Poincaré Schemes

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Abstract

We do stuff

N: Change this

Contents

1 Introduction

Theorem 1.1. *Let $\underline{A}$ be an affine Poincaré scheme with underlying $\mathbb{E}_\infty$-ring spectrum with involution A. Then the natural maps*

$$\pi_i(\mathrm{PnPic}(\underline{A})) \to \pi_i(\mathrm{Pic}(A))$$

are surjective on 2-torsion.

N: I think there is some interaction with the homotopy fixed points, or maybe even the genuine fixed points

Theorem 1.2. *Let A be an $\mathbb{E}_\infty$ ring with involution, and let $\underline{NA}$ be the associated Tate affine Poincaré scheme. Let $\mathrm{Br}_\nu(A)$ be the Brauer group of Azumaya algebras over A with involution. Then the natural map*

$$\mathrm{PnBr}(\underline{NA}) \to \mathrm{Br}_\nu(A)$$

is an equivalence

N: I think we need to define this for ring spectra. For A discrete this is done in [FW20].

Theorem 1.3. *The functors* $\mathrm{PnPic}, \mathrm{PnBr} : \mathrm{APS} \to \mathrm{Sp}$ *are fppf sheaves.*

N: probably of $\mathbb{E}_\infty$ dodads

Theorem 1.4. *There is a Poincaré group scheme $\mathbb{G}_m^{♀}$ such that*

$$B\mathbb{G}_m^{♀} \simeq \mathrm{PnPic}$$

as fppf stacks.

1.1 Conventions

Br^{p}	Poincaré Brauer space
CAlg	∞-categoriy of $\mathbf{E}_\infty$-ring spectra
$\mathrm{CAlg}(\mathcal{S})$	∞-categoriy of $\mathbf{E}_\infty$-spaces
$\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$	∞-categoriy of grouplike $\mathbf{E}_\infty$-spaces
$\mathrm{CAlg}^{\mathrm{p}}$	∞-categoriy of Poincaré ring spectra
$\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$	∞-category of small stable ∞-categories and exact functors
$\mathrm{Cat}_\infty^{\mathrm{p}}$	∞-category of Poincaré ∞-categories
$\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}}$	∞-category of idempotent complete Poincaré ∞-categories
$\mathrm{Pic}^{\mathrm{p}}$	Poincaré Picard space
$\mathcal{S}$	∞-category of spaces
Sp	∞-category of spectra

2 Poincaré Structures on Compact Modules

We will use this section to recall notions and results about Poincaré ∞-categories which we require in the sections to follow. This section can safely be skipped by anyone who posses extensive knowledge of Poincaré ∞-categories, as found in [Cal+20a].

Notation 2.1. Let R be an $\mathbf{E}_\infty$-ring spectrum. We will drop $\mathbf{E}_\infty$ from our notation and simply call R a *ring spectrum*. Moreover, we will denote the ∞-category CAlg(Sp) of commutative algebra objects in the ∞-category of spectra Sp by CAlg.

Let R be a ring spectrum and let Mod_R be the ∞-category of modules over R. We will study Poincaré structures on the ∞-category Mod_R^ω of compact modules over R.

V: - characterizati in terms of modules with genuine involution, - characterizati of symmetric monoidal structures, -Pn

3 Poincaré Ring Spectra

In this section we will define the ring theoretic building blocks of Poincaré schemes and the corresponding category they live in. Affine Poincaré Schemes will then be the dual objects, similar to how affine schemes are dual to commutative rings.

Definition 3.1. Let R be a ring spectrum. A *Poincaré structure* on R is a symmetric monoidal Poincaré ∞-category $Ϙ : (\mathrm{Mod}_R^\omega)^{\mathrm{op}} \to \mathrm{Sp}$. We call such a symmetric monoidal Poincaré ∞-category a *Poincaré ring spectrum*. We will denote the full subcategory of $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{p}})$ spanned by Poincaré ring spectra by $\mathrm{CAlg}^{\mathrm{p}}$ and call it the *∞-category of Poincaré ring spectra*.

Remark 3.2. Poincaré ring spectra, as defined in Definition 3.1, were studied in . Note that we chose a different notation. In [Cal+20a, discussion immediately preceding Examples 5.4.10], Poincaré ring spectra are referred to as *$\mathbf{E}_\infty$-ring spectra with genuine involution*.

V: cite 9 authored paper

Remark 3.3. Let R be a ring spectrum. By there is a natural equivalence between symmetric monoidal Poincaré structures on Mod_R^ω and certain algebra objects over the genuine C_2-spectrum NR [Cal+20a, Corollary 5.4.8]. In particular, a Poincaré structure on R can be identified with the following data:

- A C_2-action on R via maps of ring spectra, i.e. a functor $\lambda : BC_2 \to \mathrm{CAlg}$.
- An R-algebra $R \to C$.
- An R-algebra map $C \to R^{tC_2}$.

Here R^{tC_2} is the Tate construction with respect to the above action. Since the Tate construction is lax symmetric monoidal, R^{tC_2} is naturally an R-algebra via the Tate-valued norm. A ring spectrum equipped with a Poincaré structure will be called a *Poincaré ring spectrum*.

Remark 3.4. By Remark 3.3, a Poincaré structure on a ring spectrum R with a C_2-action via maps of ring spectra is a factorization $R \to C \to R^{tC_2}$ in CAlg of the Tate Frobenius $R \to R^{tC_2}$.

Remark 3.5. Let $\mathcal{M}$ be the full subcategory of $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ spanned by Poincaré ∞-categories with underlying ∞-category $\mathrm{Mod}^{\omega}_{R}$ for some ring spectrum R. Then the symmetric monoidal structure of $\mathrm{Cat}^{\mathrm{p}}_{\infty\infty}$ restricts to a symmetric monoidal structure on $\mathcal{M}$ by Example 3.9 and . Then we have $\mathrm{CAlg}^{\mathrm{p}} \simeq \mathrm{CAlg}(\mathcal{M})$. In particular, the symmetric monoidal structure of $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{p}}_{\infty})$ restricts to a symmetric monoidal structure on $\mathrm{CAlg}^{\mathrm{p}}$.

V: cite 9-authors I.5.1.5 and I.5.1.6

Notation 3.6. Let R be a ring spectrum. We will denote by $\underline{R}$ the spectrum R with trivial action. More precisely, $\underline{R} : BC_2 \to \mathrm{Sp}$ is the constant functor.

L: This is commonly used for constant Mackey functors–could be ambiguous

Example 3.7. Let R be a ring spectrum. If $2 \in \pi_0(R)$ is invertible, we have $\underline{R}^{tC_2} \simeq 0$. A Poincaré structure on R with the trivial action is then given by an R-algebra $R \to C$.

V: explain/reference

Example 3.8. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The Tate-valued norm endows R^{tC_2} with a natural R-algebra structure, which induces a Poincaré structure on R given by the factorization $R \xrightarrow{\mathrm{id}} R \to R^{tC_2}$. We will call this Poincaré structure the *Tate Poincaré structure on* R and will denote it by $(R, \Qoppa^{\mathrm{t}}_{R})$.

Example 3.9. The sphere spectrum $\mathbb{S}$ together with the Tate Poincaré structure will be called the *universal Poincaré ring spectrum*. We will denote it by $(\mathbb{S}, \Qoppa_u)$.

V: expain why/translate universality statement to poincare ring spectra

Remark 3.10. Let $(R, \Qoppa)$ be a ring spectrum associated to a factorization $R \to C \to R^{tC_2}$. A factorization of the map $C \to R^{tC_2}$ through R^{hC_2} induces a section of the canonical map $\Qoppa(R) \to \hom_R(R, C) \simeq C$. In that case, we have a splitting $\Qoppa(R) \simeq R_{hC_2} \oplus C$.

V: reference pullback that characterizes all quadratic functors

Example 3.11. The Tate Frobenius for the sphere spectrum factors through $\mathbb{S}^{hC_2}$. Therefore, Remark 3.10 implies $\Qoppa_u(\mathbb{S}) \simeq \mathbb{S}_{hC_2} \oplus \mathbb{S} \simeq \Sigma^{\infty}(\mathbb{P}^{\infty}_{\mathbb{R}}) \oplus \mathbb{S}$.

Example 3.12. Let R be a ring spectrum equipped with a C_2-action via maps of ring spectra. The identity map $\mathrm{id} : R^{tC_2} \to R^{tC_2}$ induces a Poincaré structure on R given by the factorization $R \to R^{tC_2} \xrightarrow{id} R^{tC_2}$. We will call this Poincaré structure the *symmetric Poincaré structure on* R.

Example 3.13. Let R be a connective ring spectrum equipped with a C_2-action via maps of ring spectra. The connective cover $\tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$ of R^{tC_2} induces a Poincaré structure on R given by the factorization $R \to \tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$. We will call this Poincaré structure the *genuine symmetric Poincaré structure on* R.

