1 Bond & Time Value of Money

Compound interest, and continuous compounding:

$$FV = PV\left(1+\frac{r}{n}\right)^{nt} \qquad PV = FV\left(1+\frac{r}{n}\right)^{-nt}$$
$$P_t = P_0 e^{rt} \qquad P_o = P_t e^{-rt}$$

Cash flow, with interest rate r_i at period i:

$$PV = \sum_{i=1}^{n} \frac{CF_i}{(1+r_i)^i}$$

Excel calculates for constant CF and r with `=PV(r,n,CF)`.
Perpetual bond: Fixed payment CF each period forever, and assumed r constant, $n \to \infty$:

$$PV = \sum_{i=1}^{\infty} \frac{CF}{(1+r)^i} = \frac{CF/(1+r)}{1-\frac{1}{1+r}} = \frac{CF}{r}$$

Rate r in perpetual bond is the hurdle rate of a corp such that it is the min IRR required for any investment.
Gordon growth model: payment grow at a constant rate $CF_i = CF_{i-1}(1+g)$. For finite term:

$$PV = \sum_{i=1}^{n} \frac{CF_0(1+g)^i}{(1+r_i)^i} = CF_0 \sum_{i=1}^{n} \frac{(1+g)^n}{(1+r)^n}$$

As $n \to \infty$, $PV = CF_0/(r-g)$

Bond pricing

Bond with coupon C, constant discount rate y and repaid \$1 after n terms,

$$PV = \sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{1}{(1+y)^n} = \frac{C}{y}\left(1-\frac{1}{(1+y)^n}\right) + \frac{1}{(1+y)^n}.$$

If PV is the price, y here is the yield for the bond. Price sensitivity to the yield:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{Par}{(1+y)^n}\right) = \sum_{i=1}^{n} \frac{-iC}{(1+y)^{i+1}} + \frac{-nPar}{(1+y)^{n+1}}$$
$$\frac{1}{P}\frac{\partial P}{\partial y} = -\frac{1}{1+y}\frac{1}{P}\left(\sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nPar}{(1+y)^n}\right)$$

which we define the Macauley duration and modified duration as

$$D_{\text{Mac}} = \frac{1}{P}\left(\sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nPar}{(1+y)^n}\right)$$
$$D_{\text{mod}} = \frac{1}{1+y} D_{\text{Mac}}$$

respectively. Then px sensitivity $\frac{1}{P}\frac{\partial P}{\partial y} = -D_{\text{mod}}$. Given D_{mod}, a bond at P will see price change ΔP at yield change Δy by $\Delta P = -P\Delta y D_{\text{mod}}$. A zero-coupon bond will have $P = Par/(1+y)^n$ and

$$D_{\text{Mac}} = \frac{1}{P}\frac{nPar}{(1+y)^n} = \frac{1}{P}nP = n,$$

which is most sensitive to Δy. Taylor expansion of price function

$$P(y+\Delta y) = P(y) + \frac{\partial P}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\Delta y^2$$

where the second order term accounts for the convexity (as yield increases, P is less sensitive to Δy). Define dollar convexity

$$C = \frac{\partial^2 P}{\partial y^2} = \sum_{i=1}^{n} \frac{i(i+1)C}{(1+y)^{i+2}} + \frac{n(n+1)Par}{(1+y)^{n+2}},$$

then price change is

$$\Delta P = -P\Delta y D_{\text{mod}} + \frac{1}{2}C\Delta y^2 + o(y^3).$$

Numerical approximation of duration and convexity:

$$D_{\text{mod}} = -\frac{P(y+\Delta y) - P(y-\Delta y)}{2P\Delta y}$$
$$C = \frac{P(y+\Delta y) - 2P(y) + P(y-\Delta y)}{\Delta y^2}$$

Weakness of duration: It only measure interest rate risk, not yield curve risk (i.e., if yield curve changes shape, not only shifted).

