6 - Jan 25 Lecture

- Linear transformations
- Rules of linear transformations

Definitions: transformation, domain, codomain, linear transformation

Definition of Transformations	"Transformation" is another word for function. ↳ Takes something from one algorithm structure and converts to another.
Transformation Notation	$T: U \to V$, denoting transformations from U to V In this course, we mostly care about transformations of column vectors $T: \mathbb{R}^n \to \mathbb{R}^m$
Ex.	$T: \mathbb{R}^2 \to \mathbb{R}^3$ denotes a transformation from column vectors of length 2 to column vectors of length 3.
Domain, Codomain, Range	For $F: U \to V$, U is called the "domain" and V is called the "codomain". (Range would be all possible values)
Transformation Notation	$T(\vec{x}) = \vec{y}$ (T will be defined in all questions.)
Ex.	If T is defined by $T\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = \begin{bmatrix} w_2 \\ w_1 \\ w_1 + w_2 \end{bmatrix}$, then $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$
Definition of Linear Transformations	A transformation T is called linear if both: 1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain. A transformation applied to the sum of 2 vectors is equal to the sum of the result of 2 transformations. 2) $T(c\vec{u}) = cT(\vec{u})$ If you have a scalar multiple of a vector, you can take the scalar multiple out of the transformation.
Textbook rules	(1) and 2) above)
More practical rules	a) If T is linear then $T(\vec{0}) = \vec{0}$. (Rule a) is only useful when showing that non-linear functions are non-linear) If $T(\vec{0}) \neq \vec{0}$, T is not linear. (contrapositive of first statement) b) T is linear if and only if (goes both ways $\Leftrightarrow$) $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$ for all scalars c, d and all $\vec{u}, \vec{v}$ in the domain.
	These rules are used to identify if a transformation is linear. Prove transformation is linear: either prove 1) and 2), or just b). Prove transformation is not linear: Show $T(\vec{0}) \neq \vec{0}$ (fail a)) or find one counter-example to any of 1), 2), or b).

doesn't always work, but fastest way

Ex. 1 Consider $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined as follows:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \\ 2+y \end{bmatrix}$$

Is this a linear transformation?

Not linear because it violates rule a).

$$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \neq \vec{0}_3$$

It also breaks rule b).

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\begin{aligned} \text{LHS} &= T(c\vec{u} + d\vec{v}) \\ &= T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + d\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} + \begin{bmatrix} dv_1 \\ dv_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} cu_1 + dv_1 \\ cu_2 + dv_2 \end{bmatrix}\right) \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix} \\ &= \begin{bmatrix} cu_2 + dv_2 \\ cu_1 + dv_1 \\ 2 + cu_2 + dv_2 \end{bmatrix} \begin{matrix} \leftarrow y \\ \leftarrow x \\ \leftarrow 2+y \end{matrix} \end{aligned}$$

LHS $\neq$ RHS, so not linear.

$$\begin{aligned} \text{RHS} &= cT(\vec{u}) + dT(\vec{v}) \\ &= cT\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + dT\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \\ &= c\begin{bmatrix} u_2 \\ u_1 \\ 2+u_2 \end{bmatrix} + d\begin{bmatrix} v_2 \\ v_1 \\ 2+v_2 \end{bmatrix} \\ &= \begin{bmatrix} cu_2 \\ cu_1 \\ 2c + cu_2 \end{bmatrix} + \begin{bmatrix} dv_2 \\ dv_1 \\ 2d + dv_2 \end{bmatrix} \\ &= \begin{bmatrix} cu_2 + dv_2 \\ cu_1 + dv_1 \\ 2c + 2d + cu_2 + dv_2 \end{bmatrix} \end{aligned}$$

Ex.2 Consider $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ 2x+y \end{bmatrix}$$

Is this linear?

Rule a) is satisfied...

$T(\vec{0}_2) = \vec{0}_3$

... but that doesn't confirm that it's linear.

Need to do test b).

