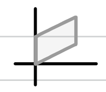
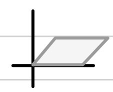
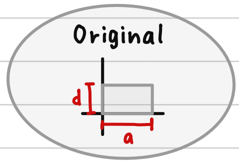
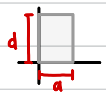
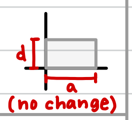
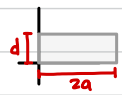
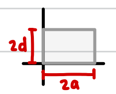
16 - Mar 8 Lecture

- effects of row replacement, row interchange, and scaling on rectangle
- eigenvalues
- eigenvectors
- eigenspaces
- how to find eigenvalues of a matrix

Addition and correction to lecture 15		$\begin{bmatrix} a \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ d \end{bmatrix}$	$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$	
	Row Replacement $R_1' = R_1 + R_2$			
	Row Interchange $R_1 \leftrightarrow R_2$			
	Scaling $R_1 = 2R_1$			

Eigenvalues

An eigenvalue of A is a scalar λ (lambda) such that $A\vec{x} = \lambda\vec{x}$ has a non-trivial $(\vec{x} \neq \vec{0})$ solution $\vec{x}$.

↳ vectors that, when multiplied by A, get scaled

Eigenvectors

Such an $\vec{x}$ is called an "eigenvector corresponding to λ."

Eigenspaces

The set of all eigenvectors of A corresponding to λ union (including) the zero vector is known as the "eigenspace of A corresponding to λ."

Note: eigenvalues / eigenvectors only exist for square matrices.

Ex.

$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\vec{x}$ and $\vec{y}$ are eigenvectors of A.

$A\vec{x} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1(1)+3(-1) \\ 1(1)-1(-1) \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2\vec{x}$

$\vec{x}$ is an eigenvector corresponding to -2.

eigenvalue: -2

$A\vec{y} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(3)+3(1) \\ 1(3)-1(1) \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 2\vec{y}$

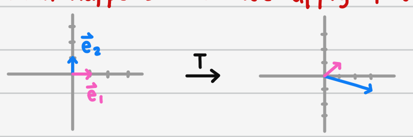
$\vec{y}$ is an eigenvector corresponding to 2.

eigenvalue: 2

Note: You can think of $A\vec{x}$ like a linear transformation on $\vec{x}$.

Ex: Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{z}) = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \vec{z}$.

What happens when we apply T on a vector that is not an eigenvector?

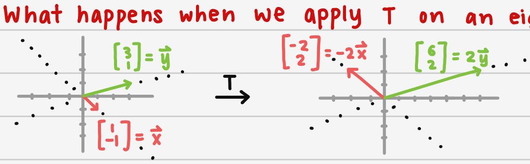


$$T(\vec{e}_1) = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(1)+3(0) \\ 1(1)-1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_2) = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(0)+3(1) \\ 1(0)-1(1) \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$\vec{z}$ does not get scaled. $\vec{z}$ has an unpredictable transformation in a random direction.

What happens when we apply T on an eigenvector? (From example on previous page)



$\vec{z}$ gets scaled and stays along the same "line"

Ex. Given A and 3 of its eigenvectors, find the corresponding eigenvalues.

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \vec{v}_1 \quad \Longrightarrow \lambda_1 = 1$$

$$A\vec{v}_2 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 \cdot \vec{v}_2 \quad \Longrightarrow \lambda_2 = 2$$

$$A\vec{v}_3 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = -3 \cdot \vec{v}_3 \quad \Longrightarrow \lambda_3 = -3$$

Note: order doesn't matter, just specify which is which.
i.e. $\lambda_1 = -3$, $\lambda_2 = 2$, $\lambda_3 = 1$ works as well

Ex: Identify the eigenvalues corresponding to $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A\vec{e}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\vec{e}_1 \implies \lambda_1 = 2 \text{ with corresponding eigenvector } \vec{e}_1$$

$$A\vec{e}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -1\vec{e}_2 \implies \lambda_2 = -1 \text{ with corresponding eigenvector } \vec{e}_2$$

$$A\vec{e}_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\vec{e}_3 \implies \lambda_3 = 0 \text{ with corresponding eigenvector } \vec{e}_3$$

Note: while an eigenvector cannot be a zero vector, an eigenvalue CAN be zero.

