# 14 - Mar 1 Lecture

- Upper and lower triangular square matrices
- Finding the determinant of an upper or lower triangular matrix
- Relationship between the determinant and row reduction
- Common mistakes regarding determinants

# Determinants Pt. 2 "less error-prone method"

**Upper Triangular**
**Lower Triangular**

A $n \times n$ matrix $A$ is called upper triangular if $A_{i,j} = 0$ for all $i > j$.
A $n \times n$ matrix $B$ is called lower triangular if $A_{i,j} = 0$ for all $i < j$.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is upper triangular

Ex: $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 5 \end{bmatrix}$ is lower triangular

Ex: zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is both upper and lower triangular

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is both upper and lower triangular

Ex: What is Det(A)?

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 & 7 \\ 0 & 2 & \pi & \sqrt{2} & -12 \\ 0 & 0 & 1 & 7 & 18 \\ 0 & 0 & 0 & -3 & 22 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

Let $j = 1$.

$$\text{Det}(A) = (-1)^{1+1}(2)\,\text{Det}\begin{bmatrix} 2 & \pi & \sqrt{2} & -12 \\ 0 & 1 & 7 & 18 \\ 0 & 0 & -3 & 22 \\ 0 & 0 & 0 & -3 \end{bmatrix} + 0 + 0 + 0 + 0$$

Let $j = 1$

$$= (2)\left((-1)^2 (2)\,\text{Det}\begin{bmatrix} 1 & 7 & 18 \\ 0 & -3 & 22 \\ 0 & 0 & -3 \end{bmatrix} + 0 + 0 + 0\right)$$

$$= (2)(2)\left((-1)^2 (1)\,\text{Det}\begin{bmatrix} -3 & 22 \\ 0 & -3 \end{bmatrix} + 0 + 0\right)$$

$$= (2)(2)(1)\left[(-3)(-3) - \cancelto{0}{(22)(0)}\right]$$

$$= \underline{(2)(2)(1)(-3)(-3)}$$ multiply terms on the diagonal

$$= \boxed{36}$$

| | |
|---|---|
| Determinant of Upper or Lower Triangular Matrix | The determinant of an upper <u>or</u> lower triangular matrix is the <u>product</u> of the entries on the main diagonal. $$\operatorname{Det}(A) = \prod_{i=1}^{n} A_{i,i} = A_{1,1} \cdot A_{2,2} \cdot \ldots \cdot A_{n,n}$$ multiplication equivalent of $\Sigma$ |
| Relationship between Row Reduction and the determinant | Ex: If $A \sim B$ $R_3' = 2R_3$ then $\operatorname{Det}(B) = 2\operatorname{Det}(A)$ <br> Why? $$\operatorname{Det}(A) = \underbrace{\sum_{j=1}^{n} (-1)^{i+j}}_{\text{unchanged}} \underbrace{A_{i,j}}_{\text{doubled}} \underbrace{\operatorname{Det}(A(i,j))}_{\text{unchanged}}$$ |
| general formula | $A \sim C$ $R_i = sR_i$ $\operatorname{Det}(C) = s \operatorname{Det} A$ — double row $\rightarrow$ det doubles <br> $A \sim D$ $R_i = R_i - sR_j$ $\operatorname{Det}(D) = \operatorname{Det}(A)$ — repl. operation $\rightarrow$ det unchanged <br> make sure you're not scaling $R_i$ <br> $A \sim E$ $R_i \leftrightarrow R_j$ $\operatorname{Det}(E) = -\operatorname{Det}(A)$ — row interchange $\longrightarrow$ negates Det. <br> odd # interchanges $\longrightarrow$ negate det <br> even # interchanges $\longrightarrow$ same det <br> All REF square matrices are upper triangular. |

Ex: What is Det (A) ?

