10 - Feb 8 Lecture

- 16 “equivalence statements” to use when solving problems
- information about test 2
- subspaces

1) goes through origin $\vec{0} \in V$

2) $u, v \in V \Rightarrow u + v \in V$

3) $c \in \mathbb{R}$, $u \in V \Rightarrow cu \in V$

a) $\{\vec{0}\} = V$ $\therefore$ subspace

b) $x_1 = 2x_2 + 3$

$0 = 2(0) + 3$

$0 \neq 3$

$\therefore$ not a subspace

c) $\ker(T)$ $T: \mathbb{R}^2 \to \mathbb{R}^2$

$\ker(T) = \{ v \in \mathbb{R}^2 \mid T(v) = 0 \}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x - 2y \\ 2x + 3y \end{bmatrix}$$

prove 2) take $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

prove 3) $c = 10, 20,$ anything

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3x - 2y \\ 2x + 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

solve $\to$ $x, y = 0$

$\therefore$ proves 1)

zero

zero

zero

zero

Q2. Rank-Nullity Thm

Rank: # non-zero rows in RREF ?

Q3.

16 Equivalence Statements	Let A be a $n \times n$ matrix.
	The following statements are <u>equivalent</u> (all true or all false):
	1) A is invertible.
	2) A is row equivalent to I_n.
	3) A has n pivot positions.
	4) $A\vec{x} = \vec{0}$ implies $\vec{x} = \vec{0}$
	5) $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$.
	6) The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$ is one-to-one.
	7) The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$ is onto.
	8) The set of columns of A is linearly independent.
	9) The set of columns of A span $\mathbb{R}^n$.
	10) There exists a $n \times n$ matrix C such that $CA = I_n$.
	11) There exists a $n \times n$ matrix D such that $AD = I_n$.
	12) A^T is invertible.
We haven't covered these topics yet.	13) The columns of A form a basis of $\mathbb{R}^n$. 14) $\text{Rank}(A) = n$ 15) $\text{Det}(A) \neq 0$ 16) All eigenvalues of A are non-zero.
Test 2 Information	Knowing these is not necessary for test 2, but it may help for 1 or 2 of the questions.
	Format:
	Q1-4: Multiple choice (4 marks each) * Marks subject to change
	Q5-7: Long answer (12 / 14 / 8 marks)
	The test will cover linear transformations and matrix "stuff"
	The corresponding textbook sections (should be) 1.8, 1.9, 2.1, and 2.2.

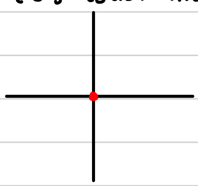
Getting ahead with some lecture 11 material...

Subspaces of $\mathbb{R}^n$	Subspaces are sets of (same-length) vectors in $\mathbb{R}^n$ that often show up when dealing with matrices and linear transformations.
	• Ex. $\text{Ker}(T)$ and $\text{Range}(T)$ are subspaces.

A subspace of $\mathbb{R}^m$ is a set of vectors in $\mathbb{R}^m$ such that all are true:

1) The zero vector is in the set.
2) If $\vec{u}$ and $\vec{v}$ are in the set, then $\vec{u} + \vec{v}$ is in the set.
3) If $\vec{u}$ is in the set and c is a scalar, $c\vec{u}$ is in the set.

$\{\vec{0}\}$ (just the zero vector) is a subspace.



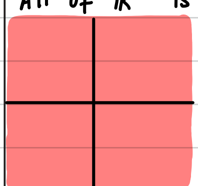
Condition 1): ✓
Condition 2): ✓
Condition 3): ✓

2D, 3D, 12D, etc...

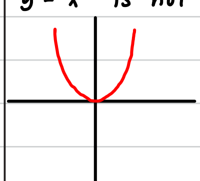
Any line through the origin (in <u>any</u> dimension) is a subspace.

Any plane / hyperplane through the origin is a subspace.

All of $\mathbb{R}^m$ is a subspace.



$y = x^2$ is not a subspace.



