13 - Feb 27 Lecture

- Determinants
- Submatrices
- Algorithm for finding the determinant of a square matrix

Note that test 3 will cover content up to Lecture 14 (March 1)

Determinants

Determinants are a property of / number associated with square matrices.

↳ The determinant of any $n \times n$ (square) matrix is <u>defined</u>.

↳ The determinant of any $n \times m$, $n \neq m$ (non-square) matrix is <u>not defined</u>.

$$\operatorname{Det}(\underbrace{[a]}_{1\times 1}) = a \quad \text{(1 term)}$$

$$\operatorname{Det}\left(\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{2\times 2}\right) = ad - bc \quad \text{(2 terms)}$$

$$\operatorname{Det}\left(\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{3\times 3}\right) = aei - afh - bdi + bfg + cdh - ceg \quad \text{(6 terms)}$$

$$\operatorname{Det}([4\times 4]) = \ldots \quad \text{(24 terms)}$$

$$\operatorname{Det}([5\times 5]) = \ldots \quad \text{(120 terms)}$$

Det ([2x2]) is easy to find. If the matrix is larger, though, using formulas is no longer feasible because of the sheer number of terms involved in the calculation. So, we will use an algorithm to find the determinant of larger matrices (next page).

Submatrices

Let $A(i,j)$ denote a submatrix, the matrix obtained by <u>removing</u> the i^{th} row and j^{th} column of A.

↳ If A is a $n \times m$ matrix, $A(i,j)$ is a $(n-1)\times(m-1)$ matrix $(n \neq m)$.

↳ If A is square, $A(i,j)$ is square.

↳ Note : textbook uses $A_{i,j}$ instead of $A(i,j)$

Ex: Given A, what is :

$$\begin{array}{c c} & \begin{matrix} & & j=3 & j=4 \end{matrix} \\ \begin{matrix} i=1 \\ i=2 \\ \\ \end{matrix} & A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \end{array}$$

rewrite A without highlighted columns/ rows

a) $A(2,3)$?

$$A(2,3) = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$$

b) $A(1,4)$?

$$A(1,4) = \begin{bmatrix} 5 & 6 & 7 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{bmatrix}$$

Algorithm for finding determinant of a matrix

To find the determinant of a nxn matrix A, use either one of the following two formulas. You will choose either a value of i between 1 to n or a value of j between 1 to n. If you choose a value for i, use formula ①. If you choose a value for j, use formula ②.

Formula ①	Formula ②
For any i between 1 and n:	For any j between 1 and n:
$\text{Det}(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot A_{i,j} \cdot \text{Det}(A(i,j))$	$\text{Det}(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot A_{i,j} \cdot \text{Det}(A(i,j))$

This formula reduces matrix size. This algorithm continues until Det(A(i,j)) is Det([2x2]). Then, we can use Det([2x2]) = ad − bc to solve everything.

Ex. What is the determinant of A?

$$\begin{array}{r} i=1\rightarrow \\ A= \\ \\ \end{array}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

Tip: choose i or j so that there are the most zeroes in that row / column. For this example, i = 1, i = 2, j = 1, j = 3 are all equally good choices. Each zero means one less term to calculate!

Let i = 1. (use formula ①)

$$\text{Det}(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot A_{i,j} \cdot \text{Det}(A(i,j))$$

we will sum j from 1 to 3 since A is a 3×3 matrix (∴ n is 3)

$$\text{Det}(A) = \underbrace{(-1)^{1+1} \cdot A_{1,1} \cdot \text{Det}(A(1,1))}^{i=1,\,j=1} + \underbrace{(-1)^{1+2} \cdot A_{1,2} \cdot \text{Det}(A(1,2))}^{i=1,\,j=2} + \underbrace{(-1)^{1+3} \cdot A_{1,3} \cdot \text{Det}(A(1,3))}^{i=1,\,j=3}$$

$$= (-1)^2 \cdot (1) \cdot \text{Det}\left(\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}\right) + (-1)^3 \cdot (1) \cdot \text{Det}\left(\begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}\right) + \cancel{(-1)^4 \cdot (0) \cdot \text{Det}\left(\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\right)}^{0}$$

$$= (1)(1)[(2)(1) - (-3)(-1)] + (-1)(1)[(0)(1) - (-3)(1)] + 0$$

$$= (1)(-1) + (-1)(3) + 0$$

$$= -1 - 3 + 0$$

$$= -4$$

Note: you will get the same determinant value no matter what value of i or j you pick (between 1 and n). So, j = 3 and using formula 2 would also result in a determinant of −4. Demonstrating on next page.