If $\vec{v}_1, \dots, \vec{v}_p$ are eigenvectors corresponding to distinct eigenvalues, then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent.

$$A\vec{x} = \lambda\vec{x} \implies A\vec{x} = \lambda I\vec{x} \implies A\vec{x} - (\lambda I\vec{x}) = \vec{0} \implies (A - \lambda I)\vec{x} = \vec{0}$$

Recall: if $B\vec{x} = \vec{0}$, B is not invertible

↳ λ is an eigenvalue of A if and only if $(A - \lambda I)$ is not invertible.

↳ λ is an eigenvalue of A if and only if $(A - \lambda I)\vec{x} = \vec{0}$.

↳ λ is an eigenvalue of A if and only if $\operatorname{Det}(A - \lambda I) = 0$.

The eigenspace corresponding to λ is the null space of $A - \lambda I$.

Eigenspaces are subspaces. → this is why we had to add $\vec{0}$ to eigenspace

To find eigenvectors for a given eigenvalue, identify the null space of $A - \lambda I$.

A is invertible if and only if 0 is not an eigenvalue

Finding eigenvalues → hard

Finding eigenvalues w/ eigenvectors → easy (algorithm)

Ex: Find the null space of $A-\lambda I$.

$$A=\begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

(A from previous example)

$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -3$

$$\begin{bmatrix} 1-1 & 3 & -2 \\ 3 & -2-1 & 1 \\ 2 & -1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -2 \\ 3 & -3 & 1 \\ 2 & -1 & 0 \end{bmatrix} \sim \dots \sim \begin{array}{c} \begin{array}{ccc} B & B & F \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \leftarrow \text{RREF}$$

$A-\lambda_1 I \longrightarrow -1$ on main diagonal

Let $x_3 = t$

$$\begin{aligned} x_1 - \tfrac{1}{3}x_3 &= 0 \\ x_2 - \tfrac{2}{3}x_3 &= 0 \\ 0 &= 0 \end{aligned} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$$

$$\text{Eigenspace} = \text{Null space} = \text{Span}\left\{\begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}\right\} \underset{\text{so integers}}{\overset{\times 3}{\Longrightarrow}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$ is an eigenvector corresponding to λ_1

Ex: Find an eigenvector of the matrix below.

$$\left[\begin{array}{ccc|c} 4 & 3 & -2 & 0 \\ 3 & 1 & 1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right] \sim \dots \sim \begin{array}{c} \begin{array}{ccc} B & B & F \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$A-\lambda_3$

Add 3 to main diagonal since $\lambda_3 = -3$

Eigenspace is: $\left\{\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} \text{ for all } t \in \mathbb{R}\right\}$

$= \text{Span}\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\} \Longrightarrow \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector

$\overset{\times -2}{\Longrightarrow} \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$ is another

Let $x_3 = t$

$$\begin{aligned} x_1 + x_3 = 0 &\longrightarrow x_1 = -t \\ x_2 - 2x_3 = 0 &\longrightarrow x_2 = 2t \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \\ 0 = 0 & \end{aligned}$$

"Evil" method for finding eigenvalues of a matrix :

1. Find det of $A-\lambda I$
2. Factor polynomial

Ex: For what values of λ is A invertible? (aka when det $\neq 0$)

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \qquad A-\lambda I = \begin{bmatrix} 1-\lambda & 3 & -2 \\ 3 & -2-\lambda & 1 \\ 2 & -1 & 1-\lambda \end{bmatrix}$$

