

# Et cetera

Viktor Burghardt, Noah Riggenbach, Lucy Yang

## Abstract

Dumping ground for other stuff: Notes, one-off observations, stuff that we can collectively use when preparing talks, etc.

L: I make no promises re: organization but I will do my best to keep it reasonably readable

## Contents

|           |  |           |
|-----------|--|-----------|
| <b>1</b>  | <b>Talk prep</b>   | <b>1</b>  |
| <b>2</b>  | <b>References</b>  | <b>1</b>  |
| <b>3</b>  | <b>Questions and directions</b>  | <b>2</b>  |
| <b>4</b>  | <b>Thoughts &amp; observations</b>   | <b>2</b>  |
| <b>5</b>  | <b>Desperate Flailing</b>  | <b>3</b>  |
| <b>6</b>  | <b>Modules with genuine involution</b>                                       | <b>3</b>  |
| 6.1       | Step (a) . . . . .   | 4         |
| 6.2       | Part (b) . . . . .   | 11        |
| 6.3       | Towards (e) . . . . .  | 12        |
| 6.4       | Endomorphisms . . . . .  | 13        |
| <b>7</b>  | <b>Categorification and structure–new leaf</b>                               | <b>14</b> |
| <b>8</b>  | <b>Comparing involutive classical Brauer and involutive higher Brauer</b>    | <b>18</b> |
| <b>9</b>  | <b>Other</b>   | <b>19</b> |
| 9.1       | Algebras with genuine involution . . . . .                                   | 21        |
| <b>10</b> | <b>Speculative norm for Brauer group at the level of infinity categories</b> | <b>23</b> |

## 1 Talk prep

## 2 References

- [Involutions of Azumaya algebras](#) by First and Williams (2020 *Documenta*)
- [Counterexamples in involutions of Azumaya algebras](#) by First and Williams; much more readable than the 2020 *Documenta* paper
- [Azumaya algebras without involution](#) by Auel, First, and Williams: the introduction of this provides a very helpful historical overview of the connection between involutions on Azumaya algebras and 2-torsion/kernel of  $\mathrm{coRes}$

### 3 Questions and directions

**Question 3.1** (Literature). • In [PS92] Parimala–Srinivas assume that 2 is invertible in the ring of functions. Has anyone been able to extend their results to the 2 not necessarily invertible case in the meantime?

**Question 3.2** (Morita theory for  $\text{Cat}_\infty^{\text{P}}$ ). Let  $R$  be a Poincaré ring. Suppose given two  $R$ -algebras (suitably interpreted so their module categories are canonically endowed with  $R$ -linear Poincaré structures—perhaps  $\mathbb{E}_\sigma$ )  $A, B$ . Can we characterize

$$\text{hom}_{\text{Cat}_\infty^{\text{P}} R}((\text{Mod}_A^\omega, \mathfrak{P}_A), (\text{Mod}_B^\omega, \mathfrak{P}_B))$$

in terms of something bimodule-like?

**Question 3.3.** On page 2 of the *Counterexamples* paper, First and Williams write that “existence of an extraordinary involution means classification of Azumaya algebras with involution...*cannot* be reduced to questions about projective modules and hermitian forms on them.”

What if we replaced projective modules by perfect complexes?

**Question 3.4.** First–Williams show (see discussion in §4 of the *Counterexamples* paper) that coarse type classify many (most?) Azumaya algebras up to (étale-local) *isomorphism*.

What is a suitable derived version of “coarse type”?

**Question 3.5** (asked by Andrew Nov 2, 2024). C. Schlichtkrull shows in [this paper](#) that a map  $BGL_1(R) \rightarrow K(R) \rightarrow THH(R) \rightarrow R$  in terms of the Hopf map  $\eta$ .

Is there a “Poincaré” version of this result?

**Question 3.6.** Are there some general conditions for a ring with involution  $R$  so that the inclusion  $R^{C_2} \rightarrow R$  is ‘nice’?

There’s some stuff in section 4 [here](#), idk. Also see M. Hochster and J. L. Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay.

#### Applications to Hodge theory?

- [A Deligne pairing for Hermitian Azumaya modules](#)
- [Burt Totaro’s paper](#) on Hodge structures of type  $(n, 0, \dots, 0, n)$
- There’s some stuff about endomorphisms of Hodge structures [here](#)

### 4 Thoughts & observations

**Question 4.1.** When  $R$  has the Tate Poincaré structure and  $(\text{Mod}_A^\omega, M_A, N_A, N_A \rightarrow M_A^{tC_2})$  is invertible, then by invertibility have an equivalence  $\text{hom}_R(A, R) \simeq N_A \otimes_R N_{A^{\text{op}}}$  of  $A \otimes_R A^{\text{op}}$ -modules. Restricting the left-hand side along the unit map  $R \rightarrow A$  gives a map  $N_A \otimes_R N_{A^{\text{op}}} \rightarrow \text{hom}_R(R, R) \simeq R$ . Is this a perfect ( $R$ -linear) pairing?

I *think* using that  $R^{\varphi C_2} \simeq R$  and combining the linear and bilinear part conditions, we get something like

$$M_A \otimes_R M_{A^{\text{op}}} \simeq (N_A \otimes_R N_{A^{\text{op}}})^{\otimes_{R^2}} \quad \text{as } A \otimes_R A^{\text{op}}\text{-bimodules.}$$

Is this useful?

**Brauer-Severi schemes** We know there is a correspondence between Azumaya algebras  $A$  over  $X$  and Brauer-Severi schemes. What does a Poincaré structure on  $\text{Mod}_A^\omega$  mean ‘geometrically’ for  $D_{\text{coh}}^b$  of the corresponding Brauer-Severi scheme? (Lucy: I didn’t get very far here, but just typing up what I had)

- $\text{Mod}_A^\omega$  corresponds to  $\alpha$ -twisted sheaves on  $X$  (see Proposition 3.2.2.1 of Max Lieblich’s thesis)
- The bounded derived category of  $\alpha$ -twisted sheaves on  $X$  includes as one ‘piece’ of a semiorthogonal decomposition on  $D_{\text{coh}}^b$  of the corresponding Brauer-Severi scheme (see Theorem 5.1 [here](#))

## 5 Desperate Flailing

This section is a cronical of my thoughts about  $\mathbb{G}_m^\circ$ .

**Goal** The goal is to build a Poincaré ring  $\mathbb{G}_m^\circ := (\text{Mod}_R, \mathbb{Y}_R)$  such that  $B\mathbb{G}_m^\circ(\underline{S}) = \text{Pic}^P(\underline{S})$  for any Poincaré ring  $\underline{S}$ .

**Lemma 5.1.** *Let  $\underline{S}$  be a Poincaré ring. Then  $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u)) = \{s \in \pi_0(S)^\times \mid s = 1 \text{ in } \pi_0(S^{C_2})\}$ .*

*Proof.* Since the functor  $\text{Pn}(\text{Mod}_S) \rightarrow \text{Mod}_S$  is conservative it follows that an element of  $\pi_0(\text{Aut}_{\text{Pn}(\text{Mod}_S)}(S, u))$  must have underlying map an element of  $\pi_0 \text{Aut}(S) = \pi_0(S)^\times$ . Then in order for  $s \in \pi_0(S)^\times$  to induce a map  $(S, u) \rightarrow (S, u)$ , the induced map  $s^* : S^{C_2} \rightarrow S^{C_2}$  must satisfy  $s^*(u) = u$ . The pullback is given by multiplication by  $s$ , so this requirement translates into  $s$  being the unit, as desired.  $\square$

The problem I thought existed maybe doesn't. Here is a candidate construction:

**Construction 5.2.** Define  $R$  to be the  $\mathbb{E}_\infty$  ring given by  $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\} \otimes_{\mathbb{S}\{z\}} \mathbb{S}$  where the map  $\mathbb{S}\{z\} \rightarrow \mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$  is induced by the map  $z \mapsto xy$ , and the map  $\mathbb{S}\{z\} \rightarrow \mathbb{S}$  is induced by  $z \mapsto 1$ . We can give  $R$  an  $\mathbb{E}_\infty$  ring structure in  $\text{Sp}^{BC_2}$  by taking the trivial action on  $\mathbb{S}\{z\}$  and  $\mathbb{S}$ , and taking the action induced by  $x \mapsto y$  and  $y \mapsto x$  on  $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}$ . Thus in  $\text{CAlg}(\text{Sp}^{BC_2})$  the ring  $R$  corepresents the functor  $S \mapsto \{s \in \pi_0(S)^\times \mid s\sigma(s) = 1\}$ .

Now take  $\underline{R}$  to be the Poincaré ring with underlying Borel  $C_2$  structure as described in the previous paragraph and geometric fixed points  $R^{\varphi C_2} = \mathbb{S}$  and the map  $R^{\varphi C_2} \rightarrow R^{tC_2}$  given by the unit map. Endowing  $R^{\varphi C_2}$  with the  $R$ -module structure given by  $x, y \mapsto 1$ , it remains to show that the unit map  $R^{\varphi C_2} \rightarrow R^{tC_2}$  factors the Tate valued Frobenius  $R \rightarrow R^{tC_2}$  in order to promote  $\underline{R}$  to a Poincaré ring. By construction of  $R$  it is then enough to show that on  $\pi_0$  the Tate valued Frobenius sends  $x, y \mapsto 1$  in  $\pi_0(R^{tC_2})$ . This map sends both  $x$  and  $y$  to  $xy \in \pi_0(R^{tC_2})$ . These are equal to 1 in  $\pi_0(R^{tC_2})$  since the functor  $(-)^{tC_2}$  is lax-monoidal so  $R^{tC_2}$  is a module over  $\mathbb{S}\{x^{\pm 1}, y^{\pm 1}\}^{tC_2} \otimes_{\mathbb{S}\{z\}^{tC_2}} \mathbb{S}^{tC_2}$  which has the image of  $xy$  equal to 1.

Now consider another Poincaré ring  $\underline{S}$ . We then have that maps  $\pi_0(\text{Maps}(\underline{R}, \underline{S}))$  is the data of a unit  $s \in \pi_0(S)^\times$ , a path  $s\sigma(s) \rightarrow 1$  in  $\Omega^\infty S$ , and paths  $x, y \rightarrow 1$  in  $\Omega^\infty S^{\varphi C_2}$ . This then agrees with  $\mathbb{G}_m^\circ$  by the following lemma.

**Lemma 5.3.** *Let  $S \in \text{CAlg}(\text{Sp}^{BC_2})$  and  $s \in \pi_0(S)^\times$ . Then  $s\sigma(s) = 1$  in  $\pi_0(S)$  if and only if  $(s \otimes s)^*$  acts by 1 on  $\pi_0(S^{hC_2}) = \pi_0(\text{Hom}_{S \otimes S}(S \otimes S, S)^{hC_2})$ .*

*Proof.* The 'only if' direction follows from the fact that the map  $S^{hC_2} \rightarrow S$  is an  $S$ -bimodule map. Now suppose that  $s\sigma(s) = 1$  in  $S$ . Then before taking homotopy fixed points the induced map  $s^* = \text{id}$  because  $S$  is  $\mathbb{E}_\infty$ .<sup>1</sup>  $\square$

## 6 Modules with genuine involution

**Remark 6.1** (Lucy). I'm just going to put drafts of stuff pertaining to hermitian modules here. Eventually when it gets to be more complete, I will hopefully move this entire section over to the main file.

L: or whatever we want to keep calling these

**Meta-commentary** There are (at least) three things we want to do:

- Define a category of 'bimodules with involution over algebras with anti-involution' equipped with a forgetful functor  $\Theta : \text{BMod}_{\text{inv}}(-) \rightarrow \mathbb{E}_1 \text{Alg}(-)^{hC_2}$ .
- Show that  $\Theta$  is a coCartesian fibration. For this, it suffices to show that it is a *Cartesian* fibration and that it satisfies the hypotheses of [Lur09, Corollary 5.2.2.5]
  - I used to think that we could obtain this by 'bootstrapping' a result from Higher Algebra, plus some facts about assembly. This doesn't seem to be working, so I'm just going to try to do this directly (imitating certain aspects of Chapter 4 of higher algebra.)

<sup>1</sup>Or just  $\mathbb{E}_2$ .

- (c) Define a relative tensor product for hermitian bimodules
- (d) Show that the formula for the cocartesian pushforward along a map  $A \rightarrow B$  in  $\mathbb{E}_1 \text{Alg}(-)^{hC_2}$  is something like  $-\otimes_{A \otimes A^{\text{op}}} (B \otimes B^{\text{op}}) \otimes_{B \otimes B^{\text{op}}} B$ .
- In Higher Algebra, the formula for the cocartesian pushforward is proven in [Lur17, §4.6]; in particular, this is in the section on duality. In particular, see Proposition 4.6.2.17 and the paragraph immediately preceding this.
  - I don't know how to do this yet—while (a) and (b) are not useful if I can't show (c), I can't suss out the feasibility of (c) without (a) and (b) already in place.
- (e) Towards an adjunction between  $\mathbb{E}_\sigma$ -algebras and categories with additional structure.
- Involutive version of statement that, for a monoidal  $\infty$ -category  $\mathcal{C}$  and an  $\mathbb{E}_1$ -algebra  $A$ ,  $\text{LMod}_A(\mathcal{C})$  is right-tensored over  $\mathcal{C}$ ?
  - Involutive version of endomorphism categories? [Lur17, §4.7.1]

I think that the equivalence of part (b) of the definition of an Azumaya algebra with genuine involution follows from the property of being Azumaya; see Lemma 1(b) (and p.216 for the ‘type 2’ case) of [PS92].

**Lemma 6.2.** *Let  $R$  be an  $\mathbb{E}_\infty$ -ring with an involution  $\sigma: R \xrightarrow{\sim} R$  and suppose  $A$  is an  $\mathbb{E}_1$ - $R$ -algebra with an anti-involution  $\lambda: A \xrightarrow{\sim} \sigma^* A^{\text{op}}$ . Suppose  $A$  is further Azumaya in the sense of . Then the bilinear pairing*

$$A \otimes_R \sigma^* A \xrightarrow{\text{id} \otimes \sigma^* \lambda} A \otimes_R A^{\text{op}} \simeq \text{End}_R(A) \xrightarrow{\text{tr}} R$$

*is perfect, i.e. its adjoint  $A \rightarrow (\sigma^* A)^\vee$  is an equivalence.*

**Question 6.3.** Does the map in part (e) of the definition of an Azumaya algebra with genuine involution follow from property of being Azumaya?

## 6.1 Step (a)

**Definition 6.4.** Define a colored operad  $\text{Assoc}_\sigma$  as follows:

- (i) The colored operad has a single object, which we denote by  $\mathfrak{a}$ .
- (ii) For every finite set  $I$ , the set of operations  $\text{Mul}_{\text{Assoc}_\sigma}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$ , where  $\mathcal{L}I$  is the set of linear orderings on  $I$  and an element of  $\{\pm 1\}^I$  is a function  $I \rightarrow \{\pm 1\}$ .
- (iii) Suppose given a map of finite sets  $\alpha: I \rightarrow J$ , together with operations  $(\preceq_J, f_J: I_J \rightarrow \{\pm 1\}) \in \text{Mul}_{\text{Assoc}_\sigma}(\{\mathfrak{a}_i\}_{\alpha(i)=j}, \mathfrak{a})$  and  $(\preceq_J, g: J \rightarrow \{\pm 1\}) \in \text{Mul}_{\text{Assoc}_\sigma}(\{\mathfrak{a}_j\}_{j \in J}, \mathfrak{a})$ . Define a linear ordering on the set  $I$  as follows:  $i \leq i'$  if  $\alpha(i) \preceq_J \alpha(i')$  or  $\alpha(i) = \alpha(i') = j$  and  $i \preceq_j i'$  and  $g(j) = +1$  or  $\alpha(i) = \alpha(i') = j$  and  $i \succeq_j i'$  and  $g(j) = -1$ . Finally, define a function

$$\begin{aligned} I &\rightarrow \{\pm 1\} \\ i &\mapsto f_{\alpha(i)}(i) \cdot g(\alpha(i)), \end{aligned}$$

where the multiplication on  $\{\pm 1\}$  is the usual one.

**Remark 6.5.** There is a map of colored operads  $\iota: \text{Assoc} \rightarrow \text{Assoc}_\sigma$  which is the identity on objects and on operations  $\text{Mul}_{\text{Assoc}}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\text{Assoc}_\sigma}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{id}_{\mathcal{L}I} \times \{c_1\}$  where  $c_1$  is the constant function on  $I$  with value 1.

There is another map of colored operads  $\iota^{\text{rev}}: \text{Assoc} \rightarrow \text{Assoc}_\sigma$  which is the identity on objects and on operations  $\text{Mul}_{\text{Assoc}}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\text{Assoc}_\sigma}(\{\mathfrak{a}_i\}_{i \in I}, \mathfrak{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  sends a linear ordering  $\ell$  to  $(\ell^{\text{rev}}, c_{-1})$  where  $c_{-1}$  is the constant function on  $I$  with value 1.

**Definition 6.6.** Let  $\text{Assoc}_\sigma^\otimes$  denote the associated  $\infty$ -operad (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

L: reference

L: This is just an imitation of [Lur17, Definition 4.1.1.1], modified in accordance with ideas from §5.4.2.

**Remark 6.7.** Unwinding definitions

- Objects  $\text{Assoc}_\sigma^\otimes$  are finite pointed sets  $\langle n \rangle \in \text{Fin}_*$
- Morphisms  $\langle m \rangle \rightarrow \langle n \rangle$  consist of
  - $\alpha: \langle m \rangle \rightarrow \langle n \rangle$  a map of finite pointed sets
  - for each  $i \in \langle n \rangle^\circ$ , a linear ordering  $\preceq_i$  on the inverse image  $\alpha^{-1}(\{i\})$
  - a map of sets  $s: \alpha^{-1}(\langle m \rangle^\circ) \rightarrow \{\pm 1\}$
- For each pair of morphisms

$$(\beta: \langle \ell \rangle \rightarrow \langle m \rangle, \preceq_j, s) \quad (\alpha: \langle m \rangle \rightarrow \langle n \rangle, \preceq_i, t),$$

the composite is the triple  $(\alpha \circ \beta, \preceq_j'', u)$  where  $\preceq_j''$  is the ordering on  $(\alpha \circ \beta)^{-1}(\{i\})$  so that if  $a, b \in \langle \ell \rangle$  so that  $\alpha(\beta(a)) = \alpha(\beta(b))$ , then  $a \preceq_j'' b$  if  $\beta(a) \preceq_i \beta(b)$  or  $\beta(a) = \beta(b) = i$  and  $a \preceq_i b$  if  $s(i) = 1$  or  $a \succeq_i b$  if  $s(i) = -1$ . Finally  $u(l) = s(l) \cdot t(\beta(l))$ .

