

# Et cetera

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## Abstract

Dumping ground for other stuff: Notes, one-off observations, stuff that we can collectively use when preparing talks, etc.

L: I make no promises re: organization but I will do my best to keep it reasonably readable

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## 1 Talk prep

## 2 References

- [Involutions of Azumaya algebras](#) by First and Williams (2020 *Documenta*)
- [Counterexamples in involutions of Azumaya algebras](#) by First and Williams; much more readable than the 2020 *Documenta* paper
- [Azumaya algebras without involution](#) by Auel, First, and Williams: the introduction of this provides a very helpful historical overview of the connection between involutions on Azumaya algebras and 2-torsion/kernel of  $\mathrm{coRes}$

### 3 Questions and directions

**Question 3.1** (Literature). • In [PS92] Parimala–Srinivas assume that 2 is invertible in the ring of functions. Has anyone been able to extend their results to the 2 not necessarily invertible case in the meantime?

**Question 3.2** (Morita theory for  $\text{Cat}_\infty^{\text{P}}$ ). Let  $R$  be a Poincaré ring. Suppose given two  $R$ -algebras (suitably interpreted so their module categories are canonically endowed with  $R$ -linear Poincaré structures—perhaps  $\mathbb{E}_\sigma$ )  $A, B$ . Can we characterize

$$\text{hom}_{\text{Cat}_\infty^{\text{P}} R}((\text{Mod}_A^\omega, \varphi_A), (\text{Mod}_B^\omega, \varphi_B))$$

in terms of something bimodule-like?

**Question 3.3.** On page 2 of the *Counterexamples* paper, First and Williams write that “existence of an extraordinary involution means classification of Azumaya algebras with involution...*cannot* be reduced to questions about projective modules and hermitian forms on them.”

What if we replaced projective modules by perfect complexes?

**Question 3.4.** First–Williams show (see discussion in §4 of the *Counterexamples* paper) that coarse type classify many (most?) Azumaya algebras up to (étale-local) *isomorphism*.

What is a suitable derived version of “coarse type”?

**Question 3.5** (asked by Andrew Nov 2, 2024). C. Schlichtkrull shows in [this paper](#) that a map  $BGL_1(R) \rightarrow K(R) \rightarrow THH(R) \rightarrow R$  in terms of the Hopf map  $\eta$ .

Is there a “Poincaré” version of this result?

### 4 Thoughts & observations

**Question 4.1.** When  $R$  has the Tate Poincaré structure and  $(\text{Mod}_A^\omega, M_A, N_A, N_A \rightarrow M_A^{tC_2})$  is invertible, then by invertibility have an equivalence  $\text{hom}_R(A, R) \simeq N_A \otimes_R N_{A^{\text{op}}}$  of  $A \otimes_R A^{\text{op}}$ -modules. Restricting the left-hand side along the unit map  $R \rightarrow A$  gives a map  $N_A \otimes_R N_{A^{\text{op}}} \rightarrow \text{hom}_R(R, R) \simeq R$ . Is this a perfect ( $R$ -linear) pairing?

I *think* using that  $R^{\varphi^{C_2}} \simeq R$  and combining the linear and bilinear part conditions, we get something like

$$M_A \otimes_R M_{A^{\text{op}}} \simeq (N_A \otimes_R N_{A^{\text{op}}})^{\otimes_{R^2}} \quad \text{as } A \otimes_R A^{\text{op}}\text{-bimodules.}$$

Is this useful?

**Brauer-Severi schemes** We know there is a correspondence between Azumaya algebras  $A$  over  $X$  and Brauer-Severi schemes. What does a Poincaré structure on  $\text{Mod}_A^\omega$  mean ‘geometrically’ for  $D_{\text{coh}}^b$  of the corresponding Brauer-Severi scheme? (Lucy: I didn’t get very far here, but just typing up what I had)

- $\text{Mod}_A^\omega$  corresponds to  $\alpha$ -twisted sheaves on  $X$  (see Proposition 3.2.2.1 of Max Lieblich’s thesis)
- The bounded derived category of  $\alpha$ -twisted sheaves on  $X$  includes as one ‘piece’ of a semiorthogonal decomposition on  $D_{\text{coh}}^b$  of the corresponding Brauer-Severi scheme (see Theorem 5.1 [here](#))

### 5 Desperate Flailing

This section is a cronical of my thoughts about  $\mathbb{G}_m^\circ$ .

**Goal** The goal is to build a Poincaré ring  $\mathbb{G}_m^q := (\text{Mod}_R, \mathbb{Y}_R)$  such that  $B\mathbb{G}_m^q(\underline{S}) = \text{Pic}^P(\underline{S})$  for any Poincaré ring  $\underline{S}$ .

**Lemma 5.1.** *Let  $\underline{S}$  be a Poincaré ring. Then  $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u)) = \{s \in \pi_0(S)^\times \mid s = 1 \text{ in } \pi_0(S^{C_2})\}$ .*

*Proof.* Since the functor  $\text{Pn}(\text{Mod}_S) \rightarrow \text{Mod}_S$  is conservative it follows that an element of  $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u))$  must have underlying map an element of  $\pi_0 \text{Aut}(S) = \pi_0(S)^\times$ . Then in order for  $s \in \pi_0(S)^\times$  to induce a map  $(S, u) \rightarrow (S, u)$ , the induced map  $s^* : S^{C_2} \rightarrow S^{C_2}$  must satisfy  $s^*(u) = u$ . The pullback is given by multiplication by  $s$ , so this requirement translates into  $s$  being the unit, as desired.  $\square$

The problem I thought existed maybe doesn't. Here is a candidate construction:

**Construction 5.2.** Define  $R$  to be the  $\mathbb{E}_\infty$  ring given by  $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z\}} \mathbb{S}$  where the map  $\mathbb{S}\{z\} \rightarrow \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$  is induced by the map  $z \mapsto xy$ , and the map  $\mathbb{S}\{z\} \rightarrow \mathbb{S}$  is induced by  $z \mapsto 1$ . We can give  $R$  an  $\mathbb{E}_\infty$  ring structure in  $\text{Sp}^{BC_2}$  by taking the trivial action on  $\mathbb{S}\{z\}$  and  $\mathbb{S}$ , and taking the action induced by  $x \mapsto y$  and  $y \mapsto x$  on  $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ . Thus in  $\text{CAlg}(\text{Sp}^{BC_2})$  the ring  $R$  corepresents the functor  $S \mapsto \{s \in \pi_0(S)^\times \mid s\sigma(s) = 1\}$ .

Now take  $\underline{R}$  to be the Poincaré ring with underlying Borel  $C_2$  structure as described in the previous paragraph and geometric fixed points  $R^{\varphi C_2} = \mathbb{S}$  and the map  $R^{\varphi C_2} \rightarrow R^{tC_2}$  given by the unit map. Endowing  $R^{\varphi C_2}$  with the  $R$ -module structure given by  $x, y \mapsto 1$ , it remains to show that the unit map  $R^{\varphi C_2} \rightarrow R^{tC_2}$  factors the Tate valued Frobenius  $R \rightarrow R^{tC_2}$  in order to promote  $\underline{R}$  to a Poincaré ring. By construction of  $R$  it is then enough to show that on  $\pi_0$  the Tate valued Frobenius sends  $x, y \mapsto 1$  in  $\pi_0(R^{tC_2})$ . This map sends both  $x$  and  $y$  to  $xy \in \pi_0(R^{tC_2})$ . These are equal to 1 in  $\pi_0(R^{tC_2})$  since the functor  $(-)^{tC_2}$  is lax-monoidal so  $R^{tC_2}$  is a module over  $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$  which has the image of  $xy$  equal to 1.

Now consider another Poincaré ring  $\underline{S}$ . We then have that maps  $\pi_0(\text{Maps}(\underline{R}, \underline{S}))$  is the data of a unit  $s \in \pi_0(S)^\times$ , a path  $s\sigma(s) \rightarrow 1$  in  $\Omega^\infty S$ , and paths  $x, y \rightarrow 1$  in  $\Omega^\infty S^{\varphi C_2}$ . This then agrees with  $\mathbb{G}_m^q$  by the following lemma.

**Lemma 5.3.** *Let  $S \in \text{CAlg}(\text{Sp}^{BC_2})$  and  $s \in \pi_0(S)^\times$ . Then  $s\sigma(s) = 1$  in  $\pi_0(S)$  if and only if  $(s \otimes s)^*$  acts by 1 on  $\pi_0(S^{hC_2}) = \pi_0(\text{Hom}_{S \otimes S}(S \otimes S, S)^{hC_2})$ .*

*Proof.* The 'only if' direction follows from the fact that the map  $S^{hC_2} \rightarrow S$  is an  $S$ -bimodule map. Now suppose that  $s\sigma(s) = 1$  in  $S$ . Then before taking homotopy fixed points the induced map  $s^* = id$  because  $S$  is  $\mathbb{E}_\infty$ .<sup>1</sup>  $\square$

## 6 Modules with genuine involution

**Remark 6.1** (Lucy). I'm just going to put drafts of stuff pertaining to hermitian modules here. Eventually when it gets to be more complete, I will hopefully move this entire section over to the main file.

L: or whatever we want to keep calling these

**Meta-commentary** There are (at least) three things we want to do:

- (a) Define a category of 'bimodules with involution over algebras with anti-involution' equipped with a forgetful functor  $\Theta : \text{BMod}_{\text{inv}}(-) \rightarrow \mathbb{E}_1 \text{Alg}(-)^{hC_2}$ .
- (b) Show that  $\Theta$  is a coCartesian fibration. For this, it suffices to show that it is a *Cartesian* fibration and that it satisfies the hypotheses of [Lur09, Corollary 5.2.2.5]
  - I used to think that we could obtain this by 'bootstrapping' a result from Higher Algebra, plus some facts about assembly. This doesn't seem to be working, so I'm just going to try to do this directly (imitating certain aspects of Chapter 4 of higher algebra.)
- (c) Define a relative tensor product for hermitian bimodules
- (d) Show that the formula for the cocartesian pushforward along a map  $A \rightarrow B$  in  $\mathbb{E}_1 \text{Alg}(-)^{hC_2}$  is something like  $- \otimes_{A \otimes A^{\text{op}}} (B \otimes B^{\text{op}}) \otimes_{B \otimes B^{\text{op}}} B$ .