V: copy more examples from notes

Definition 3.14. Let A and R be Poincaré ring spectra. A *map of Poincaré ring spectra* between A and R is a map of ring spectra $f : A \to R$ compatible with the corresponding Poincaré structures via the following additional data:

-

V: this should become a remark and go below the definition of calgp

3.1 Algebras with genuine involution

Recollection 3.15. Assume $\mathcal{C}$ is a presentable monoidal ∞-category such that the monoidal product $-\otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then there is an ∞-category $\mathrm{LMod}(\mathcal{C})$ [Lur17, Example 4.2.1.18] whose objects are pairs (A, M) where A is an associative algebra object of $\mathcal{C}$ and M is a left A-module. Write a, m respectively for the canonical forgetful functors $\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$, $\mathrm{LMod}(\mathcal{C}) \to \mathcal{C}$ which send (A, M) to A and M, resp. Then a is a cocartesian fibration [Lur17, Corollary 4.2.3.7], hence it is classified by a functor $\mathrm{mod}\colon \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_{\infty}$.

The functor s of [Lur17, Example 4.2.1.17] determines a natural transformation $\eta\colon * \to \mathrm{mod}$, where $*\colon \mathrm{Alg}(\mathcal{C}) \to \{*\} \hookrightarrow \mathcal{C}\mathrm{at}_{\infty}$ is the constant functor at the trivial category, or equivalently

$$\begin{tikzcd} & \mathcal{U} \arrow[d] \\ \mathrm{Alg}(\mathcal{C}) \arrow[ur, "\eta"] \arrow[r, "\mathrm{mod}"'] & \mathcal{C}\mathrm{at}_{\infty} \end{tikzcd} \tag{3.16}$$

where $\mathcal{U}$ is the universal cocartesian fibration. Now consider the functor $o\colon \mathcal{U} \to \mathcal{C}\mathrm{at}_\infty$ which sends $(\mathcal{D}, d \in \mathcal{D})$ to the undercategory $\mathcal{D}_{d/-}$. Define $\mathrm{LMod}(\mathcal{C})_{*/-}$ to be the cocartesian fibration over $\mathrm{Alg}(\mathcal{C})$ classified by $o \circ \eta \circ \mathrm{mod}$.

Variant 3.17. Let C be as in Recollection 3.15. There is a similar construction where left modules is replaced by *bimodules* [Lur17, Definition 4.3.1.12].

Construction 3.18. Regard $\mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})$ as a category with C_2-action given by taking the opposite/reverse algebra. There are functors $b\colon \mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}\right)^{hC_2} \to \mathrm{LMod}(\mathrm{Sp})$ and $b\colon \mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}\right)^{hC_2} \to \mathrm{BiMod}(\mathrm{Sp})_{*/-}$ so that $(a,m)\circ b$ and $(a,m)\circ b_*$ are (canonically) equivalent to $(-^e)\otimes(-^e)^{\mathrm{op}}, (-)^e$. Informally, an $\mathbb{E}_1$-algebra with involution B can be regarded as a $B\otimes B^{\mathrm{op}}$-module in a canonical way, and there is a canonical $B\otimes B^{\mathrm{op}}$-module map $B \otimes B^{\mathrm{op}} \to B$.

Definition 3.19. The category of $\mathbb{E}_1$-*algebras with genuine involution* is defined to be the limit of the $\mathcal{C}\mathrm{at}_\infty$-valued diagram

$$\begin{tikzcd}
& \mathrm{LMod}(\mathrm{Sp}) \arrow[dd, "a{,}m"] & \\
& & \mathrm{LMod}\left(\mathrm{Sp}^{C_2}\right) \arrow[ul, "(-)^e"'] \arrow[dd, "a{,}m"] \\
& \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp}) \times \mathrm{Sp} & \\
\mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}\right)^{hC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \arrow[uuur, bend left, "b\circ(-^e)"] \arrow[ur, "(-^e)\otimes(-^e)^{\mathrm{op}}{,}(-)^e"] \arrow[rr, "N^{C_2}(-^e)\times U"'] & & \mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \times \mathrm{Sp}^{C_2} \arrow[ul, "(-)^e\times(-)^e"]
\end{tikzcd} \tag{3.20}$$

where

- b is the functor/section of Construction 3.18
- U is the 'underlying' C_2-spectrum functor $\mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}\right)^{BC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \to \mathrm{Sp}^{BC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \simeq \mathrm{Sp}^{C_2}$
- The upper right trapezoid commutes canonically by definition of LMod (and the fact that the functors a, m are given by restriction to subcategories of $LM^{\otimes}$).

Write $\mathbb{E}_1\mathrm{Alg}^{\mathrm{gi}}\left(\mathrm{Sp}^{C_2}\right)$ for the ∞-category of $\mathbb{E}_1$-algebras with genuine involution.

Definition 3.21. The category of $\mathbb{E}_\sigma$-*algebras* is defined to be the limit of the $\mathcal{C}\mathrm{at}_\infty$-valued diagram

$$\begin{tikzcd}
& \mathrm{BiMod}(\mathrm{Sp})_{*/-} \arrow[dd, "a{,}m"] & \\
& & \mathrm{BiMod}\left(\mathrm{Sp}^{C_2}\right)_{*/-} \arrow[ul, "(-)^e"'] \arrow[dd, "a{,}m"] \\
& \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp}) \times \mathrm{Sp} & \\
\mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}\right)^{hC_2} \times_{\mathrm{Sp}} \mathrm{Sp}^{\Delta^1} \arrow[uuur, bend left, "b_*\circ(-^e)"] \arrow[ur, "(-^e)\otimes(-^e)^{\mathrm{op}}{,}(-)^e"] \arrow[rr, "N^{C_2}(-^e)\times U"'] & & \mathbb{E}_1\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right) \times \mathrm{Sp}^{C_2} \arrow[ul, "(-)^e\times(-)^e"]
\end{tikzcd} . \tag{3.22}$$

Write $\mathbb{E}_\sigma\mathrm{Alg}\left(\mathrm{Sp}^{C_2}\right)$ for the ∞-category of $\mathbb{E}_\sigma$-algebras.

Variant 3.23. Let the base be R an $\mathbb{E}_\infty$-algebra or Poincaré ring instead of $\mathbb{S}^0$.

Remarks 3.24. 1. Compare [AGH21, Corollary 3.10].

2. There are canonical forgetful functors $\mathbb{E}_\sigma\mathrm{Alg} \to \mathbb{E}_1\mathrm{Alg}^{\mathrm{gi}} \to \mathbb{E}_1\mathrm{Alg}^{hC_2} \to \mathbb{E}_1\mathrm{Alg}(\mathrm{Sp})$.

Construction 3.25. Let $R, R^{\varphi C_2} \to R^{tC_2}$ be a Poincaré ring. There is a functor $\left(\mathrm{Mod}^{\omega}_{(-)}, \Qoppa_{(-)}\right) : \mathbb{E}_\sigma\mathrm{Alg}_R \to \left(\mathcal{C}\mathrm{at}^h_R\right)_{(\mathrm{Mod}^\omega_R, \Qoppa_R)/-}$.

Lemma 3.26. *Let $R, R^{\varphi C_2} \to R^{tC_2}$ be a Poincaré ring. The functor of Construction 3.25 is fully faithful.*

Proof. ◻

Now we observe that given a hermitian object (x, q) of $(\mathcal{C}, \Qoppa_{\mathcal{C}})$, its endomorphism algebra admits a canonical lift to a $\mathbb{E}_\sigma$-algebra.

Construction 3.27. There is a functor $\mathrm{End}(-)\colon \left(\mathcal{C}\mathrm{at}^h_R\right)_{(\mathrm{Mod}^\omega_R, \Qoppa_R)} \to \mathbb{E}_\sigma\mathrm{Alg}$ lifting the functor $\left(\mathcal{C}\mathrm{at}^h_R\right)_{(\mathrm{Mod}^\omega_R, \Qoppa_R)} \to \mathbb{E}_1\mathrm{Alg}^{hC_2}$ of [Cal+20a, Proposition 3.1.16].

Theorem 3.28. *The functors of Construction 3.25 and 3.27 form an adjoint pair.*

4 Modules over Poincaré Ring Spectra

Let A be a Poincaré ring spectrum. Then A is a commutative algebra object in the ∞-category of Poincaré ∞-categories $\mathrm{Cat}^{\mathrm{p}}_\infty$. We may thus consider modules over it. In this section we will use modules over Poincaré ring spectra to define analogues of the Brauer and Picard groups for Poincaré ring spectra. V: ref

4.1 The Poincaré Picard Group

Recall that the Poincaré space functor $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_\infty \to \mathrm{CAlg}(\mathcal{S})$ is lax symmetric monoidal with respect to tensor product of Poincaré ∞-categories and smash product of $\mathbf{E}_\infty$-spaces [Cal+20a, Corollary 5.2.8]. In particular, we can consider invertible objects in $\mathrm{Pn}(A)$ for a Poincaré ring spectrum A.