**Remark 6.8.** The maps  $\iota, \iota^{\text{rev}}$  of Remark 6.5 induce maps of  $\infty$ -operads  $\text{Assoc}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ . There is a canonical identification  $\iota^{\text{rev}} = \sigma \circ \iota$ , where  $\sigma$  is the automorphism of the associative operad considered in [Lur17, Remark 4.1.1.7].

Note that each object  $\langle n \rangle \in \text{Assoc}_\sigma^\otimes$  has a distinguished automorphism  $\text{rev}_{\langle n \rangle}$  of order two given by the identity map on  $\langle n \rangle$  and the constant map  $c_{-1}: \langle n \rangle^\circ \rightarrow \{\pm 1\}$  at  $-1$ . There is a canonical natural equivalence  $\iota \xrightarrow{\sim} \iota^{\text{rev}}$  whose component at  $\langle n \rangle$  is  $\text{rev}_{\langle n \rangle}$ .

**Definition 6.9.** Let  $\mathcal{C}^\otimes$  be a  $\infty$ -operad equipped with the data of a fibration  $p: \mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ . Let  $\text{Alg}^\sigma(\mathcal{C})$  denote the  $\infty$ -category  $\text{Alg}_{/\text{Assoc}_\sigma}(\mathcal{C})$  of  $\infty$ -operad sections of  $p$ . We will refer to  $\text{Alg}^\sigma(\mathcal{C})$  as the  $\infty$ -category of *involutive algebra objects of  $\mathcal{C}$* .

An *involutive monoidal  $\infty$ -category* is the data of a cocartesian fibration  $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$ .

**Remark 6.10.** Suppose given a cocartesian fibration  $f: \mathcal{D}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  of  $\infty$ -operads. Write  $\mathcal{C}^\otimes := \mathcal{D}^\otimes \times_{\text{Assoc}_\sigma^\otimes, \iota} \text{Assoc}^\otimes$ ;  $\mathcal{C}^\otimes$  is a monoidal  $\infty$ -category in the sense of [Lur17, Definition 4.1.1.10]. Furthermore,  $\mathcal{C}_{\text{rev}}^\otimes := \mathcal{D}^\otimes \times_{\text{Assoc}_\sigma^\otimes, \iota^{\text{rev}}} \text{Assoc}^\otimes$  is a monoidal  $\infty$ -category. By Remark 6.8, this notation is consistent with that of [Lur17, Remark 4.1.1.7]. In particular, a  $\text{Assoc}_\sigma$ -monoidal  $\infty$ -category  $\mathcal{D}^\otimes$  determines a monoidal  $\infty$ -category  $\mathcal{C}^\otimes$  equipped with a monoidal equivalence  $\sigma_{\mathcal{C}}: \mathcal{C}^\otimes \xrightarrow{\sim} \mathcal{C}_{\text{rev}}^\otimes$ . Pullback along the involution of  $\text{Assoc}^\otimes$  determines another monoidal equivalence  $\sigma_{\mathcal{C}}^{\text{rev}}: \mathcal{C}_{\text{rev}}^\otimes \xrightarrow{\sim} \mathcal{C}^\otimes$ , and our assumptions imply that  $\sigma_{\mathcal{C}}^{\text{rev}} \circ \sigma_{\mathcal{C}}$  is equivalent to the identity on  $\mathcal{C}^\otimes$ .

Now suppose that  $A$  is an involutive algebra object of  $\mathcal{D}$ . With the same notation as before, pullback along  $\iota$  (resp.  $\iota^{\text{rev}}$ ) determines associative algebra objects  $u(A), u^{\text{rev}}(A)$  of  $\mathcal{C}$  and  $\mathcal{C}_{\text{rev}}$ , respectively. Note that  $\sigma_{\mathcal{C}}(u(A))$  is an algebra object of  $\mathcal{C}_{\text{rev}}$ , which we may regard as an algebra object of  $\mathcal{C}$  by precomposing with the autoequivalence  $\sigma: \text{Assoc}^\otimes \xrightarrow{\sim} \text{Assoc}^\otimes$ . It follows from Remark 6.8 that  $A$  determines an equivalence  $\sigma_A: u(A) \xrightarrow{\sim} \sigma_{\mathcal{C}}(u(A))^{\text{rev}}$  of algebra objects in  $\mathcal{C}$ .

Now suppose furthermore that  $\mathcal{D}^\otimes$  is of the form  $\mathcal{E}^\otimes \times_{\text{Fin}_*} \text{Assoc}_\sigma^\otimes$  for some symmetric monoidal  $\infty$ -category  $\mathcal{E}$ . Then the associated involution  $\sigma_{\mathcal{C}}$  is the identity, and for any involutive algebra object  $A$  of  $\mathcal{D}$ ,  $\sigma_A$  is an equivalence  $u(A) \simeq u(A)^{\text{rev}}$  satisfying  $\sigma_A^{\text{rev}} \circ \sigma_A \simeq \text{id}_A$ .

**Remark 6.11.** Suppose given a cocartesian fibration  $f: \mathcal{D}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  of  $\infty$ -operads, and write  $\mathcal{C}$  for its underlying monoidal  $\infty$ -category with equivalence  $\sigma_{\mathcal{C}}$  as in Remark 6.10. The functors of Remark 6.8 induce a functor

$$\text{Alg}^\sigma(\mathcal{D}) \rightarrow \text{Alg}(\mathcal{C}) \times \text{Alg}(\mathcal{C}_{\text{rev}});$$

moreover, the aforementioned functor factors through the fixed points of the  $C_2$ -action on the right-hand side given by  $\text{Alg}(\mathcal{C}) \times \text{Alg}(\mathcal{C}_{\text{rev}}) \xrightarrow{\text{swap}} \text{Alg}(\mathcal{C}_{\text{rev}}) \times \text{Alg}(\mathcal{C}) \xrightarrow{\sigma_{\mathcal{C}} \times \sigma_{\mathcal{C}}} \text{Alg}(\mathcal{C}) \times \text{Alg}(\mathcal{C}_{\text{rev}})$ .

**Definition 6.12.** Define a category  $\Delta_\sigma$

- objects are pairs  $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$

L: Note that when  $s, t$  are identically one, the resulting order  $\preceq_j''$  agrees with the lexicographic order defined in [Lur17, Remark 4.1.1.4].

L: do we need weaker than cocartesian fibration?

L: maybe better to write  $s$  as a function defined on the set of morphisms  $i < i+1$  in  $[n]$ .

- a morphism from  $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\})$  to  $([m], t: \{0, 1, \dots, m\} \rightarrow \{\pm 1\})$  is an order-preserving map  $[n] \rightarrow [m]$  in  $\Delta$ .

**Construction 6.13.** Define a functor  $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$ :

- For each  $([n], s)$ , we have  $\text{Cut}([n], s) = \langle n \rangle$ .
- Given a morphism  $\alpha: ([n], s) \rightarrow ([m], t)$ , the associated morphism  $\text{Cut}([n], s) \rightarrow \text{Cut}([m], t)$  consists of
  - On underlying finite pointed sets  $\langle m \rangle \rightarrow \langle n \rangle$ , Cut agrees with that appearing in [Lur17, Construction 4.1.2.9]
  - Identifying the cut  $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$  with the morphism  $j - 1 < j$ , we may regard  $s: \langle n \rangle^\circ \rightarrow \{\pm 1\}$  and likewise  $t: \langle m \rangle^\circ \rightarrow \{\pm 1\}$ . Define  $u: \text{Cut}(\alpha)^{-1}(\langle n \rangle^\circ) \rightarrow \{\pm 1\}$  to be the unique function so that  $u(j)t(j) = s(\text{Cut}(\alpha)(j))$ .

**Lemma 6.14.** *The functor  $\text{Cut}: \Delta_\sigma^{\text{op}} \rightarrow \text{Assoc}_\sigma^\otimes$  exhibits  $\Delta_\sigma^{\text{op}}$  as an approximation to the  $\infty$ -operad  $\text{Assoc}_\sigma^\otimes$ .*

*L: I think the proof of this lemma is not too different from the proof of Proposition 4.1.2.11 of [Lur17]; the point here is just to unravel the definitions of locally coCartesian and Cartesian; the morphisms in  $\Delta_\sigma^{\text{op}}$  are a little more complicated than  $\Delta^{\text{op}}$ , but not by much.*

**Notation 6.15.** Let  $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  exhibit  $\mathcal{C}$  as  $\mathbb{E}_\sigma$ -monoidal. Let  $\mathcal{C}^\otimes$  denote the fiber product  $\mathcal{C}^\otimes \times_{\text{Assoc}_\sigma^\otimes} \Delta_\sigma^{\text{op}}$ .

**Definition 6.16.** Say that a morphism  $([n], s) \rightarrow ([m], t)$  is *inert* if the induced map  $\text{Cut}([m], t) \rightarrow \text{Cut}([n], s)$  is an inert morphism in  $\text{Assoc}_\sigma^\otimes$ .

**Definition 6.17.** A  $\mathbb{R}^\sigma$ -planar operad is an  $\infty$ -category  $\mathcal{O}^\otimes$  equipped with a functor  $q: \mathcal{O}^\otimes \rightarrow \Delta_\sigma^{\text{op}}$  so that

1. For every object  $X \in \mathcal{O}^\otimes$  and every inert morphism  $\alpha: ([n], s) \rightarrow q(X)$  in  $\Delta_\sigma$ , there is a  $q$ -cocartesian morphism  $\bar{\alpha}: X \rightarrow Y$  satisfying  $q(\bar{\alpha}) = \alpha$
2. Let  $X$  be an object satisfying  $q(X) = ([n], s)$ , and choose  $q$ -cocartesian morphisms  $\bar{\alpha}_i: X \rightarrow X_i$  corresponding to the morphism  $([i - 1 < i], s_i) \rightarrow ([n], s)$  which is the inclusion on underlying sets and satisfies  $s_i(i) = s(i)$ . Then the morphisms  $\bar{\alpha}_i$  exhibit  $X$  as the  $q$ -product of the  $X_i$ .
3. For each  $n \geq 0$ , the construction  $C \mapsto \{C_i\}_{1 \leq i \leq n}$  induces an equivalence of  $\infty$ -categories

$$\mathcal{O}^\otimes \times_{\Delta_\sigma^{\text{op}}} \{([n], s)\} \xrightarrow{\sim} (\mathcal{O}^\otimes \times_{\Delta_\sigma^{\text{op}}} \{([1], s|_{\{i\}})\})^{\times n}$$

We say that a morphism  $\alpha$  in  $\mathbb{R}^\sigma$ -planar operad is *inert* if it is  $q$ -cocartesian and  $q(\alpha)$  is inert in  $\Delta_\sigma^{\text{op}}$  in the sense of Definition 6.16.

**Definition 6.18.** Let  $q: \mathcal{O}^\otimes \rightarrow \Delta_\sigma^{\text{op}}$  be a  $\mathbb{R}^\sigma$ -planar operad. An  $\mathbb{A}_\infty^\sigma$ -algebra object of  $\mathcal{O}^\otimes$  is a section of  $q$  which carries inert morphisms to inert morphisms. Write  $\text{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O})$  for the full subcategory of  $\text{Fun}_{\Delta_\sigma^{\text{op}}}(\Delta_\sigma^{\text{op}}, \mathcal{O}^\otimes)$  on  $\mathbb{A}_\infty^\sigma$ -algebra objects.

**Proposition 6.19.** *Let  $\mathcal{O}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  be a fibration of  $\infty$ -operads. Then precomposition with the functor Cut of Construction 6.13 induces an equivalence of  $\infty$ -categories*

$$\text{Alg}_{\text{Assoc}_\sigma^\otimes}(\mathcal{O}) \xrightarrow{\sim} \text{Alg}_{\mathbb{A}_\infty^\sigma}(\mathcal{O}) .$$

*Proof.* Combine Lemma 6.14 with [Lur17, Theorem 2.3.3.23]. □

**Definition 6.20.** Define a colored operad  $\mathbf{LM}_{\text{inv}}$

- (i) The set of objects of  $\mathbf{LM}_{\text{inv}}$  has two elements, which we denote by  $\mathbf{a}, \mathbf{m}$ .

- (ii) Let  $\{X_i\}_{i \in I}$  be a finite collection of objects of  $\mathbf{LM}_{\text{inv}}$  and let  $Y$  be another object of  $\mathbf{LM}_{\text{inv}}$ . If  $Y = \mathbf{a}$ , then  $\text{Mul}_{\mathbf{LM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is the set of pairs consisting of a linear ordering on  $I$  and a function  $I \rightarrow \{\pm 1\}$  if  $X_i = \mathbf{a}$  for all  $i$ , and empty otherwise. If  $Y = \mathbf{m}$ , then  $\text{Mul}_{\mathbf{LM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is a subset of the set of pairs  $(\lambda, c)$  consisting of a linear ordering  $\lambda = \{i_1 < i_2 < \dots < i_n\}$  on  $I$  and a function  $c: I \rightarrow \{\pm 1\}$  satisfying either

- $X_{i_n} = \mathbf{m}$  and  $c(i_n) = 1$  and  $X_j = \mathbf{a}$  otherwise
- $X_{i_1} = \mathbf{m}$  and  $c(i_n) = -1$  and  $X_j = \mathbf{a}$  otherwise

- (iii) The composition law on  $\mathbf{LM}_{\text{inv}}$  is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4

**Remark 6.21.** There is a colored operad  $\mathbf{RM}_{\text{inv}}$  defined exactly in the same way as  $\mathbf{LM}_{\text{inv}}$  in Definition 6.20. In the interest of precision:  $\mathbf{RM}_{\text{inv}}$  has the same objects  $\mathbf{a}, \mathbf{m}$ . Let  $\{X_i\}_{i \in I}$  be a finite collection of objects of  $\mathbf{RM}_{\text{inv}}$  and let  $Y$  be another object of  $\mathbf{RM}_{\text{inv}}$ . If  $Y = \mathbf{m}$ , then  $\text{Mul}_{\mathbf{RM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is a subset of the set of pairs  $(\lambda, c)$  consisting of a linear ordering  $\lambda = \{i_1 < i_2 < \dots < i_n\}$  on  $I$  and a function  $c: I \rightarrow \{\pm 1\}$  satisfying either

- $X_{i_n} = \mathbf{m}$  and  $c(i_n) = -1$  and  $X_j = \mathbf{a}$  otherwise
- $X_{i_1} = \mathbf{m}$  and  $c(i_n) = 1$  and  $X_j = \mathbf{a}$  otherwise

**Remark 6.22.** Restricting to the objects which are both called  $\mathbf{a}$ , we see that both  $\mathbf{LM}_{\text{inv}}$  and  $\mathbf{RM}_{\text{inv}}$  have a sub-colored operad which is canonically identified with  $\mathbf{Assoc}_{\text{inv}}$  of Definition 6.4.

**Remark 6.23.** There is a map of colored operads  $\iota: \mathbf{LM} \rightarrow \mathbf{LM}_\sigma$  which sends  $\mathbf{m}$  to  $\mathbf{m}$  and sends  $\mathbf{a}$  to  $\mathbf{a}$ . On  $\text{Mul}_{\mathbf{LM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{LM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{id}_{\mathcal{L}I} \times \{c_1\}$ , this map agrees with  $\iota$  of Remark 6.5. On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is the restriction of the map  $\text{id}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$  where  $c_1$  is the constant function on  $I \sqcup \{j\}$  with value 1.

There is a map of colored operads  $\iota^{\text{rev}}: \mathbf{RM} \rightarrow \mathbf{LM}_\sigma$  which sends  $\mathbf{m}$  to  $\mathbf{m}$  and sends  $\mathbf{a}$  to  $\mathbf{a}$ . On  $\text{Mul}_{\mathbf{RM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{LM}_\sigma}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{rev}_{\mathcal{L}I} \times \{c_1\}$ , this map agrees with  $\iota^{\text{rev}}$  of Remark 6.5. On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is the restriction of the map  $\text{rev}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$  where  $c_1$  is the constant function on  $I \sqcup \{j\}$  with value 1.

**Definition 6.24.** Define colored operads  $\mathbf{BM}_{\text{inv}}$  and  $\mathbf{BM}_\sigma$

- (i) The set of objects of  $\mathbf{BM}_{\text{inv}}$  has three elements, which we denote by  $\mathbf{a}_\ell, \mathbf{a}_r, \mathbf{m}$ .  $\mathbf{BM}_\sigma$  has the same objects
- (ii) Let  $\{X_i\}_{i \in I}$  be a finite collection of objects of  $\mathbf{BM}_{\text{inv}}$  and let  $Y$  be another object of  $\mathbf{BM}_{\text{inv}}$ . If  $Y = \mathbf{a}_\ell$  (resp.  $Y = \mathbf{a}_r$ ), then  $\text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is the set of pairs consisting of a linear ordering on  $I$  and a function  $I \rightarrow \{\pm 1\}$  if  $X_i = \mathbf{a}_\ell$  (resp.  $X_i = \mathbf{a}_r$ ) for all  $i$ , and empty otherwise. If  $Y = \mathbf{m}$ , then  $\text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  is the subset of pairs  $(\lambda, c)$  consisting of a linear ordering  $\lambda = \{i_1 < i_2 < \dots < i_n\}$  on  $I$  and a function  $c: I \rightarrow \{\pm 1\}$  satisfying: if there is exactly one index  $i_k$  so that  $X_{i_k} = \mathbf{m}$ , either

- $c(i_k) = 1$ ,  $X_j = \mathbf{a}_\ell$  for  $j < i_k$  and  $X_j = \mathbf{a}_r$  for  $j > i_k$ ; or
- $c(i_k) = -1$ ,  $X_j = \mathbf{a}_\ell$  for  $j > i_k$  and  $X_j = \mathbf{a}_r$  for  $j < i_k$

$\text{Mul}_{\mathbf{BM}_\sigma}(\{X_i\}_{i \in I}, Y)$  is the subset of pairs  $(\lambda, c)$  consisting of a linear ordering  $\lambda = \{i_1 < i_2 < \dots < i_n\}$  on  $I$  and a function  $c: I \rightarrow \{\pm 1\}$  satisfying: if there is exactly one index  $i_k$  so that  $X_{i_k} = \mathbf{m}$ ,  $c(i_k) = 1$ ,  $X_j = \mathbf{a}_\ell$  for  $j < i_k$  and  $X_j = \mathbf{a}_r$  for  $j > i_k$ .

- (iii) The composition law on  $\mathbf{BM}_{\text{inv}}$  is determined by the composition of linear orderings, with reversal of linear orderings according to Definition 6.4. The composition law on  $\mathbf{BM}_\sigma$  is determined by the composition of linear orderings

**Remark 6.25.** The colored operad  $\mathbf{BM}_{\text{inv}}$  has a canonical involution  $\sigma$  which fixes  $\mathbf{m}$ , exchanges  $\mathbf{a}_\ell$  and  $\mathbf{a}_r$ , and sends a morphism  $(\lambda, c)$  to  $(\lambda^{\text{rev}}, I \xrightarrow{c} \{\pm 1\} \xrightarrow{(-1)} \{\pm 1\})$ .



