You should be able to comment! Feel free to leave a comment here with any textbook questions you would like the solutions for. Please include the question number and page number! If you have questions, email me at belwang@cmail.carleton.ca :)

Contents

MATH 1009 Tutorial 5 Notes
Mock Test

Formulas

Exponent and Logarithm Rules

Exponent Rules	Logarithm Rules	
$a^x \cdot a^y = a^{x+y}$ $\frac{a^x}{a^y} = a^{x-y}$ and $a^{-y} = \frac{1}{a^y}$ $(a^x)^y = a^{xy}$ $(ab)^x = a^x b^x$ $(\frac{a}{b})^x = \frac{a^x}{b^x}$ $a^{x/y} = \sqrt[y]{a^x}$ $a^0 = 1$	$x = log_a y \Leftrightarrow y = a^x$ $(if\ y > 0)$ $log_a(xy) = log_a x + log_a y$ $log_a(\frac{x}{y}) = log_a x - log_a y$ $ylog_a x = log_a x^y$ $log_a b = \frac{log_c b}{log_c a}$, if $c = e$ then $log_a b = \frac{ln\, b}{ln\, a}$ $log_b a = \frac{1}{log_a b}$ $a^{log_a x} = x\ (if\ x > 0)$ $log_a a^x = x$	euler's number, $e \simeq 2.71828$ $log = log_{10}$ $ln = log_e$ $log_a 1 = 0$ $log_a a = 1$

Instantaneous rate of change of f at x: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Average rate of change of f with respect to x over $[x, x + h]$: $f'(x) = \lim_{h \to h} \frac{f(x+h)-f(x)}{h}$

$\frac{dy}{dx}|_{(a,\, b)}$ denotes the value of $\frac{dy}{dx}$ at (a, b)

Equation of the tangent line to the curve $y = y(x)$ at the point (x_0, y_0) is given by:

$y - y_0 = y'(x_0)\,(x - x_0)$

Elasticity of Demand: $E(p) = -\frac{p\, f'(p)}{f(p)}$

- Demand is **elastic** at p_0 when $E(p_0) > 1$.
  - small increase in price ⇔ bigger decrease in revenue
- Demand is **unitary** at p_0 when $E(p_0) = 1$.
  - small increase in price ⇔ same increase in revenue (1:1 ratio)
- Demand is **inelastic** at p_0 when $E(p_0) < 1$.
  - small increase in price ⇔ smaller increase in revenue

Implicit differentiation:
Step ①: Replace y with $y(x)$.
Step ②: Differentiate both sides.
Step ③: Isolate for $\frac{dy}{dx}$.

What's the difference between $\frac{d}{dx}$ and $\frac{dy}{dx}$? Think of $\frac{d}{dx}$ like a verb, it is telling you to find the derivative. $\frac{dy}{dx}$ is like a noun, it is the result after taking the derivative.

Rules of Differentiation

Derivative of a Constant	$\frac{d}{dx}c = 0$
Power Rule	$\frac{d}{dx}(x^r) = r\,x^{r-1}$
Derivative of a Constant Multiple of a Function	$\frac{d}{dx}[c\,f(x)] = c\,\frac{d}{dx}f(x)$
Sum Rule	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$
Derivative of Exponential Function	$\frac{d}{dx}(a^x) = a^x ln\, a$ $\frac{d}{dx}e^x = e^x ln\, e = e^x$
Derivative of Logarithmic Function	$\frac{d}{dx}log_a\,\|x\| = \frac{1}{x\,ln\,a}$ when $x \neq 0$ $\frac{d}{dx}ln\,\|x\| = \frac{1}{x\,ln\,e} = \frac{1}{x}$ when $x \neq 0$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$
Quotient Rule	$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$ when $g(x) \neq 0$
Chain Rule (for composite functions) $y(x) = f(g(x))$	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$ If you decompose $y(x)$ into $g(x) = u$ and $f(u)$, then: $\frac{d}{dx}[f(g(x))] = f'(u) \cdot g'(x)$

Applications of Chain Rule

Chain Rule for Power functions $y(x) = [f(x)]^n$	$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x)$
Chain Rule for Exponential functions $y(x) = a^{f(x)}$ where $a > 0,\ a \neq 1$	$\frac{d}{dx}a^{f(x)} = a^{f(x)} \cdot ln\,a \cdot f'(x)$ $\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'(x)$
Chain Rule for Logarithmic functions $y(x) = log_a f(x)$ where $a > 0,\ a \neq 1$	$\frac{d}{dx}log_a\,y(x) = \frac{1}{f(x) \cdot ln\,a} \cdot y'(x) = \frac{f'(x)}{f(x) \cdot ln\,a}$ $\frac{d}{dx}ln\,f(x) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$

Function is **increasing** when $f'(x) > 0$, and is **decreasing** when $f'(x) < 0$.

Critical points: x values where $f'(x) = 0$
x values where $f'(x)$ not defined (IF AND ONLY IF $f(x)$ IS DEFINED at x)

Exercises 2.3 (pg. 50)

18. John is building a rectangular herb planter. He wishes to fence the planter in order to protect the plants from squirrels. With 12 meters of fencing and the area of the planter 8 square meters, what are the dimensions of the planter?

Solution:

Let's start with the equations for the area and perimeter of a rectangle.

$Area = length \times width \quad \rightarrow \quad A = l \cdot w$

$perimeter = length + length + width + width \quad \rightarrow \quad p = 2l + 2w$

We will sub in the values $A = 8$ and $p = 12$, as given in the question.

$A = l \cdot w$	$p = 2l + 2w$
$(8) = l \cdot w$	$(12) = 2l + 2w$
	$6 = l + w$

This gives us the following system of linear equations:

$$\begin{cases} 8 = l \cdot w & ① \\ 6 = l + w & ② \end{cases}$$

We will use the method of substitution to solve for l and w.

Isolate l in equation ②

$6 = l + w$

$6 - w = l$

$\therefore l = 6 - w$

Substitute $l = 6 - w$ into equation ① to solve for w

$8 = l \cdot w$

$8 = (6 - w) \cdot w$

$8 = 6w - w^2$

$0 = - w^2 + 6w - 8$

Use the quadratic equation to solve for value(s) of w

From above, $a = -1$, $b = 6$, $c = -8$.

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(6) \pm \sqrt{(6)^2 - 4(-1)(-8)}}{2(-1)} = \frac{-6 \pm \sqrt{36-32}}{-2} = \frac{-6 \pm \sqrt{4}}{-2} = \frac{-6 \pm 2}{-2}$$

We need to consider both cases, and reject any negative values of w.

