

Et cetera

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Abstract

Dumping ground for other stuff: Notes, one-off observations, stuff that we can collectively use when preparing talks, etc.

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1 Talk prep

2 References

- [Involutions of Azumaya algebras](#) by First and Williams (2020 *Documenta*)
- [Counterexamples in involutions of Azumaya algebras](#) by First and Williams; much more readable than the 2020 *Documenta* paper

3 Questions and directions

Question 3.1 (Morita theory for $\text{Cat}_\infty^{\text{P}}$). Let R be a Poincaré ring. Suppose given two R -algebras (suitably interpreted so their module categories are canonically endowed with R -linear Poincaré structures—perhaps \mathbb{E}_σ) A, B . Can we characterize

$$\text{hom}_{\text{Cat}_\infty^{\text{P}}_R}((\text{Mod}_A^\omega, \mathfrak{P}_A), (\text{Mod}_B^\omega, \mathfrak{P}_B))$$

in terms of something bimodule-like?

Question 3.2. On page 2 of the *Counterexamples* paper, First and Williams write that “existence of an extraordinary involution means classification of Azumaya algebras with involution...*cannot* be reduced to questions about projective modules and hermitian forms on them.”

What if we replaced projective modules by perfect complexes?

Question 3.3. First–Williams show (see discussion in §4 of the *Counterexamples* paper) that coarse type classify many (most?) Azumaya algebras up to (étale-local) *isomorphism*.

What is a suitable derived version of “coarse type”?

L: I make no promises re: organization but I will do my best to keep it reasonably readable

4 Thoughts & observations

Question 4.1. When R has the Tate Poincaré structure and $(\text{Mod}_A^\omega, M_A, N_A, N_A \rightarrow M_A^{tC_2})$ is invertible, then by invertibility have an equivalence $\text{hom}_R(A, R) \simeq N_A \otimes_R N_{A^{\text{op}}}$ of $A \otimes_R A^{\text{op}}$ -modules. Restricting the left-hand side along the unit map $R \rightarrow A$ gives a map $N_A \otimes_R N_{A^{\text{op}}} \rightarrow \text{hom}_R(R, R) \simeq R$. Is this a perfect (R -linear) pairing?

I *think* using that $R^{\varphi C_2} \simeq R$ and combining the linear and bilinear part conditions, we get something like

$$M_A \otimes_R M_{A^{\text{op}}} \simeq (N_A \otimes_R N_{A^{\text{op}}})^{\otimes_R 2} \quad \text{as } A \otimes_R A^{\text{op}}\text{-bimodules.}$$

Is this useful?

Brauer-Severi schemes We know there is a correspondence between Azumaya algebras A over X and Brauer-Severi schemes. What does a Poincaré structure on Mod_A^ω mean ‘geometrically’ for D_{coh}^b of the corresponding Brauer-Severi scheme? (Lucy: I didn’t get very far here, but just typing up what I had)

- Mod_A^ω corresponds to α -twisted sheaves on X (see Proposition 3.2.2.1 of Max Lieblich’s thesis)
- The bounded derived category of α -twisted sheaves on X includes as one ‘piece’ of a semiorthogonal decomposition on D_{coh}^b of the corresponding Brauer-Severi scheme (see Theorem 5.1 [here](#))

5 Desperate Flailing

This section is a cronical of my thoughts about \mathbb{G}_m^ω .

Goal The goal is to build a Poincaré ring $\mathbb{G}_m^\omega := (\text{Mod}_R, \Omega_R)$ such that $B\mathbb{G}_m^\omega(\underline{S}) = \text{Pic}^P(\underline{S})$ for any Poincaré ring \underline{S} .

Lemma 5.1. *Let \underline{S} be a Poincaré ring. Then $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u)) = \{s \in \pi_0(S)^\times \mid s = 1 \text{ in } \pi_0(S^{C_2})\}$.*

Proof. Since the functor $\text{Pn}(\text{Mod}_S) \rightarrow \text{Mod}_S$ is conservative it follows that an element of $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u))$ must have underlying map an element of $\pi_0 \text{Aut}(S) = \pi_0(S)^\times$. Then in order for $s \in \pi_0(S)^\times$ to induce a map $(S, u) \rightarrow (S, u)$, the induced map $s^* : S^{C_2} \rightarrow S^{C_2}$ must satisfy $s^*(u) = u$. The pullback is given by multiplication by s , so this requirement translates into s being the unit, as desired. \square

The problem I thought existed maybe doesn’t. Here is a candidate construction:

Construction 5.2. Define R to be the \mathbb{E}_∞ ring given by $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z\}} \mathbb{S}$ where the map $\mathbb{S}\{z\} \rightarrow \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ is induced by the map $z \mapsto xy$, and the map $\mathbb{S}\{z\} \rightarrow \mathbb{S}$ is induced by $z \mapsto 1$. We can give R an \mathbb{E}_∞ ring structure in Sp^{BC_2} by taking the trivial action on $\mathbb{S}\{z\}$ and \mathbb{S} , and taking the action induced by $x \mapsto y$ and $y \mapsto x$ on $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$. Thus in $\text{CAlg}(\text{Sp}^{BC_2})$ the ring R corepresents the functor $S \mapsto \{s \in \pi_0(S)^\times \mid s\sigma(s) = 1\}$.

Now take \underline{R} to be the Poincaré ring with underlying Borel C_2 structure as described in the previous paragraph and geometric fixed points $R^{\varphi C_2} = \mathbb{S}$ and the map $R^{\varphi C_2} \rightarrow R^{tC_2}$ given by the unit map. Endowing $R^{\varphi C_2}$ with the R -module structure given by $x, y \mapsto 1$, it remains to show that the unit map $R^{\varphi C_2} \rightarrow R^{tC_2}$ factors the Tate valued Frobenius $R \rightarrow R^{tC_2}$ in order to promote \underline{R} to a Poincaré ring. By construction of R it is then enough to show that on π_0 the Tate valued Frobenius sends $x, y \mapsto 1$ in $\pi_0(R^{tC_2})$. This map sends both x and y to $xy \in \pi_0(R^{tC_2})$. These are equal to 1 in $\pi_0(R^{tC_2})$ since the functor $(-)^{tC_2}$ is lax-monoidal so R^{tC_2} is a modules over $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$ which has the image of xy equal to 1.

Now consider another Poincaré ring \underline{S} . We then have that maps $\pi_0(\text{Maps}(\underline{R}, \underline{S}))$ is the data of a unit $s \in \pi_0(S)^\times$, a path $s\sigma(s) \rightarrow 1$ in $\Omega^\infty S$, and paths $x, y \rightarrow 1$ in $\Omega^\infty S^{\varphi C_2}$. This then agrees with \mathbb{G}_m^ω by the following lemma.

Lemma 5.3. *Let $S \in \text{CAlg}(\text{Sp}^{BC_2})$ and $s \in \pi_0(S)^\times$. Then $s\sigma(s) = 1$ in $\pi_0(S)$ if and only if $(s \otimes s)^*$ acts by 1 on $\pi_0(S^{hC_2}) = \pi_0(\text{Hom}_{S \otimes S}(S \otimes S, S)^{hC_2})$.*

Proof. The ‘only if’ direction follows from the fact that the map $S^{hC_2} \rightarrow S$ is an S -bimodule map. Now suppose that $s\sigma(s) = 1$ in S . Then before taking homotopy fixed points the induced map $s^* = id$ because S is \mathbb{E}_∞ .¹ \square

6 Modules with genuine involution

Remark 6.1 (Lucy). I’m just going to put drafts of stuff pertaining to hermitian modules here. Eventually when it gets to be more complete, I will hopefully move this entire section over to the main file.

L: or whatever we want to keep calling these

Meta-commentary There are three things we want to do:

- (a) Define a category of ‘bimodules with involution over algebras with anti-involution’ equipped with a forgetful functor $\Theta: \mathbf{BMod}_{\text{inv}}(-) \rightarrow \mathbb{E}_1 \mathbf{Alg}(-)^{hC_2}$.
- (b) Show that Θ is a coCartesian fibration. For this, it suffices to show that it is a *Cartesian* fibration and that it satisfies the hypotheses of [Lur09, Corollary 5.2.2.5]
 - I used to think that we could obtain this by ‘bootstrapping’ a result from Higher Algebra, plus some facts about assembly. This doesn’t seem to be working, so I’m just going to try to do this directly (imitating certain aspects of Chapter 4 of higher algebra.)
- (c) Show that the formula for the cocartesian pushforward along a map $A \rightarrow B$ in $\mathbb{E}_1 \mathbf{Alg}(-)^{hC_2}$ is something like $- \otimes_{A \otimes A^{\text{op}}} (B \otimes B^{\text{op}}) \otimes_{B \otimes B^{\text{op}}} B$.
 - In Higher Algebra, the formula for the cocartesian pushforward is proven in [Lur17, §4.6]; in particular, this is in the section on duality. In particular, see Proposition 4.6.2.17 and the paragraph immediately preceding this.
 - I don’t know how to do this yet—while (a) and (b) are not useful if I can’t show (c), I can’t suss out the feasibility of (c) without (a) and (b) already in place.

