HANDBOOK OF SET-THEORETIC TOPOLOGY

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Foreword

This Handbook is designed as an introduction to recent work in set-theoretic topology. It is intended both for students who plan to enter the field and for researchers in other areas for whom results in set-theoretic topology may be relevant.

As to prerequisites, our general guideline has been to make this Handbook as self-contained as possible without repeating material that can easily be found in standard texts. In fact, the articles are quite varied in their degree of difficulty. Many of them can be read with only a basic knowledge of set theory and general topology; the reader with only this minimum knowledge might try starting at the articles of Hodel and Weiss. However, other articles require some more advanced knowledge of special areas. For example, the articles by Baumgartner, Juhász, and Kunen require a knowledge of forcing, the article by Negrepontis requires a knowledge of Banach spaces, and the article by Gardner and Pfeffer requires a knowledge of measure theory; detailed references for these subjects are found in the articles themselves.

In general, the articles may be read in any order, except that in a few cases, they occur in pairs, with the first one giving an elementary treatment of a subject and second one giving more advanced results. These pairs are: Hodel and Juhász on cardinal functions, Roitman and Abraham–Todorčević on S- and L- spaces, Weiss and Baumgartner on versions of Martin's axiom, and Vaughan and Stephenson on compactness properties.

It is intended that the Handbook be usable either as a text or a reference; to this end, the articles contain many detailed proofs of core results, as well as references to the literature for peripheral results for which space seemed insufficient to include all details. The articles also state many open problems of current interest, and it is our hope that the Handbook will provide a stimulus for future research.

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CHAPTER 1

Cardinal Functions I

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J. E. Vaughan

Introduction

Let X be a Hausdorff space with a countable base. With such a strong hypothesis, one can easily obtain a great deal of quantitative information about X. For example, 2^ω is a bound on both the number of open sets in X and the cardinality of X. Indeed, every open set is the union of elements of a base, and $p \to X - \{p\}$ is a one-one function from X into the open sets in X.

Now weaken the hypothesis and assume that X is a first countable, separable, Hausdorff space. In this case, 2^ω is still a bound on the cardinality of X. To see this, let S be a countable dense subset of X, and for each $p \in X$ let S_p be an infinite sequence from S which converges to p. Since X is Hausdorff, $p \neq q$ implies $S_p \neq S_q$. The number of such sequences is at most $\omega^\omega = 2^\omega$, so $|X| \leq 2^\omega$. (This argument shows, in fact, that a first countable Hausdorff space with a dense subset of cardinality $\leq 2^\omega$ has cardinality $\leq 2^\omega$.) Since $|X| \leq 2^\omega$ and X is first countable, X has a base of cardinality $\leq 2^\omega$. So the number of open sets in X is at most 2^{2^ω}. Moreover, 2^{2^ω} is the best bound on the number of open sets in a first countable, separable, Hausdorff space. (*Example.* $S \times S$, where S is the Sorgenfrey line.)

Now weaken the hypothesis still further and assume that X is a separable, Hausdorff space. In this case, the best bound on the cardinality of X is 2^{2^ω} and the best bound on the number of open sets in X is $2^{2^{2^\omega}}$! On the other hand, 2^ω is a bound on the number of continuous, real-valued functions on X. Indeed, let S be a countable dense subset of X. The number of functions from S into $\mathbb{R}$ is at most $(2^\omega)^\omega = 2^\omega$. If f and g are continuous functions from X into $\mathbb{R}$, and f and g agree on S, then f and g agree on X. (The fact that X is Hausdorff is not used in this argument.)

What are cardinal functions and why are they useful? Roughly speaking, cardinal functions extend such important topological properties as countable base, separable, and first countable to higher cardinality. Cardinal functions then allow one to formulate, generalize, and prove results of the type just discussed in a systematic and elegant manner. In addition, cardinal functions allow one to make precise quantitative comparisons between certain topological properties. For example, it is well known that a space with a countable base has a countable dense set. A 'converse' of this result from the theory of cardinal functions states that a regular space with a countable dense set has a base of cardinality $\leq 2^\omega$. In summary, experience indicates that the idea of a cardinal function is one of the most useful and important unifying concepts in all of set-theoretic topology.

It is perhaps accurate to say that a *systematic* study of cardinal functions did not begin until the mid-1960's. But of course many fundamental techniques and isolated results were obtained long before then. Prerequisite for work in cardinal functions is a knowledge of cardinal and ordinal numbers and transfinite constructions. These ideas were developed long ago by such notable researchers as Cantor, Alexandroff, Urysohn, Kuratowski, Sierpiński, Hausdorff, and others.

Let me quickly review some highlights in the theory of cardinal functions from

1920 to 1970. During the 1920's, Alexandroff and Urysohn developed the basic theory of compact spaces. One result obtained during that time states that a compact, perfectly normal space has cardinality at most 2^ω. Alexandroff and Urysohn quite naturally asked if every compact, first countable space has cardinality at most 2^ω.

In 1937, Čech and Pospíšil proved a now classical result on the cardinality of compact spaces. One consequence of their theorem is that every compact, first countable space is countable or has cardinality $\geq 2^\omega$.

By 1937, F.B. Jones had published his results on the cardinality of closed discrete subsets of normal, separable spaces. It was at this time that the famous normal Moore space problem was introduced.

In the mid-1940's, Hewitt, Marczewski, and Pondiczery independently proved a remarkable theorem on the density of product spaces. The countable version of their result states that the product of at most 2^ω separable spaces is separable!

In 1965, de Groot published a paper in which he introduced several new and important cardinal functions. The countable version of one of his results states that a Hausdorff space in which every subspace is Lindelöf has cardinality at most 2^ω. (This generalizes the result that compact, perfectly normal spaces have cardinality at most 2^ω.) He also raised the following question. Is the number of open sets in a Hausdorff space equal to 2^κ for some cardinal κ?

In the late 1960's, Hajnal and Juhász published two important papers which considerably extended de Groot's work. Among other results, they obtained three inequalities which are now regarded as fundamental to the theory of cardinal functions. The countable versions state that (1) every Hausdorff, first countable, ccc space has cardinality at most 2^ω; (2) every T_1 space with countable pseudo-character (= every point is a G_δ) and countable spread (= every discrete subspace is countable) has cardinality at most 2^ω; (3) every Hausdorff space with countable spread has cardinality at most 2^{2^ω}. Assuming GCH + ($\nexists$ inaccessible cardinal), Hajnal and Juhász also gave an affirmative answer to de Groot's problem on the number of open sets in a Hausdorff space.

In 1969, Arhangel'skiĭ solved the old problem of Alexandroff and Urysohn. The countable version of his inequality, also fundamental to the theory of cardinal functions, states that every Hausdorff, Lindelöf, first countable space has cardinality at most 2^ω.

How does one account for the great activity and progress in cardinal functions which was well underway by 1970? Certainly influential was the work of de Groot, Hajnal–Juhász, and Arhangel'skiĭ just discussed. But there are at least two other reasons which deserve mention. Firstly, certain results of combinatorial set theory, essential to the study of cardinal functions, were well developed and understood by the mid-1960's. The results I have in mind are all provable in ZFC and are due to Erdös, Hajnal, Rado, Hausdorff, Tarski, and others; see Section 2. It should be noted that the results of Hajnal–Juhász discussed above depended heavily on such combinatorial principles. Secondly, by the mid-1960's there was a great deal of activity in set theory as a result of Cohen's solution of the continuum problem and the introduction of the method of forcing. A systematic study of cardinal functions suggests many natural problems whose solution seems to require combinatorial principles of set theory which cannot be proved in ZFC. Fortunately, the method of forcing can often be used to establish the consistency of these combinatorial results.

This paper is a brief introduction to cardinal functions. The major themes are the following.

(1) The most important cardinal functions are carefully defined, and the relationships between these functions are stated, usually with proofs.

(2) Cardinal functions are used to obtain bounds on the cardinality of a space X. Emphasis will also be given to finding bounds on the number of open sets in X, the number of regular open sets in X, the number of compact subsets of X, and the number of continuous, real-valued functions on X. Also of interest is a bound on the smallest cardinal which arises as the cardinality of a base for X.

(3) Cardinal functions on the two most important classes of abstract topological spaces, namely compact spaces and metrizable spaces, are treated in detail.

A fuller account of cardinal functions is given in the two treatises by JUHÁSZ [1971, 1980]. Also see the recent book by COMFORT and NEGREPONTIS [1982]. The first text in general topology to use cardinal functions as an important unifying device is Engelking's *General Topology*.

I would like to thank W.W. Comfort, E.K. van Douwen and J.E. Vaughan for many helpful comments.

1. Notation and definitions

The following set-theoretic notation is adopted: κ, λ and τ are cardinal numbers; α, β, γ, and δ are ordinal numbers; i and n are non-negative integers; ω is the smallest infinite ordinal and cardinal; ω_1 is the smallest uncountable ordinal and cardinal; κ^+ is the smallest cardinal after κ. A cardinal number is the set of all ordinals which precede it. Thus $\alpha < \kappa$ and $\alpha \in \kappa$ are the same. To avoid typesetting problems, 2^κ is often written $\exp \kappa$.

A cardinal κ is a *successor cardinal* if $\kappa = \lambda^+$ for some cardinal λ. For example, ω_1 is a successor cardinal, since $\omega_1 = \omega^+$. A cardinal which is not a successor cardinal is a *limit cardinal*. Thus, κ is a limit cardinal if $\lambda < \kappa$ implies $\lambda^+ < \kappa$. A cardinal κ such that $\lambda < \kappa$ implies $2^\lambda < \kappa$ is called a *strong limit cardinal*. For example, ω is a strong limit cardinal. GCH implies that every limit cardinal is a strong limit cardinal.

The *cofinality* of κ, denoted $\mathrm{cf}(\kappa)$, is the smallest cardinal λ such that κ has a cofinal subset of cardinality λ. For example, $\mathrm{cf}(\omega) = \omega$ and $\mathrm{cf}(\kappa^+) = \kappa^+$ for $\kappa \geqslant \omega$. A cardinal κ is *regular* if $\kappa \geqslant \omega$ and $\mathrm{cf}(\kappa) = \kappa$. Thus, ω and all infinite successor cardinals are regular. A regular cardinal κ has the important property that if

$A \subseteq \kappa$ and $|A| < \kappa$, then $\sup A < \kappa$. For $\kappa \geqslant \omega$, $\mathrm{cf}(\kappa)$ is always a regular cardinal and $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.

An infinite cardinal which is not regular is said to be *singular*. Note that a singular cardinal is always a limit cardinal. A singular strong limit cardinal of cofinality ω is obtained as follows: let $\lambda_0, \lambda_1, \ldots$ be a sequence of infinite cardinals with $\lambda_{n+1} = 2^{\lambda_n}$ for all $n < \omega$, and let $\kappa = \sup\{\lambda_n : n < \omega\}$.

A regular limit cardinal $>\omega$ is said to be *weakly inaccessible*, and a regular strong limit cardinal $>\omega$ is *strongly inaccessible*. Assuming GCH, the two are the same. If ZFC is consistent, then so is ZFC + GCH + ($\nexists$ inaccessible cardinal).

Let E be a set. Then $|E|$ is the cardinality of E, $P(E)$ is the power set of E, $P_\kappa(E)$ is the collection of all subsets of E of cardinality $\leqslant\kappa$, and $[E]^n = \{A : A \subseteq E, |A| = n\}$. Recall that $|P_\kappa(E)| \leqslant |E|^\kappa$ for $|E| > 1$. For $p \in E$ and $\mathscr{A}$ a cover of E, $\mathrm{ord}(p, \mathscr{A})$ is the cardinality of the collection $\{A : A \in \mathscr{A}, p \in A\}$, and $\mathrm{st}(p, \mathscr{A}) = \bigcup\{A : A \in \mathscr{A}, p \in A\}$.

If A and B are sets, then ${}^AB$ denotes the collection of all functions from A into B. Recall that $|{}^AB| = |B|^{|A|}$. If κ is a cardinal, α and β are ordinals with $\beta < \alpha \leqslant \kappa$, and $f \in {}^\alpha\kappa$, then $(f \mid \beta) \in {}^\beta\kappa$ and is the restriction of f to β.

The following topological notation and conventions are used: X always denotes a non-empty topological space; compact spaces are Hausdorff; the class of T_0, T_1, Hausdorff, and T_3 ($= T_1$ + regular) spaces are denoted $\mathscr{T}_0$, $\mathscr{T}_1$, $\mathscr{T}_2$, and $\mathscr{T}_3$ respectively.

An open subset R of X is *regular open* if $(\bar{R})^\circ = R$. If A is any subset of X, then $(\bar{A})^\circ$ is a regular open set. The collection of all regular open sets is denoted $\mathrm{RO}(X)$. It is well known that $\mathrm{RO}(X)$ is a complete Boolean algebra under suitable definitions of sup and inf. Pierce has shown that $|B|^\omega = |B|$ for every infinite complete Boolean algebra B. COMFORT and HAGER [1972] have shown that this result holds if B is just countably complete. So if $\mathrm{RO}(X)$ is infinite, and this is the case whenever X is an infinite Hausdorff space, then $|\mathrm{RO}(X)|^\omega = |\mathrm{RO}(X)|$.

Let p be a point and A an infinite subset of X. Then p is a *complete accumulation point* of A if $|V \cap A| = |A|$ for every open neighborhood V of p. Every infinite subset of a compact space has a complete accumulation point.

A subset D of X is *discrete* if D is a discrete space when given the subspace topology. Thus, D is discrete if and only if every point $p \in D$ has an open neighborhood V_p such that $D \cap V_p = \{p\}$. If X has a discrete subset of cardinality κ, then X has at least 2^κ open sets.

Three standard examples in topology are frequently mentioned in this paper. One is $\beta\omega$, the Stone–Čech compactification of ω with the discrete topology. Important facts to remember are (1) $\beta\omega$ is the set of all ultrafilters on ω; (2) ω is dense in $\beta\omega$ and each point of ω is isolated; (3) every infinite closed subset of $\beta\omega$ has cardinality 2^{2^ω}. For a further discussion see van Mill's paper in this Handbook.

Another useful example is the space obtained by taking the product of $\kappa \geqslant \omega$ copies of $\{0, 1\}$. (Here $\{0, 1\}$ has the discrete topology while the product has the product topology.) This space is called the *Cantor cube of weight* κ and is denoted

$D(2)^\kappa$. The Cantor cube of weight κ has cardinality 2^κ, has a base of cardinality κ, and has no base of cardinality $<\kappa$.

Still another useful example is the Sorgenfrey line S. Recall that S is the real line with the collection $\{[a, b): a < b\}$ of half-open intervals as a base for the topology. The space S has the remarkable property that every subspace is both separable and Lindelöf, yet S does not have a countable base. A useful fact is that $D = \{(x, -x): x \in S\}$ is a closed discrete subset of $S \times S$ of cardinality 2^ω.

For other topological results, the reader is referred to Engelking's text.

2. Combinatorial principles

Theorems from combinatorial set theory play an important role in cardinal functions. Those results which are used in this paper are collected together in this section. Useful references are COMFORT–NEGREPONTIS [1974, 1982], KUNEN [1977], JUHÁSZ [1971], RUDIN [1975], and WILLIAMS [1977]. The reader may want to skip this section for now and refer to it later when appropriate. All results in this section can be proved in ZFC.

2.1. THEOREM (Tarski). *Let E be an infinite set. Then there is a collection $\mathscr{A}$ of subsets of E such that $|\mathscr{A}| = |E|^\omega$, $|A| = \omega$ for each $A \in \mathscr{A}$, and the intersection of any two distinct elements of $\mathscr{A}$ is finite.*

PROOF. This version of Tarski's theorem on almost disjoint collections is not difficult to prove, and a proof for the case $|E| = \omega$ is especially easy. Indeed, suppose E is the set $\mathbb{Q}$ of rational numbers, and for each irrational number x let Q_x be a sequence of rational numbers which converges to x. The number of such sequences is 2^ω, and $Q_x \cap Q_y$ is finite whenever x and y are distinct irrationals.

To prove the general case, it suffices to construct the desired collection on the set of all finite subsets of $\omega \times E$. For each $f \in {}^\omega E$, let $A(f) = \{(f \mid n): n < \omega\}$. One can easily check that $A(f) \cap A(g)$ is finite whenever f and g are distinct elements of ${}^\omega E$, so $\{A(f): f \in {}^\omega E\}$ is the desired collection of sets.

A collection $\mathscr{A}$ of subsets of a set E is said to be *independent* if for any finite collection $A_1, \ldots, A_i, B_1, \ldots, B_n$ of distinct elements of $\mathscr{A}$,

$$A_1 \cap \cdots \cap A_i \cap (E - B_1) \cap \cdots \cap (E - B_n) \neq \emptyset .$$

2.2. THEOREM (Hausdorff). *Let κ be an infinite cardinal, let E be a set with $|E| = \kappa$. Then there is an independent collection $\mathscr{A}$ of subsets of E with $|\mathscr{A}| = 2^\kappa$.*

Theorem 2.2 was first proved by Fichtenholz and Kantorovich for $\kappa = \omega$ and $\kappa = 2^\omega$ and later by Hausdorff for all κ. See van Mill's article in this Handbook

(Lemma 3.12) for a nice proof of the case $\kappa=\omega$ using Tarski's theorem. The general case of Hausdorff's theorem is proved in Section 11 using the Hewitt–Marczewski–Pondiczery theorem.

Independent collections can be used to prove the existence of 2^{2^κ} ultrafilters on $\kappa\geqslant\omega$. Indeed, suppose $\{A_\alpha: 0\leqslant\alpha<2^\kappa\}$ is an independent collection on κ, for each $f\in{}^{2^\kappa}2$, let $A_f=\{A(f,\alpha): 0\leqslant\alpha<2^\kappa\}$, where $A(f,\alpha)=A_\alpha$ if $f(\alpha)=0$ and $A(f,\alpha)=\kappa-A_\alpha$ if $f(\alpha)=1$. Then A_f has the finite intersection property, hence is contained in an ultrafilter p_f. Clearly $f\neq g$ implies $p_f\neq p_g$.

The next theorem is often paraphrased as follows. Suppose one has a set E much larger than κ, and that all of the unordered pairs of E are placed in one (or more) of κ pots. Then there is a subset A of E larger than κ such that all of the unordered pairs of A are in one of the pots.

2.3. THEOREM (Erdös, Rado). *Let κ be an infinite cardinal, let E be a set with $|E|>2^\kappa$, and suppose $[E]^2=\bigcup_{\alpha<\kappa}P_\alpha$. Then there exists $\alpha<\kappa$ and a subset A of E with $|A|>\kappa$ such that $[A]^2\subseteq P_\alpha$.*

PROOF. Let p be some point in E. For each $\alpha<\kappa^+$, construct a collection $\{R_f: f\in{}^\alpha\kappa\}$ of subsets of E and a collection $\{x_f: f\in{}^\alpha\kappa\}$ of points in E such that for each $f\in{}^\alpha\kappa$:

(1) if α is a limit ordinal, $R_f=\bigcap_{\beta<\alpha}R_{(f\mid\beta)}$;

(2) if $\alpha=\gamma+1$, $R_f=\{x: x\in R_{(f\mid\gamma)}, x\neq x_{(f\mid\gamma)}, \{x, x_{(f\mid\gamma)}\}\in P_{f(\gamma)}\}$;

(3) $x_f\in R_f$ if $R_f\neq\emptyset$ and $x_f=p$ otherwise;

(4) if $\beta<\alpha$, then $x_{(f\mid\beta)}\notin R_f$;

(5) if $\beta<\alpha$ and $x\in R_f$, then $\{x, x_{(f\mid\beta)}\}\in P_{f(\beta)}$.

The construction is by transfinite induction. For $\alpha=0$, let $R_\emptyset=E$, $x_\emptyset=p$. Now let $0<\alpha<\kappa^+$, and assume the collections constructed for each $\beta<\alpha$ so that (1)–(5) hold. Let $f\in{}^\alpha\kappa$. Then R_f is defined by (1) or (2), and x_f by (3). Moreover, (4) and (5) hold by the induction hypothesis.

Let $H=\{x_f: f\in{}^\alpha\kappa, \alpha<\kappa^+\}$; $|H|\leqslant 2^\kappa$ and $|E|>2^\kappa$, so there exists $y\in E$ such that $y\neq x_f$ for all f. Now construct a sequence $\{f_\alpha: 0\leqslant\alpha<\kappa^+\}$ such that for all $\alpha<\kappa^+$:

(6) $f_\alpha\in{}^\alpha\kappa$ and $(f_\alpha\mid\beta)=f_\beta$ for $\beta<\alpha$;

(7) $y\in R_{f_\alpha}$.

The construction is by transfinite induction. Let $0<\alpha<\kappa^+$, and assume that f_β has been constructed for each $\beta<\alpha$ so that (6), (7) hold. If α is a limit ordinal, define f_α by $f_\alpha(\beta)=f_{\beta+1}(\beta)$ for all $\beta<\alpha$; (7) follows from (1) and the induction hypothesis. Suppose $\alpha=\gamma+1$. Now $y\neq x_{f_\gamma}$, hence there exists $\delta<\kappa$ such that $\{y, x_{f_\gamma}\}\in P_\delta$. Define f_α by $(f_\alpha\mid\gamma)=f_\gamma$ and $f_\alpha(\gamma)=\delta$; (7) follows from (2) and the induction hypothesis.

Let $g:\kappa^+\to\kappa$ be such that $(g\mid\alpha)=f_\alpha$ for all α. Choose $\delta<\kappa$ and a subset T of κ^+ with $|T|=\kappa^+$ such that $g(\alpha)=\delta$ for all $\alpha\in T$. Let $\alpha,\beta\in T$ with $\beta<\alpha$. Now

$R_{(g|\alpha)} \neq 0$ ($y \in R_{f_\alpha}$); so by (3) and (4) one has $x_{(g|\beta)} \neq x_{(g|\alpha)}$, and by (3) and (5) one has $\{x_{(g|\beta)}, x_{(g|\alpha)}\} \in P_\delta$. Hence $\{x_{(g|\alpha)}: \alpha \in T\}$ is the desired subset of E.

2.4. Δ-SYSTEM LEMMA (Šanin). *Let κ be a regular cardinal with $\kappa > \omega$, let $\mathscr{A}$ be a collection of finite sets with $|\mathscr{A}| = \kappa$. Then there is a set F (possibly empty) and a subcollection $\mathscr{A}'$, of $\mathscr{A}$ with $|\mathscr{A}'| = \kappa$ such that the intersection of any two distinct elements of $\mathscr{A}'$ is F.*

PROOF. Since $\kappa > \omega$, one may assume that there is an integer $n \geqslant 0$ such that each element of $\mathscr{A}$ has exactly n elements. The proof is by induction on n. If $n = 0$, take $F = \emptyset$. Now assume $n > 0$ and that the conclusion holds for $(n-1)$. Let $\mathscr{A}'$ be a maximal pairwise disjoint subcollection of $\mathscr{A}$. If $|\mathscr{A}'| = \kappa$, take $F = \emptyset$ and the proof is complete. Suppose $|\mathscr{A}'| < \kappa$. By the maximality of $\mathscr{A}'$ and the regularity of κ, there is some $p \in \bigcup \mathscr{A}'$ such that p belongs to κ elements of $\mathscr{A}$. To complete the proof, use the induction hypothesis and the collection $\{A - \{p\}: A \in \mathscr{A}, p \in A\}$.

The next combinatorial principle was suggested by a theorem in topology. In 1962, Miščenko proved that a compact space with a point-countable base (i.e., a base $\mathscr{B}$ such that every point is in just countably many members of $\mathscr{B}$) has a countable base. In 1968, Filippov generalized this result by proving that a Hausdorff space is metrizable if and only if it is a paracompact p-space with a point-countable base. In proving this theorem, Filippov used a combinatorial result, now called Miščenko's lemma, which he abstracted from Miščenko's proof. The reader will find it instructive to write out a proof of Miščenko's theorem (every compact space with a point-countable base has a countable base) using Miščenko's lemma.

2.5. MIŠČENKO'S LEMMA. *Let κ be an infinite cardinal, let E be a set, let $\mathscr{A}$ be a collection of subsets of E such that* $\operatorname{ord}(p, \mathscr{A}) \leqslant \kappa$ *for all $p \in E$. Then the number of finite minimal covers of E by elements of $\mathscr{A}$ is at most κ.*

PROOF. Juhász has observed that Miščenko's lemma can be obtained from the Δ-system lemma. To see this, suppose Miščenko's lemma is false, and let $\{\mathscr{A}_\alpha: 0 \leqslant \alpha < \kappa^+\}$ be a collection of distinct finite minimal covers of E by elements of $\mathscr{A}$. By the Δ-system lemma, there is a subset T of κ^+ with $|T| = \kappa^+$, and a subset $\mathscr{F}$ of $\mathscr{A}$ such that $\mathscr{A}_\alpha \cap \mathscr{A}_\beta = \mathscr{F}$ for $\alpha, \beta \in T$, $\alpha \neq \beta$. Let $p \in E$, $p \notin \bigcup \mathscr{F}$. (To show $\bigcup \mathscr{F} \neq E$, choose $\alpha \in T$ such that $\mathscr{A}_\alpha \neq \mathscr{F}$ and use the fact that $\mathscr{A}_\alpha$ is a *minimal* cover of E.) For each $\alpha \in T$ choose $A_\alpha \in \mathscr{A}_\alpha$ such that $p \in A_\alpha$. Let $\alpha, \beta \in T$, $\alpha \neq \beta$. Since $p \notin \bigcup \mathscr{F}$ and $\mathscr{A}_\alpha \cap \mathscr{A}_\beta = \mathscr{F}$, it follows that $A_\alpha \neq A_\beta$. Hence $\operatorname{ord}(p, \mathscr{A}) = \kappa^+$, a contradiction.

2.6. CANONIZATION LEMMA (Erdös, Hajnal, Rado). *Let κ be a singular strong limit cardinal with $\lambda = \operatorname{cf}(\kappa)$, let $[\kappa]^n = \bigcup_{\beta<\tau} P_\beta$, where $n < \omega$ and $\tau < \kappa$. Then there is a* *subset* D *of* κ *and a partition* $\{D_\alpha : 0 \leq \alpha < \lambda\}$ *of* D *such that*

(1) $|D_\alpha| = \kappa_\alpha < \kappa$ *and* $\Sigma_{\alpha<\lambda}\kappa_\alpha = \kappa$;

(2) *if* $0 \leq \alpha < \beta < \lambda$, *then* $D_\alpha < D_\beta$ (i.e., *the ordinals in* D_α *precede those in* D_β);

(3) *if* $a, b \in [D]^n$ *and* $|a \cap D_\alpha| = |b \cap D_\alpha|$ *for all* $\alpha < \lambda$, *then* $a, b \in P_\beta$ *for some* $\beta < \tau$.

For a proof of the canonization lemma, see JUHÁSZ [1971].

3. Definitions of cardinal functions and elementary inequalities

A *cardinal function* is a function ϕ from the class of all topological spaces (or some precisely defined subclass) into the class of all infinite cardinals such that $\phi(X) = \phi(Y)$ whenever X and Y are homeomorphic. The requirement that cardinal functions take on only infinite cardinals as values simplifies statements of theorems and places the emphasis on infinite cardinal arithmetic. An obvious example of a cardinal function is *cardinality*, denoted $|X|$ and equal to the number of points in X plus ω. Perhaps the most useful cardinal function is *weight*, defined by

$$w(X) = \min\{|\mathscr{B}| : \mathscr{B} \text{ a base for } X\} + \omega .$$

Neither of these cardinal functions dominates the other; i.e., neither $w(X) \leq |X|$ nor $|X| \leq w(X)$ holds for all X. On the other hand, assuming appropriate separation axioms, $|X|$ and $w(X)$ never differ by more than one exponent. One usually expects $w(X) \leq |X|$, and this is the case for X compact or X metrizable (even first countable). But occasionally $|X| < w(X)$, as happens for a countable space not first countable. An example of a countably compact T_3 space X with $|X| = 2^\omega$ and $w(X) > 2^\omega$ is given in Section 14.

A cardinal function which always dominates weight and usually dominates $|X|$ (X must be T_0) is $o(X)$, defined as the number of open sets in X plus ω. To summarize thus far, one has the following relationships between $|X|$, $w(X)$, and $o(X)$.

3.1. THEOREM. *Let X be a topological space.*

(a) $w(X) \leq o(X) \leq 2^{|X|}$;

(b) *for* $X \in \mathscr{T}_0$, $|X| \leq 2^{w(X)}$ *and* $|X| \leq o(X)$.

PROOF. Let X be a T_0 space. To prove the first part of (b), let $\mathscr{B}$ be a base for X with $|\mathscr{B}| \leq w(X)$. Define $\Phi: X \to P(\mathscr{B})$ by $\Phi(p) = \{B : B \in \mathscr{B}, p \in B\}$. Since X is T_0, Φ is one-one; thus $|X| \leq 2^{w(X)}$. To prove the second part of (b), define a function from X into the open sets of X by $\Phi(p) = X - \{p\}^-$. Again Φ is one-one, so $|X| \leq o(X)$.

Another important cardinal function is *density*, defined as follows:

$$d(X) = \min\{|S|: S \subseteq X, \bar{S} = X\} + \omega .$$

Thus X is separable iff $d(X) = \omega$. Clearly $d(X) \leqslant w(X)$ and $d(X) \leqslant |X|$, and the gap can be arbitrarily large for non-Hausdorff spaces. (Let X be the space of cardinality κ with the cofinite topology. Then $d(X) = \omega$ and $w(X) = |X| = \kappa$.) The situation for H ausdorff spaces is more reasonable.

3.2. THEOREM (POSPÍŠIL [1937]). *For* $X \in \mathcal{T}_2$, $|X| \leqslant 2^{2^{d(X)}}$ *and* $w(X) \leqslant o(X) \leqslant 2^{2^{2^{d(X)}}}$. *In particular, every separable Hausdorff space has cardinality at most* 2^{2^ω} *and at most* $2^{2^{2^\omega}}$ *open sets.*

PROOF. The second part follows from the first part and 3.1(a). To prove the first inequality, let $d(X) = \kappa$ and let S be a dense subset of X with $|S| \leqslant \kappa$. Note that for distinct points p, q in X, there exists $A \subseteq S$ such that $p \in \bar{A}$, $q \notin \bar{A}$. It follows that the function Φ from X into $P(P(S))$ defined by $\Phi(p) = \{A: A \subseteq S, p \in \bar{A}\}$ is one-one, consequently $|X| \leqslant \exp \exp \kappa$.

The results in 3.2 are the best possible. For example, $\beta\omega$ is a compact separable space of cardinality $\exp \exp \omega$. See Section 14 for an example of a Hausdorff space X with $o(X) = \exp \exp \exp d(X)$. Juhász and Kunen have constructed an example of a Hausdorff space X with $w(X) = \exp \exp \exp d(X)$!

There is a much better bound for $w(X)$ in terms of $d(X)$ when X is regular. Incidently, in a regular space the collection of all regular open sets is a base for the topology. Consequently, in obtaining bounds for weight, it is often convenient to first obtain a bound on the number of regular open sets. One then has a bound for $w(X)$ when X is regular.

3.3. THEOREM (DE GROOT [1965]). *Let X be a topological space.*

(a) $|\mathrm{RO}(X)| \leqslant 2^{d(X)}$;

(b) *for X regular,* $w(X) \leqslant 2^{d(X)}$.

PROOF. Let $d(X) = \kappa$, let S be a dense subset of X with $|S| \leqslant \kappa$. We prove $\mathrm{RO}(X) \subseteq \{\bar{A}^\circ: A \subseteq S\}$, from which $|\mathrm{RO}(X)| \leqslant 2^\kappa$ follows. Let R be an open set in X with $\bar{R}^\circ = R$. Let $A = R \cap S$. Since R is open and S is dense, $\bar{A} = \bar{R}$, hence $\bar{A}^\circ = R$.

A pairwise disjoint collection of non-empty open sets in X is called a *cellular family*. This leads to a new cardinal function, the *cellularity of* X, defined as follows:

$$c(X) = \sup\{|\mathcal{V}|: \mathcal{V} \text{ a cellular family in } X\} + \omega .$$

One says that X is a *ccc space* (ccc = countable chain condition) if $c(X) = \omega$. If $d(X) = \kappa$, then every cellular family in X has cardinality $\leqslant\kappa$, so $c(X) \leqslant d(X)$. Moreover, $c(X) < d(X)$ is possible. Indeed, for $\kappa \geqslant \exp\exp\omega$, the Cantor cube of weight κ, denoted $D(2)^\kappa$, is a compact ccc space which is not separable. ($D(2)^\kappa$ cannot be separable since otherwise one has $|D(2)^\kappa| \leqslant \exp\exp\omega$; see Section 11 for a proof that $D(2)^\kappa$ is a ccc space.) This example shows that cellularity alone places no restrictions on $|X|$ (and hence no restriction on density, weight, or $o(X)$.)

The following proposition about cellularity is very useful.

3.4. PROPOSITION. *Let $c(X) = \kappa$, let $\mathscr{V}$ be an open collection in X. Then there is a subcollection $\mathscr{W}$ of $\mathscr{V}$ such that $|\mathscr{W}| \leqslant \kappa$ and $\bigcup\mathscr{V} \subseteq \overline{\bigcup\mathscr{W}}$.*

PROOF. Let $\mathscr{G}$ be the collection of all non-empty open sets in X which are subsets of some element of $\mathscr{V}$. Use Zorn's lemma to obtain a maximal cellular family $\mathscr{G}' \subseteq \mathscr{G}$. Then $|\mathscr{G}'| \leqslant c(X) = \kappa$, and $\bigcup\mathscr{V} \subseteq \overline{\bigcup\mathscr{G}'}$ by the maximality of $\mathscr{G}'$. One can now use $\mathscr{G}'$ to obtain $\mathscr{W} \subseteq \mathscr{V}$ with $|\mathscr{W}| \leqslant \kappa$ and $\bigcup\mathscr{V} \subseteq \overline{\bigcup\mathscr{W}}$.

An unexpectedly useful and ubiquitous cardinal function is *spread*, defined as follows:

$$s(X) = \sup\{|D| : D \subseteq X,\ D \text{ is discrete}\} + \omega\,.$$

If $s(X) = \omega$, one says that X has *countable spread*. Clearly $c(X) \leqslant s(X) \leqslant \min\{|X|, w(X)\}$. Spread and density are not directly comparable (see Section 14), although they differ by at most one exponent assuming appropriate separation axioms. Specifically, $s(X) \leqslant 2^{d(X)}$ for X regular (see 3.3b) and $d(X) \leqslant 2^{s(X)}$ for X Hausdorff (see 5.2).

The *Lindelöf degree of* X, denoted $L(X)$, is defined as the smallest infinite cardinal κ such that every open cover of X has a subcollection of cardinality $\leqslant\kappa$ which covers X. Thus X is Lindelöf iff $L(X) = \omega$. One has $L(X) \leqslant \min\{|X|, w(X)\}$, but Lindelöf degree is not directly comparable with either density or cellularity. Moreover, Lindelöf degree alone puts no bounds on $|X|$; e.g., the one-point compactification of a discrete space of cardinality κ has Lindelöf degree ω and cardinality κ.

A generalization of Lindelöf degree is *extent*, defined as follows:

$$e(X) = \sup\{|D| : D \subseteq X,\ D \text{ is closed and discrete}\} + \omega\,.$$

Clearly $e(X) \leqslant L(X)$ and $e(X) \leqslant s(X)$. On the other hand, if X is the set of all ordinals $<\omega_1$ with the order topology, then $e(X) = \omega$ (X is countably compact) and $L(X) = s(X) = \omega_1$. It is easy to prove that $e(X) \leqslant \kappa$ if and only if every infinite subset of X of cardinality greater than κ has a limit point. Thus, $e(X) = \omega$ is the familiar property of being ω_1-compact.

At this point it is appropriate to mention the subtle relationship between density and the cardinality of closed discrete sets in normal spaces. This topic was first investigated by F.B. Jones in the 1930's. One of his results states that a normal, separable space X cannot have a closed discrete set of cardinality $\geqslant 2^{\omega}$. This theorem is often used to prove that a space is not normal. For example, $S \times S$ (S is the Sorgenfrey line) is separable and has a closed discrete set of cardinality 2^{ω}, namely $\{(x, -x): x \in S\}$, hence cannot be normal. A modern version of Jones' result is now given. See VAN DOUWEN [1981] on the possibility of proving $2^{e(X)} \leqslant 2^{d(X)}$ for X normal; see Section 10 for an extension of Jones' lemma.

3.5. JONES' LEMMA. *If X is normal, then $2^{|D|} \leqslant 2^{d(X)}$ for every closed discrete $D \subseteq X$. In particular, if X is a normal, separable space, then* (1) *X cannot have a closed discrete set of cardinality $\geqslant 2^{\omega}$*; (2) *$2^{\omega} < 2^{\omega_1}$ implies that X cannot have a closed discrete set of cardinality ω_1.*

PROOF. Let S be a dense subset of X with $|S| \leqslant d(X)$. For each subset E of D, let U_E be an open set such that $E \subseteq U_E$ and $\bar{U}_E \cap (D - E) = \emptyset$. Let $V_E = U_E \cap S$. One can easily check that $V_E \neq V_F$ whenever E and F are distinct subsets of D. Thus $\{V_E: E \subseteq D\}$ is a collection of $2^{|D|}$ subsets of S and hence $2^{|D|} \leqslant 2^{|S|} \leqslant 2^{d(X)}$.

A cardinal function ϕ is *monotone* if $\phi(Y) \leqslant \phi(X)$ for every subspace Y of X. Monotone cardinal functions discussed so far are cardinality, weight, and spread. On the other hand, density, cellularity, Lindelöf degree, and extent are not. (Cellularity is monotone for open subsets or dense subsets; density is monotone for open subsets; Lindelöf degree and extent are monotone for closed subsets.) For each cardinal function ϕ which is not monotone, one can introduce a new cardinal function $h\phi$ defined by $h\phi(X) = \sup\{\phi(Y): Y \subseteq X\}$. Now $hc(X) = he(X) = s(X)$, so one has two new cardinal functions hd (*hereditary density*) and hL (*hereditary Lindelöf degree*). If $hd(X) = \omega$, X is *hereditarily separable*; if $hL(X) = \omega$, X is *hereditarily Lindelöf*. The Sorgenfrey line S is an example of a T_3 space which is both hereditarily Lindelöf and hereditarily separable but does not have a countable base. Every compact, perfectly normal space is hereditarily Lindelöf. See Section 14 for examples of (1) a Hausdorff, hereditarily separable space not Lindelöf; (2) a Hausdorff, hereditarily Lindelöf space not separable. The relationship between these properties for T_3 spaces is extremely delicate and is further discussed by Roitman and Abraham–Todorčević in this Handbook.

The functions hd and hL are useful in giving bounds on $o(X)$.

3.6. THEOREM. *For any space X, $o(X) \leqslant |X|^{hd(X)}$ and $o(X) \leqslant w(X)^{hL(X)}$*

PROOF. Let $hd(X) = \kappa$, let H be a closed set in X. Then $d(H) \leqslant \kappa$, so there exists $S \subseteq H$, $|S| \leqslant \kappa$, such that $\bar{S} = H$. Thus every closed set in X is in the collection $\{\bar{S}: S \subseteq X, |S| \leqslant \kappa\}$, so $o(X) \leqslant |X|^{\kappa}$. Now let $hL(X) = \kappa$, and let $\mathcal{B}$ be a base for X

with $|\mathscr{B}| \leqslant w(X)$. Then every open set in X is the union of $\leqslant\kappa$ elements of $\mathscr{B}$, so $o(X) \leqslant |\mathscr{B}|^\kappa$.

The cardinal functions defined so far (with the possible exception of extent) are 'classical'. Six additional cardinal functions of more recent vintage are now introduced. The first three, namely net weight, π-weight, and separating weight, are natural generalizations of weight. The next two, namely point separating weight and diagonal degree, characterize weight in compact spaces and yield interesting quantitative information about X when combined with extent. The last cardinal function, namely weak covering number, is useful as a simultaneous generalization of cellularity and Lindelöf degree and in obtaining bounds on the number of continuous, real-valued functions on X.

A *net* for a topological space X is a collection $\mathcal{N}$ of subsets of X such that every open set in X is the union of elements of $\mathcal{N}$. A net is like a base except that the elements of a net need *not* be open. For example, $\{\{p\}: p \in X\}$ is always a net for X. A remarkable fact about nets is that a compact space with a net of cardinality $\kappa \geqslant \omega$ has a base of cardinality κ. This is proved in Section 7. The *net weight* of X is defined as follows:

$$nw(X) = \min\{|\mathcal{N}|: \mathcal{N} \text{ a net for } X\} + \omega .$$

It is easy to see that net weight is monotone and that

$$\max\{hL(X), hd(X)\} \leqslant nw(X) \leqslant \min\{|X|, w(X)\} .$$

Moreover, bounds using weight can often be generalized to net weight; for example, $o(X) \leqslant 2^{nw(X)}$ and $|X| \leqslant 2^{nw(X)}$ for $X \in \mathscr{T}_0$. The relationship between the cardinal functions defined so far can be summarized as shown in Fig. 1. (Assume $X \in \mathscr{T}_0$ for $|X| \leqslant o(X)$.)

A *π-base* for X is a collection $\mathscr{V}$ of non-empty open sets in X such that if R is any non-empty open set in X, then $V \subseteq R$ for some $V \in \mathscr{V}$. For example, $\{\{n\}: n < \omega\}$ is a π-base for $\beta\omega$. The *π-weight* of X is defined as follows:

$$\pi w(X) = \min\{|\mathscr{V}|: \mathscr{V} \text{ a } \pi\text{-base for } X\} + \omega .$$

Note that $d(X) \leqslant \pi w(X) \leqslant w(X)$. The cardinal function π-weight is not monotone; e.g., $\pi w(\beta w) = \omega$ but $\pi w(\beta\omega - \omega) = 2^\omega$ (see Section 7).

Net weight and π-weight are not directly comparable. In Section 11 the Hewitt–Marczewski–Pondiczery theorem is used to construct a countable space X which is not first countable at any point. This space has the property that $nw(X) = \omega$, $\pi w(X) = 2^\omega$. On the other hand, $\pi w(\beta\omega) = \omega$ while $nw(\beta\omega) = 2^\omega$ (see Section 7). But assuming appropriate separation axioms, net weight and π-weight never differ by more than one exponent. Specifically, $nw(X) \leqslant 2^{d(X)} \leqslant 2^{\pi w(X)}$ for X regular and $\pi w(X) \leqslant w(X) \leqslant 2^{nw(X)}$ for any X.

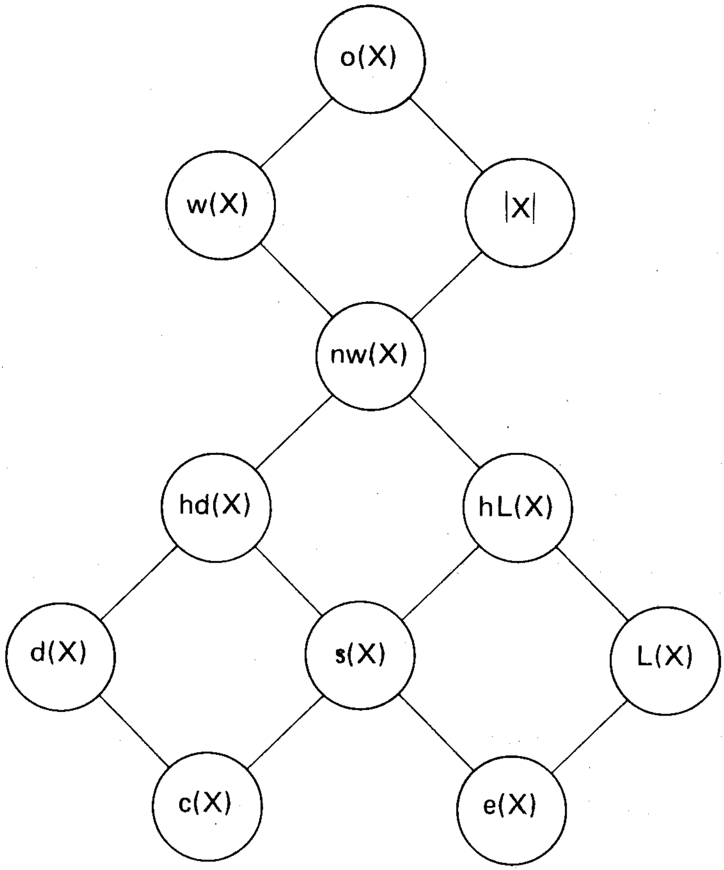

Fig. 1.

A cover $\mathscr{A}$ of a set E is *separating* if for each $p \in E$, $\bigcap\{A: A \in \mathscr{A}, p \in A\} = \{p\}$. The *separating weight* of a topological space X, denoted $sw(X)$, is the smallest infinite cardinal κ such that X has a separating open cover $\mathscr{V}$ with $|\mathscr{V}| \leq \kappa$. Separating weight is defined only for T_1 spaces (the existence of a separating open cover being equivalent to T_1). This cardinal function will not be treated in detail in this paper. But it does deserve mention since it is a useful technical device and appears frequently in the literature. Some basic facts about separating weight are summarized below. Proofs are left to the reader.

3.7. THEOREM. *Let X be a T_1 space.*
(a) $|X| \leq 2^{sw(X)}$ *and* $w(X) \leq o(X) \leq 2^{2^{sw(X)}}$;
(b) *for* $X \in \mathscr{T}_2$, $sw(X) \leq |\mathrm{RO}(X)|$ (*hence* $sw(X) \leq 2^{d(X)}$ *and* $|X| \leq 2^{2^{d(X)}}$);
(c) *for* $X \in \mathscr{T}_2$, $sw(X) \leq nw(X)$.

The *point separating weight* of X, denoted $psw(X)$, is the smallest infinite cardinal κ such that X has a separating open cover $\mathscr{V}$ with $\mathrm{ord}(p, \mathscr{V}) \leq \kappa$ for each

$p \in X$. If $psw(X) = \omega$, one says that X has a *point-countable separating open cover.* Note that $psw(X)$ is defined only for T_1 spaces. A σ-locally finite base for a T_1 space is always a point-countable separating open cover, so $psw(X) = \omega$ when X is metrizable. This shows that point separating weight alone puts no restrictions on the cardinality of a space.

The *diagonal degree* of X, denoted $\Delta(X)$, is the smallest infinite cardinal κ such that X has a collection $\{\mathcal{V}_\alpha : 0 \leq \alpha < \kappa\}$ of open covers such that $\bigcap_{\alpha<\kappa} st(p, \mathcal{V}_\alpha) = \{p\}$ for each $p \in X$. Again, $\Delta(X)$ is defined only for T_1 spaces. One has $\Delta(X) = \omega$ for X metrizable. Indeed, if $\mathcal{V}_n$ is the collection of all open balls in X of radius $1/n$, then $\bigcap_{n<\omega} st(p, \mathcal{V}_n) = \{p\}$ for each $p \in X$. This shows that diagonal degree alone puts no restrictions on the cardinality of a space. One can show that $\Delta(X)$ is the smallest infinite cardinal κ such that the diagonal Δ of X is the intersection of κ open sets in $X \times X$. Thus, if $\Delta(X) = \omega$, one says that X has a *G_δ-diagonal.*

The *weak covering number* of X, denoted $wc(X)$, is the smallest infinite cardinal κ such that every open cover of X has a subcollection of cardinality $\leq \kappa$ whose union is dense in X. Clearly $wc(X) \leq L(X)$, and by Proposition 3.4 one has $wc(X) \leq c(X)$. If $wc(X) = \omega$, one says that X is *weakly Lindelöf.* Note that $hwc(X) = s(X)$ for any space X.

Each of the cardinal functions defined so far is global; i.e., its definition is based on a topological property which gives global information about the space. Several important cardinal functions based on local topological properties are now introduced. Let X be a topological space, let $\mathcal{V}$ be a collection of non-empty open sets in X, let $p \in X$. Then $\mathcal{V}$ is a *local π-base for p* if for each open neighborhood R of p, one has $V \subseteq R$ for some $V \in \mathcal{V}$. If in addition one has $p \in V$ for all $V \in \mathcal{V}$, then $\mathcal{V}$ is a *local base for p.* Finally, if $p \in V$ for all $V \in \mathcal{V}$, and $\bigcap\{V : V \in \mathcal{V}\} = \{p\}$, then $\mathcal{V}$ is a *pseudo-base* for p. The local cardinal functions will be defined in terms of the following cardinal numbers:

$$\chi(p, X) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a local base for } p\}\,;$$

$$\pi\chi(p, X) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a local } \pi\text{-base for } p\}\,;$$

$$\psi(p, X) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a pseudo-base for } p\}\,;$$

$$\begin{aligned} t(p, X) = \min\{\kappa :\ & \text{for all } Y \subseteq X \text{ with } p \in \bar{Y}, \text{ there is} \\ & A \subseteq Y \text{ with } |A| \leq \kappa \text{ and } p \in \bar{A}\}\,. \end{aligned}$$

The *character*, the *π-character*, the *pseudo-character*, and the *tightness* of X are now defined as follows:

$$\chi(X) = \sup\{\chi(p, X) : p \in X\} + \omega\,;$$

$$\pi\chi(X) = \sup\{\pi\chi(p, X) : p \in X\} + \omega\,;$$

$$\psi(X) = \sup\{\psi(p, X) : p \in X\} + \omega\,;$$

$$t(X) = \sup\{t(p, X) : p \in X\} + \omega\,.$$

Thus, X is first countable iff $\chi(X) = \omega$; also, one says that X has *countable π-character* if $\pi\chi(X) = \omega$, *countable pseudo-character* if $\psi(X) = \omega$, and *countable tightness* if $t(X) = \omega$. Note that pseudo-character is defined for T_1 spaces only. Let X be the one-point compactification of the discrete space of cardinality κ. Then $t(X) = \pi\chi(X) = \omega$ while $\chi(X) = \psi(X) = \kappa$. Example 14.7 is a Hausdorff space with countable pseudo-character but not countable tightness.

Obviously a local cardinal function cannot by itself bound $|X|$. Character dominates the other local cardinal functions. Except for π-character, the local cardinal functions are monotone. (In Section 7 it is shown that $t(X) = h\pi\chi(X)$ for X compact.) Some useful inequalities are summarized in the following two theorems.

3.8. THEOREM. *Let X be a topological space.*
(a) $\chi(X) \leq w(X) \leq \chi(X) \cdot |X|$;
(b) $\pi w(X) = d(X) \cdot \pi\chi(X)$;
(c) $t(X) \leq hd(X)$;
(d) $t(X) \leq h\pi\chi(X)$;
(e) *for* $X \in \mathcal{T}_1$, $\psi(X) \leq \min\{psw(X), \Delta(X)\}$;
(f) *for* $X \in \mathcal{T}_2$, $\psi(X) \leq hL(X)$.

3.9. THEOREM. *Let X be a topological space, let S be a dense subset of X, let $p \in S$.*
(a) $d(X) \leq d(S) \leq d(X) \cdot t(X)$;
(b) $\pi w(S) \leq \pi w(X)$ *and* $\pi\chi(p, S) \leq \pi\chi(p, X)$;
(c) *for X regular,* $\pi w(S) = \pi w(X)$, $\chi(p, S) = \chi(p, X)$, *and* $\pi\chi(p, S) = \pi\chi(p, X)$.

4. Bounds on the cardinality of X

The aim of this section is to obtain bounds on $|X|$ in terms of other cardinal functions. Elementary bounds already obtained or easily proved are $|X| \leq \exp nw(X)$ for $X \in \mathcal{T}_0$ and $|X| \leq \exp\exp d(X)$ for $X \in \mathcal{T}_2$. Generally speaking, combinations of a *global* cardinal function and a *local* cardinal function give the sharpest and most useful bounds on $|X|$. (Of course, some global cardinal functions give local information, for example $\chi(X) \leq w(X)$, $\psi(X) \leq \min\{hL(X), \exp d(X)\}$ for $X \in \mathcal{T}_2$, etc.)

The bounds on $|X|$ proved in this section fall in one of two categories, namely *easy* and *difficult* (to prove). In the easy category one has $|X| \leq nw(X)^{\psi(X)}$ for $X \in \mathcal{T}_1$, $|X| \leq \exp(d(X) \cdot \psi(X))$ for $X \in \mathcal{T}_3$, and $|X| \leq d(X)^{\chi(X)}$ for $X \in \mathcal{T}_2$. Although easy to prove, these inequalities are often useful.

In the difficult category one has $|X| \leq \exp(L(X) \cdot \chi(X))$ for $X \in \mathcal{T}_2$, $|X| \leq \exp(c(X) \cdot \chi(X))$ for $X \in \mathcal{T}_2$, $|X| \leq \exp(s(X) \cdot \psi(X))$ for $X \in \mathcal{T}_1$, and $|X| \leq \exp\exp s(X)$ for $X \in \mathcal{T}_2$. The first of these is due to ARHANGEL'SKIĬ [1969]; the others to HAJNAL and JUHÁSZ [1967], [1969]. As noted in Section 3, Lindelöf degree alone, or cellularity alone, cannot bound $|X|$. (These two functions are described by ARHANGEL'SKIĬ [1978] as "pure" global cardinal functions.) But if either is combined with the local cardinal function character, one obtains an elegant bound on $|X|$. Spread alone bounds $|X|$ by two exponents, but a sharper bound is obtained when spread is combined with the local cardinal function pseudo-character. A comparison of $|X| \leq \exp(c(X) \cdot \chi(X))$ and $|X| \leq \exp(s(X) \cdot \psi(X))$ is useful. One weakens character to pseudo-character but compensates by strengthening cellularity to spread.

Perhaps the most exciting and dramatic of the difficult inequalities is Arhangel'skiĭ's theorem that $|X| \leq \exp(L(X) \cdot \chi(X))$ for $X \in \mathcal{T}_2$. The countable version of this result, namely that every Lindelöf, first countable, Hausdorff space has cardinality at most 2^ω, answers the following fifty-year old question of Alexandroff and Urysohn. Does there exist a compact, first countable space having cardinality greater than the continuum?

4.1. THEOREM. *For $X \in \mathcal{T}_1$, $|X| \leq nw(X)^{\psi(X)}$. In particular, every T_1 space with countable pseudo-character and a net of cardinality $\leq 2^\omega$ has cardinality at most 2^ω.*

PROOF. Let $\psi(X) = \kappa$, let $\mathcal{N}$ be a net for X with $|\mathcal{N}| \leq nw(X)$, let $p \in X$. Since $\psi(p, X) \leq \kappa$, one can choose $\mathcal{N}_p \subseteq \mathcal{N}$ with $|\mathcal{N}_p| \leq \kappa$ such that $\bigcap \mathcal{N}_p = \{p\}$. The number of subcollections of $\mathcal{N}$ chosen in this way is $\leq nw(X)^\kappa$, so $|X| \leq nw(X)^\kappa$.

4.2. THEOREM. *For $X \in \mathcal{T}_3$, $|X| \leq 2^{d(X) \cdot \psi(X)}$. In particular, every separable T_3 space with countable pseudo-character has cardinality at most 2^ω.*

PROOF. $|X| \leq nw(X)^{\psi(X)} \leq w(X)^{\psi(X)} \leq 2^{d(X) \cdot \psi(X)}$.

The following lemma is useful in obtaining bounds on $|X|$ when density is combined with one or more other cardinal functions. The lemma is used to prove $|X| \leq d(X)^{\chi(X)}$ for $X \in \mathcal{T}_2$. Other applications include a direct proof of $|X| \leq 2^{d(X) \cdot \psi(X)}$ and a proof of the inequality $|X| \leq d(X)^{L(X) \cdot t(X) \cdot \psi(X)}$ for $X \in \mathcal{T}_2$ (see Remark 4.6 below).

4.3. LEMMA. *Let κ be an infinite cardinal, let X be a topological space such that* (1) *for each $p \in X$, there is a collection $\mathcal{V}_p$ of open neighborhoods of p such that $|\mathcal{V}_p| \leq \kappa$ and $\bigcap\{\bar{V}: V \in \mathcal{V}_p\} = \{p\}$;* (2) *there is a subset S of X such that $X = \bigcup\{\bar{A}: A \subseteq S, |A| \leq \kappa\}$. Then $|X| \leq |S|^\kappa$.*

PROOF. For each $p \in X$, let A_p be a subset of S with $p \in \bar{A}_p$ and $|A_p| \leq \kappa$. Note that for any open neighborhood V of p, $(A_p \cap V) \subseteq S$, $|A_p \cap V| \leq \kappa$, and $p \in (A_p \cap V)^-$. Define $\Phi: X \to P_\kappa(P_\kappa(S))$ by $\Phi(p) = \{(A_p \cap V): V \in \mathcal{V}_p\}$. Since $\bigcap \{(A_p \cap V)^-: V \in \mathcal{V}_p\} = \{p\}$, Φ is one-one. Hence $|X| \leq (|S|^\kappa)^\kappa = |S|^\kappa$.

4.4. THEOREM (POSPÍŠIL [1937]). *For $X \in \mathcal{T}_2$, $|X| \leqslant d(X)^{\chi(X)}$. In particular, every first-countable Hausdorff space with a dense subset of cardinality $\leqslant 2^{\omega}$ has cardinality $\leqslant 2^{\omega}$.*

PROOF. Use Lemma 4.3 with $\kappa = \chi(X)$.

4.5. THEOREM (Arhangel'skiĭ). *For $X \in \mathcal{T}_2$, $|X| \leqslant 2^{L(X)\cdot\chi(X)}$. In particular, every Lindelöf, first countable, Hausdorff space has cardinality at most 2^{ω}.*

PROOF. Let $L(X) \cdot \chi(X) = \kappa$, and for each $p \in X$ let $\mathcal{V}_p$ be a local base for p with $|\mathcal{V}_p| \leqslant \kappa$. Construct an increasing sequence $\{H_\alpha : 0 \leqslant \alpha < \kappa^+\}$ of closed sets in X and a sequence $\{\mathcal{V}_\alpha : 0 < \alpha < \kappa^+\}$ of open collections in X such that

(1) $|H_\alpha| \leqslant 2^{\kappa}$, $0 \leqslant \alpha < \kappa^+$;

(2) $\mathcal{V}_\alpha = \{V : V \in \mathcal{V}_p, p \in \bigcup_{\beta<\alpha} H_\beta\}$, $0 < \alpha < \kappa^+$;

(3) if W is the union of $\leqslant\kappa$ elements of $\mathcal{V}_\alpha$, and $W \neq X$, then $H_\alpha - W \neq \emptyset$.

The construction is by transfinite induction. Let $0 < \alpha < \kappa$, and assume that H_β has been constructed for each $\beta < \alpha$. Note that $\mathcal{V}_\alpha$ is defined by (2) and $|\mathcal{V}_\alpha| \leqslant 2^{\kappa}$. For each set W which is the union of $\leqslant\kappa$ elements of $\mathcal{V}_\alpha$ and for which $W \neq X$, choose one point of $X - W$. Let A_α be the set of points chosen in this way ($|A_\alpha| \leqslant 2^{\kappa}$), and let $H_\alpha = [A_\alpha \cup (\bigcup_{\beta<\alpha} H_\beta)]^-$. Clearly H_α is closed and $H_\beta \subseteq H_\alpha$ for all $\beta < \alpha$; $|H_\alpha| \leqslant 2^{\kappa}$ follows from 4.4. This completes the construction of $\{H_\alpha : 0 \leqslant \alpha < \kappa^+\}$.

Now let $H = \bigcup_{\alpha<\kappa^+} H_\alpha$; H is closed since $\{H_\alpha : 0 \leqslant \alpha < \kappa^+\}$ is an increasing sequence of closed sets and $\chi(X) \leqslant \kappa$. The proof is complete if $H = X$. Suppose not, and let $q \in (X - H)$. For each $p \in H$ choose $V_p \in \mathcal{V}_p$ such that $q \notin V_p$. Then $\{V_p : p \in H\}$, together with $(X - H)$, covers X, so there exists a subset A of H with $|A| \leqslant \kappa$ such that $\{V_p : p \in A\}$ covers H. Let $W = \bigcup_{p\in A} V_p$, and note that $H \subseteq W$ and $q \notin W$. Choose $\alpha < \kappa^+$ such that $A \subseteq \bigcup_{\beta<\alpha} H_\beta$. Since $W \neq X$, it follows from (3) that $H_\alpha - W \neq \emptyset$. This contradicts $H \subseteq W$.

4.6. REMARK. Arhangel'skiĭ's theorem can be generalized to $|X| \leqslant \exp(L(X) \cdot \psi(X) \cdot t(X))$. In the proof above one replaces 4.4 by the inequality $|X| \leqslant d(X)^{L(X)\cdot\psi(X)\cdot t(X)}$, which in turn is a consequence of Lemma 4.3. Arhangel'skiĭ has asked if the pair $\{L, \psi\}$ bounds $|X|$. This difficult question is discussed in this Handbook by Juhász. A partial solution, due to Charlesworth, states that $|X| \leqslant psw(X)^{L(X)\cdot\psi(X)}$ for $X \in \mathcal{T}_1$.

As one might guess, Arhangel'skiĭ's original proof of 4.5 was quite difficult. The argument given above is due to Pol. The countable version of this proof should be within the reach of any first-year graduate student in mathematics. The theorem is sufficiently important to be included in any introductory graduate course in set-theoretic topology, and provides exposure to modern topology at an early level of mathematical training. The proof that every compact, first countable space has cardinality at most 2^ω is especially accessible. In this case condition (3) is easier to formulate and the proof of 4.4 for $\chi(X)=\omega$, $d(X)\leqslant 2^\omega$ is very easy (see the introduction).

We now turn to the Hajnal–Juhász theorems on cellularity and spread. These results are strongly motivated by the fundamental paper of de Groot on cardinal functions.

4.7. THEOREM (Hajnal–Juhász). *For $X\in\mathcal{T}_2$, $|X|\leqslant 2^{c(X)\cdot\chi(X)}$. In particular, every first countable, ccc, Hausdorff space has cardinality at most 2^ω.*

PROOF. Let $c(X)\cdot\chi(X)=\kappa$, and for each $p\in X$ let $\mathcal{V}_p$ be a local base for p with $|\mathcal{V}_p|\leqslant\kappa$. Construct a sequence $\{A_\alpha: 0\leqslant\alpha<\kappa^+\}$ of subsets of X and a sequence $\{\mathcal{V}_\alpha: 0<\alpha<\kappa^+\}$ of open collections in X such that

(1) $|A_\alpha|\leqslant 2^\kappa$, $0\leqslant\alpha<\kappa^+$;

(2) $\mathcal{V}_\alpha=\{V: V\in\mathcal{V}_p, p\in\bigcup_{\beta<\alpha}A_\beta\}$, $0<\alpha<\kappa^+$;

(3) if $\{G_\gamma: 0\leqslant\gamma<\kappa\}$ is a collection of $\leqslant\kappa$ open sets in X, each of which is the union of $\leqslant\kappa$ elements of $\mathcal{V}_\alpha$, and $\bigcup_{\gamma<\kappa}\bar{G}_\gamma\neq X$, then $A_\alpha-(\bigcup_{\gamma<\kappa}\bar{G}_\gamma)\neq\emptyset$.

Let $A=\bigcup_{\alpha<\kappa^+}A_\alpha$. The proof is complete if $A=X$. Suppose not, let $q\in(X-A)$, and let $\{B_\gamma: 0\leqslant\gamma<\kappa\}$ be a local base at q. For each $\gamma<\kappa$ let $\mathcal{W}_\gamma=\{V: V\in\mathcal{V}_p, p\in A, V\cap B_\gamma=\emptyset\}$. Note that for each $p\in A$, there exist $\gamma<\kappa$ such that $p\in\bigcup\mathcal{W}_\gamma$. By Proposition 3.4, there exists $\mathcal{G}_\gamma\subseteq\mathcal{W}_\gamma$ with $|\mathcal{G}_\gamma|\leqslant\kappa$ such that $\bigcup\mathcal{W}_\gamma\subseteq\overline{\bigcup\mathcal{G}_\gamma}$. Let $G_\gamma=\bigcup\mathcal{G}_\gamma$, and note that $A\subseteq\bigcup_{\gamma<\kappa}\bar{G}_\gamma$ and $q\notin\bigcup_{\gamma<\kappa}\bar{G}_\gamma$. Choose $\alpha<\kappa^+$ such that $\mathcal{G}_\gamma\subseteq\mathcal{V}_\alpha$ for all $\gamma<\kappa$. By (3), $A_\alpha-(\bigcup_{\gamma<\kappa}G_\gamma)\neq\emptyset$. This contradicts $A\subseteq\bigcup_{\gamma<\kappa}\overline{G_\gamma}$.

4.8. PROPOSITION (Šapirovskiĭ). *Let $\mathcal{V}$ be an open cover of a topological space X, let $s(X)\leqslant\kappa$. Then there is a subset A of X with $|A|\leqslant\kappa$ and a subcollection $\mathcal{W}$ of $\mathcal{V}$ with $|\mathcal{W}|\leqslant\kappa$ such that $X=\bar{A}\cup(\bigcup\mathcal{W})$.*

PROOF. Suppose not. Construct sequences $\{x_\alpha:\ 0\leqslant\alpha<\kappa^+\}\subseteq X$ and $\{V_\alpha: 0\leqslant\alpha<\kappa^+\}\subseteq\mathcal{V}$ such that $x_0\in V_0$ and $x_\alpha\in V_\alpha-[(\bigcup_{\beta<\alpha}V_\beta)\cup(\{x_\beta:\beta<\alpha\}^-)]$ for $0<\alpha<\kappa^+$. Then $\{x_\alpha: 0\leqslant\alpha<\kappa^+\}$ is a discrete set in X of cardinality κ^+, a contradiction.

4.9. THEOREM (Hajnal–Juhász). *For $X\in\mathcal{T}_1$, $|X|\leqslant 2^{s(X)\cdot\psi(X)}$. In particular, every T_1 space with countable spread and countable pseudo-character has cardinality at most 2^ω.*

PROOF. Let $s(X)\cdot\psi(X)=\kappa$, and for each $p\in X$ let $\mathcal{V}_p$ be a pseudo-base for p with $|\mathcal{V}_p|\leqslant\kappa$. Construct a sequence $\{A_\alpha: 0\leqslant\alpha<\kappa^+\}$ of subsets of X and a sequence $\{\mathcal{V}_\alpha: 0<\alpha<\kappa^+\}$ of open collections in X such that

(1) $|A_\alpha|\leqslant 2^\kappa$, $0\leqslant\alpha<\kappa^+$;

(2) $\mathcal{V}_\alpha=\{V: V\in\mathcal{V}_p, p\in\bigcup_{\beta<\alpha}A_\beta\}$, $0<\alpha<\kappa^+$;

(3) if W is the union of $\leqslant\kappa$ elements of $\mathcal{V}_\alpha$, and $\{B_\gamma: 0\leqslant\gamma<\kappa\}$ is a collection

of $\leq\kappa$ subsets of X with $|B_\gamma|\leq\kappa$ and $B_\gamma\subseteq\bigcup_{\beta<\alpha}A_\beta$ for all $\gamma<\kappa$, and $W\cup(\bigcup_{\gamma<\kappa}\overline{B_\gamma})\neq X$, then $A_\alpha-(W\cup(\bigcup_{\gamma<\kappa}\bar{B}_\gamma))\neq\emptyset$.

Let $A=\bigcup_{\alpha<\kappa^+}A_\alpha$. The proof is complete if $A=X$. Suppose not, and let $q\in(X-A)$. Since $\psi(X)\leq\kappa$, $X-\{q\}=\bigcup_{\gamma<\kappa}H_\gamma$, where each H_γ is a closed set in X. Let $L_\gamma=H_\gamma\cap A$, and for each $p\in L_\gamma$ let $V_p\in\mathcal{V}_p$ be such that $q\not\in V_p$. Apply Proposition 4.8 to L_γ and $\{V_p: p\in L_\gamma\}$ to obtain subsets B_γ and C_γ of L_γ, each of cardinality $\leq\kappa$, such that $L_\gamma\subseteq\bar{B}_\gamma\cup(\bigcup_{p\in C_\gamma}V_p)$. Let $W=\bigcup\{V_p: p\in C_\gamma, \gamma<\kappa\}$, and note that $A\subseteq(W\cup(\bigcup_{\gamma<\kappa}\bar{B}_\gamma))$ and $q\not\in(W\cup(\bigcup_{\gamma<\kappa}\bar{B}_\gamma))$. Choose $\alpha<\kappa^+$ such that $(B_\gamma\cup C_\gamma)\subseteq\bigcup_{\beta<\alpha}A_\beta$ for all $\gamma<\kappa$. By (3), $A_\alpha-(W\cup(\bigcup_{\gamma<\kappa}\bar{B}_\gamma))\neq\emptyset$, a contradiction of $A\subseteq(W\cup(\bigcup_{\gamma<\kappa}\bar{B}_\gamma))$.

4.10. COROLLARY (de Groot). *For $X\in\mathcal{T}_2$, $|X|\leq 2^{hL(X)}$. In particular, every hereditarily Lindelöf Hausdorff space has cardinality at most 2^ω.*

4.11. PROPOSITION. *For $X\in\mathcal{T}_2$, $\psi(X)\leq 2^{s(X)}$.*

PROOF. Let $s(X)=\kappa$, let $p\in X$. For each $q\neq p$ let V_q be an open neighborhood of q such that $p\not\in\bar{V}_q$. Apply proposition 4.8 to $X-\{p\}$ and $\{V_q: q\neq p\}$ to obtain subsets A and B of $X-\{p\}$, each of cardinality $\leq\kappa$, such that $X-\{p\}\subseteq\bar{A}\cup(\bigcup_{q\in B}V_q)$. Let $\mathcal{V}_A=\{X-\bar{C}: C\subseteq A, p\not\in\bar{C}\}$, $\mathcal{V}_B=\{X-\bar{V}_q: q\in B\}$, and $\mathcal{V}=\mathcal{V}_A\cup\mathcal{V}_B$. Then $\mathcal{V}$ is a pseudo-base for p of cardinality $\leq 2^\kappa$.

4.12. THEOREM (Hajnal–Juhász). *For $X\in\mathcal{T}_2$, $|X|\leq 2^{2^{s(X)}}$. In particular, every Hausdorff space with countable spread has cardinality at most 2^{2^ω}.*

PROOF. $|X|\leq\exp(s(X)\cdot\psi(X))\leq\exp(s(X)\cdot\exp s(X))=\exp\exp s(X)$.

The technique used in the proofs of 4.5, 4.7, and 4.9 is due to POL [1974] and ŠAPIROVSKIĬ [1972]; also see HODEL [1976] and the proof by Rudin in the paper of Corson and Michael. This technique provides a unified approach to the difficult inequalities in the theory of cardinal functions, and will be used again in the proofs of 4.13, 5.1, 6.2, 7.17 and 9.3.

The original proofs of 4.7 and 4.9 use the Erdös–Rado theorem. These proofs are so elegant that one is included here (the other being quite similar). Let $X\in\mathcal{T}_2$ and $c(X)\cdot\chi(X)=\kappa$, but suppose $|X|>2^\kappa$. For each $p\in X$ let $\{V(p,\alpha): 0\leq\alpha<\kappa\}$ be a local base for p. For each pair α, β in κ (not necessarily distinct), let $W(p,\{\alpha,\beta\})=V(p,\alpha)\cap V(p,\beta)$, and let $P(\{\alpha,\beta\})=\{\{p,q\}: W(p,\{\alpha,\beta\})\cap W(q,\{\alpha,\beta\})=\emptyset\}$. Since X is Hausdorff, $[X]^2\subseteq\bigcup\{P(\{\alpha,\beta\}): \alpha,\beta\in\kappa\}$. By the Erdös–Rado theorem, there exists $A\subset X$ with $|A|>\kappa$ and $\alpha,\beta\in\kappa$ such that $[A]^2\subseteq P(\{\alpha,\beta\})$. One can easily check that $\{W(p,\{\alpha,\beta\}): p\in A\}$ is a cellular family in X of cardinality $>\kappa$, a contradiction.

The Pol–Šapirovskiĭ technique can be used to prove that $|X|\leq\exp wc(X)\cdot\chi(X)$ for X a normal Hausdorff space. Except for the added hypothesis that X is normal, this gives a simultaneous generalization of 4.5 and 4.7.

4.13. THEOREM (Bell, Ginsburg, Woods). *If X is a normal, Hausdorff space, then $|X| \leq 2^{wc(X)\cdot\chi(X)}$. In particular, every first countable, weakly Lindelöf, normal Hausdorff space has cardinality at most 2^ω.*

PROOF. Let $wc(X)\cdot\chi(X)=\kappa$, and for each point $p\in X$ let $\mathcal{V}_p$ be a local base for p with $|\mathcal{V}_p|\leq\kappa$. Construct an increasing sequence $\{H_\alpha: 0\leq\alpha<\kappa^+\}$ of closed subsets of X and a sequence $\{\mathcal{V}_\alpha: 0\leq\alpha<\kappa^+\}$ of open collections in X such that

(1) $|H_\alpha|\leq 2^\kappa$, $0\leq\alpha<\kappa^+$;

(2) $\mathcal{V}_\alpha=\{V\colon V\in\mathcal{V}_p, p\in\bigcup_{\beta<\alpha}H_\beta\}$, $0<\alpha<\kappa^+$;

(3) if W is the union of $\leq\kappa$ elements of $\mathcal{V}_\alpha$, and $W\neq X$, then $H_\alpha-\bar{W}\neq\emptyset$.

Let $H=\bigcup_{\alpha<\kappa^+}H_\alpha$, and note that H is closed. If $H=X$, the proof is complete. Suppose not, and let $q\in(X-H)$. Use regularity to obtain an open set R such that $H\subset R$ and $q\notin\bar{R}$. Let $\mathcal{G}=\{V\colon V\in\mathcal{V}_p, p\in H, V\subseteq R\}$, and let $G=\bigcup\mathcal{G}$. Clearly $H\subseteq G$ and $q\notin\bar{G}$. Use normality of X to obtain an open set L with $H\subseteq L\subseteq\bar{L}\subseteq G$. Now $\mathcal{G}$, together with $(X-\bar{L})$, covers X, so by $wc(X)\leq\kappa$ there exists $\mathcal{W}\subseteq\mathcal{G}$ with $|\mathcal{W}|\leq\kappa$ such that $X=\overline{\bigcup\mathcal{W}\cup(X-\bar{L})}$. Let $W=\bigcup\mathcal{W}$. Since $H\cap\overline{(X-\bar{L})}=\emptyset$, $H\subseteq\bar{W}$. Now choose $\alpha<\kappa^+$ such that $\mathcal{W}\subseteq\mathcal{V}_\alpha$. By (3), $H_\alpha-\bar{W}\neq\phi$, a contradiction of $H\subseteq\bar{W}$.

5. Bounds using spread

A somewhat surprising result proved in Section 4 is the Hajnal–Juhász theorem that spread alone bounds $|X|$ (by two exponents). In this section it is proved that spread bounds hereditary density and net weight by just one exponent (assuming appropriate separation axioms). The inequality $|X|\leq\exp\exp s(X)$ is also generalized to $o(X)\leq\exp\exp s(X)$. The key to these results is a fundamental theorem of ŠAPIROVSKIĬ [1972] on spread.

5.1. THEOREM (Šapirovskiĭ). *Let $X\in\mathcal{T}_2$, let $s(X)\leq\kappa$. Then there is a subset S of X with $|S|\leq 2^\kappa$ such that $X=\bigcup\{\bar{A}\colon A\subseteq S, |A|\leq\kappa\}$.*

PROOF. For each $p\in X$, let $\mathcal{V}_p$ be a pseudo-base for p with $|\mathcal{V}_p|\leq 2^\kappa$ (use 4.11). Construct a sequence $\{S_\alpha: 0\leq\alpha<\kappa^+\}$ of subsets of X and a sequence $\{\mathcal{V}_\alpha: 0<\alpha<\kappa^+\}$ of open collections in X such that

(1) $|S_\alpha|\leq 2^\kappa$, $0\leq\alpha<\kappa^+$;

(2) $\mathcal{V}_\alpha=\{V\colon V\in\mathcal{V}_p, p\in\bigcup_{\beta<\alpha}S_\beta\}$, $0<\alpha<\kappa^+$;

(3) if $A\subseteq(\bigcup_{\beta<\alpha}S_\beta)$ with $|A|\leq\kappa$, and if W is the union of $\leq\kappa$ elements of $\mathcal{V}_\alpha$, and $\bar{A}\cup W\neq X$, then $S_\alpha-(\bar{A}\cup W)\neq\emptyset$.

Let $S=\bigcup_{\alpha<\kappa^+}S_\alpha$, let $q\in X$. The proof is complete if $q\in\bar{A}$ for some $A\subseteq S$,

$|A| \leqslant \kappa$. One may assume $q \notin S$. For each $p \in S$ choose $V_p \in \mathcal{V}_p$ such that $q \notin V_p$. Apply 4.8 to S and $\{V_p: p \in S\}$ to obtain subsets A and B of S, each of cardinality $\leqslant \kappa$, such that $S \subseteq \bar{A} \cup W$, where $W = \bigcup_{p \in B} V_p$. Then $q \in \bar{A}$. If not, choose $\alpha < \kappa^+$ such that $(A \cup B) \subseteq \bigcup_{\beta<\alpha} S_\beta$. By (3), $S_\alpha - (\bar{A} \cup W) \neq \emptyset$, a contradiction of $S \subseteq (\bar{A} \cup W)$.

5.2. THEOREM (Hajnal–Juhász). *For $X \in \mathcal{T}_2$, $hd(X) \leqslant 2^{s(X)}$.*

PROOF. Since spread is monotone, it suffices to prove $d(X) \leqslant \exp s(X)$. This follows immediately from 5.1.

5.3. THEOREM. *For $X \in \mathcal{T}_3$, $nw(X) \leqslant 2^{s(X)}$.*

PROOF. Let $s(X) = \kappa$, let S be a subset of X with $|S| \leqslant 2^\kappa$ and $X = \bigcup\{\bar{A}: A \subseteq S, |A| \leqslant \kappa\}$. Then $\mathcal{N} = \{\bar{N}: N \subseteq S, |N| \leqslant \kappa\}$ is a net in X of cardinality $\leqslant 2^\kappa$. To see this, let $p \in X$, let R be an open neighborhood of p. Let V be an open set such that $p \in V \subseteq \bar{V} \subseteq R$, let A be a subset of S with $p \in \bar{A}$ and $|A| \leqslant \kappa$, and let $N = A \cap V$. Then $\bar{N} \in \mathcal{N}$ and $p \in \bar{N} \subseteq R$.

5.4. REMARK. Theorem 5.3 allows an easy proof of the inequality $|X| \leqslant \exp(s(X) \cdot \psi(X))$ for $X \in \mathcal{T}_3$:

$$|X| \leqslant nw(X)^{\psi(X)} \leqslant (2^{s(X)})^{\psi(X)} = 2^{s(X) \cdot \psi(X)} .$$

5.5. THEOREM. *For $X \in \mathcal{T}_2$, $o(X) \leqslant 2^{2^{s(X)}}$. In particular, the number of open sets in a Hausdorff space with countable spread is at most 2^{2^ω}.*

PROOF. $o(X) \leqslant |X|^{hd(X)} \leqslant (\exp \exp s(X))^{\exp s(X)} = \exp \exp s(X)$.

It is natural to ask if $hd(X) \leqslant \exp s(X)$ for $X \in \mathcal{T}_2$ or $hL(X) \leqslant \exp s(X)$ for $X \in \mathcal{T}_3$ can be sharpened to $hd(X) = s(X)$ or $hL(X) = s(X)$ for $X \in \mathcal{T}_3$. (This cannot be done for $X \in \mathcal{T}_2$; see Section 14.) These questions are closely related to S and L problems, and for a detailed discussion the reader is referred to the papers in this Handbook by Roitman, Abraham–Todorčević, and Juhász (Section 2). The situation, briefly stated, is this: (1) CH implies the existence of a hereditarily Lindelöf T_3 space not separable, and a hereditarily separable T_3 space not Lindelöf; (2) there is no known 'real' example of a T_3 space where spread differs from either hereditary density or hereditary Lindelöf degree. From (1) it follows that if ZFC is consistent, then one cannot prove $hd(X) = s(X)$ or $hL(X) = s(X)$ for $X \in \mathcal{T}_3$. On the other hand, Šapirovskiĭ has proved the beautiful result that $hd(X) \leqslant s(X)^+$ for X compact. A proof is given in Section 7.

The following fact is used in proving Šapirovskiĭ's theorem: *if $s(X) \leqslant \kappa$, then X has a dense subset Y with $hL(Y) \leqslant \kappa$.* It is appropriate to prove this result now, since it allows an alternate proof of the inequality $hd(X) \leqslant \exp s(X)$ which is closer in spirit to the original proof of Hajnal–Juhász. Indeed, let $X \in \mathcal{T}_2$, let $s(X) = \kappa$; it suffices to show that $d(X) \leqslant 2^\kappa$. Let Y be a dense subset of X with $hL(Y) \leqslant \kappa$. By 4.10, $|Y| \leqslant 2^\kappa$, hence $d(X) \leqslant 2^\kappa$.

5.6. PROPOSITION (ŠAPIROVSKIĬ [1972]). *If $s(X) \leqslant \kappa$, then X has a dense subspace Y with $hL(Y) \leqslant \kappa$.*

PROOF. Let $d(X) = \lambda \geqslant \kappa^+$. (If $d(X) \leqslant \kappa$, there is nothing to prove.) Let $S = \{x_\alpha : 0 \leqslant \alpha < \lambda\}$ be a dense subset of X. Construct a subset $Y = \{x_{\alpha_\beta} : 0 \leqslant \beta < \lambda\}$ of S such that for each $\beta < \lambda$, α_β is the smallest ordinal $<\lambda$ such that $x_{\alpha_\beta} \notin \{x_{\alpha_\gamma} : 0 \leqslant \gamma < \beta\}^-$. One can show that $\alpha_\beta \geqslant \beta$ for all $\beta < \lambda$. Using this fact, one can then show that Y is dense in X. By construction, the space Y has the property that if Z is any subset of Y of cardinality κ^+, then Z has a subset A of density κ^+.

The proof is complete if $hL(Y) \leqslant \kappa$. If not, then there is a collection $\{G_\alpha : 0 \leqslant \alpha < \kappa^+\}$ of open sets in Y and a subset $Z = \{y_\alpha : 0 \leqslant \alpha < \kappa^+\}$ of Y such that $y_\alpha \in (G_\alpha - \bigcup_{\beta<\alpha} G_\beta)$ for all $\alpha < \kappa^+$. Let $A \subseteq Z$ with $d(A) = \kappa^+$. We now obtain a contradiction by showing that $d(A) \leqslant \kappa$. For each $\alpha < \kappa^+$ let $V_\alpha = \{y_\beta : \beta < \alpha\}$; note that V_α is open in Z, $|V_\alpha| \leqslant \kappa$, and $\{V_\alpha : 0 \leqslant \alpha < \kappa^+\}$ covers A. By 4.8, there is a subset B of A with $|B| \leqslant \kappa$ and $\alpha_0 < \kappa^+$ such that $A \subseteq \bar{B} \cup V_{\alpha_0}$. The set $B \cup (V_{\alpha_0} \cap A)$ is a dense subset of A of cardinality $\leqslant\kappa$.

6. Bounds using cellularity and π-character

The aim of this section is to use cellularity and π-character (or π-weight) to obtain bounds on weight, $|\mathrm{RO}(X)|$, and $|X|$. In particular, the following three inequalities are proved: $|\mathrm{RO}(X)| \leqslant \pi w(X)^{c(X)}$ for any space X; $w(X) \leqslant |\mathrm{RO}(X)| \leqslant \pi\chi(X)^{c(X)}$ for X regular; $|X| \leqslant \pi\chi(X)^{c(X)\cdot\psi(X)}$ for $X \in \mathcal{T}_3$. The inequality $w(X) \leqslant \pi\chi(X)^{c(X)}$, due to ŠAPIROVSKIĬ [1974], improves the result $w(X) \leqslant 2^{d(X)}$, since $\pi\chi(X)^{c(X)} \leqslant w(X)^{c(X)} \leqslant (2^{d(X)})^{c(X)} = 2^{d(X)}$. A consequence of the inequality $|X| \leqslant \pi\chi(X)^{c(X)\cdot\psi(X)}$ is $|X| \leqslant \exp(c(X)\cdot\pi\chi(X)\cdot\psi(X))$ for $X \in \mathcal{T}_3$, a result which should be compared to $|X| \leqslant \exp(c(X)\cdot\chi(X))$ for $X \in \mathcal{T}_2$. Pseudo-character cannot be omitted here; for example, $\pi w(\beta\omega) = \omega$ but $|\beta\omega| = 2^{2^\omega}$.

6.1. THEOREM (Efimov). *For any space X, $|\mathrm{RO}(X)| \leqslant \pi w(X)^{c(X)}$.*

PROOF. Let $c(X) = \kappa$, let $\mathcal{V}$ be a π-base for X with $|\mathcal{V}| \leqslant \pi w(X)$. Let $\mathcal{G} = \{\bar{G}^\circ : G$ is the union of $\leqslant\kappa$ elements of $\mathcal{V}\}$. We prove $\mathrm{RO}(X) \subseteq \mathcal{G}$, from which $|\mathrm{RO}(X)| \leqslant \pi w(X)^{c(X)}$ follows. Let R be an open set in X with $\bar{R}^\circ = R$. Let $\mathcal{V}_R = \{V : V \in \mathcal{V}, V \subseteq R\}$. By Proposition 3.4, there is a subcollection $\mathcal{W}$ of $\mathcal{V}_R$ with $|\mathcal{W}| \leqslant \kappa$ and $\bigcup \mathcal{V}_R \subseteq \overline{\bigcup \mathcal{W}}$. Let $G = \bigcup \mathcal{W}$. Using the fact that R is open and $\mathcal{V}$ is a π-base for X, one can show that $R \subseteq \overline{\bigcup \mathcal{V}_R}$. From this it easily follows that $\bar{R} = \bar{G}$, hence $R = \bar{G}^\circ$.

6.2. THEOREM. *For X regular, $w(X) \leqslant |\mathrm{RO}(X)| \leqslant \pi\chi(X)^{c(X)}$.*

PROOF. It suffices to prove $|\mathrm{RO}(X)| \leqslant \pi\chi(X)^{c(X)}$. Assume, for a moment, that $d(X) \leqslant \pi\chi(X)^{c(X)}$ for X regular. Then

$$|\mathrm{RO}(X)| \leqslant \pi w(X)^{c(X)} = (\pi\chi(X) \cdot d(X))^{c(X)} \leqslant \pi\chi(X)^{c(X)}.$$

It remains to prove $d(X) \leqslant \pi\chi(X)^{c(X)}$. The proof we give is taken from CHARLESWORTH [1977]. Let $c(X) = \kappa$, and for each $p \in X$ let $\mathcal{V}_p$ be a local π-base for p with $|\mathcal{V}_p| \leqslant \pi\chi(X)$. Construct a sequence $\{A_\alpha : 0 \leqslant \alpha < \kappa^+\}$ of subsets of X and a sequence $\{\mathcal{V}_\alpha : 0 < \alpha < \kappa^+\}$ of open collections in X such that

(1) $|A_\alpha| \leqslant \pi\chi(X)^\kappa$, $0 \leqslant \alpha < \kappa^+$;

(2) $\mathcal{V}_\alpha = \{V : V \in \mathcal{V}_p, p \in \bigcup_{\beta<\alpha} A_\beta\}$, $0 < \alpha < \kappa^+$;

(3) if W is the union of $\leqslant\kappa$ elements of $\mathcal{V}_\alpha$, and $\bar{W} \neq X$, then $A_\alpha - \bar{W} \neq \emptyset$.

Let $S = \bigcup_{\alpha<\kappa^+} A_\alpha$; then $|S| \leqslant \pi\chi(X)^\kappa$, so the proof is complete if $\bar{S} = X$. Suppose not, and let R be a non-empty open set in X with $\bar{R} \cap S = \emptyset$. Let $\mathcal{G} = \{V : V \in \mathcal{V}_p, p \in S, V \cap R = \emptyset\}$, and let $G = \bigcup \mathcal{G}$. One can check that $S \subseteq \bar{G}$ and $\bar{G} \cap R = \emptyset$. By Proposition 3.4, there is a subcollection $\mathcal{W}$ of $\mathcal{G}$ with $|\mathcal{W}| \leqslant \kappa$ such that $G \subseteq \overline{\bigcup \mathcal{W}}$. Let $W = \bigcup \mathcal{W}$, and note that $S \subseteq \bar{W}$ and $\bar{W} \cap R = \emptyset$. Now choose $\alpha < \kappa^+$ such that $\mathcal{W} \subseteq \mathcal{V}_\alpha$. By (3), $A_\alpha - \bar{W} \neq \emptyset$, a contradiction of $S \subseteq \bar{W}$.

6.3. THEOREM (Šapirovskiĭ). *For $X \in \mathcal{T}_3$, $|X| \leqslant \pi\chi(X)^{c(X)\cdot\psi(X)}$.*

PROOF. $|X| \leqslant nw(X)^{\psi(X)} \leqslant w(X)^{\psi(X)} \leqslant \pi\chi(X)^{c(X)\cdot\psi(X)}$.

6.4. COROLLARY. *For $X \in \mathcal{T}_3$, $|X| \leqslant 2^{c(X)\cdot\pi\chi(X)\cdot\psi(X)}$. In particular, every T_3* ccc *space with countable pseudo-character and countable π-character has cardinality at most 2^ω.*

7. Cardinal functions on compact spaces

Most topologists agree that compact spaces and metrizable spaces are pre-eminent among abstract topological spaces. Accordingly, it is natural to examine cardinal functions on these two special classes of topological spaces. This section is devoted to compact spaces. (Recall that all compact spaces are assumed Hausdorff.)

The first two results, due to Alexandroff, state that $\chi(X) = \psi(X)$ and $w(X) \leqslant |X|$ for X compact. Note that the second follows easily from the first. The third theorem of this section is $|X| \leqslant 2^{\psi(X)}$ for X compact, an easy consequence of $\chi(X) = \psi(X)$ and Arhangel'skiĭ's famous inequality.

7.1. THEOREM. *For X compact, $\psi(X) = \chi(X)$. In particular, every compact space with countable pseudo-character is first countable.*

7.2. THEOREM. *For X compact, $w(X) \leq |X|$. In particular, every compact, countable space has a countable base and hence is metrizable.*

7.3. THEOREM. *For X compact, $|X| \leq 2^{\psi(X)}$. In particular, every compact space with countable pseudo-character has cardinality at most 2^{ω}.*

The next result states that $psw(X) = nw(X) = w(X)$ for X compact. The countable version of the equality $psw(X) = w(X)$, due to Arhangel'skiĭ and Proizvolov, states that a compact space with a point-countable separating open cover has a countable base and hence is metrizable. This result simultaneously generalizes a theorem of Miščenko (every compact space with a point-countable base has a countable base) and Šneĭder (every compact space with a G_δ-diagonal has a countable base). It should be noted that Nagata proved that a Hausdorff space is metrizable if and only if it is a paracompact p-space with a point-countable separating open cover.

The equality $nw(X) = w(X)$ for X compact, due to ARHANGEL'SKIĬ [1959], has an interesting corollary. *If a compact space X is the union of $\leq\kappa$ sets, each having weight $\leq\kappa$, then X itself has weight $\leq\kappa$.* This result is called the Weight Addition Theorem, and was used by Arhangel'skiĭ to solve the following problem. If X is a compact space which is the union of a countable number of sets, each having a countable base, does X itself have a countable base? Arhangel'skiĭ's solution to this problem classifies as a mathematical gem and beautifully illustrates the principle that selecting the correct definition (in this case generalizing 'base' to 'net') is often the key to solving a mathematical problem.

7.4. THEOREM. *For X compact, $psw(X) = nw(X) = w(X)$.*

PROOF. To prove $w(X) \leq nw(X)$, let $nw(X) = \kappa$ and let $\mathcal{N}$ be a net for X with $|\mathcal{N}| \leq \kappa$. The collection $\mathcal{N}$ can be used to construct an open collection $\mathcal{V}$ in X with $|\mathcal{V}| \leq \kappa$ and having this property:

(*) if p and q are distinct points of X, then there exists $V_p, V_q \in \mathcal{V}$ such that $p \in V_p$, $q \in V_q$, and $V_p \cap V_q = \emptyset$.

The collection $\mathcal{V}$ is obtained as follows. For each pair of sets N_1, N_2 in $\mathcal{N}$ with $N_1 \cap N_2 = \emptyset$, pick, if possible, disjoint open sets V_1 and V_2 with $N_i \subseteq V_i$ for $i = 1, 2$. The collection of all open sets chosen in this way has cardinality $\leq\kappa$ and satisfies (*). Now let $\mathcal{B}$ be the collection of all finite intersections of elements of $\mathcal{V}$. Using the compactness of X, one can show that $\mathcal{B}$ is a base for X. Since $|\mathcal{B}| \leq \kappa$, $w(X) \leq \kappa$.

It remains to prove $psw(X) = nw(X)$. Now $psw(X) \leq nw(X)$ holds for any Hausdorff space X. In fact, the argument in the preceding paragraph shows that any Hausdorff space X has a separating open cover $\mathcal{V}$ with $|\mathcal{V}| \leq nw(X)$. It remains to prove $nw(X) \leq psw(X)$ for X compact. Let $psw(X) = \kappa$, and let $\mathcal{V}$ be a separating open cover of X with $\mathrm{ord}(p, \mathscr{V}) \leqslant \kappa$ for all $p \in X$. The compactness of X implies that the cardinality of $\mathscr{V}$ is $\leqslant\kappa$! To see this, let $\{\mathscr{V}_\alpha : 0 \leqslant \alpha < \kappa\}$ be all finite minimal covers of X by elements of $\mathscr{V}$ (use Miščenko's lemma). Then $\mathscr{V} \subseteq \bigcup_{\alpha<\kappa} \mathscr{V}_\alpha$. Indeed, let $V_0 \in \mathscr{V}$ and let $p \in V_0$. For each $q \neq p$, choose an element of $\mathscr{V}$ which contains q but not p. This collection of sets, together with V_0, covers X, so there is a finite minimal subcollection which covers X, say $\mathscr{V}_\alpha$. Clearly $V_0 \in \mathscr{V}_\alpha$. Now let $\mathscr{N}$ be the collection of all sets in X of the form $X - W$, where W is the union of finitely many elements of $\mathscr{V}$. Then $\mathscr{N}$ is a net for X and $|\mathscr{N}| \leqslant \kappa$.

7.5. COROLLARY. *Every compact space with a point-countable separating open cover has a countable base and hence is metrizable.*

7.6. COROLLARY. *For X compact, $\Delta(X) = w(X)$. In particular, every compact space with a G_δ-diagonal has a countable base and hence is metrizable.*

PROOF. It is not difficult to see that $\Delta(X) \leqslant nw(X)$ for any regular space X. In addition, $psw(X) \leqslant L(X) \cdot \Delta(X)$ holds for any space X.

7.7. COROLLARY. *For X compact, $w(X) \leqslant |\mathrm{RO}(X)| \leqslant 2^{s(X)}$.*

PROOF. $|\mathrm{RO}(X)| \leqslant \pi w(X)^{c(X)} \leqslant w(X)^{c(X)} = nw(X)^{c(X)} \leqslant (2^{s(X)})^{c(X)} = 2^{s(X)}$.

Most of the local cardinal functions defined in Section 3 are monotone; the one exception is π-character. Similarly, one expects a cardinal function defined in terms of a base-like condition to be monotone, but π-weight is an exception. Thus there are two additional cardinal functions which have not yet been investigated, namely $h\pi\chi$ and $h\pi w$. ŠAPIROVSKIĬ [1975] has proved that $h\pi\chi(X) = t(X)$ and $h\pi w(X) = hd(X)$ for X compact. (The second equality is an easy consequence of the first.) Now $t(X) \leqslant h\pi\chi(X)$ holds for any space X, so the emphasis is on reversing this inequality for X compact. The key idea is that of a free sequence. By definition, a sequence $\{x_\alpha : 0 \leqslant \alpha < \kappa\}$ in X is a *free sequence of length* κ if for all $\beta < \kappa$, $\{x_\alpha : \alpha < \beta\}^- \cap \{x_\alpha : \alpha \geqslant \beta\}^- = \emptyset$. Note that a free sequence in X is always a discrete subset of X. We begin with some fundamental facts about free sequences in compact spaces.

7.8. THEOREM. *If X is compact and $t(X) \leqslant \kappa$, then X does not have a free sequence of length κ^+.*

PROOF. The compactness of X guarantees that every infinite subset of X has a complete accumulation point.

7.9. LEMMA. *Let $X \in \mathscr{T}_3$, let K be a compact subset of X, let $p \in X$, $p \notin K$. Then there exist closed G_δ-sets A and B in X such that $p \in A$, $K \subseteq B$, and $A \cap B = \emptyset$.*

PROOF. Use the fact that if $\{V_n : n \in \omega\}$ is a sequence of open sets in X with $\bar{V}_{n+1} \subseteq V_n$ for all $n \in \omega$, then $\bigcap_{n<\omega} \bar{V}_n = \bigcap_{n<\omega} V_n$ is a closed G_δ-set in X.

7.10. THEOREM. *If X is compact and $h\pi\chi(X) > \kappa$, then X has a free sequence of length κ^+.*

PROOF. The proof is taken from ARHANGEL'SKIĬ's paper [1978]. Since $\pi\chi(Y) \leqslant \pi\chi(\bar{Y})$ (see 3.9(b)), it suffices to prove the result assuming $\pi\chi(X) > \kappa$. Let $p \in X$ be such that $\pi\chi(p, X) \geqslant \kappa^+$. Let $\mathscr{G}$ be the collection of all non-empty closed G_δ-sets in X. Since $\pi\chi(p, X) > \kappa$ and X is compact, the collection $\mathscr{G}$ has this property:

(*) if $\mathscr{H} \subseteq \mathscr{G}$ and $|\mathscr{H}| \leqslant \kappa$, then there is an open neighborhood R of p such that $H - R \neq \emptyset$ for all $H \in \mathscr{H}$.

Construct subcollections $\{A_\alpha : 0 \leqslant \alpha < \kappa^+\}$ and $\{B_\alpha : 0 \leqslant \alpha < \kappa^+\}$ of $\mathscr{G}$ such that

(1) $p \in A_\alpha$ and $A_\alpha \cap B_\alpha = \emptyset$, $0 \leqslant \alpha < \kappa^+$;

(2) if H is a non-empty finite intersection of $\{A_\beta : 0 \leqslant \beta < \alpha\} \cup \{B_\beta : 0 \leqslant \beta < \alpha\}$, then $H \cap B_\alpha \neq \emptyset$, $0 < \alpha < \kappa^+$. The construction is by transfinite induction. To obtain A_0, B_0, use Lemma 7.9. Now let α be fixed, $0 < \alpha < \kappa^+$, and assume that $\{A_\beta : \beta < \alpha\}$ and $\{B_\beta : \beta < \alpha\}$ have been constructed. Let $\mathscr{H}$ be the collection of all non-empty finite intersections of elements of $\{A_\beta : \beta < \alpha\} \cup \{B_\beta : \beta < \alpha\}$. Then $\mathscr{H} \subseteq \mathscr{G}$ and $|\mathscr{H}| \leqslant \kappa$, so by (*) there is an open neighborhood R of p such that $H - R \neq \emptyset$ for all $H \in \mathscr{H}$. By Lemma 7.9, there exist A_α, B_α in $\mathscr{G}$ such that $p \in A_\alpha$, $(X - R) \subseteq B_\alpha$, and $A_\alpha \cap B_\alpha = \emptyset$. Now $H - R \neq \emptyset$ implies $H \cap B_\alpha \neq \emptyset$, so both (1) and (2) are now satisfied.

Now let α be fixed, $0 \leqslant \alpha < \kappa^+$. Using the principle of finite induction, one can show that any finite intersection of elements of $\{A_\beta : \beta \leqslant \alpha\} \cup \{B_\beta : \beta > \alpha\}$ is non-empty. (In proving this, use the first part of (1) and (2).) By the compactness of X, there exists $x_\alpha \in (\bigcap_{\beta \leqslant \alpha} A_\beta) \cap (\bigcap_{\beta > \alpha} B_\beta)$. Then $\{x_\alpha : 0 \leqslant \alpha < \kappa^+\}$ is a free sequence in X of length κ^+, since for any $\beta < \kappa^+$ one has $\{x_\alpha : \alpha < \beta\}^- \subseteq B_\beta$, $\{x_\alpha : \alpha \geqslant \beta\}^- \subseteq A_\beta$, and $A_\beta \cap B_\beta = \emptyset$.

7.11. COROLLARY. *If X is compact and $t(X) > \kappa$, then X has a free sequence of length κ^+.*

The results in 7.8 and 7.11 can be summarized as follows.

7.12. THEOREM (ARHANGEL'SKIĬ [1971]). *For X compact, $t(X) = F(X)$, where $F(X) = \sup\{\lambda : X$ has a free sequence of length $\lambda\} + \omega$.*

7.13. THEOREM (Šapirovskiĭ). *For X compact, $h\pi\chi(X) = t(X)$. In particular, every subspace of a compact space with countable tightness has countable π-character.*

PROOF. For any space X one has $t(X) \leqslant h\pi\chi(X)$. Let $t(X) = \kappa$. If $h\pi\chi(X) > \kappa$, then by 7.10 there is a free sequence in X of length κ^+. This contradicts 7.8.

7.14. THEOREM. *For X compact, $h\pi w(X) = hd(X)$. In particular, every subspace of a compact, hereditarily separable space has a countable π-base.*

PROOF. For any space Y one has $\pi w(Y) = \pi\chi(Y) \cdot d(Y)$. Hence $h\pi w(X) = h\pi\chi(X) \cdot hd(X) = t(X) \cdot hd(X) = hd(X)$.

One has $t(X) \leq hd(X)$ for any space X. But for compact spaces, this inequality can be improved.

7.15. THEOREM (Arhangel'skiĭ). *For X compact, $t(X) \leq s(X)$. In particular, every compact space with countable spread has countable tightness.*

PROOF. Let $s(X) = \kappa$. If $t(X) > \kappa$, then by 7.11 X has a free sequence of length κ^+. But a free sequence is always a discrete subset, and this contradicts $s(X) = \kappa$.

The inequality $hd(X) \leq \exp s(X)$ for $X \in \mathcal{T}_2$ was proved in Section 5. We now prove Šapirovskiĭ's theorem that $hd(X) \leq s(X)^+$ for X compact. First we need a lemma.

7.16. LEMMA. *Let $X \in \mathcal{T}_3$, let $A \subseteq Y \subseteq X$ with $hL(Y) \leq \kappa$. Then there is an open collection $\mathcal{V}$ in X with $|\mathcal{V}| \leq \kappa$ such that $\bar{A} \subseteq \bigcap \mathcal{V}$ and $Y \cap \bar{A} = Y \cap (\bigcap \mathcal{V})$.*

PROOF. Let $Z = Y - \bar{A}$. For each $x \in Z$ let R_x be an open neighborhood of x such that $\bar{R}_x \cap \bar{A} = \emptyset$. Since $\{R_x : x \in Z\}$ covers Z and $hL(Y) \leq \kappa$, there is a subset E of Z with $|E| \leq \kappa$ such that $\{R_x : x \in E\}$ covers Z. Take $V = \{X - \bar{R}_x : x \in E\}$.

7.17. THEOREM (ŠAPIROVSKIĬ [1974]). *For X compact, $hd(X) \leq s(X)^+$.*

PROOF. Let $s(X) = \kappa$, and note that $t(X) \leq \kappa$ by 7.15. Let $Z \subseteq X$. Since $d(Z) \leq d(\bar{Z}) \cdot t(\bar{Z})$ (see 3.9(a)), it suffices to prove $d(\bar{Z}) \leq \kappa^+$. Now $s(\bar{Z}) \leq \kappa$, so by 5.6 there is a dense subset Y of $\bar{Z}$ such that $hL(Y) \leq \kappa$. Since Y is dense in $\bar{Z}$, $d(\bar{Z}) \leq d(Y)$. In summary, the proof is complete if we can prove this assertion: if X is compact, $t(X) \leq \kappa$, $Y \subseteq X$, and $hL(Y) \leq \kappa$, then $d(Y) \leq \kappa^+$.

The proof that $d(Y) \leq \kappa^+$ is reminiscent of the Pol–Šapirovskiĭ technique already used in Sections 4, 5, 6. Construct an increasing sequence $\{A_\alpha : 0 \leq \alpha < \kappa^+\}$ of subsets of Y an a sequence $\{\mathcal{V}_\alpha : 0 \leq \alpha < \kappa^+\}$ of open collections in X such that for $0 \leq \alpha < \kappa^+$:

(1) $|A_\alpha| \leq \kappa$ and $|\mathcal{V}_\alpha| \leq \kappa$;

(2) $\bar{A}_\alpha \subseteq \bigcap \mathcal{V}_\alpha$ and $Y \cap \bar{A}_\alpha = Y \cap (\bigcap \mathcal{V}_\alpha)$;

(3) if G is a finite union of elements of $\bigcup_{\beta<\alpha} \mathcal{V}_\beta$ and $Y - G \neq \emptyset$, then $A_\alpha \cap (Y - G) \neq \emptyset$.

Let $S = \bigcup_{\alpha<\kappa^+} A_\alpha$, and note that $|S| \leq \kappa^+$. The proof is complete if $Y \subseteq \bar{S}$. Now $t(X) \leq \kappa$ and $\{\bar{A}_\alpha : 0 \leq \alpha < \kappa^+\}$ is an increasing sequence of closed sets in X, so $\bigcup_{\alpha<\kappa^+} \bar{A}$ is closed. It follows that $\bar{S} = \bigcup_{\alpha<\kappa^+} \bar{A}_\alpha$. Suppose $Y \subseteq \bar{S}$ is false, and let

$q \in (Y - \bar{S})$. For each $\alpha < \kappa^+$, $q \notin \bar{A}_\alpha$, so by the second part of (2), there exists $V_\alpha \in \mathcal{V}_\alpha$ such that $q \notin V_\alpha$. By the first part of (2), $\{V_\alpha : 0 \leq \alpha < \kappa^+\}$ covers $\bar{S}$, so by compactness there exist $\beta_1 < \cdots < \beta_n$ such that $\{V_{\beta_1}, \ldots, V_{\beta_n}\}$ covers $\bar{S}$. Let $G = \bigcup_{i=1}^n V_{\beta_i}$, let $\alpha > \beta_n$. Then (3) is contradicted.

7.18. COROLLARY. *Every compact, perfectly normal space has a π-base of cardinality at most ω_1.*

Many of the inequalities obtained so far in this section hold for spaces more general than compact spaces. For example, if X is a Hausdorff space of point-countable type, one has $\psi(X) = \chi(X)$, $w(X) \leq |X|$, $h\pi\chi(X) = t(X)$, $h\pi w(X) = hd(X)$, $t(X) \leq s(X)$, and $|\mathrm{RO}(X)| \leq 2^{s(X)}$. (To prove the last result, note that $|\mathrm{RO}(X)| \leq \pi\chi(X)^{c(X)} \leq 2^{s(X)}$. Recall that a space X is of *point-countable type* if for each point $p \in X$, there is a compact set K such that $p \in K$ and K has countable character. The following lemma is useful in extending certain results from compact spaces to spaces of point-countable type. *If X is a Hausdorff space, K is a compact subset of X, and $p \in K$, then $\pi\chi(p, X) \leq \pi\chi(p, K) \cdot \chi(K, X)$.*

If X is a p-space, one has $nw(X) = w(X)$, and so the Weight Addition Theorem holds for p-spaces. (This result is proved in Gruenhage's paper in this Handbook.) The equality $psw(X) = w(X)$ holds for Lindelöf p-spaces. The countable version of this result states that a Lindelöf p-space with a point-countable separating open cover has a countable base. This is a special case of the metrization theorem of Nagata which states that a Hausdorff space is metrizable if and only if it is a paracompact p-space with a point-countable separating open cover. By definition, a *p-space* is a T_3 space X having a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers such that for each $p \in X$, if $p \in G_n \in \mathcal{G}_n$ for $n = 1, 2, \ldots$, then (a) $C_p = \bigcap_{n<\omega} \bar{G}_n$ is compact; (b) $\{\bigcap_{i \leq n} \bar{G}_i : n < \omega\}$ is a "base" for C_p.

One can even go further and introduce cardinal functions $p(X)$ and $h(X)$ which extend the concept of a p-space and a space of point-countable type to higher cardinality. One then has, for example, $psw(X) \cdot L(X) \cdot p(X) = w(X)$ for $X \in \mathcal{T}_3$, $\psi(X) \cdot h(X) = \chi(X)$ for $X \in \mathcal{T}_2$, and $w(X) \leq h(X) \cdot |X|$ for $X \in \mathcal{T}_2$.

The last major result in this section is the classical theorem of Čech and Pospíšil on the cardinality of compact spaces. This theorem, when combined with Arhangel'skiĭ's inequality $|X| \leq \exp(L(X) \cdot \chi(X))$, gives very precise information about the cardinality of compact, first-countable spaces.

7.19. THEOREM (Čech–Pospíšil). *Let X be a compact space such that $\chi(p, X) \geq \kappa$ for each $p \in X$. Then $|X| \geq 2^\kappa$.*

PROOF. The idea is to construct a tree with κ levels such that each point at level $\alpha < \kappa$ branches twice. Each path running through the tree determines a decreasing sequence of non-empty closed sets in X, which in turn determines a point of X. Distinct paths determine distinct points, and the number of paths is 2^κ. The countable version of this construction is the familiar Cantor tree. The details now follow.

Recall this notation: ${}^{\alpha}2$ is the collection of all functions from α into $\{0, 1\}$. For each $\alpha < \kappa$, construct a collection $\{K_f : f \in {}^{\alpha}2\}$ of non-empty closed sets in X such that

(1) if $f \in {}^{\alpha}2$ and $\beta < \alpha$, then $K_f \subseteq K_{(f|\beta)}$;

(2) If $f, g \in {}^{\alpha}2$ and $f \neq g$, then $K_f \cap K_g = \emptyset$.

Assume, for a moment, that the collection $\{K_f : f \in {}^{\alpha}2\}$ has been constructed for each $\alpha < \kappa$ so that (1) and (2) hold. For each $f \in {}^{\kappa}2$, let $K_f = \bigcap_{\alpha<\kappa} K_{(f|\alpha)}$. By (1), $\{K_{f(\alpha)} : \alpha < \kappa\}$ is a decreasing sequence of non-empty closed sets in X, so by compactness of X, $K_f \neq \emptyset$. By (2), if $f, g \in {}^{\kappa}2$ and $g \neq f$, then $K_f \cap K_g = \emptyset$. Since $|{}^{\kappa}2| = 2^{\kappa}$, one has $|X| \geq 2^{\kappa}$.

It remains to construct the collections $\{K_f : f \in {}^{\alpha}2\}$. There are two cases.

