18 - Mar 15 Lecture

- Find diagonalization of A
- Checking eigenvectors for linear independence
- Non-diagonalizable matrix example
- $A^k = PD^kP^{-1}$

2 final exam reviews.
1 recorded (asynchronous)
1 in-person (synchronous)

Recall: A $n \times n$ matrix B is diagonalizable iff there exists a linearly independent set of n eigenvectors of B.

Recall: For a $n \times n$ matrix B, if the columns of P are a set of n lin. ind. eigenvectors of B and D is a diagonal matrix with each diagonal entry being the eigenvalue of the corresponding column of P, then $B = PDP^{-1}$.

Find diagonalization of matrix A:
1. Eigenvalues $\longrightarrow$ $\text{Det}(A - \lambda I)$
2. Eigenvectors $\longrightarrow$ Nullspace of $A - \lambda I$ (want $\vec{x} \in \text{Null}(A - \lambda_1 I)$, $\vec{x} \neq \vec{0}$)
3. Find P and D

Ex: Find diagonalization of A.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$$

① $\text{Det}(A - \lambda I) = \text{Det}\begin{bmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - (3)(1) = \lambda^2 - 4 = (\lambda+2)(\lambda-2)$

$\Longrightarrow \lambda_1 = -2,\ \lambda_2 = 2$

② $A - \lambda_1 I = \begin{bmatrix} 1+2 & 3 \\ 1 & -1+2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $R_1' = \frac{1}{3} R_1$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $R_2' = R_2 - R_1$

find eigen $\vec{v}$ by inspection:
$\vec{v}$ that when mult. by matrix gives $\vec{0}$

by inspection $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ works

$\text{Null}\left(\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\right) = \text{Null}\left(\begin{array}{c} \begin{array}{cc} B & F \end{array} \\ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \end{array}\right)$ RR algo does not change Null Space.

Easiest way is to set free vars to 1.

Let $x_2 = 1$

$x_1 + x_2 = 0 \longrightarrow x_1 = -1$

$0 = 0$

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$A - \lambda_2 I = \begin{bmatrix} 1-2 & 3 \\ 1 & -1-2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \quad R_1' = \tfrac{1}{-1} R_1$$

$$\sim \begin{array}{c} \begin{array}{cc} B & F \end{array} \\ \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \end{array} \quad R_2' = R_2 - R_1$$

Let $x_2 = 1$

$x_1 - 3x_2 = 0 \longrightarrow x_1 = 3x_2 \longrightarrow x_1 = 3$ $\quad \Rightarrow \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$0 = 0$

$\lambda_1 = -2 \qquad \lambda_2 = 2$

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (or any scalar multiple of $\vec{v}$)

③ $P = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

Can also write as:

$$A = \underset{\vec{v}_1 \quad \vec{v}_2}{\overset{P}{\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}}} \underset{\lambda_1 \quad \lambda_2}{\overset{D}{\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}}} \underset{\vec{v}_1 \quad \vec{v}_2}{\overset{P^{-1}}{\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}^{-1}}} = \underset{\vec{v}_2 \quad \vec{v}_1}{\overset{P}{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}}} \underset{\lambda_2 \quad \lambda_1}{\overset{D}{\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}}} \underset{\vec{v}_2 \quad \vec{v}_1}{\overset{P^{-1}}{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}^{-1}}}$$

A $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Reduction for alg. mult. bigger than 1

Suppose x_1 is basic and x_2, x_3 are free variables.

$\vec{v}_2$: $x_2 = 1, \; x_3 = 0$

$\vec{v}_3$: $x_2 = 0, \; x_3 = 1$

(you will not always find the same # of $\vec{v}$ as alg. mult.)

Ex. Find the diagonalization of A, given $\lambda_1, \lambda_2, \vec{v}_1, \vec{v}_2, \vec{v}_3$.

aside: should be able to find eigenvalues of A

$$A = \begin{bmatrix} 20 & -6 & 24 \\ -9 & 5 & -12 \\ -18 & 6 & -22 \end{bmatrix}$$

$\lambda_1 = -1$ alg. mult. 1

$\lambda_2 = 2$ alg. mult. 2

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

associated with: λ_1 $\quad \lambda_2 \quad \lambda_2$

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & -1 \end{bmatrix}^{-1}$$

← diagonalization of A

Checking for Linear Independence

When checking for linear independence (LI), we don't need to check if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is LI. It suffices to check that both $\{\vec{v}_1\}$ and $\{\vec{v}_2, \vec{v}_3\}$ are LI.

Ex: Cont. from example above, check for LI of eigenvectors.

Is $\{\vec{v}_1\}$ LI? Yes, since set only contains a single non-zero vector.

Is $\{\vec{v}_2, \vec{v}_3\}$ LI? Yes, since set contains a pair of non-zero vectors that are not scalar multiples of each other.

What does it look like if A is not diagonalizable?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$ A not diagonalizable

$\lambda_1 = 1$ alg. mult. 2 by inspection (upper triangular)

$$A - \lambda_1 I = \begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{Null}(A - \lambda_1 I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

← 1D space + nothing else ⇒ not diagonalizable

Would need to pull 2 lin. dep. eigen$\vec{v}$ from a 1-dim subspace (impossible)

∴ A is not diagonalizable

A $n \times n$ matrix B is diagonalizable iff the sum of the eigenspace dimensions is equal to n.

Alternatively: A $n \times n$ matrix B is diagonalizable iff (both).
1) The characteristic polynomial factors to linear terms. (in reals)
2) The dimension of all eigenspaces is equal to the algebraic multiplicity of the corresponding eigenvalue.

Why diagonalize?

Suppose we have $A = PDP^{-1}$.

Then $A^2 = PD\cancel{P^{-1}}\cancel{P}DP^{-1}$

$= PD^2P^{-1}$

$\Rightarrow A^k = PD^kP^{-1}$ (k is a positive integer)

$$D = \begin{bmatrix} d_1 & & \emptyset \\ & d_2 & \\ \emptyset & & \ddots \; d_n \end{bmatrix} \Longrightarrow D^k = \begin{bmatrix} d_1^k & & \emptyset \\ & d_2^k & \\ \emptyset & & \ddots \; d_n^k \end{bmatrix}$$

Ex: Given $A = PDP^{-1}$, find $A^k = PD^kP^{-1}$.

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1/4 & 1/4 \\ 1/4 & -3/4 \end{bmatrix}$$

inverse of $\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$

(result from earlier but with $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ which was by inspection)

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}^k = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}^k\begin{bmatrix} 1/4 & 1/4 \\ 1/4 & -3/4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} 1/4 & 1/4 \\ 1/4 & -3/4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2^k(1) + \cancel{0(1)} & 2^k(1) + \cancel{0(-3)} \\ \cancel{0(1)} + (-2)^k(1) & \cancel{0(1)} + (-2)^k(-3) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2^k & 2^k \\ (-2)^k & (-3)(-2)^k \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3(2^k) + 1(-2)^k & 3(2^k) + 1(-3)(-2)^k \\ 1(2^k) - 1(-2^k) & 1(2)^k - 1(-3)(-2)^k \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3(2^k) + (-2)^k & 3(2^k) - 3(-2)^k \\ 2^k - (-2^k) & 2^k + 3(-2)^k \end{bmatrix}$$