Case 1: addition	Case 2: subtraction
$w = \frac{-6+2}{-2} = \frac{-4}{-2} = 2$ (meters)	$w = \frac{-6-2}{-2} = \frac{-8}{-2} = 4$ (meters)

Both solutions are valid! Why? Well, a rectangle's width becomes its length if you rotate it 90 degrees. So, the width of the rectangle can be 2 meters (with a length of 4 meters) or 4 meters (with a length of 2 meters)!
∴ **The rectangle's dimensions are 2 meters by 4 meters.**

We can double check this!
$A = 2 \times 4 = 8$ meters ✔️
$p = 2 + 2 + 4 + 4 = 12$ meters ✔️

Note that you will get the same result if you:
<u>Isolate l in equation ①</u>
$8 = l \cdot w$
$\frac{8}{w} = l$
$\therefore l = \frac{8}{w}$

$6 = l + w$
$6 - w = l$
$\therefore l = 6 - w$

<u>Substitute $l = \frac{8}{w}$ into equation ② to solve for w</u>
$6 = l + w$
$6 = (\frac{8}{w}) + w$
$6w = 8 + w^2$ ← by multiplying both sides by w
$0 = w^2 - 6w + 8$ ← if you divide this equation by -1, you get the result from above.
$0 = - w^2 + 6w - 8$ ← This is the same equation as the first solution, I won't go further but hopefully you can see how this will get you the same answer.

Of course, you can choose to isolate w initially instead. That should give you the same answer as well! 🙂

22. Approximately 3,900 passengers per day take city buses and pay a fare of $2.75 per ride. The city considers raising the fare to $3.25 per ride to generate a larger revenue. The Study reveals that, for each $0.25 increase in fare, the number of passengers will be reduced by 300 persons per day. Find the fare which yields the maximum revenue. Based on your computations, will you recommend that the city increase the fare to $3.25?

Solution:
Let x represent the number of $0.25 increases to the fare. Then, the fare price is $(2.75 + 0.25x)$ and the number of passengers is $(3900 - 300x)$.

The revenue is given by the product of the fare price and the number of passengers:

$$\begin{aligned} R(x) &= (2.75 + 0.25x)(3900 - 300x) \\ &= 10725 - 825x + 975x - 75x^2 \\ &= -75x^2 + 150x + 10725 \end{aligned}$$

The graph of $R(x)$ is a parabola which opens downward. The maximum of $R(x)$ is found at the vertex of the parabola.

The x-coordinate of the vertex is given by:

$x = -\frac{b}{2a} = -\frac{150}{2(-75)} = -\frac{150}{-150} = 1$ (increase)

This tells us that the maximum revenue occurs with one $0.25 increase to the fare. Thus, the fare price should be:

$p(1) = 2.75 + 0.25(1) = 3$ (dollars)

∴ **Since a fare of $3 yields the maximum revenue, I do not recommend that the city increases the fare to $3.25.**

23. Let the cost (in dollars) of producing x items be C(x) = $-0.1x^2 + 10x + 3.1$, where x is in units of a hundred. How many items must be produced in order for the average cost of production to be 7 dollars?

Solution:
From Example 21 of Section 2.3 (page 50): the average cost function $AC(x)$ is given by:

$$AC(x) = \frac{C(x)}{x}$$
$$= \frac{-0.1x^2+10x+3.1}{x}$$
$$= \frac{-0.1x^2}{x} + \frac{10x}{x} + \frac{3.1}{x}$$
$$= -0.1x + 10 + \frac{3.1}{x} \text{ (dollars per } x \text{ hundred items)}$$

We want the average cost to be 7 dollars, so $AC(x) = 7$ (dollars per x hundred items).
Set $AC(x) = 7$ equal to $AC(x) = -0.1x + 10 + \frac{3.1}{x}$:

$$7 = -0.1x + 10 + \frac{3.1}{x}$$
$$7x = -0.1x^2 + 10x + 3.1 \text{ (by multiplying both sides by } x\text{)}$$
$$0 = -0.1x^2 + 3x + 3.1$$
$$0 = -x^2 + 30x + 31 \text{ (by multiplying both sides by 10, so all coefficients are integers since we are using the quadratic formula)}$$

Using the quadratic formula, we will solve for x.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \text{ where } a = -1,\ b = 30,\ c = 31$$
$$= \frac{-(30) \pm \sqrt{(30)^2-4(-1)(31)}}{2(-1)}$$
$$= \frac{-30 \pm \sqrt{900+124}}{-2}$$
$$= \frac{-30 \pm \sqrt{1024}}{-2}$$
$$= \frac{-30 \pm 32}{-2}$$

Addition	Subtraction
$= \frac{-30+32}{-2} = \frac{2}{-2} = -1$ units of a hundred REJECT NEGATIVE! ❌	$= \frac{-30-32}{-2} = \frac{-62}{-2} = 31$ units of a hundred We accept this answer. ✅

∴ **The average cost of production will be 7 dollars when <u>31 hundred items</u> are produced.**

Note: Another way to approach the problem is to realize that if the average cost is 7 dollars, the total cost to produce x hundred items is $7x$. Therefore, the cost function is $C(x) = 7x$. If you set $C(x) = 7x$ equal to the cost function we're given, $C(x) = -0.1x^2 + 10x + 3.1$, we get: $7x = -0.1x^2 + 10x + 3.1$ which is the equation highlighted in pink above. Hopefully you can see how this will lead to the same result.

Exercises 2.5 (pg. 67)

24. Expand and simplify $log10x(x^2 + 2)^{-3}$**.**

$= log(10) + log(x) + log(x^2 + 2)^{-3}$ since $log_a xyz = log_a x + log_a y + log_a z$

$= log_{10}(10) + log(x) + (-3)log(x^2 + 2)$ since log with unspecified base is log_{10}

since $log_a(x^y) = ylog_a x$

$= 1 + log\, x - 3log(x^2 + 2)$ since $log_a a = 1$

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25. Expand and simplify $ln\frac{\sqrt{x}}{x^2(1+x)}$**.**

$= ln\sqrt{x} - ln(x^2(1 + x))$ since $log_a(\frac{x}{y}) = log_a x - log_a y$

$= ln\, x^{\frac{1}{2}} - [ln\, x^2 + ln(1 + x)]$ since $\sqrt[n]{x} = x^{\frac{1}{n}}$