Case 1: $\kappa = \omega$. For this case add the following condition: (3) for each $f \in {}^{\alpha}2$, $(K_f)^{\circ} \neq \emptyset$. Now proceed by induction. For $\alpha = 0$, ${}^{\alpha}2 = \{\emptyset\}$ and $K_{\emptyset} = X$. Now let $0 < \alpha < \omega$, and assume that $\{K_f : f \in {}^{\beta}2\}$ has been constructed for each $\beta < \alpha$ so that (1)–(3) hold. Let $\alpha = \gamma + 1$, $g \in {}^{\gamma}2$, and let $f_0, f_1 \in {}^{\alpha}2$ be such that $(f_0 | \gamma) = (f_1 | \gamma) = g$ and $f_0(\gamma) = 0$, $f_1(\gamma) = 1$. We proceed to construct K_{f_0}, K_{f_1}. By condition (3), $(K_g)^{\circ} \neq \emptyset$, and since X has no isolated points, there exist non-empty open sets V_0 and V_1 such that $(\bar{V}_0 \cup \bar{V}_1) \subseteq K_g$ and $\bar{V}_0 \cap \bar{V}_1 = \emptyset$. Take $K_{f_i} = \bar{V}_i$, $i = 0, 1$.

Case 2: $\kappa > \omega$. For this case add the following condition: (3) for each $f \in {}^{\alpha}2$, K_f is the intersection of $\leq |\alpha| + \omega$ open sets. Let $0 < \alpha < \kappa$, and assume that $\{K_f : f \in {}^{\beta}2\}$ has been constructed for each $\beta < \alpha$ so that (1)–(3) hold. If α is a limit ordinal, take $K_f = \bigcap_{\beta<\alpha} K_{(f|\beta)}$ for each $f \in {}^{\alpha}2$. Suppose α is a successor ordinal, say $\alpha = \gamma + 1$. Let $g \in {}^{\gamma}2$, and let $f_0, f_1 \in {}^{\alpha}2$ be such that $(f_0 | \gamma) = (f_1 | \gamma) = g$ and $f_0(\gamma) = 0$, $f_1(\gamma) = 1$. We proceed to construct K_{f_0}, K_{f_1}. Now $\chi(p, X) \geq \kappa$ for each $p \in X$, and K_g is the intersection of $<\kappa$ open sets by (3), so K_g must have more than one point. Use Lemma 7.9 to obtain disjoint, closed G_{δ}-sets G_0 and G_1 such that $G_0 \cap K_g \neq \emptyset$ and $G_1 \cap K_g \neq \emptyset$. Take $K_{f_i} = G_i \cap K_g$, $i = 0, 1$.

7.20. THEOREM. *Every compact, first-countable space is countable or has cardinality* 2^{ω}.

PROOF. Assume X is uncountable. By Arhangel'skiĭ's theorem one has $|X| \leq 2^{\omega}$, so it suffices to prove $|X| \geq 2^{\omega}$. Let $A = \{p$: every open neighborhood of p has cardinality $>\omega\}$. Then A is a non-empty closed subset of X, and no point of A is an isolated point of A. (Suppose one has $p \in A$ and $\{p\} = V \cap A$, V open in X. Let $\{V_n : n \in \omega\}$ be a local base for p with $\bar{V}_0 \subseteq V$ and $V_{n+1} \subseteq V_n$ for all $n \in \omega$. Then $V_0 - \{p\} = \bigcup_{n<\omega}(V_n - V_{n+1})$, so for some $i < \omega$ one has $|V_i - V_{i+1}| > \omega$. Let q be a complete accumulation point of $(V_i - V_{i+1})$. Then q is a point of $V \cap A$ distinct from p.) By the Čech–Pospíšil theorem, $|A| \geq 2^{\omega}$, hence $|X| \geq 2^{\omega}$.

7.21. COROLLARY. *Every uncountable closed subset of* $\mathbb{R}$ *has cardinality* 2^{ω}.

PROOF. Every closed subset of $\mathbb{R}$ is the union of countably many compact subsets of $\mathbb{R}$.

7.22. EXAMPLE. *The following hold for* $\beta\omega$:

(a) $\phi(\beta\omega)=\omega$ *for* $\phi\in\{d, c, L, e, wc, \pi w, \pi\chi\}$;

(b) $\phi(\beta\omega)=2^{\omega}$ *for* $\phi\in\{s, hd, hL, nw, w, h\pi w, psw, \Delta, t, h\pi\chi, \psi, \chi\}$;

(c) $|\beta\omega|=o(\beta\omega)=2^{2^{\omega}}$;

(d) *every infinite closed set in* $\beta\omega$ *has cardinality* $2^{2^{\omega}}$.

PROOF. To prove (b), first note that $w(\beta\omega)\leq 2^{\omega}$ by the inequality $|X|\leq 2^{d(X)}$. Next we show that ω^* ($=\beta\omega-\omega$, the set of free ultrafilters on ω) has a cellular family of cardinality 2^{ω}. This implies $s(\beta\omega)\geq 2^{\omega}$, hence $s(\beta\omega)=w(\beta\omega)=2^{\omega}$. This notation is used: for $A\subseteq\omega$, $A^*=\{p: p\in\omega^*, A\in p\}$. Recall that $\{A^*: A\subseteq\omega\}$ is a base for ω^*. By Tarski's theorem, there is a collection $\mathscr{A}$ of subsets of ω such that $|\mathscr{A}|=2^{\omega}$, $|A|=\omega$ for each $A\in\mathscr{A}$, and $A_1\cap A_2$ is finite for $A_1\neq A_2$. The collection $\{A^*: A\in\mathscr{A}\}$ is a cellular family in ω^* of cardinality 2^{ω}.

The proof of (b) is complete if there is some $p\in\beta\omega$ with $t(p,\beta\omega)=2^{\omega}$. This is not easy! For an elegant proof of this fact, see van Mill's paper in this Handbook. (Specifically, see Lemma 3.3.4, where he proves the existence of an R-point in ω^*.) We shall be content to prove the easier result that there is some $p\in\beta\omega$ with $\chi(p,\beta\omega)=2^{\omega}$. It suffices to construct an ultrafilter p on ω which has no base of cardinality $<2^{\omega}$. Let $\mathscr{A}$ be an independent collection on ω with $|\mathscr{A}|=2^{\omega}$ (see 2.2). Let $\mathscr{L}=\{\omega-\bigcap\mathscr{B}: \mathscr{B}\subseteq\mathscr{A}, |\mathscr{B}|\geq\omega\}$. The collection $\mathscr{A}\cup\mathscr{L}$ has the finite intersection property. Let p be any ultrafilter on ω with $\mathscr{A}\cup\mathscr{L}\subseteq p$. Then p cannot have a base of cardinality $<2^{\omega}$. (This combinatorial proof is due to Hajnal–Juhász; the result is originally due to POSPÍŠIL [1939].)

The fact that $|\beta\omega|=2^{2^{\omega}}$ is proved in Section 2 using independent collections. Independent collections are now used to prove (d). It suffices to prove that $|\bar{E}|\geq 2^{2^{\omega}}$ for every countably infinite discrete subset E of $\beta\omega$. Note that if $A\subseteq E$, then $\bar{A}\cap(E-A)=A\cap(E-A)^{-}=\emptyset$, hence $\bar{A}\cap(E-A)^{-}=\emptyset$. (Every countable subset of $\beta\omega$ is C^*-embedded.) Let $\mathscr{A}=\{A_{\alpha}: 0\leq\alpha<2^{\omega}\}$ be an independent collection in E. For each $f\in{}^{2^{\omega}}2$, let $\mathscr{A}_f=\{A(f,\alpha): 0\leq\alpha<2^{\omega}\}$, where $A(f,\alpha)=A_{\alpha}$ if $f(\alpha)=0$ and $A(f,\alpha)=E-A_{\alpha}$ otherwise. Since $\mathscr{A}_f$ has the finite intersection property and $\beta\omega$ is compact, there exists $x_f\in\bigcap_{\alpha<2^{\omega}}\overline{A(f,\alpha)}$. If $f\neq g$, say $A(f,\alpha)=A_{\alpha}$ and $A(g,\alpha)=E-A_{\alpha}$, then $\overline{A(f,\alpha)}\cap\overline{A(g,\alpha)}=\emptyset$, hence $x_f\neq x_g$. Thus $\{x_f: f\in{}^{2^{\omega}}2\}$ is a collection of $2^{2^{\omega}}$ distinct points in $\bar{E}$.

8. Cardinal functions on metrizable spaces

The fundamental facts about cardinal functions on metrizable spaces are summarized in the first theorem of this section. Since metrizable spaces are so nice, inequalities among cardinal functions tend to be dull.

8.1. THEOREM. *Let* X *be an infinite metrizable space.*

(a) $w(X)\leq|X|\leq w(X)^{\omega}$;

(b) $\psi(X)=t(X)=\pi\chi(X)=\chi(X)=psw(X)=\Delta(X)=\omega$;

(c) $w(X) = nw(X) = \pi w(X) = hd(X) = hL(X) = s(X) = L(X) = d(X) = c(X) = e(X) = wc(X)$;

(d) *X has a cellular family of cardinality $c(X)$*;

(e) $o(X) = 2^{w(X)}$.

PROOF. Let $\mathscr{B} = \bigcup_{n<\omega} \mathscr{B}_n$ be a σ-discrete base for X. To prove (c), it suffices to show that $w(X) \leqslant c(X)$, $w(X) \leqslant e(X)$, and $d(X) \leqslant wc(X)$. Let $c(X) = \kappa$. Then $|\mathscr{B}_n| \leqslant \kappa$ for each $n \in \omega$, hence $|\mathscr{B}| \leqslant \kappa$, so $w(X) \leqslant \kappa$. Essentially the same argument shows that $w(X) = e(X)$. The proof that $d(X) \leqslant wc(X)$ is left to the reader.

Part (d) states that cellularity is achieved for metrizable spaces. This general problem is discussed in Section 12, where the following result is proved. *For any space X, if $c(X)$ is a singular cardinal, then X has a cellular family of cardinality* $c(x)$. We assume this result to prove (d). Let $c(X) = \kappa$, and recall that $w(X) = \kappa$ as well. Assume $\kappa > \omega$. (One can prove that any infinite Hausdorff space has a cellular family of cardinality ω.) First suppose $cf(\kappa) > \omega$. Then for some $n < \omega$, $|\mathscr{B}_n| = \kappa$, and $\mathscr{B}_n$ is desired cellular family. If $cf(\kappa) = \omega$, then κ is singular and the theorem from Section 12 applies.

To prove (e), first recall that $o(X) \leqslant 2^{w(X)}$ always holds. To establish equality, use (d) to obtain a cellular family $\mathscr{V}$ with $|\mathscr{V}| = w(X)$. By taking all possible unions of elements of $\mathscr{V}$, one obtains $2^{w(X)}$ open sets in X.

The next result is an analogue of the Čech–Pospíšil theorem for completely metrizable spaces. This theorem allows one to obtain precise information about the cardinality of completely metrizable spaces.

8.2. THEOREM (F.K. Schmidt; A.H. Stone). *Let X be a completely metrizable space such that every non-empty open set is infinite and has weight $\geqslant \kappa$. Then $|X| \geqslant \kappa^{\omega}$.*

PROOF. The idea is to construct a tree with ω levels such that each point at level $n < \omega$ branches κ times. Each path running through the tree determines a decreasing sequence of non-empty closed sets in X, which in turn determines a point of X. Distinct paths determine distinct points, and the number of paths is κ^{ω}. The details follow.

Let d be a complete metric for X. For each $n < \omega$, construct a collection $\{V_f : f \in {}^n\kappa\}$ of non-empty open sets in X such that

(1) for all $f \in {}^n\kappa$, diameter $(V_f) \leqslant 1/n$;

(2) if f, g are distinct elements of ${}^n\kappa$, then $\bar{V}_f \cap \bar{V}_g = \emptyset$;

(3) if $f \in {}^n\kappa$ and $i < n$, then $\bar{V}_f \subseteq V_{(f|i)}$.

Assume, for a moment, that the collection $\{V_f : f \in {}^n\kappa\}$ has been constructed for each $n < \omega$. For each $f \in {}^\omega\kappa$, let $K_f = \bigcap_{n<\omega} \bar{V}_{(f|n)}$. By (1) and (3), $\{\bar{V}_{(f|n)} : n \in \omega\}$ is a decreasing sequence of non-empty closed sets with diameters approaching zero, so by completeness $K_f \neq \emptyset$. By (2), if $f, g \in {}^\omega\kappa$ and $g \neq f$, then $K_f \cap K_g = \emptyset$. Hence $|X| \geqslant \kappa^{\omega}$.

It remains to construct the collections $\{V_f : f \in {}^n\kappa\}$. Let n be fixed, $0 < n < \omega$, and assume that the collection $\{V_f : f \in {}^i\kappa\}$ has been constructed for each $i < n$ so that (1)–(3) hold. Let $g \in {}^{n-1}\kappa$, and for each $\alpha < \kappa$ let $f_\alpha \in {}^n\kappa$ be such that $(f_\alpha \mid n-1) = g$ and $f_\alpha(n-1) = \alpha$. We proceed to construct $\{V_{f\alpha} : 0 \leqslant \alpha \leqslant \kappa\}$. By hypothesis, $w(V_g) \geqslant \kappa$, so V_g has a cellular family of cardinality κ, say $\{V_\alpha : 0 \leqslant \alpha < \kappa\}$. One may assume that $\bar{V}_\alpha \cap \bar{V}_\beta = \emptyset$ for $\alpha \neq \beta$, that each set has diameter $\leqslant 1/n$, and that $\bar{V}_\alpha \subseteq V_g$ for each $\alpha < \kappa$. Take $V_{f_\alpha} = V_\alpha$ for each $\alpha < \kappa$.

8.3. THEOREM (A.H. Stone). *Let X be a completely metrizable space with weight κ. Then $|X| = \kappa$ or $|X| = \kappa^\omega$.*

PROOF. Clearly $w(X) \leqslant |X| \leqslant \kappa^\omega$. Suppose $|X| > \kappa$, which means $\kappa^\omega > \kappa$. Let $A = \{p$: every open neighborhood of p has cardinality $>\kappa\}$. Note that A is closed and that every non-empty open subset of A has cardinality $>\kappa$. Let λ be the smallest cardinal such that $\lambda \leqslant \kappa$ and $\lambda^\omega > \kappa$. Then every non-empty open set in A has weight $\geqslant \lambda$. (If V is a non-empty open set in A with $w(V) < \lambda$, then $|V| \leqslant w(V)^\omega \leqslant \kappa$, a contradiction.) By theorem 8.2, $|A| \geqslant \lambda^\omega \geqslant \kappa^\omega$, so $|X| \geqslant \kappa^\omega$.

8.4. COROLLARY. *Every uncountable closed subset of $\mathbb{R}$ has cardinality 2^ω.*

The last theorem of this section is an interesting enumeration theorem of A.H. Stone. Consider the following problem. Let X be a topological space, let κ be an infinite cardinal with $\kappa < |X|$. Does X have a closed set of cardinality κ? Not necessarily, even if X is compact. For example, $\beta\omega$ is a compact space such that every infinite closed set has cardinality 2^{2^ω}. In particular, $\beta\omega$ has no closed sets of cardinality ω and no closed sets of cardinality 2^ω. The situation with regard to metrizable spaces is quite different, as the theorem of Stone indicates. (See Section 13 for a further discussion of problems of this type.)

8.5. THEOREM (A.H. Stone). *Let X be an infinite metrizable space with weight κ.*

(a) *The number of closed sets in X of cardinality $|X|$ is 2^κ.*

(b) *For $\omega \leqslant \lambda \leqslant \kappa$, the number of closed sets in X of cardinality λ is the maximum possible, namely $|X|^\lambda$ $(= \kappa^\lambda)$.*

(c) *For $\kappa < \lambda < |X|$, the number of closed sets in X of cardinality λ is 0 or 2^κ. Moreover, if X is completely metrizable, the number is 0.*

PROOF. Assume, for a moment, that the following two results hold.

I. If Y is a topological space with a cellular family of cardinality κ, then Y has at least 2^κ distinct closed sets, each of cardinality $|Y|$.

II. If Y is a metrizable space of weight κ, then Y has a discrete subset of cardinality κ with at most one limit point.

Stone's theorem follows from I and II. By (d) of theorem 8.1, X has a cellular family of cardinality κ. By I, X has 2^κ distinct closed sets of cardinality $|X|$. This proves (a). To prove (b), let $\omega \leqslant \lambda \leqslant \kappa$. By II, there is a discrete subset D of X of cardinality κ which has at most one limit point. By taking subsets of D of cardinality λ, one can construct κ^λ distinct closed sets in X, each of cardinality λ. But $\kappa^\lambda \leqslant |X|^\lambda \leqslant (\kappa^\omega)^\lambda = \kappa^\lambda$, i.e., $\kappa^\lambda = |X|^\lambda$. This proves (b). To prove (c), let $\kappa < \lambda < \kappa^\omega$, and suppose X has a closed set H with $|H| = \lambda$. Now $w(H) = \kappa$ or $w(X - H) = \kappa$ (use the fact that weight and net weight are the same for metrizable spaces). First suppose $w(H) = \kappa$. Then H has a cellular family of cardinality κ. By I, H has 2^κ distinct closed sets, each of cardinality $|H|$. Since H is closed, there are 2^κ distinct closed sets in X, each of cardinality λ. Next suppose $w(X - H) = \kappa$. By II, there is a discrete subset D of $(X - H)$ of cardinality κ having at most one limit point in $(X - H)$. Then $\{H \cup \bar{A}: A \subseteq D\}$ is a collection of 2^κ distinct closed sets, each of cardinality λ.

Proof of I. Let $|Y| = \lambda$, let $\{V_t: t \in T\}$ be a cellular family in Y with $|T| = \kappa$. Let $T = T_1 \cup T_2$, where $|T_1| = |T_2| = \kappa$ and $T_1 \cap T_2 = \emptyset$, and let $V = \bigcup_{t \in T_1} V_t$. First suppose that $|V| = \lambda$. For each $t \in T_2$ let $x_t \in V_t$, and for each $A \subseteq T_2$ let $H(A) = \bar{V} \cup \{x_t: t \in A\}^-$. Then $\{H(A): A \subseteq T_2\}$ is a collection of 2^κ closed sets in Y, each of cardinality λ. Next assume that $|V| < \lambda$. For each $A \subseteq T_1$ let $H(A) = Y - \bigcup_{t \in A} V_t$. Then $\{H(A): A \subseteq T_1\}$ is a collection of 2^κ closed sets in Y, each of cardinality λ.

Proof of II. Let D be a discrete subset of Y of cardinality κ. The strategy in most cases is to find a subset H of D of cardinality κ which has no limit points. First assume that $cf(\kappa) \neq \omega$. For each $x \in D$, let V_x be an open neighborhood of x such that $V_x \cap D = \{x\}$, and let $V = \bigcup_{x \in D} V_x$. Since Y is metrizable, $V = \bigcup_{n<\omega} F_n$, where each F_n is closed. For each n let $H_n = F_n \cap D$; note that each H_n has no limit points. Since $D = \bigcup_{n<\omega} H_n$ and $cf(\kappa) \neq \omega$, some H_n has cardinality κ. Observe that this argument also establishes the following fact: *for any cardinal κ, if Y has a discrete subset A of cardinality $>\kappa$, then there is a subset B of A of cardinality $>\kappa$ which has no limit points.*

Now assume that $cf(\kappa) = \omega$. Let $\kappa = \sup\{\lambda_n: 0 \leqslant n < \omega\}$, where $\lambda_0 < \lambda_1 < \cdots < \kappa$. Consider two cases.

(a) *There is a cardinal $\tau < \kappa$ such that every point $x \in Y$ has an open neighborhood V_x such that $|V_x \cap D| \leqslant \tau$.* Let $\mathscr{W}$ be a point-finite open refinement of $\{V_x: x \in Y\}$. By Zorn's lemma, there is a subset H of D which is maximal with respect to the property that $q \notin \mathrm{st}(p, \mathscr{W})$ whenever p and q are distinct points of H. Note that H has no limit points. The maximal property of H implies that $D = \bigcup_{p \in H}[\mathrm{st}(p, \mathscr{W}) \cap D]$. Since $|D| = \kappa$ and $|\mathrm{st}(p, \mathscr{W}) \cap D| \leqslant \tau$ for each $p \in H$, the set H must have cardinality κ.

(b) *There is a sequence $\langle x_n \rangle$ in Y such that $|V \cap D| > \lambda_n$ whenever V is any open neighborhood of x_n.* First assume that $\langle x_n \rangle$ has no cluster points. Then there is a subsequence $\langle x_{n_i} \rangle$ of $\langle x_n \rangle$ and a discrete open collection $\{V_i: i < \omega\}$ such that $x_{n_i} \in V_i$ for all $i < \omega$. Since $(V_i \cap D)$ is a discrete subset of Y cardinality $>\lambda_{n_i} \geqslant \lambda_i$, there exists $B_i \subseteq (V_i \cap D)$ such that $|B_i| > \lambda_i$ and B_i has no limit points. Let

$H = \bigcup_{i<\omega} B_i$; clearly $|H| = \kappa$ and H has no limit points. Now assume that $\langle x_n \rangle$ has a cluster point p. Let $\langle x_{n_i} \rangle$ be a subsequence of $\langle x_n \rangle$ such that $d(p, x_{n_i}) < 1/i$ for all i. Let $V_i = B(x_{n_i}, 1/i)$. Then $|V_i \cap D| > \lambda_{n_i} \geq \lambda_i$, hence there exists $B_i \subseteq (V_i \cap D)$ such that $|B_i| > \lambda_i$, and B_i has no limit points. Let $H = \bigcup_{i<\omega} B_i$. Then $|H| = \kappa$, H is discrete ($H \subseteq D$), and H has at most one limit point, namely p.

9. Bounds on the number of compact sets in X; bounds using extent

The results in this section are largely motivated by two natural questions. (1) Let $K(X)$ denote the collection of all compact subsets of X. Are bounds for $|X|$ also bounds for $|K(X)|$? In particular, can $|X|$ be replaced by $|K(X)|$ in the inequalities in Section 4? (2) Can extent be combined with another cardinal function, perhaps character, to obtain bounds on $|X|$? For example, can $L(X)$ be replaced by $e(X)$ in Arhangel'skiĭ's theorem? The first systematic study of such problems is due to Burke and Hodel.

With regard to the first question, there is a compact, first countable, separable space in which the number of closed (hence compact) sets is 2^{2^ω}. This example (14.5) shows that the bounds for $|X|$ given by $nw(X)^{\psi(X)}$, $\exp(d(X) \cdot \psi(X))$, $d(X)^{\chi(X)}$, $\exp(L(X) \cdot \chi(X))$, and $\exp(c(X) \cdot \chi(X))$ are not bounds for $|K(X)|$. It is not known if $\exp(\psi(X) \cdot s(X))$ is a bound for $|K(X)|$, although it is consistent to assume so in the countable case. In Section 5 it was proved that $\exp \exp s(X)$ is a bound not only for $|X|$ but also for $o(X)$. Finally, it turns out that $\exp hL(X)$ is a bound not only for $|X|$ but also for $|K(X)|$.

9.1. THEOREM. *For $X \in \mathcal{T}_2$, $|K(X)| \leq 2^{hL(X)}$. In particular, every hereditarily Lindelöf Hausdorff space has at most 2^ω compact subsets.*

PROOF. Let $hL(X) = \kappa$. By Corollary 4.10, $|X| \leq 2^\kappa$. One can show that for each $p \in X$, there is a collection $\mathcal{V}_p$ of open neighborhoods of p such that $|\mathcal{V}_p| \leq \kappa$ and $\bigcap\{\bar{V}: V \in \mathcal{V}_p\} = \{p\}$. Let $\mathcal{V} = \{V: V \in \mathcal{V}_p, p \in X\}$, let $\mathcal{W}$ be all finite intersections of elements of $\mathcal{V}$, and let $\mathcal{G}$ be all unions of $\leq\kappa$ elements of $\mathcal{W}$. Since $|\mathcal{G}| \leq 2^\kappa$, it suffices to show that the complement of each element of $K(X)$ belongs to $\mathcal{G}$. Let K be a compact subset of X. For each $p \in (X - K)$, there exists $W_p \in \mathcal{W}$ such that $p \in W_p$ and $W_p \cap K = \emptyset$. This gives an open cover of $X - K$, and by $hL(X) = \kappa$ there is a subcollection of cardinality $\leq\kappa$ which covers $(X - K)$. The union of the elements of this subcollection is an element of $\mathcal{G}$, call it G, and clearly $G = X - K$.

Now consider the question of obtaining bounds for $|X|$ in terms of extent. The following example illustrates a difficulty here. Let κ be a cardinal with $cf(\kappa) > \omega$, and assume κ has the usual order topology. Let $X = \{\alpha: \alpha \in \kappa, \chi(\alpha, \kappa) \leq \omega\}$. Then X is a countably compact, first countable, T_3 space of cardinality κ. Note that

$e(X)=\chi(X)=\omega$. Since character is the strongest local invariant, this example suggests that extent must be combined with a global cardinal function to obtain bounds on $|X|$. It turns out that extent, when combined with either point separating weight or diagonal degree, gives a bound not only on $|X|$ but also on $|K(X)|$.

9.2. LEMMA. *Let $X \in \mathcal{T}_1$ with $e(X) \leqslant \kappa$, let $\mathcal{V}$ be an open cover of X such that* $\operatorname{ord}(p, \mathcal{V}) \leqslant \kappa$ *for each $p \in X$. Then $\mathcal{V}$ has a subcollection of cardinality $\leqslant\kappa$ which covers X.*

PROOF. Use Zorn's lemma to obtain a subset D of X maximal with respect to the property that if p and q are distinct elements of D, then $q \notin \operatorname{st}(p, \mathcal{V})$. Note that D is closed and discrete. From $e(X) \leqslant \kappa$ one has $|D| \leqslant \kappa$. Let $\mathcal{W}= \{V: V \in \mathcal{V}, V \cap D \neq \emptyset\}$. Then $\mathcal{W}$ is a subcollection of $\mathcal{V}$ with $|\mathcal{W}| \leqslant \kappa$, and the maximal property of D implies that $\mathcal{W}$ covers X.

9.3. THEOREM. *For $X \in \mathcal{T}_1$, $|K(X)| \leqslant 2^{e(X)\cdot psw(X)}$. In particular, every ω_1-compact space with a point-countable separating open cover has at most 2^ω compact subsets.*

PROOF. Let $e(X) \cdot psw(X)=\kappa$. The first step is to show that X has a dense subset of cardinality at most 2^κ. Let $\mathcal{V}$ be a separating open cover of X such that $\operatorname{ord}(p, \mathcal{V}) \leqslant \kappa$ for each $p \in X$. Construct a sequence $\{A_\alpha: 0 \leqslant \alpha < \kappa^+\}$ of subsets of X and a sequence $\{\mathcal{V}_\alpha: 0 < \alpha < \kappa^+\}$ of open collections in X such that

(1) $|A_\alpha| \leqslant 2^\kappa$, $0 \leqslant \alpha < \kappa^+$;
(2) $\mathcal{V}_\alpha = \{V: V \in \mathcal{V}, V \cap (\bigcup_{\beta<\alpha} A_\beta) \neq \emptyset\}$;
(3) if W is the union of $\leqslant\kappa$ elements of $\mathcal{V}_\alpha$, and $W \neq X$, then $A_\alpha - W \neq \emptyset$.

Let $S=\bigcup_{\alpha<\kappa^+} A_\alpha$. Clearly $|S| \leqslant 2^\kappa$, so it suffices to prove that $\bar{S}=X$. Suppose not, and let $q \in (X-\bar{S})$. Let $\mathcal{W}=\{V: V \in \mathcal{V}, V \cap S \neq \emptyset, q \notin V\}$. Since $\mathcal{V}$ is a separating open cover of X, $\mathcal{W}$ covers $\bar{S}$. Now $e(\bar{S}) \leqslant \kappa$, so by Lemma 9.2 one may assume that $|\mathcal{W}| \leqslant \kappa$. Choose $\alpha < \kappa^+$ such that $\mathcal{W} \subseteq \mathcal{V}_\alpha$, and let $W=\bigcup \mathcal{W}$. By (3), $A_\alpha - W \neq \emptyset$, a contradiction that $S \subseteq W$.

Since $\operatorname{ord}(p, \mathcal{V}) \leqslant \kappa$ for all $p \in X$, and X has a dense subset of cardinality $\leqslant 2^\kappa$, it follows that $|\mathcal{V}| \leqslant 2^\kappa$. At this point it is easy to see that $|X| \leqslant 2^\kappa$. (Define $\Phi: X \to P_\kappa(\mathcal{V})$ by $\Phi(p)=\{V: V \in \mathcal{V}, p \in V\}$.) However, one can actually prove more, namely $|K(X)| \leqslant 2^\kappa$. Let $\mathcal{W}$ be all finite unions of elements of $\mathcal{V}$, let $\mathcal{G}$ be all intersections of $\leqslant\kappa$ elements of $\mathcal{W}$. Now $|\mathcal{G}| \leqslant 2^\kappa$, so it suffices to show $K(X) \subseteq \mathcal{G}$. Let K be a compact subset of X. Let $\{\mathcal{A}_\alpha: 0 \leqslant \alpha < \kappa\}$ be all finite minimal covers of K by elements of $\mathcal{V}$ (use Miščenko's lemma), and for each α let $W_\alpha=\bigcup \mathcal{A}_\alpha$. Then $K=\bigcap_{\alpha<\kappa} W_\alpha$, so $K \in \mathcal{G}$.

The next result is due to Ginsburg and Woods. The proof uses the Erdös–Rado theorem; a proof using the Pol–Šapirovskiĭ technique is also possible (see HODEL [1979]).

9.4. THEOREM. *For $X \in \mathcal{T}_2$, $|K(X)| \leq 2^{e(X)\cdot \Delta(X)}$. In particular, every Hausdorff ω_1-compact space with a G_δ-diagonal has at most 2^ω compact subsets.*

PROOF. Let $e(X)\cdot \Delta(X) = \kappa$. We first prove $|X| \leq 2^\kappa$ for $X \in \mathcal{T}_1$. Let $\{\mathcal{V}_\alpha : 0 \leq \alpha < \kappa\}$ be a collection of open covers of X such that if $p, q \in X$ and $p \neq q$, then $q \notin \mathrm{st}(p, \mathcal{V}_\alpha)$ for some $\alpha < \kappa$. (This implies that $p \notin \mathrm{st}(q, \mathcal{V}_\alpha)$ as well.) Suppose $|X| > 2^\kappa$. For each $\alpha < \kappa$ let $P_\alpha = \{\{p, q\}: p, q \in X, q \notin \mathrm{st}(p, \mathcal{V}_\alpha)\}$. Then $[X]^2 \subseteq \bigcup_{\alpha<\kappa} P_\alpha$, so there exists $D \subseteq X$ with $|D| > \kappa$ and $\alpha < \kappa$ such that $[D]^2 \subseteq P_\alpha$. One can show that D is a discrete subset of X with no limit points, a contradiction of $e(X) \leq \kappa$.

Now let K be a compact subset of X. Then $w(K) \leq \kappa$ (use Theorem 7.6), so there is a subset A of K with $|A| \leq \kappa$ such that $\bar{A} = K$. The number of subsets of X of cardinality $\leq\kappa$ is at most 2^κ, so $|K(X)| \leq 2^\kappa$.

The last theorem in this section is motivated by the old result that a compact, perfectly normal space has cardinality at most 2^ω. This theorem is a consequence of any one of the basic inequalities $|X| \leq 2^{L(X)\cdot\chi(X)}$, $|X| \leq 2^{c(X)\cdot\chi(X)}$, $|X| \leq 2^{s(X)\cdot\psi(X)}$. In fact, by 9.1, the number of compact subsets is at most 2^ω. This suggests the problem of finding a bound on the number of compact subsets of a countably compact, perfectly normal space. To begin with, one is hard pressed to think of an example of such a space which is not compact. (See STEPHENSON [1972].) And no wonder, since a "real" example does not exist! (See the paper by Weiss in this Handbook; also recall Ostaszewski's famous construction using $\diamondsuit$.) But the original question is easily answered by the result below. The following cardinal function extends *perfect* (= every closed set is a G_δ) to higher cardinality. Let $\Psi(X)$ be the smallest infinite cardinal κ such that every closed set in X is the intersection of $\leq\kappa$ open sets.

9.5. THEOREM. *For $X \in \mathcal{T}_2$, $|K(X)| \leq 2^{e(X)\cdot\Psi(X)}$. In particular, every Hausdorff, ω_1-compact, perfect space has at most 2^ω compact sets.*

PROOF. Let $e(X)\cdot\Psi(X) = \kappa$. One then has $s(X) \leq \kappa$. (See the first part of the proof of 8.5(II).) Hence $|X| \leq 2^{s(X)\cdot\psi(X)} \leq 2^\kappa$. For each $p \in X$, let $\mathcal{V}_p$ be a collection of open neighborhoods of p, closed under finite intersections, such that $|\mathcal{V}_p| \leq 2^\kappa$ and $\bigcap\{\bar{V}: V \in V_p\} = \{p\}$. Let $\mathcal{V} = \bigcup_{p\in X}\mathcal{V}_p$, let $\mathcal{W}$ be all unions of $\leq\kappa$ elements of $\mathcal{V}$, and let $\mathcal{G} = \{W \cup \bar{A}: W \in \mathcal{W}, A \in P_\kappa(X)\}$. The proof is complete if the complement of every compact subset of X is the union of $\leq\kappa$ elements of $\mathcal{G}$.

Let $K \subseteq X$ be compact. By $\Psi(X) \leq \kappa$, one has $X - K = \bigcup\{F_\alpha : 0 \leq \alpha < \kappa\}$, each F_α closed. Fix $\alpha < \kappa$. For each $p \in F_\alpha$, use compactness of K to obtain $V_p \in \mathcal{V}_p$ such that $K \cap V_p = \emptyset$. Apply 4.8 to F_α and $\{V_p : p \in F_\alpha\}$ to obtain $W_\alpha \in \mathcal{W}$ and $A_\alpha \subseteq F_\alpha$ with $|A_\alpha| \leq \kappa$ such that $F_\alpha \subseteq W_\alpha \cup \bar{A}_\alpha$. Let $G_\alpha = W_\alpha \cup \bar{A}_\alpha$, and note that $G_\alpha \in \mathcal{G}$ and $G_\alpha \cap K = \emptyset$. Clearly $X - K = \bigcup_{\alpha<\kappa} G_\alpha$.

A general principle, formulated in Section 4, states that the sharpest and most elegant bounds on $|X|$ are obtained by combining a local and a global cardinal function. A variation on that principle is that the sharpest bounds on $|K(X)|$ are obtained by combining extent with a global cardinal function which is equal to ω on metrizable spaces (hence cannot by itself bound $|X|$). This is the case for the bounds $\exp(e(X)\cdot psw(X))$, $\exp(e(X)\cdot \Delta(X))$, and $(\exp e(X)\cdot \Psi(X))$ on $|K(X)|$.

10. Bounds on the number of continuous, real-valued functions on X

Let $C(X)$ denote the collection of all continuous, real-valued functions on X. Since $X\neq\emptyset$, $C(X)$ has at least 2^{ω} elements. Moreover, as proved in the introduction, 2^{ω} is a bound on $|C(X)|$ for X separable. That argument generalizes so that one has the following result.

10.1. THEOREM. *For any space X, $|C(X)|\leq 2^{d(X)}$. In particular, $|C(X)|=2^{\omega}$ for any separable space X.*

Before obtaining bounds on $|C(X)|$, we prove an extension of Jones' lemma which is suggested by 10.1.

10.2. THEOREM. *If X is normal, then $2^{|D|}\leq|C(X)|$ for every closed discrete $D\subseteq X$.*

PROOF. For each $E\subseteq D$, use Urysohn's lemma to obtain a continuous function $f_E: X\to[0,1]$ such that $f_E(E)=1$ and $f_E(D-E)=0$. Clearly $\Phi: P(D)\to C(X)$ defined by $\Phi(E)=f_E$ is one-one, hence $2^{|D|}\leq|C(X)|$.

The main results in this section are: $|C(X)|\leq|\mathrm{RO}(X)|$ for any infinite Hausdorff space X; $|C(X)|\leq w(X)^{wc(X)}$ for any space X; $|C(X)|^{\omega}=|C(X)|$ for any space X. Assuming appropriate separation axioms, the first two inequalities generalize $|C(X)|\leq 2^{d(X)}$ (see Theorem 3.3).

10.3. THEOREM. *For any infinite Hausdorff space X, $|C(X)|\leq|\mathrm{RO}(X)|$.*

PROOF. Since X is an infinite Hausdorff space, the number of regular open sets in X is infinite. Consequently, it suffices to show that $|C(X)|\leq|\mathrm{RO}(X)|^{\omega}$. Reason: $\mathrm{RO}(X)$ is a complete Boolean algebra, and Pierce has proved that $|B|^{\omega}=|B|$ for any infinite complete Boolean algebra B. For $f\in C(X)$, let f^* denote the function from $\mathbb{Q}$ (the set of rational numbers) into $\mathrm{RO}(X)$ defined by $f^*(r)=(f^{-1}((-\infty,r))^-)^\circ$. Define $\Phi: C(X)\to {}^{\mathbb{Q}}\mathrm{RO}(X)$ by $\Phi(f)=f^*$. Since Φ is one-one, $|C(X)|\leq|\mathrm{RO}(X)|^{\omega}$.

10.4. COROLLARY. *For any infinite Hausdorff space X, $|C(X)|\leq o(X)$. In other words, the number of continuous, real-valued functions on an infinite Hausdorff space X is always bounded by the number of open sets in X.*

10.5. COROLLARY. *Let X be an infinite, perfectly normal Hausdorff space. Then* $|C(X)| = |\mathrm{RO}(X)| = o(X)$.

PROOF. It suffices to prove $o(X) \leq |C(X)|$. Let H be a non-empty closed set in X. By perfect normality, there is a continuous function $f_H: X \to \mathbb{R}$ such that $f_H^{-1}(0) = H$. Define Φ: {closed sets} $\to C(X)$ by $\Phi(H) = f_H$. Since Φ is one-one, $o(X) \leq |C(X)|$.

10.6. COROLLARY. *Let X be a topological space.*

(a) *If $X \in \mathcal{T}_2$, $|C(X)| \leq \pi w(X)^{c(X)}$.*

(b) *If $X \in \mathcal{T}_3$, $|C(X)| \leq \pi\chi(X)^{c(X)}$.*

PROOF. Use 6.1 and 6.2. (If X is finite, $|C(X)| = 2^\omega$ and likewise for $\pi w(X)^{c(X)}$ and $\pi\chi(X)^{c(X)}$.)

The following example motivates the next inequality, namely $|C(X)| \leq w(X)^{wc(X)}$. Let X be the one-point compactification of a discrete space of cardinality 2^ω. Then $d(X) = c(X) = 2^\omega$ and $|\mathrm{RO}(X)| = 2^{2^\omega}$, so the inequalities obtained so far give $|C(X)| \leq 2^{2^\omega}$. However, since X is compact, $wc(X) = \omega$ and $|C(X)| \leq w(X)^{wc(X)} = 2^\omega$.

10.7. LEMMA. *Let D and E be sets, let $\mathcal{F}$ be a collection of functions from D into E, let $\mathcal{S}$ be a separating cover of E. Suppose there is a collection $\mathcal{A}$ of subsets of D such that $f^{-1}(S) \in \mathcal{A}$ whenever $f \in \mathcal{F}$ and $S \in \mathcal{S}$. Then $|\mathcal{F}| \leq |\mathcal{A}|^{|\mathcal{S}|}$.*

PROOF. Define $\Phi: \mathcal{F} \to {}^{\mathcal{S}}\mathcal{A}$ by $\Phi(f) = f^*$, where $f^*(S) = f^{-1}(S)$. Since $\mathcal{S}$ is separating, Φ is one-one, hence $|\mathcal{F}| \leq |\mathcal{A}|^{|\mathcal{S}|}$.

10.8. THEOREM. *For any space X, $|C(X)| \leq w(X)^{wc(X)}$.*

PROOF. Let $wc(X) = \kappa$, let $\mathcal{B}$ be a base for X with $|\mathcal{B}| \leq w(X)$. Let $\mathcal{G} = \{\bar{G}$: G is union of $\leq\kappa$ elements of $\mathcal{B}\}$, and let $\mathcal{A}$ be all countable unions of elements of $\mathcal{G}$. Then $|\mathcal{A}| \leq w(X)^\kappa$, so by Lemma 10.7 it suffices to show $f^{-1}((-\infty, r)) \in \mathcal{A}$ for $f \in C(X)$ and $r \in Q$. Let $V = f^{-1}((-\infty, r))$. One may assume $V = \bigcup_{n\in\omega} W_n$, where each W_n is open and $\bar{W}_n \subseteq W_{n+1}$ for all $n < \omega$. For each $n < \omega$, $\{B: B \in \mathcal{B}, B \subseteq W_{n+1}\}$, together with $(X - \bar{W}_n)$, covers X, so there is a subcollection of cardinality $\leq\kappa$ whose union is dense in X. Let $\mathcal{B}_n$ be all elements of this subcollection other than $(X - \bar{W}_n)$, and let $G_n = \bigcup \mathcal{B}_n$. Then $W_n \subseteq \bar{G}_n \subseteq \bar{W}_{n+1} \subseteq W_{n+2}$, consequently $V = \bigcup_{n\in\omega} \bar{G}_n$, and so $V \in \mathcal{A}$.

10.9. COROLLARY. *Let X be a first-countable Hausdorff space which is either Lindelöf or* ccc. *Then* $|C(X)| = 2^\omega$.

PROOF. One has $|X| \leq 2^\omega$ by 4.5 or 4.7, hence by first countability, $w(X) \leq 2^\omega$. Since $wc(X) = \omega$, $|C(X)| \leq 2^\omega$.

The following example shows that the bound for $|C(X)|$ given by $w(X)^{wc(X)}$ is not always sharper than the one given by $|\mathrm{RO}(X)|$. Let κ be a cardinal such that $2^{\omega_1} \leqslant \kappa = \kappa^\omega < \kappa^{\omega_1}$. (To obtain κ, let $\{\lambda_\alpha : 0 \leqslant \alpha < \omega_1\}$ be a sequence of cardinals such that $\lambda_0 \geqslant 2^{\omega_1}$ and $\lambda_\beta^\omega < \lambda_\alpha$ for $\beta < \alpha$, and let $\kappa = \sup\{\lambda_\alpha : 0 \leqslant \alpha < \omega_1\}$.) Let X be the topological sum of the discrete space of cardinality ω_1 and a ccc space of weight κ (e.g., the product of κ copies of $\{0, 1\}$). Then $|\mathrm{RO}(X)| \leqslant \kappa^\omega$ and $w(X)^{wc(X)} = \kappa^{\omega_1}$. (To see that the number of regular open sets in a ccc space of weight κ is at most κ^ω, use Theorem 6.1.)

One has $|\mathrm{RO}(X)|^\omega = |\mathrm{RO}(X)|$ for every infinite Hausdorff space X. The cardinal $|C(X)|$ also has this property.

10.10. THEOREM. *For every space X, $|C(X)|^\omega = |C(X)|$.*

PROOF. Let $C^*(X)$ denote the collection of all continuous, bounded functions from X into $\mathbb{R}$. Since $|C^*(X)| = |C(X)|$, it suffices to prove the result for $C^*(X)$. Recall that $C^*(X)$ is a Banach space with norm defined by $\|f\| = \sup\{|f(x)| : x \in X\}$. Let $\kappa = w(C^*(X))$; it suffices to prove that $|C^*(X)| = \kappa^\omega$. Clearly $|C^*(X)| \leqslant \kappa^\omega$; to prove $|C^*(X)| \geqslant \kappa^\omega$, it suffices, by the theorem of Schmidt and Stone, to prove that every non-empty open set in $C^*(X)$ has weight $\geqslant \kappa$. This is an easy consequence of the following observation: let $\varepsilon > 0$, let $V = \{f : f \in C^*(X), \|f\| < \varepsilon\}$; then $C^*(X)$ is homeomorphic to V (e.g., define $\Phi(f) = \varepsilon f/(1 + \|f\|)$).

The first systematic study of the cardinality of $C(X)$ is due to Comfort and Hager, and most of the results in this section are due to them; also see the book by Comfort and Negrepontis (especially Lemma 16.19, p. 445) and the survey paper by Comfort. The paper by Comfort and Hager has numerous examples illustrating the sharpness of various bounds for $|C(X)|$. Krivoručko has given a systematic study of $|C(X, Y)|$; Lemma 10.7 is taken from his papers. Krause has proved a generalization of $|C(X)|^\omega = |C(X)|$, namely $|E| = w(E)^\omega$ for any Banach space E. The proof of 10.10 is taken from Comfort's survey paper on cardinal invariants.

11. Density and cellularity of product spaces

The main result in this section is the remarkable Hewitt-Marczewski– Pondiczery theorem on the density of a product space. The countable version states that the product of at most 2^ω separable spaces is separable! The Hewitt–Marczewski–Pondiczery theorem has numerous applications. In particular, it can be used to obtain examples of (1) a countable space not first countable at any point; (2) a compact ccc space not separable and of large cardinality. We also discuss the subtle question of the cellularity of a product space. The reader is referred to JUHÁSZ [1971 or 1980] and COMFORT–NEGREPONTIS [1982] for a systematic discussion of the following general problem on product spaces. Given a cardinal function ϕ, a collection $\{X_t: t \in T\}$ of topological spaces, and $X = \Pi_{t\in T}X_t$, evaluate $\phi(X)$ in terms of $|T|$ and $\sup\{\phi(X_t): t \in T\}$.

The following notation and terminology is used. Let $\{X_t: t \in T\}$ be a collection of topological spaces, let $X = \Pi_{t\in T}X_t$. For $t \in T$, π_t denotes the projection of X onto X_t. A *canonical open set* in X is a non-empty open set of the form $\Pi_{t\in T}V_t$, where each V_t is open in X_t and $V_t = X_t$ for all but finitely many $t \in T$. The collection of all canonical open sets of X is a base for the topology of X. One calls $\{t: V_t \neq X_t\}$ the set of *restricted coordinates* of $\Pi_{t\in T}V_t$. Let $A \subseteq T$, let $Y = \Pi_{t\in A}X_t$. If $V = \Pi_{t\in T}V_t$ is a canonical open set in X, then $\Pi_{t\in A}V_t$ is a canonical open set in Y, called the *projection of V into Y*.

11.1. LEMMA. *Let T be a set with $|T| \leq 2^\kappa$. Then there is a cover $\mathscr{A}$ of T such that (1) $|\mathscr{A}| \leq \kappa$; (2) if $t_1, \ldots, t_n$ are distinct elements of T, there is a pairwise disjoint subcollection $\{A_1, \ldots, A_n\}$ of $\mathscr{A}$ such that $t_i \in A_i$, $1 \leq i \leq n$.*

PROOF. It suffices to construct such a cover of $D(2)^\kappa$, the product of κ copies of $\{0, 1\}$. Now $D(2)^\kappa$ with the product topology is a compact space, so any base for $D(2)^\kappa$ satisfies (2). Since $D(2)^\kappa$ has a base of cardinality κ, (1) can also be satisfied.

11.2. THEOREM (Hewitt, Marczewski, Pondiczery). *If $X = \Pi_{t\in T}X_t$, where $|T| \leq 2^\kappa$ and $d(X_t) \leq \kappa$ for all $t \in T$, then $d(X) \leq \kappa$. In particular, the product of no more than 2^ω separable spaces is separable.*

PROOF. Let $\mathscr{A}$ be a cover of the index set T which satisfies (1) and (2) of the lemma. For each $t \in T$ let $\{x(t, \alpha): 0 \leq \alpha < \kappa\}$ be a dense subset of X_t. For $n < \omega$, for each ordered n-tuple $\Gamma = (A_1, \ldots, A_n)$ of pairwise disjoint elements of $\mathscr{A}$, and for each ordered n-tuple $\Delta = (\beta_1, \ldots, \beta_n)$ of ordinals $<\kappa$, define an element of X as follows:

$$f(n, \Gamma, \Delta)(t) = \begin{cases} x(t, \beta_i) & t \in A_i, \\ x(t, 0) & t \notin \bigcup_{i=1}^{n} A_i . \end{cases}$$

Let S be the set of all elements of X defined in this way. Clearly $|S| \leq \kappa$. To see that S is dense, let $V = \Pi_{t\in T}V_t$ be a canonical open set in X with restricted coordinates $t_1 < t_2 < \cdots < t_n$. (Assume T is linearly ordered.) Let $\Gamma = (A_1, \ldots, A_n)$, where $\{A_1, \ldots, A_n\}$ is a pairwise disjoint collection in $\mathscr{A}$ with $t_i \in A_i$ for $1 \leq i \leq n$. For $1 \leq i \leq n$, let $x(t_i, \beta_i) \in V_{t_i}$, and let $\Delta = (\beta_1, \ldots, \beta_n)$. Then $f(n, \Gamma, \Delta) \in S \cap V$.

11.3. COROLLARY. *If $X = \Pi_{t\in T}X_t$, and $d(X_t) \leq \kappa$ for all $t \in T$, then $c(X) \leq \kappa$. In particular, any product of separable spaces is a* ccc *space.*

PROOF. Suppose false, and let $\{V_\alpha : 0 \leq \alpha < \kappa^+\}$ be a cellular family in X. Assume each V_α is a canonical open set, and let F_α be the finite set of restricted coordinates of V_α. Let $A = \bigcup_{\alpha<\kappa^+} F_\alpha$ and let $Y = \Pi_{t\in A} X_t$. Now $|A| \leq 2^\kappa$, so $d(Y) \leq \kappa$. On the other hand, if V_α^* denotes the projection of V_α into Y, then $\{V_\alpha^* : 0 \leq \alpha < \kappa^+\}$ is a cellular family in Y of cardinality κ^+, a contradiction.

We now give some applications of the Hewitt–Marczewski–Pondiczery theorem and its corollary.

11.4. EXAMPLE. *There is a countable T_3 space X such that $\pi\chi(p, X) = 2^\omega$ for each $p \in X$.* Let X be a countable dense subset of $D(2)^{2^\omega}$. The π-character of each point of $D(2)^{2^\omega}$ is 2^ω (see the proof of 11.8). Since X is dense in $D(2)^{2^\omega}$, it follows by 3.9(c) that $\pi\chi(p, X) = 2^\omega$ for each $p \in X$.

11.5. EXAMPLE. *For each cardinal $\kappa \geq \omega$ there is a compact ccc space of cardinality* 2^κ. Let X be the product of κ copies of $\{0, 1\}$. Then X is a ccc space of cardinality 2^κ. If $\kappa \geq (2^\omega)^+$, then X is not separable (since otherwise $(2^\omega)^+ \leq \kappa = w(X) \leq 2^{d(X)} = 2^\omega$).

Recall that a collection $\mathscr{A}$ of subsets of E is *independent* if for every finite subcollection $A_1, \ldots, A_i, B_1, \ldots, B_n$ of distinct elements of $\mathscr{A}$, one has $A_1 \cap \cdots \cap A_i \cap B_1^c \cap \cdots \cap B_n^c \neq \emptyset$. Another application of the Hewitt–Marczewski–Pondiczery theorem is an easy proof of Hausdorff's theorem that an infinite set of cardinality κ has an independent collection of cardinality 2^κ. Take the product of 2^κ copies of $\{0, 1\}$, and let E be a dense subset of cardinality κ. Then $\{E \cap \pi_\alpha^{-1}(\{0\}) : 0 \leq \alpha < 2^\kappa\}$ is an independent collection of subsets of E of cardinality 2^κ. As proved in Section 2, an independent collection of cardinality 2^κ on κ can be used to construct 2^{2^κ} ultrafilters on κ.

Let $\{X_t : t \in T\}$ be a collection of ccc spaces, let $X = \Pi_{t\in T} X_t$. Is X a ccc space? The answer is "yes" if each X_t is also assumed separable (see 11.3). Moreover, one can conclude from the two theorems proved below that (1) $c(X) \leq 2^\omega$; (2) X is a ccc space if each finite product of $\{X_t : t \in T\}$ is a ccc space. Nevertheless, the question cannot be settled in ZFC. KUREPA [1950] has proved that if X is a Souslin line (a linearly ordered ccc space not separable), then $X \times X$ is not a ccc space. On the other hand, if one assumes $\mathrm{MA}(\omega_1)$, then the product of any finite number (and hence any number) of ccc spaces is ccc. (See KUNEN [1980], p. 61.)

11.6. THEOREM. *Let $X = \Pi_{t\in T} X_t$, and suppose that for each finite subset F of T, $c(\Pi_{t\in F} X_t) \leq \kappa$. Then $c(X) \leq \kappa$.*

PROOF. Suppose false, and let $\{V_\alpha : 0 \leq \alpha < \kappa^+\}$ be a cellular family in X of cardinality κ^+. Assume each V_α is a canonical open set, and let F_α be the set of restricted coordinates of V_α. Apply the Δ-system lemma to $\{F_\alpha : 0 \leq \alpha < \kappa^+\}$ to obtain a finite subset F of T and a subset A of κ^+ of cardinality κ^+ such that $F_\alpha \cap F_\beta = F$ for α and β distinct elements of A. First suppose $F = \emptyset$. Choose $\alpha, \beta \in A$, $\alpha \neq \beta$. Since $F_\alpha \cap F_\beta = \emptyset$, one can easily construct $f \in X$ such that $f \in V_\alpha \cap V_\beta$, a contradiction. Suppose $F \neq \emptyset$. By hypothesis $c(\Pi_{t\in F} X_t) \leq \kappa$. On the other hand, if V_α^* denotes the projection of V_α into $\Pi_{t\in F} X_t$, then $\{V_\alpha^*: \alpha \in A\}$ is a cellular family in $\Pi_{t\in F} X_t$ of cardinality κ^+, a contradiction.

11.7. THEOREM (KUREPA [1962]). *If $X = \Pi_{t\in T} X_t$, and $c(X_t) \leq \kappa$ for each $t \in T$, then $c(X) \leq 2^\kappa$. In particular, any product of* ccc *spaces has cellularity at most 2^ω.*

PROOF. By the previous theorem it suffices to consider the case when T is finite, say $T = \{1, 2, \ldots, n\}$. Suppose $c(X) > 2^\kappa$, and let $\mathcal{V}$ be a cellular family in X with $|\mathcal{V}| > 2^\kappa$. Assume each $V \in \mathcal{V}$ is a canonical open set, and for $i = 1, \ldots, n$ let V_i be the projection of V into X_i. For $i = 1, 2, \ldots, n$ let $P_i = \{\{V, W\}: V, W \in \mathcal{V}, V_i \cap W_i = \emptyset\}$. Since $\mathcal{V}$ is a cellular family, $[\mathcal{V}]^2 = \bigcup_{i\leq n} P_i$. By the Erdös–Rado partition theorem, there is an $i \leq n$ and a subset $\mathcal{W}$ of $\mathcal{V}$ with $|\mathcal{W}| > \kappa$ such that $[\mathcal{W}]^2 \subseteq P_i$. Hence $\{V_i: V \in \mathcal{W}\}$ is a cellular family in X_i of cardinality $>\kappa$, a contradiction.

11.8. EXAMPLE. *The following hold for $D(2)^\kappa$, the Cantor cube of weight κ.*

(a) $|D(2)^\kappa| = 2^\kappa$;

(b) $\phi(D(2)^\kappa) = \kappa$ *for* $\phi \in \{s, hL, hd, nw, w, \pi w, h\pi w, psw, \Delta, t, \pi\chi, h\pi\chi, \psi, \chi\}$;

(c) $\phi(D(2)^\kappa) = \omega$ *for* $\phi \in \{c, L, e, wc\}$;

(d) $d(D(2)^\kappa) = \log \kappa$ $(= \min\{\lambda : 2^\lambda \geq \kappa\})$.

PROOF. To simplify notation, write X for $D(2)^\kappa$. To prove that $w(X) = s(X) = \kappa$, first note that the collection of all canonical open sets of X has cardinality κ and is a base for X, hence $s(X) \leq w(X) \leq \kappa$. For each $\alpha < \kappa$ let f_α be the function from κ into $\{0, 1\}$ whose value is zero everywhere except at α. Then $\{f_\alpha: \alpha < \kappa\}$ is a discrete subset of X of cardinality κ, so $\kappa \leq s(X)$.

To prove the rest of (b), we show that $\pi\chi(f, X) \geq \kappa$ for each $f \in X$. (Since X is compact, $psw(X) = \Delta(X) = w(X)$, $\chi(X) = \psi(X)$, and $\pi\chi(X) \leq t(X)$.) Let $f \in X$, and suppose f has a local π-base $\mathcal{V}$ consisting of canonical open sets such that $|\mathcal{V}| < \kappa$. For each $V \in \mathcal{V}$, let $R(V)$ be the set of restricted coordinates of V. Since $|\mathcal{V}| < \kappa$, there exists $\alpha < \kappa$ such that $\alpha \notin R(V)$ for all $V \in \mathcal{V}$. Clearly $\pi_\alpha^{-1}(\{f(\alpha)\})$ is an open neighborhood of f and $V \not\subseteq \pi_\alpha^{-1}(\{f(\alpha)\})$ for all $V \in \mathcal{V}$.

The fact that $c(X) = \omega$ follows from Corollary 11.3. By the Hewitt–Marczewski–Pondiczery theorem, $d(X) \leq \log \kappa$. On the other hand, $\kappa = w(X) \leq 2^{d(X)}$, hence $\log \kappa \leq d(X)$.

12. Achieving cellularity and spread

The cardinal functions cellularity and spread are defined as the supremum of the cardinality of certain sets. One can ask if this supremum is actually achieved. In other words, one can ask:

I. If $c(X) = \kappa$, does X have a cellular family of cardinality κ?

II. If $s(X) = \kappa$, does X have a discrete subset of cardinality κ?

Before answering these two questions, we briefly indicate why they are important. De Groot, in his fundamental paper on cardinal functions, asked if the number of open sets in a Hausdorff space is always 2^κ for some cardinal κ. He observed that this is so for metrizable spaces. Indeed, the equality $o(X) = 2^{w(X)}$ for X metrizable is proved in Section 8, and the key step in the proof is the fact that an infinite metrizable space always has a cellular family of cardinality $c(X)$. De Groot's question will be discussed in further detail in the next section; we note here that II is the key to an affirmative solution (under appropriate set-theoretic assumptions). To see why II is useful, note that if a space X has a discrete set of cardinality $\kappa \geq \omega$, then X obviously has at least 2^κ open sets.

One can ask questions similar to I and II for any cardinal function which is defined as a supremum; e.g., extent, hereditary density, hereditary Lindelöf degree, and the local cardinal functions. Such questions are called *sup = max problems.* The reader is referred to JUHÁSZ [1971, 1980] for a detailed treatment of such problems. The rest of this section is devoted to a discussion of I and II. Note that one need only consider those cases where cellularity or spread is a singular cardinal or a regular limit cardinal. For example, if $c(X) = \kappa^+$, then clearly X has a cellular family of cardinality κ^+.

12.1. THEOREM. *Every infinite Hausdorff space has a cellular family of cardinality ω.*

12.2. THEOREM (Erdös–Tarski). *Let $c(X) = \kappa$, where κ is a singular cardinal. Then X has a cellular family of cardinality κ.*

PROOF. By hypothesis, $\kappa > \omega$ and $\mathrm{cf}(\kappa) < \kappa$. Let $A = \{\lambda_\alpha : 0 \leq \alpha < \mathrm{cf}(\kappa)\}$, where each λ_α is a cardinal $<\kappa$ and $\sup A = \kappa$. There are two cases.

First, suppose there is a non-empty open set V in X such that every non-empty open subset of V has cellularity κ. Since $c(V) = \kappa$ and $\mathrm{cf}(\kappa) < \kappa$, there is a cellular family in V of cardinality $\mathrm{cf}(\kappa)$, say $\{V_\alpha : 0 \leq \alpha < \mathrm{cf}(\kappa)\}$. For each $\alpha < \mathrm{cf}(\kappa)$, $c(V_\alpha) = \kappa$ and $\lambda_\alpha < \kappa$, so there is a cellular family in V_α of cardinality $\geq \lambda_\alpha$, say $\mathscr{W}_\alpha$. Then $\{W : W \in \mathscr{W}_\alpha, 0 \leq \alpha < \mathrm{cf}(\kappa)\}$ is a cellular family in X of cardinality κ.

Now suppose that every non-empty open set in X has a non-empty open subset whose cellularity is $<\kappa$. Let $\mathscr{V}$ be a maximal cellular family in X such that $c(V) < \kappa$ for all $V \in \mathscr{V}$. Since $\mathscr{V}$ is maximal, $\bigcup \mathscr{V}$ is dense in X. Assume $|\mathscr{V}| < \kappa$, since otherwise we are finished. Then

(*) $\sup\{c(V): V \in \mathcal{V}\} = \kappa$.

To see this, let λ be a cardinal with $|\mathcal{V}| \leq \lambda < \kappa$. There is a cellular family $\mathcal{W}$ in X such that $|\mathcal{W}| = \lambda^+$. Now each element of $\mathcal{W}$ intersects some element of $\mathcal{V}$; $|\mathcal{V}| < \lambda^+$ and λ^+ regular implies that some $V \in \mathcal{V}$ intersects λ^+ distinct elements of $\mathcal{W}$. Thus $c(V) \geq \lambda^+$ and the proof of (*) is complete. Use (*) to construct a collection $\{V_\alpha: 0 \leq \alpha < \mathrm{cf}(\kappa)\}$ of distinct elements of $\mathcal{V}$ such that $c(V_\alpha) > \lambda_\alpha$ for $0 \leq \alpha < \mathrm{cf}(\kappa)$. For each α let $\mathcal{W}_\alpha$ be a cellular family in V_α with $|\mathcal{W}_\alpha| \geq \lambda_\alpha$. Then $\{W: W \in \mathcal{W}_\alpha, 0 \leq \alpha < \mathrm{cf}(\kappa)\}$ is a cellular family in X of cardinality κ.

The restriction on κ in 12.2 is essential. Suppose κ is a weakly inaccessible cardinal. For each cardinal $\lambda < \kappa$ let X_λ be the one-point compactification of a discrete space of cardinality λ and let $X = \Pi\{X_\lambda: \lambda < \kappa,\ \lambda \text{ a cardinal}\}$. Then X is a compact space with $c(X) = \kappa$, but X has no cellular family of cardinality κ. (To prove this last statement, use the Δ-system lemma as in the proof of 11.6.)

We now turn to the question of achieving spread. This problem is considerably more difficult and subtle than the corresponding problem for cellularity. So in many cases proofs are omitted; see JUHÁSZ [1971, 1980] for details.

12.3. THEOREM (HAJNAL–JUHÁSZ [1969]). *Let X be a Hausdorff space,* let $s(X) = \kappa$, *where κ is a singular strong limit cardinal. Then X has a discrete subset of cardinality κ.*

PROOF. Assume $|X| = \kappa$, and let $<$ be a well ordering on X such that $(X, <)$ and κ are order isomorphic. For $x, y \in X$ with $x < y$, let U_{xy} and V_{xy} be disjoint open sets in X with $x \in U_{xy}$ and $y \in V_{xy}$. Partition $[X]^3$ into four sets as follows:

$$P_1 = \{\{x, y, z\}: x < y < z,\ x \in U_{yz},\ z \in V_{xy}\};$$

$$P_2 = \{\{x, y, z\}: x < y < z,\ x \in U_{yz},\ z \notin V_{xy}\};$$

$$P_3 = \{\{x, y, z\}: x < y < z,\ x \notin U_{yz},\ z \in V_{xy}\};$$

$$P_4 = \{\{x, y, z\}: x < y < z,\ x \notin U_{yz},\ z \notin V_{xy}\}.$$

Let $\lambda = \mathrm{cf}(\kappa)$. By the canonization lemma, there is a subset D of X and a partition $\{D_\alpha: 0 \leq \alpha < \lambda\}$ of D such that

(1) $|D_\alpha| = \kappa_\alpha < \kappa$ and $\Sigma_{\alpha<\lambda}\kappa_\alpha = \kappa$;

(2) if $0 \leq \alpha < \beta < \kappa$ and $x \in D_\alpha$, $y \in D_\beta$, then $x < y$;

(3) if $a, b \in [D]^3$ and $|a \cap D_\alpha| = |b \cap D_\alpha|$ for all $\alpha < \lambda$, then $a, b \in P_i$ for some i, $1 \leq i \leq 4$.

Let $\alpha < \lambda$ be fixed. Let y be a point of D_α such that y has an immediate $<$-predecessor x in D_α and an immediate $<$-successor z in D_α. The set of points in D_α having this property has cardinality κ_α. Consequently the proof is complete if one can show that y is an isolated point of D.

To prove this, first note that from (2) it follows that x is the $<$-immediate prececessor of y in D and likewise z is the $<$-immediate successor of y in D. Let

$R_y = V_{xy} \cap U_{yz}$, and let us show that $R_y \cap D = \{y\}$. Clearly $x, z \notin R_y$. Let $p \in D$, and first assume $p < x$. If $p \notin V_{xy}$, we are finished. If $p \in V_{xy}$, then $p \notin U_{xy}$, and so $a \in P_3$ or $a \in P_4$, where $a = \{p, x, y\}$. Let $b = \{p, y, z\}$. Then a, b satisfy (3), so $b \in P_3$ or $b \in P_4$. In either case, $p \notin U_{yz}$, so $p \notin R_y$. A similar argument applies for $z < p$.

12.4. COROLLARY. *Let X be a Hausdorff space with $|X| \geqslant \kappa$, where κ is a singular strong limit cardinal. Then X has a discrete subset of cardinality κ.*

PROOF. If $s(X) < \kappa$, then $|X| \leqslant \exp\exp s(X) < \kappa$, a contradiction.

If one assumes GCH + ($\exists$ inaccessible cardinal), and such an assumption is consistent if ZFC is consistent, then the Hajnal–Juhász theorem gives a complete solution to the problem of achieving spread for Hausdorff spaces. What happens at a regular limit cardinal? By suitably modifying the above proof, one can show that for Hausdorff spaces the spread is achieved at a regular limit cardinal which is weakly compact. On the other hand, Jensen has shown that $V = L$ implies that for every regular limit cardinal κ which is not weakly compact, there is a linearly ordered space X with $|X| = \kappa$ but X does not contain κ pairwise disjoint open intervals. Now $V = L$ implies GCH, hence κ is a strong limit cardinal. It follows from $|X| \leqslant \exp\exp s(X)$ that $s(X) = \kappa$. However, X does not have a discrete set of cardinality κ. Reason: *any* linearly ordered space which has a discrete set of cardinality λ has λ pairwise disjoint open intervals (see p. 14 of JUHÁSZ [1971]).

What happens at a singular cardinal if GCH is not assumed? The situation for a singular cardinal with countable cofinality is summarized in the two theorems below. Roitman has shown that if $\omega_{\omega_1} \leqslant 2^\omega$ and there is a first countable Lusin space, then there is a zero-dimensional Hausdorff space X such that $s(X) = \omega_{\omega_1}$ but X has no discrete subset of cardinality ω_{ω_1}. Note that ω_{ω_1} is a singular cardinal with cofinality ω_1. For a description of this space, see Section 8 of the paper by Juhász in this Handbook.

12.5. THEOREM (HAJNAL–JUHÁSZ [1969a]). *Let κ be a singular cardinal with* $\mathrm{cf}(\kappa) = \omega$. *If X is strongly Hausdorff (a condition between Hausdorff and T_3) and $s(X) = \kappa$, then X has a discrete subset of cardinality κ.*

12.6. THEOREM (Kunen–Roitman). *Let κ be a singular cardinal with* $\mathrm{cf}(\kappa) = \omega$, *let X be a Hausdorff space with $s(X) = \kappa$. If $\kappa > 2^\omega$, or if $\kappa < 2^\omega$ and* MA *holds, then X has a discrete subset of cardinality κ.*

13. The cardinal number $o(X)$ and related results

There are two fundamental problems about $o(X)$. (1) Find bounds on $o(X)$. (2) What can one say about the nature of the cardinal number $o(X)$? Numerous bounds for $o(X)$ have already been obtained; e.g., $o(X) \leqslant |X|^{hd(X)}$, $o(X) \leqslant w(X)^{hL(X)} \leqslant 2^{nw(X)}$, and $o(X) \leqslant \exp\exp s(X)$ (for $X \in \mathcal{T}_2$). Additional bounds on $o(X)$ appear in HODEL [1978]. So emphasis will be given to the second question. De Groot, in his fundamental paper on cardinal functions, asked if $o(X)$ is necessarily of the form 2^κ for X a Hausdorff space. He observed that this is the case for X metrizable (see Section 8). De Groot's problem has been completely solved by HAJNAL and JUHÁSZ [1969], [1973a], and it turns out that the problem cannot be settled in ZFC. There is a model of ZFC in which there is a normal, Hausdorff, zero-dimensional space X such that $2^\omega < o(X) < 2^{\omega_1}$. This space is described in Roitman's paper in this Handbook. On the other hand, De Groot's problem has an affirmative solution if one assumes GCH + ($\nexists$ inaccessible cardinal).

13.1. THEOREM (Hajnal, Juhász). *Assume* GCH + ($\nexists$ *inaccessible cardinal*), *let X be a Hausdorff space. Then $o(X) = 2^\kappa$ for some cardinal κ.*

PROOF. Let $|X| = \kappa$; then by GCH $o(X) = \kappa$ or $o(X) = 2^\kappa$. Only the case $o(X) = \kappa$ need be considered. Now κ cannot be singular. (If it is, then by 12.4, X has a discrete set of cardinality κ, hence 2^κ open sets.) Thus κ must be a regular successor cardinal, say $\kappa = 2^\lambda$, and $o(X) = 2^\lambda$.

A related question about $o(X)$, due to Juhász, is the following. Is $o(X)^\omega = o(X)$ for X Hausdorff? (X is always assumed infinite when discussing this question.) Since $|C(X)|^\omega = |C(X)|$ for any space X, and $|\mathrm{RO}(X)|^\omega = |\mathrm{RO}(X)|$ for any infinite Hausdorff space X, the question is quite natural and one can hope for an affirmative answer. (Van Douwen and Zhou Hao-xuan have recently proved that the number of co-zero sets is an ω-power.) Several partial results have been obtained. For example, JUHÁSZ [1977] has proved that $o(X)^\omega = o(X)$ for X a hereditarily paracompact Hausdorff space or a Hausdorff topological group (see Comfort's paper in this Handbook). HAJNAL and JUHÁSZ [1973] have proved that $o(X)^\omega = o(X)$ whenever $X \in \mathcal{T}_3$ and $o(X) < \omega_{\omega_1+\omega}$. One also has $o(X)^\omega = o(X)$ for X perfectly normal and Hausdorff (see Corollary 10.5 and the paper by van Douwen and Hao-xuan.) However, it is not known if $o(X)^\omega = o(X)$ if X is compact.

An interesting problem, related to questions about $o(X)$, is the following. Let $\kappa < |X|$. Does X have a closed set of cardinality κ? As noted in Section 8, A.H. Stone has given a complete and elegant solution to this question for X metrizable. Also, $\beta\omega$ is a compact space with no infinite closed sets of cardinality $< 2^{2^\omega}$. The following terminology is useful when discussing this question. For $\kappa < |X|$, one says that X *omits* κ if X has no closed set of cardinality κ. Thus, $\beta\omega$ omits all infinite cardinals $< 2^{2^\omega}$, while a metrizable space of weight κ cannot omit any cardinal $\leqslant \kappa$.

Assuming GCH, no Hausdorff space can omit three consecutive cardinals κ, κ^+, κ^{++}. Indeed, if $|H| = \kappa$, then $\kappa \leqslant |\bar{H}| \leqslant \kappa^{++}$ by the inequality $|X| \leqslant \exp\exp d(X)$. JUHÁSZ [1977] has proved that under GCH no compact space can omit both κ^+

and κ^{++}. It follows from Theorem 13.2 below that under GCH no compact space can omit both ω and ω_2. Recall, however, that $\beta\omega$ omits both ω and ω_1. It is not known if under GCH a compact space can omit κ^{++}, even in the case $\kappa = \omega$. For a further discussion of this and related problems, see Juhász's paper in this Handbook. VAN DOUWEN [1981a] has shown that for every singular strong limit cardinal κ with $\mathrm{cf}(\kappa) = \omega$, there is a compact space which omits κ. His construction is given in Section 14. Thus, the situation for compact spaces can be summarized as follows: assuming GCH, ω_1 and all cardinals κ with $\mathrm{cf}(\kappa) = \omega$ can be omitted; it is not known if any other cardinal can be omitted.

A special but interesting case of the problem under discussion is that of omitting ω. There is a "large" compact space which omits ω, namely $\beta\omega$. Moreover, $\beta\omega$ can be used to construct a countably compact T_3 space of cardinality 2^ω which omits ω. Indeed, let $X = \bigcup_{\alpha<\omega_1} A_\alpha$, where $\{A_\alpha : 0 \leq \alpha < \omega_1\}$ is a collection of subsets of $\beta\omega$ such that (1) $|A_\alpha| = 2^\omega$, $0 \leq \alpha < \omega_1$; (2) if $E \subseteq \bigcup_{\beta<\alpha} A_\beta$ and $|E| = \omega$, then $|\bar{E} \cap A_\alpha| = 2^\omega$.

The following question is now suggested. If X is compact and $\omega < |X| \leq 2^\omega$, can X omit ω? This question is related to the older problem of when a countably compact or compact space must be sequentially compact. For example, it is known that every countably compact T_3 space of cardinality $<2^\omega$ is sequentially compact (see the papers of van Douwen or Vaughan in this Handbook). So our original question can be simplified as follows. If X is compact and $|X| = 2^\omega$, can X omit ω?

This question cannot be decided in ZFC. FEDORČUK [1977] has proved that it is consistent with ZFC to assume the existence of a compact space X of cardinality 2^ω which has no (non-trivial) convergent sequences. Such a space cannot have a closed set of cardinality ω. (Suppose $H \subseteq X$ is closed and $|H| = \omega$. Then H has a limit point and is metrizable, therefore has a convergent sequence.) On the other hand, *CH* implies that a compact space of cardinality 2^ω cannot omit ω. This is a consequence of the following theorem of HAJNAL and JUHÁSZ [1976]. (A somewhat stronger result states that every compact space X with $|X| < 2^{\omega_1}$ is sequentially compact; again, see van Douwen or Vaughan in this Handbook.)

13.2. THEOREM. *If X is a compact space with $|X| < 2^{\omega_1}$, then X cannot omit ω.*

PROOF. Let Y be the set of all non-isolated points of X. Note that Y is closed and non-empty. There exists $p \in Y$ such that $\chi(p, Y) \leq \omega$. (If $\chi(p, Y) \geq \omega_1$ for all $p \in Y$, $|Y| \geq 2^{\omega_1}$ by the Čech–Pospíšil theorem.) If $\chi(p, Y) = \omega$, one can easily obtain a closed set in X of cardinality ω. So assume $\chi(p, Y) = 1$. Let V be an open set in X such that $p \in V$ and $\bar{V} \cap Y = \{p\}$. Since p is a non-isolated point of X, V is infinite. Let $A \subseteq V$, $|A| = \omega$. Then $A \cup \{p\}$ is closed in $\bar{V}$ (its complement in $\bar{V}$ consists of isolated points of X). Thus $A \cup \{p\}$ is a closed set in X of cardinality ω.

It is clear that a regular space with countable pseudo-character cannot omit ω. In fact, such a space has as many closed sets of cardinality ω as possible.

13.3. THEOREM. *Let X be an infinite T_3 space with countable pseudo-character. Then the number of closed sets in X of cardinality ω is the maximum possible, namely $|X|^\omega$.*

PROOF. Let $\mathscr{A}$ be a collection of subsets of X such that $|\mathscr{A}| = |X|^\omega$, $|A| = \omega$ for each $A \in \mathscr{A}$, and the intersection of any two distinct elements of $\mathscr{A}$ is finite (use Tarski's theorem). For each $A \in \mathscr{A}$, construct a countably infinite closed subset A^* of X as follows. (1) If A has no limit points, $A^* = A$. (2) If A has a limit point p, let $\{V_n : n \in \omega\}$ be a decreasing collection of open neighborhoods of p with $\bigcap_{n<\omega} \bar{V}_n = \{p\}$, let $\{x_n : n \in \omega\}$ be a countably infinite subset of A such that $x_n \in V_n$ for all $n \in \omega$, and let $A^* = \{x_n : n \in \omega\} \cup \{p\}$.

Now let A_1 and A_2 be distinct elements of $\mathscr{A}$. Since $A_1 \cap A_2$ is finite, so is $A_1^* \cap A_2^*$, hence $A_1^* \neq A_2^*$. Thus $\{A^* : A \in \mathscr{A}\}$ is a collection of $|X|^\omega$ closed sets, each of cardinality ω.

14. Examples

This section is devoted to examples, many of whieh illustrate the sharpness of inequalities. Proofs and constructions are often omitted. A discussion of $\beta\omega$ and $D(2)^\kappa$ appears at the end of Sections 7 and 11 respectively.

14.1. EXAMPLE. *For each cardinal $\kappa \geq \omega$ there is a Hausdorff space X with $d(X) = \kappa$, $|X| = 2^{2^\kappa}$, and $o(X) = 2^{2^{2^\kappa}}$.* (See 3.2.) Let U be the collection of all free ultrafilters on κ, let $X = \kappa \cup U$. A base for a Hausdorff topology on X is described as follows: each element of κ is open; for each $p \in U$, a base of open neighborhoods of p is the collection of all sets of the form $\{p\} \cup A$, where $A \subseteq \kappa$ and $A \in p$.

14.2. EXAMPLE. *For each cardinal $\kappa \geq \omega$ there is a Hausdorff space X with $d(X) = \kappa$ and $w(X) = 2^{2^{2^\kappa}}$.* Such a space has been constructed by JUHÁSZ and KUNEN [1973].

14.3. EXAMPLE (Amirdžanov). *For each cardinal $\kappa \geq \omega$ there is a T_3 space X with $|X| = \kappa$ and $\psi(X) = c(X) = t(X) = \omega$.* This example shows that $\chi(X)$ cannot be replaced by $\psi(X)$ in $|X| \leq \exp(c(X) \cdot \chi(X))$, that $\pi\chi(X)$ cannot be replaced by $t(X)$ in $|X| \leq \exp(c(X) \cdot \pi\chi(X) \cdot \psi(X))$, and that $L(X)$ cannot be replaced by $c(X)$ in $|X| \leq \exp(L(X) \cdot t(X) \cdot \psi(X))$. The space X is obtained as follows. Assume $\kappa \geq 2^\omega$, and for each $\alpha < \kappa$ let X_α be a copy of the unit interval $[0, 1]$ with the usual topology. Let $\mathscr{F}$ be the collection of all finite subsets of κ, and let Φ be a one-one function from $\mathscr{F}$ into κ such that $\Phi(F) \notin F$ for all $F \in \mathscr{F}$. For each $F \in \mathscr{F}$, let X_F be those elements f of $\Pi_{\alpha<\kappa} X_\alpha$ such that $0 < f(\alpha) < 1$ whenever $\alpha \in F$, $f(\alpha) = 1$ for $\alpha = \Phi(F)$, and $f(\alpha) = 0$ otherwise. Finally, let $X = \bigcup\{X_F : F \in \mathscr{F}\}$. Clearly $|X| = \kappa$, and $c(X) = \omega$ since $c(\Pi_{\alpha<\kappa} X_\alpha) = \omega$ and X is dense in $\Pi_{\alpha<\kappa} X$. The space X is a subspace of a Σ-product of first countable spaces, so $t(X) = \omega$ by a theorem of Noble. The following observation is the key to constructing a countable pseudo-base for each $f \in X$. Suppose f and g are distinct points of X with $f \in X_F$. Then there is some $\alpha \in F \cup \{\Phi(F)\}$ such that $f(\alpha) \neq g(\alpha)$. (The construction of X given here is taken from ARHANGEL'SKIĬ [1978]; also see Terada.)

14.4. EXAMPLE. *There is a compact space X such that $hd(X) = hL(X) = \chi(X) = \omega$ and $|X| = w(X) = nw(X) = 2^\omega$.* X is the double arrow space of Alexandroff and Urysohn; i.e., $X = [0, 1] \times \{0, 1\}$ and the topology is the order topology given by the lexicographic order.

14.5. EXAMPLE. *There is a compact space X such that $d(X) = \pi w(X) = \chi(X) = \omega$, $|X| = w(X) = s(X) = |C(X)| = |\mathrm{RO}(X)| = 2^\omega$, and $|K(X)| = o(X) = 2^{2^\omega}$.* This example shows that $nw(X)^{\psi(X)}$, $\exp(d(X) \cdot \psi(X))$, $d(X)^{\chi(X)}$, $\exp(L(X) \cdot \chi(X))$, and $\exp(c(X) \cdot \chi(X))$, all bounds for $|X|$, are not bounds for $|K(X)|$. Also see 10.5. The space X is $Y \times Y$, where Y is the space described in 14.4. To see that X has a discrete subset of cardinality 2^ω, note that the Sorgenfrey line S is a subspace of Y so $S \times S \subseteq X$.

14.6. EXAMPLE. *There is a Moore space X such that $d(X) = \omega$, $s(X) = w(X) = L(X) = |C(X)| = |X| = |\mathrm{RO}(X)| = 2^\omega$, and $o(X) = 2^{2^\omega}$.* (See 10.5.) The required space is the tangent disc space, the traditional example of a separable Moore space not normal.

14.7. EXAMPLE (Sierpiński). *There is a hereditarily Lindelöf Hausdorff space which is not separable.* Let $\mathscr{T}$ denote the usual topology on $\mathbb{R}$, let $\mathscr{B} = \{V - A : V \in \mathscr{T}, A \subseteq \mathbb{R}$ and $|A| \leq \omega\}$. The collection $\mathscr{B}$ is a base for a new topology on $\mathbb{R}$ which has the desired properties.

14.8. EXAMPLE. *There is a hereditarily separable Hausdorff space which is not Lindelöf.* Let $X = \{x_\alpha : 0 \leq \alpha < \omega_1\}$ be a subset of $\mathbb{R}$ of cardinality ω_1, let $\mathscr{T}$ be the usual topology on $\mathbb{R}$. For each $\alpha < \omega_1$ let $V_\alpha = \{x_\beta : 0 \leq \beta \leq \alpha\}$ and let $\mathscr{S} = \{V \cap X : V \in \mathscr{T}\} \cup \{V_\alpha : 0 \leq \alpha < \omega_1\}$. Then $\mathscr{S}$ is a subbase for a new topology on X which has the desired properties (see HAJNAL and JUHÁSZ [1968]).

14.9. EXAMPLE. *The existence of a Souslin line implies the existence of a compact, hereditarily Lindelöf space not separable.* Let L be a Souslin line (a linearly ordered ccc space not separable), let $\tilde{L}$ be the Dedekind completion of L. Since L

is dense in $\tilde{L}$, $\tilde{L}$ is ccc. Any linearly ordered ccc space is hereditarily Lindelöf. Moreover, $\tilde{L}$ is compact, hence first countable. By 3.9(a), $\tilde{L}$ cannot be separable.

14.10. EXAMPLE (HAJNAL–JUHÁSZ [1973b]). *It is consistent with* ZFC *that for each cardinal* $\kappa \geqslant \omega$ *there is a* T_3 *space* X *with* $hL(X) = \kappa$ *and* $w(X) = 2^{2^\kappa}$. Note that $d(X) > \kappa$ by the inequality $w(X) \leqslant \exp d(X)$.

14.11. EXAMPLE (Van Douwen, Tall, Weiss). CH *implies the existence of a hereditarily Lindelöf* T_3 *space with a point-countable base which is not separable.*

14.12. EXAMPLE (Ostaszewski). $\diamond$ *implies the existence of a countably compact, perfectly normal, locally compact, hereditarily separable space which is not compact.* This example is described in Roitman's paper in this Handbook.

14.13. EXAMPLE (FEDORČUK [1975]). $\diamond$ *implies the existence of a compact, completely normal, hereditarily separable space* X *in which every infinite closed set has cardinality* 2^{2^ω}. Note that $\pi w(X) = \omega$ and that X cannot be hereditarily Lindelöf.

14.14. EXAMPLE (HAJNAL–JUHÁSZ [1973a]). *It is consistent with* ZFC *that for each cardinal* $\kappa \geqslant \omega$ *there is a* T_3 *space* X *with* $hd(X) = \kappa$ *and* $|X| = 2^{2^\kappa}$. Since $|X| \leqslant \exp hL(X)$, one has $hL(X) > \kappa$. The construction is given in Juhász's paper in this Handbook.

14.5. EXAMPLE (Juhász, Kunen, Rudin). CH *implies the existence of a hereditarily separable, perfectly normal, locally compact space not Lindelöf.* This space is described in the paper by Gardner and Pfeffer in this Handbook.

14.16. EXAMPLE (Comfort). *For each* cardinal $\kappa \geqslant \omega$ with $\kappa^\omega = \kappa$, *there is a countably compact* T_3 *space* X *with* $|X| = \kappa < w(X)$. Let q be a uniform ultrafilter on κ (recall that every base for q has cardinality $>\kappa$), and let $\beta\kappa$ be the Stone–Čech compactification of κ with the discrete topology. Define an increasing sequence $\{A_\alpha : 0 \leqslant \alpha < \omega_1\}$ of subsets of $\beta\kappa$ such that (1) $|A_\alpha| \leqslant \kappa$, $0 \leqslant \alpha < \omega_1$; (2) $A_0 = \kappa \cup \{q\}$; (3) if E is a countably infinite subset of $\bigcap_{\beta<\alpha} A_\beta$, then E has a limit point in A_α, $0 < \alpha < \kappa$. Let $X = \bigcup_{\alpha<\omega_1} A_\alpha$. By (3), X is countably compact; by (1) and (2), $|X| = \kappa$; by (2), X is dense in $\beta\kappa$. Since $\chi(q, X) > \kappa$, $w(X) > \kappa$.

14.17. EXAMPLE (VAN DOUWEN [1981a]). *For each singular strong limit cardinal* κ *with* $\mathrm{cf}(\kappa) = \omega$, *there is a compact space which omits* κ. The required space is $X = \beta\kappa - \kappa$. Let E be a subset of X with $|E| = \kappa$. It suffices to show that E has at least $\kappa^\omega (>\kappa)$ limit points. By 12.4, one may assume that E is discrete. Note that if A_0, B_0 are disjoint subsets of E, then $\bar{A}_0 \cap B_0 = A_0 \cap \bar{B}_0 = \emptyset$; if in addition A_0, B_0 are countable, then $\bar{A}_0 = \bar{B}_0 = \emptyset$. (Any countable set is C^*-embedded in $\beta\kappa$). By Tarski's theorem, there is a collection $\mathscr{A}$ of subsets of E such that $|\mathscr{A}| = \kappa^\omega$, $|A| = \omega$ for each $A \in \mathscr{A}$, and the intersection of any two distinct elements of $\mathscr{A}$ is finite.

For each $A \in \mathscr{A}$ let p_A be a limit point of A. The proof is complete if $p_A \neq p_B$ for $A \neq B$. Let $F = A \cap B$, $A_0 = A - F$, $B_0 = B - F$. One still has $p_A \in \bar{A}_0$, $p_B \in \bar{B}_0$, but $\bar{A}_0 \cap \bar{B}_0 = \emptyset$, hence $p_A \neq p_B$.

15. Summary of definitions and inequalities.

Definitions of cardinal functions. Let X be a topological space.

cardinality
$|X|$ = number of points in $X + \omega$;

weight
$w(X) = \min\{|\mathscr{B}|: \mathscr{B}$ a base for $X\} + \omega$;

......
$o(X)$ = number of open sets in $X + \omega$;

density
$d(X) = \min\{|S|: S \subseteq X, \bar{S} = X\} + \omega$;

hereditary density
$hd(X) = \sup\{d(Y): Y \subseteq X\}$;

cellularity
$c(X) = \sup\{|\mathscr{V}|: \mathscr{V}$ a cellular family in $X\} + \omega$;

spread
$s(X) = \sup\{|D|: D \subseteq X$, D is discrete$\} + \omega$;

Lindelöf degree
$L(X) = \min\{\kappa$: every open cover of X has a subcover of cardinality $\leqslant \kappa\} + \omega$;

hereditary Lindelöf degree
$hL(X) = \sup\{L(Y): Y \subseteq X\}$;

extent
$e(X) = \sup\{|D|: D \subseteq X$, D is closed and discrete$\} + \omega$;

net weight
$nw(X) = \min\{|\mathscr{N}|: \mathscr{N}$ a net for $X\} + \omega$;

π-weight
$\pi w(X) = \min\{|\mathscr{V}|: \mathscr{V}$ a π-base for $X\} + \omega$;

hereditary π-weight
$h\pi w(X) = \sup\{\pi w(Y): Y \subseteq X\}$;

separating weight
$sw(X) = \min\{|\mathscr{V}|: \mathscr{V}$ a separating open cover of $X\} + \omega$;

point separating weight

$psw(X) = \min\{\kappa$: X has separating open cover $\mathscr{V}$ with $\mathrm{ord}(p, \mathscr{V}) \leqslant \kappa$ for all $p \in X\} + \omega$;

diagonal degree

$\Delta(X) = \min\{\kappa$: X has sequence $\{\mathscr{G}_\alpha : \alpha < \kappa\}$ of open covers with $\bigcap_{\alpha<\kappa} st(p, \mathscr{G}_\alpha) = \{p\}$ for all $p \in X\} + \omega$;

weak covering number

$wc(X) = \min\{\kappa$: every open cover of X has a subcollection of cardinality $\leqslant \kappa$ whose union is dense in $X\} + \omega$;

. . . . . .

$\Psi(X) = \min\{\kappa$: every closed set in X is the intersection of $\leqslant\kappa$ open sets$\} + \omega$;

character

$\chi(p, X) = \min\{|\mathscr{V}|$: $\mathscr{V}$ is a local base for $p\}$;

$\chi(X) = \sup\{\chi(p, X)$: $p \in X\} + \omega$;

π-character

$\pi\chi(p, X) = \min\{|\mathscr{V}|$: $\mathscr{V}$ is a local π-base for $p\}$;

$\pi\chi(X) = \sup\{\pi\chi(p, X)$: $p \in X\} + \omega$;

hereditary π-character

$h\pi\chi(X) = \sup\{\pi\chi(Y)$: $Y \subseteq X\}$;

pseudo-character

$\psi(p, X) = \min\{|\mathscr{V}|$: $\mathscr{V}$ is a pseudo-base for $p\}$;

$\psi(X) = \sup\{\psi(p, X)$: $p \in X\} + \omega$;

tightness

$t(p, X) = \min\{\kappa$: for all $Y \subseteq X$ with $p \in \bar{Y}$, there is $A \subseteq Y$ with $|A| \leqslant \kappa$ and $p \in \bar{A}\}$;

$t(X) = \sup\{t(p, X)$: $p \in X\} + \omega$.

Elementary inequalities

(3.1) Let X be a topological space.
 (a) $w(X) \leqslant o(X) \leqslant 2^{|X|}$;
 (b) for $X \in \mathscr{T}_0$, $|X| \leqslant 2^{w(X)}$ and $|X| \leqslant o(X)$.

(3.2) For $X \in \mathscr{T}_2$, $|X| \leqslant 2^{2^{d(X)}}$ and $w(X) \leqslant o(X) \leqslant 2^{2^{2^{d(X)}}}$.

(3.3) Let X to a topological space.
 (a) $|\mathrm{RO}(X)| \leqslant 2^{d(X)}$;
 (b) for X regular, $w(X) \leqslant 2^{d(X)}$.

(3.6) For any space X, $o(X) \leqslant |X|^{hd(X)}$ and $o(X) \leqslant w(X)^{hL(X)}$.

(3.7) Let X be a T_1 space.
(a) $|X| \leq 2^{sw(X)}$ and $w(X) \leq o(X) \leq 2^{2^{sw(X)}}$;
(b) for $X \in \mathscr{T}_2$, $sw(X) \leq |\mathrm{RO}(X)|$;
(c) for $X \in \mathscr{T}_2$, $sw(X) \leq nw(X)$.

(3.8) Let X be a topological space.
(a) $\chi(X) \leq w(X) \leq \chi(X) \cdot |X|$;
(b) $\pi w(X) = d(X) \cdot \pi\chi(X)$;
(c) $t(X) \leq hd(X)$;
(d) $t(X) \leq h\pi\chi(X)$;
(e) for $X \in \mathscr{T}_1$, $\psi(X) \leq \min\{psw(X), \Delta(X)\}$;
(f) for $X \in \mathscr{T}_2$, $\psi(X) \leq hL(X)$.

(3.9) Let X be a topological space, let S be a dense subset of X, let $p \in S$.
(a) $d(X) \leq d(S) \leq d(X) \cdot t(X)$;
(b) $\pi w(S) \leq \pi w(X)$ and $\pi\chi(p, S) \leq \pi\chi(p, X)$;
(c) For X regular, $\pi w(S) = \pi w(X)$, $\chi(p, S) = \chi(p, X)$, and $\pi\chi(p, S) = \pi\chi(p, X)$.

Bounds on the cardinality of X

(4.1) For $X \in \mathscr{T}_1$, $|X| \leq nw(X)^{\psi(X)}$.

(4.2) For $X \in \mathscr{T}_3$, $|X| \leq 2^{d(X) \cdot \psi(X)}$.

(4.4) For $X \in \mathscr{T}_2$, $|X| \leq d(X)^{\chi(X)}$.

(4.5) For $X \in \mathscr{T}_2$, $|X| \leq 2^{L(X) \cdot \chi(X)}$.

(4.6) For $X \in \mathscr{T}_2$, $|X| \leq 2^{L(X) \cdot t(X) \cdot \psi(X)}$.

(4.7) For $X \in \mathscr{T}_2$, $|X| \leq 2^{c(X) \cdot \chi(X)}$.

(4.9) For $X \in \mathscr{T}_1$, $|X| \leq 2^{s(X) \cdot \psi(X)}$.

(4.10) For $X \in \mathscr{T}_2$, $|X| \leq 2^{hL(X)}$.

(4.11) For $X \in \mathscr{T}_2$, $\psi(X) \leq 2^{s(X)}$.

(4.12) For $X \in \mathscr{T}_2$, $|X| \leq 2^{2^{s(X)}}$.

(4.13) For X normal and Hausdorff, $|X| \leq 2^{wc(X) \cdot \chi(X)}$.

Bounds using spread

(5.2) For $X \in \mathscr{T}_2$, $hd(X) \leq 2^{s(X)}$.

(5.3) For $X \in \mathscr{T}_3$, $nw(X) \leq 2^{s(X)}$.

(5.5) For $X \in \mathscr{T}_2$, $o(X) \leq 2^{2^{s(X)}}$.

Bounds using cellularity and π-character

(6.1) For any space X, $|\mathrm{RO}(X)| \leqslant \pi w(X)^{c(X)}$.

(6.2) For X regular, $w(X) \leqslant |\mathrm{RO}(X)| \leqslant \pi\chi(X)^{c(X)}$.

(6.3) For $X \in \mathcal{T}_3$, $|X| \leqslant \pi\chi(X)^{c(X)\cdot\psi(X)}$.

Cardinal functions on compact spaces

(7.1) For X compact, $\psi(X) = \chi(X)$.

(7.2) For X compact, $w(X) \leqslant |X|$.

(7.3) For X compact, $|X| \leqslant 2^{\psi(X)}$.

(7.4) For X compact, $psw(X) = nw(X) = w(X)$.

(7.6) For X compact, $\Delta(X) = w(X)$.

(7.7) For X compact, $w(X) \leqslant |\mathrm{RO}(X)| \leqslant 2^{s(X)}$.

(7.12) For X compact, $t(X) = F(X)$, where $F(X) = \sup\{\kappa\colon X$ has free sequence of length $\kappa\} + \omega$.

(7.13) For X compact, $h\pi\chi(X) = t(X)$.

(7.14) For X compact, $h\pi w(X) = hd(X)$.

(7.15) For X compact, $t(X) \leqslant s(X)$.

(7.17) For X compact, $hd(X) \leqslant s(X)^+$.

(7.19) Let X be a compact space such that $\chi(p, X) \geqslant \kappa$ for each $p \in X$. Then $|X| \geqslant 2^\kappa$.

Cardinal functions on metrizable spaces

(8.1) Let X be an infinite metrizable space.
   (a) $w(X) \leqslant |X| \leqslant w(X)^\omega$;
   (b) $\psi(X) = t(X) = \pi\chi(X) = \chi(X) = psw(X) = \Delta(X) = \omega$;
   (c) $w(X) = nw(X) = \pi w(X) = hd(X) = hL(X) = s(X) = L(X) = d(X) = c(X) = e(X) = wc(X)$;
   (d) X has a cellular family of cardinality $c(X)$;
   (e) $o(X) = 2^{w(X)}$.

(8.2) Let X be a completely metrizable space such that every non-empty open set is infinite and has weight $\geqslant\kappa$. Then $|X| \geqslant \kappa^\omega$.

(8.3) Let X be a completely metrizable space of weight κ. Then $|X| = \kappa$ or $|X| = \kappa^\omega$.

Bounds on the number of compact sets in X; bounds using extent

(9.1) For $X \in \mathscr{T}_2$, $|K(X)| \leqslant 2^{hL(X)}$.

(9.3) For $X \in \mathscr{T}_1$, $|K(X)| \leqslant 2^{e(X) \cdot psw(X)}$.

(9.4) For $X \in \mathscr{T}_2$, $|K(X)| \leqslant 2^{e(X) \cdot \Delta(X)}$.

(9.5) For $X \in \mathscr{T}_2$, $|K(X)| \leqslant 2^{e(X) \cdot \Psi(X)}$.

Bounds on the number of continuous, real-valued function on X

(10.1) For any space X, $|C(X)| \leqslant 2^{d(X)}$.

(10.3) For any infinite Hausdorff space X, $|C(X)| \leqslant |\mathrm{RO}(X)|$.

(10.5) Let X be an infinite, perfectly normal Hausdorff space. Then $|C(X)| = |\mathrm{RO}(X)| = o(X)$.

(10.6) Let X be a topological space.
(a) If $X \in \mathscr{T}_2$, $|C(X)| \leqslant \pi w(X)^{c(X)}$.
(b) If $X \in \mathscr{T}_3$, $|C(X)| \leqslant \pi\chi(X)^{c(X)}$.

(10.8) For any space X, $|C(X)| \leqslant w(X)^{wc(X)}$.

(10.10) For any space X, $|C(X)|^{\omega} = |C(X)|$.

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CHAPTER 2

Cardinal Functions II

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J. E. Vaughan

Introduction

It is a familiar feature of problems of set-theoretic character that they, more often than not, tend to be independent of the usual axioms of set theory, i.e. ZFC. This is of course well-documented also by this Handbook itself in which many of the chapters, e.g. those on the normal Moore space problem or the S- and L-space problem, deal with independence results almost exclusively. The present chapter falls into this same category, in fact its main purpose is to familiarize the reader with the particular methods used to obtain independence results concerning problems that involve cardinal functions (and that are not treated in other chapters of the Handbook).

In doing this we have chosen a different guiding principle than in JUHÁSZ [1982]. There the emphasis was on the use in topology of various combinatorial principles proven to be consistent within set theory. Here we have tried to show mainly how actual forcing arguments can be used for the same end. Of course, this requires from the reader the knowledge of the basics at least of forcing. The choice of this approach naturally reflects our belief that this will soon become indispensable for those who want to work in set-theoretic topology. In this we agree with the opinion expressed in the similar survey TALL [1980].

The contents of this paper fall into two groups. The first, in Sections 1–3, concerns the sharpness of certain inequalities between cardinal functions that are presented in Cardinal Functions I. In the second, Sections 4–8, the unifying theme is 'subspaces'. We have made an effort to present together with the results also the main open problems that are left open.

Naturally, our notation and basic definitions follow those of Part I by Hodel, The nth item from the kth section in there will be simply referred to as I. n. k. The basic reference for forcing is KUNEN [1980], but we write $D(f)$ instead of $\mathrm{dom}(f)$ for the domain of a function f.

1. The sharpness of bounds on the cardinality of X

To warm up, we start with an absolute result that will show that I.4.2 and I.4.4 cannot be sharpened to $|X| \leqslant d(x)^{\psi(X)}$, even for $X \in \mathcal{T}_3$.

1.0. THEOREM. *For every regular cardinal κ there is a 0-dimensional T_2 (hence T_3) space X such that $|X| > \kappa$ but $d(X) = \kappa$ and $\psi(X) = \omega$.*

PROOF. By a well known result of TARSKI [1928], there is an almost disjoint family $\mathcal{A} \subset [\kappa]^\kappa$ with $|\mathcal{A}| = \kappa^+$. Let us write, for technical reasons,

$$\mathcal{A} = \{A_\alpha : \alpha \in \kappa^+ \backslash \kappa\},$$

where $A_\alpha \neq A_\beta$ if $\alpha \neq \beta$.

Now, the underlying set of our space X is κ^+. Every point $\alpha \in \kappa$ is declared isolated.

If, on the other hand, $\alpha \in \kappa^+ \backslash \kappa$ then we first pick a *uniform* ultrafilter $\mathscr{u}_\alpha$ on the set A_α which is not ω_1-complete. The basic neighbourhoods of α in X will have the form $\{\alpha\} \cup \mathscr{U}$ for $\mathscr{U} \in \mathscr{u}_\alpha$. If $\beta \in \kappa^+ \backslash \kappa$ and $\beta \neq \alpha$, then clearly

$$\{\beta\} \cup (A_\beta \backslash A_\alpha)$$

is a neighbourhood of β in X disjoint from $\{\alpha\} \cup \mathscr{U}$, showing that $\{\alpha\} \cup \mathscr{U}$ is clopen, consequently that X is 0-dimensional and T_2. Furthermore, $\psi(\alpha, X) = \omega$ follows immediately from the fact that $\mathscr{u}_\alpha$ is not ω_1-complete.

Now, if we put e.g. $\kappa = (2^\omega)^+$, then also $\kappa^\omega = \kappa$, hence for the above X we have

$$d(X)^{\psi(X)} < |X| .$$

Results, like 1.0, that are not completely trivial and show the sharpness of certain inequalities concerning cardinal functions without any extra set-theoretic assumptions are rather rare. Thus turning to the independence results is unavoidable at the present state of development of this subject.

Now we shall present an example which shows that in the inequalities I.4.7 and I.4.9 the upper bounds $2^{c(X)\cdot\chi(X)}$ and $2^{s(X)\cdot\psi(X)}$ for $|X|$ cannot be improved to $c(X)^{\chi(X)}$ and $s(X)^{\psi(X)}$, in fact not even to $s(X)^{\chi(X)}$. However, as we have pointed out above, we have to understand this to mean "they cannot be improved in ZFC". Indeed, the example is constructed from an ω_2-Suslin line with the simultaneous assumption of CH, this holds e.g. if $V = L$, but might consistently fail, see LAVER and SHELAH [198·]. The construction, in a slightly different form, is due to R. Jensen, see also FLEISSNER [1978].

1.1. THEOREM. *If* CH *holds and there is an ω_2-Suslin line, then there is a generalized ordered space S for which*

$$s(S)^{\chi(S)} < |S| .$$

PROOF. We assume CH and consider an ω_2-Suslin line $\langle L, \prec \rangle$, i.e. a linear order with no ω_2 disjoint intervals but no dense set of size ω_1. By standard arguments, we may assume that L is continuously ordered. A point $x \in L$ is said to have left character ω in L, in symbols $\chi_\ell(x, L) = \omega$, if the 'half line'

$$L_x = \{y \in L : y \prec x\}$$

is cofinal with ω, i.e. x is the limit of an ω-sequence 'from the left'. The continuity of our ordering obviously implies that

$$S = \{x \in L: \chi_\ell(x, L) = \omega\}$$

is dense in L. This immediately implies $|S| > \omega_1$. Now consider the good old 'Sorgenfrey topology' on S, for which the half open intervals

$$\{(y, x]: y < x \in S\}$$

form a base. This is not necessarily an ordered space, but is a subspace of one: take $S \times 2$ with the lexicographic order! As such it is hereditarily collectionwise normal, hence (cf. e.g. JUHÁSZ [1980], 2.23) we have

$$c(S) = s(S) = c(L) = \omega_1 ,$$

and clearly $\chi(S) = \psi(S) = \omega$. Consequently we have

$$s(S)^{\chi(S)} = \omega_1 < |S| .$$

Let us remark that, with some extra work, an ordered example with the same properties was produced in FLEISSNER [1978].

The next example, due to S. Shelah, shows that in Archangel'skiĭ's inequality I.4.5 one cannot replace $\chi(X)$ by $\psi(X)$ (see I.4.6). The approach we take here is that of HAJNAL and JUHÁSZ [1980], which allows us to obtain some extra results that we will need later. First we need some definitions.

1.2. DEFINITION. A map $f: X^2 \to 2$ is called a directed graph on X. The graph f is called *flexible* if for any distinct $x, y \in X$ and $i, j \in 2$ there is a $z \in X$ with $f(z, x) = i$ and $f(z, y) = j$. For any $x \in X$ and $i \in 2$ we put

$$A_x^i = \{y \in X: y \neq x \;\&\; f(x, y) = i\}$$

and for any $s \in H(X) = \mathrm{Fn}(X, 2, \omega)$

$$U_s = \bigcap \{A_x^{s(x)} \cup \{x\}: x \in D(s)\} .$$

Finally we put $\mathcal{U}_f = \{U_s: s \in H(X)\}$.

Any directed graph f on X determines two 'natural' topologies on X:

1.3. DEFINITION. Given f, a directed graph on X, and $i \in 2$ we denote by τ_f^i the topology determined by the subbase

$$\mathcal{B}_f^i = \{A_x^i \cup \{x\}: x \in X\} \cup \{A_x^{1-i}: x \in X\} .$$

Clearly, τ_f^i is 0-dimensional and is easily seen to be Hausdorff if f is flexible. Note that in all the above the values of f on $\Delta_X = \{\langle x, x\rangle : x \in X\}$ are irrelevant.

1.4. LEMMA. *If $f: X^2 \to 2$ is flexible and τ_f^0 is Lindelöf, then τ_f^1 has countable pseudocharacter.*

PROOF. By flexibility, given $x \in X$ and $y \in X \setminus \{x\}$ there is $z \in X \setminus \{x, y\}$ such that $f(z, y) = 0$ and $f(z, x) = 1$. Thus, putting $Z = \{z \in X \setminus \{x\} : x \notin A_z^0\}$, the family

$$\{A_z^0 \cup \{z\} : z \in Z\} \cup [A_x^0 \cup \{x\}] \subset \tau_f^0$$

covers X. By the Lindelöfness of τ_f^0 then there is a countable $Z' \subset Z$ such that

$$\bigcup \{A_z^0 \cup \{z\} : z \in Z'\} \cup (A_x^0 \cup \{x\}) = X .$$

Consequently we have

$$X \setminus \{x\} = \bigcup \{A_z^0 \cup \{z\} : z \in Z'\} \cup A_x^0 ,$$

which shows that $X \setminus \{x\}$ is the union of countably many τ_f^1-closed sets, i.e. $\{x\}$ is a G_δ.

1.5. LEMMA. *If $f: X^2 \to 2$ is flexible and the family $\mathcal{U}_f$ is Lindelöf, then both τ_f^0 and τ_f^1 are Lindelöf and have countable pseudocharacter.*

PROOF. By Lemma 1.4 it suffices to show that the τ_f^i are Lindelöf. For this it suffices to show that any cover of X by sets which are finite intersections of members of $\mathcal{B}_f^i$ has a countable subcover. It is obvious, however, that any such (non-empty) finite intersection can be obtained from a member U_t of $\mathcal{U}_f$ by deleting from it finitely many of its elements. Thus our claim follows immediately from the Lindelöfness of $\mathcal{U}_f$.

These preliminaries show that our aim, i.e. to show that $|X| \leq 2^{L(X) \cdot \psi(X)}$ is not provable even for regular X, is achieved if the following is proven.

1.6. THEOREM. Con(ZF) $\to$ Con(ZFC + CH + $\exists$ *a flexible $F: \omega_2^2 \to 2$ for which $\mathcal{U}_F$ is Lindelöf*).

PROOF. The desired graph F will be constructed in a generic extension of V, where we assume at the start that $V \models$ ZFC + CH.

The conditions in our notion of forcing will determine a countable fragment of F. Thus, a condition p will decide a countable set $A^p \subset \omega_2$ and a function $f^p: (A^p)^2 \to 2$. Notice that if $s \in H(A^p)$, then f^p contains enough information to determine $U_s \cap A^p$. In other words, if we put for $s \in H(A^p)$

$$U_s^p = \{x \in A^p : \forall z \in D(s)(z \neq x \to f^p(z, x) = s(z))\},$$

then for any graph $f \supset f^p$ on ω_2 we have $U_s \cap A^p = U_s^p$.

Using this notation we can now define our notion of forcing P more precisely.

1.7. DEFINITION. A condition $p \in P$ will be a triple $p = \langle A, f, T\rangle$ such that

(i) $A \subset \omega_2$ and $|A| \leqslant \omega$;

(ii) $f: A^2 \to 2$;

(iii) $|T| \leqslant \omega$ and $\forall B \in T(B \subset H(A)$ & $\bigcup\{U_s^p : s \in B\} = A\}$;

(iv) $\forall B \in T\ \forall \delta \in A\ \forall y \in (A\backslash\delta)\ \forall h \in H(A\backslash\delta)\ \exists s \in B\ (y \in U_{s\restriction\delta}^p$ & $s \cup h \in H(A))$.

The reason for the complicated technical condition (iv) will become clear later.

We next define a 3-place relation E^p on A by

$$E^p(\delta, y, z) \leftrightarrow [\delta \leqslant y, z\ \&\ \forall x \in A \cap \delta(f(x, y) = f(x, z))].$$

A condition $p' = \langle A', f', T'\rangle \in P$ will extend p, i.e. $p' \leqslant p$, if and only if $A \subset A'$, $f \subset f'$, $T \subset T'$ and $E^p \subset E^{p'}$. Clearly, then $\leqslant$ is a partial order on P.

Next we establish certain properties of $\langle P, \leqslant\rangle$.

1.8. LEMMA. *$\langle P, \leqslant\rangle$ is ω_1-complete. More precisely, if $p_n = \langle A_n, f_n, T_n\rangle \in P$ and $p_{n+1} \leqslant p_n$ for each $n \in \omega$, and we put*

$$A = \bigcup_n A_n, \qquad f = \bigcup_n f_n, \qquad T = \bigcup_n T_n,$$

then $p = \langle A, f, T\rangle \in P$ and $p \leqslant p_n$ for all $n \in \omega$.

The easy proof of this is left to the reader.

1.9. LEMMA. *$\langle P, \leqslant\rangle$ satisfies the ω_2-antichain condition.*

PROOF. Using a standard Δ-system argument and the fact that CH holds in V, one can see that it suffices to show for this the compatibility in $\langle P, \leqslant\rangle$ of any two conditions $p = \langle A, f, T\rangle$ and $p' = \langle A', f', T'\rangle$ such that $\Delta = A \cap A' < A\backslash\Delta < A'\backslash\Delta$, $f\restriction\Delta^2 = f'|\Delta^2$, $tp(A\backslash\Delta) = tp(A'\backslash\Delta)$ and satisfying $f(x, z) = f'(x, z')$ for all $x \in \Delta$ whenever $z \in A\backslash\Delta$ and $z' \in A'\backslash\Delta$ are such that $tp(A \cap z) = tp(A' \cap z)$ (in this case z and z' are called corresponding members of $A\backslash\Delta$ and $A'\backslash\Delta$, respectively). We let γ be the first member of $A\backslash\Delta$, hence γ' is the corresponding member of $A'\backslash\Delta$.