<sup>1</sup>Or just  $\mathbb{E}_2$ .

- In Higher Algebra, the formula for the cocartesian pushforward is proven in [Lur17, §4.6]; in particular, this is in the section on duality. In particular, see Proposition 4.6.2.17 and the paragraph immediately preceding this.
- I don't know how to do this yet—while (a) and (b) are not useful if I can't show (c), I can't suss out the feasibility of (c) without (a) and (b) already in place.

(e) Towards an adjunction between  $\mathbb{E}_\sigma$ -algebras and categories with additional structure.

- Involutive version of statement that, for a monoidal  $\infty$ -category  $\mathcal{C}$  and an  $\mathbb{E}_1$ -algebra  $A$ ,  $\mathrm{LMod}_A(\mathcal{C})$  is right-tensored over  $\mathcal{C}$ ?
- Involutive version of endomorphism categories? [Lur17, §4.7.1]

I think that the equivalence of part (b) of the definition of an Azumaya algebra with genuine involution follows from the property of being Azumaya; see Lemma 1(b) (and p.216 for the ‘type 2’ case) of [PS92].

**Lemma 6.2.** *Let  $R$  be an  $\mathbb{E}_\infty$ -ring with an involution  $\sigma: R \xrightarrow{\sim} R$  and suppose  $A$  is an  $\mathbb{E}_1$ - $R$ -algebra with an anti-involution  $\lambda: A \xrightarrow{\sim} \sigma^* A^{\mathrm{op}}$ . Suppose  $A$  is further Azumaya in the sense of . Then the bilinear pairing*

$$A \otimes_R \sigma^* A \xrightarrow{\mathrm{id} \otimes \sigma^* \lambda} A \otimes_R A^{\mathrm{op}} \simeq \mathrm{End}_R(A) \xrightarrow{\mathrm{tr}} R$$

*is perfect, i.e. its adjoint  $A \rightarrow (\sigma^* A)^\vee$  is an equivalence.*

**Question 6.3.** Does the map in part (e) of the definition of an Azumaya algebra with genuine involution follow from property of being Azumaya?

## 6.1 Step (a)

**Definition 6.4.** Define a colored operad  $\mathrm{Assoc}_\sigma$  as follows:

- The colored operad has a single object, which we denote by  $\mathfrak{a}$ .
- For every finite set  $I$ , the set of operations  $\mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ , where  $\mathcal{L}I$  is the set of linear orderings on  $I$  and an element of  $\{\pm 1\}^I$  is a function  $I \rightarrow \{\pm 1\}$ .
- Suppose given a map of finite sets  $\alpha: I \rightarrow J$ , together with operations  $(\preceq_j, f_j: I_j \rightarrow \{\pm 1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathfrak{a}_i\}_{\alpha(i)=j}, \mathfrak{a})$  and  $(\preceq_J, g: J \rightarrow \{\pm 1\}) \in \mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathfrak{a}_j\}_{j \in J}, \mathfrak{a})$ . Define a linear ordering on the set  $I$  as follows:  $i \leq i'$  if  $\alpha(i) \preceq_J \alpha(i')$  or  $\alpha(i) = \alpha(i') = j$  and  $i \preceq_j i'$  and  $g(j) = +1$  or  $\alpha(i) = \alpha(i') = j$  and  $i \succeq_j i'$  and  $g(j) = -1$ . Finally, define a function

$$\begin{aligned} I &\rightarrow \{\pm 1\} \\ i &\mapsto f_{\alpha(i)}(i) \cdot g(\alpha(i)), \end{aligned}$$

where the multiplication on  $\{\pm 1\}$  is the usual one.

**Remark 6.5.** There is a map of colored operads  $\iota: \mathrm{Assoc} \rightarrow \mathrm{Assoc}_\sigma$  which is the identity on objects and on operations  $\mathrm{Mul}_{\mathrm{Assoc}}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \rightarrow \mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\mathrm{id}_{\mathcal{L}I} \times \{c_1\}$  where  $c_1$  is the constant function on  $I$  with value 1.

There is another map of colored operads  $\iota^{\mathrm{rev}}: \mathrm{Assoc} \rightarrow \mathrm{Assoc}_\sigma$  which is the identity on objects and on operations  $\mathrm{Mul}_{\mathrm{Assoc}}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \rightarrow \mathrm{Mul}_{\mathrm{Assoc}_\sigma}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  sends a linear ordering  $\ell$  to  $(\ell^{\mathrm{rev}}, c_{-1})$  where  $c_{-1}$  is the constant function on  $I$  with value 1.

**Definition 6.6.** Let  $\mathrm{Assoc}_\sigma^\otimes$  denote the associated  $\infty$ -operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

**Remark 6.7.** Unwinding definitions

- Objects  $\mathrm{Assoc}_\sigma^\otimes$  are finite pointed sets  $\langle n \rangle \in \mathrm{Fin}_*$

L: reference

L: This is just an imitation of [Lur17, Definition 4.1.1.1], modified in accordance with ideas from §5.4.2.

- Morphisms  $\langle m \rangle \rightarrow \langle n \rangle$  consist of
  - $\alpha: \langle m \rangle \rightarrow \langle n \rangle$  a map of finite pointed sets
  - for each  $i \in \langle n \rangle^\circ$ , a linear ordering  $\preceq_i$  on the inverse image  $\alpha^{-1}(\{i\})$
  - a map of sets  $s: \alpha^{-1}(\langle m \rangle^\circ) \rightarrow \{\pm 1\}$
- For each pair of morphisms

$$(\beta: \langle \ell \rangle \rightarrow \langle m \rangle, \preceq_j, s) \quad (\alpha: \langle m \rangle \rightarrow \langle n \rangle, \preceq_i, t),$$

the composite is the triple  $(\alpha \circ \beta, \preceq_j'', u)$  where  $\preceq_j''$  is the ordering on  $(\alpha \circ \beta)^{-1}(\{i\})$  so that if  $a, b \in \langle \ell \rangle$  so that  $\alpha(\beta(a)) = \alpha(\beta(b))$ , then  $a \preceq_j'' b$  if  $\beta(a) \preceq_i \beta(b)$  or  $\beta(a) =_i \beta(b) = i$  and  $a \preceq_i b$  if  $s(i) = 1$  or  $a \succeq_i b$  if  $s(i) = -1$ . Finally  $u(l) = s(l) \cdot t(\beta(l))$ .

**Remark 6.8.** The maps  $\iota, \iota^{\text{rev}}$  of Remark 6.5 induce maps of  $\infty$ -operads  $\text{Assoc}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ . There is a canonical identification  $\iota^{\text{rev}} = \sigma \circ \iota$ , where  $\sigma$  is the automorphism of the associative operad considered in [Lur17, Remark 4.1.1.7].

Note that each object  $\langle n \rangle \in \text{Assoc}_\sigma^\otimes$  has a distinguished automorphism  $\text{rev}_{\langle n \rangle}$  of order two given by the identity map on  $\langle n \rangle$  and the constant map  $c_{-1}: \langle n \rangle^\circ \rightarrow \{\pm 1\}$  at  $-1$ . There is a canonical natural equivalence  $\iota \xrightarrow{\sim} \iota^{\text{rev}}$  whose component at  $\langle n \rangle$  is  $\text{rev}_{\langle n \rangle}$ .

**Definition 6.9.** Let  $\mathcal{C}^\otimes$  be a  $\infty$ -operad equipped with the data of a fibration  $p: \mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ . Let  $\text{Alg}^\sigma(\mathcal{C})$  denote the  $\infty$ -category  $\text{Alg}/_{\text{Assoc}_\sigma^\otimes}(\mathcal{C})$  of  $\infty$ -operad sections of  $p$ . We will refer to  $\text{Alg}^\sigma(\mathcal{C})$  as the  $\infty$ -category of *involutive algebra objects* of  $\mathcal{C}$ .

An *involutive monoidal  $\infty$ -category* is the data of a cocartesian fibration  $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ .

**Remark 6.10.** Suppose given a cocartesian fibration  $f: \mathcal{D}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  of  $\infty$ -operads. Write  $\mathcal{C}^\otimes := \mathcal{D}^\otimes \times_{\text{Assoc}_\sigma^\otimes, \iota} \text{Assoc}^\otimes$ ;  $\mathcal{C}^\otimes$  is a monoidal  $\infty$ -category in the sense of [Lur17, Definition 4.1.1.10]. Furthermore,  $\mathcal{C}_{\text{rev}}^\otimes := \mathcal{D}^\otimes \times_{\text{Assoc}_\sigma^\otimes, \iota^{\text{rev}}} \text{Assoc}^\otimes$  is a monoidal  $\infty$ -category. By Remark 6.8, this notation is consistent with that of [Lur17, Remark 4.1.1.7]. In particular, a  $\text{Assoc}_\sigma$ -monoidal  $\infty$ -category  $\mathcal{D}^\otimes$  determines a monoidal  $\infty$ -category  $\mathcal{C}^\otimes$  equipped with a monoidal equivalence  $\sigma_{\mathcal{C}}: \mathcal{C}^\otimes \xrightarrow{\sim} \mathcal{C}_{\text{rev}}^\otimes$ . Pullback along the involution of  $\text{Assoc}^\otimes$  determines another monoidal equivalence  $\sigma_{\mathcal{C}}^{\text{rev}}: \mathcal{C}_{\text{rev}}^\otimes \xrightarrow{\sim} \mathcal{C}^\otimes$ , and our assumptions imply that  $\sigma_{\mathcal{C}}^{\text{rev}} \circ \sigma_{\mathcal{C}}$  is equivalent to the identity on  $\mathcal{C}^\otimes$ .