since $log_a(xy) = log_a x + log_a y$

$= \frac{1}{2}ln\, x - [2ln\, x + ln(1 + x)]$ since $log_a(x^y) = ylog_a x$

$= \frac{1}{2}ln\, x - 2ln\, x - ln(1 + x)]$

$= \frac{1}{2}ln\, x - \frac{4}{2}ln\, x - ln(1 + x)]$ since $2 = \frac{4}{2}$

$= -\frac{3}{2}ln\, x - ln(1 + x)]$ by combining like terms

---

26. Expand and simplify $ln(x\, e^{-2x})$**.**

$= ln(x) + ln(e^{-2x})$ since $log_a(xy) = log_a x + log_a y$

$= ln\, x + (-2x)ln\, e$ since $ylog_a(x) = log_a(x^y)$

$= ln\, x - 2xlog_e e$ since $ln = log_e$

$= ln\, x - 2x(1)$ since $log_a a = 1$

$= ln\, x - 2x$

---

27. Expand and simplify $ln\,\frac{e^{x^2}}{1+e^{-x}}$**.**

$= ln(e^{x^2}) - ln(1 + e^{-x})$ since $log_a(\frac{x}{y}) = log_a x - log_a y$

$= x^2 ln\, e - ln(1 + e^{-x})$ since $ylog_a(x) = log_a(x^y)$

$= x^2 log_e\, e - ln(1 + e^{-x})$ since $ln = log_e$

$= x^2(1) - ln(1 + e^{-x})$ since $log_a a = 1$

$= x^2 - ln(1 + e^{-x})$

---

28. Simplify $e^{2 + ln\,x}$.

$= e^2 \cdot e^{ln\,x}$ since $a^{x+y} = a^x \cdot a^y$

$= e^2 \cdot e^{log_e x}$ since $ln = log_e$

$= e^2 \cdot x$ since $a^{log_a x} = x$

$= \boxed{x\,e^2}$ to match the textbook solution (can stop at previous step)

---

29. Simplify $2^{-log_2 1}$.

$= 2^{log_2(1^{-1})}$ since $y log_a x = log_a(x^y)$

$= 1^{-1}$ since $a^{log_a x} = x$

$= \frac{1}{1^1}$

$= \boxed{1}$

---

30. Simplify $3^{log_3 x^2}$.

$= \boxed{x^2}$ since $a^{log_a x} = x$

---

31. Simplify $e^{-2 ln\,5}$.

$= e^{ln(5^{-2})}$ since $y log_a x = log_a(x^y)$

$= e^{log_e(5^{-2})}$ since $ln = log_e$

$= 5^{-2}$ since $a^{log_a x} = x$

$= \frac{1}{5^2}$ since $a^{-y} = \frac{1}{a^y}$

$= \boxed{\frac{1}{25}}$

---

32. Solve $e^{2x} = 4$ for x, if possible.

We have been given an equation in the form $a^x = y$. Recall that if $a^x = y$, then $x = log_a y$.

In this question, $a = e$, $x = 2x$, and $y = 4$. Let's substitute these values into $x = log_a y$.

$\Rightarrow 2x = log_e 4$ since $a^x = y \Leftrightarrow x = log_a y$

$\Rightarrow 2x = ln\,4$ since $log_e = ln$

$\Rightarrow x = \boxed{\frac{ln\,4}{2}}$ by dividing both sides by 2

---

33. Solve $3^{x-1} = 6$ for x, if possible.

We have been given an equation in the form $a^x = y$. Recall that if $a^x = y$, then $x = log_a y$.

In this question, $a = 3$, $x = x - 1$, and $y = 6$. Let's substitute these values into $x = log_a y$.

$\Rightarrow x - 1 = log_3 6$

Simplify by making x "match the base" and then using the property $log_a a = 1$.

$\Rightarrow x - 1 = log_3(3 \cdot 2)$ since $a^x = y \Leftrightarrow x = log_a y$

$\Rightarrow x - 1 = log_3 3 + log_3 2$ since $log_a(xy) = log_a x + log_a y$

$\Rightarrow x - 1 = 1 + log_3 2$ since $log_a a = 1$

$\Rightarrow x = 2 + log_3 2$ by adding 1 to each side

---

34. Solve $ln\, 5x = 2$ for x, if possible.

$\Rightarrow 5x = e^2$ since $log_a y = x \Leftrightarrow y = a^x$ $(a = e, y = 5x, x = 2)$

$\Rightarrow x = \frac{e^2}{5}$ by dividing both sides by 5

---

35. Solve $ln(x^2 + 1) = 0$ for x, if possible.

$\Rightarrow log_e(x^2 + 1) = 0$ since $ln = log_e$

We need to know that $log_a 1 = 0$. In this example, $a = e$ but this applies to logarithms with any base. The only way to have a logarithm equal 0 is if x in $log_a x$ equals 1. Since the logarithm equals 0, we can conclude that $(x^2 + 1)$ must equal 1.

$\Rightarrow x^2 + 1 = 1$ since $log_a 1 = 0$

$\Rightarrow x^2 = 0$ by subtracting 1 from both sides

$\Rightarrow x = 0$ since $x^n = 0 \Rightarrow x = 0$

---

36. Solve $3e^{0.2x} - 1 = 5$ for x, if possible.

$\Rightarrow 3e^{0.2x} = 6$ by adding 1 to each side

$\Rightarrow e^{0.2x} = 2$ by dividing both sides by 3

Same process as Question 33. If $a^x = y$, then $x = log_a y$. $(a = e, x = 0.2x, y = 2)$

$\Rightarrow 0.2x = log_e 2$

$\Rightarrow 0.2x = ln\, 2$ since $log_e = ln$

$\Rightarrow x = 5ln\, 2$ by multiplying both sides by 5

---

37. **Solve $\frac{3}{1+e^{x/2}} = 10$ for x, if possible.**

$\Rightarrow 3 = 10(1 + e^{x/2})$ by multiplying both sides by $1 + e^{x/2}$

$\Rightarrow 3 = 10(1) + 10(e^{x/2})$ using the distributive property

$\Rightarrow 3 = 10 + 10e^{x/2}$

$\Rightarrow -7 = 10e^{x/2}$

$\Rightarrow -\frac{7}{10} = e^{x/2}$

At this point, we can conclude that there is no solution. This is because solving for x would lead to trying to find the value of $log_a x$ when $x \leq 0$, and this is undefined.

We will try to apply the formula $y = a^x \Leftrightarrow log_a y = x$, so we can see this for ourselves. Note that this formula has the restriction $y > 0$, for this exact reason.

$\Rightarrow log_e(-\frac{7}{10}) = \frac{x}{2}$

As stated above, the logarithm of a negative value is undefined. Therefore, there is no solution (we cannot solve for x).

Exercises 4.1 (pg. 97)

2. Find the slope of the tangent line to the graph of $f(x) = \frac{1}{x}$ at $(2, \frac{1}{2})$. Write the equation of the tangent line at $(2, \frac{1}{2})$.

Step ①: Find slope by finding $f'(x)$.

$f(x) = x^{-1}$

$f'(x) = -1x^{-1-1}$

$f'(x) = -x^{-2}$

$f'(x) = -\frac{1}{x^2}$

Step ②: Calculate value of $f'(2)$.

$f'(2) = -\frac{1}{(2)^2}$

$f'(2) = -\frac{1}{4}$

∴ Slope of the tangent line to the graph of $f(x) = \frac{1}{x}$ at $(2, \frac{1}{2})$ is $-\frac{1}{4}$.

Step ③: Find equation of the tangent line by plugging values into $y - y_0 = y'(x_0)(x - x_0)$.