To this end we shall construct $g: (A \cup A')^2 \to 2$ such that $g \supset f \cup f'$, $q = \langle A \cup A', g, T \cup T'\rangle \in P$, and q extends both p and p'. Of course, we only have to define g on

$$[(A\backslash\Delta)\times(A'\backslash\Delta)\cup(A'\backslash\Delta)\times(A\backslash\Delta)],$$

where the two summands are obviously disjoint.

Let us show how g is defined on $(A\backslash\Delta)\times(A'\backslash\Delta)$. The definition is carried out by induction in $\omega+1$ steps. To start with, we fix an ω-type enumeration of all quadruples $\langle B, \delta, z', h\rangle$ with $B\in T$, $\delta\in A\backslash\Delta$, $z'\in A'\backslash\Delta$, $h\in H(A\backslash\delta)$. Assume $i<\omega$ and we are to do step i of our construction, and that so far only finitely many $x\in A\backslash\Delta$ exist such that $g(x, y')$ has been defined for some $y'\in A'\backslash\Delta$. Let $\langle B, \delta, z', h\rangle$ be the ith term of the above enumeration. For any $x\in A\backslash\Delta$ with $g(x, z')$ already defined we put $k(x)=g(x, z')$, this defines $k\in H(A\backslash\Delta)$. Let z be the member of $A\backslash\Delta$ corresponding to z', then condition (iv) applied to p and B, γ, z and $(k\restriction\delta)\cup h\in H(A\backslash\gamma)$ (or to $k\in H(A\backslash\gamma)$ if $h=\emptyset$) yields us a $t\in B$ with $z\in U^p_{t\restriction\gamma}$ and $t\cup(k\restriction\delta)\cup h\in H(A\backslash\Delta)$ (or $t\cup k\in H(A\backslash\Delta)$ if $h=\emptyset$). Now, what we actually do in step i is putting $g(x, y')=t(x)$ whenever $x\in D(t)\backslash D[(k\restriction\delta)\cup h]$ (or $x\in D(t)\backslash D(k)$ if $h=\emptyset$) and $y'\in A'\backslash\Delta$ is such that $E^{p'}(\gamma', z', y')$ holds.

Having completed all steps $i<\omega$, in step ω we put $g(x, y')=0$ for any pair $\langle x, y'\rangle\in(A\backslash\Delta)\times(A'\backslash\Delta)$ for which $g(x, y')$ has not been previously defined. The definition of g on $(A'\backslash\Delta)\times(A\backslash\Delta)$ is similar, but even simpler. Details are left to the reader.

It remains to show that $q\in P$ and $q\le p, p'$. Now (i) and (ii) are obvious for q, to show (iii) pick e.g. $B\in T$ and $z'\in A'\backslash\Delta$ and consider $\langle B, \gamma, z', \emptyset\rangle$ which, let us say, occurs as the i-th term of the above enumeration. Then in step i of the construction of g on $(A\backslash\Delta)\times(A'\backslash\Delta)$ we made sure that $z'\in U^q_t$ for some $t\in B$. The proof of (iii) in the remaining cases is either similar or trivial.

To see (iv), consider $B\in T\cup T'$, $\delta\in A\cup A'$, $h\in H(A\cup A'\backslash\delta)$ and e.g. $z'\in A'\backslash\Delta$ with $\delta\leq z'$. If $B\in T'$, then we can choose $t\in B$ with $z'\in U^{p'}_{t\restriction\delta}$ and $t\cup(h\restriction A')\in H(A')$ (note that $t\restriction\delta=t\restriction\gamma'$ if $\delta\in A\backslash\Delta$), consequently $t\cup h\in H(A\cup A')$, hence we are done.

If $B\in T\backslash T'$ we have to distinguish three cases.

First, if $\delta\in\Delta$ we choose $t\in B$ such that $z\in U^p_{t\restriction\delta}$ and $t\cup(h\restriction A)\in H(A)$ with $z\in A\backslash\Delta$ corresponding to z'. Clearly then $z'\in U^{p'}_{t\restriction\delta}$ and $t\cup h\in H(A\cup A')$.

Next, if $\delta\in A\backslash\Delta$ we consider $\langle B, \delta, z', h\restriction A\rangle$ which occurs, say, as the ith term above. Then in step i of the construction of g on $(A\backslash\Delta)\times(A'\backslash\Delta)$ we insured the validity of $z'\in U^q_{t\restriction\delta}$ and $t\cup h\in H(A\cup A')$.

Finally, if $\delta\in A'\backslash\Delta$ then $t\restriction\delta=t$ for each $t\in B$, hence as $t\cup h\in H(A\cup A')$ holds trivially, (iv) follows from (iii) that we have already checked.

The proof of (iv) for $z\in A$ is trivial if $z\in\Delta$ and analogous to the above if $z\in A\backslash\Delta$.

Now $q\leq p$ is trivial, while $q\leq p'$ is valid because, in our construction of q, $E^q(\gamma', y', z')$ was made to hold whenever $E^{p'}(\gamma', y', z')$ did.

The next result provides two 'extension properties' of $\langle P, \leq\rangle$.

1.10. LEMMA. *Suppose $p = \langle A, f, T\rangle \in P$. Then*

(a) *for every $z \in \omega_2 \backslash A$ there is an extension $g: (A \cup \{z\})^2 \to 2$ of f such that $q = \langle A \cup \{z\}, g, T\rangle \in P$ and $q \leq p$;*

(b) *if $z \in \omega_2 \backslash (\bigcup A + 1)$, δ, $y \in A$ with $\delta \leq y$ and $h \in H(A \backslash \delta)$, there is an extension $g: (A \cup \{z\})^2 \to 2$ of f such that $q = \langle A \cup \{z\}, g, T\rangle \in P$, $q \leq p$, $g(x, z) = h(x)$ for each $x \in D(h)$, moreover $E^q(\delta, y, z)$ holds.*

PROOF. (a) First of all we put $g(z, y) = 0$ for all $y \in A$, this will insure $q \leq p$ once $q \in P$ has been shown. The values $g(y, z)$ will be defined by induction in $\omega + 1$ steps. To start with, we fix an ω-type enumeration of all triples $\langle B, \delta, h\rangle$ satisfying $B \in T$, $\delta \in (A \cap z) \cup \{z\}$ and $h \in H(A \backslash \delta)$. For each $i < \omega$ only finitely many values $g(y, z)$ will be defined.

Suppose we are to do step $i < \omega$. Put $k(y) = g(y, z)$ for each $y \in A$ with $g(y, z)$ already defined, and let $\langle B, \delta, h\rangle$ be the ith term above. Since p satisfies (iv), there is a $t \in B$ with $t \cup (k \restriction \delta) \cup h \in H(A)$ (or $t \cup k \in H(A)$ if $h = \emptyset$), hence we can put, in step i, $g(y, z) = t(y)$ for $y \in D(t) \backslash D[(k \restriction \delta) \cup h]$ (or for $y \in D(t) \backslash D(k)$ if $h = \emptyset$). This will insure $z \in U^q_{t \restriction \delta}$ (or $z \in U^q_t$ if $h = \emptyset$) and $t \cup h \in H(A \cup \{z\})$. Finally, in step ω we put $g(y, z) = 0$ for each $y \in A$ that has not occurred before.

Next we have to show (iii) and (iv) for q. Condition (iii) follows immediately from our above remark because the triple $\langle B, z, \emptyset\rangle$ was considered in some step i.

To see (iv) let $B \in T$, $\delta \in A \cup \{z\}$, $y \in A \cup \{z\}$ with $\delta \leq y$ and $h \in H(A \cup \{z\} \backslash \delta)$. If $y \neq z$ everything follows because p satisfies (iv). If $y = z$ we made $z \in U^q_{t \restriction \delta}$ and $t \cup h \in H(A \cup \{z\})$ valid for some $t \in B$ in the step in which the triple $\langle B, \delta, h \restriction A\rangle$ was considered.

(b) Now $g(z, x)$ for $x \in A$ can be chosen arbitrarily, moreover we have to put $g(x, z) = h(x)$ for $x \in D(h)$, and $g(x, z) = f(x, y)$ for $x \in A \cap \delta$, to insure $E^q(\delta, y, z)$. The values $g(x, z)$ for $x \in A \backslash (\delta \cup D(h))$ are then defined by an $(\omega + 1)$-step induction similarly as in part (a). The details are left to the reader.

Now, from Lemmas 1.8, 1.9 and 1.10(a) it follows that if G is $\langle P, \leq\rangle$-generic over V, then in $V[G]$ cardinals are preserved, CH holds, there are no new countable subsets of V, moreover if

$$F = \bigcup\{f: \exists A, T(\langle A, f, T\rangle \in G)\},$$

then $F: \omega_2^2 \to 2$. It is also immediate e.g. from 1.10(b) that F is flexible.

1.11. LEMMA. *If $p = \langle A, f, T\rangle \in P$, then $p \Vdash \check{\omega}_2 = \bigcup\{U_t: t \in \check{B}\}$ for each $B \in T$.*

PROOF. It is immediate from property (iii) of p that $p \Vdash \check{A} \subset \bigcup\{U_t: t \in \check{B}\}$. But for every $p' \leq p$ and $z \in \omega_2$, by 1.10(a), there is a $p'' = \langle A'', f'', T''\rangle \leq p'$ with $z \in A''$, hence

$$p'' \Vdash \check{z} \in \bigcup\{U_t : t \in \check{B}\},$$

which shows that indeed

$$p \Vdash \check{\omega}_2 = \bigcup\{U_t : t \in \check{B}\}.$$

Now we have everything necessary to prove that $\mathcal{U}_F$ is Lindelöf (in $V[G]$). Let $g: \omega_2 \to H(\omega_2)$ be a map in $V[G]$ such that $x \in U_{g(x)}$ for each $x \in \omega_2$. Then there is a $p = \langle A, f, T\rangle \in G$ and a P-name σ for g such that

$$(*) \qquad p \Vdash \sigma: \check{\omega}_2 \to H(\check{\omega}_2) \,\&\, \forall y \in \check{\omega}_2 (y \in U_{\sigma(y)}).$$

Clearly, it will suffice to show that whenever p satisfies $(*)$ it has an extension which forces "$\check{\omega}_2 = \bigcup\{U_{\sigma(y)} : y \in \check{A}^*\}$" for some $A^* \in [\omega_2]^\omega$.

Let us note that if $q \in P$ satisfies $(*)$ (e.g. if $q \leqslant p$), then for every $y \in \omega_2$ we can fix an $r = r(q, y) \leqslant q$ and a $t \in H(\omega_2)$ such that $r \Vdash \sigma(\check{y}) = \check{t}$. It is easy to see that then $t \in H(A^r)$.

Next we construct a decreasing ω-sequence of conditions as follows. Put $p_0 = p$, and if $p_i = \langle A_i, f_i, T_i\rangle$ has been defined we fix an enumeration $\{\langle \delta_j^{(i)}, x_j^{(i)}, h_j^{(i)}\rangle : j \in \omega\}$ of the set of all triples $\langle \delta, x, h\rangle$ such that $\delta, x \in A_i$, $\delta \leqslant x$ and $h \in H(A_i \setminus \delta)$. We also fix a bijection $\varphi: \omega \leftrightarrow \omega^2$ such that $\varphi(n) = \langle i, j\rangle$ implies $i \leqslant n$ for each $n \in \omega$. Suppose that, $p = p_0 \geqslant \cdots \geqslant p_n$ having been defined, we want to choose p_{n+1}. For this consider $\varphi(n) = \langle i, j\rangle$, pick a $z_n \in \omega_2 \setminus (\bigcup A_n + 1)$, and applying Lemma 1.10(b) take an extension $q_n \leqslant p_n$ such that $E^{q_n}(\delta_j^{(i)}, x_j^{(i)}, z_n)$ holds, moreover $f^{q_n}(x, z_n) = h_j^{(i)}(x)$ for each $x \in D(h_j^{(i)})$. Then we apply our above remark and put $p_{n+1} = r(q_n, x_j^{(i)})$.

Since $p_{n+1} \leqslant p_n$ holds for $n \in \omega$, by Lemma 1.8 we have $q^* = \langle A^*, f^*, T^*\rangle \in P$ and $q^* \leqslant p_n$ for each n, where

$$A^* = \bigcup_n A_n, \qquad f^* = \bigcup_n f_n, \qquad T^* = \bigcup_n T_n.$$

Now put

$$B^* = \{t \in H(A^*) : \exists y \in A^* (q^* \Vdash \sigma(\check{y}) = \check{t})\}.$$

We claim that $p^* = \langle A^*, f^*, T^* \cup \{B^*\}\rangle \in P$, i.e. that (iii) and (iv) are satisfied by B^*. To see (iii), pick any $y \in A^*$ and $\langle i, j\rangle \in \omega^2$ with $y = x_j^{(i)}$. Thus, if $\varphi(n) = \langle i, j\rangle$, then we have $p_{n+1} = r(q_n, y)$, hence

$$p_{n+1} \Vdash \sigma(\check{y}) = \check{t}$$

for some $t \in H(A_{n+1}) \subset H(A^*)$ and then

$$q^* \Vdash \sigma(\check{y}) = \check{t}$$

as well. But q^* also forces "$\check{y} \in U_{\sigma(\check{y})}$", hence

$$q^* \Vdash \text{“}\check{y} \in U_{\check{t}}\text{”},$$

which clearly implies $y \in U_t^{q^*} = U_t^{p^*}$.

To verify (iv), consider $\delta, y \in A^*$ with $\delta \leqslant y$ and $h \in H(A^* \setminus \delta)$ and let $\langle \delta, y, h \rangle = \langle \delta_j^{(i)}, y_j^{(i)}, h_j^{(i)} \rangle$ with $\varphi(n) = \langle i, j \rangle$. Then $z_n \in A_{n+1} \setminus A_n \subset A^*$ and $q_n \leqslant p_n$ were chosen in such a way that $E^{q_n}(\delta, y, z_n)$ and $f_{n+1}(x, z_n) = h(x)$ be valid for $x \in D(h)$. Since (iii) has already been established for B^*, we have a $t \in B^*$ with $z_n \in U_t^{p^*}$. By $q^* \leqslant q_n$ we also have $E^{q^*}(\delta, y, z_n)$ and $f^*(x, z_n) = h(x)$ for each $x \in D(h)$, which clearly imply $y \in U_{t \upharpoonright \delta}^{p^*}$ and $t \cup h \in H(A)$.

Thus, to sum up, we have $p^* \leqslant p$ and by Lemma 1.11

$$p^* \Vdash \omega_2 = \bigcup \{U_t \colon t \in \check{B}^*\},$$

hence, by the definition of B^*,

$$p^* \Vdash \omega_2 = \bigcup \{U_{\sigma(y)} \colon y \in \check{A}^*\}.$$

This completes our proof.

Let us note that the existence of the "graph" F from 1.6, thus also all its topological consequences, follows from a combinatorial principle introduced in VELLEMAN [1980], the existence of an $(\omega_1, 1)$-morass with built in $\diamond$ sequence, which is valid in L, the constructible universe.

In ARCHANGEL'SKIĬ [1978] it was asked what happens if to $L(X) = \psi(X) = \omega$ we also add $c(X) = \omega$, i.e. whether $|X| \leqslant 2^\omega$, or more generally, $|X| \leqslant 2^{L(X) \cdot \psi(X) \cdot c(X)}$ is provable for regular X. Our next result shows that the above examples decide this question also in the negative.

1.12. THEOREM. *If F is as in* 1.6, *then $c(\langle \omega_2, \tau_F^i \rangle) = \omega$ for both $i \in 2$.*

PROOF. As was noticed in 1.5, every basic open set of τ_F^i differs from some $U_t \in \mathcal{U}_F$ in a finite set only. Consequently, it suffices to show that there is no uncountable subfamily $\mathcal{V}$ of $\mathcal{U}_F$ such that any two members of $\mathcal{V}$ have a finite intersection. To this end we first show that $|U_t| = \omega_2$ for every $t \in H(\omega_2)$. In fact, what we show is that

$$1 \Vdash |U_{\check{t}}| = \check{\omega}_2.$$

This will follow from the following: for every $p \in P$ and $\alpha \in \omega_2$ there is a $q \leqslant p$ and $z \in \omega_2 \setminus \alpha$ with $q \Vdash \check{z} \in U_{\check{t}}$, hence

$$q \Vdash U_i \not\subset \check{\alpha}\,.$$

Indeed, by 1.10(a) we can first find $r = \langle A, f, T\rangle \leqslant p$ such that $D(t) \subset A$, then by 1.10(b) for any $z \in \omega_2 \setminus (\bigcup A + 1) \cup \alpha$ we can extend f to a map $g\colon (A \cup \{z\})^2 \to 2$ such that $q = \langle A \cup \{z\}, g, T\rangle$ extends p and $g(x, z) = t(x)$ for each $x \in D(t)$. But clearly $q \Vdash \check{z} \in U_i$, hence $q \Vdash U_i \not\subset \check{\alpha}$.

But if $\mathscr{H} \in [H(\omega_2)]^{\omega_1}$ is given in $V[G]$, then, as is well known, we can always find distinct $t, s \in \mathscr{H}$ such that $t \cup s \in H(\omega_2)$, and then

$$H_t \cap H_s = H_{t\cup s} \to |H_t \cap H_s| = \omega_2\,.$$

Theorem 1.6, while it does solve Archangel'skiĭ's problem, leaves us really in limbo with the question: what is the (an) exact upper bound for the size of a regular (or Hausdorff) Lindelöf space of countable pseudo-character? Our construction does not seem to be modifiable even to yield such a space of cardinality ω_3 (the reason being that the Δ-system argument in 1.9 which allows us to assume $\Delta < A\setminus\Delta < A'\setminus\Delta$ works only in ω_2).

The following is easy to see.

1.13. PROPOSITION. *If X is any Lindelöf space of countable pseudo-character, then $|X|$ is less than the first measurable cardinal and $|X|$ cannot be weakly compact.*

On the other hand example 7.2 of JUHÁSZ [1980] shows that for every cardinal κ less than the first measurable there is a T_1-space X with $L(X) = \psi(X) = \omega$ and $|X| > \kappa$, hence the case here, i.e. for T_1-spaces, is quite different. This only makes the above question (namely, where is the exact upper bound actually located in the distressingly long interval between $\omega_2 + 2^\omega$ and the first measurable cardinal) even more interesting.

Another question that is left open here is whether Lindelöf T_1-spaces of countable character do necessarily have cardinality $\leqslant 2^\omega$? This should also be compared with 4.1.

2. The sharpness of bounds using spread

The most significant inequality involving spread is without doubt I.4.12, i.e. $|X| \leqslant 2^{2^{s(X)}}$ for $X \in \tau_2$. Moreover the presence of two exponents in it suggests the question of its sharpness very strongly. We start this section with a result of TODORČEVIĆ [1981] showing that, at least in the very important special case of $s(X) = \omega$, one exponent may be dropped—consistently.

2.1. THEOREM. PFA *implies that every T_2-space of countable spread has cardinality $\leqslant 2^\omega$.*

PROOF. We shall actually make use of the following combinatorial consequence (H) of PFA, also established by TODORČEVIĆ (we could also use instead BAUMGARTNER'S principle G_2).

(H): Whenever $[\omega_1]^2 = K_0 \cup K_1$ is a 2-partition of ω_1 there is either an $A \in [\omega_1]^{\omega_1}$ with $[A]^2 \subset K_0$ or there are $A \in [\omega_1]^{\omega_1}$ and $\mathscr{B} \subset [\omega_1]^{<\omega}$ such that $|\mathscr{B}| = \omega_1$ and $\mathscr{B}$ is disjoint, moreover for each $\alpha \in A$ and $B \in \mathscr{B}$ with $\alpha < \min B$ there is a $\beta \in B$ with $\{\alpha, \beta\} \in K_1$.

2.2. DEFINITION. A space X is called *quasi-Lindelöf* if every open cover $\mathscr{U}$ of X has a countable subfamily $\mathscr{V}$ such that $X = \bigcup\{\bar{V}: V \in \mathscr{V}\}$.

2.3. LEMMA. *If X is Hausdorff and hereditarily quasi-Lindelöf, then $\psi(X) \leq \omega$.*

PROOF. Since X is T_2, every point $p \in X$ is the intersection of its closed neighbourhoods, hence $X \setminus \{p\}$ is the union of their (open) complements. But $x \setminus \{p\}$ is quasi-Lindelöf, hence it is also the union of the closures of countably many of them (because p does not belong to any such closure). Hence $X \setminus \{p\}$ is a F_σ and thus $\psi(p, X) \leq \omega$.

2.4. LEMMA. (H) (*and thus* PFA) *implies that every space of countable spread is hereditarily quasi-Lindelöf.*

PROOF. Since having countable spread is hereditary, it clearly suffices to show that it implies quasi-Lindelöfness. Assume, indirectly, that $s(X) = \omega$ but X is not quasi-Lindelöf, i.e. $\mathscr{U}$ is an open cover of X and $X \neq \bigcup\{\bar{V}: V \in \mathscr{V}\}$ for every $\mathscr{V} \in [\mathscr{U}]^{\leq\omega}$. By an easy transfinite induction we can define then $Y = \{p_\alpha: \alpha \in \omega_1\} \subset X$ and $\{V_\alpha: \alpha \in \omega_1\} \subset \mathscr{U}$ such that

$$p_\alpha \in V_\alpha \setminus \bigcup\{\bar{V}_\beta: \beta \in \alpha\}$$

for each $\alpha \in \omega_1$. Note that Y is right separated and also $s(Y) = \omega$, consequently Y must be hereditarily separable.

Let us now define a 2-partition

$$[\omega_1]^2 = K_0 \cup K_1$$

as follows: for $\alpha < \beta < \omega_1$ put

$$\{\alpha, \beta\} \in K_0 \leftrightarrow p_\alpha \notin V_\beta .$$

Now, $[A]^2 \subset K_0$ clearly implies that $\{p_\alpha: \alpha \in A\}$ is discrete, hence $|A| \leq \omega$. Thus, we have $A \in [\omega_1]^{\omega_1}$ and $\mathscr{B} \subset [\omega_1]^{<\omega}$ as stipulated above. Now $Z =$

$\{p_\alpha: \alpha \in A\} \subset Y$, hence by our above remark Z is separable, there is a $\gamma \in \omega_1$ such that $Z_0 = \{p_\alpha: \alpha \in \gamma \cap A\}$ is dense in Z. We can choose a $B \in \mathscr{B}$ with $\gamma < \min B$, then for every $\alpha \in \gamma \cap A$ there is a $\beta \in B$ with $\{\alpha, \beta\} \in K_1$, i.e. $p_\alpha \in V_\beta$. Consequently, we have $Z_0 \subset \bigcup\{V_\beta: \beta \in B\}$, thus $Z \subset \bigcup\{\bar{V}_\beta: \beta \in B\}$, which is clearly impossible.

Now, to finish the proof of 2.1, observe that if X is T_2 and $s(X) = \omega$, then by 2.4 and 2.3, we also have $\psi(X) \leq \omega$, hence by I.4.9

$$|X| \leq 2^{s(X)\cdot\psi(X)} = 2^\omega .$$

Next we look at the other side of this same coin and look for models in which I.4.12 is sharp. For the sake of simplicity we shall again restrict ourselves to the case $s(X) = \omega$, though contrary to 2.1, these results easily generalize to higher cardinals.

The main tool we use is a slight generalization of the notion of HFD-spaces as formulated in the article "Basic S and L" by Judy Roitman in this volume.

2.5. DEFINITION. Let I be uncountable, then a set $A \subset 2^I$ in the I-th power of the two-point discrete space is called *almost dense* if there is a $J \in [I]^{\leq\omega}$ such that

$$A \upharpoonright (I \backslash J) = \{f \upharpoonright (I\backslash J): f \in A\}$$

is dense in $2^{I\backslash J}$, i.e. for avery $\varepsilon \in H(I\backslash J)$ there is an $f \in A$ with $\varepsilon \subset f$. We say that an infinite set $X \subset 2^I$ is HFD if every $A \in [X]^\omega$ is almost dense.

In the following proposition we mention several basic properties of HFD's whose proofs can be found in Roitman's article or in HAJNAL and JUHÁSZ [1974].

2.6. PROPOSITION. *If $X \subset 2^I$ is* HFD, *then X is hereditarily separable (i.e. $hd(X) = \omega$), and if for each $f \in X$ we have an $f' \in 2^I$ such that*

$$|\{i \in I: f(i) \neq f'(i)\}| \leq \omega ,$$

then $X' = \{f': f \in X\}$ is also HFD. *Every* HFD *is hereditarily collectionwise normal.*

Our aim is to construct large HFD's. But how large can an HFD be?

2.7. PROPOSITION. *If $X \subset 2^I$ is* HFD, *then $|I| \leq 2^\omega$ and $|X| \leq 2^{\omega_1}$.*

PROOF. For any $A \in [X]^\omega$ there is a $J \in [I]^{\leq\omega}$ such that $A \upharpoonright (I\backslash J)$ is dense in $2^{I\backslash J}$, hence, e.g. by I.3.3(b), we have

$$|I| = |I\backslash J| = w(2^{I\backslash J}) \leq 2^\omega .$$

To show $|X| \leqslant 2^{\omega_1}$, fix a set $J \in [I]^{\omega_1}$ and note that for any $g \in 2^J$

$$|\{f \in X : f \upharpoonright J = g\}| < \omega,$$

since otherwise X could not be an HFD. But then the map $f| \to f|J$ of X into 2^J is finite-to-one, thus clearly $|X| \leqslant 2^{\omega_1}$.

We propose to show that the bounds in 2.7 are sharp. To this end we first state a result of BAUMGARTNER [1976] without proof.

2.8. PROPOSITION. *If $\omega_1 \leqslant \kappa < \lambda$ are regular cardinals, then there is a generic extension of the universe in which cardinals are preserved, $2^\omega = \kappa$, $2^{\omega_1} = \lambda$, and there is a family $\mathcal{A} = \{A_\alpha : \alpha \in \lambda\} \subset [\kappa]^\kappa$ such that $|A_\alpha \cap A_\beta| \leqslant \omega$ for any $\alpha < \beta < \lambda$.*

2.9. THEOREM. *Suppose V satisfies the conclusion of 2.8 and consider $V[G]$ obtained by adding κ Cohen reals to V. Then in $V[G]$ there is an* HFD*-set $X \subset 2^\kappa$ with $|X| = \lambda$.*

PROOF. Let us take as our Cohen notion of forcing not the usual $H(\kappa)$ but the isomorphic $H(\kappa^2)$ (with the extension of functions as partial order, of course). Now, for G, a $H(\kappa^2)$-generic set over V, put $F = \bigcup G : \kappa^2 \to 2$. Next, for each $\alpha \in \lambda$ we fix a one-one map $g_\alpha : \kappa \to \kappa$ such that $A_\alpha = R(g_\alpha)$. Then the HFD-set X is defined in $V[G]$ as follows: $X = \{h_\alpha : \alpha \in \lambda\}$, where

$$h_\alpha(\nu) = F(\nu, g_\alpha(\nu)).$$

To see that X is indeed HFD fix a set $a \in [\lambda]^\omega$ in $V[G]$. Then, as is well known (see e.g. VIII.2.2 in KUNEN [1980]), one can find a $b \in [\kappa]^\omega$ such that $a \in V[G \cap H(b^2)]$. Since the $R(g_\alpha) = A_\alpha$'s have pairwise countable intersections we can also assume that $g_\alpha(\nu) \neq g_\beta(\nu)$ whenever $\nu \in \kappa \backslash b$ and α, β are distinct members of a. We want to show that under these conditions

$$\{h_\alpha \upharpoonright (\kappa \backslash b) : \alpha \in a\}$$

is dense in $2^{\kappa \backslash b}$. To this end first note that

$$H(\kappa^2) \approx H(b^2) \times H(\kappa^2 \backslash b^2)$$

holds in V, hence we can use the product lemma for forcing (or more specifically VIII.2.1 of KUNEN [1980]) by which

$$V[G] = V[G_0][G_1],$$

where

$$G_0 = G \cap H(b^2) \quad \text{and} \quad G_1 = G \cap H(\kappa^2 \backslash b^2).$$

Thus we can use forcing with $H(\kappa^2\backslash b^2)$ over $V[G_0]$, the corresponding forcing relation will be denoted by $\Vdash_1$.

Now pick any $\varepsilon \in H(\kappa\backslash b)$ and an arbitrary condition $p \in H(\kappa^2\backslash b^2)$. Clearly, it will suffice to show that p has an extension q such that for some $\alpha \in a$

$$q \Vdash_1 \check{\varepsilon} \subset h_{\check{\alpha}}.$$

By our above condition on b, for each $\nu \in D(\varepsilon)$ there are only finitely many $\alpha \in a$ with $\langle \nu, g_\alpha(\nu)\rangle \in D(p)$. Thus we can find an $\alpha \in a$ for which $\langle \nu, g_\alpha(\nu)\rangle \notin D(p)$ for every $\nu \in D(\varepsilon)$. Hence we can put

$$q = p \cup \{\langle\langle \nu, g_\alpha(\nu)\rangle, \varepsilon(\nu)\rangle : \nu \in D(\varepsilon)\},$$

and clearly $q \in H(\kappa^2\backslash b^2)$ is as required.

2.10. COROLLARY. *If $\lambda = 2^\kappa$ (e.g. if $\kappa = \omega_1$), then in $V[G]$ there is a hereditarily collectionwise normal space X with $s(X) = hd(X) = \omega$ and $|X| = \lambda = 2^{2^\omega}(=2^{\omega_1})$.*

2.10 is as far as at present we can go in showing the sharpness of I.4.12. However, if we compare 2.1 and 2.10 then, remembering that $2^\omega = 2^{\omega_1}$ under PFA, we cannot avoid wondering whether 2^{ω_1} is the real upper bound for a T_2 (or $T_3 \ldots$) space of countable spread?

There are also other uses of 2.9 than 2.10. First we shall use it to show that I.5.3 is sharp.

2.11. THEOREM. *If there is an* HFD *set $X \subset 2^\kappa$ with $|X| = \kappa$, then we also have a hereditarily separable and right separated $Y \subset 2^\kappa$ with $|Y| = \kappa$. In particular, then $hL(Y) = w(Y) = \kappa$.*

PROOF. Suppose $X = \{f_\alpha : \alpha \in \kappa\}$ is a one-one enumeration of X, then Y is defined as $Y = \{g_\alpha : \alpha \in \kappa\}$, where g_α is determined by the following stipulations:

$$g_\alpha(\nu) = \begin{cases} 0 & \text{if } \nu < \alpha, \\ 1 & \text{if } \nu = \alpha, \\ f_\alpha(\nu) & \text{if } \nu > \alpha. \end{cases}$$

It is obvious that Y is right separated in the well-ordering given by the indices of its elements, so it only remains to show that Y is hereditarily separable. (Observe that if $\kappa > \omega_1$ then Y is not an HFD!)

Assume, indirectly, that Y is not hereditarily separable, i.e. there is an $a \subset \kappa$ with $tp(a) = \omega_1$ such that $Z = \{g_\alpha : \alpha \in a\}$ is left separated. By standard arguments, we can assume that the natural ω_1-type order on a is the left separating order of Z. Thus if $a = \{\alpha_\xi : \xi \in \omega_1\}$ is the increasing enumeration of a, for each $\xi \in \omega_1$

there is a $D_\xi \in [\kappa]^{<\omega}$ such that if $\eta < \xi$, then

$$g_{\alpha_\eta} \restriction D_\xi \neq g_{\alpha_\xi} \restriction D_\xi .$$

Let L_1 be the set of limit ordinals $\lambda < \omega_1$ and for each $\lambda \in L_1$ put

$$\beta_\lambda = \bigcup\{\alpha_\xi : \xi \in \lambda\} .$$

Let us define the function $\varphi : L_1 \to \omega_1$ as follows: for $\lambda \in L_1$

$$\varphi(\lambda) = \min\{\xi : D_\lambda \cap \beta_\lambda \subset \alpha_\xi\} .$$

Clearly φ is regressive, hence by Neumer's theorem there are an $S \in [L_1]^{\omega_1}$ and a $\xi \in \omega_1$ such that for each $\sigma \in S$

$$\varphi(\sigma) = \xi \quad \text{and} \quad \xi < \sigma .$$

Thus for each $\sigma \in S$ and $\zeta < \alpha_\xi$ we have $g_{\alpha_\sigma}(\zeta) = 0$, hence if $\rho, \sigma \in S$ and $\rho < \sigma$ we must have by the above

$$g_{\alpha_\rho} \restriction (D_\sigma \backslash \alpha_\xi) \neq g_{\alpha_\sigma} \restriction (D_\sigma \backslash \alpha_\xi) .$$

But we also have for $\sigma \in S$: $D_\sigma \backslash \alpha_\xi = D_\sigma \backslash \beta_\sigma$, hence for $\rho < \sigma$

$$g_{\alpha_\rho} \restriction (D_\sigma \backslash \alpha_\xi) = f_{\alpha_\rho} \restriction (D_\sigma \backslash \alpha_\xi) .$$

This shows that if we put for $\sigma \in S$

$$h_\sigma(\nu) = \begin{cases} g_{\alpha_\sigma}(\nu) & \text{if } \nu \in D_\sigma \backslash \alpha_\xi , \\ f_{\alpha_\sigma}(\nu) & \text{if } \nu \in \kappa \backslash (D_\sigma \backslash \alpha_\xi) , \end{cases}$$

then again for $\rho, \sigma \in S$, $\rho < \sigma$ we have

$$h_\rho \restriction (D_\sigma \backslash \alpha_\xi) \neq h_\sigma \restriction (D_\sigma \backslash \alpha_\xi) ,$$

hence $\{h_\sigma : \sigma \in S\}$ is left separated. This, however, is impossible because now

$$|\{\nu : h_\sigma(\nu) \neq f_{\alpha_\sigma}(\nu)\}| < \omega$$

for each $\sigma \in S$, hence by 2.6 $\{h_\sigma : \sigma \in S\}$ is HFD.

2.12. REMARK. There is a notion, sort of dual to the HFD's, that of the HFC subsets of 2^I. Now, 2.9 and its consequences all 'dualize' for the HFC case, thus we obtain e.g. hereditarily Lindelöf T_3 spaces of weight 2^{2^ω} and one that is left-separated and of size 2^ω which can be as big as you wish. The consistency of the existence of spaces with the latter properties was also shown by SHELAH [1978], using a completely different method. This is of special interest in view of a result of ŠAPIROVSKII [1974], which says that $hd(X) \leq s(X)^+$ if X is compact T_2 (see I.7.17).

3. On products

Our first result here will show the sharpness of Kurepa's result I.11.7 about the cellularity of product spaces. The question whether the exponentiation occurring in this result could not be replaced with just taking the successor had been around for a very long time, since already in KUREPA [1950] it was shown that for a Suslin line S one has $c(S \times S) = c(S)^+$ and for a long time only this type of example had been known showing that the cellularity can actually be raised in products. The problem was eventually solved in FLEISSNER [1978].

3.1. THEOREM. *If κ is an infinite cardinal in V and $V[G]$ is the generic extension of V obtained by adding κ Cohen reals to V, then in $V[G]$ there are compact 0-dimensional (even extremally disconnected) spaces X and Y such that $c(X) = c(Y) = \omega$ but $c(X \times Y) = \kappa$.*

PROOF. We shall actually construct partial orders P and Q with the required properties, i.e. P and Q are ccc but $c(P \times Q) = \kappa$. Then P and Q are embeddable into complete Boolean algebras B and D as dense subsets and the Stone spaces of B and D will be as required.

Now similarly as in 2.9, we shall not take G as $H(\kappa)$-generic, but in this case as $P = H([\kappa]^2)$-generic over V. We put $F = \bigcup G\colon [\kappa]^2 \to 2$, then let

$$C = \{\{\alpha, \beta\} \in [\kappa]^2\colon F(\{\alpha, \beta\}) = 0\}.$$

Next we define R and Q:

$$R = \{s \in [\kappa]^{<\omega}\colon [s]^2 \subset C\} \quad \text{and} \quad Q = \{s \in [\kappa]^{<\omega}\colon [s]^2 \subset [\kappa]^2 \setminus C\}.$$

Both R and Q are ordered by reverse inclusion. Note that $s, t \in R$ (Q) are compatible if and only if $s \cup t \in R$ (Q). We first show that R is ccc, the same for Q follows by symmetry.

Let $p \in P$ and τ be a P-name such that

$$p \Vdash \tau\colon \check{\omega}_1 \to R.$$

Then for each $\xi \in \omega_1$ we can define, in V, a $p_\xi \leq p$ and an $s_\xi \in [\kappa]^{<\omega}$ with

$$p_\xi \Vdash \tau(\check{\xi}) = \check{s}_\xi .$$

We can obviously assume that $D(p_\xi) = [t_\xi]^2$ for some $t_\xi \in [\kappa]^{<\omega}$, and obviously we must have then $[s_\xi]^2 \subset D(p_\xi)$, hence $s_\xi \subset t_\xi$. Using a standard Δ-system and counting argument we can find distinct $\xi, \eta \in \omega_1$ such that

$$p_\xi \restriction [t_\xi \cap t_\eta]^2 = p_\eta \restriction [t_\xi \cap t_\eta]^2 .$$

Thus we can define an extension q of $p_\xi \cup p_\eta$ to $[t_\xi \cup t_\eta]^2$ by putting $q(\{\alpha, \beta\}) = 0$ whenever $\alpha \in t_\xi \backslash t_\eta$ and $\beta \in t_\eta \backslash t_\xi$. Clearly, then

$$q \Vdash F''[s_\xi \cup s_\eta]^2 = \{0\} ,$$

i.e. p has an extension that forces $\tau(\check{\xi})$ and $\tau(\check{\eta})$ to be compatible in R.

Next, since $|R| = |Q| = \kappa$ is obvious, hence $c(R \times Q) \leq \kappa$, for the second half it suffices to show that there is an antichain of cardinality κ in $R \times Q$.

To this end observe that $\{\alpha\} \in R \cap Q$ for each $\alpha \in \kappa$ trivially, hence

$$\{\langle\{\alpha\}, \{\alpha\}\rangle : \alpha \in \kappa\} \subset R \times Q .$$

But if $\{\alpha, \beta\} \in [\kappa]^2$ and $F(\{\alpha, \beta\}) = 0$ $(=1)$ then $\{\alpha, \beta\} \notin Q$ $(\notin R)$, hence $\langle\{\alpha\}, \{\alpha\}\rangle$ and $\langle\{\beta\}, \{\beta\}\rangle$ must be incompatible in $R \times Q$.

3.2. REMARK. It is well known by now that under $\mathrm{MA}(\omega_1)$ any product of ccc spaces is ccc, thus for the case in which every factor of a product has countable cellularity everything is settled (i.e. is independent). Not much is known however for higher cardinals. Todorčević has recently shown in ZFC that if $\kappa = 2^\omega$ or $\kappa = \mathrm{cf}(2^\omega)$, then there exist spaces with the κ-cc whose product is not κ-cc.

Another problem of which very little is known concerns the Lindelöf degree of products. It has been always known that, unlike compactness, the Lindelöf property is not productive, the Sorgenfrey line S for instance satisfies $L(S) = hL(S) = \omega$ but $L(S^2) = 2^\omega$. It was natural to ask whether this increase with one exponent gives the maximum possible increase of the Lindelöf degree in products, similarly as for the cellularity c, or for a number of other cardinal functions including hL, see §5 of JUHÁSZ [1980]. Our next result here, taken from HAJNAL and JUHÁSZ [1980], shows that the answer to this question is negative.

3.3. THEOREM. $\mathrm{Con(ZF)} \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{CH} + \exists$ *Lindelöf* 0*-dimensional* T_2*-spaces* X^0 *and* X^1 *such that* $L(X^0 \times X^1) = \omega_2 > 2^\omega = \omega_1)$.

PROOF. Consider the model of ZFC + CH constructed in 1.6, where a flexible $F: \omega_2^2 \to 2$ with $\mathcal{U}_F$ Lindelöf was constructed. We know from 1.5 that then both τ_F^0 and τ_F^1 are Lindelöf, 0-dimensional and T_2 topologies on ω_2. Put $X^0 = \langle \omega_2, \tau_F^0 \rangle$ and $X^1 = \langle \omega_2, \tau_F^1 \rangle$, we claim that

$$L(X^0 \times X^1) = \omega_2 .$$

In fact, we show that the diagonal

$$\Delta = \{\langle x, x \rangle : x \in \omega_2\}$$

is closed discrete in $X^0 \times X^1$. Since for each $x \in \omega_2$ we have

$$\Delta \cap (A_x^0 \cup \{x\}) \times (A_x^1 \cup \{x\}) = \{\langle x, x \rangle\} ,$$

Δ is indeed discrete. Now, if $\langle x, y \rangle \in X^0 \times X^1$ with $x \neq y$, then there is a $z \in \omega_2 \setminus \{x, y\}$ such that $x \in A_z^0$ and $y \in A_z^1$ since F is flexible. Consequently $(A_z^0 \cup \{z\}) \times (A_z^1 \cup \{z\})$ is a neighbourhood of $\langle x, y \rangle$ in $X^0 \times X^1$ which meets Δ in the single point $\langle z, z \rangle$, hence Δ is also closed.

3.4. REMARK. Of course, here we are again confronted with the question about the exact upper bound for $L(X \times Y)$ with X and Y Lindelöf. Here the situation is even worse than with 1.6 and 1.13, because the only upper bound we do have at present is the first strongly compact cardinal which is, usually, much bigger than the first measurable. Thus the interval of uncertainty is here much bigger.

4. Subspaces of compact spaces

Let us warn the reader first of all that we do not assume that compact spaces are automatically Hausdorff. In fact, we start this section by showing that I.7.20 actually holds for all compact T_1 spaces of countable pseudo-character. (*Warning.* Pseudo-character is not necessarily equal to the character in compact spaces that are not Hausdorff!) The proof of this naturally splits into two, the first result, due to Gryzlov, generalizes I.7.3.

THEOREM 4.1. *If X is compact and T_1, then $|X| \leq 2^{\psi(X)}$.*

PROOF. The proof of this runs very much like that of I.4.5: Putting $\psi(X) = \kappa$ we first fix for each $p \in X$ a local pseudo-base $\mathcal{V}_p$ of size $\leq \kappa$, and then construct an increasing sequence $\{H_\alpha : \alpha \in \kappa^+\}$ of *compact* subsets of X and a sequence $\{\mathcal{V}_\alpha : \alpha \in \kappa^+\}$ of open collections in X such that

(1) $|H_\alpha| \leqslant 2^\kappa$;
(2) $\mathcal{V}_\alpha = \bigcup\{\mathcal{V}_p : p \in \bigcup_{\beta<\alpha} H_\beta\}$;
(3) if $\mathcal{W} \in [\mathcal{V}_\alpha]^{<\omega}$ and $X \setminus \bigcup \mathcal{W} \neq \emptyset$, then $H_\alpha \setminus \bigcup \mathcal{W} \neq \emptyset$ as well.

That this can be done follows immediately from the following two observations.

4.2. LEMMA. *If X is compact T_1 and $H \in [X]^{2^\kappa}$, then there is an initially κ-compact subspace $K \in [X]^{2^\kappa}$ with $H \subset K$.*

PROOF. This is an easy consequence of the well known fact that a T_1-space Y is initially κ-compact if and only if every $A \in [Y]^{\leqslant\kappa}$ has a complete accumulation point in Y.

4.3. LEMMA. *If X is compact with $\psi(X) \leqslant \kappa$, then every initially κ-compact subspace $Y \subset X$ is also compact.*

PROOF. Suppose, indirectly, that Y is not compact, hence there is an X-open family $\mathcal{U}$ such that $\bigcup \mathcal{U} \supset Y$ but no finite subfamily of $\mathcal{U}$ covers Y. By Zorn's lemma, we may assume that $\mathcal{U}$ is maximal with respect to this property. Of course, such a $\mathcal{U}$ cannot cover X itself, which is compact, hence we can pick a point $p \in X \setminus \bigcup \mathcal{U}$. Since $\psi(p, X) \leqslant \kappa$, there is a family $\mathcal{V}$ of open sets such that $\{p\} = \bigcap \mathcal{V}$ and $|\mathcal{V}| \leqslant \kappa$. Then $V \notin \mathcal{U}$ for each $V \in \mathcal{V}$, hence, by the maximality of $\mathcal{U}$, there is a finite subfamily $\mathcal{U}_V \in [\mathcal{U}]^{<\omega}$ with $\bigcup \mathcal{U}_V \cup V \supset Y$. Put

$$\tilde{\mathcal{U}} = \bigcup\{\mathcal{U}_V : V \in \mathcal{V}\},$$

clearly $|\tilde{\mathcal{U}}| \leqslant \kappa$, moreover $\bigcap \mathcal{V} = \{p\} \subset X \setminus Y$ implies that $Y \subset \bigcup \tilde{\mathcal{U}}$. Since Y is initially κ-compact, however, this implies the existence of a finite $\mathcal{W} \subset \tilde{\mathcal{U}} \subset \mathcal{U}$ with $Y \subset \mathcal{W}$, a contradiction.

The proof of 4.1 is now finished again similarly as that of I.4.5. We take

$$H = \bigcup\{H_\alpha : \alpha \in \kappa^+\}$$

and claim that $H = X$. Since $|H| \leqslant 2^\kappa$ holds trivially, this will do the job.

Suppose on the contrary that $q \in X \setminus H$. Then observe that H is initially κ-compact, hence by 4.3 it is also compact. Thus if we pick for each $p \in H$ a set $V_p \in \mathcal{V}_p$ with $q \notin V_p$, then there is a finite set $A \in [H]^{<\omega}$ such that $\mathcal{W} = \{V_p : p \in A\}$ covers H. Now if $\alpha \in \kappa^+$ is such that $A \subset H_\alpha$ then by (3) we have $H_{\alpha+1} \setminus \bigcup \mathcal{W} \neq \emptyset$ since $q \notin \bigcup \mathcal{W}$, which is clearly impossible.

The second result we need states that no compact space X with $\psi(X) \leqslant \omega$ can satisfy $\omega < |X| < 2^\omega$. This is due to Malyhin and we do not give its proof here because it also appears as 3.17 in JUHÁSZ [1980]. Putting this together with GRYZLOV's result for $\kappa = \omega$ we immediately get the following.

4.4. THEOREM. *If X is compact T_1 and $\psi(X)\leq\omega$, then either $|X|\leq\omega$ or $|X|=2^\omega$.*

The main aim of this section is to investigate whether this beautiful result could be generalized to higher cardinals. To make things simpler we shall only consider the question for ω_1, moreover we shall also assume that CH holds, i.e. $2^\omega=\omega_1$, but on the other hand $2^{\omega_1}>\omega_2$. Several reasonable generalizations of 4.4 (or I.7.20) then would read as follows:

(*) If X is compact with $\psi(X)=\omega_1$ (or $\chi(X)=\omega_1$, or even $w(X)=\omega_1$), then either $|X|\leq\omega_1$ or $|X|=2^{\omega_1}$.

We shall next present several results of KUNEN [1975] concerning (*). Not surprisingly, they will show that CH + $2^{\omega_1}>\omega_2$ does not decide the question(s), even for compact T_2 spaces.

4.5. DEFINITION. Let A_1 (A_2) be the statement: If X is compact T_2 with $w(X)=\omega_1$ ($\chi(X)=\omega_1$), then either $|X|=\omega_1$ or $|X|=2^{\omega_1}$.

Before we show that A_1 and A_2 are consistent, we first reformulate A_1 and A_2 to equivalent statements about trees. For any tree T we denote by $\operatorname{br}(T)$ the set of all branches (i.e. maximal chains) of T.

4.6. THEOREM (CH). (i) *A_1 holds if and only if for every tree T with $|T|=\omega_1$ we have either $|\operatorname{br}(T)|\leq\omega_1$ or $|\operatorname{br}(T)|2^{\omega_1}$.*

(ii) *A_2 holds if and only if A_1 holds and for every ω_2-Aronszajn tree $|\operatorname{br}(T)|=2^{\omega_1}$.*

PROOF. (i) Suppose first that T is a tree with $|T|=\omega_1$ but

$$\omega_1<|\operatorname{br}(T)|<2^{\omega_1}.$$

Consider the family

$$\mathscr{C}=\operatorname{br}(T)\cup\{\hat{t}: t\in T\}\subset P(T),$$

clearly we have $|\mathscr{C}|=|\operatorname{br}(T)|$. Then we define a subspace X of 2^T as follows:

$$X=\{h_C: C\in\mathscr{C}\},$$

where h_C is the characteristic function of C. We claim that X is closed in 2^T, hence compact T_2. Indeed, if $H\subset T$ and $H\notin\mathscr{C}$, then either H is not a chain, i.e. there are $s,t\in H$ such that s and t are incomparable in T, or there is a $t\in H$ and an $s\in T\backslash H$ such that s is below t in T. In the first case

$$\{f\in 2^T: f(s)=f(t)=1\},$$

in the second

$$\{f\in 2^T: f(s)=0\ \&\ f(t)=1\}$$

is an open neighbourhood of h_H in 2^T that obviously does not intersect X. Since $|T| = \omega_1$, we clearly have $w(X) = \omega_1$ (and also $\chi(X) = \omega_1$), furthermore $|X| = |\mathscr{C}|$, hence $\omega_1 < |X| < 2^{\omega_1}$.

On the other hand, if X is compact T_2, $\omega_1 < |X| < 2^{\omega_1}$ and $w(X) = \omega_1$, we can assume that X is actually embedded as a closed subspace in I^{ω_1}, where $I = [0, 1]$. Let us define then the tree T as follows:

$$T = \{f \restriction \alpha : f \in X \subset I^{\omega_1} \; \& \; \alpha \in \omega_1\},$$

with inclusion as the tree ordering. By CH we have $|T| = \omega_1$, and obviously $|\mathrm{br}(T)| = |X|$. This proves the other half of the equivalence.

Observe that we have actually shown that if $\omega_1 < \kappa < 2^{\omega_1}$ then the existence of a tree T with $|T| = \omega_1$ and $|\mathrm{br}(T)| = \kappa$ is equivalent to the existence of a compact T_2 space X with $w(X) = \omega_1$ and $|X| = \kappa$.

(ii) We have seen already what happens if A_1 fails. Now, assume that T is an ω_2-Aronszajn tree with $|\mathrm{br}(T)| < 2^{\omega_1}$. (Note that $\omega_2 \leq |\mathrm{br}(T)|$ is trivial.) Let us define again

$$\mathscr{C} = \mathrm{br}(T) \cup \{\hat{t} : t \in T\} \quad \text{and} \quad X = \{h_C \in 2^T : C \in \mathscr{C}\},$$

then similarly as above X is closed in 2^T and $|X| = |\mathscr{C}| = |\mathrm{br}(T)|$. For every $C \in \mathscr{C}$ fix a $B \in \mathrm{br}(T)$ with $C \subset B$ and define the function $k_C \in 2^B$ as follows:

$$k_C(s) = \begin{cases} 1 & \text{if } s \in C, \\ 0 & \text{if } s \in B \setminus C. \end{cases}$$

It is easy to see that if $h \in X$ and $h \restriction B = k_C$ then $h = h_C$, hence, as $|B| \leq \omega_1$ by assumption, we have $\psi(h_C, X) = \chi(h_C, X) \leq |B| \leq \omega_1$. By $|X| > \omega_1 = 2^\omega$ we cannot have $\chi(X) \leq \omega$, hence indeed $\chi(X) = \omega_1$.

Now assume that A_1 holds, moreover X is compact T_2, $\chi(X) = \omega_1$ and $\omega_1 < |X| < 2^{\omega_1}$. We may assume that $d(X) \leq \omega_2$, since otherwise we could just replace X with the closure of a subset of size ω_2. Thus we have $X = \bar{S}$, where $|S| = \omega_2$.

Our aim now is to define an ω_2-Aronszajn tree T with $|\mathrm{br}(T)| < 2^{\omega_1}$. T will consist of non-empty subsets of S, i.e. $T \subset P(S)$, with (proper) containment as the tree ordering. To make the (inductive) construction of T easier, we first define an operation Ω on subsets $H \subset S$ with $|H| > 1$ as follows.

Given such an H, pick distinct x_0, $x_1 \in H$ and then fix a continuous map $f: \bar{H} \to I$ such that $f(x_0) = 0$ and $f(x_1) = 1$. Then put

$$\Omega(H) = \{H \cap f^{-1}(c) : c \in I\} \setminus \{\emptyset\}.$$

Clearly, the members of $\Omega(H)$ have disjoint closures, $2 \leq |\Omega(H)| \leq \omega_1 = |I|$, and $H = \bigcup \Omega(H)$.

To start our induction, we put $T_0 = \{S\}$. Next, suppose $\alpha \in \omega_2$ and we have already defined

$$T \restriction \alpha = \bigcup\{T_\beta : \beta \in \alpha\} \subset P(S)\backslash\{\emptyset\}$$

in such a way that the following inductive hypotheses hold: If $\beta < \alpha$, then the members of T_β have disjoint closures in X, $|T_\beta| \leqslant \omega_1$ and $|S \backslash \bigcup T_\beta| \leqslant \omega_1$. If $\alpha = \beta + 1$, take any $H \in T_\beta$ with $|H| > 1$ (of course, if $|H| = 1$ then H will not have any successor at level α), and appoint the members of $\Omega(H)$ as the immediate successors of H in T_α. Then the inductive hypotheses hold trivially.

If α is limit, consider first

$$\mathcal{B}_\alpha = \{b \in \mathrm{br}(T \restriction \alpha) : \forall \beta \in \alpha (b \cap T_\beta \neq \emptyset)\},$$

then $|\mathcal{B}_\alpha| \leqslant \omega_1$ since we have assumed A_1 and, clearly, different branches of $T \restriction \alpha$ give rise to disjoint non-empty sets by taking the intersection of the closures of their members.

For each $b \in \mathcal{B}_\alpha$, we let

$$H_b = \bigcap\{\bar{H} : H \in b\} \cap S.$$

In a manner, to be clarified later, we might have at step α a certain point $p_\alpha \in X$ 'to be taken care of' in such a way that if $p_\alpha \in \bar{H}_b$ for some $b \in \mathcal{B}_\alpha$ then $\chi(p_\alpha, \bar{H}_b) \leqslant \omega$. By our construction the sets $\{H_b : b \in \mathcal{B}_\alpha\}$ have disjoint closures, hence if $p_\alpha \in \bigcup\{\bar{H}_b : b \in \mathcal{B}_\alpha\}$ then $b_\alpha \in \mathcal{B}_\alpha$ with $p_\alpha \in \bar{H}_{b_\alpha}$ is uniquely determined. Now we define T_α as follows: for every $b \in \mathcal{B}_\alpha \backslash \{b_\alpha\}$ with $H_b \neq \emptyset$ we put on top of the branch b the node H_b. Moreover, *if b_α exists and* $|H_{b_\alpha}| > 1$, we fix a continuous function $f : \overline{H_{b_\alpha}} \to I$ with $f^{-1}(0) = \{p_\alpha\}$, this is possible by $\chi(p_\alpha, \overline{H_{b_\alpha}}) \leqslant \omega$, and put on top of the branch b_α the members of

$$\{H_{b_\alpha} \cap f^{-1}(c) : c \in I \backslash \{0\}\}.$$

The inductive hypotheses are clearly kept alive this way, and thus T_α will be defined and non-empty for each $\alpha \in \omega_2$. We put $T = \bigcup\{T_\alpha : \alpha \in \omega_2\}$, then T is clearly an ω_2-tree.

Next we explain how the points p_α have to be chosen. For each $H \in T$ we have $|H| \leqslant \omega_2$, hence obviously

$$|\{p \in \bar{H} : \chi(p, \bar{H}) \leqslant \omega\}| \leqslant \omega_2^\omega = \omega_2.$$

Consequently, if

$$R = \{p \in X : \exists H \in T(\chi(p, \bar{H}) \leqslant \omega)\},$$

then $|R| \leqslant \omega_2$ as well. What we have to do then is to insure that every $p \in R$ be taken as p_α for a limit $\alpha \in \omega_2$ in such a way that if β is the smallest ordinal for which there is an $H \in T_\beta$ with $\chi(p, \bar{H}) \leqslant \omega$, then $\beta < \alpha$. The easy, but tedious, details of this are left to the reader.

We still have to show that T has no branch of length ω_2. For this we need the following lemma.

4.7. LEMMA. *If X is compact T_2 with $\chi(X) = \omega_1$, $\{F_\alpha : \alpha \in \omega_2\}$ is a decreasing sequence of closed sets in X, and $p \in F = \bigcap\{F_\alpha : \alpha \in \omega_2\}$ with $\chi(p, F) \leqslant \omega$, then there is an $\alpha \in \omega_2$ with $\chi(p, F_\alpha) \leqslant \omega$ as well.*

PROOF. Assume, indirectly, that $\chi(p, F_\alpha) = \omega_1$ for each $\alpha \in \omega_2$. By $\chi(p, F) \leqslant \omega$ we can pick open neighbourhoods $\{U_n : n \in \omega\}$ of p in X such that $\overline{U_{n+1}} \subset U_n$ for each $n \in \omega$ and

$$\{p\} = \bigcap\{U_n : n \in \omega\} \cap F.$$

Then the set $A = \bigcap\{U_n : n \in \omega\}$ is a closed G_δ in X. Let $\mathscr{V}$ be an open neighbourhood base of p in X with $|\mathscr{V}| \leqslant \omega_1$.

According to our assumption, for every $\alpha \in \omega_2$ there is a point $q_\alpha \neq p$ in $A \cap F_\alpha$, hence we can also pick a $V_\alpha \in \mathscr{V}$ with $q_\alpha \notin V_\alpha$. But $|\mathscr{V}| \leqslant \omega_1$ then implies the existence of a set $C \in [\omega_2]^{\omega_2}$ and a $V \in \mathscr{V}$ with $V_\alpha = V$ for each $\alpha \in C$. This shows that

$$A \cap F_\alpha \setminus V \neq \emptyset$$

for each $\alpha \in C$, hence for each $\alpha \in \omega_2$ as the F_α are decreasing, thus by the compactness of X,

$$\bigcap\{A \cap F_\alpha \setminus V : \alpha \in \omega_2\} = A \cap F \setminus V \neq \emptyset$$

as well, a contradiction.

Now, if b were a branch of T of length ω_2 and, say, $b \cap T_\alpha = \{H_\alpha\}$ for each $\alpha \in \omega_2$, then 4.7 applied to $\{\overline{H_\alpha} : \alpha \in \omega_2\}$ and $F = \bigcap\{\overline{H_\alpha} : \alpha \in \omega_2\}$ would yield $\chi(p, F) \geqslant \omega_1$ for each $p \in F$, since every point of countable character in some $\overline{H_\alpha}$ is eliminated by our construction from $\bigcup\{\bar{K} : K \in T_\gamma\}$ for a suitable $\gamma \in \omega_2$. This, however, implies by I.7.19 that $|F| = 2^{\omega_1}$, contradicting $|X| < 2^{\omega_1}$.

This shows that T is indeed an ω_2-Aronszajn tree, hence we have

$$\mathrm{br}(T) \subset \bigcup\{\mathrm{br}(T \upharpoonright \alpha) : \alpha \in \omega_2\},$$

and the set on the right-hand side has cardinality ω_2 because A_1 is assumed.

The next result shows that A_1 and A_2 can be proved consistent modulo some appropriate large cardinals. Following the notation of KUNEN [1980], we denote by $\mathrm{Lv}'(\kappa)$ the "Lévy collapsing p.o. of κ to ω_2 by countable functions", i.e.

$$\mathrm{Lv}'(\kappa) = \{p \in \mathrm{Fn}(\kappa \times \omega_1, \kappa, \omega_1) : \forall \langle \alpha, \xi \rangle \in D(p)(p(\alpha, \xi) \in \alpha)\} .$$

We shall also put for any λ

$$H'(\lambda) = \mathrm{Fn}(\lambda, 2, \omega_1) .$$

Finally, we let $Q(\kappa, \lambda) = \mathrm{Lv}'(\kappa) \times H'(\lambda)$.

4.8. THEOREM. *Suppose $\kappa < \lambda$ and $V \models \mathrm{GCH}$. Let G be $Q(\kappa, \lambda)$-generic over V.*
(i) *If κ is inaccessible (in V), then $V[G] \models A_1$.*
(ii) *If κ is also weakly compact (in V), then $V[G] \models A_2$.*

Before we give the proof of 4.8 we need a "general" lemma on forcing and trees.

4.9. LEMMA. *If P is countably closed, G is P-generic over V and $T \in V$ is a tree such that T has new branches in $V[G]$, i.e.*

$$V[G] \models \mathrm{br}(T) \setminus V \neq \emptyset ,$$

then T has at least 2^{ω_1} branches in $V[G]$.

PROOF. Suppose $p \in P$ and

$$p \Vdash \tau \in \mathrm{br}(\check{T}) \setminus V .$$

By straightforward induction we can define in V for all $s \in 2^{<\omega_1}$ ordinals α_s, conditions p_s and tree elements t_s such that $p_\emptyset = p$,

$$p_{\widehat{si}} \Vdash \check{t}_{\widehat{si}} \in \check{T}_{\alpha_s} \cap \tau, \quad \text{and} \quad t_{\widehat{s0}} \neq t_{\widehat{s1}}$$

for $s \in 2^{<\omega_1}$ and $i \in 2$, moreover $s \supset s'$ implies $p_s \leq p_{s'}$ and $\alpha_s > \alpha_{s'}$. Of course, we have to use here that P is ω_1-closed.

Now, if $f: \omega_1 \to 2$ in $V[G]$, then

$$c_f = \{t_{f \restriction \alpha+1} : \alpha \in \omega_1\}$$

is clearly a chain in T, thus we can extend it to a branch $b_f \in \mathrm{br}(T)$, and obviously $f \neq f'$ will then imply $b_f \neq b_{f'}$.

PROOF OF THEOREM 4.8. (i) By 4.6(i) we have to show that if $T \in V[G]$ is a tree with $|T| = \omega_1$, then either $|\mathrm{br}(T)| \leqslant \omega_1$ or $|\mathrm{br}(T)| = 2^{\omega_1}$ holds in $V[G]$. Assuming, as we may, that the underlying set of T is ω_1, standard arguments using that $Q(\kappa, \lambda)$ has the κ-chain condition show that there are a $\vartheta < \kappa$ and an $A \in [\lambda]^{<\kappa}$ such that actually $T \in V[G_0]$, where $G_0 = G \cap Q(\vartheta, A)$ with $Q(\vartheta, A) = \mathrm{Lv}'(\vartheta) \times H'(A)$. Moreover, we have

$$Q(\kappa, \lambda) = Q(\vartheta, A) \times Q^*$$

where the exact form of Q^*, except that it is ω_1-closed, is of no interest to us. By standard calculations again we have

$$V[G_0] \models |\mathrm{br}(T)| \leqslant 2^{\omega_1} < \kappa\,.$$

Thus if T will acquire no new branch after forcing with Q^*, then

$$V[G] \models |\mathrm{br}(T)| < \kappa = \omega_2\,.$$

If, on the other hand, new branches will occur, then by 4.9 $|\mathrm{br}(T)| = 2^{\omega_1} = \lambda$ will be valid in $V[G]$.

(ii) We know from (i) that A_1 holds in $V[G]$. Thus we consider an $\omega_2 = \kappa$-Aronszajn tree T in $V[G]$. Since κ is weakly compact this 'reflects down' to some $\mu < \kappa$, more precisely, there are $\mu < \kappa$ and $U \in [\lambda]^{<\kappa}$ such that for $G_0 = G \cap Q(\mu, U)$

$$T \restriction \mu \in V[G_0] \quad \text{and} \quad V[G_0] \models T \restriction \mu \text{ is a } \mu\text{-Aronszajn tree}\,.$$

Thus $T \restriction \mu$ has no branch of length μ in $V[G_0]$, while it clearly has many such branches in $V[G]$, hence 4.9 can be applied again, similarly as above, moreover all but ω_1 branches of $T \restriction \mu$ will be branches of T as well, hence $|\mathrm{br}(T \restriction \mu)| = |\mathrm{br}(T)| = 2^{\omega_1}$ in $V[G]$, which was to be shown by 4.6(ii).

The following result of Kunen that we mention without proof shows that the assumption of the large cardinals in 4.8 was in fact essential.

4.10. THEOREM. *Assume* $\mathrm{CH} + 2^{\omega_1} > \omega_2$.

(i) A_1 *implies that* ω_2 *is inaccessible in L.*

(ii) A_2 *implies that* ω_2 *is weakly compact in L.*

5. Subspaces of Lindelöf spaces

It is obvious that 4.4 does not generalize to Lindelöf spaces, just think of the real line for instance in which every subspace is Lindelöf. However, if we think of the fact that Lindelöfness is like 'compactness one cardinal higher' we may ask whether Lindelöf spaces of countable (pseudo) character could have cardinality strictly between ω_1 and 2^{ω_1}, assuming e.g. CH.

Of course, 1.6 shows that the answer to this is consistently negative, even for T_3 spaces. But the results of Section 4 give us the hope that a consistent affirmative answer is not unreasonable. And indeed, S. SHELAH [1978] has shown this. That his solution involves collapsing a weakly compact cardinal should not come as a surprise, especially in view of 1.14. In fact, we shall use the same notion of forcing $Q(\kappa, \lambda)$ as in 4.8. The result we prove is actually stronger and involves no topology really.

THEOREM 5.1. *Suppose κ is weakly compact, $\kappa < \lambda$ and $V \models$ GCH, moreover G is $Q(\kappa, \lambda)$-generic over V. Then the following holds in $V[G]$: If*

$$\mathcal{F} = \{U_{x,n} : \langle x, n\rangle \in \kappa \times \omega\} \subset P(\kappa)$$

is such that

$$\bigcap\{U_{x,n} : n \in \omega\} = \{x\}$$

for each $x \in \kappa = \omega_2^{V[G]}$, then $\mathcal{F}$ is not Lindelöf.

PROOF. We start by proving two lemmas that are of independent interest. To see that the next lemma applies, use the fact that $V[G] \models$ CH since $Q(\kappa, \lambda)$ is ω_1-closed.

5.2. LEMMA (CH). *If $\mathcal{F} \subset P(\omega_2)$ is as above and Lindelöf, then there is a closed unbounded set $C \subset \omega_2$ such that if $\alpha \in C$ and $\mathrm{cf}(\alpha) > \omega$, then*

$$\mathcal{F} \restriction \alpha = \{U_{x,n} \cap \alpha : \langle x, n\rangle \in \alpha \times \omega\}$$

is not Lindelöf.

PROOF. Let us define $C \subset \omega_2$ as follows: $\alpha \in C \leftrightarrow$ for every countable set $D \subset \alpha$ with $\sup D < \alpha$ and for every map $g: D \to \omega$, if

$$\omega_2 \setminus \bigcup\{U_{x,g(x)} : x \in D\} \neq \emptyset, \quad \text{then} \quad \alpha \setminus \bigcup\{U_{x,g(x)} : x \in D\} \neq \emptyset$$

as well. It is obvious that C is closed. Now, if $\beta \in \omega_2$ is arbitrary, then using CH one can easily define an increasing sequence $\{\beta_n : n \in \omega\} \subset \omega_2$ such that $\beta_0 = \beta$ and for each $n \in \omega$ if $D \subset \beta_n$ and $g: D \to \omega$ satisfies $\omega_2 \setminus \bigcup\{U_{x,g(x)} : x \in D\} \neq \emptyset$ then

$$\beta_{n+1} \setminus \bigcup\{U_{x,g(x)} : x \in D\} \neq \emptyset$$

as well. Clearly $\alpha = \bigcup\{\beta_n : n \in \omega\} \in C$.

Now let $\alpha \in C$ with $\mathrm{cf}(\alpha) > \omega$. For each $x \in \alpha$ we can pick $g(x) \in \omega$ such that $\alpha \notin U_{x,g(x)}$. Since $\{U_{x,g(x)}: x \in \alpha\}$ covers α, if $\mathcal{F} \upharpoonright \alpha$ were Lindelöf we could choose a $D \in [\alpha]^{\leq \omega}$ such that

$$\alpha \subset \bigcup \{U_{x,g(x)}: x \in D\}.$$

Since D is necessarily bounded in α and $\alpha \in C$, this would imply $\alpha \in \omega_2 = \bigcup \{U_{x,g(x)}: x \in D\}$, a contradiction.

The following lemma is again of independent interest.

5.3. LEMMA. *Suppose (in V) $\mathcal{F}$ is a Lindelöf family on some set X, and P is an ω_1-closed notion of forcing such that $\mathcal{F}$ is no longer Lindelöf in a generic extension by P. Then $\mathcal{F}$ also ceases to be Lindelöf in the generic extension by* $\mathrm{Fn}(\omega_1, \omega, \omega_1)$, *or what amounts to the same, by $H'(\omega_1)$.*

PROOF. Suppose $p \in P$ and τ is a P-name such that

$$p \Vdash \tau \subset \check{\mathcal{F}} \;\&\; \bigcup \tau = \check{X} \;\&\; \forall s \in [\tau]^{\check{\omega}} (\bigcup s \neq \check{X}).$$

Next we define in V, by induction on $\alpha \in \omega_1$, conditions $p_\varepsilon \in P$ for all $\varepsilon \in {}^\alpha\omega$ as follows. We first put $p_\emptyset = p$.

Suppose $\alpha \in \omega_1$ and for $\beta \in \alpha$ and $\eta \in {}^\beta\omega$ we have already defined p_η in such a way that if $\eta \subset \eta'$ then $p_{\eta'} \leq p_\eta$. If α is limit, then for every $\varepsilon \in {}^\alpha\omega$ we can choose, by the ω_1-closedness of P, a condition p_ε satisfying $p_\varepsilon \leq p_{\varepsilon \upharpoonright \beta}$ for each $\beta \in \alpha$. If on the other hand $\alpha = \beta + 1$ and $\eta \in {}^\beta\omega$, we can consider in V

$$\mathcal{U}_\eta = \{U \in \mathcal{F}: \exists q \leq p_\eta (q \Vdash \check{U} \in \tau)\}.$$

Since p_η forces τ to cover X, it is obvious that $\mathcal{U}_\eta$ is a cover of X in V, hence by the Lindelöfness of $\mathcal{F}$ it has a countable subcover $\mathcal{V}_\eta = \{U_\eta^n: n \in \omega\}$. Now for each $n \in \omega$, $p_{\eta \frown n}$ is chosen from the non-empty set of extensions $q \leq p$ satisfying

$$q \Vdash \check{U}_\eta^n \in \tau.$$

Let f be an $\mathrm{Fn}(\omega_1, \omega, \omega_1)$-generic map of ω_1 into ω and, in $V[f]$, define for each $\beta \in \omega_1$

$$U_\beta = U_{f \upharpoonright \beta}^{f(\beta)}.$$

We claim that $\{U_\beta: \beta \in \omega_1\}$ is a cover of X with no countable subcover. Indeed, given $x \in X$ and $h \in \mathrm{Fn}(\omega_1, \omega, \omega_1)$, the condition h has an extension of the form $\eta \in {}^\beta\omega$ for some $\beta \in \omega_1$, hence there is an $n \in \omega$ with $x \in U_\eta^n$, and then $\eta \cup \{\langle \beta, n \rangle\}$ forces $x \in U_\beta$.

On the other hand, for every $\beta \in \omega_1$ we have

$$p_{f\restriction\beta} \Vdash \{U_\gamma : \gamma \in \check{\beta}\} \subset \tau ,$$

hence by our assumption

$$p_{f\restriction\beta} \Vdash \bigcup\{U_\gamma : \gamma \in \check{\beta}\} \neq \check{X} .$$

But $\{U_\gamma : \gamma \in \beta\} \in V$ since P is ω_1-closed, hence we actually have

$$\bigcup\{U_\gamma : \gamma \in \beta\} \neq X.$$

Let us now return to the proof of 5.1. Since $\mathscr{F}$ is uniquely determined, in $V[G]$, by the set

$$F = \{\langle \alpha, x, n\rangle \in \omega_2^3 : \alpha \in U_{x,n}\} ,$$

where $|F| = \omega_2 = \kappa$, and $Q(\kappa, \lambda)$ has the κ-CC, there is an $A \in [\lambda]^\kappa$ in V such that $\mathscr{F} \in V[G_0]$, where $G_0 = G \cap (\mathrm{Lv}'(\kappa) \times H'(A))$. Without loss of generality we may assume that actually $A = \kappa$. Since

$$Q(\kappa, \lambda) = Q(\kappa, \kappa) \times H'(\lambda \setminus \kappa) ,$$

by the product lemma $V[G]$ is an $H'(\lambda\setminus\kappa)$-generic extension of $V[G_0]$. Consequently, $\mathscr{F}$ is also Lindelöf in $V[G_0]$, since $H'(\lambda\setminus\kappa)$ is ω_1-closed and thus $\mathscr{F}$ has no new countable subsets in $V[G]$. By the same token, we also get that in the $H'(\omega_1)$-generic extension of $V[G_0]$ the family $\mathscr{F}$ remains Lindelöf.

Let τ be a $Q(\kappa, \kappa)$-name for $\mathscr{F}$, then the following can be shown to be a valid Π_1^1-statement in V_κ: "κ is inaccessible and τ names a Lindelöf family $\mathscr{F}$ on $\kappa = \omega_2$ in the $Q(\kappa, \kappa)$-generic extension of V, and $\mathscr{F}$ remains Lindelöf when further extending by $H'(\omega_1)$".

Since κ is weakly compact in V, there is a stationary set $S \subset \kappa$ such that the above sentence reflects down to each $\mu \in S$, i.e. in the $Q(\mu, \mu)$-extension of V the $Q(\mu, \mu)$-name $\tau \restriction \mu$ names the Lindelöf family $\mathscr{F} \restriction \mu$ on μ which remains Lindelöf in the $H'(\omega_1)$-extension.

Consider now in $V[G]$ the closed unbounded set $C \subset \kappa = \omega_2^{V[G]}$ that is given by Lemma 5.2. Since $Q(\kappa, \lambda)$ is ω_1-closed, however, there is a closed unbounded set $D \subset \kappa$ in V such that $D \subset C$. Now, let $\mu \in D \cap S$, then $\mathrm{cf}(\mu) > \omega$, hence, in $V[G]$, $\mathscr{F} \restriction \mu$ is not Lindelöf. But $Q(\mu, \mu)$ is also a factor of $Q(\kappa, \lambda)$, i.e.

$$Q(\kappa, \lambda) = Q(\mu, \mu) \times Q^* ,$$

where Q^* is ω_1-closed. This, however, means that forcing with Q^* over $V[G \restriction Q(\mu, \mu)]$ destroys the Lindelöfness of $\mathscr{F} \restriction \mu$, hence by 5.3 so does $H'(\omega_1)$, a contradiction to $\mu \in S$.

In view of our remark after 1.13 we certainly cannot replace ω_2 in 5.1 with all cardinals greater than ω_1. Of course, according to our present state of ignorance, this could conceivably be done for the case of T_2 or T_3 topologies of countable pseudo-character. Let us now point out a similar such problem.

5.4. DEFINITION. A space X is said to be *subbase Lindelöf* if it has an (open) subbase which is a Lindelöf family.

The question we are interested in reads as follows: For what cardinals κ is $D(\kappa)$, the discrete space of size κ, subbase Lindelöf? It is easy to see that every $\kappa \leqslant 2^{\omega}$ is such, moreover from 1.6 and 5.1, respectively, we immediately obtain the following.

5.5. PROPOSITION. (i) (CH + $D(\omega_2)$ *is subbase Lindelöf*) *is consistent*;

(ii) (CH + $D(\omega_2)$ *is not subbase Lindelöf*) *is consistent, modulo a weakly compact cardinal.*

This is all that we know about this problem, in addition to the obvious analogue of 1.13. Even the following question seems to be quite intriguing: Can we have $\kappa < \lambda$ such that $D(\lambda)$ is subbase Lindelöf but $D(\kappa)$ is not?

6. Omitting cardinals

Let us recall briefly from I.13 that a space X is said to omit a cardinal κ if $|X| > \kappa$ but there is no closed subset of X of size equal to κ. There are some very basic unsolved problems concerning this notion. Perhaps the most interesting of these is whether there is, in ZFC, a T_3 space that omits ω_2? Some other questions were mentioned in I.13. We remark here that HUŠEK [1982] proved that there is, in ZFC, a T_2 space that omits ω_2. This is of course trivial if $2^{2^{\omega}} > \omega_2$, since then $\beta\omega$ does this, and he actually constructs with the help of CH, in a very complicated way, such a T_2 space, which, unfortunately, is hopelessly not regular.

It is easy to see that no T_2 space can omit every cardinal in the interval $[\kappa, 2^{2^{\kappa}}]$ since the closure of a set of size κ falls into this interval by I.3.2. (Assuming GCH this means that no T_2 space can omit simultaneously three consecutive cardinals.) Our next result will show that, in some sense, this cannot be further improved. In this we shall make use of the higher cardinal versions of the ubiquitous HFD-sets, compare what follows with 2.5.

6.1. DEFINITION. Given a cardinal κ and an index set I with $|I| > \kappa$, we call $X \subset 2^I$ a κ-HFD-set if $|X| \geqslant \kappa$ and for every $A \in [X]^{\kappa}$ there is a $J \in [I]^{\kappa}$ such that $A \restriction (I \setminus J)$ is dense in $2^{I \setminus J}$.

We leave it to the reader to verify that the basic facts about ordinary, i.e.

ω-HFD-sets easily generalize for all κ. Thus, for instance, every κ-HFD-set X is hereditarily κ-separable, i.e. $hd(X) \leqslant \kappa$, moreover, if for each $f \in X$ we have an $f' \in 2^I$ with

$$|\{i \in I: f(i) \neq f'(i)\}| \leqslant \kappa$$

then $X' = \{f': f \in X\}$ is also κ-HFD. Also, if $X \subset 2^I$ is κ-HFD then $|I| \leqslant 2^\kappa$ and $|X| \leqslant 2^{\kappa^+}$, and these bounds are sharp, as was shown in 2.8 for $\kappa = \omega$. Finally, $2^\kappa = \kappa^+$ implies the existence of a κ-HFD-set $X \subset 2^{\kappa^+}$ with $|X| = \kappa^+$, and if in addition $V = L$, then we even have such an X with $|X| = 2^{\kappa^+} = \kappa^{++}$, see HAJNAL and JUHÁSZ [1974] and JUHÁSZ [1977].

6.2. DEFINITION. A κ-HFD $X \subset 2^I$ is said to be *full* if for every $h \in \mathrm{Fn}(I, 2, \kappa^+)$, $|\{f \in X: h \subset f\}| = |X|$ holds.

6.3. PROPOSITION. *If $\lambda \geqslant 2^\kappa$ and there is a κ-HFD-set of cardinality λ, then there is also a full κ-HFD of size λ.*

PROOF. Let $X \subset 2^I$ with $|X| = \lambda$ be a κ-HFD and write $X = \{f_\xi: \xi \in \lambda\}$. Since $2^\kappa = |\mathrm{Fn}(I, 2, \kappa^+)| \leqslant \lambda$ we may also write

$$\mathrm{Fn}(I, 2, \kappa^+) = \{h_\xi: \xi \in \lambda\},$$

where for each $h \in \mathrm{Fn}(I, 2, \kappa^+)$ we have

$$|\{\xi \in \lambda: h_\xi = h\}| = \lambda .$$

Now, we define for $\xi \in \lambda$ a map $g_\xi \in 2^I$ as follows:

$$g_\xi(i) = \begin{cases} h_\xi(i) & \text{if} \quad i \in D(h_\xi);\\ f_\xi(i) & \text{if} \quad i \in \setminus D(h_\xi). \end{cases}$$

It is obvious then, by the above, that $\{g_\xi: \xi \in \lambda\}$ is a full κ-HFD of cardinality λ.

Now, we shall see what this has to do with omitting cardinals.

6.4. THEOREM. *If X is a full κ-HFD, then for every closed subset $F \subset X$ either $|F| < \kappa$ or $|F| = |X|$.*

PROOF. For simplicity we assume that $X \subset 2^{\kappa^+}$, this does not affect our result. Suppose $F \subset X$ and $|F| \geqslant \kappa$. Then F has the form $F = \bar{A}$ for some $A \in [F]^\kappa$, where the closure is taken in X. For any $\varepsilon \in H(\kappa^+)$ we put $[\varepsilon] = \{g \in 2^{\kappa^+}: \varepsilon \subset g\}$, the elementary open set in 2^{κ^+} determined by ε. Using this notation, we put

$$\mathcal{J}(A) = \{\alpha \in \kappa: \exists \nu \in \kappa^+\, \forall \varepsilon \in H(\nu)\, (|A \cap [\varepsilon]| = \kappa \rightarrow A \cap [\varepsilon] \text{ is dense in } 2^{\kappa^+ \setminus \nu})\}$$

6.5. LEMMA. *$\mathcal{J}(A)$ is closed and unbounded in κ^+.*

PROOF. That $\mathcal{J}(A)$ is closed is obvious. Now, given $\beta \in \kappa^+$ we define by induction on $n \in \omega$ an increasing sequence of ordinals $\beta_n \in \kappa^+$ as follows. Put $\beta_0 = \beta$, and if β_n has already been chosen consider

$$H_n = \{\varepsilon \in H(\beta_n): |A \cap [\varepsilon]| = \kappa\}.$$

Then $|H_n| \leq \kappa$, hence applying that A is κ-HFD, we clearly can pick $\beta_{n+1} \in \kappa^+ \setminus \beta_n$ such that $A \cap [\varepsilon] \restriction (\kappa \setminus \beta_{n+1})$ is dense in $2^{\kappa^+ \setminus \beta_{n+1}}$ for every $\varepsilon \in H_n$. It is easy to see then that $\alpha = \bigcup\{\beta_n : n \in \omega\} \in \mathcal{J}(A)$, hence $\mathcal{J}(A)$ is also unbounded in κ^+.

Now, returning to the proof of 6.4, choose $\alpha \in \mathcal{J}(A)$ in such a way that $|A \restriction \alpha| = \kappa$, which is possible by $|A| = \kappa$. Since 2^α is compact, there is an $h \in 2^\alpha$ which is a complete accumulation point of $A \restriction \alpha$ in 2^α. We claim that for every $f \in X$ with $h \subset f$ we have $f \in \bar{A}$. Since X is full, this will prove that $|F| = |\bar{A}| = |X|$.

Indeed, let $\varepsilon \in H(\kappa^+)$ be arbitrary such that $f \in [\varepsilon]$, i.e. $\varepsilon \subset f$. Then $\varepsilon \restriction \alpha \subset h$, consequently

$$|A \cap [\varepsilon \restriction \alpha]| = \kappa$$

by the choice of h. This, however, immediately implies

$$A \cap [\varepsilon] = A \cap [\varepsilon \restriction \alpha] \cap [\varepsilon \restriction \kappa^+ \setminus \alpha] \neq \emptyset$$

because $\alpha \in \mathcal{J}(A)$, hence $f \in \bar{A}$ and our claim is verified.

As a consequence of 6.4 and some preceding remarks we obtain the following.

6.6. COROLLARY. *If* GCH *holds then every* $\kappa \geq \omega$ *is omitted by a* 0*-dimensional* T_2 *(and thus* T_3*) space. Moreover, if* $V = L$ *then, for any* κ*, both* κ *and* κ^+ *can be simultaneously omitted by a* 0*-dimensional* T_2*-space.*

It was shown in I.13.3 that a T_3-space of countable pseudo-character does not omit ω. Moreover, it is easy to see that a first countable T_2-space cannot omit 2^ω, because in such a space the closure of a set of size 2^ω also has cardinality 2^ω. These facts motivate the following problem raised by HODEL [1978]: Can a T_3-space of countable pseudo-character omit 2^ω? Quite recently a consistent affirmative answer to this question has been given by CIESELSKI [198].
We shall next describe this result.

The example needs a generalization of the graph space construction from 1.3.

6.7. DEFINITION. Consider a set X and a map $f: X^2 \to \omega$ (actually, the values $f(x, x)$ are irrelevant just like in 2.3). For each $x \in X$ and $n \in \omega$ put

$$A_x^n = \{y \in X: y \neq x \;\&\; f(x, y) = n\}.$$

To simplify notation, we write

$$\mathcal{A} = [\omega]^{<\omega}, \qquad \mathcal{B} = \{\omega \setminus a : a \in \mathcal{A}\},$$

moreover $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. Then for $x \in X$ and $a \in \mathcal{A}$ we put

$$U_x^a = \bigcup\{A_x^n : n \in a\}$$

and for $b \in \mathcal{B}$

$$U_x^b = \bigcup\{A_x^n : n \in b\} \cup \{x\}.$$

Clearly, for any $a \in \mathcal{A}$ and $x \in X$

$$U_x^a = X \setminus U_x^{\omega \setminus a},$$

hence the family $\{U_x^c : x \in X \text{ and } c \in \mathcal{C}\}$ determines, as a subbase, a 0-dimensional topology on X. We shall denote this topology by τ_f. A base for this topology is formed by the sets

$$U_\varepsilon = \bigcap\{U_x^{\varepsilon(x)} : x \in D(\varepsilon)\},$$

where ε runs through $H(X, \mathcal{C}) = \mathrm{Fn}(X, \mathcal{C}, \omega)$. Finally, τ_f has countable pseudo-character, since for each $x \in X$

$$\bigcap\{U_x^{\omega \setminus n} : n \in \omega\} = \{x\}.$$

Our aim now is to find a generic extension of V in which, for a suitable f as above, τ_f omits 2^ω. Before we do that, however, we introduce the following piece of notation: For any set S we put

$$\tilde{H}(S) = \mathrm{Fn}(S, \omega, \omega_1).$$

Let us emphasize that $\tilde{H}(S)$ will be *always* meant in the sense of the ground model V!

6.8. THEOREM. *Suppose $V \models \mathrm{CH}$ and consider in V the product notion of forcing*

$$P = H(\lambda) \times \tilde{H}(\kappa \times \kappa),$$

where $\omega_1 < \lambda < \kappa$. Then, if G is P-generic over V, there is in $V[G]$ a 0-dimensional T_2 space of countable pseudo-character that omits $\lambda = (2^\omega)^{V[G]}$.

PROOF. In view of the product theorem for forcing, $G = H \times K$, where H is $H(\lambda)$-generic over V and K is $\tilde{H}(\kappa \times \kappa)$-generic over $V[H]$, moreover $V[G] = V[H][K]$. It is also well known that cardinals are preserved and $\lambda = 2^\omega$ in $V[G]$, see e.g. KUNEN [1980], VII.6.18(c). Moreover, using CH in V it is quite easy to show that P is ω_2-CC.

Let us put in $V[G]$, $f = \bigcup K$, then clearly $f: \kappa \times \kappa \to \omega$, hence we can consider the topology τ_f on κ defined in 6.7. It is quite obvious from the genericity of K that f is "flexible", hence τ_f is also T_2. We claim that $\langle \kappa, \tau_f \rangle$ also satisfies the other requirements of our theorem.

Given any $h \in \tilde{H}(\kappa)$ we shall write (see 6.7)

$$A_h = \bigcap \{A_\alpha^{h(\alpha)}: \alpha \in D(h)\}.$$

Quite similarly as in 2.22, one can show that $|A_h| = \kappa$ for each $h \in \tilde{H}(\kappa)$.

Now let C be an arbitrary closed set in $\langle \kappa, \tau_f \rangle$ such that $|C| < \kappa$. Observe that by our above remark $A_h \setminus C \neq \emptyset$ for every $h \in \tilde{H}(\kappa)$. Since τ_f clearly has a base of size κ, namely the U_ε's for $\varepsilon \in H(\kappa, \mathscr{C})$, and $\kappa \setminus C$ is open, there is in $V[G]$ a function $\delta: \kappa \to H(\kappa, \mathscr{C})$ such that

$$\kappa \setminus C = \bigcup \{U_{\delta(\alpha)}: \alpha \in \kappa\}.$$

Let us now define an increasing sequence $\{Z_\xi: \xi \in \omega_1\}$ of subsets of κ in $V[G]$ such that the following conditions hold:

(i) $Z_\xi \in V$ and $|Z_\xi| \leq \omega_1$;

(ii) $\alpha \in Z_\xi \to D(\delta(\alpha)) \subset Z_{\xi+1}$;

(iii) $h \in \tilde{H}(Z_\xi) \to \exists \alpha \in Z_{\xi+1}(A_h \cap U_{\delta(\alpha)} \neq \emptyset)$.

To see that this can be done assume that $\xi \in \omega_1$ and $\{Z_\eta: \eta \in \xi\}$ satisfying (i)–(iii) have already been defined. If ξ is limit, consider $Z'_\xi = \bigcup \{Z_\eta: \eta \in \xi\}$, then $|Z'_\xi| \leq \omega_1$, hence as P satisfies the ω_2-CC we can find a $Z_\xi \in [\kappa]^{\leq \omega_1}$ in V such that $Z'_\xi \subset Z_\xi$. If, on the other hand, $\xi = \eta + 1$, then first note that, as CH holds in V, we have $|\tilde{H}(Z_\eta)| \leq \omega_1$, hence if we put for each $h \in \tilde{H}(Z_\eta)$

$$\alpha_h = \min\{\alpha \in \kappa: A_h \cap U_{\delta(\alpha)} \neq \emptyset\},$$

then

$$Z'_\xi = Z_\eta \cup \bigcup \{D(\delta(\alpha)): \alpha \in Z_\eta\} \cup \{\alpha_h: h \in H(Z_\eta)\},$$

satisfies $|Z'_\xi| \leq \omega_1$, hence again we can choose $Z_\xi \in V$ with $|Z_\xi| \leq \omega_1$ and $Z'_\xi \subset Z_\xi$. This completes the construction of the sets Z_ξ.

Let us put $Z = \bigcup \{Z_\xi: \xi \in \omega_1\}$, then $|Z| \leq \omega_1$ as well. Hence, by the ω_2-CC property of P, one can find $B \in [\kappa]^{\omega_1}$ in V such that $Z \subset B$ and

$$\delta \upharpoonright Z \in V[H][K \cap \tilde{H}(B^2)].$$

We claim that $C \subset B$, hence $|C| \leq \omega_1$. Indeed, let $\eta \in \kappa \setminus B$ be arbitrary. Noting that

$$V[G]=V[H][K\cap \tilde{H}(B^2)][K\cap \tilde{H}(\kappa^2\backslash B^2)]\,,$$

we consider an arbitrary $s\in \tilde{H}(\kappa^2\backslash B^2)$ and show that s has an extension t in $\tilde{H}(\kappa^2\backslash B^2)$ that forces $\eta\not\in C$. To this end let

$$E=\{\sigma\in Z\colon \langle\sigma,\eta\rangle\in D(s)\}\,.$$

Since s is countable there must be a $\xi\in\omega_1$ for which actually

$$E=\{\sigma\in Z_\xi\colon \langle\sigma,\eta\rangle\in D(s)\}\,.$$

Since $s\in V$ and $Z_\xi\in V$ this implies $E\in V$, and if the function $h\colon E\to\omega$ is defined by $h(\sigma)=s(\sigma,\eta)$ for all $\sigma\in E$, then $h\in\tilde{H}(Z_\xi)$.

Consequently, using (iii), there is an $\alpha\in Z_{\xi+1}$ with $A_h\cap U_{\delta(\alpha)}\neq\emptyset$. Note that then by (ii) we also have $D(\delta(\alpha))\subset Z\subset B$. Now, let $\rho\in A_h\cap U_{\delta(\alpha)}$ and consider any $\sigma\in D(\delta(\alpha))$. If σ is also in E, then $\rho\in A_\sigma^{h(\sigma)}$, which implies $\rho\neq\sigma$, hence $\rho\in U_\sigma^{\delta(\alpha)(\sigma)}$ can only hold because $h(\sigma)=s(\sigma,\eta)\in\delta(\alpha)(\sigma)$. If, on the other hand, $\sigma\in D(\delta(\alpha))\backslash E$, then $\langle\sigma,\eta\rangle\not\in D(s)$. Let us pick for each $\sigma\in D(\delta(\alpha))\backslash E$ an element $n_\sigma\in\delta(\alpha)(\sigma)$, which is clearly possible since otherwise we had $U_\sigma^{\delta(\alpha)(\sigma)}=U_{\delta(\alpha)}=\emptyset$. Now we define the extension t of s by putting

$$D(t)=D(s)\cup\{\langle\sigma,\eta\rangle\colon \sigma\in D(\delta(\alpha))\backslash E\}$$

and stipulating for each $\langle\sigma,\eta\rangle\in D(t)\backslash D(s)$

$$t(\sigma,\eta)=n_\sigma\,.$$

But then we have for each $\sigma\in D(\delta(\alpha))$

$$t(\sigma,\eta)\in U_\sigma^{\delta(\alpha)(\sigma)}\,,$$

hence t forces $\eta\in U_{\delta(\alpha)}\subset X\backslash C$.

Thus what we actually have shown is that $\langle\kappa,\tau_f\rangle$ omits every cardinal strictly between ω_1 and κ.

7. The character of ω_1 in first countable spaces

To motivate the problem dealt with in this section we start by stating a very simple proposition.

7.1. PROPOSITION. *If X is a first countable space and $Y\subset X$ is homeomorphic with a countably infinite ordinal γ (taken with its natural order topology), then $\chi(Y,X)=\omega$ if γ has a largest element and $\chi(Y,X)>\omega$ if γ has no largest element and Y is nowhere dense in X.*

Now, our question is what can be said about $\chi(Y, X)$ if Y is homeomorphic with ω_1. Note that for $\omega_1 + 1$ the answer is analogous to 7.1: if Y is homeomorphic to $\omega_1 + 1$, then $\chi(Y, X) = \omega_1$. We shall show that the analogy breaks down for ω_1. The results are taken from FLEISSNER [1977].

7.2. THEOREM. *If $\diamondsuit^+$ holds, then there is a first countable space X with $Y \subset X$ homeomorphic with ω_1 and satisfying $\chi(Y, X) > \omega_1$.*

PROOF. Let us take as the underlying set of X the set $\omega_1 \times (\omega + 1)$. Moreover we shall fix a non-empty collection $\mathcal{F} \subset {}^{\omega_1}\omega$ of functions from ω_1 to ω with the help of $\diamondsuit^+$. Then the topology of X is defined as follows: every point $\langle \alpha, n \rangle \in X$ with $n \in \omega$ will be isolated; the other points $\langle \beta, \omega \rangle \in \omega_1 \times \{\omega\}$ have basic open neighbourhoods of the form

$$U(f; \alpha, \beta) = \{\langle \delta, \nu \rangle : \alpha < \delta \leq \beta \ \&\ f(\delta) < \nu \leq \omega\},$$

where $\alpha < \beta$ and $f \in F$. For any $g \in {}^{\omega_1}\omega$ we shall use the notation

$$U(g) = \{\langle \delta, \nu \rangle : g(\delta) < \nu \leq \omega\}.$$

It is easy to see that $Y = \{\langle \alpha, \omega \rangle : \alpha \in \omega_1\}$ is then homeomorphic as a (nowhere dense) subspace to ω_1. Our aim is to construct $\mathcal{F}$ with the following two properties:

(1) $|\{f \restriction \alpha : f \in \mathcal{F}\}| \leq \omega$ for each $\alpha \in \omega_1$;

(2) for every ω_1-sequence $\{g_\alpha : \alpha \in \omega_1\} \subset {}^{\omega_1}\omega$ there is an $f \in \mathcal{F}$ such that $f(\gamma_\alpha) > g_\alpha(\gamma_\alpha)$ holds for some $\gamma_\alpha \in \omega_1$ whenever $\alpha \in \omega_1$, i.e. $U(g_\alpha) \not\subset U(f)$ for all $\alpha \in \omega_1$.

Clearly, then (1) implies that X is first countable and (2) implies that $\chi(Y, X) > \omega_1$, the easy details of this are left to the reader. We also assume that $\mathcal{F}$ contains all the constant functions because this will make X Hausdorff.

Let $\{\mathcal{A}_\alpha : \alpha \in \omega_1\}$ be a $\diamondsuit^+$-sequence, see e.g. KUNEN [1980], II.7.9, i.e. $\mathcal{A}_\alpha \subset \mathcal{P}(\alpha)$, $|\mathcal{A}_\alpha| \leq \omega$ and for each $S \subset \omega_1$ there is a closed unbounded set $C(S) \subset \omega_1$ such that both $S \cap \alpha \in \mathcal{A}_\alpha$ and $C(S) \cap \alpha \in \mathcal{A}_\alpha$ whenever $\alpha \in C(S)$. We may, of course, assume that every $C(S)$ consists of limit ordinals only.

Let us fix a bijection $\rho: \omega_1 \to \omega_1 \times \omega_1 \times \omega$ such that for each $\lambda \in L_1$, the limit ordinals below ω_1, we have $\rho[\lambda] = \lambda \times \lambda \times \omega$. Then if $G = \{g_\alpha : \alpha \in \omega_1\} \subset {}^{\omega_1}\omega$ is given as in (2), we can code G as a subset G^* of ω_1 by means of ρ as follows:

$$G^* = \{\rho^{-1}(\alpha, \beta, n) : g_\alpha(\beta) = n\}.$$

Let $C(G^*) = \{\gamma_\alpha : \alpha \in \omega_1\}$ be the increasing enumeration of $C(G^*)$. Then we define the function $d_G : \omega_1 \to \omega$ as follows:

$$d_G(\gamma) = \begin{cases} g_\alpha(\gamma) + 1 & \text{if } \exists \alpha (\gamma = \gamma_\alpha); \\ 0 & \text{otherwise}. \end{cases}$$

Clearly, (2) will be satisfied if we put $d_G \in \mathcal{F}$ for each such G. To this end we define for each $\lambda \in L_1$ a family $\mathcal{B}_\lambda \subset {}^\lambda\omega$ such that

(i) $|\mathcal{B}_\lambda| \leq \omega$;

(ii) if $\lambda \in C(G^*)$ for some G as above, then $d_G \upharpoonright \lambda \in \mathcal{B}_\lambda$;

(iii) if $\lambda' < \lambda$, $n \in \omega$ and $f \in \mathcal{B}_{\lambda'}$, then

$$f \cup \{\langle \lambda', n\rangle\} \cup \{\langle \beta, 0\rangle : \lambda' < \beta < \lambda\} \in \mathcal{B}_\lambda .$$

To see that this can be done, observe that if $\lambda \in C(G^*)$, then $G^* \cap \lambda$, $C(G^*) \cap \lambda \in \mathcal{A}_\lambda$ and that $d_G \upharpoonright \lambda$ can be recovered from these two intersections with λ. The rest is easy.

Now put in $\mathcal{F}$ every $f \in {}^{\omega_1}\omega$ satisfying

$$f \upharpoonright \lambda \in \mathcal{B}_\lambda$$

for all $\lambda \in L_1$, plus all the constants. Then (1) follows immediately from (i). Next we consider any $G = \{g_\alpha : \alpha \in \omega_1\} \subset {}^{\omega_1}\omega$ and show that $d_G \in \mathcal{F}$, i.e. $d_G | \lambda \in \mathcal{B}_\lambda$ for every $\lambda \in L_1$. For $\lambda \in C(G^*)$ this follows from (ii). If $\lambda \notin C(G^*)$, then either $d_G \upharpoonright \lambda$ is constantly 0 or there is (because $C(G^*)$ is closed) a greatest element λ' of $C(G^*) \cap \lambda$. Then $d_G \upharpoonright \lambda' \in \mathcal{B}_{\lambda'}$ by (ii), hence $d_G \upharpoonright \lambda \in \mathcal{B}_\lambda$ by (iii) and the definition of d_G. This completes the proof.

Now, it is well known, see KUNEN [1980], VII.B.9, that if Kunepa's hypothesis fails, then ω_2 is inaccessible in L. Hence it is quite natural that an inaccessible cardinal appears in the following result that shows the consistency of the impossibility of the conclusion in 7.2.

7.3. THEOREM. *Suppose $V \models$ GCH and κ is inaccessible in V. If G is* Lv$'(\kappa)$-*generic over V then, for every first countable space $X \in V[G]$, if $Y \subset X$ is homeomorphic to ω_1, then $\chi(Y, X) \leq \omega_1$.*

PROOF. Let $\alpha \mapsto y_\alpha$ be a homeomorphism of ω_1 onto Y in $V[G]$. For every $\alpha \in \omega_1$ we put $Y_\alpha = \{y_\beta : \beta \leq \alpha\}$, then by 7.1 we have $\chi(Y_\alpha, X) \leq \omega$ for all $\alpha \in \omega_1$.

We fix for each $\alpha \in \omega_1$ a countable neighbourhood base of y_α $\{B_\alpha^n : n \in \omega\}$ in X. Let us put for any $S \subset \omega_1$ and $g : S \to \omega$, $U(g) = \bigcup\{B_\alpha^{g(\alpha)} : \alpha \in S\}$. Then, by the above, we can choose for each $\alpha \in \omega_1$ a countable family $\{h_\alpha^n : n \in \omega\} \subset {}^\alpha\omega$ such that $\{U(h_\alpha^n) : n \in \omega\}$ forms a neighbourhood base of Y_α in X. Since Lv$'(\kappa)$ has the κ-CC, just like in the proof of 4.8, we can find a $\vartheta < \kappa$ such that

$$H(X, Y) = \{h_\alpha^n : \langle \alpha, n\rangle \in \omega_1 \times \omega\} \in V[G \cap \mathrm{Lv}'(\vartheta)] .$$

Since $V[G]$ is a generic extension of $V[G \cap \mathrm{Lv}'(\vartheta)]$ via a countably closed notion of forcing in which $(2^{\omega_1})^{V[G\cap L_{V'}(\vartheta)]}$ is collapsed to ω_1, the following lemma will complete the proof.

7.4. LEMMA. *If $V[G]$ is a generic extension of V via an ω_1-closed notion of forcing P, X, $Y \in V[G]$ are as above and $H(X, Y) \in V$, then $\{U(f): f \in ({}^{\omega_1}\omega)^V\}$ yields a neighbourhood base of Y in X.*

PROOF. Let τ be a P-name and suppose, indirectly, that some $p \in P$ forces "$\tau: \omega_1 \to \omega$ and $U(f) \not\subset U(\tau)$ for all $f \in {}^{\omega_1}\omega$". First we show that then, given any $q \leq p$, there is an $\alpha \in \omega_1$ such that for any $n \in \omega$ there is an extension $q_n \leq q$ for which

$$q_n \Vdash \check{B}^n_\alpha \not\subset U(\tau).$$

Indeed, otherwise for every $\alpha \in \omega_1$ we could choose (in V) an $n_\alpha \in \omega$ such that

$$q \Vdash \check{B}^{n_\alpha}_\alpha \subset U(\tau),$$

hence the function $f = \{\langle \alpha, n_\alpha \rangle: \alpha \in \omega_1\}$, which is in V, would satisfy $q \Vdash U(\check{f}) \subset U(\tau)$, contradicting that $q \Vdash U(\check{f}) \not\subset U(\tau)$. Using this statement we can easily construct for all sequences $s \in \omega^{<\omega}$ conditions $p_s \in P$ and ordinals $\alpha_s \in \omega_1$ such that $p_\emptyset = p$, $s \subset s'$ implies $p_{s'} \leq p_s$, moreover

$$p_{s^\frown n} \Vdash \check{B}^n_{\alpha_s} \not\subset U(\tau)$$

for all s and n. Having finished this construction choose $\alpha \in \omega_1$ in such a way that $\{\alpha_s: s \in \omega^{<\omega}\} \subset \alpha$.

Let us now define by recursion on $n \in \omega$ sequences $s_n \in {}^n\omega$ as follows: If we have already picked s_n, then let

$$s_{n+1} = s_n{}^\frown h^n_\alpha(\alpha_{s_n}).$$

Clearly, then

$$p_{s_{n+1}} \Vdash B^{h^n_\alpha(\alpha_{s_n})}_{\alpha_{s_n}} \not\subset U(\tau),$$

consequently

$$p_{s_{n+1}} \Vdash U(\check{h}^n_\alpha) \not\subset U(\tau).$$

But P is ω_1-closed, hence we can pick a $q \in P$ with $q \leq p_{s_n}$ for each $n \in \omega$, and then

$$q \Vdash \forall n \in \check{\omega}(U(\check{h}^n_\alpha) \not\subset U(\tau)),$$

which is impossible since, again by the ω_1-closedness of P, $1_P \Vdash \tau \restriction \check{\alpha} \in V$, hence

$$1_P \Vdash \exists n \in \check{\omega}(U(h^n_\alpha) \subset U(\tau)).$$

The proofs of 7.2 and 7.3 very closely parallel those of the corresponding results for Kurepa trees. This raises the, so far still unsolved, problem whether a direct connection could be established between Kurepa trees and the character of ω_1 in first countable spaces.

The original motivation for the latter, as is mentioned by FLEISSNER [1977], came from the Moore space problem. However, the next result of ZHOU [1981] shows its importance for other kinds of problems too. We start this with a definition due to HUŠEK [1977].

DEFINITION. A space X is said to have *small diagonal* if for every uncountable set

$$A \subset X \times X \setminus \Delta \ (=\{\langle x, y\rangle \in X \times X : x \neq y\})$$

there is a neighbourhood U of Δ such that $A \setminus U$ is also uncountable.

Clearly, a G_δ diagonal is also small, moreover compact T_2 spaces with a G_δ diagonal are known to be metrizable (see e.g. I.7.6). A natural question is then whether a compact T_2 space with a small diagonal is necessarily metrizable. Now, we show that this is consistently so. We present the proof in full details because it nicely illustrates the use of a wide variety of cardinal functions.

7.5. THEOREM. *If* CH *holds and the character of ω_1 is at most ω_1 in every first countable space* (*this is the case e.g. in the model $V[G]$ of* 7.3), *then every compact T_2 space with a small diagonal is metrizable.*

PROOF. Let us start with the trivial observation that, for any T_1 space X, if

$$\psi(\Delta, X^2) = \chi(\Delta, X^2) = \omega_1,$$

then X does not have a small diagonal. Indeed, if this holds and $\{U_\alpha : \alpha \in \omega_1\}$ is a neighbourhood base of Δ in X^2, then we can pick points p_α recursively such that

$$p_\alpha \in \bigcap\{U_\beta : \beta < \alpha\} \setminus (\Delta \cup \{p_\beta : \beta < \alpha\}).$$

for each $\alpha \in \omega_1$. Clearly, then every neighbourhood of Δ contains all but countably many of the p_α.

This immediately implies that if X is compact T_2 and $w(X) \leq \omega_1$, then X is metrizable if it has a small diagonal. In fact, then by the above and I.7.6

$$\psi(\Delta, X^2) = \chi(\Delta, X^2) = w(X) = \omega_1$$

is impossible, hence X must actually have a G_δ diagonal. Thus, if CH holds, this immediately yields that if X is compact T_2 with a small diagonal and $d(X) = \omega$, then X must be metrizable as then, by I.3.3(b), $w(X) \leqslant 2^\omega = \omega_1$.

A less trivial consequence of CH is the following result of HUŠEK [1977]: If X is compact T_2 with a small diagonal and $t(X) \leqslant \omega$ then X must be metrizable. To see this, we first show that then X must be (hereditarily) separable. That will suffice by the above. If, on the contrary, X would contain a left separated subspace $Y = \{y_\alpha : \alpha \in \omega_1\}$, then consider

$$Z = \bigcup\{\bar{Y}_\alpha : \alpha \in \omega_1\},$$

where $Y_\alpha = \{y_\beta : \beta < \alpha\}$. Then we know already that $w(\bar{Y}_\alpha) = \omega$ for each $\alpha \in \omega_1$, moreover Z is closed in X, hence compact, since we have $t(X) \leqslant \omega$. But then, by I.7.4, we already have

$$w(Z) = nw(Z) \leqslant \omega_1,$$

hence actually $w(Z) = \omega$ (here we also use the trivial fact that having a small diagonal is a property inherited by subspaces). This contradicts that Z contains an uncountable left separated subspace, namely Y.

Thus we see that if X is a non-metrizable compact T_2 space with a small diagonal under CH, then $t(X) > \omega$, hence by I.7.11 there is an uncountable free sequence, say $S = \{p_\alpha : \alpha < \omega_1\}$, in X. Putting again $S_\alpha = \{p_\beta : \beta < \alpha\}$, we set

$$Z = \bigcup\{\bar{S}_\alpha : \alpha \in \omega_1\}.$$

Then each $\bar{S}_\alpha$ is compact metrizable, moreover it is also open in Z. Indeed, if $q \in \bar{S}_\alpha$ then, since S is free, there is an open neighbourhood U of q in X such that $U \cap (S \setminus S_\alpha) = \emptyset$. Now, if $x \in U \cap Z$ then $x \notin \overline{S \setminus S_\alpha}$, hence we must have $x \in \bar{S}_\alpha$, i.e. $U \cap Z \subset \bar{S}_\alpha$, showing that $\bar{S}_\alpha$ is open, hence clopen, in Z. This clearly implies that Z is countably compact but not compact and that Z is first countable.

Let $\lambda \in L_1$ (= the limit ordinals below ω_1) and put

$$T_\lambda = \bar{S}_\lambda \setminus \bigcup\{\bar{S}_\alpha : \alpha < \lambda\}.$$

Then each T_λ is a non-empty compact G_δ in Z, and it also follows from the freeness of S that the T_λ's are disjoint from each other and from S. We claim that the quotient space of Z obtained by collapsing each T_λ to a point (i.e. the space Z/R, where R is the equivalence relation on Z with

$$\{T_\lambda : \lambda \in L_1\} \cup \{\{p_\alpha\} : \alpha \in \omega_1\}$$

as the collection of R-equivalence classes) is homeomorphic to ω_1. In fact, we claim that the one-one map

$$h: Z/R \to \omega_1 \backslash \{0\},$$

defined by

$$h(z) = \begin{cases} \alpha + 1 & \text{if } z = \{p_\alpha\}, \\ \lambda & \text{if } z = T_\lambda, \end{cases}$$

is a homeomorphism. For any $\alpha < \beta \in \omega_1$ we have

$$\bigcup h^{-1}(\alpha, \beta) = \bar{S}_\beta \backslash \bar{S}_\alpha,$$

which is clearly an R-saturated open set in Z, hence h is continuous. But both Z/R and $\omega_1 \backslash \{0\}$ are first countable and countably compact, hence it is easy to see that then h must be a homeomorphism.

