4 - Jan 18 Lecture

- The matrix equation $A\vec{x} = \vec{b}$
- Properties of the matrix equation
- The identity matrix I_n
- Solution sets of linear equations
- Linear dependence and independence

Test on Jan 30th up to and including this lecture (first 4 lectures).
Test covers sections 1.1 to 1.5, and 1.7.
Tutorials start next week.

A small addition to last lecture's content…

If all of $\mathbb{R}^m$ is in Span($\vec{v}_1, \dots, \vec{v}_k$), then:

- Span($\vec{v}_1, \dots, \vec{v}_k$) = $\mathbb{R}^m$
- the set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is said to span $\mathbb{R}^m$.

refers to vectors which contain m real number entries

Essentially: some sets of vectors span the entirety of space.

If span = space, the collection of vectors is said to span $\mathbb{R}^m$.

The matrix equation
$A\vec{x} = \vec{b}$

$$A = \begin{array}{c} \begin{matrix} a_1 & a_2 & & a_n \end{matrix} \\ \begin{bmatrix} A_{1,1}, & A_{1,2}, & \dots, & A_{1,n} \\ \vdots & & & \\ A_{m,1}, & A_{m,2}, & \dots, & A_{m,n} \end{bmatrix} \end{array} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let $\vec{a}_i$ be the i^{th} column of A.

Note that this notation is not standardized, hence why it is being defined here.

So, $\vec{a}_i = \begin{bmatrix} A_{1,i} \\ A_{2,i} \\ \vdots \\ A_{m,i} \end{bmatrix}$

Ex. $a_2 = \begin{bmatrix} A_{1,2} \\ A_{2,2} \\ \vdots \\ A_{m,2} \end{bmatrix}$

We define:

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n a_n = \vec{b}$$

Ex. 1

$$\begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \overset{x_1}{7}\overset{a_1}{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} + \overset{x_2}{8}\overset{a_2}{\begin{bmatrix} 2 \\ 5 \end{bmatrix}} + \overset{x_3}{9}\overset{a_3}{\begin{bmatrix} 3 \\ 6 \end{bmatrix}} = \begin{bmatrix} 7 \\ 28 \end{bmatrix} + \begin{bmatrix} 16 \\ 40 \end{bmatrix} + \begin{bmatrix} 27 \\ 54 \end{bmatrix} = \overset{\vec{b}}{\begin{bmatrix} 50 \\ 122 \end{bmatrix}}$$

Some things to note:

1) Length of $\vec{x}$ is equal to the number of columns in A. (3 in the example above)
2) Length of $\vec{b}$ is equal to the number of rows in A. (2 in the example above)
3) $\vec{x}$ and $\vec{b}$ are column vectors.

Ex. 2

$$\begin{bmatrix} 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = 2[1] + 4[-1] - 1[3] = [2] + [-4] + [-3] = [-5]$$

Note that this is a vector of length 1, NOT a scalar / constant.

Ex. 3

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$B\vec{x}$ = undefined

$B\vec{y}$ = undefined

Undefined because $\vec{x}$ and $\vec{y}$ are the wrong dimension. Vector must be length 3 to multiply with B.

Properties of the Matrix Equation $A\vec{x} = \vec{b}$.

1) $A\vec{0}_n = \vec{0}_m$ Note: $\vec{0}_n$ and $\vec{0}_m$ could be different zero vectors, depending on dimension of A.

Ex. 1

$$A = \begin{array}{cc} n & \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} & m \end{array} \qquad \vec{0}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad A\vec{0}_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

different zero vectors

Ex. 2

$$B = \begin{array}{cc} n & \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & m \end{array} \qquad \vec{0}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{0}_m = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

same zero vectors

2) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

3) $A(c\vec{x}) = c(A\vec{x}) = (cA)\vec{x}$ You can pull a constant "out front" when multiplying.

Identity Matrix I_n

4) $I_n\vec{x} = \vec{x}$ where

$$I_n = \begin{array}{cc} n & \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & n \end{array}$$

If you multiply this matrix by any vector, it returns the exact vector.

I_n is a $n \times n$ square matrix with 1s on the diagonal and 0s in all the other spaces.