Definition 4.1. Let A be a Poincaré ring spectrum. We define the *Picard space of A* to be

$$\mathrm{Pic}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Pn}(A)).$$

Remark 4.2. Let $(\mathrm{Mod}^\omega_R, \Qoppa_R)$ be a Poincaré ring spectrum, where $(M_R = R, N_R = R^{\varphi C_2}, R^{\varphi C_2} \to R^{tC_2})$ is the module with genuine involution associated to $\Qoppa_R$. Then a point in the Poincaré Picard space is the data of a pair $(\mathcal{L}, q)$, where $\mathcal{L}$ is an invertible module in Mod^ω_R and q is a point in $\Omega^\infty \Qoppa_R(\mathcal{L})$. By [Cal+20a, Proposition 1.3.11], the data of q is equivalent to the data of points in the lower left and upper right corner of the square

$$\begin{tikzcd} \Qoppa(\mathcal{L}) \arrow[rr] \arrow[d] & & \hom_R(\mathcal{L}, R^{\varphi C_2}) \ni \ell(q) \arrow[d] \\ b(q) \in \hom_{R\otimes R}(\mathcal{L}\otimes\mathcal{L}, R)^{hC_2} \arrow[rr] & & \hom_R(\mathcal{L}, R^{tC_2}) \end{tikzcd} \tag{4.3}$$

and a path between their images in the lower right corner. In particular, the adjoint of $b(q)$ must define a nondegenerate hermitian form on $\mathcal{L}$, that is, an equivalence $\mathcal{L} \simeq \hom_R(\mathcal{L}, R^*)$ where R^* is considered as an R-module via the action of the generator of C_2.

Write $(\mathcal{L}^\vee, q^\vee)$ is for the inverse of $(\mathcal{L}, q)$. By definition of invertibility, there exists an R-linear map $\ell(q^\vee)\colon \mathcal{L}^\vee \to R^{\varphi C_2}$ so that the following diagram commutes L: add equivariance/symmetry data

$$\begin{tikzcd} \mathcal{L}\otimes_R \mathcal{L}^\vee \arrow[r, "\ell(q)\otimes\ell(q^\vee)"] \arrow[d, "\sim"', "\mathrm{ev}"] & R^{\varphi C_2}\otimes_R R^{\varphi C_2} \arrow[d, "\text{multiplication}"] \\ R \arrow[r, "\text{given}"] & N_R \end{tikzcd} \tag{4.4}$$

Lemma 4.5. *Let (R, Ω) be a connective Poincaré ring spectrum. Then, for any integer n, the spectrum $\Omega(\Sigma^n R)$ is $(-2n)$-connective.*

V: define connectivity and make conditions here precise. As stated

Proof. This follows from the fiber sequence

$$(\Sigma^{-2n} R)_{hC_2} \to \Omega(\Sigma^n R) \to \hom_R(\Sigma^n R, C) \simeq \Sigma^{-n} C.$$

$\square$

V: write out details

connective.

Remark 4.6. The functor $\mathrm{Pic}^{\mathrm{p}} : \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$ preserves certain structures. Let A be a Poincaré ring spectrum. Since A is a module over $(\mathbb{S}, \Omega_u)$, the space $\mathrm{Pic}^{\mathrm{p}}(A)$ is something over $\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Omega_u)$.

V: there is no truth in here yet. Work in progress. Noah had an example using Witt vectors which showed that π_0 does not need to be 2-torsion

Since the forgetful functor $\mathrm{Pn}(A) \to \mathrm{Mod}_A^{\omega}$ is symmetric monoidal we get an induced map

$$U : \mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)$$

of spectra. For a point $(\mathcal{L}, q) \in \pi_0(\mathrm{Pic}^{\mathrm{p}}(A))$ we will refer to $\mathcal{L} := U(\mathcal{L}, q)$ as the *underlying invertible module*. Note that the A-module A^* is (nonequivariantly) isomorphic to A via the involution, and so the fact that $\mathcal{L} \simeq \hom_A(\mathcal{L}, A^*)$ forces $\mathcal{L}$ to be 2-torsion. In particular we get a refined map

$$\mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)[2]$$

which factors the underlying invertible module map.

Example 4.7. Let $(\mathbb{S}, \Omega_u)$ be the universal Poincaré ring spectrum from Example 3.9. The only 2-torsion element of $\mathrm{Pic}(\mathbb{S}) \simeq \mathbf{Z}$ is $\mathbb{S}$. Therefore, any element in $\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Omega_u)$ lies above $\mathbb{S}$. With Remark 3.11, we conclude $\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}, \Omega_u)) \simeq \pi_0(\mathbb{S}_{hC_2} \oplus \mathbb{S}^{\times})^{\times} \simeq (\mathbf{Z} \times \mathbf{Z/2})^{\times} \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$.

Remark 4.8. One might hope that the map $\mathrm{Pic}^{\mathrm{p}}(A) \to \mathrm{Pic}(A)[2]$ is close to an equivalence. This however is quite fare from being true. Let k be a finite field of characteristic 2, and let $\mathbb{S}_{W(k)}$ be the spherical Witt vectors on k in the sense of [Lur18, Example 5.2.7]. Then by [Nik23, Example 3.4] we know that $\mathbb{S}_{W(k)}$ must satisfy that the map $\varphi_2 : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}^{tC_2}$ is an equivalence where the action is trivial.

Consider now the Poincaré ring $(\mathrm{Mod}^{\omega}_{\mathbb{S}_{W(k)}}, \Omega^u_{\mathbb{S}_{W(k)}})$ where $\Omega^u_{\mathbb{S}_{W(k)}}$ is the Tate Poincaré structure. We have that $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) \cong \mathbb{Z}$ and is generated by $\Sigma\mathbb{S}_{W(k)}$. To see this note that for $\mathcal{L}$ an invertible module over $\mathbb{S}_{W(k)}$, $\mathcal{L}$ must be bounded below since otherwise it would not be perfect. Then for $\pi_n(\mathcal{L})$ its bottom homotopy group, we have that $\pi_n(\mathcal{L}/2) \cong k$ since it must be an invertible k-module and k is a field. Thus we get a map $\mathbb{S}^n \to \mathcal{L}$ lifting a generator of k, and by adjunction an $\mathbb{S}_{W(k)}$-module map $\Sigma^n \mathbb{S}_{W(k)} \to \mathcal{L}$ which on $\pi_n((-)/2)$ gives an isomorphism $k \cong k$. Therefore

$$\mathbb{S}_{W(k)}[n] \otimes k \simeq k[n] \to k[n] \simeq \mathcal{L} \otimes k$$

is an equivalence, where the equivalence $k[n] \simeq \mathcal{L} \otimes k$ follows from the fact that base change preserves invertible objects. The map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is then a k-local, and therefore an $\mathbb{F}_p$-local, equivalence. Both sides are connective and p-complete so it follows that the map $\mathbb{S}_{W(k)}[n] \to \mathcal{L}$ is an equivalence.

N: There is probably a reference for this fact, I'll look around for one.

Thus $\pi_0(\mathrm{Pic}(\mathbb{S}_{W(k)})) = 0$. On the other hand, we have that the unit map $\mathbb{S}_{W(k)} \to \Omega^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})$ is split by the map $\Omega^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)}) \to \mathbb{S}_{W(k)}^{\varphi C_2} = \mathbb{S}_{W(k)}$. Consequently $\pi_0(\Omega^u_{\mathbb{S}_{W(k)}}(\mathbb{S}_{W(k)})) \cong \pi_0(\mathbb{S}_{W(k)} \oplus (\mathbb{S}_{W(k)})_{hC_2}) \cong W(k) \times W(k)$. As a ring this is $W_2(W(k))$, and in order for $q \in W_2(W(k))$ to induce a Poincaré structure we must have that $q \in W_2(W(k))^{\times} \cong W(k)^{\times} \times W(k)^{\times}$.

We then have that $\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong W(k)^{\times} \times W(k)^{\times}/H$ where H is the subgroup of Poincaré structures q on $\mathbb{S}_{W(k)}$ which are identified by some automorphism $f : \mathbb{S}_{W(k)} \to \mathbb{S}_{W(k)}$. By the defining property of spherical Witt vectors there is an equivalence $\mathrm{Maps}_{\mathrm{CAlg}}(\mathbb{S}_{W(k)}, \mathbb{S}_{W(k)}) \simeq \mathrm{Maps}_{Perf}(k, k) = \mathrm{Gal}(k/\mathbb{F}_2)$ and the action on $W(k)^{\times} \times W(k)^{\times}$ is given by $g \in \mathrm{Gal}(k/\mathbb{F}_2)$ acts via $W(g) \times W(g)$. Consequently

$$\pi_0(\mathrm{Pic}^{\mathrm{p}}(\mathbb{S}_{W(k)})) \cong (W(k)^{\times} \times W(k)^{\times})/\mathrm{Gal}(k/\mathbb{F}_2)$$

which even for $k = \mathbb{F}_2$ is not zero and in fact not even 2^{∞}-torsion.

4.2 The Poincaré Brauer Group

Recall that a Poincaré ∞-category is called idempotent complete if the underlying stable ∞-category is idempotent complete. The full subcategory of $\mathrm{Cat}^{\mathrm{p}}_{\infty}$ spanned by idempotent complete Poincaré ∞-categories is denoted by $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}$ [Cal+20b, Definition 1.3.2].

