15 - Mar 6 Lecture

- $A_i(\vec{b})$
- Cramer's Rule
- 2 methods of finding the value at i,j of the inverse of a matrix
    1) Replacing column with standard basis vector
    2) Using submatrices
- Parallelopipes
- Volume of an n-dimensional parallelopipe
- Visual representation of EROs on a rectangle
- $vol(T(s))$

Addition to last lecture

Assuming A, B, C, D square:

$Det(ABC) = Det((AB)C) = Det(AB)\,Det(C) = Det(A)\,Det(B)\,Det(C)$

$Det(D_1 \times D_2 \times \ldots \times D_k) = Det(D_1) \times \ldots \times Det(D_k)$ ← general form

$Det(AB) = Det(BA)$

For those who understand dot / cross products... for 3x3 matrices:

↳ $Det(D) = \vec{a} \cdot (\vec{b} \times \vec{c})$ if $\vec{a}, \vec{b}, \vec{c}$ are the columns or rows (in order) of a matrix D.

$A_i(\vec{b})$

For a nxn matrix A and length n column vector $\vec{b}$, $A_i(\vec{b})$ is the matrix obtained by <u>replacing</u> column i of A with $\vec{b}$

Ex: Given A and $\vec{b}$, what is: a) $A_1(\vec{b})$? b) $A_2(\vec{b})$?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 11 \\ 12 \\ 13 \end{bmatrix} \quad A_1(\vec{b}) = \begin{bmatrix} 11 & 2 & 3 \\ 12 & 5 & 6 \\ 13 & 8 & 9 \end{bmatrix} \quad A_3(\vec{b}) = \begin{bmatrix} 1 & 2 & 11 \\ 4 & 5 & 12 \\ 7 & 8 & 13 \end{bmatrix}$$

Cramer's Rule

Let A be an invertible nxn matrix. The solution to the matrix vector product $A\vec{x} = \vec{b}$ is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ with } x_i = \frac{Det(A_i(\vec{b}))}{Det(A)} \text{ for all } i = 1, \ldots, n$$

Ex. 1 Use Cramer's Rule to find $\vec{x}$.

Need to calculate x_1 and x_2

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$x_1 = \frac{Det(A_1(\vec{b}))}{Det(A)} = \frac{Det\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}}{Det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \frac{[(2)(4)-(2)(2)]}{[(1)(4)-(2)(3)]} = \frac{8-4}{4-6} = \frac{4}{-2} = -2$$

$$x_2 = \frac{Det(A_2(\vec{b}))}{Det(A)} = \frac{Det\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}{Det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \frac{[(1)(2)-(2)(3)]}{[(1)(4)-(2)(3)]} = \frac{2-6}{4-6} = \frac{-4}{-2} = 2$$

Ex.2: **Use Cramer's Rule to find $\vec{x}$.**

$A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 4/8 \\ 7/8 \end{bmatrix}$

$\text{Det}(A) = 8$ $\qquad \text{Det}(A_1(\vec{b})) = \text{Det}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} = 3 \qquad x_1 = \dfrac{\text{Det}(A_1(\vec{b}))}{\text{Det}(A)} = \dfrac{3}{8}$

(skipped finding the det) to save space

$\text{Det}(A_2(\vec{b})) = \text{Det}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = 4 \qquad x_2 = \dfrac{\text{Det}(A_2(\vec{b}))}{\text{Det}(A)} = \dfrac{4}{8}$

$\text{Det}(A_3(\vec{b})) = \text{Det}\begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = 7 \qquad x_3 = \dfrac{\text{Det}(A_3(\vec{b}))}{\text{Det}(A)} = \dfrac{7}{8}$

Finding Inverse at i,j (using standard basis vector)

This is a way to find the value at i,j when A is inverted. This method is ideal for finding a specific value of A^{-1}, but it should <u>NOT</u> be used to find A^{-1} in its entirety.

$$(A^{-1})_{i,j} = \frac{\text{Det}(A_i(\vec{e}_j))}{\text{Det}(A)}$$

← standard basis vector

↑ i,jth entry of A^{-1}

Alternative Formula for Finding Inverse at i,j (using submatrix)

note the order

$$(B^{-1})_{i,j} = (-1)^{i+j}\,\frac{\text{Det}(B(j,i))}{\text{Det}(B)}$$

← Recall : B(i,j) is B with row i and column j removed (submatrix)

