9 - Feb 6 Lecture

- Inverse of a matrix (“invertible”)
- Shortcut to invert 2x2 matrices
- General method to invert all square matrices
- Properties of inversion

An addition to last lecture

If you have $B = C$, then $AB = AC$ and $BD = CD$

Note: multiply on the same side since $AB \neq CA$ in general.

Inverse of a Matrix

A $n \times n$ matrix B is "invertible" if there exists another $n \times n$ matrix C such that $BC = CB = I_n$.

If B is invertible, this C matrix is called the inverse of B and is denoted B^{-1}.

The "trivial" examples:

1) I_n is invertible with inverse $I_n^{-1} = I_n$.
2) $O_{n\times n}$ is not invertible since there is nothing we can multiply with $O_{n \times n}$ to give I_n.

Ex 1. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

Ex 2: $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ is invertible with $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

How do we know this? Keep reading... ᵕ̈

Note: non-square matrices are not invertible.

Note: the inverse matrix is unique. If $AB = BA = I_n = AC = CA$, then $B = C$.

Sometimes, textbooks will use the term "singular" instead. Singular is the opposite of invertible, so "non-invertible" = "singular"

"invertible" = "nonsingular"

Shortcut for Inverting 2×2 Matrices

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(aka "the determinant")

1) If $ad - bc = 0$, then A is <u>not</u> invertible.
2) If $ad - bc \neq 0$, then A is invertible with inverse $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

To demonstrate this, let's look at the examples from above again:

Ex 1: Why is this not invertible?

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Because $ad - bc = 0$.

$1(0) - 0(0) = 0 \Rightarrow$ not invertible

Ex 2: Why is this matrix invertible?

$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ Since $ab - bc \neq 0$.

$1(5) - 2(3) = 5 - 6 = -1 \Rightarrow$ invertible

What's the inverse?

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

Ex 3: $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ $ad-bc = 1(-4)-2(3) = -4-6 = -10 \Rightarrow$ invertible $\frac{1}{-10}\begin{bmatrix} -4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 \\ 3/10 & -1/10 \end{bmatrix}$ ← the inverse

If you were to multiply the matrix and its inverse together (on either side), you would get I_2.

$$\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2/5 & 1/5 \\ 3/10 & -1/10 \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 \\ 3/10 & -1/10 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

General Method to Find Inverse for all Square Matrices

To invert the $n \times n$ matrix A, make an augmented matrix with A to the left of the line and I_n to the right of the line. Then, row reduce to RREF.

1) If the left side reduces to I_n, then A is invertible and the right side is A^{-1}.

2) If the left side does not reduce to I_n, then A is not invertible.

Ex 1: Is this matrix invertible? If yes, what is its inverse?

$$B = \begin{bmatrix} -7 & 3 \\ -5 & 2 \end{bmatrix}$$

① Augment B with I_2

$$\left[\begin{array}{cc|cc} -7 & 3 & 1 & 0 \\ -5 & 2 & 0 & 1 \end{array}\right]$$

② Row reduce to RREF

$$\sim \left[\begin{array}{cc|cc} 1 & -3/7 & -1/7 & 0 \\ -5 & 2 & 0 & 1 \end{array}\right] \quad R_1' = \frac{1}{-7}R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & -3/7 & -1/7 & 0 \\ 0 & -1/7 & 5/7 & 1 \end{array}\right] \quad R_2' = R_2 + 5R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & -3/7 & -1/7 & 0 \\ 0 & 1 & -5 & -7 \end{array}\right] \quad R_2' = -7R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -5 & -7 \end{array}\right] \quad R_1' = R_1 + 3/7 R_2$$

← RREF

This is the inverse of B, B^{-1}

This is $I_2 \Rightarrow$ B is invertible

The RREF matrix tells us that:

① B is invertible.

② B^{-1} is $\begin{bmatrix} 2 & -3 \\ -5 & -7 \end{bmatrix}$.

Note: If you have $B\vec{x} = \vec{c}$ and you know B^{-1} and $\vec{c}$, you can easily find $\vec{x}$.

Ex: So, an alternative way to solve Q5 on test 1 would be:

Recall that we were given a system of linear equations:

$$x_1 - 3x_2 + 2x_3 = 1$$
$$-x_1 + 3x_2 = 0$$
$$2x_1 - 3x_2 + x_3 = -1$$

We can convert the equations into the following matrix, which will be our matrix B.

$$B = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 3 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

Now we can row reduce to find B^{-1}.

