# 11 - Feb 13 Lecture

- Column space
- Nullspace
- Basis
- Finding basis for Null(A)
- Finding basis for Col(A)
- Finding basis for Span($\vec{v}_1, \ldots, \vec{v}_k$)

**Addition to Last Lecture**

$H$ is a subspace if all of the following are true:

1) $\vec{0} \in H$
2) $\vec{u}, \vec{v} \in H \implies \vec{u} + \vec{v} \in H$
3) $\vec{u} \in H$ and $c \in \mathbb{R} \implies c\vec{u} \in H$

If $T$ is a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, Range(T) is a subspace of $\mathbb{R}^m$ and Ker(T) is a subspace of $\mathbb{R}^n$.

**Column Space** (Matrix version of Range)

The column space of a matrix $A$, denoted Col(A), is the set of all linear combinations of the columns of $A$.

- If $a_1, \dots, a_m$ are the columns of $A$, then $\text{Col}(A) = \text{Span}(a_1, \dots, a_n)$.
- If $T(\vec{v}) = A\vec{v}$ for all $\vec{v}$, then $\text{Col}(A) = \text{Range}(T)$.

**Nullspace** (Matrix version of Ker)

The nullspace of $A$, denoted Null(A), is the set of all vectors $\vec{v}$ such that $A\vec{v} = \vec{0}$.

- If $T(\vec{v}) = A\vec{v}$ for all $\vec{v}$, then $\text{Null}(A) = \text{Ker}(T)$.

**Basis**

Context: Span of something usually includes something redundant

↳ $\text{Span}(a_1, \dots, a_n)$ is often equivalent to $\text{Span}(a_1, \dots, a_{n-1})$.

"A basis is what's left after knocking out all the redundant information".

A basis for a subspace $H$ of $\mathbb{R}^n$ is a linearly independent set of vectors that span $H$.

- i.e. $\text{Span}(\vec{h}_1, \dots, \vec{h}_k) = H$.

You can always write a basis for a subspace of $\mathbb{R}^n$.

Ex 1: Show that $\vec{e_1}, \vec{e_2}, \vec{e_3}$ is a basis of $\mathbb{R}^3$.

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad H = \mathbb{R}^3 \rightarrow n = 3$ by definition

These vectors are linearly independent since the only solution is the trivial solution $c_1 = c_2 = c_3 = 0$.

To show spanning, we show $c_1\vec{e_1} + c_2\vec{e_2} + c_3\vec{e_3} = \vec{x}$ is consistent for all of the subspace (in this case, for all $\vec{x} \in H$).

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $\underline{x_1\vec{e_1} + x_2\vec{e_2} + x_3\vec{e_3} = \vec{x}}$

Being able to write this out shows spanning.

As $\vec{x}$ was an arbitrary vector of H, we conclude that $H = \text{Span}(\vec{e_1}, \vec{e_2}, \vec{e_3})$. Therefore, $\vec{e_1}, \vec{e_2}, \vec{e_3}$ is a basis of $\mathbb{R}^3$.

Ex 2. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is also a basis for $\mathbb{R}^3$.

## Finding a Basis for Null (A)

When solving $A\vec{x} = \vec{0}$, we can write the answer in parametric vector form.

Recall that the general form of parametric vector form is:

[constants] + free variable $[\vdots]$ + free variable $[\vdots]$ + ... + free variable $[\vdots]$

The non-constant vectors of the parametric vector form will form a basis.

[~~constants~~] + s[ ] + t[ ]

non-constant vectors form a basis

The algorithm for finding a basis for Null(A) is:
1) Solve for the solutions of A when $A\vec{x} = \vec{0}$.
2) Write the solution for $A\vec{x} = \vec{0}$ in parametric vector form.
3) Remove the constant vector.

Note: { } is a basis for $\{\vec{0}\}$.

empty set

Ex: Find a basis for Null(A).

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A\vec{x} = \vec{0} = \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & \boxed{1} & 3 & 0 \end{array}\right] \quad R_2' = R_2 + R_1$$

$$\sim \begin{array}{c} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 3 & 0 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \\ B & B & F \end{array} \end{array} \quad R_1' = R_1 - R_2$$

$x_3 = t$

$x_1 - x_3 = 0 \longrightarrow x_1 = t$

$x_2 + 3x_3 = 0 \longrightarrow x_2 = -3t$

(remove in next step)

Solution: $\vec{x} = \begin{bmatrix} t \\ -3t \\ t \end{bmatrix} = \underbrace{t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}}_{\text{parametric vector form}}$ for all $t \in \mathbb{R}$.

linearly independent spanning set for Null(A)

Therefore, $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for Null(A).

**Finding Basis for Col(A)**

The pivot columns of A form a basis for Col(A).

The algorithm for finding a basis for Col(A) is:
1) Calculate a REF of A (RREF works as well).
2) Identify the pivot positions.
3) The columns of A corresponding to those pivot positions form a basis for Col(A).

Note that the columns with the pivot positions in the RREF of matrix A are NOT what we are looking for. We are looking for the corresponding columns in the ORIGINAL matrix A.

Ex: Find a basis for Col (A)

$$A = \begin{bmatrix} \boxed{1} & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & 3 \end{bmatrix} \quad R_2' = R_2 + R_1$$

pivot columns

pivot position

these are the columns of A, NOT of the RREF of A.

