2 - Jan 11 Lecture

- Back substitution
- REF
- RREF
- Row Reduction Algorithm

Definitions: leading entry of a row, Echelon form, pivot, pivot position, pivot column, basic variable, free variable

Recall from the Jan 9 lecture that we took the matrix to the left and performed EROs to change it into the matrix on the right.

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ -x_1 - x_2 + 3x_3 &= -1 \\ -2x_1 + x_2 + x_3 &= 2 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ -1 & -1 & 3 & -1 \\ -2 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 3 & -1 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

What is back substitution?

Then we converted the resulting matrix into equations, and solved these equations! This method is known as "back substitution".

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 3 & -1 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 3x_2 - x_3 &= 8 \\ x_3 &= 1 \end{aligned}$$

correction from last lecture's notes

Solution: $x_1 = 1$, $x_2 = 3$, $x_3 = 1$

Back substitution relies on the idea that EROs do not change the solution set of the system of linear equations. So, the equations derived from the matrix in green have the same solutions as equations derived from the matrix in purple.

Limits of back substitution

Back substitution will give us the answer if there is **only 1 solution** to the system of linear equations.
If there are no solutions or infinitely many solutions, we will need to use a different approach. More about this below.

Let's continue doing EROs on the purple matrix above and see if we can simplify it further to give us the solution outright!

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 3 & -1 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{aligned} R_1' &= R_1 + R_3 \\ R_2' &= R_2 + R_3 \end{aligned}$$

Note that these EROs can be done simultaneously since one does not affect the other.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_2' = \tfrac{1}{3} R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_1' = R_1 - R_2 \rightarrow \begin{aligned} x_1 &= 1 \\ x_2 &= 3 \\ x_3 &= 1 \end{aligned}$$

We get the solution outright!

The last matrix (in red) is in what's known as "reduced row echelon form", which is a form where the solutions are immediately obvious. More on this below.

What is REF?

A matrix is in "Row Echelon Form" if: (which I highlighted in the examples below)

1) <u>All rows consisting of only zeroes</u> are placed below rows with at least 1 non-zero element. AND

(which I put a $\boxed{\text{box}}$ around in the examples below)

2) <u>The first non-zero entry of a row</u> is to the right of the first non-zero entry of the previous row. Think of it like a "staircase pattern"! This "staircase" does not have to have evenly-sized "steps". staircase "to the <u>right</u>"!!

Note: When determining if a matrix is REF (or RREF), <u>ignore</u> the "augmented line" (the vertical line present in augmented matrices, in red below).

Some Exs: REF or Not REF?

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 3 \\ 0 & \boxed{3} & -1 & 8 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

Rule 1 · ✓ (There are no rows consisting of only zeroes.)

Rule 2 · ✓

This matrix is in REF.

ignore this line!

$$\begin{bmatrix} \boxed{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rule 1 : ✓

Rule 2 : ✓

This matrix is in REF.

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & 0 & 8 \\ 0 & \boxed{2} & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

Rule 1 : ✓

Rule 2 : ✓

This matrix is in REF.

$$\begin{bmatrix} 0 & \boxed{1} & 2 \\ \boxed{1} & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rule 1 : ✓

Rule 2 : X

This matrix is <u>not</u> in REF.

$$\begin{bmatrix} \boxed{7} & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{2} \end{bmatrix}$$

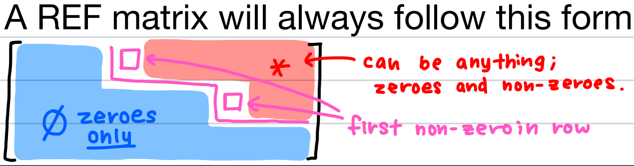
Rule 1 · X

Rule 2 : ✓

This matrix is <u>not</u> in REF.

Generalized form of REF matrices

A REF matrix will always follow this format:



What is RREF?

RREF is a subgroup of REF. If a matrix is in RREF, it is also in REF.

A matrix is in "Reduced Row Echelon Form" if:

1) It is in REF. AND
2) The first non-zero element in a row is 1. AND
3) The first non-zero entry of every non-zero row is the only non-zero element in that column.

