Proof of Collatz Conjecture Using Division Sequence V

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Abstract

This paper is positioned as a sequel edition of [1]. First, as in [1], define "division sequence", "complete division sequence", and "star conversion". Next, we consider loops and divergences in the Collatz conjecture, respectively. Through elementary arguments, the probability of there being a counterexample to the Collatz conjecture is almost 0. Also, Theorem Proving is not used in this paper.

Keywords

Collatz conjecture, Division sequence, Probability

1 Introduction

1.1 Collatz Conjecture

The Collatz conjecture poses the question: “What happens if one repeats the operations of taking any positive integer n,

- Divide n by 2 if n is even, and
- Multiply n by 3 and then add 1 if n is odd

The Collatz conjecture affirms that “for any initial value, one always reaches 1 (and enters a loop of 1 to 4 to 2 to 1) in a finite number of operations.”
We call “**(one) Collatz operation**” an operation of performing (3x+1) on an odd number and dividing by 2 as many times as one can.
The “**initial value**” is the number on which the Collatz operation is performed. This initial value is called the “**Collatz value**.”

1.2 Division Sequence and Complete Division Sequence

Definition 1.1 A division sequence is the sequence given by arranging the numbers of division by 2 in each operation when the Collatz operation is continuously performed with a positive odd number, n, as the initial value.
For example, in the case of 9, the arrangement of numbers given by continuously performing 3x + 1, and dividing by 2 provides
9,28,14,7,22,11,34,17,52,26,13,40,20,10,5,16,8,4,2,1 (stops when 1 is reached).
Therefore, the division sequence of 9 is [2,1,1,2,3,4].
The division sequence of 1 is an empty list []. Further, [6] is a division sequence of 21, but [6,2] and [6,2,2] ... that repeat the loop of 1 to 4 to 2 to 1 are not division sequences.
When the division sequence is finite, it is equivalent to reaching 1 in a series of Collatz operations.
When the division sequence is infinite, it does not reach 1 in a series of Collatz operations.
It is equivalent to entering a loop other than 4-2-1 or increasing the Collatz value endlessly.
Definition 1.2 A complete division sequence is a division sequence of multiples of 3.

- 9[2,1,1,2,3,4] is a complete division sequence of 9.
- 7[1,1,2,3,4] is a division sequence of 7.

Definition 1.3 Supposing that only one element exists in the division sequence of n, no Collatz operation can be applied to n.
Theorem 1.1 When the Collatz operation is applied to x in the complete division sequence of x (two or more elements), (some) y and its division sequence are obtained.
Proof: This follows the Collatz operation and definition of a division sequence. □
Theorem 1.2 When the Collatz operation is applied to y in the division sequence of y (two or more elements), (some) y and its division sequence are obtained.
Proof: It is self-evident from the Collatz operation and definition of a division sequence. □

1.3 One Only Looks at Odd Numbers of Multiples of 3

There is no need to look at even numbers.

By continuing to divide all even numbers by 2, one of the odd numbers is achieved.
Therefore, it is only necessary to check "whether all odd numbers reach 1 by the Collatz operation."
One only needs to look at multiples of 3.
For a number x that is not divisible by 3, the Collatz inverse operation is defined as obtaining a positive integer by $(x \times 2^k - 1)/3$. Multiple numbers can be obtained using the Collatz reverse operation.
Here, we consider the Collatz reverse operation on x.
The remainder of dividing x by 9 is one of 1,2,4,5,7,8, i.e.:
$1 \times 2^6 \equiv 1$
$2 \times 2^5 \equiv 1$
$4 \times 2^4 \equiv 1$
$5 \times 2^1 \equiv 1$
$7 \times 2^2 \equiv 1$
$8 \times 2^3 \equiv 1 \pmod 9$
This indicates that multiplying any number by 2 appropriate number of times provides an even number with a reminder of 1 when divided by 9.
By subtracting 1 from this and dividing by 3, we get an odd number that is a multiple of 3.
Performing the Collatz reverse operation once from x provides an odd number y that is a multiple of 3.
If y reaches 1, then x, which was once given by the Collatz operation of y, also reaches 1. Therefore, the following can be stated.
Theorem 1.3 One only needs to check "whether an odd number that is a multiple of 3 reaches 1 by the Collatz operation." □

2 Star Conversion

A star conversion is defined for a complete division sequence.
A complete division sequence of length, n, is copied to a complete division sequence of length, n or n+1.
The remainder, which is given by dividing the Collatz value x by 9 is
$x \equiv 3 \bmod 9$
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[6, a_1 - 4, a_2, a_3...]$ is described as A[6, −4].
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[1, a_1 - 2, a_2, a_3...]$ is described as B[1, −2].
$x \equiv 6 \bmod 9$
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[4, a_1 - 4, a_2, a_3...]$ is described as C[4, −4].
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[3, a_1 - 2, a_2, a_3...]$ is described as D[3, −2].
$x \equiv 0 \bmod 9$
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[2, a_1 - 4, a_2, a_3...]$ is described as E[2, −4].
The conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[5, a_1 - 2, a_2, a_3...]$ is described as F[5, −2].
Furthermore, the conversion to copy a finite or infinite sequence $[a_1, a_2, a_3...]$
to a sequence $[a_1 + 6, a_2, a_3...]$ is described as G[+6].
Conversions in which the elements of the division sequence are 0 or negative are prohibited.
If the original first term is 0 or negative, G[+6] is performed in advance.
Example
$117 \equiv 0 \pmod 9$, 117[5,1,2,3,4]
can be converted to E[2, −4] → 9[2, 5−4, 1, 2, 3, 4] and F[5, −2] → 309[5, 5−2, 1, 2, 3, 4].
Table 1 shows the functions corresponding to each star conversion.
The function represents a change in the Collatz value.