This tells us that $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis for Col(A).

Find Basis for Span$(\vec{v}_1, \dots, \vec{v}_k)$

The algorithm for finding a basis for Span$(\vec{v}_1, \dots, \vec{v}_k)$ is:

1) Make a matrix with $\vec{v}_1, \dots, \vec{v}_k$ as the columns (even if they're row vectors).
2) Row reduce the matrix.
3) The vectors corresponding to pivot columns form a basis.
4) Rewrite the vectors as row vectors if they were originally row vectors.

| | |
|---|---|
| Add. to last lec? | H is a subspace of $\mathbb{R}^n$ if (all of the following):<br>1) $\vec{0} \in H$<br>2) $\vec{u}, \vec{v} \in H \Rightarrow \vec{u}+\vec{v} \in H$<br>3) $\vec{u} \in H$ $c \in \mathbb{R}$ $\Rightarrow c\vec{u} \in H$<br><br>If T is a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, Range(T) is a subspace of $\mathbb{R}^m$ and Ker(T) is a subspace of $\mathbb{R}^n$. |
| Column Space (matrix equiv. of Range) | The column space of a matrix A, denoted Col(A), is the set of all linear combinations of the columns of A.<br><br>If $a_1, \dots, a_m$ are the columns of A, then Col(A) = Span $(a_1, \dots, a_n)$<br><br>If $T(\vec{v}) = A\vec{v}$ for all $\vec{v}$ then Col(A) = Range(T). |
| Nullspace (matrix equiv. of Ker) | The nullspace of A, Null(A), is the set of all vectors $\vec{v}$ s.t. $A\vec{v} = \vec{0}$.<br><br>If $T(\vec{v}) = A\vec{v}$ for all $\vec{v}$, then Null(A) = Ker(T) |
| Basis | Span usually has something redundant.<br>span $(a_1, \dots, a_n)$ span $(a_1, \dots, a_{n-1})$<br><br>Basis: what's left after knocking out all redundant info<br>(only in regards to a supspace)<br>A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set of vectors that span H.<br>i.e. span $(\vec{h_1}, \dots, \vec{h_k}) = H$<br><br>You can always write a basis for a subspace of $\mathbb{R}^n$. |
| aside: (last lec.)<br>$\mathbb{R}^n$ is a subspace of $\mathbb{R}^m$ | Standard basis of $\mathbb{R}^3$:<br>$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ($H = \mathbb{R}^3, n = 3$ in definition) Linearly independent ✓<br>These vectors are linearly independent. $(c_1 = c_2 = c_3 = 0)$<br>To show spanning, we show $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = \vec{x}$ is consistent for all $\vec{x} \in H$.<br>(for all of subspace) |

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3 = \vec{x}$

As $\vec{x}$ was an arbitrary vector of H, we conclude that $H = \text{Span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$
"H is contained inside $\text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$"

revised: As $\vec{x}$ was an arbitrary vector of H, $\vec{e}_1, \vec{e}_2, \vec{e}_3 \in H$, and H is a subspace, then $H = \text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.

$\therefore$ $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is a basis of $\mathbb{R}^3$

Ex: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is also a basis for $\mathbb{R}^3$

Finding a basis for Null(A) ... we have an algo for this.

When solving $A\vec{x} = \vec{0}$ we can write the answer in parametric vector form.

reminder of general form: $[\text{constant}] + \text{free var}[\;] + \text{free var}[\;]$

$s[\;] + t[\;]$

vectors form a basis

If we obtain the parametric vector form using free vars as shown in class, non-constant vectors will form a basis.

Finding basis for Null(A) Ex.
"get nullspace"

Find a basis for Null (A)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \quad R_2' = R_2 + R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \quad R_1' = R_1 - R_2$$

B B F

$x_3 = t$

$x_1 - x_3 = 0 \rightarrow x_1 = t$

$x_2 + 3x_3 = 0 \rightarrow x_2 = -3t$

In parametric vector form, the solution set is:

$$t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \text{ for all } t \in \mathbb{R}$$

$$\therefore \left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\} \text{ is a basis for Null (A).}$$

vector
(lin. indp.)

"Method gives lin. ind., spanning set for Null (A)."

$\{\ \}$ is a basis for $\{\vec{0}\}$.

## Finding basis for Col(A)

For finding a basis for Col(A), the <u>pivot columns of A</u> form a basis for Col(A).

The algo is to calculate a REF of A (RREF works) and identify the pivot positions. The <u>columns</u> of A corresponding to those pivot positions form a basis for Col (A).

Ex:

pivot positions

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 3 \end{bmatrix}$$

columns of A!!
not of RREF of A.

common mistake!!

This tells us that $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis for Col (A).

## Find basis for Span $(\vec{v}_1, \ldots, \vec{v}_k)$

Finding a basis for span $(\vec{v}_1, \ldots, \vec{v}_k)$

1) make a matrix w/ $\vec{v}_1, \ldots, \vec{v}_k$ as the <u>columns</u>. (even if they're row vectors)
2) Row reduce the matrix
3) The vectors corresponding to pivot columns form a basis. (rewrite as row vectors if they were row vectors)