Definition 6.2. Define a colored operad Assoc_σ as follows:

- (i) The colored operad has a single object, which we denote by \mathbf{a} .
- (ii) For every finite set I , the set of operations $\text{Mul}_{\text{Assoc}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$, where $\mathcal{L}I$ is the set of linear orderings on I and an element of $\{\pm 1\}^I$ is a function $I \rightarrow \{\pm 1\}$.
- (iii) Suppose given a map of finite sets $\alpha: I \rightarrow J$, together with operations $(\preceq_j, f_j: I_j \rightarrow \{\pm 1\}) \in \text{Mul}_{\text{Assoc}_\sigma}(\{\mathbf{a}_i\}_{\alpha(i)=j}, \mathbf{a})$ and $(\preceq_J, g: J \rightarrow \{\pm 1\}) \in \text{Mul}_{\text{Assoc}_\sigma}(\{\mathbf{a}_j\}_{j \in J}, \mathbf{a})$. Define a linear ordering on the set I as follows: $i \leq i'$ if $\alpha(i) \preceq_J \alpha(i')$ or $\alpha(i) = \alpha(i') = j$ and $i \preceq_j i'$ and $g(j) = +1$ or $\alpha(i) = \alpha(i') = j$ and $i \succeq_j i'$ and $g(j) = -1$. Finally, define a function

$$\begin{aligned} I &\rightarrow \{\pm 1\} \\ i &\mapsto f_{\alpha(i)}(i) \cdot g(\alpha(i)), \end{aligned}$$

where the multiplication on $\{\pm 1\}$ is the usual one.

Definition 6.3. Let $\text{Assoc}_\sigma^\otimes$ denote the associated ∞ -operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.4. Unwinding definitions

- Objects $\text{Assoc}_\sigma^\otimes$ are finite pointed sets $\langle n \rangle \in \mathbf{Fin}_*$
- Morphisms $\langle m \rangle \rightarrow \langle n \rangle$ consist of

¹Or just \mathbb{E}_2 .

L: This is just an imitation of [Lur17, Definition 4.1.1.1], modified in accordance with ideas from §5.4.2.

- $\alpha: \langle m \rangle \rightarrow \langle n \rangle$ a map of finite pointed sets
- for each $i \in \langle n \rangle^\circ$, a linear ordering \preceq_i on the inverse image $\alpha^{-1}(\{i\})$
- a map of sets $s: \alpha^{-1}(\langle m \rangle^\circ) \rightarrow \{\pm 1\}$

- For each pair of morphisms

$$(\beta: \langle \ell \rangle \rightarrow \langle m \rangle, \preceq_j, s) \quad (\alpha: \langle m \rangle \rightarrow \langle n \rangle, \preceq_i, t),$$

the composite is the triple $(\alpha \circ \beta, \preceq_j'', u)$ where \preceq_j'' is the ordering on $(\alpha \circ \beta)^{-1}(\{i\})$ so that if $a, b \in \langle \ell \rangle$ so that $\alpha(\beta(a)) = \alpha(\beta(b))$, then $a \preceq_j'' b$ if $\beta(a) \preceq_i \beta(b)$ or $\beta(a) = \beta(b) = i$ and $a \preceq_i b$ if $s(i) = 1$ or $a \succeq_i b$ if $s(i) = -1$. Finally $u(l) = s(l) \cdot t(\beta(l))$.

Definition 6.5. Define a category Δ_σ

- objects are pairs $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$
- a morphism from $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$ to $([m], t: \{0, 1, \dots, m\} \rightarrow \{\pm 1\})$ is an order-preserving map $[n] \rightarrow [m]$ in Δ .

Construction 6.6. Define a functor $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$:

- For each $([n], s)$, we have $\text{Cut}([n], s) = \langle n \rangle$.
- Given a morphism $\alpha: ([n], s) \rightarrow ([m], t)$, the associated morphism $\text{Cut}([n], s) \rightarrow \text{Cut}([m], t)$ consists of
 - On underlying finite pointed sets $\langle m \rangle \rightarrow \langle n \rangle$, Cut agrees with that appearing in [Lur17, Construction 4.1.2.9]
 - Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j - 1 < j$, we may regard $s: \langle n \rangle^\circ \rightarrow \{\pm 1\}$ and likewise $t: \langle m \rangle^\circ \rightarrow \{\pm 1\}$. Define $u: \text{Cut}(\alpha)^{-1}(\langle n \rangle^\circ) \rightarrow \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\text{Cut}(\alpha)(j))$.