2 Probability & Statistics

Correlation between two RVs X and Y:

$$r_{X,Y} = \frac{\sum_{i=1}^{n}(X-\bar{X})(Y-\bar{Y})}{(n-1)\sigma_X\sigma_Y} \in [-1,1]$$

which $\rho_{X,Y} = r_{X,Y}^2 \in [0,1]$ is the *coefficient of determination*.
Sum of RVs: Let $Z = w_1X + w_2Y$, and $\text{Cov}(X,Y) = \rho_{X,Y}\sigma_X\sigma_Y$ be the covariance,

$$\text{E}(Z) = w_1\bar{X} + w_2\bar{Y}$$
$$\text{Var}(Z) = w_1^2\sigma_X^2 + w_2^2\sigma_Y^2 + 2w_1w_2\,\text{Cov}(X,Y)$$

Lognormal distribution: Common to model stock price
Wiener process: $W_t + u - W_t \sim N(0,u)$, i.e., standard deviation scales with sq root of time

Regression

Linear regression: Find line $y = a + bx$ that is best linear unbiased estimator (BLUE) that relates x_i to y_i. Error of regresion: SSR (sum of sq error due to regression), SSE (sum of sq errors), SST (sum of sq total error, equiv to sample variance times sample size):

$$SSR = \sum_i (y(x_i) - \bar{y})^2 \quad \text{with } \bar{y} = \frac{1}{n}\sum_i y_i$$
$$SST = \sum_i (y_i - \bar{y})^2$$
$$SSE = SST - SSR$$
$$R^2 = SSR/SST \in [0,1]$$

R^2 measures the goodness of fit, i.e., percentage of variation of the data explained by the regression equation.
Test for *significance* of regression coefficients a and b: by t-statistics. Simple linear regression on n samples, the degree of freedom is $n-2$ for two coefficients are derived from them. With desired level of significance α, a corresponding t value (t_α) is set. We test if $|t'| > t_\alpha$ to determine the regression coefficient is significant, i.e., the data is not random. Which (usually $\tilde{a} = 0$ as null hypothesis)

$$t' = \left|\frac{r_{X,Y}\sqrt{n-2}}{\sqrt{1-r_{X,Y}^2}}\right| \qquad \text{for sig of correlation coeff } r_{X,Y}$$
$$t' = \left|\frac{a-\tilde{a}}{s_a}\right| \qquad \text{for sig of regression coeff } a \text{ (or similarly } b\text{)}$$

Time series

Regression on time series: $AR(n)$, auto-regression of lag n periods. For $AR(1)$, the linear model is $Y_t = a + bY_{t-1}$, i.e. value at t depends only on value at $t-1$.
Random walk: $AR(1)$ model of $Y_t = a + bY_{t-1} + \epsilon$ where error term *epsilon* $\sim N(0,\sigma^2)$ is independent of any Y_τ. Here b is the *drift* term,

- if $b \in [0,1)$, the series is *mean-reversion* (pull Y_t back to the mean). Mean of Y_t is the solution to $\mu = a + b\mu$ or $\mu = a/(1+b)$
- if $b \in (-1,0]$, the series is oscillating, with the same mean as above
- if $|b| = 1$, the series is a Weiner process
- if $|b| > 1$, the series is explosive

3 Derivatives

Forward contract

Risk-free rate r, spot price S_t, price of forward contract in no-arbitrage argument:

$$F_{t,T} = S_t e^{r(T-t)}$$

Forward price should equal to future value of spot price. The price of forward contract can be extended to include the cost of carry s, i.e., all storage, insurance, etc.: $F_{t,T} = (S_t + s)e^{r(T-t)}$.
Forward contract: No up-front cost, settled at maturity, but no clearing house to regulate or monitor, with exposure to counterparty credit risk.

Future contract

Standardized contract, with clearing house. Counterparties deposit an initial margin, with position marked to market every day and investors shall keep balance above the maintenance margin.
Spot-future parity (no-arbitrage argument for pricing):

$$F_{t,T} = (S_t + s)e^{r(T-t)}$$

extension to account for discrete dividents δ_i paid at times t_i:

$$F_{t,T} = (S_t + s)e^{r(T-t)} - \sum_{i=1}^{n} \delta_i e^{r(T-t_i)}$$

or with continuous dividend δ:

$$F_{t,T} = (S_t + s)e^{(r-\delta)(T-t)}$$

Example: interest rate parity. Currency exchange rate E_t at spot and forward rate $F_{t,T}$ at T. Domestic interest rate r_D and foreign interest rate r_F. No arbitrage argument says that if one country's interest rate is higher than the other, it is balanced by the exchange rate. Then forward rate is

$$F_{t,T} = E_t e^{(r_D - r_F)(T-t)}.$$

Example: implied interest rate. Given future $F_{t,T}$ and spot S_t are known, we can invert for the interest rate

$$r = \frac{1}{T-t} \ln\left(\frac{F_{t,T}}{S_t + s}\right).$$

Options

Options are traded for hedging, for speculation, and by arbitrageurs.
Put-call parity: Hedge a stock by long a put option and short a call option with prices S, p, c respectively, both options with strike price K and expiry t. Stock and put option are financed by borrowed money Ke^{-rt} at rate r and the sale of call option:

$$S + p - c = Ke^{-rt}$$

Put-call parity can be solved to create synthetic position:

- put position = borrowing money, long call and short stock
  $p = Ke^{-rt} + c - S$
- short on stock = lending money, long put and short call
  $-S = p - c - Ke^{-rt}$
- long on stock = borrow money, short put and long call
  $S = c - p + Ke^{-rt}$

Combining options:

- Naked call: selling call without owning the asset, payoff to the writer is $c - \max(S-K, 0)$
- Naked put: similarly, payoff to the writer is $p - \max(K-S, 0)$
- Straddle: buy both put and a call with same K and expiry, payoff function: $\max(K-S,0) + \max(S-K,0) - c - p$, will lose money (part of premium) if asset price close to K at expiry
- Naked straddle: sell both put and call with same K and expiry, payoff is reverse of straddle
- Calendar spread: buy call at strike K with expiry t_1 and sell call at K with expirt t_2, use the proceed from sales to finance purchase
- Bull spread: Long a call with low strike K_1 and short a call with higher strike K_2 at the same expiry, use the proceed from sale to finance purchase and produce a profit if stock price rises.

Nick Leeson & Barings Bank: Naked straddle on Nikkei 225 at end of 1994.

Option Valuation

Wiener process for variable X_t: $X_t = \mu t + \sigma B_t$ with μ and σ^2 the drift rate and variance rate respectively, $B_t \sim N(0,t)$.
Ito's lemma: for $f(t, X_t)$, which $dX_t = \mu_t dt + \sigma_t dB_t$:

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2})dx^2 + O(dt^2) \\ &= \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t \frac{\partial f}{\partial x}dB_t. \end{aligned}$$

Asset price model: $dS/S = \mu dt + \sigma dB_t$ with $dB_t^2 \approx dt$,

$$\begin{aligned} dS &= \mu S dt + \sigma S dB_t \\ dS^2 &= (\mu S dt + \sigma S dB_t)^2 \\ &= \mu^2 S^2 dt^2 + 2\mu\sigma S^2 dB_t dt + \sigma^2 S^2 dB_t^2 \\ &\approx \sigma^2 S^2 dt \end{aligned}$$

Assume option price V be a function of S and t, using Ito's lemma,

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt + \sigma S \frac{\partial V}{\partial S} dB_t$$

Construct a portfolio Π with option V and asset S of quantity $-\Delta$,

$$\begin{aligned} \Pi &= V - \Delta S \\ d\Pi &= dV - \Delta dS = dV - \Delta(\mu S dt + \sigma S dB_t) \\ &= \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \Delta \mu S\right)dt + \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dB_t \end{aligned}$$

We set

$$\Delta = \frac{\partial V}{\partial S}$$

("delta", change in value of option for a change in value of the underlying asset) and $d\Pi = \Pi r dt$ to make the portfolio deterministic, and in a no-arbitrage argument with risk-free rate r, then we have the Black-Scholes model in p.d.e.

$$\begin{aligned} d\Pi = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \Delta \mu S\right)dt &= \left(V - \frac{\partial V}{\partial S}S\right) r dt \\ \therefore \quad \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \frac{\partial V}{\partial S}\mu S &= Vr - \frac{\partial V}{\partial S} S r \\ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= 0 \end{aligned}$$

pic at p.100

The model assumes:

- European option
- investor can borrow or lend at rate r
- asset volatility remains constant
- no dividends paid, no transaction costs, short sales allowed

If dividends r_D are paid in the asset, replace rS part with $(r - r_D)S$.