Ex 1: **Given A, find $(A^{-1})_{2,1}$ using method 1.**

$A = \begin{bmatrix} 1 & -3 & 2 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$ $\qquad \text{Det}(A) = -2$

$$(A^{-1})_{2,1} = \frac{\text{Det}(A_2(\vec{e}_1))}{\text{Det}(A)} = \frac{\text{Det}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}}{-2} = \frac{2}{-2} = -1$$

↑ e_1 in 2nd col

Find $(A^{-1})_{2,1}$ using method 2.

$$(A^{-1})_{2,1} = (-1)^{i+j}\frac{\text{Det}(A(j,i))}{\text{Det}(A)} = (-1)^{2+1}\frac{\text{Det}(A(1,2))}{-2} = (-1)^3\frac{\text{Det}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}}{-2} = (-1)\frac{(1)(0)-(1)(2)}{-2} = (-1)\frac{-2}{-2} = -1$$

Ex 2: Given A and $Det(A)$, what is A^{-1}? (just to show concept – will never use this to find entire inverse) using method 1;

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad Det(A) = ad - bc = (1)(4) - (2)(3) = 4 - 6 = -2$$

$$(A^{-1})_{1,1} = \frac{Det(A_1(\vec{e}_1))}{Det(A)} = \frac{Det\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}}{-2} = \frac{(1)(4)-(2)(0)}{-2} = \frac{4}{-2} = -2$$

e_1 in 1st col

$$(A^{-1})_{1,2} = \frac{Det(A_1(\vec{e}_2))}{Det(A)} = \frac{Det\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}}{-2} = \frac{(0)(4)-(2)(1)}{-2} = \frac{-2}{-2} = 1$$

e_2 in 1st col

$$(A^{-1})_{2,1} = \frac{Det(A_2(\vec{e}_1))}{Det(A)} = \frac{Det\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}}{-2} = \frac{(1)(0)-(1)(3)}{-2} = \frac{-3}{-2} = \frac{3}{2}$$

e_1 in 2nd col

$$(A^{-1})_{2,2} = \frac{Det(A_2(\vec{e}_2))}{Det(A)} = \frac{Det\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}{-2} = \frac{(1)(1)-(0)(3)}{-2} = \frac{1}{-2} = -\frac{1}{2} \quad \Rightarrow A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

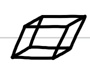
e_2 in 2nd col

Parallelopipe and vol(parallelopipe)

If A is a $n \times n$ matrix, then the n-dimensional volume of the n-dimensional parallelopipe defined by the columns of A is $\boxed{|Det(A)|}$.

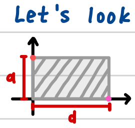
<u>2D parallelopipe</u> (parallelogram)

<u>3D parallelopipe</u>



Why does volume $= |Det(A)|$?

Let's look at a 2D case, a rectangle.



This rectangle is created by the vectors $\begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ d \end{bmatrix}$ ← y axis ← x axis

We know that Area of Rectangle $= |a \times d|$.

Let's confirm this using $|Det(A)|$.

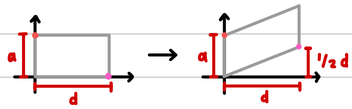
$$|Det(A)| = \left| Det\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right| = |(a)(d) - (0)(0)| = \boxed{|ad|}$$

Visual Representation of Row Replacement	Think of EROs as row operations on the shape.

Visual Representation of Row Replacement

Think of EROs as row operations on the shape.

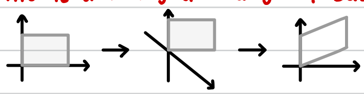
Visually, what does $R_1' = R_1 + \frac{1}{2}R_2$ look like?



$$\begin{bmatrix} a \\ 0 \end{bmatrix} \sim \begin{bmatrix} a + 1/2(0) \\ 0 \end{bmatrix} \; R_1' = R_1 + 1/2 R_2 \quad = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ d \end{bmatrix} \sim \begin{bmatrix} 0 + 1/2\, d \\ d \end{bmatrix} \; R_1' = R_1 + 1/2 R_2 \quad = \begin{bmatrix} 1/2\, d \\ 0 \end{bmatrix}$$

This is actually a change of base.

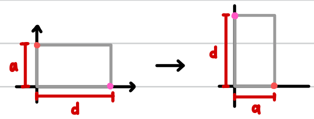


Row replacement does not change the area and <u>does not change the determinant</u>.

