20 - Mar 22 Lecture

- Finding solutions to equations over complex numbers
- 16 angles in radians
- De Moivre's Theorem
- Finding complex number z given z^k
- Complex eigenvectors and eigenvalues

Solving eqns over Complex Numbers

Ex: Find z.

$(1+i)z + |z|^2 = 2 + \mathrm{Re}(z)i$

① Replace $z = a+bi$, $a, b \in \mathbb{R}$

can't split this up into real and imaginary before multiplying

since $(a+bi)^2 = a^2+b^2$

$(1+i)(a+bi) + (a^2+b^2) = 2 + ai$

since $\mathrm{Re}(a+bi) = a$

since $bi^2 = b(-1) = -b$

$a + bi + \cancel{ai} - b + a^2 + b^2 = 2 + \cancel{ai}$

$a + bi - b + a^2 + b^2 = 2$

2 complex numbers are equal if and only if their real components are the same and their imaginary components are the same.

② Need to Split into real and imaginary components

$a - b + a^2 + b^2 = 2$ and $bi = 0i$

$a - (0) + a^2 + (0)^2 = 2$ $\Rightarrow$ $b = 0$, sub into first eqn

$a + a^2 = 2$

$a^2 + a - 2 = 0$

$(a+2)(a-1) = 0$

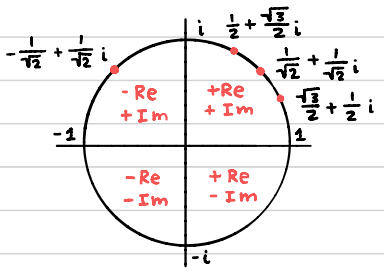
$\downarrow$ $\downarrow$

$a = -2$ $a = 1$

$\Rightarrow$ $z_1 = -2 + 0i = -2$

$z_2 = 1 + 0i = 1$

Expected to know 16 angles in radians.



Angle for i is $\frac{\pi}{2}$

Angle for 1 is 0

Angle for $-i$ is $\frac{3\pi}{2}$ or $-\frac{\pi}{2}$

Angle for $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ is $\frac{\pi}{6}$ $(30°)$

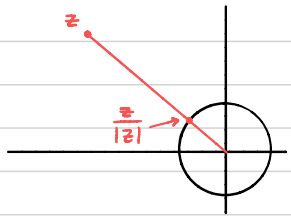
Angle for $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ is $\frac{\pi}{4}$ $(45°)$

Angle for $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is $\frac{\pi}{3}$ $(60°)$

$\cos\theta + i\sin\theta$

$\uparrow$ real component $\quad$ $\uparrow$ imaginary component

To find the point on the unit circle with the same angle as a complex number z, calculate $\boxed{\frac{z}{|z|}}$.



$z = -2+2i$

$$\frac{z}{|z|} = \frac{-2+2i}{\sqrt{(-2)^2+(2)^2}} = \frac{-2+2i}{\sqrt{4+4}} = \frac{-2+2i}{\sqrt{8}} = \frac{-2+2i}{2\sqrt{2}} = \frac{-1+i}{\sqrt{2}} = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

Logic Behind De Moivre's Theorem

$z_1 = |z_1|(\cos\phi_1 + i\sin\phi_1)$, $\quad z_2 = |z_2|(\cos\phi_2 + i\sin\phi_2)$

$z_1 z_2 = |z_1||z_2|[\cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2)]$

$z_1 z_2 \ldots z_k = |z_1|\ldots|z_k|[\cos(\phi_1+\ldots+\phi_k) + i\sin(\phi_1+\ldots+\phi_k)]$

What if $z_1 = z_2 = \ldots = z_k$?