Solving Black-Scholes equation

European call/put at expiry: $V = \max(S-K, 0)$, or value of put $V = \max(K-S, 0)$. Then the value of a call/put is

$$\begin{aligned}
c(S,t) &= SN(d_1) - Ke^{-rt}N(d_2) \\
p(S,t) &= Ke^{-rt}N(-d_2) - SN(-d_1) \\
\text{where} \qquad N(d) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-s^2/2} ds \\
d_1 &= \frac{\log(S/K) + (r+\sigma^2/2)t}{\sigma\sqrt{t}} \\
d_2 &= \frac{\log(S/K) + (r-\sigma^2/2)t}{\sigma\sqrt{t}}
\end{aligned}$$

Solving σ^2 from option price = Implied volatility.

Proof: With $\frac{\partial d_1}{\partial S}$ and $\frac{\partial d_2}{\partial S}$, fit $\frac{\partial c}{\partial t}$, $\frac{\partial c}{\partial S}$, and $\frac{\partial^2 c}{\partial S^2}$ into the Black-Scholes p.d.e. with boundary condition at $\lim_{S\to 0} c(S,t) = 0$ and $\lim_{S\to\infty} c(S,t) = S - Ke^{-rt}$. Then derive $p(S,t)$ from $c(S,t)$ from put-call parity. □

Delta of call and put are respectively, $\frac{\partial c}{\partial S} = N(d_1)$ and $\frac{\partial p}{\partial S} = N(d_1) - 1$

Black-Scholes model is good at pricing options at the money but less so for out of the money and deep in the money options.

Risk-neutrality approach (?): Assume asset price modeled with drift at interest rate r, $dS_t = rS_t dt + \sigma S_t dB_t$. Then the density function of future values of S_t

$$\Pr[S_t = s] = \frac{1}{\sigma s\sqrt{2\pi t}} \exp\left(-\frac{\left(\log(\frac{s}{S_0}) - (r - \frac{1}{2}\sigma^2)t\right)^2}{2\sigma^2 t}\right)$$

With payoff function $V(S_T, T)$ at time T, the expected payoff is

$$E[V(S_T,T)] = \frac{1}{\sigma\sqrt{2\pi T}}\int_0^\infty \frac{V(s)}{s} \exp\left(-\frac{\left(\log(\frac{s}{S_0}) - (r - \frac{1}{2}\sigma^2)T\right)^2}{2\sigma^2 T}\right) ds$$

discounting to present value:

$$E[V(S_0,0)] = \frac{e^{-rT}}{\sigma\sqrt{2\pi T}}\int_0^\infty \frac{V(s)}{s} \exp\left(-\frac{\left(\log(\frac{s}{S_0}) - (r - \frac{1}{2}\sigma^2)T\right)^2}{2\sigma^2 T}\right) ds$$

Which satisfies the p.d.e.

Binomial tree method

For handling path-dependent options and situations that Black-Scholes equation cannot handle, e.g., interest rate options. Consider asset price S at discrete time period Δt, the probability of upward ($S \to Su$) and downward ($S \to Sd$) move are π_u and π_d respectively, $\pi_u + \pi_d = 1$. To make the model fit $dS = \mu S dt + \sigma S dB_t$,

$$u = e^{\sigma\sqrt{\Delta t}} \qquad d = \frac{1}{u} \qquad \pi_u = \frac{e^{r\Delta t} - d}{u - d} \qquad \pi_d = 1 = \pi_u = \frac{u - e^{r\Delta t}}{u - d}$$

Algorithm:

1. set up a tree with S_0 at root, each time step branch into upward and downward price until expiry
2. at leaf nodes, derive option value based on the price
3. trace backward in time the expected option value, using the upward and downward probabilities
4. expected option value at root = option price

Trinomial method is similar, with possibility of sideway move at probability π_m:

$$\begin{aligned}
\mathrm{E}[dS/S] &= \pi_u u + \pi_m(0) + \pi_d d = \mu\Delta t \\
\mathrm{Var}[dS/S] &= \pi_u u^2 + \pi_m(0) + \pi_d d^2 = \mu\Delta t^2 + \sigma^2\Delta z^2 \approx \sigma^2\Delta t \\
1 &= \pi_u + \pi_m + \pi_d
\end{aligned}$$

which gives the following:

$$\begin{aligned}
u &= e^{\sigma\sqrt{2\Delta t}} & d &= \frac{1}{u} = e^{-\sigma\sqrt{2\Delta t}} \\
\pi_u &= \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2 & \pi_d &= \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2 \\
\pi_m &= 1 - \pi_u - \pi_d
\end{aligned}$$

