$\mathbf{A}$ is a **stochastic matrix**, that is, a square matrix with all entries nonnegative and all column sums equal to 1. Our example concerns a **Markov process**,[1] that is, a process for which the probability of entering a certain state depends only on the last state occupied (and the matrix $\mathbf{A}$), not on any earlier state.

Solution. From the matrix $\mathbf{A}$ and the 2004 state we can compute the 2009 state,

$$\begin{array}{c} \mathrm{C} \\ \mathrm{I} \\ \mathrm{R} \end{array} \begin{bmatrix} 0.7\cdot 25 + 0.1\cdot 20 + 0\cdot 55 \\ 0.2\cdot 25 + 0.9\cdot 20 + 0.2\cdot 55 \\ 0.1\cdot 25 + 0\cdot 20 + 0.8\cdot 55 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix}.$$

To explain: The 2009 figure for C equals 25% times the probability 0.7 that C goes into C, plus 20% times the probability 0.1 that I goes into C, plus 55% times the probability 0 that R goes into C. Together,

$$25\cdot 0.7 + 20\cdot 0.1 + 55\cdot 0 = 19.5\,[\%]. \qquad \text{Also} \qquad 25\cdot 0.2 + 20\cdot 0.9 + 55\cdot 0.2 = 34\,[\%].$$

Similarly, the new R is 46.5%. We see that the 2009 state vector is the column vector

$$\mathbf{y} = [19.5 \quad 34.0 \quad 46.5]^{\mathsf{T}} = \mathbf{Ax} = \mathbf{A}\,[25 \quad 20 \quad 55]^{\mathsf{T}}$$

where the column vector $\mathbf{x} = [25 \quad 20 \quad 55]^{\mathsf{T}}$ is the given 2004 state vector. Note that the sum of the entries of $\mathbf{y}$ is 100 [%]. Similarly, you may verify that for 2014 and 2019 we get the state vectors

$$\mathbf{z} = \mathbf{Ay} = \mathbf{A}(\mathbf{Ax}) = \mathbf{A}^2\mathbf{x} = [17.05 \quad 43.80 \quad 39.15]^{\mathsf{T}}$$

$$\mathbf{u} = \mathbf{Az} = \mathbf{A}^2\mathbf{y} = \mathbf{A}^3\mathbf{x} = [16.315 \quad 50.660 \quad 33.025]^{\mathsf{T}}.$$

Answer. In 2009 the commercial area will be 19.5% (11.7 mi^2), the industrial 34% (20.4 mi^2), and the residential 46.5% (27.9 mi^2). For 2014 the corresponding figures are 17.05%, 43.80%, and 39.15%. For 2019 they are 16.315%, 50.660%, and 33.025%. (In Sec. 8.2 we shall see what happens in the limit, assuming that those probabilities remain the same. In the meantime, can you experiment or guess?) ■

PROBLEM SET 7.2

1–10 GENERAL QUESTIONS

1. **Multiplication.** Why is multiplication of matrices restricted by conditions on the factors?
2. **Square matrix.** What form does a 3×3 matrix have if it is symmetric as well as skew-symmetric?
3. **Product of vectors.** Can every 3×3 matrix be represented by two vectors as in Example 3?
4. **Skew-symmetric matrix.** How many different entries can a 4×4 skew-symmetric matrix have? An $n \times n$ skew-symmetric matrix?
5. Same questions as in Prob. 4 for symmetric matrices.
6. **Triangular matrix.** If $\mathbf{U}_1$, $\mathbf{U}_2$ are upper triangular and $\mathbf{L}_1$, $\mathbf{L}_2$ are lower triangular, which of the following are triangular?

   $\mathbf{U}_1 + \mathbf{U}_2$, $\mathbf{U}_1\mathbf{U}_2$, $\mathbf{U}_1^2$, $\mathbf{U}_1 + \mathbf{L}_1$, $\mathbf{U}_1\mathbf{L}_1$, $\mathbf{L}_1 + \mathbf{L}_2$

7. **Idempotent matrix,** defined by $\mathbf{A}^2 = \mathbf{A}$. Can you find four 2×2 idempotent matrices?
8. **Nilpotent matrix,** defined by $\mathbf{B}^m = \mathbf{0}$ for some m. Can you find three 2×2 nilpotent matrices?
9. **Transposition.** Can you prove (10a)–(10c) for 3×3 matrices? For $m \times n$ matrices?
10. **Transposition.** (a) Illustrate (10d) by simple examples. (b) Prove (10d).