PROOF: $\left| \operatorname{Det} \begin{bmatrix} a & 1/2\, d \\ 0 & d \end{bmatrix} \right| = |(a)(d) - (1/2\, d)(0)| = |ad|$

Visual Representation of Row Interchange

Visually, what does $R_1 \longleftrightarrow R_2$ look like?

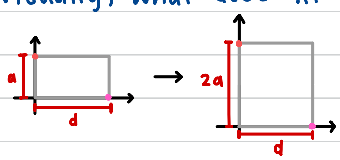


$$\begin{bmatrix} a \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ a \end{bmatrix} \; R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 0 \\ d \end{bmatrix} \sim \begin{bmatrix} d \\ 0 \end{bmatrix} \; R_1 \leftrightarrow R_2$$

Visual Representation of Scaling

Visually, what does $R_1' = 2R_1$ look like?



$$\begin{bmatrix} a \\ 0 \end{bmatrix} \sim \begin{bmatrix} 2a \\ 0 \end{bmatrix} \; R_1' = 2R_1$$

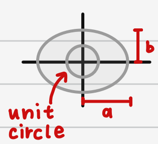
$$\begin{bmatrix} 0 \\ d \end{bmatrix} \sim \begin{bmatrix} 0 \\ d \end{bmatrix} \; R_1' = 2R_1$$

$\text{vol}(T(s))$

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with corresponding matrix A and s is any bounded region in $\mathbb{R}^n$ (i.e. not infinite), then $\text{vol}(T(s)) = |\operatorname{Det}(A)| \cdot \text{vol}(s)$.

Ex. We know that the area of the unit circle is π. $(A = \pi r^2 = \pi (1)^2 = \pi)$

What is the area of an ellipse (unit circle after transformation below)?



The transformation being applied is:

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\begin{aligned} \text{vol}(\text{ellipse}) &= |\operatorname{Det}(A)| \cdot \text{vol}(\text{unit circle}) \\ &= |(a)(b) - (0)(0)| \cdot \pi \\ &= |ab| \cdot \pi \\ &= ab\pi \end{aligned}$$

end

Assuming A, B, C, D square:

Add. to last lecture

$\text{Det}((AB)C) = \text{Det}(AB)\,\text{Det}(C) = \text{Det}(A)\,\text{Det}(B)\,\text{Det}(C)$

$\text{Det}(D_1 D_2 \dots D_k) = \text{Det}(D_1) \dots \text{Det}(D_k)$

$\text{Det}(AB) = \text{Det}(BA)$

Cramer's Rule

For a $n \times n$ matrix A and length n column vector $\vec{b}$, $A_i(\vec{b})$ is the matrix obtained by replacing column i of A with $\vec{b}$.

Ex.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 11 \\ 12 \\ 13 \end{bmatrix} \quad A_1(\vec{b}) = \begin{bmatrix} 11 & 2 & 3 \\ 12 & 5 & 6 \\ 13 & 8 & 9 \end{bmatrix} \quad A_3(\vec{b}) = \begin{bmatrix} 1 & 2 & 11 \\ 4 & 5 & 12 \\ 7 & 8 & 13 \end{bmatrix}$$

<u>Cramer's Rule</u>: Let A be an invertible $n \times n$ matrix, then the solution to the matrix vector product $A\vec{x} = \vec{b}$ is:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ with } x_i = \frac{\text{Det}(A_i(\vec{b}))}{\text{Det}(A)} \text{ for all } i = 1, \dots, n.$$

Ex. Use Cramer's Rule to find $\vec{x}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\text{Det}\underbrace{\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}}_{A_1(\vec{b})} = 4 \qquad \text{Det}\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}_{A_2(\vec{b})} = -4 \qquad \text{Det}(A) = -2$$

$$x_1 = \frac{4}{-2} = -2 \qquad x_2 = \frac{-4}{-2} = 2 \quad \Rightarrow \quad \vec{x} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Ex. Use Cramer's Rule to find $\vec{x}$

$$A\vec{x} = \vec{b} \text{ with } A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\text{Det}(A) = 8$