$y - \frac{1}{2} = (-\frac{1}{4})(x - 2)$

$y - \frac{1}{2} = -\frac{1}{4}x + \frac{2}{4}$

$y = -\frac{1}{4}x + \frac{1}{2} + \frac{1}{2}$

$y = -\frac{1}{4}x + 1$

∴ The equation of the tangent line at $(2, \frac{1}{2})$ is $y = -\frac{1}{4}x + 1$.

3. Suppose that a car is moving along a straight road, and the distance travelled t hours after starting from rest is given by $f(t) = 2t^2 + 60t$. Find the average velocity of the car over the time intervals $[0, 2]$ and $[1, 2]$. Find the instantaneous velocity at $t = 1$ and $t = 2$.

Recall that average rate of change of f with respect to x over $[x, x + h]$ is given by…

$f'(x) = \lim_{h \to h} \frac{f(x+h)-f(x)}{h}$

… and that instantaneous rate of change of f at x is given by…

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

a) Average velocity over $[0, 2]$ ($t = 0$ and $h = 2$)

$f'(t) = \lim_{h \to h} \frac{f(t+h)-f(t)}{h}$

$f'(t) = \lim_{h \to 2} \frac{[2(t+h)^2+60(t+h)]-[2t^2+60t]}{h}$ by subbing in $f(t) = 2t^2 + 60t$

$f'(0) = \frac{[2(0+2)^2+60(0+2)]-[2(0)^2+60(0)]}{2}$

$f'(0) = \frac{[2(2)^2+60(2)]-[0+0]}{2}$

$f'(0) = \frac{2(4)+120}{2}$

$f'(0) = \frac{8+120}{2}$

$f'(0) = \frac{128}{2}$

$f'(0) = 64$

b) Average velocity over $[1, 2]$ ($t = 1$ and $h = 1$)

$f'(t) = \lim_{h \to h} \frac{f(t+h)-f(t)}{h}$

$f'(t) = \lim_{h \to 1} \frac{[2(t+h)^2+60(t+h)]-[2t^2+60t]}{h}$

$f'(1) = \frac{[2(1+1)^2+60(1+1)]-[2(1)^2+60(1)]}{1}$

$f'(1) = [2(2)^2 + 60(2)] - [2 + 60]$

$f'(1) = [2(4) + 120] - [62]$

$f'(1) = 128 - 62$

$f'(1) = 66$

c) Instantaneous velocity at $t = 1$

$$f'(t) = \lim_{h \to 0} \frac{f(t+h)-f(t)}{h}$$

$$f'(t) = \lim_{h \to 0} \frac{[2(t+h)^2+60(t+h)]-[2t^2+60t]}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{[2(1+h)^2+60(1+h)]-[2(1)^2+60(1)]}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{[2(1^2+2h+h^2)+60+60h)]-[2+60]}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{2+4h+2h^2+60+60h-62}{h}$$

$$f'(1) = \lim_{h \to 0} \frac{64h+2h^2}{h}$$

$$f'(1) = \lim_{h \to 0} 64 + 2h$$

$$f'(1) = 64 + 2(0)$$

$$f'(1) = 64$$

d) Instantaneous velocity at $t = 2$

$$f'(t) = \lim_{h \to 0} \frac{f(t+h)-f(t)}{h}$$

$$f'(t) = \lim_{h \to 0} \frac{[2(t+h)^2+60(t+h)]-[2t^2+60t]}{h}$$

$$f'(2) = \lim_{h \to 0} \frac{[2(2+h)^2+60(2+h)]-[2(2)^2+60(2)]}{h}$$

$$f'(2) = \lim_{h \to 0} \frac{[2(2^2+4h+2h^2)+120+60h)]-[8+120]}{h}$$

$$f'(2) = \lim_{h \to 0} \frac{8+8h+4h^2+120+60h-128}{h}$$

$$f'(2) = \lim_{h \to 0} \frac{68h+4h^2}{h}$$

$$f'(2) = \lim_{h \to 0} 68 + 4h$$

$$f'(2) = 68 + 4(0)$$

$$f'(2) = 68$$

Exercises 4.2 (pg. 105)

6. Find the derivative of $f(x) = \frac{x^4+2x^2-x+3}{x}$.

$f'(x) = \frac{(x^4+2x^2-x+3)'(x)-(x^4+2x^2-x+3)(x)'}{x^2}$

$f'(x) = \frac{(4x^3+4x-1)(x)-(x^4+2x^2-x+3)(1)}{x^2}$

$f'(x) = \frac{(4x^4+4x^2-x)-(x^4+2x^2-x+3)}{x^2}$

$f'(x) = \frac{3x^4+2x^2-3}{x^2}$

$f'(x) = \frac{3x^4}{x^2} + \frac{2x^2}{x^2} - \frac{3}{x^2}$

$f'(x) = 3x^{4-2} + 2x^{2-2} - 3x^{-2}$

$f'(x) = 3x^2 + 2(1) - 3x^{-2}$

$f'(x) = 3x^2 + 2 - 3x^{-2}$

8. Find the derivative of $f(x) = \frac{1}{2}x^2 + (x^3 - 2)(x^2 + \frac{3}{x}) + 115$**.**

$f(x) = \frac{1}{2}x^2 + (x^3 - 2)(x^2 + 3x^{-2}) + 115$ change $\frac{3}{x}$ to $3x^2$

$f'(x) = \frac{1}{2}(2)x^{2-1} + (x^3 - 2)'(x^2 + 3x^{-1}) + (x^3 - 2)(x^2 + 3x^{-1})' + 0$

$f'(x) = x + ((3)x^{3-1} - 0)(x^2 + 3x^{-1}) + (x^3 - 2)((2)x^{2-1} + (-1)3x^{-1-1})$

$f'(x) = x + (3x^2)(x^2 + 3x^{-1}) + (x^3 - 2)(2x^1 - 3x^{-2})$

$f'(x) = x + (3x^4 + 9x^1) + 2x^4 - 3x^1 - 4x + 6x^{-2}$

$f'(x) = 5x^4 + 3x + 6x^{-2}$

$f'(x) = 5x^4 + 3x + \frac{6}{x^2}$

12. Find the equation of the tangent line to the graph of $f(x) = \frac{2x}{x^2+3}$ at $(-1, -\frac{1}{2})$. Determine the point(s) on the graph where the tangent line is horizontal.