This is a "characteristic polynomial."

$$\text{Det}\left(\begin{bmatrix} 1-\lambda & 3 & -2 \\ 3 & -2-\lambda & 1 \\ 2 & -1 & 1-\lambda \end{bmatrix}\right) = -\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$$

Eigenvalues : zeroes of the characteristic polynomial

Ex: Find the eigenvalues of matrix A.

$$A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \qquad A-\lambda I = \begin{bmatrix} 3-\lambda & 5 \\ 1 & -1-\lambda \end{bmatrix}$$

$$\text{Det}\begin{bmatrix} 3-\lambda & 5 \\ 1 & -1-\lambda \end{bmatrix} = (3-\lambda)(-1-\lambda) - (5)(1) = -3-3\lambda+\lambda+\lambda^2-5 = \lambda^2-2\lambda-8$$

Find roots using the quadratic formula.

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(-8)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4+32}}{2}$$

$$= \frac{2 \pm \sqrt{36}}{2}$$

$$= \frac{2 \pm 6}{2}$$

$$= 1 \pm 3 \quad \overset{-}{\Longrightarrow} \lambda_1 = 1-3 = -2$$

$$\overset{+}{\Longrightarrow} \lambda_2 = 1+3 = 4$$

end

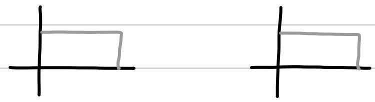
	$\begin{bmatrix} a \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ d \end{bmatrix}$	$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$
Corr. to last lec. Add. to last lec.		

Row replacement
$R' = R_1 + R_2$

Row Interchange
$R_1 \leftrightarrow R_2$

(no change)

Scaling
$R_1 = 2R_1$



"$\vec{v}$ that when multiplied by A, get scaled."

Exs: $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ "eigenvectors of A"

$A\vec{x} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2\vec{x}$ ← eigenvector corresponding to -2

eigenvalue : -2

like a lin. trans. on $\vec{x}$

$A\vec{y} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 2\vec{y}$ ← eigenvector corresponding to 2

eigenvalue : 2

swap order

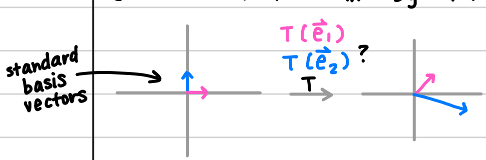
lamda

An **eigenvalue** of A is a scalar λ such that $A\vec{x} = \lambda\vec{x}$ has a non-trivial (not $\vec{0}$) solution $\vec{x}$.
Such an $\vec{x}$ is called an "**eigenvector** corresponding to λ".

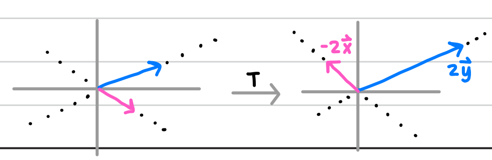
The set of all eigenvectors of A corresponding to λ (including) union the zero vector is known as the **eigenspace** of A corresponding to λ.

Eigenvalues / vectors only appear for square matrices.

Ex: Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{z}) = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \vec{z}$ $\qquad \vec{z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T(\vec{z}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



T on (Non-eigenvectors)
↳ random direction



T on (eigenvectors)
↳ stay along same line

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$A\vec{v}_1 = \underline{1} \cdot \vec{v}_1$ $\quad A\vec{v}_2 = \underline{2}\,\vec{v}_2$ $\quad A\vec{v}_3 = \underline{-3}\,\vec{v}_3$

$\lambda_1 = 1$ $\quad \lambda_2 = 2$ $\quad \lambda_3 = -3$

Order does not matter, just specify which is which.
$\rightarrow \lambda_1 = -3,\ \lambda_2 = 2,\ \lambda_3 = 1$ is fine too