$(z_1)^k = |z_1|^k[\cos(k\phi_1) + i\sin(k\phi_1)]$

De Moivre's Theorem

$(\cos\phi + i\sin\phi)^k = \cos(k\phi) + i\sin(k\phi)$ for k a positive integer

i^k for k positive integer

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ so $i^k = [\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}]^k$

$= \cos(k\frac{\pi}{2}) + i\sin(k\frac{\pi}{2})$

$i^1 = i$, $\quad i^2 = \cos(\pi) + i\sin(\pi) = -1$

$i^3 = \cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}) = -i$

$i^4 = \cos(2\pi) + i\sin(2\pi) = 1$

$$\begin{aligned} i^{2023} &= \cos\left(\frac{2023\pi}{2}\right) + i\sin\left(\frac{2023\pi}{2}\right) \\ &= \cos\left(\frac{2023\pi}{2} - 505(2\pi)\right) + i\sin\left(\frac{2023\pi}{2} - 505(2\pi)\right) \\ &= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) \\ &= -i \end{aligned}$$

505 rotations

360° rotation

want to get to an angle between 0 and 2π or $-\pi$ and π

$-506(2\pi) \longrightarrow \frac{-\pi}{2}$

cos and sin are cyclic.

$\pi + 2\pi$ = angle of π

How to find z given z^k

Ex: $z^3 = i$, then $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ \frac{-\sqrt{3}}{2} + \frac{1}{2}i,\ -i$

3 solutions

$|r| = |z^k|$

$z^k = |r|(\cos\phi + i\sin\phi) \Longrightarrow z_j = \sqrt[k]{|z|}\left[\cos\left(\frac{\phi}{k} + \frac{2j\pi}{k}\right) + i\sin\left(\frac{\phi}{k} + \frac{2j\pi}{k}\right)\right]$ for $j = 0, \ldots, k-1$

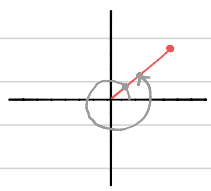
Will be k solutions.

$j = 0 \rightarrow$ 1st root

$j = 1 \rightarrow$ 2nd root

$j = 2 \rightarrow$ 3rd root

The $\frac{\phi}{k}$ part and $\sqrt{|r|}$ are just the reverse of exponentiation.



Ex Find z such that $z^3 = i$ and $i = 1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$z_j = \sqrt[3]{1}\left(\cos\left(\frac{\pi}{6} + \frac{2j\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2j\pi}{3}\right)\right)$ for $j = 0, 1, 2$

Find the 3 solutions by substituting $j = 0, 1, 2$.

$$z_0 = \cos\left(\frac{\pi}{6} + \frac{2(0)\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2(0)\pi}{3}\right)$$
$$= \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$z_1 = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2} + \frac{1}{2}i$$

$$z_2 = \cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right) = -i$$

1 MCQ on this will be on test 4

Complex Eigenvalues / Eigenvectors

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ← 90° rotation matrix

Characteristic Polynomial:

$\text{Det}(A - \lambda I)$

$= \text{Det}\begin{bmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{bmatrix}$

$= \text{Det}\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

$= (-\lambda)(-\lambda) - (-1)(1)$

$= \lambda^2 + 1$ ← characteristic poly

$= (\lambda + i)(\lambda - i)$ $\quad \lambda_1 = i \quad \lambda_2 = -i$ are eigenvalues

eigenvalues always exist over $\mathbb{C}$

Note: don't try to visualize complex eigenvalues / eigenvectors

Ex: $B = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$ $\quad \text{Det}\left(\begin{bmatrix} 1-\lambda & -2 \\ 4 & -3-\lambda \end{bmatrix}\right) = \lambda^2 + 2\lambda + 5$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = -1 \pm \frac{\sqrt{-16}}{2} = -1 \pm \frac{\sqrt{16}}{2}\sqrt{-1}$$

Practically, you can set $\sqrt{-1} = i$ in 99.99% of situations (not actually mathimatically correct but it works)

$= -1 \pm 2i$ $\quad \lambda_1 = -1 + 2i$

$\lambda_2 = -1 - 2i$

<u>steps</u>

1. charact. poly
2. quad. formula
3. ↳ find solutions