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\begin{aligned} LHS &= T(c\vec{u} + d\vec{v}) \\ &= T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + d\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} + \begin{bmatrix} dv_1 \\ dv_2 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} cu_1 + dv_1 \\ cu_2 + dv_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} cu_2 + dv_2 \\ cu_1 + dv_1 \\ 2(cu_1 + dv_1) + cu_2 + dv_2 \end{bmatrix} \\ &= \begin{bmatrix} cu_2 + dv_2 \\ cu_1 + dv_1 \\ 2cu_1 + 2dv_1 + cu_2 + dv_2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} RHS &= cT(\vec{u}) + dT(\vec{v}) \\ &= cT\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + dT\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \\ &= c\begin{bmatrix} u_2 \\ u_1 \\ 2u_1 + u_2 \end{bmatrix} + d\begin{bmatrix} v_2 \\ v_1 \\ 2v_1 + v_2 \end{bmatrix} \\ &= \begin{bmatrix} cu_2 \\ cu_1 \\ 2cu_1 + cu_2 \end{bmatrix} + \begin{bmatrix} dv_2 \\ dv_1 \\ 2dv_1 + dv_2 \end{bmatrix} \\ &= \begin{bmatrix} cu_2 + dv_2 \\ cu_1 + dv_1 \\ 2cu_1 + 2dv_1 + cu_2 + dv_2 \end{bmatrix} \end{aligned}$$

LHS = RHS, so T is a linear transformation.

Ex.3 Consider $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x^2 \end{bmatrix}$$

Is this linear?

Checking rule a), we find the following, which doesn't tell us anything.

$T(\vec{0}_2) = \vec{0}_2$

Let's use rule 2) instead. I suspect that T is not linear, so I will try to pick c and $\vec{u}$ such that the equation $T(c\vec{u}) = cT(\vec{u})$ does not hold.

Let $c = 2$, $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

LHS	RHS
$LHS = T(c\vec{u})$	$RHS = cT(\vec{u})$
$= T\left(2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$	$= 2T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\begin{matrix}\leftarrow x \\ \leftarrow y\end{matrix}$
$= T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)\begin{matrix}\leftarrow x \\ \leftarrow y\end{matrix}$	$= 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{matrix}\leftarrow y \\ \leftarrow x^2\end{matrix}$
$= \begin{bmatrix} 0 \\ 4 \end{bmatrix}\begin{matrix}\leftarrow y \\ \leftarrow x^2\end{matrix}$	$= \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$LHS \neq RHS$, therefore T is not linear.

Multiplication by a matrix is a linear transformation.

Note that rules 1), 2), a), b) are similar to the matrix equation rules from lecture 4.

$T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by:

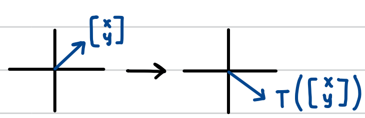
$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$ for only $A \in \mathbb{R}^{2\times 3}$ is always linear.

Transformations:

$T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$

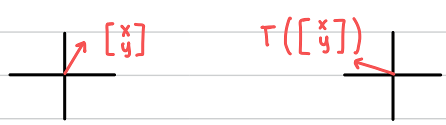
Reflection across x-axis

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$



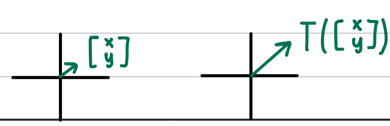
90° Rotation Counter-Clockwise

$A\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$



Scale width ×2 and height ×3

$A\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$



Another rule	If T is linear then: $T(c_1\vec{v}_1 + \ldots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + \ldots + c_pT(\vec{v}_p)$
Standard Basis Vector j	standard basis vector j $T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = T\left(x_1\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right) = x_1T(e_1) + \ldots + x_nT(e_n)$ where e_j is the vector with all zeroes except a 1 in position j.
	This $x_1T(e_1) + \ldots + x_nT(e_n)$ is a linear combination of vectors.
	$x_1T(e_1) + \ldots + x_nT(e_n) = A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ for $A = [T(e_1), \ldots, T(e_n)]$.
	Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. There exists a unique matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. (T equal to multiplication by that matrix)
	This A will be the $m \times n$ matrix whose j^{th} column is equal to $T(e_j)$.
	$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2y \\ -x+y \\ x-3y \end{bmatrix} \longrightarrow A\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 1 & -3 \end{bmatrix}$