11–20 MULTIPLICATION, ADDITION, AND TRANSPOSITION OF MATRICES AND VECTORS

Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 4 \\ 1 & 2 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -1 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \\ 2 & 0 \end{bmatrix}, \qquad \mathbf{a} = [-1 \quad -2 \quad 0], \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

Showing all intermediate results, calculate the following expressions or give reasons why they are undefined:

11. $\mathbf{AB}$, $\mathbf{AB}^{\mathsf{T}}$, $\mathbf{BA}$, $\mathbf{B}^{\mathsf{T}}\mathbf{A}$
12. $\mathbf{AA}^{\mathsf{T}}$, $\mathbf{A}^2$, $\mathbf{BB}^{\mathsf{T}}$, $\mathbf{B}^2$
13. $\mathbf{CC}^{\mathsf{T}}$, $\mathbf{BC}$, $\mathbf{CB}$, $\mathbf{C}^{\mathsf{T}}\mathbf{B}$
14. $3\mathbf{A} - 2\mathbf{B}$, $(3\mathbf{A} - 2\mathbf{B})^{\mathsf{T}}$, $3\mathbf{A}^{\mathsf{T}} - 2\mathbf{B}^{\mathsf{T}}$, $(3\mathbf{A} - 2\mathbf{B})^{\mathsf{T}}\mathbf{a}^{\mathsf{T}}$
15. $\mathbf{Aa}$, $\mathbf{Aa}^{\mathsf{T}}$, $(\mathbf{Ab})^{\mathsf{T}}$, $\mathbf{b}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$
16. $\mathbf{BC}$, $\mathbf{BC}^{\mathsf{T}}$, $\mathbf{Bb}$, $\mathbf{b}^{\mathsf{T}}\mathbf{B}$
17. $\mathbf{ABC}$, $\mathbf{ABa}$, $\mathbf{ABb}$, $\mathbf{Ca}^{\mathsf{T}}$
18. $\mathbf{ab}$, $\mathbf{ba}$, $\mathbf{aA}$, $\mathbf{Bb}$
19. $1.5\mathbf{a} + 3.0\mathbf{b}$, $1.5\mathbf{a}^{\mathsf{T}} + 3.0\mathbf{b}$, $(\mathbf{A} - \mathbf{B})\mathbf{b}$, $\mathbf{Ab} - \mathbf{Bb}$
20. $\mathbf{b}^{\mathsf{T}}\mathbf{Ab}$, $\mathbf{aBa}^{\mathsf{T}}$, $\mathbf{aCC}^{\mathsf{T}}$, $\mathbf{C}^{\mathsf{T}}\mathbf{ba}$
21. **General rules.** Prove (2) for 2×2 matrices $\mathbf{A} = [a_{jk}]$, $\mathbf{B} = [b_{jk}]$, $\mathbf{C} = [c_{jk}]$, and a general scalar.
22. **Product.** Write $\mathbf{AB}$ in Prob. 11 in terms of row and column vectors.
23. **Product.** Calculate $\mathbf{AB}$ in Prob. 11 columnwise. See Example 1.
24. **Commutativity.** Find all 2×2 matrices $\mathbf{A} = [a_{jk}]$ that commute with $\mathbf{B} = [b_{jk}]$, where $b_{jk} = j + k$.
25. **TEAM PROJECT. Symmetric and Skew-Symmetric Matrices.** These matrices occur quite frequently in applications, so it is worthwhile to study some of their most important properties.

    (a) Verify the claims in (11) that $a_{kj} = a_{jk}$ for a symmetric matrix, and $a_{kj} = -a_{jk}$ for a skew-symmetric matrix. Give examples.