Step ①: Find the derivative of $f(x) = \frac{2x}{x^2+3}$ using the Quotient Rule.

$$f'(x) = \frac{(2x)'(x^2+3)-(2x)(x^2+3)'}{(x^2+3)^2}$$

$$f'(x) = \frac{(2)(x^2+3)-(2x)(2x^1)}{(x^2+3)^2}$$

$$f'(x) = \frac{2x^2+6-4x^2}{(x^2+3)^2}$$

$$f'(x) = \frac{-2x^2+6}{(x^2+3)^2}$$

Step ②: Find the slope of the tangent line by evaluating $\frac{dy}{dx}$ at $(-1, -\frac{1}{2})$.

$$\frac{dy}{dx}\Big|_{(-1,-1/2)} = \frac{-2(-1)^2+6}{((-1)^2+3)^2} = \frac{-2(1)+6}{(1+3)^2} = \frac{-2+6}{(4)^2} = \frac{4}{16} = \frac{1}{4}$$

Step ③: Plug values into $y - y_0 = y'(x_0)(x - x_0)$.

$x_0 = -1,\ y_0 = -\frac{1}{2},\ f'(x_0) = \frac{1}{4}$

$y - (-\frac{1}{2}) = \frac{1}{4}(x - (-1))$

$y + \frac{1}{2} = \frac{1}{4}(x + 1)$

$y + \frac{1}{2} = \frac{1}{4}x + \frac{1}{4}$

$y = \frac{1}{4}x - \frac{1}{4}$

Step ④: Find value(s) of x in $f'(x) = \frac{-2x^2+6}{(x^2+3)^2}$ where $y = 0$.

$$0 = \frac{-2x^2+6}{(x^2+3)^2}$$

$0 = -2x^2 + 6$ by multiplying both sides by $(x^2 + 3)^2$

$2x^2 = 6$

$x^2 = 3$

$x = \pm\sqrt{3}$

Step ⑤: Find values of y when $x = \pm\sqrt{3}$.

$$f'(\pm\sqrt{3}) = \frac{-2(\pm\sqrt{3})^2+6}{((\pm\sqrt{3})^2+3)^2} = \frac{-2(3)+6}{((3)+3)^2} = \frac{-6+6}{(6)^2} = \frac{0}{36} = 0$$

∴ The tangent line is horizontal at $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$.

Note: the textbook solution is $(\sqrt{3}, \frac{1}{\sqrt{3}})$ and $(-\sqrt{3}, -\frac{1}{\sqrt{3}})$ and I don't think it's correct.

26. Find $\frac{dy}{dx}$ by implicit differentiation.

$x^4 + xy + y^3 = x$

Step ①: Rewrite the equation, replacing y by $y(x)$.

$x^4 + x[y(x)] + [y(x)]^3 = x$

Step ②: Differentiate both sides of the equation with respect to x.

$\frac{d}{dx}(x^4 + x[y(x)] + [y(x)]^3) = \frac{d}{dx}(x)$

$\frac{d}{dx}(x^4) + \frac{d}{dx}(x[y(x)]) + \frac{d}{dx}([y(x)]^3) = \frac{d}{dx}(x)$

$4x^{4-1} + \frac{d}{dx}(x)[y(x)] + (x)\frac{d}{dx}[y(x)] + (3\frac{dy}{dx}[y(x)]^{3-1}) = (1)$

$4x^3 + (1 \cdot y) + (x \cdot 1\frac{dy}{dx}) + (3 \cdot y^2\frac{dy}{dx}) = 1$

$4x^3 + (y) + (x\frac{dy}{dx}) + (3y^2\frac{dy}{dx}) = 1$

Step ③: Solve for $\frac{dy}{dx}$.

$x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 1 - 4x^3 - y$

$\frac{dy}{dx}(x + 3y^2) = 1 - 4x^3 - y$

$\frac{dy}{dx} = \frac{1-4x^3-y}{x+3y^2}$

27. Find $\frac{dy}{dx}$ by implicit differentiation.

$x^2y + xy^2 = 1$

Step ①: Rewrite the equation, replacing y by $y(x)$.

$x^2[y(x)] + x[y(x)]^2 = 1$

Step ②: Differentiate both sides of the equation with respect to x.

$\frac{d}{dx}(x^2 \cdot y(x) + x \cdot y(x)^2) = \frac{d}{dx}(1)$

$\frac{d}{dx}(x^2 \cdot y(x)) + \frac{d}{dx}(x \cdot y(x)^2) = \frac{d}{dx}(1)$

$\frac{d}{dx}(x^2) \cdot y(x) + x^2 \cdot \frac{d}{dx}[y(x)] + \frac{d}{dx}(x) \cdot y(x)^2 + x \cdot \frac{d}{dx}([y(x)]^2) = \frac{d}{dx}(1)$

$2x^{2-1} \cdot y(x) + x^2 \cdot \frac{dy}{dx} + (1) \cdot y(x)^2 + x \cdot 2[y(x)]^{2-1} \cdot \frac{dy}{dx} = 0$

$2x \cdot y + x^2 \cdot \frac{dy}{dx} + y^2 + x \cdot 2y \cdot \frac{dy}{dx} = 0$

Step ③: Solve for $\frac{dy}{dx}$.

$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} =- 2xy - y^2$

$\frac{dy}{dx}(x^2 + 2xy) =- 2xy - y^2$

$\frac{dy}{dx} = \frac{-2xy-y^2}{x^2+2xy} =$

$\frac{dy}{dx} =- \frac{y^2+2xy}{x^2+2xy}$

28. Find $\frac{dy}{dx}$ by implicit differentiation.

$\sqrt{x + y} = x^2$

Step ①: Rewrite the equation, replacing y by $y(x)$.

$[x + y(x)]^{\frac{1}{2}} = x^2$

Step ②: Differentiate both sides of the equation with respect to x.

$\frac{d}{dx}([x + y(x)]^{\frac{1}{2}}) = \frac{d}{dx}(x^2)$

I will be using the Chain Rule to find $\frac{d}{dx}([x + y(x)]^{\frac{1}{2}})$.

$g(x) = x + y(x) = u$

$f(u) = u^{\frac{1}{2}}$

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= f'(u) \cdot g'(x)\\ &= \frac{d}{dx}(u^{\frac{1}{2}}) \cdot \frac{d}{dx}(x + y(x))\\ &= \frac{1}{2}u^{\frac{1}{2}-1} \cdot (1 + \frac{dy}{dx})\\ &= \frac{1}{2}(x + y)^{-\frac{1}{2}} \cdot (1 + \frac{dy}{dx})\\ &= \frac{1}{2\sqrt{x+y}} \cdot (1 + \frac{dy}{dx})\\ &= \frac{1 + \frac{dy}{dx}}{2\sqrt{x+y}}\end{aligned}$$

Putting this result back in the equation, we get:

$\frac{1 + \frac{dy}{dx}}{2\sqrt{x+y}} = 2x^{2-1}$

Step ③: Solve for $\frac{dy}{dx}$.

$1 + \frac{dy}{dx} = 2x \cdot 2\sqrt{x + y}$

$\frac{dy}{dx} = 4x\sqrt{x + y} - 1$

29. Find $\frac{dy}{dx}$ by implicit differentiation.

$e^{2y} - x^2 - xy = 0$

Step ①: Rewrite the equation, replacing y by $y(x)$.