Now suppose that  $A$  is an involutive algebra object of  $\mathcal{D}$ . With the same notation as before, pullback along  $\iota$  (resp.  $\iota^{\text{rev}}$ ) determines associative algebra objects  $u(A), u^{\text{rev}}(A)$  of  $\mathcal{C}$  and  $\mathcal{C}_{\text{rev}}$ , respectively. Note that  $\sigma_{\mathcal{C}}(u(A))$  is an algebra object of  $\mathcal{C}_{\text{rev}}$ , which we may regard as an algebra object of  $\mathcal{C}$  by precomposing with the autoequivalence  $\sigma: \text{Assoc}^\otimes \xrightarrow{\sim} \text{Assoc}^\otimes$ . It follows from Remark 6.8 that  $A$  determines an equivalence  $\sigma_A: u(A) \xrightarrow{\sim} \sigma_{\mathcal{C}}(u(A))^{\text{rev}}$  of algebra objects in  $\mathcal{C}$ .

Now suppose furthermore that  $\mathcal{D}^\otimes$  is of the form  $\mathcal{E}^\otimes \times_{\text{Fin}_*} \text{Assoc}_\sigma^\otimes$  for some symmetric monoidal  $\infty$ -category  $\mathcal{E}$ . Then the associated involution  $\sigma_{\mathcal{C}}$  is the identity, and for any involutive algebra object  $A$  of  $\mathcal{D}$ ,  $\sigma_A$  is an equivalence  $u(A) \simeq u(A)^{\text{rev}}$  satisfying  $\sigma_A^{\text{rev}} \circ \sigma_A \simeq \text{id}_A$ .

**Definition 6.11.** Define a category  $\Delta_\sigma$

- objects are pairs  $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$
- a morphism from  $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$  to  $([m], t: \{0, 1, \dots, m\} \rightarrow \{\pm 1\})$  is an order-preserving map  $[n] \rightarrow [m]$  in  $\Delta$ .

**Construction 6.12.** Define a functor  $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$ :

- For each  $([n], s)$ , we have  $\text{Cut}([n], s) = \langle n \rangle$ .
- Given a morphism  $\alpha: ([n], s) \rightarrow ([m], t)$ , the associated morphism  $\text{Cut}([n], s) \rightarrow \text{Cut}([m], t)$  consists of
  - On underlying finite pointed sets  $\langle m \rangle \rightarrow \langle n \rangle$ , Cut agrees with that appearing in [Lur17, Construction 4.1.2.9]

L: Note that when  $s, t$  are identically one, the resulting order  $\preceq_j''$  agrees with the lexicographic order defined in [Lur17, Remark 4.1.1.4].

L: do we need weaker than cocartesian fibration?

L: maybe better to write  $s$  as a function defined on the set of morphisms  $i < i+1$  in  $[n]$ .

- Identifying the cut  $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$  with the morphism  $j - 1 < j$ , we may regard  $s: \langle n \rangle^\circ \rightarrow \{\pm 1\}$  and likewise  $t: \langle m \rangle^\circ \rightarrow \{\pm 1\}$ . Define  $u: \text{Cut}(\alpha)^{-1}(\langle n \rangle^\circ) \rightarrow \{\pm 1\}$  to be the unique function so that  $u(j)t(j) = s(\text{Cut}(\alpha)(j))$ .

**Lemma 6.13.** *The functor  $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$  exhibits  $\Delta_\sigma^{\text{op}}$  as an approximation to the  $\infty$ -operad  $\text{Assoc}_\sigma^\otimes$ .*

*L: I think the proof of this lemma is not too different from the proof of Proposition 4.1.2.11 of [Lur17]; the point here is just to unravel the definitions of locally coCartesian and Cartesian; the morphisms in  $\Delta_\sigma^{\text{op}}$  are a little more complicated than  $\Delta^{\text{op}}$ , but not by much.*

**Notation 6.14.** Let  $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  exhibit  $\mathcal{C}$  as  $\mathbb{E}_\sigma$ -monoidal. Let  $\mathcal{C}^\otimes$  denote the fiber product  $\mathcal{C}^\otimes \times_{\text{Assoc}_\sigma^\otimes} \Delta_\sigma^{\text{op}}$ .

**Definition 6.15.** Say that a morphism  $([n], s) \rightarrow ([m], t)$  is *inert* if the induced map  $\text{Cut}([m], t) \rightarrow \text{Cut}([n], s)$  is an inert morphism in  $\text{Assoc}_\sigma^\otimes$ .

**Definition 6.16.** A  $\mathbb{R}^\sigma$ -planar operad is an  $\infty$ -category  $\mathcal{O}^\otimes$  equipped with a functor  $q: \mathcal{O}^\otimes \rightarrow \Delta_\sigma^{\text{op}}$  so that

1. For every object  $X \in \mathcal{O}^\otimes$  and every inert morphism  $\alpha: ([n], s) \rightarrow q(X)$  in  $\Delta_\sigma$ , there is a  $q$ -cocartesian morphism  $\bar{\alpha}: X \rightarrow Y$  satisfying  $q(\bar{\alpha}) = \alpha$
2. Let  $X$  be an object satisfying  $q(X) = ([n], s)$ , and choose  $q$ -cocartesian morphisms  $\bar{\alpha}_i: X \rightarrow X_i$  corresponding to the morphism  $([i-1 < i], s_i) \rightarrow ([n], s)$  which is the inclusion on underlying sets and satisfies  $s_i(i) = s(i)$ . Then the morphisms  $\bar{\alpha}_i$  exhibit  $X$  as the  $q$ -product of the  $X_i$ .
3. For each  $n \geq 0$ , the construction  $C \mapsto \{C_i\}_{1 \leq i \leq n}$  induces an equivalence of  $\infty$ -categories

$$\mathcal{O}^\otimes \times_{\Delta_\sigma^{\text{op}}} \{([n], s)\} \xrightarrow{\sim} (\mathcal{O}^\otimes \times_{\Delta_\sigma^{\text{op}}} \{([1], s|_{\{i\}})\})^{\times n}$$

We say that a morphism  $\alpha$  in  $\mathbb{R}^\sigma$ -planar operad is *inert* if it is  $q$ -cocartesian and  $q(\alpha)$  is inert in  $\Delta_\sigma^{\text{op}}$  in the sense of Definition 6.15.

**Definition 6.17.** Let  $q: \mathcal{O}^\otimes \rightarrow \Delta_\sigma^{\text{op}}$  be a  $\mathbb{R}^\sigma$ -planar operad. An  $\mathbb{A}_\infty^\sigma$ -algebra object of  $\mathcal{O}^\otimes$  is a section of  $q$  which carries inert morphisms to inert morphisms. Write  $\text{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O})$  for the full subcategory of  $\text{Fun}_{\Delta_\sigma^{\text{op}}}(\Delta_\sigma^{\text{op}}, \mathcal{O}^\otimes)$  on  $\mathbb{A}_\infty^\sigma$ -algebra objects.

**Proposition 6.18.** *Let  $\mathcal{O}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  be a fibration of  $\infty$ -operads. Then precomposition with the functor  $\text{Cut}$  of Construction 6.12 induces an equivalence of  $\infty$ -categories*

$$\text{Alg}_{\text{Assoc}_\sigma^\otimes}(\mathcal{O}) \xrightarrow{\sim} \text{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O}).$$

*Proof.* Combine Lemma 6.13 with [Lur17, Theorem 2.3.3.23]. □

**Definition 6.19.** Define a colored operad  $\mathbf{LM}_{\text{inv}}$

- (i) The set of objects of  $\mathbf{LM}_{\text{inv}}$  has two elements, which we denote by  $\mathbf{a}, \mathbf{m}$ .
- (ii) Let  $\{X_i\}_{i \in I}$  be a finite collection of objects of  $\mathbf{LM}_{\text{inv}}$  and let  $Y$  be another object of  $\mathbf{LM}_{\text{inv}}$ . If  $Y = \mathbf{a}$ , then  $\text{Mul}_{\mathbf{LM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is the set of pairs consisting of a linear ordering on  $I$  and a function  $I \rightarrow \{\pm 1\}$  if  $X_i = \mathbf{a}$  for all  $i$ , and empty otherwise. If  $Y = \mathbf{m}$ , then  $\text{Mul}_{\mathbf{LM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is a subset of the set of pairs  $(\lambda, c)$  consisting of a linear ordering  $\lambda = \{i_1 < i_2 < \dots < i_n\}$  on  $I$  and a function  $c: I \rightarrow \{\pm 1\}$  satisfying either
  - $X_{i_n} = \mathbf{m}$  and  $c(i_n) = 1$  and  $X_j = \mathbf{a}$  otherwise
  - $X_{i_1} = \mathbf{m}$  and  $c(i_1) = -1$  and  $X_j = \mathbf{a}$  otherwise
- (iii) The composition law on  $\mathbf{LM}_{\text{inv}}$  is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4

**Remark 6.20.** There is a colored operad  $\mathbf{RM}_{\text{inv}}$  defined exactly in the same way as  $\mathbf{LM}_{\text{inv}}$  in Definition 6.19. In the interest of precision:  $\mathbf{RM}_{\text{inv}}$  has the same objects  $\mathbf{a}, \mathbf{m}$ . Let  $\{X_i\}_{i \in I}$  be a finite collection of objects of  $\mathbf{RM}_{\text{inv}}$  and let  $Y$  be another object of  $\mathbf{RM}_{\text{inv}}$ . If  $Y = \mathbf{m}$ , then  $\text{Mul}_{\mathbf{RM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is a subset of the set of pairs  $(\lambda, c)$  consisting of a linear ordering  $\lambda = \{i_1 < i_2 < \dots < i_n\}$  on  $I$  and a function  $c: I \rightarrow \{\pm 1\}$  satisfying either

- $X_{i_n} = \mathbf{m}$  and  $c(i_n) = -1$  and  $X_j = \mathbf{a}$  otherwise
- $X_{i_1} = \mathbf{m}$  and  $c(i_1) = 1$  and  $X_j = \mathbf{a}$  otherwise

**Remark 6.21.** Restricting to the objects which are both called  $\mathbf{a}$ , we see that both  $\mathbf{LM}_{\text{inv}}$  and  $\mathbf{RM}_{\text{inv}}$  have a sub-colored operad which is canonically identified with  $\mathbf{Assoc}_{\text{inv}}$  of Definition 6.4.

**Remark 6.22.** There is a map of colored operads  $\iota: \mathbf{LM} \rightarrow \mathbf{LM}_\sigma$  which sends  $\mathbf{m}$  to  $\mathbf{m}$  and sends  $\mathbf{a}$  to  $\mathbf{a}$ . On  $\text{Mul}_{\mathbf{LM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{LM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{id}_{\mathcal{L}I} \times \{c_1\}$ , this map agrees with  $\iota$  of Remark 6.5. On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is the restriction of the map  $\text{id}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$  where  $c_1$  is the constant function on  $I \sqcup \{j\}$  with value 1.