Definition 4.9. Let A be a Poincaré ring spectrum. We define the *Poincaré Brauer space of A* as

$$\mathrm{Br}^{\mathrm{p}}(A) := \mathrm{Pic}(\mathrm{Mod}_A(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})).$$

The assignment $A \mapsto \mathrm{Br}^{\mathrm{p}}(A)$ defines a functor

$$\mathrm{Br}^{\mathrm{p}} \colon \mathrm{CAlg}^{\mathrm{p}} \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$$

valued in grouplike $\mathbf{E}_\infty$-spaces.

Remark 4.10. The symmetric monoidal forgetful functor $\mathrm{Mod}_A(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_A(\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty})$ induces a map $\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A)$ of grouplike $\mathbf{E}_\infty$-spaces, where $\mathrm{Br}(A)$ is the Brauer space $\mathrm{br}_{\mathrm{alg}}(A)$ of [AG14, pp. 1154–1155].

Proposition 4.11. *Let A be a Poincaré ring spectrum. Then we have a canonical equivalence*

$$\Omega\,\mathrm{Br}^{\mathrm{p}}(A) \simeq \mathrm{Pic}^{\mathrm{p}}(A).$$

L: What else do we need to do to show that we have an equivalence of *functors*?

V: todo

Proof. Since $\Omega\,\mathrm{Br}^{\mathrm{p}}(R)$ is given by the space of automorphisms of any object in $\mathrm{Br}^{\mathrm{p}}(R)$, it suffices to determine the space of autoequivalences of $(\mathrm{Mod}^{\omega}_R, Ϙ_R)$. An autoequivalence is the data of a pair (f, η) where $f\colon \mathrm{Mod}^{\omega}_R \xrightarrow{\simeq} \mathrm{Mod}^{\omega}_R$ is an exact R-linear autoequivalence and $\eta\colon Ϙ_R \xrightarrow{\sim} Ϙ_R \circ f^{\mathrm{op}}$ is a natural equivalence. Since $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}$ is symmetric monoidal, f is of the form $- \otimes_R \mathcal{L}$ where $\mathcal{L}$ is an invertible R-module. Since taking bilinear and linear parts is functorial/by [Cal+20a, Proposition 1.3.11], η is equivalently the data of a pair of equivalences

$$\begin{aligned} b(\eta)\colon\ & \hom_{R\otimes R}((-\otimes\mathcal{L})\otimes(-\otimes\mathcal{L}), R)^{hC_2} \simeq \hom_{R\otimes R}(-\otimes -, R)^{hC_2} \\ \ell(\eta)\colon\ & \hom_R(-\otimes\mathcal{L}, R^{\varphi C_2}) \simeq \hom_R(-, R^{\varphi C_2}) \end{aligned}$$

plus a path between their images in $\hom_R(\mathcal{L}, R^{tC_2})$. The transformation $b(\eta)$ is equivalent to the data of an R-bilinear equivalence $R \simeq \mathcal{L}^\vee \otimes \mathcal{L}^\vee$, and the transformation $\ell(\eta)$ is equivalent to the data of an $R^{\varphi C_2}$-linear equivalence $\ell(\eta)\colon R^{\varphi C_2} \otimes_R \mathcal{L}^\vee \xrightarrow{\sim} R^{\varphi C_2}$.

L: maybe one of these should be conjugate dual here?

L: is the $R^{\varphi C_2}$-linearity of this $\simeq$ correct?

Now consider the composites

$$R \otimes_R \mathcal{L}^\vee \xrightarrow{\mathrm{unit}\otimes\mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L}^\vee \xrightarrow{\ell(\eta)} R^{\varphi C_2}$$

$$R \otimes_R \mathcal{L} \xrightarrow{\mathrm{unit}\otimes\mathrm{id}} R^{\varphi C_2} \otimes \mathcal{L} \xrightarrow{\ell(\eta)^{-1}\otimes\mathrm{id}_{\mathcal{L}}} R^{\varphi C_2}\,.$$

These correspond to the $\ell(q^\vee), \ell(q)$ of Remark **??**, respectively. In particular, the condition that $\ell(q^\vee), \ell(q)$ make the diagram (4.4) commute is equivalent to the condition that $\ell(\eta)$ is an equivalence by an adjunction argument. $\square$

L: under construction– not sure what to say about the $(-)^{tC_2}$ part yet.

Proposition 4.12. *Let $(\mathrm{Mod}^{\omega}_R, Ϙ_R)$ be a Poincaré ring spectrum.*

1. *The ∞-category $\mathrm{Mod}_{(\mathrm{Mod}^{\omega}_R, Ϙ_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ admits all small limits and colimits, and it inherits a canonical symmetric monoidal structure, and for every morphism $\left(R, R^{\varphi C_2} \to R^{tC_2}\right) \to (S, S^{\varphi C_2} \to S^{tC_2})$, the functor $\mathrm{Mod}_{(\mathrm{Mod}^{\omega}_R, Ϙ_R)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_{(\mathrm{Mod}^{\omega}_S, Ϙ_S)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}})$ is a symmetric monoidal left adjoint.*

2. *Let A be an $\mathbb{E}_1$-R-algebra in spectra, and regard the category of compact right A-modules Mod_A^ω as left-tensored over Mod_R^ω in the canonical way. Then the pullback*

$$\begin{tikzcd} & \mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}\left(\mathrm{Cat}_\infty^{\mathrm{h}}\right) \arrow[d] \\ \{\mathrm{Mod}_A^\omega\} \arrow[r] & \mathcal{C}\mathrm{at}_{\infty R}^{\mathrm{ex}} \end{tikzcd} \tag{4.13}$$

is canonically equivalent to $\mathrm{Mod}_{N_R A \otimes_{N_R R} R^L}\left(\mathrm{Sp}^{C_2}\right)$ where R^L is the $\mathbb{E}_\infty$-$N_R R$-algebra with $(R^L)^e \simeq R$ and $(R^L)^{\varphi C_2} \simeq C$.

A $N_R A \otimes_{N_R R} R^L$-module classifies a $(\mathrm{Mod}_R^\omega, \Qoppa_R)$-module in Poincaré ∞-categories if its underlying R-module is invertible in the sense of [Cal+20a, Definition 3.1.4].

3. *Let A, B be R-algebras with associated (R-linear) modules with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and $(M_B, N_B, N_B \to M_B^{tC_2})$, respectively so that (under item 2) $(\mathrm{Mod}_A^\omega, \Qoppa_A)$ and $(\mathrm{Mod}_B^\omega, \Qoppa_B)$ are objects of $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}_{\infty,\mathrm{idem}}^{\mathrm{p}})$. Then the symmetric monoidal structure of item 1 is so that the underlying R-linear ∞-category with perfect duality $(\mathrm{Mod}_A^\omega, \Qoppa_A) \otimes_{(\mathrm{Mod}_R^\omega, \Qoppa_R)} (\mathrm{Mod}_B^\omega, \Qoppa_B)$ is $\mathrm{Mod}_A^\omega \otimes_{\mathrm{Mod}_R^\omega} \mathrm{Mod}_B^\omega \simeq \mathrm{Mod}_{A \otimes_R B}^\omega$, and the associated module with genuine involution is given by $M_A \otimes_R M_B$, $N_A \otimes_{R^{\varphi C_2}} N_B$, and the structure map is $N_A \otimes_{R^{\varphi C_2}} N_B \to M_A^{tC_2} \otimes_{R^{tC_2}} M_B^{tC_2} \to (M_A \otimes_R M_B)^{tC_2}$ where the latter map arises canonically from lax monoidality of the Tate construction.*

4. *Let $(\mathcal{C}, \Qoppa_{\mathcal{C}}), (\mathcal{D}, \Qoppa_{\mathcal{D}})$ be objects of $\mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}_\infty^{\mathrm{h}})$. Then the forgetful functor induces $\hom_{\mathrm{Cat}_{\infty R}^{\mathrm{h}}}((\mathcal{C}, \Qoppa_{\mathcal{C}}), (\mathcal{D}, \Qoppa_{\mathcal{D}})) \to \hom_{\mathcal{C}\mathrm{at}_{\infty R}^{\mathrm{ex}}}(\mathcal{C}, \mathcal{D})$ on mapping spaces so that the fiber over an R-linear functor $F \colon \mathcal{C} \to \mathcal{D}$ is the mapping space $\mathrm{map}_{\Qoppa_R}(F_! \Qoppa_{\mathcal{C}}, \Qoppa_{\mathcal{D}}) \simeq \mathrm{map}_{\Qoppa_R}(\Qoppa_{\mathcal{C}}, \Qoppa_{\mathcal{D}} \circ F^{\mathrm{op}})$, where the mapping space is taken in $\mathrm{Fun}^q_{\Qoppa_R}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp})$ and $\mathrm{Fun}^q_{\Qoppa_R}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$, respectively.*[1]

5. *The symmetric monoidal forgetful functor $\theta \colon \mathrm{Mod}_{(\mathrm{Mod}_R^\omega, \Qoppa_R)}(\mathrm{Cat}_\infty^{\mathrm{h}}) \to \mathrm{Mod}_{\mathrm{Mod}_R^\omega}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}})$ is a (co)cartesian fibration.*

L: what is it classified by?

Remark 4.14. A special case of part 2 is [Cal+20a, Example 5.4.13].

Proof. 1. The first part of the statement follows from [Cal+20a, §6.1] and [Lur17, §4.2.3].

