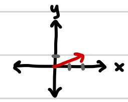
12 - Feb 15 Lecture

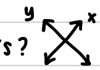
- Different coordinate systems (different basis)
- How to find coordinate vector
- Dimension of a subspace
- Rank of a matrix
- Properties of Rank
- Rank-Nullity Theorem

Different Coordinate Systems	How would you describe $\begin{bmatrix}2\\1\end{bmatrix}$ in $\mathbb{R}^2$?

Most ppl would describe it as 2 units right, 1 unit up.



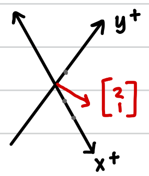
Why right then up? Why not up then right? Could the coordinate system look like this?



It's just convention.

We will look at some alternate coordinate systems!

Like this:

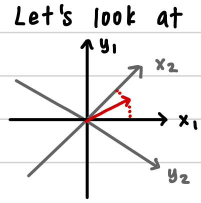


$\vec{e}_1$ is the vector along the first axis.

$\vec{e}_2$ is the vector along the second axis.

$$\begin{bmatrix}a\\b\end{bmatrix} = a\vec{e}_1 + b\vec{e}_2$$

Let's look at vectors with different basis.



(Overlay the coordinate systems)

How to Find Same Vector in Different Basis

Suppose $B = \{\vec{b}_1, \dots, \vec{b}_k\}$ is a basis for a subspace H.

Then, every $\vec{u} \in H$ can be <u>uniquely</u> written as a linear combination of the basis vectors.

$$\vec{u} = c_1\vec{b}_1 + \dots + c_k\vec{b}_k \text{ for } c_1, \dots, c_k \in \mathbb{R}$$

notation means to decompose $\vec{u}$ into basis B

Then $\begin{bmatrix}c_1\\c_2\\\vdots\\c_k\end{bmatrix} = [\vec{u}]_B$ is the B-coordinate vector of $\vec{u}$.

coordinate vector

A just defines the subspace.

Recall from lecture 11

Nullspace(A) is vector multiplication to the right of A that gives $\vec{0}$.

Ex:

Let $A = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $H = \text{Null}(A)$.

Then $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ are bases of H.

Note that $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \in H$.

① What is $[\vec{x}]_B$?

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $[\vec{x}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② What is $[\vec{x}]_C$?

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So $[\vec{x}]_C = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

"reverse process"

③ Suppose $[\vec{y}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, what is $\vec{y}$?

This is given as $[\vec{y}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. We know that $\vec{y} = c_1 \vec{b_1} + c_2 \vec{b_2}$.

$$\vec{y} = \overset{c_1}{2} \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{b_1}} \overset{c_2}{-1} \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\vec{b_2}} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

In this example, H is a 2-dimensional subspace of a 4-dimensional space (since there are 2 vectors in the basis of K and 4 elements per vector).

If you plotted H, you would find that you have a 2D plane going through the origin of a 4-dimensional space.

0D : point

1D : line

2D : plane

3D + : hyper space

Dimension of a Subspace

The dimension of a subspace K, denoted <u>$\dim(K)$</u>, is the <u>number of vectors</u> in any basis of K.

For any subspace K, all bases of K have the same number of elements / vectors

Ex. we had basis with 2 elts above $\Rightarrow$ 2 dimensional subspace.

(Subspaces have the same origin as the space since addition needs to be defined properly).

Recall from lec. 10 that: $\{\vec{0}\}$ has $\{\}$ as a basis, which is a set with 0 elements.

$\Rightarrow \{\vec{0}\}$ is a 0-dimensional subspace.

What is $\dim(\mathbb{R}^n)$?

Recall that $\{\vec{e}_1, \dots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$.

Therefore, $\dim(\mathbb{R}^n) = n$.

Length n vector w/ no restrictions $\Rightarrow$ n-dimensional space.

Recall rule 13) from lec. 10: 13) A is invertible $\Longleftrightarrow$ The columns of A form a basis of $\mathbb{R}^n$.

Ex:

$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ -2 & 0 & 1 \end{bmatrix}$ is invertible. What is its basis?

columns of A

This tells us that $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Ex:

Let $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

What is $[\vec{x}]_B$?

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow [\vec{x}]_B = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$ ← coordinate vector

The coordinate vector of the standard basis is the <u>vector itself</u>.

Ex: Let E be the standard basis. $\vec{x}$ is still $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

What is $[\vec{x}]_E$?

$[\vec{x}]_E = \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ← coordinate vector

Columns of A are linearly independent (A is $n \times n$)
$\Leftrightarrow$ columns of A span $\mathbb{R}^n$.
$\Leftrightarrow$ Columns of A form a basis for $\mathbb{R}^n$.

More general theorem

- Any set of n linearly independent vectors in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$.
- Any linearly independent set of k vectors in a k-dimensional subspace H is a basis for H.

Rank of a Matrix

$\text{Rank}(A) = \dim(\text{Col}(A))$

The rank of a matrix is equal to the dimension of the column space of that matrix.

Properties of Rank

1) $\text{Rank}(A) = \text{Rank}(A^T)$
2) $\text{Rank}(A) \leq n, m$ (A is $n \times m$) "If A is $n \times m$, Rank cannot exceed n or m."

What is the Rank of matrix A?

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 7 & 1 & 3 \\ 0 & 1 & 3 & 3 \end{bmatrix} \qquad \text{RREF}(A) = \begin{bmatrix} \boxed{1} & 0 & -10 & -9 \\ 0 & \boxed{1} & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Col}(A)$

$\Rightarrow \text{Rank}(A) = 2$ (since 2 vectors in basis)

The rank of a matrix is equal to the <u>number of pivot positions</u> in that matrix.

If you have a $n \times n$ matrix A and $\text{Rank}(A) = n$, A is invertible. (Rule 14)).

Rank-Nullity Theorem

If A is a matrix with n columns, then $\text{Rank}(A) + \dim(\text{Null}(A)) = n$

Usually, you know Rank and n, and need to find $\dim(\text{Null}(A))$.

$\dim(\text{Null}(A)) = n - \text{Rank}(A)$

Ex:

If $A = \begin{bmatrix} \boxed{1} & -2 & 0 & -1 \\ 0 & 0 & \boxed{1} & -1 \end{bmatrix}$ ← RREF, what is $\dim(\text{Null}(A))$?

2 pivots $\Rightarrow \text{Rank}(A) = 2$ A has 4 columns $\Rightarrow n = 4$

$\dim(\text{Null}(A)) = 4 - 2 = 2 \Rightarrow$ Nullspace of A is dimension 2

Recall: Any linearly independent set of K vectors in a k-dimensional subspace is a basis.

Ex: Is B a basis?

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

2 vectors in linearly independent space

The vectors of B are linearly independent.

$\Longrightarrow$ B is a basis

Any two linearly-independent vectors will form a basis if vectors are in the subspace.

There are infinitely-many "basis combos".