**Remark 6.26.** There is a map of colored operads  $\iota: \mathbf{BM} \rightarrow \mathbf{BM}_{\text{inv}}$  which sends  $\mathbf{m}$  to  $\mathbf{m}$  and sends  $\mathbf{a}_-$  to  $\mathbf{a}_\ell$  and  $\mathbf{a}_+$  to  $\mathbf{a}_r$ . On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}_\pm) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{id}_{\mathcal{L}I} \times \{c_1\}$ , this map agrees with  $\iota$  of Remark 6.5. On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is the restriction of the map  $\text{id}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_1\}$  where  $c_1$  is the constant function on  $I \sqcup \{j\}$  with value 1.

There is *also* a map of colored operads  $\iota^{\text{rev}}: \mathbf{BM} \rightarrow \mathbf{BM}_{\text{inv}}$  which sends  $\mathbf{m}$  to  $\mathbf{m}$  and sends  $\mathbf{a}_-$  to  $\mathbf{a}_r$  and  $\mathbf{a}_+$  to  $\mathbf{a}_\ell$ . On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I}, \mathbf{a}_\pm) \simeq \mathcal{L}I \rightarrow \text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{\mathbf{a}_i\}_{i \in I}, \mathbf{a}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is  $\text{id}_{\mathcal{L}I} \times \{c_1\}$ , this map agrees with  $\iota^{\text{rev}}$  of Remark 6.5. On  $\text{Mul}_{\mathbf{BM}}(\{(\mathbf{a}_\pm)_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \subseteq \mathcal{L}(I \sqcup \{j\}) \rightarrow \text{Mul}_{\mathbf{BM}_\sigma}(\{\mathbf{a}_i\}_{i \in I} \sqcup \{\mathbf{m}\}, \mathbf{m}) \simeq \mathcal{L}I \times \{\pm 1\}^I$  is the restriction of the map  $\text{rev}_{\mathcal{L}(I \sqcup \{j\})} \times \{c_{-1}\}$  where  $c_{-1}$  is the constant function on  $I \sqcup \{j\}$  with value  $-1$ .

**Definition 6.27.** Let  $\mathcal{LM}_{\text{inv}}^\otimes$ ,  $\mathcal{RM}_{\text{inv}}^\otimes$ ,  $\mathcal{BM}_{\text{inv}}^\otimes$ , and  $\mathcal{BM}_\sigma^\otimes$  denote the  $\infty$ -operads associated to the colored operads of Definitions 6.20 and 6.24 (via Construction 2.1.1.7 and Example 2.1.1.21 of [Lur17]).

**Remark 6.28.** We can describe the category  $\mathcal{LM}_{\text{inv}}^\otimes$  as follows:

- (1) An object of  $\mathcal{LM}_{\text{inv}}^\otimes$  is a pair  $(\langle n \rangle, S)$  where  $S$  is a subset of  $\langle n \rangle^\circ$ .
- (2) Morphisms  $(\langle m \rangle, T) \rightarrow (\langle n \rangle, S)$  consist of a map  $(\alpha: \langle m \rangle \rightarrow \langle n \rangle, \lambda: \langle m \rangle^\circ \rightarrow \{\pm 1\})$  in  $\text{Assoc}_\sigma^\otimes$  satisfying:
  - The map  $\alpha$  takes  $T \cup \{*\}$  to  $S \cup \{*\}$
  - For each  $s \in S$ , then  $\alpha^{-1}(\{s\})$  contains exactly one element  $t_s$  of  $T$ , and it is maximal (resp. minimal) with respect to the linear ordering on  $\alpha^{-1}(\{s\})$  if  $\lambda(t_s) = 1$  (resp.  $\lambda(t_s) = -1$ ).

**Remark 6.29.** We can describe the category  $\mathcal{BM}_{\text{inv}}^\otimes$  as follows:

- (1) An object of  $\mathcal{BM}_{\text{inv}}^\otimes$  is a triple  $(\langle n \rangle, c_+, c_-)$  where  $c_\pm$  are functions  $\langle n \rangle^\circ \rightarrow \{0, 1\}$  and  $c_-(i) \leq c_+(i)$  for all  $i \in \langle n \rangle^\circ$ .
- (2) Morphisms  $(\langle m \rangle, c_+, c_-) \rightarrow (\langle n \rangle, c'_+, c'_-)$  consist of a map  $(\alpha: \langle m \rangle \rightarrow \langle n \rangle, \lambda: \langle m \rangle^\circ \rightarrow \{\pm 1\})$  in  $\text{Assoc}_\sigma^\otimes$  satisfying: if  $j \in \langle n \rangle^\circ$  and  $\alpha^{-1}(j) = \{i_1 < i_2 < \dots < i_\ell\}$ ,
  - If  $c_-(j) = c_+(j)$ , then
$$c'_-(j) = c_-(i_1) \leq c_+(i_1) = c_-(i_2) \leq c_+(i_2) \dots c_-(i_{m-1}) \leq c_+(i_m) = c'_+(j)$$
  - If  $c_-(j) < c_+(j)$ , then there exists a unique  $k$  so that  $c_-(i_k) < c_+(i_k)$  and
$$\lambda(i_k) \cdot c'_-(j) = \lambda(i_k) \cdot c_-(i_1) \leq \lambda(i_k) \cdot c_+(i_1) = \lambda(i_k) \cdot c_-(i_2) \leq \lambda(i_k) \cdot c_+(i_2) \dots$$

$$\lambda(i_k) \cdot c_-(i_{m-1}) \leq \lambda(i_k) \cdot c_+(i_m) = \lambda(i_k) \cdot c'_+(j)$$

**Remark 6.30.** Each morphism  $\varphi \in \text{Mul}_{\mathbf{BM}_{\text{inv}}}(\{X_i\}_{i \in I}, Y)$  determines a linear ordering  $\ell$  on the set  $I$  and a function  $s: I \rightarrow \{\pm 1\}$ . Passing from  $\varphi$  to the pair  $(\ell, s)$  determines a map of colored operads  $j: \mathbf{BM}_{\text{inv}} \rightarrow \mathbf{Assoc}_{\text{inv}}$ . The map  $j$  induces a morphism of  $\infty$ -operads  $\mathcal{BM}_{\text{inv}}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  which we will also denote by  $j$ .

**Remark 6.31.** The maps  $\iota, \iota^{\text{rev}}$  of Remark 6.23 induce maps of  $\infty$ -operads  $\iota: \mathcal{LM}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$  and  $\iota^{\text{rev}}: \mathcal{RM}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$ .

**Remark 6.32.** The maps  $\iota, \iota^{\text{rev}}$  of Remark 6.26 induce maps of  $\infty$ -operads  $\iota, \iota^{\text{rev}}: \mathcal{BM}^\otimes \rightarrow \mathcal{BM}_\sigma^\otimes$ . There are canonical identifications  $\iota \circ \text{rev} \simeq \sigma \circ \iota^{\text{rev}}$  where  $\sigma$  is the involution on  $\mathcal{BM}_\sigma^\otimes$  induced by Remark 6.25 and  $\text{rev}$  is the involution on  $\mathcal{BM}^\otimes$  of [Lur17, Construction 4.6.3.1].

**Remark 6.33.** There are canonical maps of  $\infty$ -operads  $\mathcal{LM}_{\text{inv}}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  and  $\mathcal{RM}_{\text{inv}}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  sending  $\mathbf{a}$  to  $\mathbf{a}_\ell$ , resp.  $\mathbf{a}_r$  and making the diagram

$$\begin{array}{ccc} \text{Assoc}^\otimes & \longrightarrow & \mathcal{LM}_{\text{inv}}^\otimes \\ \downarrow \sigma & & \downarrow \text{rev} \\ \text{Assoc}^\otimes & \longrightarrow & \mathcal{RM}_{\text{inv}}^\otimes \end{array} \quad \begin{array}{c} \nearrow \\ \nearrow \end{array} \quad \mathcal{BM}_{\text{inv}}^\otimes$$

commute, where  $\text{rev}$  is (an involutive version of) the reversal involution of [Lur17, Remark 4.6.3.2].



**Definition 6.34.** Let  $\mathcal{C}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  and  $\mathcal{D}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  be fibrations of  $\infty$ -operads and let  $\mathcal{M}$  be an  $\infty$ -category. Suppose given a fibration of  $\infty$ -operads  $q: \mathcal{O}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$  together with equivalences  $\mathcal{O}_a^\otimes \simeq \mathcal{C}^\otimes$  and  $\mathcal{O}_m^\otimes \simeq \mathcal{M}$ . Let  $L^\sigma \text{Mod}(\mathcal{M})$  denote the  $\infty$ -category  $\text{Alg}_{/\mathcal{LM}_{\text{inv}}^\otimes}(\mathcal{O})$ . We will refer to  $L^\sigma \text{Mod}(\mathcal{M})$  as the  *$\infty$ -category of left hermitian module objects of  $\mathcal{M}$* .

Suppose given a fibration of  $\infty$ -operads  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  together with equivalences  $\mathcal{O}_{a_\ell}^\otimes \simeq \mathcal{C}^\otimes$ ,  $\mathcal{O}_{a_r}^\otimes \simeq \mathcal{D}^\otimes$  and  $\mathcal{O}_m^\otimes \simeq \mathcal{M}$ . Let  ${}^\sigma \text{Mod}(\mathcal{M})$  denote the  $\infty$ -category  $\text{Alg}_{/\mathcal{BM}_{\text{inv}}^\otimes}(\mathcal{O})$ . We will refer to  ${}^\sigma \text{Mod}(\mathcal{M})$  as the  *$\infty$ -category of hermitian bimodule objects of  $\mathcal{M}$* . Composition with the inclusions  $\text{Assoc}_\sigma^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  induces a categorical fibration

$${}^\sigma \text{Mod}(\mathcal{M}) = \text{Alg}_{/\mathcal{BM}_{\text{inv}}^\otimes}(\mathcal{O}) \rightarrow \text{Alg}_{\text{Assoc}_\sigma}(\mathcal{C}) \times \text{Alg}_{\text{Assoc}_\sigma}(\mathcal{D}).$$

If  $A$  is an  $\text{Assoc}_\sigma$ -algebra object of  $\mathcal{C}$ , we let  ${}^\sigma \text{Mod}_A(\mathcal{M})$  denote the fiber  ${}^\sigma \text{Mod}(\mathcal{M}) \times_{\text{Alg}_{\text{Assoc}_\sigma}(\mathcal{C})} \{A\}$ . We will refer to  ${}^\sigma \text{Mod}_A(\mathcal{M})$  as the  *$\infty$ -category of hermitian  $A$ -bimodule objects of  $\mathcal{M}$* .

**Definition 6.35.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads. We say that  $q$  exhibits  $\mathcal{O}_m$  as  $\mathbb{E}_\sigma$ -bintensored over  $\mathcal{O}_{a_\ell}$  and  $\mathcal{O}_{a_r}$  if  $q$  is a cocartesian fibration.

**Remark 6.36.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads. Then  $q$  is classified by a map  $\chi: \mathcal{BM}_{\text{inv}}^\otimes \rightarrow \text{Cat}_\infty$ . By Remark 6.32, we can think of  $q$  as giving two  $\mathbb{E}_\sigma$  algebras  $\mathcal{C}, \mathcal{D}$  in  $\text{Cat}_\infty$  with an  $\infty$ -category  $\mathcal{M}$  equipped with both the structure of a  $\mathcal{C}$ - $\mathcal{D}$ -bimodule (equivalently, the structure of a left  $\mathcal{C} \times \mathcal{D}_{\text{rev}}$ -module) and of a  $\mathcal{D}$ - $\mathcal{C}$ -bimodule, and an autoequivalence  $\sigma_{\mathcal{M}}: \mathcal{M} \simeq \mathcal{M}$  of order two which is linear with respect to the autoequivalence  $\mathcal{C} \times \mathcal{D}_{\text{rev}} \xrightarrow{\text{flip}} \mathcal{D}_{\text{rev}} \times \mathcal{C} \xrightarrow{\sigma_{\mathcal{D}}^{-1} \times \sigma_{\mathcal{C}}} \mathcal{D} \times \mathcal{C}_{\text{rev}}$ .

**Remark 6.37.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{LM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads. Consider a left hermitian module object  $F: \mathcal{LM}_{\text{inv}}^\otimes \rightarrow \mathcal{O}^\otimes$ . By Remark 6.33,  $F$  determines an associative algebra  $A$  of  $\mathcal{C}$  with an equivalence of algebras  $\sigma_A: A \simeq \sigma_{\mathcal{C}}(A)^{\text{rev}}$ , an object  $M \in \mathcal{M}$  so that  $M$  (resp.  $\sigma_{\mathcal{M}}(M)$ ) is equipped with the structure of a left  $A$ -module (resp. right  $\sigma_{\mathcal{C}}(A)$ -module). Furthermore, we have an equivalence  $\sigma_M: M \simeq \sigma_{\mathcal{M}}(M)$  which is linear with respect to the equivalence  $A \xrightarrow{\sigma_A} \sigma_{\mathcal{C}}(A)^{\text{rev}}$ .

L: is this related to “modules with involution” from [Cal+20, §3.1]?

**Remark 6.38.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{BM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads. Consider a hermitian module object  $F: \mathcal{BM}_{\text{inv}}^\otimes \rightarrow \mathcal{O}^\otimes$  and keep the notation of Remark 6.36. By Remark 6.33,  $F$  determines an associative algebra  $A$  of  $\mathcal{C}$  with an equivalence of algebras  $\sigma_A: A \simeq \sigma_{\mathcal{C}}(A)^{\text{rev}}$  and an associative algebra  $B$  of  $\mathcal{D}$  with an equivalence of algebras  $\sigma_B: B \simeq \sigma_{\mathcal{D}}(B)^{\text{rev}}$ , an object  $M \in \mathcal{M}$  so that  $M$  (resp.  $\sigma_{\mathcal{M}}(M)$ ) is equipped with the structure of a  $A$ - $B$ -bimodule (resp.  $\sigma_{\mathcal{D}}(B)$ - $\sigma_{\mathcal{C}}(A)$ -bimodule). Furthermore, we have an equivalence  $\sigma_M: M \simeq \sigma_{\mathcal{M}}(M)$  which is linear with respect to the equivalence  $A \otimes B \xrightarrow{\text{flip}} B \otimes A \xrightarrow{\sigma_B^{-1} \otimes \sigma_A} \sigma_{\mathcal{D}}(B)^{\text{rev}} \otimes \sigma_{\mathcal{C}}(A)^{\text{rev}}$ .

L: when  $\mathcal{C} = \mathcal{D}$  and  $\sigma_{\mathcal{M}}$  and  $\sigma_{\mathcal{C}}$  are both the identity and  $A = B$ , I think this recovers the “module with involution” from [Cal+20, §3.1].

**Remark 6.39.** Let  $q: \mathcal{O}^\otimes \rightarrow \text{Assoc}_\sigma$  be a cocartesian fibration of  $\infty$ -operads exhibiting  $\mathcal{C}$  as a monoidal  $\infty$ -category with an involution  $\sigma_{\mathcal{C}}$ . Let us write  $\mathcal{C}_\pm^\otimes \rightarrow \mathcal{BM}^\otimes$  for the pullback of  $q$  the functors of Remark 6.26 and 6.30. Restriction along the functors of Remark 6.26 and 6.30 defines functors  $\text{Alg}^\sigma(\mathcal{D}) \rightarrow {}^\sigma \text{Mod}(\mathcal{D}) \rightarrow \text{BMod}(\mathcal{C}_\pm^\otimes)$  which is equivariant with respect to the action of  $\sigma_{\mathcal{C}}$  on the target. Moreover, there is a commutative diagram

$$\begin{array}{ccccc} \text{Alg}^\sigma(\mathcal{D}) & \longrightarrow & {}^\sigma \text{Mod}(\mathcal{D}) & \longrightarrow & \text{BMod}(\mathcal{C}_\pm^\otimes) \\ \downarrow \Delta_\tau & \swarrow & & & \downarrow \\ \text{Alg}^\sigma(\mathcal{D}) \times \text{Alg}^\sigma(\mathcal{D}) & \xrightarrow{(A,B) \mapsto A} & \text{Alg}^\sigma(\mathcal{D}) & \longrightarrow & \text{Alg}(\mathcal{C}) \times \text{Alg}(\mathcal{C}_{\text{rev}}) \end{array}$$

where the right vertical arrow is the categorical fibration of [Lur17, Definition 4.3.1.12], the central diagonal arrow is the categorical fibration of Definition 6.34, and the left vertical arrow  $\Delta_\tau$  is the identity on one component and precomposition with the involution of Remark 6.25. The total composite from the upper left to the lower right agrees with the forgetful functor of Remark 6.11. If  $A: \text{Assoc}_\sigma \rightarrow \mathcal{O}^\otimes$  is an involutive algebra object (Definition 6.9), informally the functor sends  $A$  to  $A$ , regarded as an  $A$ - $\sigma_{\mathcal{C}}(A)$ -bimodule in the notation of Remark 6.38 (compare [Lur17, Remark 4.2.1.17]).

L: Lurie gives this a name (Definition 4.2.1.12 *weakly enriched*)—not sure what to call this. something *bi-enriched*?

L: explain the action on  $\mathcal{C}_\pm$  or delete the last half of the sentence.

**Construction 6.40.** Define a functor  $\text{MCut}: \Delta_\sigma^{\text{op}} \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ :

- For each  $([n], s)$ , we have  $\text{MCut}([n], s) = \langle n+1 \rangle \simeq \text{RCut}_0([n])$  where  $\text{RCut}$  is from [Lur17, Construction 4.8.4.4].
- Given a morphism  $\alpha: ([n], s) \rightarrow ([m], t)$ , the associated morphism  $\text{MCut}([m], t) \rightarrow \text{MCut}([n], s)$  consists of
  - On underlying finite pointed sets  $\langle m+1 \rangle \rightarrow \langle n+1 \rangle$ ,  $\text{MCut}$  agrees with (the reverse of) that appearing in [Lur17, Construction 4.2.2.6]
  - Identifying the cut  $\{k \mid k < j\} \sqcup \{k \mid k \geq j\}$  with the morphism  $j-1 < j$ , we may regard  $s: \langle n+1 \rangle^\circ \rightarrow \{\pm 1\}$  and likewise  $t: \langle m+1 \rangle^\circ \rightarrow \{\pm 1\}$ . Define  $u: \text{MCut}(\alpha)^{-1}(\langle n+1 \rangle^\circ) \rightarrow \{\pm 1\}$  to be the unique function so that  $u(j)t(j) = s(\text{MCut}(\alpha)(j))$ .

L: maybe this overloaded notation is not good. I'm running out of ideas.

