# 19 - Mar 20 Lecture

- Intro to complex numbers
- Vector addition and multiplication
- Adding and multiplying complex numbers
- Properties of complex numbers (addition, multiplication, inverses, complex conjugates, magnitudes)
- Taking real or imaginary component of a complex number
- Complex conjugate
- Magnitude of a complex number
- Inverse of a complex number
- Polar coordinates
- Multiplication of polar coordinates

**Review**

(+ irrationals)

Natural numbers ⟶ integers ⟶ rationals ⟶ reals
1, 4 — $-3, 0$ — $\frac{1}{2}, \frac{-7}{4}$ — $\sqrt{2}, \pi$

⟶ complex numbers ⟶ quaternions ⟶ . . .
$2+i$, $3-2i$

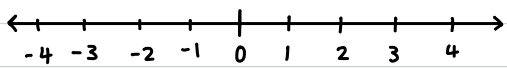
"$\sqrt{-1}$ doesn't exist" ... in $\mathbb{R}$
↳ it does exist in $\mathbb{C}$

**Complex Number**

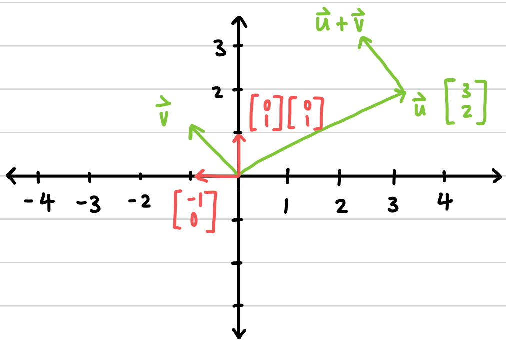
Written as $a+bi$ such that $i^2=-1$ and $a, b \in \mathbb{R}$

**Visual Representation**

Take a number line...




... and add another dimension. For now, you can think of this as $\mathbb{R}^2$.




vector addition exists as normal,
but multiplication is defined differently.

**Vector Multiplication**

Define multiplication of vectors as follows (no longer in $\mathbb{R}^2$):

$$\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}ac-bd\\ad+bc\end{bmatrix}$$

Ex: $$\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}(1)(2)-(3)(4)\\(1)(4)+(2)(3)\end{bmatrix} = \begin{bmatrix}-10\\10\end{bmatrix}$$

**Results** ①

think of as "real number a"

$$\begin{bmatrix}a\\0\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix} = a\begin{bmatrix}c\\d\end{bmatrix}$$

(unlike matrix multiplication)
Note: multiplication is <u>commutative</u>. Can multiply vectors in <u>any order</u> and get the <u>same result</u>.

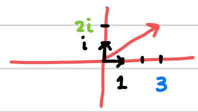
②
$$\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

(technically wrong) but shows concept

We can decompose vectors into basis vector components.

$\begin{bmatrix} a \\ b \end{bmatrix} \rightsquigarrow a + bi$ (slightly wrong notation)

$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ ← real number line, ← imaginary component




$\rightsquigarrow 3 + 2i$

**Adding Complex Numbers**

Consider $a+bi$, treat $i$ as a variable in a polynomial.

$a, b, c, d \in \mathbb{R}$

$(a+bi) + (c+di) = (a+c) + (b+d)i$

**Multiplying Complex Numbers**

$(a+bi)(c+di) = ac + adi + bci + bdi^2$

$= ac + (ad+bc)i + (bd)i^2$

<u>Set $i^2 = -1$.</u>

$= ac + (ad+bc)i + (bd)(-1)$

$= (ac-bd) + (ad+bc)i$

$\Longrightarrow \begin{bmatrix} ac - bd \\ ad - bc \end{bmatrix}$ Same as formula given on prev. page!

Steps:
1) Treat as polynomial until simplified to include $i^2$.
2) Set $i^2$ to $-1$.

$i^2 = -1$

$i^3 = i^2 \cdot i = -1i$

Ex:

$(2+i)(1-i)$
$= 2 - 2i + i - i^2$
$= 2 - i - i^2$
$= 2 - i - (-1)$
$= 2 - i + 1$
$= 3 - i$

Ex: $(3+2i)(-1+2i) = -3 + 6i - 2i + 4i^2$
$= -3 + 4i + 4i^2$
$= -3 + 4i + 4(-1)$
$= -7 + 4i$

Ex: $(4+i)(1-i) = 4 - 4i + i - i^2$
$= 5 - 3i$

<u>Other method</u>:
Replace $i$ with $\sqrt{-1}$ before multiplication.

Ex: $(4+\sqrt{-1})(1-\sqrt{-1}) = 4 - 4\sqrt{-1} + \sqrt{-1} - \sqrt{-1}^2$
$= 4 - 3\sqrt{-1} - (-1)$
$= 5 - 3\sqrt{-1}$
<u>Change $\sqrt{-1}$ back to $i$</u>
$= 5 - 3i$

**Properties of Complex Numbers**

If $z, w, u$ are complex numbers:

① $u + (w + z) = (u + w) + z$, $w + z = z + w$

② $u(wz) = (uw)z$, $wz = zw$

③ If $z \neq 0$, $z^{-1}$ exists.

$\mathbb{R}$ : reals

$\mathbb{C}$ : complex numbers

All polynomials with coefficients in $\mathbb{C}$ factor to linear terms in $\mathbb{C}$.

Even something like $7x^{12} + 7x^{11} + 3x^4 - \pi x^3 + (2-i)x^2 - 3$ will factor into the form $7(x+c_1)(x+c_2) + \ldots + (x+c_{12})$ where $c_1, \ldots, c_{12} \in \mathbb{C}$.