$e^{2y} - x^2 - x \cdot y(x) = 0$

Step ②: Differentiate both sides of the equation with respect to x.

$\frac{d}{dx}(e^{2y} - x^2 - x \cdot y(x)) = \frac{d}{dx}(0)$

$\frac{d}{dx}(e^{2y}) - \frac{d}{dx}(x^2) - \frac{d}{dx}(x \cdot y(x)) = \frac{d}{dx}(0)$

$\frac{d}{dx}(e^{2y}) - \frac{d}{dx}(x^2) - [\frac{d}{dx}(x) \cdot y(x) + x \cdot \frac{d}{dx}(y(x))] = \frac{d}{dx}(0)$

$2e^{2y} \cdot \frac{dy}{dx} - 2x - [1 \cdot y(x) + x \cdot \frac{d}{dx}] = 0$

Step ③: Solve for $\frac{dy}{dx}$.

$2e^{2y}\frac{dy}{dx} - 2x - [y + x\frac{dy}{dx}] = 0$

$2e^{2y}\frac{dy}{dx} - 2x - y - x\frac{dy}{dx}] = 0$

$2e^{2y}\frac{dy}{dx} - x\frac{dy}{dx} = 2x + y$

$\frac{dy}{dx}(2e^{2y} - x) = 2x + y$

$\frac{dy}{dx} = \frac{2x+y}{2e^{2y} - x}$

30. Find the equation of the tangent line to the curve in the xy-plane, given by $x^3 + y^3 = 9$, at $(1, 2)$.

Step ①: Find $\frac{dy}{dx}$ by implicit differentiation.

$\frac{d}{dx}(x^3 + y(x)^3) = \frac{d}{dx}(9)$

$\frac{d}{dx}(x^3) + \frac{d}{dx}[y(x)^3] = \frac{d}{dx}(9)$

$3x^{3-1} + 3[y(x)^{3-1}] \cdot \frac{dy}{dx} = 0$

$3x^2 + 3y^2\frac{dy}{dx} = 0$

$3y^2\frac{dy}{dx} = - 3x^2$

$\frac{dy}{dx} = \frac{-3x^2}{3y^2}$

$\frac{dy}{dx} = - \frac{x^2}{y^2}$

Step ②: Find the slope of the tangent line by evaluating $\frac{dy}{dx}$ at $(1, 2)$.

$\frac{dy}{dx}|_{(1,2)} = - \frac{(1)^2}{(2)^2} = - \frac{1}{4}$

Step ③: Plug values into $y - y_0 = y'(x_0)(x - x_0)$.

$y - 2 = (-\frac{1}{4})(x - 1)$

$y - 2 = - \frac{1}{4}x + \frac{1}{4}$

$y = - \frac{1}{4}x + \frac{1}{4} + 2$

$y = - \frac{1}{4}x + \frac{9}{4}$

$y = \frac{9}{4} - \frac{1}{4}x$ to match the textbook solution

Exercises 4.3 (pg. 107)

4. Find the third derivative of $f(x) = ln(x^2 + 1)$ and state its domain.

First derivative

$f'(x) = \frac{1}{x^2+1} \cdot 2x$ by the chain rule for logarithmic functions

$f'(x) = \frac{2x}{x^2+1}$

Second derivative

$f''(x) = \frac{(2)(x^2+1)-(2x)(2x)}{(x^2+1)^2}$ by the Quotient Rule

$f''(x) = \frac{2x^2+2-4x^2}{(x^2+1)^2}$

$f''(x) = \frac{-2x^2+2}{(x^2+1)^2}$

Third derivative

$f'''(x) = \frac{(-4x)(x^2+1)^2-(-2x^2+2)[2(x^2+1)\cdot 2x]}{(x^2+1)^4}$ by the chain rule

$f'''(x) = \frac{(-4x)(x^2+1)^2-(-2x^2+2)[4x(x^2+1)]}{(x^2+1)^4}$

$f'''(x) = \frac{(-4x)(x^2+1)^2+(-4x)(-2x^2+2)(x^2+1)}{(x^2+1)^4}$

$f'''(x) = \frac{(x^2+1)[(-4x)(x^2+1)+(-4x)(-2x^2+2)]}{(x^2+1)^4}$ by pulling out $(x^2 + 1)$ from each term

$f'''(x) = \frac{(-4x)(x^2+1)+(-4x)(-2x^2+2)}{(x^2+1)^3}$ by cancelling out $(x^2 + 1)$

$f'''(x) = \frac{(-4x)[(x^2+1)+(-2x^2+2)]}{(x^2+1)^3}$ by pulling out $(-4x)$ from each term

$f'''(x) = \frac{(-4x)(x^2+1-2x^2+2)}{(x^2+1)^3}$

$f'''(x) = \frac{(-4x)(-x^2+3)}{(x^2+1)^3}$

$f'''(x) = \frac{(-4x)(3-x^2)}{(x^2+1)^3}$ to match textbook solution

Exercises 5.3 (pg. 116)

In Exercises 1-3, compute the elasticity of demand and determine whether the demand is elastic, inelastic, or unitary, at the given unit price p_0. If the unit price is increased slightly from p_0, will the revenue increase or decrease?

1. $x = -\frac{4}{3}p + 24$; $p_0 = 6$.

Find $f'(p)$:

$f(p) = -\frac{4}{3}p + 24$

$f'(p) = -\frac{4}{3}$

The elasticity of demand is:

$E(p) = -\frac{p\,f'(p)}{f(p)} = -\frac{p(-\frac{4}{3})}{-\frac{4}{3}p+24} = -\frac{(-\frac{4}{3})p}{(-\frac{4}{3})p+(-\frac{4}{3})(-18)} = -\frac{(-\frac{4}{3})p}{(-\frac{4}{3})(p-18)} = -\frac{p}{p-18} = \frac{p}{18-p}$

Determine the elasticity of demand at $p_0 = 6$:

$E(6) = \frac{(6)}{18-(6)} = \frac{6}{12} = \frac{1}{2}$

Compare the result to 1.

A brief review:

- If $E(p) > 1$, demand is elastic at p.
- If $E(p) = 1$, demand is unitary at p.
- If $E(p) < 1$, demand is inelastic p.

$E(6) = \frac{1}{2} < 1$, so the demand is inelastic at $p_0 = 6$.

If the unit price is increased slightly from p_0, the revenue will increase.

2. $x + \frac{1}{4}p - 10 = 0$**;** $p_0 = 20$**.**

Find $f'(p)$:

$x = -\frac{1}{4}p + 10 = f(p)$

$f'(p) = -\frac{1}{4}$

The elasticity of demand is:

$$E(p) = -\frac{p\,f'(p)}{f(p)} = -\frac{p(-\frac{1}{4})}{-\frac{1}{4}p + 10} = -\frac{(-\frac{1}{4})p}{(-\frac{1}{4})p + (-\frac{1}{4})(-40)} = -\frac{(-\frac{1}{4})p}{(-\frac{1}{4})(p-40)} = -\frac{p}{p-40} = \frac{p}{40-p}$$

Determine the elasticity of demand at $p_0 = 20$:

$$E(20) = \frac{20}{40-20} = \frac{20}{20} = 1$$

Compare the result to 1.