2. Let $\mathcal{LM}^\otimes$ denote the ∞-operad of [Lur17, Definition 4.2.1.7]. Our strategy of proof will be similar to that of [Cal+20a, §5.3]: First, we show that an $\mathcal{LM}^\otimes$-algebra object in $\mathrm{Cat}_\infty^{\mathrm{h}}$ is equivalent to an $\mathcal{LM}^\otimes$-algebra object in an operad of functor categories. Then, we use a (suitably coherent version of) the classification of hermitian structures on module categories as categories of modules over the Hill–Hopkins–Ravenel norm [Cal+20a, Theorem 3.3.1] to conclude. Recall that the action of Mod_R^ω on Mod_A^ω is given by a functor $\mathcal{LM}^\otimes \to \mathcal{C}\mathrm{at}_\infty^\times$, and define $\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes$ via the following pullback square of ∞-operads:

$$\begin{tikzcd} \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes \arrow[r, "p"] \arrow[d] & \mathcal{LM}^\otimes \arrow[d, "{\mathrm{Mod}_R^\omega, \mathrm{Mod}_A^\omega}"] \\ (\mathcal{C}\mathrm{at}_\infty)^\otimes_{\mathrm{op}/-/\,\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}_\infty^\times \end{tikzcd} . \tag{4.15}$$

Informally, an object $F \in \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^\otimes_{\mathfrak{a}}$ is a functor $F \colon \mathrm{Mod}_R^{\omega,\mathrm{op}} \to \mathrm{Sp}$ and an object G over the fiber of $\mathfrak{m}$ is a functor $G \colon \mathrm{Mod}_A^{\omega,\mathrm{op}} \to \mathrm{Sp}$. The p-cocartesian edge over the canonical map

[1]The proof of 2 in particular shows that $\mathrm{Fun}^q(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is left-tensored over $\mathrm{Fun}^q(\mathrm{Mod}_R^{\omega,\mathrm{op}}, \mathrm{Sp})$ in the sense of [Lur17, Definition 4.2.1.19], so this makese sense.

$(\mathfrak{a}, \mathfrak{m}) \to \mathfrak{m}$ in $\mathcal{LM}^{\otimes}$ sends (F, G) to the lower arrow in the diagram

$$\begin{tikzcd}
\mathrm{Mod}_R^{\omega,\mathrm{op}} \times \mathrm{Mod}_A^{\omega,\mathrm{op}} \arrow[r, "F\times G"] \arrow[d, "-\otimes_R-"'] & \mathrm{Sp}\times\mathrm{Sp} \arrow[d, "\otimes_{\mathrm{Sp}}"] \\
\mathrm{Mod}_A^{\omega,\mathrm{op}} \arrow[r, "F\otimes G:=\mathrm{LKE}_{\otimes_R}(\otimes_{\mathrm{Sp}}\circ(F\times G))"] & \mathrm{Sp}
\end{tikzcd} .$$

Now define $\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes}$ to consist of the full subcategory of $\mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes}$ consisting of those tuples of functors which are all quadratic. The inclusion $\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \to \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes}$ exhibits the former as an ∞-operad, and moreover the localization is compatible with the $\mathcal{LM}^{\otimes}$-monoidal structure in the sense of [Lur17, Definition 2.2.1.6]. We can extend the previous diagram to

$$\begin{tikzcd}
\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r, "p"] \arrow[d] & \mathcal{LM}^{\otimes} \arrow[d, "\mathrm{Mod}_R^{\omega},\mathrm{Mod}_A^{\omega}"] \\
\mathrm{Cat}_\infty^{\mathrm{h}\ \otimes} \arrow[r] & (\mathcal{C}\mathrm{at}_\infty)^{\otimes}_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}_\infty^{\otimes}
\end{tikzcd} . \tag{4.16}$$

Modifying [Cal+20a, Construction 5.3.15 & Lemma 5.3.15] slightly (note that Corollary 5.1.4 did not assume the tensor factors to be equivalent), we obtain an analogous commutative diagram of ∞-operads

$$\begin{tikzcd}
\mathrm{Fun}^p_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes} \arrow[r, "p"] \arrow[d] & \mathcal{LM}^{\otimes} \arrow[d, "\mathrm{Mod}_R^{\omega},\mathrm{Mod}_A^{\omega}"] \\
\mathrm{Cat}_\infty^{\mathrm{p}\ \otimes} \arrow[r] & \mathrm{Cat}_\infty^{\mathrm{h}\ \otimes} \arrow[r] & (\mathcal{C}\mathrm{at}_\infty)^{\otimes}_{\mathrm{op}/-/\mathrm{Sp}} \arrow[r] & \mathcal{C}\mathrm{at}_\infty^{\otimes}
\end{tikzcd} \tag{4.17}$$

in which all squares are pullbacks. Now suppose A is given a module with genuine involution $(M_A, N_A, N_A \to M_A^{tC_2})$ and call the associated Poincaré ∞-category $\overline{\mathrm{Mod}}_A$. Then to lift $\overline{\mathrm{Mod}}_A$ to a module over $(\mathrm{Mod}_R^{\omega}, \Qoppa_R)$ compatibly with the Mod_R^{ω}-module structure on Mod_A^{ω} is to give a map of ∞-operads $\mathcal{LM}^{\otimes} \to \mathrm{Cat}_\infty^{\mathrm{h}\ \otimes}$ so that the restriction along the canonical inclusion $\mathrm{Assoc}^{\otimes} \to \mathcal{LM}^{\otimes}$ gives the algebra object $(\mathrm{Mod}_R^{\omega}, \Qoppa_R)$ and postcomposing with the canonical projection to $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}\times}$ recovers the given Mod_R^{ω}-module structure on Mod_A^{ω}. By the pullback square (4.16), this is equivalent to giving an object of $\mathrm{Alg}_{\mathcal{LM}/\mathcal{LM}}\left(\mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathrm{Mod}_A^{\omega,\mathrm{op}}, \mathrm{Sp})^{\otimes}\right)$. Now let us identify the bilinear functor $\mathrm{Mod}_R^{\omega} \times \mathrm{Mod}_A^{\omega} \xrightarrow{-\otimes_R-} \mathrm{Mod}_A^{\omega}$ with the exact functor $\mathrm{Mod}_R^{\omega} \otimes \mathrm{Mod}_A^{\omega} \simeq \mathrm{Mod}_{R\otimes A}^{\omega} \to \mathrm{Mod}_A^{\omega}$ which is induction along the action map $R \otimes A \to A$. Using [Cal+20a, Corollary 3.4.1] and unravelling definitions gives the claim for R-linear hermitian structures. The proof for R-linear Poincaré structures considers (4.17) instead but otherwise proceeds in an identical fashion.

3. By [Lur17, Theorem 4.4.2.8], the relative tensor product $(\mathrm{Mod}_A^{\omega}, \Qoppa_A) \otimes_{(\mathrm{Mod}_R^{\omega}, \Qoppa_R)} (\mathrm{Mod}_B^{\omega}, \Qoppa_B)$ is computed as the geometric realization of the bar construction

$$\begin{aligned} p\colon \Delta^{\mathrm{op}} &\to \mathrm{Cat}_\infty^{\mathrm{h}} \\ [n] &\mapsto (\mathrm{Mod}_A^{\omega}, \Qoppa_A) \otimes (\mathrm{Mod}_R^{\omega}, \Qoppa_R)^{\otimes n} \otimes (\mathrm{Mod}_B^{\omega}, \Qoppa_B) \end{aligned}$$

Write $f\colon \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ for the forgetful functor. Then $f \circ p$ has a colimit with value $\mathrm{Mod}_A^{\omega} \otimes_{\mathrm{Mod}_R^{\omega}} \mathrm{Mod}_B^{\omega} \simeq \mathrm{Mod}_{A\otimes_R B}^{\omega}$. Writing $g\colon \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \to \{*\}$, by Example 4.3.1.3 of [Lur09] we see that $f \circ p$ is a g-colimit. By Proposition 4.3.1.5(2) and Example 4.3.1.3 of [Lur09], p admits a colimit in $\mathrm{Cat}_\infty^{\mathrm{h}}$ if and only if it admits an f-colimit. Now recall that f is a cocartesian fibration with pushforward given by left Kan extension [Cal+20a, Corollary 1.4.2]. We show that f satisfies the conditions of [Lur09, Corollary 4.3.1.11].

- Condition (1) follows from Theorem 6.1.1.10 of [Lur17] applied to $\mathrm{Sp}^{\mathrm{op}}$ (see the end of [Cal+20a, Construction 1.1.26]).
- Condition (2) follows from [Cal+20a, Corollary 1.4.2], the adjoint functor theorem, and presentability of $\mathrm{Fun}^q(\mathcal{C})$, which is discussed in the proof of [Cal+20a, Lemma 5.3.3] (also see [Lur17, Remark 6.1.1.11]).