Lemma 6.7. The functor $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$ exhibits $\Delta_\sigma^{\text{op}}$ as an approximation to the ∞ -operad $\text{Assoc}_\sigma^\otimes$.

L: I think the proof of this lemma is not too different from the proof of Proposition 4.1.2.11 of [Lur17]; the point here is just to unravel the definitions of locally coCartesian and Cartesian; the morphisms in $\Delta_\sigma^{\text{op}}$ are a little more complicated than Δ^{op} , but not by much.

L: Note that when s, t are identically one, the resulting order \preceq_j'' agrees with the lexicographic order defined in [Lur17, Remark 4.1.1.4].

L: maybe better to write s as a function defined on the set of morphisms $i < i + 1$ in $[n]$.

Definition 6.8. Define a colored operad \mathbf{BM}_{inv}

- The set of objects of \mathbf{BM}_{inv} has two elements, which we denote by \mathbf{a}, \mathbf{m} .
- Let $\{X_i\}_{i \in I}$ be a finite collection of objects of \mathbf{BM}_{inv} and let Y be another object of \mathbf{BM}_{inv} . If $Y = \mathbf{a}$, then $\text{Mul}(\{X_i\}_{i \in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \rightarrow \{\pm 1\}$ if $X_i = \mathbf{a}$ for all i , and empty otherwise. If $Y = \mathbf{m}$, then $\text{Mul}(\{X_i\}_{i \in I}, Y)$ is the set of pairs consisting of a linear ordering on I and a function $I \rightarrow \{\pm 1\}$ IF there is exactly one index i_k so that $X_{i_k} = \mathbf{m}$ and $X_j = \mathbf{a}$ for all $j \neq i_k$, and $\text{Mul}(\{X_i\}_{i \in I}, Y)$ is empty otherwise.
- The composition law on \mathbf{BM}_{inv} is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.2

Definition 6.9. Let $\mathcal{BM}_{\text{inv}}^\otimes$ denote the associated ∞ -operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

Remark 6.10. We can describe the category $\mathcal{BM}_{\text{inv}}^\otimes$ as follows:

- (1) An object of $\mathcal{BM}_{\text{inv}}^\otimes$ is a pair $(\langle n \rangle, S)$ where S is a subset of $\langle n \rangle^\circ$.
- (2) Morphisms $(\langle m \rangle, T) \rightarrow (\langle n \rangle, S)$ consist of a map $\alpha: \langle m \rangle \rightarrow \langle n \rangle$ in $\text{Assoc}_\sigma^\otimes$ satisfying:
 - The map α takes $T \cup \{*\}$ to $S \cup \{*\}$

L: compare Higher Algebra Notation 4.2.1.6

- For each $s \in S$, then $\alpha^{-1}(\{s\})$ contains exactly one element of t , and that element is minimal with respect to the linear ordering on $\alpha^{-1}(\{s\})$.

L: I've changed things a little so that S (in the notation of Higher Algebra) has been replaced by S^c —this way, we can regard $[n]$ as representing the ordered set $\{-n < -n+1 < \dots < -1 < 0 < 1 < \dots < n-1 < n\}$ where C_2 acts by $\cdot(-1)$ (or something along these lines). This is really a generalization of Notation 4.2.1.7 but for RM^\otimes .

Construction 6.11. Define a functor $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow BM^\otimes$:

- For each $([n], s)$, we have $\text{Cut}([n], s) = \langle n+1 \rangle \simeq \text{RCut}_0([n])$ where RCut is from [Lur17, Construction 4.8.4.4].
- Given a morphism $\alpha: ([n], s) \rightarrow ([m], t)$, the associated morphism $\text{Cut}([n], s) \rightarrow \text{Cut}([m], t)$ consists of
 - On underlying finite pointed sets $\langle m+1 \rangle \rightarrow \langle n+1 \rangle$, Cut agrees with (the reverse of) that appearing in [Lur17, Construction 4.2.2.6]
 - Identifying the cut $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$ with the morphism $j-1 < j$, we may regard $s: \langle n+1 \rangle^\circ \rightarrow \{\pm 1\}$ and likewise $t: \langle m+1 \rangle^\circ \rightarrow \{\pm 1\}$. Define $u: \text{Cut}(\alpha)^{-1}(\langle n+1 \rangle^\circ) \rightarrow \{\pm 1\}$ to be the unique function so that $u(j)t(j) = s(\text{Cut}(\alpha)(j))$.

L: maybe this overloaded notation is not good. I'm running out of ideas.

L: check later

References

- [Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: [10.1515/9781400830558](https://doi.org/10.1515/9781400830558). URL: <https://doi.org/10.1515/9781400830558>.
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