5 - Jan 23 Lecture

- Theorems of linear dependence

Problems covered:

1) Given 3 vectors, find dependence relation

Theorems of linear dependence	1) A set of 2 vectors $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if and only if they are scalar multiples of each other. $\rightarrow$ $c\vec{v}_1 = \vec{v}_2$ for a constant c 2) Any set of vectors containing $\vec{0}$ is linearly dependent. 3) A set of vectors if linearly dependent if and only if at least one of the vectors in the set can be expressed as a linear combination of the other vectors. 4) If a set of vectors has more vectors than the length of those vectors, then the set is linearly dependent. $\{\vec{v}_1, \ldots, \vec{v}_k\}$, $\vec{v}_j \in \mathbb{R}^k$ if $j > k$ then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependant. The reverse is not true. You can have linear dependence with $j \leq k$, it's just not guaranteed. $j > k$: linear dependence guaranteed $j \leq k$: do more work to determine if set is linearly dependent or not
Ex. 1	$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ $j = 4$, $k = 3$ Since $j > k$, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependant.
Ex. 2	$\vec{v}_1 = [1]$, $\vec{v}_2 = [2]$ $j = 2$, $k = 1$ $\rightarrow$ $j > k$, so $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependant.

END OF TEST 1 MATERIAL

	Some notes for the problem below: Pick a value for $t \in \mathbb{R}$ and evaluate c_1, c_2, and c_3. ↳ Don't pick all zeroes for your free variables (you'll get the "trivial eqn" – not dependance relation). No free variables tells you the vectors are linearly independant. • RR Algorithm $\rightarrow$ no free vars $\rightarrow$ linearly independant • RR Algorithm $\rightarrow$ some free vars $\rightarrow$ linearly dependant

Given the 3 following vectors, determine the dependance relation.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -3 \end{bmatrix}$$

Need to find c_1, c_2, c_3 such that the following equation holds.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

Substitute the vectors into the equation above.

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} c_1 \\ c_1 \\ 0 \\ -2c_1 \end{bmatrix} + \begin{bmatrix} -2c_2 \\ c_2 \\ 2c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3c_3 \\ 2c_3 \\ -3c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} c_1 - 2c_2 \\ c_1 + c_2 + 3c_3 \\ 2c_2 + 2c_3 \\ -2c_1 + c_2 - 3c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$c_1 - 2c_2 = 0$$
$$c_1 + c_2 + 3c_3 = 0$$
$$2c_2 + 2c_3 = 0$$
$$-2c_1 + c_2 - 3c_3 = 0$$

The system of linear equations above corresponds to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \begin{array}{l} \\ R_2' = R_2 - R_1 \\ \\ R_4' = R_4 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \begin{array}{l} \\ R_2' = \frac{1}{3} R_2 \\ \\ \\ \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} B & B & F \end{array} \\ \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \end{array} \begin{array}{l} R_1' = R_1 + 2R_2 \\ \\ R_3' = R_3 - 2R_2 \\ R_4' = R_4 + 3R_2 \end{array}$$

This matrix is in RREF, and corresponds to the following equations:

$$c_1 + 2c_3 = 0$$
$$c_2 + c_3 = 0$$
$$0 = 0$$
$$0 = 0$$

Let c_3, the free variable, be equal to t.

$$c_1 = -2t \qquad \text{for all } t \in \mathbb{R}$$
$$c_2 = -t$$

The above holds true for all $t \in \mathbb{R}$, so we can pick any valid value for t

Let's set $t = 1$. $\rightarrow$ $c_1 = -2$, $c_2 = -1$, $c_3 = 1$

Plugging c_1, c_2, c_3 into $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, we find that the equation holds.