Finite difference method

Solve Black-Scholes p.d.e. by discretization: option value $V(S,t)$, solved on grid points of (S,t) for $S \in [S_{\min}, S_{\max}]$ and $t \in [0,T]$ with increments ΔS and Δt respectively. We substitute with finite difference form of partial derivatives:

$$\begin{aligned}
\frac{\partial V}{\partial t} &\approx \frac{V(S, t+\Delta t) - V(S,t)}{\Delta t} \\
\frac{\partial V}{\partial S} &\approx \frac{V(S+\Delta S, t) - V(S-\Delta S, t)}{2\Delta S} \\
\frac{\partial^2 V}{\partial S^2} &\approx \frac{V(S+\Delta S, t) - 2V(S,t) + V(S-\Delta S, t)}{\Delta S^2}
\end{aligned}$$

into the p.d.e. and get

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \\
\frac{V(S,t+\Delta t) - V(S,t)}{\Delta t} + \frac{1}{2}\sigma^2 S^2 \frac{V(S+\Delta S,t) - 2V(S,t) + V(S-\Delta S,t)}{\Delta S^2} \\
+ rS\frac{V(S+\Delta S,t) - V(S-\Delta S,t)}{2\Delta S} - rV(S,t) = 0
\end{aligned}$$

Solve for $V(S, t+\Delta t)$ gives a system of linear equations of $V(\cdot, t)$:

$$\begin{aligned}
V(S, t+\Delta t) &= aV(S-\Delta S, t) + bV(S,t) + cV(S+\Delta S, t) \\
\text{where} \qquad a &= -\frac{1}{2}\sigma^2\Delta t\frac{S^2}{\Delta S^2} + \frac{r\Delta tS}{2\Delta S} \\
b &= 1 + \frac{\sigma^2\Delta t S^2}{\Delta S^2} + r\Delta t \\
c &= -\frac{1}{2}\sigma^2\Delta t\frac{S^2}{\Delta S^2} - \frac{r\Delta tS}{2\Delta S}
\end{aligned}$$

Boundary conditions:

- for a call option, $S << K$, $V(S,t) = 0$ and $V(S,t) = S$ as $S \to \infty$
- for a put option, $S >> K$, $V(S,t) = 0$ and $V(S,t) = K$ as $S \to 0$
- at expiry, $V(S,t) = \max(S-K, 0)$ for a call and $V(S,t) = \max(K-S, 0)$ for a put

Solution by matrix: Compute $V(S,t)$ on a grid such that $S = S_{\min} + i\Delta S$ and $t = t_0 + j\Delta t$ with $i = 0, \cdots, M$ and $j = 0, \cdots, N$. Then the above equation will become

$$a_i V_{i-1,j-1} + b_i V_{i,j-1} + cV_{i+1,j-1} = V_{i,j}$$

And $V_{\cdot,j-1}$ is related to $V_{\cdot,j}$ by a tridiagonal matrix (which can be LU-factorized). We can derive $V(S,t_0)$ backward from $V(S,T)$ one step at a time by matrix multiplication.

Solution by explicit derivation: The p.d.e. can be alternatively written as

$$\begin{aligned}
\frac{V_{i,j+1} - V_{i,j}}{\Delta t} + \frac{1}{2}\sigma^2 S^2 \frac{V_{i+1,j+1} - 2V_{i,j+1} + V_{i-1,j+1}}{\Delta S^2} \\
+ rS\frac{V_{i+1,j+1} - V_{i-1,j+1}}{2\Delta S} - rV_{i,j} = 0
\end{aligned}$$

by evaluating $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ at $t = t_0 + (j+1)\Delta t$. Solving for $V_{i,j}$ term

$$V_{i,j} = d_i V_{i-1,j+1} + e_i V_{i,j+1} + f_i V_{i-1,j+1}$$

where
$$d_i = \frac{1}{1+r\Delta t}\left(-\frac{1}{2}r\Delta t\frac{S_{\min}+i\Delta S}{\Delta S} + \frac{1}{2}\Delta t\sigma^2\left(\frac{S_{\min}+i\Delta S}{\Delta S}\right)^2\right)$$
$$e_i = \frac{1}{1+r\Delta t}\left(1-\Delta t\sigma^2\left(\frac{S_{\min}+i\Delta S}{\Delta S}\right)^2\right)$$
$$f_i = \frac{1}{1+r\Delta t}\left(\frac{1}{2}r\Delta t\frac{S_{\min}+i\Delta S}{\Delta S} + \frac{1}{2}\Delta t\sigma^2\left(\frac{S_{\min}+i\Delta S}{\Delta S}\right)^2\right)$$