L: check later

**Remark 6.41.** We can identify  $\text{Assoc}_\sigma^\otimes$  with the full subcategory of  $\mathcal{RM}_{\text{inv}}^\otimes$  spanned by objects of the form  $(\langle n \rangle, \langle n \rangle^\circ)$ . We can regard Construction 6.13 as defining a functor  $\Delta_\sigma^{\text{op}} \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ . For each  $([n], s) \in \Delta_\sigma^{\text{op}}$ , there is a map of sets  $\theta: \text{MCut}([n], s) \rightarrow \text{Cut}([n], s)$  defined as in [Lur17, Remark 4.2.2.8]. Concretely, on underlying pointed sets,  $\theta$  takes the form

$$\theta: \langle n+1 \rangle \rightarrow \langle n \rangle$$

$$k \mapsto \begin{cases} k-1 & \text{if } k > 0 \\ * & \text{if } k = 0, * \end{cases}$$

L: check that the signs  $s$  work out!

This construction determines a morphism  $\gamma$  in the  $\infty$ -category  $\text{Fun}(\Delta_\sigma^{\text{op}}, \mathcal{RM}_{\text{inv}}^\otimes)$ , or equivalently a map  $\gamma: \Delta_\sigma^{\text{op}} \times \Delta^1 \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$ .

**Lemma 6.42.** The morphism  $\gamma: \Delta_\sigma^{\text{op}} \times \Delta^1 \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  defined in Remark 6.41 exhibits  $\Delta_\sigma^{\text{op}} \times \Delta^1$  as an approximation to the  $\infty$ -operad  $\mathcal{RM}_{\text{inv}}^\otimes$ .

**Definition 6.43.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as weakly bi-enriched over  $\mathcal{O}_a^\otimes$ . Let  $\gamma$  be as in Remark 6.41. Let  $R^\sigma \text{Mod}^{\mathbb{A}_\sigma^\otimes}(\mathcal{M})$  denote the full subcategory of  $\text{Fun}_{\mathcal{RM}_{\text{inv}}^\otimes}(\Delta_\sigma^{\text{op}} \times \Delta^1, \mathcal{O}^\otimes)$  spanned by those maps  $f: \Delta_\sigma^{\text{op}} \times \Delta^1 \rightarrow \mathcal{O}^\otimes$  satisfying

1. The restriction of  $f$  to  $\Delta_\sigma^{\text{op}} \times \{1\}$  belongs to  $\text{Alg}_{\mathbb{A}_\sigma^\otimes}(\mathcal{O})$  of Definition 6.18
2. If  $\alpha: ([m], s) \rightarrow ([n], t)$  so that  $\alpha(0) = 0$ , then the induced map  $f([m], s, 0) \rightarrow f([n], t, 0)$  is an inert map in  $\mathcal{O}^\otimes$
3. for each object  $([n], s)$  in  $\Delta_\sigma^{\text{op}}$ , the induced map  $f([n], s, 0) \rightarrow f([n], s, 1)$  is an inert map in  $\mathcal{O}^\otimes$

**Example 6.44.** Let  $\mathcal{C}^\otimes \rightarrow \mathcal{RM}^\otimes$  be a fibration of  $\infty$ -operads. Restriction along the map of  $\infty$ -operads  $\mathcal{RM}_{\text{inv}}^\otimes \rightarrow \text{Assoc}_\sigma^\otimes$  induced by Remark 6.30 induces a map  $\mathbb{E}_\sigma \text{Alg}(\mathcal{C}) \rightarrow R^\sigma \text{Mod}(\mathcal{C})$  which is a section of the projection map  $R^\sigma \text{Mod}(\mathcal{C}) \rightarrow \mathbb{E}_\sigma \text{Alg}(\mathcal{C})$ .

L: see Example 4.2.1.17 of higher algebra

**Notation 6.45.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as weakly bi-enriched over  $\mathcal{O}_a^\otimes$ . Define a new simplicial set  $\overline{\mathcal{M}}^\otimes$  by the following universal property

L: fibration?

$$\text{hom}_{\text{sSet}/\Delta_\sigma^{\text{op}}} (K, \overline{\mathcal{M}}^\otimes) \simeq \text{hom}_{\text{sSet}/\mathcal{RM}_{\text{inv}}^\otimes} (K \times \Delta^1, \mathcal{O}^\otimes).$$

Here we regard  $K \times \Delta^1$  as a simplicial set over  $\mathcal{RM}_{\text{inv}}^\otimes$  via the composite  $K \times \Delta^1 \rightarrow \Delta_\sigma^{\text{op}} \times \Delta^1 \xrightarrow{\gamma} \mathcal{RM}_{\text{inv}}^\otimes$  where  $\gamma$  is from Remark 6.41.

Unwinding definitions, we see that a vertex in  $\overline{\mathcal{M}}^\otimes$  lying over an object  $([n], s: \{1, \dots, n\} \rightarrow \{\pm 1\}) \in \Delta_\sigma^{\text{op}}$  corresponds to a morphism  $\alpha$  in  $\mathcal{O}^\otimes$  whose image in  $\mathcal{RM}_{\text{inv}}^\otimes$  is the map  $(\langle n+1 \rangle, \{0\}) \rightarrow (\langle n \rangle, \emptyset)$ . Now let  $\mathcal{M}^\otimes$  denote the full simplicial subset of  $\overline{\mathcal{M}}^\otimes$  spanned by those vertices for which  $\alpha$  is inert.

L: this might be off-revisit later!

**Remark 6.46.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as weakly enriched over  $\mathcal{O}_a^\otimes$ . By [Lur09, Example 4.3.1.4 & Proposition 4.3.2.15], composition with the inclusion  $\{0\} \rightarrow \Delta^1$  induces a trivial Kan fibration  $\mathcal{M}^\otimes \xrightarrow{\sim} \mathcal{O}^\otimes \times_{\mathcal{RM}_{\text{inv}}^\otimes} \Delta_{\sigma^\text{op}}^\text{op}$ . In particular, the fiber of  $\mathcal{M}^\otimes$  over an object  $([n], s) \in \Delta_{\sigma^\text{op}}^\text{op}$  is canonically equivalent to  $\mathcal{M} \times \mathcal{C}^{\times n}$ .

Finally, since  $q$  is a categorical fibration and categorical fibrations are closed under pullback and composition with trivial fibrations,  $q$  induces categorical fibrations  $\mathcal{M}^\otimes \rightarrow \mathcal{C}^\otimes \rightarrow \Delta_{\sigma^\text{op}}^\text{op}$ .

**Lemma 6.47.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as tensored over  $\mathcal{O}_a^\otimes$ . Then the associated functor  $\mathcal{M}^\otimes \rightarrow \mathcal{C}^\otimes$  (Notation 6.15) is a locally coCartesian fibration.

**Proposition 6.48.** Let  $q: \mathcal{O}^\otimes \rightarrow \mathcal{RM}_{\text{inv}}^\otimes$  be a cocartesian fibration of  $\infty$ -operads, so  $q$  exhibits  $\mathcal{M} := \mathcal{O}_m^\otimes$  as tensored over  $\mathcal{O}_a^\otimes$ . Then precomposition with the functor  $\text{MCut}$  of Construction 6.40 induces an equivalence of  $\infty$ -categories

$$R^\sigma \text{Mod}(\mathcal{M}) \simeq \text{Alg}_{/\mathcal{RM}_{\text{inv}}}(\mathcal{O}) \xrightarrow{\sim} R^\sigma \text{Mod}^{\text{A}_\infty^\sigma}(\mathcal{M}).$$

*Proof.* Combine Lemma 6.42 with [Lur17, Theorem 2.3.3.23].  $\square$

**Definition 6.49.** Introduction to section 3.4 here gives a recipe for defining  $\mathbb{E}_\sigma^\otimes$  as a genuine/parametrized  $\infty$ -operad. Relate the ‘underlying’  $\infty$ -operad with  $C_2$ -action to  $\text{Assoc}_\sigma$ ?

L: Unravel what it means to be  $\mathbb{E}_\sigma$ -monoidal in a genuine way!

**Lemma 6.50.** Let  $R$  be a Poincaré ring. Then  $\text{He}(\text{Mod}_R^\omega, \mathfrak{Y}_R) \rightarrow \text{He}(\text{Hyp}(\text{Mod}_R^\omega))$  is canonically endowed with the structure of an  $\mathbb{E}_\sigma$ -monoidal  $C_2$ - $\infty$ -category. Recall that  $\text{He}(\text{Hyp}(\text{Mod}_R^\omega)) \simeq \text{TwAr}(\text{Mod}_R^\omega)$  by [Cal+20, §2.2].

L: What does it mean for a  $C_2$ - $\infty$ -category to be tensored over a  $\mathbb{E}_\sigma$ -monoidal  $C_2$ - $\infty$ -category? Show that if  $(\mathcal{C}, \mathfrak{Y})$  is an  $R$ -linear Poincaré  $\infty$ -category, then  $\text{He}(\mathcal{C}, \mathfrak{Y}) \rightarrow \text{He}(\mathcal{C}^e)$  is an  $\mathbb{E}_\sigma$ -linear/-tensored over  $\underline{\text{He}}(\text{Mod}_R^\omega)$ ?

## 6.2 Part (b)

**Proposition 6.51.** Let  $\mathcal{C}$  be an involutive monoidal  $\infty$ -category and let  $\mathcal{M}$  be an  $\infty$ -category which is bitensored over  $\mathcal{C}$ . Let  $K$  be a simplicial set so that  $\mathcal{M}$  admits  $K$ -indexed limits, and let  $\theta: R^\sigma \text{Mod}(\mathcal{M}) \rightarrow \text{Alg}^\sigma(\mathcal{C})$  be the forgetful functor. Then

(1) For every commutative square

$$\begin{array}{ccc} K & \longrightarrow & R^\sigma \text{Mod}(\mathcal{M}) \\ \downarrow & \nearrow & \downarrow \theta \\ K^\triangleleft & \longrightarrow & \text{Alg}^\sigma(\mathcal{C}), \end{array}$$

there exists a dashed arrow which is a  $\theta$ -limit diagram.

(2) An arbitrary map  $\bar{g}: K^\triangleleft \rightarrow R^\sigma \text{Mod}(\mathcal{M})$  is a  $\theta$ -limit diagram if and only if the induced map  $K^\triangleleft \rightarrow \mathcal{M}$  is a limit diagram.

*Proof.*  $\square$

**Corollary 6.52.**  $\theta$  is a cartesian fibration, and a morphism  $f: \Delta^1 \rightarrow R^\sigma \text{Mod}(\mathcal{M})$  is  $\theta$ -cartesian if and only if the image of  $f$  in  $\mathcal{M}$  is an equivalence.

**Corollary 6.53.** Let  $\mathcal{C}$  be an involutive monoidal  $\infty$ -category and let  $\mathcal{M}$  be an  $\infty$ -category which is bitensored over  $\mathcal{C}$ . Let  $K$  be a simplicial set so that  $\mathcal{M}$  admits  $K$ -indexed limits, and let  $\theta: R^\sigma \text{Mod}(\mathcal{M}) \rightarrow \text{Alg}^\sigma(\mathcal{C})$  be the forgetful functor. Let  $A$  be an involutive algebra object of  $\mathcal{C}$ . Then

(1)  $R^\sigma \text{Mod}_A(\mathcal{M})$  admits  $K$ -indexed limits.

(2) A diagram  $K^\triangleleft \rightarrow R^\sigma \text{Mod}_A(\mathcal{M})$  is a limit diagram if and only if the induced diagram  $K^\triangleleft \rightarrow \mathcal{M}$  is a limit diagram.

(3) Given a morphism  $A \rightarrow B$  of involutive algebra objects of  $\mathcal{C}$ , the induced functor  $R^\sigma \text{Mod}_B(\mathcal{M}) \rightarrow R^\sigma \text{Mod}_A(\mathcal{M})$  preserves  $K$ -indexed limits.

L: Jacob explains this in a really terse way—just by citing Prop 4.3.2.15 of HTT. It does just follow from definitions/observations but there are many (for instance, definition of inert edge).

L: This statement is [Lur17, Proposition 4.2.3.1] with some words changed; no claim of originality here.

L: todo

### 6.3 Towards (e)

**Construction 6.54.** Define a functor  $\text{Pr}: \mathbf{LM}_{\text{inv}}^{\otimes} \times \mathbf{RM}_{\text{inv}}^{\otimes} \rightarrow \mathbf{BM}_{\text{inv}}^{\otimes}$ .

- (1) Let  $(\langle m \rangle, S)$  be an object of  $\mathbf{LM}_{\text{inv}}^{\otimes}$  and let  $(\langle n \rangle, T)$  be an object of  $\mathbf{RM}_{\text{inv}}^{\otimes}$ . Let  $\text{Pr}((\langle m \rangle, S), (\langle n \rangle, T)) = (X_*, c_-, c_+)$  where  $X_*, c_-, c_+$  are described in [Lur17, Construction 4.3.2.1(1)].
- (2) Let  $(\alpha, \lambda): (\langle m \rangle, S) \rightarrow (\langle m' \rangle, S')$  be a morphism in  $\mathbf{LM}_{\text{inv}}^{\otimes}$  and let  $(\beta, \mu): (\langle n \rangle, T) \rightarrow (\langle n' \rangle, T')$  be a morphism in  $\mathbf{RM}_{\text{inv}}^{\otimes}$ . Write  $\text{Pr}((\langle m' \rangle, S'), (\langle n' \rangle, T')) = (X'_*, c'_-, c'_+)$ . Then  $\text{Pr}((\alpha, \lambda), (\beta, \mu))$  is the unique morphism in  $\mathbf{BM}_{\text{inv}}^{\otimes}$  lying over the map  $\gamma: X_* \rightarrow X'_*$  described by
  - (i)  $\gamma(i, j) = \begin{cases} (\alpha(i), \beta(j)) & \text{if } \alpha(i) \in \langle m' \rangle^{\circ}, \beta(j) \in \langle n' \rangle^{\circ} \\ * & \text{otherwise.} \end{cases}$
  - (ii) Let  $i' \in \langle m' \rangle^{\circ} \setminus S'$  and  $j' \in T'$  so  $j' = \beta(j)$  for a unique  $j \in T$ . Then the linear ordering on  $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\}$  is (a) determined by the map  $\alpha$  if  $\mu(j) = 1$ , and (b) it is the reverse of the linear ordering determined by  $\alpha$  if  $\mu(j) = -1$ . The map  $\gamma^{-1}(i', j') = \alpha^{-1}(i') \times \{j\} \rightarrow \{\pm 1\}$  is determined by  $\lambda$  if  $\mu(j) = 1$  and it is  $-\lambda$  if  $\mu(j) = -1$ .
  - (iii) Likewise if  $i' \in S'$  and  $j' \in \langle n' \rangle^{\circ} \setminus T'$
  - (iv) Let  $i' \in S'$  and  $j' \in T'$  so  $i' = \alpha(i)$  for a unique  $i \in S$  and  $j' = \beta(j)$  for a unique  $j \in T$ . Then  $\gamma^{-1}\{(i', j')\} = \{i\} \times \beta^{-1}\{(j')\} \sqcup_{\{(i, j)\}} \alpha^{-1}\{(i')\} \times \{j\}$ . Define  $\gamma^{-1}\{(i', j')\} \rightarrow \{\pm 1\}$  by  $\lambda \times \mu$ . Endow  $\gamma^{-1}\{(i', j')\}$  with the linear ordering from [Lur17, Construction 4.3.2.1(2)(iv)] if  $\lambda(i) = \mu(j)$  and endow  $\gamma^{-1}\{(i', j')\}$  with the opposite ordering if  $\lambda(i) \neq \mu(j)$  (or equivalently, if  $\lambda(i) = -\mu(j)$ ).

Write  $\text{Pr}$  for the induced map  $\mathcal{LM}_{\sigma}^{\otimes} \times \mathcal{RM}_{\sigma}^{\otimes} \rightarrow \mathcal{BM}_{\sigma}^{\otimes}$  of  $\infty$ -categories.

**Construction 6.55.** Let  $q: \mathcal{C}^{\otimes} \rightarrow \mathcal{BM}_{\sigma}^{\otimes}$  be a fibration of  $\infty$ -operads. We define a map of simplicial sets  $\overline{L}^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes} \rightarrow \mathcal{RM}_{\sigma}^{\otimes}$  by the universal property: For any simplicial set  $K \rightarrow \mathcal{RM}_{\sigma}^{\otimes}$ , there is a bijection

$$\text{Hom}_{\text{Set}}_{/\mathcal{RM}_{\sigma}^{\otimes}}(K, \overline{L}^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes}) \simeq \text{Hom}_{\text{Set}}_{/\mathcal{BM}_{\sigma}^{\otimes}}(\mathcal{LM}_{\sigma}^{\otimes} \times K, \mathcal{C}^{\otimes}).$$

Let  $L^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes}$  denote the full simplicial subset of  $\overline{L}^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes}$  spanned by those vertices which correspond to a vertex  $X \in \mathcal{RM}_{\sigma}^{\otimes}$  and a functor  $F: \mathcal{LM}_{\sigma}^{\otimes}\{X\} \rightarrow \mathcal{BM}_{\sigma}^{\otimes}$  which takes inert morphisms in  $\mathcal{LM}_{\sigma}^{\otimes}$  to inert morphisms in  $\mathcal{BM}_{\sigma}^{\otimes}$ .

**Remark 6.56.** The composite  $\mathcal{LM}_{\sigma}^{\otimes} \times \{\mathbf{m}\} \hookrightarrow \mathcal{LM}_{\sigma}^{\otimes} \times \mathcal{RM}_{\sigma}^{\otimes} \xrightarrow{\text{Pr}} \mathcal{BM}_{\sigma}^{\otimes}$  agrees with the inclusion of Remark 6.33. Taking  $K \rightarrow \mathcal{RM}_{\sigma}^{\otimes}$  to be the inclusion  $\{\mathbf{m}\} \hookrightarrow \mathcal{RM}_{\sigma}^{\otimes}$ , we have an isomorphism of simplicial sets  $L^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes} \times_{\mathcal{RM}_{\sigma}^{\otimes}} \{\mathbf{m}\} \simeq L^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes}$  where  $L^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes}$  is the  $\infty$ -category of left modules associated to the fibration of  $\infty$ -operads  $\mathcal{C}^{\otimes} \times_{\mathcal{BM}_{\sigma}^{\otimes}} \mathcal{LM}_{\sigma}^{\otimes} \rightarrow \mathcal{LM}_{\sigma}^{\otimes}$ .