Ex.

$$I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \qquad I_5\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Application of Non-Augmented Matrices with $A\vec{x} = \vec{b}$

Given the following systems of equations... we can write them as a non-augmented matrix:

$$\boxed{\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ -x_1 - x_2 + 3x_3 &= -1 \\ -2x_1 + x_2 + x_3 &= 2 \end{aligned}} \rightarrow A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

If we let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $A\vec{x} = x_1\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - x_3 \\ -x_1 - x_2 + 3x_3 \\ -2x_1 + x_2 + x_3 \end{bmatrix} = \vec{b}$

To summarize: If A is the <u>non-augmented matrix</u> associated with a system of linear equations, $\vec{x}$ is the column vector of the variables and $\vec{b}$ is the column vector of the constants, then $A\vec{x} = \vec{b}$.

So, solving for $\vec{x}$ in $A\vec{x} = \vec{b}$ is <u>equivalent</u> to solving a system of linear equations.

Some different ways to find the solution set.

In each problem, certain methods may be easier

MATRIX VECTOR PRODUCT	⟷	SYSTEM OF LINEAR EQUATIONS	⟷	VECTOR EQUATION $c_1v_1 + \ldots + c_nv_n = \vec{w}$
↓ solve for $\vec{x}$ ↓		↓ solve system ↓		↓
solution set of vectors	=	solution set in vector form	=	solutions to c_i
↓		↓		↓
number of solutions	=	number of solutions	=	number of solutions

Solution Sets of Linear Equations

Suppose we have a system of linear equations that corresponds to $A\vec{x} = \vec{b}$.

Consider $A\vec{y} = \vec{0}$. By the property $A\vec{0}_n = \vec{0}_m$, $\vec{y} = \vec{0}$ is a solution, so the corresponding system of linear equations is consistent.

Why does it matter if the system is consistent?

If the system is consistent, then we can write the solution set in parametric vector form.

We can switch out column vector of coefficients with the zero vector

$$\vec{y} = \vec{z}_0 + c_1 \vec{z}_1 + \dots + c_n \vec{z}_n \quad \text{for all } c_i \in \mathbb{R}$$

$$= \vec{0} + c_1 \vec{z}_1 + \dots + c_n \vec{z}_n \quad \text{for all } c_i \in \mathbb{R}$$

Let $\vec{y}_1$ be a solution to $A\vec{y} = 0$.

Assume $A\vec{x} = \vec{b}$ is consistent and let $\vec{x}_1$ be a solution. $A\vec{x}_1 = b$

Then $A(\vec{x}_1 + \vec{y}_1) = A\vec{x}_1 + A\vec{y}_1 = \vec{b} + \vec{0} = \vec{b}$.

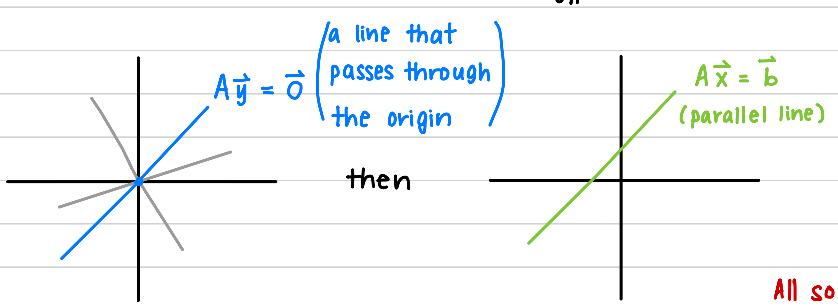
Therefore for all solutions $\vec{y}_h$ to $A\vec{y} = \vec{0}$, $\vec{x}_1 + \vec{y}_h$ is a solution to $A\vec{x} = \vec{b}$.

So $\vec{x}_1 + c_1\vec{z}_1 + c_2\vec{z}_2 + \dots + c_n\vec{z}_n$ are all solutions to $A\vec{x} = \vec{b}$

Doing a similar argument, we can find that these are all of the solutions to $A\vec{x} = \vec{b}$.

General Theorem

If $A\vec{x} = \vec{b}$ is consistent then the solution set for $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{x}_1 + \vec{y}_h$, where $\vec{x}$ is a solution to $A\vec{x} = \vec{b}$ and $\vec{y}_h$ is any solution to $A\vec{y} = \vec{0}$.