$E(20) = 1$, so the demand is unitary at $p_0 = 20$.

If the unit price is increased slightly from p_0, the revenue will increase.

3. $x = \sqrt{121 - p}$**;** $p_0 = 88$**.**

Find $f'(p)$:

$f(p) = \sqrt{121 - p} = (121 - p)^{\frac{1}{2}}$

$f'(p) = \frac{1}{2}(121 - p)^{-\frac{1}{2}} \cdot (-1)$ by the chain rule

$f'(p) = -\frac{1}{2\sqrt{121-p}}$

The elasticity of demand is:

$E(p) = -\frac{p\,f'(p)}{f(p)}$

$= -\frac{p(-\frac{1}{2\sqrt{121-p}})}{\sqrt{121-p}} = -\frac{-\frac{p}{2\sqrt{121-p}}}{\sqrt{121-p}} = \frac{p}{2\sqrt{121-p}} \div \sqrt{121 - p} = \frac{p}{2\sqrt{121-p}} x \frac{1}{\sqrt{121-p}} = \frac{p}{2(121-p)}$

Determine the elasticity of demand at $p_0 = 88$:

$E(88) = \frac{88}{2(121-88)} = \frac{88}{2(33)} = \frac{88}{66} = \frac{4}{3}$

Compare the result to 1:

$E(88) = \frac{4}{3} > 1$, so the demand is elastic at $p_0 = 88$.

If the unit price is increased slightly from p_0, the revenue will decrease.

4. An art gallery offers prints by a famous artist. If each print is priced at p dollars, it is expected that $x = \sqrt{7500 - 0.03p^2}$ prints will be sold.

(a) Find the elasticity of demand. Determine the values of p for which the demand is elastic, inelastic, and of unit elasticity.

Find $f'(p)$:

$f(p) = \sqrt{7500 - 0.03p^2} = (7500 - 0.03p^2)^{\frac{1}{2}}$

$f'(p) = \frac{1}{2}(7500 - 0.03p^2)^{-\frac{1}{2}} \cdot (2)(-0.03p^{2-1})$ by the chain rule

$f'(p) = \frac{1}{2\sqrt{7500-0.03p^2}} \cdot (-0.06p)$

$f'(p) = -\frac{0.06p}{2\sqrt{7500-0.03p^2}}$

$f'(p) = -\frac{0.03p}{\sqrt{7500-0.03p^2}}$

Find the elasticity of demand, $E(p)$:

$E(p) = -\frac{p\,f'(p)}{f(p)}$

$E(p) = -\frac{p(-\frac{0.03p}{\sqrt{7500-0.03p^2}})}{\sqrt{7500-0.03p^2}}$

$E(p) = -\frac{-\frac{0.03p^2}{\sqrt{7500-0.03p^2}}}{\sqrt{7500-0.03p^2}}$

$E(p) = \frac{\frac{0.03p^2}{\sqrt{7500-0.03p^2}}}{\sqrt{7500-0.03p^2}}$

$E(p) = \frac{0.03p^2}{\sqrt{7500-0.03p^2}} \div \sqrt{7500 - 0.03p^2}$

$E(p) = \frac{0.03p^2}{\sqrt{7500-0.03p^2}} \cdot \frac{1}{\sqrt{7500-0.03p^2}}$

$E(p) = \frac{0.03p^2}{7500-0.03p^2}$

Determine where demand is elastic (where $E(p) > 1$), which will also tell you where demand is inelastic (where $E(p) < 1$):

$\frac{0.03p^2}{7500-0.03p^2} > 1$

$0.03p^2 > 7500 - 0.03p^2$

$0.06p^2 > 7500$

$p^2 > 125000$

$p > \sim 353.6$

∴ Demand is elastic for $p > \sim 353.6$, and demand is inelastic for $p < \sim 353.6$.

(b) If you were the owner of the gallery, what price would you charge for each print?
A brief review:

- When demand is elastic: as the price increases, revenue decreases.
- When demand is inelastic: as the price increases, revenue increases.

We want the revenue to increase as the price increases, so we should charge a price such that the demand is inelastic.

∴ I would charge no more than $353.6, since demand is inelastic for $p < \sim 353.6$.

Final exam review

1. The amount of $1, 000 is deposited in a bank that pays interest at the rate of 6% per year compounded monthly. Find how many years it would take to triple the investment.

$A(t) = P(1 + \frac{r}{m})^{mt}$

Given: $P = \$1000$

$r = 0.06$

$m = 12$

$A = \$1000 \times 3 = \$3000$

$t = ?$

$3000 = 1000(1 + \frac{0.06}{12})^{12t}$

$3 = (1.005)^{12t}$

$ln(3) = ln(1.005)^{12t}$

$ln(3) = 12t \cdot ln(1.005)$

$12t = \frac{ln(3)}{ln(1.005)}$

$t = \frac{ln(3)}{12 \cdot ln(1.005)} = 18.356 \simeq 18\ years$ (b)

2. The half-life of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of 1/2. The half-life of aspirin in a human body is 5 hours. Find its decay constant.

$P(t) = P(0)e^{-rt}$

Given: $P(5) = \frac{1}{2}P(0)$

$t = 5$

$r = ?$

$P(5) = P(0)e^{-5r}$

$\frac{1}{2}P(0) = P(0)e^{-5r}$

$\frac{1}{2} = e^{-5r}$

$ln(\frac{1}{2}) = ln(e^{-5r})$

$ln(\frac{1}{2}) = - 5r \cdot ln(e)$

$ln(\frac{1}{2}) = - 5r(1)$

$r = \frac{ln(\frac{1}{2})}{-5} = 0.1386 \simeq 0.14$ (c)

5. The expression $\frac{25^{x+1}\cdot 2^x\cdot 5^{-2}}{10^x}$ simplifies to:

$$\frac{25^{x+1}\cdot 2^x\cdot 5^{-2}}{10^x} = \frac{(5^2)^{x+1}\cdot(\frac{10}{5})^x\cdot 5^{-2}}{10^x} = \frac{5^{2x+2}\cdot\frac{10^x}{5^x}\cdot 5^{-2}}{10^x} = \frac{5^{2x+2}\cdot\frac{1}{5^x}\cdot 5^{-2}}{1} = 5^{2x+2}\cdot\frac{1}{5^x}\cdot 5^{-2}$$

$= 5^{2x+2}\cdot 5^{-x}\cdot 5^{-2} = 5^{(2x+2)+(-x)+(-2)} = 5^{2x+2-x-2} = 5^x$ (a)