Thus the preceding discussion shows that there exists a map of simplicial sets p' making the diagram commute

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}^{\mathrm{h}}_{\infty} \arrow[d, "f"] \\ (\Delta^{\mathrm{op}})^{\triangleright} \arrow[r] \arrow[ur, "p'"] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \end{tikzcd} .$$

Since $\{0\} \to \Delta^1$ is left anodyne, by [Lur09, Corollary 2.1.2.7] the inclusions

$$\{0\} \times \Delta^{\mathrm{op}} \to \Delta^1 \times \Delta^{\mathrm{op}}$$
$$\iota\colon \left(\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}\right) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) \to \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright}$$

are left anodyne. The former implies that there exists a map p'' making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[r, "p"] \arrow[d] & \mathrm{Cat}^{\mathrm{h}}_{\infty} \arrow[d, "f"] \\ \Delta^1 \times \Delta^{\mathrm{op}} \arrow[r] \arrow[ur, "p''"] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \end{tikzcd}$$

commute. The maps p' and p'' assemble to give a map $p''' := p' \sqcup_p p''$ making the diagram

$$\begin{tikzcd} \{0\} \times \Delta^{\mathrm{op}} \arrow[rr, "p"] \arrow[d] & & \mathrm{Cat}^{\mathrm{h}}_{\infty} \arrow[d, "f"] \\ \left(\{0\} \times (\Delta^{\mathrm{op}})^{\triangleright}\right) \sqcup_{\{0\}\times\Delta^{\mathrm{op}}} \left(\Delta^1 \times \Delta^{\mathrm{op}}\right) \arrow[r, "\iota"] \arrow[urr, "p'''"] & \Delta^1 \times (\Delta^{\mathrm{op}})^{\triangleright} \arrow[r] \arrow[ur, bend right, "\overline{p}"'] & \mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty} \end{tikzcd}$$

commute, and likewise $\overline{p}$ exists making the diagram commute since ι is left anodyne. Now we show that $\overline{p}$ satisfies the conditions of [Lur09, Proposition 4.3.1.9]. By (the opposite/dual/cocartesian version of) [Lur09, Remark 3.1.1.10] and Proposition 3.1.1.5(2") *ibid.* and the fact that f is a cocartesian fibration, we can choose $\overline{p}$ so that for all $k \in (\Delta^{\mathrm{op}})^{\triangleright}$, $\overline{p}|_{\Delta^1\times\{k\}}$ is f-cocartesian. Furthermore, since we can choose Δ^{op}, $(\Delta^{\mathrm{op}})^{\triangleright}$ to have the markings $(-)^{\flat}$ in [Lur09, Remark 3.1.1.10], $f \circ \overline{p}|_{\Delta^1\times\{\infty\}}$ is a degenerate edge in $\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty}$.

Now [Lur09, Proposition 4.3.1.9] implies that $\overline{p}_0$ is an f-colimit diagram if and only if $\overline{p}_1$ is an f-colimit diagram. Now notice that $\overline{p}|_{\{1\}\times(\Delta^{\mathrm{op}})^{\triangleright}}$ has image contained in the fiber of f over $\mathrm{Mod}^{\omega}_{A\otimes_R B}$. By [Lur09, Proposition 4.3.1.10], it suffices to show that $\overline{p}_1$ is a colimit diagram in $\mathrm{Fun}^q\left(\mathrm{Mod}^{\omega}_{A\otimes_R B}\right)$. Write $\overline{M}_A \in \mathrm{Mod}_{N^{C_2}A}$ and $\overline{M}_B \in \mathrm{Mod}_{N^{C_2}B}$ for the corresponding modules (see introduction to §3.3 of [Cal+20a]). Unraveling definitions and using [Cal+20a, Theorem 3.3.1 & Corollary 3.4.1 & Lemma 5.4.6], it follows that the diagram $\overline{p}_1|_{\{1\}\times\Delta^{\mathrm{op}}}$ is the bar construction

$$[n] \mapsto \overline{M}_A \otimes_{N^{C_2}R} R^{\otimes_{N^{C_2}R} n} \otimes_{N^{C_2}R} \overline{M}_B \,.$$

This proves the result.

4. Let $(\mathcal{C}, Ϙ_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{\left(\mathrm{Mod}^{\omega}_R, Ϙ_R\right)}(\mathrm{Cat}^{\mathrm{h}}_{\infty})$ and let $F\colon \mathcal{C} = \theta\,(\mathcal{C}, Ϙ_{\mathcal{C}}) \to \mathcal{D}$ be an R-linear functor. Now define $Ϙ_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp}$ to be the left Kan extension of $Ϙ_{\mathcal{C}}$ along F^{op}. Now $(\mathcal{D}, Ϙ_{\mathcal{D}}) \in \mathrm{Cat}^{\mathrm{h}}_{\infty}$ and there

is a canonical map $(f,\eta)\colon (\mathcal{C},\Omega_{\mathcal{C}}) \to (\mathcal{D},\Omega_{\mathcal{D}})$. Now F is classified by a functor $\Delta^1 \times \mathcal{LM}^{\otimes} \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}\otimes}$, and we may form the pullback

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \Delta^1 \times \mathcal{LM}^{\otimes} \arrow[d] \\ {\mathrm{Cat}_{\infty}^{\mathrm{h}}}^{\otimes} \arrow[r, "p"] & \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}\otimes} \end{tikzcd} . \tag{4.18}$$

Since p is a cocartesian fibration [Cal+20a, Theorem 5.2.7], $\mathcal{N} \to \Delta^1 \times \mathcal{LM}^{\otimes}$ is a cocartesian fibration, and the nontrivial morphism in Δ^1 classifies a map $F_!\colon \mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathcal{C}^{\mathrm{op}},\mathrm{Sp})^{\otimes} \to \mathrm{Fun}^q_{\mathrm{Mod}_R^{\omega,\mathrm{op}}}(\mathcal{D}^{\mathrm{op}},\mathrm{Sp})^{\otimes}$ of ∞-operads over $\mathcal{LM}^{\otimes}$. Passing to algebra objects, we obtain the desired result on mapping spaces.

5. By [Lur09, Proposition 2.4.2.8], it suffices to show that θ is a locally (co)cartesian fibration, and that locally (co)cartesian edges are closed under composition. We give the proof that θ is a cocartesian fibration; the proof that θ is a cartesian fibration is formally dual and will be left to the reader.

   Let $(\mathcal{C},\Omega_{\mathcal{C}})$ be an object of $\mathrm{Mod}_{(\mathrm{Mod}_R^{\omega},\Omega_R)}(\mathrm{Cat}_{\infty}^{\mathrm{h}})$ and let $F\colon \mathcal{C} = \theta(\mathcal{C},\Omega_{\mathcal{C}}) \to \mathcal{D}$ be an R-linear functor. Now define $\Omega_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Sp}$ to be the left Kan extension of $\Omega_{\mathcal{C}}$ along F^{op}. By the proof of 4, we see that the image of $\Omega_{\mathcal{C}}$ under $F_!$ is a lift of $(\mathcal{D},\Omega_{\mathcal{D}})$ to an object of $\mathrm{Mod}_{(\mathrm{Mod}_R^{\omega},\Omega_R)}(\mathrm{Cat}_{\infty}^{\mathrm{h}})$ and (f,η) to a morphism in $\mathrm{Mod}_{(\mathrm{Mod}_R^{\omega},\Omega_R)}(\mathrm{Cat}_{\infty}^{\mathrm{h}})$.

   Now by Lemma 2.4.4.1 and the locally cocartesian version of Proposition 2.4.1.10 of [Lur09], we must show that for all choices $\Omega'_{\mathcal{D}}$ of an R-linear Hermitian structure on $\mathcal{D}$, precomposition with $F_!$ induces a pullback square

$$\begin{tikzcd} \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}((\mathcal{D},\Omega_{\mathcal{D}}),(\mathcal{D},\Omega'_{\mathcal{D}})) \arrow[r] \arrow[d] & \hom_{\mathrm{Cat}^{\mathrm{h}}_{\infty R}}((\mathcal{C},\Omega_{\mathcal{C}}),(\mathcal{D},\Omega'_{\mathcal{D}})) \arrow[d] \\ \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}(\mathcal{D},\mathcal{D}) \arrow[r] & \hom_{\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty R}}(\mathcal{C},\mathcal{D}) \end{tikzcd} . \tag{4.19}$$

   By 4, $F_!$ induces equivalences on the fibers of the vertical maps, hence (f,η) is locally θ-cocartesian. The locally θ-cocartesian maps are manifestly closed under composition, hence we are done. ☐

Corollary 4.20. *Let R be a Poincaré ring, and let A, B be $\mathbb{E}_1$-R-algebras with genuine involution. Then there is an equivalence* $\hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}}((\mathrm{Mod}_A^{\omega},\Omega_A),(\mathrm{Mod}_B^{\omega},\Omega_B)) \simeq (\mathrm{BiMod}_{A\otimes_R B^{\mathrm{op}}})_{A^{\varphi C_2}\otimes_R B^{\varphi C_2}/-}$.

Proof. ☐

L: todo–probably need to fix the statement with *duals* when the proof is written

As in the Picard group case, the forgetful functor θ induces a map of spectra $\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A)$ which will again factor through the 2-trosion on π_0. As a consequence of Proposition 2 we can identify the fiber of this map.