Knowing this, the solution is a parallel line that passes through the origin.
(or no solution)

All solutions are "parallel spaces"

Ex 1. Suppose a linear system has solution:

$$\begin{bmatrix} 1 + r - 2s + 3t \\ 3 + 3s - 4t \\ r - 3t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix} \text{ for all } r, s, t \in \mathbb{R}$$

The solution to $A\vec{y} = \vec{0}$ is :

we zeroed the constant vector

$$\boxed{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} + r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix} \text{ for all } r, s, t \in \mathbb{R}$$

If $\vec{z}$ is a solution to $A\vec{w} = \vec{c}$, the general solution set is :

$$\vec{z} + r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix}$$

Just need to find one solution

Linear Dependance and Independance

?

Given the following (2D space)...

$$s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ for all } s, t \in \mathbb{R} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

... if this set were to exist in a 1-D space, we could write it as :

$$= u \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ for all } u \in \mathbb{R} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Or, if we are given the following in a 3D space...

$$r \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + t \boxed{\begin{bmatrix} 0 \\ 3 \\ 2 \\ -3 \end{bmatrix}} \text{ for all } r, s, t \in \mathbb{R}$$

.. in a 2D space we could write it as :

$$= r \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \text{ for all } r, s \in \mathbb{R}$$

We can actually "drop" any of the vectors - it's not important which one. In this case we dropped the highlighted vector above.

"Linearly independant"

A set of vectors $\{\vec{v}_1, .., \vec{v}_k\}$ for which the only solution to $c_1\vec{v}_1 + .. + c_k\vec{v}_k = \vec{0}$ is the "trivial solution" $c_1 = c_2 = \ldots = c_n = 0$.

"trivial solution" → always exists and easy to find

$c_1 = c_2 = \ldots = c_n = 0$ → all constants equal to 0

"Linearly dependant"

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ for which $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$ has non-trivial solutions.

non-trivial → non-zero constants

Alternatively, if the set is not linearly independant (sets have to be one or the other)

Any $c_1, \ldots, c_n$ not all zero such that $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = 0$ is called a linear dependance relation.

Ex.1

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$c_1 = 1$, $c_2 = -1$ is a dependance relation.

This is a dependance relation since:

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \rightarrow 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 0 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

Therefore, $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependant.

We only care about finding one solution to prove that it's linearly dependant.
We already knew that $c_1 = c_2 = 0$ was a solution as well, so if we aren't able to find one non-trivial solution, the set is linearly independant.

Proving the trivial solution:

$$0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

Ex. 2

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -3 \end{bmatrix}$

$c_1 = 2, c_2 = 1, c_3 = -1$ is a dependance relation

(More on how to find the right solution in Lec. 6).

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0} \longrightarrow 2\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 3 \\ 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 0 \\ -4 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

Therefore $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependant.

End of lecture material

Test on Jan 30^{th} up to and including today's lecture. (First 4 lectures)

Textbook sections 1.1 – 1.5 & 1.7

Tutorial starts next week.

Addition to last lecture

If all of $\mathbb{R}^m$ is in Span$(\vec{v_1}, \dots, \vec{v_k})$, then Span$(\vec{v_1}, \dots, \vec{v_k}) = \mathbb{R}^m$ and the set $\{\vec{v_1}, \dots, \vec{v_k}\}$ is said to span $\mathbb{R}^m$.

Vectors that span entirety of space.

If span = space, collection of vectors is said to span $\mathbb{R}^m$.

The matrix equation $A\vec{x} = \vec{b}$

$$A = \begin{bmatrix} A_{1,1}, & A_{1,2}, & \dots, & A_{1,n} \\ \vdots & & & \\ A_{m,1}, & A_{m,2}, & \dots, & A_{m,n} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let $\vec{a_i}$ be the i^{th} column of A.

Not standard notation, which is why it's defined here.

$$\text{so, } \vec{a_i} = \begin{bmatrix} A_{1,i} \\ A_{2,i} \\ \vdots \\ A_{m,i} \end{bmatrix}$$

We define:

$$A\vec{x} = \underbrace{x_1}_{\text{scalar}}\underbrace{\vec{a_1}}_{\text{vector}} + x_2\vec{a_2} + \dots + x_n\vec{a_n}$$

Ex.

$$\underset{A}{\begin{array}{ccc} a_1 & a_2 & a_3 \\ \end{array}} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\overset{\vec{x}}{\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}} = 7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \overset{\vec{b}}{\begin{bmatrix} 50 \\ 122 \end{bmatrix}}$$

Length of $\vec{x}$ is equal to the number of columns in A.