**Proposition 6.57.** Let  $q: \mathcal{C}^{\otimes} \rightarrow \mathcal{BM}_{\sigma}^{\otimes}$  be a fibration of  $\infty$ -operads. Then

- (1) the induced map  $p: L^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes} \rightarrow \mathcal{RM}_{\sigma}^{\otimes}$  is a fibration of  $\infty$ -operads
- (2) a morphism  $\alpha$  in  $L^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes}$  is inert if and only if  $p(\alpha)$  is inert in  $\mathcal{RM}_{\sigma}^{\otimes}$  and for all  $X \in \mathcal{LM}_{\sigma}$ ,  $\alpha(X)$  is an inert morphism in  $\mathcal{C}^{\otimes}$ .
- (3) if  $q$  is a cocartesian fibration of  $\infty$ -operads, then so is  $p$
- (4) if  $q$  is a cocartesian fibration of  $\infty$ -operads, a morphism  $\alpha$  in  $L^{\sigma}\text{Mod}(\mathcal{C}_{\mathbf{m}})^{\otimes}$  is  $p$ -cocartesian if and only if, for all  $X \in \mathcal{LM}_{\sigma}^{\otimes}$ ,  $\alpha(X)$  is  $q$ -cocartesian in  $\mathcal{C}^{\otimes}$ .

*Proof.* Similar to [Lur17, Proposition 4.3.2.5]. □

**Theorem 6.58.** Let  $\mathcal{C}$  be an  $\mathbb{E}_{\sigma}$ -monoidal  $\infty$ -category, and let  $A$  be an  $\mathbb{E}_{\sigma}$ -algebra in  $\mathcal{C}$ . Then  $L^{\sigma}\text{Mod}_A(\mathcal{C})$  is right  $\mathbb{E}_{\sigma}$ -tensored over  $\mathcal{C}$ .

## 6.4 Endomorphisms

Let  $\mathcal{C}$  be an  $\mathbb{E}_\sigma$ -monoidal  $\infty$ -category, and write  $\sigma_{\mathcal{C}}: \mathcal{C} \xrightarrow{\sim} \mathcal{C}$  for its involution. Suppose  $M \in \mathcal{C}$  is an object equipped with an equivalence  $\sigma_M: M \simeq \sigma_{\mathcal{C}}(M)$ . By [Lur17, §4.7.1], endomorphisms of  $M$  can be regarded as an  $\mathbb{E}_1$ -algebra in  $u(\mathcal{C})^\otimes$ , where  $u$  is from Remark 6.10. Now  $\sigma_M$  induces an equivalence  $\text{End}_{\mathcal{C}}(M) \simeq \text{End}_{\mathcal{C}}(\sigma_{\mathcal{C}}(M))$ . On the other hand,  $\sigma_{\mathcal{C}}$  induces an equivalence  $\text{End}_{\mathcal{C}}(\sigma_{\mathcal{C}}(M)) \simeq \text{End}_{\mathcal{C}}(M)^{\text{rev}}$ . In particular, for any  $\infty$ -category  $\mathcal{M}$  left  $\mathbb{E}_\sigma$ -tensored over  $\mathcal{C}$  and any object  $M \in \mathcal{M}$  which is fixed by the involution on  $\mathcal{M}$ , we expect the endomorphisms of  $M$  to admit the structure of an  $\mathbb{E}_\sigma$ -algebra in  $\mathcal{C}$ .

To this end, we will define an  $\infty$ -category of objects acting on  $M$ , show that it has an  $\mathbb{E}_\sigma$ -monoidal structure, and locate endomorphisms of  $M$  as the final object in this  $\infty$ -category. Informally, we may define a category  $\mathcal{C}[M]$  whose objects consist of either

- pairs  $(C, \eta)$  where  $C \in \mathcal{C}$  and  $\eta: C \otimes M \rightarrow M$  is a morphism in  $\mathcal{M}$ ; or
- pairs  $(C', \eta')$  where  $C' \in \mathcal{C}$  and  $\eta': \sigma_{\mathcal{M}}(M) \otimes C' \rightarrow \sigma_{\mathcal{M}}(M)$ .
- pairs  $(D, \xi)$  where  $D \in \mathcal{C}$  and  $\xi: M \otimes D \rightarrow \sigma_{\mathcal{M}}(M)$
- pairs  $(D, \xi')$  where  $D \in \mathcal{C}$  and  $\xi': D \otimes \sigma_{\mathcal{M}}(M) \rightarrow M$

L: added July 26th

The monoidal structure is as described in [Lur17, §4.7.1]. Note that given an object  $(C, \eta)$ , the involution  $\sigma_{\mathcal{M}}$  on  $\mathcal{M}$  sends  $\eta$  to the map  $\sigma_{\mathcal{M}}(C \otimes M) \simeq \sigma_{\mathcal{M}}(M) \otimes \sigma_{\mathcal{C}}(C) \rightarrow \sigma_{\mathcal{M}}(M)$ . Given an object  $(D, \xi)$ , the involution  $\sigma_{\mathcal{M}}$  on  $\mathcal{M}$  we may consider the map  $\xi': \sigma_{\mathcal{C}}(D) \otimes M \xrightarrow{\text{id} \otimes \sigma_M} \sigma_{\mathcal{C}}(D) \otimes \sigma_{\mathcal{M}}^2(M) \simeq \sigma_{\mathcal{M}}(M \otimes D) \xrightarrow{\sigma_{\mathcal{M}}(\xi)} \sigma_{\mathcal{M}}(M)$ . The assignment  $(D, \xi) \mapsto (\sigma_{\mathcal{C}}(D), \xi') \dots$  This is the involution on  $\mathcal{C}[M]$ .

L: is an involution?

**Definition 6.59.** Let  $p: \mathcal{M}^\otimes \rightarrow \Delta^1 \times \Delta_\sigma^{\text{op}}$  exhibit  $\mathcal{M}^\otimes$  as weakly enriched over  $\mathcal{C}^\otimes$ . An *enriched morphism* of  $\mathcal{M}$  is a diagram

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

satisfying either

- $p(\alpha)$  is the morphism  $(0, [1], c_1) \rightarrow (0, [0])$  in  $\Delta_\sigma^{\text{op}}$  determined by the embedding  $[0] \simeq \{0\} \hookrightarrow [1]$  and  $c_1: \{1\} \rightarrow \{\pm 1\}$  is the constant function at  $+1$ , and
- the map  $\beta$  is inert, and  $p(\beta)$  is the morphism  $(0, [1], c_1) \rightarrow (0, [0])$  in  $\Delta^1 \times \Delta_\sigma^{\text{op}}$  determined by the embedding  $[0] \simeq \{1\} \hookrightarrow [1]$

or

- $p(\alpha)$  is the morphism  $(0, [1], c_{-1}) \rightarrow (0, [0])$  in  $\Delta_\sigma^{\text{op}}$  determined by the embedding  $[0] \simeq \{0\} \hookrightarrow [1]$  and  $c_{-1}: \{1\} \rightarrow \{\pm 1\}$  is the constant function at  $-1$ .
- the map  $\beta$  is inert, and  $p(\beta)$  is the morphism  $(0, [1], c_{-1}) \rightarrow (0, [0])$  in  $\Delta^1 \times \Delta_\sigma^{\text{op}}$  determined by the embedding  $[0] \simeq \{1\} \hookrightarrow [1]$

Let  $\text{Str } \mathcal{M}_{[1]}^{\text{en}}$  denote the full subcategory of  $\text{Fun}_{\Delta^1 \times \Delta_\sigma^{\text{op}}}(\Lambda_0^2, \mathcal{M}^\otimes)$  spanned by the enriched morphisms of  $\mathcal{M}$ .

Note that there are two evaluation functors  $\text{Str } \mathcal{M}_{[1]}^{\text{en}} \rightarrow \mathcal{M}$ . Given  $M \in \mathcal{M}$ , write  $\mathcal{C}[M] := \{M\} \times_{\mathcal{M}} \text{Str } \mathcal{M}_{[1]}^{\text{en}} \times_{\mathcal{M}} \{M\}$  and refer to it as the endomorphism  $\infty$ -category of  $M$ .

**Definition 6.60.** *enriched  $n$ -string*

**Proposition 6.61** (Segal condition).

In the course of thinking about the ‘involutive’ generalization of the statement that given an  $\mathbb{E}_1$ -algebra, its category of modules is  $\mathbb{E}_0$  (and conversely, that given an object in a stable  $\infty$ -category, that its endomorphism spectrum is an  $\mathbb{E}_1$ -algebra), I have run up against some questions.

**Question 6.62.** • Can we sidestep an involutive version of the construction of endomorphism categories of [Lur17, §4.7.1]?

- Suppose  $\mathcal{C}$  is a monoidal  $\infty$ -category and  $\mathcal{M}$  is an  $\infty$ -category which is enriched over  $\mathcal{C}$  in the sense of [Lur17, §4.2.1]. The opposite category  $\mathcal{M}^{\text{op}}$  is enriched over  $\mathcal{C}$  by [Hei23, §10].

## 7 Categorification and structure—new leaf

Use [genuine operadic nerve here](#) to go from simplicial genuine operads to parametrized operads.

**Notation 7.1.** Let  $S, T$  be finite  $C_2$ -sets, and let  $f: S \rightarrow T$  be a  $C_2$ -equivariant map. A *t-ordering* on  $f$  is the data of, for all free orbits  $V \subset S$ ,  $V \simeq C_2$  so that  $f(V) = \{*\}$ , a choice of an ordering  $\leq$  on  $V$ . A *t-ordered map*  $S \rightarrow T$  is the data of  $f$  and a t-ordering on  $f$ .

Note that if  $f_i: S_i \rightarrow T_i$  are all t-ordered, there is a canonical t-ordering on  $\bigsqcup_i f_i$ . If  $f: S \rightarrow T$  is t-ordered and  $g: T \rightarrow U$  is t-ordered, there is a canonical t-ordering on  $g \circ f$ . Let us spell this out: Suppose given an orbit  $V \subset S$  on which  $C_2$  acts freely so that  $g \circ f(V)$  is trivial. There are two cases

- Suppose  $f$  restricts to an isomorphism  $f|_V: V \xrightarrow{\sim} f(V)$  and  $g$  sends  $f(V)$  to  $*$ . Then the t-ordering on  $g$  means we have an ordering on  $f(V)$ , which we transport to an ordering on  $V$  via  $f|_V$ .
- Suppose  $f(V) = *$ . Then the t-ordering on  $f$  endows  $V$  with a canonical ordering.

Finally, we define a t-ordering on pullbacks as follows: Suppose given a pullback square of  $C_2$ -sets.

$$\begin{array}{ccc} Z \times_X W & \xrightarrow{g^*f} & W \\ \downarrow \pi_1 & & \downarrow g \\ Z & \xrightarrow{f} & X \end{array}$$

and a t-ordering on  $f$ . Suppose given a free orbit  $C_2 \simeq U \subseteq Z \times_X W$  so that  $g^*(f)(U)$  is a singleton (with trivial  $C_2$ -action). Because the square above is a pullback,  $\pi_1(U) \subset Z$  is acted on by  $C_2$  freely, and  $\pi_1|_U$  defines an isomorphism  $U \simeq \pi_1(U)$ . By  $C_2$ -equivariance of  $g$ ,  $g(g^*(f)(U)) = f(\pi_1(U))$  is a singleton. The given t-ordering on  $f$  means  $\pi_1(U)$  has a given ordering, which we use to endow  $U$  with an ordering using the isomorphism  $\pi_1|_U$ .

**Remark 7.2.** In the following diagram, both t-orderings on  $f$  induce the same t-ordering on  $g^*(f)$

$$\begin{array}{ccc} C_2 \times C_2 & \longrightarrow & C_2 \\ \downarrow & & \downarrow g \\ C_2 & \xrightarrow{f} & C_2/C_2 \end{array}$$

(in fact the set of t-orderings on  $g^*(f)$  is a singleton.)

In what follows, we will refer to the following diagram

$$\begin{array}{ccccc} U & \xleftarrow{g} & Z & \xrightarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & Y & \xlongequal{\quad} & Y \end{array} \tag{7.3}$$

repeatedly; in particular, the induced map  $Z \rightarrow U \times_V Y$  is always assumed to be a summand inclusion.

**Definition 7.4.** Define a  $C_2$ - $\infty$ -operad  $\mathbb{E}_p^\otimes \rightarrow \underline{\mathbf{Fin}}_{C_2,*}$  as having

- objects consist of those arrows  $U \rightarrow V$  so that  $C_2$  acts freely on  $U$  and transitively on  $V$
- morphisms consist of spans (7.3) in  $\underline{\mathbf{Fin}}_{C_2,*}$ , plus the data of a t-ordering on  $f$  in the sense of Notation 7.1.
- composition is defined to agree with that in  $\underline{\mathbf{Fin}}_{C_2,*}$ .

there is a canonical map to  $\underline{\mathbf{Fin}}_{C_2,*}$  which is an inclusion on objects and on morphisms forgets the t-ordering.

**Definition 7.5.** Define a  $C_2$ - $\infty$ -operad  $\underline{\mathbf{Assoc}}_{\text{naive}}^\otimes \rightarrow \underline{\mathbf{Fin}}_{C_2,*}$  as having

L: I think this is the  $C_2$ - $\infty$ -operad whose algebras give Poincaré rings.



- objects consist of those arrows  $U \rightarrow V$  so that  $C_2$  acts freely on  $U$  and transitively on  $V$
- morphisms consist of spans (7.3) in  $\underline{\text{Fin}}_{C_2,*}$  plus the data of, for all orbits  $T \subset X$  an ordering on the orbits in the preimage  $f^{-1}(T) \subseteq Z$ .
- composition is defined to agree with that in  $\underline{\text{Assoc}}^\otimes$ .

there is a canonical map to  $\underline{\text{Fin}}_{C_2,*}$  which is an inclusion on objects and on morphisms forgets the ordering.

**Remark 7.6.** Let  $\underline{\text{Free}}_{C_2,*}$  be the full subcategory of  $\underline{\text{Fin}}_{C_2,*}$  on those arrows  $U \rightarrow V$  where  $U$  is a free  $C_2$ -set. It is evident that  $\underline{\text{Free}}_{C_2,*}$  is a sub  $C_2$ - $\infty$ -operad of  $\underline{\text{Fin}}_{C_2,*}$  in the sense of [NS22, Definition 2.2.7]. By [BHS22, Lemma 4.1.13], there is an equivalence of  $C_2$ - $\infty$ -operads  $\underline{\text{Free}}_{C_2,*} \times_{\underline{\text{Fin}}_{C_2,*}} \underline{\text{Assoc}}^\otimes \simeq \underline{\text{Assoc}}_{\text{naive}}^\otimes$ . Now for any  $C_2$ - $\infty$ -operad  $\mathcal{C}^\otimes$ , there is an equivalence

$$\text{hom}_{\underline{\text{Fin}}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}^\otimes, \mathcal{C}^\otimes) \simeq \text{hom}_{\underline{\text{Free}}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}^\otimes, \mathcal{C}^\otimes \times_{\underline{\text{Fin}}_{C_2,*}} \underline{\text{Free}}_{C_2,*}).$$

The functor  $\underline{\text{Free}}_{C_2,*} \rightarrow \underline{\text{Fin}}_*$  which sends a (pointed, finite) free  $C_2$ -set to its set of orbits exhibits  $\underline{\text{Free}}_{C_2,*}$  as a generalized  $\infty$ -operad in the sense of [Lur17, Definition 2.3.2.1]. Now choose a free  $C_2$ -set with transitive action; this determines a functor  $BC_2 \rightarrow \underline{\text{Free}}_{C_2,*}$ . This induces a functor  $p: BC_2 \times \underline{\text{Fin}}_* \rightarrow \underline{\text{Free}}_{C_2,*}$ . By [Lur17, Corollary 2.3.2.13], there is an equivalence

$$\text{hom}_{\underline{\text{Free}}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}^\otimes, \mathcal{C}_{\text{naive}}^\otimes) \simeq \text{hom}_{BC_2 \times \underline{\text{Fin}}_*}(p^*(\underline{\text{Assoc}}_{\text{naive}}^\otimes), p^*(\mathcal{C}_{\text{naive}}^\otimes))$$

where the left hand side denotes morphisms in generalized  $\infty$ -operads over  $\underline{\text{Free}}_{C_2,*}$  and the right hand side denotes morphisms of  $BC_2$ -families of  $\infty$ -operads in the sense of [Lur17, Definition 2.3.2.10]. Now the latter is equivalent to  $\mathbb{E}_1$ -algebra objects in  $(\mathcal{C}^e)^{hC_2}$ .

**Remark 7.7.** Note that in the definition of  $\underline{\text{Assoc}}_{\text{naive}}^\otimes$  (7.3),  $f$  is equivalent to a fold map because  $X$  is acted upon freely by  $C_2$ .

**Definition 7.8.** Define a  $C_2$ - $\infty$ -operad  $\mathcal{HM}^\otimes \rightarrow \underline{\text{Fin}}_{C_2,*}$  as having

- objects consist of pairs  $(h: U \rightarrow V, \ell: U \rightarrow \{a, m\})$  where  $h: U \rightarrow V$  is an object of  $\underline{\text{Fin}}_{C_2,*}$  and  $\ell$  is a function which is constant on orbits (think ‘label’) satisfying the condition that  $C_2$  acts freely on  $\ell^{-1}(\{a\})$ .
- morphisms from  $(h: U \rightarrow V, \ell_U: U \rightarrow \{a, m\})$  to  $(h: X \rightarrow Y, \ell_X: X \rightarrow \{a, m\})$  consists of
  - a span (7.3) in  $\underline{\text{Fin}}_{C_2,*}$
  - $Z$  is equipped with a labeling  $\ell_Z$  satisfying  $\ell_U = g \circ \ell_Z$
  - a t-ordering on  $f$  in the sense of Notation 7.1
  - for all orbits  $T \subset X$ , an ordering  $\leq$  on the orbits in the preimage  $f^{-1}(T) \subseteq Z$ .

These are required to satisfy the conditions

- if  $x \in X$  is so that  $\ell_X(x) = m$ , there exists at exactly one orbit, call it  $Z_x$  in  $f^{-1}(x)$  which  $\ell_Z$  sends to  $m$ , which is maximal with respect to the ordering.
- let  $x \in X$  be so that  $\ell_X(x) = m$ , and write  $Z_x$  for the orbit in  $f^{-1}(x)$  which  $\ell_Z$  sends to  $m$  of the previous bullet point. Then  $f$  restricts to an isomorphism  $f|_{Z_x}$ .

there is a canonical map to  $\underline{\text{Fin}}_{C_2,*}$  which is an inclusion on objects and on morphisms forgets the ordering.

**Remark 7.9.** There is a map of  $C_2$ - $\infty$ -operads  $s: \mathcal{HM}^\otimes \rightarrow \mathbb{E}_p^\otimes$  which forgets the data of the labelings and the orderings on the preimages of the orbits (but remembers the t-orderings!). Furthermore, there is a map of  $C_2$ - $\infty$ -operads  $\iota_a: \underline{\text{Assoc}}_{\text{naive}}^\otimes \rightarrow \mathcal{HM}^\otimes$  which sends  $U \rightarrow V$  to the pair  $(U \rightarrow V, U \rightarrow \{a\} \hookrightarrow \{a, m\})$ .