Let $j = 3$. (use formula ②)

$$\text{Det}(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot A_{i,j} \cdot \text{Det}(A(i,j))$$

As a reminder, $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 1 & -1 & 1 \end{bmatrix}$

$$\text{Det}(A) = \underbrace{(-1)^{1+3} \cdot A_{1,3} \cdot \text{Det}(A(1,3))}_{i=1,\, j=3} + \underbrace{(-1)^{2+3} \cdot A_{2,3} \cdot \text{Det}(A(2,3))}_{i=2,\, j=3} + \underbrace{(-1)^{3+3} \cdot A_{3,3} \cdot \text{Det}(A(3,3))}_{i=3,\, j=3}$$

$$= \cancelto{0}{(-1)^4 \cdot 0 \cdot \text{Det}(A(1,3))} + (-1)^5 \cdot (-3) \cdot \text{Det}\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) + (-1)^6 \cdot (1) \cdot \text{Det}\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right)$$

$$= 0 + (-1)(-3)\left[(1)(-1) - (1)(1)\right] + (1)(1)\left[(1)(2) - (1)(0)\right]$$

$$= 0 + 3(-2) + 1(2)$$

$$= 0 - 6 + 2$$

$$= -4 \quad \leftarrow \text{same answer :)}$$

So, how exactly does this algorithm work?

1) The formula expands along the row/ column you select.

Ex: selecting $i = 2$ would expand along the second row

Ex: selecting $j = 1$ would expand along the first column

2) The formula adds together all terms associated with the selected row / column.

Breaking down the formula:

$$\text{Det}(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot A_{i,j} \cdot \text{Det}(A(i,j))$$

$$(-1)^{i+j} = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

changes sign of term based on i and j

$A_{i,j}$: term of A. Pick i or j to maximize the number of zeroes in that column/ row.

$\text{Det}(A(i,j))$: if A is $n \times n$, $A(i,j)$ is $(n-1) \times (n-1)$.

Ex: What is Det(A)?

$$A = \begin{bmatrix} 3 & -2 & 1 & 0 & 4 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 4 & 1 & 3 & 1 \\ 0 & -1 & -5 & 2 & 2 \\ 0 & 2 & 3 & 1 & 4 \end{bmatrix}$$

Let $j = 1$ since that column has the most zeroes.

$$\text{Det}(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot A_{i,j} \cdot \text{Det}(A(i,j))$$ since $A_{2,1}, A_{3,1}, A_{4,1}$, and $A_{5,1}$ are all 0

$$\text{Det}(A) = (-1)^{1+1} \cdot A_{1,1} \cdot \text{Det}(A(1,1)) + 0 + 0 + 0 + 0$$

$$= (1)(3) \cdot \text{Det} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 4 & 1 & 3 & 1 \\ -1 & -5 & 2 & 2 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$ ← let $B = A(1,1)$

$$= (3)(-234)$$

$$= \boxed{-702}$$ ← Det(A)

Let $i = 1$.

$$\text{Det}(B) = \overset{i=1,\ j=1}{(-1)^{1+1} \cdot B_{1,1} \cdot \text{Det}(B(1,1))} + \overset{i=1,\ j=2}{(-1)^{1+2} \cdot B_{1,2} \cdot \text{Det}(B(1,2))} + \overset{i=1,\ j=3}{(-1)^{1+3} \cdot B_{1,3} \cdot \text{Det}(B(1,3))} + 0$$

since $B_{1,4}$ is 0

$$= (1)(1) \cdot \text{Det} \begin{bmatrix} 1 & 3 & 1 \\ -5 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix} + (-1)(3) \cdot \text{Det} \begin{bmatrix} 4 & 3 & 1 \\ -1 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix} + (1)(2) \cdot \text{Det} \begin{bmatrix} 4 & 1 & 1 \\ -1 & -5 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

← let $E = B(1,1)$ ← let $F = B(1,2)$ ← let $G = B(1,3)$

$$= (1)(73) + (-3)(43) + (2)(-89)$$

$$= 73 - 129 - 178$$

$$= -234$$

Let $i = 1$.

