7 - Jan 30 Lecture

- Review of standard basis vector
- Linear transformations as matrix multiplication
- Rotation & projection transformations
- Kernel(T) & Range(T)
- Onto & one-to-one

Standard Basis Vector

Recall from Lecture 6 that:

where $0 < i < \mathbb{R}^i$ ← space

The standard basis vector $\vec{e}_i$ is a column vector with 1 in the i^{th} position and 0s in all other positions.

In $\mathbb{R}^2$: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In $\mathbb{R}^3$: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

As well, it is true that.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \ldots + x_n \vec{e}_n$$

We can write linear transformations as matrix multiplication.

Let $T \cdot \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then, there exists a matrix A such that $T(\vec{v}) = A\vec{v}$.

This $A = [T(e_1) \quad T(e_2) \quad \ldots \quad T(e_n)]$. So, $T(\vec{e}_i)$ is the i^{th} column of matrix A.

Consider $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$T_r \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ where r is a constant.

We want to find the matrix A which corresponds to this.

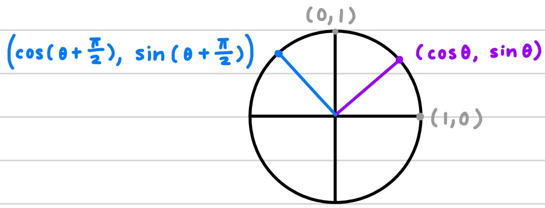
We know $A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)]$, so $T(\vec{e}_1) = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$.

Then, $A = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$. ← get by "squishing" $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$ together into one matrix

Rotation Transformation

Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation of rotating vectors around the origin counter-clockwise by an angle θ

Think about the unit circle



$$\Rightarrow T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\Rightarrow T(\vec{e}_2) = \begin{bmatrix} \cos(\theta+\frac{\pi}{2}) \\ \sin(\theta+\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

From this, we get the general matrix for rotations:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Projection Transformation

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the "projection transformation" which zeroes out at least one component (in this case, the $\vec{z}$).

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \rightarrow T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad T(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Kernel

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

The set of all vectors $\vec{x}$ such that $T(\vec{x}) = \vec{0}$ is called the "kernel" and is denoted $Ker(T)$.

1) $\vec{0}$ is always in the kernel — zero vector is always in $Ker(T)$.
2) $\vec{u}$ in $Ker(T) \Rightarrow c\vec{u}$ in $Ker(T)$ — if vector $\vec{u}$ is in $Ker(T)$, all multiples of $\vec{u}$ are in $Ker(T)$.
3) $\vec{u}, \vec{v}$ in $Ker(T) \Rightarrow \vec{u}+\vec{v}$ in $Ker(T)$ — if vectors $\vec{u}$ and $\vec{v}$ are in $Ker(T)$, sum $\vec{u}+\vec{v}$ is in $Ker(T)$

Ex: As an example, let's look again at the projection transformation above:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Vectors like $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$ are in the kernel.

Note: $Ker(T)$ and $Range(T)$ are always subspaces.

Range

The set of all vectors $\vec{b}$ such that there exists $\vec{y}$ in $\mathbb{R}^n$ with $T(\vec{y}) = \vec{b}$ is called the range of T and is denoted Range(T).

In other words:

→ all the output vectors ($\vec{b}$) such that you can write $\vec{b}$ as some transformation of $\vec{y}$.

→ all possible outputs of a transformation.

Ex: As an example, let's look again at the projection transformation above:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad T(\vec{y}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ has no solution. Therefore, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in range(T) (even though it's in the codomain)

In this case, the codomain is all column vectors with 3 rows

$T(\vec{y}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is possible if $(\vec{y}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in Range(T).

Onto

A transformation is "onto" if for every $\vec{b}$ in the codomain, there exists at least one vector $\vec{x}$ in the domain such that $T(\vec{x}) = \vec{b}$.