Now consider $Z \times Z$ and its diagonal Δ_Z which, of course, is homeomorphic to Z. Let Q be the equivalence relation on $Z \times Z$ that collapses all the subsets Δ_{T_λ} of $\Delta_Z \subset Z \times Z$ for $\lambda \in L_1$ to points, and let

$$k: Z \times Z \to Z \times Z/Q$$

be the corresponding quotient map. Since

$$Z \times Z = \bigcup \{\bar{S}_\alpha \times \bar{S}_\alpha : \alpha \in \omega_1\},$$

moreover the sets $\bar{S}_\alpha \times \bar{S}_\alpha$ are compact, metrizable and open in $Z \times Z$, we have on one hand that

$$\chi(\Delta_{T_\lambda}, Z^2) = \chi(\Delta_{T_\lambda}, \bar{S}^2_\lambda) = \omega$$

for each λ, hence $Z \times Z/Q$ is first countable, moreover both $Z \times Z$ and $Z \times Z/Q$ are countably compact. From the first part of this and from our assumption about the character of ω_1 in first countable spaces it follows that

$$\chi(k(\Delta_Z), Z^2/Q) \leq \omega_1,$$

since by the above $k(\Delta_Z)$ is homeomorphic to ω_1. But it also follows that k must be a closed map, hence

$$\chi(\Delta_Z, Z^2) \leq \chi(k(\Delta_Z), Z^2/Q) \leq \omega_1.$$

But Z is countably compact, T_3 and non-metrizable, hence by CHABER [1976] it cannot have a G_δ diagonal. Consequently we must have

$$\psi(\Delta_Z, Z^2) = \chi(\Delta_Z, Z^2) = \omega_1\,,$$

contradicting that Z has a small diagonal. The proof is now completed.

To end this section we mention an interesting related problem. Let us observe that having a small diagonal is determined by the subspaces of size ω_1. Hence if every subspace Y of X with $|Y| = \omega_1$ is metrizable then X has a small diagonal. The question, to which we were actually led by completely different motivations, now reads as follows: Is it true in ZFC that if X is compact T_2 and every $Y \subset X$ with $|Y| = \omega_1$ is metrizable then so is X? To indicate that this is not unreasonable to expect we mention here, without proof, two related results: If X is as above and every $Y \in [X]^{\omega_1}(Y \in [X]^{\leq 2^\omega})$ is first countable (metrizable) then so is X.

8. On sup ≠ max

In this concluding section we wish to present a result of ROITMAN [1978] which shows that in I.12.5 the requirement $\mathrm{cf}(\kappa) = \omega$ is indeed necessary, by showing that sup ≠ max can occur for a regular X with e.g. $s(X) = \omega_{\omega_1}$.

8.1. THEOREM. *If $2^\omega \geq \omega_{\omega_1}$ and there is a first countable Luzin space then there is a 0-dimensional T_2 space X such that $s(X) = \omega_{\omega_1}$ but no discrete subspace of X has cardinality ω_{ω_1}.*

PROOF. Let us first consider a first countable, 0-dimensional and T_2 topology τ on ω_1. We also assume that no point in ω_1 is isolated in τ. Thus for each $\alpha \in \omega_1$ we can fix a strictly decreasing sequence $\{U_\alpha^k : k \in \omega\}$ of clopen sets forming a τ-neighbourhood base at α. We also put

$$V_\alpha^k = U_\alpha^k \setminus U_\alpha^{k+1}\,.$$

then the V_α^k are clopen and non-empty with $\alpha \notin V_\alpha^k$.

Next, using $2^\omega \geq \omega_{\omega_1}$, we choose an independent family $\mathscr{A} \subset P(\omega)$ with $|\mathscr{A}| = \omega_{\omega_1}$ and let

$$\mathscr{A} = \{A_\nu : \nu \in \omega_{\omega_1}\}\,,$$

a one–one enumeration of $\mathscr{A}$. Our space X will have ω_{ω_1} as its underlying set. We shall also fix some disjoint family

$$\{D_\alpha : \alpha \in \omega_1\} \subset P(\omega_{\omega_1})$$

such that $|D_\alpha| = \omega_\alpha$ and $\bigcup\{D_\alpha : \alpha \in \omega_1\} = \omega_{\omega_1}$.

We shall use the following piece of notation: if $H \subset \omega_1$ then

$$D(H) = \bigcup\{D_\alpha : \alpha \in H\}.$$

The topology of X will be defined in such a way that every such D_α is discrete, hence $s(X) = \omega_{\omega_1}$.

Now, given $\nu \in D_\alpha \subset X$, a neighbourhood base of ν in X is formed by the sets of the form $B_\nu(F, k)$ with $F \in [D_\alpha \setminus \{\nu\}]^{<\omega}$ and $k \in \omega$, where

$$B_\nu(F, k) = D\Big(\bigcup\Big\{V_\alpha^l : l \in A_\nu \setminus \bigcup_{\mu \in F} A_\mu \setminus k\Big\}\Big) \cup \{\nu\}$$

Note that as $\mathscr{A}$ is independent the set $A_\nu \setminus \bigcup_{\mu \in F} A_\mu$ is infinite, moreover

$$B_\nu(F, k) \cap D_\alpha = \{\nu\}$$

holds because $\alpha \notin V_\alpha^l$ for all l. It is also easy to see that for $F, F' \in [D_\alpha \setminus \{\nu\}]^{<\omega}$ and k, $k' \in \omega$ we have

$$B_\nu(F, k) \cap B_\nu(F', k') = B_\nu(F \cup F', k \cup k').$$

Now, let $\rho \in B_\nu(F, k)$ and $\rho \neq \nu$, then $\rho \in D_\beta$ with $\beta \in V_\alpha^l$ for some $l \in A_\nu \setminus \bigcup_{\mu \in F} A_\mu \setminus k$. But then there is an $m \in \omega$ such that $U_\beta^m \subset V_\alpha^l$, for V_α^l is open. Obviously, this implies

$$B_\rho(\emptyset, m) \subset B_\nu(F, k),$$

showing that the sets $B_\nu(F, k)$ indeed form an open base for a topology on X. This topology is T_2, for if $\mu, \nu \in X$ and $\mu \neq \nu$, then either $\mu \in D_\alpha$, $\nu \in D_\beta$ with $\alpha \neq \beta$ and then

$$B_\mu(\emptyset, k) \cap B_\nu(\emptyset, l) = \emptyset \quad \text{if} \quad U_\alpha^k \cap U_\beta^l = \emptyset,$$

or $\mu, \nu \in D_\alpha$ and then

$$B_\mu(\{\nu\}, 0) \cap B_\nu(\emptyset, 0) = \emptyset.$$

Finally, we show it is 0-dimensional by showing that each $B_\nu(F, k)$ is also closed. Suppose $\nu \in D_\alpha$ and $\mu \in X \setminus B_\nu(F, k)$. If $\mu \in D_\alpha$ then

$$B_\mu(\{\nu\}, 0) \cap B_\nu(F, k) = \emptyset$$

as we have seen above. If, on the other hand, $\mu \in D_\beta$ with $\beta \neq \alpha$, then we also have

$$\beta \notin \bigcup\left\{V_\alpha^l : l \in A_\nu \setminus \bigcup_{\mu \in F} A_\mu \setminus k\right\},$$

hence there is an $m \in \omega$ such that

$$U_\beta^m \cap \bigcup\left\{V_\alpha^l : l \in A_\nu \setminus \bigcup_{\mu \in F} A_\mu \setminus k\right\}.$$

Obviously, then

$$B_\mu(\emptyset, m) \cap B_\nu(F, k) = \emptyset,$$

hence the complement of $B_\nu(F, k)$ is indeed open.

Now it only remains to show that X has no discrete subspace of cardinality ω_{ω_1}. This is where the assumption that $\langle \omega_1, \tau \rangle$ is Luzin will be used. We recall from KUNEN [1976] that a space is Luzin if it is uncountable, T_3, without isolated points and such that every nowhere dense subset in it is countable. It is trivial to show that any Luzin space has a Luzin subspace of size ω_1. It is less trivial, but nonetheless true that every Luzin space is 0-dimensional, see KUNEN [1976]. Let us also note that a Suslin line always has a Luzin subspace and so does the real line in a generic extension obtained by adjoining uncountably many Cohen reals. Hence our simultaneous assumption of $2^\omega \geq \omega_{\omega_1}$ with the existence of a first countable Luzin space is clearly consistent. Let us note however that KUNEN [1976] has shown that under MA(ω_1) there are no Luzin spaces at all.

Given any set $S \subset X$ let us put

$$E(S) = \{\alpha \in \omega_1 : S \cap D_\alpha \neq \emptyset\}.$$

Our claim now clearly follows from the following lemma.

8.2. LEMMA. *If $S \subset X$ is discrete as a subspace, then $E(S)$ is nowhere dense in $\langle \omega_1, \tau \rangle$.*

PROOF. Suppose, on the contrary, that $E(S)$ is dense in some $G \in \tau \setminus \{\emptyset\}$, hence we can pick $\alpha \in G \cap E(S)$ and $\nu \in S \cap D_\alpha$. Then there is a $k \in \omega$ with $U_\alpha^k \subset G$. Now, if $F \in [D_\alpha \setminus \{\nu\}]^{<\omega}$ and $n \in \omega$ are arbitrary, then $|A_\nu \setminus \bigcup_{\mu \in F} A_\mu| = \omega$ implies that

$$D(V_\alpha^l) \subset B_\nu(F, n)$$

holds for some $l \geq k$. But $V_\alpha^l \subset U_\alpha^k \subset G$, hence $E(S) \cap V_\alpha^l \neq \emptyset$, for $E(S)$ is dense in G. But then we have $(S \setminus \{\nu\}) \cap D(V_\alpha^l) \neq \emptyset$, consequently

$$(S \setminus \{\nu\}) \cap B_\nu(F, n) \neq \emptyset,$$

showing that $\nu \in \overline{S \setminus \{\nu\}}$, and this contradicts the fact that S is discrete.

8.3. REMARK. By I.12.3 we always have sup = max if $s(X)$ is a singular strong limit cardinal and X is T_2, hence something like the assumption $2^\omega \geqslant \omega_{\omega_1}$ is necessary in a sup $\neq$ max result like 8.1. However $2^\omega \geqslant \omega_{\omega_1}$ is the strongest possible violation of the strong limit property at ω_{ω_1}, hence the question naturally arises what happens if only a strictly weaker violation occurs. Nothing is known about this at present, however let us note that by KUNEN and ROITMAN [1977] (see I.12.6 or JUHÁSZ [1980], 4.4), $s(X) = \lambda$ with $\mathrm{cf}(\lambda) = \omega$ and $2^\omega < \lambda$ imply that sup = max if X is T_2.

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CHAPTER 3

The Integers and Topology

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"Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk."
(Kronecker)

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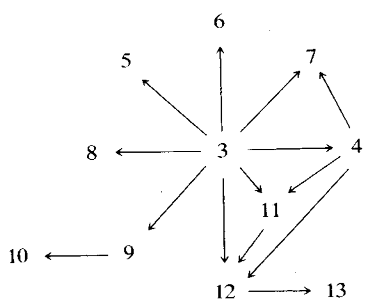

Connections between sections

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

1. Introduction

Two well known cardinals associated with ω are ω_1, the successor of ω, and $\mathfrak{c}$, the cardinality of $\mathcal{P}(\omega)$, or, equivalently, of ${}^{\omega}\omega$. These cardinals are different, but only potentially: One can easily show that $\mathfrak{c} \geqslant \omega_1$ (Cantor), and both $\mathfrak{c} \geqslant \omega_1$ and $\mathfrak{c} = \omega_1$ are consistent with ZFC (Cohen, Gödel).

In this article we study the role in topology of certain other cardinals associated with ω. We limit our discussion to problems involving first countability, convergence and separable metrizable spaces. Each of our cardinals, $\mathfrak{m}$ say, is defined to be the minimal cardinality of some special subfamily of $\mathcal{P}(\omega)$ or ${}^{\omega}\omega$. The definition is purely set theoretic, and this makes consistency results about $\mathfrak{m}$ feasible. Our $\mathfrak{m}$'s are different from both ω_1 and $\mathfrak{c}$, but only potentially: One has $\omega_1 \leqslant \mathfrak{m} \leqslant \mathfrak{c}$ (so $\mathfrak{m} = \omega_1 \wedge \mathfrak{m} = \mathfrak{c}$ if $\mathfrak{c} = \omega_1$), and each of $\omega_1 < \mathfrak{m} < \mathfrak{c}$ and $\omega_1 = \mathfrak{m} < \mathfrak{c}$ and $\omega_1 < \mathfrak{m} = \mathfrak{c}$ is consistent with ZFC. (So both $\mathfrak{m} = \omega_1$ and $\mathfrak{m} = \mathfrak{c}$ are weakenings of $\mathfrak{c} = \omega_1$.) (One should be careful: one can define cardinals associated with ω which look different than both ω_1 and $\mathfrak{c}$ but which in fact are ω_1, see Corollaries 4.2 and 4.4.)

A typical use of these set theoretic cardinals associated with ω involves topologically defined cardinals, like the minimum cardinality of a space having a certain property. In the cases we are interested in the topological cardinal, μ say, always satisfies $\omega_1 \leqslant \mu \leqslant \mathfrak{c}$, but consistency results about μ seem hard because of μ being not set theoretic. However, surprisingly often μ is equal to one of those cardinals $\mathfrak{m}$ associated with ω, a cardinal which we already know consistency results about, or which is known to be ω_1 in disguise. This is quite satisfactory, since any result of the form $\mu = \mathfrak{m}$ tells us what μ is as exactly as we can hope for, in ZFC. (See Section 8 for some examples.)

(The preceding is important for appraising certain consistency results. One may have a consistency result where some topological statement S is derived from $\mathfrak{c} = \omega_1$. It may not be clear just how important the assumption $\mathfrak{c} = \omega_1$ is. However, analysis of the proof that $\mathfrak{c} = \omega_1$ implies S may show that what one really uses in the proof is $\mathfrak{m} = \mathfrak{c}$. This does not help to decide whether or not S is independent from ZFC, but it does show that the proof uses $\mathfrak{m} = \mathfrak{c}$ in an essential way. (See Section 13 for a good example.) Of course this also shows that S is strictly weaker than $\mathfrak{c} = \omega_1$, but I feel this has been overemphasized by some: since the proof that $\mathfrak{m} = \mathfrak{c}$ implies S is (usually) a trivial modification of the proof that $\mathfrak{c} = \omega_1$ implies S, one wants to know whether or not S is strictly weaker than $\mathfrak{m} = \mathfrak{c}$, not $\mathfrak{c} = \omega_1$.)

Another use of these set theoretic cardinals associated with ω is that certain topological results hold iff one of these cardinals equals ω_1. (See Section 10 for an example.)

Despite the fact that this chapter concerns the cardinals associated with ω, we feel that the main objects of study are $\mathcal{P}(\omega)$ and ${}^{\omega}\omega$. The cardinals arise only because they capture certain aspects of $\mathcal{P}(\omega)$ and ${}^{\omega}\omega$, and everything that can not be captured by cardinals has been ignored in this article.

I am indebted to Peter Nyikus for improvements and comments.

2. Conventions and notation

Set theory. An *ordinal* is the set of smaller ordinals (but we write occasionally $[0, \eta)$ for η to avoid confusion), and a cardinal is an initial ordinal. ω is ω_0, and $\mathfrak{c}$ is 2^ω. '*Countable*' means 'at most countable'. We use κ and λ for cardinals, ξ, η, ζ for ordinals, and i, j, k, n for integers.

For a set X we frequently use the subcollections

$$[X]^\omega = \{A \in \mathscr{P}(X): |A| = \omega\}$$

and

$$[X]^{<\omega} = \{A \in \mathscr{P}(X): |A| < \omega\}$$

of the power set $\mathscr{P}(X)$ of X.

If A is a set, then $\mathscr{F} \restriction A$ denotes $\{F \cap A: F \in \mathscr{F}\}$ if $\mathscr{F}$ is a collection of sets, and $f \restriction A$ denotes the restriction of f to A if f is a function. This use of '$\restriction$' in two formally different meanings will not lead to confusion.

For sets A and B the set of functions $A \to B$ is denoted by ${}^A B$, and the domain and range of a function f are denoted by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$. Also, the preimage and image of a set X under a function f are denoted by $f^{\leftarrow} X$ and $f^{\rightarrow} X$ respectively. (So $f^{\rightarrow} X = \mathrm{ran}(f \restriction X)$.) As always, a function is its graph. In particular, $D \times \{d\}$ is the (constant) function with domain D and range $\{d\}$. We often write f_x for $f(x)$.

A *quasi-order* is a pair $\langle D, \leqslant \rangle$ of a set D and a reflexive transitive relation $\leqslant$ on D. As usual we write D for $\langle D, \leqslant \rangle$ if $\leqslant$ is clear from the context. We write $x < y$ if $x \leqslant y$ and $y \not\leqslant x$. (So $x < y$ is at least as strong as $x \leqslant y$ and $x \neq y$; it is stronger iff $\exists x, y \in D$ $[x \neq y$ and $x \leqslant y$ and $y \leqslant x]$.) (In particular, $\subset$ denotes *strict* inclusion.) A set $L \subseteq D$ is called *cofinal* in $\langle D, \leqslant \rangle$ if $\forall x \in D\ \exists y \in L\ [x \leqslant y]$, and $\mathrm{cof}(D)$ denotes the smallest cardinality of a cofinal subset of D.

We index a set if that seems unavoidable, and we write e.g. $\langle A_x: x \in X \rangle$ rather than $\{A_x: x \in X\}$ just to emphasize that we use the indexing.

Topology. All spaces considered are at least regular, hence T_1, and occasionally completely regular; this will be clear from the context. (In fact all examples are *zero-dimensional*, i.e. the family of *clopen* (= closed and open) sets is a base.

We use $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ for the irrationals, the rationals, and the reals.

For a (sub)space X we use X' to denote the derived set of X, i.e. the set of nonisolated points of X.

All ordinals always carry the order topology. In particular, κ is the discrete space with κ points if $\kappa \leqslant \omega$.

If A is a set and X is a space, then ${}^A X$ carries the product topology. We identify $\mathbb{P}$ with ${}^\omega \omega$.

An indexed collection $\langle A_X: x \in D \rangle$ of subsets of a space X will be called an *indexed discrete family* if each point of X has a neighborhood which intersects A_x for at most one $x \in D$, and is called an *indexed pairwise disjoint family* if $\forall x, y \in D\ [x \neq y \Rightarrow U_x \cap U_y = \emptyset]$.

3. Six cardinals

In this section we define six cardinals associated with ω, and give purely set theoretic information about them. Frequently this information, or the ideas behind the proof, are useful in topological applications of these cardinals, and therefore we have not limited the information to exactly what we need.

Our cardinals describe what happens if one ignores finite sets, and we begin with defining various ways in which one can do that.

Call two countably infinite sets *almost disjoint* if their intersection is finite, and for a set X call a subfamily of $[X]^\omega$ a *mad* family on X if it is a maximal (with respect to inclusion) pairwise almost disjoint subfamily of $[X]^\omega$.

Define the quasi-order $\leqslant^*$ on ${}^\omega\omega$ by

$$f \leqslant^* g \quad \text{if} \quad f(n) \leqslant g(n) \text{ for all but finitely many } n \in \omega .$$

Note that $\langle {}^\omega\omega, \leqslant^* \rangle$ has no maximal elements. A subset of ${}^\omega\omega$ is called *unbounded* if it is unbounded in $\langle {}^\omega\omega, \leqslant^* \rangle$. We follow tradition and call a subset of ${}^\omega\omega$ *dominating* if it is cofinal in $\langle {}^\omega\omega, \leqslant^* \rangle$, and call it a *scale* if it is both dominating and well-ordered by $\leqslant^*$.

Define the quasi-order $\subseteq^*$ on $\mathcal{P}(\omega)$, and occasionally on $\mathcal{P}(X)$ for other countably infinite X, by

$$\begin{aligned} &F \subseteq^* G \quad \text{if } F - G \text{ is finite}, \qquad \text{or, equivalently},\\ &x \in G \quad \text{for all but finitely many } x \in F. \end{aligned}$$

We say that A is a *pseudo-intersection* of a family $\mathcal{F}$ if $A \subseteq^* F$ for each $F \in \mathcal{F}$; note that each set is a pseudo-intersection of the empty family. We call $\mathcal{T} \subseteq [\omega]^\omega$ a *tower* if $\mathcal{T}$ is well-ordered by $\supseteq^*$ and has no infinite pseudointersection. We say that a family of countable sets has the *sfip* (strong finite intersection property) if every nonempty finite subfamily has infinite intersection. Finally, we call $\mathcal{S} \subseteq [\omega]^\omega$ *splitting* if $\neg\exists A \in [\omega]^\omega\ \forall S \in \mathcal{S}\ [A \subseteq^* S \text{ or } A \subseteq^* \omega - S]$ or, equivalently, and this explains the term, if

$$\forall A \in [\omega]^\omega \quad \exists S \in \mathcal{S} \quad [|A \cap S| = |A - S| = \omega] .$$

With these definitions available we now give the standard definition of our cardinals:

$$\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is an infinite mad on } \omega\},$$

$$\mathfrak{b} = \min\{|B| : B \text{ is an unbounded subset of } {}^\omega\omega\},$$

$$\mathfrak{d} = \min\{|D| : D \text{ is a dominating subset of } {}^\omega\omega\},$$

$$\mathfrak{p} = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a subfamily of } [\omega]^\omega \text{ with the } sfip \text{ which has no infinite pseudo-intersection}\},$$

$$\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a splitting family in } [\omega]^\omega\},$$

$$\mathfrak{t} = \min\{|\mathcal{T}| : \mathcal{T} \text{ is a tower}\}.$$

We omit the routine verification that these cardinals are well-defined. (This we also do below if we give alternative definitions.) Of course they are all at most equal to $\mathfrak{c}$. They also are all at least ω_1. This is a good exercise if you have not seen it yet. It also follows from the following result.

3.1. Theorem. (a) *$\omega_1 \leqslant \mathfrak{p} \leqslant \mathfrak{t} \leqslant \mathfrak{b} \leqslant \mathfrak{a}$ and $\mathfrak{t} \leqslant \mathfrak{s} \leqslant \mathfrak{d}$ and $\mathfrak{b} \leqslant \mathfrak{d}$.*

(b) *If $\mathfrak{p} = \omega_1$, then $\mathfrak{t} = \omega_1$.*

(c) *If $\omega \leqslant \kappa < \mathfrak{t}$, then $2^\kappa = \mathfrak{c}$.*

(d) *$\mathfrak{t}$ and $\mathfrak{b}$ are regular, and $\mathrm{cof}(\mathfrak{d}) \geqslant \mathfrak{b}$.*

(e) *$\mathfrak{p}$ is regular.*

We show in Section 5 that $\mathfrak{t} < \mathfrak{b}$ and $\mathfrak{t} < \mathfrak{s}$ consistent via (c) by proving $2^{\omega_1} > \mathfrak{c}$ consistent with $\mathfrak{b} = \omega_2 = \mathfrak{c}$ or $\mathfrak{s} = \omega_2 = \mathfrak{c}$ (which suggests an obvious question), and also prove $\mathfrak{a} < \mathfrak{d}$ and (hence) $\mathfrak{b} < \mathfrak{d}$ consistent. We do not know whether or not any of $\mathfrak{p} < \mathfrak{t}$, $\mathfrak{b} < \mathfrak{s}$ or $\mathfrak{b} < \mathfrak{a}$ is consistent (see 12.5 for consequences of $\mathfrak{a} = \mathfrak{b}$), nor whether or not $\mathfrak{a}$ and $\mathfrak{s}$, or $\mathfrak{a}$ and $\mathfrak{d}$ are unrelated.

The proof of 5.2 shows it is consistent to have $\mathfrak{d} > \omega_1 = \mathrm{cof}(\mathfrak{d})$. We have no information about $\mathrm{cof}(\mathfrak{a})$ and $\mathrm{cof}(\mathfrak{s})$.

Before we prove Theorem 3.1 we give alternative formulations of our six cardinals.

3.2. Fact. $\mathfrak{a}_1 = \mathfrak{a}_1 = \mathfrak{a}_2$, *where*

$$\mathfrak{a}_1 = \min\{|\mathscr{A}|: \textit{there is a decomposition } \mathscr{D} \textit{ of } \omega \textit{ such that } \mathscr{D} \cap \mathscr{A} = \emptyset \textit{ and such that } \mathscr{D} \cup \mathscr{A} \textit{ is an infinite mad on } \omega\},$$

$$\mathfrak{a}_2 = \min\{|\mathscr{A}|: \forall k \in \omega[\{k\} \times \omega \notin \mathscr{A}], \textit{ and } \mathscr{A} \cup \{\{k\} \times \omega : k \in \omega\} \textit{ is a mad family on } \omega \times \omega\}.$$

Proof. The inequality $\mathfrak{a}_1 \leqslant \mathfrak{a}_2$ is trivial, and to see that $\mathfrak{a}_2 \leqslant \mathfrak{a}_1$ one only has to observe that $\mathscr{D}$ as in the definition of $\mathfrak{a}_1$ must be countably infinite. Also, trivially $\mathfrak{a} \leqslant \mathfrak{a}_1 + \omega$, hence $\mathfrak{a} \leqslant \mathfrak{a}_1$ since $\mathfrak{a} \geqslant \omega_1$. It remains to prove that $\mathfrak{a}_1 \leqslant \mathfrak{a}$.

Let $\mathscr{B}$ be any infinite mad on ω. Let $\beta: \omega \to \mathscr{B}$ be an injection, and define $\beta': \omega \to [\omega]^\omega$ with recursion by $\beta'(n) = (\{n\} \cup \beta(n)) - \bigcup_{k \in n} \beta'(k)$. Then $\mathrm{ran}(\beta')$ is a decomposition of ω into infinite sets. Also, for each $n \in \omega$ we have: $\beta(n) \subseteq^* \beta'(n) \subseteq \beta(n)$, hence $\forall X \in [\omega]^\omega [|X \cap \beta(n)| = \omega$ iff $|X \cap \beta'(n)| = \omega]$. It follows that $\mathscr{A} = \mathscr{B} - \mathrm{ran}(\beta)$ and $\mathscr{D} = \mathrm{ran}(\beta')$ show that $\mathfrak{a}_1 \leqslant \mathfrak{a}$. $\square$

For our next result we need some terminology. If $\mathscr{F}$ and $\mathscr{G}$ are families of countably infinite sets we write $\mathscr{F} \perp \mathscr{G}$ if $\forall F \in \mathscr{F}\ \forall G \in \mathscr{G}\ [|F \cap G| < \omega]$, and we say that $\mathscr{F}$ and $\mathscr{G}$ can be *separated* if there is a set S with $\forall F \in \mathscr{F}\ [F \subseteq^* S]$ and $\forall G \in \mathscr{G}\ [|G \cap S| < \omega]$; note that if S separates $\mathscr{F}$ and $\mathscr{G}$, then $(\bigcup(\mathscr{F} \cup \mathscr{G})) - S$ separates $\mathscr{G}$ and $\mathscr{F}$, hence 'can be separated' is symmetric. We write $F \perp \mathscr{G}$ if $\{F\} \perp \mathscr{G}$. Finally, if $B \subseteq {}^\omega\omega$ and $I \in [\omega]^\omega$ we call B *unbounded on I* if

$$\forall f \in {}^{\omega}\omega \quad \exists g \in B \quad [|\{n \in I: f(n) < g(n)\}| = \omega],$$

i.e. if $\{f \upharpoonright I: f \in B\}$ is unbounded in $\langle {}^{I}\omega, \leqslant^* \rangle$.

3.3. THEOREM. *Let $\mathcal{V}$ be the collection of vertical lines in $\omega \times \omega$, i.e. $\mathcal{V} = \{\{k\} \times \omega : k \in \omega\}$. We have $\mathfrak{b} = \mathfrak{b}_i$ for $i \in \{1, \ldots, 7\}$, where*

$\mathfrak{b}_1 = \min\{|B|$: *$B$ is an unbounded subset of ${}^{\omega}\omega$ consisting of strictly increasing functions, which is well-ordered by $<^*$*$\}$,

$\mathfrak{b}_2 = \min\{|B|$: *$B \subseteq {}^{\omega}\omega$, and B is unbounded on each infinite subset of ω*$\}$,

$\mathfrak{b}_3 = \min\{|\mathcal{B}|$: *$\mathcal{B} \subseteq [\omega \times \omega]^{\omega}$ is well-ordered by $\subset^*$, and $\mathcal{B} \perp \mathcal{V}$, and $\forall X \in [\omega \times \omega]^{\omega} [X \perp \mathcal{V} \Rightarrow \exists B \in \mathcal{B}\ [|X \cap B| = \omega]]$*$\}$,

$\mathfrak{b}_4$: *as $\mathfrak{b}_3$ but without "well-ordered by $\subset^*$"*,[1]

$\mathfrak{b}_5 = \min\{|\mathcal{B}|$: *$\mathcal{B} \subseteq [\omega]^{\omega}$, and $\exists \mathcal{C} \subseteq [\omega]^{\omega} [|\mathcal{C}| = \omega$ and $\mathcal{B} \cap \mathcal{C} = \emptyset$ and $\mathcal{B} \cup \mathcal{C}$ is pairwise almost disjoint and $\forall \mathcal{D} \in [\mathcal{C}]^{\omega}$ [$\mathcal{B}$ and $\mathcal{D}$ can not be separated]*$\}$,

$\mathfrak{b}_6 = \min\{|\mathcal{B}|$: *$\mathcal{B} \subseteq [\omega]^{\omega}$, and $\exists \mathcal{C} \subseteq [\omega]^{\omega} [|\mathcal{C}| = \omega$ and $\mathcal{B} \perp \mathcal{C}$ and $\mathcal{B}$ and $\mathcal{C}$ are well-ordered by $\subseteq^*$ and $\mathcal{B}$ and $\mathcal{C}$ can not be separated*$\}$,

$\mathfrak{b}_7$: *as $\mathfrak{b}_6$, but without "$\mathcal{B}$ and $\mathcal{C}$ are well-ordered by $\subseteq^*$"*.

PROOF. The inequalities $\mathfrak{b}_2 \geqslant \mathfrak{b}$ and $\mathfrak{b}_3 \geqslant \mathfrak{b}_4$ and $\mathfrak{b}_6 \geqslant \mathfrak{b}_7$ are obvious, so it remains to prove $\mathfrak{b} \geqslant \mathfrak{b}_1 \geqslant \mathfrak{b}_2$, and $\mathfrak{b}_1 \geqslant \mathfrak{b}_3 \geqslant \mathfrak{b}_6$ and $\mathfrak{b}_1 \geqslant \mathfrak{b}_5 \geqslant \mathfrak{b}_7$ and $\mathfrak{b}_4 \geqslant \mathfrak{b}_7 \geqslant \mathfrak{b}$.

For $f \in {}^{\omega}\omega$ define L_f (lower f) to be $\{\langle k, n \rangle \in \omega \times \omega : n \leqslant f(k)\}$, and note that

(1) $\quad \forall f, g \in {}^{\omega}\omega \quad [f <^* g \Leftrightarrow L_f \subset^* L_g]$.

Also, for $X \in [\omega \times \omega]^{\omega}$ define $K_X = \{k \in \omega : X \cap \{k\} \times \omega \neq \emptyset\}$, and if $X \perp \mathcal{V}$ define $f_X \in {}^{\omega}\omega$ by $f_X(k) = \max\{n \in \omega : n = 0 \text{ or } \langle k, n \rangle \in X\}$, and note that

(2) $\quad X \subseteq L_{f_X} \cap K_X \times \omega$.

Proof that $\mathfrak{b} \geqslant \mathfrak{b}_1$. Let $F = \{f_\xi : \xi \in \mathfrak{b}\}$ be unbounded in ${}^{\omega}\omega$. Since the set S of strictly increasing functions clearly is dominating we can with recursion pick $\mathfrak{b}_\eta \in S$ for $\eta \in \mathfrak{b}$ so that $\forall f \in \{\mathfrak{b}_\xi : \xi \in \eta\} \cup \{f_\eta\} [f <^* \mathfrak{b}_\eta]$. Then $B = \{\mathfrak{b}_\eta : \eta \in \mathfrak{b}\}$ shows $\mathfrak{b} \geqslant \mathfrak{b}_1$.

Proof that $\mathfrak{b}_1 \geqslant \mathfrak{b}_2$. It suffices to prove the following fact.

3.4. FACT. *If $B \subseteq {}^{\omega}\omega$ is unbounded and consists of nondecreasing functions, then B is unbounded on each infinite subset of ω.*

[1] Note the similarity between $\mathfrak{a}_2$ and $\mathfrak{b}_4$.

Indeed, consider any $f \in {}^\omega\omega$ and $I \in [\omega]^\omega$. We can define $\hat{f} \in {}^\omega\omega$ by

$$\hat{f}(k) = \max\{f(n)\colon n \leqslant \min\{i \in I\colon i \geqslant k\}\}.$$

There is $g \in B$ such that $g \not\leqslant^* \hat{f}$, i.e. $J = \{j \in \omega\colon \hat{f}(j) < g(j)\}$ is infinite. For each $j \in J$, if $i = \min\{k \in I\colon k \geqslant j\}$, then $g(i) \geqslant g(j) > \hat{f}(j) \geqslant f(i)$ since g is nondecreasing, hence $\{i \in I\colon f(i) < g(i)\}$ is infinite.

Proof that $\mathfrak{b}_1 \geqslant \mathfrak{b}_3$: Let B be as in the definition of $\mathfrak{b}_1$, and put $\mathscr{B} = \{L_f\colon f \in B\}$. From (1) we see that $\mathscr{B}$ is well-ordered by $\subset^*$, and clearly $\mathscr{B} \perp \mathscr{V}$. Consider any $X \in [\omega \times \omega]^\omega$ with $X \perp \mathscr{V}$. Clearly K_X is infinite, hence by Fact 3.4 there is $g \in B$ such that $J = \{k \in K_X\colon f_X(k) \leqslant g(k)\}$ is infinite. Then $f_X \restriction J$ is an infinite subset of both X and L_g.

Proof that $\mathfrak{b}_1 \geqslant \mathfrak{b}_5$. Define $\mathfrak{b}_5'$ as $\mathfrak{b}_5$, but with '$[\omega \times \omega]^\omega$' instead of '$[\omega]^\omega$'. Then $\mathfrak{b}_5' = \mathfrak{b}_5$, hence it suffices to prove $\mathfrak{b}_1 \geqslant \mathfrak{b}_5'$. Let B be as in the definition of $\mathfrak{b}_1$ and put $\mathscr{B} = B$ and $\mathscr{C} = \mathscr{V}$. Obviously $|\mathscr{C}| = \omega$ and $\mathscr{B} \cap \mathscr{C} = \emptyset$ and $\mathscr{B} \cup \mathscr{C}$ is almost disjoint. (This is where we use that $\mathscr{B}$ is well-ordered by $<^*$.) Consider any S such that $\forall C \in \mathscr{C}[C \subseteq^* S]$ and any $\mathscr{D} \in [\mathscr{V}]^\omega$. Let $D = \{k \in \omega\colon \{k\} \times \omega \in \mathscr{D}\}$. Obviously there is $f \in {}^\omega\omega$ such that $\forall k, n \in \omega\ [n \geqslant f(k) \Rightarrow \langle k, n\rangle \in S]$, e.g. define f by $f(k) = \max\{n \in \omega\colon n = 0$ or $(n \geqslant 1$ and $\langle k, n-1\rangle \notin S)\}$, for $k \in \omega$. By Fact 3.4 there is $g \in B$ such that $I = \{k \in D\colon g(k) \geqslant f(k)\}$ is infinite. Clearly $g \restriction I \subseteq S$, hence $S \cap B$ is infinite. It follows that $\mathscr{B}$ and $\mathscr{D}$ can not be separated.

Proof that $\mathfrak{b}_3 \geqslant \mathfrak{b}_6$. This is like the proof that $\mathfrak{b}_1 \geqslant \mathfrak{b}_5$, but with $\mathscr{C} = \{k \times \omega\colon 1 \leqslant k \in \omega\}$. (Recall that $k = \{0, \ldots, k-1\}$ if $1 \leqslant k \in \omega$.)

Proof that $\mathfrak{b}_4 \geqslant \mathfrak{b}_7$. This is like the proof that $\mathfrak{b}_3 \geqslant \mathfrak{b}_6$.

Proof that $\mathfrak{b}_5 \geqslant \mathfrak{b}_7$. If $\mathscr{B}$ and $\mathscr{C}$ are as in the definition of $\mathfrak{b}_5$ then $\mathscr{B} \perp \mathscr{C}$.

Proof that $\mathfrak{b}_7 \geqslant \mathfrak{b}$. Define $\mathfrak{b}_7'$ as $\mathfrak{b}_7$ but with '$[\omega \times \omega]^\omega$' instead of '$[\omega]^\omega$'. Put $\mathscr{A} = \{\bigcup \mathscr{F}\colon \mathscr{F} \in [\mathscr{C}]^{<\omega} - \{\emptyset\}\}$. No member of $\mathscr{A}$ separates $\mathscr{C}$ and $\mathscr{B}$, hence there is $\alpha\colon\omega \to \mathscr{A}$ such that

$$\forall k \in \omega[\alpha(k) \subset \alpha(k+1) \text{ and } |\alpha(k+1) - \alpha(k)| = \omega\},$$

and

$$\forall C \in \mathscr{C}\ \exists k \in \omega[C \subseteq \alpha(k)].$$

Then clearly $\operatorname{ran}(\alpha) \perp \mathscr{B}$ and $\operatorname{ran}(\alpha)$ and $\mathscr{B}$ can not be separated, hence without loss of generality we can assume $\mathscr{C} = \operatorname{ran}(\alpha)$. We also can assume without loss of generality that $\alpha(k) = (k+1) \times \omega$, for $k \in \omega$. Under this assumption $\mathscr{V} \perp \mathscr{B}$ and $\mathscr{V}$ and $\mathscr{B}$ can not be separated. Then f_X is defined for all $X \in \mathscr{B}$ hence we can define $B = \{f_X\colon X \in \mathscr{B}\}$. But $\mathscr{B}' = \{L_f\colon f \in B\}$. Trivially $\mathscr{B}' \perp \mathscr{V}$, and it follows from (2) that $\mathscr{B}'$ and $\mathscr{C}$ can not be separated.

We now show that B is unbounded: Consider any $f \in {}^\omega\omega$. Then $L_f \perp \mathscr{V}$. Since $\mathscr{B}'$ and $\mathscr{V}$ can not be separated there must be $g \in B$ such that $L_g \not\subseteq^* L_f$, hence such that $g \not\leqslant^* f$ because of (1). $\square$

3.5. COROLLARY TO PROOF (of $\mathfrak{b} \geq \mathfrak{b}_1$). $\mathfrak{b} = \mathfrak{d}$ *iff* $^{\omega}\omega$ *has a scale.*

In the following theorem we say that $D \subseteq {}^{\omega}\omega$ is *cofinal on* $\mathcal{A} \subseteq [\omega]^{\omega}$ if

$$\forall f \in {}^{\omega}\omega \quad \exists g \in D \quad \exists A \in \mathcal{A} \quad [f \leq g \text{ on } A, \text{ i.e. } \forall a \in A\ [f(a) \leq g(a)]] .$$

Remember that $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many $n \in \omega$, while $f \leq g$ means that $f(n) \leq g(n)$ for all $n \in \omega$; this explains the $\mathfrak{d}_1$ below.

3.6. THEOREM. $\mathfrak{d} = \mathfrak{d}_1 = \mathfrak{d}_2$, *where*

$$\mathfrak{d}_1 = \min\{|D| \colon D \textit{ is cofinal in } \langle {}^{\omega}\omega, \leq \rangle\} ;$$

$$\mathfrak{d}_2 = \min\{|D| + |\mathcal{A}| \colon D \subseteq {}^{\omega}\omega \textit{ and } \mathcal{A} \subseteq [\omega]^{\omega} \textit{ and } D \textit{ is cofinal on } \mathcal{A}\} .$$

PROOF. The inequality $\mathfrak{d} \leq \mathfrak{d}_1$ is trivial, and the inequality $\mathfrak{d}_2 \leq \mathfrak{d}$ follows from the fact that $D \subseteq {}^{\omega}\omega$ is dominating iff it is cofinal on $\{A \subseteq \omega \colon \omega - A \text{ is finite}\}$. (In this context note that D is cofinal in $\langle {}^{\omega}\omega, \leq \rangle$ iff D is cofinal on $\{\omega\}$.)

Proof that $\mathfrak{d}_1 \leq \mathfrak{d}_2$. Let $D \subseteq {}^{\omega}\omega$ be cofinal on $\mathcal{A} \subseteq [\omega]^{\omega}$, with $|D| + |\mathcal{A}| = \mathfrak{d}_2$. For $d \in D$ and $A \in \mathcal{A}$ define $d_A \in {}^{\omega}\omega$ by: $d_A(k) = d(\min\{a \in A \colon k \leq a\})$, for $k \in \omega$. The set $\{d_A \colon d \in D, A \in \mathcal{A}\}$ obviously has cardinality at most $\mathfrak{d}_2$; we show it is cofinal in $\langle {}^{\omega}\omega, \leq \rangle$: Consider any $f \in {}^{\omega}\omega$. Let g be a nondecreasing function with $f \leq g$. There are $d \in D$ and $A \in \mathcal{A}$ such that $g \leq d$ on A. Then $f \leq d_A$, for if $k \in \omega$ is arbitrary, and if $n = \min\{a \in A \colon k \leq a\}$, then since g is nondecreasing, $d_A(k) = d_A(m) \geq g(m) \geq g(k) \geq f(k)$. □

3.7. THEOREM. $\mathfrak{t} = \mathfrak{t}_1 = \mathfrak{t}_2$, *where*

$$\mathfrak{t}_1 = \min\{\kappa \colon \textit{there is } T \colon \kappa \to [\omega]^{\omega} \textit{ such that } \forall \xi, \eta \in \kappa\ [\xi < \eta \Rightarrow T_\eta \subseteq^* T_\xi] \textit{ and } \operatorname{ran}(T) \textit{ has no infinite pseudo-intersection}\} ,$$[2]

$$\mathfrak{t}_2 = \min\{|\mathcal{T}| \colon \mathcal{T} \subseteq [\omega]^{\omega} \textit{ is well-ordered by } \subset^*, \textit{ and } \forall T \in \mathcal{T}\ [\omega - T \textit{ infinite}], \textit{ and } \forall X \in [\omega]^{\omega}\ \exists T \in \mathcal{T}\ [X \cap T \textit{ infinite}]\} .$$[3]

PROOF. It should be clear that $\mathfrak{t} = \mathfrak{t}_2$, and clearly $\mathfrak{t}_1 \leq \mathfrak{t}$. Let T be as in the definition of $\mathfrak{t}_1$, put $I = \{\eta \in \kappa \colon \exists \xi \in \eta\ [T_\xi \subseteq^* T_\eta]\}$. Then $\operatorname{ran}(T \restriction (\kappa - I))$ is a subfamily of $[\omega]^{\omega}$, well-ordered by $^*\supset$, which has no infinite pseudointersection, hence $\mathfrak{t} \leq \mathfrak{t}_1$. □

3.8. THEOREM. $\mathfrak{p} = \mathfrak{p}_1 = \mathfrak{p}_2$, *where*

$$\mathfrak{p}_1 = \min\{|\mathcal{T}| \colon \mathcal{T} \subseteq [\omega]^{\omega}, \textit{ and } \forall \mathcal{F} \in [\mathcal{T}]^{<\omega} [\omega - \bigcup \mathcal{F} \textit{ is infinite}], \textit{ and } \forall X \in [\omega]^{\omega}\ \exists T \in \mathcal{T}\ [X \cap T \textit{ infinite}]\} ,$$[4]

$$\mathfrak{p}_2 = \min\{|\mathcal{A}| + |\mathcal{B}| \colon \mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\omega),\ \mathcal{B} \neq \emptyset, \textit{ and } \forall B \in \mathcal{B}\ [\mathcal{A} \restriction B \textit{ has the sfip}], \textit{ and } \mathcal{A} \textit{ has no pseudo-intersection } I \textit{ such that } \forall B \in \mathcal{B}\ [I \cap B \textit{ is infinite}]\} .$$

[2] The difference with the definition of $\mathfrak{t}$ is that we have '$\subseteq^*$', not '$\subset^*$'.

[3] Note the similarity between $\mathfrak{t}_2$ and $\mathfrak{b}_3$.

[4] Note the similarity between $\mathfrak{p}_1$, $\mathfrak{t}_2$ and $\mathfrak{b}_4$.

PROOF. It should be clear that $\mathfrak{p}_1 = \mathfrak{p} \geq \mathfrak{p}_2$. To prove $\mathfrak{p}_2 \geq \mathfrak{p}$ consider $\mathscr{A}, \mathscr{B} \subseteq \mathscr{P}(\omega)$ with $|\mathscr{A}| + |\mathscr{B}| < \mathfrak{p}$ and $\mathscr{B} \neq \emptyset$ such that $\forall B \in \mathscr{B}$ [$\mathscr{A} \restriction B$ has the sfip]. Define subfamilies $\mathscr{H}_1$ and $\mathscr{H}_2$ of $\mathscr{P}([\omega]^{<\omega})$ by

$$\mathscr{H}_1 = \{[A]^{<\omega} : A \in \mathscr{A}\},$$

and

$$\mathscr{H}_2 = \{\{F \in [\omega - n]^{<\omega} : F \cap B \neq \emptyset\} : B \in [\omega]^{<\omega} \text{ and } n \in \omega\}.$$

It is easy to check that $\mathscr{H}_1 \cup \mathscr{H}_2$ has the sfip, and that $|\mathscr{H}_1 \cup \mathscr{H}_2| < \mathfrak{p}$ since $\mathfrak{p} > \omega$. Since $|[\omega]^{<\omega}| = \omega$ it follows that there is an infinite $\mathscr{I} \subseteq [\omega]^{<\omega}$ which is a pseudo-intersection of $\mathscr{H}_1 \cup \mathscr{H}_2$. Consequently, for each $A \in \mathscr{A}$ we have $I \subseteq A$ for all but finitely many $I \in \mathscr{I}$, hence $\bigcup \mathscr{I} \subseteq^* A$, and for each $B \in \mathscr{B}$, and $n \in \omega$, we have $\min(I) \geq n$ and $I \cap B \neq \emptyset$ for all but finitely many $I \in \mathscr{I}$, hence $(\bigcup \mathscr{I}) \cap B$ is infinite. $\square$

3.9. PROOF OF THEOREM 3.1. We begin with part (a).

Proof that $\omega_1 \leq \mathfrak{p}$. First note that $\mathfrak{p} > 0$ since ω is an almost intersection of $\emptyset$. Let $\mathscr{F} = \{F_n ; n \in \omega\}$ have the sfip. Define $a : \omega \to \omega$ with recursion by $a_n = \min(\bigcap_{k \leq n} F_k - \{a_k : k \in n\})$. Then $\operatorname{ran}(a)$ is an infinite pseudo-intersection of $\mathscr{F}$.

Proof that $\mathfrak{p} \leq \mathfrak{t}$. Trivial.

Proof that $\mathfrak{t} \leq \mathfrak{s}$. Let $\kappa < \mathfrak{t}$, consider $\mathscr{S} \subseteq [\omega]^\omega$ with $|\mathscr{S}| = \kappa$, and enumerate $\mathscr{S}$ as $\langle S_\eta : \eta \in \kappa + 1\rangle$. (This is slightly more convenient than $\langle S_\eta : \eta \in \kappa\rangle$.) With recursion on $\eta \in \kappa + 1$ pick $T_\eta \in [\omega]^\omega$ such that

(1) $\forall \xi \in \eta [T_\eta \subseteq^* T_\xi]$, and

(2) $T_\eta \subseteq S_\eta$ or $T_\eta \subseteq \omega - S_\eta$.

At stage η first pick an infinite pseudo-intersection A of $\{T_\xi : \xi \in \eta\}$, which exists since $\mathfrak{t} = \mathfrak{t}_1$, and then let T_η be $A \cap S_\eta$ if this is infinite, otherwise let T_η be $A - S_\eta$. Then $\forall S \in \mathscr{S}$ [$T_\kappa \subseteq^* S$ or $T_\kappa \subseteq^* \omega - S$] hence $\mathscr{S}$ is not splitting.

Proof that $t \leq \mathfrak{b}$. Let $\kappa < \mathfrak{t}$, and consider any $B = \langle b_\eta : \eta \in \kappa + 1\rangle \subseteq {}^\omega\omega$. With recursion on $\eta \in \kappa + 1$ pick strictly increasing $f_\eta \in {}^\omega\omega$ such that

(3) $\forall \xi \in \eta$ [$\operatorname{ran}(f_\eta) \subseteq^* \operatorname{ran}(f_\xi)$], and

(4) $\forall n \in \omega$ [$f_\eta(n) \geq \max\{b_\eta(k) : k \leq 2n\}$].

At stage η first pick an infinite pseudo-intersection A of $\{\operatorname{ran}(f_\xi) : \xi \in \eta\}$ and then put $f_\eta(n) = \min\{a \in A : \forall k \leq 2n\ [b_\eta(k) \leq a]\}$ $(n \in \omega)$.

We claim that f_κ bounds B: Consider any $\xi \in \kappa + 1$. Since f_κ is injective, being strictly increasing, and since $\operatorname{ran}(f_\kappa) \subseteq^* \operatorname{ran}(f_\xi)$ there must be $m \in \omega$ such that $\forall n \in \omega$ [$f_\kappa(m + n) \in \operatorname{ran}(f_\xi)$]. Since both f_κ and f_ξ are strictly increasing it follows that $\forall n \in \omega$ [$f_\kappa(m + n) \geq f_\kappa(n)$]. Because of (4) we now have for all $n \in \omega$: if $n \geq 2m$, hence if $2(n - m) \geq n$, then

$$f_\kappa(n) = f_k(n - m + m) \geq f_\xi(n - m) \geq b_\xi(n).$$

Proof that $\mathfrak{s} \leqslant \mathfrak{d}$. For $A \in [\omega]^\omega$ let b_A be the (unique) strictly increasing bijection $\omega \to A$. For $f \in {}^\omega\omega$ and $n \in \omega$ let f^n denote the n-fold composition of f. For strictly increasing $f \in {}^\omega\omega$ (or even f satisfying $\forall n \in \omega\ [f^n(0) < f^{n+1}(0)]$ define

$$S_f = \bigcup_{n\in\omega}[f^{2n}(0), f^{2n+1}(0)) \qquad \text{(intervals in } \omega).$$

There is dominating $F \subseteq {}^\omega\omega$ with $|F| = \mathfrak{d}$ consisting of strictly increasing functions; the following lemma implies $\{S_f : f \in F\}$ is splitting for such F.

LEMMA. *Let* $A \in [\omega]^\omega$, *and let* $f \in {}^\omega\omega$ *be strictly increasing. If* $b_A <^* f$ *then* S_f *splits* A, *i.e.* $|A \cap S_f| = |A - S_f| = \omega$.

To prove this consider $m \in \omega$ such that $\forall n \in \omega\ [n \geqslant m \Rightarrow f(n) > b_A(n)]$. As $\forall n \in \omega\ [b_A(n) \geqslant n]$ we have for all $n \geqslant m$ that

$$f^n(0) \leqslant b_A(f^n(0)) < f(f^n(0)) = f^{n+1}(0),$$

hence $b_A(f^n(0)) \in S_f$ if n is even and $b_A(f^n(0)) \notin S_f$ if n is odd.

Aside. Define

$$\mathfrak{s}' = \min\{|F| : F \subseteq {}^\omega\omega,\ \forall f \in F\ [f \text{ increases strictly}],\ \{S_f : f \in F\} \text{ splitting}\}$$

$$\mathfrak{s}'' = \min\{|F| : F \subseteq {}^\omega\omega, \forall f \in F\, \forall n \in \omega\ [f^n(0) < f^{n+1}(0)], \{S_f : f \in F\} \text{ splitting}\}.$$

Then $\mathfrak{d} \geqslant \mathfrak{s}' \geqslant \mathfrak{s}'' = \mathfrak{s}$: The lemma shows $\mathfrak{d} \geqslant \mathfrak{s}'$, and $\mathfrak{s} \geqslant \mathfrak{s}''$ since for each $S \subseteq \omega$ with $|S| = |\omega - S| = \omega$ there is $f \in {}^\omega\omega$ with $\forall n \in \omega\ [f^n(0) < f^{n+1}(0)]$ and $S_f = S$.

Proof that $\mathfrak{b} \leqslant \mathfrak{a}$. Footnote 1.

Aside. Footnote 3 does not seem to prove that $\mathfrak{t} \leqslant \mathfrak{b}$; one only gets $\mathfrak{p} \leqslant \mathfrak{b}$ from footnote 4.

Proof that $\mathfrak{b} \leqslant \mathfrak{d}$. Each dominating subset of ${}^\omega\omega$ is unbounded.

Proof of (b). If $\mathfrak{b} = \omega_1$, then $\mathfrak{t} = \omega_1$ since $\omega_1 \leqslant \mathfrak{t} \leqslant \mathfrak{b}_3$ and if $\mathfrak{b} > \omega_1$, then $\mathfrak{t} = \omega_1$ because of the following fact.

3.10. FACT. $[\mathfrak{b} > \omega_1]$ *For each* $\mathcal{U} \subseteq [\omega]^\omega$ *with the sfip with* $|\mathcal{U}| \leqslant \omega_1$ *there is* $T: \omega_1 \to [\omega]^\omega$ *such that* $\forall \xi, \gamma \in \omega_1\ [\xi < \eta \Rightarrow T_\eta \subseteq^* T_\xi]$ *and such that* $\forall U \in \mathcal{U}\ \exists \eta \in \omega_1\ [T_\eta \subseteq^* U]$ (*hence each pseudo-intersection of* $\operatorname{ran}(T)$ *is a pseudo-intersection of* $\mathcal{U}$).

To prove this enumerate $\mathcal{U}$ as $\langle U_\eta : \eta \in \omega_1\rangle$. Since $\mathfrak{p} \geqslant \omega_1$ we can choose an infinite pseudo-intersection A_η of $\{U_\xi : \xi \in \eta\}$ for each $\eta \in \omega_1$. With recursion on $\eta \in \omega_1$ pick $T_\eta \in [\omega]^\omega$ satisfying

(5) $\forall \xi \in \eta\ [T_\eta \subseteq^* T_\xi$ and $T_\eta \subseteq^* U_\xi]$, and

(6) $\forall \xi \in \omega_1 - \eta\ [A_\xi \subseteq T_\eta]$.

At stage 0 set $T_0 = U_0$. At stage γ, for $0 < \gamma \in \omega_1$ assume T_η to be picked for

$\eta \in \gamma$ in such a way that (5) and (6) hold for $\eta \in \gamma$. Put

$$\mathscr{B} = \{A_\xi \colon \gamma < \xi \in \omega_1\} \quad \text{and} \quad \mathscr{C} = \{\omega - U_\gamma\} \cup \{\omega - T_\eta \colon \eta \in \gamma\}.$$

Then $\mathscr{B} \perp \mathscr{C}$, and $|\mathscr{C}| \leqslant \omega$, hence since $\mathfrak{b}_7 > \omega_1$ it follows that $\mathscr{B}$ and $\mathscr{C}$ can be separated. Pick $T_\gamma \in [\omega]^\omega$ such that $\forall B \in \mathscr{B}\ [B \subseteq^* T_\gamma]$ and $\forall C \in \mathscr{C}\ [C \cap T_\gamma$ is finite (so $T_\gamma \subseteq^* \omega - C)]$.

Remark. Theorem 4.1 shows that one must have $|\mathscr{C}| \leqslant \omega$ in $\mathfrak{b}_7$, even if $\mathfrak{b}_7 > \omega_1$.

Proof of (c). Let $\omega \leqslant \kappa < \mathfrak{t}$. Then $2^\kappa \geqslant \mathfrak{c}$, hence we must prove $2^\kappa \leqslant \mathfrak{c}$. We will show that for each $\eta \in \kappa + 1$ there are $T_f \in [\omega]^\omega$ for all $f \in {}^\eta 2$ satisfying

(7) $\forall f \in {}^\eta 2\, \forall \xi \in \eta\ [T_f \subseteq^* T_{f \restriction \xi}]$, and

(8) $\forall f, g \in {}^\eta 2\, [f \neq g \Rightarrow T_f \cap T_g$ finite].

The special case $\eta = \kappa$ of (8) implies $2^\kappa \leqslant |[\omega]^\omega| = \mathfrak{c}$.

We construct all $T_f \in [\omega]^\omega$ for $f \in {}^\eta 2$ simultaneously, with recursion on η. If $\eta = 0$, then ${}^0 2 = \{\emptyset\}$; let $T_\emptyset = \omega$. Now let $0 < \gamma \leqslant \kappa + 1$ and assume T_f known for $f \in {}^\eta 2$ if $\eta \in \gamma$.

Case 1. *γ is a limit.* For $f \in {}^\gamma 2$ choose an infinite pseudointersection T_f of $\{T_{f \restriction \xi} \colon \xi \in \gamma\}$.

Case 2. *γ is a successor, $\gamma = \delta + 1$ say.* First pick infinite disjoint subsets $T_{f,0}$ and $T_{f,1}$ of T_f for $f \in {}^\delta 2$. Then define $T_f = T_{f \restriction \delta, f(\delta)}$ for $f \in {}^\gamma 2$.

To see that (8) holds consider distinct $f, g \in {}^\gamma 2$. Define $\mu = \min\{\xi \in \gamma \colon f(\xi) \neq g(\xi)\}$. If $\gamma = \mu + 1$, then $T_f \cap T_g = \emptyset$ by the construction under Case 2, and if $\gamma > \mu + 1$, then $T_{f \restriction \mu+1} \cap T_{g \restriction \mu+1}$ is finite (and in fact is empty) since (8) holds for $\eta = \mu + 1 < \gamma$, hence $T_f \cap T_g$ is finite since $T_f \subseteq^* T_{f \restriction \mu+1}$ and $T_g \subseteq^* T_{g \restriction \mu+1}$.

Proof of (d). Left as an exercise.

Proof of (e). The key to the proof is the following immediate consequence of the equality $\mathfrak{p} = \mathfrak{p}_2$ from Theorem 3.8.

(9) If $\mathscr{A}, \mathscr{B} \subseteq [\omega]^\omega$ satisfy $|\mathscr{A}| + |\mathscr{B}| < \mathfrak{p}$, and if each member of $\mathscr{B}$ is a pseudointersection of $\mathscr{A}$, then $\mathscr{A}$ has an infinite pseudointersection I with $\forall B \in \mathscr{B}$ $[A \cap I$ infinite].

To prove $\mathfrak{p}$ regular we must show that if $\mathscr{F} \subseteq [\omega]^\omega$ has the sfip, and if $|\mathscr{F}|$ is singular and $|\mathscr{F}| \leqslant \mathfrak{p}$, then $\mathscr{F}$ has an infinite almost intersection. Consider such an $\mathscr{F}$; it is convenient to assume

(10) $\mathscr{F}$ is closed under finite intersections.

For some $\kappa < |\mathscr{F}| \leqslant \mathfrak{p}$ we can write $\mathscr{F}$ as $\bigcup_{\xi \in \kappa} \mathscr{F}_\xi$, with $\forall \xi \in \kappa [|\mathscr{F}_\xi| < \mathfrak{p}]$. With recursion on η we pick infinite $T_\eta \subseteq \omega$ satisfying

(11) T_η is a pseudointersection of $\mathscr{F}_\eta$;

(12) $\forall \xi \in \eta\ [T_\eta \subseteq^* T_\xi]$;

(13) $\forall F \in \mathscr{F}\ [T_\xi \cap F$ infinite].

Let $\eta \in \kappa$, and assume $\mathscr{T} = \{T_\xi \colon \xi \in \eta\}$ known. For each $\xi \in \kappa$ we can choose a pseudointersection P_ξ of $\mathscr{F}_\xi \cup \mathscr{F}_\eta \cup \mathscr{T}$ since $\mathscr{F}_\xi \cup \mathscr{F}_\eta \cup \mathscr{T}$ has the sfip because of (10), (11) and (12). Because of (9) we can now find a pseudointersection T_η of $\mathscr{F}_\eta \cup \mathscr{T}$

such that $\forall \xi \in \kappa\ [T_\eta \cap P_\xi$ infinite]. Clearly (11) and (12) hold, and (13) holds since $\forall F \in \mathcal{F}\ \exists \xi \in \kappa\ [P_\xi \subseteq^* F]$.

[Note that the use of the P_ξ's is similar to the use of the A_η's in the proof of (b).]

The family $\{T_\eta : \eta \in \kappa\}$ has an infinite pseudointersection because of (12), and this is a pseudointersection of $\mathcal{F}$ because of (13).

Notes to Section 3

$\mathfrak{b}$, $\mathfrak{p}$ and $\mathfrak{t}$ are due to ROTHBERGER [1939, 1948], $\mathfrak{d}$ to KATĚTOV [1960A], $\mathfrak{a}$ to HECHLER [1972b] and SOLOMON, and $\mathfrak{s}$ to BOOTH. (For simplicity we give someone credit for a cardinal $\mathfrak{k}$ even if he or she only considers the possibility that $\mathfrak{k}$ is or is not equal to ω_1 or $\mathfrak{c}$. Also, "is due to" abbreviates some laborious phrase with "the earliest reference we are aware of".) There are other cardinals associated with ω, e.g.

$$\mathfrak{i} = \min\{|\mathcal{I}| : \mathcal{I} \subseteq [\omega]^\omega \text{ is a maximal family which is } \textit{independent}, \text{ i.e. } \forall \mathcal{A}, \mathcal{B} \in [\mathcal{I}]^{<\omega}\ [\mathcal{A} \neq \emptyset \text{ and } \mathcal{A} \cap \mathcal{B} = \emptyset \Rightarrow \bigcap \mathcal{A} - \bigcup \mathcal{B} \neq \emptyset]\},$$

$$\mathfrak{u} = \min\{|\mathcal{U}| : \mathcal{U} \text{ generates a free ultrafilter on } \omega\},$$

see also HECHLER [1972], but those are currently not useful enough to warrant inclusion. (In connection with Theorem 3.1 we mention the inequalities $\mathfrak{b} \leq \mathfrak{u}$, due to SOLOMON (which immediately follows from the Lemma in the proof of $\mathfrak{s} \leq \mathfrak{d}$ in Theorem 3.1) and $\mathfrak{b} \leq \mathfrak{i}$, which follows from the Lemma on p. 131.)

There have been many notations for these cardinals. For example, $\mathfrak{b}$ is called $\aleph_\eta$ by ROTHBERGER [1939], K_8 by HECHLER [1972], λ_3 by SOLOMON and ξ by BURKE and VAN DOUWEN. Moreover, ROTHBERGER [1939, 1941] uses $B(\kappa)$ for "every $F \subseteq [\omega]^\omega$ with $|F| = \kappa$ is bounded", i.e. for $\kappa^+ \leq \mathfrak{b}$ (B for 'bornée'), TALL [2000] uses $P(\kappa)$ for $\kappa \leq \mathfrak{p}$ (P because Rudin's proof that $\beta\omega - \omega$ has a P-point under $\mathfrak{c} = \omega_1$ works if $P(\mathfrak{c})$), and VAN DOUWEN [1976A] uses $\mathrm{BF}(\kappa)$ for $\kappa \leq b$ (BF for 'bounded functions').

We here introduce yet another notation which we hope will be definitive. Our letters were chosen to be mnemonic (of course $\mathfrak{p}$ was inspired by $P(\kappa)$, and 'pseudo-intersection' was created to make $\mathfrak{p}$ look mnemonic), and they are lower case German (whenever available) because $\mathfrak{c}$ is lower case German. (Note that $\mathfrak{c}$ is mnemonic: the cardinality of the $\mathfrak{c}$ontinuum.) Jerry Vaughan [1979] has independently come up with this lower case German convention, and, with one exception, even chose the same letters; we now agree about the letters for the eight cardinals mentioned.

In Theorem 3.1 $\omega_1 \leq \mathfrak{p}$ is due to HAUSDORFF [p. 244], $\mathfrak{t} \leq \mathfrak{b}$ and (b) and (c) to ROTHBERGER [1948, Theorem 3a, Lemma 7, Theorem 7a], $\mathfrak{b} \leq \mathfrak{a}$ to SOLOMON, $\mathfrak{s} \leq \mathfrak{d}$ is due to P.J. Nyikos (the proof given is due to G. Gruenhage and is included with his permission and has been found independently by Galvin and Miller), the second part of (d) is a simple observation about quasi-orders in general, and (e) is due to SZYMAŃSKI.

Theorem 3.2 is implicit in SOLOMON's proof that $\mathfrak{b} \leqslant \mathfrak{a}$.

In Theorem 3.3 the equalities $\mathfrak{b} = \mathfrak{b}_1$, $\mathfrak{b} = \mathfrak{b}_6$ and $\mathfrak{b}_6 = \mathfrak{b}_7$ are due to ROTHBERGER [1939, Lemme 1; 1941, Thme 6; 1948, Lemme 6], $\mathfrak{b} = \mathfrak{b}_5$ is due to HECHLER [1970, Theorem 1], and $\mathfrak{b} = \mathfrak{b}_3$ and Fact 3.4 are implicit in BURKE and VAN DOUWEN.

Corollary 3.5 is folklore.

In Theorem 3.8 $\mathfrak{d} = \mathfrak{d}_1$ was pointed out by KATĚTOV [1960A, 2.1], and $\mathfrak{d} = \mathfrak{d}_2$ is due to ROITMAN; note that $\mathfrak{b}_2$ compares to $\mathfrak{b}$ as $\mathfrak{d}$ compares to $\mathfrak{d}_2$.

In Theorem 3.8 $\mathfrak{p}_2$ captures what has been called Solovay's Lemma, i.e. $S_\aleph$ of MARTIN and SOLOVAY [p. 154]. I proved $\mathfrak{p} = \mathfrak{p}_2$ around 1976; the proof is published here to have a quotable reference for it. The result has been strengthened by BELL, who proved that MA_κ-for-σ-centered-posets holds if (and trivially only if) $\kappa < \mathfrak{p}$. (KUNEN and TALL have shown that this weaker form of MA_κ is strictly weaker than MA_κ.)

Each of our cardinals κ is defined to be the minimum of a certain set $C(\kappa)$ of cardinals $\leqslant \mathfrak{c}$. In this paper we will not consider the question of what $C(\kappa)$ is; HECHLER gives information about $C(\mathfrak{a})$ and $C(\mathfrak{t})$ in [1972a], and about $C(\mathfrak{b}_1)$ in [1970] (of course $C(\mathfrak{b})$ is without interest).

One can also define analogous cardinals associated with other cardinals than ω.

The study of the structure of $\mathscr{P}(\omega)$ and ${}^\omega\omega$ that underlies cardinals associated with ω owes much to Rothberger. We will not mention all his work in this area; for the interested reader we have included all his set theoretic papers in the references.

4. The cardinal ω_1

In Theorem 3.3 we saw that $\mathfrak{b}$ is the smallest cardinality of a subfamily $\mathscr{B}$ of $[\omega]^\omega$ such that there is a countable $\mathscr{C} \subseteq [\omega]^\omega$ such that $\mathscr{B}$ and $\mathscr{C}$ can not be separated, but either $\mathscr{B} \cap \mathscr{C} = \emptyset$ and $\mathscr{B} \cup \mathscr{C}$ is pairwise almost disjoint or $\mathscr{B} \perp \mathscr{C}$ and both $\mathscr{B}$ and $\mathscr{C}$ are well-ordered by $\subset^*$ (with $\langle \mathscr{C}, \subset^* \rangle$ having type ω). We here investigate what the smallest cardinality of $\mathscr{B}$ is if $|\mathscr{C}| = \omega_1$ (with $\langle \mathscr{C}, \subset^* \rangle$ having type ω_1 in the second case). Obviously, in both cases the minimum cardinality of $\mathscr{B}$ is ω iff $\mathfrak{b} = \omega_1$: just interchange $\mathscr{B}$ and $\mathscr{C}$. In this section we show that if $\mathfrak{b} > \omega_1$ then, again in both cases, the minimum cardinality of $\mathscr{B}$ is ω_1. Indeed, we have the following two theorems, which hold no matter what $\mathfrak{b}$ is.

4.1. THEOREM. *There are disjoint $\mathscr{B}, \mathscr{C} \subseteq [\omega]^\omega$ with $|\mathscr{B}| = |\mathscr{C}| = \omega_1$ such that $\mathscr{B} \cup \mathscr{C}$ is pairwise almost disjoint but $\mathscr{B}$ and $\mathscr{C}$ can not be separated. In fact there is a pairwise almost disjoint $\mathscr{A} \subseteq [\omega]^\omega$ with $|\mathscr{A}| = \omega_1$ such that if $\mathscr{B}$ and $\mathscr{C}$ are any two uncountable disjoint subcollections of $\mathscr{A}$, then $\mathscr{B}$ and $\mathscr{C}$ can not be separated.*

Proof. $\mathscr{A}$ will be $\{A_\eta : \eta \in \omega_1\}$, with the A_η's satisfying

(1) $\forall \xi \in \eta\ [|A_\xi \cap A_\eta| < \omega]$, and

(2) $\forall k \in \omega\ [\{\xi \in \eta : \max(A_\xi \cap A_\eta) < k\}$ is finite$]$.

Let us first check that such $\mathscr{A}$ is as required. Consider any two uncountable disjoint $B, C \subseteq \omega_1$, and consider any $S \subseteq \omega$. We wish to show that S does not separate $\{A_\eta : \eta \in B\}$ and $\{A_\eta : \eta \in C\}$. Without loss of generality we assume $\forall \xi \in C\ [|S \cap A_\xi| < \omega]$; our objective is to find $\eta \in B$ such that $A_\eta \not\subseteq^* S$, i.e. $|A_\eta - S| = \omega$. As C is uncountable there is $m \in \omega$ such that

$$H = \{\xi \in C : \max(S \cap A_\xi) = m\}$$

is infinite. As B is uncountable there is $\eta \in b$ such that $H \cap \eta$ is infinite. By (2) there is for each $k \in \omega$ a $\xi \in H \cap \eta$ such that $\max(A_\eta \cap A_\xi) > k$. It follows that $|A_\eta - S| = \omega$ since $\max(A_\eta \cap A_\xi) \notin S$ for $\eta \in H$ with $\max(A_\eta \cap A_\xi) > m$.

Remark. For later use we point out that in this proof we only needed to know that (2) holds for $\eta \in B$.

We now construct $\langle A_\eta : \eta \in \omega_1\rangle$. Let $\langle A_k : k \in \omega\rangle$ enumerate any (necessarily) countably infinite pairwise disjoint subcollection of $[\omega]^\omega$. Now let $\zeta \in \omega_1 - \omega$, and assume A_η known for $\eta \in \zeta$. Reenumerate $\langle A_\eta : \eta \in \zeta\rangle$ as $\langle Z_n : n \in \omega\rangle$. Clearly $\forall n \in \omega\ [|Z_n - \bigcup_{k \in n} Z_k| = \omega]$, hence there is a: $\omega \to \omega$ satisfying

$$\forall n \in \omega \quad [a(n) \in (Z_n - \textstyle\bigcup_{k\in n} Z_k) - n]\,.$$

Let $A_\zeta = \mathrm{ran}(a)$. Then $A_\zeta \in [\omega]^\omega$, and (1) and (2) hold for $\eta = \zeta$ since $\forall n \in \omega$ $[|\mathrm{ran}(a) \cap Z_n| \leqslant n+1]$ and since $\forall n \in \omega[\max(\mathrm{ran}(a) \cap Z_n) \geqslant a_n \geqslant n]$. □

4.2. COROLLARY. $\omega_1 = \min\{|\mathscr{B}| : \mathscr{B} \subseteq [\omega]^\omega$ *is pairwise almost disjoint, and* $\exists \mathscr{C} \in \mathscr{B}$ $[\mathscr{C}$ *and* $\mathscr{B}$-$\mathscr{C}$ *can not be separated*$]\}$.

4.3. THEOREM. *There are* $\mathscr{B}, \mathscr{C} \subseteq [\omega]^\omega$, *both well-ordered by* $\subseteq^*$ *in type* ω_1, *such that* $\mathscr{B} \perp \mathscr{C}$ *and such that* $\mathscr{B}$ *and* $\mathscr{C}$ *can not be separated.*

PROOF. We will construct $A : 2 \times \omega_1 \to [\omega]^\omega$ such that for all $\eta \in \omega_1$

(1) $A_{0,\eta} \cup A_{1,\eta}$ is coinfinite;

(2) $A_{0,\eta} \cap A_{1,\eta} = \emptyset$;

(3) $\forall i \in 2\, \forall \xi \in \eta\ [A_{i,\xi} \subset^* A_{i,\eta}]$;

(4) $\forall k \in \omega\ [\{\xi \in \eta : \max(A_{0,\eta} \cap A_{1,\xi}) < k\}$ is finite$]$.

Then $\mathscr{B} = A^{\to}\{0\} \times \omega_1$ and $\mathscr{C} = A^{\to}\{1\} \times \omega_1$ are as required: The proof that they can not be separated is very similar to the first part of the proof of Theorem 4.1 above, in view of the Remark, and it should be clear that the other properties hold.

We now construct A. Let $\zeta \in \omega_1$, and suppose $A_{0,\eta}$ and $A_{1,\eta}$ to be constructed for $\eta \in \zeta$.

Case 1. *ζ is a successor, or $\zeta = 0$.* If $\zeta = 0$ put $A_{0,-1} = A_{1,-1} = \emptyset$. Next, in both cases split $\omega - (A_{0,\zeta-1} \cup A_{1,\zeta-1})$ into three infinite pairwise disjoint sets I_0, I_1 and I_2. For $i \in 2$ put $A_{i,\zeta} = A_{i,\zeta-1} \cup I_i$.

Case 2. *ζ is a limit.* Using $\mathfrak{t} > \omega$ and $\mathfrak{b}_6 > \omega$ we first find coinfinite $X \subseteq \omega$ and then $S \subseteq X$ such that

(5) $\forall \eta \in \zeta\, [A_{0,\eta} \cup A_{1,\eta} \subseteq^* X]$;

(6) $\forall \eta \in \zeta\, [A_{0,\eta} \subseteq^* S$ and $|A_{1,\eta} \cap^* S| < \omega]$.

For $H \subseteq \omega$ and $Z \subseteq \zeta$ we let $H \perp Z$ abbreviate $H \perp \{A_{1,\eta}: \eta \in Z\}$ and we write $H(\mathrm{ct})Z$ (H is close to $\{A_{1,\eta}: n \in Z\}$) if $\{\eta \in Z$: $\max(H \cap A_{1,\eta}) < k\}$ is finite for each $k \in \omega$. So (4) says $A_{0,\eta} \perp \eta$ for $\eta \in \zeta$. It follows from (6) that $S \perp \eta$ for $\eta \in \zeta$ since evidently

(7) $\forall I, J, K \subseteq \omega\, [\max(I \cap K) > \max(I - J) \Rightarrow \max(J \cap K) \geqslant \max(I \cap K)]$.

For $k \in \omega$ define the bad set B_k by

$$B_k = \{\eta \in \zeta: \max(S \cap A_{1,\eta}) < k\}\,.$$

With recursion we construct a sequence $\langle S_n: n \in \omega\rangle$ of subsets of X satisfying $S_0 = S$, and $S_k \perp \eta$, and $S_k \subseteq S_{k+1}$, and $S_{k+1}(\mathrm{ct})B_k$, for $k \in \omega$. Let $k \in \omega$, and suppose S_k known. If B_k is finite let $S_{k+1} = S_k$. Now assume B_k is infinite. Since $S \perp \eta$ for $\eta \in \zeta$, as just observed, $B_k \cap \eta$ is finite for $\eta \in \zeta$. Hence B_k is cofinal in η and has order type ω. Let b: $\omega \to B_k$ be the strictly increasing surjection. From (3) and (5) we see that for each $n \in \omega$ the set $(A_{1,b(n)} - \bigcup_{i<n} A_{1,b(i)}) \cap X$ is infinite, hence we can pick p_n in it with $p_n \geqslant n$. Clearly $\mathrm{ran}(p) \perp B_k$, hence $\mathrm{ran}(p) \perp \eta$ since B_k is cofinal in η. Since $S_k \perp \eta$ it follows that $S_{k+1} \perp \eta$. Furthermore, $\mathrm{ran}(p)(\mathrm{ct})B_k$ by construction, hence $S_{k+1}(\mathrm{ct})B_k$. This completes the construction.

Using $\mathfrak{b}_6 > \omega$ once more we find $A_{0,\zeta} \subseteq X$ with $A_{0,\zeta} \perp \zeta$ such that $S_n \subseteq^* A_{0,\zeta}$ for $n \in \omega$. We may even have $S \subseteq A_{0,\zeta}$. Let $A_{1,\zeta} = \omega - (X \cup A_{0,\zeta})$. Clearly (1) and (2) hold for $\eta = \zeta$. Also, (3) holds for $\eta = \zeta$ with "$\subseteq$" instead of "$\subseteq^*$", hence it holds as stated since ζ is a limit and since (3) holds for $\eta \in \zeta$. To prove (4) for $\eta = \zeta$, suppose there is $l \in \omega$ such that $B' = \{\xi \in \zeta$: $\max(A_{0,\zeta} \cap A_{1,\xi}) < l\}$ is infinite. As $S \subseteq A_{0,\zeta}$ it follows that $B' \cap B_k$ is infinite for some $k \leqslant l$. But $S_{k+1}(\mathrm{ct})B_k$, hence it follows from (7) and the fact that $S_{k+1} \subset^* A_{0,\zeta}$ that $A_{0,\zeta}(\mathrm{ct})B_k$, which leads to the absurdity that $B' \cap B_k$ is finite. $\square$

4.5. EXERCISE. Use Theorem 4.3 to find a collection $\mathscr{G}$ of G_δ subsets of ${}^\omega 2$ such that $\subset$ well-orders $\mathscr{G}$ in type ω_1 and such that $\bigcup \mathscr{G} = {}^\omega 2$. [*Hint.* For $\eta \in \omega_1$ let $G_\eta = \{s \in {}^\omega 2$: $A_{0,\xi} \not\subseteq^* s^{\leftarrow}\{1\}$ or $|s^{\leftarrow}\{1\} \cap A_{1,\xi}| = \omega\}$.]

Notes to Section 4

Theorem 4.3 and Exercise 4.5 are due to HAUSDORFF [1936], and a pair $\langle \mathscr{A}, \{\omega - B: B \in \mathscr{B}\}\rangle$ is nowadays called a *Hausdorff gap*; we prefer to use the pair $\langle \mathscr{A}, \mathscr{B}\rangle$ instead, since it is the fact that $\mathscr{A}$ and $\mathscr{B}$ can not be separated that we will use in Example 7.4.

Theorem 4.1 is credited to LUSIN [1943], who seems to have been unaware of HAUSDORFF. It will be used topologically in Proposition 11.7.

A question that naturally arises after Theorem 4.3 is whether there is a pairwise almost disjoint $\mathscr{A} \subseteq [\omega]^\omega$ with $|\mathscr{A}| = \omega_1$ such that every two disjoint subfamilies of $\mathscr{A}$ can be separated. We refer to Taylor for a recent discussion of this question, and only point out that the existence of such an $\mathscr{A}$ implies $2^{\omega_1} = 2^\omega$ (for $|\mathscr{P}(\mathscr{A})| = 2^{\omega_1}$ but there are only 2^ω many separating sets).

5. Consistency results

We begin with showing that if κ is any of our six cardinals, then each of $\omega_1 = \kappa < \mathfrak{c}$ and $\omega_1 < \kappa < \mathfrak{c}$ and $\omega_1 < \kappa = \mathfrak{c}$ is consistent with ZFC.

5.1. THEOREM. *Let κ and λ be regular cardinals with $\omega_1 \leqslant \kappa \leqslant \lambda$. It is consistent with* ZFC *that $\mathfrak{c} = \lambda$ and $\mathfrak{a} = \mathfrak{b} = \mathfrak{d} = \mathfrak{s} = \mathfrak{t} = \mathfrak{p} = \kappa$.*

PROOF. (Rough Outline) If $\kappa = \lambda$ we use the fact that it is consistent to have $\mathrm{MA} + \mathfrak{c} = \kappa$, and that $\mathfrak{p} = \mathfrak{c}$ under MA, see MARTIN and SOLOVAY. Then the result follows from Theorem 3.1.

Now assume $\kappa < \lambda$. First use the fact that it is consistent that $\mathfrak{c} = \lambda$ to see that we may assume our ground model M satisfies $\mathfrak{c} = \lambda$. Let $\mathscr{D}$ be a countably infinite disjoint family in $M \cap [\omega]^\omega$. Obtain an iterated ccc extension $\langle M_\eta : \eta \leqslant \lambda \rangle$ adding $f_\eta \in {}^\omega\omega$ and $X_\eta, Y_\eta \in [\omega]^\omega$ at stage $\eta \in \lambda$ as follows: First extend M_η to M_η^* satisfying $\mathrm{MA} + \mathfrak{c} = \lambda$. Then:

force $f_\eta \in {}^\omega\omega$ satisfying $\forall f \in M_\eta^* \cap {}^\omega\omega\ [f \leqslant^* f_\eta]$;

force $X_\eta \in [\omega]^\omega$ such that if $\mathscr{D}_\eta = \mathscr{D} \cup \{X_\xi : \xi \in \eta\}$, then $\forall D \in \mathscr{D}_\eta\ [|D \cap X_\eta| < \omega]$ and $\forall Y \in M_\eta^*\ [\forall D \in \mathscr{D}_\eta [|D \cap Y| < \omega] \Rightarrow |Y \cap X_\eta| = \omega]$;

let Y_η be any Cohen subset of ${}^\omega\omega$ over M_η^*.

Let $M_{\eta+1} = M_\eta^*\ [f_\eta, X_\eta, Y_\eta]$. In M_κ we have $\mathfrak{c} = \lambda$. Moreover, since $\mathrm{cf}(\lambda) \geqslant \omega_1$ we have

$$(1) \qquad M_\kappa \cap {}^\omega\omega = \bigcup_{\eta \in \kappa} M_\eta \cap {}^\omega\omega \quad \text{and} \quad M_\kappa \cap [\omega]^\omega = \bigcup_{\eta \in \kappa} M_\eta \cap [\omega]^\omega .$$

It follows that in M_κ, $\{f_\eta : \eta \in \lambda\}$ is a dominating subset of ${}^\omega\omega$, and $\mathscr{D} \cup \{X_\eta : \eta \in \lambda\}$ is a mad, and $\{Y_\eta : \eta \in \lambda\}$ is splitting.

It follows that $\mathfrak{a}, \mathfrak{d}, \mathfrak{s} \leqslant \lambda$. At the other hand, since each M_η^* satisfies MA, hence has $\mathfrak{p} = \mathfrak{c}$, and since κ is regular, we see from (1) that $\mathfrak{p} \geqslant \kappa$. Hence $\mathfrak{a} = \mathfrak{b} = \mathfrak{d} = \mathfrak{s} = \mathfrak{t} = \mathfrak{p} = \kappa$, because of Theorem 3.1. □

We next show that not all six cardinals are equal.

5.2. THEOREM. *It is consistent with* ZFC *that $\mathfrak{a} < \mathfrak{d}$ (hence $\mathfrak{b} < \mathfrak{d}$ since $\mathfrak{b} \leqslant \mathfrak{a}$).*

PROOF. Let M be a model of ZFC + CH. Let $\kappa > \omega_1$ satisfy $\kappa^\omega = \kappa$, and let N be a model obtained from M by adding κ many Cohen reals. Then $\mathfrak{a} = \omega_1$ in M, see KUNEN [1980, VIII, Theorem 2.3], but $\mathfrak{d} = \mathfrak{c}$. (A direct proof that $\mathfrak{b} = \omega_1$ is that $M \cap {}^\omega\omega$ is unbounded in $N \cap {}^\omega\omega$, see KUNEN [1980, VI, Ex. G7].) □

5.3. THEOREM. *It is consistent with* ZFC *that* $\mathfrak{b} > \mathfrak{t}$.

PROOF. Assume your ground model M satisfies $\mathfrak{c} = \omega_2$ and $2^{\omega_1} > \omega_2$. Iterate ccc forcing ω_2 many times to get a model N in which $\mathfrak{c} = \omega_2$ and $2^{\omega_1} > \omega_2$ and where there is a dominating set $\langle f_\eta \colon \eta \in \omega_2\rangle$ such that $\forall \eta \in \omega_2 \, \forall \xi \in \eta \, [f_\xi <^* f_\eta]$. In N we have $\mathfrak{b} = \mathfrak{d} = \omega_2$ but $\mathfrak{t} = \omega_1$ since $2^{<\omega_2} = 2^{\omega_1} > \mathfrak{c}$, see Theorem 3.1. □

5.4. THEOREM. *It is consistent with* ZFC *that* $\mathfrak{s} > \mathfrak{t}$.

PROOF. Start with a ground model M satisfying $\mathfrak{c} = \omega_2$ and $2^{\omega_1} = \omega_3$. Obtain an iterated ccc extension $\langle M_\eta \colon \eta \leqslant \omega_2\rangle$ adding $X_\eta \in [\omega]^\omega$ at stage $\eta \in \omega_2$ such that $\forall A \in M \cap [\omega]^\omega \, [X_\eta \subseteq^* A$ or $X_\eta \subseteq^* \omega - A]$. In M_{ω_2} we have $c = \omega_2 < 2^{\omega_1}$, hence $\mathfrak{t} = \omega_1$ by Theorem 3.1. Moreover, $M_{\omega_2} \cap [\omega]^\omega = \bigcup_{\eta \in \omega_2} M_\eta \cap [\omega]^\omega$, so for each $S \subseteq M_{\omega_2} \cap [\omega]^\omega$, if $|\mathcal{S}| \leqslant \omega_1$ there is $\eta \in \omega_2$ with $\mathcal{S} \subseteq M_\eta$, and then $\forall S \in \mathcal{S}$ $[X_\eta \not\subseteq^* S$ or $X_\eta \subseteq^* \omega - S]$. This shows $\mathfrak{s} \geqslant \omega_2$. □

5.5. COROLLARY (TO PROOF). *It is consistent with* ZFC *that* $2^{\mathfrak{t}} < 2^{\mathfrak{s}}$.

PROOF. Let M also satisfy $2^{\omega_2} = \omega_4$. □

Notes to Section 5

Theorem 5.1, with $\kappa < \lambda$, was proved for me by Arnold Miller.

HECHLER [1974] proved much more than Theorem 5.1: Any ω-directed poset of cardinality at most $\mathfrak{c}$ can be isomorphic to a cofinal subset of ${}^\omega\omega$. For example, if β and δ are regular cardinals with $\omega_1 \leqslant \beta < \delta \leqslant \mathfrak{c}$ one can have a cofinal subset isomorphic to $\beta \times \delta$ (ordered by $\langle \xi, \eta\rangle \leqslant \langle \xi', \eta'\rangle$ if $\xi \leqslant \xi'$ and $\eta \leqslant \eta'$), and then $\mathfrak{b} = \beta < \delta = \mathfrak{d}$.

Theorems 5.3 and 5.4 are due to SOLOMON, and BOOTH (whose outline of proof we don't understand), respectively.

Added in proof. Baumgartner has pointed out that $\mathfrak{s} = \omega_1 < \mathfrak{b}$ in any model obtained from a model of $\mathfrak{b} > \omega_1$ by adding ω_1 random reals. Note that this shows $\mathfrak{t} < \mathfrak{b}$ consistent with $2^{\mathfrak{t}} = \mathfrak{c}$.

6. Sequential compactness and countable compactness

A countable set A of a space X is said to *cluster* at $x \in X$ if each neighborhood of x contains infinitely many points of A, and is said to *converge* to $x \in X$ if each neighborhood of x contains all but finitely many points of A. A space is called *countably compact* if each countably infinite set clusters at some point, and is called *sequentially compact* if each countably infinite set has an infinite subset that converges somewhere. Finally, a space X is called *subsequential* if for every countably infinite $A \subseteq X$ and for every cluster point x of A there is an infinite subset of A that converges to x. Clearly

$$\begin{array}{rcl} \text{subsequential} + \text{countably compact} & \Rightarrow & \text{sequentially compact} \\ & & \Downarrow \\ \text{compact} & \Rightarrow & \text{countably compact} \end{array}$$

6.1. THEOREM. $\mathfrak{s} = \mathfrak{s}_p = \mathfrak{s}_c = \mathfrak{s}_{cc}$, *where*

$\mathfrak{s}_p = \min\{\kappa$: *the power* ${}^{\kappa}2$ *is not sequentially compact*$\}$,

$\mathfrak{s}_c = \min\{\kappa$: *there is a compact* X *with* $w(X) = \kappa$ *which is not sequentially compact*$\}$,

$\mathfrak{s}_{cc} = \min\{\kappa$: *there is a countably compact* X *with* $w(X) = \kappa$ *which is not sequentially compact*$\}$.

PROOF. Clearly $\mathfrak{s}_p \geqslant \mathfrak{s}_c \geqslant \mathfrak{s}_{cc}$.

Proof that $\mathfrak{s} \leqslant \mathfrak{s}_{cc}$. Let X be a countably compact space with $w(X) < \mathfrak{s}$. We prove X is sequentially compact. So consider any countably infinite subset N of X. Let $\mathcal{B}$ be a base for X with $|\mathcal{B}| < \mathfrak{s}$, and define $\mathcal{B}' = \{B \in \mathcal{B}: |B \cap N| = \omega\}$. Then $|\mathcal{B}' \restriction N| \leqslant |\mathcal{B}| < \mathfrak{s}$, hence $\mathcal{B}' \restriction N$ does not split $[N]^{\omega}$. So there is an infinite $A \subseteq N$ such that $\forall B \in \mathcal{B}'$ $[A \subseteq^* B \cap N$ or $A \subseteq^* N - B]$. By countable compactness A has a cluster point x. We claim A converges to x. Indeed, consider any $B \in \mathcal{B}$ with $x \in B$. Then clearly $B \in \mathcal{B}'$ but $A \not\subseteq^* N - B$ since $|B \cap A| = \omega$. It follows from our choice of A that $A \subseteq^* B \cap N$, i.e. B contains all but finitely many points of A.

Proof that $\mathfrak{s}_p \leqslant \mathfrak{s}$. Consider any splitting $\mathcal{S}$. We will show that the power ${}^{\mathcal{S}}2$ is not sequentially compact. Define $\sigma: \omega \to {}^{\mathcal{S}}2$ by $\sigma(n)_S = 1 \Leftrightarrow n \in S$ for $S \in \mathcal{S}$, for $n \in \omega$. We claim ran(σ) is (necessarily countably) infinite but has no infinite convergent subset. If not there would be $I \in [\omega]^{\omega}$ such that

$$\forall S \in \mathcal{S} \quad \exists k \in 2 \quad [|\{i \in I: \sigma(i)_S = k\}| < \omega]$$

(if $|\mathrm{ran}(\sigma \restriction I)| = 1$ we could even have $\forall S \in \mathcal{S}\, \exists k \in 2\, \forall i \in I$ $[\sigma(i)_S = k]$), i.e. $\forall S \in \mathcal{S}$ $[|I \cap S| < \omega \vee |I - S| < \omega]$, which would contradict the fact that $\mathcal{S}$ is splitting. $\square$

6.2. THEOREM. $\mathfrak{p} = \mathfrak{p} = \mathfrak{p}_c = \mathfrak{p}_{cc}$, *where*

$\mathfrak{p}_p = \min\{\kappa$: *the power* ${}^{\kappa}2$ *is not subsequential*$\}$;

$\mathfrak{p}_c = \min\{\kappa$: *there is a compact* X *with* $\chi(X) = \kappa$ *which is not subsequential*$\}$;

$$\mathfrak{p}_{cc} = \min\{\kappa\colon \textit{there is a countably compact } X \textit{ with } \chi(X) = \kappa \textit{ which is not subsequential}\};$$

$$\mathfrak{p}_{\chi} = \min\{\kappa\colon \textit{there is a space } X \textit{ with } \chi(X) = \kappa \textit{ which is not subsequential}\}.$$

PROOF. Clearly $\mathfrak{p}_p \geq \mathfrak{p}_c \geq \mathfrak{p}_{cc} \geq \mathfrak{p}_\chi$.

Proof that $\mathfrak{p} \leq \mathfrak{p}_\chi$. Consider any countable subset N of a space X and any cluster point x of N such that $\chi(x, X) < \mathfrak{p}$. Let $\mathcal{B}$ be a neighborhood base of x with $|\mathcal{B}| < \mathfrak{p}$. Clearly $\mathcal{B} \restriction N$ has the sfip, hence there is infinite $A \subseteq N$ which is an almost intersection of $\mathcal{B} \restriction N$. Trivially A converges to x.

Proof that $\mathfrak{p}_p \leq \mathfrak{p}$. Since every zero-dimensional X with $w(X) \leq \kappa$ embeds in ${}^{\kappa}2$, cf. ENGELKING [1977, 6.2.12], it suffices to find a zero-dimensional X with $w(X) = \mathfrak{p}$ which is not subsequential: consider any $\mathcal{U} \subseteq [\omega]^\omega$ with $|\mathcal{U}| = \mathfrak{p}$ which has the sfip but has no infinite almost intersection. We may assume $\mathcal{U}$ is closed under finite intersections, and that $\bigcap \mathcal{U} = \emptyset$. Topologize $X = \omega \cup \{\mathcal{U}\}$ the obvious way: points of ω are isolated, and basic neighborhoods of $\mathcal{U}$ have the form $\{\mathcal{U}\} \cup U$, with $U \in \mathcal{U}$. It should be clear that X is as required. □

The preceding two results put a best possible lower bound on the weight (or character) of (countably) compact spaces which are not sequentially compact. We now give a lower bound on the cardinality, see also 6.8.

6.3. THEOREM. *If X is a compact space which is not sequentially compact, then* $|X| \geq 2^{\mathfrak{t}}$.

PROOF. For $A \subseteq X$ let A' denote the set of cluster points of X. Note that

$$\text{(1)} \qquad \forall A, B \in \mathcal{P}(X) \quad [A \subseteq^* B \text{ then } A' \subseteq B'].$$

Let N be a countably infinite subset of X which has no infinite subset that converges, or, equivalently, which satisfies

$$\text{(2)} \qquad \forall T \in [N]^\omega \quad \exists T_0, T_1 \in [T]^\omega \quad [T_0' \cap T_1' = \emptyset].$$

Using (1) and (2) one can show that for each $\eta \in \mathfrak{t}$ there are $T_f \in [\omega]^\omega$ for all $f \in {}^{\eta}2$ such that

$$\text{(3)} \qquad \forall f \in {}^{\eta}2 \quad \forall \xi \in \eta \quad [T_f \subseteq^* T_{f\restriction\xi}], \qquad \text{and}$$

$$\text{(4)} \qquad \forall f, g \in {}^{\eta}2 \quad [f \neq g \Rightarrow T_f' \cap T_g' = \emptyset].$$

This is a straightforward modification of the proof of Theorem 3.1(c).

Since X is compact and since T' is a nonempty closed subset of X for each infinite $T \subseteq X$ it follows from (2) and (3) that we can define $S\colon {}^{\mathfrak{t}}2 \to \mathcal{P}(X) - \{\emptyset\}$ by

$S(f) = \bigcap_{\eta \in \mathfrak{t}} T'_{f\restriction\eta}$. It follows from (3) and (4) that $\forall F, g \in {}^{\mathfrak{t}}2 [f \neq g \Rightarrow S(f) \cap S(g) = \emptyset]$. Hence $|X| \geqslant 2^{\mathfrak{t}}$. □

6.4. COROLLARY. ($2^{\mathfrak{t}} > \mathfrak{c}$) *The following conditions on a compact space X are equivalent*:

(a) *Every countably compact subspace is closed*;

(b) *X is* sequential, *i.e. for each $A \subseteq X$, if A is not closed, then some countable subset of A converges to a point of $X - A$.*

PROOF. (b)$\Rightarrow$(a). If $A \subseteq X$ has a countable subset that converges to a point of $X - A$, then A is not closed.

(a)$\Rightarrow$(b). Assume $A \subseteq X$ is not closed. Then A is not countably compact, hence it has a countably infinite closed discrete set N. We claim some infinite subset of N converges, necessarily to a point of $X - A$. Suppose not. Then the compact subspace $\bar{N}$ of X is not sequentially compact, hence $|\bar{N}| > \mathfrak{c}$ since we assume $2^{\mathfrak{t}} > \mathfrak{c}$. However, we also have $|\bar{N}| \leqslant \mathfrak{c}$, since the following known fact implies that there is a countably compact Y, hence closed, set with $|Y| \leqslant \mathfrak{c}$ and $N \subseteq Y \subseteq \bar{N}$.

6.5. FACT. *If Z is countably compact, then for each $S \subseteq Z$ there is a countably compact T with $S \subseteq T (\subseteq \bar{S})$ and $|T| \leqslant |S|^{\omega}$.*

Indeed, pick a cluster point $c(N)$ of N for $N \in [Z]^{\omega}$. Next, with recursion define S_η for $\eta \in \omega_1$ as follows: $S_0 = S$, and

$$S_\eta = \bigcup_{\xi \in \eta} S_\eta \cup \{c(N) \colon N \in [\bigcup_{\xi \in \eta} S_\xi]^{\omega}\}, \quad \text{if } 0 < \eta < \omega_1 .$$

Finally, let $T = \bigcup_{\eta \in \omega_1} S_\eta$. □

6.6. QUESTION. Is Theorem 6.3 best possible, i.e. is there a compact space of cardinality $2^{\mathfrak{t}}$ which is not sequentially compact? (By Theorem 6.1 there is a compact space of cardinality $2^{\mathfrak{s}}$ that is not sequentially compact. However, by Corollary 5.5 it is consistent that $2^{\mathfrak{s}} > 2^{\mathfrak{t}}$.)

6.7. QUESTION. Does Corollary 6.4 hold in ZFC?

For countably compact spaces we have an exact version of Theorem 6.3.

6.8. THEOREM. $\mathfrak{c} = \min\{|X| \colon$ *X is countably compact but not sequentially compact*$\}$.

PROOF. *Proof of* "$\leqslant$". Do the first countably many steps of the proof of Theorem 6.3.

Proof of "$\geqslant$". Use Fact 6.5 and the fact that there exists a compact space which is not sequentially compact, e.g. ${}^{\mathfrak{s}}2$. □

We now turn our attention to products. TERESAKA and NOVÁK have shown that a product of two countably compact spaces need not be countably compact, and in fact need not be pseudocompact. By contrast we have the following result; see also Vaughan's Handbook article.

6.9. THEOREM. (a) *Any product of fewer than* $\mathfrak{t}$ *sequentially compact spaces is sequentially compact.*

(b) *Any product of at most* $\mathfrak{t}$ *sequentially compact spaces is countably compact.*

PROOF. Let $\kappa \leq \mathfrak{t}$, let X_η be a sequentially compact space for $\eta \in \kappa$, let Π denote $\Pi_{\eta \in \kappa} X_\eta$, and for $\eta \in \kappa$ let $\pi_\eta \colon \Pi \to X_\eta$ denote the projection. Let N be any countably infinite subset of Π. It is easy to find $T_\eta \in [N]^\omega$ and $x_\eta \in X_\eta$ for $\eta \in \kappa$, and $T \in [N]^\omega$ if $\kappa < \mathfrak{t}$, such that

(1) $\forall \xi \in \eta\ [T_\eta \subseteq^* T_\xi]$, and if $\kappa < \mathfrak{t}$, then $\forall \eta \in \kappa\ [T \subseteq^* T_\eta]$,

(2) $\pi_\eta^{\rightarrow} T_\eta = \{x_\eta\}$ or $\pi_\eta^{\rightarrow} T_\eta$ is infinite and converges to x_η.

Consider any basic open set B in Π containing $x = \langle x_\eta \colon \eta \in \kappa \rangle$, i.e. any set of the form $\bigcap_{\eta \in F} \pi_\eta^{\leftarrow} B_\eta$, with $F \in [\kappa]^{<\omega} - \{\emptyset\}$ and with B_η open and containing x_η for $\eta \in F$. For each $\eta \in F$ all but finitely many members of T_η belong to $\pi_\eta^{\leftarrow} B_\eta$, by (2). It follows from (1) that B contains all but finitely many points of $T_{\max(F)}$ (hence $|B \cap N| = \omega$), and also that B contains all but finitely many points of T if $\kappa < \mathfrak{t}$. This proves both (a) and (b). $\square$

6.10. QUESTION. By Theorem 6.1 no product of $\mathfrak{s}$ nondegenerate spaces is sequentially compact, hence $\mathfrak{t} \leq \mu \leq \mathfrak{s}$, where

$$\mu = \min\{\kappa \colon \text{some product of } \kappa \text{ sequentially compact spaces is not sequentially compact}\}.$$

Can μ be expressed as a set theoretically defined cardinal?

6.11. QUESTION. It is too early to ask the analogue of the preceding question in ZFC: We don't know if there is a product of sequentially compact spaces that is not countably compact, in ZFC. (There is such a product if $\mathfrak{b} = \mathfrak{c}$, see Example 13.1.)

Notes to Section 6

BOOTH [Theorem 2,1] proved $\mathfrak{s} = \mathfrak{s}_\mathrm{p} = \mathfrak{s}_\mathrm{c}$ of Theorem 6.1 and $\mathfrak{p} = \mathfrak{p}_\mathrm{p} = \mathfrak{p}_\mathrm{c}$ of Theorem 6.2.

MALYHIN and ŠAPIROVSKIĬ essentially prove $\mathfrak{p}_X \geq \mathfrak{p}$ of Theorem 6.2, and conclude that Theorem 6.3 holds with "$2^\mathfrak{p}$" instead of "$2^\mathfrak{t}$".

Corollary 6.4, with "$2^{\omega_1} > \mathfrak{c}$ or MA" instead of "$2^\mathfrak{t} > \mathfrak{c}$" is due to ISMAIL and NYIKOS [Theorem 1.24], who also asked Question 6.7.

In connection with Question 6.6 we point out that FEDORČUK has shown that

"$\mathfrak{s} = \omega_1$ and $2^{\omega_1} = \mathfrak{c}$" implies the existence of a compact space of cardinality $\mathfrak{c}$ in which no countably infinite set converges.

HECHLER [1975, p. 86] has pointed out that Theorem 6.9 is essentially due to SCARBOROUGH and STONE [5.2 and 5.5].

7. Spaces from $\subset^*$-chains

Certain set theoretic structures have the pleasant property that there is an interesting space naturally associated with them. In this section we give three closely related examples of this phenomenon.

7.1. EXAMPLE. There is a noncompact separable sequentially compact locally compact normal space.

PROOF. By Theorem 3.6 there is $\mathcal{T} \subseteq [\omega]^\omega$ such that $\subset^*$ well-orders $\mathcal{T}$ in type $\mathfrak{t}$, and such that

$$(1) \qquad \forall X \in [\omega]^\omega \quad \exists T \in \mathcal{T} \quad [|X \cap T| = \omega]\,.$$

Let $Y = \omega \cup \mathcal{T}$, and topologize Y as follows: Points of ω are isolated, and a basic neighborhood of $T \in \mathcal{T}$ has the form

$$B(S, T, F) = \{A \in \mathcal{T} : S \subset^* A \subseteq^* T\} \cup ((T - S) - F)\,,$$

where $S \in \mathcal{T} \cup \{\emptyset\}$ satisfies $S \subset^* T$ and where $F \in [\omega]^{<\omega}$.

The closed subspace $\mathcal{T}$ of Y is homeomorphic to $\mathfrak{t}$, hence is not compact. It follows that Y is not compact.

Obviously ω is dense in Y, hence Y is separable.

To prove Y is sequentially compact (defined in Section 6) consider any $I \in [\omega]^\omega$. It follows from (1) that we can define

$$M = \min\{T \in \mathcal{T} : |B(\emptyset, T, \emptyset) \cap I| = \omega\}$$

where the minimum is with respect to $\subset^*$. We claim that the infinite subset $D = I \cap B(\emptyset, M, \emptyset)$ if I converges to M. Indeed, consider any $S \in \mathcal{T} \cup \{\emptyset\}$ with $S \subset^* M$ and any $F \in [\omega]^\omega$. Then

$$B(S, M, F) = B(\emptyset, M, \emptyset) - (B(\emptyset, S, \emptyset) \cup F)$$

hence $I - B(S, M, F)$ is finite since $B(\emptyset, S, \emptyset) \cap I$ is finite by the minimality of M. Since $\mathcal{T}$, being homeomorphic to $\mathfrak{t}$, is sequentially compact this proves that X is sequentially compact.

We prove Y locally compact by proving basic open sets are compact. This is trivial for the singleton subsets of ω, so consider any $S, T \in \mathcal{T} \cup \{\emptyset\}$ with $S \subset^* T$ and any $F \in [\omega]^{\omega}$. (Note that the displayed formula above, with $M = T$, shows $B(S, T, F)$ is closed.) Let $\mathcal{U}$ be any open cover of $B(S, T, F)$. Since $\subset^*$ well-orders $\mathcal{T} \cup \{\emptyset\}$ we can find $n \in \omega - \{0\}$, and $S_i \in \mathcal{T}$ and $F_i \in [\omega]^{\omega}$ in such a way that

$$S_0 = T, \quad \text{and} \quad \forall k \in n\ [S_{k+1} \subset^* S_k], \quad \text{and} \quad S_n = S$$

and $\{B(S_i, T_i, F_i) : i \in n\}$ refines $\mathcal{U}$. Clearly $B(S, T, F) \cap \mathcal{T} \subseteq \bigcup_{i \leq n} B(S_{i+1}, S_i, F_i)$ and

$$B(S, T, F) \cap \omega = (S - T) - F = (\bigcup_{i \leq n} (S_i - S_{i+1})) - F$$

$$\subseteq (\bigcup_{i \leq n} B(S_{i+1}, S_i, F_i) - F) \cup \bigcup_{i \leq n} F_i .$$

Since $\bigcup_{i \leq n} F_i$ is finite it follows that $\mathcal{U}$ has a finite subcover.

In order to prove that Y is normal it suffices to prove that any two disjoint closed subsets of $\mathcal{T}$ have disjoint neighborhoods in Y since the points of $Y - \mathcal{T}$ are isolated. To this end it suffices to observe that of any two disjoint closed subsets of $\mathcal{T}$, in the order topology, at least one is not cofinal, hence is compact.

In fact, of any two disjoint closed subsets F and G at least one is compact. [By the preceding we may assume $F \cap \mathcal{T}$ is compact. Let $\mathcal{U}$ be an open cover of F. There is a finite $\mathcal{V} \subseteq \mathcal{U}$ with $F \cap \mathcal{T} \subseteq \bigcup \mathcal{V}$. Since $F - \bigcup \mathcal{V}$ is a closed discrete set in the countably compact space Y is finite. Hence F is compact.] This simultaneously proves that Y is normal and that Y is locally compact (since it implies that $|\beta Y - Y| = 1$). We gave the separate arguments since we need them below. □

7.2. REMARK. Upon considering the one point compactification of our example we see that for $\kappa = \mathfrak{t}$ we obtain

$(\mathrm{a})_\kappa$ there is a compactification $b\omega$ of ω such that there is a homeomorphism h from $\kappa + 1$ onto $b\omega - \omega$ and such that $b\omega - \{h(\kappa)\}$ is sequentially compact.

For any κ the statement $(\mathrm{b})_\kappa$ holds iff

$(\mathrm{b})_\kappa$ there is $\mathcal{T} \subseteq [\omega]^{\omega}$, well-ordered by $\subset^*$ in type κ, such that

$$\forall X \in [\omega]^{\omega} \quad \exists T \in \mathcal{T} \quad [|X \cap T| = \omega] .$$

Since $(\mathrm{b})_\kappa$ implies $\mathrm{cof}(\kappa) \geq \omega_1$, the construction of Example 7.1 proves $(\mathrm{b})_\kappa \Rightarrow (\mathrm{a})_\kappa$. To prove the reverse implication pick for each $\eta \in \kappa$ a closed neighborhood U_η of the closed set $h^{\rightarrow}[0, \eta]$ which misses the closed set $h^{\rightarrow}(\eta, \kappa]$, and note that U_η is open. It easily follows that $\forall \eta \in \kappa\ \forall \xi \in \eta\ [\omega \cap U_\xi \subseteq^* \omega \cap U_\eta]$. Hence $\subset^*$ well-orders $\mathcal{T} = \{\omega \cap U_\eta : \eta \in \kappa\}$ in type κ. The second part of $(\mathrm{b})_\kappa$ is an easy consequence of the fact that each infinite subset of ω has a cluster point in $h^{\rightarrow}[0, \kappa)$.

Subremark. The minimum value of κ for which (b)$_\kappa$ holds is $\mathfrak{t}$. However, it is consistent with ZFC to have a κ for which (b)$_\kappa$ holds with $\kappa > \mathfrak{t}$. To prove this modify the proof of Theorem 5.4 and have $\forall \eta \in \omega_2 \forall \xi \in \eta\ [X_\eta \subseteq^* X_\xi]$, and let $\mathcal{T} = \{\omega - X_\eta : \eta \in \omega_2\}$.

7.3. EXAMPLE. There is a separable locally compact normal space X such that

(a) X is an M-space, i.e. X admits a *quasi-perfect* (= continuous and closed with countably compact fibers) map onto a metrizable space,

(b) if Y is any countably compact space with $Y \supseteq X$, then X has a closed countably compact subset which is not closed in Y.

PROOF. Let

$$\mathcal{A} = \{A \in [\omega \times \omega]^\omega : \forall k \in \omega\ [|A \cap \{k\} \times \omega| < \omega]\}.$$

By Theorem 3.3 there is $\mathcal{B} \subseteq \mathcal{A}$ such that $\subset^*$ well-orders $\mathcal{B}$ in type $\mathfrak{b}$, and

(1) $\forall A \in \mathcal{A}\ \exists B \in \mathcal{B}\ [|A \cap B| = \omega]$.

Topologize $X = \omega \times \omega \cup \mathcal{B}$ in a way that should be obvious after Example 7.1. Then X is separable, locally compact and normal, and $\mathcal{B}$ is a countably compact closed subspace (since $\mathfrak{b}$, being regular and uncountable, is countably compact. Clearly (1) translates as

(2) each neighborhood of $\mathcal{B}$ in X includes $\{k\} \times \omega$ for all but finitely many $k \in \omega$.

To prove (a) consider the quotient $X/\mathcal{B}$ obtained from X by collapsing $\mathcal{B}$ to a single point. The underlying set of $X/\mathcal{B}$ is $\omega \times \omega \cup \{\mathcal{B}\}$, and from (2) we see that $\{\{\mathcal{B}\} \cup (\omega - k) \times \omega : k \in \omega\}$ is a neighborhood base at $\mathcal{B}$. Hence $X/\mathcal{B}$ is first countable at its nonisolated point, hence $X/\mathcal{B}$ is metrizable. The quotient map obviously has countably compact fibers, and it follows easily from (1) or (2) that the quotient map is closed. (We will meet this quotient space again, as $\mathbb{L}$, in Section 8.)

To prove (b) consider any countably compact $Y \supseteq X$. We may assume $\bar{X} = Y$. Then $Y - X$ is closed in Y, hence is countably compact, since X is locally compact. For each $k \in \omega$ the set $\{k\} \times \omega$ is infinite and closed discrete in X, hence it has a cluster point p_k in $Y - X$. Since $Y - X$ is countably compact there is $q \in Y - X$ such that $\{k \in \omega : p_k \in U\}$ is infinite for each neighborhood U of q. We claim that q is in the closure of the countably compact closed subset $\mathcal{B}$ of X: Consider any $y \in Y$ with $y \notin \bar{\mathcal{B}}$. Let U be an open neighborhood of y with $\bar{U} \cap \mathcal{B} = \emptyset$. Then $X - \bar{U}$ is a neighborhood of $\mathcal{B}$ in X, hence $I = \{k \in \omega : \{k\} \times \omega \not\subseteq X - \bar{U}\}$ is finite by (2). Then $p_k \notin U$ for $k \in \omega - I$, hence $\{k \in \omega : p_k \in U\}$ is finite. This shows $y \neq q$. $\square$

7.4. EXAMPLE. There is a first countable separable countably paracompact cwH (defined in 12.6) locally compact space that is not normal.