There is a map of colored operads  $\iota^{\text{rev}}: \mathbf{RM} \rightarrow \mathbf{LM}_\sigma$  which sends  $\mathbf{m}$  to  $\mathbf{m}$  and sends  $\mathbf{a}$  to  $\mathbf{a}$ . On  $\text{Mul}_{\mathbf{RM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{LM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{rev}_{\mathcal{L}I} \times \{c_1\}$ , this map agrees with  $\iota^{\text{rev}}$  of Remark 6.5. On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is the restriction of the map  $\text{rev}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$  where  $c_1$  is the constant function on  $I \sqcup \{j\}$  with value 1.

**Definition 6.23.** Define a colored operad  $\mathbf{BM}_{\text{inv}}$

- The set of objects of  $\mathbf{BM}_{\text{inv}}$  has three elements, which we denote by  $\mathbf{a}_\ell, \mathbf{a}_r, \mathbf{m}$ .
- Let  $\{X_i\}_{i \in I}$  be a finite collection of objects of  $\mathbf{BM}_{\text{inv}}$  and let  $Y$  be another object of  $\mathbf{BM}_{\text{inv}}$ . If  $Y = \mathbf{a}_\ell$  (resp.  $Y = \mathbf{a}_r$ ), then  $\text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is the set of pairs consisting of a linear ordering on  $I$  and a function  $I \rightarrow \{\pm 1\}$  if  $X_i = \mathbf{a}_\ell$  (resp.  $X_i = \mathbf{a}_r$ ) for all  $i$ , and empty otherwise. If  $Y = \mathbf{m}$ , then  $\text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is the subset of pairs  $(\lambda, c)$  consisting of a linear ordering  $\lambda = \{i_1 < i_2 < \dots < i_n\}$  on  $I$  and a function  $c: I \rightarrow \{\pm 1\}$  satisfying: if there is exactly one index  $i_k$  so that  $X_{i_k} = \mathbf{m}$ , either
  - $c(i_k) = 1$ ,  $X_j = \mathbf{a}_\ell$  for  $j < i_k$  and  $X_j = \mathbf{a}_r$  for  $j > i_k$ ; or
  - $c(i_k) = -1$ ,  $X_j = \mathbf{a}_\ell$  for  $j > i_k$  and  $X_j = \mathbf{a}_r$  for  $j < i_k$
- The composition law on  $\mathbf{BM}_{\text{inv}}$  is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4

**Remark 6.24.** The colored operad  $\mathbf{BM}_{\text{inv}}$  has a canonical involution  $\sigma$  which fixes  $\mathbf{m}$ , exchanges  $\mathbf{a}_\ell$  and  $\mathbf{a}_r$ , and sends a morphism  $(\lambda, c)$  to  $(\lambda^{\text{rev}}, I \xrightarrow{c} \{\pm 1\} \xrightarrow{\cdot(-1)} \{\pm 1\})$ .

**Remark 6.25.** There is a map of colored operads  $\iota: \mathbf{BM} \rightarrow \mathbf{BM}_\sigma$  which sends  $\mathbf{m}$  to  $\mathbf{m}$  and sends  $\mathbf{a}_-$  to  $\mathbf{a}_\ell$  and  $\mathbf{a}_+$  to  $\mathbf{a}_r$ . On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}_\pm) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{id}_{\mathcal{L}I} \times \{c_1\}$ , this map agrees with  $\iota$  of Remark 6.5. On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is the restriction of the map  $\text{id}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$  where  $c_1$  is the constant function on  $I \sqcup \{j\}$  with value 1.

There is *also* a map of colored operads  $\iota^{\text{rev}}: \mathbf{BM} \rightarrow \mathbf{BM}_\sigma$  which sends  $\mathbf{m}$  to  $\mathbf{m}$  and sends  $\mathbf{a}_-$  to  $\mathbf{a}_r$  and  $\mathbf{a}_+$  to  $\mathbf{a}_\ell$ . On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}_\pm) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{id}_{\mathcal{L}I} \times \{c_1\}$ , this map agrees with  $\iota^{\text{rev}}$  of Remark 6.5. On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is the restriction of the map  $\text{rev}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_{-1}\}$  where  $c_{-1}$  is the constant function on  $I \sqcup \{j\}$  with value  $-1$ .

**Definition 6.26.** Let  $\mathcal{LM}_{\text{inv}}^\otimes$ ,  $\mathcal{RM}_{\text{inv}}^\otimes$ , and  $\mathcal{BM}_{\text{inv}}^\otimes$  denote the associated  $\infty$ -operads (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

**Remark 6.27.** We can describe the category  $\mathcal{LM}_{\text{inv}}^\otimes$  as follows:

- (1) An object of  $\mathcal{LM}_{\text{inv}}^\otimes$  is a pair  $(\langle n \rangle, S)$  where  $S$  is a subset of  $\langle n \rangle^\circ$ .



(2) Morphisms  $(\langle m \rangle, T) \rightarrow (\langle n \rangle, S)$  consist of a map  $(\alpha: \langle m \rangle \rightarrow \langle n \rangle, \lambda: \langle m \rangle^\circ \rightarrow \{\pm 1\})$  in  $\text{Assoc}_\sigma^\otimes$  satisfying:

- The map  $\alpha$  takes  $T \cup \{*\}$  to  $S \cup \{*\}$
- For each  $s \in S$ , then  $\alpha^{-1}(\{s\})$  contains exactly one element  $t_s$  of  $T$ , and it is maximal (resp. minimal) with respect to the linear ordering on  $\alpha^{-1}(\{s\})$  if  $\lambda(t_s) = 1$  (resp.  $\lambda(t_s) = -1$ ).

**Remark 6.28.** We can describe the category  $\mathcal{BM}_{\text{inv}}^\otimes$  as follows:

- (1) An object of  $\mathcal{BM}_{\text{inv}}^\otimes$  is a triple  $(\langle n \rangle, c_+, c_-)$  where  $c_\pm$  are functions  $\langle n \rangle^\circ \rightarrow \{0, 1\}$  and  $c_-(i) \leq c_+(i)$  for all  $i \in \langle n \rangle^\circ$ .
- (2) Morphisms  $(\langle m \rangle, c_+, c_-) \rightarrow (\langle n \rangle, c'_+, c'_-)$  consist of a map  $(\alpha: \langle m \rangle \rightarrow \langle n \rangle, \lambda: \langle m \rangle^\circ \rightarrow \{\pm 1\})$  in  $\text{Assoc}_\sigma^\otimes$  satisfying: if  $j \in \langle n \rangle^\circ$  and  $\alpha^{-1}(j) = \{i_1 < i_2 < \dots < i_\ell\}$ ,

- If  $c_-(j) = c_+(j)$ , then

$$c'_-(j) = c_-(i_1) \leq c_+(i_1) = c_-(i_2) \leq c_+(i_2) \cdots \cdots c_-(i_{m-1}) \leq c_+(i_m) = c'_+(j)$$

- If  $c_-(j) < c_+(j)$ , then there exists a unique  $k$  so that  $c_-(i_k) < c_+(i_k)$  and

$$\begin{aligned} \lambda(i_k) \cdot c'_-(j) &= \lambda(i_k) \cdot c_-(i_1) \leq \lambda(i_k) \cdot c_+(i_1) = \lambda(i_k) \cdot c_-(i_2) \leq \lambda(i_k) \cdot c_+(i_2) \cdots \\ &\quad \lambda(i_k) \cdot c_-(i_{m-1}) \leq \lambda(i_k) \cdot c_+(i_m) = \lambda(i_k) \cdot c'_+(j) \end{aligned}$$

**Remark 6.29.** Each morphism  $\varphi \in \text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  determines a linear ordering  $\ell$  on the set  $I$  and a function  $s: I \rightarrow \{\pm 1\}$ . Passing from  $\varphi$  to the pair  $(\ell, s)$  determines a map of colored operads  $j: \mathbf{BM}_{\text{inv}} \rightarrow \mathbf{Assoc}_{\text{inv}}$ . The map  $j$  induces a morphism of  $\infty$ -operads  $\mathcal{BM}_{\text{inv}}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  which we will also denote by  $j$ . For any  $\mathbb{E}_\sigma$ -monoidal  $\infty$ -category  $\mathcal{C}$ , restriction along  $j$  sends an  $\mathbb{E}_\sigma$ -algebra  $A: \text{Assoc}_\sigma \rightarrow \mathcal{C}^\otimes$  to the pair  $(A, A)$  where  $A$  is regarded as an involutive bimodule over itself.

L: hermitian

**Remark 6.30.** The maps  $\iota, \iota^{\text{rev}}$  of Remark 6.22 induce maps of  $\infty$ -operads  $\iota: \mathcal{LM}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$  and  $\iota^{\text{rev}}: \mathcal{RM}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$ .

**Remark 6.31.** The maps  $\iota, \iota^{\text{rev}}$  of Remark 6.25 induce maps of  $\infty$ -operads  $\iota, \iota^{\text{rev}}: \mathcal{BM}^\otimes \rightarrow \mathcal{BM}_\sigma^\otimes$ . There are canonical identifications  $\iota \circ \text{rev} \simeq \sigma \circ \iota^{\text{rev}}$  where  $\sigma$  is the involution on  $\mathcal{BM}_\sigma^\otimes$  induced by Remark 6.24 and  $\text{rev}$  is the involution on  $\mathcal{BM}^\otimes$  of [Lur17, Construction 4.6.3.1].

**Remark 6.32.** There are canonical maps of operads  $\mathcal{LM}_{\text{inv}}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  and  $\mathcal{RM}_{\text{inv}}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  sending  $\mathbf{a}$  to  $\mathbf{a}_\ell$ , resp.  $\mathbf{a}_r$  and making the diagram

$$\begin{array}{ccc} \text{Assoc}^\otimes & \longrightarrow & \mathcal{LM}_{\text{inv}}^\otimes \\ \downarrow \sigma & & \downarrow \text{rev} \\ \text{Assoc}^\otimes & \longrightarrow & \mathcal{RM}_{\text{inv}}^\otimes \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \mathcal{BM}_{\text{inv}}^\otimes$$

commute, where  $\text{rev}$  is (an involutive version of) the reversal involution of [Lur17, Remark 4.6.3.2].