Corollary 4.21. *Let $(\mathrm{Mod}_A^{\omega},\Omega_A)$ be a Poincaré ring with underlying genuine C_2 spectrum A^L as in Proposition 2. Then the fiber of the map*

$$\mathrm{Br}^{\mathrm{p}}(A) \to \mathrm{Br}(A)$$

can be naturally identified with $\mathrm{Pic}(\mathrm{Mod}_{A^L}(\mathrm{Sp}^{C_2}))$.

Proof. Since $\theta : \mathrm{Mod}_{(\mathrm{Mod}_A^{\omega},\Omega_A)}(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}) \to \mathrm{Mod}_{\mathrm{Mod}_A}(\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}})$ is (co)cartesian, symmetric monoidal, and conservative, it follows that the induced functor on the groupoid core of invertible objects is a Kan fibration by [Lur24, Proposition 01EZ]. Consequently we need only identify the fiber instead of the homotopy fiber, which follows from the identifiaction of the fiber in Proposition 2. ☐

N: Needs a lot more detail I know, but I think the skeleton is here.

Write Pn for the composite $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \xrightarrow{U} \mathrm{Cat}^{\mathrm{p}}_{\infty} \xrightarrow{\mathrm{Pn}} \mathcal{S}$.

Proposition 4.22. *Let $(R, R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring. Then $(\mathrm{Mod}_R^{\omega},\Omega_R)$ corepresents the functor* $\mathrm{Pn}\colon \mathrm{Cat}^{\mathrm{p}}_{\infty R} \to \mathcal{S}$.

Proof. Recall that Proposition 4.12.1 furnishes an adjoint pair $\mathrm{Cat}^{\mathrm{p}}_{\infty R} \rightleftarrows \mathrm{Cat}^{\mathrm{p}}_{\infty}$ of functors. Write $\overline{\mathcal{C}} = (\mathcal{C}, Ϙ_{\mathcal{C}}) \in \mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R}$. Then

$$\mathrm{Pn}(\mathcal{C}) = \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty}}\left((\mathrm{Sp}^f, Ϙ^u), U(\overline{\mathcal{C}})\right) \simeq \hom_{\mathrm{Cat}^{\mathrm{p}}_{\infty R}}\left((\mathrm{Mod}^{\omega}_R, Ϙ_R) \otimes (\mathrm{Sp}^f, Ϙ^u), \overline{\mathcal{C}}\right),$$

where the first equivalence is [Cal+20a, Proposition 4.1.3]. □

4.3 Azumaya algebras with genuine involution

Let R be an $\mathbb{E}_\infty$-ring spectrum.

Recollection 4.23. Recall [BRS12; AG14] that an $\mathbb{E}_1$-R-algebra is said to be *Azumaya* if it is a compact generator of Mod_R and if the natural R-algebra map giving the bimodule structure on A

$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$$

is an equivalence of R-algebras.

Definition 4.24. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring spectrum. An *Azumaya algebra with genuine involution* over R is the data of

L: For context, compare Examples 3.1.9 and 3.2.9 of the first 9-author paper.

(a) An $\mathbb{E}_1$-R-algebra A equipped with an anti-involution $\tau \colon A \to A^{\mathrm{op}}$ so that A is an Azumaya R-algebra in the sense of Recollection 4.23

(b) A left $A \otimes_R R^{\varphi C_2}$-module $A^{\varphi C_2}$

(c) An equivalence of $A \otimes_R A^{\mathrm{op}}$-modules

L: terminology: "involution" or "anti-involution"?

$$\hom_R(A, R^{\varphi C_2}) \simeq A^{\varphi C_2} \otimes_{R^{\varphi C_2}} \hom_R(A^{\varphi C_2}, R^{\varphi C_2})$$

(d) An A-linear map $A^{\varphi C_2} \to A^{tC_2}$ where A^{tC_2} is regarded as an A-module via the Tate-valued Frobenius.

L: compatibility with Tate?

Remark 4.25. If A is an Azumaya algebra with genuine involution over R, then in particular $M_A = A$, $N_A = A^{\varphi C_2}$ is a module with genuine involution over A in the sense of [Cal+20a, Definition 3.2.3].

L: Tate-valued norm?

With ordinary Azumaya algebras, the prototypical Azumaya algebra with anti-involution arises from endomorphism rings of perfect modules. Choosing a (nondegenerate symmetric bilinear) form on a perfect module P endows its endomorphism algebra with additional structure.

L: define the category!

Example 4.26. Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(P, q) \in \mathrm{Pn}\left(\mathrm{Mod}^{\omega}_R, Ϙ_R\right)$.

Then $A := \mathrm{End}_R(P)$ admits a canonical lift to an Azumaya algebra with genuine involution over R with $A^{\varphi C_2} := \hom_R(P, R^{\varphi C_2})$.

L: to-do: explain! Also $A^{\varphi C_2}$ could also be $P \otimes_R R^{\varphi C_2}$?

By [Cal+20a, Proposition 3.1.16], A inherits a canonical anti-involution. Since P is compact, it is dualizable with respect to the symmetric monoidal structure on Mod^{ω}_R [Elm+07, Theorem III.7.9]. Since $\otimes_R R^{\varphi C_2}$ is symmetric monoidal, in particular it takes P to a dualizable object–call it $\overline{P}$. Now there is a canonical choice of equivalence (c) since both sides are canonically equivalent to $\overline{P} \otimes_{R^{\varphi C_2}} \overline{P}^{\vee}$.

Proposition 4.27. *Let R be the Eilenberg–Mac Lane spectrum associated to a discrete ring, and suppose R has a given C_2-action. Let A be a classical Azumaya algebra over R with an involution of type 2. Regard R as a Poincaré ring spectrum with the genuine symmetric Poincaré structure of Example 3.13.*

Then there is a canonical Azumaya algebra with genuine involution over R^{gs} so that $A^{\varphi C_2} := \tau_{\geq 0} A^{tC_2}$.

L: todo: Make some noises about an R-linear enhancement of Proposition 3.1.16?

Proof. □

Proposition 4.28. *Let $(R, R \to R^{\varphi C_2} \to R^{tC_2})$ be a Poincaré ring, and let $(A, A^{\varphi C_2} \to A^{tC_2})$ be an Azumaya algebra with genuine involution over R. Then $(\mathrm{Mod}^{\omega}_A, Ϙ_A)$ is an invertible object in $\mathrm{Mod}_{(\mathrm{Mod}^{\omega}_R, Ϙ_R)}\left(\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}}\right)$.*

L: todo

Remark 4.29. Contrast Proposition 4.28 with [AG14, Theorem 3.15], where it is shown that an R-linear stable ∞-category is invertible *if and only if* it is equivalent to modules over an Azumaya R-algebra. The difference lies in the fact that not every R-linear (Morita anti-)equivalence $\mathrm{Mod}_A^\omega \simeq \mathrm{Mod}_{A^{\mathrm{op}}}^\omega$ is induced by a map of $\mathbb{E}_1$-rings $A \to A^{\mathrm{op}}$.

L: this is not quite the same (in a literal sense), but I think similar in spirit to the "counterexamples" paper by First–Williams

Proof. First, by [Cal+20a, Example 3.2.9], we see that $(\mathrm{Mod}_A^\omega, \Qoppa_A)$ is indeed an R-linear Poincaré ∞-category (and not merely hermitian). To show that the associated Poincaré ∞-category is invertible, we must identify a dual $(\mathrm{Mod}_A^\omega, \Qoppa_A)^\vee$ and exhibit an equivalence $(\mathrm{Mod}_A^\omega, \Qoppa_A) \otimes (\mathrm{Mod}_A^\omega, \Qoppa_A)^\vee \simeq (\mathrm{Mod}_R^\omega, \Qoppa_R)$. Since $\mathrm{Cat}^{\mathrm{p}}_{\infty,\mathrm{idem}\,R} \to \mathcal{C}at^{\mathrm{ex}}_{\infty\,R}$ is symmetric monoidal, we see that the underlying R-linear ∞-category associated to the dual must be $\mathrm{Mod}_{A^{\mathrm{op}}}^\omega$. Moreover, the canonical evaluation map $\mathrm{ev}\colon \mathrm{Mod}_A^\omega \otimes \mathrm{Mod}_{A^{\mathrm{op}}}^\omega \xrightarrow{\simeq} \mathrm{Mod}_R^\omega$ sends $A \otimes A^{\mathrm{op}}$ to A. Endow $\mathrm{Mod}_{A^{\mathrm{op}}}$ with a Poincaré structure corresponding to the module with genuine involution $M_{A^{\mathrm{op}}} := A^{\mathrm{op}}$, $N_{A^{\mathrm{op}}} := \hom_R(A^{\varphi C_2}, R^{\varphi C_2})$. It remains to exhibit a natural equivalence

$$\eta\colon\ (\Qoppa_A \otimes \Qoppa_{A^{\mathrm{op}}}) \xrightarrow{\simeq} \mathrm{ev}^*\Qoppa_R \tag{4.30}$$

of [quadratic] functors $\mathrm{Mod}_A^\omega \otimes \mathrm{Mod}_{A^{\mathrm{op}}}^\omega \to \mathrm{Sp}$. By [Cal+20a, Theorem 3.3.1], it suffices to exhibit equivalences on the bilinear and linear parts of (4.30) which glue compatibly. $\square$

L: finish

5 Poincaré schemes

Definition 5.1. Let APS be the $(\infty,1)$-category defined by the pullback

$$\begin{tikzcd}
\mathrm{APS} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \arrow[d, "d_1^*"] \\
\mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "U(-)\to(-)^{tC_2}"] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp}))
\end{tikzcd}$$

where $U : \mathrm{Sp}^{BC_2} \to \mathrm{Sp}$ is the functor which forgets the C_2-action.