PROOF. (Outline) Let $\mathscr{B}$ and $\mathscr{C}$ be as in Theorem 4.1, and topologize $G = \omega \cup \mathscr{B} \cup \mathscr{C}$, the gap space, in a way which should be obvious after Example 7.1. Then G is separable and first countable.

$\mathscr{B}$ and $\mathscr{C}$ are disjoint (necessarily closed) copies of ω_1 (in the order topology), hence are countably compact. It follows that G is countably paracompact and cwH.

The proof that G is locally compact is as above. G is not normal since the fact that $\mathscr{B}$ and $\mathscr{C}$ can not be separated prevents them from having disjoint neighborhoods; this we leave as an exercise. □

Notes to Section 7

Example 7.1 is due to FRANKLIN and RAJAGOPALAN. Remark 7.2 is due to ENGELKING [1972], the subremark is like Exercise VIII.A10 in KUNEN. Example 7.3 is due to BURKE and VAN DOUWEN; we have simplified the proof of (b). Example 7.4 is due to VAN DOUWEN [1976B]. We refer to the references for the reason d'être of Examples 7.3 and 7.4.

RUDIN has constructed a separable normal space Z which is not paracompact since Z' is homeomorphic to ω_1 in the order topology; as pointed out by MCAULEY, such an example was earlier constructed by JONES. Each of the examples in this section has a subspace like Z which has the additional property of being locally compact and first countable.

8. Separable metrizable spaces

For a space X let $\mathscr{K}(X)$ denote the family of all compact subsets of X. We will consider three cardinal invariants on X associated with $\mathscr{K}(X)$ which tell how far X is from being compact. We calculate these cardinal invariants for $\mathbb{P}$, the irrationals, and $\mathbb{Q}$, the rationals, and deduce values for these cardinal invariants for some other separable metrizable spaces. We also establish the connection with realcompactness and $\mathbb{N}$-compactness.

Our first cardinal invariant is $\mathrm{cof}(\mathscr{K}(X))$ (see Section 2), where $\mathscr{K}(X)$ is considered a poset under inclusion. The second invariant is

$$\mathrm{kc}(X) = \min\{|\mathscr{L}| : \mathscr{L} \subseteq \mathscr{K}(X) \text{ and } \bigcup \mathscr{L} = X\},$$

this invariant we call the *compact covering number* of X. For our third invariant we need some terminology. If $\mathscr{D}$ is a collection of closed subsets of a space X we say that $\mathscr{D}$ *determines* X if $\forall F \subseteq X$ [F is closed $\Leftrightarrow \forall D \in \mathscr{D}$ [$F \cap D$ is closed]]. A space X is called a *k-space* if $\mathscr{K}(X)$ determines X. Clearly first countable spaces, in particular metrizable spaces, are k-spaces. For a space X we define the *k-ness* of X to be

$$k(X) = \min\{|\mathscr{L}| : \mathscr{L} \subseteq \mathscr{K}(X) \text{ determines } X \text{ and } \bigcup \mathscr{L} = X\}.$$

By convention $\min(\emptyset)=\infty$, which is greater than each cardinal. (We require $\bigcup\mathscr{L}=X$ in the definition of $k(X)$ since otherwise $k(X)=0$ for a discrete space.)

Before we start our calculations we make the following observation.

8.1. FACT. (a) *X is not compact* $\Leftrightarrow \mathrm{kc}(X)\geqslant\omega \Leftrightarrow k(X)\geqslant\omega\Leftrightarrow \mathrm{cof}(\mathscr{K}(X))\geqslant\omega$.

(b) *If X is a k-space, then* $\mathrm{kc}(X)\leqslant k(X)\leqslant \mathrm{cof}(\mathscr{K}(X))$.

(c) *If X is closed in Y, then* $\mathrm{kc}(X)\leqslant\mathrm{kc}(Y)$ *and* $k(X)\leqslant k(Y)$ *and* $\mathrm{cof}(\mathscr{K}(X))\leqslant \mathrm{cof}(\mathscr{K}(Y))$.

(d) *If $k(X)=\omega$, then* $\mathrm{kc}(X)=\mathrm{cof}(\mathscr{K}(X))=\omega$.

PROOF. We prove (d). Consider any countable $\mathscr{L}\subseteq\mathscr{K}(X)$ which determines X. We claim that the countable subcollection $\mathscr{C}=\{\bigcup\mathscr{F}\colon \mathscr{F}\subseteq[\mathscr{L}]^{<\omega}\}$ of $\mathscr{K}(X)$ is cofinal in $\mathscr{K}(X)$. Consider any $K\subseteq X$ such that $\forall C\in\mathscr{C}\ [K\not\subseteq C]$. Since $X\notin\mathscr{C}$ (for otherwise $k(X)=1$ or 0) we can find countably infinite $I\subseteq K$ such that $\forall L\in\mathscr{L}\ [|I\cap L|<\omega]$. This I is an infinite closed discrete subset of X since $\mathscr{L}$ determines X. It follows that K is not compact. □

We now do our calculations.

8.2. THEOREM. $\mathrm{kc}(\mathbb{P})=k(\mathbb{P})=\mathrm{cof}(\mathscr{K}(\mathbb{P}))=\mathfrak{d}$.

PROOF. Because of Fact 8.1(a) it suffices to prove $\mathfrak{d}\leqslant\mathrm{kc}(\mathbb{P})$ and $\mathrm{cof}(\mathscr{K}(\mathbb{P}))\leqslant\mathfrak{d}$. Recall that $\mathbb{P}$ is identified with the power ${}^{\omega}\omega$, and that $\mathfrak{d}=\mathfrak{d}_1=\mathrm{cof}(\langle{}^{\omega}\omega,\leqslant\rangle)$.

For $K\in\mathscr{K}(\mathbb{P})-\{\emptyset\}$ we can define $f_K\in{}^{\omega}\omega$ by $f_K(n)=\max(\pi_n^{\rightarrow}K)$ $(n\in\omega)$ since for each $n\in\omega$ the projection $\pi_n^{\rightarrow}K$ of K into the nth factor is compact, hence is finite. Note that

$$\text{(1)}\qquad \forall K\in\mathscr{K}(\mathbb{P})\ \left[K\subseteq\prod_{n\in\omega}[0,f_K(n)]\right].$$

Proof that $\mathfrak{d}_1\leqslant\mathrm{kc}(\mathbb{P})$. Consider $\mathscr{L}\subseteq\mathscr{K}(\mathbb{P})$ with $\bigcup\mathscr{L}=\mathbb{P}$. Then $\{f_L\colon L\in\mathscr{L}\}$ is cofinal in $\langle{}^{\omega}\omega,\leqslant\rangle$: Consider any $f\in{}^{\omega}\omega$. There is $L\in\mathscr{L}$ with $f\in L$, and $f\leqslant f_L$ by (1).

Proof that $\mathrm{cof}(\mathscr{K}(\mathbb{P}))\leqslant\mathfrak{d}$. It is clear from (1) that if D is cofinal in $\langle{}^{\omega}\omega,\leqslant\rangle$, then $\{\prod_{n\in\omega}[0,d_n]\colon d\in D\}$ is cofinal in $\mathscr{K}(\mathbb{P})$. □

We introduce an auxiliary space $\mathbb{L}$. The underlying set of $\mathbb{L}$ is $\omega\times\omega\cup\{\infty\}$, where $\infty\notin\omega\times\omega$, and $\mathbb{L}$ is topologized as follows: Points of $\omega\times\omega$ are isolated, and basic neighborhoods of ∞ have the form $\{\infty\}\cup(\omega-k)\times\omega$, where $k\in\omega$. The following known result, whose easy proof we omit, shows that $\mathbb{L}$ is quite abundant.

8.3. LEMMA. *A first countable space is* not *locally countably compact iff it has a closed subspace homeomorphic to* $\mathbb{L}$.

Our cardinal invariants are rather easy to calculate for $\mathbb{L}$.

8.4. LEMMA. *$k(\mathbb{L})=\mathfrak{b}$, $\mathrm{cof}(\mathcal{K}(\mathbb{L}))=\mathfrak{d}$, and (trivially) $\mathrm{kc}(\mathbb{L})=\omega$.*

PROOF. Define

$$\mathcal{A}=\{A\in[\omega\times\omega]^{\omega}:\forall k\in\omega\ [|A\cap\{k\}\times\omega|<\omega]\}.$$

Note that

(1) $\forall K\subseteq\mathbb{L}\quad [K \text{ is compact}\Leftrightarrow|K|<\omega \text{ or } (\infty\in K \text{ and } K-\{\infty\}\in\mathcal{A})]$.

Proof that $k(\mathbb{L})=\mathfrak{b}$. It is easy to see that

(2) $\forall F\subseteq\mathbb{L}\quad [F \text{ is not closed}\Leftrightarrow\infty\notin F \text{ and } \exists A\in\mathcal{A}\ [A\subseteq F]]$.

From (1) and (2) we see that

$$k(\mathbb{L})=\min\{|\mathcal{B}|: \mathcal{B}\subseteq\mathcal{A}, \text{ and } \forall A\in\mathcal{A}\ \exists B\in\mathcal{B}\ [|A\cap B|=\omega]\}.$$

The right hand side is $\mathfrak{b}_4$, which equals $\mathfrak{b}$ by Theorem 3.3.

Proof that $\mathrm{cof}(\mathcal{K}(\mathbb{L}))=\mathfrak{b}$. For $f\in{}^{\omega}\omega$ define L_f (lower f) to be $L_f=\{\langle k,n\rangle\in\omega\times\omega: n\leqslant f(k)\}$. From (1) we see that $\{L_f\cup\{\infty\}: f\in{}^{\omega}\omega\}$ is cofinal in $\mathcal{K}(X)$. Clearly

$$\mathrm{cof}(\langle\{L_f\cup\{\infty\}: f\in{}^{\omega}\omega\},\subseteq\rangle)=\mathrm{cof}(\langle\{L_f: f\in{}^{\omega}\omega\},\subseteq\rangle)=\mathrm{cof}(\langle{}^{\omega}\omega,\leqslant\rangle).$$

The right hand side is $\mathfrak{d}_1$, which equals $\mathfrak{d}$ by Theorem 3.6. $\square$

The following is a simple consequence of 8.1(c), 8.3 and 8.4.

8.5. PROPOSITION. *Let X be metrizable. (More generally, let X be first countable and isocompact, i.e. closed countably compact subsets are compact.) Then $k(X)\leqslant\omega$ iff $\mathrm{cof}(\mathcal{K}(X))\leqslant\omega$ iff X is σ-compact and locally compact.*

We now continue our calculations.

8.6. THEOREM. $k(\mathbb{Q})=\mathfrak{b}$.

PROOF. It follows from 8.1(c), 8.3 and 8.4 that $k(\mathbb{Q})\geqslant b$. We prove that $k(\mathbb{Q})\leqslant b$ by slightly modifying the proof that $k(\mathbb{L})\leqslant b$.

Since $|\mathbb{Q}|=\omega$, in order to prove $k(\mathbb{Q})\leqslant\mathfrak{b}$ it suffices to find for each $q\in\mathbb{Q}$ a $\mathcal{D}\subseteq\mathcal{K}(\mathbb{Q})$ with $|\mathcal{D}|\leqslant\mathfrak{b}$ such that

(1) $\forall F \subseteq X \ [q \in \bar{F} - F \Rightarrow \exists D \in \mathscr{D} \ [F \cap D \text{ is not closed}]]$.

Let $\langle B_n : n \in \omega \rangle$ be a neighborhood base for q such that

(2) $\forall n \in \omega \quad [B_{n+1} \subset B_n] \qquad (\text{hence } \forall n \in \omega \ [|B_n - B_{n+1}| = \omega])$.

Define

$$\mathscr{A} = \{A \in [B_0]^\omega : \forall n \in \omega \ [|A \cap (B_n - B_{n+1})| < \omega]\}.$$

Then $\forall A \in \mathscr{A}$ [A is not compact but $A \cup \{q\}$ is], because of (2). Also,

$$\forall F \subseteq X \quad [q \in \bar{F} - F \Leftrightarrow q \notin F \text{ and } \exists A \in \mathscr{A} \ [A \subseteq F]],$$

hence there is $\mathscr{D} \subseteq \mathscr{K}(\mathbb{Q})$ satisfying (1) with

$$|\mathscr{D}| = \min\{|\mathscr{B}| : \mathscr{B} \subseteq \mathscr{A}, \text{ and } \forall A \in \mathscr{A} \ \exists B \in \mathscr{B} \ [|A \cap B| = \omega]\}.$$

It follows from (2) that the right hand side is $\mathfrak{b}_4$, which equals $\mathfrak{b}$. □

8.7. THEOREM. $\mathrm{cf}(\mathscr{K}(\mathbb{Q})) = \mathfrak{d}$.

PROOF. It follows from 8.1(c), 8.3 and 8.4 that $\mathrm{cf}(\mathscr{K}(\mathbb{Q})) \geqslant \mathfrak{d}$. The proof of the reverse inequality is rather unpleasant.

Intermezzo. We review the facts about scattered spaces that we need for the proof. A space is called *scattered* if each subspace (or, equivalently, each closed subspace) has an isolated point. The relevance of this notion lies in the fact that compact subsets of $\mathbb{Q}$ are scattered since countable compact spaces are scattered. [The fastest way to prove this is to observe that in a countable space without isolated points one can find a sequence $\langle U_n : n \in \omega \rangle$ of nonempty open sets such that $\bigcap_{n\in\omega} U_n = \emptyset$ and $\forall n \in \omega \ [U_n \supseteq \bar{U}_{n+1}]$.]

For a space X let X' denote the set of nonisolated points of X and recursively define $X^{(\eta)}$ for η an ordinal by

$$X^{(0)} = X, \quad X^{(\eta+1)} = (X^{(\eta)})' \quad \text{and} \quad X^{(\eta)} = \bigcap_{\xi\in\eta} X^{(\xi)} \quad \text{if } \eta \text{ is a limit},$$

or, more succinctly,

$$X^{(0)} = X, \quad \text{and} \quad X^{(\eta)} = \bigcap_{\xi\in\eta} (X^{(\xi)})' \quad \text{if } \eta > 0.$$

Clearly each $X^{(\eta)}$ is closed in X. It is routine to show that X is scattered iff there is η, necessarily with $\eta < |X|^+$, such that $X^{(\eta)} = \emptyset$. This η can not be a limit if X is compact. It follows that for nonempty scattered X there is $h(X) < |X|^+$, the height of X, such that $X^{(h(X))}$ is a nonempty finite set. Let $h(\emptyset) = -1$.

Continuation of proof. With recursion on $\eta \in \omega_1$ we will construct a compact $L(\eta, f)$ in $\mathbb{Q}$ for each $f \in {}^\omega\omega$ in such a way that

(1) $\quad \forall f, g \in {}^\omega\omega \quad [f \leq g \Rightarrow L(\alpha, f) \subseteq L(\alpha, g)]$;

(2) $\quad \forall K \in \mathcal{K}(\mathbb{Q}) \quad [h(K) < \alpha \Rightarrow \exists f \in {}^\omega\omega \ [K \subseteq L(\alpha, f)]]$.

Then $\{L(\eta, f): \eta \in \omega_1, f \in D\}$ is cofinal in $\mathcal{K}(\mathbb{Q})$ for each cofinal D in $\langle {}^\omega\omega, \leq \rangle$ since $\forall K \in \mathcal{K}(\mathbb{Q})\ [h(K) < \omega_1]$. Since $\mathrm{cof}(\langle {}^\omega\omega, \leq \rangle) = \mathfrak{d} \geq \omega_1$ this proves that $\mathrm{cof}(\mathcal{K}(\mathbb{Q})) \leq \mathfrak{d}$.

For $\eta = 0$ put $L(\eta, f) = \emptyset$ for $f \in {}^\omega\omega$.

Now let $0 < \eta < \omega_1$, and assume $L(\xi, f)$ to be known for $\xi \in \eta$ and $f \in {}^\omega\omega$. Partially order $\omega \times {}^\omega\omega \times {}^\omega({}^\omega\omega)$ the obvious way:

$$\langle m, f, \langle g_n \rangle_n \rangle \leq \langle m', f', \langle g'_n \rangle_n \rangle \quad \text{if} \quad m \leq m' \text{ and } f \leq f' \text{ and } \forall n \in \omega\ [g_n \leq g'_n] .$$

Then $\langle \omega \times {}^\omega\omega \times {}^\omega({}^\omega\omega), \leq \rangle$ is isomorphic to $\langle {}^\omega\omega, \leq \rangle$ (since both posets have ω many factors $\langle \omega, \leq \rangle$), hence in order to construct $L(\eta, f)$ for $f \in {}^\omega\omega$ it suffices to construct $L(\eta; m, f, \langle g_n \rangle_n)$ for $\langle m, f, \langle g_n \rangle_n \rangle \in \omega \times {}^\omega\omega \times {}^\omega({}^\omega\omega)$ in such a way that the obvious analogues of (1) and (2) hold.

Let $q: \omega \to \mathbb{Q}$ be a bijection. For $i \in \omega$ let $\langle B_{i,k}: k \in \omega \rangle$ be a decreasing neighborhood base at q_i consisting of clopen sets with $B_{i,0} = \mathbb{Q}$. Let $e: \omega \to \eta$ be a surjection. For $\langle m, f, \langle g_n \rangle_n \rangle \in \omega \times {}^\omega\omega \times {}^\omega({}^\omega\omega)$ define

$$L(\eta; m, f, \langle g_n \rangle_n) = \bigcup_{i \in m} (\{q_i\} \cup \bigcup_{k \in \omega} \bigcup_{j \leq f(k)} [L(e(j), g_k) \cap (B_{i,k} - B_{i,k+1})]) .$$

To see that this set is compact note that for each $i \leq m$ the ith summand is compact since it intersects the closed set $B_{i,k} - B_{i,k+1}$ in a compact set. [Here we use the evident fact that if C_k is a compact subset of $B_{i,k}$ for $k \in \omega$, then $\{q_i\} \cup \bigcup_{k \in \omega} C_k$ is compact since $\langle B_{i,k}: k \in \omega \rangle$ is a decreasing neighborhood base at q_i.]

It is obvious that the analogue of (1) holds. (We use q and e to get around the fact that we do not have $L(\xi, f) \subseteq L(\xi', f)$ whenever $\xi \leq \xi'$.)

It remains to show that the analogue of (2) holds, so consider an arbitrary $K \in \mathcal{K}(\mathbb{Q})$ with $h(K) < \eta$. We are to find $m \in \omega$ and $f \in {}^\omega\omega$ and $g_n \in {}^\omega\omega$ for $n \in \omega$ in such a way that $K \subseteq L(\eta; m, f, \langle g_n \rangle_n)$. Since $K^{(h(K))}$ is finite we can pick $m \in \omega$ so that $K^{(h(K))} \subseteq \{q_i: i \in m\}$. Pick $l \in \omega$ so large that $U_{i,l} \cap U_{j,l} = \emptyset$ for every two distinct $i, j \in m$, and define

$$H = \mathbb{Q} - \bigcup_{i \in m} B_{i,l}, \quad \text{and} \quad \mathcal{F} = \{H\} \cup \{ \bigcup_{i \in m} (B_{i,k} - B_{i,k+1}): l \leq k \in \omega\} .$$

Then $\bigcup \mathcal{F} = \mathbb{Q} - \{q_i: i \in m\}$ since each $\langle B_{i,k}: k \in \omega \rangle$ is decreasing. As $\{q_i: i \in m\} \subseteq L(\eta; m, f, \langle g_n \rangle_n)$ for each choice of f and $\langle g_n \rangle_n$ it follows that it remains to pick f

and $\langle g_n\rangle_n$ in such a way that for each $F \in \mathscr{F}$ we have

(3) $\quad K \cap F \subseteq L(\eta; m, f, \langle g_n\rangle_n)$.

For each $F \in \mathscr{F}$ the following holds: F is closed in $\mathbb{Q}$ (since the $B_{i,k}$'s are clopen) and misses $K^{(h(K))}$. Hence $K \cap F$ is compact and $h(K \cap F) < h(K)$. [For clearly $(K \cap F)^{(\xi)} \subseteq K^{(\xi)} \cap F$ for each ordinal ξ, hence $(K \cap F)^{(h(K))} \subseteq K^{(h(K))} \cap F = \emptyset$.] Hence there are $\phi(F) \in \omega$ and $\gamma(F) \in {}^{\omega}\omega$ such that $(h(K \cap F) < e(\phi(F))$ and)

(4) $\quad K \cap F \subseteq L(e(\phi(F)), \gamma(F)), \qquad$ (hence $K \cap F \subseteq L(e(\phi(F)), \gamma(F)) \cap F)$.

We now use ϕ and γ to construct f and $\langle g_k\rangle_k$.

For $k < l$ put $f(k) = \phi(H)$ and $g_k = \gamma(H)$. This makes (3) true for $F = H$. To see this consider any $i \in m$. Then $H \subseteq \mathbb{Q} - B_{i,l} = \bigcup_{k \in l} (B_{i,k} - B_{k+1})$ since $\langle B_{i,k}\colon k \in \omega\rangle$ is decreasing and since $B_{i,0} = \mathbb{Q}$. Hence, from (4),

$$K \cap H \subseteq L(e(\phi(H)), \gamma(H)) \cap H \subseteq L(e(\phi(H)), \gamma(H)) \cap \bigcup_{k \in l} (B_{i,k} - B_{i,k+1})$$

$$= \bigcup_{k \in l} [L(e(\phi(H)), \gamma(H)) \cap (B_{i,k} - B_{i,k+1})]$$

$$= \bigcup_{k \in l} [L(f(k), g_k) \cap (B_{i,k} - B_{i,k+1})] \subseteq L(\eta; m, f, \langle g_k\rangle_k).$$

For $k \geqslant l$, if $F = \bigcup_{i \in m} (B_{i,k} - B_{i,k+1})$, put $f(k) = \phi(F)$ and $g_k = \gamma(F)$; it should be clear that (3) holds for this F. □

In order to use the preceding results to calculate our invariants for more general spaces we need the following.

8.8. LEMMA. *Let Y be a metrizable space which is a continuous image of a separable completely metrizable space. Then Y has a closed subspace homeomorphic to $\mathbb{P}$ if (and, trivially, only if) Y is not σ-compact.*

PROOF. Let X be a separable completely metrizable space which admits a continuous map f onto Y. Define

$$\Sigma = \{A \subseteq Y\colon \text{there is a } \sigma\text{-compact } S \subseteq Y \text{ with } A \subseteq S\}.$$

Step 1. We construct for each $n \in \omega$ a closed family $\langle A_s\colon s \in {}^{n}\omega\rangle$ in X such that
(1) $\forall s \in {}^{n}\omega\ [\operatorname{diam}(A_s) < 2^{-n}]$, if $n \geqslant 1$;
(2) $\forall s \in {}^{n}\omega\ [\operatorname{diam}(f^{\rightarrow}A_s) < 2^{-n}]$, if $n \geqslant 1$;
(3) $\langle A_s\colon s \in {}^{n}\omega\rangle$ is an indexed discrete family in X;
(4) $\langle f^{\rightarrow}A_s\colon s \in {}^{n}\omega\rangle$ is an indexed discrete family in Y;

(5) $\forall s \in {}^n\omega\ \forall t \in {}^{n+1}\omega\ [s \subseteq t \Rightarrow A_s \supseteq A_t]$; and
(6) $\forall s \in {}^n\omega\ [f^{\rightarrow}A_s \notin \Sigma]$.
Let $A_\emptyset = X$. This defines A_s for $s \in {}^02 = \{\emptyset\}$. Now let $m \in \omega$ and assume A_s known for $s \in {}^m\omega$. For $s \in {}^m\omega$ define

$$B_s = \{x \in A_s : \forall \text{ neighborhood } U \text{ of } x \text{ in } A_s\ [f^{\rightarrow}U \notin \Sigma]\}.$$

Then $A_s - B_s$ is Lindelöf, being separable metrizable, hence $f^{\rightarrow}(A_s - B_s) \in \Sigma$. Since $f^{\rightarrow}A_s \notin \Sigma$ it follows that the superset $f^{\rightarrow}B_s$ of $f^{\rightarrow}A_s - f^{\rightarrow}(A_s - B_s)$ does not have compact closure in Y. Hence $f^{\rightarrow}B_s$ has a countably infinite subset D which is closed discrete in Y. Consider any $\phi_s : \omega \to B_s$ such that $f \circ \phi_s$ is a bijection from ω onto D. It is not hard to find a closed neighborhood $A_{s,k}$ of $\phi_s(k)$ for $k \in \omega$ in such a way that the obvious analogues of (1) through (4) hold for $\langle A_{s,k} : k \in \omega \rangle$ with $n = m + 1$. Now for $t \in {}^{m+1}\omega$ define $A_t = A_{t\restriction m, t(m)}$. Then (1) through (6) hold for $n = m + 1$.

Step 2. We construct a closed homeomorph of $\mathbb{P}$ in Y. Since the A_s's are closed it follows from (1) and (5) that $\forall x \in {}^\omega\omega\ [|\bigcap_{n\in\omega} A_{s\restriction n}| = 1]$. Hence we can define a function $h : \mathbb{P} = {}^\omega\omega \to Y$ by

$$h = \{\langle x, y \rangle : y \in \bigcap_{n\in\omega} f^{\rightarrow}A_{x\restriction n}\}.$$

For each $n \in \omega$ and $s \in {}^n\omega$ we have $h^{\rightarrow}\{x \in {}^\omega\omega : x \supseteq s\} = (h^{\rightarrow}\mathbb{P}) \cap (f^{\rightarrow}A_s)$. Since $\langle (h^{\rightarrow}\mathbb{P}) \cap (f^{\rightarrow}A_s) : s \in {}^n\omega \rangle$ is an indexed pairwise disjoint *open* family in $h^{\rightarrow}\mathbb{P}$ (for $\forall s \in {}^\omega n\ [f^{\rightarrow}A_s$ has closed complement in $\bigcup\{f^{\rightarrow}A_t : t \in {}^\omega n - \{s\}\}]$, because of (4)) whose members have diameter less than 2^{-n}, for $n \in \omega$, it follows that h is an embedding of $\mathbb{P}$ into Y.

To prove $h^{\rightarrow}\mathbb{P}$ is closed in Y consider any $y \in \overline{h^{\rightarrow}\mathbb{P}}$. There is an infinite $D \subseteq h^{\rightarrow}\mathbb{P}$ which converges to y (i.e. each neighborhood of y contains all but finitely many members of D). Evidently $\forall n \in \omega\ [\{s \in {}^n\omega : D \cap f^{\rightarrow}A_s \neq \emptyset\}$ is finite$]$ because of (2). Hence there is $x \in {}^\omega\omega$ such that $\forall n \in \omega\ [|D \cap f^{\rightarrow}A_{x\restriction n}| = \omega]$. Then $y = h(x)$ since each neighborhood of y contains some $A_{x\restriction n}$, hence contains points of D. □

We also need the following fact about perfect maps. (Recall that a map f from a space X onto a space Y is called *perfect* if it is continuous and closed and if its fibers are compact.

8.9. LEMMA. *Let Y be the image of X under a perfect map. Then* $\mathrm{kc}(Y) = \mathrm{kc}(X)$ *and* $\mathrm{cof}(\mathcal{K}(Y)) = \mathrm{cof}(\mathcal{K}(X))$ *and* $k(Y) = k(X)$.

PROOF. Let f be a perfect map from X onto Y. It is well-known that $\forall K \in \mathcal{K}(Y)$ $[f^{\leftarrow}K \in \mathcal{K}(X)]$, cf. ENGELKING [1977, 3.72]. From these the first two equalities easily follow. The proof that $k(Y) = k(X)$ is a straightforward extension of the proof that Y is a k-space iff X is a k-space. (The proof of "if" can be found in ARHANGEL'SKIĬ [Theorem 2.5], and for the proof of "only if" the only needs f to be quotient (i.e. $F \subseteq Y$ [F is closed $\Leftrightarrow f^{\leftarrow}F$ is closed]). $\square$

We now calculate our cardinal invariants for some other separable metrizable spaces. We need some terminology. We call a separable metrizable space *absolutely* F_σ ($G_\delta, \ldots,$ Borel) if it is an F_σ ($G_\delta, \ldots,$ Borel) subset of each metrizable compactification, or, equivalently, of some metrizable compactification. A separable metrizable space is absolutely F_σ iff it is σ-compact, and is absolutely G_δ if it is completely metrizable. We call a space *analytic* if it is a continuous image of $\mathbb{P}$. It is known that every separable metrizable space that is absolutely Borel is analytic, but not conversely, cf. KURATOWSKI and MOSTOWSKI [XIII, Theorems 6, 11].

8.10. THEOREM. *Let X be a separable metrizable space.*

(a) *If X is locally compact noncompact, then* $\mathrm{kc}(X) = k(X) = \mathrm{cof}(\mathscr{K}(X)) = \omega$.

(b) *If X is σ-compact but not locally compact, then* $k(X) = \mathfrak{b}$ *and* $\mathrm{cof}(\mathscr{K}(X)) = \mathfrak{d}$.

(c) *If X is completely metrizable but not σ-compact, then* $\mathrm{kc}(X) = k(X) = \mathrm{cof}(\mathscr{K}(X)) = \mathfrak{d}$.

(d) *If X is absolutely $F_{\sigma\delta}$ but not σ-compact, then* $\mathrm{kc}(X) = k(X) = \mathrm{cof}(\mathscr{K}(X)) = \mathfrak{d}$.

(e) *If X is analytic but not σ-compact, then* $\mathrm{kc}(X) = \mathfrak{d}$.

PROOF. (a) X is σ-compact, being locally compact and second countable, hence the result follows from 8.5.

(e) Since X is a continuous image of $\mathbb{P}$ we have $\mathrm{kc}(X) \leqslant \mathrm{kc}(\mathbb{P})$, and since X has a closed subspace homeomorphic to $\mathbb{P}$, by 8.8, we see from 8.1(c) that $\mathrm{kc}(X) \geqslant \mathrm{kc}(\mathbb{P})$. Hence $\mathrm{kc}(X) = \mathfrak{d}$ by 8.2.

(b), (c) and (d). Note that (c) follows from (d) since if X is absolutely G_δ, then it is absolutely $F_{\sigma\delta}$. We include a proof anyway.

We have $k(X) \geqslant \mathfrak{b}$ and $\mathrm{cof}(\mathscr{K}(X)) \geqslant \mathfrak{d}$ in (b) because of 8.1(c), 8.3 and 8.4, and have $\mathrm{kc}(X) \geqslant \mathfrak{d}$ in (c) and (d) because of 8.1(c), 8.2 and 8.8.

To prove the reverse inequalities we first point out

(1) $k({}^\omega 2 \times \mathbb{Q}) = \mathfrak{b}$ and $\mathrm{cof}(\mathscr{K}({}^\omega 2 \times \mathbb{Q})) = \mathfrak{d}$, and

(2) $\mathrm{cof}(\mathscr{K}({}^\omega \mathbb{Q})) = \mathfrak{d}$.

Indeed, the projection ${}^\omega 2 \times \mathbb{Q} \to \mathbb{Q}$, being the projection along a compact factor, is perfect, hence (1) follows from 8.6, 8.7 and 8.9. Moreover from the proof of 8.9 we see that $X = \mathbb{Q}$ satisfies

$$\text{there is } L\colon \omega_1 \times {}^\omega\omega \to \mathscr{K}(X) \text{ such that}$$

$$(*) \qquad \forall \eta \in \omega_1 \ \forall f, g \in {}^\omega\omega \ [f \leqslant g \Rightarrow L(\eta, f) \subseteq L(\eta, g)] \quad \text{and}$$

$$\forall K \in \mathscr{K}(X) \ \exists \eta \in \omega_1 \ \forall \xi \in \omega_1 - \eta \ \exists f \in {}^\omega\omega \ [K \subseteq L(\xi, f)].$$

Now any product of countably many spaces satisfying (*) is easily seen to satisfy (*) since $\langle {}^{\omega}({}^{\omega}\omega), \leqslant\rangle$ and $\langle {}^{\omega}\omega, \leqslant\rangle$ are isomorphic. But obviously any X satisfying (*) has $\mathrm{cof}(\mathcal{K}(X)) \leqslant \mathfrak{d}$.

From (1), 8.2 and (2), when combined with 8.1(c), 8.9 and 8.1(b) we see that the reverse inequalities in (b), (c) and (d) follow from:

(B) if X is σ-compact, then X is a perfect image of some closed subspace of ${}^{\omega}2 \times \mathbb{Q}$,

(C) if X is completely metrizable, then X is a perfect image of some closed subspace of $\mathbb{P}$,

(D) if X is absolutely $F_{\sigma\delta}$, then X is a perfect image of some closed subspace of ${}^{\omega}\mathbb{Q}$,

provided X is separable metrizable. (Note that the converse statements also hold.) To prove this we may assume X is a subspace of the Hilbert cube ${}^{\omega}I$, where I is the closed unit interval. Now ${}^{\omega}2$ admits a continuous map onto ${}^{\omega}I$, hence it admits a continuous map f onto ${}^{\omega}I$. Then f is perfect, and $f^{\leftarrow}X$ admits $f \restriction f^{\leftarrow}X$ as perfect map onto X, and $f^{\leftarrow}X$ is an F_σ (or G_δ, or $F_{\sigma\delta}$) in ${}^{\omega}2$ if (and only if) X is an F_σ (or G_δ, or $F_{\sigma\delta}$) in ${}^{\omega}I$. This reduces our proof to proving

(B′) if X is σ-compact, then X embeds as a closed subspace into ${}^{\omega}2 \times \mathbb{Q}$,

(C′) if X is completely metrizable, then X embeds as a closed subspace in $\mathbb{P}$,

(D′) if X is absolutely $F_{\sigma\delta}$, then X embeds as a closed subspace in ${}^{\omega}\mathbb{Q}$,

provided X is zero-dimensional and separable metrizable. Let $X \approx Y$ denote that X and Y are homeomorphic. Then (B′) and (C′) follow from

(B″) if X is σ-compact, then $X \times {}^{\omega}2 \times \mathbb{Q} \approx {}^{\omega}2 \times \mathbb{Q}$,

(C″) if X is completely metrizable, then $X \times \mathbb{P} \approx \mathbb{P}$,

provided X is zero-dimensional and separable metrizable. These follow from the facts that each zero-dimensional separable metrizable space X which is nowhere locally compact is homeomorphic to ${}^{\omega}2 \times \mathbb{Q}$ if it is σ-compact and nowhere countable, and is homeomorphic to $\mathbb{P}$ if it is completely metrizable, by classical results of ALEXANDROFF and URYSOHN [1928]. [Actually Alexandroff and Urysohn use a different space than ${}^{\omega}2 \times \mathbb{Q}$, but that is irrelevant of course.]

Finally, since $2 \times \mathbb{Q} \approx \mathbb{Q}$, hence ${}^{\omega}({}^{\omega}2 \times \mathbb{Q}) \approx {}^{\omega}2 \times {}^{\omega}\mathbb{Q} \approx {}^{\omega}(2 \times \mathbb{Q}) \approx {}^{\omega}\mathbb{Q}$, statement (D′) is a consequence of (B′) since if $\mathcal{F}$ is a family of spaces then $\bigcap \mathcal{F}$ is homeomorphic to a closed subspace of the product $\Pi \mathcal{F}$, namely the diagonal $\{x \in \Pi \mathcal{F} \colon \forall F, G \in \mathcal{F}\ [x_F = x_G]\}$. $\square$

8.11. QUESTION. If X is separable metrizable, is $\mathrm{cf}(\mathcal{K}(X)) = k(X) = \mathfrak{d}$ if X is analytic, or at least if X is absolutely Borel?

The condition that X be analytic in 8.11 is essential: It is well known that there is a separable metrizable X with $|X| = \mathfrak{c}$ every compact subset of which is countable (a Bernstein set), and for such X we have $\mathrm{kc}(X) = k(X) = \mathrm{cof}(\mathcal{K}(X)) = \mathfrak{c}$, no matter that size $\mathfrak{d}$ has.

Having performed our calculations we use our results to calculate some other invariants. Recall that for a subset S of a space X the character and pseudocharacter of S in X are defined to be

$$\chi(S, X) = \min\{|\mathcal{U}|: \mathcal{U} \text{ is a neighborhood base for } S \text{ in } X\},$$

$$\psi(S, X) = \min\{|\mathcal{U}|: \mathcal{U} \text{ is an open family in } X \text{ with } \bigcap \mathcal{U} = S\}.$$

We omit the proof of the following.

8.12. FACT. *Let S be a subset of a compact space X. Then $\psi(S, X) = \mathrm{kc}(X - S)$ and $\chi(S, X) = \mathrm{cof}(\mathcal{K}(X - S))$.*

This will be used if we calculate some characters as follows.

8.13. THEOREM. *Let S be a subset of a separable metrizable space X.*

(a) *If $S \cap \overline{X - S}$ is compact, then $\chi(S, X) \leq \omega$.*

(b) *If $S \cap \overline{X - S}$ is noncompact, then $\chi(S, X) \geq \mathfrak{d}$.*

(c) *If $S \cap \overline{X - S}$ is noncompact and if S is absolutely G_σ, F_δ or $G_{\delta\sigma}$, then $\chi(S, X) = \mathfrak{d}$.*

PROOF. (a) If $\mathcal{U}$ is a neighborhood base of $S \cap \overline{X - S}$, then $\{U \cup S: U \in \mathcal{U}\}$ is a neighborhood base of S. Compact sets have countable neighborhood bases.

(b) The subspace $S \cap \overline{X - S}$ of X has a countably infinite closed discrete set D. The set $U = X - (\bar{D} - D)$ is a neighborhood of S in X, hence $\chi(S, X) = \chi(S, U)$. So without loss of generality we assume D is closed in X. Since $D \subseteq \overline{X - S}$ we may assume the product $\omega \times (\omega + 1)$ is a closed subspace of X satisfying $\omega \times \{\omega\} = D$ and $\omega \times \omega \subseteq X - S$. Then for each open U in $\omega \times (\omega + 1)$, if $\omega \times \{\omega\} \subseteq U$ then $X - ((\omega \times (\omega + 1) - U)$ is a neighborhood of S. It follows that $\chi(S, X) \geq \chi(\omega \times \{\omega\}, \omega \times (\omega + 1))$. It is pretty straightforward to check that $\chi(\omega \times \{\omega\}, \omega \times (\omega + 1)) = \mathfrak{d}$.

(c) Let bX be a separable metrizable compactification of X. Then $bX - S$ is absolutely F_σ, G_δ or $F_{\sigma\delta}$, hence $\mathrm{cof}(\mathcal{K}(bX - S)) \leq \mathfrak{d}$ by Theorem 8.10. It follows from 8.12 that $\chi(S, bX) \leq \mathfrak{d}$, and of course $\chi(S, X) \leq \chi(S, bX)$. Hence $\chi(S, X) = \mathfrak{d}$ by 8.13(b). □

8.14. QUESTION. Does 8.13(c) hold if S is absolutely Borel?

We conclude this section with calculating

$$\mathrm{Exp}_{\mathbb{R}}(X) = \min\{\kappa: X \text{ embeds as a closed subspace in } {}^{\kappa}\mathbb{R}\},$$

$$\mathrm{Exp}_{\omega}(X) = \min\{\kappa: X \text{ embeds as a closed subspace in } {}^{\kappa}\omega\}$$

for certain separable metrizable X. (As before, $\min(\emptyset) = \infty$, which is greater than each cardinal.) Our results are

8.15. THEOREM. *Let X be a separable metrizable space. Then*

(a) *If X is locally compact, then* $\mathrm{Exp}_{\mathbb{R}}(X) \leq \omega$.

(b) *If X is completely metrizable but not locally compact, then* $\mathrm{Exp}_{\mathbb{R}}(X) = \omega$.

(c) *If X is absolutely Borel but not completely metrizable, then* $\mathrm{Exp}_{\mathbb{R}}(X) = \mathfrak{d}$.

8.16. THEOREM. *If X is separable metrizable, then* $\mathrm{Exp}_{\omega}(X) = \mathrm{Exp}_{\mathbb{R}}(X)$ *if (and, trivially, only if) X is zero-dimensional.*

8.17. REMARK AND QUESTION. One always has $\mathrm{Exp}_{\mathbb{R}}(X) \leq \mathrm{Exp}_{\omega}(X)$, and there is a zero-dimensional metrizable space X with $\mathrm{Exp}_{\mathbb{R}} < \mathrm{Exp}_{\omega}(X) = \infty$, see NYIKOS. Is there a (preferable metrizable) nonlocally compact space X with $\mathrm{Exp}_{\mathbb{R}}(X) < \mathrm{Exp}_{\omega}(X) < \infty$?

8.18. LEMMA. *Let X be a space.*

(a) $\mathrm{Exp}_{\mathbb{R}}(X) + \omega = w(X) + \min\{\kappa$: *$X$ has a compactification bX so that $bX - X$ is covered by κ many closed G_δ's of bX*$\}$.

(b) *If X is zero-dimensional, then* $\mathrm{Exp}_{\omega}(X) + \omega = w(X) + \min\{\kappa$: *$X$ has a zero-dimensional compactification bX such that $bX - X$ is covered by κ many closed G_δ's of bX*$\}$.

PROOF. We prove (a) only since the proof of (b) is quite similar. Call the minimum of the right hand side μ. Let $\alpha\mathbb{R} = \mathbb{R} \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}$.

Proof that $\mathrm{Exp}_{\mathbb{R}}(X) \leq w(X) + \mu$. Let bX be a compactification of X as in the definition of μ. There are sets F_0 and F_1 of continuous functions $bX \to \alpha\mathbb{R}$ with $|F_0| = w(X)$ and $|F_1| = \mu$ such that

(1) $\forall x \in X$ [$\{f^{\leftarrow}(-1, 1)$: $x \in f^{\leftarrow}(-1, 1)$ and $f \in F_0\}$ is a neighborhood base at x in $bX\}$],

(2) $\forall f \in F_0$ [$\mathrm{ran}(f) \subseteq [0, 1]$],

(3) $bX - X = \bigcup\{f^{\leftarrow}\{\infty\}: f \in F_1\}$.

(For (3) use functions $bX \to [0, \infty]$.) Let $F = F_0 \cup F_1$. Define the diagonal map d: $bX \to {}^F\alpha\mathbb{R}$ by $d(x)_f = f(x)$, for $x \in X$, $f \in F$. Then $e \restriction X$ is an embedding of X into ${}^F\alpha\mathbb{R}$ because of (1), and $e^{\rightarrow}X = {}^F\mathbb{R} \cap e^{\rightarrow}bX$ because of (2) and (3). This equation implies that $e^{\rightarrow}X$ is closed in ${}^F\mathbb{R}$ since $e^{\rightarrow}bX$, being compact, is closed in ${}^F\alpha\mathbb{R}$. Finally, $\omega \leq w(X)$ since X is infinite.

Proof that $w(X) + \mu \leq \mathrm{Exp}_{\mathbb{R}}(X)$. We may assume that X is a closed subspace of ${}^\kappa\mathbb{R}$ for some κ. Clearly $w(X) \leq w({}^\kappa\mathbb{R}) = \kappa + \omega$. Also, if bX denotes the closure of X in ${}^\kappa\alpha\mathbb{R}$, then bX is a compactification of X such that

$$bX - X \subseteq {}^\kappa\alpha\mathbb{R} - {}^\kappa\mathbb{R} = \bigcup_{\eta \in \kappa} \{x \in {}^\kappa\alpha\mathbb{R}: x_\eta = \infty\}$$

since X is closed in ${}^\kappa\mathbb{R}$. Since $\{x \in {}^\kappa\alpha\mathbb{R}: x_\eta = \infty\}$ obviously is a closed G_δ of ${}^\kappa\alpha\mathbb{R}$, for $\eta \in \mathbb{R}$, it follows that $\mu \leq \kappa$. $\square$

8.19. LEMMA. *Let X be a separable metrizable space (or, more generally, a space that has a perfectly normal compactification) that is not locally compact. Then* $\exp_{\mathbb{R}}(X) = \mathrm{kc}(bX - X)$ *for each compactification bX of X.*

PROOF. If cX is any compactification of X define

$$\mu_c = \min\{|\mathcal{G}|\colon \mathcal{G} \text{ is a family of closed } \mathcal{G}_\delta\text{'s of } bX \text{ with } \mathcal{G} = bX - X\}\,,$$

and define

$$\mu = \min\{\mu_c\colon cX \text{ is a compactification of } X\}\,.$$

Since X is not locally compact we have $\mathrm{Exp}_{\mathbb{R}}(X) \geqslant \omega$ and $\mu \geqslant \omega$. It follows from 8.18 that we prove (a) if we prove $\mu = \mathrm{kc}(bX - X)$.

If cX is any compactification of X let f_c denote the (unique) continuous map from βX onto cX which extends id_X. As is well known, $f_c \restriction \beta X - X$ is a map from $\beta X - X$ onto $cX - X$ which is perfect. Using this and 8.9 we see that

$$\mu_\beta \leqslant \mu_c\,, \quad \text{hence } \mu = \mu_\beta\,; \quad \text{and} \quad \mathrm{kc}(cX - X) = \mathrm{kc}(\beta X - X)\,.$$

So we prove $\mu = \mathrm{kc}(bX - X)$ if we prove $\mu_\beta = \mathrm{kc}(\beta X - X)$.

Obviously, if cX is any compactification of X, then $\mathrm{kc}(cX - X) \leqslant \mu_c$, and even $\mathrm{kc}(cX - X) = \mu_c$ if cX is perfectly normal. Since X has a perfectly normal compactification cX we see that

$$\mathrm{kc}(\beta X - X) \leqslant \mu_\beta \leqslant \mu_c = \mathrm{kc}(cX - X) = \mathrm{kc}(\beta X - X)\,.$$

This proves $\mu_\beta = \mathrm{kc}(\beta X - X)$.

Remark. If X is separable metrizable then $\mu_c = \mu$ for each compactification cX of X. □

8.20. LEMMA. *Let X be strongly zero-dimensional, i.e. βX is zero-dimensional. Then* $\mathrm{Exp}_\omega(X) = \mathrm{Exp}_{\mathbb{R}}(X)$ *if X is not locally compact.*

PROOF. If μ_c is defined as above, we find that

$$\mu_\beta = \min\{\mu_c\colon cX \text{ is a zero-dimensional compactification of } X\}\,.$$

The result now follows from 8.8(b). □

8.21. PROOF OF THEOREM 8.15. *Proof of* (a). Lemma 8.18.

Proof of (b) *and* (c). Let bX be any metrizable compactification of X. Then $\mathrm{Exp}_{\mathbb{R}}(X) = \mathrm{kc}(bX - X)$ by Lemma 8.19, and of course $bX - X$ is absolutely Borel, hence the result follows from Theorem 8.10.

Remark. The argument shows it suffices to know X is coanalytic, i.e. $bX - X$ is analytic for some (or, equivalently, each) compactification bX of X. □

8.22. PROOF OF THEOREM 8.16. Zero-dimensional Lindelöf spaces are strongly zero-dimensional, by ENGELKING [1977, 6.2.12 and 6.2.7]. Now use Lemma 8.20. □

8.23. EXERCISE. For every cardinal κ with $\omega \leqslant \kappa \leqslant \mathfrak{c}$ there is a zerodimensional separable metrizable space X with $\mathrm{Exp}_{\mathbb{R}}(X) = \kappa$. [*Hint.* I has a dense subset B with $|B| = \mathfrak{c}$ every compact subset of which is countable.]

Notes to Section 8

Results 8.2 (except k), 8.12, 8.13(b) and 8.13(c) (for S σ-compact) are due to KATĚTOV [1960A; 3.6, 2.3, 2.4], who also proved 8.7 [1960B]. KATĚTOV [1960A, 1.7] also proves 8.5 (in a different manner), but states it only for $\mathrm{cof}(\mathcal{K}(X))$; earlier ARENS [1946, p. 486, (b)] proved 8.5 for $k(X)$.

HECHLER [1975] rediscovered KATĚTOV's result that $\mathrm{kc}(\mathbb{P}) = \mathfrak{d}$, proved that $\mathrm{kc}(X) \leqslant \mathfrak{d}$ for all $X \subseteq \mathbb{R}$ which are F_α for some $\alpha \in \omega$, and asked if one can prove $\mathrm{kc}(X) \leqslant \mathfrak{d}$ for all Borel $X \subseteq \mathbb{R}$. Our 8.2(e) answers his question affirmatively. Lemma 8.8 is due to HUREWICZ; we have given a fresh proof since his seems rather indirect.

Statements (a) and (b) of Theorem 8.15 are trivial consequences of the known fact that a separable metrizable space is completely metrizable iff it embeds as a closed subspace in ${}^{\omega}\mathbb{R}$, cf. ENGELKING [1977, 4.2.25]. HECHLER proves the special case $X = \mathbb{Q}$ of 8.15(c) in [1975, 1974]. Lemma 8.18 is a special case of results of MRÓWKA [1966, 3.3 and 4.1].

9. G_δ's and F_σ's

A space X is called a λ*-set* if each countable subset is a G_δ, and is called a σ*-set* if each F_σ-subset is a G_δ. Clearly each σ-set is a λ-set, and clearly each σ-set is *perfect* (= closed sets are G_δ's), while each λ-set satisfies $\psi(X) \leqslant \omega$ (i.e. points are G_δ's). A subset X of $\mathbb{R}$ is called a λ'*-set* if for each countable $A \subseteq \mathbb{R}$ the subspace $X \cup A$ of $\mathbb{R}$ is a λ-set.

Recall that $\mathbb{P}$ and $\mathbb{Q}$ the irrationals and the rationals, and that $\mathbb{P}$ is identified with the power ${}^{\omega}\omega$.

9.1. THEOREM. $\mathfrak{b} = \mathfrak{b}_c = \mathfrak{b}_{\sigma c} = \mathfrak{b}_\lambda = \mathfrak{b}'_\lambda = \mathfrak{b}_{\lambda'} = \mathfrak{b}_\sigma = \mathfrak{b}'_\sigma = \mathfrak{b}_G$, *where*

$$\mathfrak{b}_{\sigma c} = \min\{|A|: A \subseteq \mathbb{P}, \textit{ there is no } \sigma\textit{-compact } S \subseteq \mathbb{P} \textit{ with } A \subseteq S\}\,;$$

$$\mathfrak{b}_\lambda = \min\{|X|: X \textit{ is separable metrizable but is not a } \lambda\textit{-set}\}\,;$$

$\mathfrak{b}'_\lambda$ = min{$|X|$: $\psi(X) = \omega$ *but* X *is not a* λ*-set*} ;

$\mathfrak{b}_{\lambda'}$ = min{$|X|$: $X \subseteq \mathbb{R}$ *is a* λ*-set but is not a* λ'*-set*} ;

$\mathfrak{b}_\sigma$ = min{$|X|$: X *is separable metrizable but is not a* σ*-set*} ;

$\mathfrak{b}'_\sigma$ = min{$|X|$: X *is perfect but is not a* σ*-set*} ;

$\mathfrak{b}_G$ = min{$|X|$: *there is a countable collection* $\mathcal{G}$ *of* G_δ*-subsets of* X *such that* $\bigcup\mathcal{G}$ *is not a* G_δ*-subset of* X} .

9.2. COROLLARY. *There is a subspace of* $\mathbb{R}$ *which is a* λ*-set but not a* λ'*-set.*

For the proof we will need the following.

9.3. LEMMA. *The following conditions on* $B \subseteq \mathbb{P}$ *are equivalent*:
(a) *B, when seen as subset of* ${}^\omega\omega$, *is bounded*;
(b) *there is a* σ*-compact* $S \subseteq {}^\omega\omega$ *with* $B \subseteq S$;
(c) Q *is a* G_δ *in the subspace* $B \cup \mathbb{Q}$ *of* $\mathbb{R}$.

PROOF. (a)$\Rightarrow$(b). If $\forall f \in B\ [f \leqslant^* g]$ define countable $H \subseteq {}^\omega\omega$ by

$$H = \{h \in {}^\omega\omega\colon h(n) = g(n) \text{ for all but finitely many } n \in \omega\}.$$

Then $\forall f \in B\ \exists h \in H\ [f \leqslant h]$. Hence $S = \bigcup_{h\in H} \prod_{n\in\omega} [0, h_n]$ is as required.

(b)$\Rightarrow$(a). From the proof of Theorem 8.2 we recall that for each compact $K \subseteq \mathbb{P}$ there is a $g \in {}^\omega\omega$ with $K \subseteq \prod_{n\in\omega} [0, g_n]$. Hence there is a countable $G \subseteq {}^\omega\omega$ such that $\forall f \in B\ \exists g \in G\ [f \leqslant g]$. As G is bounded, being countable, it follows that B is bounded.

(b)$\Leftrightarrow$(c). Since a subset of $\mathbb{R}$ is an F_σ of $\mathbb{R}$ iff it is σ-compact we have: (b)$\Leftrightarrow B$ is an F_σ in $B \cup \mathbb{Q} \Leftrightarrow$ there is an F_σ S in $\mathbb{R}$ with $S \cap (B \cup \mathbb{Q}) = B \Leftrightarrow$ there is a σ-compact S in $\mathbb{R}$ with $B \subseteq S$ and $S \cap \mathbb{Q} = \emptyset \Leftrightarrow$ (c). $\square$

9.4. PROOF OF THEOREM 9.1. From Theorem 9.3 we see $\mathfrak{b} = \mathfrak{b}_{\sigma c} \geqslant \mathfrak{b}_\lambda$. It is clear that $\mathfrak{b}_{\lambda'} \geqslant \mathfrak{b}_\lambda \geqslant \mathfrak{b}'_\lambda \geqslant \mathfrak{b}_G$ and that $\mathfrak{b}_\lambda \geqslant \mathfrak{b}_\sigma \geqslant \mathfrak{b}'_\sigma \geqslant \mathfrak{b}_G$. Hence it suffices to prove that $\mathfrak{b}_G \geqslant \mathfrak{b}$ and that $\mathfrak{b} \geqslant \mathfrak{b}_{\lambda'}$.

Proof that $\mathfrak{b}_G \geqslant \mathfrak{b}$. For $k \in \omega$ let G_k be a G_δ-subset of a space X. We will prove that $G = \bigcup_{k\in\omega} G_k$ is a G_δ-subset of X if $|X - G| < \mathfrak{b}$. For $k \in \omega$ choose a sequence $\langle G_{k,n}\colon n \in \omega\rangle$ of open subsets of X such that

(1) $\bigcap_{n\in\omega} G_{k,n} = G_k$, and
(2) $\forall n \in \omega\ [G_{k,n} \supseteq G_{k,n+1}]$.

For each $f \in {}^\omega\omega$ define $U(f) = \bigcup_{k\in\omega} G_{k,f(k)}$. It is clear from (1) that for each $y \in X - G$ we can choose $f_y \in {}^\omega\omega$ such that $y \notin U(f_y)$. Since $|X - G| < \mathfrak{b}$ there is $g \in {}^\omega\omega$ such that $\forall y \in X - G\ [f_y \leqslant^* g]$. Define countable $H \subseteq {}^\omega\omega$ by

$$H = \{h \in {}^\omega\omega\colon h(n) = g(n) \text{ for all but finitely many } n \in \omega\}.$$

For each $y \in X - G$ there is $h \in H$ with $f_y \leqslant h$, and $y \notin U(h)$ for this h because of (2). Hence $G = \bigcap_{h \in H} U(h)$.

Proof that $\mathfrak{b} \geqslant \mathfrak{b}_{\lambda'}$. Since $\mathfrak{b} = \mathfrak{b}_1$, by Theorem 3.3, there is an unbounded $X \subseteq {}^\omega\omega$ such that $<^*$ well-orders X in type $\mathfrak{b}$. This X, when seen as subset of $\mathbb{P}$, hence of $\mathbb{R}$, is not a λ'-set, by Lemma 9.3. We show it is a λ-set. For $g \in X$ define $X_g = \{f \in X\colon f <^* g\}$. We have

(1) $\quad \forall A \in [X]^\omega \quad \exists g \in X \quad [A \subseteq X_g]$

since $<^*$ well-orders X in type $\mathfrak{b}$, a regular uncountable cardinal. We also have

(2) $\quad \forall g \in X \quad [X_g \text{ is a } \lambda\text{-set}]$

since we already know that $\mathfrak{b} \leqslant \mathfrak{b}_G \leqslant \mathfrak{b}_\lambda$. Hence in order to prove that X is a λ-set it suffices to prove that $\forall g \in X$ $[X_g$ is a G_δ in $X]$. So consider any $g \in X$; then

$$\begin{aligned} X_g &= \{f \in X\colon f <^* g = X - \{f \in {}^\omega\omega\colon g \leqslant^* f\} \\ &= X - \{f \in {}^\omega\omega\colon \exists k \in \omega\, \forall n > k\, [f(n) \geqslant g(k)] \\ &= X - \bigcup_{n \in \omega} \bigcap_{k \geqslant n} \{f \in {}^\omega\omega\colon f(n) \geqslant g(k)\} \end{aligned}$$

since $<^*$ linearly orders X. This shows $X - X_g$ is an $F_{\delta\sigma}$, i.e. an F_σ. □

Notes to Section 9

Corollary 9.2 is the main result of ROTHBERGER [1939B], who also proved Lemma 9.2 and $\mathfrak{b} = \mathfrak{b}_{\lambda'}$ in [1939B, Lemme 3, Theorème 4], and $\mathfrak{b} = \mathfrak{b}_{\sigma c} = \mathfrak{b}_\lambda = \mathfrak{b}_\sigma$ in [1941, Theorème 1].

Theorem 9.1 gives an uncountable separable metrizable λ-set in ZFC, but only gives consistent uncountable λ'-sets in ${}^\omega 2$ (or in $\mathbb{R}$), and only gives consistent uncountable separable metrizable σ-sets. SIERPIŃSKI has pointed out that the existence of an uncountable λ'-set in ${}^\omega 2$ (or in $\mathbb{R}$) easily follows from Exercise 4.5. At the other hand MILLER has shown it is consistent that no uncountable separable metrizable σ-set exists.

See Miller's Handbook article for additional information on λ-sets, σ-sets and related sets or spaces.

10. Covering properties and the Michael line

The Michael line $\mathbb{M}$ is the set $\mathbb{R}$ retopologized by declaring

$$\{U \subseteq \mathbb{M}\colon U \text{ is open in } \mathbb{R}\} \cup \{\{x\}\colon x \in \mathbb{P}\}$$

to be a base. A space is called *κ-compact* if it has no closed discrete subset of cardinality κ. Recall from Section 8 that a space is called analytic if it is a continuous image of $\mathbb{P}$, and that all separable metrizable absolutely Borel spaces are analytic.

10.1. THEOREM. $\mathfrak{b} = \mathfrak{b}_{\mathbb{P}} = \mathfrak{b}_a$, *where*

$$\mathfrak{b}_{\mathbb{P}} = \min\{\kappa\colon \mathrm{cf}(\kappa) \geq \omega_1 \text{ and there is a } \kappa\text{-compact space whose product with } \mathbb{P} \text{ is not } \kappa\text{-compact}\}$$

$$\mathfrak{b}_a = \min\{\kappa\colon \mathrm{cf}(\kappa) \geq \omega_1 \text{ and there are a } \kappa\text{-compact space and an analytic space whose product is not } \kappa\text{-compact}\}.$$

PROOF. *Proof that* $\mathfrak{b} \leq \mathfrak{b}_{\mathbb{P}}$. Consider κ with $\mathrm{cf}(\kappa) \geq \omega_1$ and $\kappa < \mathfrak{b}$, and a κ-compact space X and any $A \in [X \times \mathbb{P}]^\kappa$. Let π_2 denote the projection $X \times \mathbb{P} \to \mathbb{P}$. As $\mathfrak{b} = \mathfrak{b}_{\sigma c}$, by Theorem 9.1, there is a σ-compact $S \subseteq \mathbb{P}$ with $\pi_2^{\to} A \subseteq S$. Since $\mathrm{cf}(|A|) > \omega$ it follows that there is $B \in [A]^\kappa$ such that $\overline{\pi_2^{\to} B}$ is compact. It is not hard to prove that the product of a κ-compact space and a compact space is κ-compact. It follows that A is not closed discrete in $X \times \mathbb{P}$ since its subset B is not closed discrete in $X \times \overline{\pi_2^{\to} B}$.

Proof that $\mathfrak{b}_{\mathbb{P}} \leq \mathfrak{b}_a$. κ-compactness is preserved by continuous maps.

Proof that $\mathfrak{b}_a \leq \mathfrak{b}_{\mathbb{P}}$. $\mathbb{P}$ is analytic.

Proof that $\mathfrak{b}_{\mathbb{P}} \leq \mathfrak{b}$. Consider any unbounded $B \subseteq {}^\omega\omega$ which $<^*$ well-orders in type $\mathfrak{b}$. Then $\forall A \subseteq B$ $[|A| = \mathfrak{b} \Rightarrow A$ is unbounded$]$. Let X be $B \cup \mathbb{Q}$ as subspace of the Michael line $\mathbb{M}$.

We show X is $\mathfrak{b}$-compact: Consider any $A \subseteq X$ with $|A| = \mathfrak{b}$. Since $\mathfrak{b} > \omega$ we may assume $A \subseteq B$. Then A is unbounded, hence A is not an F_σ, leave alone a closed set, of the subspace $A \cup \mathbb{Q}$ of $\mathbb{R}$, by Lemma 9.3. Clearly each rational is in the $\mathbb{M}$-closure of A if (and only if) it is in the $\mathbb{R}$-closure of A. Hence A is not closed discrete in X.

We show $X \times \mathbb{P}$ is not $\mathfrak{b}$-compact: The set $D = \{\langle x, x\rangle\colon x \in B\}$ clearly is relatively discrete. Now consider $\Delta = \{\langle x, x\rangle\colon x \in \mathbb{R}\}$. This set is closed discrete in $\mathbb{R} \times \mathbb{R}$, hence in $\mathbb{M} \times \mathbb{R}$. Evidently $\Delta \cap X \times \mathbb{P} = D$ since $X \cap \mathbb{P} = B$. It follows that D is closed discrete. Of course $|D| = |B| = \mathfrak{b}$. $\square$

For our next result we need some terminology. A space X is said to be *concentrated about* $A \subseteq X$ if each neighborhood of A contains all but countably many points of X, and is said to be *concentrated* if it is concentrated about some countable set. As usual, if X is a space then X' denotes the set of non-isolated points of X.

10.2. THEOREM. *The following conditions are equivalent*:

(a) $\mathfrak{b} = \omega_1$;

(b) *there is an uncountable* $X \subseteq \mathbb{R}$ *with* $\mathbb{Q} \subseteq X$ *such that* X *is concentrated about* $\mathbb{Q}$;

(c) $\mathbb{M}$ *has an uncountable Lindelöf subspace*;

(d) *there is an uncountable concentrated space* X *with* $\psi(X) = \omega$; *and*

(e) *there is an uncountable Lindelöf space* X *with* $\psi(X) = \omega$ *such that* $|X'| = \omega$.

PROOF. (a)$\Rightarrow$(b). Consider an unbounded $B \subseteq {}^\omega\omega$ which $<^*$ well-orders in type ω_1. If $A \subseteq B$ is uncountable, then A is unbounded, hence A is not an F_σ, leave alone a closed set, in the subspace $A \cup \mathbb{Q}$ of $\mathbb{R}$ by Lemma 9.3. This tells us that $B \cup \mathbb{Q}$ is concentrated about $\mathbb{Q}$.

(b)$\Rightarrow$(c). The X furnished by (b) is easily seen to be Lindelöf.

It is clear that (c)$\Rightarrow$(e).

Proof that (e)$\Rightarrow$(d). It is clear that X is concentrated about X'.

Proof that (d)$\Rightarrow$(b). First we point out that each countable space Y admits a continuous injection into $\mathbb{Q}$. The fastest proof is this: We may assume Y is infinite, let $y: \omega \to Y$ be a bijection. Since Y is zero-dimensional one can easily find a continuous function $f_n: Y \to \mathbb{R}$ for $n \in \omega$ such that

$$\sum_{k \leq n} f_k(y_n) \in \mathbb{Q} - \left\{\sum_{k \leq i} f_k(y_i): i \in n\right\}$$

and $\forall i \in n\ [f_n(y_i) = 0]$, and $\|f_n\| \leq 2^{-n}$. Then $\Sigma_{n \in \omega} f_n$ is a continuous injection $Y \to \mathbb{Q}$. (In fact there is a continuous surjection if Y is infinite.)

Since X is normal, being Lindelöf, it follows from the Tietze–Urysohn Extension Theorem that there is a continuous $f: X \to R$ such that $f \restriction X'$ is an injection of X into $\mathbb{Q}$. It should be clear that $f^{\to}X$ is concentrated about $\mathbb{Q}$, and in fact about $f^{\to}X'$. To prove that $f^{\to}X$ is uncountable it suffices to prove that f is countable-to-one. So consider any $y \in \mathbb{R}$. Then $f^{\leftarrow}\{y\}$ is Lindelöf, being closed in X, and has at most one isolated point. Since points are G_δ's in X it follows that $|f^{\leftarrow}\{y\}| \leq \omega$.

Proof that (b)$\Rightarrow$(a). Consider any uncountable $B \subseteq X - \mathbb{Q}$. We claim B is unbounded, when seen as subset of ${}^\omega\omega$. Indeed, if not then B is an F_σ-subset of the subspace $B \cup \mathbb{Q}$ of $\mathbb{R}$. As B is uncountable it follows that $B \cup \mathbb{Q}$ has uncountable closed subset which misses $\mathbb{Q}$. This contradicts X being concentrated about $\mathbb{Q}$. □

10.3. COROLLARY TO PROOF. *Let* (a)$_\kappa$ *be the statement that there is a member of* $[{}^\omega\omega]^\kappa$ *every uncountable subset of which is unbounded. For* $i \in \{b, c, d, e\}$ *let* $(i)_\kappa$ *be as* (i) *with 'of cardinality* κ*' instead of 'uncountable'. Then* (a)$_\kappa$, (b)$_\kappa$, (c)$_\kappa$, (d)$_\kappa$ *and* (e)$_\kappa$ *are equivalent.* □

10.4. COROLLARY TO 10.1 AND 10.2. *Either every product of an analytic space and an* ω_1*-compact space is* ω_1*-compact or there is an uncountable first countable Lindelöf space with only countably many nonisolated points.* □

10.5. REMARK. $\mathbb{M}$ and $\mathbb{P}$ are the first example of a (hereditarily) paracompact space and a (separable) metrizable space whose product is not normal. This example is due to MICHAEL [1963], who in a footnote mentions that if $\mathfrak{c} = \omega_1$, then "one can even find a Lindelöf, hereditarily paracompact space whose product with the irrationals is not normal." As he mentions in [1971, p. 203], that Lindelöf space is any Lindelöf subspace of $\mathbb{M}$ which includes $\mathbb{Q}$. From Theorem 10.2 we see that this Lindelöf space exists iff $\mathfrak{b} = \omega_1$. It is unknown if there is in ZFC a Lindelöf X such that $X \times \mathbb{P}$ is not normal; by Theorem 10.2 we must have $|X'| > \omega$ or $\psi(X) > \omega$ if $\mathfrak{b} > \omega_1$. (However, as pointed out by MICHAEL [1963], there are in ZFC a Lindelöf space and a separable metrizable space whose product is not normal; the metrizable space is far from being $\mathbb{P}$: it is not even analytic.)

10.6. REMARK. In connection with the preceding remark we wish to point out the oddity that $\mathfrak{b} = \mathfrak{b}_n = \mathfrak{b}_p$, where

$$\mathfrak{b}_n = \min\{|X|: X \subseteq \mathbb{M}, \text{ and } X \times \mathbb{P} \text{ is not normal}\};$$

$$\mathfrak{b}_p = \min\{|X|: |X'| \leqslant \omega \text{ and } \psi(X) \leqslant \omega \text{ and there is a paracompact space whose product with } X \text{ is not paracompact}\}.$$

Proof that $\mathfrak{b}_n \leqslant \mathfrak{b}$. If $B \subseteq {}^{\omega}\omega$ is unbounded, then B is not an F_σ-subset of the subspace $B \cup \mathbb{Q}$ of $\mathbb{R}$. The proof by MICHAEL [1963] that $\mathbb{M} \times \mathbb{P}$ is not normal shows that X(as subspace of $\mathbb{M}) \times \mathbb{P}$ is not normal.

Proof that $\mathfrak{b}_p \leqslant \mathfrak{b}_n$. Paracompact spaces are normal, and $\mathbb{P}$ is paracompact.

Proof that $\mathfrak{b} \leqslant \mathfrak{b}_p$. Let X satisfy $|X| < \mathfrak{b}$ and $|X'| \leqslant \omega$ and $\psi(X) \leqslant \omega$, let Y be paracompact and let $\mathscr{U}$ be an open cover of $X \times Y$. We see from Theorem 9.1 that $|X| < b'$, hence $X - X'$ is an F_σ consisting of isolated points. It easily follows that there is a σ-locally finite open family $\mathscr{V}$ in X with $\bigcup \mathscr{V} = X - X'$ which refines $\mathscr{U}$. For each $x \in X'$ the subspace $\{x\} \times Y$ of $X \times Y$ is a retract of $X \times Y$; it is not hard to exploit this to find a locally finite open family $\mathscr{V}_x$ in X with $\bigcup \mathscr{V}_x \supset \{x\} \times Y$ which refines $\mathscr{U}$. Then $\mathscr{V} \cup \bigcup_{x \in X'} \mathscr{V}_x$ is a σ-locally finite open cover of $X \times Y$ which refines $\mathscr{U}$. $\square$

Notes to Section 10

Theorem 10.1 was mentioned in [VAN DOUWEN, 1976].

BESICOVITCH [§7] derived (b) of Theorem 10.2 from $\mathfrak{c} = \omega_1$ by implicitly deriving it from $\mathfrak{d} = \omega_1$ and in fact $\mathfrak{b} = \omega_1$ is all his proof requires. ROTHBERGER [1941, Lemme 2′] proved $(\mathrm{a})_\kappa \Leftrightarrow (\mathrm{b})_\kappa$ in Corollary 10.3, and MICHAEL [1971, Lemma 3.1] proved (b)$\Rightarrow$(c) in Theorem 10.2, and BELL and GINSBURG proved (e)$\Rightarrow$(a) of Theorem 10.2.

See Miller's Handbook article for more information about concentrated sets and related sets.

11. Spaces from almost disjoint families

Throughout this section D is an infinite set and $\mathscr{A}$ is a pairwise almost disjoint subfamily of $[D]^\omega$. Define a space $\Psi(D, \mathscr{A})$ as follows: the underlying set of $\Psi(D, \mathscr{A})$ is $D \cup \mathscr{A}$, the points of D are isolated and a basic neighborhood of $A \in \mathscr{A}$ has the form $\{A\} \cup (A - F)$, with F finite. (So a basic neighborhood of the *point* A of $\Psi(D, \mathscr{A})$ consists of A and all but finitely many points of the *subset* A of $\Psi(D, \mathscr{A})$.) In this section we study the interplay between set theoretic properties of $\mathscr{A}$ and topological properties of $\Psi(D, \mathscr{A})$.

11.1. EXERCISE. The family of compact open subsets of $\Psi(D, \mathscr{A})$ is a base for $\Psi(D, \mathscr{A})$. (Hence $\Psi(D, \mathscr{A})$ is first countable and locally compact.) Also D is dense in $\psi(D, \mathscr{A})$ and $\mathscr{A}$ is closed discrete in $\Psi(D, \mathscr{A})$. Finally, $\Psi(D, \mathscr{A})$ is a Moore space if $|D| = \omega$.

11.2. EXERCISE. Every first countable locally compact space in which the derived set is discrete is homeomorphic to some $\Psi(D, \mathscr{A})$.

Most results about the interaction between $\mathscr{A}$ and $\Psi(D, \mathscr{A})$ follow from the following result whose simple proof we omit.

11.3. LEMMA. *$\mathscr{F}, \mathscr{G} \subseteq \mathscr{A}$ can be separated* (*defined after Fact* 3.2) *iff they have disjoint neighborhoods in $\Psi(D, \mathscr{A})$.*

11.4. PROPOSITION. *For each of the following statements there is a pairwise almost disjoint $\mathscr{A} \subseteq [\omega]^\omega$ such that $\Psi = \Psi(\omega, \mathscr{A})$ satisfies it.*

(a) $|\Psi| = \omega_1$ *and Ψ is not normal, indeed*;

(b) $|\Psi| = \omega_1$ *and no two disjoint uncountable subsets of the uncountable closed discrete subset Ψ' of Ψ have disjoint neighborhoods*;

(c) $|\Psi| = \mathfrak{b}$ *and* $\exists C \in [\Psi']^\omega$ [*C and $\Psi' - C$ do not have disjoint neighborhoods*], *or even*

(d) $|\Psi| = \mathfrak{b}$ *and* $\exists C \in [\Psi']^\omega\, \forall D \in [C]^\omega$ [*D and $\Psi' - D$* (*or even D and $\Psi' - C$*) *do not have disjoint neighborhoods*];

(e) $|\Psi| = \mathfrak{b}$ *and* $\forall C \in [\Psi']^\omega$ [*C and $\Psi - C$ have disjoint neighborhoods*].

PROOF. *Proof of* (a) *and* (b). Lemma 11.3 and Theorem 4.1.

Proof of (c) *and* (d). Lemma 11.3 and the equality $\mathfrak{b} = \mathfrak{b}_5$ of Theorem 3.3.

Proof of (e). Since $\mathfrak{b} = \mathfrak{b}_1$, by Theorem 3.3., there is $B \subseteq {}^\omega\omega$ which is well-ordered by $<^*$ in type $\mathfrak{b}$. We prove (e) if we prove

$$\forall C \in [B]^\omega \quad [C \text{ and } B - C \text{ can be separated}]$$

since $\Psi(\omega \times \omega, B)' = B$. So consider $C \in [B]^\omega$. There is $g \in B$ such that $\forall f \in C$ $[f <^* g]$. Define L_g and B^* by

$$L_g = \{\langle k, n\rangle \in \omega \times \omega : n \leq g(k)\}\,, \qquad B^* = \{f \in B : f \leq^* g\}\,.$$

Then $|B^*| < \mathfrak{b}$. Since $\mathfrak{b} = \mathfrak{b}_5$, by Theorem 3.3, there is $S \subseteq \omega \times \omega$ which separates C and $B^* - C$. Then $L_g \cap S$ separates C and $B - C$. $\square$

11.5. REMARK. From the proof of (e) we see that $\mathfrak{b}$ is best possible in (c) and (d).

Another example of the interaction between $\mathscr{A}$ and $\Psi(D, \mathscr{A})$ is the following.

11.6. PROPOSITION. *$\Psi(D, \mathscr{A})$ has no infinite discrete open families* (*or, equivalently, $\Psi(D, \mathscr{A})$ is pseudocompact*) *iff $\mathscr{A}$ is mad.*

PROOF. Clearly $\mathscr{A}$ is mad iff each infinite subset of D has a limit point in $\Psi(D, \mathscr{A})$ iff $\{\{x\}: x \in D\}$ has no infinite discrete subfamily iff $\Psi(D, \mathscr{A})$ has no infinite discrete open families. $\square$

Notes to Section 11

Ψ, and Proposition 11.6, are due to MRÓWKA [1954], and Isbell (credited in GILLMAN and JERISON). Proposition 11.4(a) is mentioned by TALL [1972]. Proposition 11.4(e) is a variation on an idea of PROCTOR.

12. Weakenings of normality

A space is called *pseudo-normal* if every countable closed subset has arbitrarily small closed neighborhoods. A space is said to have *property D* if every countable closed discrete set has arbitrarily small closed neighborhoods, and is said to have *property wD* (for weak D) if every infinite closed discrete set has an infinite subset which has arbitrarily small closed neighborhoods. Clearly

$$\text{normal} \Rightarrow \text{pseudonormal} \Rightarrow \text{property D} \Rightarrow \text{property wD}.$$

The following known observation is frequently useful when one works with property D or wD.

12.1. PROPOSITION. *The following conditions on a countable closed discrete set D in a space X are equivalent*:

(a) *D has arbitrarily small closed neighborhoods in X*;

(b) *there is an indexed discrete open family $\langle U_x: x \in D\rangle$ in X satisfying $\forall x \in D$ $[x \in U_x]$.*

PROOF. (a)$\Rightarrow$(b). There is an indexed pairwise disjoint open family $\langle V_x: x \in D\rangle$ in X such that $\forall x \in D\ [x \in V_x]$: Let $<$ well-order D in type $|D|$ (so each member of D has only finitely many predecessors), and with recursion pick a neighborhood V_x for $x \in D$, disjoint from $\bigcup\{V_y: y \in D, y < x\}$ such that $\bar{V}_x \cap \{y \in D: y > x\} = \emptyset$. Let W be any open set in X with $D \subseteq W$ and $\bar{W} \subseteq \bigcup_{x\in D} V_x$. For $x \in D$ define $U_x = V_x \cap W$.

(b)$\Rightarrow$(a). Let V be open in X with $V \supseteq D$. For $x \in D$ choose a closed neighborhood W_x of x with $W_x \subseteq V \cap U_x$. Then $\bigcup_{x\in D} W_x$ is a neighborhood of D, included in U, which is closed if $\langle W_x: x \in D\rangle$ is an indexed discrete family. $\square$

The following result tells that small first countable regular spaces are pseudonormal; it will be used in Section 13 when we construct certain first countable spaces from the axiom $\mathfrak{b} = \mathfrak{c}$.

12.2. THEOREM. $\mathfrak{b} = \mathfrak{b}_p = \mathfrak{b}_D = \mathfrak{b}_{wD} = \mathfrak{b}'_{wD}$, *where*

$\mathfrak{b}_p$ $= \min\{|X'|$: *X is a first countable nonpseudonormal space*$\}$;

$\mathfrak{b}_D$ $= \min\{|X'|$: *X is a first countable space without property D*$\}$;

$\mathfrak{b}_{wD} = \min\{|X'|$: *X is a first countable space without property wD*$\}$;

$\mathfrak{b}'_{wD} = \min\{|X|$: *X is a first countable separable locally compact locally countable space without property wD*$\}$.

PROOF. Clearly $\mathfrak{b}_p \leqslant \mathfrak{b}_D \leqslant \mathfrak{b}_{wD} \leqslant \mathfrak{b}'_{wD}$.

Proof that $\mathfrak{b} \leqslant \mathfrak{b}_p$. We prove that if X is any first countable space with $|X'| < b$, then X is pseudonormal. So, given X, let F be a countable closed subset of X and let U be a neighborhood of F in X. We are to find a neighborhood V of F with $\bar{V} \subseteq U$. This is trivial if F is infinite, hence, for convenience, assume $F = \omega$. For each $k \in \omega$ we can pick a neighborhood base $\langle B_{k,n}: n \in \omega\rangle$ for k in X such that

(1) $\forall n \in \omega\ [B_{k,n} \supseteq B_{k,n+1}]$, and

(2) $\bar{B}_{k,0} \subseteq U$.

For $f \in {}^{\omega}\omega$ define $B(f) = \bigcup_{k\in\omega} B_{k,f(k)}$. Then $\langle B(f): f \in {}^{\omega}\omega\rangle$ is a neighborhood base for ω in X. For $y \in X' - U$ choose $f_y \in {}^{\omega}\omega$ such that $y \notin \overline{B(f_y)}$. Since $|X - U| \leqslant |X| < \mathfrak{b}$ there is $g \in {}^{\omega}\omega$ such that $\forall y \in X' - Y\ [f_y \leqslant^* g]$. We claim that the neighborhood $B(g)$ of ω satisfies $\overline{B(g)} \subseteq U$. Indeed, consider any $y \in X' - U$. Pick $m \in \omega$ such that $\forall k \in \omega\ [k \geqslant m \Rightarrow f_y(k) \leqslant g(k)]$. Then (1) implies

$$\overline{B(g)} \subseteq \overline{\bigcup_{k\in m} B_{k,0} \cup B(f_y)} = \bigcup_{k\in m} \bar{B}_{k,0} \cup \overline{B(f_y)}$$

It follows from (2) and our choice of f_y that $y \notin \overline{(B(g)}$.

Proof that $\mathfrak{b}'_{wD} \leqslant \mathfrak{b}$. Proposition 11.4(d) and Proposition 12.1. For later use we tell what the example is. By Theorem 3.3 there is an unbounded $B \subseteq {}^{\omega}\omega$ which $<^*$ well-orders in type $\mathfrak{b}$. The latter implies that B is pairwise almost disjoint. Topologize $X = \omega \times \omega \cup B \cup \omega$ as follows. Points of $\omega \times \omega$ are isolated. A basic neighborhood of $f \in B$ has the form $\{f\} \cup (f - F)$, for finite F. A basic neighborhood of $k \in \omega$ has the form $\{k\} \cup \{k\} \times [n, \omega)$, with $n \in \omega$. The proof that $\mathfrak{b}_5 \leqslant \mathfrak{b}_1$, Theorem 3.3, shows that the disjoint closed (even closed discrete) subsets B and ω of X do not have disjoint neighborhoods. □

12.3. EXAMPLE. First countability is essential in Theorem 12.2 and Corollary 12.3.

PROOF. Consider the subspace $T = ((\omega_1 + 1) \times (\omega + 1)) - \{\langle\omega_1, \omega\rangle\}$ of $(\omega_1 + 1) \times (\omega + 1)$, the well known Tychonoff Plank. Clearly $\{\omega_1\} \times \omega$ is a countably infinite closed discrete set in T. However, T has no infinite discrete open family: If $\mathcal{U}$ is any countably infinite open family in T note that for $U \in \mathcal{U}$ we can pick $\langle\alpha_U, x_U\rangle \in U$ with $\alpha_U < \omega_1$; if $\alpha = \sup_{U\in\mathcal{U}}$, then $\alpha < \omega_1$, hence $(\alpha + 1) \times (\omega + 1)$ is a compact subset of T which meets each member of $\mathcal{U}$.

(It is amusing to note that T is first countable at all points except those of that countable closed discrete set $\{\omega_1\}\times\omega$.) □

12.4. REMARK. It is well known that every two disjoint closed subsets F and G in a space X have disjoint neighborhoods provided they have compactness number ω_1 (the *compactness number* of a space X is the smallest cardinal κ such that every open cover of X has a subcover of cardinality strictly less than κ), i.e. provided they are Lindelöf, cf. ENGELKING [1977, proof of 1.5.14]. The proof that $\mathfrak{b}\geqslant\mathfrak{b}_p$ shows that one may weaken (if $\mathfrak{b}>\omega_1$) the condition on F to "F has compactness number at most $\mathfrak{b}$" provided one simultaneously strengthens the conditions on G to "G is countable and X is first countable at each point of G". In particular, if X (or even X') has compactness number at most $\mathfrak{b}$ then a closed subset G has arbitrarily small closed neighborhoods provided G is countable and X is first countable at each point of G. Proposition 11.4(a) and Example 12.3 (via Proposition 12.1) show that both conditions on G are essential.

12.5. REMARK AND QUESTION. Call a space *Ψ-like* if it is separable, first countable, locally compact and has a discrete derived set. Say that a subset S of a space X is *relatively pseudocompact* in X if every continuous real valued function on X is bounded on S, or, equivalently, if every discrete open family $\mathcal{U}$ such that $\forall U\in\mathcal{U}$ $[U\cap S\neq\emptyset]$ is finite. Define

$\mathfrak{a}_p = \min\{|X|$: X is Ψ-like and pseudocompact$\}$;

$\mathfrak{a}'_p = \min\{|X|$: X is first countable and pseudocompact but not countably compact$\}$;

$\mathfrak{b}_{rp} = \min\{|X|$: X *is* Ψ-like and has a countably infinite closed discrete relatively pseudocompact subset$\}$;

$\mathfrak{b}'_{rp} = \min\{|X|$: X is first countable and has a countably closed discrete relatively pseudocompact subset$\}$;

From Section 11 and Theorem 12.2 we see that $\mathfrak{a}_p=\mathfrak{a}$ and that $\mathfrak{b}_{rp}=\mathfrak{b}'_{rp}=\mathfrak{b}$. (This can be construed as a topological proof that $\mathfrak{a}\geqslant\mathfrak{b}$ since obviously $\mathfrak{a}_p\geqslant\mathfrak{b}_{rp}$.) This leaves open the question of what $\mathfrak{a}'_p$ is; we only have the trivial inequalities $\mathfrak{a}\geqslant\mathfrak{a}'_p\geqslant\mathfrak{b}$. (In this context we recall that it is unknown if $\mathfrak{a}\neq\mathfrak{b}$ is consistent with ZFC.)

12.6. QUESTION. The fact that $\Psi(D,\mathcal{A})$, when pseudocompact, has a closed discrete set of cardinality at least $\mathfrak{a}$ suggests the question of whether there is an honest first countable (preferably separable and locally compact) pseudocompact space that is not countably compact but that has no uncountable closed discrete subset. (Exercise 13.3 gives a consistent example.) (Mike Reed has pointed out such a space can not be a Moore space: it can not be subparacompact.) (Example 12.3 gives an example that is not first countable.)

12.6. REMARK. A space X is called *cwH* (for collectionwise Hausdorff) if every closed discrete set D in X can be *screened*, i.e. if there is an indexed pairwise disjoint open family $\langle U_x: x \in D\rangle$ in X such that $\forall x \in D\ [x \in U_x]$. One defines $<\kappa$-*cwH* and $\leqslant\kappa$-*cwH* analogously with the restriction $|D| < \kappa$, or $|D| \leqslant \kappa$.

Work on the normal Moore space problem has led to the question of whether $\leqslant\omega_1$-cwH Moore spaces are cwH, and more generally of when a $<\kappa$-cwH Moore space is $\leqslant\kappa$-cwH; consistent examples are known for all regular not weakly compact $\kappa > \omega_1$, see FLEISSNER [1978, 1977]. We here modify the example X of the proof that $\mathfrak{b}'_{\mathrm{wD}} \leqslant b$ to give an example of a $<\mathfrak{b}$-cwH Moore space that is not $\mathfrak{b}$-cwH. (Recall that $\mathfrak{b} > \omega_1$ is consistent.) It is unknown if there is a real example of a non-cwH Moore space that is $\leqslant\omega_1$-cwH; SHELAH [1977] has shown that it is consistent that there is no locally small such example.

Topologize $Y = {}^\omega\omega \cup \omega \cup {}^\omega\omega \times \omega \times \omega$ as follows: Points of ${}^\omega\omega \times \omega \times \omega$ are isolated, a basic neighborhood of $f \in {}^\omega\omega$ has the form $\{f\} \cup \{f\} \times (f - F)$, with F finite, and a basic neighborhood of $k \in \omega$ has the form $\{k\} \cup {}^\omega\omega \times \{k\} \times (\omega - n)$, with $n \in \omega$. Then Y is a Moore space.

(The major difference between Y and X is not that we use all of ${}^\omega\omega$—for we could have used B instead of ${}^\omega\omega$, but that each function in ${}^\omega\omega$ gets its own copy of $\omega \times \omega$.)

We wish to show a closed discrete set D in Y can be separated if $|D| < \mathfrak{b}$, but not necessarily if $|D| = \mathfrak{b}$. Without loss of generality we may assume that $D \subseteq {}^\omega\omega \cup \omega$, since all points of $Y - ({}^\omega\omega \cup \omega)$ are isolated, and that $\omega \subseteq D$ since ω is closed discrete in Y (and since $\omega < \mathfrak{b}$). It should be clear that both ${}^\omega\omega$ and ω can be screened in Y. Hence a necessary and sufficient condition that D can be screened is that $D \cap {}^\omega\omega$ and ω have disjoint neighborhoods. It should be clear that this condition holds iff $D \cap {}^\omega\omega$ is bounded.

Notes to Section 12

The results of this section are due to VAN DOUWEN [1976A] (except 12.1 and 12.3 or course). Theorem 12.2 answers TALL's [1972] question of whether it is consistent with ZFC that every regular separable first countable space has the property that every two disjoint closed discrete sets F and G with $|F| = \omega$ and $|G| = \omega_1$ have disjoint neighborhoods: this is equivalent to $\mathfrak{b} > \omega_1$.

It is unknown if there is an honest example of a first countable space with property D which is not pseudonormal; VAN DOUWEN and WAGE have constructed an example from $\mathfrak{p} = \mathfrak{c}$. The Sorgenfrey plane is an easy example of a first countable space with property wD which does not have property D. To see it does not have property D consider $\{\langle q, -q\rangle: q \in \mathbb{Q}\}$. To prove it has property D one proves that every realcompact space has property wD. (Use the fact that X is realcompact iff it has a compactification bX such that $bX - X$ is a union of G_δ-subsets of bX, or that X is realcompact iff it embeds as a closed subspace in a power of $\mathbb{R}$.) This was also observed, independently, by VAUGHAN [1978, 4.4].

13. Recursive construction of topologies

We here present two constructions, and a variation of one of them, of certain first countable spaces. These constructions can be performed precisely when $\mathfrak{b}_D = \mathfrak{c}$, i.e. when each first countable space X with $|X'| < \mathfrak{c}$ has property D; by Theorem 12.2 this happens iff $\mathfrak{b} = \mathfrak{c}$. (Actually what one really needs for the constructions (or some obvious modification) is $b'_{wD} = \mathfrak{c}$ and $|X| < \mathfrak{c}$, in particular one only needs property wD. However, $b'_{wD} = \mathfrak{b}_D$ by Theorem 12.2.)

Our first construction yields an affirmative answer to the question of whether a product of sequentially compact spaces can fail to be countably compact, i.e. to Question 6.11.

13.1. EXAMPLE. $[\mathfrak{b} = \mathfrak{c}]$ There is a family of $2^{\mathfrak{c}}$ first countable countably compact (hence sequentially compact) locally compact spaces whose product is not countably compact.

PROOF. Let ω^* denote the set of free ultrafilters on ω. We plan to construct a first countable countably compact locally compact space X_p with underlying set $\mathfrak{c}$ such that $\Pi = \Pi_{p \in \omega^*} X_p$ is not countably compact. (Recall that $|\omega^*| = 2^{\mathfrak{c}}$.)

We begin with finding a simple condition on the X_p's separately which ensures that Π is not countably compact. For each $n \in \omega$ the function $\omega^* \times \{n\}$ $(= \{\langle p, n\rangle : n \in \omega^*\})$ belongs to Π since $\forall p \in \omega^*\ [\omega \subseteq X_p]$. Suppose the countably infinite subset $\{\omega^* \times \{n\} : n \in \omega\}$ of Π has a cluster point x. Then

$$\mathcal{F} = \{I \subseteq \omega : \exists \text{ open } U \subseteq \Pi\ [x \in U \text{ and } I = \{n \in \omega : \omega^* \times \{n\} \in U\}]\}$$

is a collection of nonempty subsets of ω with the finite intersection property such that $\bigcap \mathcal{F} = \emptyset$. By the Kuratowski–Zorn Lemma there is $p \in \omega^*$ with $\mathcal{F} \subseteq p$. Clearly $\forall P \in p\ [x \in \overline{\{\omega^* \times \{n\} : n \in P\}}]$ (in Π). Hence $\forall P \in p\ [x_q \in \bar{P}]$ (in X_q) holds for all $q \in \omega^*$, in particular for $q = p$. Consequently $\{\omega^* \times \{n\} : n \in \omega\}$ will not have a cluster point, and hence Π will not be countably compact, if for each $p \in \omega^*$ we have

$$(p) \qquad \forall x \in X_p \quad \exists P \in p \quad [x \notin \bar{P}].$$

For the construction we need the following simple lemma.

13.2. LEMMA. *There is a surjection* $K : \mathfrak{c} - \omega \to [\mathfrak{c}]^\omega$ *such that* $\forall \eta \in \mathfrak{c} - \omega\ [K_\eta \subseteq \eta]$.

Indeed, since $|[\mathfrak{c}]^\omega \times \mathfrak{c}| = \mathfrak{c} = |\mathfrak{c} - \omega|$ there is an $L: \mathfrak{c} - \omega \to [\mathfrak{c}]^\omega$ such that $\forall T \in [\mathfrak{c}]^\omega$ $[|L^{\leftarrow}T| = \mathfrak{c}]$. Define K by

$$K_\eta = L_\eta \quad \text{if } L_\eta \subseteq \eta\,, \qquad \text{else } K_\eta = \omega\,.$$

To see K is as required consider $T \in [\mathfrak{c}]^\omega$. Then $\sup(T) < \mathfrak{c}$ since $\mathrm{cof}(\mathfrak{c}) > \omega$, hence there is $\eta \in L^{\leftarrow}T$ with $\eta > \sup(T)$. Clearly $K_\eta = T$.

Now fix $p \in \omega^*$. Recall that the underlying set of X_p will be $\mathfrak{c}$. We construct X_p by constructing the topology $\mathcal{T}_\eta$ of the subspace η of X_p with recursion on $\eta \in \mathfrak{c} + 1$. For each $\eta \in \mathfrak{c} + 1$ we will have

(1) $\langle \eta, \mathcal{T}_\eta \rangle$ is a T_1-space in which the collection of compact countable subsets is a base,

(2) $\forall \xi \in \eta$ $[\langle \xi, \mathcal{T}_\xi \rangle$ is an open subspace of $\langle \eta, \mathcal{T}_\eta \rangle]$,

(3) $\forall \xi \in \eta - \omega$ $[K_\xi$ has a cluster point in $\langle \eta, \mathcal{T}_\eta \rangle]$,

(4) $\forall \xi \in \eta - \omega\ \exists P \in p\ [\xi \notin \bar{P}]$ (in $\langle \eta, \mathcal{T}_\eta \rangle$).

Note that (1) implies that each $\langle \eta, \mathcal{T}_\eta \rangle$ is first countable.

From the special case $\eta = \mathfrak{c}$ we see that $X_p = \langle \mathfrak{c}, \mathcal{T}_\mathfrak{c} \rangle$ is as required. (We have (p) for all $x \in \mathfrak{c}$, and not just for $x \in \mathfrak{c} - \eta$, since (1) implies that the points of ω are isolated.)

We now proceed to the construction. For $n \in \omega$ let $\mathcal{T}_n$ be the only topology on n satisfying (1), the discrete topology. Next consider $\zeta \in \mathfrak{c} + 1$ with $\zeta \geq \omega$, and assume $\mathcal{T}_\eta$ known for $\eta \in \zeta$.

Case 1. *ζ is a limit ordinal.* Since $\forall \eta \in \zeta\ \forall \xi \in \eta\ [\mathcal{T}_\xi \subseteq \mathcal{T}_\eta]$ the family $\mathcal{B} = \bigcup_{\eta \in \zeta} \mathcal{T}_\eta$ is the base for some topology $\mathcal{T}_\zeta$ on ζ. We prove that (2) holds for $\eta = \zeta$. Consider any $\xi \in \zeta$. Then $\mathcal{T}_\xi \subseteq \mathcal{B}$, hence each set open in $\langle \xi, \mathcal{T}_\xi \rangle$, in particular ξ, is open in $\langle \zeta, \mathcal{T}_\zeta \rangle$. Moreover, for each $B \in \mathcal{B}$ there is $\eta \in \zeta$ with $B \in \mathcal{T}_\eta$, and hence $B \cap \xi \in \mathcal{T}_\xi$ because of (2) (and even $B \in \mathcal{T}_\xi$ if $\eta \leq \xi$). This proves $\langle \xi, \mathcal{T}_\xi \rangle$ is an open subspace of $\langle \zeta, \mathcal{T}_\zeta \rangle$.

It is easy to see that the fact that (2) holds for $\eta = \zeta$ implies that (1), (3) and (4) hold for $\eta = \zeta$.

Case 2. *ζ is a successor ordinal, $\zeta = \xi + 1$ say.*

Subcase A. *K_ξ has a cluster point in $\langle \xi, \mathcal{T}_\xi \rangle$.* Let $\mathcal{T}_\zeta$ be the topology on ζ which has $\mathcal{T}_\xi \cup \{\xi\}$ as a base. It should be clear that (1) through (4) hold for $\eta = \zeta$.

Subcase B. *Not Subcase* A, *i.e. K_ξ is closed discrete in $\langle \xi, \mathcal{T}_\xi \rangle$.* Since $\langle \xi, \mathcal{T}_\xi \rangle$ has property D, because of (*) mentioned above, there is an indexed discrete open family $\langle U_x : x \in K_\eta \rangle$ with $\forall x \in K_\eta\ [x \in U_x]$, and we may take the U_x's to be compact and countable since (1) holds for $\eta = \xi$. Let H and I be two disjoint infinite subsets of K_ξ. Then $\bigcup_{x \in H} U_x$ and $\bigcup_{x \in I} U_x$ are disjoint, hence their traces on ω are disjoint. Since p is an ultrafilter it follows that there is $J \in \{H, I\}$ such that $\omega \cap \bigcup_{x \in J} U_x \notin p$. Define

$$\mathcal{B} = \{\{\xi\} \cup \bigcup_{x \in J - F} U_x : F \in [J]^{<\omega}\}\,.$$

We topologize $\zeta = \xi \cup \{\xi\}$ by declaring $\mathcal{T}_\xi \cup \mathcal{B}$ to be a base for it; in other words,

$\langle \xi, \mathcal{T}_\xi \rangle$ is an open subspace of $\langle \zeta, \mathcal{T}_\zeta \rangle$, and $\mathcal{B}$ is a neighborhood base for ξ in $\langle \zeta, \mathcal{T}_\zeta \rangle$. This is well-defined since $\forall B \in \mathcal{B}\ [B \cap \xi \in \mathcal{T}_\xi]$.

By our construction x is a cluster point of K_ξ, and $P = \omega - \bigcup_{x \in J} U_x$ belongs to p but has $\xi \notin \bar{P}$. Moreover, $\bigcap \mathcal{B} = \{\xi\}$, and the members of $\mathcal{B}$ are compact and countable in $\langle \zeta, \mathcal{T}_\zeta \rangle$ since the U_x's are compact and countable in $\langle \xi, \mathcal{T}_\xi \rangle$. Since (1) through (4) hold for $\eta = \xi$, and since $\langle \xi, \mathcal{T}_\xi \rangle$ is an open subspace of $\langle \zeta, \mathcal{T}_\zeta \rangle$, it follows that (1) through (4) hold for $\eta = \zeta$.

This completes the construction of X_p. Note that X_p probably is not separable, since Subcase A might happen uncountably many times, but it is easy to find a separable X_p: let the new X_p be the closure of ω in the old X_p. □

13.3. EXERCISE. Assume $\mathfrak{b} = \mathfrak{c}$. Construct a first countable separable zero-dimensional locally compact pseudocompact space which is not countably compact but which is ω_1-compact, i.e. which has no uncountable closed discrete subset. (Recall Question 13.3.) [*Hint.* First construct a space first countable zero-dimensional locally compact space X with underlying set $\mathfrak{c}$ such that $D = \{\xi : \omega \leq \xi < \omega + \omega\}$ is a closed discrete set of cluster points of ω and such that each countably infinite subset of $X - D$ has a cluster point.]

The fact that both locally compact spaces and completely metrizably space are Baire spaces has led to the introduction of several completeness properties, i.e. properties $\mathcal{P}$ such that each locally compact space and each completely metrizable space has $\mathcal{P}$ and such that each space with $\mathcal{P}$ is a Baire space. AARTS and LUTZER [1974, 5.3.5.] ask if several of these completeness properties are preserved by continuous closed maps. The following result provides a consistent counter example, even if one considers *quasi-perfect* (= continuous and closed and countably compact fibers) maps: since $\mathbb{Q}$ is not a Baire space it shows that even the quasi-perfect image of even a locally compact space need not be a Baire space. (This can not happen with perfect maps since perfect maps preserve local compactness in two directions.)

13.4. THEOREM. $[\mathfrak{b} = \mathfrak{c}]$. *Each first countable space of cardinality at most $\mathfrak{c}$ is a quasi-perfect image of some locally compact space.*

PROOF. Let X be any first countable space with $|X| \leq \mathfrak{c}$; we may assume $|X| \geq \omega$. We will construct a topology for the *set* $X \times \mathfrak{c}$ such that the projection $\pi : X \times \mathfrak{c} \to X$ is quasi-perfect. We need a simple condition for a map to be quasi-perfect. If A is a countable subset of a space Y we say that A *converges* to $y \in Y$ if either $A = \{x\}$ or if $|A| = \omega$ and each neighborhood of x contains all but finitely many points of A. With this terminology we have the following result, whose straightforward proof we omit.

13.5. LEMMA. *If T is first countable (or even sequential), then a continuous map f from any space S onto T is quasi-perfect iff*

$$\forall A \in [S]^\omega \quad [f^\to A \text{ converges (in } T) \Rightarrow \exists B \in [A]^\omega \ [B \text{ has a cluster point (in } S)]] .$$

Since $|X| \geq \omega$ a simple modification of the proof of Lemma 13.2 yields a function K: $\mathfrak{c} - \{0\} \to \{A \in [X \times \mathfrak{c}]^\omega : \pi^\to A \text{ converges}\}$ such that $\forall \eta \in \mathfrak{c} - \{0\}\ [K_\eta \subseteq X \times \eta]$. With recursion on $\eta \in \mathfrak{c} + 1$ we construct a topology $\mathcal{T}_\eta$ on $X \times \eta$ such that

(1) $\langle X \times \eta, \mathcal{T}_\eta \rangle$ is a T_1-space in which the collection of compact countable sets is a base;

(2) $\pi \restriction X \times \eta$ is continuous as map $\langle X \times \eta, \mathcal{T}_\eta \rangle \to X$;

(3) $\forall \xi \in \eta\ [\langle X \times \xi, \mathcal{T}_\xi \rangle$ is an open subspace of $\langle X \times \eta, \mathcal{T}_\eta \rangle]$;

(4) $\forall \xi \in \eta\ [K_\xi$ has a cluster point in $\langle X \times \eta, \mathcal{T}_\eta \rangle]$;

(5) $\langle X \times \eta, \mathcal{T}_\eta \rangle$ has at most $|\eta|$ many nonisolated points.

Then $\mathcal{T}_\mathfrak{c}$ is our required topology on $X \times \mathfrak{c}$.

Let $\mathcal{T}_1$ be the discrete topology. (Of course $\mathcal{T}_0 = \{\emptyset\}$.) Now let $\zeta \in (\mathfrak{c} + 1) - \{0\}$, and assume $\mathcal{T}_\eta$ known for $\eta \in \zeta$. If ζ is a limit ordinal then, as above, we let $\mathcal{T}_\zeta$ be the topology on $X \times \zeta$ which has $\bigcup_{\eta \in \zeta} \mathcal{T}_\eta$ as a base. Now assume ζ is a successor ordinal, $\zeta = \xi + 1$ say.

Case A. *K_ξ has a cluster point in $\langle X \times \xi, \mathcal{T}_\xi \rangle$.* Let $\mathcal{T}_\zeta$ be the topology on $X \times \zeta = X \times \xi \cup X \times \{\xi\}$ which has $\mathcal{T}_\xi \cup \{\{\langle x, \xi \rangle\} : x \in X\}$ as a base. (So all points of $X \times \{\xi\}$ are isolated.)

Case B. *K_ξ has no cluster point in $\langle X \times \xi, \mathcal{T}_\xi \rangle$ and $\pi^\to K_\xi$ converges.* Let $\pi^\to K_\xi$ converge to $x \in X$. Let $\langle U_n : n \in \omega \rangle$ be a countable decreasing neighborhood base of x. Choose an injection s: $\omega \to K_\xi$ such that $\forall n \in \omega\ [\pi(s_n) \in U_n]$. (Any injection will do if $|\pi^\to K_\xi| = 1$, but not necessarily if $|\pi^\to K_\xi| \neq 1$.) Because of (5) $\langle X \times \xi, \mathcal{T}_\xi \rangle$ has property D, hence there is an indexed discrete collection $\langle V_n : n \in \omega \rangle$ of open sets in $\langle X \times \eta, \mathcal{T}_\eta \rangle$ with $\forall_n \in \omega\ [s_n \in V_n]$. Because of (1), (2) and the fact that $\forall n \in \omega\ [\pi(s_n) \in U_n]$ we can made sure that $\forall n \in \omega\ [V_n$ is compact and countable, and $V_n \subseteq \pi^\leftarrow U_n]$. Define $\mathcal{B} = \{\{\langle x, \xi \rangle\} \cup \bigcup_{k \in \omega - n} V_k\}$, and topologize $X \times \zeta$ by declaring $\mathcal{T}_\xi \cup \mathcal{B} \cup \{\{\langle y, \xi \rangle\} : y \in X - \{x\}\}$ to be a base. (So all points of $X \times \{\xi\}$ are isolated, except for $\langle x, \xi \rangle$ which gets $\mathcal{B}$ as a neighborhood base.) Note that (2) holds for $\eta = \zeta$ because $\forall n \in \omega\ [V_n \subseteq \pi^\leftarrow U_n]$. $\square$

13.6. Questions. Is 13.4 true in ZFC? Is the condition "of cardinality at most $\mathfrak{c}$" essential? How far can "first countable" be weakened?

If $\mathcal{P}$ is a property of functions, a function f from a space X onto a space Y is called *inductively* $\mathcal{P}$ if there is a closed $F \subseteq X$ such that $f^\to F = Y$ and such that $f \restriction F$ has $\mathcal{P}$. A surjection f: $X \to Y$ is called *irreducible* if $f^\to F \neq Y$ for each proper closed subset F of X. V.I. Ponomarev has reportedly asked if continuous closed maps with normal domain are inductively irreducible. The final result of this section is a consistent counter example, even with a quasi-perfect map. (At the other hand, it is a well-known straightforward consequence of the Kuratowski–Zorn Lemma that each perfect map is inductively perfect.)

13.7. EXAMPLE. [$\diamondsuit$] There is a locally compact perfectly normal space which admits a quasi-perfect map onto $\mathbb{Q}$ that is not inductively perfect.

PROOF. We begin with pointing out that this example can not exist in ZFC: It is consistent with ZFC that each perfectly normal countably compact space be compact, see Vaughan's Handbook article, hence that every quasi-perfect map from a perfectly normal space be perfect. (In fact this happens if $\mathfrak{p} > \omega_1$, hence it is consistent with $\mathfrak{b} = \mathfrak{c} > \omega_1$.) As we pointed out above, perfect maps are inductively perfect.

Our example is based on the observation that a continuous closed map f from a scattered space S onto a space T which has no isolated points is not inductively irreducible. (See the Intermezzo in the proof of Theorem 7.5 for a discussion of scattered spaces.) Indeed, consider any closed F in S with $f^{\rightarrow}F = T$. Pick an isolated point x of the subspace F. Then $F - \{x\}$ is closed, hence so is $f^{\rightarrow}(F - \{x\})$. Since $f^{\rightarrow}(F - \{x\}) \supseteq T - \{f(x)\}$ and since $f(x)$ is not isolated it follows that $f^{\rightarrow}(F - \{x\}) = T$. Consequently $f \restriction F$ is not irreducible.

From (the proof of) Theorem 13.4 we get a space X such that

(1) the collection of compact countable subsets of X is a base for X, and a quasi-perfect map f from X onto $\mathbb{Q}$. Clearly X is scattered, since compact countable spaces are scattered, and zero-dimensional (this simplifies life below). We use $\diamondsuit$ to make the fibers of f very perfectly normal: we will have for each $q \in \mathbb{Q}$ that

(2) each closed set of $f^{\leftarrow}\{q\}$ is countable or cocountable in $f^{\leftarrow}\{q\}$.

The following exercise shows there are such X and f.

13.8. EXERCISE. Assume $\diamondsuit$, i.e. $\mathfrak{c} = \omega_1 \wedge \clubsuit$. Let $\pi: \mathbb{Q} \times \omega_1 \to \mathbb{Q}$ the projection. Construct a topology $\mathcal{T}$ on the set $\mathbb{Q} \times \omega_1$ such that

(3) the collection of compact countable subsets of $\langle \mathbb{Q} \times \omega, \mathcal{T} \rangle$ is a base for $\mathcal{T}$,

(4) π is a quasi-perfect map from $\langle \mathbb{Q} \times \omega_1, \mathcal{T} \rangle$ onto $\mathbb{Q}$,

(5) $\forall q \in \mathbb{Q}$ [each closed subset (in $\langle \mathbb{Q} \times \omega_1, \mathcal{T} \rangle$ of $\{q\} \times \omega_1$ is countable or cocountable in $\{q\} \times \omega_1$].

[*Hint.* Combine the construction of Theorem 13.4 with the construction of Ostaszewski's example given by Roitman in her Hand Book article. Let Λ be the set of limit ordinals in ω_1, and with recursion on $\lambda \in \Lambda \cup \{\omega_1\}$ construct your topology $\mathcal{T}_\lambda$ on $\mathbb{Q} \times \lambda$. Let $\langle S_\lambda : \lambda \in \Lambda \rangle$ be the $\clubsuit$-sequence. For $\lambda \in \Lambda$ the following holds: S_λ is cofinal in λ, hence for $q \in \mathbb{Q}$ one can pick $S_{\lambda,q} \subseteq S_\lambda$, still cofinal in λ, such that $\forall \eta \in \lambda$ $[|\{q \in \mathbb{Q} : \min(S_{\lambda,q}) < \eta|\} < \omega]$. Then $\bigcup_{q \in \mathbb{Q}} \{q\} \times S_{\lambda,q}$ is closed discrete in $\langle \mathbb{Q} \times \lambda, \mathcal{T}_\lambda \rangle$ since you made sure that $\forall \eta \in \lambda$ $[\mathbb{Q} \times \eta \in \mathcal{T}_\lambda]$.]

Each fiber of f is like Ostaszewski's example, hence is perfectly normal. In order to prove X is perfectly normal it suffices to prove that each open set in X is the union of some countable collection of clopen sets, see ENGELKING [1977, 1.5.14 and 1.5.19]. Since X is the union of countably many perfectly normal spaces we prove this if each closed subset of each fiber has a neighborhood base in X

consisting of clopen sets. So consider closed F in X and $q \in \mathbb{Q}$ with $F \subseteq f^{\leftarrow}\{q\}$. If F is countable, then F is compact since $f^{\leftarrow}\{q\}$ is countably compact. (Recall that f is quasi-perfect.) In that case F has a compact countable neighborhood, because of (3), hence then F has a (countable) neighborhood base of clopen neighborhoods. So assume F is not countable. Then $f^{\leftarrow}\{q\} - F$ is countable, because of (5), hence there is a decreasing collection $\langle U_n \colon n \in \omega \rangle$ of clopen sets in X such that $f^{\leftarrow}\{q\} \cap \bigcap_{n \in \omega} U_n = F$. Let $\langle V_n \colon n \in \omega \rangle$ be a decreasing collection of clopen sets in $\mathbb{Q}$ which is a neighborhood base at q, and for $n \in \omega$ define $W_n = U_n \cap f^{\leftarrow} V_n$. Clearly $\langle W_n \colon n \in \omega \rangle$ is a decreasing collection of clopen sets with $\bigcap_{n \in \omega} W_n = F$. Suppose it is not a neighborhood base of F. Then there is an open H in X with $F \subseteq H$ such that $\forall n \in \omega\ [W_n \not\subseteq U]$. There is an infinite $D \subseteq X - H$ such that $\forall n \in \omega\ [|D - W_n| < \omega]$; then each cluster point of D lies in $X - H \subseteq X - F$ or in $\bigcap_{n \in \omega} W_n = F$, hence D has no cluster points. We may assume $|f^{\rightarrow} D| = 1$ or $|f^{\rightarrow} D| = \omega$. But then $f^{\rightarrow} D$ converges to D since $\forall n \in \omega\ [|D - f^{\leftarrow} V_n| < \omega]$. Since f is quasi-perfect it follows from Lemma 13.5 that D has a cluster point. This contradiction proves that $\langle W_n \colon n \in \omega \rangle$ is a neighborhood base for F in X. $\square$

Notes to Section 13

The constructions of this section are based on Ostaszewski's technique.