$\text{Det}(A_1(\vec{b})) = 3$

$\text{Det}(A_2(\vec{b})) = 4$

$\text{Det}(A_3(\vec{b})) = 7$

$$\Rightarrow \vec{x} = \begin{bmatrix} 3/8 \\ 4/8 \\ 7/8 \end{bmatrix}$$

i,j th entry of A

standard basis vector

$$(A^{-1})_{i,j} = \frac{Det(A_i(\vec{e}_j))}{Det(A)}$$

$$A = \begin{bmatrix}1&2\\3&4\end{bmatrix} \quad Det(A) = -2$$

$$(A^{-1})_{1,1} = \frac{Det\begin{bmatrix}1&2\\0&4\end{bmatrix}}{-2} = -2$$

1st col 1st row

$$(A^{-1})_{2,1} = \frac{Det\begin{bmatrix}1&1\\3&0\end{bmatrix}}{-2} = \frac{3}{2}$$

★ backwards ??
(why? inversion swaps rows and cols)

2nd col 1st row

repl. 1st col w/ 2nd $\vec{e}$

$$(A^{-1})_{1,2} = \frac{Det\begin{bmatrix}0&2\\1&4\end{bmatrix}}{-2} = 1$$

1st row 2nd col

$$(A^{-1})_{2,2} = \frac{Det\begin{bmatrix}1&0\\3&1\end{bmatrix}}{-2} = -\frac{1}{2}$$

2nd row 2nd col

$$A^{-1} = \begin{bmatrix}-2&1\\3/2&-1/2\end{bmatrix}$$

Good for finding specific value of A^{-1}.
Not used to find A^{-1} in its entirety.

Find $(A^{-1})_{2,1}$ for $A = \begin{bmatrix}1&-3&2\\1&-1&1\\2&0&0\end{bmatrix}$

2nd col 1st row

$$A^{-1} = \begin{bmatrix}\circ&\circ&\circ\\\circ&\circ&\circ\\\circ&\circ&\circ\end{bmatrix}$$

$Det(A) = -2$

$$Det\begin{bmatrix}1&1&2\\1&0&1\\2&0&0\end{bmatrix} = 2 \Rightarrow (A^{-1})_{2,1} = -1$$

$$= (-1)^{i+j} Det(A(j,i))$$

Formula $$(B^{-1})_{i,j} = (-1)^{i+j}\frac{Det(B(j,i))}{Det(B)}$$ to find inverse

☆☆ ☆ ☆☆

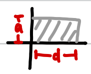
$$(A^{-1})_{2,3} = (-1)^{2+3}\frac{Det\begin{bmatrix}1&2\\1&1\end{bmatrix}}{-2} = -\frac{1}{2}$$

+ do this for all i,j in matrix

If A is a $n \times n$ matrix, then the n-dimensional volume of the n-dimensional parallelopipe defined by the columns of A is $|\mathrm{Det}(A)|$

2D parallelopipe  (parallelogram)

3D parallelopipe 

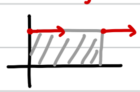
2D - case (rectangle) $\begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ d \end{bmatrix}$ 

Area : $|a \times d|$

$$\left| \mathrm{Det} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right| = |ad|$$

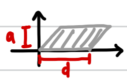
Think of EROS as row operations on shape.

Visually, what does $R_1' = R_1 + \frac{1}{2} R_2$ look like?

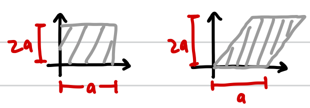


moves point to the right

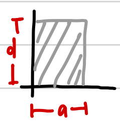
Row repl. does not change area and does not change det.



Visually, what does $R_1' = 2R_1$ look like?



$R_1 \longleftrightarrow R_3$



?

a, b, c must be in order

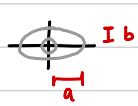
$\vec{a} \cdot (\vec{b} \times \vec{c})$ if $\vec{a}, \vec{b}, \vec{c}$ are the columns (or rows) of a matrix D, then: $\mathrm{Det}(D) = \vec{a} \cdot (\vec{b} \times \vec{c})$

only for 3×3

Dot / Cross product

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with corresponding matrix A and S is any bounded region in $\mathbb{R}^n$ (i.e. not infinite), then $\mathrm{vol}(T(S)) = |\mathrm{Det}(A)| \cdot \mathrm{vol}(S)$

Area of unit circle is π.

Area of ellipse 

$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $|\mathrm{Det}(A)| = |ab|$

$\Rightarrow ab\pi$

?