17 - Mar 13 Lecture

- Review of eigenvalues and eigenvectors
- Algebraic multiplicity
- Similar matrices
- Characteristic polynomial
- Matrix diagonalization

Test originally scheduled for next week has been moved to Mar 27.
↳ mostly covers eigenvalues / eigenvectors (no Cramer's Rule)

Final exam: Apr 27 9AM – 12PM

① Final exam questions are slightly easier than test questions but cover the whole course.
② There will be review sessions (details to be announced)
③ Multiple choice and long answer (same structure as term tests and ~3x longer)

Review (Find eigenvalues)
1. Calculate $\det(A-\lambda I)$
2. Factor $\det(A-\lambda I)$

Find eigenvector w/ eigenvalue
1. RR $A-\lambda I$
2. Set F vars to 1
3. Solve for B vars

Ex: Find eigenvalues of A.

$$A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A-\lambda I) &= (-3-\lambda)(3-\lambda) - (4)(4) \\ &= -9 + 3\lambda - 3\lambda + \lambda^2 - 16 \\ &= \lambda^2 - 25 \end{aligned}$$

Factor
$(\lambda+5)(\lambda-5)$
$\downarrow \qquad \downarrow$
$\lambda_1 = -5 \quad \lambda_2 = 5$

Find $\vec{v}_1 \in \text{Null}(A-\lambda I)$, $\vec{v}_1 \neq 0$ (find eigenvalue corresponding to λ_1)

$$A - \lambda_1 I = \begin{bmatrix} -3+5 & 4 \\ 4 & 3+5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \quad R_1' = \tfrac{1}{2} R_1$$

$$\sim \begin{array}{cc} B & F \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \leftarrow \text{RREF} \quad R_2' = R_2 - 4R_1$$

Let $x_2 = 1$

$x_1 + 2x_2 = 0 \rightarrow x_1 = -2 \qquad \Rightarrow \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$0 = 0$

Check ans. by $A\vec{v}_1 = \lambda_1 \vec{v}_1$.

Matrix upper tri. $\Rightarrow$ $A-\lambda I$ upper tri.

Find eigenvalues

Ex:

$$A = \begin{bmatrix} 1 & 7 & 1 & 2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix} \xrightarrow{\text{upper triangular}} A-\lambda I = \begin{bmatrix} 1-\lambda & 7 & 1 & 2 \\ 0 & 2-\lambda & 4 & 0 \\ 0 & 0 & 2-\lambda & 3 \\ 0 & 0 & 0 & -3-\lambda \end{bmatrix}$$

$$\text{Det}(A-\lambda I) = (1-\lambda)\underbrace{(2-\lambda)(2-\lambda)}(-3-\lambda)$$ ← already factored for you

$$= (1-\lambda)\underbrace{(2-\lambda)^2}_{\text{repeated roots}}(-3-\lambda)$$

$\lambda_1 = 1$ with algebraic multiplicity 1.

$\lambda_2 = 2$ with algebraic multiplicity 2.

$\lambda_3 = -3$ with algebraic multiplicity 1.

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue is the number of times the root is repeated in the factorization.

Thm.

The eigenvalues of an upper or lower triangular matrix are the entries on the main diagonal. The algebraic multiplicity is how many times that value appears on the main diagonal.

Ex: Find eigenvalues.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$ ← both upper triangular

Both A and B have a single eigenvalue (1) with alg. mult. 2.

"all $\mathbb{R}$ vectors of length 2"

The eigenspace of A corresponding to $\lambda_1 = 1$ is $\mathbb{R}^2$. (dim 2)

The eigenspace of B corresponding to $\lambda_1 = 1$ is Span $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ (dim 1)

Rlt'n btwn Eigenspace and alg. mult.

The dimension of the eigenspace of a matrix corresponding to an eigenvalue is at most the alg. mult. of that eigenvalue.

deg n. polynomial

$$\pm\lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0 = \text{Det}(A-\lambda I)$$

The sums of the alg. mult. of the eigenvalues is at most n.

If we work in $\mathbb{C}$ ← complex #s instead of $\mathbb{R}$, at most becomes equals.

Polynomials always factor to linear terms over complex numbers (not the case for $\mathbb{R}$)

Similar Matrices

Let A and B be nxn matrices. A and B are "similar" if there exists an invertible nxn matrix P such that $P^{-1}AP = B$ (or $A = PBP^{-1}$)

Ex. $\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} -22 & -30 \\ 16 & 22 \end{bmatrix}$ are similar b/c $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -22 & -30 \\ 16 & 22 \end{bmatrix}$

Characteristic Polynomial

Characteristic poly of A is $\det(A - \lambda I)$

Characteristic poly of $\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ is $\lambda^2 - 4 = (\lambda-2)(\lambda+2)$ same!

Characteristic poly of $\begin{bmatrix} -22 & -30 \\ 16 & 22 \end{bmatrix}$ is $\lambda^2 - 4 = (\lambda-2)(\lambda+2)$

If A and B are similar, the eigenvalues, alg. muls. and char. polys. will be the same.
Eigenvectors in general are not the same.

Note that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ share Eigenvalues, alg. mult, and char. polys, but they are NOT similar.

Matrix Diagonalizations

A nxn matrix B is called diagonalizable if B is similar to a diagonal matrix.

A diagonal matrix is a matrix with no non-zero entries off the main diagonal. Aka. both upper and lower triangular

Ex: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ can have 0s on main diagonal

Check if diagonalizable

A nxn matrix B is diagonalizable iff there exists a linearly independent set of n eigenvectors of B.

set these as cols of matrix P

WT construct 2 matrices

If the columns of P are a set of n linearly independent eigenvectors of a matrix A and D is a diagonal matrix with diagonal entries equal to the eigenvalues of A with each eigenvalue corresponding to the eigenvector of P in the same column.

(order must match P)

eigen$\vec{v}$ of λ_1 $\Rightarrow$ entry 1

Ex: $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$ $\lambda_1 = 2$ $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\lambda_2 = -2$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ $A = PDP^{-1}$