Example 13.1 was constructed from $\mathfrak{c} = \omega_1$ by Rajagopalan and Woods in a needlessly complicated manner; VAUGHAN [1976] shows one can use Ostaszewski's technique, and after being informed of that result I showed one can weaken '$\mathfrak{c} = \omega_1$' to '$\mathfrak{b} = \mathfrak{c}$'. Nyikos and Vaughan have a consistent example with $\mathfrak{t}^+$ factors, and $\mathfrak{t}$ and $\mathfrak{c}$ anything reasonable; cf. 6.9.

After being informed of Exercise 13.3 Peter Nyikos has come up with another consistent example which we describe with his permission: Take the space $X = \omega \times \omega \cup \mathscr{B}$ of Example 7.3. Topologize $Y = X \cup \omega$ as follows: X is an open subspace of Y, and a basic neighborhood of $k \in \omega$ has the form $\{k\} \cup \{k\} \times [n, \omega)$, with $n \in \omega$. Then ω is closed discrete in Y, and no infinite discrete subset of $Y - \omega$ is closed discrete in Y, because of (1) of 7.3, hence Y is pseudocompact. Clearly Y is first countable iff $\mathfrak{b} = \omega_1$.

Theorem 13.4 (for the rationals) was proved by VAN DOUWEN [1975] from $\mathfrak{c} = \omega_1$; my attempts to eliminate, or at least weaken, the assumption $\mathfrak{c} = \omega_1$ were my motivation for discovering the results of Section 12, which show one can get by with $\mathfrak{b} = \mathfrak{c}$.

Example 13.7 is published here for the first time.

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CHAPTER 4

Box Products

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

Introduction

The topology first considered [TIETZE 1923] on the set product of infinitely many spaces is that which is currently named the *box topology*. As this topology proved inefficient at preserving complex properties (e.g., compactness) central to the field it was soon replaced by the successful topology of Tychonov. Thus, the study of box topologies remained in infancy for nearly a half century.

The discovery [RUDIN 1977a] that box topologies could provide the site for important counter-examples engendered today's scrutiny of the subject. We now know that box products are never metrizable [1.3] and often have a non-normal subspace (2.6). We know that a box product of a 'small' family of realcompact spaces is realcompact (1.15), but the box product of a countable family of compact spaces need not be normal (5.3). However, we have only partial answers (3.6, 5.7, 6.6) to the simplest questions, such as 'Is the box product of countably many copies of the real line, or the unit interval, a paracompact space?'

1. Preliminaries and elementary results

Suppose $(X(i)\colon i \in I)$ is a family of sets. Then we designate the Cartesian set-product of this family by $\Pi_i X(i)$. If, in addition, each $X(i)$ is identical to a fixed set, say X, we write ${}^I X$ (the set of functions from I into X) for $\Pi_i X(i)$. $\pi_j\colon \Pi_i X(i) \to X(j)$ will be the projection map, however, given $x \in \operatorname{dom}(\pi_j)$ we often write $x(j)$ for $\pi_j(x)$. Whenever each $X(i)$ is a topological space, $T_i X(i)$ (and $T^I X$) designate $\Pi_i X(i)$ as possessing the Tychonov product topology.

1.1. DEFINITION. Suppose $(X(i); i \in I)$ is a family of spaces. Then a set of the shape $\Pi_i G(i)$ is called an *open box* of $\Pi_i X(i)$ whenever $G(i)$ is an open set of $X(i)$ $\forall i \in I$. The (full) *box topology* on $\Pi_i X(i)$ is the topology generated by the base of all open boxes. $\Box_i X(i)$ or $\Box^I X$ designates $\Pi_i X(i)$ with the box topology. (If partial box topologies are of interest, the reader should see [VAN DOUWEN 1980].)

We make several conventions: *The indexing set for a product is infinite*; and, unless modified with 'topological' the term space means *infinite non-discrete Hausdorff and completely regular topological space.*

1.2. PROPOSITION. *Suppose $(X(i)\colon i \in I)$ is a family of topological spaces. Then*

(i) $\pi_j\colon \Box_i X(i) \to X(j)$ *is continuous and open for each* $j \in I$.

(ii) $\Pi_i F(i)$ *is a closed (open) subset of* $\Box_i X(i)$ *iff for each* $i \in I$ $F(i)$ *is a closed (open) subset of* $X(i)$.

(iii) $\Box_i X(i)$ *is Hausdorff (regular, completely regular) iff for each* $i \in I$ $X(i)$ *is Hausdorff (respectively, regular, completely regular).*

PROOF. (i) is an immediate consequence of 1.1.

(ii) Use de Morgan's laws and the identity

$$\prod_i (A(i) \cap B(i)) \subseteq \Big(\prod_i A(i)\Big) \cap \Big(\prod_i B(i)\Big). \tag{1}$$

(iii) For "Hausdorff" use (ii) and (1). For "regular" use (ii) and the inequality

$$\prod_i A(i) \subseteq \prod_i B(i) \quad \text{iff} \quad \text{for each } i \in I \quad A(i) \subseteq B(i). \tag{2}$$

'Completely regular' has a straight-forward proof; however, we will find it to be a consequence of 1.8(i) and 1.10. □

Since the product of just two normal spaces may fail to be normal, we can expect difficulties with preservation of properties stronger than completely regular.

1.3. THEOREM. *For a family* $(X(i): i \in I)$ *of spaces* $\square_i X(i)$ *is not any of the following*: (i) *locally compact,* (ii) *separable,* (iii) *connected or locally connected,* (iv) *first countable,* (v) *perfect* ($\equiv$ *each closed set is a* G_δ*-set*).

PROOF. Let $\square$ designate $\square_i X(i)$ and choose $p \in \square$ such that $p(i)$ is not an isolated point of $X(i)$ for each $i \in I$.

(i) Given an open box $\Pi_i G(i)$ containing p, we may find $x(i) \in G(i) - \{p(i)\}$ $\forall i \in I$. Since each $X(i)$ is regular Hausdorff, there is $\forall i \in I$ an open set $H(i)$ of $X(i)$ such that $p(i) \in H(i) \subseteq H(i)^- \subseteq G(i) - \{x(i)\}$. Therefore,

$$\mathscr{H} = \Big\{\prod_i K(i): K(i) \in \{H(i), G(i) - H(i)^-\} \forall i \in I\Big\}$$

is a pairwise-disjoint open covering of the uncountable closed set

$$\{y \in \square: y(i) \in \{p(i), x(i)\} \forall i \in I\}.$$

(ii) If $\square$ has a countable dense set, then the family $\mathscr{H}$ of (i) would be countable.

(iii) Again suppose G is an open box containing p. For each $i \in I$ there is a family $\{G(i, n): n \in \omega\}$ of open sets of $X(i)$ such that

$$p(i) \in G(i, n+1)^- \subsetneqq G(i, n) \subseteq G(i) \quad \forall n \in \omega.$$

Let $\varphi: \omega \to I$ be an injection, and define a closed box E_k^m, for $k, m \in \omega$, by

$$E_k^m(i) = \begin{cases} G(i, n+k+1)^- & \text{if } \varphi(n) = i \text{ and } m < n, \\ G(i, 1)^- & \text{otherwise}. \end{cases}$$

Let $E_k = \bigcup_{m \in \omega} E_k^m$ and $E = \bigcap_{k \in \omega} E_k$. Then $p \in E \subseteq G$.

$\square - E$ is open because it is the union of $\square - \Pi_i G(i, 1)^-$ with all boxes of the shape

$$(X(j) - G(j, n+1)^-) \times \prod_{i \neq j} X(i) \quad \text{with } j \in \varphi(\omega).$$

We show E is also open.

Let $x \in E$ and define an open box U by

$$U(i) = \begin{cases} \bigcap\{G(i, n+k+1) \colon k \leq n,\ x(i) \in G(i, n+k+1)\} \\ \text{if } i = \varphi(n) \text{ and } x(i) \in G(i, 2), \\ G(i, 1), \quad \text{otherwise}. \end{cases}$$

Then $x \in U$. Further, given $k \in \omega$, $x \in E_{k+2}$. So $\exists m(k) \in \omega$ such that $\varphi(n) = i$ and $m(k) \leq n$ implies $x(i) \in G(i, n+k+3)$. Hence, $U \subseteq E_k^{m(k)} \forall k \in \omega$, and $U \subseteq E$.

(iv) Suppose $\{\Pi_i G(i, n) \colon n \in \omega\}$ and E are as given in (iii). Then $\Pi_i G(i, n) \not\subseteq E$ $\forall n \in \omega$.

(v) It will be sufficient (and technically easier) to show this claim in the case $I = \omega$. Here we let $G(i) = X(i) - \{p(i)\}$ $\forall i \in I$ and show $G = \Pi_i G(i)$ is not an F_σ-set.

Suppose F_n is closed in $\square$ $\forall n \in \omega$, and $F_n \subseteq F_{n+1} \subseteq G \forall n \in \omega$. From 1.2 (iii) $\square$ is regular, so there is an open neighborhood $G(i, 0)$ of each $p(i)$ such that $F_0 \cap \Pi_i G(i, 0) = \emptyset$. Since no coordinate of p is isolated $\exists x_0 \in \square$ such that $x_0(0) \in G(0, 0) - \{p(0)\}$ and $\forall i > 0$ $x_0(i) = p(i)$. Similarly we may construct, inductively on $n \in \omega$, open boxes $\Pi_i G(i, n)$ and points $x_n \in \square$ subject to the restrictions:

(1) $\Pi_i G(i, n)$ is a neighborhood of x_{n-1} disjoint from F_n,

(2) $G(i, n)^- \subseteq G(i, n-1)$ $\forall i, n \in \omega$, and

(3) $\forall i, n \in \omega \quad x_n(i) \in \begin{cases} G(i, n) - \{p(i)\} & \text{if } n = i, \\ \{x_{n-1}(i)\}, & \text{otherwise}. \end{cases}$

If $x(i) = x_i(i)$ $\forall i \in \omega$, then $x \in (\bigcap_n \Pi_i G(i, n)) - (\bigcup_n F_n)$. $\square$

The notions of uniform continuity, uniform convergence, and completeness are central to much mathematics. Consequently they have supported numerous extensions of the notion of distance. The most successful of these extensions (at least as a tool for topics exterior to General Topology) is the 'uniformity'.

Our motivation for introducing the uniformity is to aid in the study of paracompactness. However, we will discover that 'complete uniform space' is the richest general structure preserved by box products. A few standard facts about uniformities have been included for the non-topologist reader.

1.4. NOTATION. For a set X we make the following conventions:

(i) $X^2 = X \times X$, and $\{(x, y) \in X^2 : x = y\}$ is called the *diagonal* of X.

(ii) For $x \in X$ and $D \subseteq X^2$, $D[x] = \{y \in X : (x, y) \in D\}$.

(iii) For $D \subseteq X^2$, $D^{-1} = \{(y, x) : (x, y) \in D\}$. D is *symmetric* if $D = D^{-1}$.

(iv) For $C, D \subseteq X^2$, $C \circ D = \{(x, z) \in X^2 : \exists y \in x, (x, y) \in C, (y, z) \in D\}$.

1.5. DEFINITION. Suppose X is a set. Then a filter $\mathscr{D}$ on X^2 is called a (Hausdorff) *uniformity* on X whenever each of (i) through (iii) is satisfied:

(i) $\bigcap \mathscr{D}$ is the diagonal of X.

(ii) $D \in \mathscr{D} \Rightarrow \exists C \in \mathscr{D}, C^{-1} \subseteq D$.

(iii) $D \in \mathscr{D} \Rightarrow \exists C \in \mathscr{D}, C \circ C \subseteq D$.

A *uniform space* is a pair $(X, \mathscr{D})$ consisting of a set X and a uniformity $\mathscr{D}$ on X. A filterbase on X^2 satisfying (i) through (iii) is called a *uniformity base* (abbreviated *unif. base*) on X. Observe that unif. base can be characterized as a co-final, under $\supseteq$, subset of some uniformity. The *uniform topology* on X induced by a unif. base $\mathscr{D}$ is the set

$$\tau(\mathscr{D}) = \{G \subseteq X : x \in G \Rightarrow \exists D \in \mathscr{D}, D[x] \subseteq G\}.$$

A *normal sequence* in a unif. base $\mathscr{D}$ is a family $\langle D(n) : n \in \omega \rangle$ of members of $\mathscr{D}$ such that $\forall n \in \omega D(n+1) \circ D(n+1) \subseteq D(n) \cap D(n)^{-1}$.

1.6. BASIC FACTS I. *Suppose X is a set and $\mathscr{D}$ is a uniformity on X. Then the following hold*:

(i) *$\tau(\mathscr{D})$ is a Hausdorff topology on X.*

(ii) *If $\mathscr{E}$ is cofinal in $(\mathscr{D}, \supseteq)$, than $\tau(\mathscr{E}) = \tau(\mathscr{D})$.*

(iii) *$\{D \in \mathscr{D} : D$ is symmetric and open in $T^2(X, \tau(\mathscr{D}))\}$ is cofinal in $(\mathscr{D}, \supseteq)$, and hence, is a unif. base on X).*

(iv) *If $Y \subseteq X$ and $\mathscr{D} \restriction Y = \{D \cap Y^2 : D \in \mathscr{D}\}$, then $\tau(\mathscr{D} \restriction Y)$ is the subspace topology on Y induced by $\tau(\mathscr{D})$).*

1.7. DEFINITION. Suppose X is topological space. A unif. base $\mathscr{D}$ on X is said to be compatible with X provided $\tau(\mathscr{D})$ is the topology of X. If X possesses a compatible unif. base, we define the *uniform weight* $uw(X)$ to be the least infinite cardinality of some compatible unif. base on X.

1.8. BASIC FACTS II. *For a topological space X the following hold*:

(i) *X possesses a compatible (Hausdorff) unif. base iff X is completely regular (and Hausdorff).*

(ii) *X is metrizable iff X has a compatible normal sequence.*

(iii) *If X is completely regular, then $w(X) = uw(X) \cdot L(X)$ ($w(X)$ and $L(X)$* are, respectively the weight and Lindelöf number of X, see [HODEL 1982]).

We consider uniformities in box products.

1.9. DEFINITION. Suppose that $\forall i \in I \mathscr{D}(i)$ is a compatible unif. base on a space $X(i)$. If $\forall i \in I\ D(i) \in \mathscr{D}(i)$, we define

$$\square_i D(i) = \left\{(x, y) \in \left(\prod_i X(i)\right)^2 : \forall i \in I\ (x(i), y(i) \in D(i)\right\},$$

and for each $j \in I$ we let $(\square_i D(i))(j) = D(j)$. So $\square_i \mathscr{D}(i)$, called the *box product* of $(\mathscr{D}(i): i \in I)$, will be the set of all $\square_i D(i)$.

1.10. THEOREM. *If, for each $i \in I$, $\mathscr{D}(i)$ is a compatible unif. base on a space $X(i)$, then $\square_i \mathscr{D}(i)$ is a compatible unif. base on $\square_i X(i)$.*

PROOF. This is an immediate consequence of the following applications of equations (1) and (2) in the proof of 1.2:

(1) $D \in \square_i \mathscr{D}(i) \Rightarrow (D^{-1})(i) = D(i))^{-1} \forall i \in I$.

(2) $D, E \in \square_i \mathscr{D}(i) \Rightarrow (D \circ E)(i) = D(i) \circ E(i)\ \forall i \in I$.

(3) $D \in \square_i \mathscr{D}(i)$ and $x \in \Pi_i X(i) \Rightarrow D[x] = \Pi_i D(i)[x(i)]$. $\square$

A similar proof yields:

1.11. COROLLARY. *If, for each $i \in I$, $X(i)$ is a topological group, then $\square_i X(i)$ is a topological group.*

The extension of completeness to uniform spaces is natural. For example, a filter $\mathscr{F}$ on X is said to be *$\mathscr{D}$-Cauchy* (where D is compatible unif. base on X) provided $D \in \mathscr{D} \Rightarrow \exists x \in X$ with $D[x] \in \mathscr{F}$. Thus, $\mathscr{D}$ is *complete* whenever every $\mathscr{D}$-Cauchy filter coverages ($\equiv$ contains a neighborhood base at some point). A space is *topologically complete* whenever it possesses a compatible complete unif. base.

1.12. COROLLARY. *If, for each $i \in I$, $X(i)$ is topologically complete, then $\square_i X(i)$ is topologically complete.*

PROOF. Suppose that for each $i \in I$, $\mathscr{D}(i)$ is a compatible complete uniformity on $X(i)$. In order to see $\square_i \mathscr{D}(i)$ is complete we suppose $\mathscr{F}$ is a $\square_i \mathscr{D}(i)$-Cauchy filter on $\square_i X(i)$. For each $i \in I$ define

$$\mathscr{F}_i = \{F \subseteq X(i): \pi_i^{-1}(F) \in \mathscr{F}\}.$$

According to 1.10 (3), $\mathscr{F}_i$ is a $\mathscr{D}(i)$-Cauchy filter on $X(i)$ $\forall_i \in I$. Let $x \in \Pi_i X(i)$ such that $\forall i \in I$ $\mathscr{F}_i$ converges to $x(i)$. For $D \in \Box_i \mathscr{D}(i)$ there is a symmetric $E \in \Box_i \mathscr{D}(i)$ such that $E \circ E \circ E \subseteq D$. Since $\mathscr{F}$ is $\Box_i \mathscr{D}(i)$-Cauchy, $\exists y \in \Pi_i X(i)$ with $E[y] \in \mathscr{F}$. From 1.10 (3) and 1.2 (1), $E[y] \cap E[x] \neq \emptyset$. So $E[y] \subseteq D[x]$. Therefore, $\mathscr{F}$ converges to x. $\Box$

Within the class of topologically complete spaces we have the *realcompact spaces* ($\equiv$ closed subspaces of Tychonov products of copies of the real line). Before proving a 1.12-like result for this class we observe the connection between realcompactness and measureable cardinals.

1.13. DEFINITION. An uncountable cardinal κ is said to be *measureable* provided there is a $\{0, 1\}$-valued measure m whose domain is the power set of κ such that $\mathrm{m}(\kappa) = 1$, $\forall \alpha \in \kappa$ $\mathrm{m}(\{\alpha\}) = 0$, and such that each union of less than κ sets of measurable 0 has measure 0.

If $V = L$, then there are no measurable cardinals [BELL–SLOMSON 1969, pg. 300–305]. However, the consistency of 'there is a measurable cardinal' has not been proved. Pertinant to our needs in Section 4 is: If κ is a measurable cardinal and if $\lambda < \kappa$, then $2^\lambda < \kappa$,

1.14. LEMMA. *For a space X, the following are equivalent:*

(i) *X is realcompact.*

(ii) *Each zero-set ultrafilter on X closed under countable intersections converges in X.*

(iii) *X is topologically complete and contains no closed discrete subspace of measurable cardinality.*

1.15. THEOREM. *Suppose that, for each $i \in I$, $X(i)$ is realcompact. Then $\Box_i X(i)$ is realcompact iff $|I|$ contains no measurable cardinals.*

PROOF. Suppose $J \subseteq I$ and $|J|$ is measurable. Then from the proof of 1.3 (i), $\Box_{i \in J} X(i)$, and hence $\Box_{i \in I} X(i)$ possesses a closed discrete subspace of cardinality $2^{|J|}$. Thus, 1.14(iii) shows $\Box_i X(i)$ is not realcompact.

Now let us suppose $|I|$ is less than the first (if it exists) measurable cardinal, and by way of contradiction, $\Box_i X(i)$ is not realcompact. According to 1.12 and 1.14(iii) there is a closed discrete subspace $K \subseteq \Box_i X(i)$ of measureable cardinality. If m is the measure on K given in 1.13, then set $u = \{A \subseteq K\colon \mathrm{m}(A) = 1\}$. Clearly, u is closed under intersections of $|I|$ many of its members, and, since K is closed and discrete, u does not converge in $\Box_i X(i)$.

Let $v = \{Z\colon Z$ is a $\Box_i X(i)$ zero-set and $Z \cap K \in u\}$. Then v is a non-convergent zero-set ultrafilter on $\Box_i X(i)$ closed under intersections of $|I|$ many of its members. We consider $w = \{Z \in v\colon Z$ is a $T_i X(i)$ zero-set$\}$.

Since countable intersections of zero-sets are zero-sets, w is closed under countable intersections. Therefore given f: $T_iX(i) \to \mathbb{R}$ a continuous map satisfying $f^{-1}(0) \in w$, there is in $r \in \mathbb{R}$ $r > 0$ such that $f^{-1}([-r, r]) \notin w$. As the identity from $\Box$ to T is continuous, $f^{-1}(\mathbb{R} - [-r, r]) \in v$, and hence, belongs to w. So w is a zero-set ultrafilter on $T_iX(i)$ converging to, say x.

Suppose G is an open box containing x. Then for $i \in I$, there is $Z_i \in v$ with $Z_i \subseteq \pi_i^{-1}(G(i))$. Since $\bigcap_i Z_i \in v$ and $\bigcap_i Z_i \subseteq \Pi_i G(i)$, we see, contradictorally, that v converges in $\Box_i X(i)$ to x. $\Box$

1.16. NOTES. 1.2 is proved in [KNIGHT 1964]. 1.3 (except (v)) can be found in [RUDIN 1975], while 1.3 (v) is in [VAN DOUWEN 1975]. Adequate details on uniformities are given in [KELLY 1955]; however, [ISBELL 1964] is more complete and suggested, implicitly, 1.10. Page 36 of [JUHASZ 1971] contains 1.8(iii). 1.11 is in [VAN DOUWEN 1980] as an alternative to proving 1.2(iii). 1.12 and 1.15 is first proved in [KATO 1979]. For a proof of 1.14 see [COMFORT–NEGREPONTIS 1975].

2. Paracompact spaces

Since its inception as a tool of Analysis, paracompactness has received more attention from topologists than any other single property. Therefore, questions concerning its preservation in box products is one of primary interest.

2.1. DEFINITION. Suppose that X is a space and $\mathscr{S}$ is a collection of subsets of X. $\mathscr{S}$ is *locally-finite* if for each $x \in X$ there is a neighborhood G_x of x intersecting at most finitely many members of $\mathscr{S}$. $\mathscr{S}$ is *σ-locally-finite* if it is the countable union of locally-finite collections. $\mathscr{S}$ *refines* (or is a refinement of) a collection of $\mathscr{R}$ is sets if $\bigcup \mathscr{S} = \bigcup \mathscr{R}$ and if for each $S \in \mathscr{S}$ there is an $R_S \in \mathscr{R}$ with $S \subseteq R_S$. If $\mathscr{S}$ refines $\mathscr{R}$ and $\mathscr{S} = \{D[x]: x \in X\}$ for some $D \subseteq X^2$, we also say that D *refines* $\mathscr{R}$.

A space X is said to be *paracompact* provided each open covering of X has an open locally-finite refinement.

There are, at least, twenty non-trivial characterizations of paracompactness. For our purposes we need only two.

2.2. THEOREM. *For a space X, the following are equivalent*:

(i) *X is paracompact.*

(ii) *Each open covering of X has an open σ-locally-finite refinement.*

(iii) *Each open covering of X is refined by an entourage ($\equiv$ an X^2 neighborhood of the diagonal).*

2.3. PROPOSITION. *Suppose that $(X(i)$: $i \in I)$ is a family of paracompact spaces. Then $\Box_i X(i)$ is topologically complete.*

PROOF. This is an immediate consequence of 1.12 and a lemma whose proof allows practice with entourages:

2.4. LEMMA. *If X is a paracompact space, then the filter of entourages of X is a complete uniformity compatible with X.*

PROOF. Let $\mathcal{N}$ be the filter of entourages of X. We first show $\mathcal{N}$ is a uniformity. Suppose $D \in \mathcal{N}$. For each $x \in X$ choose an open neighborhood G_x of x such that $G_x^2 \subseteq D$. From 2.2(iii) there is an $E \in \mathcal{N}$ refining $\{G_x\colon x \in X\}$. Let $C = E \cap E^{-1}$. Then C satisfies 1.5(ii). Now suppose $(x, y), (y, z) \in C$. Then there is a $w \in X$ such that $x, z \in C[y] \subseteq G_w$. So $(x, z) \in D$. Then C satisfies 1.5(iii).

In order to see that $\mathcal{N}$ is compatible with X. First observe that $D[x]$ is open whenever $D \in \mathcal{N}$ is open (in X^2). On the other hand, given an open neighborhood G of $y \in X$, we find $F[y] \subseteq G$, where

$$F = G \times G \cup (X - \{y\}) \times (X - \{y\}).$$

Finally, we suppose $\mathcal{F}$ is a non-convergent filter on X. Then, by definition, there is for each $y \in X$ a neighborhood G_y of y such that $G_y \notin \mathcal{F}$. According to 2.2(iii) $\exists D \in \mathcal{N}$ refining $\{G_x\colon x \in X\}$. Clearly $\mathcal{F}$ is not $\mathcal{N}$-Cauchy. □

If the filter of entourages of a space X is a uniformity, then (as shown in the proof of 2.4) it is compatible with X. A similar proof shows that X will also be a normal space. However, we cannot expect paracompactness, in general, to be preserved by box products since the square of a Lindelöf space may not be normal [PRZYMUSINSKI 1982].

It is now standard to use the adverb 'hereditarily' to modify a property whenever we wish to say each subspace has that property. Instances of hereditarily paracompact box products are particularly pathological, as revealed below.

2.5. PROPOSITION. *Suppose $(X(i)\colon i \in I)$ is a family of spaces such that $\square_i X(i)$ is hereditarily normal. Then $\square_i X(i)$ is a P-space ($\equiv$ every G_δ-set is open).*

PROOF. Fix $j \in I$ and set $\square' = \square_{i \neq j} X(i)$. Then associativity of products allows $\square_i X(i) = X(j) \times \square'$. According to 1.3(v) $\square'$ has a closed set not a G_δ-set. Now apply the theorem of Katětov proved in [PRZYMUSINSKI 1982], 'If $X \times Y$ is hereditarily normal, then either X is a P-space or each closed subspace of Y is a G_δ-set'. Obviously, a box product of P-spaces is a P-space □

2.6. COROLLARY. *Suppose $(X(i)\colon i \in I)$ is a family of compact or first countable spaces. Then $\square_i X(i)$ has a non-normal subspace.*

Even a product of two hereditarily paracompact P-spaces may have a non-normal subspace. [For $i \in \{1,2\}$, suppose that X_i is the space generated on the ordinal $\omega_i + 1$ by adding to the order topology all bounded sets of ω_i. Then $\{(\alpha, \omega_2): \alpha \in \omega_1\}$ and $\{(\omega_1, \beta): \beta \in \omega_2\}$ cannot be separated in $X_1 \times X_2 - \{\omega_1, \omega_2)\}$.] So we introduce a special class of hereditarily paracompact P-spaces which will also be of importance in the study of the box product of compact metrizable spaces (see 5.6).

2.7. DEFINITION. Suppose κ is an infinite regular cardinal. A space X is said to be *κ-metrizable* whenever X possesses a compatible unif. base $\mathscr{D}$ such that $(\mathscr{D}, \subseteq)$ has order type κ. Observe that a space X is κ-metrizable iff $uw(X) \leqslant \kappa$ and $\bigcap_i G_i$ is open whenever $(G_i: i \in I)$ is a family of open sets of X with $|I| < \kappa$. Therefore, κ-metrizable $\equiv$ hereditarily κ-metrizable (see 1.6(iv)). Further, 1.8(ii) shows metrizable $\equiv \aleph_0$-metrizable.

2.8. LEMMA. *A κ-metrizable space is paracompact.*

PROOF. Since A.H. Stone's theorem is standard [KELLY 1955, pg. 160], we assume κ is also an uncountable regular ordinal. Suppose that X is a space with a compatible unif. base $\mathscr{D} = \{D_\alpha: \alpha \in \kappa\}$ such that $\beta < \alpha \Rightarrow D_\alpha \subseteq D_\beta$. Then, by 1.5(ii) and (iii), $\forall \alpha \in \kappa\ \exists \langle D_\alpha(n): n \in \omega \rangle$ a normal sequence in $\mathscr{D}$ such that $D_\alpha(0) = D_\alpha$. Set $E_\alpha = \bigcap \{D_\beta(n): \beta \leqslant \alpha,\ n \in \omega\}$. Then for each $\alpha \in \kappa$. E_α is an equivalence relation on X. Further, since κ is regular uncountable there exists a $\lambda \in \kappa$ with $D_\lambda \subset E_\alpha$. Therefore $\{E_\alpha: \alpha \in \kappa\}$ is a unif. base compatible with X.

Suppose $\mathscr{R}$ is an open covering of X. Then $\mathscr{R}$ is refined by a collection

$$\mathscr{S} \subseteq \{E_\alpha[x]: \alpha \in \kappa, x \in X\}.$$

For each $S \in \mathscr{S}$ define $G_S = \bigcup \{S' \in \mathscr{S}: S \subseteq S'\}$. Since $\beta < \alpha < \kappa \Rightarrow E_\alpha \subseteq E_\beta$, and since each E_α is an equivalence relation, $G_S \cap G_{S'} \neq \emptyset$ iff $G_S = G_{S'}$, whenever $S, S' \in \mathscr{S}$. Therefore $\{G_S: S \in \mathscr{S}\}$ is a pairwise-disjoint refinement of $\mathscr{R}$. $\square$

2.9. THEOREM. *Suppose κ is a regular cardinal and for each $i \in I$ $X(i)$ is a κ-metrizable space. If $|I| < \kappa$, then $\square_i X(i)$ is κ-metrizable, and therefore, hereditarily paracompact.*

PROOF. Of course the 'and therefore' follows from 2.8, so we suppose that for each $i \in I$ $\mathscr{D}(i) = \{D_\alpha(i): \alpha \in \kappa\}$ is a unif. base compatible with $X(i)$ such that $\beta < \alpha \Rightarrow D_\alpha(i) \subseteq D_\beta(i)$. If $|I| < \kappa$, then $\{\square_i D_\alpha(i): \alpha \in \kappa\}$ is cofinal in $\square_i \mathscr{D}(i)$ and the theorem follows from 1.10. $\square$

Although we now understand that $\square_i X(i)$ is never metrizable (1.3)(iv) or (v)), we might conjecture that $\square_{i \in \omega} X(i)$ is paracompact whenever $X(i)$ is metrizable for each $i \in \omega$. Such a conjecture was made and unanswered for nearly a decade.

2.10. EXAMPLE. *There is a family* $(X(i): i \in \omega)$ *of separable complete metric spaces such that* $\square_i X(i)$ *is not normal.*

PROOF. We consider two subspaces of the real line,

$$R = \left\{\frac{1}{n+1} : n \in \omega\right\} \cup \{0\} \quad \text{and} \quad L = \{-r : r \in R\}.$$

d will denote a metric compatible with the compact metrizable space $T^{\omega}R$. P will be $\{t \in T^{\omega}R : t(n) \neq 0 \ \forall_n \in \omega\}$ considered as a subspace of $T^{\omega}R$. Then P is homeomorphic to the space of irrationals [ENGLEKING 1977, pg. 348], and so it is separable and completely metrizable. L is, of course, homeomorphic to the space $\omega + 1$. The actual box product we consider is the space $X = P \times \square^{\omega}L$.

For $t \in {}^{\omega}R$, $-t \in {}^{\omega}L$ is defined by $(-t)(n) = -(t(n))$ $\forall n \in \omega$. Let $F = \{(p, -p) : p \in P\}$ and $G = P \times {}^{\omega}(L - \{0\})$. Clearly F is closed in X, G is open in X, and $F \subseteq G$. We show that there do not exist disjoint open sets separating F and $X - G$.

Suppose, by way of contradiction, there is an openset U of X such that $F \subseteq U \subseteq U^- \subseteq U$. For each $n \in \omega$ define

$$U_n = \{-p : p \in P \text{ and } \{t \in P : d(p, t) < 2^{-n}\} \times \{-p\} \subseteq U\}.$$

Then $\bigcup_n U_n = {}^{\omega}(L - \{0\})$, and $n < m \Rightarrow U_n \subseteq U_m$. Following the proof of 1.3(v), we see that ${}^{\omega}(L - \{0\})$ is not an F_σ-set of $\square^{\omega}L$. Therefore, $\exists m \in \omega\ \exists a \in {}^{\omega}R$ such that $-a \in U_m^- - {}^{\omega}(L - \{0\})$ in $\square^{\omega}L$.

Since $\square^{\omega}R$ and $\square^{\omega}L$ are homeomorphic by $t \mapsto -t$,

$$\text{in } T^{\omega}R, \quad a \in \{t : -t \in U_m\}^- - P. \tag{1}$$

Now P is dense in $T^{\omega}R$ so there exists $b \in P$ with $d(a, b) < 2^{-m-1}$. Because $(b, -a) \notin G$, there exists $V \subseteq \square^{\omega}L$ an open box such that $-a \in V$ and there is $k \geq m + 1$ such that

$$U^- \cap (\{p \in P : d(p, b) < 2^{-k}\} \times V) = \emptyset. \tag{2}$$

Since $\{-p : p \in P\}$ is dense in $T^{\omega}R$, (1) implies there exists $c \in P$ such that $d(b, c) < 2^{-k-1}$ and $-c \in U_{k+1} \cap V$. Therefore,

$$\{p \in P : d(p, c) < 2^{-m}\} \times \{-c\} \subseteq U. \tag{3}$$

However, $d(a, c) < d(a, b) + d(b, c) < 2^{-m}$. So (3) implies $(a, -c) \in U$. Thus, (2) is contradicted. $\square$

2.11. NOTES. [KELLY 1955] contains the standard 2.2 (on pages 155–157) and the problem 2.4 (on pg. 208). Also see [BURKE 1982]. 2.5 was noticed in [VAN DOUWEN 1975]. The notion of κ-(or rather $\aleph_\mu$-) metrizability is due to Hausdorff; however, recent characterizations of this property can be found in [STEVENSON–THRON 1969]. Weak versions of 2.9 appeared in [RUDIN 1971b] and [VAUGHAN

1975], while its present form is noticed in [VAN DOUWEN 1977]. 2.10 comes from [VAN DOUWEN 1975]. For efforts to study properties weaker or parallel to paracompactness and normality in box products, see [VAN DOUWEN 1980].

3. The Nabla Lemma (for countable products only)

Since all but one of the factors of the Example 2.10 are compact, one begins to question whether a box product of non-pathological spaces can ever be paracompact. In case the indexing set is uncountable and the factors are locally compact metrizable this question remains unanswered under any set-theoretic assumption. However, the situation is different when there are only countably many factors. In this section we introduce an object, the nabla product, which allows for a considerable simplification of proof.

3.1. DEFINITION. Suppose $(X(i): i \in \omega)$ is a family of spaces. Define a relation $\equiv$ on $\Pi_i X(i)$ by $x \equiv y$ iff $\{i: x(i) \neq y(i)\}$ is finite. The relation $\equiv$ is an equivalence which induces a quotient space, denoted by $\nabla_i X(i)$ (or $\nabla^\omega X$, if $X(i) = X\ \forall i \in \omega$), of $\square_i X(i)$. $\nabla_i X(i)$ is called the *nabla product* of the $X(i)$'s. Let $q: \square_i X(i) \to \nabla_i X(i)$ designate the quotient map.

3.2. LEMMA. *Suppose $(X(i): i \in \omega)$ is a family of spaces and for each $i \in \omega$ $A(i)$, $B(i) \subseteq X(i)$. Then*

(i) *$q(\Pi_i A(i)) \subseteq q(\Pi_i B(i))$ iff $\{i: A(i) \not\subseteq B(i)\}$ is finite.*

(ii) *$q(\Pi_i(A(i))$ is open (or closed) iff there exists $n \in \omega$ with $A(i)$ open (respectively, closed) in $X(i)$ for each $i > n$.*

(iii) *If $A(i)$ has the subspace topology for each $i \in \omega$, then $h: a \to a$ is a homeomorphism from $\nabla_i A(i)$ onto $q(\Pi_i A(i))$, where the latter has the subspace topology inherited from $\nabla_i X(i)$.*

(iv) *$\nabla_i X(i)$ is a P-space.*

PROOF. All advertisements (i)–(iii) have direct proofs. (iv) follows from the observation: If $\forall i \in \omega\ \langle A(i, n): n \in \omega\rangle$ is a sequence such that $A(i, n+1) \subseteq A(i, n) \subseteq X(i)\ \forall n \in \omega$, then $q(\Pi_i A(i, i)) \subseteq \bigcap_n q(\Pi_i A(i, n))$. $\square$

With the exception of (iv), 3.2 is the nabla version of 1.2. There is a nabla version of 1.9 and 1.10, as well.

3.3. DEFINITION. Suppose that for each $i \in \omega$ $\mathscr{D}(i)$ is a compatible unif. base on a space $X(i)$. For $D \subseteq \Pi_i X(i))^2$ define

$$\mathfrak{q}(D) = \{(q(x), q(y)): (x, y) \in D\}. \text{ Let } \nabla_i \mathscr{D}(i) = \{\mathfrak{q}(D): D \in \square_i \mathscr{D}(i)\}.$$

3.4. LEMMA. *Suppose that for each $i \in \omega$ $\mathscr{D}(i)$ is a compatible unif. base on a space $X(i)$. Then*

(i) *$\nabla_i\mathscr{D}(i)$ is a unif. base compatible with $\nabla_i X(i)$.*

(ii) *The uniformity generated by $\nabla_i\mathscr{D}(i)$ is closed under countable intersections, and hence has a cofinal family of clopen equivalence relations.*

PROOF. (i) This follows immediately from the following three easily established facts. Suppose $C, D \in \Box_i\mathscr{D}(i)$. Then

(1) $\mathfrak{q}(D^{-1}) = \mathfrak{q}(D)^{-1}$.

(2) $\mathfrak{q}(C \circ D) = \mathfrak{q}(C) \circ \mathfrak{q}(D)$.

(3) $\mathfrak{q}(D)[q(x)] = q(D[x])$.

(ii) Suppose $\mathscr{N} = \langle D_n : n \in \omega \rangle \subseteq \Box_i\mathscr{D}(i)$ is a normal sequence. Then (1) and (2) above show $q(\mathscr{N})$ to be a normal sequence in $\nabla_i\mathscr{D}(i)$. So $\bigcap \mathfrak{q}(\mathscr{N})$ is a closed equivalence relation on $\nabla_i X(i)$. Since $\mathfrak{q}(\Box_i D_i(i)) \subseteq \bigcap \mathfrak{q}(\mathscr{N})$, $\bigcap \mathfrak{q}(\mathscr{N})$ is open. The proof is completed by observing that for each collection $\{D_n : n \in \omega\} \subseteq \mathscr{D}$ there is a normal sequence $\langle D'_n : n \in \omega \rangle$ in $\mathscr{D}$ such that $D'_n \subseteq D_n$. $\Box$

The clause (3) of 3.4 demonstrates the quotient map q to be both uniformly continuous and uniformly open, with respect to $\Box_i\mathscr{D}(i)$ and $\nabla_i\mathscr{D}(i)$. When the factors are σ-compact and locally compact, q has two much stronger properties.

3.5. LEMMA. *Suppose that for each $i \in \omega$ $X(i)$ is a σ-compact locally compact space. Then each of the following hold:*

(i) *If D is an entourage of $\Box_i X(i)$, then there is an entourage E of $\nabla_i X(i)$ such that for each $x \in \Pi_i X(i)$ $E[q(x)] \subseteq q(D[x])$.*

(ii) *If $\equiv$ refines an open cover $\mathscr{R}$ of $\Box_i X(i)$, then there is an open cover $\mathscr{S}$ of $\nabla_i X(i)$ such that $q^{-1}(\mathscr{S})$ refines $\mathscr{R}$.*

PROOF. We prove (i) and (ii) simultaneously. Since we could always take $D = (\Pi_i X(i))^2$ or $\mathscr{R} = \{\Pi_i X(i)\}$, we will assume D and $\mathscr{R}$ are given as above.

Because each $X(i)$ is both σ-compact and locally compact, it is covered by a countable family $\mathscr{U}(i)$ of open sets with compact closure. Therefore, we may write $X(i) = \bigcup_n X(i, n)$ such that for each $n \in \omega$ $X(i)$ is open and $X(i, n)^-$ is a compact subset of $X(i, n+1)$. Let $\mathscr{X}$ denote the set of classes of $\equiv$.

Fix $Z \in \mathscr{X}$ and fix $z \in Z$. There is an increasing $f \in {}^{\omega}\omega$ such that $z(i) \in X(i, f(i))$ $\forall i \in \omega$. Now suppose $Z \subseteq R_Z \in \mathscr{R}$. For $j \in \omega$ set

$$L_j = \{x \in \Pi_i X(i) : \forall i > n, x(i) = z(i)\}.$$

Then given $x \in L_j$, there is an open box G_x such that

$$(x, x) \in G_x^2 \subseteq D \cap R_Z^2. \tag{1}$$

Since L_j, as a subspace of $\Box_i X(i)$, is homeomorphic to $T_{i \leqslant j} X(i)$, compactness allows us to choose, for each $n \in \omega$, $\mathscr{G}_{(j,n)} \in [\{G_x: x \in L_j\}]^{<\omega}$ satisfying

$$\{x \in L_j: \forall i \leq j\ x(i) \in X(i,n)^-\} \subseteq \bigcup \mathscr{G}(j,n).$$

We define an open box H_Z as follows: Arbitrarily choose H with $z \in H \in \mathscr{G}_{(0,f(0))}$ and set $H_Z(0) = H(0)$. If $i > 0$, we use the finiteness of $\bigcup\{\mathscr{G}_{(j,n)}: j < i, n \leq f(i)\}$ to define

$$H_Z(i) = \bigcap\{G(i) \cap X(i, f(i)): G \in \mathscr{G}_{(j,n)}, j < i, n \leq f(i)\}. \tag{3}$$

The claim is that by letting Z vary in $\mathscr{Z}$ and by defining

$$E = \bigcup\{q(H_Z)^2: Z \in \mathscr{Z}\} \quad \text{and} \quad \mathscr{S} = \{q(H_Z): Z \in \mathscr{Z}\}$$

will be as advertised.

To see the claim we suppose $q(x)$, $q(y) \in q(H_Z)$. Find a k so large that $i > k$ implies $x(i)$, $y(i) \in H_Z(i)$. Now find $j > k$ so that $i \leq j$ implies $x(i) \in X(i, f(j))$. Choose $u \in L_j$ such that for $i \leq j$ we have $u(i) = x(i)$ and define $v \in \Pi_i X(i)$, so that $v \equiv y$, by

$$v(i) = \begin{cases} x(i) & \text{if } i \leq j, \\ y(i) & \text{if } i > j. \end{cases}$$

According to (2) there is a $G \in \mathscr{G}_{(j,f(j))}$ such that $u \in G$. From (3) $\forall i > j$, $x(i)$, $v(i) \in G(i)$. Applying (1) we find $(x, v) \in D \cap (R_Z^2)$. Therefore, $E[q(x)] \subseteq q(D[x])$ and for each $Z \in \mathscr{Z}$ $q^{-1}q(Z) \subseteq R_Z$. $\square$

We are now ready to exhibit the principal reason for introducing the nabla product: In order to prove $\square_i X(i)$ is paracompact for each family $(X(i): i \in \omega)$ contained in a class of locally compact paracompact spaces, it is necessary and sufficient to prove $\nabla_i K(i)$ is paracompact whenever $K(i)$ is a compact member of the class for each $i \in \omega$. Explicitly there is

3.6. THE NABLA LEMMA. *Suppose that for each $i \in \omega$ $X(i)$ is a paracompact locally compact space. Then the following are equivalent*:

(i) $\square_i X(i)$ *is paracompact.*

(ii) $\nabla_i X(i)$ *is paracompact.*

(iii) $\nabla_i K(i)$ *is paracompact whenever $K(i)$ is a compact subspace of $X(i)$ for each $i \in \omega$.*

PROOF. Applying the standard results, "Every paracompact locally compact space is the topological sum of σ-compact spaces" and "the topological sum of paracompact spaces is paracompact" [KELLY 1955, pg. 215] along with 3.2(iv) allows us to assume, without loss of generality $X(i)$ is σ-compact $\forall i \in \omega$.

(i)$\Rightarrow$(ii). Suppose $\mathscr{R}$ is an open cover of $\nabla_i X(i)$. Since q is continuous, 2.2((i)$\Rightarrow$(iii)) produces an entourage D of $\Box_i X(i)$ refining $q^{-1}(\mathscr{R})$. From 3.5(i) there is an entourage E of $\nabla_i X(i)$ such that for each $x \in \Pi_i X(i)$ $E[q(x)] \subseteq q(D[x])$. Therefore, E refines $\mathscr{R}$. Now apply 2.2 ((iii)$\Rightarrow$(i)).

(ii)$\Rightarrow$(i). Suppose $\mathscr{R}$ is an open cover of $\Box_i X(i)$. Let $\mathscr{Z}$ be the set of classes of $\equiv$. Since each $X(i)$ is σ-compact, Z is σ-compact $\forall Z \in \mathscr{Z}$. So for each $Z \in \mathscr{Z}$ $\exists \mathscr{R}^Z \in [\mathscr{R}]^{\leq\omega}$, $Z \subseteq \bigcup \mathscr{R}^Z$. From 3.5(ii) $\exists$ an open cover $\mathscr{S}$ of $\nabla_i X(i)$ such that $q^{-1}(\mathscr{S})$ refines $\{\bigcup \mathscr{R}^Z : Z \in \mathscr{Z}\}$. Since $\nabla_i X(i)$ is paracompact, 2.2(iii) and 2.4 imply the existence of a normal sequence $\mathscr{N}$ of open entourages of $\nabla_i X(i)$ such that D refines $\mathscr{S} \forall D \in \mathscr{N}$. Since $\nabla_i X(i)$ is a P-space (3.2(iv)), $\bigcap \mathscr{N}$ is an open equivalence relation on $\nabla_i X(i)$. Therefore,

$$\{q^{-1}(\bigcap \mathscr{N}[q(Z)]) \cap R : R \in \mathscr{R}^Z, Z \in \mathscr{Z}\}$$

is a σ-locally finite refinement of $\mathscr{R}$. Now apply 2.2((ii)$\Rightarrow$(i)).

(ii)$\Rightarrow$(iii). If $K(i) \subseteq X(i)$ is compact $\forall i \in \omega$, then 3.2((iii) and (iv)) imply $\nabla_i K(i)$ is homeomorphic to a closed subspace of $\nabla_i X(i)$. So $\nabla_i K(i)$ is paracompact.

(iii)$\Rightarrow$(ii). According to 2.4, the σ-compactness of $X(i)$ implies $\mathscr{D}(i)$, the filter of entourages on $X(i)$, is a compatible uniformity for each $i \in \omega$. Since $X(i)$ is locally compact we may choose $D \in \Box_i \mathscr{D}(i)$ such that $\forall x \in \Pi_i X(i)$ $\forall j \in \omega$ $D(j)[x(j)]$ is compact. From 3.2(iii), and from (3) of 3.4(i) $\mathfrak{q}(D)[q(x)]$ is paracompact $\forall x \in \Pi_i X(i)$. Now 3.4(ii) says $\exists$ a clopen entourage E of $\nabla_i X(i)$ such that $E \subseteq \mathfrak{q}(D)$. Applying (ii)$\Rightarrow$(i) above to each $q^{-1}(q(D[x])$, shows $q^{-1}(E[q(x)])$ is paracompact. From the second quote in the first paragraph $\Box_i X(i)$ is paracompact. $\square$

Unfortunately, the analog of the nabla product for an uncountable index set appears to be valueless for our goal. In fact the corresponding Lemma 3.5 is false.

3.7. NOTES. Although implicit in [RUDIN 1972] and [RUDIN 1974], the isolation of the nabla product and the lemmae 3.2, 3.5, and 3.6 are due to K. KUNEN, see [RUDIN 1975]. The variation in the proofs of 3.5 and 3.6 is similar to that in [VAN DOUWEN 1980]. It is unknown whether normality implies paracompactness in box products of compact spaces.

4. Large compact factors

In order for a box product spaces to be paracompact, the factors must be 'small'. Our sense of 'small' comes from cardinal function theory. Since each Tychonov product of compact spaces is compact, the following extension of Arhangel'skii's theorem on compact first countable spaces is helpful.

4.1. LEMMA. *No compact space may be partitioned into more than $\mathfrak{c}$ closed G_δ-sets.*

PROOF. Suppose that X is a compact space, E is an equivalence relation on X, and suppose that for each class C of E $\{G_n(C): n \in \omega\}$ is a collection of open sets satisfying

(1) $\forall n \in \omega \forall C,\ G_{n+1}(C)^- \subseteq G_n(C)$, and

(2) $\forall C,\ C = \bigcap_n G_n(C)$.

Then the following hold:

(3) If G is open and if the class $C \subseteq G$, then $\exists n \in \omega,\ G_n(C)^- \subseteq G$.

Observe that (3) is true because $\{G\} \cup \{X - G_n(C)^-: n \in \omega\}$ has a finite subcover.

Now we will construct, recursively, a collection $\{X_\alpha: \alpha \in \mathfrak{c}+1\}$ of compact subsets of X subject to the following restrictions:

(4) $|X_\alpha/E| \leqslant \mathfrak{c}$.

(5) If α is a limit ordinal, then $X_\alpha = E[(\bigcup_{\beta<\alpha} X_\beta)]^-$.

(6) If $\beta < \alpha$ and if $\mathscr{F} \in [\{G_n(E[x]): x \in X_\beta,\ n \in \omega\}]^{<\omega}$, then either $X = \bigcup \mathscr{F}$ or $X_{\beta+1} - \bigcup \mathscr{F} \neq \emptyset$.

Let $X_0 = \emptyset$ and suppose $\lambda \leqslant \mathfrak{c}$ is such that $\forall \alpha < \lambda X_\alpha$ has been constructed to satisfy (4), (5) and (6). Then we claim:

(7) If $Y \subseteq X$ and if $|Y/E| \leqslant \mathfrak{c}$, then $|Y^-/E| \leqslant \mathfrak{c}$.

In order to see (7) define a function Φ: $Y^-/E \to {}^\omega(Y/E)$ by arbitrarily choosing $(\varphi(E[y])(n)$ to be $E[y_n]$, where $y_n \in Y \cap G_n(E[y])$. Applying (3) to distinct elements of $\mathrm{dom}(\varphi)$ shows φ is a bijection. Therefore, $|\mathrm{ran}(\varphi)| \leqslant |Y/E|^{\aleph_0} \leqslant \mathfrak{c}^{\aleph_0} = \mathfrak{c}$.

If λ is a limit ordinal, we define $X_\lambda = E[(\bigcup_{\alpha<\lambda} X_\alpha)]^-$. So (7) proves this to be sufficient. If λ is a non-limit ordinal, then (7) shows

(8) $|[\{G_n(E[x]): x \in X_{\lambda-1},\ n \in \omega\}]^{<\omega}| \leqslant \mathfrak{c}^{\aleph_0} = \mathfrak{c}$.

Call the set in (8) Γ and for each $\mathscr{F} \in \Gamma$ arbitrarily choose an

$$x_{\mathscr{F}} \in \begin{cases} X - \bigcup \mathscr{F} & \text{if } \bigcup \mathscr{F} \neq X, \\ X & \text{otherwise}. \end{cases}$$

Then (4) and (7) show $X_\lambda = X_{\lambda-1} \cup \{x_{\mathscr{F}}: \mathscr{F} \in \Gamma\}^-$ satisfies our recursion hypothesis.

We wish to show $X = X_\mathfrak{c}$ which, by (4), completes the proof. So suppose, by way of contradiction, $y \in X - X_\mathfrak{c}$. From (5) $X_\mathfrak{c} = E[X_\mathfrak{c}]$ so we obtain, from (3), for each $x \in X_\mathfrak{c}$ an $m(x) \in \omega$ such that $E[y] \cap G_{m(x)}(E[x]) = \emptyset$. Since it is compact, $X_\mathfrak{c}$ is covered by a finite subcollection $\mathscr{F}$ of $\{G_{m(x)}(E[x]): x \in X_\mathfrak{c}\}$. Applying (5) we obtain an $\alpha < \mathfrak{c}$ such that $\mathscr{F} \subseteq \{G_n(E[x]): x \in X_\alpha\}$. Since $y \notin \bigcup \mathscr{F}$, (4) implies the contradicting $X_{\alpha+1} - \bigcup \mathscr{F} \neq \emptyset$. $\square$

4.2. THEOREM. *Suppose that for each $i \in \omega$ $X(i)$ is compact. If $\Box_i X(i)$ is paracompact, then $L(\Box_i X(i)) = \mathfrak{c}$ (the cardinal function L is the Lindelöf degree, see* [HODEL 1982]).

PROOF. Suppose that $\mathscr{R}_0$ is an open covering of $\Box_i X(i)$. Then apply 2.2(iii) and 2.4 to construct, recursively, a normal sequence $\langle D_n : n \in \omega \rangle$ of entourages of $\Box_i X(i)$, and a sequence $\langle \mathscr{R}_{n+1} : n \in \omega \rangle$ of open box covers of $\Box_i X(i)$ subject to the restrictions:

(1) D_n refines $\mathscr{R}_n$, and

(2) $\{R^- ; R \in \mathscr{R}_{n+1}\}$ refines $\{D_n[x] : x \in \Pi_i X(i)\}$.

Since it is clear that $E = \bigcap_n D_n$ is an equivalence relation on $\Pi_i X(i)$ and E refines $\mathscr{R}_0$. We can apply 4.1 to complete the proof by claiming

(3) $\forall x \in \Pi_i X(i)$, $E[x]$ is a closed G_δ-set in $T_i X(i)$.

In order to see (3) fix $x \in \Pi_i X(i)$ and $n \in \omega$. According to (1) we may find an $R_{n+3} \in \mathscr{R}_{n+3}$ such that $D_{n+3}[x] \subseteq R_{n+3}$. From (2) $\exists y \in \Pi_i X(i)$ such that $\Pi_i (R_{n+3}(i)^-) \subseteq D_{n+2}[y]$. Since $D_{n+1} \circ D_{n+1} \subseteq D_n \cap D_n^{-1}$, $\Pi_i (R_{n+3})(i)^-) \subseteq D_n[x]$. Now allow n to vary and define an open box G_n by

$$G_n(i) = \begin{cases} R_n(i) & \text{if } i \leq n\,, \\ X(i) & \text{if } n < 1\,. \end{cases}$$

Then claim (3) follows from the identity (in $T_i X(i)$)

$$(\bigcap_n G_n)^- = \bigcap_n G_n = E[x]\,. \quad \Box$$

We are now ready to study the easiest of two examples of a non-paracompact box product of compact spaces (also see 5.3). However, we again consider a P-space.

4.3. DEFINITION. For a space X, let X_δ denote the *G_δ-modification* of X; i.e., the space with the same ground set as X whose topology is generated by the set of all G_δ-sets of X.

4.4. LEMMA. *Suppose X is a space. Then X_δ is homeomorphic to a closed subspace of $\nabla^\omega X$.*

PROOF. The desired map is $x \mapsto q(\bar{x})$ where $\bar{x}(i) = x$ $\forall i \in \omega$. $\Box$

4.5. EXAMPLE. *A compact space K such that $\Box^\omega K$ is not paracompact.*

PROOF. Let Z be a discrete topological space of cardinality $\mathfrak{c}^+$ and K its Stone–Čech compactification. We claim Z is a closed discrete subspace of K_δ. If the claim is true, then 4.4 shows $L(\Box^\omega K) \geq L(\nabla^\omega K) \geq L(K_\delta) \geq \mathfrak{c}^+$. So $\Box^\omega K$ is not paracompact by virtue of 4.2.

Of course Z is discrete in K_δ so we show $K_\delta - Z$ is open. It is convenient to recognize K as the space of all ultrafilters on Z (see [VAN MILL 1982]). Thus, each $u \in K - Z$ is a non-principal ultrafilter on Z. According to 1.14, for each $u \in K - Z$ $\exists \{U_n : n \in \omega\} \subseteq u$ such that $\bigcap_n U_n = \emptyset$. Therefore, $\bigcap_n \{v \in K : U_n \in v\}$ is a G_δ-set of K disjoint from Z and containing u. □

4.5 directs us to investigate families $\mathscr{F} = (X(i): i \in \omega)$ of compact spaces for which $L(\Box_i X(i)) = \mathfrak{c}$. Obviously, $\mathscr{F}$ is such a family whenever $w(X(i)) \leqslant \mathfrak{c}$ for each $i \in \omega$. In particular, it is sufficient that each $X(i)$ be separable, or, by virtue of 4.1, each $X(i)$ be first countable. Our next goal is to determine that the weight condition of members of $\mathscr{F}$ is not always necessary.

4.6. THEOREM. *Suppose that for each $i \in \omega$ $X(i)$ is a compact ordinal space. Then*
(i) *$\Box_i X(i)_\delta$ is paracompact and $L(\Box_i X(i)_\delta) = \mathfrak{c}$.*
(ii) *$L(\Box_i X(i)) = \mathfrak{c}$.*

PROOF. As (i) implies (ii), we just prove the former. By a *special box* in $\Pi_i X(i)$ we mean a set of the form $G = \Pi_i G(i)$ such that for each $i \in \omega$ $G(i)$ is a closed interval of $X(i)$ and a G_δ-set of $X(i)$. Further, given a special box G we define $s_G \in \Pi_i X(i)$ by $s_G(i) = \sup G(i)$ $\forall i \in \omega$. Observe three facts:
(1) The set of all special boxes is a base for $\Box_i X(i)_\delta$.
(2) Each countable intersection of special boxes is a special box.
(3) If G and H are special boxes such that $s_G \in H \underset{+}{\subset} G$, then $G - H$ is the disjoint union of special boxes.

In order to see (3) we define for each $i \in \omega$ $K_n(i) \subseteq G(i)$ $\forall n \in \omega$ as follows: If $G(i) = H(i)$, let $K_n(i) = H(i)$ $\forall n \in \omega$. If $\inf H(i) = \alpha + 1$ for $\alpha \in G(i)$, let $K_n(i) = [\inf G(i), \alpha]$ $\forall n > 0$ and $K_0(i) = H(i)$. Otherwise, find an increasing sequence $\{\alpha_n : n \in \omega\} \subseteq G(i) - H(i)$ such that $\alpha_0 = \inf G(i)$, $\sup_n \alpha_n = \inf H(i)$, and α_n is a non-limit ordinal $\forall n > 0$. In the case we set $K_n(i) = [\alpha_{n-1}, \alpha_n)$ for $n > 0$ and $K_0(i) = H(i)$. So the desired family is $\{\Pi_i K_{f(i)}(i): f \in {}^\omega\omega\} - \{H\}$.

Suppose $\mathscr{R}$ is an open covering of $\Box_i X(i)_\delta$. By virtue of (1)–(3) above, we may construct a family $\{\mathscr{G}_\alpha : \alpha < \omega_1\}$ of pairwise-disjoint special box coverings of $\Pi_i X(i)$ such that if $\beta < \alpha$ and $G \in \mathscr{G}_\beta$, then the conditions (4)–(6) below hold:
(4) $G = \bigcup \{G' \in \mathscr{G}_\alpha : G' \cap G \neq \emptyset\}$.
(5) If there is an $R \in \mathscr{R}$ with $G \subseteq R$, then $G \in \mathscr{G}_\alpha$.
(6) If $G \nsubseteq R$ for each $R \in \mathscr{R}$, then there is an $G' \in \mathscr{G}_\alpha$ and an $S \in \mathscr{R}$ such that $s_G \in G' \subseteq G \cap S$.

We claim

$$\mathscr{G} = \{G : \exists \alpha \in \omega_1\ \exists R \in \mathscr{R} \text{ such that } G \in \mathscr{G}_\alpha \text{ and } G \subseteq R\}$$

is a pairwise-disjoint refinement of $\mathscr{R}$. Since (4) implies no two members of $\mathscr{G}$ intersect, we need only show $\mathscr{G}$ covers $\Pi_i X(i)$. By way of contradiction, suppose $x \notin \bigcup \mathscr{G}$ and $\forall \alpha \in \omega_1$ choose $G\alpha \in \mathscr{G}_\alpha$ with $x \in G\alpha'$. Then, by applying (4), $\beta < \alpha$

implies $s_{G\alpha}(i) \leqslant s_{G\beta}(i)$ $\forall i \in \omega$. So for each $\alpha \in \omega_1$ let

$$A_\alpha = \{i \in \omega \colon s_{G(\alpha+1)}(i) < s_{G\alpha}(i)\}\,.$$

According to (6) $A_\alpha \neq \emptyset$ $\forall \alpha \in \omega_1$. So $\exists j \in \omega$ belonging to uncountably many of the A_α. Therefore, $\{s_{G\alpha}(j) \colon \alpha \in \omega_1\}$ contains an uncountable descending sequence of ordinals—a contradiction.

The Lemma 4.1 and a simple variant of 4.2 shows $L(\Box_i X(i)_\delta) = \mathfrak{c}$. $\Box$

4.7. NOTES. 4.1 is an extension (due to R. POL) of the result for compact first countable spaces (see [HODEL 1982]). 4.2 is in [KUNEN 1978]. 4.4 is essentially due to K. Kunen (see [VAN DOUWEN 1980]). 4.5 comes from an observation of J. Isbell and S. Mrowka (see [WILLIAMS–FLEISCHMAN 1974]). 4.6 is essentially in [RUDIN 1974]; however, in [KUNEN 1978] it is proved for a larger class of spaces which includes all compact dispersed spaces. I do not know if 4.6 remains true when 'ordinal' is replaced with 'linearly ordered'. For further study of box products of ordinal spaces the reader should investigate [ERDÖS–RUDIN 1973], [KUNEN 1973], [RUDIN 1974], [WILLIAMS 1977], and [KUNEN 1999].

5. CH and lighter axioms

Each of the so-called 'positive' results on paracompactness of box products of compact spaces are consistency results. The strongest axiom used is the continuum hypothesis which makes things easy by exploiting a P-space version of "Lindelof implies paracompact".

5.1. LEMMA. *If X is a P-space such that $L(X) = \aleph_1$, then X is paracompact.*

PROOF. If $\mathcal{R} = \{R_\alpha \colon \alpha < \omega_1\}$ consists of clopen G_δ-sets, then $\{R_\alpha - \bigcup_{\beta<\alpha} R_\beta \colon \alpha > \omega_1\}$ is a pairwise-disjoint open refinement of $\mathcal{R}$. $\Box$

Applying the Nable Lemma (3.6), 3.2(v), 5.1, and 4.2, results in

5.2. THEOREM. (CH). *Suppose that for each $i \in \omega$ $X(i)$ is a compact space. Then $\Box_i X(i)$ is paracompact iff $L(\Box_i X(i)) = \mathfrak{c}$.*

Perhaps it is imprecise to use $\mathfrak{c}$ in 5.2. For in this form the next example shows 5.2 is equivalent to CH.

5.3. EXAMPLE. *A compact space X such that $w(X) = \aleph_2$ and $\Box^\omega X$ is not normal.*

PROOF. Let $X = T^{\omega_2}[0, 1]$. Then X is compact and because $[0, 1]$ possesses a countable base, $w(X) = \aleph_2$. Since X_δ is homeomorphic to a closed subspace of $\Box_i X(i)$ (similar to 4.4), it will be sufficient to prove X_δ is not normal. First we observe X_δ and $(T^{\omega_2}[0,1]_\delta)_\delta$ are homeomorphic (because $[0,1]_\delta$ is discrete), and consider a subspace $Z=\{z_\alpha\colon \alpha\in\omega_1\}$ of $[0,1]$. Let $Y=T^{\omega_2}Z$. Then Y_δ is a closed subspace of X_δ. We prove Y_δ is not normal.

For each $i\in\{0,1\}$ define

$$C(i)=\{y\in Y\colon |y^{-1}(z_\delta)|\leq 1\ \forall\beta\in\omega_1-\{i\}\}\,.$$

Then $C(0)$ and $C(1)$ are disjoint closed subsets of Y_δ. We suppose, by way of contradiction, G is open in Y_δ and

$$C(0)\subseteq G\subseteq G^-\subseteq Y_\delta-c(1)$$

For $U=\Pi_\nu U(\nu)\subseteq Y_\delta$, let $\mathrm{spt}(u)=\{\nu\in\omega_2\colon U(\nu)\neq Z\}$. It is clear that Y_δ has a base of open sets of the form $U=\Pi_\nu U(\nu)$ satisfying:

(1) $U(\nu)\neq Z\Rightarrow |U(\nu)|=1$.

(2) $|\mathrm{spt}(U)|=\aleph_0$ and $\omega_1\cap\mathrm{spt}(U)$ is an infinite ordinal.

We will call such sets U basic sets.

For every $\nu\in\omega_2$ define $u_0(\nu)=z_0$. Choose a basic set U_0 such $u_0\in U_0\subseteq G$. Let $\eta_0=\omega_1\cap\mathrm{spt}(U_0)$. From (2) there is a bijection $\varphi_0\colon\eta_0\to\mathrm{spt}(U_0)$. Recursively, we construct for each $\alpha\in\omega_1-\{0\}$ a basic set U_α, an ordinal η_α, a function $\varphi_\alpha\colon\eta_\alpha\to\omega_2$, and a point $u_\alpha\in C(0)$ all subject to the restrictions (3) through (7) below:

(3) $\omega_1\cap\mathrm{spt}(U_\alpha)=\eta_\alpha$.

(4) $\alpha<\eta_\alpha<\omega_1$, and if $\beta<\alpha$, then $\eta_\beta<\eta_\alpha$.

(5) φ_α is an injection, $\beta<\alpha\Rightarrow\varphi_\beta\subseteq\varphi_\alpha$, $\mathrm{ran}(\varphi_\alpha)=\mathrm{spt}(U_\alpha)$.

(6) $u_\alpha\in U_\alpha\subseteq G$.

(7) $\forall\nu\in\omega_2,\quad u_\alpha(\nu)=\begin{cases} z_\gamma & \text{if } \exists\beta<\alpha\ \exists\gamma\in\eta_\beta \text{ with } \varphi_\beta(\gamma)=\nu\,,\\ z_0 & \text{otherwise}\,.\end{cases}$

Suppose $\lambda\in\omega_1$ is given, and for each $\alpha<\lambda$ the objects U_α, η_α, φ_α, and u_α have been constructed to respect (3) to (7). Let $S_\lambda=\bigcup_{\alpha<\lambda}\mathrm{spt}(U_\alpha)$, $\theta=\bigcup_{\alpha<\lambda}\eta_\alpha$, and $\varphi=\bigcup_{\alpha<\lambda\varphi_\alpha}$. Then (3) and (5) show $\varphi\colon\theta\to S_\lambda$ is a bijection. So (7) allows

$$u_\lambda(\nu)=\begin{cases} z_\gamma & \text{if } \exists\gamma<\theta \text{ with } \varphi(\gamma)=\nu\,,\\ z_0 & \text{otherwise}\end{cases}$$

to define an element $u_\lambda\in C(0)-\{u_\alpha\colon\alpha<\lambda\}$.

Since λ and S_λ are countable, there is a basic set U_λ containing u_λ such that $S_\lambda\cup(\theta+\omega)\subseteq\mathrm{spt}(U_\lambda)$. Let $\eta_\lambda=\omega_1\cap\mathrm{spt}(U_\lambda)$. Since $\mathrm{spt}(U_\lambda)-S_\lambda$ is countably infinite, there is a bijection $\varphi_\lambda\colon\eta_\lambda\to\mathrm{spt}(U_\lambda)$ extending φ. Our recursion hypothesis is now easily shown to hold at λ. So we consider the construction complete $\forall\alpha\in\omega_1$.

Let $S=\bigcup_\alpha\mathrm{spt}(U_\alpha)$ and $\Phi=\bigcup_\alpha\varphi_\alpha$. According to (4) and (5) $\mathrm{dom}(\Phi)=\omega_1$. According (5) Φ is a bijection with $\mathrm{ran}(\Phi)=S$. So

$$x(\nu)=\begin{cases} z_\gamma & \text{if } \nu\in S \text{ and } \Phi(\gamma)=\nu,\\ z_1 & \text{otherwise}\end{cases}$$

defines an element $x\in C(1)$. Since $C(1)\cap G^-=\emptyset$, there is a basic set V with $x\in V\subseteq Y_\delta - G^-$. From (4) S is uncountable so (3) allows us to find a first $\alpha\in\omega_1$ such that $S\cap\operatorname{spt}(V)\subseteq\operatorname{spt}(U_\alpha)$. If $\nu\notin\operatorname{spt}(U_\alpha)$, then $V(\nu)\subseteq U_\alpha(\nu)$. If $\nu\notin\operatorname{spt}(V)$, then $U_\alpha(\nu)\subseteq V(\nu)$. If $\nu\in\operatorname{spt}(V)\cap\operatorname{spt}(U_\alpha)$, then $x(\nu)=u_\alpha(\nu)\in V(\nu)\cap{}_\alpha(\nu)$. Therefore, $V\cap U_\alpha\neq\emptyset$ which gives a contradiction. □

We do not know whether the paracompactness of box products of compact ordinal spaces can be established without CH (recall 4.6 and 5.2); however, for compact metrizable spaces much less than CH can be used. The crucial axioms concern the structure of a directed quotient of ${}^\omega\omega$ when the latter has the product point-wise order.

5.4. Definition. For r, $s\in{}^\omega\omega$ we write $r\leqslant^* s$ if $\{i\in\omega: s(i)<r(i)\}$ is finite, and write $r\not\leqslant^* s$ otherwise. We say $W\subseteq{}^\omega\omega$ is *unbounded* if $r\in{}^\omega\omega\Rightarrow\exists s\in W$ with $s\not\leqslant^* r$. $W\subseteq{}^\omega\omega$ is called *dominating* provided $r\in{}^\omega\omega\Rightarrow\exists s\in W$ such that $r\leqslant^* s$. There are cardinals associated with these notions:

$$\mathfrak{b}=\inf\{|W|: W \text{ is unbounded in } {}^\omega\omega\}\,,$$

$$\mathfrak{d}=\inf\{|W|: W \text{ is dominating in } {}^\omega\omega\}\,.$$

CH and even Martin's Axiom imply implies $\mathfrak{b}=\mathfrak{d}=\mathfrak{c}$ [Rudin, 1975, pg. 40]. If one simultaneously adds $\aleph_2$ random reals [Kunen 1982] to a model of CH, then $\aleph_1=\mathfrak{d}<\aleph_2=\mathfrak{c}$ in the resulting model. If one simultaneously adds $\aleph_2$ Cohen reals [Kunen 1982] to a model of CH, then $\aleph_1=\mathfrak{b}<\aleph_2=\mathfrak{d}=\mathfrak{c}$. However, in Section 6 we shall establish that $\mathfrak{b}$ and $\mathfrak{d}$ can be nearly anything subject to certain restrictions given (i) and (ii) below. [van Douwen 1982] also studies the cardinals $\mathfrak{b}$ and $\mathfrak{d}$.

5.5. Lemma. *The following are true*:

(i) $\aleph_1\leqslant\mathfrak{b}\leqslant\mathfrak{d}\leqslant\mathfrak{c}$.

(ii) *There is an unbounded family well-ordered by* $\leqslant^*$ *and of order type* $\mathfrak{b}$; *hence* $\mathfrak{b}$ *is regular.*

(iii) *If* $W\subseteq{}^\omega\omega$ *and if* $\mathscr{A}\subseteq[\omega]^\omega$ *are such that* $|W|\cdot|A|<\mathfrak{d}$, *then* $\exists s\in{}^\omega\omega$ *such that* $\{i\in A: r(i)<s(i)\}$ *is infinite* $\forall r\in W$ $\forall A\in\mathscr{A}$.

Proof. (i) If $\{r_n: n\in\omega\}\subseteq{}^\omega\omega$ and if $i\in\omega$, define $s(i)=\Sigma_{n<i}r_n(i)$. So $r_n\leqslant^* s$ $\forall n\in\omega$.

(ii) If $\{r_\alpha: \alpha<\mathfrak{b}\}$ lists an unbounded family, we may recursively define $s_\alpha\in{}^\omega\omega$ so that s_α bounds $\{r_\beta+s_\beta: \beta<\alpha\}$. Then, $\{s_\alpha: \alpha<\mathfrak{b}\}$ is the desired family.

(iii) For $(r, A) \in W \times \mathscr{A}$ define $r_A \in {}^\omega\omega$ by

$$r_A(i) = \Sigma(r(j): j \leqslant \min\{k \in A: i < k\}) \quad \forall i \in \omega .$$

Since $|W \times A| < \mathfrak{d}$, there is an increasing $s \in {}^\omega\omega$ such that $s \not\leqslant^* r_A$ $\forall (r, A) \in W \times \mathscr{A}$. Note that $r_A(i) < s(i)$ and $i \leqslant j \leqslant \min\{k \in A: i < k\}$ implies $r(j) < s(i) < s(j)$. $\square$

5.6. LEMMA. *Suppose that for each $i \in \omega$ $\mathscr{N}(i)$ is an infinite normal sequence of entourages in a space $X(i)$. Then $(\nabla_i \mathscr{N}(i), \supseteq)$ and $({}^\omega\omega, \leqslant^*)$ are order-isomorphic directed sets.*

5.7. THEOREM. *Suppose that for each $i \in \omega$ $X(i)$ is a compact space. Then $\square_i X(i)$ is paracompact if one of the following holds:*

(i) $[\mathfrak{b} = \aleph_1]$ $\sup_i w(X(i)) \leqslant \aleph_1$.

(ii) $[\mathfrak{d} = \mathfrak{b}]$ *each $X(i)$ is first countable and* $\sup_i w(X(i)) \leqslant \aleph_1$.

(iii) $[\mathfrak{d} = \mathfrak{c}]$ *each $X(i)$ is first countable.*

PROOF. According to the Nabla Lemma, it is sufficient to prove $\nabla_i X(i)$ is paracompact. Suppose that for each $i \in \omega$ $\mathscr{D}(i)$ is a unif. base compatible with $X(i)$.

(i) Since a compatible uniformity on a space has at least $\mathfrak{c}$ members, we may assume $\forall i \in \omega$ $|\mathscr{D}(i)| = \aleph_1$ and $\mathscr{D}(i)$ is the union of an increasing family $\{\mathscr{E}_\alpha(i): \alpha < \omega_1\}$ of countable sub-families satisfying

$$C, D \in \mathscr{E}_\alpha(i) \Rightarrow \exists E \in \mathscr{E}_\alpha(i), \quad E \circ E \subseteq C \cap D \cap D^{-1} .$$

It is clear that $\nabla_i \mathscr{D}(i) = \bigcup_{\alpha < \omega_1} \nabla_i \mathscr{E}_\alpha(i)$. Further, each $(\nabla_i \mathscr{E}_\alpha(i), \supseteq)$ has a cofinal normal sequence. So 5.6 and $\mathfrak{b} = \aleph_1$ imply $\nabla_i \mathscr{D}(i)$ has a cofinal set of cardinality $\aleph_1$. Since $\nabla_i X(i)$ is a P-space, it is now seen to be $\aleph_1$-metrizable. 2.8, therefore, completes the proof.

(ii) We wish to prove $\nabla_i X(i)$ is $\mathfrak{d}$-metrizable so that 2.8 applies. First observe that the weight restriction allows us to show $uw(\nabla_i X(i)) \leqslant \mathfrak{d}$ by imitating the proof of (i). Thus, it suffices to prove every intersection of less than $\mathfrak{d}$ open sets of $\nabla_i X(i)$ is open.

Suppose $x \in \square_i X(i)$ and $A = \{i \in \omega: \{x(i)\}$ is not open$\}$ is infinite. Since each $X(i)$ is first countable, there is $\forall i \in A$ an infinite normal sequence $\mathscr{N}(i)$ of entourages of $X(i)$ such that $\mathscr{N}(i)[x(i)]$ is a local base at $x(i)$. Since $\mathfrak{b} = \mathfrak{d}$, the proof of 5.5(ii) (replace unbounded with dominating) and 5.6 show that in $\nabla_i X(i)$, $q(x)$ has a well-ordered local base of order type $\mathfrak{d}$.

(iii) From 4.1 we assume $|\mathscr{D}(i)| \leqslant \mathfrak{c}$ $\forall i \in \omega$. Let

$$\mathscr{E} = \{\bigcap q(\mathscr{N}): \mathscr{N} \text{ is a normal sequence in } \square_i \mathscr{D}(i)\} .$$

Then $|\mathscr{E}| = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$ and $\mathscr{E}$ is a compatible unif. base of clopen equivalence relations on $\nabla_i X(i)$ (see the proof of 3.4(ii)). We claim

(1) If $Y \subseteq \Pi_i X(i)$, $|Y| < \mathfrak{c}$, and if $\varphi: Y \to \mathscr{E}$ is a function, then

$$\bigcup\{\varphi(y)[q(y)]: y \in Y\} \quad \text{is closed.}$$

Assume (1) is true, and suppose $\mathscr{R}$ is an open cover of $\nabla_i X(i)$. Listing $\nabla_i X(i)$ by $\{z_\alpha: \alpha < \mathfrak{c}\}$, we choose for $\alpha < \mathfrak{c}$ $E_\alpha \in \mathscr{E}$ such that $\exists R \in \mathscr{R}$ with $E_\alpha[z_\alpha] \subseteq R$. Clearly (1) implies

$$\{E_\alpha[z_\alpha] - \bigcup_{\beta<\alpha} E_\beta[z_\beta]: \alpha < \mathfrak{c}\}$$

is a pairwise-disjoint open refinement of $\mathscr{R}$.

In order to see (1), we let U be the union in (1) and suppose $q(x) \notin U$. Then for each $y \in Y$ and associated normal sequences $\{N_n^y: n \in \omega\}$ with $\varphi(y) = \bigcap_n \mathfrak{q}(N_n^y)$, there is an $m(y) \in \omega$ such that $(x, y) \notin \equiv[N_{m(y)}^y]$. So the set

$$A^y = \{i \in \omega: N_{m(y)+1}^y(i)[x(i)] \cap N_{m(y)+1}^y(i)[y(i)] = \emptyset\}$$

is infinite for each $y \in Y$. Since $X(i)$ is first countable $\forall i \in \omega$, $x(i)$ has a non-decreasing neighborhood base $\{G_n(i): n \in \omega\}$ $\forall i \in \omega$. So there is an $r(y) \in {}^\omega\omega$ such that $\forall i \in \omega$ $G_{r(y)(i)} \subseteq N_{m(y)+1}^y(i)[x(i)]$. In accord with 5.5(iii), $\exists s \in {}^\omega\omega$ such that $\{i \in A^y: r(y)(i) < s(i)\}$ is infinite $\forall y \in Y$. Therefore,

$$q\Big(\prod_i G_{s(i)}\Big) \cap q(D_{m(y)+1}^y[y]) = \emptyset \quad \forall y \in Y.$$

So $q(\Pi_i G_{s(i)}(i)) \cap U = \emptyset$. $\square$

In light of our example 5.3 it is interesting to note that $w(T^{\omega_1}[0, 1]) = \aleph_1$, and hence, 5.7(i) shows $\square^\omega T^{\omega_1}[0, 1]$ can be paracompact even though $\aleph_1 < \mathfrak{c}$. Aside from the compact metrizable spaces, there are a few other interesting spaces which come under the hypothesis of 5.7. For example compact Souslin Lines, when they exist (see [ROITMAN 1982]) fit the hypothesis.

Many of the 'usual' models for $\aleph_1 < \mathfrak{c}$ actually have $\mathfrak{c} = \aleph_2$. Therefore, the following consequence of 5.7(i) and (iii) quite interesting.

5.8. COROLLARY ($\mathfrak{c} \leq \aleph_2$). *Suppose that for each $i \in \omega$ $X(i)$ is a compact first countable space of weight at most $\aleph_1$. Then $\square_i X(i)$ is paracompact.*

5.9. NOTES. The observations 5.1 and 5.2 [KUNEN 1978] significantly extend the principal result in [RUDIN 1972]. The proof of 5.3 is essentially in [BORGES 1969], although [VAN DOUWEN 1977] first noticed its application to box products. The importance of $({}^\omega\omega, \leq^*)$ in box products was first identified by K. Kunen in [RUDIN 1975] and further developed in [WILLIAMS 1976], where weak versions of 5.7(i) and

(ii) were obtained. In the present form 5.5(iii), 5.7(iii), and 5.8 all come from [ROITMAN 1979], and 5.7(ii) (in the metrizable case) comes from [VAN DOUWEN 1980]. More recently parts of 5.7 were obtained from [MILLER 1982] for partial box products of uncountably many spaces.

6. In forcing extensions

The axioms $\mathfrak{b} = \mathfrak{d}$ and $\mathfrak{d} = \mathfrak{c}$ each have topological characterizations. For example, the statement $\mathfrak{b} = \mathfrak{d}$ is equivalent to the statement 'there is a cardinal κ such that $\nabla^\omega[0, 1]$ is κ-metrizable', while $\mathfrak{d} = \mathfrak{c}$ is equivalent to '$\nabla^\omega[0, 1]$ has a base for which each union of less than $\mathfrak{c}$ members is closed'. On the other hand, there are many models of $\mathfrak{b} < \mathfrak{d} < \mathfrak{c}$. Since we have not been able to prove in ZFC that $\Box^\omega[0, 1]$ is paracompact, we illustrate in this section that $\Box^\omega[0, 1]$ can be paracompact in the presence of any 'legal' aleph values of $\mathfrak{b}$ and $\mathfrak{d}$.

6.1. THEOREM. *Suppose that $\mathcal{M}$ is a* (countable transitive) *model of* ZFC *and, in $\mathcal{M}$, W is a directed set of cardinality at most $\mathfrak{c}$ whose every countable subset is bounded. Then there is an extension $\mathcal{N}$ of $\mathcal{M}$ such that*

(i) *the aleph value of $\mathfrak{c}$ in $\mathcal{N}$ is the aleph value of $\mathfrak{c}$ in $\mathcal{M}$, and*
(ii) *in $\mathcal{N}$ W can be embedded cofinally in $({}^\omega\omega, \leqslant^*)$.*

The consequences of 6.1 are interesting. For example, suppose $\mathcal{M}$ is a model of $\mathfrak{c} = \aleph_5$. Then the candidate for W might be $\omega_2 \times \omega_4$ with the point-wise product partial order. In this case

$$\aleph_1 < \aleph_2 = \mathfrak{b} < \aleph_3 < \aleph_4 = \mathfrak{d} < \aleph_5 = \mathfrak{c}$$

holds in $\mathcal{N}$.

We will not attempt to prove 6.1 here; however, we remark its proof in [HECHLER 1974] obtains $\mathcal{N}$ as the limit model of a ccc interated with finite support forcing construction of uncountable length. The main result of this section illustrates (via the nabla lemma) that box products of countably many locally compact metrizable spaces will be paracompact whenever such constructions are employed.

All undefined notions and terms of logic in this section are to be found in [KUNEN 1980] (especially chapter VIII). Therefore, all p.o. (partial orders) $\mathbb{P}$ posses a largest element $\mathbb{1}_{\mathbb{P}}$ (or $\mathbb{1}$ when no confusion results). In addition, we make the convention that *if a p.o. $\mathbb{P}$ has more than one element, then each element of $\mathbb{P}$ has at least two* (and hence, infinitely many) *incompatible predecessors.* Since all our interated forcing constructions

$$\langle\langle \mathbb{P}_\xi : \xi \leqslant \alpha \rangle, \quad \langle \pi_\xi : \xi < \alpha \rangle\rangle$$

(see [KUNEN 1980, VIII. 5.8]) have finite support, *we assume,* without loss of generality, $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$ *is an ascending chain under* $\subset_c$. So once we have identified a particular $\mathbb{P}_\alpha$-generic filter G over $\mathcal{M}$, *we can use* $\mathcal{M}_\xi$ *as an abbreviation for the intermediate models* $\mathcal{M}[G \cap \mathbb{P}_\xi]$.

By interpreting elements of ${}^\omega\omega$ as subsets of $\omega \times \omega$ we can see the next lemma as an immediate corollary of [KUNEN 1980, VIII. 5.14].

6.2. LEMMA. *Assume that in* $\mathcal{M}$, α *is a limit ordinal of uncountable cofinality and*

$$\langle\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle, \quad \langle \pi_\xi : \xi < \alpha \rangle\rangle$$

is an α-stage interated forcing construction with finite support. Suppose G is a $\mathbb{P}_\alpha$-generic filter over $\mathcal{M}$. Then for each $r \in {}^\omega\omega \cap \mathcal{M}_\alpha$ and each $A \in [\omega]^\omega \cap \mathcal{M}_\alpha$, there is a $\xi < \alpha$ such that $r, A \in \mathcal{M}_\xi$. $\square$

To be clear about 6.2 we note that for $\xi \leq \alpha$ the objects ${}^\omega\omega \cap \mathcal{M}_\xi$ and $[\omega]^\omega \cap \mathcal{M}_\xi$ are precisely the objects which the model $\mathcal{M}_\xi$ thinks as being ${}^\omega\omega$ and $[\omega]^\omega$. Since $\xi \leq \xi'$ implies $\mathcal{M}_\xi \subseteq \mathcal{M}'_\xi$, $\mathcal{M}_\alpha$ views ${}^\omega\omega$ and $[\omega]^{\leq\omega}$ to be the union of an increasing α-sequence of subalgebras belonging to the intermediate models of $\mathcal{M}_\xi$. The construction below shows that $\mathcal{M}_\alpha$ can be made to view a compact metrizable space as the union of an increasing sequence of subspaces whose elements belong to the intermediate models.

6.3. CONSTRUCTION. Under the hypothesis of 6.2 we suppose, in addition, that X is a space in $\mathcal{M}_\alpha$ with a countable base $\{B_n : n \in \omega\}$. For each $x \in X$ let $\bar{x} = \{n \in \omega : x \in B_n\}$.

Of course if $\bar{x}$ is finite, then $\bar{x} \in \mathcal{M} = \mathcal{M}_0$. However, 6.2 says if $\bar{x}$ is infinite there is a $\xi < \alpha$ such that $\bar{x} \in \mathcal{M}_\xi$. In $\mathcal{M}_\alpha$ define for each $\xi \leq \alpha$

$$X_\xi = \{\bar{x} \in \mathcal{M}_\xi : x \in X\}.$$

Then $X_\alpha = \bigcup_\xi X_\xi$. Further, the map $x \mapsto \bar{x}$ defines a homeomorphism of X onto X_α when the latter is topologized via the base $\{\{\bar{x} : x \in B_n\} : n \in \omega\}$.

Although $\mathcal{M}_\alpha$ recognizes each X_ξ as the subspace of X_α whose points are elements of $\mathcal{M}_\xi$, it is possible (whenever $\mathrm{cf}(\xi) \leq \omega$) that X_ξ is not a set in $\mathcal{M}_\xi$. In any case, *we will identify X with X_α in the sequel.* $\square$

6.4. LEMMA. *Under the hypothesis of* 6.2 *we suppose, in addition, that in $\mathcal{M}_\alpha$ $(D(i, n) : n \in \omega)$ is a normal sequence compatible with a second countable space $X(i)$ for each $i \in \omega$. If $x, y \in \pi_i X(i)_\xi$, then*

(i) $\mathcal{M}_\alpha \models \{i \in \omega : x(i) \neq y(i)\} \in \mathcal{M}_\xi$, *and*

(ii) *there is an $r \in \mathcal{M}_\xi \cap {}^\omega\omega$ such that for each $i \in \omega$ either $x(i) = y(i)$ or $(x(i), y(i)) \notin D(i, r(i))$.*

PROOF. Of course to say $x, y \in \pi_i X(i)_\xi$ means we are applying the construction 6.3 for each $i \in \omega$ with some countable base for $X(i)$.

(i) Assume $\xi < \alpha$, then by way of the identification in 6.3, $\mathcal{M}_\alpha \Vdash x, y \in {}^\omega([\omega]^{\leq\omega})$. Thus, $\{i \in \omega: x(i) \neq y(i)\}$ has definition in $\mathcal{M}_\xi$. Since $\omega \cap \mathcal{M}_\xi = \omega \cap \mathcal{M}_\alpha$ and since '$=$' and '$\neq$', as defined on $[\omega]^{\leq\omega}$, are absolute (remember that $\mathcal{M}$ is transitive), $x(i) \neq y(i)$ in $\mathcal{M}_\alpha$ iff $x(i) \neq y(i)$ in $\mathcal{M}_\xi$.

(ii) For each $i \in \omega$ $\{x(i), y(i)\}$ is a set in $\mathcal{M}_\xi$ and a discrete subspace of $X(i)$ in $\mathcal{M}_\alpha$; and hence, it is a topological space in $\mathcal{M}_\xi$. So in $\mathcal{M}_\xi$

$$(D(i, n) \cap \{x(i), y(i)\}^2 \colon n \in \omega)$$

is a normal sequence compatible with $\{x(i), y(i)\}$. Thus if $x(i) \neq y(i)$ we choose, in $\mathcal{M}_\xi$, $r(i)$ to be the first $n \in \omega$ such that

$$(x(i), y(i)) \notin D(i, n) \cap \{x(i), y(i)\}^2 .$$

Otherwise we set $r(i) = 0$. Since $r(i)$ is defined in $\mathcal{M}_\xi$ for each $i \in \omega$, $r \in \mathcal{M}_\xi$. □

The next two lemmas exhibit, in the interated forcing case, a situation very similar to that under the axiom $\mathfrak{d} = \mathfrak{c}$. In fact 6.6 should be compared with 5.5(iii) and it implies that $\mathrm{cf}(\alpha) \leq \mathfrak{d}$.

6.5. LEMMA. *In a model $\mathcal{M}$ consider the set $\mathbb{C} = \bigcup_{n\in\omega} {}^n\omega$ ordered by reverse inclusion. If G is a $\mathbb{C}$-generic filter over $\mathcal{M}$, then*

(i) *$\bigcup G \in \mathcal{M}[G] \cap {}^\omega\omega$, and*

(ii) *if $r \in \mathcal{M} \cap {}^\omega\omega$ and $A \in \mathcal{M} \cap [\omega]^\omega$, then*

$$\{i \in A \colon r(i) < (\bigcup G)(i)\} \in \mathcal{M}[G] \cap [\omega]^\omega .$$

PROOF. For $k \in \omega$, $A \in \mathcal{M} \cap [\omega]^\omega$, and for $r \in \mathcal{M} \cap {}^\omega\omega$ define

$$\langle k, A, r\rangle = \{g \in \mathbb{C} \colon \exists j \in A \cap \mathrm{dom}(g),\, k < j,\, r(j) < g(j)\} .$$

Then $\langle k, A, r\rangle$ is dense in $\mathbb{C}$. To see this suppose $n \in \omega$, $f \in {}^n\omega$, $m \in A$, and suppose $k + n \leq m$. Then

$$f \subseteq (f \cup \{(i, r(i) + 1) \colon n \leq i \leq m\}) \in \langle k, A, r\rangle .$$

In order to prove (i) let k vary and take $A = \omega$. (ii) follows from the definition of the arbitrary $\langle k, A, r\rangle$. □

6.6. LEMMA. *Under the hypothesis of* 6.2 *we suppose, in addition, that $\mathbb{P}_\alpha$ is ccc and $\nu \in \mathcal{M} \cap \alpha$ has $\mathrm{cf}(\nu) = \omega$. Then there is an $s_\nu \in \mathcal{M}_\nu \cap {}^\omega\omega$ such that for each $\xi < \nu$, each $r \in \mathcal{M}_\xi \cap {}^\omega\omega$, and each $A \in \mathcal{M}_\xi \cap [\omega]^\omega$, we have*

$$\{i \in A \colon r(i) < s(i)\} \in \mathcal{M}_\nu \cap [\omega]^\omega .$$

PROOF. Suppose $(\nu(n): n \in \omega)$ is an increasing sequence of ordinals in $\mathcal{M}$ cofinal with ν. We claim there is a complete embedding $\psi: \mathbb{C} \to \mathbb{P}_\nu$, where $\mathbb{C}$ is as given in 6.5. If our claim holds, then 6.5 shows there is a $t_0 \in \mathcal{M}_\nu \cap {}^\omega\omega$ satisfying the conclusion of this lemma for $\xi = 0$ and $s_\nu = t_0$. Since $\mathrm{cf}(\alpha) > \omega$ and

$$\langle\langle \mathbb{P}_\zeta: \nu(n) \leq \zeta \leq \alpha\rangle, \langle \pi_\zeta: \nu(n) \leq \zeta < \alpha\rangle\rangle$$

is an α-stage iterated forcing construction in $\mathcal{M}_{\nu(n)}$ for each $n \in \omega$, the claim shows that for each $n \in \omega$ there is $t_n \in \mathcal{M}_\nu \cap {}^\omega\omega$ such that for each $r \in \mathcal{M}_{\nu(n)} \cap {}^\omega\omega$ and each $A \in \mathcal{M}_{\nu(n)} \cap [\omega]^\omega$, we have

$$\{i \in A: r(i) < t_n(i)\} \in \mathcal{M}_\nu \cap [\omega]^\omega .$$

Therefore, $s_\nu \in \mathcal{M}_\nu$ defined by $s_\nu(i) = \Sigma_{n \leq i} t_n(i)\ \forall i \in \omega$ will be the desired function.

In order to prove our claim, we first recall that $\mathbb{1}$, the largest element of $\mathbb{P}_\alpha$, is the only element of $\mathbb{P}_0$. Let $\psi_0: {}^0\omega \to \mathbb{P}_0$ be defined by $\psi_0(\emptyset) = \mathbb{1}$. Suppose that for a given $n \in \omega$ we have constructed $\psi_m: \bigcup_{k \leq m} {}^k\omega \to \mathbb{P}_{\nu(m)}$ for each $m \leq n$ such that

(1) $\psi_m \subseteq \psi_n$, and

(2) if $m < n$, $f \in {}^m\omega$, and if $f \subseteq g \in {}^{m+1}\omega$, then $\psi_n(g) \leq \psi_n(f)$ and (as elements of $\mathbb{P}_\nu$) $\psi_m(g)\restriction_n = \psi_n(f)\restriction_n$, and

(3) $\psi_m({}^m\omega)$ is a maximal pairwise-incompatible family of $\mathbb{P}_{\nu(m)}$.

In order to construct ψ_{n+1} observe that for each $g \in {}^n\omega$, $\psi_n(g)$ has a maximal (with respect to $\subseteq$) family I_g of infinitely many pairwise incompatible predecessors in $\mathbb{P}_{\nu(n+1)}$ such that $\psi_n(g)\restriction_n = x\restriction_n$ for all $x \in I_g$. Since $\mathbb{P}_\alpha$ is ccc and $\mathbb{P}_{\nu(n+1)} \subset_c \mathbb{P}_\alpha$, I_g is countable. So we define $\psi_{n+1} = \psi_n \cup (\bigcup_g \varphi_g)$, where φ_g is, for each $g \in {}^n\omega$, a bijection from $\{h \in {}^{n+1}\omega: g \subseteq h\}$ onto I_g. In order to see that the maximality in (3) is satisfied we suppose $q \in \mathbb{P}_{\nu(n+1)}$. Then $\mathbb{P}_{\nu(n)} \subset_c \mathbb{P}_{\nu(n+1)}$ implies there is a $p \in \mathbb{P}_{\nu(n)}$ such that p' and q are compatible whenever $p' \in \mathbb{P}_{\nu(n)}$ and $p' \leq p$. According to (3) there is a $G \in {}^n\omega$ such that $\psi_n(g)$ and p are compatible in $\mathbb{P}_{\nu(n)}$. So there is a $p'' \in \mathbb{P}_{\nu(n)}$ with $p'' \leq \psi_n(g)$ and $p'' \leq p$. Thus, there is a q' in $\mathbb{P}_{\nu(n+1)}$ such that $q' \leq q$ and $q' \leq p''$. By definition q', and hence q, is compatible with some member of I_g. Therefore, (3) is satisfied for $n + 1$.

By induction ψ_n is defined for each $n \in \omega$. So (1) implies $\psi = \bigcup_n \psi_n: \mathbb{C} \to \mathbb{P}_\nu = \bigcup_n \mathbb{P}_{\nu(n)}$ (recall that support is finite) is a function. Condition (2) shows ψ is order-preserving, while condition (3) taken inductively shows ψ is incompatible-preserving and that each $p \in \mathbb{P}$ has a reduction to $\psi(\mathbb{C})$. Thus, ψ is a complete embedding. $\square$

6.7. THEOREM. *Assume that $\mathcal{M}$ is a* (countable transitive) *model of* ZFC, *and in $\mathcal{M}$ α is a limit ordinal of uncountable cofinality. Suppose G is a $\mathbb{P}_\alpha$-generic filter over $\mathcal{M}$, where*

$$\langle\langle \mathbb{P}_\xi: \xi \leq \alpha\rangle, \langle \pi_\xi: \xi < \alpha\rangle\rangle$$

is an α-stage ccc iterated forcing construction in $\mathcal{M}$ with finite support. Then $\mathcal{M}_\alpha \models \Box_i X(i)$ is paracompact whenever $X(i)$ is a compact metrizable space for each $i \in \omega$.

PROOF. For simplicity set $\mathcal{A}_\xi = \mathcal{M}_\xi \cap [\omega]^\omega$, $W_\xi = \mathcal{M}_\xi \cap {}^\omega\omega$, and $\nabla_\xi = q(\Pi_i X(i)_\xi)$. Then 6.3 implies $\mathcal{A}_\alpha = \bigcup_{\xi<\alpha} \mathcal{A}_\xi$ and $W_\alpha = \bigcup_{\xi<\alpha} W_\xi$. Since $\omega < \mathrm{cf}(\alpha)$, $\nabla_i X(i) = \bigcup_{\xi<\alpha} \nabla_\xi$. In addition, since $\{\xi < \alpha: \mathrm{cf}(\xi) = \omega\}$ is cofinal in α, we may apply 6.6 to assume, without loss of generality, that for each $\xi < \alpha$ $\exists s_\xi \in W_{\xi+1}$ such that

$$\{i \in A: r(i) < s_\xi(i)\} \in \mathcal{A}_{\xi+1} \quad \forall (A, r) \in \mathcal{A}_\xi \times X_\xi .$$

In $\mathcal{M}_\alpha$ we suppose that $\forall_i \in \omega$ $(D(i, n): n \in \omega)$ is a compatible normal sequence of entourages of $X(i)$. Given an $r \in W_\alpha$ we define $Er = \bigcap_{n\in\omega} \nabla_i D(i, r(i) + n)$. Then the proof of 3.4(ii) shows $\{Er: r \in W_\alpha\}$ to be compatible unif. base of clopen equivalence relations in $\nabla_\alpha = \nabla_i X(i)$.

Suppose $\mathcal{R}$ is an open cover of ∇_α. In order to complete the proof, we define, by recursion on α, an increasing sequence $(\mathcal{S}_\xi: \xi < \alpha)$ of families of pairwise-disjoint clopen sets of ∇_α such that $\bigcup_{\xi<\alpha} \mathcal{S}_\xi$ refines $\mathcal{R}$.

For $u \in \nabla_\alpha$ define

$$\lambda(u) = \inf\{\xi < \alpha: \exists r \in W_\xi \ \exists R \in \mathcal{R} \text{ with } Er[u] \subseteq R\} .$$

Suppose $\lambda < \alpha$ is given such that for each $\xi < \lambda$ $\mathcal{S}_\xi$ has been defined subject to the restrictions (1)–(4) below.

(1) $\mathcal{S}_\xi$ is a pairwise-disjoint family of clopen sets of ∇_α.

(2) $\zeta < \xi \Rightarrow \mathcal{S}_\zeta \subseteq \mathcal{S}_\xi$.

(3) $\{u \in \nabla_\xi: \lambda(u) \leq \xi\} \subseteq \bigcup \mathcal{S}_\xi$.

(4) $S \in \mathcal{S}_\xi \Rightarrow \exists \zeta \leq \xi \ \exists \mu \in \nabla_\zeta \ \exists r, \ s \in W_{\zeta+1}$, and $\exists R \in \mathcal{R}$ such that $(Er \cap Es)[u] = S \subseteq R$.

Let $\mathcal{S} = \bigcup_{\xi<\lambda} \mathcal{S}_\xi$ and $U = \{u \in \nabla_\lambda - \bigcup \mathcal{S}: \lambda(u) \leq \lambda\}$. Given $u \in U$ we choose $r_u \in W_{\lambda(u)}$ with $Er_u[u] \subseteq R$ for some $R \in \mathcal{R}$. Define $\mathcal{S}' = \{(Er_u \cap Es_\lambda)[u]: u \in U\}$, and define $\mathcal{S}_\lambda = \mathcal{S} \cup \mathcal{S}'$.

In order to see $\mathcal{S}'$ is pairwise-disjoint we consider two distinct members of ∇_λ; i.e., suppose that $x, \ y \in \Pi_i X(i)_\lambda$ are such that $q(x) \neq q(y)$. Let $A = \{i \in \omega: x(i) \neq y(i)\}$. According to 6.4, $A \in \mathcal{A}_\lambda$ and there is an $r \in W_\lambda$ such that $(x(i), y(i)) \notin D(i, r(i))$ $\forall i \in A$. Therefore, $(x(i), y(i)) \notin D(i, s_\lambda(i))$ for each i belonging to some infinite subset of A. So $(q(x), q(y)) \notin \nabla_i D(i, s_\lambda(ii))$. Since Es_λ is an equivalence relation, $Es_\lambda[q(x)] \cap Es_\lambda[q(y)] = \emptyset$.

In order to complete the proof that $\mathcal{S}_\lambda$ is pairwise-disjoint, we suppose $x \in \Pi_i X(i)_\lambda$, $\xi < \lambda$, $y \in \Pi_i X(i)_\xi$, $r \in W_{\xi+1}$, and we suppose $q(x) \notin Er[q(y)]$. Analogous to the argument of the previous paragraph, $\exists n \in \omega \ \exists A \in \mathcal{A}_\lambda$ such that

$$\begin{aligned} i \in A \ &\Rightarrow \ (x(i), y(i)) \notin D(i, r(i) + n) \\ &\Rightarrow D(i, r(i) + n + 1)[x(i)] \cap D(i, r(i) + n + 1)[y(i)] = \emptyset \end{aligned}$$

But $\{i \in A: r(i) + n < s_\lambda(i)\} \in \mathscr{A}_{\lambda+1}$, so

$$D(i, s_\lambda(i))[x(i)] \cap D(i, r(i) + n + 1)[y(i)] = \emptyset$$

for infinitely many $i \in \omega$. Therefore, $Er[q(x)] \cap Es_\lambda[q(y))] = \emptyset$. According to (4), $\mathscr{S}_\lambda$ is pairwise-disjoint.

Our recursion hypothesis for α is now determined to be complete. For $\bigcup_{\xi<\alpha} \mathscr{S}_\xi$, we see that (1) and (2) implies it is pairwise-disjoint family of clopen sets, (3) implies it covers ∇_α, and (4) implies it is a refinement of $\mathscr{R}$. □

6.8. REMARKS. (i) The famous example of ccc iterated forcing is the consistency proof of $MA + \rceil CH$ [KUNEN 1980, pg. 278–281]. In fact $MA \Rightarrow \mathfrak{b} = \mathfrak{c}$ (and hence, $\nabla_i X(i)$ is $\mathfrak{c}$-metrizable whenever $X(i)$ is compact first countable for each $i \in \omega$, as can be determined from an adaptation of [RUDIN 1975, pgs. 40 and 60].

(ii) In order to obtain models of $\mathfrak{b} < \mathfrak{d} < \mathfrak{c}$ one does not have to rely exclusively upon 6.1 or even iterated forcings of uncountable length. [ROITMAN 1979] has investigated a few of these models and found enough similarity to the conclusion of 6.2, 6.3, 6.4, and 6.6 exist to allow the method of proof of 6.7 to go through.

(iii) At the time of this writing it is still unknown whether '$\square^\omega \omega + 1$ is normal' is a theorem of ZFC.

6.9. NOTES. With some variance in proof, 6.3 through 6.7 may be all found in [ROITMAN 1978].

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Note added in proof

2.5 is incorrect unless "P-space" is replaced by the statement "each countable set is closed" (2.6 remains unaffected).

In 4.1 $X_\mathfrak{c}$ should be $(\bigcup_{\alpha<\mathfrak{c}} E[X_\alpha])^-$ instead of $E[X_\mathfrak{c}]$.

CHAPTER 5

Special Subsets of the Real Line

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

1. Introduction

The purpose of this chapter is to discuss some peculiar sets of real numbers and some of the methods for obtaining them. The first such set was constructed by Bernstein in 1908. He constructed a set of reals of cardinality the continuum which is neither disjoint from nor contains an uncountable closed set. His construction used transfinite induction and the fact that every uncountable closed set has cardinality the continuum. The kinds of singular sets which we will discuss can almost all be found in KURATOWSKI (1966, §40) "Totally imperfect spaces and other singular spaces".

We are going to be concerned with the topological notion of first category set or meager set and the fundamental properties of Lebesgue measure. Recall that a set is nowhere dense iff its closure has no interior. A set of reals is meager iff it is the countable union of nowhere dense sets. A set of reals is comeager iff it is the complement of a meager set. The Baire category theorem says that no complete metric space is meager in itself.

Let us establish some of the terminology to be used. $\mathbb{R}$ denotes the real line, $\mathbb{Q}$ the set of rationals, and $[0, 1]$ the closed unit interval. The symbol ω stands for the set $\{0, 1, 2, \ldots\}$ and 2 for the set $\{0, 1\}$. We will use ω_1 to denote the first uncountable cardinal and c to denote the cardinality of the continuum. The space ω^ω (Baire space) consists of the set of all functions from ω to ω. It is given the product topology where ω is given the discrete topology. This topology is most conveniently described as follows. Let $\omega^{<\omega}$ be the set of finite sequences of elements of ω. For any $s \in \omega^{<\omega}$ let

$$[s] = \{f \in \omega^\omega : s \subseteq f\},$$

and for any $n < \omega$ let $s\hat{\ }n$ be the finite sequence which begins like s and ends in n. The topology on ω^ω is given by letting $\{[s] : s \in \omega^{<\omega}\}$ be the basic open sets. Similarly 2^ω (Cantor space) is the space of all functions from ω to 2 given the product topology. The space 2^ω also has on it the product measure which is determined by declaring that for each $s \in 2^n$, $[s]$ has measure $(\frac{1}{2})^n$. Of course the space 2^ω is homeomorphic to Cantor's "middle thirds" set which is the set of all $x \in [0, 1]$ whose ternary expansion

$$x = \sum_{n=1}^{\infty} \frac{i_n}{3^n}$$

has only $i_n = 0$ or $i_n = 2$. There is a useful continuous map $\rho : 2^\omega \to [0, 1]$ defined by

$$\rho(x) = \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}.$$

This map is onto, one-to-one except on a countable set (the set of $x \in 2^\omega$ which are eventually one or eventually zero), and takes each $[s]$ onto an interval of the same measure. Furthermore if we identify (via characteristic functions) 2^ω with $P(\omega)$ (the set of all subsets of ω), then $X \subseteq Y$ implies $\rho(X) \leq \rho(Y)$. Let $[\omega]^\omega$ be the subspace of $P(\omega) = 2^\omega$ of infinite subsets of ω. Then ω^ω is homeomorphic to $[\omega]^\omega$ via the natural embedding $\sigma\colon \omega^\omega \to [\omega]^\omega$ where $\sigma(g)$ is the set contained in ω whose characteristic function is given by the sequence of zeros and ones:

$$0^{g(0)}1\,0^{g(1)}1\,0^{g(2)}1 \cdots$$

($g(0)$ zeros, then a one, then $g(1)$ zeros, then a one, etc.).

The space ω^ω is also homeomorphic to the set of irrationals. Let $\mathbb{Z}$ be the set of integers. Then clearly $\mathbb{Z}^\omega$ is homeomorphic to ω^ω. Construct a family of open intervals I_s for $s \in \mathbb{Z}^{<\omega}$ as follows. For each $n \in \mathbb{Z}$ let $I_{\langle n\rangle} = (n, n+1)$. Suppose we have already found I_s for some $s \in \mathbb{Z}^{<\omega}$. Let $\{I_{s^\frown n}\colon n \in \mathbb{Z}\}$ be a family of disjoint open subintervals of I_s ordered like $\mathbb{Z}$ and lying next to each other (i.e. the right hand end point of $I_{s^\frown n}$ is the left hand end point of $I_{s^\frown(n+1)}$, with union dense in I_s, and each having diameter less than $\frac{1}{2}$ the diameter of I_s. If we define $\tau\colon \mathbb{Z}^\omega \to R$ by

$$\{\tau(g)\} = \bigcap\{I_{g\restriction n}\colon n < \omega\},$$

then it is not hard to check that τ is a homeomorphism of $\mathbb{Z}^\omega$ and $R - H$ where H is the countable dense set of end points of the I_s's. If we give $\mathbb{Z}^\omega$ the lexicographical order, then τ is order preserving. Since H is order isomorphic to $\mathbb{Q}$ there is an order isomorphism

$$f\colon \mathbb{R} \to \mathbb{R}$$

taking H to $\mathbb{Q}$. I think this argument is roughly equivalent to the classical one using continued fractions (see ALEXANDROFF and URYSOHN (1927)).

2. Luzin and Sierpiński sets

Arguments using transfinite induction to construct singular sets of reals are certainly the most plentiful. Until Cohen's method of forcing arrived on the scene this was practically the only method used. Most of the time such a construction requires the continuum hypothesis (CH) or at least Martin's Axiom (MA). Most of the time I have refrained from pointing out the obvious generalization of an argument or definition under CH to one that works under MA.

In 1914 Luzin constructed, using the continuum hypothesis, an uncountable set of reals having countable intersection with every meager set. The same construction had been published in 1913 by Mahlo. But (as is not unusual in mathematics) such a set has become universally known as a Luzin set.

2.1. THEOREM (MAHLO (1913), LUZIN (1914)). *Assuming the continuum hypothesis there is a set of reals of cardinality the continuum which has countable intersection with every meager set.*

PROOF. Let $\{C_\alpha : \alpha < \omega_1\}$ be the set of all closed nowhere dense sets. Inductively choose x_α a real not in:

$$\{x_\beta : \beta < \alpha\} \cup \bigcup \{C_\beta : \beta < \alpha\}.$$

We can always make such a choice by the Baire category theorem. Then

$$X = \{x_\alpha : \alpha < \omega_1\}$$

is the required set. □

A similar construction using Borel sets of measure zero results in a set called a Sierpiński set. For some applications of Luzin and Sierpiński sets to topology see VAN DOUWEN, TALL, and WEISS (1977), TALL (1976), and TALL (1978).