Any line not going through the origin is not a subspace.

end

(Will be posted online)

Let A be a nxn matrix The following statements are equivalent:

ALL TRUE or ALL FALSE

1) A is invertible.
2) A is row equiv. to I_n.
3) A has n pivot positions.
4) $A\vec{x} = \vec{0}$ implies $\vec{x} = 0$ #4 false ⟺ there exists a nonzero $\vec{x}$ s.t. $A\vec{x} = 0$
5) $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$
6) The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$ is one-to-one.
7) The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$ is onto.
8) The set of columns of A is linearly independent.
9) The set of columns of A span $\mathbb{R}^n$.
10) There exists a nxn matrix C s.t. $CA = I_n$
11) There exists a nxn matrix D s.t. $AD = I$
12) A^T is invertible
13) The columns of A form a basis of $\mathbb{R}^n$
14) Rank (A) = n
15) Det (A) ≠ 0
16) All eigen values of A are non-zero

have not covered yet

So $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ is not invertible because $\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\text{matrix-vector}} = \vec{0}$ 4

one of the fastest methods to show A is not invertible

⇕

satisfies 4) =
satisfies 8) =
satisfies 6)
$-2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \vec{0}$ 8

linear dependance rltn

$\begin{bmatrix} 1 & 0 & \pi \\ 2 & 0 & 3 \\ \sqrt{2} & 0 & 1 \end{bmatrix}$

↪ linearly dependant ⟹ not invertible ⟹ all other rules true

5, 7, 9 are connected (Range (T) = $\mathbb{R}^n$)

$T: \mathbb{R}^n \to \mathbb{R}^n$ implies T is either both one-to-one and onto or neither

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ → both

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → neither

9): If $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are the columns of A then $\text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \mathbb{R}^n$

10/11) If A is square and $AC = I_n$ then $CA = I_n$ and A is invertible.
and $C = A^{-1}$, $C^{-1} = A$

Columns of A Linearly independant $\Leftrightarrow$ A^T is invertible
Let $B = A^T$
$\Leftrightarrow$ B is invertible
$\Leftrightarrow$ columns of B are linearly independent
$\Leftrightarrow$ rows of A are linearly independent

This lets you replace "columns" with rows in 8 and 9

Alt. way of writing 4)

A invertible $\Leftrightarrow$ A^T invertible $(\vec{x} = \vec{0})$
$\Leftrightarrow$ $A^T\vec{x} = \vec{0}$ only has trivial solution
$\Leftrightarrow$ $(A^T\vec{x})^T = \vec{0}^T$ only has trivial solution
$\Leftrightarrow$ $\vec{x}A = \vec{0}^T$ only has trivial solution

In 4) and 5) you can switch the eqns to $\vec{x}^TA = \vec{b}^T$

Not necessary for test, but will help for 1-2 of the questions.

Q1,2 → lin. trans.
Q3,4 → matrix stuff

Q 1 - 4 Multiple choice (4 each)
Q 5 - 7 Long answer (12 / 14 / 8)

marks subject to change

5, 6 7?
lin trans / matrix stuff

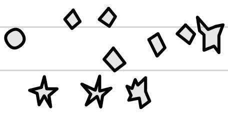
half got Q4 right

Textbook Sections
1.8, 1.9, 2.1, 2.2, (2.3)

Don't need to know specific trans. like rotations.

(same length vectors)

Subspaces of $\mathbb{R}^n$



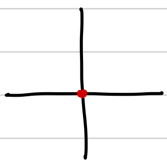
Subspaces are sets of vectors in $\mathbb{R}^n$ that often show up when dealing with matrices and linear transformations.

Ker(T) and Range (T) are subspaces

"range of T lies in the codomain"

A subspace of $\mathbb{R}^m$ is a set of vectors in $\mathbb{R}^m$ such that all are true:

1) The zero vector is in the set
2) If $\vec{u}$ and $\vec{v}$ are in the set then $\vec{u}+\vec{v}$ is in the set
3) If $\vec{u}$ is in the set and c is a scalar, $c\vec{u}$ is in the set

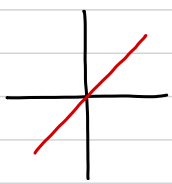


Just the zero vector: $\{\vec{0}\}$ is a subspace

1 ✓

2 ✓

3 ✓



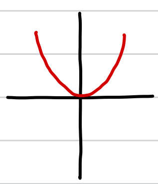
Any line through the origin (in any dimension) is a subspace. (2D, 3D, 12D, etc.)

Any plane / hyperplane through the origin is a subspace

plane → defined by 2 vectors in 3D

hyperplane → ex. 5D space in 12D space

All of $\mathbb{R}^m$ is called a subspace



$y = x^2$ not a subspace

Any line not through the origin is not a subspace