

# Proof of Collatz Conjecture Using Division Sequence V

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## Abstract

This paper is positioned as a sequel edition of [1]. First, as in [1], define "division sequence", "complete division sequence", and "star conversion". Next, we consider loops and divergences in the Collatz conjecture, respectively. Through elementary arguments, the probability of there being a counterexample to the Collatz conjecture is almost 0. Also, Theorem Proving is not used in this paper.

## Keywords

Collatz conjecture, Division sequence, Probability

# 1 Introduction

## 1.1 Collatz Conjecture

The Collatz conjecture poses the question: “What happens if one repeats the operations of taking any positive integer  $n$ ,

- Divide  $n$  by 2 if  $n$  is even, and
- Multiply  $n$  by 3 and then add 1 if  $n$  is odd

The Collatz conjecture affirms that “for any initial value, one always reaches 1 (and enters a loop of 1 to 4 to 2 to 1) in a finite number of operations.”

We call “**(one) Collatz operation**” an operation of performing  $(3x+1)$  on an odd number and dividing by 2 as many times as one can.

The “**initial value**” is the number on which the Collatz operation is performed. This initial value is called the “**Collatz value**.”

## 1.2 Division Sequence and Complete Division Sequence

**Definition 1.1** A division sequence is the sequence given by arranging the numbers of division by 2 in each operation when the Collatz operation is continuously performed with a positive odd number,  $n$ , as the initial value.

For example, in the case of 9, the arrangement of numbers given by continuously performing  $3x + 1$ , and dividing by 2 provides

9,28,14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1 (stops when 1 is reached).

Therefore, the division sequence of 9 is [2,1,1,2,3,4].

The division sequence of 1 is an empty list []. Further, [6] is a division sequence of 21, but [6,2] and [6,2,2] ... that repeat the loop of 1 to 4 to 2 to 1 are not division sequences.

When the division sequence is finite, it is equivalent to reaching 1 in a series of Collatz operations.

When the division sequence is infinite, it does not reach 1 in a series of Collatz operations.

It is equivalent to entering a loop other than 4-2-1 or increasing the Collatz value endlessly.

**Definition 1.2** A complete division sequence is a division sequence of multiples of 3.

- 9[2,1,1,2,3,4] is a complete division sequence of 9.
- 7[1,1,2,3,4] is a division sequence of 7.

**Definition 1.3** Supposing that only one element exists in the division sequence of  $n$ , no Collatz operation can be applied to  $n$ .

**Theorem 1.1** When the Collatz operation is applied to  $x$  in the complete division sequence of  $x$  (two or more elements), (some)  $y$  and its division sequence are obtained.

Proof: This follows the Collatz operation and definition of a division sequence. □

**Theorem 1.2** When the Collatz operation is applied to  $y$  in the division sequence of  $y$  (two or more elements), (some)  $y$  and its division sequence are obtained.

Proof: It is self-evident from the Collatz operation and definition of a division sequence. □

## 1.3 One Only Looks at Odd Numbers of Multiples of 3

*There is no need to look at even numbers.*

By continuing to divide all even numbers by 2, one of the odd numbers is achieved. Therefore, it is only necessary to check “whether all odd numbers reach 1 by the Collatz operation.”

*One only needs to look at multiples of 3.*

For a number  $x$  that is not divisible by 3, the Collatz inverse operation is defined as obtaining a positive integer by  $(x \times 2^k - 1)/3$ . Multiple numbers can be obtained using the Collatz reverse operation.

Here, we consider the Collatz reverse operation on  $x$ .

The remainder of dividing  $x$  by 9 is one of 1,2,4,5,7,8, i.e.:

$$1 \times 2^6 \equiv 1$$

$$2 \times 2^5 \equiv 1$$

$$4 \times 2^4 \equiv 1$$

$$5 \times 2^1 \equiv 1$$

$$7 \times 2^2 \equiv 1$$

$$8 \times 2^3 \equiv 1 \pmod{9}$$

This indicates that multiplying any number by 2 appropriate number of times provides an even number with a remainder of 1 when divided by 9.

By subtracting 1 from this and dividing by 3, we get an odd number that is a multiple of 3.

Performing the Collatz reverse operation once from  $x$  provides an odd number  $y$  that is a multiple of 3.

If  $y$  reaches 1, then  $x$ , which was once given by the Collatz operation of  $y$ , also reaches 1.

Therefore, the following can be stated.

**Theorem 1.3** One only needs to check “whether an odd number that is a multiple of 3 reaches 1 by the Collatz operation.”  $\square$

## 2 Star Conversion

A star conversion is defined for a complete division sequence.

A complete division sequence of length,  $n$ , is copied to a complete division sequence of length,  $n$  or  $n+1$ .

