{\Delta^1 \times \Delta^{\mathrm{op}}_{\sigma}}\left(\Lambda^2_0, \mathcal{M}^{\circledast}\right)$ spanned by the enriched morphisms of $\mathcal{M}$.

Note that there are two evaluation functors $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]} \to \mathcal{M}$. Given $M \in \mathcal{M}$, write $\mathcal{C}[M] := \{M\} \times_{\mathcal{M}} \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]} \times_{\mathcal{M}} \{M\}$ and refer to it as the endomorphism ∞-category of M.

Definition 6.54. *enriched n-string*

Proposition 6.55 (Segal condition)**.**

7 Categorification and structure

In the course of thinking about the 'involutive' generalization of the statement that given an $\mathbb{E}_1$-algebra, its category of modules is $\mathbb{E}_0$ (and conversely, that given an object in a stable ∞-category, that its endomorphism spectrum is an $\mathbb{E}_1$-algebra), I have run up against some questions.

Question 7.1.
- Can we sidestep an involutive version of the construction of endomorphism categories of [Lur17, §4.7.1]?
- Suppose $\mathcal{C}$ is a monoidal ∞-category and $\mathcal{M}$ is an ∞-category which is enriched over $\mathcal{C}$ in the sense of [Lur17, §4.2.1]. The opposite category $\mathcal{M}^{\mathrm{op}}$ is enriched over $\mathcal{C}$ by [Hei23, §10].

8 Comparing involutive classical Brauer and involutive higher Brauer

Question 8.1.
- If, for a Poincaré ∞-category $(\mathcal{C}, Ϙ)$, there exists a Poincaré object (E, q) so that E is a compact generator, can we rewrite both the category and its Poincaré structure in terms of $\mathrm{End}_{\mathcal{C}}(E)$?
- Can the *property* of an existence of a Poincaré object (E, q) in $(\mathrm{Perf}_X, Ϙ_L)$ so that E is a compact generator be checked Zariski-locally? See Toën's paper §3.

9 Other

Tried showing that the geometric fixed points of the structure sheaf is supported on the ramification locus of an involution–did not succeed in doing so, but am saving the partial writeup here in case the parts of it which do work are useful later.

Guess..? Assume that X has a *good quotient* Y in the sense of [FW20, Remark 4.20], and write $p\colon X \to Y$ for the quotient map. Let $i\colon U \subseteq Y$ be the largest open subscheme on which $\pi|_{X_U}$ is étale [FW20, Proposition 4.45]. Write $\mathrm{RamLoc}(\pi)$ for the closed complement of U regarded as a topological space, and let $j\colon \mathrm{RamLoc}(\pi) \to Y$ denote the inclusion. Then $\underline{\mathcal{O}}^{\varphi C_2}$ is in the essential image of $j_*\colon \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{RamLoc}(\pi)) \to \mathrm{Shv}_{\mathrm{Zar}}(Y)$. In other words, there exists a sheaf $\mathcal{Q}$ of $\mathbb{E}_\infty$-rings on $\mathrm{RamLoc}(\pi)$ so that $j_*\mathcal{Q} \simeq \underline{\mathcal{O}}^{\varphi C_2}$.

L: finish this

Proof attempt. Recall that the open-closed decomposition of Y induces a symmetric monoidal récollement

$$\mathrm{Shv}_{\mathrm{Zar}}(U) \xleftarrow{i^*} \mathrm{Shv}_{\mathrm{Zar}}(Y) \xrightarrow{j^*} \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{RamLoc}(\pi))\,.$$

Therefore, to show that $\underline{\mathcal{O}}^{\varphi C_2}$ is in the essential image of j_*, it suffices to show that $i^*\left(\underline{\mathcal{O}}^{\varphi C_2}\right) \simeq 0$ as a sheaf on U.

By [FW20, Proposition 4.45], it suffices to show that if y is a point in U, then $\underline{\mathcal{O}}^{\varphi C_2}_y = 0$. Since $\underline{\mathcal{O}}^{\varphi C_2}_y = \tau_{\geq 0}\left(\mathcal{O}^{tC_2}_{Y,y}\right)$ if $A \to B$ is a quadratic étale map so that B has an involution λ and $A = B^\lambda$ is a local ring with maximal ideal $\mathfrak{m}_A$ (therefore B is semilocal by [FW20, Proposition 3.15]), then $\pi_0 B^{tC_2} = 0$. By [NS18, Lemma I.2.9], we may without loss of generality replace A and B by their 2-completions. Let $J \subseteq B$ denote the Jacobson radical of B. We claim that $B \simeq \lim_i B/J^i$ induces an equivalence $B^{tC_2} \to \lim_i \left(B/\mathfrak{m}_B^i\right)^{tC_2}$. Thus it suffices to show that $\left(B/\mathfrak{m}_B^i\right)^{tC_2}$ is zero for each i. Since $(-)^{tC_2}$ is exact and each $B/\mathfrak{m}_B^i$ can be written as an extension of finitely many $B/\mathfrak{m}_B$-modules, it suffices to show that $(B/\mathfrak{m}_B)^{tC_2}$ is zero. Now we have an exact sequence $B/\mathfrak{m}_B \xrightarrow{\mathrm{Tr}} A/\mathfrak{m}_A = (B/\mathfrak{m}_B)^{C_2} \to \pi_0(B/\mathfrak{m}_B)^{tC_2} \to 0$. Now $B/\mathfrak{m}_B$ is an étale $A/\mathfrak{m}_A$-algebra, hence it is either of the form $A/\mathfrak{m}_A \times A/\mathfrak{m}_A$ or a separable field extension of $A/\mathfrak{m}_A$ of degree 2. Since C_2 acts freely on $B/\mathfrak{m}_B$ as an $A/\mathfrak{m}_A$-module by the normal basis theorem.

L: Need hypercompleteness to reduce to checking on points?

Since homotopy fixed points commute with limits, it suffices to show that $B \simeq \lim_i B/\mathfrak{m}_B^i$ induces an equivalence $B_{hC_2} \to \lim_i \left(B/\mathfrak{m}_B^i\right)_{hC_2}$. This is true because the $B/\mathfrak{m}_B^i$ are uniformly bounded below.

L: double-check later

Recall that there is an exact sequence $B \xrightarrow{\mathrm{Tr}} B^{C_2} = A \to \pi_0 B^{tC_2} \to 0$. We have an exact sequence $B/\mathfrak{m}_B \xrightarrow{\mathrm{Tr}} A/\mathfrak{m}_A \to \pi_0(B/\mathfrak{m}_B)^{tC_2} \to 0$. Is there an isomorphism $\pi_0 B^{tC_2}/\pi_0\mathfrak{m}_B^{tC_2} \simeq \pi_0(B/\mathfrak{m}_B)^{tC_2}$? There IS an exact sequence $\mathfrak{m}_B^{tC_2} \to B^{tC_2} \to (B/\mathfrak{m}_B)^{tC_2}$ which gives a LES on π_*. Maybe $\pi_0 B^{tC_2}$ is a complete local ring with maximal ideal $\pi_0\mathfrak{m}_B^{tC_2}$? If so, then it suffices to show that $\pi_0(B/\mathfrak{m}_B)^{tC_2} = 0$. This follows from the normal basis theorem. □

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