Monte Carlo analysis

Algorithm:

1. Generate random walks (usually lognormal) from t to expiry with increment Δt
2. At the end of each path, evaluate the payoff at expiry
3. Compuate average of all such payoff, and discount to PV by e^{-rt}

Model stock price with lognormal distribution:

$$df = \sigma S\frac{\partial f}{\partial S}dB_t + \left(\mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right)dt$$

with $f = \log S$:

$$\begin{aligned}
d(\log S) &= \sigma S\frac{1}{S}dB_t + \left(\mu S\frac{1}{S} + \frac{1}{2}\sigma^2 S^2(-\frac{1}{S^2})\right)dt \\
&= \sigma dB_t + \left(\mu - \frac{1}{2}\sigma^2\right)dt \\
\log S_T - \log S_0 &= \int_{t=0}^{t=T}\sigma dB_t + \int_0^T(\mu - \frac{1}{2}\sigma^2)dt \\
S_T &= S_0\exp\left((\mu - \frac{1}{2}\sigma^2)T + \int_{t=0}^{t=T}\sigma dB_t\right) \\
\implies \qquad \Delta S &= \exp\left((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}z\right)
\end{aligned}$$

with $z \sim N(0,1)$. Random walk is generated from steps $DeltaS$ of random size.

Option greeks

$$\Delta = \frac{\partial V}{\partial S} \qquad \Gamma = \frac{\partial^2 V}{\partial S^2} \qquad \theta = \frac{\partial V}{\partial t} \qquad v = \frac{\partial V}{\partial \sigma} \qquad \rho = \frac{\partial V}{\partial r}$$

Delta Δ always have positive slope, with call $\Delta \geq 0$ and put $\Delta \leq 0$. Call's delta is 0 on extremely low stock price and 1 on extremely high price, for option is worthless if deep out of money and $\lim_{S\to\infty} V = S$ if deep into the money.
Gamma Γ is the rate of change of delta, always positive. Gamma for call and put are the same.
Theta θ is most sensitive when the option is close to the money, with sensitivity decreasing as the option approaches expiry.
Vega v is the option sensitivity to change in volatility. Vega for call and put are the same.
Rho ρ is the sensitivity of option price to interest rate.

4 Fixed income

Bond equivalent yield is computed on the basis of a 365-day year.
Probability of default estimation: A corporate bond with yield y has probability of default p, which assumed default will lost all value, if a treasury bond has yield y' and $(1-p)(1+y) = (1+y')$
Par yield: Coupon rate, the yield if priced at par.
Forward yield: Yield of a loan expected for future time.
Spot yield: Yield of the market today. Always between par and forward.
For an upward sloping yield curve (i.e., yield goes up for par yield and spot yield for longer time or forward yield further in the future), forward > par.
For a downward sloping yield curve, forward < par.
PCA finds that 90% of price change of bonds was due to changes in interest levels; 8.5% due to changing yield curve slope; 1.5% due to change of curvature of the yield curve.
Binomial tree method for valuation: Black-Derman-Toy model, Ho-Lee model, Heath-Jarrow-Morton model
Morgage-backed securities: Common to assume an annualized constant prepayment rate (CPR) and find the equivalent single month mortality rate (SMM):

$$SMM = 1 - (1 - CPR)^{1/12}$$

5 Equity markets

Systematic risks: All stocks are subject to
Unsystematic risk: Company-specific portion of the risk
Price of stock calculated using perpetuity model: $P = \frac{DIV}{r_{CE}}$ where DIV is the dividend, expected to remain stable over time, r_{CE} is the (unobservable) cost of equity capital (e.g. hurdle rate for new projects). If dividend is expected to grow at constant rate g, we have the Gordon growth model

$$P = \frac{DIV_0(1+g)}{r_{CE} - g}$$

Common multiples for analysis:

- price/earnig ratio
- dividend/price (aka dividend yield): high dividend yield if price is undervalued
- price/sales
- price/book value
- price/cashflow
- return on asset