The remainder, which is given by dividing the Collatz value  $x$  by 9 is

$$x \equiv 3 \pmod{9}$$

The conversion to copy a finite or infinite sequence  $[a_1, a_2, a_3 \dots]$  to a sequence  $[6, a_1 - 4, a_2, a_3 \dots]$  is described as  $A[6, -4]$ .

The conversion to copy a finite or infinite sequence  $[a_1, a_2, a_3 \dots]$  to a sequence  $[1, a_1 - 2, a_2, a_3 \dots]$  is described as  $B[1, -2]$ .

$$x \equiv 6 \pmod{9}$$

The conversion to copy a finite or infinite sequence  $[a_1, a_2, a_3 \dots]$  to a sequence  $[4, a_1 - 4, a_2, a_3 \dots]$  is described as  $C[4, -4]$ .

The conversion to copy a finite or infinite sequence  $[a_1, a_2, a_3 \dots]$  to a sequence  $[3, a_1 - 2, a_2, a_3 \dots]$  is described as  $D[3, -2]$ .

$$x \equiv 0 \pmod{9}$$

The conversion to copy a finite or infinite sequence  $[a_1, a_2, a_3 \dots]$  to a sequence  $[2, a_1 - 4, a_2, a_3 \dots]$  is described as  $E[2, -4]$ .

The conversion to copy a finite or infinite sequence  $[a_1, a_2, a_3 \dots]$  to a sequence  $[5, a_1 - 2, a_2, a_3 \dots]$  is described as  $F[5, -2]$ .

*Furthermore*, the conversion to copy a finite or infinite sequence  $[a_1, a_2, a_3 \dots]$  to a sequence  $[a_1 + 6, a_2, a_3 \dots]$  is described as  $G[+6]$ .

Conversions in which the elements of the division sequence are 0 or negative are prohibited.

If the original first term is 0 or negative,  $G[+6]$  is performed in advance.

*Example*

$$117 \equiv 0 \pmod{9}, 117[5, 1, 2, 3, 4]$$

can be converted to  $E[2, -4] \rightarrow 9[2, 5-4, 1, 2, 3, 4]$  and  $F[5, -2] \rightarrow 309[5, 5-2, 1, 2, 3, 4]$ .

Table 1 shows the functions corresponding to each star conversion.

The function represents a change in the Collatz value.

**Table 1.** Star conversion in mod 9.

When	star conversion 1	star conversion 2
$x \equiv 3 \pmod{9}$	$A[6, -4] \ y = 4x/3 - 7$	$B[1, -2] \ y = x/6 - 1/2$
$x \equiv 6 \pmod{9}$	$C[4, -4] \ y = x/3 - 2$	$D[3, -2] \ y = 2x/3 - 1$
$x \equiv 0 \pmod{9}$	$E[2, -4] \ y = x/12 - 3/4$	$F[5, -2] \ y = 8x/3 - 3$
Always	$G[+6] \ y = 64x + 21$	none

The star conversion A for  $21[6]$  replaces  $[6]-A \rightarrow [6, 2]$  with  $[6]-A \rightarrow [6]$ . The Collatz value is 21, and it does not change.

### 3 Collatz Tree

#### 3.1 Collatz Tree

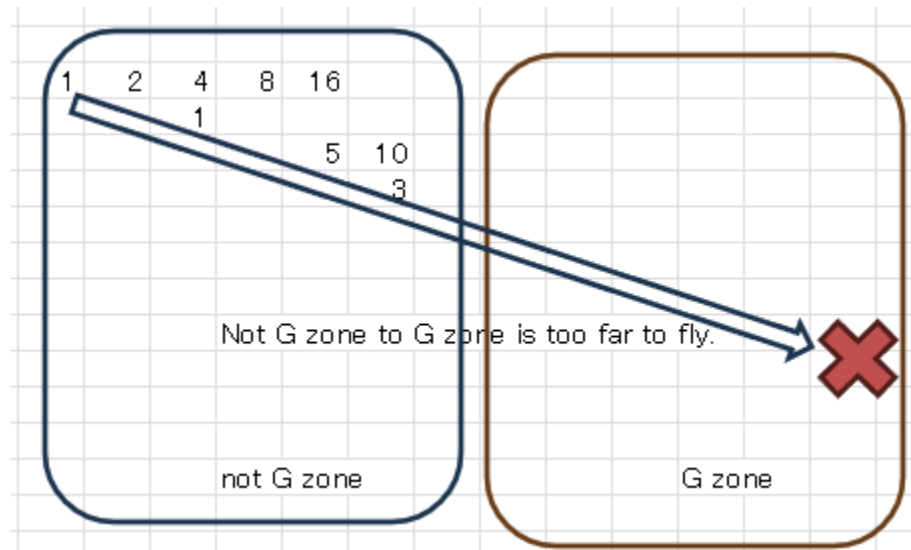
**Definition 3.1** A directed graph of the Collatz operation is defined as a **Collatz tree**. Place Collatz value 1 in the top left corner. From right to left it is  $/2$ , and from bottom to top it is  $3x+1$ .