2.2. THEOREM (SIERPIŃSKI (1924)). *Assuming the continuum hypothesis there is a set of reals of cardinality the continuum which has countable intersection with every measure zero set.*

HUREWICZ (1932) also used this argument to construct (assuming the continuum hypothesis) an uncountable set $X \subseteq \mathbb{R}^\omega$ with the property that every uncountable subset of X has infinite dimension. See also WALSH (1979).

A modern day construction of a Luzin set is to note that the generic set of reals in a Cohen extension is a Luzin set. Similarly the generic set of reals in Solovay's random real extension is a Sierpiński set. See Kunen's chapter for the details on this.

Assuming MA + $\neg$CH (Martin's axiom plus the failure of the continuum hypothesis) neither Luzin nor Sierpiński sets exist. This is because under MA any set of reals of cardinality less than the continuum is both meager and has measure zero (MARTIN–SOLOVAY (1970) see also SHOENFIELD (1975) and RUDIN (1977)).

KUNEN (1976) generalized this to show that assuming MA + $\neg$CH, if Y is a Hausdorff space without isolated points, then there are no Luzin sets in Y. It is easy to show that it is consistent with $\neg$CH that there exists a Luzin set and a Sierpiński set. One way to do this is to start with a model of $\neg$CH and iteratively with finite support forcing, add a Cohen real and a random real. In the resulting model the Cohen reals will be a Luzin set and the random reals will be a Sierpiński set. Another way is to note that a Luzin or Sierpiński set remains such when Sacks reals (perfect set forcing) are added. Already in 1938, ROTHBERGER knew that ω_1 was the best one could do.

2.3. THEOREM (ROTHBERGER (1938)). *Suppose X is a Luzin set and Y is a Sierpiński set. Then*

$$|X| = |Y| = \omega_1 .$$

PROOF. The following lemma is necessary to prove Theorem 2.3.

LEMMA. *If X is not meager (not measure zero) and X has cardinality κ, then the real line is the union of κ many measure zero sets (meager sets).*

PROOF. Let G be a comeager set of measure zero (see OXTOBY (1971), Corollary 1.7). Then the real line is covered by the family:

$$\{x + G : x \in X\} .$$

Because if z is not in $x + G$ for any $x \in X$, then $z - G$ is disjoint from X. But $z - G$ is comeager. A similar argument works for the dual statement. □

Since every uncountable subset of a Luzin set is a Luzin set, and Luzin sets are not meager, it follows that the existence of a Luzin set implies that the real line is the union of ω_1 many measure zero sets. Thus any Sierpiński set must have cardinality ω_1. Dually the existence of a Sierpiński set implies any Luzin set must have cardinality ω_1. □

It is possible to generalize the construction of Luzin sets as follows. Suppose I is a countably additive proper ideal of subsets of the real line containing all singletons. We call a set of reals I-Luzin iff it is uncountable and it has countable intersection with every element of I. A family $J \subseteq I$ generates I iff for all $X \in I$ there is a $Y \in J$ with $X \subseteq Y$. Assuming the continuum hypothesis and that the ideal I is generated by a family J of cardinality $\leqslant \omega_1$, it is easy to inductively construct an I-Luzin set. For example, if I is the ideal of meager sets (which is generated by the meager F_σ sets), then an I-Luzin set is just what Luzin constructed. Similarly if I is the ideal of measure zero sets (which is generated by the G_δ sets of measure zero), then an I-Luzin set is a Sierpiński set. An interesting extension of Luzin's construction is given by TALL (19 · ·). He shows that assuming Baumgartner's axiom and the continuum hypothesis an I-Luzin set can be constructed for any ideal I generated by fewer than 2^{ω_1} sets.

In MILLER (1976b) I give a result of Kunen and myself that for each α, $2 < \alpha < \omega_1$, there is an ideal I_α (generated by its Borel members) such that any I_α-Luzin set has Baire order α.

If we assume Martin's axiom and the failure of the continuum hypothesis it is often the case that in order to generalize results proved under the continuum hypothesis you must replace 'countable' by 'less than continuum' (see KUNEN

(1968)). Thus we might define a c-I-Luzin set as a set of reals X of cardinality the continuum meeting each member of I in a set of cardinality less than the continuum. In MILLER (1979a) I show that assuming Martin's axiom a c-I_α-Luzin set has Borel order between α and $\alpha+2$.

Clearly under Martin's axiom c-Luzin sets and c-Sierpiński sets exist. In MARTIN–SOLOVAY (1970) it is shown that Martin's axiom is equivalent to saying that for any σ-ideal I generated by its Borel members and satisfying the countable chain condition, the real line cannot be covered by fewer than continuum many elements of I. Thus Martin's axiom is equivalent to saying for any ideal I as above there is a c-I-Luzin set (assuming c is regular).

What should be the complete negation of Martin's axiom? More specifically consider the following question.

Question. Is the failure of the continuum hypothesis consistent with either of the following?

(1) For any partial order $\mathbb{P}$ with the countable chain condition and cardinality $\leq c$ there exists $\langle G_\alpha : \alpha < \omega_1 \rangle$ a sequence of $\mathbb{P}$-filters such that for any dense $D \subseteq \mathbb{P}$ all but countably many G_α meet D.

(2) For any non-trivial ideal I in the Borel sets with the countable chain condition there is an I-Luzin set of cardinality ω_1.

This question was motivated by the definable forcing axiom (DFA) of VAN DOUWEN–FLEISSNER (19··). The models of BELL–KUNEN (1981) and STEPRĀNS (1982) in which the continuum is $\aleph_{\omega_1}$ may be relevant. Recently BAUMGARTNER (19··) has shown that the following sentence is true in the side-by-side Sacks model. For every partial order $\mathbb{P}$ with the countable chain condition there are ω_1 dense sets in $\mathbb{P}$ such that no filter in $\mathbb{P}$ meets them all.

3. Concentrated sets and sets of strong measure zero

A set of reals X has strong measure zero iff given any sequence $\varepsilon_n > 0$ for $n < \omega$, X can be covered by a sequence of sets X_n each having diameter less than ε_n (BOREL (1919)). A set of reals X is concentrated on a set D iff for any open set G if $D \subseteq G$, then $X - G$ is countable (BESICOVITCH (1934)). BOREL (1919) conjectured that every strong measure zero set is countable. This conjecture is now known to be independent.

3.1. THEOREM. (i) (SZPILRAJN (MARCZEWSKI) (1938b)). *A set of reals X is a Luzin set iff X is uncountable and concentrated on every countable dense set of reals.*

(ii) (SIERPIŃSKI (1928)). *A set of reals X concentrated on a countable set has strong measure zero.*

PROOF. Part (i) is easy to prove since X is a Luzin set iff for every dense open G, $X - G$ is countable.

Part (ii) is proved as follows. Suppose X is concentrated on the set $D = \{d_N : n < \omega\}$. Given any sequence $\varepsilon_n > 0$ for $n < \omega$ let I_n be an open interval about d_n of diameter less than ε_{2n}. Let:

$$G = \bigcup \{I_n : n < \omega\}.$$

We have that $X - G$ is countable. Use the ε_{2n+1} for $n < \omega$ to cover $X - G$. □

3.2. THEOREM (LAVER (1976)). *It is consistent that every strong measure zero set is countable.*

Thus special set theoretic axioms or models of set theory must be used to construct uncountable strong measure zero sets, concentrated sets, etc. Assuming Martin's axiom c-Luzin sets exist and it is not hard to see that c-Luzin implies c-concentrated implies strong measure zero. The axiom $V = L$ implies the continuum hypothesis, thus Luzin and Sierpiński sets exist. In fact, since there is a Δ^1_2-well-ordering of the reals in L it is easy to show that there are Δ^1_2 Luzin sets and Δ^1_2 Sierpiński sets. See MOSCHOVAKIS (1980) Chapter 5 for descriptive set theory in L. Since a Sierpiński set is not measurable and a Luzin set does not have the property of Baire, neither of these sets can be $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$. An uncountable concentrated set cannot be $\mathbf{\Sigma}^1_1$ since it cannot contain a perfect subset. However the following is true.

3.3. THEOREM (ERDÖS, KUNEN, MAULDIN (1981)). *If $V = L$, then there is an uncountable Π^1_1 set which is concentrated on the rationals.*

Sierpiński asked whether every strong measure zero set is concentrated on some countable set. This was answered by BESICOVITCH (1942).

3.4. THEOREM (BESICOVITCH (1942)). *Assuming the continuum hypothesis there is a set of reals which has strong measure zero but is not concentrated on any countable set.*

PROOF. Construct (as in the construction of a Luzin set) a sequence P_α for $\alpha < \omega_1$ of disjoint nowhere dense perfect sets (perfect = closed, nonempty, and no isolated points) with the property that for any meager set M, M meets at most countably many of the P_α. For each $\alpha < \omega_1$ let $E_\alpha \subseteq P_\alpha$ be a (relativized to P_α) Luzin set. Then:

$$E = \bigcup \{E_\alpha : \alpha < \omega_1\}$$

has strong measure zero, but is not concentrated on any countable set. For any sequence $\varepsilon_n > 0$ for $n < \omega$ let G be an open dense set which is the union of intervals I_n each of diameter less than ε_{2n}. Then there is an $\alpha < \omega_1$ such that:

$$\bigcup\{E_\beta : \alpha < \beta < \omega_1\} \subseteq \bigcup\{P_\beta : \alpha < \beta < \omega_1\} \subseteq G .$$

But $\bigcup\{E_\beta : \beta \leqslant \alpha\}$ being the countable union of strong measure zero sets can be covered by a union of intervals J_n each of diameter less than ε_{2n+1}. E is not concentrated anywhere since for any countable D there is an α such that $P_\alpha \cap D = \emptyset$, and so the complement of P_α is an open set containing D but disjoint from the uncountable set E_α. $\square$

GARDNER (1979) generalized Besicovitch's argument and studies a hierarchy of sets which are not concentrated on any countable set, yet which do have strong measure zero.

An interesting characterization of strong measure zero sets was found by GALVIN, MYCIELSKI, and SOLOVAY (1979).

3.5. THEOREM (GALVIN, MYCIELSKI, and SOLOVAY (1979)). *A set X has strong measure zero iff for any meager set G there exists a real x such that $(x + X) \cap g = \emptyset$.*

The implication from right to left is trivial. Given any sequence of $\varepsilon_n > 0$ let Q be an open dense set which is a union of intervals I_n of length ε_n. By assumption there exists x such that $x + X \subseteq Q$ and therefore $X \subseteq \bigcup_{n<\omega}(I_n - x)$. The implication from left to right is harder. For simplicity let's assume that $X \subseteq [0, 1]$. We need the following lemma.

3.5.1. LEMMA. *For any closed nowhere dense C and closed interval J there exists an $\varepsilon > 0$ and a finite family F of closed subintervals of J such that for any interval $I \subseteq [0, 1]$ of length ε there exists $J' \in F$ such that $(J' + I) \cap C = \emptyset$.*

PROOF. Since C is nowhere dense, for any $x \in [0, 1]$ there exists I_x with $x \in I_x$ and $J_x \subseteq J$ such that

$$(J_x + I_x) \cap C = \phi .$$

By compactness finitely many I_x cover $[0, 1]$. Thus for some sufficiently small $\varepsilon > 0$ every interval I of length ε is contained in some I_x (i.e. ε smaller than the length of overlap of any two I_x in the finite cover will do). $\square$

Now we prove Theorem 3.5.

Suppose $G = \bigcup_{n<\omega} C_n$ where each C_n is closed nowhere dense and $C_n \subseteq C_{n+1}$. Using the Lemma construct a finitely branching tree $T \subseteq \omega^{<\omega}$ (ordered by inclusion) along with J_s and ε_s for each $s \in T$ satisfying:

(1) $J_{s\hat{}n}$ is a closed subinterval of J_s;
(2) $\varepsilon_s > 0$; and
(3) for any $s \in \omega^n \cap T$ if $I \subseteq [0,1]$ has length less than ε_s, then for some $s\hat{}m \in T$

$$(J_{s\hat{}m} + I) \cap C_n = \emptyset.$$

For $n < \omega$ define $\delta_n = \min\{\varepsilon_s : s \in \omega^n \cap T\}$. Since T is finitely branching $\delta_n > 0$. Since X has strong measure zero there exists I_n of length less than δ_n such that

$$X \subseteq \bigcap_{m<\omega} \bigcup_{n>m} I_n$$

By the construction of T there exists $f \in \omega^\omega$ such that for all n

$$(J_{f\restriction(n+1)} + I_n) \cap C_n = \emptyset .$$

Letting $x \in \bigcap_{n<\omega} J_{f\restriction n}$ we get that

$$(x + \bigcap_{m<\omega} \bigcup_{n>m} I_n) \cap \bigcup_{n<\omega} C_n = \emptyset . \quad \square$$

This result suggests we define X to have strong first category or to be strongly meager iff for every set H of measure zero there exists a real x such that

$$(x + X) \cap H = \emptyset .$$

The consistency of the dual Borel conjecture has been shown by Carlson

3.6. THEOREM (CARLSON (19· ·a)). *It is consistent that every strong first category set is countable.*

In fact, he shows this holds in the Cohen real model. Although it is not difficult to show that some uncountable Sierpiński sets have strong first category, I don't know if all do.

Question (Galvin). Does every Sierpiński set have strong first category?

4. σ-sets and Q-sets

A set of reals X is a σ-set iff every subset of X which is a relative G_δ is also a relative F_σ.

4.1. THEOREM (SZPILRAJN (MARCZEWSKI) (1930)). *If X is a Sierpiński set, then X is a σ-set.*

PROOF. Suppose A is any Borel (indeed measurable) set. Then there is an F_σ set $F \subseteq A$ such that $A - F$ has measure zero (see ROYDEN (1971), Chapter 3. Section 3, Proposition 15). Since X is a Sierpiński set: $(A - F) \cap X$ is countable, and therefore an F_σ set F_0. Thus we have:

$$A \cap X = (F \cup F_0) \cap X$$

and so A is a relative F_σ. $\square$

A natural generalization of a σ-set is the notion of Q-set. A set of reals X is a Q-set iff every subset of X is a relative F_σ. Q-sets are also studied because of their connection with the normal Moore space problem (see Fleissner's chapter and also FLEISSNER (1978) and PRZYMUSIŃSKI (1977)).

4.2. THEOREM (SILVER, see MARTIN–SOLOVAY (1970) or RUDIN (1977)). *Assuming Martin's axiom every set of reals of cardinality less than the continuum is a Q-set.*

4.3. THEOREM (MILLER (1979b), Theorem 22). *It is consistent that there are no uncountable σ-sets, in fact, it is consistent that every uncountable set of reals has Baire order ω_1.*

Under MA + $\neg$CH + $\omega_1^V = \omega_1^L$ there are Π_1^1 sets of cardinality ω_1, therefore it is consistent to have uncountable Π_1^1 sets which are Q-sets. Also, it is easy to show that it is consistent with the continuum hypothesis that there are uncountable Π_1^1 sets which are σ-sets.

While a σ-set need not have measure zero (e.g. a Sierpiński set is not measurable), Q-sets must have measure zero.

4.4. THEOREM (LUZIN, see FLEISSNER (1978)). *Every Q-set has universal measure zero.*

However σ-sets are meager (see the next section).

Question (FLEISSNER (1978)). Do all Q-sets have strong measure zero?

For any countable ordinals α, we say that a set of reals X has Baire order $\leqslant\alpha$ iff for every Borel set C there is a Borel set B of rank $\leqslant\alpha$ such that $X \cap C = X \cap B$. If there is no such countable α, then X has Baire order ω_1. POPROUGÉNKO (1930) showed that the Baire order of a Luzin set is 3. A Q-set is a Q_2-set. A Q_α-set is an uncountable set of reals of Baire order α with the property that every subset is a relative Borel set.

4.5. THEOREM (MILLER (1979a)). (i) *It is consistent that for every* α, $2 \leq \alpha < \omega_1$, *there is an uncountable* Q_α*-set.*

(ii) *If* X *is a set of reals such that every subset of* X *is relatively Borel, then* X *is a* Q_α*-set for some* $\alpha < \omega_1$.

In HANSELL (1980) a Q_A set is defined to be any set of reals such that every subset is (relatively) analytic. He, in fact, is concerned there with arbitrary topological spaces. For further work in this direction see BALOGH–JUNNILA (19··) and FREMLIN–HANSELL–JUNNILA (19··).

In MILLER (1981a) I show that it is consistent to have a Q_A set which has Baire order ω_1. Theorem 4.3 of MILLER (1979b) shows that it is consistent to have a set of reals X of Baire order ω_1 such that every relatively analytic set is relatively Borel. One natural question here is:

Question (Mauldin). Is it consistent to have a set of reals X of bounded Baire order, but not every relatively analytic set is relatively Borel?

In BROWN (1977) it is shown that the continuum hypothesis implies that there is a set of reals concentreated on the rationals with Baire order ≥ 4. In FLEISSNER–MILLER (1980) we show that it is consistent to have an uncountable Q-set which is concentrated on the rationals. Using a similar argument we can show for any $\alpha \leq \omega_1$, that it is consistent with the continuum hypothesis to have a set of reals of Baire order α which is concentrated on the rationals. In fact, the continuum hypothesis implies there is an uncountable σ-set concentrated on a countable set. (See Theorem 5.7.)

5. Universal measure zero sets, perfectly meager sets, λ, λ' and s_0-sets, and Hausdorff gaps

In this section we consider the weakest properties of Luzin and Sierpiński sets. A set of reals X has universal measure zero iff for all measures μ on the Borel sets there is a Borel set of μ-measure zero covering X. By measure we always mean a countably additive, atomless (i.e. points have measure zero), finite measure. Alternatively let $\mathscr{B}$ be the σ-algebra of (relative) Borel subsets of a set X. Then X has universal measure zero iff for any measure μ on $\mathscr{B}$, $\mu(X) = 0$. The following result is classical.

5.1. THEOREM (SZPILRAJN (MARCZEWSKI) (1934)). *Every strong measure zero set has universal measure zero.*

PROOF. Let μ be any measure on the Borel sets. For any $\varepsilon > 0$ there is a $\delta > 0$ such that if I is any closed subinterval of $[0, 1]$ of diameter less than δ, then $\mu(I) < \varepsilon$. To prove this note that every point has neighborhoods of arbitrarily small μ-measure, since the point has measure zero. So for each $x \in [0, 1]$ let

$x \in I_x$, $\mu(I_x) < \varepsilon$ and let I_x^* be the middle one third of I_x. By compactness there is a finite $F \subseteq [0, 1]$ such that $\{I_x^*: x \in F\}$ covers $[0, 1]$. Let δ be the minimum diameter of I_x^* for $x \in F$. If J is any interval of diameter less than δ there is an I_x for some $x \in F$ with $J \subseteq I_x$ and so $\mu(J) < \varepsilon$.

From this it is easy to see that any strong measure zero set has μ-measure zero. □

Somewhat analogously (to the notion of universal measure zero set) we say that a set of reals X is perfectly meager iff for all perfect sets P the set $X \cap P$ is meager relative to the topology of P. This is also sometimes called 'always of first category' (LUZIN (1933)). A slight weakening of the notion of σ-set is that of a λ-set (or rarefied set, introduced by KURATOWSKI (1933)). A set X is a λ-set iff every countable subset of X is a relative G_δ in X. λ-sets were used by F.B. JONES (1937) to construct a pseudonormal Moore space. This served as the inspiration for nonmetrizable normal Moore spaces based on Q-sets (see also MCAULEY (1956)).

5.2. THEOREM. *If X is a λ-set, then X is perfectly meager.*

PROOF. Suppose P is any perfect set and let $F \subseteq P \cap X$ be a countable set dense in $P \cap X$. Since X is a λ-set it is easy to get a G_δ set G comeager in P with $G \cap X = F$. And hence Y is meager in P. □

The existence of uncountable sets of universal measure zero and uncountable perfectly meager sets does not require any axioms beyond the usual Zermelo–Fraenkel with the axiom of choice.

5.3. THEOREM. *There exists a set of reals X of cardinality ω_1 which has universal measure zero and is perfectly meager.*

PROOF. Select one element from each of the constituents of a nontrivial coanalytic set, then this set will have universal measure zero and will be perfectly meager. More explicitly, let WO be the set of elements of $2^{\omega\times\omega}$ which are the characteristic functions of well-orderings of ω. Then WO is coanalytic (Π_1^1) (see MOSCHOVAKIS (1980), p. 192) and hence universally measurable and has the property of Baire everywhere. For each countable ordinal α choose $x_\alpha \in$ WO of order type α. Then $X = \{x_\alpha: \alpha < \omega_1\}$ is perfectly meager and has universal measure zero. To see this suppose μ is any Borel measure. Since WO is μ-measurable there exists a Borel set B and a μ-measure zero set M such that WO $= B \cap M$. But by the boundedness theorem (see MOSCHOVAKIS (1980) p. 196) there is an $\alpha < \omega_1$ such that:

$$B \subseteq \{x \in \mathrm{WO}: \text{order type of } x \text{ is less than } \alpha\}.$$

Thus X has μ-measure zero.

A similar argument shows X is perfectly meager. □

LUZIN (1921) was the first to construct an uncountable perfectly meager set. The idea of using analytic sets seems to be a joint result of Sierpiński and Luzin (see SIERPIŃSKI (1934)). HAUSDORFF (1934) also gave a proof of this theorem using his famous (ω_1, ω_1^*)-gap (see Theorem 5.5). An interesting proof due to Todorčević uses an Aronszajn tree of perfect sets (see Todorčević's Chapter).

One cannot in general do better than ω_1 since it is a theorem of Baumgartner and Laver that in the random real model every universal measure zero set has cardinality less than or equal to ω_1 (see MILLER (19··a) for a proof). The continuum can be made as large as desired in this model. Similarly in the iterated perfect set model (see BAUMGARTNER and LAVER (1979)), every perfectly meager set has cardinality less than or equal to ω_1. In this model $c = 2^{\omega_1}$ so there are only continuum many universal measure zero sets and continuum many perfectly meager sets.

Question (Mauldin). Are there always more than c absolutely measurable sets and more than c sets with the restricted Baire property?

See GRZEGOREK and RYLL–NARDZEWSKI (19··) and FENSTAD and NORMANN [1974] for some related results.

The following theorem of Grzegorek was known assuming the continuum hypothesis or Martin's axiom. The point here is that he uses nothing beyond the usual axioms of set theory (ZFC).

5.4. THEOREM (GRZEGOREK 1980) (1981) (19··). (i) *If κ is the cardinality of the smallest nonmeasurable set of reals, then there is a universal measure zero set of cardinality κ.*

(ii) *If κ is the cardinality of the smallest nonmeager set, then there is a perfectly meager set of cardinality κ.*

(iii) *There is a set of reals X which has universal measure zero but does not have strong measure zero.*

We say that X is a λ'-set iff for every countable set of reals F, contained in X or not, F is a relative G_δ in $X \cup F$. It is easy to show that X is a λ'-set iff for every countable set F, $X \cup F$ is a λ-set (i.e. if $H \subseteq X \cup F$ is countable and G is a G_δ set such that

$$G \cap (X \cup H) = H,$$

then

$$G - (F - H) \cap (X \cup F) = H.)$$

Hausdorff actually proved that there is a set of reals of cardinality ω_1 which has universal measure zero and is a λ'-set (hence perfectly meager). The argument for λ'-set was pointed out by SIERPIŃSKI (1945). Define for X and Y sets of natural numbers, $X \subseteq^* Y$ iff $Y - X$ is finite.

5.5. THEOREM (HAUSDORFF (1934)). *There exist* $\langle X_\alpha : \alpha < \omega_1 \rangle$ *and* $\langle Y_\alpha : \alpha < \omega_1 \rangle$ *such that*:

(i) *for* $\alpha < \beta < \omega_1$, $X_\alpha \subseteq^* X_\beta \subseteq^* Y_\beta \subseteq^* Y_\alpha$;

(ii) *there does not exist* Z *with* $X_\alpha \subseteq^* Z \subseteq^* Y_\alpha$ *for all* $\alpha < \omega_1$.

The proof is also given in LAVER (1976). Laver also gives the proof that the Hausdorff gap has universal measure zero. Let us now see that the gap is a λ'-set. Recall that we identify subsets of ω with their characteristic functions. Let:

$$X = \{X_\alpha : \alpha < \omega_1\} \cup \{Y_\alpha : \alpha < \omega_1\}$$

be the Hausdorff gap from Theorem 5.5. Since the Cantor set is a closed subset of the real line we only have to worry about countable F contained in 2^ω. Define for each $\alpha < \omega_1$:

$$F_\alpha = \{A \subseteq \omega : X_\alpha \subseteq^* A \subseteq^* Y_\alpha\}.$$

Note that for any $B \subseteq \omega$, $\{A \subseteq \omega : A \subseteq B\}$ and $\{A \subseteq \omega : B \subseteq A\}$ are closed sets. Hence each F_α is an F_σ-set.

Let G_α be the complement in 2^ω of F_α. The G_α are strictly increasing G_δ sets whose union is all of 2^ω and for any $\alpha < \omega_1$, $X \cap G_\alpha$ is countable. For any countable $F \subseteq 2^\omega$ there is an $\alpha < \omega_1$ with $F \subseteq G_\alpha$. Now:

$$K = (X \cup F) \cap G_\alpha$$

is countable and hence $K - F$ is F_σ and

$$F = (X \cup F) \cap (G_\alpha - (K - F)).$$

Since F was arbitrary we see that X is a λ'-set. For some other uses of Hausdorff gaps see VAN DOUWEN (1976) and NYIKOS and VAUGHAN (19··).

Rothberger showed that not every λ-set is a λ'-set. The following is a key observation: From the introduction we know that every closed subset of the unit interval disjoint from the rationals corresponds to a compact subset of ω^ω. It is easy to show that for every compact subset C of ω^ω there is an $f \in \omega^\omega$ such that:

$$C \subseteq \{g \in \omega^\omega \mid \text{for all } n < \omega,\ g(n) < f(n)\}.$$

Define for $f, g \in \omega^\omega$, $f <^* g$ iff for all but finitely many $n < \omega$, $f(n) < g(n)$. Note that for any countable set $F \subseteq \omega^\omega$ there exists $f \in \omega^\omega$ such that for all $g \in F$, $g <^* f$. For any $f \in \omega^\omega$ let

$$C_f = \{g \in \omega^\omega \mid g <^* f\}.$$

Then C_f is an F_σ (in fact the countable union of compact sets). And for any F_σ set F disjoint from the rationals we have that for some $f \in \omega^\omega$, $F \subseteq C_f$.

5.6. (THEOREM) (ROTHBERGER (1939)). (i) *Not every λ-set is a λ'-set.*

(ii) *Assuming the continuum hypothesis there is an uncountable λ-set concentrated on the rationals.*

PROOF. Assuming the continuum hypothesis we can find a set:

$$X = \{f_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$$

such that

(a) $\alpha < \beta$ implies $f_\alpha <^* f_\beta$;

(b) for all $f \in \omega^\omega$ there exists $\alpha < \omega_1$ with $f <^* f_\alpha$.

To see that X is a λ-set consider the G_δ sets, $G_\alpha = \{f \in \omega^\omega$: for infinitely many n, $f(n) < f_\alpha(n)\}$ (just use (a)). However by (b) and earlier remarks we see that for any G_δ set G containing the rationals, $X - G$ is countable. So X is concentrated on the rationals and thus not a λ'-set. This proves (ii).

Part (i) is proved using similar ideas, but no hypothesis beyond ZFC is used. Rothberger shows that the least cardinality of an unbounded subset of ω^ω is also the cardinality of a λ-set which is not a λ'-set. See VAN DOUWEN'S Chapter (10.2) for some similar arguments. □

It is worth noting that the set X we have constructed while being concentrated on the rationals is a λ'-set with respect to the irrationals. The construction of an order type ω_1 subset of $(\omega^\omega, <^*)$ is in fact LUZIN'S (1921) original construction of an uncountable perfectly meager set.

By a slight modification of Rothberger's argument we can show:

5.7. THEOREM. *Assuming the continuum hypothesis, there exists an uncountable σ-set which is concentrated on a countable set.*

PROOF. Construct $X_\alpha \subseteq \omega$ infinite for $\alpha < \omega_1$ such that $\alpha < \beta$ implies $X_\alpha \supseteq^* X_\beta$. By an argument similar to the one used in Theorem 5.6 if the X_α grow fast enough, then

$$X = \{X_\alpha : \alpha < \omega_1\}$$

will be concentrated on the set

$$[\omega]^{<\omega} = \{A : A \subseteq \omega \text{ is finite}\}.$$

By fast enough we mean, look at the natural map showing ω^ω and $[\omega]^\omega$ are homeomorphic (see introduction), then the image of X should satisfy (a) and (b) in the proof of Theorem 5.6. For any set A, $[A]^\omega$ is the set of (countably) infinite subsets of A. The GALVIN–PRIKRY Theorem (1973) says that for every Borel set $B \subseteq [\omega]^\omega$ and $X \in [\omega]^\omega$ there exists $Y \in [X]^\omega$ such that

$$[Y]^\omega \subseteq B \quad \text{or} \quad [Y]^\omega \cap B = \emptyset .$$

(This result has been extended from BOREL sets to $\mathbf{\Sigma}^1_1$ (analytic) sets by SILVER (1970) see also ELLENTUCK (1974)). The following lemma easily gives us Theorem 5.7.

5.7.1. LEMMA. *For any Borel set $B \subseteq [\omega]^\omega$ and $X \in [\omega]^\omega$ there exists $Y \in [X]^\omega$ such that B is a relative F_α set in $\{Z \in [\omega]^\omega : Z \subseteq^* Y\}$.*

PROOF. Construct a sequence $Y_{n+1} \in [Y_n]^\omega$ with $Y_0 = X$ as follows. Let a_n be the least element of Y_n. Repeatedly apply the Galvin–Prikry Theorem to obtain $H_n \subseteq \{A : A \subseteq a_n\}$ and X_{n+1} such that for all $Z \in [\omega]^\omega$ if $Z - a_n \subseteq X_{n+1}$, then $Z \in B$ iff $Z \cap a_n \in H_n$. Now let $Y = \{a_n : n < \omega\}$. Then for any $Z \subseteq^* Y$, $Z \in B$ iff there exists n such that $Z \cap a_n \in H_n$ and $Z - a_n \subseteq Y$. □

Since we are assuming the continuum hypothesis the theorem follows easily from the lemma. □

Next we consider the Baire order of λ-sets.

5.8. THEOREM (MAULDIN (1977)). *Assuming the continuum hypothesis there is a λ-set of Baire order ω_1.*

The proof uses the σ-algebra of abstract rectangles in the plane and also a key lemma on universal sets proved in BING, BLEDSOE, and MAULDIN (1974).

Note that if X is a λ-set in a model M of set theory and N is a countable chain condition forcing extension of M, then X remains a λ-set in N. Thus as a corollary to Theorem 3.5 (MILLER [1979a]) we see that it is consistent to have λ-sets of all possible Baire orders.

I conclude this section with a notion which is weaker than both universal measure zero and perfectly meager. We say that X is an s_0-set iff for every perfect set P there is a perfect set $Q \subseteq P$ such that Q is disjoint from X.

5.9. THEOREM. (SZPILRAJN (MARCZEWSKI) (1935a)). *If X has universal measure zero or X is perfectly meager, then X is an s_0-set.*

PROOF. Suppose x has universal measure zero. If P is any perfect set, then transfer the product measure on 2^ω via any homeomorphism with P to P and call it μ. Then there is a Borel set $B \subseteq P$ of ω measure zero such that $X \cap P \subseteq B$. But since $P - B$ is an uncountable Borel set it contains a perfect subset. A similarly argument works if X is perfectly meager since any comeager set contains an uncountable Borel set. □

In contrast to the case of perfectly meager and universal measure zero there are always large s_0-sets.

5.10. THEOREM. *There exists a s_0-set of cardinality the continuum.*

PROOF. Let P_α for $\alpha < c$ be all perfect subsets of the plane. For any set P in the plane and real x let:

$$P^x = \{y : \langle x, y\rangle \in P\}\,.$$

We construct an (s_0)-set x as follows. Let $\{x_\alpha : \alpha < c\}$ be the set of all real numbers. For each α choose y_α not an element of:

$$\bigcup\{P_\beta^{x_\alpha} : \beta < \alpha \text{ and } P_\beta^{x_\alpha} \text{ is countable}\}\,.$$

Now let $X = \{(x_\alpha, y_\alpha) : \alpha < c\}$. Suppose P is any perfect subset of the plane. If for some x_α, P^{x_α} is uncountable, then since $\{x_\alpha\} \times P^{x_\alpha} \cap X \subseteq \{(x_\alpha, y_\alpha)\}$ it is easy to get a perfect $Q \subseteq \{x_\alpha\} \times P^{x_\alpha}$ disjoint from X. On the other hand if P^{x_α} is countable for all $\alpha < c$, then by our construction $P \cap X$ has cardinality less than c. Partition P into continuum many disjoint perfect sets; then one of them misses X. □

For more on (s_0)-sets see MORGAN (1978) Example 3B.

6. Order type of the real line

Baumgartner generalized Cantor's theorem that any two countable dense linear orders are isomorphic. A set of reals X is ω_1-dense iff between any two reals there are ω_1 elements of X.

6.1. THEOREM (BAUMGARTNER (1973)). *It is consistent with Martin's axiom that any two ω_1 dense sets of reals are order isomorphic.*

His forcing argument was rather unusual in that it requires that the continuum hypothesis be true in intermediate models but in the final model it must fail. A similar argument occurs in BAUMGARTNER (1980).

Abraham and Shelah showed that Martin's axiom is not sufficient for this result.

6.2. THEOREM (ABRAHAM–SHELAH (1981)). *It is consistent with Martin's Axiom and the failure of the continuum hypothesis that not every two ω_1-dense sets are order isomorphic.*

Most of the classical work on order types contained in the real line is in SIERPIŃSKI (1950).

7. Unions

It is easy to show that the families of universal measure zero sets, perfectly meager sets, strong measure zero sets, Luzin sets, Sierpiński sets, and sets concentrated on the rationals are closed under countable union. Assuming Martin's Axiom in the case of universal measure zero and perfectly meager, countable can be replaced by less than continuum. This is also true for strong measure zero, but is not so obvious.

7.1. THEOREM (CARLSON 19· ·a)). *Assuming Martin's Axiom the union of less than continuum many strong measure zero sets has strong measure zero.*

We saw in Section 5 that there is a λ-set whose union with the rationals is not a λ-set. However, it is true that the family of λ' sets is closed under countable union.

7.2. THEOREM (SIERPIŃSKI (1937a)). *The countable union of λ' sets is a λ' set.*

PROOF. Suppose X_n for $n < \omega$ are λ'-sets and F is a countable set. Since X_n is a λ'-set there is a G_δ set G_n such that:

$$F = G_n \cap (X_n \cup F).$$

But then

$$F = \bigcap_{n<\omega} G_n \cap \Big(\bigcup_{n<\omega} X_n \cup F \Big). \quad \square$$

7.3. THEOREM (FLEISSNER–MILLER (1980)). *It is consistent that there is an uncountable Q-set which is concentrated on the rationals. So neither the family of Q-sets nor the family of σ-sets need be closed under finite union.*

To prove this theorem instead of constructing the set of reals we construct the model of set theory over the top of a set of Cohen reals.

8. Products

8.1. THEOREM (SZPILRAJN (MARCZEWSKI) (1937)). *The product of two universal measure zero sets has universal measure zero.*

PROOF. Suppose X and Y have universal measure zero and let μ be any measure on $X \times Y$. Define a measure ν on Y by:

$$\nu(B) = \mu(X \times B)\,.$$

Since X has universal measure zero so does $X \times \{y\}$ for any $y \in Y$ and so ν is atomless. Since Y has universal measure zero

$$\mu(X \times Y) = \nu(Y) = 0\,. \quad \square$$

8.2. THEOREM. *If X and Y are λ-sets (λ'-sets), then $X \times Y$ is a λ-set (λ'-set).*

PROOF. Suppose F is a countable subset of $X \times Y$. Then let $F_x \subseteq X$ and $F_y \subseteq Y$ be countable with $F \subseteq F_x \times F_y$. Since $F_x \times F_y$ is a relative G_δ so is $F = F_x \times F_y - (F_x \times F_y - F)$. A similar argument works for λ'-sets. $\square$

This contrasts sharply with the following theorem.

8.3. THEOREM (FLEISSNER (19··)). *It is consistent that there is a Q-set whose square is not a Q-set.*

However, the following result is true.

8.4. THEOREM (PRZYMUSIŃSKI (19··)). *If there is an uncountable Q-set, then there is one whose square is also a Q-set.*

PROOF. Let $R = \{A \times B\colon A, B \subseteq \omega_1\}$. The next two lemmas are needed to prove Theorem 8.4.

8.4.1. LEMMA. *The graph of any function from ω_1 to ω_1 is a countable intersection of finite unions of elements of R.*

PROOF. Let $f\colon \omega_1 \to \omega_1$ be an arbitrary function. Let $\{x_\alpha\colon \alpha < \omega_1\}$ be any set of distinct real numbers. Let q be the set of rational numbers. Then $f(\alpha) = \beta$ iff

$$\forall r \in Q \quad (r < x_{f(\alpha)} \leftrightarrow r < x_\beta)\,.$$

Hence the graph of f is

$$\bigcap_{r \in Q} ([\{\alpha\colon r < x_{f(\alpha)}\} \times \{\beta\colon r < x_\beta\}] \cup [\{\alpha\colon x_{f(\alpha)} \leqslant r\} \times \{\beta\colon x_\beta \leqslant r\}])\,. \qquad \square$$

8.4.2. LEMMA. *There exists a Q-set of cardinality ω_1 iff there exists a countable family $\mathscr{A}$ of subsets of ω_1 such that every subset of ω_1 is a countable union of countable intersections of elements of $\mathscr{A}$.*

PROOF. Let $Y = \{x_\alpha : \alpha < \omega_1\}$ be a Q-set and $\{C_n : n < \omega\}$ a clopen basis for Y closed under finite union and complementation. Then $\{C_n^* : n < \omega\}$ defined by

$$C_n^* = \{\alpha : x_\alpha \in C_n\}$$

has the required property. □

Now we prove Theorem 8.4.

Choose $f_n : \omega_1 \in \omega_1$ so that for each $\alpha < \omega_1$

$$\{f_n(\alpha) : n < \omega\} = \{B : \beta \leqslant \alpha\}.$$

For each $n < \omega$ let

$$D_n = \{(\alpha, f_n(\alpha)) : \alpha < \omega_1\} \quad \text{and} \quad E_n = \{(f_n(\alpha), \alpha) : \alpha < \omega_1\}.$$

From the lemmas we can find $\{A_n : n < \omega\}$ a family of subsets of ω_1 such that for each $n < \omega$, D_n and E_n are the countable intersections of finite unions of $\{A_n \times A_m : n, m < \omega\}$ and every subset of ω_1 is the countable union of countable intersections of elements of $\{A_n : n < \omega\}$. Now let us see that every subset of $\omega_1 \times \omega_1$ is the countable union of countable intersections of finite unions of $\{A_n \times A_m : n, m < \omega\}$ (i.e. F_σ). Suppose $A \subseteq \omega_1 \times \omega_1$. For any n let

$$X_n = \{\alpha : \exists \beta \ (\alpha, \beta) \in D_n \cap A\}.$$

Then since

$$D_n \cap A = (X_n \times \omega_1) \cap D_n$$

we see that $D_n \cap A$ is the countable union of countable intersections of finite unions of elements of $\{A_n \times A_m : n, n < \omega\}$. Similarly for $E_n \cap A$. But by the choice of the f_n,

$$\omega_1 \times \omega_1 = \bigcup_{n<\omega} D_n \cup \bigcup_{n<\omega} E_n$$

and so

$$A = \bigcup_{n<\omega} (D_n \cap A) \cup \bigcup_{n<\omega} (E_n \cap A).$$

The mapping $\sigma : \omega_1 \to 2^\omega$ defined by

$$\sigma(x)(n) = 0 \quad \text{iff} \quad x \in A_n$$

takes ω_1 onto a Q-set whose square is also a Q-set. □

8.5. THEOREM (SIERPIŃSKI (1935)). *Assuming the continuum hypothesis there exists a Luzin set whose square does not have strong measure zero.*

PROOF. We need the following lemma.

8.5.1. LEMMA. *For any real z and comeager set G there are points x and y in G with $z = x + y$.*

PROOF. Since G is comeager so is $z - G = \{z - g : g \in G\}$, let Y be an element of their intersection. Then $y = z - x$ for some $x \in G$. □

Using this lemma (and the continuum hypothesis) it is now easy to construct a Luzin set X such that for any real z there is an x and y in X such that $z = x + y$. I claim that X^2 is not of strong measure zero. To see this define π from the plane onto the reals by $\pi(x, y) = x + y$. Geometrically this is just a 45° projection onto the real axis. Thus a disk of diameter ε is taken to an interval of diameter $\sqrt{2} \cdot \varepsilon$. Hence the image under π of any strong measure zero set must have strong measure zero. But π takes X^2 onto the real line. □

Clearly the product of two uncountable sets of reals cannot be a Luzin set, a Sierpiński set, or a concentrated set, since horizontal lines are closed measure zero sets.

Question (SZPILRAJN (MARCZEWSKI) (1935b)). Is the product of two perfectly meager sets perfectly meager?

9. Continuous and homeomorphic images and C'' and C'-sets

It is not hard to see that every set homeomorphic to a perfectly meager set is perfectly meager. This is also true for universal measure sets and in fact it characterizes them.

9.1. THEOREM (SZPILRAJN–MARCZEWSKI (1937)). *A set of reals X has universal measure zero iff every set homeomorphic to X has Lebesgue measure zero.*

PROOF. Suppose X has universal measure zero and $\phi: X \to Y$ is a homeomorphism. For any μ a measure on Y define ν a measure on X by letting $\nu(B) = \mu(\phi(B))$. If ν vanishes so does μ. It follows that Y has Lebesgue measure zero. Now to prove the converse suppose X is any set of reals (which for simplicity we will assume is in the unit interval) and suppose every homeomorphic image of X has Lebesgue measure zero. Suppose μ is any measure on the reals such that X does not have μ-measure zero. We may assume that μ does not vanish on any interval since we could replace μ by $\frac{1}{2}(\mu + \lambda)$ where λ is Lebesgue measure. Now define f by

$$f(x) = \mu([0, x]).$$

Since μ vanishes on no intervals and is atomless we see that f is strictly increasing and continuous and thus a homeomorphism. Define

$$\nu(B) = \mu(f^{-1}(B))$$

for any Borel set B. We are done if we show that ν is Lebesgue measure. Suppose $I = [a, b]$ is any interval. Then

$$f^{-1}(I) = \{y: \mu([0, y]) \in I\} = [c, d]$$

where $\mu([0, c]) = a$ and $\mu([0, d]) = b$. But then

$$\nu(I) = \mu(f^{-1}(I)) = \mu([c, d]) = b - a.$$

Since ν agrees with Lebesgue measure on the intervals and the intervals generate the Borel sets we have that ν is Lebesgue measure. □

This characterization is not true for perfectly meager sets.

9.2. THEOREM (MORGAN (1979)). *There is a set of reals which is not perfectly meager but every set homeomorphic to it is meager.*

PROOF. Let K be the Cantor set and Q the rationals. Let $A \subseteq K$ be a set such that A and $K - A$ meet every perfect subset of K. Let $S = Q \cup A$. S is not perfectly meager since it cannot be meager relative to K. On the other hand S has the property that every nonempty open set U contains a nonempty open set V such that V is countable. But this is true of any homeomorphic image of S and is easily seen to imply first categoricity. □

Clearly any property which is purely topological is preserved by homeomorphisms. The homeomorphic image of a Luzin set is a ν-set—i.e. a set in which every (relatively) meager set is countable. The homeomorphic image of a Sierpiński set is still a Sierpiński set with respect to a different Borel measure. Since λ-sets, σ-sets, and Q-sets are all defined topologically these properties are all preserved by homeomorphisms (but not necessarily one-to-one continuous mappings). Before taking up the homeomorphism problem for λ'-sets, concentrated sets, and strong measure zero sets, we consider one-to-one continuous images.

9.3. THEOREM (KURATOWSKI (1933), SZPILRAJN (MARCZEWSKI) (1937)). *Every set of reals of cardinality ω_1 is the one-to-one continuous image of a set which is both perfectly meager (in fact, a λ'-set) and of universal measure zero. Thus assuming the continuum hypothesis the real line is the continuous image of a set which is both perfectly meager and of universal measure zero.*

PROOF. Let $X = \{x_\alpha : \alpha < \omega_1\}$ be an arbitrary set of reals, $Y = \{y_\alpha : \alpha < \omega_1\}$ be any λ'-set, and let $Z = \{z_\alpha : \alpha < \omega_1\}$ be any set of universal measure zero. Define

$$Q = \{(x_\alpha, y_\alpha, z_\alpha) : \alpha < \omega_1\} .$$

X is the projection of Q onto the first coordinate. We need the following lemma to prove Theorem 9.3.

9.3.1. LEMMA. *Suppose P and T are sets of reals, $f: P \to T$ is a one-to-one continuous map, and T is a* (a) *set of universal measure zero*; (b) *λ-set*; (c) *λ'-set*; *or* (d) *Q-set. Then P is also.*

PROOF. For universal measure zero note that if μ is any measure on P, then

$$\nu(B) = \mu(f^{-1}(B))$$

defines a measure on T. For a λ-set suppose $D \subseteq P$ is countable. Then $f(D)$ is a countable subset of R and so there is a G_δ set G with $G \cap R = f(D)$. Since f is one-to-one $f^{-1}(G) \cap P = D$. This same argument works for Q-sets. For λ'-sets, the following is what we mean. Suppose $f: X \to Y$ is continuous, with $P \subseteq X$ and $T \subseteq Y$, and f takes P one-to-one onto T. If T is a λ'-set with respect to Y, then P is a λ'-set with respect to X. To show that P is a λ'-set with respect to X we must show that for every countable $D \subseteq X$ there is a G_δ set G such that

$$G \cap (P \cup D) = D .$$

Suppose $D \subseteq X$ is countable and let G be a G_δ set in Y such that

$$G \cap (T \cup f(D)) = f(D) .$$

But then $f(D) \subseteq G$ implies $D \subseteq f^{-1}(G)$ and $f^{-1}(G) \cap P = \emptyset$ (since $f^{-1}(G) \cap P \subseteq f^{-1}(G) \cap f^{-1}(T) = f^{-1}(G \cap T) = \emptyset$. Thus

$$f^{-1}(G) \cap (P \cup D) = D .$$

This proves the lemma and the theorem immediately follows. □

This same quasi-diagonal type argument as in 9.3 shows that if we have a Q set of cardinality κ, then every set of reals of cardinality κ is the continuous image of a Q-set. Thus by FLEISSNER–MILLER (1980) it is consistent to have a Q-set whose continuous image is not a Q-set.

Lemma 9.3.1 is not true for perfectly meager sets since we will see (Theorem 9.7) a Luzin set can be mapped one-to-one to a perfectly meager set.

This lemma is also false for σ-sets, i.e. assuming X is any σ-set of cardinality the continuum there exists Y a non σ-set which can be continuously mapped one-to-one onto X. Suppose $X = \{x_\alpha : \alpha < c\}$. To construct Y, let H be any countable dense subset of 2^ω (i.e. any F_σ which is not G_δ) and let $\{G_\alpha : \alpha < c\}$ be all G_δ subsets of $X \times 2^\omega$. For each α choose y_α such that

$$y_\alpha \in (G_\alpha^{x_\alpha} - H) \cup (H - G_\alpha^{x_\alpha}) \quad \text{where } G_\alpha^{x_\alpha} = \{y : (x_\alpha, y) \in G_\alpha\}.$$

This is always possible since H cannot be G_δ in 2^ω. Now let $Y = \{(x_\alpha, y_\alpha) : \alpha < c\}$. The projection map takes Y onto X. The set Y is not a σ-set since $(X \times H) \cap Y$ is not G_δ in Y.

Question. Is it consistent to have a σ-set X which can be mapped continuously onto the reals?

9.4. THEOREM (ROTHBERGER (1941)). *Assuming the continuum hypothesis there exists a set concentrated on the rationals which can be mapped continuously onto 2^ω.*

PROOF. Let $X = \{f_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$ be a set of order type ω_1 under $<^*$ and for every $g \in \omega^\omega$ there exists $\alpha < \omega_1$ with $g <^* f_\alpha$. We saw in Theorem 5.6(ii) that any such set is concentrated on the rationals. Let $\{x_\alpha : \alpha < \omega_1\} = 2^\omega$. Define $f_\alpha^* \in \omega^\omega$ by

$$f_\alpha^*(n) = 2 f_\alpha(n) + x_\alpha(n)$$

for all $n < \omega$. Let $X^* = \{f_\alpha^* : \alpha < \omega_1\}$ and define $\pi : \omega^\omega \to 2^\omega$ by $\pi(f) = x$ iff for all $n < \omega$ $\pi(x)(n) = f(n) \pmod 2$. Then π maps X^* onto 2^ω. $\square$

Since a concentrated set has strong measure zero it follows that the continuous image of a strong measure zero set need not have strong measure zero.

9.5. THEOREM (SIERPIŃSKI (1945)). *Assuming the continuum hypothesis there is a concentrated set X which is homeomorphic to a set Y which is a λ'-set without strong measure zero.*

PROOF. Using Theorem 9.4 find $X \subseteq 2^\omega$ a concentrated set and a continuous $f: X \to S$ which is one-to-one and onto an uncountable Sierpiński set $S \subseteq 2^\omega$. Let $G = \{(x, f(x)) : x \in X\} \subseteq 2^\omega \times 2^\omega$. G is homeomorphic to X via the map $x \mapsto$

$(x, f(x))$. Let $\pi: 2^\omega \times 2^\omega \to 2^\omega$ be projection onto the 2^{nd} coordinate. Since π is continuous and one-to-one on G it follows from Lemma 9.3.1 that G is a λ'-set (since S is). On the other hand since π is uniformly continuous if G had strong measure zero, then so would $\pi(G) = S$. But S does not even have measure zero. Since the map which shows $2^\omega \times 2^\omega$ is homeomorphic to 2^ω is uniformly continuous we get a subset Y of 2^ω which is homeomorphic to G, is a λ'-set and does not have strong measure zero. □

This shows that the property of being strong measure zero is not topological but depends on the metric. In light of this it is perhaps surprising that the following is true.

9.6. THEOREM (CARLSON (19··b)). *If every strong measure zero set of reals is countable, then for every metric space X if X has strong measure zero, then X is countable.*

Sierpiński's question of whether or not the family of strong measure zero sets was closed under continuous image lead ROTHBERGER (1938) to consider two other classes of sets. Strong measure zero sets were also called C-sets or sets with property C. A set of reals has property C' iff for every family $\mathcal{G}_n$ of finite open covers there is a diagonal sequence $U_n \in \mathcal{G}_n$ such that

$$X \subseteq \bigcup_{n<\omega} U_n .$$

Rothberger showed that a set X is a C'-set iff every continuous image of X has strong measure zero. If we drop the condition that the covers $\mathcal{G}_n$ be finite we get the notion of a C''-set. It isn't hard to show that the continuous image of any C''-set is a C''-set. Also, any set concentrated on a countable subset of itself is a C''-set.

Question (Rothberger). *Is every C' set a C'' set?*

Recently there has been some work on very singular sets of cardinality the continuum. These sets can be looked at as generalizations of C'' sets. For reference see GALVIN–MILLER (19··).

Now let us consider the continuous image of a Luzin or Sierpiński set.

9.7. THEOREM (LUZIN (1933)). *There exists a one-to-one continuous function from ω^ω to ω^ω which takes every Luzin set to a perfectly meager set.*

PROOF. Let $\omega^{<\omega}$ be the set of finite sequences of ω. Let $\{P_s : s \in \omega^{<\omega}\}$ be a family of perfect subsets of ω^ω constructed as follows. Suppose we have P_s. Let $\{P_{s^\frown\langle n\rangle} : n < \omega\}$ be a family of disjoint perfect subsets of P_s each of which is nowhere dense relative to P_s and for any clopen set C if $P \cap C \neq \emptyset$, then there exists n such that

$$P_{s^\frown\langle n\rangle} \subseteq C \cap P .$$

Define f by

$$\{f(x)\} = \bigcap\{P_{x\restriction n} \colon n < \omega\} .$$

Now suppose Q is any perfect subset of ω^ω and let

$$R = \bigcup\{P_s \colon P_s \text{ is nowhere dense in } Q\} .$$

For any $s \in \omega^{<\omega}$ if P_s is not contained in Q, then for some n, $P_{s^\frown\langle n\rangle}$ is disjoint from Q. Otherwise, if P_s is contained in Q, then any $P_{s^\frown\langle n\rangle}$ is nowhere dense in Q. Hence $\{x \in \omega^\omega \colon f(x) \in R\}$ contains an open dense set. The result follows. □

The next theorem is proved in MILLER (19· ·a).

9.8. THEOREM. (a) *It is consistent that every set of reals of cardinality the continuum contains the one-to-one continuous image of a Luzin set of cardinality the continuum.*

(b) *It is consistent that every set of reals of cardinality the continuum contains the one-to-one continuous image of a Sierpiński set of cardinality the continuum.*

The model for (a) is the Cohen real model and the model for (b) is the random real model. Since the one-to-one preimage of a universal measure zero set has universal measure zero, (b) implies there are no universal measure zero sets of cardinality the continuum. Also, since the one-to-one preimage of a λ-set is a λ-set, (a) implies there are no λ-sets of cardinality the continuum.

10. Implications and definitions

10.1. IMPLICATIONS.

Luzin set ⇒ concentrated set ⇒ c''-set ⇒ c'-set ⇒

⇒ strong measure zero set ⇒ universal measure zero set ⇒ $(S)_0$-set.

$$\begin{tikzcd}
 & Q\text{-set} \arrow[d, Rightarrow] & & & \\
 & \sigma\text{-set} \arrow[dr, Rightarrow] & & & \\
\text{Sierpinski set} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \lambda\text{-set (rarefied)} \arrow[r, Rightarrow] & \text{Perfectly meager set} \arrow[r, Rightarrow] & (S)_0\text{-set.} \\
 & \lambda'\text{-set} \arrow[ur, Rightarrow] & & &
\end{tikzcd}$$

10.2. DEFINITIONS. In the following definitions X is a set of reals.

Luzin set (property L). For every meager set M, $X \cap M$ is countable.

Concentrated set. There exist a countable set of reals D such that for every open $G \supseteq D$, $X - G$ is countable.

C''-set (ROTHBERGER property). For every family $\{\mathcal{G}_n : n < \omega\}$ of open covers of X there exists $U_n \in \mathcal{G}_n$ for $n < \omega$ such that $X \subseteq \bigcup_{n<\omega} U_n$.

C'-set. (Same as C'' but each $\mathcal{G}_n$ a finite open cove or X.)

Strong measure zero set (property C). For a sequence of reals $\varepsilon_n > 0$ for $n < \omega$ there exists a set I_n of diameter less than ε_n such that $X \subseteq \bigcup_{n<\omega} I_n$.

Universal measure zero set. For any atomless countably additive measure μ, $\mu(X) = 0$.

$(S)_0$-set. For any perfect set P there exists a perfect set $Q \subseteq P$ disjoint from X.

Sierpiński set (Property S). For every measure zero set M, $X \cap M$ is countable.

Q-set. for every $A \subseteq X$ there exists a G_δ-set G such that $A = X \cap G$.

σ-set. For every F_σ-set F there is a G_δ-set G such that $F \cap X = G \cap X$.

λ-set (rarefied). For every countable set $F \subseteq X$ there is a G_δ-set G such that $F = G \cap X$.

λ'-set. For every countable set F, $X \cup F$ is a λ-set.

Perfectly meager (always of first category). For any perfect set P, $X \cap P$ is meager in P.

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CHAPTER 6

Trees and Linearly Ordered Sets

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
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Introduction

The purpose of this chapter is to give a short introduction to several problems concerning trees, linearly ordered sets and dualities between them. Several classes of trees and linearly ordered sets are presented and considered as set-theoretical, topological and algebraical structures. Our initial intention was not to give priority to any particular aspect of consideration, however, the reader will notice a predomination of the set-theoretical method. This is a result of the fact that many fundamental problems concerning trees and linearly ordered sets are set-theoretical in nature, and that most of them are undecidable on the basis of the usual axioms of set theory. Thus, a great number of results we present here are consistency results. However, we do not choose the main purpose of our chapter giving the full consistency proofs, but rather mentioning almost all basic problems and ideas. We do not consider linearly ordered sets as linearly ordered topological spaces nor do we consider trees as bases of nonarchimedian spaces which is done in several well-known survey papers, but rather present many examples of trees and linearly ordered sets which the topologically interested reader can easily translate into his own language and use for his own purposes.

The first three sections are introductory in character. They contain some basic definitions and notation as well as a classification of problems we are going to consider in the rest of the chapter. No special set-theoretical knowledge is presupposed. However, at some places in Sections 4–9 we assume familiarity with some basic set-theoretical results which can be found in any standard text of set theory. Each section ends with some historical remarks and references for further reading. All uncredited results are considered as a part of the folklore of the subject except for a very few trivial observations of the author.

1. Preliminaries

We are using standard notation and terminology. Greek letters $\alpha, \beta, \gamma, \delta, \xi, \ldots$ are reserved for ordinals and $\kappa, \lambda, \nu, \theta, \ldots$ for (infinite) cardinals. By On we denote the class of all ordinals, and by Lim and Succ the classes of all limit and successor ordinals, respectively. By Λ we denote $\mathrm{Lim} \cap \omega_1 = \{\alpha < \omega_1 \mid \lim(\alpha)\}$. We assume the reader is familiar with the notions of closed and unbounded sets (*club sets*) and stationary sets, as well as with the Pressing Down Lemma (see JECH [1978; p. 56] and KUNEN [1980; p. 76]). If $\mathrm{cf}\, \lambda > \omega$ and $A, B \subseteq \lambda$, then $A \subseteq^* B$ means that $A - B$ is a non-stationary subset of λ. We shall need the following well-known fact about stationary sets.

1.1. PROPOSITION. *Let κ be a regular uncountable cardinal. Then there is a family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ of cardinality 2^κ such that $A - B$ is stationary in κ for all $A, B \in \mathcal{A}$, $A \neq B$.*

PROOF. Let $\langle E_\alpha \mid \alpha < \kappa \rangle$ be a sequence of disjoint stationary subsets of κ (see JECH [1978; p. 59]). Let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ be a family of cardinality 2^κ such that $A - B \neq \emptyset$ for every $A, B \in \mathcal{F}$ with $A \neq B$. For $A \in \mathcal{F}$, we define $E_A = \bigcup\{E_\alpha \mid \alpha \in A\}$. Then $\{E_A \mid A \in \mathcal{F}\}$ satisfies the conclusion of the proposition.

In Section 6 we shall use the following well-known combinatorial principle.

($\diamondsuit$) There is a sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ of subsets of ω_1 such that for every $A \subseteq \omega_1$, $\{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$ is a stationary subset of ω_1.

$\diamondsuit$ holds in the Gödel's constructible universe L, and implies CH (see JECH [1978; p. 226] and KUNEN [1980; p. 80]). The following weak form of $\diamondsuit$ of DEVLIN–SHELAH [1979] is an equivalent of $2^{\aleph_0} < 2^{\aleph_1}$. To state this principle, let $\mathcal{I}$ be the set of all $A \subseteq \omega_1$ such that there is an $F: {}^{\underline{\omega_1}}2 \to 2$ such that for all $g \in {}^{\omega_1}2$ there is an $f \in {}^{\omega_1}2$ for which $\{\alpha \in A \mid F(f \restriction \alpha) = g(\alpha)\}$ is not stationary. Then $\mathcal{I}$ is a normal σ-complete ideal on ω_1. The *weak diamond principle* of DEVLIN and SHELAH [1978] states that $\mathcal{I}$ is a proper ideal on ω_1, that is, $\omega_1 \notin \mathcal{I}$. They proved that the weak diamond principle is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$.

We shall also need the following combinatorial principle of Jensen denoted by $\square_\kappa$: There is a sequence $\langle C_\alpha \mid \alpha \text{ limit} < \kappa^+ \rangle$ such that:

(i) C_α is closed and unbounded in α;

(ii) if $\operatorname{cf} \alpha < \kappa$, then $\operatorname{tp} C_\alpha < \kappa$;

(iii) if β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.

Jensen proved that $\square_\kappa$ holds in L for every cardinal κ (see DEVLIN [1973; §8]).

By PFA we denote the Proper Forcing Axiom. The statement and many details about this axiom can be found in Baumgartner's chapter of the Handbook.

By $\mathbb{R}$ we denote the set of all real numbers; $\mathbb{Q}$ is the set of rational numbers. $\mathcal{P}(A)$ denotes the set of all subsets of A, and ${}^BA$ denotes the set of all functions from B to A. By ${}^{\underline{\alpha}}A$ we denote the set $\bigcup\{{}^\beta A \mid \beta < \alpha\}$. If $f \in {}^A2$, then $\operatorname{supp}(f) = \{a \in A \mid f(a) = 1\}$.

We shall often identify a family $\mathcal{F} \subseteq \mathcal{P}(A)$ with the set $\{\chi_B \mid B \in \mathcal{F}\} \subseteq {}^A2$ of characteristic functions of elements of $\mathcal{F}$. Thus, if A is a well-ordered set, we may consider the lexicographical ordering on $\mathcal{F}$ induced by this identification. We may also consider $\mathcal{F}$ as a subset of ${}^A2$ with the TYCHONOFF topology. The κ-*topology* on ${}^A2$ is the topology generated by the $<\kappa$ intersections of the TYCHONOFF open sets. A topological space X is κ-compact (= finally κ-compact) iff every open cover of X has a subcover of cardinality $<\kappa$.

Let κ be a regular cardinal, and let G be an ordered abelian group which has a strictly decreasing sequence $\langle g_\alpha \mid \alpha < \kappa \rangle$ of elements converging to the unit elements $0 \in G$. A function $\rho: X^2 \to \{g \in G \mid g \geqslant 0\}$ is called a κ-*metric* iff: $\rho(x, y) = 0$ iff $x = y$; $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$; and $\rho(x, y) = \rho(y, x)$. A topological space X is κ-*metrizable* iff for some κ-metric ρ on X the family of all sets of the form $\{y \in X \mid \rho(x, y) < g\}$, where $x \in X$, $g \in G$ and $g > 0$ form a basis of X. If X is a κ-metrizable space, then we say that a sequence $\langle x_\alpha \mid \alpha < \kappa \rangle$ *converges* to a point

$x \in X$ iff for every neighborhood U of x there is an $\alpha_0 < \kappa$ such that $x_\alpha \in U$ for every α with $\alpha_0 \leq \alpha < \kappa$. From the theory of κ-metrizable spaces we need to know only the fact that ${}^\kappa 2$ with the κ-topology is a κ-metrizable space (see SIKORSKI [1950]). A *Corson space* is a compact subspace of

$$\{f \in {}^A[0,1] \mid \{\alpha \in A \mid f(\alpha) \neq 0\} \text{ is countable}\}.$$

2. Trees

A *tree* is a partially ordered set $(T, \leq_T)$ such that for every $t \in T$, the set $(\cdot, t)_T = \{s \in T \mid s <_T t\}$ is well-ordered. So we may consider trees as a very natural generalization of ordinals. The *height* of t in $(T, \leq_T)$, $\mathrm{ht}_T(t)$, is the order type of $(\cdot, t)_T$. The αth *level* of T is the set $R_\alpha T = \{t \in T \mid \mathrm{ht}_T(t) = \alpha\}$. We shall often use T_α to denote $R_\alpha T$, and we shall identify $(T, \leq_T)$ with its domain T. The *height* of T, ht T, is the ordinal $\min\{\alpha \mid R_\alpha T = \emptyset\}$. Since every tree T is a well-founded partially ordered set, the levels of T can be introduced also in the following way. If X is any subset of T then by $R_0 X$ we denote the set of all minimal elements of X. By induction on α it is easily checked that

$$R_\alpha T = R_0(T - \bigcup\{R_\beta T \mid \beta < \alpha\}).$$

Every subset of T will also be considered as a *subtree* of T. Note that if U is a subtree of T then $\mathrm{ht}_U(t) \leq \mathrm{ht}_T(t)$ for every $t \in U$, and that in general $R_\alpha U \not\subseteq R_\alpha T$. However, if U is an *initial part* of T, i.e., if $(\cdot, t)_T \subseteq U$ for every $t \in U$, then for every α we have $R_\alpha U = R_\alpha T \cap U$, that is, $\mathrm{ht}_U(t) = \mathrm{ht}_T(t)$ for every $t \in U$. If A is a set of ordinals then a typical example of a subtree of T is the set $T \restriction A = \bigcup\{R_\alpha T \mid \alpha \in A\}$. If $t \in T$, then $T^t = \{s \in T \mid t \leq_T s\}$ is another example of a subtree of T. Note that in both cases $T \restriction A$ and T^t it is quite easy to compute the height function.

A *branch* of a tree T is a maximal chain of T. A *path* of T is any chain of T which is also an initial part of T. An α*-chain* of T is a chain of order type α. Similarly we define an α*-branch* and α*-path*.

Probably the most natural example of an infinite tree is the set S of all finite sequences of 0's and 1's ordered by $\subseteq$, that is,

$$S = \bigcup\{{}^n 2 \mid n < \omega\}.$$

Clearly $\mathrm{ht}_S(s) = |s|$ for $s \in S$, hence $R_n S = {}^n 2$ for every $n < \omega$, and ht $S = \omega$. Note that every branch of S has the form $\{f \restriction n \mid n < \omega\}$ where $f \in {}^\omega 2$. Hence the set of all branches of S has cardinality $2^{\aleph_0}$ and can be identified with the Cantor set. So we have here an interesting phenomenon: A tree S which has height ω and all levels finite, but $2^{\aleph_0}$ ω-branches. We shall see later that a generalization of this phenomenon already to the first uncountable cardinal leads to a statement undecidable in ZFC. If E is a subset of ${}^{\omega}2$, then we let

$$S_E = S \cup E,$$

ordered by $\subseteq$. If E is uncountable, then S_E is called a *Cantor tree.* We extend this definition by saying that T is a Cantor tree iff T is isomorphic to an uncountable subtree of $S_{(\omega_2)}$.

If T is a tree and if $s, t \in T$, then we define $\rho(s, t) = (\cdot, s)_T \cap (\cdot, t)_T$. Clearly, $\rho(\cdot, \cdot)$ has the following easy and useful property.

2.1. LEMMA. *If $s, t, u \in T$, then $R_{s,t,u} = \{\rho(s, t), \rho(t, u), \rho(s, u)\}$ has at most two elements, and $p, q \in R_{s,t,u}$ implies $p \subseteq q$ or $q \subseteq p$.*

A *node* of T is any equivalence class of the relation $\sim$ defined on T by $s \sim t$ iff $(\cdot, s)_T = (\cdot, t)_T$. So every node N of T is a subset of some level $R_\alpha T$ of T in which case we call α the height of N in T. Note that for every $t \in T$, the set of all immediate successors of t in T is a node of T, but in general not every node has this form. If p is a bounded path of T, then the set

$$N_p = R_0\{t \in T \mid s <_T t \text{ for every } s \in p\}$$

of all immediate successors of p in T form a node of T. Conversely, if N is a node of T, then

$$\rho(N) = \{s \in T \mid s <_T t \text{ for all } t \in N\}$$

is the path of all predecessors of N. It is clear that $N_{\rho(N)} = N$ for every node N of T. If N is a node of T, and if $t \in \bigcup\{T^s \mid s \in N\}$, then by t_N we denote the unique element of $\{s \in T \mid s \leqslant_T t\} \cap N$.

Now we are ready to define the first fundamental operation which connects trees and linearly ordered sets. So let T be a tree, and let $\mathcal{N}(T)$ be the set of all nodes of T. For every $N \in \mathcal{N}(T)$, we fix a linear ordering $\leqslant_N$ of N. Then the *lexicographical ordering* $\preccurlyeq$ of T induced by $\{\leqslant_N \mid N \in \mathcal{N}(T)\}$ is defined by $s \preccurlyeq t$ iff

(i) $s \leqslant_T t$, or

(ii) $s \nleqslant_T t$ and $t \nleqslant_T s$ and $s_N \leqslant_N t_N$, where $N = N_{\rho(s,t)}$.

2.2. LEMMA. *$\preccurlyeq$ is a linear ordering of T which extends $\leqslant_T$.*

PROOF. We prove only that $\preccurlyeq$ is transitive. So let $s \preccurlyeq t$ and $t \preccurlyeq u$. By 2.1, it follows that either $\rho(s, t) = \rho(s, u)$, or $\rho(s, t) = \rho(t, u)$, or $\rho(s, u) = \rho(t, u)$ holds. We consider only the case $\rho(s, t) = \rho(s, u)$ since the other two cases are similar. Note that $s >_T u$ contradicts $s \preccurlyeq t$ and $t \preccurlyeq u$, hence we may assume s and u are incomparable since $s \leqslant_T u$ implies $s \leqslant u$. Let $N = N_{\rho(s,t)} = N_{\rho(s,u)}$. If $t_N = u_N$, then s and t are incomparable, hence $s_N \leqslant t_N = u_N$, and so $s \leqslant u$. If $t_N \neq u_N$, then t and u are incomparable, $\rho(s,t) = \rho(s,u) = \rho(t,u)$, and $t_N \leqslant_N u_N$. Since $s_N \leqslant_N t_N$ and since $\leqslant_N$ is transitive, we have $s_N \leqslant_N u_N$, hence $s \leqslant u$. This finishes the proof.

The proof of the following fact is similar.

2.3. LEMMA. *For every $t \in T$, T^t is a convex set in $(T, \leqslant)$.*

For $N \in \mathcal{N}(T)$, let $\leqslant_N^*$ be the converse of $\leqslant_N$. Let $\leqslant^*$ be the lexicographical ordering of T induced by $\{\leqslant_N^* \mid N \in \mathcal{N}(T)\}$. Then it is easily seen that $\leqslant$ and $\leqslant^*$ are linear extensions of $\leqslant_T$ with the property $\leqslant_T = \leqslant \cap \leqslant^*$. Note that $\leqslant^*$ is not the converse of $\leqslant$ unless T is an antichain. In the course of this chapter we shall see that many properties of $(T, \leqslant)$ depend on the structure of the tree $(T, \leqslant_T)$. However, some properties of $(T, \leqslant)$ may depend on the way we choose the orderings $\leqslant_N$ for $N \in \mathcal{N}(T)$, especially in the case when T has large nodes. For example, one can find a tree T (see Section 6) such that no lexicographical ordering $(T, \leqslant)$ of T contains a subset isomorphic to an uncountable set or real numbers (= uncountable real ordering). On the other hand, one can find a tree T and two lexicographical orderings $\leqslant_0$ and $\leqslant_1$ of T so that $(T, \leqslant_0)$ contains an uncountable real subordering but $(T, \leqslant_1)$ does not. A trivial example would be when T is an uncountable antichain. However, if T has all nodes countable then if one lexicographical ordering of T contains an uncountable real subordering then all of them have this property. This follows from the following fact.

2.4. PROPOSITION. *Under the above notation, suppose that no $(N, \leqslant_N)$ with $N \in \mathcal{N}(T)$ contains an uncountable real ordering. Then $(T, \leqslant)$ contains an uncountable real ordering iff $(T, \leqslant_T)$ contains a Cantor subtree.*

A fixed family $\{\leqslant_N \mid N \in \mathcal{N}(T)\}$ of linear orderings of nodes of a tree T also introduces a lexicographical ordering $\leqslant$ on the set B_T of all branches of T defined by

$$l \leqslant m \quad \text{iff} \quad l_N \leqslant_N m_N,$$

where $N = N_{l \cap m}$, $\{l_N\} = l \cap N$ and $\{m_N\} = m \cap N$. Note that this is a special case of the above definition of lexicographical orderings of trees since we may consider the tree $(T \cup B_T, \leqslant_T^*)$ defined by: $\leqslant_T^* \restriction T = \leqslant_T$ and $t \leqslant_T^* m$ iff $t \in T$, $m \in B_T$ and $t \in m$. Hence, $\leqslant$ is a linear ordering of B_T, and for every $t \in T$, $B_t = \{m \in B_T \mid t \in m\}$ is a convex set in $(B_T, \leqslant)$ (see 2.2 and 2.3).

2.5. PROPOSITION. *Under the above notation, if $(N, \leqslant_N)$ is a complete linear ordering for each $N \in \mathcal{N}(T)$, then $(B_T, \leqslant)$ is also complete.*

PROOF. Let A be a nonempty initial segment of $(B_T, \leqslant)$. We show that A has a least upper bound (l.u.b) in $(B_T, \leqslant)$. Let m be the set of all $t \in T$ such that $B_t \cap A \neq \emptyset$ while $\{l \in B_T \mid B_t < l\} \cap A = \emptyset$. Then using the completeness of $(N, \leqslant_N)$ for $N \in \mathcal{N}(T)$, one easily shows that m is a branch of T, and that m is the l.u.b of A. This finishes the proof.

Let us now start with a review of several natural questions that can be asked about trees, and that will be considered during the course of this chapter. Perhaps the first fact one notices when looking at trees is that thin and high trees have long branches. Thus one naturally asks when exactly a given tree T has to contain a *cofinal branch*, that is, a branch which intersects each level of T. It is clear that we may restrict ourselves to the case when ht $T = \kappa$ is an initial ordinal. A trivial example shows that we may as well assume $|R_\alpha T| < \kappa$ for all $\alpha < \kappa$. Thus we may restrict ourselves to κ-*trees*, i.e., trees of height κ and of levels of cardinality $<\kappa$. If κ is a singular cardinal then it is clear that there is a κ-tree with no cofinal branches. Hence the only nontrivial case is when κ is a regular infinite cardinal. In the case $\kappa = \aleph_0$ we have the following well-known positive answer.

2.6. THEOREM (König). *Every $\aleph_0$-tree has a cofinal branch.*

PROOF. Let T be an $\aleph_0$-tree. By induction on $n < \omega$, we choose $t_n \in R_n T$ such that T^{t_n} is infinite and $t_n <_T t_{n+1}$. Then $\{t_n \mid n < \omega\}$ is a cofinal branch of T.

It turns out that a direct generalization of König's theorem for the case $\kappa = \aleph_1$ is false, i.e., that there is an $\aleph_1$-tree without cofinal branches. Such a tree is called an *Aronszajn tree*. In general, a κ-*Aronszajn tree* is a κ-tree with no cofinal branches. However, some generalization of König's theorem is still possible.

2.7. THEOREM (Kurepa). *Let κ be a regular cardinal and let T be a tree of height κ such that for some $\lambda < \kappa$, $|R_\alpha T| < \lambda$ for all $\alpha < \kappa$. Then T has a cofinal branch.*

PROOF. We may assume λ is also a regular cardinal. For each $\delta < \kappa$ with cf $\delta = \lambda$ we choose $t_\delta \in R_\delta T$ arbitrarily. Now for each $\delta < \kappa$ with cf $\delta = \lambda$ we pick $s_\delta <_T t_\delta$ with the property $T^{s_\delta} \cap R_\delta T = \{t_\delta\}$. Clearly such an s_δ exists, since otherwise we would have $|R_\delta T| \geqslant \lambda$ which is a contradiction. By the Pressing Down Lemma, we can find stationary $A \subseteq \{\delta < \kappa \mid \text{cf}\,\delta = \lambda\}$ and $\gamma < \kappa$ so that $\text{ht}_T(s_\delta) = \gamma$ for all $\delta \in A$. Since $|R_\gamma T| < \lambda < \kappa$, we may assume that for some $s \in R_\gamma T$, $s_\delta = s$ for $\delta \in A$. Now it is easily seen that $\{t_\delta \mid \delta \in A\}$ is a κ-chain of T. This completes the proof.

Some more details on the problem of generalizing König's theorem to higher cardinals can be found in Sections 5 and 7.

The next natural problem is to find a bound on the cardinality of a tree T using the cardinalities of its chains and antichains. So let chain(T) be the supremum of all cardinals of the form $|c|$ where c is a chain of T, and let antichain(T) be the supremum of all cardinals of the form $|A|$ where A is an antichain of T. Since each level of T is also an antichain of T, and since ht $T \leq (\text{chain}(T))^+$, we have the following estimate

$$|T| \leq (\text{chain}(T))^+ \cdot \text{antichain}(T)\,.$$

So one naturally asks whether the above bound can be improved to

(RH) $\quad |T| \leq \text{chain}(T) \cdot \text{antichain}(T)\,.$

In particular, we would like to have the following sentence true:

(SH) *Every uncountable tree has an uncountable chain or antichain.*

In Section 6 we shall see that SH is undecidable on the basis of the usual axioms of set theory. A counterexample to SH is called a *Suslin tree*. In general, a tree T is a κ-Suslin tree iff $|T| = \kappa$ and T has no chain nor antichain of cardinality κ. Note that every κ-Suslin tree is also κ-Aronszajn. Some information concerning the κ-Suslin trees can be found in Sections 6 and 7.

Instead of asking whether a given tree T has a cofinal branch, we may also ask how many cofinal branches it can have. Again it is easily seen that this question is nontrivial only in the case when T is a κ-tree and when κ is not a strong limit cardinal. An example of a κ-tree with exactly κ cofinal branches is

$$\{s \in {}^{\leq\kappa}2 \mid \{\alpha \mid s(\alpha) = 1\} \text{ is finite}\}$$

ordered by $\subseteq$. Hence, the simplest question which naturally arises is whether there can be an $\aleph_1$-tree with $>\aleph_1$ cofinal branches. Such a tree is called a *Kurepa tree.* In general, a κ-Kurepa tree is a κ-tree with $>\kappa$ cofinal branches. Thus, a Kurepa tree is an $\aleph_1$-Kurepa tree. The *Kurepa Hypothesis* (KH) is the statement that there is a Kurepa tree. In Section 8 we shall show that KH is not decidable on the basis of the usual axioms of set theory. However, this undecidability is in a sense different from that of SH mentioned earlier. We shall also see a much finer generalization of KH to higher cardinals.

Now we are going to introduce an important technical notion concerning trees which is very useful in most of the problems we consider in this chapter. So let us say that a tree T is a *normal tree* iff:

N(i) $|R_0 T| = 1$;

N(ii) if $s, t \in T$, if $\text{ht}_T(s) = \text{ht}_T(t)$ is a limit ordinal, and if $(\cdot, s)_T = (\cdot, t)_T$, then $s = t$;

N(iii) if $\alpha < \beta < \text{ht}\, T$, and if $t \in R_\alpha T$, then $|T^t \cap R_\beta T| \geq 2$.

A typical example of a normal tree is the *complete binary tree* of height α, $({}^{\underline{\alpha}}2, \subseteq)$.

In general, we say that T is a *λ-ary tree* iff $|N| = \lambda$ for every node N of T of successor height. Note that every tree T can be made to satisfy N(i) and N(ii) without increasing cardinality of T. It suffices to add a new point between $\rho(N)$ and N for each node N of T of limit (or zero) height. On the other hand, if T satisfies the conditions N(i) and N(ii), then by induction on the levels of T, we can embed T as an initial part of ${}^{\underline{\alpha}}\lambda$, where $\alpha = \operatorname{ht} T$ and $\lambda = n(T) = \sup\{|N| \mid N$ is a node of $T\}$. Note that this proves the following estimate

$$|T| \leq \sum \{n(T)^{|\beta|} \mid \beta < \operatorname{ht} T\}.$$

This also shows that every normal T can be identified with an initial part of ${}^{\underline{\alpha}}\lambda$, where $\alpha = \operatorname{ht} T$ and $\lambda = n(T)$. This identification is very convenient. For example, we may identify branches of T with elements of ${}^{\alpha+1}\lambda$ and consider the lexicographical ordering on B_T induced by this identification. If $t \in T$ and if $\alpha \leq \operatorname{ht}_T(t)$, then we may use $t \restriction \alpha$ to denote the unique element of the αth level of T below t. What is more important, in many problems concerning trees, we may restrict ourselves to normal trees without losing any generality. For example, the next fact shows that this is the case with the problems we have already mentioned.

2.8. PROPOSITION. *Let $2 \leq \lambda \leq \aleph_0$ and assume that there is an Aronszajn (or Suslin or Kurepa) tree. Then there is one which is normal and λ-ary.*

PROOF. We prove the lemma only for the case of Aronszajn trees; the proofs of the other two cases are similar. So let T be any Aronszajn tree, i.e., an $\aleph_1$-tree with no uncountable chains. By adding a new point between $\rho(N)$ and N for each node N of limit or zero height, we may assume that T satisfies N(i) and N(ii) from the definition of normal trees. For each $t \in T$ we choose an ordinal $\alpha = \alpha(t) < \omega_1$ with the property that either $T^t \cap R_\alpha T$ is empty or infinite. Since T is an Aronszajn tree such an $\alpha(t)$ exists (see 2.7). Let

$$C = \{\delta < \omega_1 \mid \lim(\delta) \text{ and } \alpha(t) < \delta \text{ for all } t \in T \restriction \delta\}.$$

Then C is a closed and unbounded subset of ω_1 and $T \restriction C$ is a normal $\aleph_0$-ary Aronszajn tree.

Assume now $2 \leq \lambda < \aleph_0$. Let T be a normal $\aleph_0$-ary Aronszajn tree. By induction on the levels of T we can embed T into an initial segment T' of $\bigcup\{{}^{\alpha}\lambda \mid \lim(\alpha)\}$ such that $T^* = \{s \in {}^{\underline{\omega_1}}\lambda \mid s \subseteq t$ for some $t \in T'\}$ is a normal λ-ary Aronszajn tree. This completes the proof.

We shall also consider trees as topological and algebraic structures, so we need to introduce some terminology in this respect. For example, a given tree T will also be considered as a topological space with the *interval topology* generated by the intervals of the form $(s, t]_T$, where $s, t \in T \cup \{-\infty\}$. Note that T with the interval topology is not necessarily Hausdorff unless T satisfies the condition N(ii) from the definition of normal trees. So from now on, when talking about the interval topology on a given tree T, we always assume that T satisfies the condition N(ii). Note that we have here two different notions which use the same word 'normal'. Namely, T can be a normal topological space and a normal tree. It will be clear from a particular context about which kind of normality we are talking. In Sections 5 and 6 we shall consider the interval topologies on Aronszajn and Suslin trees. But let us here just mention that Cantor trees are also very interesting when considered as topological spaces. Every Cantor tree is a separable, locally compact, non-collectionwise Hausdorff, Moore space. The tree S_E is a normal topological space iff every subset of E is relatively G_δ in E. Hence if MA_κ holds then every Cantor tree of cardinality $\leqslant\kappa$ is a normal topological space. The proofs of these results can be found in Fleissner's and Tall's chapters.

Two separation properties of the interval topology on a given $\aleph_1$-tree T are closely related to the following two combinatorial properties of T (see Section 6). We say that T has *property* γ if whenever A is an antichain of T, there is a club $C \subseteq \omega_1$ such that $T - T\restriction C$ contains a closed neighborhood of A. An antichain $A \subseteq T$ is called a *stationary antichain* of T iff $\{\mathrm{ht}_T(t) \mid t \in A\}$ is a stationary subset of ω_1.

When considering isomorphism types of trees, to avoid some trivial discussions, we shall always work with a particular class of normal λ-ary trees. Usually we shall be interested in the class of all normal λ-ary $\aleph_1$-trees where λ is equal to 2 or $\aleph_0$. We say that T is a *rigid tree* iff T has no non-trivial automorphisms. If T has a fixed lexicographical ordering $\preccurlyeq$, then we say that an automorphism σ of T is a *lexicographical* automorphism if σ also preserves the lexicographical ordering $\preccurlyeq$ of T. Note that in this case σ is also an automorphism of $(T, \preccurlyeq)$. Note also that in general, not every automorphism of $(T, \leqslant_T)$ is an automorphism of $(T, \preccurlyeq)$. However, in Section 6 we shall see that for some trees every automorphism of $(T, \preccurlyeq)$ introduces an automorphism of $T\restriction C$ where C is a closed and unbounded subset of ht T. Let T and U be given trees of the same limit height δ. Then we say that T and U are *club-isomorphic* iff there is a club $C \subseteq \delta$ such that $T\restriction C$ and $U\restriction C$ are isomorphic. T is *club-embeddable* into U iff T is club-isomorphic to some initial part of U.

Let us now define two operations which are very useful in producing trees. If P is a partially ordered set, then by σP we denote the set of all bounded well-ordered subsets of P ordered by $s \leqslant t$ iff s is an initial segment to t. Then σP is a tree which contains a lot of information about P. Let $\sigma' P = \{t \in \sigma P \mid t$ has a maximal element$\}$. Note that the max$(\cdot)$ is a strictly increasing mapping from $\sigma' P$ into P. This shows that $\sigma'\mathbb{Q}$ is a tree of height ω_1 which is the union of countably many antichains. In Section 5 we shall construct an Aronszajn subtree of $\sigma'\mathbb{Q}$, hence an Aronszajn tree which is the union of countable many antichains. So let us introduce some terminology concerning this interesting phenomenon. We say that a tree T is *P-embeddable* iff there is a strictly increasing mapping $f \colon T \to P$.

Note that f need not be 1–1. We say T is *special* iff T is the union of $\leqslant\aleph_0$ antichains. In Section 9 we shall prove that T is special iff T is $\mathbb{Q}$-embeddable. In general, a tree T is *κ-special* iff T is the union of $\leqslant\kappa$ antichains.

If T and U are trees then by $T \otimes U$ we denote the set $\{\langle t, u\rangle \mid t \in T,\ u \in U$ and $\mathrm{ht}_T(t) = \mathrm{ht}_U(u)\}$ partially ordered by $\langle t, u\rangle \leqslant \langle t', u'\rangle$ iff $t \leqslant_T t'$ and $u \leqslant_U u'$. Then $T \otimes U$ is also a tree. If $1 \leqslant n < \omega$, then by $T^{(n)}$ we denote the product $T \otimes T \otimes \cdots \otimes T$ (n times). Note that if T and U are $\aleph_1$-trees and one of them is Aronszajn, then $T \otimes U$ is also Aronszajn. However $\otimes$ does not preserve the Suslin property of trees as the next lemma shows.

2.9. PROPOSITION. *If T is an uncountable tree, then $T \otimes T$ has an uncountable chain or antichain.*

PROOF. It T has an uncountable chain or antichain, we are done. Otherwise, we can find a sequence $\langle t_\alpha \mid \alpha < \omega_1\rangle$ of elements of T such that every t_α has two incomparable successors s_α^0 and s_α^1 such that $\mathrm{ht}_T(s_\alpha^0) = \mathrm{ht}_T(s_\alpha^1) < \mathrm{ht}_T(t_\beta)$ for every $\alpha < \beta < \omega_1$. Now it is easily checked that $\{\langle s_\alpha^0, s_\alpha^1\rangle \mid \alpha < \omega_1\}$ is an uncountable antichain of $T \otimes T$.

In Section 6 we shall show that there exist two Suslin trees T and U so that $T \otimes U$ is a Suslin tree.

Clearly, no Suslin tree is a special tree. In fact, every normal Suslin tree has the following property which is very far from being special. So we shall say that a tree T is a *Baire tree* if the intersection of countably many dense final parts of T is also dense in T. A set $D \subseteq T$ is *dense* in T if for every $s \in T$ there exists $t \in D$ such that $s \leqslant_T t$. Note that the word 'dense' here refers to a topology which is different from the interval topology on T. This new topology will not be considered in this chapter so that there is no danger of ambiguity.

2.10. REMARKS. (i) The first systematic study of trees appeared in KUREPA [1935]. The notion of the lexicographical orderings of trees was introduced in this paper, as well as the notion of normal trees. The name 'normal' was apparently first used by GAIFMAN–SPECKER [1964].

(ii) Aronszajn, Suslin and Kurepa trees were introduced by KUREPA [1935], [1937a] and [1942], Theorem 2.6 was proved by KÖNIG [1927]. Theorems 2.7 and 2.9 were proved by KUREPA [1935; p. 80] and [1950], respectively. RH is the well-known Kurepa's 'rectangle hypothesis' (see KUREPA [1968]).

(iii) Of larger works or survey papers which appeared after KUREPA [1935] and which are concerned with trees, we mention RICABARRA [1958], KUREPA [1968], RUDIN [1969], JECH [1971], DEVLIN–JOHNSBRÅTEN [1974], and LUTZER [1980]. Some information about trees can also be found in DEVLIN [1973] [19··a], JECH [1978] and KUNEN [1977], [1980].

3. Linearly ordered sets

Let $(L, \leqslant_L)$ be a linearly ordered set. We shall often identify $(L, \leqslant_L)$ with its domain L, and consider only infinite L. We shall say that L is *densely ordered* if whenever $l, m \in L$ and $l <_L m$ there is a $d \in L$ such that $l <_L d <_L m$. A *Dedekind cut* of L is an initial segment of L. We say that L is *complete* if every Dedekind cut of L has a least upper bound (l.u.b) in L. The *Dedekind completion* of L is obtained by adding l.u.b. to every cut in L without a l.u.b. L is a *linearly ordered continuum* if L is densely ordered and complete. A subset D of L is *order-dense* in L if for every $l, m \in L$ with $l <_L m$ there is a $d \in D$ such that $l \leqslant_L d \leqslant_L m$. The minimal cardinality of an order-dense subset of L is called the *order-density* of L and denoted by $d(L, \leqslant_L)$. It is easily seen that $d(L, \leqslant)$ is equal to the weight of L as a topological space with the order topology. It is also clear that if L is a densely ordered set, then the order-density and the topological density of L coincide. Almost any example of a linearly ordered set presented in this chapter can be assumed to be densely ordered without any loss of generality, hence the notation of $d(L, \leqslant_L)$ will not lead to any ambiguity. The next fact gives an equivalent definition of $d(L, \leqslant_L)$.

3.1. PROPOSITION. *Let $(L, \leqslant_L)$ be a linearly ordered set. Then $d(L, \leqslant_L) = \min\{\kappa \mid (L, \leqslant_L)$ is embeddable into $(\mathcal{P}(\kappa), \subseteq)\}$.*

PROOF. Let $\kappa = d(L, \leqslant_L)$ and let D be a dense subset of L of cardinality κ. Then $l \to (\cdot, l]_L \cap D$ is a strictly increasing mapping from $(L, \leqslant_L)$ into $(\mathcal{P}(D), \subseteq)$. Conversely, let us assume L is a chain in $\mathcal{P}(\kappa)$. For each $\alpha \in \kappa$, let $d_\alpha = \bigcap\{l \mid \alpha \in l \in L\}$. Then it is easily seen that $D = \{d_\alpha \mid \alpha < \kappa\}$ is a dense subset of $(D \cup L, \subseteq)$. Hence $d(L, \subseteq) \leqslant d(D \cup L, \subseteq) \leqslant \kappa$. This finishes the proof.

3.2. COROLLARY. $d(L, \leqslant_L) \leqslant \aleph_0$ *iff $(L, \leqslant_L)$ is isomorphic to a set of real numbers.*

PROOF. Let $\leqslant$ be the lexicographical ordering of $\mathcal{P}(\omega)$ (see Section 1). Then $A \subseteq B$ implies $A \leqslant B$, and $(\mathcal{P}(\omega), \leqslant)$ is isomorphic to the Cantor set.

Let us now introduce some terminology about isomorphism types of linearly ordered sets. By $\mathrm{tp}(L, \leqslant_L)$ we denote the order type of $(L, \leqslant_L)$, that is, the class (in the sense of D. Scott) of all linearly ordered sets isomorphic to $(L, \leqslant_L)$. If $\varphi = \mathrm{tp}(L, \leqslant_L)$, then we let $|\varphi| = |L|$, $d(\varphi) = d(L, \leqslant_L)$, and $\varphi^* = \mathrm{tp}(L, \geqslant_L)$. If $\varphi = \mathrm{tp}(L, \leqslant_L)$ and $\psi = \mathrm{tp}(K, \leqslant_K)$ then by $\psi \leqslant \varphi$ we denote the fact that there exists a strictly increasing mapping from K into L. We say that L is *rigid* if L has no non-trivial automorphisms. L is *homogeneous* iff for any four non-end points l_0, l_1, m_0, m_1 of L with $l_0 <_L l_1$ and $m_0 <_L m_1$ there exists an automorphism σ of L such that $\sigma(l_i) = m_i$ for $i < 2$. L is *reversible* iff $(L, \leqslant_L)$ is isomorphic to $(L, \geqslant_L)$. By L^2

we shall denote the set $\{\langle l, m\rangle \mid l, m \in L\}$ partially ordered by $\langle l, m\rangle \leqslant \langle l', m'\rangle$ iff $l \leqslant_L l'$ and $m \leqslant_L m'$.

Let us now define the second fundamental operation which connects trees and linearly ordered sets. This operation is called a *process of atomization* of a linearly ordered set $(L, \leqslant_L)$, and it is a process of inductive construction of families T_α, $\alpha \in \mathrm{On}$ of nonempty convex subsets of L such that:

(i) If $\alpha = 0$, then $T_\alpha = \{L\}$.

(ii) If $\alpha = \beta + 1$, then for each nontrivial $I \in T_\beta$ there exist disjoint $I_0, I_1 \in T_\alpha$ such that $I = I_0 \cup I_1$, and

$$T_\alpha = \bigcup\{\{I_0, I_1\} \mid I \in T_\beta \text{ and } |I| \geqslant 2\}.$$

(iii) If α is a limit ordinal, then

$$T_\alpha = \{\bigcap b \mid b \subseteq \bigcup\{T_\beta \mid \beta < \alpha\},\ b \cap T_\beta \neq \emptyset \text{ for all } \beta < \alpha, \text{ and } \bigcap b \neq \emptyset\}.$$

It is clear that for some α, $T_\alpha = \emptyset$, hence we may define

$$\operatorname{ht} T = \min\{\alpha \mid T_\alpha = \emptyset\} \quad \text{and} \quad T = \bigcup\{T_\alpha \mid \alpha < \operatorname{ht} T\}.$$

Then $(T, \supseteq)$ is a tree and $R_\alpha T = T_\alpha$ for all $\alpha < \operatorname{ht} T$. Any tree which is a result of an atomization of L is called a *partition tree* of L. Our process of atomization is binary, but one may also consider atomizations with larger splittings. Note that in this case the corresponding partition tree may loose some information about L. For example, an extreme possibility would be to define $T_0 = \{L\}$ and $T_1 = \{\{l\} \mid l \in L\}$. In this case the partition tree $\bigcup\{T_\alpha \mid \alpha < 2\}$ does not contain any information about L. On the other hand, there are problems where it is convenient to consider atomizations with larger splittings. In this chapter we restrict ourselves to binary atomizations.

The next two remarks show that the process of atomization of linearly ordered sets and the process of lexicographical ordering of trees are in some sense dual. So, let L be a linearly ordered set and let T be a partition tree of L. If N is a node of T, and if $I_0, I_1 \in N$, then we let $I_0 <_N I_1$ iff $l <_L m$ for every $l \in I_0$ and $m \in I_1$. Let $\leqslant$ be the lexicographical ordering of T induced by $\{\leqslant_N \mid N \in \mathcal{N}(T)\}$. Then the mapping $l \to \{l\}$ is an isomorphical embedding of $(L, \leqslant_L)$ into $(T, \leqslant)$. For suppose $l <_L m$, and let $I = \bigcap\{J \in T \mid \{l, m\} \subseteq J\}$. Then $I \in T$. Let I_0 and I_1 be the immediate successors of I in T, and let us assume $I_0 <_N I_1$, where N is the node $\{I_0, I_1\}$ of T. Then we must have $l \in I_0$ and $m \in I_1$, hence $\{l\} < \{m\}$. Suppose now that $(T, \leqslant_T)$ is a tree and that $\leqslant$ is a lexicographical ordering of T. Then it is easily seen that $t \to T^t$ is an isomorphical embedding of $(T, \leqslant_T)$ into a partition tree of $(T, \leqslant)$ (see 2.3). Note that if T satisfies the first two conditions of normality, and if we allow atomizations with larger splittings, then, in fact, for every lexicographical ordering $\leqslant$ of T, $\{T^t \mid t \in T\}$ *is* a partition tree of $(T, \leqslant)$.

3.3. PROPOSITION. *Assume φ and ψ are order types such that $\psi \leqslant \varphi$. Then any partition tree of φ contains a subtree isomorphic to a partition tree of ψ.*

PROOF. Let L be a linearly ordered set, let K be a subset of L, and let T be a partition tree of L. Let $T_K = \{I \cap K \mid I \in T \text{ and } I \cap K \neq \emptyset\}$. Then it is easily checked that T_K is a partition tree of K. For $J \in T_K$ let $h(J)$ be the $\supseteq$-minimal element of T such that $J = h(J) \cap K$. Then h is an isomorphical embedding of $(T_K, \supseteq)$ into $(T, \supseteq)$.

Let us now present an immediate application of the notion of partition trees of linearly ordered sets. In this respect, for any linearly ordered set, we let $\mathrm{wo}(L)$ be the supremum of all cardinals of the form $|A|$, where A is a well-ordered or conversely well-ordered subset of L.

3.4. THEOREM (Hausdorff–Urysohn). *Let L be a linearly ordered set. Then $|L| \leqslant 2^{\mathrm{wo}(L)}$.*

PROOF. Let T be a partition tree of L. Then $|L| \leqslant |T|$, since $\{l\} \in T$ for every $l \in L$. Since T is embeddable as an initial part of ${}^{\underline{\mathrm{ht}\,T}}2$ (see Section 2), it suffices to show that $\mathrm{ht}\, T \leqslant \mathrm{wo}(L)^+$. We show the stronger result, i.e., that every chain of T has cardinality $\leqslant \mathrm{wo}(L)$. Let c be a chain of T and let $\{I_\beta \mid \beta < \alpha\}$ be the $\supseteq$-increasing enumeration of c. For each $\beta < \alpha - 1$, we pick $l_\beta \in I_\beta - I_{\beta+1}$ and put $A_0 = \{l_\beta \mid l_\beta <_L I_{\beta+1}\}$ and $A_1 = \{l_\beta \mid I_{\beta+1} <_L l_\beta\}$. Then A_0 and A_1 are well-ordered and conversely well-ordered subsets of L, respectively, and $|c| = |A_0| + |A_1| + 1 \leqslant \mathrm{wo}(L)$. This completes the proof.

Let us now introduce some classes of linearly ordered sets we are going to consider later in this chapter. An order type $\varphi = \mathrm{tp}(L, \leqslant_L)$ is called an *uncountable real type* iff $(L, \leqslant_L)$ is isomorphic to an uncountable set of real numbers. Thus by 3.2, φ is an uncountable real type iff $|\varphi| > \aleph_0$ and $d(\varphi) \leqslant \aleph_0$. It is clear that every uncountable real type φ has a partition tree which is a Cantor tree and that every partition tree of φ contains a Cantor subtree. This with 3.3 shows that the following is true.

3.5. PROPOSITION. *Let φ be an order type and let T be a partition tree of φ. Then φ contains an uncountable real type iff T contains a Cantor subtree.*

It is clear that every uncountable real type φ satisfies $\omega_1, \omega_1^* \not\leqslant \varphi$, hence one may ask whether every uncountable order type ψ such that $\omega_1, \omega_1^* \not\leqslant \psi$ contains an uncountable real type. It turns out that the answer is negative and that the counterexample can be constructed without any additional set-theoretic assumptions. Such a type is called an *Aronszajn type*, i.e., an order type φ with the following properties:

(i) $|\varphi| > \aleph_0$;
(ii) $\omega_1, \omega_1^* \not\leqslant \varphi$;
(iii) φ contains no uncountable real type.

In Section 5 we shall show that every Aronszajn type has cardinality $\aleph_1$, hence (iii) is equivalent to the statement that $d(\psi) = |\psi|$ for every $\psi \leqslant \varphi$. This will be used in Section 7 in generalizing the notion of an Aronszajn type to higher cardinals. At this moment we say that φ is a *κ-Aronszajn type* iff $|\varphi| \geqslant \kappa$; κ, $\kappa^* \not\leqslant \psi$; and $d(\psi) \geqslant \kappa$ for every $\psi \leqslant \varphi$ with $|\psi| \geqslant \kappa$. Thus, an Aronszajn type is an $\aleph_1$-Aronszajn type. A linearly ordered set L is called an *Aronszajn line* iff $\mathrm{tp}(L, \leqslant_L)$ is a Aronszajn type. Note that if L is an Aronszajn line, then $L/\sim$ is a densely ordered Aronszajn line, where $\sim$ is defined on L by $l \sim m$ iff $[l, m]_L$ is countable. To show this one first show that each equivalence class of $\sim$ is countable which follows from $\omega_1, \omega_1^* \not\leqslant \mathrm{tp}(L, \leqslant_L)$. A linearly ordered continuum $\mathbb{K}$ is called an *Aronszajn continuum* iff

(i) $\mathbb{K}$ is not separable;
(ii) $\mathbb{K}$ is first countable;
(iii) if $M \subseteq \mathbb{K}$ is countable, then $\bar{M}$ is second countable.

3.6. PROPOSITION. *Let $\mathbb{K}$ be a linearly ordered continuum. Then $\mathbb{K}$ is an Aronszajn continuum iff $\mathbb{K}$ is the Dedekind completion of a densely ordered Aronszajn line.*

PROOF. Assume $\mathbb{K}$ is Aronszajn. Let T be a partition tree of $\mathbb{K}$. Let T' be the set of all nontrivial intervals of T, and let L be the set of all end-points of intervals from T'. Then L is dense in $\mathbb{K}$ and L is an Aronszajn line. L is uncountable because $\mathbb{K}$ is not separable, ω_1, $\omega_1^* \not\leqslant \mathrm{tp}\, L$ because $\mathbb{K}$ is first countable, and $\mathrm{tp}\, L$ contains no uncountable real type, since for every countable $M \subseteq \mathbb{K}$, $\bar{M}$ is second countable.

Conversely, let L be a densely ordered Aronszajn line and let $\mathbb{K}$ be the Dedekind completion of L. Then $\mathbb{K}$ is not separable because L is not separable. $\mathbb{K}$ is first countable since $\omega_1, \omega_1^* \not\leqslant \mathrm{tp}\, L$. Assume $M \subseteq \mathbb{K}$ is countable, but $w(\bar{M}) > \aleph_0$. This means that the number of convex components of $\mathbb{K} - \bar{M}$ is uncountable. Since L is dense in $\mathbb{K}$, from each convex component of $\mathbb{K} - \bar{M}$ we can pick a member of L and form an uncountable subset L' of L. Then it is easily seen that M is order-dense in $L' \cup M$, hence $\mathrm{tp}(L' \cup M)$ is an uncountable real type (see 3.2), and so is $\mathrm{tp}\, L'$, a contradiction. This finishes the proof.

Since every Aronszajn line has cardinality $\aleph_1$ (see Section 5), 3.6 shows that every Aronszajn continuum has weight $\aleph_1$. In general, we say that $\mathbb{K}$ is a *κ-Aronszajn continuum* iff $w(\mathbb{K}) = \kappa$, every point of $\mathbb{K}$ has character $<\kappa$ (i.e., $\kappa, \kappa^* \leqslant \mathrm{tp}\, \mathbb{K}$), and $w(\bar{M}) < \kappa$ for every $M \subseteq \mathbb{K}$ with $|M| < \kappa$. In Section 7 we shall find a κ-Aronszajn continuum $\mathbb{K}$ with the following stronger property: $w(\bar{M}) \leqslant |M|$ for every $M \subseteq \mathbb{K}$ with $|M| < \kappa$.

An order type φ is called a *Kurepa type* iff

(i) $|\varphi|>\aleph_1$;
(ii) $d(\varphi)=\aleph_1$;
(iii) φ contains no uncountable real type.

Note the following equivalent form of (iii): $d(\psi)=|\psi|$ for every $\psi\leqslant\varphi$ with $|\psi|\leqslant\aleph_1$. This will be used in Section 8 in finding more exact generalization of the notion of a Kurepa type to higher cardinals. At this point we say that φ is a *κ-Kurepa type* if $|\varphi|>\kappa$, $d(\varphi)=\kappa$, and $d(\psi)<\kappa$ for every $\psi\leqslant\varphi$ with $|\psi|<\kappa$. A linearly ordered set L is a (κ-) Kurepa line iff $\operatorname{tp} L$ is a (κ-) Kurepa type. In Section 8 we shall show that there is a Kurepa line iff there is a Kurepa tree. Hence the problem about the existence of a Kurepa line cannot be decided in ZFC. It is clear that if there is a Kurepa line then there is one which is densely ordered. If we take the Dedekind completion of a densely ordered Kurepa line, then the resulting continuum $\mathbb{K}$ has the following properties:

(i) $\mathbb{K}$ has $>\aleph_1$ points of uncountable character;
(ii) $w(\mathbb{K})=\aleph_1$;
(iii) if $M\subseteq\mathbb{K}$ is countable then $\bar{M}$ is second countable. Any linearly ordered continuum $\mathbb{K}$ with properties (i)–(iii) is called a *Kurepa continuum*. Using the ideas from the proof of 3.6 (and also some ideas from Section 8) one easily shows the following.

3.7. PROPOSITION. *Let $\mathbb{K}$ be a linearly ordered continuum. Then $\mathbb{K}$ is a Kurepa continuum iff $\mathbb{K}$ is the Dedekind completion of a densely ordered Kurepa line.*

Thus in general, we say that $\mathbb{K}$ is a *κ-Kurepa continuum* iff $\mathbb{K}$ is the Dedekind completion of a densely ordered κ-Kurepa line.

The next problem we are going to consider in this chapter is the famous *Suslin problem* which can be stated as follows:

Let $\mathbb{K}$ be a linearly ordered continuum with no uncountable family of pairwise disjoint open intervals. Is $\mathbb{K}$ isomorphic to the unit interval $[0,1]$?

We say that a linearly ordered set L has the Suslin property (or satisfies the countable chain condition (ccc)) iff every family of pairwise disjoint open intervals in L is countable. The *Suslin Hypothesis* (SH) is the assertion that the answer to the Suslin problem is positive. Since every separable linearly ordered continuum is isomorphic to $[0,1]$, the Suslin problem asks whether every linearly ordered continuum with the Suslin property is, in fact, separable. Note that a linearly ordered continuum $\mathbb{K}$ with the Suslin property is separable iff every atomization of $\mathbb{K}$ stops at some countable stage. Hence SH holds iff every atomization of a linearly ordered continuum with the Suslin property stops at some countable stage. In Section 6 we shall show that the Suslin Hypothesis is undecidable in ZFC. A counterexample to the Suslin Hypothesis is called a *Suslin continuum*, i.e., a nonseparable linearly ordered continuum with Suslin property. More generally, a linearly ordered set L is called a *Suslin line* iff L has the Suslin property but it is not (topologically) separable. Let L be a Suslin line. For

$l, m \in L$, we let $l \sim m$ iff $[l, m]_L$ is separable. Then $\sim$ is an equivalence relation on L each class of which is convex and separable subset of L (since $\omega_1, \omega_1^* \not\leq \text{tp}\, L$). Thus $L/\!\sim$ is a nowhere separable densely ordered Suslin line. Let $\mathbb{K}$ be the Dedekind completion of $L/\!\sim$. Then it is easily seen that $\mathbb{K}$ is a Suslin continuum. Hence the following is true.

3.8. PROPOSITION. *There is a Suslin continuum iff there is a Suslin line.*

Thus SH holds iff every linearly ordered set with the Suslin property is (topologically) separable. In Section 6 we shall show that there is a Suslin line iff there is a Suslin tree. Since every Suslin tree is also Aronszajn one might expect some relationships between Suslin and Aronszajn lines. Our next result shows this.

3.9. PROPOSITION. *Every Suslin line L contains an Aronszajn line K which is topologically dense in L.*

PROOF. Let T be a partition tree of L, and let T' be the set of all elements of T with non-empty interiors. For each $I \in T'$ we pick $l(I) \in I$ arbitrarily, and let $K = \{l(I) \mid I \in T'\}$. Then it is easily seen that K is (topologically) dense in L, hence K is uncountable. Let M be an uncountable subset of K and let $D \subseteq M$ be countable. Since $\omega_1, \omega_1^* \not\leq \text{tp}\, L$ and since each level of T' is countable, we can find an $\alpha < \omega_1$ such that $D \cap (\bigcup R_\alpha T') = \emptyset$. Since $M - \bigcup R_\alpha T'$ is countable, we can find $I \in R_\alpha T'$ such that $I \cap M$ is uncountable convex subset of M disjoint from D. Hence D is not order-dense in M. This shows that $\text{tp}\, K$ contains no uncountable real type, hence K is an Aronszajn line.

3.10. COROLLARY. *Let $\mathbb{K}$ be an ordered continuum. Then $\mathbb{K}$ is a Suslin continuum iff $\mathbb{K}$ is the Dedekind completion of a densely ordered Aronszajn line with the Suslin property.*

Since every Aronszajn line has cardinality $\aleph_1$ (see Section 5), by 3.10 we have that every Suslin continuum has weight exactly $\aleph_1$.

The *Generalized Suslin Hypothesis* (GSH) is the following statement:

Every linearly ordered continuum $\mathbb{K}$ has a family of cardinality $d(\mathbb{K})$ of disjoint open intervals.

Let us note that it is still unknown whether GSH is consistent with ZFC or ZFC + GCH. A linearly ordered continuum $\mathbb{K}$ is called a κ-Suslin continuum iff $d(\mathbb{K}) = \kappa$ and $\mathbb{K}$ has no family of cardinality κ of disjoint open intervals.

3.11. REMARKS. (i) Proposition 3.1 was proved by KUREPA [1957; pp. 209–210], using an idea of Sierpiński (see, e.g., SIERPIŃSKI [1934; pp. 120–123]). Some results concerning the order density of linearly ordered sets can be found in MITCHELL [1972] and BAUMGARTNER [1976].

(ii) The notion of a partition tree of a linearly ordered set was introduced by KUREPA [1935; p. 112], although the idea seems to be much older. One may also consider atomizations in not necessarily disjoint parts and not necessarily of linearly ordered sets but also of other mathematical structures (see e.g., ALEXANDROFF–URYSOHN [1929], RAMSEY [1930], KUREPA [1937c], ERDÖS [1942], KUREPA [1953], ERDÖS–RADO [1965], KUREPA [1959], ERDÖS–HAJNAL–RADO [1965], ARHANGEL'SKIĬ [1969], . . .). The reader interested in the partition trees of linearly ordered sets may also consult RICABARRA [1958] and MAURICE [1964].

(iii) Theorem 3.5 (countable case) was proved by URYSOHN [1924] in response to a problem of SIERPIŃSKI [1921]. This result can also be deduced from an earlier result of HAUSDORFF [1914; Ch. VI] (see URYSOHN [1924b]). The proof presented here is from KUREPA [1935; p. 118].

(iv) The Suslin problem was asked by SUSLIN [1920]. This problem played a central role in the development of the theory of infinite trees. The formulation of the Generalized Suslin Hypothesis is from KUREPA [1935; p. 131].

(v) The notions of Aronszajn and Kurepa lines came from the corresponding notions of trees, but they certainly have their intrinsic interest. For example, the problem about the existence of Aronszajn types was independently of Aronszajn trees (and later) asked by ERDÖS–RADO [1956; p. 433]. A classification of order types which includes the class of Aronszajn types (called also Specker types) was given in BAUMGARTNER [1976a].

