Example 1. Find the intervals where each function f is increasing and those where it is decreasing:

(a) $f(x) = -3^{1-x}$ **(b)** $f(x) = x^4 - 2x^2$

Step ①: Take the derivative of f to get f'.
Step ②: Find the "critical points" of f, aka where $f' = 0$.
Step ③: Test the regions to find where f is increasing and decreasing.

Notes:

- f is increasing when $f' > 0$.
- f has a "critical point" when $f' = 0$.
- f is decreasing when $f' < 0$.

For (a):

Derivative of a constant multiple of a function: $\frac{d}{dx}[c\,f(x)] = c\,\frac{d}{dx}f(x)$

Chain rule for exponential functions: $\frac{d}{dx}a^{f(x)} = a^{f(x)} \cdot \ln a \cdot f'(x)$

For (b):

Sum rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Derivative of a constant multiple of a function: $\frac{d}{dx}[c\,f(x)] = c\,\frac{d}{dx}f(x)$

Power rule: $\frac{d}{dx}(x^r) = r\,x^{r-1}$

Example 2. Management estimates that the profit (in dollars) realizable by a company for the manufacture and sale of x units of thermometers each week is $P(x) = -0.001x^2 + 8x - 5000$.

(a) Find the marginal profit function. What is the marginal profit when the level of manufacturing and sales is at 1000 units? Interpret your result.

Notes:

- The marginal profit function, $P'(x)$, is the derivative of the total cost function, $P(x)$.
- Marginal cost is the change in the cost incurred when the production level is raised by one additional unit.
- $P'(1000)$ will give us the approximate profit from the manufacturing and sale of the 1001-st unit.

For (a):

Sum rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Derivative of a constant multiple of a function: $\frac{d}{dx}[c\,f(x)] = c\,\frac{d}{dx}f(x)$

Power rule: $\frac{d}{dx}(x^r) = r\,x^{r-1}$

Derivative of a constant: $\frac{d}{dx}c = 0$

(b) Find the intervals where the profit function is increasing and the intervals where it is decreasing.

- P is increasing when $P' > 0$.
- P is decreasing when $P' < 0$.

Example 3. Let the demand function be given by: $x = f(p) = \sqrt{400 - 0.01p^2}$, $(0 \leq p \leq 200)$, where p is the price in dollars and x is the quantity demanded.

(a) Is the demand elastic, inelastic, or unitary when $p = 120$?

Step ①: Find f' by taking the derivative of f.

Step ②: Find the Elasticity of Demand function, $E(p)$.

Step ③: Calculate the Elasticity of Demand at p_0 by finding the value of $E(p_0)$.

Step ④: Interpret the Elasticity of Demand by comparing it to 1.

Notes:

- $E(p) = -\frac{p \cdot f'(p)}{f(p)}$
- Demand is elastic at p_0 when $E(p_0) > 1$.
- Demand is unitary at p_0 when $E(p_0) = 1$.
- Demand is inelastic at p_0 when $E(p_0) < 1$.

<u>For (a):</u>

Chain rule for power functions: $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x)$

Sum rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Derivative of a constant: $\frac{d}{dx}c = 0$

Derivative of a constant multiple of a function: $\frac{d}{dx}[c\, f(x)] = c\,\frac{d}{dx}f(x)$

Power rule: $\frac{d}{dx}(x^r) = r\, x^{r-1}$

(b) If the price is 120, will raising it slightly cause the revenue to increase or decrease?

- If demand is elastic: small increase in price $\Leftrightarrow$ bigger decrease in revenue.
- If demand is unitary: small increase in price $\Leftrightarrow$ same increase in revenue (1:1).
- If demand is inelastic: small increase in price $\Leftrightarrow$ smaller increase in revenue.

Example 4. Find the classify the critical numbers of the following functions:

(a) $f(x) = log_2(3x^2 + 1)$ **(b)** $f(x) = (x - 3)^{5/3}$

Step ①: Identify domain of f.
Step ②: Find f' by taking derivative of f.
Step ③: Identify domain of f'.
Step ④: Compare $Dom\{f'\}$ to $Dom\{f\}$ if they aren't both $\mathbb{R}$.

- If $Dom\{f'\}$ has a restriction that DOES NOT ALSO EXIST in $Dom\{f\}$, then that value of x is a **critical value**.

Step ⑤: Determine which values of x make $f'(x) = 0$.

- These values of x are **critical values**.

Step ⑥: Classify critical values (3 types).

- Local minimum: ⊖ → ⊕
- Local maximum: ⊕ → ⊖
- Neither: sign doesn't change. Either ⊕ → ⊕ or ⊖ → ⊖.

For (a):

Chain rule for logarithmic functions: $\frac{d}{dx} log_a f(x) = \frac{1}{f(x) \cdot ln\, a} \cdot f'(x) = \frac{f'(x)}{f(x) \cdot ln\, a}$

Sum rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Derivative of a constant multiple of a function: $\frac{d}{dx}[c\, f(x)] = c\, \frac{d}{dx}f(x)$

Power rule: $\frac{d}{dx}(x^r) = r\, x^{r-1}$

Derivative of a constant: $\frac{d}{dx}c = 0$

(b) $f(x) = (x - 3)^{5/3}$

For (b):

Chain rule for power functions: $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x)$

Sum rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Power rule: $\frac{d}{dx}(x^r) = r\,x^{r-1}$

Derivative of a constant: $\frac{d}{dx}c = 0$

Example 5. Find and classify the critical numbers of the following functions:

(a) $f(x) = 2x^4 - 4x^2 + 3$ **(b)** $g(x) = \frac{x^2}{2-x}$

Step ①: Identify domain of f.
Step ②: Find f' by taking derivative of f.
Step ③: Identify domain of f'.
Step ④: Compare $Dom\{f'\}$ to $Dom\{f\}$ if they aren't both $\mathbb{R}$.

- If $Dom\{f'\}$ has a restriction that DOES NOT ALSO EXIST in $Dom\{f\}$, then that value of x is a **critical value**.

Step ⑤: Determine which values of x make $f'(x) = 0$.

- These values of x are **critical values**.

Step ⑥: Classify critical values (3 types).

- Local minimum: $\ominus \to \oplus$
- Local maximum: $\oplus \to \ominus$
- Neither: sign doesn't change. Either $\oplus \to \oplus$ or $\ominus \to \ominus$.

For (a):

Sum rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Derivative of a constant multiple of a function: $\frac{d}{dx}[c\,f(x)] = c\,\frac{d}{dx}f(x)$

Power rule: $\frac{d}{dx}(x^r) = r\,x^{r-1}$

Derivative of a constant: $\frac{d}{dx}c = 0$

(b) $g(x) = \frac{x^2}{2-x}$

<u>For (b):</u>

Quotient rule: $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$ when $g(x) \neq 0$

Power rule: $\frac{d}{dx}(x^r) = r\,x^{r-1}$

Sum rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Derivative of a constant multiple of a function: $\frac{d}{dx}[c\,f(x)] = c\,\frac{d}{dx}f(x)$

Power rule: $\frac{d}{dx}(x^r) = r\,x^{r-1}$