#### 3.2 Two Theorems

**Theorem 3.1** The probability that the loop contains  $G[+6]$  is extremely small.

proof: First case: If there is a loop that includes  $G[+6]$ , The Collatz operation on the left side of the Collatz tree has nowhere to go.

Second case, when the loop crosses not  $G[+6]$  and  $G[+6]$ : We would have to go from the loop minimum Collatz value to the  $G[+6]$  zone by  $3x+1$ , which is not possible. (Because not  $G[+6]$  and  $G[+6]$  are  $64x+21$  apart.)



**Fig 1.** Second case in Theorem 3.1.

Third case: If the number increases from the loop minimum Collatz value as  $(3x+1)/2$ , the division sequence is 1, 1, 1, .... If there are no consecutive 1, they will be cancelled out by the downward flow. Since we are repeating  $D[3, -2]$ , using the fact that loops do not exist up to the order of  $10^{11}$  steps, the probability  $p_{\text{loop}G}$  is

$$p_{\text{loop}G} < (1/3)^{(\alpha \cdot 10^{11})} \quad (1)$$

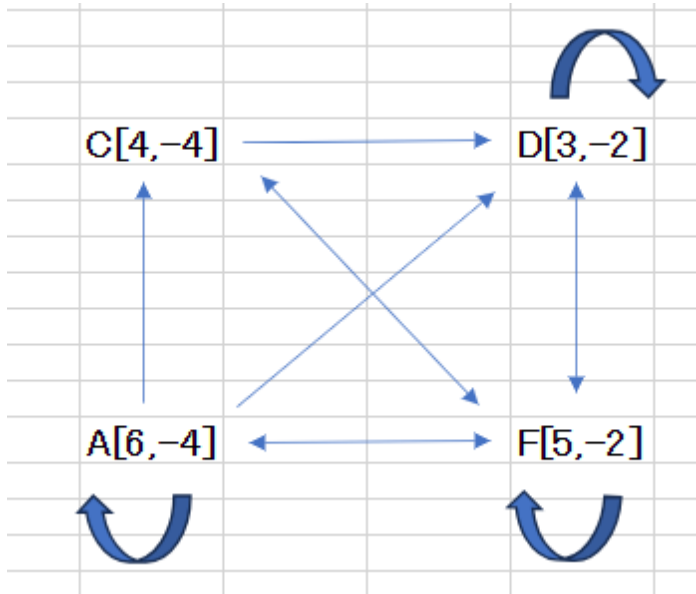
Where,  $\alpha$  represents the ratio of  $D[3, -2]$  to the total transitions. Even if we estimate  $\alpha$  conservatively, it is larger than  $1\%(10^{-2})$ . (Because it is carrying the upward flow of the loop.)  $\square$

**Theorem 3.2** If there are divergences that include  $G[+6]$ , there are also divergences that do not include  $G[+6]$ .

proof: If there is a divergence that includes  $G[+6]$ , then on the left side of the Collatz tree there is a divergence that does not include  $G[+6]$ .  $\square$

## 4 About Loops

If  $G[+6]$  is not present in the loop, then  $B[1,-2]$  and  $E[2,-4]$  are also not present. The diagram at this time is shown in Fig 2.



**Fig 2.** Restriction transition diagram for star conversion.

There are patterns where the transition ends midway.

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      3 mod 9 next
F[5,-2] → 6 mod 9 next, transition probability 1
      0 mod 9 next
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      3 mod 9 next

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A[6,-4] → 6 mod 9 next, transition probability 1  
 0 mod 9 next

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3 mod 9 fail  
 C[4,-4] → 6 mod 9 next, transition probability 2/3  
 0 mod 9 next

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3 mod 9 fail  
 D[3,-2] → 6 mod 9 next, transition probability 2/3  
 0 mod 9 next

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Here, if F, A, C, D is one period, it is known that loops do not exist up to the order of  $10^{11}$  steps. Therefore, if the probability of a loop is  $p_{\text{loop}}$ , then

$$p_{\text{loop}} \doteq (4/9)^{(10^{11}/4)} \quad (2)$$

$p_{\text{loop}}$  becomes an extremely small value. By the way,  $(1/2)^{(10^{11}/4)} \doteq 0.000000029802$ .

## 5 About Divergence

We found that the probability of looping is  $(4/9)^{(t/4)}$ . Now, let's say we apply star conversion an infinite number of times and the probability is  $p_{\text{div}}$ .

$$p_{\text{div}} \rightarrow 0 \quad (3)$$

$p_{\text{div}}$  converges to 0.

## 6 Conclusion

We have proven that the probability of loops or divergences in the Collatz conjecture is almost 0.

## Acknowledgements

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## References

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