21 - Apr 10 Lecture

- Inner product (dot product) $\vec{u} \cdot \vec{v}$
- Properties of inner product
- Length of a vector $\|\vec{v}\|$
- Find angle between vectors θ
- Testing if two vectors are orthogonal $\vec{u} \cdot \vec{v} = \vec{0}$
- Orthogonal set
- Projection $\text{proj}_w(\vec{x})$
- Making a set orthogonal
- $w^\perp$

Note: the final exam WILL cover this content.

Lecture on Orthagonality (6.1, 6.2, 6.3)

Inner Product (aka dot product)

Note: this lecture does NOT apply for $\mathbb{C}$. We will be working in $\mathbb{R}$.

Given $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$, the inner product of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \cdot \vec{v} = \sum_{j=1}^{n} u_j v_j = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

technically wrong but it works

$$\underbrace{\vec{u} \cdot \vec{v}}_{\mathbb{R}} \text{ "equals" } \underbrace{\vec{u}^T \vec{v}}_{1 \times 1 \text{ matrix}}$$

2 vs. [2]

Ex: What is $\vec{u} \cdot \vec{v}$?

$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$ $\quad \vec{u}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$$\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3 = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot (-6) = \boxed{-4}$$

"Other" method

$$\vec{u}^T \vec{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot (-6) = -4$$

Properties of Inner Product

$(\underbrace{\vec{u}, \vec{v}, \vec{w}}_{\text{real vectors of size } n} \in \mathbb{R}^n, \underbrace{c}_{\text{constant}} \in \mathbb{R})$

1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$
4) $\vec{u} \cdot \vec{u} \geq 0 \leftarrow$ non-negative number
5) $\vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0}$

Length of a Vector

length of $\vec{u}$

$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$ and $\vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \dots + u_n^2$

$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$

$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

Find Angle Between Vectors

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

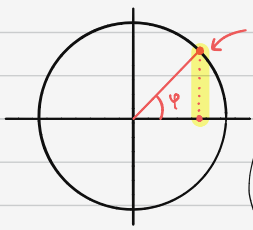
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

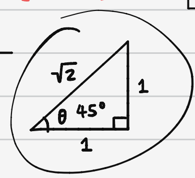
Ex: Find θ.

$\vec{u} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\sqrt{\vec{u}\cdot\vec{u}}\sqrt{\vec{v}\cdot\vec{v}}} = \frac{(1)(1)+(1)(0)}{\sqrt{(1)(1)+(1)(1)}\sqrt{(1)(1)+(0)+(0)}} = \frac{1}{\sqrt{1+1}\sqrt{1}} = \frac{1}{\sqrt{2}}$$

How do we find θ from this? Unit circle!!





$\cos\theta = \frac{1}{\sqrt{2}}$

not the internal angle

$\boxed{\theta = \frac{\pi}{4}}, \cancel{\frac{7\pi}{4}}, \cancel{-\frac{\pi}{4}}$ → angle should not be negative

same

$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ $\quad$ $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ ← from unit circle chart from last lecture

We need to eliminate all but one solution. $\theta = \frac{\pi}{4}$.

Ex: Find θ.

$\vec{u} = \begin{bmatrix}1\\-1\\0\\2\\0\end{bmatrix}$ $\quad$ $\vec{v} = \begin{bmatrix}-1\\1\\1\\-2\\1\end{bmatrix}$ $\quad$ $\cos\theta = \frac{-6}{\sqrt{6}\sqrt{8}} = \frac{-\sqrt{3}}{2}$ $\quad$ $\boxed{\theta = \frac{5\pi}{6}}, \cancel{\frac{7\pi}{6}}, \cancel{-\frac{5\pi}{6}}$

"long way around" $\quad$ negative

Note: when looking for the angle between vectors, the answer should be between 0 and π.

What if $\theta = \frac{\pi}{2}$? (90°)

$\Longrightarrow \cos\frac{\pi}{2} = 0$

Then $\vec{u}\cdot\vec{v} = 0$

Testing if Two Vectors are Orthagonal

Vectors $\vec{u}$ and $\vec{v}$ are orthagonal to each other if $\vec{u}\cdot\vec{v} = 0$.
Note that $\vec{0}$ is orthagonal to everything.

$\vec{u} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$ are orthagonal to each other since

$$\vec{u}\cdot\vec{v} = (1)(1) + (1)(2) + (1)(-3) = 1 + 2 - 3 = 0$$

Note: vectors are orthagonal to <u>other vectors</u> (in a pair). A vector cannot be orthagonal to itself.

Orthagonal Set

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is called an **orthogonal set** if all vectors in the set are orthagonal to all other vectors in the set.
↳ if $i \neq j$ then $\vec{v}_i \cdot \vec{v}_j = 0$

Ex:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$$

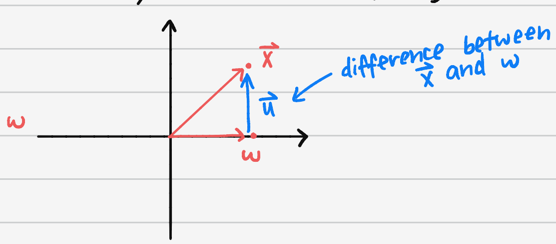
$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set.
Why? $\vec{v}_1 \cdot \vec{v}_2 = 0$ $\quad \vec{v}_1 \cdot \vec{v}_3 = 0$ $\quad \vec{v}_2 \cdot \vec{v}_3 = 0$
(Only need to check a single direction.)