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -3 & -6 & 3 \\ 2 & 7 & 2 \end{bmatrix} \sim B_1 = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & -3 \\ 0 & 3 & 6 \end{bmatrix} \begin{matrix} \\ R_2' = R_2 + 3R_1 \\ R_3' = R_3 - 2R_1 \end{matrix}$$

$$\sim B_2 = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ \\ R_3' = \frac{1}{3} R_3 \end{matrix}$$

$$\sim B_3 = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} R_2 \leftrightarrow R_3$$

$\operatorname{Det}(B_3) = (1)(1)(-3) = -3$

$\operatorname{Det}(B_3) = -\operatorname{Det}(B_2) \Longrightarrow \operatorname{Det}(B_2) = 3$

$\operatorname{Det}(B_2) = {}^1/_3 \operatorname{Det}(B_1) \Longrightarrow \operatorname{Det}(B_1) = 9$

$\operatorname{Det}(A) = \operatorname{Det}(B_1) = \boxed{9}$

Ex: Find det.

from tutorial 3

these are equal

$$\text{Det}\begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = 3\ \text{Det}\begin{bmatrix} 1 & -2/3 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad R_1' = 1/3\, R_1$$

$$= 3\ \text{Det}\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & 1 \\ 0 & 2/3 & 2 \end{bmatrix} \quad \begin{matrix} R_2' = R_2 + R_1 \\ R_3' = R_3 - R_1 \end{matrix}$$

$$= 3 \cdot 1/3\ \text{Det}\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 3 & 3 \\ 0 & 2/3 & 2 \end{bmatrix} \quad R_2' = 3R_2$$

$$= \text{Det}\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3' = R_3 - 2/3\, R_2$$

$$= (1)(1)(0)$$

$$= 0$$

$$= \text{Det}\begin{bmatrix} 3 & -1 & 1 \\ -2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ Transpose}$$

can do during row reduction. usually not helpful though

In general, lecture 13's method works better for $3 \times 3$ matrices.

This method is better for matrices $5 \times 5$ or bigger.

$4 \times 4$ w/ lots of zeroes → lecture 13

$4 \times 4$ w/o zeroes → lecture 14 method

**Property 1** $\text{Det}(A) \neq 0$ if and only if $A$ is invertible

**Property 2** $\text{Det}(A) = \text{Det}(A^T)$ (If you want to do column reductions instead of row reductions)

**Property 3** $\text{Det}(AB) = \text{Det}(A)\,\text{Det}(B)$

↳ A or B not invertible → $\text{Det}(AB) = 0$

$\text{Det}(A^{-1}) = \frac{1}{\text{Det}(A)}$ → Why?

$\text{Det}(A^k) = (\text{Det}(A))^k$ ← k is integer

$A\ A^{-1} = I_n$

det = 5 det = $\frac{1}{5}$ det = 1

Ex: $\text{Det}(A^2) = (\text{Det}(A))^2$

Most common mistake

$\text{Det}(2A) = 2^n\ \text{Det}(A)$ ($A$ is $n \times n$)

If $A$ is $n \times n$ then $\text{Det}(cA) = c^n\ \text{Det}(A)$ NOT $c\,\text{Det}(A)$

Another common mistake

$\text{Det}(A+B) =$ no formula, $\text{Det}(A+B)$ has no relationship to $\text{Det}(A)$ and $\text{Det}(B)$

NOT $\text{Det}(A) + \text{Det}(B)$

Need to sum A and B into one matrix, then calculate it

Ex: Can combine row reduction with det formula

Find determinant of matrix. (from end of lecture 13)

$$\text{Det}\begin{bmatrix} 1 & 3 & 2 & 0 \\ 4 & 1 & 3 & 1 \\ -1 & -5 & 2 & 2 \\ 2 & 3 & 1 & 4 \end{bmatrix} = \text{Det}\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & -11 & -5 & 1 \\ 0 & -2 & 4 & 2 \\ 0 & -3 & -3 & 4 \end{bmatrix} \begin{array}{l} \\ R_2' = R_2 - 4R_1 \\ R_3' = R_3 + R_1 \\ R_4' = R_4 - 2R_1 \end{array}$$

Let $j = 1$.

$$= (-1)^{1+1}(1)\,\text{Det}\begin{bmatrix} -11 & -5 & 1 \\ -2 & 4 & 2 \\ -3 & -3 & 4 \end{bmatrix} + 0 + 0 + 0$$

$$= \text{Det}\begin{bmatrix} -11 & -5 & 1 \\ 20 & 14 & 0 \\ 41 & 17 & 0 \end{bmatrix} \begin{array}{l} \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 4R_1 \end{array}$$

Let $j = 3$

$$= (-1)^{1+3}(1)\,\text{Det}\begin{bmatrix} 20 & 14 \\ 41 & 17 \end{bmatrix} + 0 + 0$$

$$= \boxed{-234}$$

↳ Det 5×5 = −702 (So much quicker to do than formula from lecture 13)

(Least error-prone method)