    (b) Show that for every square matrix $\mathbf{C}$ the matrix $\mathbf{C} + \mathbf{C}^{\mathsf{T}}$ is symmetric and $\mathbf{C} - \mathbf{C}^{\mathsf{T}}$ is skew-symmetric. Write $\mathbf{C}$ in the form $\mathbf{C} = \mathbf{S} + \mathbf{T}$, where $\mathbf{S}$ is symmetric and $\mathbf{T}$ is skew-symmetric and find $\mathbf{S}$ and $\mathbf{T}$ in terms of $\mathbf{C}$. Represent $\mathbf{A}$ and $\mathbf{B}$ in Probs. 11–20 in this form.

    (c) A **linear combination** of matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{M}$ of the same size is an expression of the form

    $$(14) \qquad a\mathbf{A} + b\mathbf{B} + c\mathbf{C} + \cdots + m\mathbf{M},$$

    where $a, \cdots, m$ are any scalars. Show that if these matrices are square and symmetric, so is (14); similarly, if they are skew-symmetric, so is (14).

    (d) Show that $\mathbf{AB}$ with symmetric $\mathbf{A}$ and $\mathbf{B}$ is symmetric if and only if $\mathbf{A}$ and $\mathbf{B}$ **commute**, that is, $\mathbf{AB} = \mathbf{BA}$.

    (e) Under what condition is the product of skew-symmetric matrices skew-symmetric?

26–30 FURTHER APPLICATIONS

26. **Production.** In a production process, let N mean "no trouble" and T "trouble." Let the transition probabilities from one day to the next be 0.8 for $N \to N$, hence 0.2 for $N \to T$, and 0.5 for $T \to N$, hence 0.5 for $T \to T$.

    If today there is no trouble, what is the probability of N two days after today? Three days after today?
27. **CAS Experiment. Markov Process.** Write a program for a Markov process. Use it to calculate further steps in Example 13 of the text. Experiment with other stochastic 3×3 matrices, also using different starting values.
28. **Concert subscription.** In a community of 100,000 adults, subscribers to a concert series tend to renew their subscription with probability 90% and persons presently not subscribing will subscribe for the next season with probability 0.2%. If the present number of subscribers is 1200, can one predict an increase, decrease, or no change over each of the next three seasons?
29. **Profit vector.** Two factory outlets F_1 and F_2 in New York and Los Angeles sell sofas (S), chairs (C), and tables (T) with a profit of \$35, \$62, and \$30, respectively. Let the sales in a certain week be given by the matrix

    $$\begin{array}{cc} & \begin{array}{ccc} \mathrm{S} & \mathrm{C} & \mathrm{T} \end{array} \\ \mathbf{A} = & \begin{bmatrix} 400 & 60 & 240 \\ 100 & 120 & 500 \end{bmatrix} \end{array} \begin{array}{c} F_1 \\ F_2 \end{array}$$

    Introduce a "profit vector" $\mathbf{p}$ such that the components of $\mathbf{v} = \mathbf{Ap}$ give the total profits of F_1 and F_2.
30. **TEAM PROJECT. Special Linear Transformations. Rotations** have various applications. We show in this project how they can be handled by matrices.

    (a) Rotation in the plane. Show that the linear transformation $\mathbf{y} = \mathbf{Ax}$ with

    $$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

    is a counterclockwise rotation of the Cartesian x_1x_2-coordinate system in the plane about the origin, where θ is the angle of rotation.

    (b) Rotation through $n\theta$. Show that in (a)

    $$\mathbf{A}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

    Is this plausible? Explain this in words.

    (c) Addition formulas for cosine and sine. By geometry we should have

    $$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}.$$

    Derive from this the addition formulas (6) in App. A3.1.

[1] ANDREI ANDREJEVITCH MARKOV (1856–1922), Russian mathematician, known for his work in probability theory.

$\mathbf{Rx} = \mathbf{f}$ is inconsistent: No solution is possible. Therefore the system $\mathbf{Ax} = \mathbf{b}$ is inconsistent as well. See Example 4, where $r = 2 < m = 3$ and $f_{r+1} = f_3 = 12$.