$-2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$ We found a single dependance relation

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -3 \end{bmatrix}$$

$2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = 0$ → dependance relation (non-trivial)

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ (Find c_1, c_2, c_3 s.t. equation holds.)

$$\begin{bmatrix} c_1 - 2c_2 \\ c_1 + c_2 + 3c_3 \\ 2c_2 + 2c_3 \\ -2c_1 + c_2 - 3c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{aligned} c_1 - 2c_2 &= 0 \\ c_1 + c_2 + 3c_3 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ -2c_1 + c_2 - 3c_3 &= 0 \end{aligned}$$

This sys of lin eqns corresponds to the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \begin{array}{l} R_2' = R_2 - R_1 \\ \\ \\ R_4' = R_4 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \quad R_2' = \tfrac{1}{3} R_2$$

$$\begin{array}{cccc} B & B & F & \\ c_1 & c_2 & c_3 & \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longleftarrow \text{RREF matrix}$$

$$\begin{array}{c} \begin{array}{ccc|c} B & B & F & \\ c_1 & c_2 & c_3 & \end{array} \\ \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \longleftarrow \text{RREF matrix}$$

Corresponds to equations:

$$c_1 + 2c_3 = 0$$
$$c_2 + c_3 = 0$$
$$0 = 0$$
$$0 = 0$$

Let $c_3 = t$. $\longrightarrow$ $\boxed{c_1 = -2t,\ c_2 = -t \text{ for all } t \in \mathbb{R}.}$

True for all $t \in \mathbb{R}$, so just pick a valid value for t.

$c_1 = -2,\ c_2 = -1,\ c_3 = 1.$

$\downarrow$

Plug in and get $-2\vec{v} - \vec{v}_2 + \vec{v}_3 = \vec{0}$, eqn holds. Found single dependance rltn

Pick a value for $t \in \mathbb{R}$ and evaluate c_1, c_2, and c_3.

* Don't pick all zeroes for your free variables.

If you pick all zeroes you will get "trivial eqn" (not dependance rltn).

No free variables tells you the vectors are linearly independant.

RR Algo $\rightarrow$ no free $\rightarrow$ linear independance

RR Algo $\rightarrow$ some free vars $\rightarrow$ linear dependance

Thms

1) A set of 2 vectors $\{v_1, v_2\}$ is linearly dependant iff they are scalar multiples of each other. $\rightarrow$ $cv_1 = v_2$ for a constant c.

2) Any set of vectors containing $\vec{0}$ is linearly dependant.

3) A set of vectors is linearly dependant iff at least one of the vectors in the set can be expressed as a linear combination of the other vectors

4) If a set of vectors has more vectors than the length of those vectors, then the set is linearly dependant

$\{\vec{v}_1, \dots, \vec{v}_k\}$, $\vec{v}_i \in \mathbb{R}^k$ if $j > k$ then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependant.

The reverse is not true. You can have linearly dependant with $j \leq k$, just not guaranteed.

$j > k$: linear dependance

$j < k$: do more work

given 3 vectors
find dependance rltn

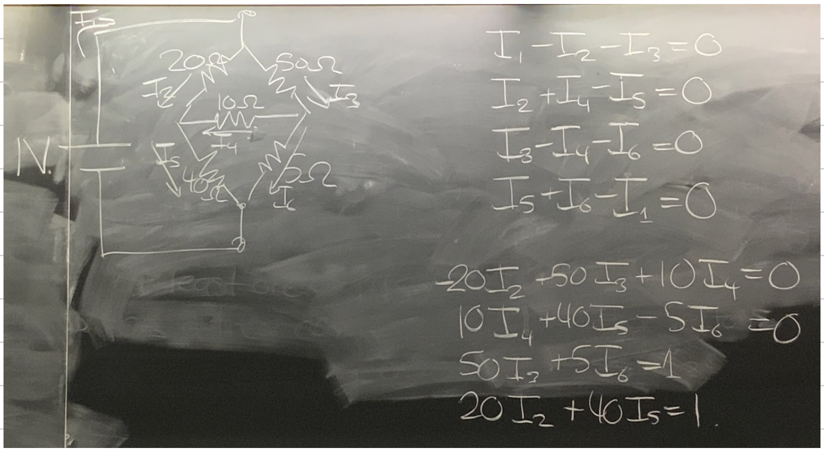
know all props.

End of test 1 material

The rest of this lecture will not be tested.

Application of linear systems

kirschov's law - uses lin alg



$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \begin{array}{l} \\ R_2' = R_2 - R_1 \\ \\ R_4' = R_4 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \quad R_2' = \frac{1}{3} R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 \end{array}\right] \begin{array}{l} R_1' = R_1 + 2R_2 \\ \\ R_3' = R_3 - 2R_2 \\ R_4' = R_4 + 3R_2 \end{array}$$