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 3 & -3 & -2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 3 & -3 & -2 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{array}\right] \quad R_2 \Leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2/3 & 0 & 1/3 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{array}\right] \quad R_2' = \tfrac{1}{3}R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -2/3 & 0 & 1/3 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{array}\right] \quad R_1' = R_1 + 3R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array}\right] \quad R_3' = \tfrac{1}{2}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1 \\ 0 & 1 & 0 & -1/6 & 1/2 & 1/3 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array}\right] \quad \begin{array}{l} R_1' = R_1 + R_3 \\ R_2' = R_2 + R_3 \end{array} \leftarrow \text{RREF}$$

B invertible $\quad B^{-1}$

$$B^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1 \\ -1/6 & 1/2 & 1/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

We were given $\vec{c} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. We know that $B\vec{x} = \vec{c} \rightarrow B^{-1}B\vec{x} = B^{-1}\vec{c} \rightarrow I_n\vec{x} = B^{-1}\vec{c} \rightarrow \vec{x} = B^{-1}\vec{c}$

$$B\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightarrow B^{-1}B\vec{x} = B^{-1}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightarrow \vec{x} = \begin{bmatrix} -1/2 & 1/2 & 1 \\ -1/6 & 1/2 & 1/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= 1\begin{bmatrix} -1/2 \\ -1/6 \\ 1/2 \end{bmatrix} + 0\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 1/3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ -1/6 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1/3 \\ 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$ ← This is the solution to Q5!

Properties of Matrix Inversion

1) A $n \times n$ matrix B is invertible if and only if the RREF of B is I_n.
2) $(A^{-1})^{-1} = A$ (the inverse of an inverse is the original matrix)
3) $(AB)^{-1} = B^{-1}A^{-1}$ (assuming A, B invertible)
4) $(A^T)^{-1} = (A^{-1})^T$ (transpose and inverse can be applied in any order)

Recall the 3 most common matrix multiplication mistakes:

1) $AB = BA$
2) $XY = XZ \Rightarrow Y = Z$
3) $CD = 0 \Rightarrow C = 0$ or $D = 0$

THESE ARE NOT TRUE

There are similar-looking rules for inverses, but these ones are TRUE:

1) $AA^{-1} = A^{-1}A$
2) $XY = XZ$ and X is invertible $\Longrightarrow Y = Z$
3) $CD = 0$ and either C invertible or D invertible $\Longrightarrow$ other matrix is the zero matrix

(This content was planned for lecture 10)

If you can find $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \vec{0}$, that proves that A is not invertible.

We can do a proof by contradiction to show this.

Suppose $A\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, then A is not invertible.

Suppose A^{-1} exists. Then $A^{-1}A\vec{x} = A^{-1}\vec{0} \rightarrow \vec{x} = \vec{0}$, which is a contradiction.

Ex 1: $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ 3 & 6 & 2 \end{bmatrix}, \quad A\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \vec{0} \Rightarrow$ A is not invertible

$\vec{x}$ represents linear dependance relation on the columns of A

Also, a $n \times n$ matrix is invertible if and only if the columns of A are linearly independant.

Ex 2. (Example from past years (2nd-year-level).)

Given an $n \times n$ matrix following the pattern, below, determine if the matrix is invertible.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1 & 0 & & 0 & 0 \\ 0 & -1 & 1 & & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ not invertible since $A\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$.

Note: If sum of each row in matrix is 0, can multiply matrix by vector with all 1s to get $\vec{0}$.

end

Add. to last lecture

If you have $B = C$ then $AB = AC$ and $BD = CD$ (note: multiply on same side)

So $AB \neq CA$ in general (if they are it's commutative)

Inverse of a Matrix

A $n \times n$ matrix B is called "invertible" if there exists another $n \times n$ matrix C such that $BC = CB = I_n$

If B is invertible, this C matrix is called the inverse of B, denoted B^{-1}.

"trivial examples"

I_n is invertible with inverse $I_n^{-1} = I_n$

$O_{n \times n}$ is not invertible, nothing multiplied by $O_{n \times n}$ gives I_n.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ not invertible

$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ is invertible with $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

must be a square matrix

Non-square matrices are not invertible.

The inverse matrix is unique. If $AB = BA = I_n = AC = CA$, then $B = C$.

Non-invertible = singular   Singular is the opposite of invertible.

Invertible = nonsingular

Shortcut for all 2x2 matrices

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc = 0$ then A is not invertible.

If $ad - bc \neq 0$ then A is invertible with inverse $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & b \\ c & a \end{bmatrix}$

try with , will cancel to I_2.

Ex 1: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $1 \cdot 0 - 0 \cdot 0 = 0 \Rightarrow$ not invertible

$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $1 \cdot 5 - 2 \cdot 3 = 5 - 6 = -1 \neq 0 \Rightarrow$ invertible with

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

mult. on either side = 2x2 identity matrix

Ex. 3 $\begin{bmatrix}1 & 2\\ 3 & -4\end{bmatrix}$, $1(-4) - 2(3) = -10 \neq 0 \Rightarrow$ invertible $\qquad \frac{1}{-10}\begin{bmatrix}-4 & -2\\ -3 & 1\end{bmatrix} = \begin{bmatrix}2/5 & 1/5\\ 3/10 & -1/10\end{bmatrix}$

This $ad-bc$ term is called "the determinant"

General method for all square matrices

If we want to invert the nxn matrix A, we make an augmented matrix with A on the left and In (nxn identity matrix) on the right. Then we row reduce.