**Definition 7.10.** Let  $\mathcal{C}^\otimes$  be a  $C_2$ -symmetric monoidal  $C_2$ - $\infty$ -category. Write  $\text{Mod}^h(\mathcal{C})$  denote the  $C_2$ - $\infty$ -category  $\underline{\text{Alg}}_{\mathcal{HM}}(\mathcal{C})$ . Note that precomposition with the map  $\iota_a$  of Remark 7.9 defines a map of  $C_2$ - $\infty$ -categories  $\theta: \text{Mod}^h(\mathcal{C}) \rightarrow \underline{\text{Assoc}}_{\text{naive}}(\mathcal{C})$ .

L: Now pass to spans of finite free  $C_2$ -sets (not spans of arrows!), and write  $\mathcal{C}_{\text{naive}}^\otimes \rightarrow \underline{\text{Free}}_{C_2,*}$  and  $\underline{\text{Assoc}}_{\text{naive}}^\otimes$ . Use Higher algebra Remark 2.3.3.4 and (Thm 2.3.3.23 or Cor 2.3.2.13).

L: (1) and (2) are clear, think about (3).



**Remark 7.11.** Let  $\mathcal{C}^\otimes$  be the  $C_2$ -symmetric monoidal  $C_2$ - $\infty$ -category of  $C_2$ -spectra equipped with the Hill–Hopkins–Ravenel norm. In view of Remark 7.6, we may regard an  $\mathcal{O}_{C_2}^{\text{op}}$ -object of  $\text{Mod}^h(\underline{\text{Sp}}^{C_2})$  as consisting of a pair  $(A, M)$  where  $A$  is an  $\mathbb{E}_1$ -algebra in  $\text{Sp}^{BC_2}$  and  $M$  is an  $N^{C_2}A$ -module in  $\text{Sp}^{BC_2}$ . Precomposition with the maps of Remark 7.9 induces a commutative diagram

$$\begin{array}{ccc} & & \text{Mod}^h \\ & \nearrow & \downarrow \\ \underline{\text{Alg}}_{\mathbb{E}_p}(\underline{\text{Sp}}^{C_2}) & \longrightarrow & \underline{\text{Alg}}_{\text{Assoc}_{\text{naive}}}(\mathcal{C}) \end{array}$$

**Construction 7.12.** Recall that by [NS22, Proposition 2.8.7(1)], there is an equivalence  $\underline{\text{Alg}}_{\text{Assoc}_{\text{naive}}}(\underline{\text{Sp}}^{C_2}) \simeq \text{Fun}_{C_2}^\otimes(\text{Env}_{\text{Fin}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}), (\underline{\text{Sp}}^{C_2})^\otimes)$ . Now recall that  $\text{Env}_{\text{Fin}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}) \simeq \underline{\text{Assoc}}_{\text{naive}} \times_{\text{Fin}_{C_2,*}} \text{Ar}^{\text{act}}(\underline{\text{Fin}}_{C_2,*})$  [NS22, Definition 2.8.4]. In the following, we will be interested in objects of  $\text{Ar}^{\text{act}}(\underline{\text{Fin}}_{C_2,*})$  in the fiber over  $C_2/C_2$ , so we omit it from notation. Define objects

$$\begin{aligned} X &= (C_2 \rightarrow C_2/C_2, C_2 \rightarrow C_2/C_2, C_2 = C_2) \\ Y &= (C_2 \rightarrow C_2/C_2, C_2 \rightarrow C_2/C_2, C_2 \rightarrow C_2/C_2) \end{aligned}$$

in  $\text{Env}_{\text{Fin}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}})$ . The collapse map gives a canonical morphism  $c: X \rightarrow Y$ ; observe that  $c$  is cocartesian over  $\underline{\text{Fin}}_{C_2,*}$ . Using semiadditivity/universal properties, define a functor which is uniquely characterized by

$$\begin{aligned} \Delta^1 \times \text{Fin}_* &\rightarrow \text{Env}_{\text{Fin}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}) \\ (0, \langle 1 \rangle) &\mapsto X \\ (0, \langle 1 \rangle) &\mapsto Y \\ (0 < 1, \text{id}_{\langle 1 \rangle}) &\mapsto (c: X \rightarrow Y). \end{aligned}$$

Precomposition with  $\text{Assoc} \rightarrow \text{Fin}_*$  defines a functor<sup>2</sup>

$$q: \Delta^1 \times \text{Assoc} \rightarrow \text{Env}_{\text{Fin}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}). \quad (7.13)$$

Write  $p$  for the composite  $\Delta^1 \times \text{Assoc} \rightarrow \text{Env}_{\text{Fin}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}) \rightarrow \underline{\text{Fin}}_{C_2,*}$ . Restriction along the functor of 7.13 defines

$$\text{Fun}_{C_2}^\otimes(\text{Env}_{\text{Fin}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}), (\underline{\text{Sp}}^{C_2})^\otimes) \rightarrow \text{Fun}_{\Delta^1 \times \text{Assoc}}(\Delta^1 \times \text{Assoc}, p^*(\underline{\text{Sp}}^{C_2})). \quad (7.14)$$

Observe that  $p^*(\underline{\text{Sp}}^{C_2}) \rightarrow \Delta^1 \times \text{Assoc}$  is a  $\Delta^1$ -cocartesian family of monoidal  $\infty$ -categories in the sense of [Lur17, Definition 4.8.3.1], and the image of the aforementioned restriction functor consists of  $\Delta^1$ -cocartesian families of associative algebra objects in  $p^*(\underline{\text{Sp}}^{C_2})$ . In particular, by [Lur17, Remark 4.8.3.7], the aforementioned restriction functor refines to

$$\text{Fun}_{C_2}^\otimes(\text{Env}_{\text{Fin}_{C_2,*}}(\underline{\text{Assoc}}_{\text{naive}}), (\underline{\text{Sp}}^{C_2})^\otimes) \rightarrow \text{Fun}(\Delta^1, \text{Cat}_\infty^{\text{Alg}}).$$

Its image in  $\text{Fun}(\Delta^1, \text{Alg}(\text{Cat}_\infty))$  consists of the single arrow  $N^{C_2}: \text{Sp} \rightarrow \text{Sp}^{C_2}$ .

**Definition 7.15.** Let  $\text{Cat}_\infty^{\text{Alg Mod}}$  denote the full subcategory of the fiber product

$$\text{Mon}_{\text{Assoc}}(\text{Cat}_\infty) \times_{\text{Fun}_{\text{Assoc}}(\text{Assoc}, \text{Assoc} \times \text{Mon}_{\text{Assoc}}(\text{Cat}_\infty))} \text{Fun}_{\text{Assoc}}(\mathcal{LM}^\otimes, \widetilde{\text{Mon}}_{\text{Assoc}}(\text{Cat}_\infty))$$

on those triples  $(\mathcal{C}^\otimes, A, M)$  where  $\mathcal{C}^\otimes$  is a monoidal  $\infty$ -category,  $A$  is an algebra object of  $\{\mathcal{C}^\otimes\} \times_{\text{Mon}_{\text{Assoc}}(\text{Cat}_\infty)} \widetilde{\text{Mon}}_{\text{Assoc}}(\text{Cat}_\infty) \simeq \mathcal{C}^\otimes$ , and  $M$  is a left  $A$ -module in  $\mathcal{C}$ .

<sup>2</sup>with values in the fiber over  $C_2/C_2$  of the target

L: todo; see e.g. [Yan23, Notation 4.1]

L: compare Higher Algebra Definition 4.8.3.7

**Remark 7.16.**

L: The **KEY** is getting an analogue of Remark 4.8.3.8 of Higher Algebra for  $\text{Cat}_\infty^{\text{Alg Mod}}$  instead of  $\text{Cat}^{\text{Alg}}$ . Here is an attempt.

For any simplicial set  $S$ , there is a bijection between equivalence classes of maps  $S \rightarrow \text{Cat}_\infty^{\text{Alg Mod}}$  and equivalence classes of diagrams

$$\begin{array}{ccc} \mathcal{LM}^\otimes \times S & \xrightarrow{M_A} & \mathcal{C}^\otimes \\ \uparrow & & \downarrow q \\ \text{Assoc}^\otimes \times S & \xrightarrow{\text{id}} & \text{Assoc}^\otimes \times S \end{array}$$

where  $q$  exhibits  $\mathcal{C}^\otimes$  as an  $S$ -cocartesian family of monoidal  $\infty$ -categories and  $M_A$  is an  $S$ -family of  $\mathcal{LM}$ -algebra objects (i.e. associative algebra objects and modules over them).

**Next:** Define a forgetful functor  $\text{Cat}_\infty^{\text{Alg Mod}} \rightarrow \text{Cat}_\infty^{\text{Mod}}$  which sends  $(\mathcal{C}^\otimes, A, M)$  to  $(\mathcal{C}^\otimes, \text{Mod}_A)$ .

Define a functor  $\sigma\text{Mod} \rightarrow \text{Fun}(\Delta^1, \text{Cat}_\infty^{\text{Mod}}) \times_{\text{Cat}_\infty^{\text{Mod}}} \text{Cat}_\infty^{\text{Alg Mod}}$  living in a commutative square

$$\begin{array}{ccc} \sigma\text{Mod} & \xrightarrow{\quad \exists? \quad} & \text{Fun}(\Delta^1, \text{Cat}_\infty^{\text{Mod}}) \times_{\text{Cat}_\infty^{\text{Mod}}} \text{Cat}_\infty^{\text{Alg Mod}} \\ \downarrow & & \downarrow \\ \text{Alg}_{\text{Assoc}_{\text{naive}}}(\underline{\text{Sp}}^{C_2}) & \longrightarrow & \text{Fun}(\Delta^1, \text{Cat}_\infty^{\text{Alg}}) \xrightarrow{\quad \Theta \quad} \text{Fun}(\Delta^1, \text{Cat}_\infty^{\text{Mod}}) \end{array}$$

where the lower arrow is Construction 7.12 and  $\Theta$  is from [Lur17, Construction 4.8.3.24].

**Construction 7.17.** Recall that by [NS22, Proposition 2.8.7(1)], there is an equivalence  $\sigma\text{Mod}(\underline{\text{Sp}}^{C_2}) \simeq \text{Fun}_{C_2}^\otimes(\text{Env}_{\text{Fin}_{C_2,*}}(\mathcal{HM}^\otimes), (\underline{\text{Sp}}^{C_2})^\otimes)$ .

Choose a  $C_2$ -set  $U$  on which  $C_2$  acts freely and transitively, and fix an ordering  $\leq$  on  $U$ . Define a functor from  $\mathcal{LM}^\otimes$  into the fiber of  $\mathcal{HM}^\otimes$  over  $C_2/C_2$

$$(\langle n \rangle, S) \mapsto \left( (\langle n \rangle^\circ \setminus S) \times U \sqcup S_+ \rightarrow C_2/C_2, \ell(i) = \begin{cases} a & i \in (\langle n \rangle^\circ \setminus S) \times U \\ m & i \in S \end{cases} \right)$$

Now recall that  $\text{Env}_{\text{Fin}_{C_2,*}}(\mathcal{HM}^\otimes) \simeq \mathcal{HM}^\otimes \times_{\text{Fin}_{C_2,*}} \text{Ar}^{\text{act}}(\text{Fin}_{C_2,*})$  [NS22, Definition 2.8.4]. Notice that the fiber of  $\text{Fin}_{C_2,*}$  over  $C_2/C_2$  has a final object given by the identity arrow at  $[C_2/C_2 = C_2/C_2]$  and for every object  $[U \rightarrow C_2/C_2]$  in  $\text{Fin}_{C_2,*}$ , the unique arrow from  $[U \rightarrow C_2/C_2]$  to  $[C_2/C_2 \rightarrow C_2/C_2]$ , hence the composite  $\mathcal{LM}^\otimes \rightarrow \mathcal{HM}_{C_2/C_2}^\otimes \rightarrow (\text{Fin}_{C_2,*})_{C_2/C_2}$  lifts to  $\mathcal{LM}^\otimes \rightarrow \mathcal{HM}_{C_2/C_2}^\otimes \rightarrow (\text{Fin}_{C_2,*})_{C_2/C_2} \rightarrow [\text{id}_{C_2/C_2}] \rightarrow \text{Ar}^{\text{act}}(\text{Fin}_{C_2,*})$ . This defines a functor

$$r: \mathcal{LM}^\otimes \rightarrow \text{Env}_{\text{Fin}_{C_2,*}}(\mathcal{HM}^\otimes) \simeq \mathcal{HM}^\otimes \times_{\text{Fin}_{C_2,*}} \text{Ar}^{\text{act}}(\text{Fin}_{C_2,*}). \quad (7.18)$$

Writing  $\iota_a$  for the canonical inclusion  $\text{Assoc}^\otimes \rightarrow \mathcal{LM}^\otimes$ , there is an equivalence<sup>3</sup>

$$r \circ \iota_a = p \circ (\iota_1 \times \text{id}_{\text{Assoc}^\otimes}) =: p_1. \quad (7.19)$$

Restriction along the functor (7.18) sends maps of  $C_2$ - $\infty$ -operads to maps of ordinary  $\infty$ -operads, hence it defines

$$\text{Fun}_{C_2}^\otimes(\text{Env}_{\text{Fin}_{C_2,*}}(\mathcal{HM}^\otimes), (\underline{\text{Sp}}^{C_2})^\otimes) \rightarrow \text{Alg}_{\mathcal{LM}/\text{Assoc}}(p_1^*(\underline{\text{Sp}}^{C_2, \otimes})) \rightarrow \text{Alg}_{/\text{Assoc}}(p_1^*(\underline{\text{Sp}}^{C_2, \otimes})) \quad (7.20)$$

<sup>3</sup>Actually, need to postcompose right hand side with the canonical map  $\text{Env}_{\text{Fin}_{C_2,*}}(\text{Assoc}_{\text{naive}}) \rightarrow \text{Env}_{\text{Fin}_{C_2,*}}(\mathcal{HM}^\otimes)$  induced by Remark 7.9.

L: to-do: cardinalities

L: do I want a pointed version of  $\text{Cat}_\infty^{\text{Mod}}$ ? could probably even work with that directly?

L: explain how the ordering on  $U$  is used to define the map!

L: observe that  $r$  sends inert maps to inert maps? What does it mean for a morphism to be inert in  $\text{Env}_{\text{Fin}_{C_2,*}}(-)$ ?

in the notation of [Lur17, Definition 2.1.3.1]. Now observe that  $p_1^* (\underline{\mathrm{Sp}}^{C_2, \otimes})$  is the ordinary  $\infty$ -category of  $C_2$ -spectra, regarded as a monoidal  $\infty$ -category via smash product, and there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{Fun}_{C_2}^{\otimes} \left( \mathrm{Env}_{\mathrm{Fin}_{C_2, *}}(\mathcal{H}\mathcal{M}^{\otimes}), (\underline{\mathrm{Sp}}^{C_2})^{\otimes} \right) & \longrightarrow & \mathrm{LMod}(\underline{\mathrm{Sp}}^{C_2, \otimes}) & \longrightarrow & \widetilde{\mathrm{Cat}}_{\infty}^{\mathrm{ex}} \\ & & \downarrow & & \downarrow \\ & & \mathrm{Alg}_{\mathbb{E}_1}(\underline{\mathrm{Sp}}^{C_2}) & \longrightarrow & \mathrm{Cat}_{\infty}^{\mathrm{ex}} \end{array} \quad (7.21)$$

where  $\widetilde{\mathrm{Cat}}_{\infty}^{\mathrm{ex}} \rightarrow \mathrm{Cat}_{\infty}^{\mathrm{ex}}$  is the restriction of the universal cocartesian fibration to  $\mathrm{Cat}_{\infty}^{\mathrm{ex}} \subseteq \mathrm{Cat}_{\infty}$  and the right hand square is a pullback by [Lur17, Corollary 4.2.3.7(3)] and the straightening-unstraightening equivalence.

There is a canonical forgetful functor  $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty}^{\mathrm{Mod}}) \times_{\mathrm{Cat}_{\infty}^{\mathrm{Mod}}} \mathrm{Cat}_{\infty}^{\mathrm{Alg Mod}} \rightarrow \mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty}) \times_{\mathrm{Cat}_{\infty}} \mathrm{Cat}_{\infty, *}$ .

Define a functor  $\mathrm{Cat}_{\infty, *}^{\mathrm{ex}} \rightarrow \mathrm{Cat}_{\infty, (-)^{\mathrm{op}}/\mathrm{Sp}}^{\mathrm{ex}}$ .

**Lemma 7.22.** *Write  $\widetilde{\mathrm{Cat}}_{\infty}^{\mathrm{ex}} \rightarrow \mathrm{Cat}_{\infty}^{\mathrm{ex}}$  for the restriction of the universal cocartesian fibration to  $\mathrm{Cat}_{\infty}^{\mathrm{ex}} \subseteq \mathrm{Cat}_{\infty}$ . Then there is a functor  $\mathrm{Cat}_{\infty}^{\mathrm{ex}} \rightarrow \mathrm{Cat}_{\infty, (-)^{\mathrm{op}}/\mathrm{Sp}}^{\mathrm{ex}}$ .*

*Proof.*

L: inspired by Yonatan's answer [here](#)

Write  $\mathrm{RFib} \subset \mathrm{Ar}(\mathrm{Cat}) \times_{\mathrm{Cat}} \mathrm{Cat}_{\infty}^{\mathrm{ex}}$  be the full subcategory of the arrow category spanned by the right fibrations. Write  $\mathrm{RFib}_* \subseteq \mathrm{RFib}$  for the full subcategory on those arrows  $\mathcal{C} \rightarrow \mathcal{D}$  so that  $\mathcal{C}$  has a final object. The maps  $\mathrm{RFib}_* \rightarrow \mathrm{Cat}_{\infty}^{\mathrm{ex}}$ ,  $\mathrm{RFib} \rightarrow \mathrm{Cat}_{\infty}^{\mathrm{ex}}$  which send an arrow to its target are cocartesian fibrations. Given a right fibration  $c: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ , consider

$$\begin{array}{ccccc} & t^*(c) & \longrightarrow & \tilde{\mathcal{C}} & \\ & \downarrow & \lrcorner & \downarrow c & \\ \mathcal{C}^{\mathrm{op}} & \longleftarrow & \mathrm{TwAr}(\mathcal{C}) & \xrightarrow{t} & \mathcal{C}. \end{array}$$

If  $\tilde{\mathcal{C}}$  has a final object  $*$  and we write  $c = c(*)$ , then the composite  $t^{-1}(c) \rightarrow \mathcal{C}^{\mathrm{op}}$  is a right fibration. Furthermore, the right fibration  $t^*(c)$  straightens to the functor  $\mathrm{hom}_{\mathcal{C}}(-, c) \in \mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$  where  $\mathrm{Fun}'$  denotes the full sub  $\infty$ -groupoid on those functors which preserve finite limits. Since  $\mathcal{C}$  is stable,  $\Omega^{\infty}$  induces an equivalence  $\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$  by [Lur17, Corollary 1.4.2.23].