**Definition 6.33.** Let  $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  and  $\mathcal{D}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  be fibrations of  $\infty$ -operads and let  $\mathcal{M}$  be an  $\infty$ -category. Suppose given a fibration of  $\infty$ -operads  $q: \mathcal{O}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$  together with equivalences  $\mathcal{O}_{\mathbf{a}}^\otimes \simeq \mathcal{C}^\otimes$  and  $\mathcal{O}_{\mathbf{m}}^\otimes \simeq \mathcal{M}$ . Let  $L^\sigma \text{Mod}(\mathcal{M})$  denote the  $\infty$ -category  $\text{Alg}_{/\mathcal{LM}_{\text{inv}}}(\mathcal{O})$ . We will refer to  $L^\sigma \text{Mod}(\mathcal{M})$  as the  $\infty$ -category of left hermitian module objects of  $\mathcal{M}$ .

Suppose given a fibration of  $\infty$ -operads  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  together with equivalences  $\mathcal{O}_{\mathbf{a}_\ell}^\otimes \simeq \mathcal{C}^\otimes$ ,  $\mathcal{O}_{\mathbf{a}_R}^\otimes \simeq \mathcal{D}^\otimes$  and  $\mathcal{O}_{\mathbf{m}}^\otimes \simeq \mathcal{M}$ . Let  ${}^\sigma \text{Mod}(\mathcal{M})$  denote the  $\infty$ -category  $\text{Alg}_{/\mathcal{BM}_{\text{inv}}}(\mathcal{O})$ . We will refer to  ${}^\sigma \text{Mod}(\mathcal{M})$  as the  $\infty$ -category of hermitian bimodule objects of  $\mathcal{M}$ . Composition with the inclusions  $\text{Assoc}_\sigma^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  induces a categorical fibration

$${}^\sigma \text{Mod}(\mathcal{M}) = \text{Alg}_{/\mathcal{BM}_{\text{inv}}}(\mathcal{O}) \rightarrow \text{Alg}_{\text{Assoc}_\sigma}(\mathcal{C}) \times \text{Alg}_{\text{Assoc}_\sigma}(\mathcal{D}).$$

If  $A$  is an  $\text{Assoc}_\sigma$ -algebra object of  $\mathcal{C}$ , we let  ${}^\sigma \text{Mod}_A(\mathcal{M})$  denote the fiber  ${}^\sigma \text{Mod}(\mathcal{M}) \times_{\text{Alg}_{\text{Assoc}_\sigma}(\mathcal{C})} \{A\}$ . We will refer to  ${}^\sigma \text{Mod}_A(\mathcal{M})$  as the  $\infty$ -category of hermitian  $A$ -bimodule objects of  $\mathcal{M}$ .

L: Lurie gives this a name (Definition 4.2.1.12 weakly enriched)–not sure what to call this. something bi-enriched?



**Definition 6.34.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads. We say that  $q$  exhibits  $\mathcal{O}_m$  as  $\mathbb{E}_\sigma$ -bitensored over  $\mathcal{O}_{a_\ell}$  and  $\mathcal{O}_{a_r}$  if  $q$  is a cocartesian fibration.

**Remark 6.35.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads. Then  $q$  is classified by a map  $\chi: \mathcal{BM}_{\text{inv}}^\otimes \rightarrow \text{Cat}_\infty$ . By Remark 6.31, we can think of  $q$  as giving two  $\mathbb{E}_\sigma$  algebras  $\mathcal{C}, \mathcal{D}$  in  $\text{Cat}_\infty$  with an  $\infty$ -category  $\mathcal{M}$  equipped with both the structure of a  $\mathcal{C}$ - $\mathcal{D}$ -bimodule (equivalently, the structure of a left  $\mathcal{C} \times \mathcal{D}_{\text{rev}}$ -module) and of a  $\mathcal{D}$ - $\mathcal{C}$ -bimodule, and an autoequivalence  $\sigma_{\mathcal{M}}: \mathcal{M} \simeq \mathcal{M}$  of order two which is linear with respect to the autoequivalence  $\mathcal{C} \times \mathcal{D}_{\text{rev}} \xrightarrow{\text{flip}} \mathcal{D}_{\text{rev}} \times \mathcal{C} \xrightarrow{\sigma_{\mathcal{D}}^{-1} \times \sigma_{\mathcal{C}}} \mathcal{D} \times \mathcal{C}_{\text{rev}}$ .

**Remark 6.36.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads. Consider a left hermitian module object  $F: \mathcal{LM}_{\text{inv}}^\otimes \rightarrow \mathcal{O}^\otimes$ . By Remark 6.32,  $F$  determines an associative algebra  $A$  of  $\mathcal{C}$  with an equivalence of algebras  $\sigma_A: A \simeq \sigma_{\mathcal{C}}(A)^{\text{rev}}$ , an object  $M \in \mathcal{M}$  so that  $M$  (resp.  $\sigma_{\mathcal{M}}(M)$ ) is equipped with the structure of a left  $A$ -module (resp. right  $\sigma_{\mathcal{C}}(A)$ -module). Furthermore, we have an equivalence  $\sigma_M: M \simeq \sigma_{\mathcal{M}}(M)$  which is linear with respect to the equivalence  $A \xrightarrow{\sigma_A} \sigma_{\mathcal{C}}(A)^{\text{rev}}$ .

L: is this related to “modules with involution” from [Cal+20, §3.1]?

**Remark 6.37.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads. Consider a hermitian module object  $F: \mathcal{BM}_{\text{inv}}^\otimes \rightarrow \mathcal{O}^\otimes$ . By Remark 6.32,  $F$  determines an associative algebra  $A$  of  $\mathcal{C}$  with an equivalence of algebras  $\sigma_A: A \simeq \sigma_{\mathcal{C}}(A)^{\text{rev}}$  and an associative algebra  $B$  of  $\mathcal{D}$  with an equivalence of algebras  $\sigma_B: B \simeq \sigma_{\mathcal{D}}(B)^{\text{rev}}$ , an object  $M \in \mathcal{M}$  so that  $M$  (resp.  $\sigma_{\mathcal{M}}(M)$ ) is equipped with the structure of a  $A$ - $B$ -bimodule (resp.  $\sigma_{\mathcal{D}}(B)$ - $\sigma_{\mathcal{C}}(A)$ -bimodule). Furthermore, we have an equivalence  $\sigma_M: M \simeq \sigma_{\mathcal{M}}(M)$  which is linear with respect to the equivalence  $A \otimes B \xrightarrow{\text{flip}} B \otimes A \xrightarrow{\sigma_B^{-1} \otimes \sigma_A} \sigma_{\mathcal{D}}(B)^{\text{rev}} \otimes \sigma_{\mathcal{C}}(A)^{\text{rev}}$ .

L: when  $\mathcal{C} = \mathcal{D}$  and  $\sigma_{\mathcal{M}}$  and  $\sigma_{\mathcal{C}}$  are both the identity and  $A = B$ , I think this recovers the “module with involution” from [Cal+20, §3.1].

**Construction 6.38.** Define a functor  $\text{MCut}: \Delta_\sigma^{\text{op}} \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ :

- For each  $([n], s)$ , we have  $\text{MCut}([n], s) = \langle n+1 \rangle \simeq \text{RCut}_0([n])$  where  $\text{RCut}$  is from [Lur17, Construction 4.8.4.4].
- Given a morphism  $\alpha: ([n], s) \rightarrow ([m], t)$ , the associated morphism  $\text{MCut}([m], t) \rightarrow \text{MCut}([n], s)$  consists of
  - On underlying finite pointed sets  $\langle m+1 \rangle \rightarrow \langle n+1 \rangle$ ,  $\text{MCut}$  agrees with (the reverse of) that appearing in [Lur17, Construction 4.2.2.6]
  - Identifying the cut  $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$  with the morphism  $j-1 < j$ , we may regard  $s: \langle n+1 \rangle^\circ \rightarrow \{\pm 1\}$  and likewise  $t: \langle m+1 \rangle^\circ \rightarrow \{\pm 1\}$ . Define  $u: \text{MCut}(\alpha)^{-1}(\langle n+1 \rangle^\circ) \rightarrow \{\pm 1\}$  to be the unique function so that  $u(j)t(j) = s(\text{MCut}(\alpha)(j))$ .

L: maybe this overloaded notation is not good. I’m running out of ideas.

L: check later

**Remark 6.39.** We can identify  $\text{Assoc}_\sigma^\otimes$  with the full subcategory of  $\mathcal{RM}_{\text{inv}}^\otimes$  spanned by objects of the form  $(\langle n \rangle, \langle n \rangle^\circ)$ . We can regard Construction 6.12 as defining a functor  $\Delta_\sigma^{\text{op}} \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ . For each  $([n], s) \in \Delta_\sigma^{\text{op}}$ , there is a map of sets  $\theta: \text{MCut}([n], s) \rightarrow \text{Cut}([n], s)$  defined as in [Lur17, Remark 4.2.2.8]. Concretely, on underlying pointed sets,  $\theta$  takes the form

$$\theta: \langle n+1 \rangle \rightarrow \langle n \rangle$$

$$k \mapsto \begin{cases} k-1 & \text{if } k > 0 \\ * & \text{if } k = 0, * \end{cases}$$

L: check that the signs  $s$  work out!

This construction determines a morphism  $\gamma$  in the  $\infty$ -category  $\text{Fun}(\Delta_\sigma^{\text{op}}, \mathcal{RM}_{\text{inv}}^\otimes)$ , or equivalently a map  $\gamma: \Delta_\sigma^{\text{op}} \times \Delta^1 \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ .

**Lemma 6.40.** The morphism  $\gamma: \Delta_\sigma^{\text{op}} \times \Delta^1 \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  defined in Remark 6.39 exhibits  $\Delta_\sigma^{\text{op}} \times \Delta^1$  as an approximation to the  $\infty$ -operad  $\mathcal{RM}_{\text{inv}}^\otimes$ .