Ex. Find the

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A\vec{e}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\vec{e}_1 \implies \lambda_1 = 2 \text{ with corresponding eigenvector } \vec{e}_1$$

$$A\vec{e}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -1\vec{e}_2 \implies \lambda_2 = -1 \text{ with corresponding eigenvector } \vec{e}_2$$

☆ eigenvector cannot be zero vector, but λ can be zero.

$$A\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\vec{e}_3 \implies \lambda_3 = 0 \text{ with corresponding eigenvector } \vec{e}_3$$

Generalization

If $\vec{v}_1, \dots, \vec{v}_p$ are eigenvectors corresponding to distinct (i.e. different) eigenvalues, then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent.

$B\vec{x} = \vec{0} \implies B$ not invertible

$$A\vec{x} = \lambda\vec{x} \implies A\vec{x} = \lambda I \vec{x} \implies A\vec{x} - (\lambda I)\vec{x} = \vec{0} \implies \underline{(A - \lambda I)}\vec{x} = \vec{0}$$

$\rightarrow \lambda$ is an eigenvalue of A iff $(A - \lambda I)$ is not invertible.
$\rightarrow \lambda$ is an eigenvalue of A iff $\text{Det}(A - \lambda I) = 0$.

The eigenspace corresponding to λ is the null space of $A - \lambda I$.
Eigenspaces are subspaces. (which is why we had to add back $\vec{0}$)

To find eigenvectors for a given eigenvalue, identify the null space of $A - \lambda I$.
A is invertible iff 0 is not an eigenvalue.

finding eigenvalues → hard

finding eigenvalues w/ eigen vectors? → easy (algo)

want to find null space of $A - \lambda I$

A from above

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad \lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = -3$$

RREF

$$\underbrace{\left[\begin{array}{ccc|c} 0 & 3 & -2 & 0 \\ 3 & -3 & 1 & 0 \\ 2 & -1 & 0 & 0 \end{array}\right]}_{A - \lambda_1 I} \sim \ldots \sim \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} B & B & F \end{array} \end{array}$$

Eigenspace is = Null space Span $\left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix} \right\} \overset{\times 3}{\Rightarrow} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigen $\vec{v}$ corresponding to λ_1

-1 on main diagonal

Let $x_3 = t$

$$x_1 = \tfrac{1}{3}t, \quad x_2 = \tfrac{2}{3}t, \quad 0 = 0 \quad \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$$

$$\underbrace{\left[\begin{array}{ccc|c} 4 & 3 & -2 & 0 \\ 3 & 1 & 1 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right]}_{A - \lambda_3 I} \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Eigenspace is: $\left\{ \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} \text{ for all } t \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

wierd result → RR Algo wrong → not eigen — ?

add 3 to main diagonal since $\lambda_3 = -3$

$x_3 = t$

$$x_1 = -t, \quad x_2 = 2t \quad \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \text{ is an eigenvector}$$

$\begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$ is another

? ☆

invertible for what values of λ? aka when $\det \neq 0$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 3 & -2 \\ 3 & -2-\lambda & 1 \\ 2 & -1 & 1-\lambda \end{bmatrix}$$

evil method :(

→ find det

→ factor polynomial

$$\text{Det}\left(\begin{bmatrix} 1-\lambda & 3 & -2 \\ 3 & -2-\lambda & 1 \\ 2 & -1 & 1-\lambda \end{bmatrix} \right) = -\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$$

General method for finding __ ? → Find det (not good method)

"characteristic polynomial"

What are the eigenvalues?

↳ zeroes of the characteristic polynomial :(

Ex: Find eigenvalues of matrix

Find det

$A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} 3-\lambda & 5 \\ 1 & -1-\lambda \end{bmatrix}$ $\text{Det}\left(\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}\right) = (3-\lambda)(-1-\lambda) - (5)(1) = \lambda^2 - 2\lambda - 8$

Find roots (quadratic formula)

$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} =$

more - look

at pic!