L: In progress: Now use  $\mathrm{Fun}'(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \rightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$  to construct a map of cocartesian fibrations.

□

Then define a functor  $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty}) \times_{\mathrm{Cat}_{\infty}} \mathrm{Cat}_{\infty, *}^{\mathrm{ex}} \rightarrow \mathrm{Fun}(\Delta^1, \mathrm{Cat}_{\infty}) \times_{\mathrm{Cat}_{\infty}} \mathrm{Cat}_{\infty, (-)^{\mathrm{op}}/\mathrm{Sp}}^{\mathrm{ex}} \xrightarrow{\mathrm{compose}} \mathrm{Cat}_{(-)^{\mathrm{op}}/\mathrm{Sp}}$ .

L: use a model for the former in terms of bifibrations and representable left fibrations! Finally use naturality of unstraightening in the base [Proposition A.31 here](#), esp. see how it is used in Paper I p.149

Finally, show that algebras over  $\mathbb{E}_p$  are equivalent to Poincaré rings.

## 8 Comparing involutive classical Brauer and involutive higher Brauer

**Question 8.1.** • If, for a Poincaré  $\infty$ -category  $(\mathcal{C}, \mathcal{Q})$ , there exists a Poincaré object  $(E, q)$  so that  $E$  is a compact generator, can we rewrite both the category and its Poincaré structure in terms of  $\mathrm{End}_{\mathcal{C}}(E)$ ?

- Can the *property* of an existence of a Poincaré object  $(E, q)$  in  $(\mathrm{Perf}_X, \mathcal{Q}_L)$  so that  $E$  is a compact generator be checked Zariski-locally? See Toën's paper §3.

L: Use Corollary A.32 [here](#).

L: Technically I should consider the space of all diagrams and consider the projection which forgets  $t^*(c)$ . This is a trivial Kan fibration, so we can choose an inverse.

## 9 Other

**Proposition 9.1.** Assume that  $X$  has a good quotient  $Y$  in the sense of [FW20, Remark 4.20], and write  $p: X \rightarrow Y$  for the quotient map. Let  $i: U \subseteq Y$  be the largest open subscheme on which  $\pi|_{X_U}$  is étale [FW20, Proposition 4.45]. Write  $Z(\pi)$  for the closed complement of  $U$  regarded as a topological space, and let  $j: Z(\pi) \rightarrow Y$  denote the inclusion<sup>4</sup>. Then  $\underline{\mathcal{Q}}^{\varphi C_2}$  is in the essential image of  $j_*: \mathrm{Shv}_{\mathrm{Zar}}(Z(\pi)) \rightarrow \mathrm{Shv}_{\mathrm{Zar}}(Y)$ . In other words, there exists a sheaf  $\mathcal{Q}$  of  $\mathbb{E}_\infty$ -rings on  $Z(\pi)$  so that  $j_*\mathcal{Q} \simeq \underline{\mathcal{Q}}^{\varphi C_2}$ .

L: How does  $\mathcal{Q}$  relate to the structure sheaf on the branch locus (as reduced subscheme of  $Y$ ) used in First–Williams?

L: hypothesis? move to main text?

*Proof.* Recall that the open-closed decomposition of  $Y$  induces a symmetric monoidal recollement

$$\mathrm{Shv}_{\mathrm{Zar}}(U) \xleftarrow{i^*} \mathrm{Shv}_{\mathrm{Zar}}(Y) \xrightarrow{j_*} \mathrm{Shv}_{\mathrm{Zar}}(Z(\pi)).$$

Therefore, to show that  $\underline{\mathcal{Q}}^{\varphi C_2}$  is in the essential image of  $j_*$ , it suffices to show that  $i^*(\underline{\mathcal{Q}}^{\varphi C_2}) \simeq 0$  as a sheaf on  $U$ .

By [FW20, Proposition 4.45], it suffices to show that if  $y$  is a point in  $U$ , then  $\underline{\mathcal{Q}}_y^{\varphi C_2} = 0$ . Since  $\underline{\mathcal{Q}}_y^{\varphi C_2} = \tau_{\geq 0}(\Gamma \mathcal{O}_{X \times_Y \{y\}}^{tC_2})$  where  $A = \Gamma \mathcal{O}_{Y,y} \rightarrow B = \Gamma \mathcal{O}_{X \times_Y \{y\}}$  is a quadratic étale map so that  $B$  has an involution  $\lambda$  and  $A = B^\lambda$  is a local ring with maximal ideal  $\mathfrak{m}_A$  (therefore  $B$  is semilocal by [FW20, Proposition 3.15]), it suffices to show that  $\pi_0 B^{tC_2} = 0$ . By [NS18, Lemma I.2.9], we may without loss of generality replace  $A$  and  $B$  by their 2-completions.

L: this is unnecessary to the proof—but shows that the support of  $\mathcal{Q}$  intersects trivially with the open subscheme  $Y[\frac{1}{2}]$ .

By the recollement of  $A$ -modules in terms of  $\mathfrak{m}_A$ -complete and  $A[\mathfrak{m}_A^{-1}]$ -modules, it suffices to show that  $(B_{\mathfrak{m}_A}^\wedge)^{tC_2} = 0$  and  $(B[\mathfrak{m}_A^{-1}])^{tC_2} = 0$ .

By [FW20, Propositions 3.4 & 3.15],  $B\mathfrak{m}_A = J \subseteq B$ , where  $J$  denotes the Jacobson radical of  $B$ . We claim that  $B \simeq \lim_i B/J^i$  induces an equivalence  $B^{tC_2} \rightarrow \lim_i (B/J^i)^{tC_2}$ . Granting the claim, it suffices to show that  $(B[\mathfrak{m}_A^{-1}])^{tC_2} = 0$  and  $(B/J^i)^{tC_2}$  is zero for each  $i$ . Since  $(-)^{tC_2}$  is exact and lax symmetric monoidal and each  $B/J^i$  can be written as an extension of finitely many  $B/J$ -modules, it suffices to show that  $(B[\mathfrak{m}_A^{-1}])^{tC_2}$  and  $(B/J)^{tC_2}$  are zero.

Now observe that  $A/\mathfrak{m}_A$  (resp.  $A[\mathfrak{m}_A^{-1}]$ -algebra) is a field and  $B/J$  (resp.  $B[\mathfrak{m}_A^{-1}]$ ) is a quadratic étale  $A/\mathfrak{m}_A$ -algebra (resp.  $A[\mathfrak{m}_A^{-1}]$ -algebra). By [FW20, Proposition 3.4(ii)],  $B/J$  (resp.  $B[\mathfrak{m}_A^{-1}]$ ) is either a separable quadratic field extension of  $A/\mathfrak{m}_A$ -algebra (resp.  $A[\mathfrak{m}_A^{-1}]$ -algebra), or it is isomorphic to  $\prod_{C_2} A/\mathfrak{m}_A$  (resp.  $\prod_{C_2} A[\mathfrak{m}_A^{-1}]$ ). In the latter case, the action of  $C_2$  on  $B/J$  (resp.  $B[\mathfrak{m}_A^{-1}]$ ) is manifestly free, hence  $(B/J)^{tC_2} = 0$  (resp.  $B[\mathfrak{m}_A^{-1}]^{tC_2} = 0$ ). Suppose instead that  $B/J$  (resp.  $B[\mathfrak{m}_A^{-1}]$ ) is a separable quadratic field extension of  $A/\mathfrak{m}_A$ -algebra (resp.  $A[\mathfrak{m}_A^{-1}]$ -algebra). By [FW20, Proposition 3.4(ii)],  $\lambda \otimes_B B/J$  (resp.  $\lambda \otimes_B B[\mathfrak{m}_A^{-1}]$ ) is nontrivial, hence by [Stacks, Lemma 9.21.2, Tag 09DU] the extension  $A/\mathfrak{m}_A \rightarrow B/J$  (resp.  $A[\mathfrak{m}_A^{-1}] \rightarrow B[\mathfrak{m}_A^{-1}]$ ) is Galois. Since  $C_2$  acts freely on  $B/J$  as an  $A/\mathfrak{m}_A$ -module by the normal basis theorem,  $(B/J)^{tC_2} = 0$  (resp.  $B[\mathfrak{m}_A^{-1}]^{tC_2} = 0$ ).

We conclude the proof by proving the claim. Since homotopy fixed points commute with arbitrary limits, it suffices to show that  $B \simeq \lim_i B/\mathfrak{m}_B^i$  induces an equivalence  $B_{hC_2} \rightarrow \lim_i (B/\mathfrak{m}_B^i)_{hC_2}$ . This is true because the  $B/\mathfrak{m}_B^i$  are uniformly bounded below. □

**Example 9.2.** If  $\lambda = \mathrm{id}_X$  and  $Y = X$ , then  $\pi_0 \underline{\mathcal{Q}}^{\varphi C_2} = \mathcal{O}_Y/2$ . On  $\pi_0$ , the norm map  $\underline{\mathcal{Q}}^e \simeq \mathcal{O}_X \rightarrow \underline{\mathcal{Q}}^{\varphi C_2}$  takes  $f \mapsto f^2$ .

L: compare [CMM21, Remark 2.8].

**pushforwards along quotient maps** Two attempts to show the pushforward preserves filtered colimits.

L: DEPRECATED JUNE 3RD: Here are some thoughts towards showing that the canonical map  $\mathrm{colim}_{[a,b]} \pi_* \mathbf{M}_{A^e}^{[a,b]} \rightarrow \pi_* \mathbf{M}_{A^e}$  is an equivalence. Later: see if Example 3.1.2 here could be useful?

<sup>4</sup> $Z(\pi)$  is referred to as the *branch locus* in [FW20]

1. Each  $\mathbf{M}_{A^e}^{[a,b]}$  and  $\mathbf{M}_{A^e}$  is a hypersheaf. For  $\mathbf{M}_{A^e}$  this follows from [AG14, Lemma 5.4]; need to prove for  $\mathbf{M}_{A^e}^{[a,b]}$ .
2.  $\pi_*$  sends hypersheaves on the small étale site of  $X$  to hypersheaves on the small étale site of  $Y$ ; this follows from the proof of [Lur09, Proposition 6.5.2.13].
3. Is  $\operatorname{colim}_{[a,b]} \pi_* \mathbf{M}_{A^e}^{[a,b]}$  still a hypersheaf? Hypersheaves are not in general closed under colimits, but maybe we can argue using an explicit model for this sheaf?
4. The hypercompletion of the étale  $\infty$ -topos of  $Y$  has enough points; this follows from [Lur18, Theorem A.4.0.5] and Proposition 3.7.3 of Exodromy.
5. It suffices to show that the canonical map  $\operatorname{colim}_{[a,b]} \pi_* \mathbf{M}_{A^e}^{[a,b]} \rightarrow \pi_* \mathbf{M}_{A^e}$  is an equivalence on points; use the explicit model from [FW20, Theorem 3.16]?

L: which I learned from Recollection 1.14 of this paper.

L: As of June 3, I am suspicious of the next argument/think something has gone wrong—(9.5) is not supposed to hold in this level of generality. I'm not sure what the problem is yet—maybe that  $\pi_*$  does not in fact define a morphism of recollements? Will revisit later.

**Lemma 9.3.** *Let  $(X, \sigma, Y, \pi)$  be a scheme with involution and good quotient. Let  $\tilde{U} \subseteq X$  be the largest open subscheme of  $X$  on which  $\pi$  is quadratic étale and let  $W \subseteq Y$  and  $Z = \pi^{-1}(W) \subseteq X$  be the branch and ramification loci of  $\pi$  in the sense of [FW20, Proposition 4.45-4.47] (in particular,  $W$  and  $Z$  are endowed with the reduced subscheme structure). Assume that  $2 \in \mathcal{O}_Y^\times$ . Then the pushforward  $\pi_*: \operatorname{Shv}_{\text{ét}}(X; \mathcal{S}) \rightarrow \operatorname{Shv}_{\text{ét}}(Y; \mathcal{S})$  preserves filtered colimits.*

*Proof.* Since  $W \subseteq Y$ ,  $Z \subseteq X$  are closed immersions, there exist recollements

$$\begin{array}{ccccc} \operatorname{Shv}_{\text{ét}}(\tilde{U}; \mathcal{S}) & \xleftarrow{j_{\tilde{U}}^*} & \operatorname{Shv}_{\text{ét}}(X; \mathcal{S}) & \xleftarrow{i_{Z*}} & \operatorname{Shv}_{\text{ét}}(Z; \mathcal{S}) \\ & \xrightarrow{j_{\tilde{U}*}} & & \xrightarrow{i_{Z*}^*} & \\ \operatorname{Shv}_{\text{ét}}(U; \mathcal{S}) & \xleftarrow{j_U^*} & \operatorname{Shv}_{\text{ét}}(Y; \mathcal{S}) & \xleftarrow{i_W^*} & \operatorname{Shv}_{\text{ét}}(W; \mathcal{S}) \\ & \xrightarrow{j_{U*}} & & \xrightarrow{i_W*} & \end{array}.$$

L: this is automatic, but [FW20, Lemma 5.36] asserts that  $W \rightarrow Y$  is a closed embedding without proof

Moreover, the pushforward functor  $\pi_*$  is a morphism of recollements in the sense of [Sha21, Definition 2.3]. In particular, the ‘components’ of  $\pi_*$  (see Observation 2.4 of *loc. cit.*) are

$$\begin{aligned} \pi_*|_{\operatorname{Shv}_{\text{ét}}(\tilde{U}; \mathcal{S})} &= j_U^* \circ \pi_* \circ j_{\tilde{U}*} = j_U^* \circ (\pi \circ j_{\tilde{U}})_* = j_U^* \circ (j_U \circ \pi|_{\tilde{U}})_* = j_U^* \circ j_{U*} \circ (\pi|_{\tilde{U}})_* \\ \pi_*|_{\operatorname{Shv}_{\text{ét}}(Z; \mathcal{S})} &= i_W^* \circ \pi_* \circ i_{Z*} = i_W^* \circ (\pi \circ i_Z)_* = i_W^* \circ (i_W \circ \pi|_Z)_* = i_W^* \circ i_{W*} \circ (\pi|_Z)_*. \end{aligned}$$

Now  $j_U^* \circ j_{U*} \circ (\pi|_{\tilde{U}})_* \simeq (\pi|_{\tilde{U}})_*$  (resp.  $i_W^* \circ i_{W*} \circ (\pi|_Z)_* \rightarrow (\pi|_Z)_*$ ) induced by the counit of the adjunction  $(j_U^*, j_{U*})$  (resp.  $(i_W^*, i_{W*})$ ) is an equivalence because  $j_{U*}$  (resp.  $i_{W*}$ ) is fully faithful, thus

$$\pi_*|_{\operatorname{Shv}_{\text{ét}}(\tilde{U}; \mathcal{S})} \simeq (\pi|_{\tilde{U}})_* \quad \pi_*|_{\operatorname{Shv}_{\text{ét}}(Z; \mathcal{S})} \simeq (\pi|_Z)_*. \quad (9.4)$$

We note for further reference the equivalences

$$\pi_*|_{\operatorname{Shv}_{\text{ét}}(\tilde{U}; \mathcal{S})} \circ j_{\tilde{U}}^* \simeq j_U^* \circ \pi_* \quad \pi_*|_{\operatorname{Shv}_{\text{ét}}(Z; \mathcal{S})} \circ i_W^* \simeq i_Z^* \circ \pi_*. \quad (9.5)$$

L: diagram instead?

Suppose given a filtered diagram  $\mathcal{F}_\bullet$  in  $\operatorname{Shv}_{\text{ét}}(X; \mathcal{S})$  and write  $\mathcal{F}$  for its colimit. We would like to show that the canonical map  $\operatorname{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \rightarrow \pi_*(\mathcal{F})$  is an equivalence. Since  $j_U^*, i_W^*$  are jointly conservative (by definition of a recollement, see [Lur17, Definition A.8.1(e)]), it suffices to show that the canonical maps

$$\begin{aligned} j_U^* \left( \operatorname{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \right) &\rightarrow j_U^* \pi_*(\mathcal{F}) \\ i_W^* \left( \operatorname{colim}_\bullet \pi_*(\mathcal{F}_\bullet) \right) &\rightarrow i_W^* \pi_*(\mathcal{F}) \end{aligned} \quad (9.6)$$

are equivalences. Since  $j_U^*$  and  $i_W^*$  preserve all colimits, the morphisms of (9.6) can be identified with the canonical maps

$$\begin{aligned} \operatorname{colim}_{\bullet} j_U^* \pi_*(\mathcal{F}_{\bullet}) &\simeq \operatorname{colim}_{\bullet} (\pi|_{\tilde{U}})_*(j_{\tilde{U}}^* \mathcal{F}_{\bullet}) \rightarrow (\pi|_{\tilde{U}})_*(j_{\tilde{U}}^* \mathcal{F}_{\bullet}) \\ \operatorname{colim}_{\bullet} i_W^* \pi_*(\mathcal{F}_{\bullet}) &\simeq \operatorname{colim}_{\bullet} (\pi|_W)_*(i_W^* \mathcal{F}_{\bullet}) \rightarrow (\pi|_Z)_*(i_Z^* \mathcal{F}_{\bullet}), \end{aligned}$$

respectively, where we have used (9.5). Now  $\pi|_W$  is an equivalence and  $\pi|_{\tilde{U}}$  is finite étale by [FW20, Proposition 4.47], hence  $(\pi|_{\tilde{U}})_*$  and  $(\pi|_W)_*$  preserve filtered colimits.  $\square$

## 9.1 Algebras with genuine involution

**Construction 9.7.** Assume  $\mathcal{C}$  is a presentable monoidal  $\infty$ -category such that the monoidal product  $- \otimes -: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  preserves small colimits separately in each variable. Then there is an  $\infty$ -category  $\operatorname{LMod}(\mathcal{C})$  [Lur17, Example 4.2.1.18] whose objects are pairs  $(A, M)$  where  $A$  is an associative algebra object of  $\mathcal{C}$  and  $M$  is a left  $A$ -module. Write  $a, m$  respectively for the canonical forgetful functors  $\operatorname{LMod}(\mathcal{C}) \rightarrow \operatorname{Alg}(\mathcal{C})$ ,  $\operatorname{LMod}(\mathcal{C}) \rightarrow \mathcal{C}$  which send  $(A, M)$  to  $A$  and  $M$ , resp. Then  $a$  is a cocartesian fibration [Lur17, Corollary 4.2.3.7], hence it is classified by a functor  $\operatorname{mod}: \operatorname{Alg}(\mathcal{C}) \rightarrow \operatorname{Cat}_{\infty}$ .