**Taking real component of a complex number**

$\mathrm{Re}(a+bi) = a$

| |
|---|
| Ex: $\mathrm{Re}(2+3i) = 2$ |
| Ex: $\mathrm{Re}(7) = 7$ |

**Taking imaginary component of a complex number**

$\mathrm{Im}(a+bi) = b$ ← NOT bi

$i$ is known as the "imaginary number"

| |
|---|
| Ex: $\mathrm{Im}(2+3i) = 3$ |
| Ex: $\mathrm{Im}(7) = \mathrm{Im}(7+0i) = 0$ |

**Complex Conjugate**

Denoted with an overline, the complex conjugate negates the imaginary component of a complex number.

overline → $\overline{a+bi} = a - bi$

graphically: flip over horiz axis

Interestingly, if you multiply a number by its complex conjugate...

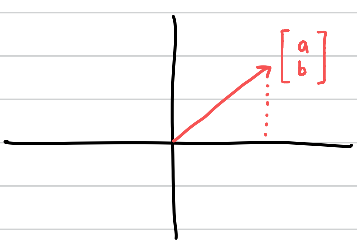
$$\begin{aligned}(a+bi)(\overline{a+bi}) &= (a+bi)(a-bi) \\ &= a^2 - \cancel{abi} + \cancel{abi} - b^2 i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2\end{aligned}$$

... you get a positive real number!

**Magnitude of a complex number**

For any complex number $z$, $|z| = \sqrt{z\bar{z}}$. ← magnitude

$|a+bi| = \sqrt{a^2+b^2}$ look familiar? pythagorean theorem



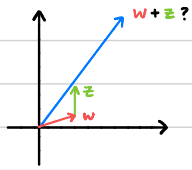

$z \rightarrow$ "distance from origin"

## More Properties of complex numbers

1. $\overline{z} = z$ if and only if $z \in \mathbb{R}$
2. $\overline{w+z} = \overline{w} + \overline{z}$
3. $\overline{wz} = \overline{w} \cdot \overline{z}$
4. $z\overline{z} = |z|^2 \geq 0$ ( if $z \neq 0$ then $|z| > 0$ ) — distance from origin
5. $|wz| = |w||z|$
6. $|w+z| \leq |w| + |z|$ (aka "triangle inequality)
7. If $z \neq 0$ then $z^{-1} = \dfrac{\overline{z}}{|z|^2}$

## Triangle Inequality (6)

Can't make triangle with side lengths 1, 1, 12, for example.




$|w+z| \leq |w| + |z|$

$12 \nleq 1 + 1$

$\therefore$ cannot form triangle with these side lengths

## Inverse of a complex number

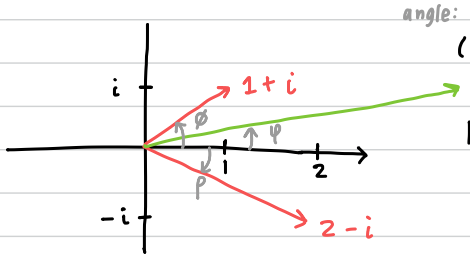
$$(a+bi)^{-1} = \frac{a-bi}{a^2+b^2}$$

Ex: $(2-3i)^{-1} = \dfrac{2-(-3)i}{(2)^2+(-3)^2} = \dfrac{2+3i}{4+9} = \dfrac{2+3i}{13} = \dfrac{2}{13} + \dfrac{3i}{13}$

Ex: $\dfrac{(1+i)}{(2-3i)} = (1+i)\left(\dfrac{1}{2-3i}\right) = (1+i)\left(\dfrac{2}{13} + \dfrac{3i}{13}\right) = \dfrac{2}{13} + \dfrac{3i}{13} + \dfrac{2i}{13} + \dfrac{3i^2}{13} = \dfrac{2}{13} + \dfrac{5i}{13} + \dfrac{3(-1)}{13} = \dfrac{-1}{13} + \dfrac{5i}{13}$

(from example above)

## Visual Representation of Polar Coordinates




angle: ø ρ φ

$(1+i)(2-i) = \underline{3+i}$

$|1+i||2-i| \quad |3+i|$ (Prop. 5)

$= \sqrt{2}\sqrt{5} \quad = \sqrt{10}$

$\phi + \underset{\text{negative}}{\underline{\rho}} = \varphi$

$|w||z| = |wz|$

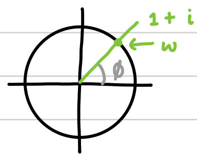
Write as <u>angle</u> and <u>distance</u>. Instead of a units right, b units up..

**Polar Coordinates** — Write positions using <u>angle</u> and <u>distance</u>.

Ex: $1 + i$ has a distance of $\sqrt{2}$ and an angle of $\frac{\pi}{4}$

**Radians** — Angle is taken as the angle <u>counter-clockwise</u> from the "positive real direction".

Consider the unit circle.




$$1 + i = |1+i|\,w$$
$$= \sqrt{2}\,(\underbrace{\cos\tfrac{\pi}{4}}_{\text{horiz.}} + i\,\underbrace{\sin(\tfrac{\pi}{4})}_{\text{vert.}})$$

In general, $z = |z|(\cos\phi_z + i\sin \underbrace{\phi_z}_{\text{angle corresponding to } z})$

"polar form" (i think)

$w = |w|(\cos\phi_w + i\sin\phi_w)$

$z = |z|(\cos\phi_z + i\sin\phi_z)$

**Polar Multiplication**

$$wz = \underbrace{|w||z|}_{\text{prod. of magnitudes}}(\cos(\phi_w+\phi_z) + i\sin(\phi_w+\phi_z))$$

sum of angles

2 Conventions for using radians:

→ angles between $-\pi$ and $\pi$

→ angles between $0$ and $2\pi$

But ultimately it's up to you. The math will work out no matter what.