Table 1. Star conversion in mod 9.

When	star conversion 1	star conversion 2
$x \equiv 3 \bmod 9$	A[6, −4] $y = 4x/3 - 7$	B[1, −2] $y = x/6 - 1/2$
$x \equiv 6 \bmod 9$	C[4, −4] $y = x/3 - 2$	D[3, −2] $y = 2x/3 - 1$
$x \equiv 0 \bmod 9$	E[2, −4] $y = x/12 - 3/4$	F[5, −2] $y = 8x/3 - 3$
Always	G[+6] $y = 64x + 21$	none

The star conversion A for 21[6] replaces [6]-A->[6, 2] with [6]-A->[6]. The Collatz value is 21, and it does not change.

3 One Theorem

Theorem 3.1 The loop does not contain G[+6].

proof: If we apply G[+6] and loop, we get the following equation.

$$64x+21 = y \Rightarrow ac64x+ac21 = acy \quad (1)$$

$$\frac{b}{a}x+\frac{d}{c} = y \Rightarrow cbx+ad = acy \quad (2)$$

Division theorem makes 64=b/a, 21=d/c true.
In this case, (2) cannot be created by inverse function of composing the functions A, B, C, D, E, F, and G. □

4 About Loops

If G[+6] is not present in the loop, then B[1,−2] and E[2,−4] are also not present. The diagram at this time is shown in Fig 1.

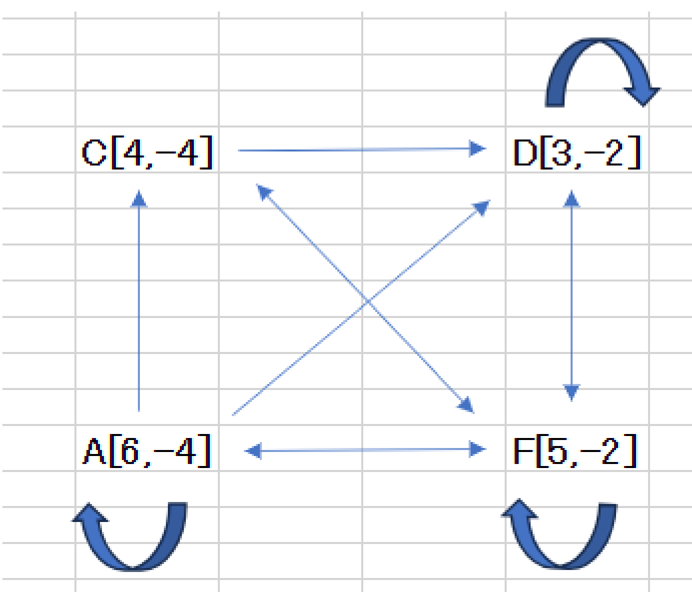

Fig 1. Restriction transition diagram for star conversion.

There are patterns where the transition ends midway.

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3 mod 9 next

F[5,−2] → 6 mod 9 next, transition probability 1

0 mod 9 next

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3 mod 9 next

A[6,−4] → 6 mod 9 next, transition probability 1

0 mod 9 next

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3 mod 9 fail

C[4,−4] → 6 mod 9 next, transition probability 2/3

0 mod 9 next

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3 mod 9 fail

D[3,−2] → 6 mod 9 next, transition probability 2/3

0 mod 9 next

Here, if F, A, C, D is one period, it is known that loops do not exist up to the order of 10^11 steps. Therefore, if the probability of a loop is p_{loop}, then

$$p_{loop} \fallingdotseq (4/9)^{(10^{11}/4)} \qquad (3)$$

p_{loop} becomes an extremely small value. By the way, $(1/2)^{(10^{2}/4)} \fallingdotseq 0.000000029802$.

5 About Divergence

When diverging, the Collatz value becomes smaller in the star conversion. One period,

$$\frac{8}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} \fallingdotseq 0.79 \qquad (4)$$

To use this to cancel out G[+6],

$$64 \cdot 0.79^{18} \fallingdotseq 0.92 \qquad (5)$$

Naturally, there are transitions that will drop out probability within the 18 periods. Now, let's say we apply star conversion an infinite number of times and the probability is p_{div}, then

$$p_{div} \rightarrow 0 \qquad (6)$$

p_{div} converges to 0.

6 Conclusion

We have proven that the probability of loops or divergences in the Collatz conjecture is almost 0.

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References

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