The functor  $s$  of [Lur17, Example 4.2.1.17] determines a commutative diagram

$$\begin{array}{ccc} & & \mathcal{U} \\ & \nearrow \eta & \downarrow \\ \operatorname{Alg}(\mathcal{C}) & \xrightarrow{\operatorname{mod}} & \operatorname{Cat}_{\infty} \end{array} \quad (9.8)$$

where  $\mathcal{U}$  is the universal cocartesian fibration. Now consider the functor  $o: \mathcal{U} \rightarrow \operatorname{Cat}_{\infty}$  which sends  $(\mathcal{D}, d \in \mathcal{D})$  to the undercategory  $\mathcal{D}_{d/-}$ . Define  $\operatorname{LMod}(\mathcal{C})_{*/-}$  to be the cocartesian fibration over  $\operatorname{Alg}(\mathcal{C})$  classified by  $o \circ \eta$ . Informally, an object of  $\operatorname{LMod}(\mathcal{C})_{*/-}$  lying over  $A \in \operatorname{Alg}(\mathcal{C})$  is the data of a left  $A$ -module  $M$  and a map of left  $A$ -modules  $A \rightarrow M$ .

**Variant 9.9.** Let  $\mathcal{C}$  be an involutive monoidal  $\infty$ -category in the sense of Definition 6.9. There is a variation on Construction 9.7 where left modules is replaced by *bimodules* [Lur17, Definition 4.3.1.12].

L: involutive  
bimodules?

**Construction 9.10.** Regard  $\mathbb{E}_1 \operatorname{Alg}(\operatorname{Sp})$  as a category with  $C_2$ -action given by taking the opposite/reverse algebra. There are functors  $b: \mathbb{E}_1 \operatorname{Alg}(\operatorname{Sp})^{hC_2} \rightarrow \operatorname{LMod}(\operatorname{Sp})$  and  $b: \mathbb{E}_1 \operatorname{Alg}(\operatorname{Sp})^{hC_2} \rightarrow \operatorname{BiMod}(\operatorname{Sp})_{*/-}$  so that  $(a, m) \circ b$  and  $(a, m) \circ b_*$  are (canonically) equivalent to  $(-)^e \otimes (-)^{\operatorname{op}}, (-)^e$ . Informally, an  $\mathbb{E}_1$ -algebra with involution  $B$  can be regarded as a  $B \otimes B^{\operatorname{op}}$ -module in a canonical way, and there is a canonical  $B \otimes B^{\operatorname{op}}$ -module map  $B \otimes B^{\operatorname{op}} \rightarrow B$ .

**Definition 9.11.** The category of  $\mathbb{E}_1$ -algebras with genuine involution is defined to be the limit of the  $\operatorname{Cat}_{\infty}$ -valued diagram

$$\begin{array}{ccccc} & & \operatorname{LMod}(\operatorname{Sp}) & & \\ & \nearrow b \circ (-)^e & \downarrow a, m & \nwarrow (-)^e & \\ & & \mathbb{E}_1 \operatorname{Alg}(\operatorname{Sp}) \times \operatorname{Sp} & & \operatorname{LMod}(\operatorname{Sp}^{C_2}) \\ & \nearrow (-)^e \otimes (-)^{\operatorname{op}}, (-)^e & \downarrow (-)^e \times (-)^e & \nwarrow & \downarrow a, m \\ \mathbb{E}_1 \operatorname{Alg}(\operatorname{Sp})^{hC_2} \times_{\operatorname{Sp}} \operatorname{Sp}^{\Delta^1} & \xrightarrow{N^{C_2}(-)^e \times U} & \mathbb{E}_1 \operatorname{Alg}(\operatorname{Sp}^{C_2}) \times \operatorname{Sp}^{C_2} & & \end{array} \quad (9.12)$$

where

- $b$  is the functor/section of Construction 9.10

- $U$  is the ‘underlying’  $C_2$ -spectrum functor  $\mathbb{E}_1 \text{Alg}(\text{Sp})^{BC_2} \times_{\text{Sp}} \text{Sp}^{\Delta^1} \rightarrow \text{Sp}^{BC_2} \times_{\text{Sp}} \text{Sp}^{\Delta^1} \simeq \text{Sp}^{C_2}$
- The upper right trapezoid commutes canonically by definition of  $\text{LMod}$  (and the fact that the functors  $a, m$  are given by restriction to subcategories of  $LM^{\otimes}$ ).

Write  $\mathbb{E}_1 \text{Alg}^{\text{gi}}(\text{Sp}^{C_2})$  for the  $\infty$ -category of  $\mathbb{E}_1$ -algebras with genuine involution.

**Definition 9.13.** The category of  $\mathbb{E}_\sigma$ -algebras is defined to be the limit of the  $\text{Cat}_\infty$ -valued diagram

$$\begin{array}{ccccc}
 & & \text{BiMod}(\text{Sp})_{*/-} & & \\
 & \nearrow^{b_* \circ (-)^e} & \downarrow^{a, m} & \nwarrow^{(-)^e} & \\
 & & \mathbb{E}_1 \text{Alg}(\text{Sp}) \times \text{Sp} & & \text{BiMod}(\text{Sp}^{C_2})_{*/-} \\
 & \nwarrow^{(-)^e \otimes (-)^{\text{op}}, (-)^e} & \downarrow^{(-)^e \times (-)^e} & \swarrow^{a, m} & \\
 \mathbb{E}_1 \text{Alg}(\text{Sp})^{hC_2} \times_{\text{Sp}} \text{Sp}^{\Delta^1} & \xrightarrow{N^{C_2}(-)^e \times U} & \mathbb{E}_1 \text{Alg}(\text{Sp}^{C_2}) \times \text{Sp}^{C_2} & & 
 \end{array} \quad (9.14)$$

Write  $\mathbb{E}_\sigma \text{Alg}(\text{Sp}^{C_2})$  for the  $\infty$ -category of  $\mathbb{E}_\sigma$ -algebras.

**Variant 9.15.** Let the base be  $R$  an  $\mathbb{E}_\infty$ -algebra or Poincaré ring instead of  $\mathbb{S}^0$ .

**Remarks 9.16.** 1. Compare [AGH21, Corollary 3.10].

2. There are canonical forgetful functors  $\mathbb{E}_\sigma \text{Alg} \rightarrow \mathbb{E}_1 \text{Alg}^{\text{gi}} \rightarrow \mathbb{E}_1 \text{Alg}^{hC_2} \rightarrow \mathbb{E}_1 \text{Alg}(\text{Sp})$ .

**Construction 9.17.** Let  $R, R^{\varphi C_2} \rightarrow R^{tC_2}$  be a Poincaré ring. There is a functor  $(\text{Mod}_{(-)}^\omega, \mathfrak{Y}_{(-)}) : \mathbb{E}_\sigma \text{Alg}_R \rightarrow (\text{Cat}_R^h)_{(\text{Mod}_R^\omega, \mathfrak{Y}_R)/-}$ .

**Lemma 9.18.** Let  $R, R^{\varphi C_2} \rightarrow R^{tC_2}$  be a Poincaré ring.

1. The functor of Construction 9.17 factors through the subcategory  $(\text{Cat}_R^p)_{(\text{Mod}_R^\omega, \mathfrak{Y}_R)/-}$ . In other words, a map of  $\mathbb{E}_\sigma$ - $R$ -algebras  $A \rightarrow B$  induces a duality-preserving map of  $R$ -linear Poincaré  $\infty$ -categories.
2. Write  $\text{Mod}^p : \mathbb{E}_1^{\text{gi}} \text{Alg}_R \rightarrow (\text{Cat}_R^p)_{(\text{Mod}_R^\omega, \mathfrak{Y}_R)/-}$  for the canonical factorization from part 1. Then  $\text{Mod}^p$  is fully faithful.

*Proof.*

L: Proof of the first point should be quite similar to/a relative variant on Corollary 3.4.2, Lemma 3.4.3 of [Cal+20].

Let  $A, B$  be  $\mathbb{E}_1$ - $R$ -algebras with genuine involution. Then there is a fiber sequence

$$\begin{array}{ccccc}
 \text{hom}_{\text{Mod}_R^p / -}(\text{Mod}_A^p, \text{Mod}_B^p) & \longrightarrow & \text{hom}_{\text{Mod}_R^p\text{-linear}}(\text{Mod}_A^p, \text{Mod}_B^p) & \longrightarrow & \text{hom}_{\text{Mod}_R^p\text{-linear}}(\text{Mod}_R^p, \text{Mod}_B^p) \\
 & & \downarrow \sim & & \downarrow \sim \\
 & & \text{Pn}(\text{Mod}_{A^{\text{op}} \otimes_R B}, \mathfrak{Y}_{A^{\text{op}} \boxtimes B}) & \longrightarrow & \text{Pn}(\text{Mod}_B, \mathfrak{Y}_B)
 \end{array}$$

where we have used Corollary ???. The fiber of the horizontal map over the point  $(B, q_B) \in \text{Pn}(\text{Mod}_B, \mathfrak{Y}_B)$  ( $q_B$  is the canonical nondegenerate form on  $B$ ) is

L: IN PROGRESS: should be some sort of  $\mathbb{E}_\sigma$  version of [AG14, Proposition 3.1].

L: this refers to main text; will be fixed when we move it.

□



Now we observe that given a Poincaré object  $(x, q)$  of  $(\mathcal{C}, \mathfrak{Q}_{\mathcal{C}})$ , its endomorphism algebra admits a canonical lift to a  $\mathbb{E}_{\sigma}$ -algebra.

**Construction 9.19.** There is a functor  $\text{End}(-): (\text{Cat}_R^h)_{(\text{Mod}_R^{\omega}, \mathfrak{Q}_R)} \rightarrow \mathbb{E}_{\sigma}\text{Alg}$  lifting the functor  $(\text{Cat}_R^h)_{(\text{Mod}_R^{\omega}, \mathfrak{Q}_R)} \rightarrow \mathbb{E}_1\text{Alg}^{hC_2}$  of [Cal+20, Proposition 3.1.16].

**Theorem 9.20.** The functors of Construction 9.17 and 9.19 form an adjoint pair.

L: make everything Poincaré

**Lemma 9.21.** The right adjoint of Construction 9.19 preserves filtered colimits.

L: similar to [AG14, Lemma 3.4].

*Proof.* □

**Proposition 9.22.** Let  $A$  be an  $\mathbb{E}_1$ - $R$ -algebra with genuine involution. Then  $A$  is compact in  $\mathbb{E}_1^{\text{gi}}\text{Alg}_R$  if and only if  $\text{Mod}_A^p$  is compact in  $\text{Cat}_{\infty R}^p$ .

L: similar to [AG14, Proposition 3.5].

*Proof.* The only if part of the statement follows from observing that  $\text{Mod}^p$  admits a right adjoint which preserves filtered colimits (Lemma 9.21) and [Lur09, Lemma 5.5.1.4]. □

L: other half of the statement

**Proposition 9.23.** Let  $A$  be an  $\mathbb{E}_1$ - $R$ -algebra with genuine involution. If  $\text{Mod}_A^p$  is dualizable in  $\text{Cat}_{\infty R}^p$ , then  $A$  is compact in  $\mathbb{E}_1^{\text{gi}}\text{Alg}_R$ .

L: similar to [AG14, Proposition 3.11].

*Proof.* □

## 10 Speculative norm for Brauer group at the level of infinity categories

We would want a functor from  $A^e$ -linear stable idempotent complete infinity categories to  $A^L$ -linear stable idempotent complete infinity categories in order to get an extension of our exact sequence to the right. Here is what I think might do it:

**Construction 10.1.** Let  $\lambda: A^e \rightarrow A^e$  denote the involution. Consider the functor

$$\text{Mod}_{\text{Mod}_{A^e}^{\omega}}(\text{Cat}_{\infty, \text{idem}}^{st}) \xrightarrow{\left(-\otimes_{\text{Mod}_{A^e}^{\omega}} \lambda^* -\right)^{hC_2}} \text{Mod}_{(\text{Mod}_{A^e}^{\omega})^{hC_2}}(\text{Cat}_{\infty, \text{idem}}^{st})$$

which we will denote by  $N_{A^{hC_2}/A}$ . This functor is symmetric monoidal and we have that the composite with the base change functor gives

$$\left(\mathcal{C} \otimes_{(\text{Mod}_{A^e}^{\omega})^{hC_2}} \text{Mod}_{A^e}^{\omega} \otimes_{\text{Mod}_{A^e}^{\omega}} \lambda^*(\mathcal{C} \otimes_{(\text{Mod}_{A^e}^{\omega})^{hC_2}} \text{Mod}_{A^e}^{\omega})\right)^{hC_2} \simeq \mathcal{C}^{\otimes_{(\text{Mod}_{A^e}^{\omega})^{hC_2}} 2}$$

For  $\mathcal{C} \in \text{Mod}_{A^e}$ , we have that

$$(\mathcal{C} \otimes_{\text{Mod}_{A^e}} \lambda^* \mathcal{C})^{hC_2} \otimes_{\text{Mod}_{A^e}^{hC_2}} \text{Mod}_{A^e} \simeq \mathcal{C}^{\otimes_{\text{Mod}_{A^e}} 2}$$

via the functor which forgets the  $C_2$ -action.

**Lemma 10.2.** The composite  $\text{PnBr}(A) \rightarrow \text{br}(A^e) \rightarrow \text{Pic}(\text{Mod}_{\text{Mod}_{A^e}^{hC_2}})$  is nullhomotopic.

*Proof.* The underlying category of a Poincaré invertible category is self-dual, and so its square will vanish. Since the functor is naturally nullhomotopic so too is the composite after applying the functor  $\text{Pic}(-)$ . □

N: Maybe... We would want this to be true but I don't see a proof immediately...

There is thus a map  $\mathrm{PnBr}(-) \rightarrow \mathcal{F}(-)$ , where  $\mathcal{F}(-)$  is the fiber. Delooping both fiber sequences we see that we get a map of fiber sequences

$$\begin{array}{ccccc} \mathrm{Pic}(\mathrm{Mod}_{A^L}(\mathrm{Sp}^{C_2})) & \longrightarrow & \mathrm{PnBr}(A) & \longrightarrow & \mathrm{br}(A^e) \\ \downarrow & & \downarrow & & \downarrow = \\ \mathrm{Pic}(\mathrm{Mod}_{A^e}^{hC_2}) & \longrightarrow & \mathcal{F}(A) & \longrightarrow & \mathrm{br}(A^e) \end{array}$$

from which we see that the middle horizontal map must be an equivalence whenever  $\frac{1}{2} \in A^e$  and  $A^{\varphi C_2} = 0$ .

## References

- [AG14] Benjamin Antieau and David Gepner. “Brauer groups and étale cohomology in derived algebraic geometry”. In: *Geom. Topol.* 18.2 (2014), pp. 1149–1244. ISSN: 1465-3060,1364-0380. DOI: [10.2140/gt.2014.18.1149](https://doi.org/10.2140/gt.2014.18.1149). URL: <https://doi.org/10.2140/gt.2014.18.1149>.
- [AGH21] Gabriel Angelini-Knoll, Teena Gerhardt, and Michael Hill. *Real topological Hochschild homology via the norm and Real Witt vectors*. 2021. arXiv: [2111.06970](https://arxiv.org/abs/2111.06970) [math.AT]. URL: <https://arxiv.org/abs/2111.06970>.
- [BHS22] Shaul Barkan, Rune Haugseng, and Jan Steinebrunner. *Envelopes for Algebraic Patterns*. To appear in *Algebraic & Geometric Topology*. 2022. arXiv: [2208.07183](https://arxiv.org/abs/2208.07183) [math.CT].
- [Cal+20] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. *Hermitian K-theory for stable  $\infty$ -categories I: Foundations*. 2020. DOI: [10.48550/ARXIV.2009.07223](https://arxiv.org/abs/2009.07223). URL: <https://arxiv.org/abs/2009.07223>.
- [CMM21] Dustin Clausen, Akhil Mathew, and Matthew Morrow. “K-theory and topological cyclic homology of henselian pairs”. In: *J. Amer. Math. Soc.* 34.2 (2021), pp. 411–473. ISSN: 0894-0347,1088-6834. DOI: [10.1090/jams/961](https://doi.org/10.1090/jams/961). URL: <https://doi.org/10.1090/jams/961>.
- [FW20] Uriya A. First and Ben Williams. “Involutions of Azumaya Algebras”. en. In: *DOCUMENTA MATHEMATICA* Vol 25 (2020 2020), p. 527–633. DOI: [10.25537/DM.2020V25.527-633](https://www.elibm.org/article/10012038). URL: <https://www.elibm.org/article/10012038>.
- [Hei23] Hadrian Heine. *An equivalence between enriched  $\infty$ -categories and  $\infty$ -categories with weak action*. 2023. arXiv: [2009.02428](https://arxiv.org/abs/2009.02428) [math.AT]. URL: <https://arxiv.org/abs/2009.02428>.
- [Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: [10.1515/9781400830558](https://doi.org/10.1515/9781400830558). URL: <https://doi.org/10.1515/9781400830558>.
- [Lur17] Jacob Lurie. *Higher algebra*. Manuscript available at <https://www.math.ias.edu/~lurie/papers/HA.pdf>. 2017.
- [Lur18] Jacob Lurie. “Spectral Algebraic Geometry”. Feb. 2018. URL: <https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf>.
- [NS18] Thomas Nikolaus and Peter Scholze. “On topological cyclic homology”. In: *Acta Math.* 221.2 (2018), pp. 203–409. ISSN: 0001-5962. DOI: [10.4310/ACTA.2018.v221.n2.a1](https://doi.org/10.4310/ACTA.2018.v221.n2.a1). URL: <https://doi.org/10.4310/ACTA.2018.v221.n2.a1>.
- [NS22] Denis Nardin and Jay Shah. *Parametrized and equivariant higher algebra*. 2022. DOI: [10.48550/ARXIV.2203.00072](https://arxiv.org/abs/2203.00072). arXiv: [2203.00072](https://arxiv.org/abs/2203.00072) [math.AT]. URL: <https://arxiv.org/abs/2203.00072>.
- [PS92] R. Parimala and V. Srinivas. “Analogues of the Brauer group for algebras with involution”. In: *Duke Math. J.* 66.2 (1992), pp. 207–237. ISSN: 0012-7094,1547-7398. DOI: [10.1215/S0012-7094-92-06606-3](https://doi.org/10.1215/S0012-7094-92-06606-3). URL: <https://doi.org/10.1215/S0012-7094-92-06606-3>.
- [Sha21] Jay Shah. *Recollements and stratification*. Preprint available at <https://arxiv.org/abs/2110.06567>. 2021. DOI: [10.48550/ARXIV.2110.06567](https://arxiv.org/abs/2110.06567).

- [Stacks] The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>. 2024.
- [Yan23] Lucy Yang. *On normed  $\mathbb{E}_\infty$ -rings in genuine equivariant  $C_p$ -spectra*. 2023. arXiv: [2308.16107](https://arxiv.org/abs/2308.16107) [[math.AT](#)].