**Definition 6.41.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as weakly bi-enriched over  $\mathcal{O}_a^\otimes$ . Let  $\gamma$  be as in Remark 6.39. Let  $R^\sigma \text{Mod}^{\mathbb{A}_\infty^\sigma}(\mathcal{M})$  denote the full subcategory of  $\text{Fun}_{\mathcal{RM}_{\text{inv}}^\otimes}(\Delta_\sigma^{\text{op}} \times \Delta^1, \mathcal{O}^\otimes)$  spanned by those maps  $f: \Delta_\sigma^{\text{op}} \times \Delta^1 \rightarrow \mathcal{O}^\otimes$  satisfying

1. The restriction of  $f$  to  $\Delta_\sigma^{\text{op}} \times \{1\}$  belongs to  $\text{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O})$  of Definition 6.17
2. If  $\alpha: ([m], s) \rightarrow ([n], t)$  so that  $\alpha(0) = 0$ , then the induced map  $f([m], s, 0) \rightarrow f([n], t, 0)$  is an inert map in  $\mathcal{O}^\otimes$
3. for each object  $([n], s)$  in  $\Delta_\sigma^{\text{op}}$ , the induced map  $f([n], s, 0) \rightarrow f([n], s, 1)$  is an inert map in  $\mathcal{O}^\otimes$

**Example 6.42.** Let  $\mathcal{C}^\otimes \rightarrow \mathcal{RM}^\otimes$  be a fibration of  $\infty$ -operads. Restriction along the map of  $\infty$ -operads  $\mathcal{RM}_{\text{inv}}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  induced by Remark 6.29 induces a map  $\mathbb{E}_\sigma \text{Alg}(\mathcal{C}) \rightarrow R^\sigma \text{Mod}(\mathcal{C})$  which is a section of the projection map  $R^\sigma \text{Mod}(\mathcal{C}) \rightarrow \mathbb{E}_\sigma \text{Alg}(\mathcal{C})$ .

**Notation 6.43.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as weakly bi-enriched over  $\mathcal{O}_a^\otimes$ . Define a new simplicial set  $\overline{\mathcal{M}}^\otimes$  by the following universal property

$$\text{hom}_{\text{sSet}/\Delta_\sigma^{\text{op}}} (K, \overline{\mathcal{M}}^\otimes) \simeq \text{hom}_{\text{sSet}/\mathcal{RM}_{\text{inv}}^\otimes} (K \times \Delta^1, \mathcal{O}^\otimes) .$$

Here we regard  $K \times \Delta^1$  as a simplicial set over  $\mathcal{RM}_{\text{inv}}^\otimes$  via the composite  $K \times \Delta^1 \rightarrow \Delta_\sigma^{\text{op}} \times \Delta^1 \xrightarrow{\gamma} \mathcal{RM}_{\text{inv}}^\otimes$  where  $\gamma$  is from Remark 6.39.

Unwinding definitions, we see that a vertex in  $\overline{\mathcal{M}}^\otimes$  lying over an object  $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\}) \in \Delta_\sigma^{\text{op}}$  corresponds to a morphism  $\alpha$  in  $\mathcal{O}^\otimes$  whose image in  $\mathcal{RM}_{\text{inv}}^\otimes$  is the map  $(\langle n+1 \rangle, \{0\}) \rightarrow (\langle n \rangle, \emptyset)$ . Now let  $\mathcal{M}^\otimes$  denote the full simplicial subset of  $\overline{\mathcal{M}}^\otimes$  spanned by those vertices for which  $\alpha$  is inert.

**Remark 6.44.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as weakly enriched over  $\mathcal{O}_a^\otimes$ . By [Lur09, Example 4.3.1.4 & Proposition 4.3.2.15], composition with the inclusion  $\{0\} \rightarrow \Delta^1$  induces a trivial Kan fibration  $\mathcal{M}^\otimes \xrightarrow{\sim} \mathcal{O}^\otimes \times_{\mathcal{RM}_{\text{inv}}^\otimes} \Delta_\sigma^{\text{op}}$ . In particular, the fiber of  $\mathcal{M}^\otimes$  over an object  $([n], s) \in \Delta_\sigma^{\text{op}}$  is canonically equivalent to  $\mathcal{M} \times \mathcal{C}^{\times n}$ .

Finally, since  $q$  is a categorical fibration and categorical fibrations are closed under pullback and composition with trivial fibrations,  $q$  induces categorical fibrations  $\mathcal{M}^\otimes \rightarrow \mathcal{C}^\otimes \rightarrow \Delta_\sigma^{\text{op}}$ .

**Lemma 6.45.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as tensored over  $\mathcal{O}_a^\otimes$ . Then the associated functor  $\mathcal{M}^\otimes \rightarrow \mathcal{C}^\otimes$  (Notation 6.14) is a locally coCartesian fibration.

**Proposition 6.46.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as tensored over  $\mathcal{O}_a^\otimes$ . Then precomposition with the functor  $\text{MCut}$  of Construction 6.38 induces an equivalence of  $\infty$ -categories

$$R^\sigma \text{Mod}(\mathcal{M}) \simeq \text{Alg}_{/\mathcal{RM}_{\text{inv}}^\otimes}(\mathcal{O}) \xrightarrow{\sim} R^\sigma \text{Mod}^{\mathbb{A}_\infty^\sigma}(\mathcal{M}) .$$

*Proof.* Combine Lemma 6.40 with [Lur17, Theorem 2.3.3.23].  $\square$

## 6.2 Part (b)

**Proposition 6.47.** Let  $\mathcal{C}$  be an involutive monoidal  $\infty$ -category and let  $\mathcal{M}$  be an  $\infty$ -category which is bitensored over  $\mathcal{C}$ . Let  $K$  be a simplicial set so that  $\mathcal{M}$  admits  $K$ -indexed limits, and let  $\theta: R^\sigma \text{Mod}(\mathcal{M}) \rightarrow \text{Alg}^\sigma(\mathcal{C})$  be the forgetful functor. Then

- (1) For every commutative square

$$\begin{array}{ccc} K & \longrightarrow & R^\sigma \text{Mod}(\mathcal{M}) \\ \downarrow & \nearrow & \downarrow \theta \\ K^\triangleleft & \longrightarrow & \text{Alg}^\sigma(\mathcal{C}) , \end{array}$$

there exists a dashed arrow which is a  $\theta$ -limit diagram.

L: see Example 4.2.1.17 of higher algebra

L: fibration?

L: this might be off-revisit later!

L: Jacob explains this in a really terse way—just by citing Prop 4.3.2.15 of HTT. It does just follow from definitions/observations but there are many (for instance, definition of inert edge).

L: This statement is [Lur17, Proposition 4.2.3.1] with some words changed; no claim of originality here.

- (2) An arbitrary map  $\bar{g}: K^\triangleleft \rightarrow R^\sigma \text{Mod}(\mathcal{M})$  is a  $\theta$ -limit diagram if and only if the induced map  $K^\triangleleft \rightarrow \mathcal{M}$  is a limit diagram.

Proof.

□

L: todo

**Corollary 6.48.**  $\theta$  is a cartesian fibration, and a morphism  $f: \Delta^1 \rightarrow R^\sigma \text{Mod}(\mathcal{M})$  is  $\theta$ -cartesian if and only if the image of  $f$  in  $\mathcal{M}$  is an equivalence.

**Corollary 6.49.** Let  $\mathcal{C}$  be an involutive monoidal  $\infty$ -category and let  $\mathcal{M}$  be an  $\infty$ -category which is bitensored over  $\mathcal{C}$ . Let  $K$  be a simplicial set so that  $\mathcal{M}$  admits  $K$ -indexed limits, and let  $\theta: R^\sigma \text{Mod}(\mathcal{M}) \rightarrow \text{Alg}^\sigma(\mathcal{C})$  be the forgetful functor. Let  $A$  be an involutive algebra object of  $\mathcal{C}$ . Then

- (1)  $R^\sigma \text{Mod}_A(\mathcal{M})$  admits  $K$ -indexed limits.
- (2) A diagram  $K^\triangleleft \rightarrow R^\sigma \text{Mod}_A(\mathcal{M})$  is a limit diagram if and only if the induced diagram  $K^\triangleleft \rightarrow \mathcal{M}$  is a limit diagram.
- (3) Given a morphism  $A \rightarrow B$  of involutive algebra objects of  $\mathcal{C}$ , the induced functor  $R^\sigma \text{Mod}_B(\mathcal{M}) \rightarrow R^\sigma \text{Mod}_A(\mathcal{M})$  preserves  $K$ -indexed limits.

### 6.3 Towards (e)

**Construction 6.50.** Define a functor  $\text{Pr}: \mathbf{LM}_{\text{inv}}^\otimes \times \mathbf{RM}_{\text{inv}}^\otimes \rightarrow \mathbf{BM}_{\text{inv}}^\otimes$ .