N: I keep trying to make this work but the technical details are actively killing me. Better, I think, to use the following definition instead.

Definition 5.2. Define the catgeory of affine Hermetian schemes, denoted AHS, to be the infinity category given by the Grothendieck construction applied to the functor

$$\mathrm{CAlg}(\mathrm{Sp})^{op} \to \mathrm{Cat}_\infty$$

given by sending a ring R to the category $\mathrm{CAlg}(\mathrm{Mod}_{NR})$ of $\mathbb{E}_\infty$ algebras in modules with genuine involutions over R. Then define the category of affine Poincare schemes, denoted by APS, to be the full subcategory of AHS spanned by the pairs (R, M) where $M \in \mathrm{CAlg}(\mathrm{Mod}_{NR})$ is invertible.

We record here a few structural results about this category.

Theorem 5.3. *The following statements about* APS *hold:*

1. *The category* APS *is a cocomplete and symmetric monoidal infinite category;*
2. *the pullback diagram above is homotopy Cartesian;*
3. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ *is symmetric monoidal and (co)continuous;*
4. *the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}$ *is lax symmetric monoidal;*
5. *and the functor* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2} \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathrm{Sp})$ *is symmetric monoidal.*

Proof. For (2) it is enough to show that d_1^* is an categorical fibration which follows from [Lur09, Corollary 2.3.2.5] and [Lur09, Corollary 2.4.6.5]. In fact $d_!^*$ is a left fibration by [(]taking $p = id$, $i = d_1 : \Delta^1 \to \Delta^2$) There is a (pseudo-)functor

L: Is this reference correct? The conclusion asserts that some map of simplicial sets is a *categorical* fibration. The follow-

$$\begin{aligned} F\colon\ \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) &\to \mathcal{C}\mathrm{at}_\infty \\ (\varphi\colon A \to B) &\mapsto \left(\left(\mathrm{CAlg}(\mathrm{Sp})_{A/-/B}\right)_{/\varphi}\right) \end{aligned}$$

which sends a square

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\psi"] & D \end{tikzcd} \tag{5.4}$$

regarded as a morphism from φ to ψ, to the functor

$$\begin{aligned} \left(\mathrm{CAlg}(\mathrm{Sp})_{A/-/B}\right)_{/\varphi} &\to \left(\mathrm{CAlg}(\mathrm{Sp})_{C/-/D}\right)_{/\psi} \\ (A \to R \to B) &\mapsto C \simeq A \otimes_A C \xrightarrow{\varphi \otimes \mathrm{id}_C} B \otimes_A C \to D \end{aligned} \tag{5.5}$$

where $B \otimes_A C \to D$ is the canonical map induced by the commuting square (5.4). The functor F classifies the cocartesian fibration d_1^*.

For (3), let $p : K \to \mathrm{APS}$ be a map of simplicial sets, K a small simplicial set. Suppose the $K^{\triangleright} \to \mathrm{APS}$ be an extension such that $K^{\triangleright} \to \mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{BC_2})$ is a colimit diagram. By [Lur09, Proposition 2.4.3.2] the diagram

$$\begin{tikzcd} \mathrm{APS}_{p/} \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}_{p/-} \arrow[d] \\ \mathrm{CAlg}(\mathrm{Sp}^{BC_2})_{p/} \arrow[r] & \mathrm{CAlg}(\mathrm{Sp})^{\Delta^1}_{p/-} \end{tikzcd}$$

is again homotopy cartesian. Then

$$\begin{aligned} \hom_{\mathrm{APS}}(p(\infty), -) &\simeq \hom_{\mathrm{CAlg}(\mathrm{Sp}^{BC_2})}(p(\infty), -) \times_{\hom_{\mathrm{CAlg}(\mathrm{Sp})^{\Delta^1}}(p(\infty),-)} \hom_{\mathrm{CAlg}(\mathrm{Sp})^{\Delta^2}}(p(\infty)) \\ &\simeq \end{aligned}$$

□

We will denote elements of APS by $\underline{A} = (A, s : A^{\Phi C_2} \to A^{tC_2})$. Here $s : A^{\Phi C_2} \to A^{tC_2}$ is the image of $\underline{A}$ under the top horizontal map above. The use of the notation $A^{\Phi C_2}$ is justified by the following.

Lemma 5.6. *Let* $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp})$ *be the composition of the functors*

$$\mathrm{APS} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \xrightarrow{ev_{[1]}} \mathrm{CAlg}(\mathrm{Sp}).$$

Then this functor factors as a composition $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2}) \xrightarrow{(-)^{\Phi C_2}} \mathrm{CAlg}(\mathrm{Sp})$.

Proof. The commutativity of the diagram

$$\begin{tikzcd} & \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \arrow[d, "d_1^*"] \arrow[dr, "d_0^*"] & \\ & & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \arrow[dd, "ev_{[1]}"] \\ \mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "U(-) \to (-)^{tC_2}"] \arrow[dr, "id"] & \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \arrow[dr, "ev_{[1]}"] & \\ & \mathrm{CAlg}(\mathrm{Sp}^{BC_2}) \arrow[r, "(-)^{tc_2}"] & \mathrm{CAlg}(\mathrm{Sp}) \end{tikzcd}$$

induces a functor on the pullback infinity categories $\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2})$ which makes the corresponding cube commute. The functor $ev_{[1]} : \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \to \mathrm{CAlg}(\mathrm{Sp})$ factors through d_0^* and so $\mathrm{APS} \to \mathrm{Fun}(\Delta^2, \mathrm{CAlg}(\mathrm{Sp})) \to \mathrm{CAlg}(\mathrm{Sp})$ is equivalent to the composition

$$\mathrm{APS} \to \mathrm{CAlg}(\mathrm{Sp}^{C_2}) \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{Sp})) \to \mathrm{CAlg}(\mathrm{Sp})$$

and the composition of the last two maps is the geometric fixed point functor as desired. □

The following Lemma gives the justification of the name Poincaré scheme.

Construction 5.7. We shall construct a functor

$$\mathrm{Perf}^{\mathrm{Pn}} : \mathrm{APS} \to \mathrm{Cat}_{\infty}^{\mathrm{Pn}}$$

to the category of Poincaré infinity categories.

L: For symmetric monoidal structure– maybe want to swap out Mod_{NA} for CAlg_{NA}?

Recall that $\mathrm{Cat}^{\mathrm{h}}_{\infty\infty} \to (\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty\infty})^{op}$ is a cocartesian fibration [Cal+20a, §1.4.] We will first construct a map of cocartesian fibrations

$$\begin{tikzcd} \mathrm{APS} \arrow[r, dotted] \arrow[d] & \mathrm{Cat}^{\mathrm{h}}_{\infty\infty} \arrow[d] \\ \mathrm{CAlg}\left(\mathrm{Sp}^{BC_2}\right) \arrow[r] & (\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty\infty})^{op} \end{tikzcd} , \tag{5.8}$$

then show that the dotted arrow factors through the subcategory $\mathcal{C}\mathrm{at}^{p}_{\infty} \subseteq \mathcal{C}\mathrm{at}^{h}_{\infty}$. To construct a map of cartesian fibrations, it suffices to exhibit a natural transformation of classifying functors. Unraveling the definitions, by Theorem 3.2.13 of [Cal+20a] we must exhibit for each $A \in \mathrm{CAlg}(\mathrm{Sp})^{BC_2}$, a functor

$$\left(\mathrm{CAlg}(\mathrm{Sp})_{A/-/A^{tC_2}}\right)_{/\varphi} \to \mathrm{Mod}_{N^{C_2}(A^e)}\left(\mathrm{Sp}^{C_2}\right) \tag{5.9}$$

(where $\varphi \colon A \to A^{tC_2}$ is the Tate-valued Frobenius and N^{C_2} is the Hill–Hopkins–Ravenel norm) which is natural in A.

That the resulting functor factors through the subcategory $\mathcal{C}\mathrm{at}^{p}_{\infty}$ follows from Proposition 3.1.3 and Lemma 3.3.3 of *loc. cit.*

Lemma 5.10. *The functor of Construction 5.7 is symmetric monoidal and has essential image the subcategory spanned by objects* $(\mathrm{Perf}(R), \Omega)$ *which are* $\mathbb{E}_\infty$*-algebras.*

Definition 5.11. A map $f : \underline{A} \to \underline{B} \in \mathrm{APS}$ is faithfully flat if the underlying map $f : A \to B$ is faithfully flat and the map $f^{\Phi C_2} : A^{\Phi C_2} \to B^{\Phi C_2}$ is also faithfully flat.

Lemma 5.12. *The fpqc covers on* APS *form a Grothendieck site.*

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