Some Exs: RREF or Not RREF?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

RREF Rule 1: ✓
↳ REF Rule 1: ✓
↳ REF Rule 2: ✓

RREF Rule 2: ✓
RREF Rule 3: ✓
This matrix is in RREF.

$$\begin{bmatrix} 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF Rule 1:
↳ REF Rule 1: ✓
↳ REF Rule 2: ✓

RREF Rule 2: ✓
RREF Rule 3: ✓
This matrix is in RREF.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

cannot be to the left of column 2

RREF Rule 1: ✗
↳ REF Rule 1: ✓
↳ REF Rule 2: ✗

RREF Rule 2: ✓
RREF Rule 3: ✓
This matrix is <u>NOT</u> in RREF (since it is not in REF).

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

must be 1

RREF Rule 1: ✓
↳ REF Rule 1: ✓
↳ REF Rule 2: ✓

RREF Rule 2: ✗
RREF Rule 3: ✓
This matrix is not in RREF, but can be easily changed using EROs so that it is in RREF.

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1' = -1 \cdot R_1$$

This resulting matrix <u>IS</u> in RREF.

must be 0

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

RREF Rule 1: ✓
↳ REF Rule 1: ✓
↳ REF Rule 2: ✓

RREF Rule 2: ✓
RREF Rule 3: ✗
This matrix is not in RREF since it does not satisfy rule 3.

Given some arbitrary n x m real matrix A…

$$A = [\,?\,] \in \mathbb{R}^{n \times m}$$

… there are some things that are true.

1) You can always reduce A to RREF (and thus REF).
The resulting RREF matrix is <u>unique</u>. No matter the order or steps taken, you'll always get the exact same matrix as a result (if you don't, you did something wrong!).

2) In general, REF matrices are <u>not unique</u>. There are infinitely-many possibilities. The only exception is all-zero matrices.

What is the "leading entry of a row"?

Leftmost non-zero entry of a non-zero row.

What is the "Echelon Form of A"?

Any REF matrix that can be obtained by applying EROs to the matrix A.

What is a "pivot"?

The leading entry of a row of a matrix in Echelon form.

What is "pivot position"?

The position of a pivot in an Echelon form of the matrix.
The pivot positions of the infinite number of REF matrices derived from the same original matrix will always be the same.

Ex:

$$\begin{bmatrix} 0 & \boxed{1} & 0 & 4 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

No matter how this matrix is row reduced:
- the pivot positions will be at the <u>same place</u>.
- the bottom row will always be <u>all zeroes</u>.

What is a "pivot column"

A column that contains a pivot position.

Ex:

pivot columns

$$\begin{array}{c} \begin{array}{ccccc} & \downarrow & \downarrow & & \downarrow \end{array} \\ \begin{bmatrix} 0 & \boxed{1} & 0 & 4 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

What is a "basic variable"?

A variable corresponding to a pivot column.

What is a "free variable"?

A variable that corresponds to a column that is not a pivot column.
Alternatively: any variable that is not a basic variable.

Note: each column corresponds to a variable, like below:

This matrix is in REF.

$$\begin{array}{ccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 3 \\ -1 & \boxed{-1} & 3 & -1 \\ -2 & 1 & \boxed{1} & 2 \end{array}\right]$$

This matrix has the same pivot positions.

Note: Put matrices in REF to find pivot positions.

Ex: basic variables vs free variables

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \text{Not a variable} \\ \end{array}$$
$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1 : basic variable since column contains a pivot

x_2 : free variable

x_3 : free variable

augmented column: not a variable, so neither basic nor free variable

How does the Row Reduction Algorithm work?

Input: Real matrix A
Output: The RREF of A

1) Find the leftmost non-zero column (this is known as the current column).
If the top row contains a zero in this column, use Interchange to switch the top row with a row that does not contain a zero in that column.
The top row is the "current row" and 's' is the first non-zero element of the row.

2) $R_i' = \frac{1}{s} R_i$ where R_i is the current row.

3) $R_j' = R_j - k \cdot R_i$ for all $j \neq i$ and k is the entry of row j in the current column (the current column is the same column that s is in).

4) Ignoring all rows that have been a current row, repeat step 1.

5) No longer ignoring rows, repeat steps 2 and 3 with the current row identified in step 4.

6) Repeat steps 4 and 5 until step 4 fails to find a row.