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$. (can prove this)
$\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an **orthogonal basis** of $\mathbb{R}^3$.

Suppose we have $\vec{x} \in \mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$.
We want to find $\vec{w} \in W$ as close as possible to $\vec{x}$.
↳ find w: "find closest vector in subspace to x"

Ex:

$$\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad W = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$



$\vec{w}$ is on the horizontal line, as close as possible to $\vec{x}$.

$\vec{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\quad \vec{u} = \vec{x} - \vec{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Projection

($\vec{v}_1, \dots, \vec{v}_k$ non-zero) since can't divide by zero

If $w = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ where $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthagonal set then:

"projection of $\vec{x}$ onto w"

$$\vec{w} = \text{proj}_w(\vec{x}) = \sum_{j=1}^{k} \frac{\vec{x}\cdot\vec{v}_j}{\vec{v}_j\cdot\vec{v}_j}\vec{v}_j$$

$\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$ is sometimes called the "projection of $\vec{x}$ onto $\vec{v}$.

Ex: What is $\text{proj}_w(\vec{x})$?

$$\vec{x} = \begin{bmatrix}1\\3\\1\end{bmatrix} \qquad w = \text{span}\left\{ \begin{array}{cc} \vec{v}_1 & \vec{v}_2 \\ \begin{bmatrix}1\\1\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{array} \right\}$$

should be able to calculate for a set that contains 3-5 vectors.

what does the projection of $\vec{x}$ look like?

! Make sure it's an orthogonal set. (It is) $\vec{v}_1\cdot\vec{v}_2 = 0$

$$\text{proj}_w(\vec{x}) = \frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{x}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \frac{4}{2}\vec{v}_1 + \frac{1}{1}\vec{v}_2$$

$$= \begin{bmatrix}2\\2\\1\end{bmatrix} = \vec{w} \qquad \vec{u} = \vec{x} - \vec{w} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$$

Note: set of one vector is automatically orthogonal.

Ex: Recall graph from page above. Calculate $\text{proj}_w(\vec{x})$.

$$\vec{x} = \begin{bmatrix}2\\2\end{bmatrix} \quad v_1 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\text{proj}_w(\vec{x}) = \frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \frac{\begin{bmatrix}2\\2\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\end{bmatrix}\right\|^2}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} = \vec{w}$$ ← what we got earlier

Note: $\vec{u}$ is orthogonal to all vectors in w.

Making a Set Orthogonal

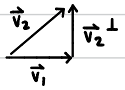
$$\text{span}\{\vec{v}_1, \vec{v}_2\} = \underbrace{\text{span}\left\{\vec{v}_1, \vec{v}_2 - \frac{\vec{v}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1\right\}}_{\text{orthogonal set}}$$

difference of 2nd vector and proj. of 2nd vector onto first vector

→ works if set is linearly independent

→ works for sets of two vectors

→ Interested in learning more? Gram-Schmidt formula w orthagonal



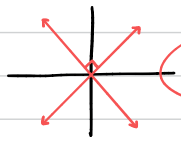
Let $w^{\perp}$ be the set of all vectors orthagonal to all vectors in w.

① $w^{\perp}$ is a subspace ($\vec{0}$ included)

② $(w^{\perp})^{\perp} = w$ (if w is a subspace)

$\{\vec{0}\}^{\perp} = \mathbb{R}^n$ and $(\mathbb{R}^n)^{\perp} = \{\vec{0}\}$

$\left(\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}\right)^{\perp} = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$



In $\mathbb{R}^2$ any two perpendicular lines meeting at the origin are orthagonal

$\vec{x} \in w^{\perp}$ if and only if it's orthogonal to every vector in a spanning set of w.

↳ meaning we only need to check a spanning set, not infinite vectors (yay!!)

Ex: Find $w^{\perp}$.

Let $w = \text{span}\left\{ \underset{\vec{v}_1}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}} \right\}$ Let $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \in w^{\perp}$ "$\vec{x}$ in w orthogonal"

$\vec{x} \in w^{\perp} \iff \vec{x}\cdot\vec{v}_1 = 0$ and $\vec{x}\cdot\vec{v}_2 = 0$

$$\iff \left.\begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 - x_4 = 0 \end{array}\right] \text{system of lin. eqns.}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} B & B & F & F \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 2 & 0 \\ 0 & \boxed{1} & 0 & -1 & 0 \end{array}\right] \end{array} R_1' = R_1 - R_2$$

$x_3 = s \quad x_4 = t \implies x_1 = -s - 2t$
$\qquad\qquad\qquad\qquad\quad x_2 = t$

$$x = \begin{bmatrix}-s-2t\\t\\s\\t\end{bmatrix} \quad s, t \in \mathbb{R}$$

$$w^{\perp} = \left\{ \begin{bmatrix}-s-2t\\t\\s\\t\end{bmatrix} \text{ fo all } s,t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\1\end{bmatrix} \right\}$$

Let A be a $m \times n$ matrix.

$(\text{Col}(A))^{\perp} = \text{Null}(A^T)$

span of columns of A

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$$