In other terms: Onto means that Range(T) = codomain of T (rotation is an onto function)

One-to-one

A transformation is one-to-one if for every $\vec{b}$ in the codomain, there exists at most one vector $\vec{x}$ in the domain such that $T(\vec{x}) = \vec{b}$.

In other words: "if the only thing that maps to $\vec{0}$ is $\vec{0}$, the function is one-to-one".

Note that functions can be:
- onto & one-to-one
- only onto
- only one-to-one
- neither onto nor one-to-one

A trick for determining if something is onto and/or one-to-one

1) For linear mappings to be onto, the dimension can't increase.
2) For linear mappings to be one-to-one, the dimensions can't decrease.
Note that these only work forwards!!

The more useful "backwards versions" of the above rules are:
1) If dimensions increase, the function is not onto.
2) If dimensions decrease, the function is not one-to-one.

Given linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ with corresponding matrix A...

Prove T is one-to-one

"iff" = "if and only if" = proving one proves the others

Prove T is one-to-one $\iff$ $\text{Ker}(T) = \{\}$

$\iff$ columns of A are linearly independant

$\iff$ each column of A has a pivot position if A is in RREF

Prove T is onto

Prove T is onto $\iff$ $\text{Range}(T) = \mathbb{R}^m$ ← codomain

just need to prove one of these!

$\iff$ columns of A span $\mathbb{R}^m$

$\iff$ each row of A has a pivot position

$\iff$ $A\vec{x} = \vec{b}$ is consistent for all $b \in \mathbb{R}^m$

Ex 1: Onto and one-to-one

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ 3z \end{bmatrix}$$

Ex 2: Onto

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+y \\ 2y \end{bmatrix}$$

Why? There is a $\vec{v}$ other than $\vec{0}$ which maps to $\vec{0}$: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, therefore not 1:1.

Also, a decrease in dimension $\Rightarrow$ not one-to-one.

(it is onto since every possible 2D vector has a 3D vector input)

Ex 3: One-to-one

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x \\ z \\ y \end{bmatrix}$$

(it is one-to-one since any 2 distinct vector inputs will produce two distinct outputs)

Not onto since dimension increased.

Ex 4: Not onto and not one-to-one

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x \\ y \end{bmatrix}$$

Not onto since the output $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is not possible.

Not one-to-one since the input $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives the same output as $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Standard basis $\vec{v}$

where $0 < i \leq \mathbb{R}^i$

space

$\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ 1 in ith pos.
0 everywhere else

In $\mathbb{R}^2$: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Addition to last lec.

In $\mathbb{R}^3$: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \ldots + x_n \vec{e}_n$$

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be <u>linear</u>, then there exists a matrix A such that $T(\vec{v}) = A\vec{v}$

"Can write lin. trans. as matrix. mult."

This $A = [T(e_1)\ T(e_2) \ldots T(e_n)]$. So $T(\vec{e}_i)$ is the i^{th} column of matrix A.

Consider $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$T_r \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ where r is a constant

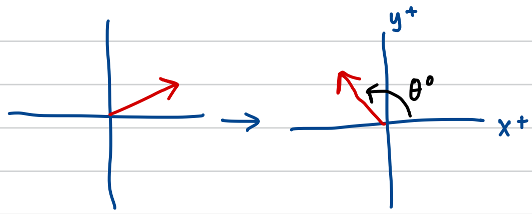
Want to Find what matrix A corresponds to this.

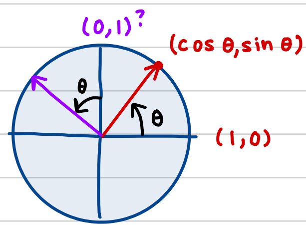
we know $A = [T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)]$, so

$T(\vec{e}_1) = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$ $T(\vec{e}_2) = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$ $T(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$

Then $A = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$

Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation of rotating vectors around the origin counter-clockwise by an angle θ.





$$\Rightarrow T(\vec{e_1}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\Rightarrow T(\vec{e_2}) = \begin{bmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

we get the general matrix for rotations

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

"projection" transformation zeroes out 1+ components (in this case the $\vec{z}$)

$T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad T(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

The set of all vectors $\vec{x}$ s.t. $T(\vec{x}) = \vec{0}$ is called the "kernel" and is denoted $\ker(T)$.