$$\text{Det}(E) = \overset{i=1,\ j=1}{(-1)^{1+1} \cdot E_{1,1} \cdot \text{Det}(E(1,1))} + \overset{i=1,\ j=2}{(-1)^{1+2} \cdot E_{1,2} \cdot \text{Det}(E(1,2))} + \overset{i=1,\ j=3}{(-1)^{1+3} \cdot E_{1,3} \cdot \text{Det}(E(1,3))}$$

$$= (1)(1) \cdot \text{Det} \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} + (-1)(3) \cdot \text{Det} \begin{bmatrix} -5 & 2 \\ 3 & 4 \end{bmatrix} + (1)(1) \cdot \text{Det} \begin{bmatrix} -5 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= (1)\left[(2)(4) - (2)(1)\right] + (-3)\left[(-5)(4) - (2)(3)\right] + (1)\left[(-5)(1) - (2)(3)\right]$$

$$= (1)(6) + (-3)(-26) + (1)(-11)$$

$$= 6 + 78 - 11$$

$$= 73$$

(Example Continued)

Let $i = 1$.

$$\text{Det}(F) = \overset{i=1,\,j=1}{\underline{(-1)^{1+1} \cdot F_{1,1} \cdot \text{Det}(F(1,1))}} + \overset{i=1,\,j=2}{\underline{(-1)^{1+2} \cdot F_{1,2} \cdot \text{Det}(F(1,2))}} + \overset{i=1,\,j=3}{\underline{(-1)^{1+3} \cdot F_{1,3} \cdot \text{Det}(F(1,3))}}$$

$$= (1)(4) \cdot \text{Det}\begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} + (-1)(3) \cdot \text{Det}\begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix} + (1)(1) \cdot \text{Det}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= (4)\left[(2)(4)-(2)(1)\right] + (-3)\left[(-1)(4)-(2)(2)\right] + (1)\left[(-1)(1)-(2)(2)\right]$$

$$= (4)(6) + (-3)(-8) + (1)(-5)$$

$$= 24 + 24 - 5$$

$$= 43$$

Let $i = 1$.

$$\text{Det}(G) = \overset{i=1,\,j=1}{\underline{(-1)^{1+1} \cdot G_{1,1} \cdot \text{Det}(G(1,1))}} + \overset{i=1,\,j=2}{\underline{(-1)^{1+2} \cdot G_{1,2} \cdot \text{Det}(G(1,2))}} + \overset{i=1,\,j=3}{\underline{(-1)^{1+3} \cdot G_{1,3} \cdot \text{Det}(G(1,3))}}$$

$$= (1)(4) \cdot \text{Det}\begin{bmatrix} -5 & 2 \\ 3 & 4 \end{bmatrix} + (-1)(1) \cdot \text{Det}\begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix} + (1)(1) \cdot \text{Det}\begin{bmatrix} -1 & -5 \\ 2 & 3 \end{bmatrix}$$

$$= (4)\left[(-5)(4)-(2)(3)\right] + (-1)\left[(-1)(4)-(2)(2)\right] + (1)\left[(-1)(3)-(-5)(2)\right]$$

$$= (4)(-26) + (-1)(-8) + (1)(7)$$

$$= -104 + 8 + 7$$

$$= -89$$

Test 3 will cover content up to this wed (Mar 1)

Determinants

Determinant is a property of / number associated with a square matrix

The determinant of any (square) $n \times n$ matrix is defined

The determinant of any (non-square) $n \times m$ matrix $(n \neq m)$ is not defined.

$\text{Det}([a]) = a$ (1 term)

$\text{Det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ (2 terms)

$\text{Det}\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = aei - afh - bdi + bfg + cdh - ceg$

(6 terms) * don't need to memorize *

$\text{Det}([4 \times 4]) = \ldots$ (24 terms)

$\text{Det}([5 \times 5]) = \ldots$ (120 terms)

We need a better method of finding the det.

Submatrices Let $A(i,j)$ denote the matrix obtained by removing the i^{th} row and j^{th} column of A.

(Note: textbook uses $A_{i,j}$)

Ex.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A(2,3) = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$$

$$A(1,4) = \begin{bmatrix} 5 & 6 & 7 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{bmatrix}$$

If A is square then $A(i,j)$ is square.