15. If $log_2(x^2 - 1) = 3$ then x is equal to:

$log_a x = y \Leftrightarrow x = a^y$

$log_2(x^2 - 1) = 3 \Leftrightarrow x^2 - 1 = 2^3$

$x^2 = 8 + 1$

$x^2 = 9$

$x = \pm\sqrt{9}$

$x = \pm 3$ (d)

16. Find the inflection point(s) of the graph of $y = f(x) = \frac{1}{6}x^4 - 9x^2 + \frac{163}{2}$.

Inflection point: where concavity changes

Step 1: find f'(x)

$f'(x) = (4)\frac{1}{6}x^{4-1} - (2)9x^{2-1} + 0$

$f'(x) = \frac{2}{3}x^3 - 18x$

Step 2: find f''(x)

$f''(x) = (3)\frac{2}{3}x^{3-1} - 18$

$f''(x) = 2x^2 - 18$

Step 3: set f''(x)=0 and find values that make it true

$0 = 2x^2 - 18$

$18 = 2x^2$

$9 = x^2$

$x^2 = \pm 3$

Step 4: find corresponding values of y

$f(-3) = \frac{1}{6}(-3)^4 - 9(-3)^2 + \frac{163}{2}$ $f(-3) = \frac{1}{6}(81) - 9(9) + \frac{163}{2}$ $f(-3) = 13.5 - 81 + 81.5$ $f(-3) = 14$	$f(3) = \frac{1}{6}(3)^4 - 9(3)^2 + \frac{163}{2}$ $f(3) = \frac{1}{6}(81) - 9(9) + \frac{163}{2}$ $f(3) = 13.5 - 81 + 81.5$ $f(3) = 14$

∴ Inflection points are (3,14) and (-3,14). (b)

19. The slope of the tangent line to the curve of $x^2y - 2^y = 2$ at the point (2,1) is given by:

Use implicit differentiation

$\frac{d}{dx}(x^2y) - \frac{d}{dx}(2^y) = \frac{d}{dx}(2)$

$\frac{d}{dx}(x^2 \cdot y(x)) - \frac{d}{dx}(2^{y(x)}) = \frac{d}{dx}(2)$

$[(x^2)'(y(x)) + (x^2)(y)'] - (2^{y(x)} \cdot ln2 \cdot \frac{dy}{dx}) = 0$

$[(2x)(y) + (x^2)(\frac{dy}{dx})] - (2^y \cdot ln2 \cdot \frac{dy}{dx}) = 0$

$2xy + (x^2 \cdot \frac{dy}{dx}) - (2^y \cdot ln2 \cdot \frac{dy}{dx}) = 0$

$(x^2 \cdot \frac{dy}{dx}) - (2^y \cdot ln2 \cdot \frac{dy}{dx}) =- 2xy$

$\frac{dy}{dx}(x^2 - 2^y \cdot ln2) =- 2xy$

$\frac{dy}{dx} = \frac{-2xy}{x^2 - 2^y \cdot ln2}$

$\frac{dy}{dx}|_{(2,1)} = \frac{-2(2)(1)}{(2)^2 - 2^{(1)} \cdot ln2} = \frac{-4}{4-2 \cdot ln2} = \frac{-2(2)}{-2(-2+ln2)} = \frac{2}{ln2-2}$ (c)

21. If the sixth term of an arithmetic sequence is a_6 = 28 and the common difference is 3, then the forty fourth term a_{44} is equal to:

$a_n = a_1 + d(n - 1)$

Find a_1

$a_6 = a_1 + d(6 - 1)$

$28 = a_1 + 3(5)$

$28 - 15 = a_1$

$a_1 = 13$

Find a_{44}

$a_{44} = 13 + 3(44 - 1)$

$a_{44} = 13 + 3(43)$

$a_{44} = 13 + 129$

$a_{44} = 142$ (e)

22. Alex makes a deposit of $1, 500 into her account. Each month she deposits $80 less than in the previous month. Find the total value of the account after 1.5 years.

$\sum_{i=1}^{n} a_i = \frac{n}{2}(2a_1 + d(n-1)$

Given: $a_{1.5\ years} = a_{18\ months}$

$a_1 = 1500$

$d = -80$

$\sum_{i=1}^{18} a_{18} = \frac{18}{2}(2 \cdot 1500 - 80(18-1)) = 9(3000 - 80(17)) = 9(3000 - 1360) = 9(1640) =$ $\$14,760$ (c)

23. Evaluate $\sum_{i=1}^{\infty} 4(-\frac{1}{2})^i$.

$\sum_{i=1}^{\infty} cr^i = \frac{cr}{1-r}$

Given: $c = 4$

$r = -\frac{1}{2}$

$\sum_{i=1}^{\infty} 4(-\frac{1}{2})^i = \frac{4(-\frac{1}{2})}{1-(-\frac{1}{2})} = \frac{-\frac{4}{2}}{\frac{2}{2}+\frac{1}{2}} = \frac{-2}{\frac{3}{2}} = -\frac{4}{3}$ (e)

24. The amount of $100 is deposited at the end of every month into a savings account that pays 1.8% compounded monthly. Find the future value of the annuity at the end of the 4-year term.

$FV = P_A \frac{(1+i)^N - 1}{i}$

Given: $i = \frac{r}{m} = \frac{0.018}{12} = 0.0015$

$N = mt = 12 \times 4 = 48$

$P_A = 100$

$FV = ?$

$FV = 100\frac{(1+0.0015)^{48} - 1}{0.0015} = 4973.158 \simeq \$4,973$ (a)

25. You are a financial adviser, and your client wants to save \$10, 000 at the end of a 3-year term. Your bank offers a savings account which pays 3.6% compounded semiannually. The client wishes to make deposits at the end of each six-month period. What is the amount of the annuity the client needs to pay in order to meet his/her goals?

$FV = P_A \frac{(1+i)^N - 1}{i}$

Given: $FV = 10000$

$N = mt = 2 \cdot 3 = 6$

$i = \frac{r}{m} = \frac{0.036}{2} = 0.018$

$P_A = ?$

$10000 = P_A \frac{(1+0.018)^6 - 1}{0.018}$

$\frac{10000 \times 0.018}{(1+0.018)^6 - 1} = P_A$

$P_A = 1593.227 \simeq \$1,593$ (c)

Q8.

V.A at $x = a$: $\lim_{x \to a+} f(x) = \infty \, or \, -\infty$ or $\lim_{x \to a-} f(x) = \infty \, or \, -\infty$

H.A at $y = b$: $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$

Q9.

Step 1: take derivative

Step 2: factor derivative so each term has x

Step 3: find values of x that make f'(x)=0

Q16:

Inflection point: where concavity changes

Step 1: find f'(x)

Step 2: find f''(x)

Step 3: set f''(x)=0 and find values that make it true