This will require a lot of practice. PRACTICE!!!!

Note: The textbook uses a slightly different algorithm. Prof Welch does not care which method you use, as long as you follow EROs and get to the correct unique RREF of matrix A.

Ex: Let's apply the Row Reduction Algorithm to a matrix. In this example, we are given the following linear equations:

$$2x_2 + 2x_3 = 0$$
$$x_2 + x_3 = 0$$
$$x_2 + 2x_3 + x_4 = 1$$

Let's start off by converting these equations into an augmented matrix:

$$\begin{matrix} 2x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \\ x_2 + 2x_3 + x_4 = 1 \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

(This was covered in lecture 1!) ↳ Jan 9

Now we're ready to use the algorithm.

STEP 1: Find the leftmost non-zero column (this is known as the current column).

current column: 2nd column from the left

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

current column

If the top row contains a zero in this column, use Interchange to switch the top row with a row that does not contain a zero in that column.

The top row contains a 2 in this column, so skip this part.

The top row is the "current row" and s is the first non-zero element of the row.

Current row: R_1

$s = 2$

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \overset{s}{} \\ \left[\begin{array}{cccc|c} 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right] \end{array} \leftarrow \text{current row}$$

STEP 2: $R_i' = \frac{1}{s} R_i$ where R_i is the current row.

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

Knowing the current row and the value of s, we performed:

$R_1' = \frac{1}{2} R_1$

and got the matrix to the left.

STEP 3: $R_j' = R_j - k \cdot R_i$ for all $j \neq i$ and k is the entry of row j in the current column (the current column is the same column that s is in).

Let's determine the value of k for all rows except the current row.

current column

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ k \text{ for } R_2 : 1 \\ k \text{ for } R_3 : 1 \end{array}$$

Now knowing k, we can use Replacement to change the matrix further.

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - 1 \cdot R_1 \\ R_3' = R_3 - 1 \cdot R_1 \end{array}$$

These EROs can be done simultaneously.

STEP 4: Ignoring all rows that have been a current row, repeat step 1.

step 1: Find leftmost non-zero column.

$$\left[\begin{array}{cccc|c} \cancel{0} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

← ignore R_1 since it has already been a current row.

current column

Use Interchange since the top value in the current column is 0.

current row : R_2

value of s: 1

$$\sim \left[\begin{array}{cccc|c} \cancel{0} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \Leftrightarrow R_3$$

s

STEP 5: No longer ignoring rows, repeat steps 2 and 3 with the current row identified in step 4.

step 2: $R_i' = \frac{1}{s} R_i$ where R_i is the current row.

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2' = \frac{1}{1} \cdot R_2$$

Note that this step is not necessary to write out since the matrix does not change.

step 3: Determine the value of k for all rows except the current row.

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} k \text{ for } R_1 : 1 \\ \\ k \text{ for } R_3 : 0 \end{array}$$

For all $j \neq i$, $R_j' = R_j + k \cdot R_i$.

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1' = R_1 - 1 \cdot R_2 \\ \boxed{R_3' = R_3 - 0 \cdot R_2} \end{array}$$

This is redundant.

STEP 6: Repeat steps 4 and 5 until step 4 fails to find a row.

The only row that has not been a current row (R_3) only contains zeroes.
Therefore, we stop the algorithm here since the matrix we have is in RREF.
Let's take a look at the resulting matrix and its equations.

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left[\begin{array}{cccc|c} 0 & \boxed{1} & 0 & -1 & -1 \\ 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{aligned} x_2 - x_4 &= -1 \\ x_3 + x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

This $0 = 0$ equation tells us that we had a <u>redundant equation</u> in our previous set of equations.

Let's look back at the previous set of equations:

$$\left.\begin{aligned} 2x_2 + 2x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\}$$ These two equations are actually the same!

$$x_2 + 2x_3 + x_4 = 1$$

We need to consider if x_1 / x_2 / x_3 / x_4 are basic or free variables.

x_1 : free

x_2 : basic

x_3 : basic

x_4 : free

Now we will set the value of the free variables to arbitrary things that exist in $\mathbb{R}$.

Let $x_1 = s$.

Let $x_4 = t$.