Length of $\vec{b}$ is equal to the number of rows of A.

$\vec{x}$ and $\vec{b}$ are column vectors.

Ex. $\begin{bmatrix} 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \end{bmatrix} + 4\begin{bmatrix} -1 \end{bmatrix} - 1\begin{bmatrix} 3 \end{bmatrix} = \begin{bmatrix} -5 \end{bmatrix}$

vector of length 1, not a scalar / constant

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $B\vec{x}$ = undefined

$\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $B\vec{y}$ = undefined

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Properties of

1) $A\vec{0}_n = \vec{0}_m$ Technically different 0 vectors.

ex. $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ vs. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

3) $A(c\vec{x}) = c(A\vec{x}) = (cA)\vec{x}$ can pull a constant out front.

Idenity matrix

4) $I_n \vec{x} = \vec{x}$ where $I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & 0 & 0 & 0 & 1 \end{bmatrix}$ ($n \times n$ matrix)

gives dimension

square $n \times n$ matrix.

all 0s, but 1 on diagonal

mult matrix by any vector, returns vector

$I_5 = 5 \times 5$

non-augmented matrix $\rightarrow$ vector of solutions

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ -x_1 - x_2 + 3x_3 &= -1 \\ -2x_1 + x_2 + x_3 &= 2 \end{aligned} \quad \rightarrow \quad A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Solve for x.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $A\vec{x} = x_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} x_1 + x_2 - x_3 \\ -x_1 - x_2 + 3x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix} = \vec{b}$$

If A is the non-augmented matrix associated with a system of linear equations, $\vec{x}$ is the column vector of the variables and $\vec{b}$ is the column vector of the constants, then $A\vec{x} = \vec{b}$.

Solving for $\vec{x}$ in $A\vec{x} = b$ is equivalent to solving a system of linear equations.

Matrix Vector product $\longleftrightarrow$ System of lin. eqns. $\longleftrightarrow$ Vector eqn

$c_1 v_1 + \ldots + c_n v_n = \vec{w}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

Solve for $\vec{x}$ $\qquad\qquad$ Solve system

sometimes easier than solving system.

$\downarrow$ $\qquad\qquad$ $\downarrow$

Solution set of vectors $=$ Solution set in vector form $=$ solutions to c_i

soln $=$ # soln $=$ # soln

Solution sets of linear systems

Suppose we have a system of lin eqns. that corresponds to

$A\vec{x} = \vec{b}$. ← mat. of constants

↑ matrix of variables

is this consistent?

Consider $A\vec{y} = \vec{0}$, $\vec{y} = \vec{0}$ is a solution, so the corresponding system of linear eqns. is consistent.

Therefore, we can write the soln set in parametric vector form.

$\vec{y} = \vec{z}_0 + c_1\vec{z}_1 + \ldots + c_n\vec{z}_n$ for all $c_i \in \mathbb{R}$.

can switch out $\vec{z}_0$ for $\vec{0}$.

$= \vec{0} + c_1\vec{z}_1 + \ldots + c_n\vec{z}_n$ for all $c_i \in \mathbb{R}$.

$A\vec{y}_1 = \vec{0}$

Let $\vec{y}_1$ be a solution to $A\vec{y} = 0$.

goal: differentiating $1, \infty$ & dimension

$A\vec{x}_1 = \vec{b}$

Assume $A\vec{x} = \vec{b}$ is consistent and let $\vec{x}_1$ be a soln

Then $A(\vec{x}_1 + \vec{y}_1) = A\vec{x}_1 + A\vec{y}_1 = \vec{b} + \vec{0} = \vec{b}$

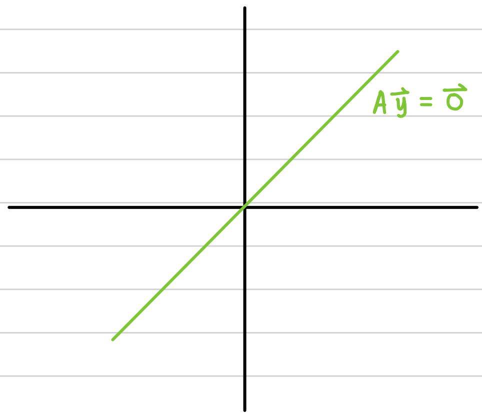
Therefore for all solutions $\vec{y}_h$ to $A\vec{y} = \vec{0}$, $\vec{x}_1 + \vec{y}_h$ ← h like house is a solution to $A\vec{x} = \vec{b}$.