The determinant of a $n \times n$ matrix A.

for any i between 1 and n:

$$\text{Det}(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{i,j} \ \det(A(i,j))$$

or for any j between 1 and n:

$$\text{Det}(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{i,j} \ \text{Det}(A(i,j))$$

Ex.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

$A_{1,2}$

review

$$\sum_{i=1}^{3} i^2 = (i^2)_{i=1} + (i^2)_{i=2} + (i^2)_{i=3}$$
$$= 1^2 + 2^2 + 3^2$$

Let $i = 1$. (selected i → use formula 1)

Sum from 1 to 3 since A is a 3×3 matrix → $n = 3$.

$$\text{Det}(A) = \underbrace{(-1)^{1+1} A_{1,1} \ \text{Det}(A(1,1))}_{j=1} + \underbrace{(-1)^{1+2} A_{1,2} \ \text{Det}(A(1,2))}_{j=2}$$
$$+ \underbrace{(-1)^{1+3} A_{1,3} \ \text{Det}(A(1,3))}_{j=3}$$

will be on test →

$$= 1(1) \cdot \text{Det}\left(\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}\right) + (-1)(1) \cdot \text{Det}\begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix} + 1(0) \cdot \text{Det}\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

Once you get to Det [2x2], use ad-bc. This formula just reduces matrix size until we get to 2x2.

$$= -1 + (-3) + 0$$
$$= -4$$

Let $j = 3$. $\rightarrow$ use formula 2

$$\text{Det}(A) = \overset{i=1}{\underline{(-1)^{1+3} A_{1,3} \text{ Det}(A(1,3))}} + \overset{i=2}{\underline{(-1)^{2+3} A_{2,3} \text{ Det}(A(2,3))}} + \overset{i=3}{\underline{(-1)^{3+3} A_{3,3} \text{ Det}(A(3,3))}}$$

$$= \cancel{(1)(0) \cdot \text{Det}\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}} + (-1)(-3) \cdot \text{Det}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + (1)(1) \cdot \text{Det}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$= 0 + (-6) + 2$

$= -4$

Will get same answer regardless of i, j picked.

How does this formula work?
- The formula expands along a row or column (the one you pick).
    - Selecting i=2 would expand along the second row.
    - Selecting j=1 would expand along the first column.
- The formula sums over all terms associated with the selected row/column (terms along column/row are added together).

$$(-1)^{i+j} = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

$A_{i,j}$ = term of A. Pick i or j to <u>maximize the number of 0s</u> in that column or row.

Ex: $\begin{bmatrix} 3 & 1 & 7 \\ 0 & 2 & 3 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow$ expand on 1st column. easiest way!

$\text{Det}(A(i,j))$ = If A is $n \times n$, $A(i,j)$ is $(n-1) \times (n-1)$ matrix.

$$A = \begin{bmatrix} 3 & -2 & 1 & 0 & 4 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 4 & 1 & 3 & 1 \\ 0 & -1 & -5 & 2 & 2 \\ 0 & 2 & 3 & 1 & 4 \end{bmatrix}$$

Let $j = 1$.

$\text{Det}(A) = (-1)^2 (3)\ \text{Det}(A(1,1)) + 0 + 0 + 0 + 0$ Let $B = A(1,1)$

$$A(1,1) \longrightarrow \text{Det}\begin{bmatrix} 1 & 3 & 2 & 0 \\ 4 & 1 & 3 & 1 \\ -1 & -5 & 2 & 2 \\ 2 & 3 & 1 & 4 \end{bmatrix} = (-1)^{1+1}(1)\cdot \text{Det}(B(1,1)) + (-1)^{1+2}(3)\cdot \text{Det}(B(1,2))$$
$$+ (-1)^{1+3}(2)\cdot \text{Det}(B(1,3)) + 0$$

Let $B = A(1,1)$

$$B(1,1) \rightarrow \text{Det}\begin{bmatrix} 1 & 3 & 1 \\ -5 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix} \quad j = 2$$
$$= (-1)^{1+2} E_{1,2}\ \text{Det}(E(1,2)) + (-1)^{2+2} E_{2,2}\cdot \text{Det}(E(2,2))$$
$$+ (-1)^{3+2} E_{3,2}\cdot \text{Det}(E(3,2))$$

$E = B(1,1)$

$= -26\ (-1)(3)\cdots$

$= 78 + 2 + (-7)$

$= 73 \leftarrow$ determinant of 1st Det ([3x3])

☆ review

$$\text{Det}\begin{bmatrix} 4 & 3 & 1 \\ -1 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix} = +4\ \text{Det}(F(1,1) - 3\ \text{Det}(F(1,2)) + 1\ \text{Det}\ (F[1,3]$$

$F(B(1,2) = 24 + 24 + 53$

$F = (B(1,2))$

Recall:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\text{Det}\begin{bmatrix} 4 & 1 & 1 \\ -1 & -5 & 2 \\ 2 & 3 & 4 \end{bmatrix} = -89$$

$= 73 - 3\ 53 + 2(89)$

? 159   -188 ?

$= -274$ (sub to top)

Review expansion!

$\Rightarrow \text{Det}(A) = -822$