We will substitute these values into the two equations above.

$$\begin{aligned} x_2 - t &= -1 \\ x_3 + t &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_2 &= t - 1 \\ x_3 &= 1 - t \end{aligned}$$

The solution set is:

$$\begin{aligned} x_1 &= s \\ x_2 &= t - 1 \\ x_3 &= 1 - t \\ x_4 &= t \end{aligned} \quad \text{for any } s, t \in \mathbb{R}$$

What conclusions can be made from RREF?	RREF will give us all of the correct solutions, unlike back substitution. **Some free variables:** infinitely many solutions **No free variables:** 1 solution $\left[\begin{array}{ccc\|c} 0 & 0 & 0 & 1 \end{array}\right]$ ✓ **0=1 as a final equation:** 0 solutions (inconsistent system) $\left[\begin{array}{ccc\|c} 0 & 0 & 0 & 4 \end{array}\right]$ ✗
Note:	If you do not get 0=1 (ex. 0=4) you either: 1) Have a system that is consistent. 2) Messed up somewhere, since by the definition of RREF, the leading entry of a row must be 1.
Fundamentals of the Row Replacement Algorithm:	1) Find a pivot using Interchange (step 1) 2) Set pivot to 1 using Scaling (step 2) 3) Zero out the rest of the column using Replacement (step 3) Repeat (steps 4, 5, 6)

(will give

From last lec:

EROs do not change the solution set of the system of linear equations.

We solved last problem using "back substitution" ans. if only 1 soln. to sys of lin eqns.)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 3 & -1 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1' = R_1 + R_3 \\ R_2' = R_2 + R_3 \end{array}$$

only changes R_1

These calcs can be done simultaneously.

only changes R_2

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_2' = \tfrac{1}{3} R_2$$

"RREF" reduced row escelon form ↳ outright gives solution

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_1' = R_1 - R_2$$

$x_1 = 1,\ x_2 = 3,\ x_3 = 1$

REF (Row Echelon Form):

A matrix is in REF if:

1) All rows consisting of only zeroes are placed below rows with at least 1 non-zero elt.

2) The first non-zero entry of a row is to the right of the first non-zero entry of the previous row.

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 3 \\ 0 & \boxed{3} & -1 & 8 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

This matrix is in REF.

$$\begin{bmatrix} \boxed{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ REF ✓

("augmented line")

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & 0 & 8 \\ 0 & \boxed{2} & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$ REF ✓

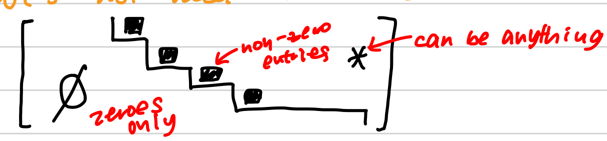
Ignore vertical line when determining if a matrix is REF or RREF.

$$\begin{bmatrix} 0 & \boxed{1} & 2 \\ \boxed{1} & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ X Not REF

$$\begin{bmatrix} 7 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ Not REF

Think of it like a "staircase pattern"

Does not need to be an even staircase



subgroup of REF

RREF (reduced row echelon form):

A matrix is in RREF if:

1) It is in REF
2) The first non-zero elt. in a row is "1"
3) The first non-zero entry of every non-zero row is the only non-zero elt in that column.

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

1) ✓
2) ✓
3) ✓

RREF ✓

$$\begin{bmatrix} 0 & \boxed{1} & 0 & 4 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1) REF ✓
2) ✓
3) ✓

RREF ✓

Something not in REF			
	$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ Not RREF X	$\begin{bmatrix} \boxed{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Not RREF X (but can quickly make it RREF)	$\begin{bmatrix} \boxed{1} & 0 & 3 & 1 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$ Does not satisfy 3rd condition. Not RREF. X

$A = [?] \in R^{n \times m}$ ← some arbitrary $n \times m$ real matrix.

1) You can always reduce A to RREF (and thus REF).

This RREF matrix is unique.