"vectors like $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are in the kernel" ??? "$\begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$ also in Ker" ????

- $\vec{0}$ is always in the kernel
- $\vec{u}$ in $\text{Ker}(T) \Rightarrow c\vec{u}$ in $\text{Ker}(T)$ one $\vec{v}$ = all multiples in Ker
- $\vec{u}, \vec{v}$ in $\ker(T) \Rightarrow \vec{u} + \vec{v}$ in $\text{Ker}(T)$ 2 $\vec{v}$ in Ker $\to$ their sum in Ker

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The set of all vectors $\vec{b}$ s.t. there exists $\vec{y}$ in $\mathbb{R}^n$ with $T(\vec{y}) = \vec{b}$ is called the range of T and is denoted Range(T).

"all the output vectors ($\vec{b}$) s.t. you can write $\vec{b}$ as some Trans of $\vec{y}$."
"all possible outputs of a transformation"

"$T(\vec{y}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has no solution"

↳ "$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ not in range even though it's not in the codomain."

"Codomain: must have 3 rows in vector"

" # cols: input

rows: output? "

$T(\vec{y}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ possible

$(\vec{y}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in Range (T)

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ not in Range (T)

T is called "<u>onto</u>" if for every $\vec{b}$ in the codomain there exists at least 1 vector $\vec{x}$ in the domain s.t. $T(\vec{x}) = \vec{b}$.

Onto means Range (T) = codomain of T.
(rotation is an onto function)

T is called "<u>one-to-one</u>" if for every $\vec{b}$ in the codomain there exists at most 1 vector $\vec{x}$ in the domain s.t. $T(\vec{x}) = \vec{b}$.

"If only thing that maps to $\vec{0}$ is $\vec{0}$, func. is 1:1"

Can be both / 1 of / neither 1:1 or onto.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ 3z \end{bmatrix}$$

this is onto
this is 1:1

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+y \\ 2y \end{bmatrix}$$

is there a $\vec{v}$ other than $\vec{0}$ that maps to $\vec{0}$?

yes, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow$ not 1:1

Every possible 2D $\vec{v}$ has a 3D vector input

Yes it is onto ??

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

output $\vec{v}$ w/ no possible input $\vec{v}$

$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x \\ y \end{bmatrix}$ not onto $\rightarrow$ output $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is not possible

not 1:1 $\rightarrow$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ same result as $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x \\ z \\ y \end{bmatrix}$ not onto (also b/c of dimensions)

is 1:1 any 2 distinct $\vec{v}$ = 2 distinct outputs

Both $\rightarrow$ same dimension

Onto, not 1:1 $\rightarrow$ dimension dec.

Not onto, 1:1 $\rightarrow$ dimension inc.

For ~~a~~ linear mappings to be "onto" the dimension can't increase.

For linear mappings to be ‘one-to-one’ the dimensions can’t decrease.
(only works forwards!!!)

“backwards versions”: (more useful)
Dimensions increase = not onto
Dimensions decrease = not 1:1

Given
$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear with corresponding matrix A

"iff"

prove one = prove other

T is 1:1 $\Leftrightarrow$ $Ker(T) = \{\vec{0}\}$ ($T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$)

$\Leftrightarrow$ columns of A are linearly independant

$\Leftrightarrow$ each column of A has a pivot position (after RR algo)

Prove

T is onto $\Leftrightarrow$ $Range(T) = \mathbb{R}^m$ codomain

prove one of these

$\Leftrightarrow$ columns of A span $\mathbb{R}^m$

$\Leftrightarrow$ Each row of A has a pivot position

$\Leftrightarrow$ $A\vec{x} = \vec{b}$ is consistent for all $b \in \mathbb{R}^m$