If the left side of the augmented matrix reduces to In, then the right side will be A^(-1).

If the left side does not reduce to In, A is not invertible.

Ex: $B = \begin{bmatrix}-7 & 3\\ -5 & 2\end{bmatrix}$

Augment B with I_2 and row reduce.

$$\left[\begin{array}{cc|cc}-7 & 3 & 1 & 0\\ -5 & 2 & 0 & 1\end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc}1 & -\frac{3}{7} & \frac{-1}{7} & 0\\ -5 & 2 & 0 & 1\end{array}\right] \quad R_1' = \frac{1}{-7}R_1$$

$$\sim \left[\begin{array}{cc|cc}1 & -3/7 & -1/7 & 0\\ 0 & -1/7 & -5/7 & 1\end{array}\right] \quad R_2' = R_2 + 5R_1$$

$$\sim \left[\begin{array}{cc|cc}1 & -3/7 & -1/7 & 0\\ 0 & 1 & 5 & -7\end{array}\right] \quad R_2' = -7R_1$$

$$\sim \left[\begin{array}{cc|cc}1 & 0 & 2 & -3\\ 0 & 1 & 5 & -7\end{array}\right] \quad R_1' = R_1 + 3/7 R_2$$

I_2 this the inverse

This matrix is invertible.

This tells us B is invertible and $B^{-1} = \begin{bmatrix}2 & -3\\ 5 & -7\end{bmatrix}$.

(Q5 from test)

$$\begin{bmatrix} 1 & -3 & 2 \\ -1 & 3 & 0 \\ 2 & -3 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 3 & -3 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 3 & -3 & -2 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{array}\right] R_2 \Leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2/3 & 0 & 1/3 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{array}\right] R_2' = \frac{1}{3}R_2$$

After row reducing you get

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1 \\ 0 & 1 & 0 & -1/6 & 1/2 & 1/3 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array}\right] \Rightarrow B \text{ invertible}, \quad B^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1 \\ -1/6 & 1/2 & 1/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Extra Note

Prof says this is harder and not really relevant

$$B(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow B^{-1}B(\vec{x}) = B^{-1}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

∵ If you have $B\vec{x} = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$ and you know B^{-1} you can easily find $\vec{x}$.

$$\Rightarrow \vec{x} = B^{-1}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$B\vec{x} = \vec{c} \Rightarrow \vec{x} = B^{-1}\vec{c}$

Properties

1) A nxn matrix B is invertible if and only if it is row equivalent to In (RREF of B is In)

2) Inverse of an inverse is the original matrix $(A^{-1})^{-1} = A$ (b/c $AB = BA = I_n$?)

3) $(AB)^{-1} = B^{-1}A^{-1}$ (assuming A, B invertible)
Inverse distributes and swaps order of matrices.

$$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AA^{-1} = I_n$$

4) $(A^T)^{-1} = (A^{-1})^T$
Transpose and inverse can be applied in any order.

Recall: 3 common matrix multiplication mistakes

$AB = BA$

$XY = XZ \rightarrow Y = Z$

$CD = 0 \Rightarrow C = 0$ or $D = 0$

These are true:

1) $AA^{-1} = A^{-1}A$
2) $XY = XZ$ and X is invertible $\rightarrow Y = Z$ (multiply $XY = XZ$ on the left by X^{-1})
3) $CD = 0$ and either C invertible or D invertible $\Rightarrow$ other matrix is the zero matrix

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(This content was planned for lecture 10)

Pf. by contradiction

non-trivial $\vec{x}$

Suppose $A\vec{x} = \vec{0}$ and $\vec{x} \neq 0$, then A is not invertible.

Why? Suppose A^{-1} exists. Then $A^{-1}A\vec{x} = A^{-1}\vec{0}$

$\vec{x} = \vec{0}$ ← contradiction

If you can find $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \vec{0}$, that proves that A is not invertible.

linear dependance rltn on columns of A

Ex: $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ 3 & 6 & 2 \end{bmatrix} = A, \quad A\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \vec{0}$

These $\vec{x}$ represent linear dependance relations on the columns of A.

A ($n \times n$ matrix) is invertible if and only if the columns of A are linearly independant.

Is this invertible?

A favoured past question

(2nd yr question)

$$\begin{bmatrix} 1 & 0 & 0 & & 0 & -1 \\ -1 & 1 & 0 & & 0 & 0 \\ 0 & -1 & 1 & & 0 & 0 \\ & \vdots & & \ddots & & \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

No :)

Small amt of Feb 8 content on test