So $\vec{x}_1 + c_1\vec{z}_1 + c_2\vec{z}_2 + \ldots + c_n\vec{z}_n$ are all solns to $A\vec{x} = \vec{b}$.

Doing a similar argument we can find these are all of the solutions to $A\vec{x} + \vec{b}$.

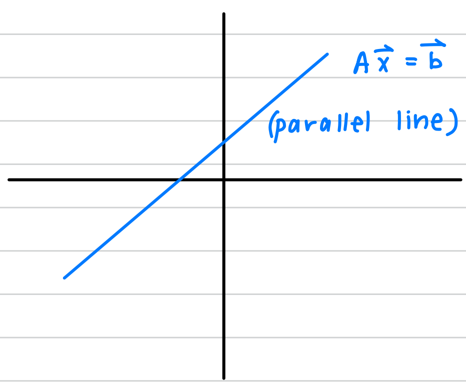
General thm.

If $A\vec{x} = \vec{b}$ is consistent then the solution set for $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{x}_1 + \vec{y}_h$, where $\vec{x}$ is a solution to $A\vec{x} = \vec{b}$ and $\vec{y}_h$ is any solution to $A\vec{y} = \vec{0}$.



(has to be a line that passes through the origin.)

then



knowing this, we know soln is a parallel line that passes through the origin.

or no solns.

All solns are "parallel spaces"

Suppose a linear system has solution:

parametric

$$\begin{bmatrix} 1 + r - 2s + 3t \\ 3 + 3s - 4t \\ r - 3t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix} \text{ for all } r,s,t \in \mathbb{R}$$

The soln to $A\vec{y} = \vec{0}$ is

we zeroed the constant vector.

$$r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix} \text{ for all } r,s,t \in \mathbb{R}$$

If $\vec{z}$ is a solution to $A\vec{w} = \vec{c}$ the general solution set is

$$\vec{z} + r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix}$$

"Just need to find one soln"???

Linear dependance and independance

redundant?

$$s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ for all } s,t \in \mathbb{R} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

1 D space

$$= u\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ for all } u \in \mathbb{R} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

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actually a 2D space

$$r\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + t\cancel{\begin{bmatrix} 0 \\ 3 \\ 2 \\ -3 \end{bmatrix}} \text{ for all } r,s,t \in \mathbb{R}$$

(could've dropped any vector - not important)

$$= r\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \text{ for all } r,s \in \mathbb{R}$$

A set of vectors $\{\vec{v_1}, \dots, \vec{v_k}\}$ is called "linearly independant" if the only solution to $c_1\vec{v_1} + \dots + c_k\vec{v_k} = \vec{0}$ is the "trivial solution" $c = c_2 = \dots = c_n = 0$

↳ always exists & easy to find

all constants equal to 0.

A set of vectors $\{\vec{v_1}, \dots, \vec{v_k}\}$ is called linearly dependant if $c_1\vec{v_1} + \dots + c_k\vec{v_k} = \vec{0}$ has non-trivial solns.

Alternatively, if the set is not linearly independant. (sets have to be one or the other).

some can be, but >1 not zero

Any $c_1, \dots, c_n$ not all zero such that $c_1\vec{v_1} + \dots + c_k\vec{v_k} = 0$ is called a linear dependance relation.

$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $c_1 = 1, c_2 = -1$ is a dependance relation.

only care abt finding 1 soln
we know 0,0 is a soln

notation for set

Therefore, $\{\vec{v_1}, \vec{v_2}\}$ is linearly dependant.

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, $V_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -3 \end{bmatrix}$ $c_1 = 2, c_2 = 1, c_3 = -1$ is a dependance relation.

Properties

- A set of 2 vectors is linearly dependant if and only if $\vec{v_1} = c\vec{v_2}$ for some $c \in \mathbb{R}$ lin. dep. or a constant multiple of each other.
- Any set of vectors containing the zero vector is linearly dependant.