Whatever order + steps you take, you'll get the exact same matrix as a result (or you fucked up)

In general, the REF matrices are not unique.
(w/ exception of all-zero matrises)

∞ - many possibilities

Leading entry of a row	Leftmost non-zero entry of a non-zero row
Echelon Form of A	Any REF matrix that can be obtained by applying EROs to the matrix A
Pivot	The leading entry of a row of a matrix in Echelon form
Pivot position	The position of a pivot in an Echelon form of the matrix The pivot positions of the infinite number of REF matrices will always be the same

$$\begin{bmatrix} 0 & \boxed{1} & 0 & 4 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

← Ex: no matter how this is row reduced, the pivot positions will be at the same place (and the bottom row will always be zeroes).

Pivot Column: A column that contains a pivot position

cols correspond to variables.

basic variables → $x_1 \quad x_2 \quad x_3$

put in REF to find pivot positions

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 3 \\ -1 & \boxed{-1} & 3 & -1 \\ -2 & 1 & \boxed{1} & 2 \end{array}\right] \end{array}$$

will have same pivot positions

Basic variable	A variable corresponding to a pivot column
Free variable	A variable that corresponds to a column that is not a pivot column Alternatively: any variable that is not a basic variable

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \text{Not a var} \end{array}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1 : basic variable b/c there is a pivot in that column

x_2 : free variable

x_3 : free variable

augmented col: neither basic nor free var (b/c it's not a var to begin with)

row reduction algo: takes any matrix and puts into RREF form

Row Reduction Algorithm

Input: Real matrix A

Output: The RREF of A

1) Find the leftmost non-zero column. If the top row contains a zero in this column, user interchange to switch a row with the top row that does not have a zero in that column. The top row is the current row and S is the first non-zero elt of the row.
Want to get a non-zero elt. into the topmost position that you can.

2) R(i)’ = 1/S R(i) where R(i) is the current row.

3) R(j)’ = R(j) - kR(i) for all j =/= i and k is the entry of row j in the current column.
   During tests, can do all of these operations at once??
   Take entire column in same pos of 1 in step 2, zero out everything in row
   k and S are same column

4) Ignoring all rows that have been a current row, repeat step 1

5) No longer ignoring rows, repeat steps 2 and 3 (with current row identified in step 4)

6) Repeat steps 4 and 5 until step 4 fails to find a row.

PRACTICE!!

Fundamentals of RR Algo:

1) Find a pivot (step 1)
2) Set pivot to 1 (step 2)
3) Zero out the rest of the column (step 3)

Repeat (steps 4, 5, 6)

prof does not care as long how you do it as you follow EROs and get to RREF of A

Note that textbook has a slightly different algorithm that can be followed.

If anything in lecture does not make sense, check course outline to find corresponding textbook section and learn from that

advantages of RREF: will give us all solutions, including infinitely many

$2x_2 + 2x_3 = 0$
$x_2 + x_3 = 0$
$x_2 + 2x_3 + x_4 = 1$

$$\rightarrow \left[\begin{array}{cccc|c} 0 & \boxed{2} & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

S

① leftmost column, does not have a zero in colum top row

②

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right] \quad R_1' = \frac{1}{2} R_1$$

④ ignore row 1

③ 3

top elt, will be pivot here

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

pivot!

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

don't need to write this step ②

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2' = \frac{1}{1} R_2$$

③ ?

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} R_1' = R_1 - R_2 \\ R_3' = R_3 - 0R_2 \end{array}$$

redundant

can stop here, in RREF form

$x_2 - x_4 = -1$
$x_3 + x_4 = 1$
$0 = 0$

tells us we had redundant eqn in previous set (first 2 mean the same thing)

set free vars to arbitrary things that exist in $\mathbb{R}$

x_1 = free
x_2 = basic
x_3 = basic
x_4 = free

set the free vars to free vars
Let $x_1 = s$
Let $x_4 = t$

$x_2 - t = -1$
$x_3 + t = 1$
$\Rightarrow$
$x_2 = -1 + t$
$x_3 = 1 - t$

solution set (every single sol that exists):
$x_1 = s$
$x_2 = t - 1$
$x_3 = 1 - t$ for any $s, t \in \mathbb{R}$
$x_4 = t$

no free vars = single solution.

If you get $0 = 1$ you have NO SOLUTIONS!

If the system is inconsistent then the RREF matrix will correspond to the equation $0 = 1$.

If you do not get $0 = 1$ you ① fucked up
② have a system that is consistent

by def of RREF will be 1.

$0\ 0\ 0 \mid 1$ ← must be 1.