If the system is consistent (either $r = m$, or $r < m$ *and* all the numbers $f_{r+1}, f_{r+2}, \cdots, f_m$ are zero), then there are solutions.

(b) Unique solution. If the system is consistent and $r = n$, there is exactly one solution, which can be found by back substitution. See Example 2, where $r = n = 3$ and $m = 4$.

(c) Infinitely many solutions. To obtain any of these solutions, choose values of $x_{r+1}, \cdots, x_n$ arbitrarily. Then solve the rth equation for x_r (in terms of those arbitrary values), then the $(r - 1)$st equation for x_{r-1}, and so on up the line. See Example 3.

Orientation. Gauss elimination is reasonable in computing time and storage demand. We shall consider those aspects in Sec. 20.1 in the chapter on numeric linear algebra. Section 7.4 develops fundamental concepts of linear algebra such as linear independence and rank of a matrix. These in turn will be used in Sec. 7.5 to fully characterize the behavior of linear systems in terms of existence and uniqueness of solutions.

PROBLEM SET 7.3

1–14 GAUSS ELIMINATION

Solve the linear system given explicitly or by its augmented matrix. Show details.

1. $\begin{aligned} -3x + 8y &= 5 \\ 8x - 12y &= -11 \end{aligned}$

2. $\begin{bmatrix} 3.0 & -0.5 & 0.6 \\ 1.5 & 4.5 & 6.0 \end{bmatrix}$

3. $\begin{aligned} 8y + 6z &= -4 \\ -2x + 4y - 6z &= 18 \\ x + y - z &= 2 \end{aligned}$

4. $\begin{bmatrix} 4 & 1 & 0 & 4 \\ 5 & -3 & 1 & 2 \\ -9 & 2 & -1 & 5 \end{bmatrix}$

5. $\begin{bmatrix} 13 & 12 & 6 \\ -4 & 7 & 73 \\ 4 & 5 & 11 \end{bmatrix}$

6. $\begin{bmatrix} 4 & -8 & 3 & 16 \\ -1 & 2 & -5 & -21 \\ 3 & -6 & 1 & 7 \end{bmatrix}$

7. $\begin{bmatrix} 4 & 0 & 6 \\ -1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}$

8. $\begin{aligned} 4y + 3z &= 8 \\ 2x \quad\quad - z &= 2 \\ 3x + 2y \quad\quad &= 5 \end{aligned}$

9. $\begin{aligned} -2y - 2z &= 8 \\ 3x + 4y - 5z &= 8 \end{aligned}$

10. $\begin{bmatrix} 5 & -7 & 3 & 17 \\ -15 & 21 & -9 & 50 \end{bmatrix}$

11. $\begin{aligned} y + z - 2w &= 0 \\ 2x - 3y - 3z + 6w &= 2 \\ 4x + y + z - 2w &= 4 \end{aligned}$

12. $\begin{bmatrix} 2 & -2 & 4 & 0 & 0 \\ -3 & 3 & -6 & 5 & 15 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$

13. $\begin{aligned} 10x + 4y - 2z &= 14 \\ -3w - 15x + y + 2z &= 0 \\ w + x + y \quad\quad &= 6 \\ 8w - 5x + 5y - 10z &= 26 \end{aligned}$

14. $\begin{bmatrix} 2 & 3 & 1 & -11 & 1 \\ 5 & -2 & 5 & -4 & 5 \\ 1 & -1 & 3 & -3 & 3 \\ 3 & 4 & -7 & 2 & -7 \end{bmatrix}$

15. Equivalence relation. By definition, an *equivalence relation* on a set is a relation satisfying three conditions: (named as indicated)

(i) Each element A of the set is equivalent to itself (*Reflexivity*).

(ii) If A is equivalent to B, then B is equivalent to A (*Symmetry*).

(iii) If A is equivalent to B and B is equivalent to C, then A is equivalent to C (*Transitivity*).