Capital asset pricing model (CAPM): Compare return of stock R_s to return of market R_m with assumed risk-free return R_f

$$R_s - R_f = \beta(R_m - R_f)$$

where β is the measure of correlation between the security and the market. The return above R_f is the excess return.
Dupont analysis: Calculate the return on equity as

$$\begin{aligned}
ROE &= \frac{\text{sales}}{\text{assets}}\frac{\text{net income}}{\text{sales}}\frac{\text{assets}}{\text{equity}} \\
&= (\text{asset turnover})(\text{profit margin})(\text{leverage ratio})
\end{aligned}$$

Markowitz efficient frontier: Return vs Risk cluster plot will show a boundary of max return on risk. The boundary is concave, monotonically increasing on increasing risk.

Valuting convertible bonds

Convertible bond has a term sheet to specify a conversion price and conversion ratio. Most convertible bonds are callable, allowing the issuer to force conversion if so desired.

1. Determine the min value of the convertible bond (comparing the bond as a straight bond without conversion feature)
2. Calculate the investment premium
3. Calculate the premium payback period
4. Calulate value of the embedded convertible bond option

Black-Scholes equation can be used to value option on the underlying stock but not be used to value the embedded call. Bonds do not have constant volatilities as BS assumed, but descreasing volatilities as bonds approach maturity, for holder receives a known par at maturity. Also bonds would never be worth more than its par value plus any accrued coupons, thus bonds does not fit a lognormal distribution which price of zero to infinity are possible.
Example:

1. 8-yr 7% coupon convertible bond, compare to 8% yield in market, its par should be $\sum_{k=1}^{16}\frac{7}{2}(1.04)^{-k} + 100(1.04)^{-16} = 94.17$. The

\$100 bond are convertible to 5.263 shares of stock when it hit \$19, which the conversion ratio is $k = 100/19 = 5.263$. If the current stock price is \$15.5, the conversion value of the bond is $15.5k = 81.576$. Taking the max of the two, the straight bond should worth \$94.17.

2. If the market price for this convertible bond is \$120, the premium is $\frac{120}{94.17} - 1 = 27\%$ over the market price of the bond. Alternatively, the expected price for stock conversion is $\frac{120}{5.263} = \$22.8$ per share, which is $\frac{22.8-15.5}{15.5} = 47\%$ premium over the market price of the shares.
3. A fair value is which the coupon/dividend interest should offset the premium we pay to hold the bond. Here the shares premium over the coupon is $\frac{120-5.263(15.5)}{7} = 5.48$ years. Normally we expect this to be 3–5 years.
4. Value of the embedded call: price of convertible bond = price of straight bond + call option. Say, $T = 8$, $S = 15.5$, $K = 19$, $r_f = 5\%$, $\sigma = 35\%$, using Black-Scholes, the call option per share worth 6.835. Thus the price of the convertible bond should be $94.17 + 6.835(5.263) = 130.1$. If the issuer can also call the bond at any time after 3 years, we should subtract the value of issuer's call option.

6 Risk management

Value at Risk (VaR): the economic loss that can be expected at a given confidence level under adverse market condtions, calculated using simulation or distribution

To measure VaR, first evaluate all possible risk exposures and choose a confidence level (e.g. 95% or 99%), and a time horizon. Then the "worst case" value is

$$VaR = V_0 \alpha \sigma \sqrt{t}$$

where α is the inv of std normal distribution at the selected confidence interval, σ is the asset's std dev, t is the time scale factor between σ and VaR (equity takes $t = 250$ to convert daily volitility to annual) and V_0 to the spot price.

A Formulae

Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(k)}{n!}(x-k)^n = f(k) + f'(k)(x-k) + \frac{f''(k)}{2!}(x-k)^2 + \cdots$$

Multivariate Taylor series:

$$\begin{aligned} df(x,y) &= f(x+dx, y+dy) - f(x,y) \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2\right) + \cdots \end{aligned}$$

Variance scaling: $Var(\alpha X) = \alpha^2 Var(X)$

CLT: $Z = \dfrac{\bar{X} - \mu_X}{\sigma_X/\sqrt{n}}$ is normally distributed regardless the distro of X

Lognormal PDF: $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2 x^2}} \exp\left(-\dfrac{(\ln x - \mu)^2}{2\sigma^2}\right)$