- (1) Let  $(\langle m \rangle, S)$  be an object of  $\mathbf{LM}_{\text{inv}}^\otimes$  and let  $(\langle n \rangle, T)$  be an object of  $\mathbf{RM}_{\text{inv}}^\otimes$ . Let  $\text{Pr}((\langle m \rangle, S), (\langle n \rangle, T)) = (X_*, c_-, c_+)$  where  $X_*, c_-, c_+$  are described in [Lur17, Construction 4.3.2.1(1)].
- (2) Let  $(\alpha, \lambda): (\langle m \rangle, S) \rightarrow (\langle m' \rangle, S')$  be a morphism in  $\mathbf{LM}_{\text{inv}}^\otimes$  and let  $(\beta, \mu): (\langle n \rangle, T) \rightarrow (\langle n' \rangle, T')$  be a morphism in  $\mathbf{RM}_{\text{inv}}^\otimes$ . Write  $\text{Pr}((\langle m' \rangle, S'), (\langle n' \rangle, T')) = (X'_*, c'_-, c'_+)$ . Then  $\text{Pr}((\alpha, \lambda), (\beta, \mu))$  is the unique morphism in  $\mathbf{BM}_{\text{inv}}^\otimes$  lying over the map  $\gamma: X_* \rightarrow X'_*$  described by

$$(i) \quad \gamma(i, j) = \begin{cases} (\alpha(i), \beta(j)) & \text{if } \alpha(i) \in \langle m' \rangle^\circ, \beta(j) \in \langle n' \rangle^\circ \\ * & \text{otherwise.} \end{cases}$$

- (ii) Let  $i' \in \langle m' \rangle^\circ \setminus S'$  and  $j' \in T'$  so  $j' = \beta(j)$  for a unique  $j \in T$ . Then the linear ordering on  $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\}$  is (a) determined by the map  $\alpha$  if  $\mu(j) = 1$ , and (b) it is the reverse of the linear ordering determined by  $\alpha$  if  $\mu(j) = -1$ . The map  $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\} \rightarrow \{\pm 1\}$  is determined by  $\lambda$  if  $\mu(j) = 1$  and it is  $-\lambda$  if  $\mu(j) = -1$ .
- (iii) Likewise if  $i' \in S'$  and  $j' \in \langle n' \rangle^\circ \setminus T'$
- (iv) Let  $i' \in S'$  and  $j' \in T'$  so  $i' = \alpha(i)$  for a unique  $i \in S$  and  $j' = \beta(j)$  for a unique  $j \in T$ . Then  $\gamma^{-1}\{(i', j')\} = \{i\} \times \beta^{-1}\{j'\} \sqcup_{\{(i, j)\}} \alpha^{-1}\{(i')\} \times \{j\}$ . Define  $\gamma^{-1}\{(i', j')\} \rightarrow \{\pm 1\}$  by  $\lambda \times \mu$ . Endow  $\gamma^{-1}\{(i', j')\}$  with the linear ordering from [Lur17, Construction 4.3.2.1(2)(iv)] if  $\lambda(i) = \mu(j)$  and endow  $\gamma^{-1}\{(i', j')\}$  with the opposite ordering if  $\lambda(i) \neq \mu(j)$  (or equivalently, if  $\lambda(i) = -\mu(j)$ ).

Write  $\text{Pr}$  for the induced map  $\mathcal{LM}_\sigma^\otimes \times \mathcal{RM}_\sigma^\otimes \rightarrow \mathcal{BM}_\sigma^\otimes$  of  $\infty$ -categories.

**Construction 6.51.** Let  $q: \mathcal{C}^\otimes \rightarrow \mathcal{BM}_\sigma^\otimes$  be a fibration of  $\infty$ -operads. We define a map of simplicial sets  $\overline{L^\sigma \text{Mod}(\mathcal{C}_\text{m})}^\otimes \rightarrow \mathcal{RM}_\sigma^\otimes$  by the universal property: For any simplicial set  $K \rightarrow \mathcal{RM}_\sigma^\otimes$ , there is a bijection

$$\text{Hom}_{\text{sSet}} /_{\mathcal{RM}_\sigma^\otimes} (K, \overline{L^\sigma \text{Mod}(\mathcal{C}_\text{m})}^\otimes) \simeq \text{Hom}_{\text{sSet}} /_{\mathcal{BM}_\sigma^\otimes} (\mathcal{LM}_\sigma^\otimes \times K, \mathcal{C}^\otimes).$$

Let  $L^\sigma \text{Mod}(\mathcal{C}_\text{m})^\otimes$  denote the full simplicial subset of  $\overline{L^\sigma \text{Mod}(\mathcal{C}_\text{m})}^\otimes$  spanned by those vertices which correspond to a vertex  $X \in \mathcal{RM}_\sigma^\otimes$  and a functor  $F: \mathcal{LM}_\sigma^\otimes \{X\} \rightarrow \mathcal{BM}_\sigma^\otimes$  which takes inert morphisms in  $\mathcal{LM}_\sigma^\otimes$  to inert morphisms in  $\mathcal{BM}_\sigma^\otimes$ .

**Remark 6.52.** The composite  $\mathcal{LM}_\sigma^\otimes \times \{\mathfrak{m}\} \hookrightarrow \mathcal{LM}_\sigma^\otimes \times \mathcal{RM}_\sigma^\otimes \xrightarrow{\text{Pr}} \mathcal{BM}_\sigma^\otimes$  agrees with the inclusion of Remark 6.32. Taking  $K \rightarrow \mathcal{RM}_\sigma^\otimes$  to be the inclusion  $\{\mathfrak{m}\} \hookrightarrow \mathcal{RM}_\sigma^\otimes$ , we have an isomorphism of simplicial sets  $L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \times_{\mathcal{RM}_\sigma^\otimes} \{\mathfrak{m}\} \simeq L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})$  where  $L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})$  is the  $\infty$ -category of left modules associated to the fibration of  $\infty$ -operads  $\mathcal{C}^\otimes \times_{\mathcal{BM}_\sigma^\otimes} \mathcal{LM}_\sigma^\otimes \rightarrow \mathcal{LM}_\sigma^\otimes$ .

**Proposition 6.53.** *Let  $q: \mathcal{C}^\otimes \rightarrow \mathcal{BM}_\sigma^\otimes$  be a fibration of  $\infty$ -operads. Then*

- (1) *the induced map  $p: L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})^\otimes \rightarrow \mathcal{RM}_\sigma^\otimes$  is a fibration of  $\infty$ -operads*
- (2) *a morphism  $\alpha$  in  $L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$  is inert if and only if  $p(\alpha)$  is inert in  $\mathcal{RM}_\sigma^\otimes$  and for all  $X \in \mathcal{LM}_\sigma$ ,  $\alpha(X)$  is an inert morphism in  $\mathcal{C}^\otimes$ .*
- (3) *if  $q$  is a cocartesian fibration of  $\infty$ -operads, then so is  $p$*
- (4) *if  $q$  is a cocartesian fibration of  $\infty$ -operads, a morphism  $\alpha$  in  $L^\sigma \text{Mod}(\mathcal{C}_\mathfrak{m})^\otimes$  is  $p$ -cocartesian if and only if, for all  $X \in \mathcal{LM}_\sigma$ ,  $\alpha(X)$  is  $q$ -cocartesian in  $\mathcal{C}^\otimes$ .*

*Proof.* Similar to [Lur17, Proposition 4.3.2.5]. □

**Theorem 6.54.** *Let  $\mathcal{C}$  be an  $\mathbb{E}_\sigma$ -monoidal  $\infty$ -category, and let  $A$  be an  $\mathbb{E}_\sigma$ -algebra in  $\mathcal{C}$ . Then  $L^\sigma \text{Mod}_A(\mathcal{C})$  is right  $\mathbb{E}_\sigma$ -tensored over  $\mathcal{C}$ .*

## 6.4 Endomorphisms

Let  $\mathcal{C}$  be an  $\mathbb{E}_\sigma$ -monoidal  $\infty$ -category, and write  $\sigma_\mathcal{C}: \mathcal{C} \xrightarrow{\sim} \mathcal{C}$  for its involution. Suppose  $M \in \mathcal{C}$  is an object equipped with an equivalence  $\sigma_M: M \simeq \sigma_\mathcal{C}(M)$ . By [Lur17, §4.7.1], endomorphisms of  $M$  can be regarded as an  $\mathbb{E}_1$ -algebra in  $u(\mathcal{C})^\otimes$ , where  $u$  is from Remark 6.10. Now  $\sigma_M$  induces an equivalence  $\text{End}_\mathcal{C}(M) \simeq \text{End}_\mathcal{C}(\sigma_\mathcal{C}(M))$ . On the other hand,  $\sigma_\mathcal{C}$  induces an equivalence  $\text{End}_\mathcal{C}(\sigma_\mathcal{C}(M)) \simeq \text{End}_\mathcal{C}(M)^{\text{rev}}$ . In particular, for any  $\infty$ -category  $\mathcal{M}$  left  $\mathbb{E}_\sigma$ -tensored over  $\mathcal{C}$  and any object  $M \in \mathcal{M}$  which is fixed by the involution on  $\mathcal{M}$ , we expect the endomorphisms of  $M$  to admit the structure of an  $\mathbb{E}_\sigma$ -algebra in  $\mathcal{C}$ .

To this end, we will define an  $\infty$ -category of objects acting on  $M$ , show that it has an  $\mathbb{E}_\sigma$ -monoidal structure, and locate endomorphisms of  $M$  as the final object in this  $\infty$ -category. Informally, we may define a category  $\mathcal{C}[M]$  whose objects consist of either

- pairs  $(C, \eta)$  where  $C \in \mathcal{C}$  and  $\eta: C \otimes M \rightarrow M$  is a morphism in  $\mathcal{M}$ ; or
- pairs  $(C', \xi)$  where  $C' \in \mathcal{C}$  and  $\xi: \sigma_\mathcal{M}(M) \otimes C' \rightarrow \sigma_\mathcal{M}(M)$ .

The monoidal structure is as described in [Lur17, §4.7.1]. Note that given an object  $(C, \eta)$ , the involution  $\sigma_\mathcal{M}$  on  $\mathcal{M}$  sends  $\eta$  to the map  $\sigma_\mathcal{M}(C \otimes M) \simeq \sigma_\mathcal{M}(M) \otimes \sigma_\mathcal{C}(C) \rightarrow \sigma_\mathcal{M}(M)$ . This is the involution on  $\mathcal{C}[M]$ .

**Definition 6.55.** Let  $p: \mathcal{M}^\otimes \rightarrow \Delta^1 \times \Delta_\sigma^{\text{op}}$  exhibit  $\mathcal{M}^\otimes$  as weakly enriched over  $\mathcal{C}^\otimes$ . An *enriched morphism* of  $\mathcal{M}$  is a diagram

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

satisfying either

- $p(\alpha)$  is the morphism  $(0, [1], c_1) \rightarrow (0, [0])$  in  $\Delta_\sigma^{\text{op}}$  determined by the embedding  $[0] \simeq \{0\} \hookrightarrow [1]$  and  $c_1: \{1\} \rightarrow \{\pm 1\}$  is the constant function at  $+1$ , and
- the map  $\beta$  is inert, and  $p(\beta)$  is the morphism  $(0, [1], c_1) \rightarrow (0, [0])$  in  $\Delta^1 \times \Delta_\sigma^{\text{op}}$  determined by the embedding  $[0] \simeq \{1\} \hookrightarrow [1]$

or

- $p(\alpha)$  is the morphism  $(0, [1], c_{-1}) \rightarrow (0, [0])$  in  $\Delta_\sigma^{\text{op}}$  determined by the embedding  $[0] \simeq \{0\} \hookrightarrow [1]$  and  $c_{-1}: \{1\} \rightarrow \{\pm 1\}$  is the constant function at  $-1$ .
- the map  $\beta$  is inert, and  $p(\beta)$  is the morphism  $(0, [1], c_{-1}) \rightarrow (0, [0])$  in  $\Delta^1 \times \Delta_\sigma^{\text{op}}$  determined by the embedding  $[0] \simeq \{1\} \hookrightarrow [1]$

Let  $\text{Str } \mathcal{M}_{[1]}^{\text{en}}$  denote the full subcategory of  $\text{Fun}_{\Delta^1 \times \Delta_{\sigma}^{\text{op}}}(\Lambda_0^2, \mathcal{M}^{\otimes})$  spanned by the enriched morphisms of  $\mathcal{M}$ .