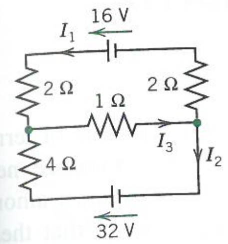
Show that row equivalence of matrices satisfies these three conditions. *Hint.* Show that for each of the three elementary row operations these conditions hold.

16. CAS PROJECT. Gauss Elimination and Back Substitution. Write a program for Gauss elimination and back substitution **(a)** that does not include pivoting and **(b)** that does include pivoting. Apply the programs to Probs. 11–14 and to some larger systems of your choice.

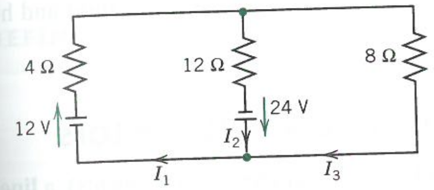
17–21 MODELS OF NETWORKS

In Probs. 17–19, using Kirchhoff's laws (see Example 2) and showing the details, find the currents:

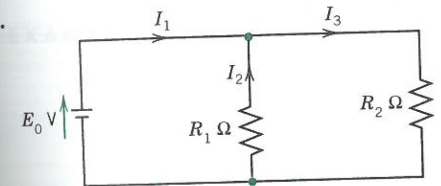
17.



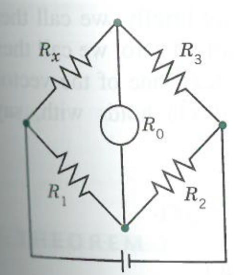
18.



19.

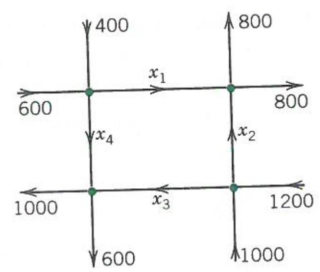


20. Wheatstone bridge. Show that if $R_x/R_3 = R_1/R_2$ in the figure, then $I = 0$. (R_0 is the resistance of the instrument by which I is measured.) This bridge is a method for determining R_x. R_1, R_2, R_3 are known. R_3 is variable. To get R_x, make $I = 0$ by varying R_3. Then calculate $R_x = R_3R_1/R_2$.



Wheatstone bridge

Problem 20



Net of one-way streets

Problem 21

21. Traffic flow. Methods of electrical circuit analysis have applications to other fields. For instance, applying the analog of Kirchhoff's Current Law, find the traffic flow (cars per hour) in the net of one-way streets (in the directions indicated by the arrows) shown in the figure. Is the solution unique?

22. Models of markets. Determine the equilibrium solution ($D_1 = S_1$, $D_2 = S_2$) of the two-commodity market with linear model (D, S, P = demand, supply, price; index 1 = first commodity, index 2 = second commodity)

$$D_1 = 33 - 2P_1 - 2P_2, \qquad S_1 = 4P_1 - 2P_2 + 3,$$

$$D_2 = 16 + 4P_1 - 3P_2, \qquad S_2 = 4P_2 + 1.$$

23. Balancing a chemical equation $x_1C_3H_8 + x_2O_2 \rightarrow x_3CO_2 + x_4H_2O$ means finding integer x_1, x_2, x_3, x_4 such that the numbers of atoms of carbon (C), hydrogen (H), and oxygen (O) are the same on both sides of this reaction, in which propane C_3H_8 and O_2 give carbon dioxide and water. Find the smallest positive integers $x_1, \cdots, x_4$.

24. PROJECT. Elementary Matrices. The idea is that elementary operations can be accomplished by matrix multiplication. If $\mathbf{A}$ is an $m \times n$ matrix on which we want to do an elementary operation, then there is a matrix $\mathbf{E}$ such that $\mathbf{EA}$ is the new matrix after the operation. Such an $\mathbf{E}$ is called an **elementary matrix**. This idea can be helpful, for instance, in the design of algorithms. (*Computationally*, it is generally preferable to do row operations *directly*, rather than by multiplication by $\mathbf{E}$.)

(a) Show that the following are elementary matrices, for interchanging Rows 2 and 3, for adding -5 times the first row to the third, and for multiplying the fourth row by 8.

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$