Note that there are two evaluation functors  $\text{Str } \mathcal{M}_{[1]}^{\text{en}} \rightarrow \mathcal{M}$ . Given  $M \in \mathcal{M}$ , write  $\mathcal{C}[M] := \{M\} \times_{\mathcal{M}} \text{Str } \mathcal{M}_{[1]}^{\text{en}} \times_{\mathcal{M}} \{M\}$  and refer to it as the endomorphism  $\infty$ -category of  $M$ .

**Definition 6.56.** *enriched  $n$ -string*

**Proposition 6.57** (Segal condition).

## 7 Categorification and structure

In the course of thinking about the ‘involutive’ generalization of the statement that given an  $\mathbb{E}_1$ -algebra, its category of modules is  $\mathbb{E}_0$  (and conversely, that given an object in a stable  $\infty$ -category, that its endomorphism spectrum is an  $\mathbb{E}_1$ -algebra), I have run up against some questions.

**Question 7.1.** • Can we sidestep an involutive version of the construction of endomorphism categories of [Lur17, §4.7.1]?

- Suppose  $\mathcal{C}$  is a monoidal  $\infty$ -category and  $\mathcal{M}$  is an  $\infty$ -category which is enriched over  $\mathcal{C}$  in the sense of [Lur17, §4.2.1]. The opposite category  $\mathcal{M}^{\text{op}}$  is enriched over  $\mathcal{C}$  by [Hei23, §10].

## 8 Comparing involutive classical Brauer and involutive higher Brauer

**Question 8.1.** • If, for a Poincaré  $\infty$ -category  $(\mathcal{C}, \mathfrak{Q})$ , there exists a Poincaré object  $(E, q)$  so that  $E$  is a compact generator, can we rewrite both the category and its Poincaré structure in terms of  $\text{End}_{\mathcal{C}}(E)$ ?

- Can the *property* of an existence of a Poincaré object  $(E, q)$  in  $(\text{Perf}_X, \mathfrak{Q}_L)$  so that  $E$  is a compact generator be checked Zariski-locally? See Toën’s paper §3.

## 9 Other

**Proposition 9.1.** *Assume that  $X$  has a good quotient  $Y$  in the sense of [FW20, Remark 4.20], and write  $p: X \rightarrow Y$  for the quotient map. Let  $i: U \subseteq Y$  be the largest open subscheme on which  $\pi|_{X_U}$  is étale [FW20, Proposition 4.45]. Write  $\text{RamLoc}(\pi)$  for the closed complement of  $U$  regarded as a topological space, and let  $j: \text{RamLoc}(\pi) \rightarrow Y$  denote the inclusion. Then  $\mathcal{Q}^{\varphi C_2}$  is in the essential image of  $j_*: \text{Shv}_{\text{Zar}}(\text{RamLoc}(\pi)) \rightarrow \text{Shv}_{\text{Zar}}(Y)$ . In other words, there exists a sheaf  $\mathcal{Q}$  of  $\mathbb{E}_{\infty}$ -rings on  $\text{RamLoc}(\pi)$  so that  $j_* \mathcal{Q} \simeq \mathcal{Q}^{\varphi C_2}$ .*

*Proof.* Recall that the open-closed decomposition of  $Y$  induces a symmetric monoidal *récollement*

$$\text{Shv}_{\text{Zar}}(U) \xleftarrow{i^*} \text{Shv}_{\text{Zar}}(Y) \xrightarrow{j^*} \text{Shv}_{\text{Zar}}(\text{RamLoc}(\pi)).$$

Therefore, to show that  $\mathcal{Q}^{\varphi C_2}$  is in the essential image of  $j_*$ , it suffices to show that  $i^*(\mathcal{Q}^{\varphi C_2}) \simeq 0$  as a sheaf on  $U$ .

By [FW20, Proposition 4.45], it suffices to show that if  $y$  is a point in  $U$ , then  $\mathcal{Q}_y^{\varphi C_2} = 0$ . Since  $\mathcal{Q}_y^{\varphi C_2} = \tau_{\geq 0} \left( \Gamma \mathcal{O}_{X \times_Y \{y\}}^{tC_2} \right)$  where  $A = \Gamma \mathcal{O}_{Y, y} \rightarrow B = \Gamma \mathcal{O}_{X \times_Y \{y\}}$  is a quadratic étale map so that  $B$  has an involution  $\lambda$  and  $A = B^{\lambda}$  is a local ring with maximal ideal  $\mathfrak{m}_A$  (therefore  $B$  is semilocal by [FW20, Proposition 3.15]), it suffices to show that  $\pi_0 B^{tC_2} = 0$ . By [NS18, Lemma I.2.9], we may without loss of generality replace  $A$  and  $B$  by their 2-completions. By the *récollement* of  $A$ -modules in terms of  $\mathfrak{m}_A$ -complete and  $A[\mathfrak{m}_A^{-1}]$ -modules, it suffices to show that  $(B_{\mathfrak{m}_A}^{\wedge})^{tC_2} = 0$  and  $(B[\mathfrak{m}_A^{-1}])^{tC_2} = 0$ .

By [FW20, Propositions 3.4 & 3.15],  $B\mathfrak{m}_A = J \subseteq B$ , where  $J$  denotes the Jacobson radical of  $B$ . We claim that  $B \simeq \lim_i B/J^i$  induces an equivalence  $B^{tC_2} \rightarrow \lim_i (B/J^i)^{tC_2}$ . Granting the claim, it suffices to show that  $(B[\mathfrak{m}_A^{-1}])^{tC_2} = 0$  and  $(B/J^i)^{tC_2}$  is zero for each  $i$ . Since  $(-)^{tC_2}$  is exact and lax symmetric monoidal and each  $B/J^i$  can be written as an extension of finitely many  $B/J$ -modules, it suffices to show that  $(B[\mathfrak{m}_A^{-1}])^{tC_2}$  and  $(B/J)^{tC_2}$  are zero.

L: hypothesis?  
move to main  
text?

L: Need hypercompleteness to reduce to checking on points?  
Appeal to Clausen–Mathew and add finite Krull dim hypothesis?

L: this is unnecessary to the proof—but shows that  $U$  is contained in the open subscheme  $Y \setminus [\frac{1}{\lambda}]$

Now observe that  $A/\mathfrak{m}_A$  (resp.  $A[\mathfrak{m}_A^{-1}]$ -algebra) is a field and  $B/J$  (resp.  $B[\mathfrak{m}_A^{-1}]$ ) is a quadratic étale  $A/\mathfrak{m}_A$ -algebra (resp.  $A[\mathfrak{m}_A^{-1}]$ -algebra). By [FW20, Proposition 3.4(ii)],  $B/J$  (resp.  $B[\mathfrak{m}_A^{-1}]$ ) is either a separable quadratic field extension of  $A/\mathfrak{m}_A$ -algebra (resp.  $A[\mathfrak{m}_A^{-1}]$ -algebra), or it is isomorphic to  $\prod_{C_2} A/\mathfrak{m}_A$  (resp.  $\prod_{C_2} A[\mathfrak{m}_A^{-1}]$ ). In the latter case, the action of  $C_2$  on  $B/J$  (resp.  $B[\mathfrak{m}_A^{-1}]$ ) is manifestly free, hence  $(B/J)^{tC_2} = 0$  (resp.  $B[\mathfrak{m}_A^{-1}]^{tC_2} = 0$ ). Suppose instead that  $B/J$  (resp.  $B[\mathfrak{m}_A^{-1}]$ ) is a separable quadratic field extension of  $A/\mathfrak{m}_A$ -algebra (resp.  $A[\mathfrak{m}_A^{-1}]$ -algebra). By [FW20, Proposition 3.4(ii)],  $\lambda \otimes_B B/J$  (resp.  $\lambda \otimes_B B[\mathfrak{m}_A^{-1}]$ ) is nontrivial, hence by [Stacks, Lemma 9.21.2, Tag 09DU] the extension  $A/\mathfrak{m}_A \rightarrow B/J$  (resp.  $A[\mathfrak{m}_A^{-1}] \rightarrow B[\mathfrak{m}_A^{-1}]$ ) is Galois. Since  $C_2$  acts freely on  $B/J$  as an  $A/\mathfrak{m}_A$ -module by the normal basis theorem,  $(B/J)^{tC_2} = 0$  (resp.  $B[\mathfrak{m}_A^{-1}]^{tC_2} = 0$ ).

We conclude the proof by proving the claim. Since homotopy fixed points commute with arbitrary limits, it suffices to show that  $B \simeq \lim_i B/\mathfrak{m}_B^i$  induces an equivalence  $B_{hC_2} \rightarrow \lim_i (B/\mathfrak{m}_B^i)_{hC_2}$ . This is true because the  $B/\mathfrak{m}_B^i$  are uniformly bounded below. □

**Example 9.2.** If  $\lambda = \text{id}_X$  and  $Y = X$ , then  $\pi_0 \underline{\mathcal{O}}^{\varphi C_2} = \mathcal{O}_Y/2$ . On  $\pi_0$ , the norm map  $\underline{\mathcal{O}}^e \simeq \mathcal{O}_X \rightarrow \underline{\mathcal{O}}^{\varphi C_2}$  takes  $f \mapsto f^2$ .

L: compare [CMM21, Remark 2.8].

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