4. A class of $\aleph_1$-trees

For each $\delta \in \Lambda$ we fix a strictly increasing sequence $\eta_\delta = \langle \eta_\delta(n) \mid n < \omega \rangle$ of ordinals cofinal to δ. For technical reasons, we let $\eta_0 = \emptyset$. Now for $A \subseteq \Lambda$, we define

$$T(A) = \{t \in {}^{\omega_1}2 \mid \operatorname{supp}(t) = F \cup \{\eta_\delta(n) \mid n < \omega\}$$
$$\text{for some } \delta \in A \cup \{0\} \text{ and finite } F \subseteq \omega_1 - \delta\}$$

The ordering on $T(A)$ is $\subseteq$. Then $T(A)$ is a normal $\aleph_1$-tree. For $\delta \in \Lambda$ we define $t_\delta \in {}^{\delta}2$ by $\operatorname{supp}(t_\delta) = \{\eta_\delta(n) \mid n < \omega\}$. Note that $\{t_\delta \mid \delta \in A\}$ is an antichain of $T(A)$. Note also that $T(\emptyset) = \{t \in {}^{\omega_1}2 \mid \operatorname{supp}(t) \text{ is finite}\}$.

4.1. THEOREM. (i) $T(A)$ *contains no Aronszajn subtrees.*
(ii) *If M is an antichain of $T(A)$, then* $\operatorname{ht}''M \subseteq^* A$.
(iii) *If $T(A)$ is club-embeddable into $T(B)$, then $A \subseteq^* B$.*

PROOF. (i) Assume, on the contrary, that $T(A)$ contains an Aronszajn subtree U. We may assume that U is an initial part of $T(A)$. It is clear that there is a club

$C \subseteq \Lambda$ such that $|U_\delta| \geq 2$ for every $\delta \in C$. Now, for each $\delta \in C$, we choose $u_\delta \in U_\delta$ different from t_δ. Hence, for each $\delta \in C$ there are $\gamma(\delta) < \delta$ and finite $F_\delta \subseteq \delta - \gamma(\delta)$ such that $\operatorname{supp}(u_\delta) = \{\eta_{\gamma(\delta)}(n) \mid n < \omega\} \cup F_\delta$. Using the Pressing Down Lemma, we can find stationary $E \subseteq C$, $\gamma < \omega_1$ and finite $F \subseteq \omega_1$ such that $\gamma(\delta) = \gamma$ and $F_\delta = F$ for each $\delta \in E$. It follows that $\{u_\delta \mid \delta \in E\}$ is an ω_1-chain of U, a contradiction.

(ii) The proof of (i) shows that $T(A)$ has the following property: If E is a stationary subset of Λ and if $\{u_\delta \mid \delta \in E\}$ is such that $t_\delta \neq u_\delta \in T(A)_\delta$ for each $\delta \in E$, then for some stationary $E' \subseteq E$, $\{u_\delta \mid \delta \in E'\}$ forms a chain of $T(A)$. It is clear that this shows that $\mathrm{ht}''X \subseteq^* A$ holds for any antichain $X \subseteq T$.

(iii) follows from (ii) since $\{t_\delta \mid \delta \in A\}$ is an antichain.

4.2. COROLLARY. (i) $T(\emptyset)$ *has no Aronszajn subtrees nor stationary antichains.*

(ii) *Every normal* $\aleph_1$*-tree has an Aronszajn subtree or a subtree isomorphic to* $T(\emptyset)$.

(iii) $T(\emptyset)$ *as a topological space is homeomorphic with the topological sum of* $\aleph_1$ *copies of* ω_1.

PROOF. Only (iii) requires a proof. For finite $F \subseteq \omega_1$ we define $b(F) = \{t \in T(\emptyset) \mid \operatorname{supp}(t) = F\}$. Then $b(F)$, $F \in [\omega_1]^{<\aleph_0}$ is a clopen disjoint partition of $T(\emptyset)$ and each $b(F)$ is homeomorphic to ω_1 with the order topology. This completes the proof.

4.3. REMARK. Let $\mathscr{A} \subseteq \mathscr{P}(\omega_1)$ be a family of cardinality $2^{\aleph_1}$ of stationary subsets of ω_1 such that $A - B$ is stationary in ω_1 for every $A, B \in \mathscr{A}$, $A \neq B$ (see 1.1). Then by Theorem 4.1, we have that $T(A)$, $A \in \mathscr{A}$ is a family of cardinality $2^{\aleph_1}$ of pairwise non-club-embeddable normal $\aleph_1$-trees. One can go further and use the trees $T(A)$ to construct a family $\mathscr{T}$ of cardinality $2^{\aleph_1}$ of normal $\aleph_1$-trees such that if $T, T' \in \mathscr{T}$ and if $H: T \upharpoonright C \to T' \upharpoonright C$ is 1-to-1 order and level preserving, where C is a club subset of ω_1, then $T = T'$ and $H =$ identity.

Now for $A \subseteq \Lambda$, we define

$$L(A) = \{x \in {}^{\omega_1}2 \mid \operatorname{supp}(x) = F \cup \{\eta_\delta(n) \mid n < \omega\},$$
$$\text{for some } \delta \in A \cup \{0\} \text{ and finite } F \subseteq \omega_1 - \delta\}$$

We consider $L(A)$ as a linearly ordered set ordered lexicographically. It is clear that we can identify $L(A)$ with the set of all ω_1-branches of $T(A)$. For $t \in T(A)$, let $B_t = \{x \in L(A) \mid t \subseteq x\}$. Then B_t, $t \in T(A)$ form a basis for the topology on $L(A)$. It will be clear from what follows that $L(A)$ is an σ-additive ω_1-metrizable Lindelöf space. But first we are going to consider the following subspace of $L(A)$

$$X(A) = \{x \in {}^{\omega_1}2 \mid \operatorname{supp}(x) \text{ is finite or}$$
$$\operatorname{supp}(x) = \{\eta_\delta(n) \mid n < \omega\}, \text{ for some } \delta \in A\}.$$

4.4. THEOREM. (i) *$X(A)$ is an ω_1-metrizable Lindelöf space.*

(ii) *If $\langle x_\alpha \mid \alpha < \omega_1\rangle$ is a convergent sequence in $X(A)$, then*

$$\{\delta \in A \mid \{\eta_\delta(n) \mid n < \omega\} = \operatorname{supp}(x_\alpha), \text{ for some } \alpha < \omega_1\} \text{ is nonstationary}.$$

(iii) *$X(A)$ is linearly orderable iff A is nonstationary.*

PROOF. (i) Since $X(A)$ is a closed subspace of $L(A)$ it suffices to show that $L(A)$ is Lindelöf. Let $\mathcal{U}$ be an open cover of $L(A)$. It is clear that we may assume that $\mathcal{U} \subseteq \{B_t \mid t \in T(A)\}$ and that, in fact, $\mathcal{U} = \{B_t \mid t \in M\}$ for some antichain $M \subseteq T(A)$. Let $U = \{s \in T(A) \mid s \subseteq t$ for some $t \in M\}$. Then U is an initial part of $T(A)$ without ω_1-branches. Since $T(A)$ contains no Aronszajn subtrees (Theorem 4.1(i)), U is countable, hence $\mathcal{U}$ is countable.

(ii) For $\delta \in A$, we define $p_\delta \in {}^{\omega_1}2$ by $\operatorname{supp}(p_\delta) = \{\eta_\delta(n) \mid n < \omega\}$. We have to prove that for no stationary $E \subseteq A$, $\{p_\delta \mid \delta \in E\}$ is a convergent sequence in $X(A)$. This follows easily from the Pressing Down Lemma.

(iii) Assume $X(A)$ is linearly orderable by an ordering $\lhd$ and assume, on the contrary, that A is stationary in ω_1. Since each p_δ, $\delta \in A$ is an isolated point of $X(A)$, we can define $q_\delta = \min\{x \in X(A) \mid p_\delta \lhd x\}$ for $\delta \in A$.

Case 1: $\{\delta \in A \mid \operatorname{supp}(q_\delta)$ *is infinite*$\}$ *is stationary.* For each $\delta \in E = \{\delta \in A \mid \operatorname{supp}(q_\delta)$ is infinite$\}$ there is a unique $j(\delta) \in A$ such that $q_\delta = p_{j(\delta)}$. By the Pressing Down Lemma, we may assume that $j(\delta) > \delta$ for each $\delta \in E$. Hence, $\operatorname{supp}(q_\delta) \cap \delta$ is finite for each $\delta \in E$ so that we can find stationary $E' \subseteq E$ and finite $F \subseteq \omega_1$ such that $\operatorname{supp}(q_\delta) \cap \delta = F$ for each $\delta \in E'$. Define $x \in {}^{\omega_1}2$ by $\operatorname{supp}(x) = F$. Then $x \in X(A)$ and $\langle q_\delta \mid \delta \in E'\rangle$ converges to x. Since $L(A)$ is linearly orderable by $\lhd$, it follows that $\langle p_\delta \mid \delta \in E'\rangle$ also converges to x, contradicting (ii).

Case 2: $\{\delta \in A \mid \operatorname{supp}(q_\delta)$ *is finite*$\}$ *is stationary.* Again, by using the Pressing Down Lemma, we can find stationary $E \subseteq \{\delta \in A \mid \operatorname{supp}(q_\delta)$ is finite$\}$ and finite $F \subseteq \omega_1$ such that $\operatorname{supp}(q_\delta) \cap \delta = F$ for each $\delta \in E$. The rest is as in the Case 1.

Assume now that A is nonstationary in ω_1. Let $C \subseteq A$ be a club such that $C \cap A = \emptyset$. The basis $\mathcal{B} = \{B_t \cap X(A) \mid t \in T(A) \upharpoonright C\} - \{\emptyset\}$ has the property that each element of $\mathcal{B}$ has infinitely many immediate $\subseteq$-successors. If we order each set of the immediate successors into the order type $\omega^* + \omega$, then the induced lexicographical ordering of $X(A)$ generates the topology on $X(A)$.

4.5. COROLLARY (Kurepa). *There is a Lindelöf ω_1-metrizable non-linearly orderable topological space.*

To mention just one more application of the line $L(A)$ we need to introduce the following definition. For a topological space X, the *point open* game on X is played as follows. There are two players: I and II. The player I starts the game by choosing a point x_1 of X. After that II chooses an open neighbourhood G_1 of x_1. Again I chooses a point x_2 of X and II chooses an open neighbourhood G_2 of x_2, and so on. I wins the play $\langle x_1, G_1, x_2, G_2, \ldots \rangle$ if $\bigcup\{G_n \mid n < \omega\} = X$; otherwise II wins. The space X is *determined* if one of the players has a winning strategy in the point open game. It is known that the existence of an undetermined metric space is consistent with and independent of ZFC. However, the next result shows that there is always an undetermined ω_1-metric space.

4.6. THEOREM. *The space $L(A)$ is undetermined iff A is a stationary subset of ω_1.*

4.7. COROLLARY. *There is an undetermined ω_1-metrizable topological space.*

4.8. REMARKS. (i) Our trees $T(A)$ are all binary. But it is clear that we may replace 2 in the above definition of $T(A)$ by any number n, $2 \leqslant n \leqslant \omega$. For example, we may define

$$T_n(\emptyset) = \{t \in {}^{\omega_1}n \mid \{\alpha \mid t(\alpha) \neq 0\} \text{ is finite}\}\,.$$

The resulting class of trees has similar properties. It should also be clear that we may replace ω_1 by any regular uncountable cardinal κ (Λ is replaced by $\{\delta < \kappa \mid \operatorname{cf} \delta = \omega\}$).

(ii) The tree $T_\omega(\emptyset)$ (under an equivalent definition) was introduced by KUREPA [1942]. Corollaries 4.2(i) and (ii) were proved by KUREPA [1942] (see also [1958]). Let us note that both 4.2(ii) and (iii) are characterizations of $T(\emptyset)$ in the class of all normal binary $\aleph_1$-trees.

(iii) Corollary 4.5 was proved by KUREPA [1963]; his space is essentially the same as $X(\Lambda)$. KUREPA [1963] also showed that every such example must have a lot of isolated points by proving that every ω_1-metrizable space without isolated points is linearly orderable.

(iv) The point-open game was independently introduced by TELGÁRSKY [1975] and GALVIN [1978]. The problem whether the point-open game is undetermined in ZFC was asked by GALVIN [1978]. A proof of the Corollary 4.7 and some historical remarks concerning this result can be found in TELGÁRSKY [1980]. Let us note that 4.6 also holds for $X(A)$ instead of $L(A)$.

5. Aronszajn trees and lines

The first result of this section shows that Aronszajn trees and Aronszajn lines are very closely related. It also shows that any Aronszajn line has cardinality exactly $\aleph_1$.

5.1. THEOREM. (i) *Every lexicographical ordering of an Aronszajn tree is an Aronszajn line.*

(ii) *Every partition tree of an Aronszajn line is an Aronszajn tree.*

PROOF. (i) Let T be an Aronszajn tree and let $<$ be a lexicographical ordering of T. Let $\varphi = \mathrm{tp}(T, <)$. We have to prove $\varphi \not\geq \omega_1, \omega_1^*$ and $\varphi \not\geq$ uncountable real type. We prove here only $\varphi \not\geq \omega_1$, since the rest is similar. Assume the contrary, i.e., that there exists a $B \subseteq T$ such that $\mathrm{tp}(B, <) = \omega_1$. Since T has countable levels, for each $\alpha < \omega_1$ we can find $t_\alpha \in T_\alpha$ such that $\{s \in B \mid t_\alpha \leq_T s\}$ is uncountable. Since $\mathrm{tp}(B, <) = \omega_1$, we must have $t_\alpha <_T t_\beta$ for $\alpha < \beta < \omega_1$, i.e., $\{t_\alpha \mid \alpha < \omega_1\}$ is an ω_1-branch of T, a contradiction.

(ii) Let $(L, <)$ be an Aronszajn line and let $(T, \supseteq)$ be a partition tree of L. Since $\mathrm{tp}\, L \not\geq \omega_1, \omega_1^*$, it follows that T has no uncountable chains. To prove that T is Aronszajn, it suffices to prove that each level of T is countable. Assume the contrary and define $\alpha = \min\{\beta \mid T_\beta$ is uncountable$\}$. Since T is a binary tree, α is a limit countable ordinal. For each $J \in T \restriction \alpha$, we choose a countable set $A(J) \subseteq J$ cofinal and coinitial with J and define $D = \bigcup\{A(J) \mid J \in T \restriction \alpha\}$. For each $I \in T_\alpha$, we pick $x(I) \in I$ and define $K = D \cup \{x(I) \mid I \in T_\alpha\}$. Then D is a countable order-dense subset of K. This shows that $\mathrm{tp}(K, <)$ is an uncountable real subtype of $\mathrm{tp}(L, <)$, a contradiction. This finishes the proof.

Using similar arguments one can show that any Aronszajn line is isomorphic to a lexicographically ordered Aronszajn tree, and conversely, any (binary) Aronszajn tree is isomorphic to a partition tree of an Aronszajn line.

The next theorem reveals an essential property of the first uncountable cardinal. At this point of our knowledge it is not known whether a similar theorem holds (in ZFC) for any other regular uncountable cardinal (see Section 7).

5.2. THEOREM (Aronszajn). *There is an Aronszajn tree.*

PROOF. Let $\sigma'\mathbb{Q} = \{t \in \sigma\mathbb{Q} \mid t$ has a maximal element$\}$ (see Section 3). Our Aronszajn tree T will be a subtree of $\sigma'\mathbb{Q}$. We shall construct T by induction on the levels T_α, $\alpha < \omega_1$. The αth induction hypothesis is:

$(*)_\alpha$ For each $\gamma < \beta < \alpha$, each $t \in T_\gamma$ and each $\mathbb{Q} \ni x > \max(t)$, there is an $s \in T_\beta$ such that $t \lessdot s$ and $x > \max(s)$.

Case $\alpha = \beta + 1$. Let $T_\alpha = \{t \cup \{x\} \mid t \in T_\beta,\ x \in \mathbb{Q}$ and $x > \max(t)\}$. Then $|T_\alpha| \leq \aleph_0$ and $(*)_{\alpha+1}$ holds.

Case $\lim(\alpha)$. Let $\langle \alpha_n \mid n < \omega\rangle$ be an increasing sequence of ordinals cofinal in α. Fix $t \in T \restriction \alpha$ and $x \in \mathbb{Q}$ such that $x > \max(t)$. Let $m = \min\{n \mid \alpha_n \geq \mathrm{ht}(t)\}$. Using $(*)_\alpha$, we inductively construct an increasing sequence $\langle t_k \mid k < \omega\rangle$ of elements of $T \restriction \alpha$ such that $t_0 = t$, $t_k \in T_{\alpha_{m+k}}$ and $\max(t_k) < x$. Moreover, we may assume that $\sup\{\max(t_k) \mid k < \omega\} = x$. Let $s_{t,k} = (\bigcup\{t_k \mid k < \omega\}) \cup \{x\}$. Then $s_{t,x} \in \sigma'\mathbb{Q}$ and $t \lessdot s_{t,x}$. Finally, let $T_\alpha = \{s_{t,x} \mid t \in T \restriction \alpha,\ x \in \mathbb{Q}$ and $x > \max(t)\}$. Then $|T_\alpha| \leq \aleph_0$ and $(*)_{\alpha+1}$ holds.

Let $T = \bigcup\{T_\alpha \mid \alpha < \omega\}$. Then T is an Aronszajn tree. This finishes the proof.

Note that the above Aronszajn tree T is $\mathbb{Q}$-embeddable, since $\max(\cdot)$ is a strictly increasing rational function on T. Whether or not every Aronszajn tree is $\mathbb{Q}$-embeddable is a question which cannot be decided on the basis of the usual axioms of set theory (see Section 9). The proof of the next theorem shows that in the above Aronszajn tree construction we can also have a considerable control on the lexicographical ordering of the tree so constructed.

5.3. THEOREM (Shelah). *There is an uncountable linearly ordered set L such that L^2 is the union of countable many chains.*

PROOF. The line L will be a lexicographically ordered Aronszajn tree T. The tree T will be as usual constructed by induction on the levels $T_\alpha \subseteq {}^\alpha\mathbb{Q}$, $\alpha < \omega_1$. Let $<$ denote the lexicographical ordering of ${}^{\omega_1}\mathbb{Q}$, as well as the ordering on the square $({}^{\omega_1}\mathbb{Q})^2$: $\bar{s} \leqslant \bar{t}$ iff $\bar{s}(i) \leqslant \bar{t}(i)$ for all $i < 2$. We shall also consider the following partial ordering on $({}^{\omega_1}\mathbb{Q})^2$: $\bar{s} \lll \bar{t}$ iff

(i) $\bar{s}(i) \subseteq \bar{t}(i)$ for all $i < 2$, and

(ii) if $l < 2$ and if $\mathrm{ht}(\bar{s}(l)) < \mathrm{ht}(\bar{s}(1-l))$, then $\bar{s}(l) = \bar{t}(l)$. We also inductively construct a function C: $T^2 \to [\omega]^{\aleph_0}$ such that $C(\bar{s}) \supseteq C(\bar{t})$ for $\bar{s} \lll \bar{t}$ in T^2. The αth induction hypothesis is:

$(1)_\alpha$ If $\gamma < \beta < \alpha$, $s_0, \ldots, s_n \in T \restriction \gamma$, $s_{n+1}, \ldots, s_m \in T_\gamma$ and $k < \omega$, then there are $s'_{n+1}, \ldots, s'_m \in T_\beta$ such that $s_l \subseteq s'_l$ for $n < l \leqslant m$, and if $s'_l = s_l$ for $l \leqslant n$, then $k \cap C(s_{i(0)}, s_{i(1)}) = k \cap C(s'_{i(0)}, s'_{i(1)})$ for all $i(0), i(1) \leqslant m$.

$(2)_\alpha$ If $\bar{s}$, $\bar{t} \in (T \restriction \alpha)^2$ and if $C(\bar{s}) \cap C(\bar{t}) \neq \emptyset$, then

(i) $\bar{s} \leqslant \bar{t}$ or $\bar{t} \leqslant \bar{s}$, and

(ii) if $i < 2$, $\bar{s}(i) = \bar{t}(i)$ and $\bar{s} \neq \bar{t}$, then

$$\mathrm{ht}(\bar{s}(i)) < \max\{\mathrm{ht}(\bar{s}(1-i)), \mathrm{ht}(\bar{t}(1-i))\}\,.$$

The condition $(1)_\alpha$ is to ensure the possibility of constructing $C(\bar{t})$ for $\bar{t} \in (T \restriction \alpha+1)^2$ in the case when α is a limit ordinal. The condition $(2)_\alpha$(i) is the key for showing that L^2 is the union of countably many chains.

To begin with, let $T_0 = \{\emptyset\}$ and $C(\emptyset, \emptyset) = \omega$.

Case $\alpha = \delta + 1$, Let $T_\alpha = \{t^\frown x \mid t \in T_\delta, x \in \mathbb{Q}\}$. We have to construct $C(\bar{t})$ for $\bar{t} \in (T \restriction \alpha+1)^2 - (T \restriction \alpha)^2$. Since it suffices to show $(1)_{\alpha+1}$ only for the case $\beta = \delta$, let $k(i)$, $n(i)$, $m(i)$, $\langle s^i_0, \ldots, s^i_{m(i)}\rangle$ for odd $i < \omega$ be a list of all possible candidates for $(1)_{\alpha+1}$ with $\beta = \delta$. Let $\bar{s}^i$, for even $i < \omega$ be a list of all $\bar{t} \in (T \restriction \alpha+1)^2 - (T \restriction \alpha)^2$ each appearing infinitely many times. Let π_α: $(T \restriction \alpha+1)^2 \to (T \restriction \alpha)^2$ be defined by $\pi_\alpha(t_0, t_1) = (t_0 \restriction \alpha, t_1 \restriction \alpha)$. Now by induction on $i < \omega$ we define a finite set Γ_i of conditions of the form $[l \in C(\bar{s})]$ so that if we define $C(\bar{t}) = \{l < \omega \mid [l \in C(\bar{t})] \in \bigcup\{\Gamma_i \mid i < \omega\}\}$ for $\bar{t} \in (T \restriction \alpha+1)^2 - (T \restriction \alpha)^2$, then $(1)_{\alpha+1}$ and $(2)_{\alpha+1}$ are satisfied.

Assume first i is even. Since $C(\pi_\alpha(\bar{s}^i))$ is infinite, there is an $l_i \in C(\pi_\alpha(\bar{s}^i))$ such that $l_i > k(j)$ for all odd $j < i$ and l_i does not appear in $\bigcup\{\Gamma_j \mid j < i\}$. Let $\Gamma_i = \{[l_i \in C(\bar{s}^i)]\}$.

Assume now i is odd. Choose a rational number x_i so that for every $j < i$ and every t appearing in Γ_j, if $t \in T_\alpha$, then $t(\delta) < x_i$. Let

$$\Gamma_i = \{[l \in C(t_{p(0)}, t_{p(1)})] \mid p(0), p(1) \leqslant m(i), l < k(i) \text{ and } l \in C(s^i_{p(0)}, s^i_{p(1)})\}$$

where $t_0, \ldots, t_{m(i)}$ are defined by: $t_j = s^i_j$ for $j \leqslant n(i)$ and $t_j = s^i_j{}^\frown x_i$ for $n(i) < j \leqslant m(i)$.

We have to check that $(1)_{\alpha+1}$ and $(2)_{\alpha+1}$ hold. The definition of Γ_i directly implies that each $C(\bar{t})$ is infinite, that $C(\bar{t}) \subseteq C(\pi_\alpha(\bar{t}))$ and that $(1)_{\alpha+1}$ holds. Thus $\bar{s}, \bar{t} \in (T \restriction \alpha + 1)^2$, $\bar{s} \leqslant \bar{t}$ implies $C(\bar{s}) \supseteq C(\bar{t})$. So, let us check $(2)_{\alpha+1}(i)$. Assume the contrary, i.e., that for some $\bar{s}, \bar{t} \in (T \restriction \alpha + 1)^2$, $C(\bar{s}) \cap C(\bar{t}) \neq \emptyset$ but, for example, $\bar{s}(0) < \bar{t}(0)$ and $\bar{s}(1) > \bar{t}(1)$. It follows that $C(\pi_\alpha(\bar{s})) \cap C(\pi_\alpha(\bar{t})) \neq \emptyset$ and $\pi_\alpha(\bar{s})(0) \leqslant \pi_\alpha(\bar{t})(0)$ and $\pi_\alpha(\bar{s})(1) \geqslant \pi_\alpha(\bar{t})(1)$. By $(2)_\alpha(i)$, we have that either $\pi_\alpha(\bar{s})(0) = \pi_\alpha(\bar{t})(0)$ or $\pi_\alpha(\bar{s})(1) = \pi_\alpha(\bar{t})(1)$ holds. By symmetry, assume that $\pi_\alpha(\bar{s})(0) = \pi_\alpha(\bar{t})(0)$. If $\pi_\alpha(\bar{s}) \neq \pi_\alpha(\bar{t})$, then by $(2)_\alpha$(ii) we have that $\pi_\alpha(\bar{s})(0) \in T \restriction \delta$, hence by the definition of π_α we have $\bar{s}(0) = \pi_\alpha(\bar{s})(0) = \pi_\alpha(\bar{t})(0) = \bar{t}(0)$, a contradiction. Now assume $\pi_\alpha(\bar{s}) = \pi_\alpha(\bar{t})$. It is clear that $\bar{s} \neq \pi_\alpha(\bar{s})$ and $\bar{t} \neq \pi_\alpha(\bar{t})$. Choose $l \in C(\bar{s}) \cap C(\bar{t})$ and assume $[l \in C(\bar{s})]$ is determined not later than $[l \in C(\bar{t})]$. This implies that $[l \in C(\bar{t})]$ is determined on some odd stage $i < \omega$ and that $\bar{t} = (t_{p(0)}, t_{p(1)})$ for some $p(0), p(1) \leqslant m(i)$. By the assumption $\bar{s}(1) > \bar{t}(1)$ and by the definition of x_i we must have $p(1) \leqslant n(i)$, hence $\bar{t}(1) = t_{p(1)} = s^i_{p(1)} \in T \restriction \delta$. But this implies that $\bar{t}(1) = \pi_\alpha(\bar{t})(1) = \pi_\alpha(\bar{s})(1) = \bar{s}(1)$, a contradiction. The checking of $(2)_{\alpha+1}$(ii) is similar.

Case $\lim(\alpha)$. Let $\langle \alpha_n \mid n < \omega \rangle$ be a strictly increasing sequence of ordinals cofinal with α. Using the hypothesis $(1)_\alpha$ we can inductively define $k_i < \omega$, $s^i_0, \ldots, s^i_{m(i)} \in T \restriction (\alpha_i + 1)$, for $i < \omega$, such that:

(a) $k_i < k_{i+1}$ and $m(i) < m(i+1)$,

(b) if $l \leqslant m(i)$ and $i < j$, then $s^i_l \subseteq s^j_l$, and if $s^i_l \in T \restriction \alpha_i$ or $s^j_l \in T \restriction \alpha_j$, then $s^i_l = s^j_l$,

(c) if $p(0), p(1) \leqslant m(i)$, then $k_i \cap C(s^i_{p(0)}, s^i_{p(1)}) = k_i \cap C(s^{i+1}_{p(0)}, s^{i+1}_{p(1)})$ and $\{l \mid k_i < l < k_{i+1}\} \cap C(s^{i+1}_{p(0)}, s^{i+1}_{p(1)}) \neq \emptyset$,

(d) for any $\gamma, \beta, k, n, m, s_0, \ldots, s_m$ as in $(1)_\alpha$ there is an $i < \omega$ so that

(i) $k \leqslant k_i$ and $\beta < \alpha_i$,

(ii) if $l \leqslant n$, then $s_l = s^{i+1}_{m(i)+l}$, and if $n < l \leqslant m$, then $s_l \subseteq s^{i+1}_{m(i)+l} \in T_{\alpha_{i+1}}$,

(iii) if $p(0), p(1) \leqslant m$, then $k \cap C(s_{p(0)}, s_{p(1)}) = k \cap C(s^{i+1}_{m(i)+p(0)}, s^{i+1}_{m(i)+p(1)})$.

Fix $j < \omega$. If the sequence $\langle s^i_j \mid j \leqslant i < \omega \rangle$ determines an α-branch of $T \restriction \alpha$, let $s_j \in {}^\alpha\mathbb{Q}$ be the unique extension of this branch. If this sequence does not determine an α-branch of $T \restriction \alpha$, let $s_j = s^j_j$. Note that in the latter case we have $s^i_j = s^j_j$ for all $i \geqslant j$. Finally, let $T_\alpha = \{s_j \mid j < \omega \text{ and } s_j \in {}^\alpha\mathbb{Q}\}$. Then (a)–(d) ensure that $T \restriction (\alpha + 1)$ is a normal tree. We have to define $C(\bar{t})$ for $\bar{t} \in (T \restriction \alpha + 1)^2 - (T \restriction \alpha)^2$. To finish this, for each $i < \omega$ we define σ_i: $T \restriction (\alpha + 1) \to T \restriction \alpha$ as follows. If $t \in T \restriction \alpha$, let $\sigma_i(t) = t$; if $s_j \in T_\alpha$, let $\sigma_i(s_j) = s^i_j$ if $j \leqslant i$ and $\sigma_i(s_j) = \emptyset$ if $j > i$. Now for each $\bar{t} \in (T \restriction \alpha + 1)^2 - (T \restriction \alpha)^2$ and $l < \omega$, we put l in $C(\bar{t})$ iff for arbitrary large $i < \omega$ we have $l \in C(\sigma_i(\bar{t}(0)), \sigma_i(\bar{t}(1)))$. By (a)–(d), we have that for each such $\bar{t}$ there are $j(0), j(1) < \omega$ such that for every large enough $i < \omega$, $\sigma_i(\bar{t}(0)) = s^i_{j(0)} = \bar{t}(0) \restriction \alpha_i$ and $\sigma_i(\bar{t}(1)) = s^i_{j(1)} = \bar{t}(1) \restriction \alpha_i$. This shows that $l \in C(\bar{t})$ is well-defined and also that $C(\bar{t})$ is infinite (by (c)). The conditions $(1)_{\alpha+1}$ and $(2)_{\alpha+1}$

follow directly using $(1)_\alpha$ and $(2)_\alpha$ respectively, and the above definition of $C(t)$. This completes the case $\lim(\alpha)$, hence the inductive construction.

Finally, let $L = \bigcup\{T_\alpha \mid \alpha < \omega_1\}$ ordered by the lexicographical ordering $<$. For $n < \omega$, we define $J_n = \{\bar{t} \in L^2 \mid n = \min C(\bar{t})\}$. Then by $(2)_\alpha$(i), we have that each J_n is a chain in L^2 and $L^2 = \bigcup\{J_n \mid n < \omega\}$. This completes the proof.

Note that we have not checked that $(\bigcup\{T_\alpha \mid \alpha < \omega_1\}, \subseteq)$ is an Aronszajn tree nor that L is an Aronszajn line. This follows from the next result.

5.4. THEOREM. *If L is an uncountable linearly ordered set such that L^2 is the union of countably many chains, then*

(i) *L is an Aronszajn line.*

(ii) *L^n is the union of countably many chains for each $n < \omega$.*

(iii) *L contains no two uncountable anti-isomorphic subsets.*

PROOF. (i) No square of an uncountable set of reals is the union of countably many chains. Also, neither $(\omega_1)^2$ nor $(\omega_1^*)^2$ is the union of countably many chains.

(ii) Let $L^2 = \bigcup\{J_m \mid m < \omega\}$ be a disjoint decomposition of L^2 into countably many chains. Fix $2 \leqslant n < \omega$. Let Γ be the set of all $\sigma: \{\langle i, j\rangle \mid i < j < n\} \to \omega$. For $\sigma \in \Gamma$, we define $J_\sigma = \{\langle t_0, \ldots, t_{n-1}\rangle \mid \langle t_i, t_j\rangle \in J_{\sigma\langle i,j\rangle}$ for all $i < j < n\}$. Then each J_σ is a chain, and $L^n = \bigcup\{J_\sigma \mid \alpha \in \Gamma\}$.

(iii) Assume that K and K' are uncountable subsets of L and that $f: K \to K'$ is an anti-isomorphism. Then $\{\langle t, f(t)\rangle \mid t \in K\}$ is an uncountable antichain of L^2, a contradiction. This finishes the proof.

Let L be an Aronszajn line such that L^2 is the union of countably many chains. Let L^* be the converse of L. Then by 5.4(iii), L and L^* have no uncountable isomorphic subsets. It is an open problem whether we can find in ZFC two Aronszajn lines with no uncountable isomorphic nor anti-isomorphic subsets.

5.5. THEOREM (Shelah). *There is a homogeneous non-reversible Aronszajn continuum.*

PROOF. The Aronszajn tree $T \subseteq {}^{\omega_1}\mathbb{Q}$ we have constructed in the proof of 5.3 may have also the following property: For all $\alpha < \omega_1$ and $s, t \in T_\alpha$ there is an automorphism σ of T such that $\sigma(s) = t$ and σ also preserves the lexicographical ordering of T. In 6.4 we shall see that this implies that the Dedekind completion $\mathbb{K}$ of the line $(T, \preccurlyeq)$ is a homogeneous Aronszajn continuum. Assume now that $\rho: \mathbb{K} \to \mathbb{K}$ is an anti-isomorphism. It is clear that $\mathbb{K}$ cannot be a Suslin continuum, hence we can find an uncountable family $\mathscr{I}$ of disjoint nonempty open intervals of $\mathbb{K}$. For each $I \in \mathscr{I}$, we choose $s_I \in I \cap T$ and $t_I \in \rho'' I \cap T$ arbitrarily. Then $\{s_I \mid I \in \mathscr{I}\}$ and $\{t_I \mid I \in \mathscr{I}\}$ are two uncountable anti-isomorphic subsets of $(T, \preccurlyeq)$ contradicting the Theorem 5.4(iii).

We have just seen that there exist two non-isomorphic Aronszajn lines. Now we are going to find a canonical construction of $2^{\aleph_1}$ *topologically* different Aronszajn lines.

Let T be a fixed Aronszajn tree such that for each $\alpha < \omega_1$ we have a fixed linear ordering $<_\alpha$ on T_α. For each $\delta \in \Lambda$ we choose a δ-branch l_δ of $T \restriction \delta$ (hence l_δ has no extension in T_δ). Now, for each uncountable $A \subseteq \Lambda$, we define $L(A) = \{l_\delta \mid \delta \in A\}$. We order $L(A)$ by the lexicographical ordering induced by $<_\alpha$, $\alpha < \omega_1$, and consider only those subsets A of Λ which have the property that if $\delta \in A$ and $t \in l_\delta$, then l_δ is neither a minimal nor a maximal point of $\{l \in L(A) \mid t \in l\}$. If B is any stationary subset of Λ, then a standard argument shows that we can find $A \subseteq B$ such that $B - A$ is nonstationary in ω_1 and $L(A)$ has the above property.

5.6. LEMMA. *If f: $L(A) \to L(B)$ is 1–1 and continuous, then $A - B$ is nonstationary in ω_1.*

PROOF. Assume not, i.e., that $E = A - B$ is stationary in ω_1. Let $C = \{\delta \in \Lambda \mid$ if $\alpha \in A$, $\beta \in B$ and $f(l_\alpha) = l_\beta$, then $\alpha < \delta$ iff $\beta < \delta\}$, and $D = \{\delta \in \Lambda \mid$ if $t \in T \restriction \delta$ is contained in a branch from $L(A)$, then it is contained in a branch from $\{l_\alpha \mid \alpha \in A \cap \delta\}\}$. Clearly, both C and D are clubs in ω_1. Let $\delta \in C \cap D \cap E$ and let $\gamma \in B$ be such that $f(l_\delta) = l_\gamma$. It follows that $\gamma > \delta$. Let $t \in T_\delta \cap l_\gamma$, and let $I = \{l \in L(B) \mid t \in l\}$. Then I is an open convex subset of $L(B)$ which contains l_γ and which is disjoint from $\{f(l_\alpha) \mid \alpha \in A \cap \delta\}$, contradicting the fact that f is continuous. This completes the proof.

5.7. THEOREM. *There is a family $\mathscr{L}$ of $2^{\aleph_1}$ Aronszajn lines, such that if $K, L \in \mathscr{L}$ and if f: $K \to L$ is 1–1 and continuous, then $K = L$ and $f =$ identity.*

PROOF. Choose $\mathscr{A} \subseteq \mathscr{P}(\omega_1)$ such that if A and B are different numbers of $\mathscr{A}$, then $A - B$ is stationary in ω_1 (see 1.1). Moreover, we can assume that each $A \in \mathscr{A}$ has the following property: For each $t \in T$, the set $\{\alpha \in A \mid t \in l_\alpha\}$ is either empty or stationary, while $\{l \in L(A) \mid t \in l\}$ has no minimal nor maximal element. Now, the fact that $L(A)$, $A \in \mathscr{A}$ satisfies the theorem follows easily from 5.6.

We have already remarked that any Aronszajn line has cardinality $\aleph_1$, hence the number $2^{\aleph_1}$ in the previous theorem is best possible.

Let us now consider the isomorphism types of (normal) Aronszajn trees. Although the Aronszajn trees and Aronszajn lines are very closely related, the above result about lines does not give a similar result for trees. However, a similar result is provable using different methods.

5.8. THEOREM. *There is a family of $2^{\aleph_1}$ nonisomorphic rigid Aronszajn trees.*

The proof of Theorem 5.8 will not be given here, but an examination of that proof will show that it does not give two Aronszajn trees which are not club-isomorphic. In fact, the next theorem shows that in ZFC, we cannot find such trees.

We prove first a lemma which will also be needed later. Let $1 \leq n < \omega$ and let $T^1, \ldots, T^n$ be a fixed sequence of trees with no uncountable chains. Let $\langle \bar{t}_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence from $T^1 \otimes \cdots \otimes T^n$ such that $\mathrm{ht}(\bar{t}_\alpha) < \mathrm{ht}(\bar{t}_\beta)$ if $\alpha < \beta < \omega_1$. If $k < \omega$ and if $\delta < \omega_1$ then we say that $\langle \bar{t}_\alpha \mid \alpha < \omega_1 \rangle$ is *k-distributed* on the δth level $(T^1 \otimes \cdots \otimes T^n)_\delta$ of $T^1 \otimes \cdots \otimes T^n$ if for any disjoint sequence $\bar{u}_1, \ldots, \bar{u}_k$ of elements of $(T^1 \otimes \cdots \otimes T^n)_\delta$ there are uncountably many $\alpha < \omega_1$ such that u_i^j is $\leq_j$-incomparable with t_α^j for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$.

5.9. LEMMA. *For any ω_1-sequence $\langle \bar{t}_\alpha \mid \alpha < \omega_1 \rangle$ of elements of $T^1 \otimes \cdots \otimes T^n$ as above, and for any $k < \omega$ there is a $\delta < \omega_1$ such that $\langle \bar{t}_\alpha \mid \alpha < \omega_1 \rangle$ is k-distributed on $(T^1 \otimes \cdots \otimes T^n)_\delta$.*

PROOF. Assume the contrary. Then there exists a $1 \leq k \leq \omega$ such that for any $\delta < \omega_1$ we can find disjoint $\bar{u}_1(\delta), \ldots, \bar{u}_k(\delta)$ in $(T^1 \otimes \cdots \otimes T^n)_\delta$ such that for all but countably many $\alpha < \omega_1$ we can find $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$ such that $u_i^j(\delta) <_j t_\alpha^j$. Let $\mathcal{U}$ be a uniform ultrafilter on ω_1. For $\delta < \omega_1$, $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$ we define $A_\delta(i, j) = \{\alpha < \omega_1 \mid u_i^j(\delta) <_j t_\alpha^j\}$. Then for each $\delta < \omega_1$ we can find i_δ and j_δ so that $A_\delta(i_\delta, j_\delta) \in \mathcal{U}$. Hence, for some uncountable $B \subseteq \omega_1$ and $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$ we have $i_\delta = i$ and $j_\delta = j$ for all $\delta \in B$. Fix $\delta < \gamma$ in B, and choose $\alpha \in A_\delta(i, j) \cap A_\gamma(i, j)$ such that $\alpha > \gamma$. Then $u_i^j(\delta), u_i^j(\gamma) <_j t_\alpha^j$, hence $u_i^j(\delta) <_j u_i^j(\gamma)$. This shows that $\{u_i^j(\delta) \mid \delta \in B\}$ is an ω_1-chain of T^j, a contradiction.

5.10. THEOREM. *Assume PFA. Then every two Aronszajn trees are club-isomorphic.*

PROOF. Let T and U be given $\aleph_0$-ary Aronszajn trees. Let $<$ be a fixed lexicographical ordering of T. Then every finite subset $\{t_1, \ldots, t_k\}$ of T is also considered as a sequence $\langle t_1, t_2, \ldots, t_k \rangle$, where $t_1 < t_2 < \cdots < t_k$.

Let $\mathbb{P}$ be the set of all pairs $p = \langle A_p, f_p \rangle$ such that

(i) A_p is a finite subset of ω_1;

(ii) f_p is a finite partial isomorphism from $T \restriction A_p$ into $U \restriction A_p$.

(iii) $\mathrm{dom}(f_p)$ is a subtree of $T \restriction A_p$ in which every branch has cardinality $|A_p|$.

$\mathbb{P}$ is ordered by $p \leq q$ iff $A_p \supseteq A_q$ and $f_p \supseteq f_q$. For $p \in \mathbb{P}$, we let $\alpha_p = \max(A_p)$. Now for $p \in \mathbb{P}$ and $\alpha \leq \alpha_p$, we define $p \restriction \alpha = \langle A_p \cap \alpha,\ f_p \restriction (T \restriction (A_p \cap \alpha)) \rangle$ and $p \restriction \{\alpha\} = \langle t_1 \restriction \alpha, \ldots, t_n \restriction \alpha; f_p(t_1) \restriction \alpha, \ldots, f_p(t_n) \restriction \alpha \rangle$, where $\{t_1, \ldots, t_n\} = \mathrm{dom}(f_p) \cap T_{\alpha_p}$.

Claim. $\mathbb{P}$ is proper.

Proof of Claim. Let λ be a large enough regular cardinal and let N be a countable elementary submodel of $H(\lambda)$ such that $T, U, \mathbb{P} \in N$. Fix $p \in \mathbb{P} \cap N$.

We have to find a $(\mathbb{P}, N)$-generic condition $q \leqslant p$ (see Baumgartner's chapter of the Handbook). Let $N \cap \omega_1 = \delta$ and let $A_q = A_p \cup \{\delta\}$. Choose f_q such that $q = \langle A_q, f_q \rangle$ is an element of $\mathbb{P}$ and $q \restriction (\alpha_p + 1) = p$. We will show that q is $(\mathbb{P}, N)$-generic. Let $r \leqslant q$, and let $D \in N$ be a dense open subset of $\mathbb{P}$. We have to show that r is compatible with a member of $D \cap N$. Let $p_0 = r \restriction \delta$. We may assume that $r \in D$. Choose $\alpha_{p_0} \leqslant \alpha < \delta$ such that $|\mathrm{dom}(r) \cap T_\delta| = |\{t \restriction \alpha \mid t \in \mathrm{dom}(r) \cap T_\delta\}|$ and $|\mathrm{rang}(r) \cap U_\delta| = |\{u \restriction \alpha \mid u \in \mathrm{rang}(r) \cap U_\delta\}|$. Extend p_0 to a condition $p_1 \in \mathbb{P} \cap N$ such that $\alpha_{p_1} = \alpha$ and $r \restriction \{\alpha\} = p_1 \restriction \{\alpha\}$. Let $n = |\mathrm{dom}(r) \cap T_\delta|$ and let

$$S = \{\bar{s} \in T^{(n)} \otimes U^{(n)} | \text{ there are } \beta \geqslant \alpha \text{ and } r' \in D \text{ such that } r' \restriction (\alpha + 1) = p_1, \\ r' \restriction \{\beta\} = s, \text{ and } A_{r'} \cap (\alpha, \beta) = \emptyset\}.$$

Then S is an initial part of $(T^{(n)} \otimes U^{(n)})^{p_1 \restriction \{\alpha\}}$ and $S \in N$. Since $r \restriction \{\beta\} \in S$ for all $\alpha \leqslant \beta < \delta$, we have that $N \models S$ is uncountable. Hence S is really uncountable. By an easy application of 5.9 and by the fact that N is an elementary submodel of $H(\lambda)$, we can find a $\beta \in (\alpha, \delta)$ such that $\{\bar{s} \in S \mid \mathrm{ht}(\bar{s}) = \beta\}$ has infinitely many disjoint sequences. Thus we can find $\bar{s} \in S$, $\mathrm{ht}(\bar{s}) = \beta$ such that $r \restriction \{\beta\}$ and $\bar{s}$ are disjoint. Since $\bar{s} \in S$, we can find $r' \in D \cap N$ such that $r' \restriction \{\beta\} = \bar{s}$, $r' \restriction (\alpha + 1) = p_1$, and $A_{r'} \cap (\alpha, \beta) = \emptyset$. Then r and r' are compatible. This proves the Claim.

For $t \in T$ and $u \in U$ we define $D_t = \{p \in \mathbb{P} \mid \mathrm{ht}(t) \notin A_p$ and $\mathrm{ht}(t) < \alpha_p$ or $\mathrm{ht}(t) \in A_p$ and $t \in \mathrm{dom}(f_p)\}$ and $E_u = \{p \in \mathbb{P} \mid \mathrm{ht}(u) \notin A_p$ and $\mathrm{ht}(u) < \alpha_p$ or $\mathrm{ht}(u) \in A_p$ and $u \in \mathrm{rang}(f_p)\}$. Then both D_t and E_u are dense open subsets of $\mathbb{P}$. By PFA, let $G \subseteq \mathbb{P}$ be a $\{D_t, E_u \mid t \in T,\ u \in U\}$-generic set. Let $A = \bigcup\{A_p \mid p \in G\}$ and $f = \bigcup\{f_p \mid p \in G\}$. Then A is uncountable and f: $T \restriction A \to U \restriction A$ is an isomorphism. It is clear that f can be extended uniquely on the closure of A in ω_1. Hence T and U are club-isomorphic. This finishes the proof.

The reader has already noticed that the conclusion of 5.10 cannot be proved in ZFC alone. For example, if T is a special Aronszajn tree and if U is a Suslin tree, then T and U are not club-isomorphic. Our next result uses, in a sense, optimal hypothesis to construct two non-club-isomorphic Aronszajn trees.

5.11. THEOREM. *Assume* $2^{\aleph_0} < 2^{\aleph_1}$. *Then for every Aronszajn tree T there is an Aronszajn tree U such that T is not club-embeddable into U.*

The proof of 5.11 will not be given here, but let us just note that one can go further and use the hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ to construct a family of $2^{\aleph_1}$ non-club-isomorphic Aronszajn trees.

In the rest of this section we consider Aronszajn tree as a topological space with the interval topology. Our first theorem shows that special Aronszajn trees are particularly interesting.

5.12. THEOREM (Jones). *Let T be an $\aleph_1$-tree. Then T is a special Aronszajn tree iff T is a Moore space.*

PROOF. Assume $T = \bigcup\{A_n \mid n < \omega\}$, where each A_n is an antichain of T. For $n < \omega$ and $t \in T$, let $G_n(t) = \{s \leq_T t \mid [s, t) \cap (\bigcup\{A_i \mid i \leq n\}) = \emptyset\}$. Let $\mathscr{G}_n = \{G_n(t) \mid t \in T\}$. Then $\{\mathscr{G}_n \mid n < \omega\}$ is a development of T. Conversely, let $\{\mathscr{H}_n \mid n < \omega\}$ be a development of T. For $t \in T$, let $h(t)$ be the least $n < \omega$ such that $\bigcup\{H \in \mathscr{H}_n \mid t \in H\} \subseteq (\cdot, t]$. Now for $n < \omega$, let $E_n = \{t \in T \mid h(t) = n\}$. Then it is easily seen that no E_n contains a chain of type $\omega + 1$, hence each E_n is a special subtree of T. Thus T is a special tree. This finishes the proof.

Let T be a special Aronszajn tree and let $f: T \to \omega$ be a specializing map. For each $\alpha < \omega$ pick $t_\alpha \in R_\alpha T$. Then for some $n < \omega$, $\{\alpha < \omega_1 \mid f(t_\alpha) = n\}$ is a stationary subset of ω_1, i.e. $A = \{t_\alpha \mid f(t_\alpha) = n\}$ is a stationary antichain of T. Using the Pressing Down Lemma, one can easily show that A is a discrete subset of T which cannot be separated. Hence no special Aronszajn tree is a collectionwise Hausdorff space, hence no special Aronszajn tree is metrizable. Hence the problem whether a given special Aronszajn tree is a normal topological space now becomes interesting. It turns out that this problem is undecidable in ZFC, as we shall see now.

5.13. THEOREM (Fleissner). *Assume* $\mathrm{MA}_{\aleph_1}$. *Then every Aronszajn tree is a normal topological space.*

PROOF. Let T be an Aronszajn tree and let F and H be closed and disjoint subsets of T. In 9.5 we shall prove that T is a special tree, so let $f: T \to \omega$ be a specializing map. For $t \in T$ and $n < \omega$, let $B_n(t) = \bigcup\{[s, t] \mid s \leq_T t$ and $[s, t) \cap f^{-1}(n) = \emptyset\}$. Then $\{B_n(t) \mid n < \omega\}$ is a neighbourhood basis of t in T.

Let $\mathbb{P}$ be the set of all finite partial functions from $F \cup H$ into ω such that $(F \cup \bigcup\{B_{p(t)}(t) \mid t \in \mathrm{dom}(p) \cap F\}) \cap (H \cup \bigcup\{B_{p(t)}(t) \mid t \in \mathrm{dom}(p) \cap H\}) = \emptyset$. The ordering on $\mathbb{P}$ is $\supseteq$. A standard Δ-system argument shows that $\mathbb{P}$ is a ccc poset.

For $t \in F \cup H$, let $D_t = \{p \in \mathbb{P} \mid t \in \mathrm{dom}(p)\}$. Then D_t is a dense open subset of $\mathbb{P}$. Let G be a $\{D_t \mid t \in F \cup H\}$-generic subset of $\mathbb{P}$ and let $h = \bigcup\{p \mid p \in G\}$. Then $\bigcup\{B_{h(t)}(t) \mid t \in F\}$ and $\bigcup\{B_{h(t)}(t) \mid t \in H\}$ are two disjoint open sets which separate F and H. Hence T is normal.

5.14. THEOREM (Devlin–Shelah). *Assume* $2^{\aleph_0} < 2^{\aleph_1}$. *Let T be a special Aronszajn tree. Then T is not a normal topological space.*

PROOF. Let $\mathscr{I}$ denote the ideal of small sets defined in Section 1. By the assumption $2^{\aleph_0} < 2^{\aleph_1}$, we have $\omega_1 \notin \mathscr{I}$ (see Section 1). Let T be a special Aronszajn tree which we identify with ω_1 so that $\alpha <_T \beta$ implies $\alpha < \beta$, and $\mathrm{ht}_T(\alpha) = \alpha$ if α is a limit ordinal. Let A_n be disjoint antichains of T with $T = \bigcup\{A_n \mid n < \omega\}$. Since $\mathscr{I}$ is σ-complete, there is some n_0 such that $A_{n_0} \notin \mathscr{I}$. Let $E = A_{n_0} \cap A$.

Define F: ${}^{\omega_1}2 \to 2$ as follows. Given $f \in {}^{\alpha}2$, let $F(f) = 0$ iff there is a $\gamma <_T \alpha$ such that $f(\beta)=0$ for all β, $\gamma <_T \beta <_T \alpha$; and let $F(f)=1$ otherwise. Since $E \notin \mathcal{I}$, there is a $g \in {}^{\omega_1}2$ such that for all $f \in {}^{\omega_1}2$, the set $\{\alpha \in E \mid F(f \restriction \alpha) = g(\alpha)\}$ is stationary. Let $J = \{\alpha \in E \mid g(\alpha)=0\}$, $K = \{\alpha \in E \mid g(\alpha) = 1\}$. Since E is an antichain of T, both J and K are closed subsets of T. Let us prove that J and K cannot be separated by disjoint open sets. Assume the contrary, that there are disjoint open sets U, $W \subseteq T$ with $J \subseteq U$, $K \subseteq W$. Define $f \in {}^{\omega_1}2$ by $f(\alpha)=1$ iff $\alpha \notin W$. Then by the property of g, the set $E' = \{\alpha \in E \mid F(f \restriction \alpha) = g(\alpha)\}$ is stationary. Fix $\alpha \in E'$. Assume $g(\alpha)=0$. Then $F(f\restriction\alpha)=0$, hence for some $\gamma <_T \alpha$, $f(\beta)=0$ for all $\gamma <_T \beta <_T \alpha$. By the definition of f, $(\gamma, \alpha)_T \subseteq W$, a contradiction, since $g(\alpha)=0$ means that $\alpha \in J$. If $g(\alpha)=1$ we get a contradiction in a similar way. This completes the proof.

5.15. REMARKS. (i) Theorem 5.2 was proved by Aronszajn in KUREPA [1935; p. 96]. The proof of this theorem is taken from KUREPA [1937a], who was probably the first to notice the existence of a special Aronszajn tree. The Aronszajn tree construction was rediscovered later by SPECKER [1949] and JONES [1953] (see also [1965]).

(ii) Theorem 5.1 is a part of the folklore of the subject. A large part of this theorem was proved in KUREPA [1935; pp. 127–9] (see also RICABARRA [1958], and ERDÖS–RADO [1956; p. 443]).

(iii) The question whether there exists an uncountable linearly ordered set whose square is the union of countably many chains was asked by COUNTRYMAN [1970]. Theorems 5.3 and 5.5 were proved by SHELAH [1976]. Theorem 5.4(i) was proved by COUNTRYMAN [1970] and Theorems 5.4(ii) and (iii) were proved by Galvin; see SHELAH [1976; p. 113].

(iv) The existence of $2^{\aleph_1}$ Aronszajn lines none of which is homomorphic to a subspace of any other, as well as the existence of a rigid Aronszajn line were proved by BAUMGARTNER [1975]. The existence of $2^{\aleph_1}$ non-isomorphic Aronszajn trees was proved by GAIFMAN–SPECKER [1964] in response to a problem of KUREPA [1935; p. 100]. The existence of a rigid Aronszajn tree was independently proved by AVRAHAM [1979], BAUMGARTNER (unpublished) and TODORČEVIĆ [1980] in response to a problem of JECH [1972; p. 70].

(v) Lemma 5.9 appeared implicitly in BAUMGARTNER [1970a; p. 85] and explicitly in DEVLIN–JOHNSBRÅTEN [1974; pp. 63–4]. Theorem 5.10 and 5.11 were proved in AVRAHAM–SHELAH [19··] where we refer the reader for some further results. A family of cardinality $2^{\aleph_1}$ of non-club-isomorphic Aronszajn trees can be constructed assuming the existence of only one Aronszajn tree T with no stationary antichains. Namely, let $\mathcal{A}$ be a family of cardinality $2^{\aleph_1}$ of stationary subsets of ω_1 such that $A-B$ is stationary for every $A, B \in \mathcal{A}$, $A \neq B$. Then $T \otimes T(A)$, $A \in \mathcal{A}$ (see Section 4) is a family of cardinality $2^{\aleph_1}$ of pairwise non-club-embeddable Aronszajn trees. This can be proved using some ideas from §4.

(v) The problem whether a given special Aronszajn tree is a normal topological space was asked by JONES [1965] who also proved the Theorem 5.12. Theorem 5.13 was proved by FLEISSNER [1975] (he also proved Theorem 5.14 under the stronger assumption $\diamondsuit$ instead of $2^{\aleph_0} < 2^{\aleph_1}$). Theorem 5.14 was proved by DEVLIN–SHELAH [1979b].

6. Suslin trees and lines

We start this section by showing that the Suslin Problem has a very easily understood translation into terms of trees. This result was one of the greatest initial motivations for considering infinite trees at all.

6.1. LEMMA. *Any partition tree of a Suslin line contains a Suslin subtree.*

PROOF. Let L be a Suslin line. By 3.3 and 3.9, we may assume that L is also an Aronszajn line. Let T be a partition tree of L. By 5.1(ii), we have that T is an Aronszajn tree. Let $T' = \{I \in T \mid |I| \geq 3\}$. Then T' is also an Aronszajn tree as an uncountable initial part of T. Let $A \subseteq T'$ be an antichain of $(T', \supseteq)$. By the definition of T', each $I \in A$ contains a non-empty open interval (l_I, m_I). Since T is a partition tree of L, A is a disjoint family of convex subsets of L, hence $\{(l_I, m_I) \mid I \in A\}$ is a disjoint family of nonempty open intervals of L. Hence A is countable. This shows that T' is a Suslin tree.

6.2. LEMMA. *Any lexicographical ordering of a Suslin tree contains a Suslin line.*

PROOF. Let T be a Suslin tree and let $<$ be a lexicographical ordering of T. Let L be a maximal subset of T with the property that $\{u \in T \mid s < u < t\}$ is uncountable for every $s < t$ in L. By 5.1(i), it suffices to show that L has the Suslin property. Otherwise, let $\{(s_\alpha, t_\alpha)_L \mid \alpha < \omega_1\}$ be a disjoint family of non-empty open intervals of L. By induction on α, we choose $u_\alpha \in (s_\alpha, t_\alpha)_L$ such that $\mathrm{ht}(u_\alpha) > \mathrm{ht}(s_\alpha)$, $\mathrm{ht}(t_\alpha)$ and $\mathrm{ht}(u_\alpha) > \mathrm{ht}(u_\beta)$ for all $\beta < \alpha$. It is clear that such u_α's exist and that they form an antichain of T, a contradiction.

6.3. THEOREM. *The Suslin Hypothesis is true if and only if every uncountable tree has an uncountable chain or antichain.*

PROOF. By 3.8, there is a Suslin continuum iff there is a Suslin line. By 6.1 and 6.2, there is a Suslin line iff there is a Suslin tree.

The Suslin Hypothesis is neither provable nor refutable in ZFC, even if we assume CH or $\neg$CH. A typical model of ZFC + $\neg$SH is the Gödel constructible universe L (see, e.g., 6.6), while a typical model of ZFC + SH is the Solovay–Tennenbaum model of ZFC + $\mathrm{MA}_{\aleph_1}$ (see 9.5). In the rest of this section we shall be interested in some algebraical and topological properties of Suslin trees and continua. Hence all results will depend on some additional set theoretical assumption which is the best we can hope for.

Let us start with the automorphism types of Suslin continua. To do this we shall look more closely at the correspondence between Suslin trees and continua. So let T be a normal binary Aronszajn tree which is an initial part of ${}^{\omega_1}2$. We assume that no point of T of limit height ends with a 0-sequence or a 1-sequence. Let $\mathbb{K}'_T$ be the set of all $s \in {}^{\omega_1}2$ such that $\{s \upharpoonright \alpha \mid \alpha < \mathrm{dom}(s)\}$ form a branch of T. $\mathbb{K}'_T$ is ordered lexicographically. Then $\mathbb{K}'_T$ is a complete linear ordering (see 2.5). Let us now remove the left hand points of all jumps of $\mathbb{K}'_T$, and obtain a linearly ordered continuum $\mathbb{K}_T$. For $t \in T$, we define $I_t = \{s \in \mathbb{K}_T \mid t \subseteq s\}$. Then I_t is a nontrivial half-open interval of $\mathbb{K}_T$. It is clear that each open nonempty interval of $\mathbb{K}_T$ contains an interval of the form I_t, $t \in T$. Thus $\mathbb{K}_T$ is a Suslin continuum iff T is a Suslin tree (see the proof of 6.1 and 6.2).

6.4. LEMMA. *Let C be a club subset of ω_1. Then every lexicographical automorphism σ of $T \upharpoonright C$ determines a unique automorphism $\bar{\sigma}$ of $\mathbb{K}_T$.*

PROOF. We may assume $C \subseteq \Lambda$. For $\alpha \in C$, let $K_\alpha = \{s \upharpoonright \alpha \mid s \in \mathbb{K}_T\}$. Then K_α is isomorphic to the unit interval $[0, 1]$ of real numbers and T_α is a countable dense subset of K_α. Thus $\sigma \upharpoonright T_\alpha$ extends uniquely to an automorphisms σ_α, of K_α. Let $\sigma' = \bigcup\{\sigma_\alpha \mid \alpha \in C\}$. Then σ' is an automorphism of $\bigcup\{K_\alpha \mid \alpha \in C\}$ which uniquely extends to an automorphism of $\mathbb{K}_T$.

6.5. LEMMA. *If T is, moreover, a Suslin tree and if π is an automorphism of $\mathbb{K}_T$, then there exist a club set $C \subseteq \omega_1$ and a lexicographical automorphism σ of $T \upharpoonright C$ such that $\bar{\sigma} = \pi$.*

PROOF. For $t \in T$, let $A_t = \{s \in T \mid I_s \subseteq \pi'' I_t$ and $I_u \not\subseteq \pi'' I_t$ for every $u \subset s\}$. Then A_t is an antichain of T, hence A_t is countable. It is clear that $\pi'' I_t = \bigcup\{I_s \mid s \in A_t\}$. Let $C_0 = \{\alpha \in \Lambda \mid A_t \subseteq T \upharpoonright \alpha$ for all $t \in T \upharpoonright \alpha\}$. Then C_0 is a club subset of ω_1, and for every $\alpha \in C_0$ and $t \in T_\alpha$ there is a $t' \in T_\alpha$ such that $I_{t'} \subseteq \pi'' I_t$. Similarly we find a club $C_1 \subseteq \omega_1$ such that for every $\alpha \in C_1$ and $t \in T_\alpha$ there is a $t' \in T_\alpha$ such that $I_{t'} \subseteq \pi^{-1}(I_t)$. Let $C = C_0 \cap C_1$. Then for each $\alpha \in C$ and $t \in T_\alpha$ there is a unique $\sigma(t) \in T_\alpha$ such that $\pi'' I_t = I_{\sigma(t)}$. This defines a lexicographical automorphism σ of $T \upharpoonright C$. It is clear (from the proof of 6.4) that the automorphism of $\mathbb{K}_T$ induced by σ is equal to π i.e., $\bar{\sigma} = \pi$. This completes the proof.

It is clear that the above arguments can also be used in proving similar results for anti-lexicographical automorphisms. From now on until Theorem 6.15, we always work with normal binary Aronszajn subtrees of ${}^{\omega_1}2$. Thus we have a fixed correspondence $T \to \mathbb{K}_T$. A Suslin tree T is called a *full Suslin tree* iff $T^{n} \otimes \cdots$

$\otimes T^{t_n}$ is a Suslin tree for every sequence $t_1, \ldots, t_n$ of distinct elements of some level of T.

6.6. THEOREM (Jensen). *Assume $\diamond$. Then there is a full Suslin tree.*

PROOF. Let $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ be a fixed $\diamond$-sequence. We construct T by induction on the levels $T_\alpha \subseteq {}^\alpha 2$, $\alpha < \omega_1$. Let $T_0 = \{\emptyset\}$ and $T_{\alpha+1} = \{s^\frown i \mid s \in T_\alpha,\ i < 2\}$ if T_α is constructed. Hence we may assume α is a limit ordinal $<\omega_1$ and that $T \restriction \alpha$ is constructed.

Let M_α be a countable transitive model of ZFC $-$ P such that $\alpha, A_\alpha, T \restriction \alpha \in M_\alpha$. Let $\mathbb{P}_\alpha$ be the set of finite functions p such that $\mathrm{dom}(p) \subseteq \omega$ and $\mathrm{rang}(p)$ is a subset of some level of $T \restriction \alpha$; order $\mathbb{P}_\alpha$ by: $p \leqslant q$ iff $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and $p(i) \supseteq q(i)$ for all $i \in \mathrm{dom}(q)$. Clearly $\mathbb{P}_\alpha \in M_\alpha$. Let G_α be a M_α-generic subset of $\mathbb{P}$. For each $i < \omega$, let $b_i = \bigcup\{p(i) \mid p \in G_\alpha\}$. It is clear that $T \restriction (\alpha + 1)$ is normal.

This defines $T = \bigcup\{T_\alpha \mid \alpha < \omega_1\}$. We claim that T is a full Suslin tree. So let $t_1 \ldots, t_n$ be distinct member of some level of T and let $T^* = T^{t_1} \otimes \cdots \otimes T^{t_n}$. Let X be a maximal antichain of T^*. Let $A \subseteq \omega_1$ code T, $\langle t_1, \ldots, t_n \rangle$ and X in some nice way, and β be the least ordinal $>\omega_1$ such that $L_\beta[A]$ is a model of ZFC $-$ P. Now choose an increasing continuous sequence $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ of countable elementary submodels of $L_\beta[A]$ such that $A \in N_0$. Let $C = \{\alpha < \omega_1 \mid N_\alpha \cap \omega_1 = \alpha\}$. By $\diamond$, we can find $\alpha \in C$ such that $A \cap \alpha = A_\alpha$. Let $\pi: N_\alpha \cong \bar{N}_\alpha$ be the transitive collapse of N_α. Then $\bar{N}_\alpha = L_{\bar{\beta}}[A_\alpha]$ and $\pi(\omega_1) = \alpha$. Moreover, $\bar{\beta}$ is the least ordinal $>\alpha$ such that $L_{\bar{\beta}}[A_\alpha]$ is a model of ZFC $-$ P. Hence $L_{\bar{\beta}}[A_\alpha] \subseteq M_\alpha$. This means that $\pi(X) \in M_\alpha$ and that $\pi(X)$ is a maximal antichain of $\pi(T^*) = T^* \restriction \alpha \in M_\alpha$. By the definition of T_α, we have that every $\langle s_1, \ldots, s_n \rangle \in T^*_\alpha$ lies above an element of $\pi(X) = X \cap (T^* \restriction \alpha)$. Hence $X \cap (T^* \restriction \alpha)$ is a maximal antichain of T^*, hence $X = X \cap (T^* \restriction \alpha)$. Hence X is countable.

6.7. LEMMA. *If T is a full Suslin tree, then*:

(i) *$\mathbb{K}_T$ is a rigid nonreversible Suslin continuum.*

(ii) *No nonempty open subset of $\mathbb{K}_T^2$ has a dense metrizable subspace.*

PROOF. (i) If s and t are two distinct elements of some level of T, then T^s is not club-embeddable into T^t, since $T^s \otimes T^t$ is a Suslin tree (see the proof of 2.9). Hence by 6.5, we have that $\mathbb{K}_T$ is a rigid nonreversible Suslin continuum.

(ii) Follows easily from the fact that every nonempty open subset of $\mathbb{K}_T^2$ contains a rectangle of the form $I_s \times I_t$ for some $\langle s, t \rangle \in T \otimes T$, $s \neq t$.

6.8. THEOREM (Jensen). *Assume $\diamond$. Then there is a rigid Suslin continuum $\mathbb{K}$ such that $\mathbb{K} \times \mathbb{K}$ has no dense metrizable subspace.*

The above Suslin continuum is also rigid in the topological sense. It is not true that every rigid Suslin continuum is non-reversible. Using $\diamond$ and similar arguments one can construct Suslin continuum which *is* reversible.

We say that a Suslin tree $T \subseteq {}^{\omega_1}2$ is a *thin Suslin tree* iff $\{\alpha \mid s(\alpha) \neq t(\alpha)\}$ is finite for all $s, t \in T$. The next result shows that a thin Suslin tree may exist.

6.9. THEOREM (Jensen). *Assume $\diamond$. Then there is a homogeneous thin Suslin tree.*

PROOF. The proof is similar to that of 6.6, hence we give only a sketch. So let $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ be a fixed $\diamond$-sequence. The steps $\alpha = 0$ and $\alpha = \beta + 1$ are the same as in 6.6, hence we assume α is limit $<\omega_1$ and that $T \restriction \alpha$ is defined.

Again let M_α be a c.t.m. of ZFC $-$ P such that $\alpha, A_\alpha, T \restriction \alpha \in M_\alpha$. Considering $T \restriction \alpha$ as a poset under the ordering $\supseteq$, we chose an M_α-generic branch $b \subseteq T \restriction \alpha$. Let $T_\alpha = \{\bigcup c \mid c$ is an α-branch of $T \restriction \alpha$ such that $\bigcup c \restriction (\alpha - \beta) = \bigcup b \restriction (\alpha - \beta)$ for some $\beta < \alpha\}$. It is clear that $T \restriction (\alpha + 1)$ is a normal tree.

Let $T = \bigcup \{T_\alpha \mid \alpha < \omega_1\}$. An argument similar to that of 6.6 shows that T is a Suslin tree. By induction on $\alpha < \omega_1$, one easily shows that $\{\beta < \omega_1 \mid s(\beta) \neq t(\beta)\}$ is finite for all $s, t \in T_\alpha$. Similarly, if $t \in T_\alpha$ and if s is obtained from t by changing only finitely many coordinates, then s is also an element of T_α. Thus T is a thin homogeneous Suslin tree.

One can go further and show that if T is the tree we have just constructed, then $T \restriction C$ has no anti-lexicographical autumorphism for any club set $C \subset \omega_1$.

6.10. LEMMA. *If T is a thin Suslin tree, then $\mathbb{K}_T$ is a Suslin continuum whose square has a dense metrizable subspace.*

PROOF. For $n < \omega$, let $D_n = \{\langle s, t \rangle \in T \otimes T \mid \{\alpha \mid s(\alpha) \neq t(\alpha)\}$ has n elements$\}$. It is clear that $\bigcap \{D_n \mid n < \omega\} = \emptyset$. Let $M = \{\langle u, v \rangle \in \mathbb{K}_T^2 \mid \mathrm{dom}(u) = \mathrm{dom}(v)$ and $\{\alpha \mid u(\alpha) \neq v(\alpha)\}$ is infinite$\}$. Then M is a dense subset of $\mathbb{K}_T^2$ and $\{(I_s \times I_t) \cap M \mid \langle s, t \rangle \in \bigcup \{R_0 D_n \mid n < \omega\}\}$ is a σ-discrete basis of M. Hence M is metrizable.

6.11. THEOREM (Jensen). *Assume $\diamond$. Then there is a homogeneous non-reversible Suslin continuum $\mathbb{K}$ such that $\mathbb{K} \times \mathbb{K}$ has a dense metrizable subspace.*

PROOF. Let T be the homogeneous thin Suslin tree which, moreover, has the property that $T \restriction C$ has no anti-lexicographical automorphisms for every club $C \subseteq \omega_1$. Then by 6.4 and 6.5 (for the case of antilexicographical automorphisms), the continuum $\mathbb{K}_T$ satisfies the conclusions of the theorem.

Using $\diamond$ and similar arguments one can construct a homogeneous Suslin continuum which *is* reversible. Also, one can use $\diamond$ to construct $2^{\aleph_1}$ pairwise non-isomorphic Suslin continua (hence, $2^{\aleph_1}$ pairwise non-club-isomorphic Suslin trees).

Let us now consider the automorphism types of Suslin trees, and let us by $\sigma(T)$ denote the cardinality of the automorphism group of a given tree T. The next two results give a quite complete description of the function $\sigma(T)$ on the class of all (say) binary $\aleph_1$-tree and show that Suslin trees have a particular place.

6.12. THEOREM (Jech). *Let T be a normal binary $\aleph_1$-tree.*

(i) *$\sigma(T)$ is either finite, or $2^{\aleph_0} \leqslant \sigma(T) \leqslant 2^{\aleph_1}$.*

(ii) *If $\sigma(T)$ is infinite, then $\sigma(T)^{\aleph_0} = \sigma(T)$.*

(iii) *If T contains no Suslin subtrees, then $\sigma(T)$ is either finite or $\sigma(T) = 2^{\aleph_0}$ or $\sigma(T) = 2^{\aleph_1}$.*

6.13. THEOREM (Jech). *Assume* CH. *Let κ be any cardinal with the property $\kappa^{\aleph_0} = \kappa$. Then there is a σ-closed $\aleph_2$-cc poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$ we have a Suslin tree with exactly κ automorphisms.*

PROOF (sketch). The set $\mathbb{P}$ consists of all pairs $p = \langle T_p, f_p \rangle$ such that

(i) T_p is a countable normal initial part of ${}^{\omega_1}2$,

(ii) f_p is a 1–1 function from a countable subset of κ into the automorphism group of T_p.

We let $p \leqslant q$ iff

(iii) T_p is an end-extension of T_q, $\mathrm{dom}(f_p) \supseteq \mathrm{dom}(f_q)$ and $f_p(\xi) \supseteq f_1(\xi)$ for all $\xi \in \mathrm{dom}(f_q)$.

It is clear that $\mathbb{P}$ is a σ-closed poset, while a standard Δ-system argument (using CH) shows that $\mathbb{P}$ satisfies the $\aleph_2$-cc. Let G be a generic subset of $\mathbb{P}$. Now, in $V[G]$, we define $T_G = \bigcup\{T_p \mid p \in G\}$. Then T_G is a Suslin tree with exactly κ automorphisms. This can be shown by a diagonalization argument in V.

Using 6.4 and 6.5, we have that in the same model there is a Suslin continuum with exactly κ automorphism. Namely, if T_G is the above generic Suslin tree, then $\mathbb{K}_{T_G}$ is such a continuum.

Suslin lines and continua are also interesting as topological spaces with the interval topologies. First of all the fact that they are regular hereditarily Lindelöf nonseparable spaces should be mentioned (see Roitman's chapter of the Handbook). We shall consider mainly Suslin continua, although a similar result can also be proved for Suslin lines.

6.14. THEOREM. *If there is a Suslin line, then there is one with a point-countable ramified basis.*

PROOF. By 6.3, we may assume to have a Suslin tree T. Furthermore, by 2.8, we may assume that T is a normal $\aleph_0$-ary $\aleph_1$-tree. For each $\delta \in \Lambda$, we can choose a δ-branch l_δ of T in such a way that $\{\delta \in \Lambda \mid t \in l_\delta\}$ is uncountable for each $t \in T$. Let $L = \{l_\delta \mid \delta \in \Lambda\}$. Now for each node N of T, we choose a linear ordering $<_N$ such that $\mathrm{tp}(N, <_N) = \omega^* + \omega$. Let $<$ be the lexicographical ordering of L induced by $\{<_N \mid N$ is a node of $T\}$. If for $t \in T$, we define $B_t = \{l \in L \mid t \in l\}$, then it is easily checked that $(L, <)$ is a Suslin line and that $\{B_t \mid t \in T\}$ is a point countable ramified basis of L.

6.15. THEOREM (Kurepa). *If $\mathbb{K}$ is a Suslin continuum, then $c(\mathbb{K}\times\mathbb{K})>\aleph_0$.*

PROOF. Let T be a partition tree of $\mathbb{K}$ such that if $I\in T$ is a nontrivial interval, then the immediate successors I_0, I_1 of I in T are both nontrivial. Let $\mathcal{R}=\{I_0\times I_1 \mid I\in T \text{ and } |I|\geq 2\}$. Then $\mathcal{R}$ is an uncountable family of nontrivial rectangles in $\mathbb{K}\times\mathbb{K}$ with disjoint interiors.

Let $\mathbb{K}$ be a nowhere separable Aronszajn continuum (see Section 3) and let T be a partition tree of $\mathbb{K}$. Then as in the proof of 5.1(ii), one can show that the set of all nontrivial intervals of T is an Aronszajn tree. Now for every $\alpha<\omega_1$, we define $N_\alpha=\{x\in\mathbb{K} \mid x \text{ is an end-point of an element of } T\restriction(\alpha+1)\}$. Then each N_α is a closed second countable (hence nowhere dense) subset of $\mathbb{K}$ and $\mathbb{K}=\bigcup\{N_\alpha \mid \alpha<\omega_1\}$. A standard argument shows that a subsequence of $\{N_\alpha \mid \alpha<\omega_1\}$ will satisfy the following theorem.

6.16. THEOREM. *If $\mathbb{K}$ is a nowhere separable Aronszajn continuum, then $\mathbb{K}$ is the union of an increasing ω_1-sequence of nowhere dense subsets homeomorphic to the Cantor set.*

Thus if $\mathbb{K}$ is, moreover, a Suslin continuum, then every first category subset of $\mathbb{K}$ is nowhere dense and second countable.

6.17. THEOREM (Weiss). *If $\mathbb{K}$ is a Suslin continuum, then there is a real function on $\mathbb{K}$ which is not continuous on any dense subset of $\mathbb{K}$.*

PROOF. We may assume that $\mathbb{K}$ is nowhere separable. By 6.16, there is a sequence $\langle M_\alpha \mid \alpha<\omega_1\rangle$ of disjoint nowhere dense subsets of $\mathbb{K}$ such that $\mathbb{K}=\bigcup\{M_\alpha \mid \alpha<\omega\}$. Let $\langle r_\alpha \mid \alpha<\omega\rangle$ be a 1–1 sequence of real numbers. Define $f:\mathbb{K}\to\mathbb{R}$ such that $f''M_\alpha=\{r_\alpha\}$ for all $\alpha<\omega_1$. It is easily checked that f is not continuous on any dense subset of $\mathbb{K}$.

A Suslin tree as a topological space with the interval topology has several properties in common with ω_1. In fact, most of them can be proved using corresponding properties of ω_1 and a forcing argument.

6.18. THEOREM. *Let T be a Suslin tree.*
(i) *If A and B are disjoint closed subsets of T, then $(\cdot, A]\cap(\cdot, B]$ is countable.*
(ii) *If $f: T\to\mathbb{R}$ is continuous, then $f''T$ is countable.*

6.19. COROLLARY. *Every Suslin tree is a normal topological space.*

One can go further and prove that every Suslin tree is also collectionwise normal. Hence the interval topologies of Suslin trees have very strong separation properties. Note that this is not the case with an arbitrary Aronszajn tree. In Section 5 we have proved that no special Aronszajn tree is collectionwise Hausdorff (and also not normal if we assume $2^{\aleph_0} < 2^{\aleph_1}$; see 5.13). But special Aronszajn trees are, in some sense, very far from being Suslin, hence one may naturally wonder whether every tree with no "large" antichains satisfies stronger separation properties. The next theorems give a quite satisfactory positive answer to this question.

6.20. THEOREM (Devlin–Shelah). *If T is an $\aleph_1$-tree, then T is collectionwise Hausdorff iff T has no stationary antichains.*

6.21. THEOREM (Fleissner). *If T is an $\aleph_1$-tree, then T is collectionwise normal iff T has property γ.*

This means that some kind of 'Suslin properties' of $\aleph_1$-trees are equivalent to some kind of separation properties of the interval topologies. The next theorem shows that being $\mathbb{Q}$-embeddable or $\mathbb{R}$-embeddable does not necessarily mean the same as not having a kind of 'Suslin properties' in the above sense.

6.22. THEOREM. (i) *Assuming $\diamond$, there is a $\mathbb{Q}$-embeddable Aronszajn tree with no club-antichains.*

(ii) *Assuming $\diamond^*$, there is an $\mathbb{R}$-embeddable Aronszajn tree with no stationary antichains.*

Note that any tree which satisfies the conclusion of 6.22(ii) is $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable.

6.23. REMARKS. (i) Theorem 6.3 was proved by KUREPA [1935; pp. 127–134]. This theorem was rediscovered later by MILLER [1943] and SIERPIŃSKI [1948].

(ii) The consistency of ZFC + ¬SH was independently proved by JECH [1967] and TENNENBAUM [1978]. They also proved the consistency of ZFC ± CH + ¬SH. The fact that ¬SH holds in L was proved by JENSEN [1968]. The consistency of ZFC + ¬CH + SH was proved by SOLOVAY–TENNENBAUM [1971]. The consistency of ZFC + CH + SH was proved by JENSEN (see DEVLIN–JOHNSBRÅTEN [1974]).

(iii) The results 6.4–6.11 were proved by Jensen in DEVLIN–JOHNSBRÅTEN [1974] where we refer the reader for many details which are not given here. Theorems 6.12 and 6.13 were proved by JECH [1972] who also proved that $\diamond$ implies the existence of $2^{\aleph_1}$ pairwise nonisomorphic Suslin trees and continua.

(iv) Theorem 6.14 was proved independently by PONOMAREV [1968] and BENNETT [1971]. Theorems 6.15 was proved by KUREPA [1950]. Theorem 6.16 for the case of Suslin continua appeared in PAPIĆ [1971]. The general fact is apparently a part of the folklore of the subject. Theorem 6.17 was proved by WEISS [1978]. For some other topological properties of Suslin lines, we refer the reader to RUDIN [1979].

(v) Theorem 6.18(i) is apparently a part of the folklore of the subject (see RUDIN [1955] and DEVLIN–SHELAH [1979a] for proof). Theorem 6.18(ii) was proved by STEPRĀNS [1981]. Theorem 6.20 was proved by DEVLIN–SHELAH [1979a]. They also proved that if $V = L$, then an $\aleph_1$-tree T is normal iff T has property γ. Theorem 6.21 was proved by FLEISSNER [1980]. The statement of Theorem 6.22(i) appeared in DEVLIN–SHELAH [1979a; p. 251]. The proofs of both parts of this theorem appeared in SHELAH [1981]. The fact that in L there exists an $\mathbb{R}$-embeddable non-$\mathbb{Q}$-embeddable Aronszajn tree was proved by BAUMGARTNER [1970b]. The existence of such a tree using $\diamondsuit$ was proved by DEVLIN [1972]. Let us also note that it is consistent with ZFC to have an Aronszajn tree T with no stationary antichains (hence T is nonspecial) but every subset of T is an F_σ set in the interval topology on T.

(vi) The generalized Suslin problem also has a translation in terms of trees. To mention the first two of the twelve equivalent formulations of GSH which appeared in KUREPA [1935; p. 130–132], we need a definition from the same paper. Let T be a tree. A subset D of T is called a *d-subset* iff $\{s \in D \mid t \leqslant_T s\}$ is a chain of T for every $t \in D$. Let $bT = \sup\{|D| \mid D$ is a d-subset of $T\}$.

(P$_1$) For every tree T the supremium bT is achieved, i.e., T contains a d-subset of cardinality bT.

(P$_2$) Every infinite tree is equinumerous with some of its d-subsets.

7. Aronszajn and Suslin trees of greater heights

In this section we consider generalizations of some results from Sections 5 and 6 to higher cardinals. All those generalizations depend on some additional set-theoretic assumptions, and in many cases it is known that some such assumptions are also necessary. Proofs of most results are omitted.

We start with the following generalizations of Theorem 5.2.

7.1. THEOREM (Specker). *Assume $\kappa^{\underline{\kappa}} = \kappa \geqslant \aleph_0$. Then there is a κ-special κ^+-Aronszajn tree.*

PROOF. The proof is a very direct generalization of the proof of Theorem 5.2, hence we mention only the main differences. Instead of rationals we consider the set $\mathbb{Q}_\kappa = \{f \in {}^\omega\kappa \mid \{n < \omega \mid f(n) \neq 0\}$ is finite$\}$ ordered lexicographically. Note that $\mathbb{Q}_\kappa$ is a densely ordered set and that each non-trivial interval of $\mathbb{Q}_\kappa$ contains each ordinal $<\kappa^+$. As before we construct our tree $T \subseteq \sigma'\mathbb{Q}_\kappa$ by induction of the levels T_α, $\alpha < \kappa^+$, each of cardinality $\leqslant\kappa$. The αth induction hypothesis $(*)_\alpha$ is as before. The case $\alpha = \beta + 1$ is also as before. If $\alpha < \kappa^+$ is a limit ordinal of cofinality $\lambda < \kappa$, then we extend in T_α each α-branch $b \subseteq T \restriction \alpha$. Note that, by the assumption $\kappa^\lambda = \kappa$, we have $|T_\alpha| \leqslant |T \restriction \alpha|^\lambda \leqslant \kappa^\lambda = \kappa$. If $\mathrm{cf}(\alpha) = \kappa$ then we choose a strictly increasing sequence $\langle \alpha_\xi \mid \xi < \kappa \rangle$ of ordinals cofinal with α, and for each $t \in T \restriction \alpha$ and each $x \in \mathbb{Q}_\kappa$, $x > \max(t)$, we construct inductively, along the sequence $\langle \alpha_\xi \mid \xi < \kappa \rangle$, an α-branch $b_{t,x} \in T \restriction \alpha$ such that $t \in b_{t,x}$ and $\sup \bigcup b_{t,x} \leqslant x$. This construction is possible by our definition of T_β for $\beta < \alpha$ and $\operatorname{cf} \beta < \kappa$. Let $s_{t,x} = (\bigcup b_{t,x}) \cup \{x\}$. Then $s_{t,x} \in \sigma' \mathbb{Q}_\kappa$. Let $T_\alpha = \{s_{t,x} \mid t \in T \restriction \alpha,\ x \in \mathbb{Q}_\kappa$ and $x > \max(t)\}$. Then $|T_\alpha| \leqslant \kappa$ and $(*)_{\alpha+1}$ remains true. Let $T = \bigcup\{T_\alpha \mid \alpha < \kappa\}$. Then T is a κ-special κ^+-Aronszajn tree. This completes the proof.

Thus, if GCH holds then for every *regular* cardinal $\kappa \geqslant \aleph_0$ there is a κ-special κ^+-Aronszajn tree. This leaves (for example) an open problem whether the existence of an $\aleph_{\omega+1}$-Aronszajn tree can be proved in ZFC or ZFC + GCH. The next results show that $V = L$ settles completely, not only the problem about the existence of κ-Aronszajn trees, but also the generalized Suslin problem.

7.2. THEOREM (Jensen). *Assume $V = L$. Then there is a κ-Suslin tree iff κ is not a weakly compact cardinal.*

It has been noticed that the actual hypotheses in Jensen's proof can be obtained using a weaker assumption than $V = L$ so that the following holds.

7.3. THEOREM. *Assume $\kappa > \aleph_0$, $2^\kappa = \kappa^+$ and $\lambda^{\aleph_0} \leqslant \kappa$, for all $\lambda < \kappa$. Then $\square_\kappa$ implies that there exists a κ^+-Suslin tree.*

Thus GCH + $\square_\kappa$ is already enough for constructing a κ^+-Suslin tree ($\kappa \geqslant \aleph_1$). It is not known whether we can go further and construct a κ^+-Suslin tree assuming only GCH. Thus the simplest open problem is whether GCH implies $\neg \mathrm{SH}_{\aleph_2}$. The next result shows that, if we want to construct only a κ^+-Aronszajn tree, then the GCH assumption in 7.3 is not needed.

7.4. THEOREM. *Assume $\square_\kappa$. Then there is a κ-special κ^+-Aronszajn tree.*

The relationships between κ-Aronszajn and κ-Suslin trees and κ-Aronszajn and κ-Suslin lines, respectively are the same as in the case $\kappa = \aleph_1$. Hence all the above results can also be regarded as results about the existence of κ-Aronszajn and κ-Suslin lines. Moreover, we can construct our κ-Aronszajn or κ-Suslin line L with the following additional property.

(Φ_κ) If $\psi \leqslant \operatorname{tp} L$ and $|\psi| < \kappa$, then for every regular $\lambda \leqslant |\psi|$, $\psi \geqslant \lambda$ or $\psi \geqslant \lambda^*$.

For example, Theorem 7.2 can be read as follows: If $V = L$ holds, then for every non-weakly compact cardinal κ there is a κ-Suslin line L with the property Φ_κ. Also, Theorem 7.4 can be read as follows: If $\square_\kappa$ holds, then there is a κ^+-Aronszajn line L with the property Φ_{κ^+}.

Note that if L is a densely ordered κ-Suslin line with the property Φ_κ, then the Dedekind completion $\mathbb{K}$ of L is a κ-Suslin continuum with the following property:

(Φ_κ) If $M \subset \mathbb{K}$ and $|M| < \kappa$, then $w(\bar{M}) \leq |M|$.

Of course, a similar result holds if $\mathbb{K}$ is the Dedekind completion of a densely ordered κ-Aronszajn line with the property Φ_κ.

A cardinal κ has the *tree property* iff there are no κ-Aronszajn trees, that is, if every κ-tree has a κ-branch. If κ is inaccessible then κ has the tree property iff κ is weakly compact. So one may ask whether the tree property is in fact equivalent to weak compactness. We shall see later that this is not the case; in fact, $\aleph_2$ may have the tree property. However, by the next result, the tree property is still, in some sense, related the weak compactness.

7.5. THEOREM (Silver). *If κ is regular and there are no κ-Aronszajn trees, then κ is weakly compact in L.*

This also shows that the assumption of the existence of a weakly compact cardinal in the next theorem is also necessary.

7.6. THEOREM. *If there is a weakly compact cardinal κ, then there is a forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ we have $\kappa = \aleph_2$ and there are no $\aleph_2$-Aronszajn trees.*

A similar result holds for any successor of a *regular* uncountable cardinal instead of $\aleph_2$. This leaves an open problem whether a similar result can be proved for $\aleph_{\omega+1}$ instead of $\aleph_2$. From 7.4 and some recent results on inner models of set theory, we have that for proving similar results for (say) $\aleph_{\omega+1}$ one has to assume the existence of a large cardinal much bigger than weakly compact. We shall prove here the next result which is weaker than 7.6, but a reader with some knowledge of iterated forcing should be able to convert this into a proof of 7.6.

7.7. THEOREM (Baumgartner). *If PFA holds, then there are no $\aleph_2$-Aronszajn trees.*

PROOF. Assume, on the contrary, that PFA holds but there is an $\aleph_2$-Aronszajn tree T. We may assume that the domain of T is ω_2 and that $R_\alpha T = [\omega_1\alpha, \omega_1(\alpha+1))$ holds for all $\alpha < \omega_2$. Let $\mathbb{P}_0$ be the poset of all countable maps from ω_1 into ω_2, ordered by $\supseteq$. Thus, $\mathbb{P}_0$ is a σ-closed poset which collapses $\aleph_2$ to $\aleph_1$. Since each level of T has cardinality $\aleph_1 < 2^{\aleph_0}$, the proof of Silver's lemma (see KUNEN [1980; Ch. VIII, Lemma 3.4]) shows that T has no ω_1-branches in $V^{\mathbb{P}_0}$. In $V^{\mathbb{P}_0}$, let c be a strictly increasing cofinal map from $\check{\omega}_1$ into $\check{\omega}_2$, and let $U = T\restriction\{c(\alpha) \mid \alpha < \check{\omega}_1\}$. (We consider U as a subtree of T.) Thus, in $V^{\mathbb{P}_0}$ U is a tree with no uncountable chains, hence by Remark 9.6(ii), the poset $\mathbb{P}^\omega(U)$ is ccc and specializes U, i.e., introduces a map $f\colon U \to \omega$ such that $\alpha <_T \beta$ in U implies $f(\alpha) \neq f(\beta)$. Let $\mathbb{P} = \mathbb{P}_0 * \mathbb{P}^\omega(U)$. Then $\mathbb{P}$ is a proper poset. For $\alpha, \xi < \omega_1$ we define $D_{\alpha,\xi} = \{p \in \mathbb{P} \mid$ there exist $\beta < \omega_2$ and $n < \omega$ such that p forces $c(\alpha) = \beta$ and

$f(\omega_1\beta+\xi)=n\}$. Then each $D_{\alpha,\xi}$ is a dense open subset of $\mathbb{P}$. Let G be a $\{D_{\alpha,\xi} \mid \alpha, \xi<\omega_1\}$-generic subset of $\mathbb{P}$. Define $d\colon \omega_1 \to \omega_2$ by: $d(\alpha)=\beta$ iff there exists $p \in G$ such that $p \Vdash c(\alpha)=\beta$. Then d is an increasing map. Let $T' = T \upharpoonright \{d(\alpha) \mid \alpha<\omega_1\}$. Then T' is a subtree of T which, clearly, has many uncountable chains. But if we define $g\colon T' \to \omega$ by $g(\xi)=n$ iff there exists $p \in G$ such that $p \Vdash f(\xi)=n$, then g specializes T', a contradiction. This completes the proof.

7.8. REMARKS. (i) Theorem 7.1 was proved by SPECKER [1949]. Theorem 7.2 was proved by JENSEN [1972]. The case cf $\kappa>\omega$ of 7.3 was proved by GREGORY [1976]. The case cf $\kappa=\omega$ of 7.3 appeared in DEVLIN [19··a; Ch. IV] (see also SHELAH [1979; p. 377]). Proof of both 7.2 and 7.3 can also be found in DEVLIN [19··a; Ch. IV]. The fact that GCH + $\square_\kappa$ implies the existence of a κ-special κ^+-Aronszajn trees was proved by Jensen (see JENSEN [1972; p. 283] and KANAMORI–MAGIDOR [1978; p. 262]). The proofs of the results about Φ_κ as well as the proof of 7.4 can be found in TODORČEVIĆ [1981a].

(ii) If κ is a regular uncountable cardinal then almost all results about the isomorphism types of Aronszajn trees and lines from Section 5 can be generalized to corresponding results for κ-Aronszajn trees and lines (assuming that there is at least one κ-Aronszajn tree). For example, 5.7 and 5.8 have exact generalizations to higher regular uncountable cardinals. However, it is not known whether 5.10 and 5.11 have generalizations to higher cardinals. For example, it is not known whether GCH implies the existence of two non-club-isomorphic σ-closed $\aleph_2$-Aronszajn trees (see AVRAHAM–SHELAH [19··]).

(iii) Theorem 7.5 was proved by SILVER (see MITCHELL [1972]). MITCHELL [1972] found a general method for collapsing cardinals and proved, for example, that if κ is a Mahlo cardinal then there is a forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ we have $\kappa=\aleph_2$ and there are no $\aleph_1$-special $\aleph_2$-Aronszajn trees. MITCHELL [1972] also proved that the assumption about the existence of a Mahlo cardinal is also necessary. Theorem 7.6 was proved by Silver using the above mentioned MITCHELL's model (see MITCHELL [1972; pp. 41–42]). Theorem 7.7 was proved by BAUMGARTNER (see DEVLIN [19··b]).

(iv) Concerning the problem whether GCH implies the existence of an $\aleph_2$-Suslin tree, we mention that LAVER–SHELAH [1981] constructed a model of ZFC + CH + $\mathrm{SH}_{\aleph_2}$. In their model $2^{\aleph_1}>\aleph_2$ holds.

8. Kurepa trees and lines

The Kurepa Hypothesis may be regarded as a statement about an 'incompactness' at the first uncountable cardinal of certain natural properties of trees, order types and families of sets. The purpose of this section is to show this in some details. But let us first see how Kurepa trees and lines are related.

8.1. LEMMA. (i) *Every lexicographical ordering of the set of all ω_1-branches of a Kurepa tree is a Kurepa line.*

(ii) *Every Kurepa line has a partition tree of height ω_1+1 the first ω_1 levels of which form a Kurepa tree.*

PROOF. (i) Let T be Kurepa tree and let L be the set of all ω_1-branches of T ordered by a lexicographical ordering $<$. For each $t\in T$ we choose $l_t\in L$ as follows. If $B_t=\{l\in L \mid t\in l\}$ is empty, we let l_t be any member of L. If $B_t\neq\emptyset$ and if B_t has a $<$-minimum, we define $l_t=\min B_t$. If $B_t\neq\emptyset$ and $\min B_t$ does not exist, we choose l_t to be any element of B_t. Let $D=\{l_t \mid t\in T\}$. Then it is easily seen that D is an order-dense subset of L, hence $d(L,\leqslant)\leqslant\aleph_1$. Now let M be any uncountable subset of L and let $N\subseteq M$ be countable. Since N is countable, for each $l\in M$ we can choose $t_l\in l$ such that $t_l\notin m$ for all $m\in N$, $m\neq l$. Since M is uncountable, we can find $t\in T$ such that $\mathrm{ht}(t)>\sup\{\mathrm{ht}(t_l) \mid l\in N\}$ and $|B_t\cap M|\geqslant\aleph_1$. Thus, $B_t\cap M$ is an uncountable convex subset of M which has at most one point in common with N, hence N is not order-dense in M. This shows that $\mathrm{tp}\,L$ contains no uncountable real subtype, hence L is a Kurepa line.

(ii) Let L be a Kurepa line and let $\{l_\alpha \mid \alpha<\omega_1\}$ be an enumeration of a dense subset of L. By induction on the levels T_α, $\alpha\leqslant\omega_1$, we define a partition tree T of L as follows. Suppose $\alpha\leqslant\omega_1$ and T_β is defined and countable for all $\beta<\alpha$.

Case $\alpha=\beta+1$. Fix $I\in T_\beta$ such that $|I|\geqslant 2$. If $l_\beta\in I$, then l_β splits I into two nonempty disjoint convex parts I_0 and I_1 such that $I_0\leqslant l_\beta\leqslant I_1$. If $l_\beta\notin I$, we split I into two disjoint nonempty convex subsets I_0 and I_1, arbitrarily. Let $T_\alpha=\bigcup\{\{I_0,I_1\} \mid I\in T_\beta \text{ and } |I|\geqslant 2\}$.

Case $\lim(\alpha)$. Let $T_\alpha=\{\bigcap b \mid b \text{ is an } \alpha\text{-branch of } T\restriction\alpha \text{ such that } \bigcap b\neq\emptyset\}$. Then T_α is countable if $\alpha<\omega_1$. This is proved using the same argument as in the proof of the corresponding part of 5.1(ii), since $\mathrm{tp}\,L$ contains no uncountable real subtype.

Finally, let $T=\bigcup\{T_\alpha \mid \alpha\leqslant\omega_1\}$. Using the way how T is defined, it is easily seen that T is a partition tree of L. Since $|T\restriction\omega_1|\leqslant\aleph_1$ and since $|L|>\aleph_1$, we have that $\{l\}\in T_{\omega_1}$ for more than $\aleph_1$ many $l\in L$. Hence $T\restriction\omega_1$ is a Kurepa tree.

8.2. THEOREM. *There is a Kurepa line iff there is a Kurepa tree.*

Let L be a Kurepa line and let N be any countable subset of L. Then the set of all Dedekind cuts of N which are realized by the elements of L is at most countable. On the other hand, if M is a dense subset of L of cardinality $\aleph_1$, then the set of all Dedekind cuts of M which are realized by the elements of L is of cardinality $>\aleph_1$. This shows that the Kurepa Hypothesis asserts a strong incompactness at the first uncountable cardinal of the property $d(\psi)=|\psi|$ of order types. A similar incompactness can be seen from the following equivalent formulation of KH.

(KH) There is a family $\mathcal{F} \subset \mathcal{P}(\omega_1)$ such that
(i) $|\mathcal{F}| > \aleph_1$, but
(ii) $|\mathcal{F} \upharpoonright X| \leqslant \aleph_0$, for all $X \subset \omega_1$, $|X| \leqslant \aleph_0$.

Any family that satisfies the above statements is called a *Kurepa family*. It is easily seen (using the proof of 8.1) that, if $\mathcal{F} \subset \mathcal{P}(\omega_1)$ is a Kurepa family, then the set of all characteristic functions of elements of $\mathcal{F}$ ordered lexicographically is a Kurepa line. Conversely, if L is a Kurepa line, and if M is a dense subset of L of cardinality $\aleph_1$ then the set of all Dedekind cuts of M realized by the elements of L is a Kurepa family on M which is, moreover, linearly ordered by $\subseteq$. This forms of KH suggests the following natural generalization ($\kappa \geqslant \aleph_0$)

(KH_κ) There is a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ such that
(i) $|\mathcal{F}| > \kappa$, but
(ii) $|\mathcal{F} \upharpoonright X| \leqslant |X|$, for every infinite $X \subseteq \kappa$, $|X| < \kappa$.

The following is a translation of KH_κ into terms of order types.

8.3. THEOREM. KH_κ *holds iff there is an order type φ such that*
(i) $|\varphi| > \kappa$;
(ii) $d(\varphi) = \kappa$;
(iii) $d(\psi) = |\psi|$, *for every $\psi \leqslant \varphi$ with $|\psi| \leqslant \kappa$.*

PROOF. Let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ be a κ-Kurepa family and let $<$ be the lexicographical ordering of $\mathcal{F}$. Then $\varphi = \mathrm{tp}(\mathcal{F}, <)$ satisfies (i)–(iii).

Conversely, let $\varphi = \mathrm{tp}(L, <)$ satisfy (i)–(iii). Let M be a dense subset of L of cardinality κ. Let $\mathcal{F} \subseteq \mathcal{P}(M)$ be the set of all Dedekind cuts of M realized by the elements of L. Then $\mathcal{F}$ satisfies the statement of KH_κ.

Note that $\mathrm{KH}_{\aleph_0}$ is true and that $\mathrm{KH}_{\aleph_1}$ is equivalent to KH. We shall see later that KH may or may not be true depending on some additional axioms of set theory, but let us first see for which cardinals κ the hypothesis KH_κ is simply false.

8.4. THEOREM (Erdös–Hajnal–Milner). *Assume $\kappa > \mathrm{cf}\,\kappa > \aleph_0$ and $\theta^{\mathrm{cf}\,\kappa} < \kappa$ for all $\theta < \kappa$. Let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ be such that $\{\alpha < \kappa \mid |\mathcal{F} \upharpoonright \alpha| \leqslant |\alpha|\}$ is stationary in κ. Then $|\mathcal{F}| \leqslant \kappa$.*

8.5. THEOREM (Jensen–Kunen). *If κ is ineffable, then* KH_κ *fails.*

Thus if GCH holds, then KH_κ fails for every singular cardinal of cofinality $>\aleph_0$. But 8.5 says that KH_κ may also fail for some regular cardinal (but very large; much larger than weakly compact). A detailed proof of 8.4 is given in both KUNEN [1977; p. 388] and JECH [1978; p. 65], while a detailed proof of 8.5 is given in DEVLIN [1973; Ch. 10]. The next theorem shows that in L the only restrictions on KH_κ are contained in 8.4 and 8.5.

8.6. THEOREM. *Assume* $V = L$. *Then* KH_κ *holds iff* $\mathrm{cf}\,\kappa \in \{\kappa, \aleph_0\}$ *and* κ *is not an ineffable cardinal.*

By 8.3 the case $\mathrm{cf}\,\kappa = \omega$ of 8.6 follows easily from the next result.

8.7. THEOREM. *Assume* $\mathrm{cf}\,\kappa = \omega$ *and* $\Box_\kappa$. *Then there exists an order type* φ *such that*:

(i) $|\varphi| > \kappa$;

(ii) $d(\varphi) = \kappa$;

(iii) *if* $\psi \leqslant \varphi$ *and* $|\psi| \leqslant \kappa$, *then* ψ *is the union of countably many well-ordered subtypes.*

PROOF. We may assume $\kappa > \omega$. Let $\langle \kappa_n \mid n < \omega \rangle$ be a strictly increasing sequence of regular cardinals which is cofinal in κ. Let $\langle C_\alpha \mid \alpha \text{ limit } {<}\kappa^+ \rangle$ be a $\Box_\kappa$-sequence. For α limit $<\kappa^+$, let S_α be the set of all limit points of C_α. Then $\langle S_\alpha \mid \alpha \text{ limit } {<}\kappa^+ \rangle$ satisfies:

(i) S_α is a closed subset of α;

(ii) if $\mathrm{cf}\,\alpha > \omega$, then S_α is unbounded in α;

(iii) $|S_\alpha| < \kappa$;

(iv) if $\beta \in S_\alpha$, then $S_\beta = \beta \cap S_\alpha$.

We shall construct a subset $L \subseteq {}^\omega\kappa$ such that $\varphi = \mathrm{tp}(L, \prec)$ satisfies the theorem, where $\prec$ denotes the lexicographical ordering of ${}^\omega\kappa$.

A pair (K, r) is called special iff

(i) $K \subseteq {}^\omega\kappa$ and $|K| = \kappa$;

(ii) $r\colon K \to \omega$ and $|\{f \in K \mid r(f) \leqslant n\}| = \kappa_n$ for each n;

(iii) if $f, g \in K$ and $f \neq g$, then $r(f), r(g) \leqslant n$ implies $f \restriction n \neq g \restriction n$.

If (K, r) and (K', r') are special pairs then we write $(K, r) < (K', r')$ iff $K \subset K'$ and $r \subset r'$. For $n < \omega$, we write $(K, r) <_n (K', r')$ iff $K \subset K'$ and $r(f) < n$ implies $r'(f) < n$, while $r(f) \geqslant n$ implies $r(f) = r'(f)$ for $f \in K$.

Now by induction on α we construct a strictly increasing sequence $\langle (L_\alpha, r_\alpha) \mid \alpha < \kappa^+ \rangle$ of special pairs such that:

(i) if $\beta < \alpha$, then $(L_\beta, r_\beta) <_n (L_\alpha, r_\alpha)$ for some n.

(ii) if $\beta \in S_\alpha$, then $(L_\beta, r_\beta) < (L_\alpha, r_\alpha)$;

(iii) if $\beta \in S_\alpha$ and $\mathrm{tp}\, S_\beta = \kappa_n$, then $r_\alpha(f) > n$ for all $f \in L_\alpha \backslash L_\beta$.

Note that $\beta \in S_\alpha$ and $\mathrm{tp}\, S_\beta = \kappa_n$ in (iii) simply means that β is the κ_nth element of S_α, since $S_\beta = S_\alpha \cap \beta$. The cases $\alpha = 0$ or $\alpha = \beta + 1$ are trivial. So we assume $\alpha < \kappa^+$ is a limit ordinal and (L_β, r_β) is defined for every $\beta < \alpha$.

Case I: S_α *is cofinal in* α. Let

$$L_\alpha = \bigcup \{L_\beta \mid \beta \in S_\alpha\} \quad \text{and} \quad r_\alpha = \bigcup \{r_\beta \mid \beta \in S_\alpha\}.$$

Then by (ii) and (iii) of the inductive hypothesis, (L_α, r_α) is a special pair. Clearly, the inductive hypotheses remain true.

Case II: *S_α is bounded in α.* Then $\mathrm{cf}\,\alpha = \omega$. Let $\langle \alpha_i \mid i < \omega\rangle$ be a strictly increasing sequence of ordinals cofinal with α such that $\alpha_0 = \max S_\alpha$. Let m be the minimal with the property $\kappa_m > \mathrm{tp}\, S_\alpha$. Let $\langle n_i \mid i < \omega\rangle$ be a strictly increasing sequence of natural numbers such that $n_0 = 0$, $n_1 > m$ and $(L_{\alpha_i}, r_{\alpha_i}) <_{n_{i+1}} (L_{\alpha_{i+1}}, r_{\alpha_{i+1}})$. Let $L_\alpha = \bigcup\{L_{\alpha_i} \mid i < \omega\}$. For $f \in L_\alpha$, let

$$r_\alpha(f) = \max(n_{i(f)}, r_{\alpha_{i(f)}}(f)),$$

where $i(f)$ is the minimal $i < \omega$ such that $f \in L_{\alpha_i}$. We show that (L_α, r_α) is a special pair and that the inductive hypotheses (i)–(iii) are satisfied. The fact that (L_α, r_α) is a special pair follows from the equality

$$\{f \in L_\alpha \mid r(f) \leqslant n\} = \{f \in L_{\alpha_i} \mid r_{\alpha_i}(f) \leqslant n\},$$

where $i < \omega$ is determined by $n_i \leqslant n < n_{i+1}$. This follows directly from our definitions of L_α and r_α. Let us check the inductive hypotheses (i)–(iii). If $f \in L_{\alpha_0}$, then $r_\alpha(f) = r_\alpha(f) = r_{\alpha_0}(f)$, hence $r_\alpha \supset r_{\alpha_0}$, so (ii) is satisfied. The hypothesis (i) is satisfied because $(L_{\alpha_i}, r_{\alpha_i}) <_{n_{i+1}} (L_\alpha, r_\alpha)$ for each $i < \omega$, and because for each n, $<_n$ is a transitive relation and $<_n \subseteq <_{n+1}$. Assume now β is the κ_nth point of S_α, and that $f \in L_\alpha \setminus L_\beta$. If $f \in L_{\alpha_0}$, then $r_\alpha(f) > n$ by the inductive hypothesis. So we may assume $f \notin L_{\alpha_0}$, hence $i(f) > 0$. Then $r_\alpha(f) \geqslant n_{i(f)} > m > n$. This shows that (iii) remains true.

Let $L = \bigcup\{L_\alpha \mid \alpha < \kappa^+\}$ and let $\varphi = \mathrm{tp}(L, <)$. Then by the construction φ satisfies the theorem, since for each n, $({}^n\kappa, <)$ is a well-ordered set. This completes the proof.

8.8. COROLLARY. *Assume* $\mathrm{cf}\,\kappa = \omega$. *Then* $\square_\kappa$ *implies* KH_κ.

PROOF. Follows from 8.3 and 8.7.

The condition (iii) from 8.7 suggests a natural strengthening of KH_κ for κ not necessarily of cofinality ω. However, it can be shown that if κ is a regular uncountable cardinal then there is no order type φ which satisfies (i)–(iii) of 8.7. But the following strengthening of KH_κ is still consistent:

(KH^*_κ) There is an order type φ such that
- (i) $|\varphi| > \kappa$;
- (ii) $d(\varphi) = \kappa$;
- (iii) if $\psi \leqslant \varphi$ and $|\psi| \leqslant \kappa$, then $\varphi \geqslant \lambda$ or $\varphi \geqslant \lambda^*$ for every regular $\lambda \leqslant |\psi|$.

Using the ideas from the proofs of 8.1 and 8.3, one can easily find translations of KH^*_κ into terms of trees and families of subsets of κ. For example, $\mathrm{KH}^*_{\aleph_1}$ holds if and only if there is a Kurepa tree with no Aronszajn subtrees. A significance of such a tree can be seen from the next theorem.

8.9. THEOREM (Juhász–Weiss). *The following are equivalent for every regular cardinal κ:*

(i) *There is a κ-compact κ-metrizable space of cardinality $>\kappa$.*

(ii) *There is a κ-Kurepa tree with no κ-Aronszajn subtrees.*

PROOF. (i)$\to$(ii). Let X be a κ-compact κ-metrizable space of cardinality $>\kappa$. A standard argument (as in the case $\kappa = \aleph_0$ and X 0-dimensional) shows that we may assume that X is a subspace of ${}^{\kappa}2$ with κ-topology. Then it is easily seen that $\{s \in {}^{\leq\kappa}2 \mid s \subset f \text{ for some } f \in X\}$ is a κ-Kurepa tree with no κ-Aronszajn subtrees.

(ii)$\to$(i). Let T be a κ-Kurepa tree with no κ-Aronszajn subtrees. We may assume that T is an initial part of ${}^{\leq\kappa}2$. Let $X = \{f \in {}^{\kappa}2 \mid f \upharpoonright \alpha \in T \text{ for all } \alpha < \kappa\}$ considered as a subspace of ${}^{\kappa}2$ with κ-topology. Then X is a κ-compact κ-metrizable space of cardinality $>\kappa$ (see the proof of 4.4(i)).

Note that a standard example of a $\aleph_0$-compact ($\equiv$compact) $\aleph_0$-metrizable ($\equiv$metrizable) space of cardinality $2^{\aleph_0}$ is the Cantor set. Similarly, if κ is a weakly compact cardinal, then a standard example of a κ-compact κ-metrizable space of cardinality 2^{κ} is ${}^{\kappa}2$ with κ-topology. (Note that this follows easily from 8.9.) So it is very natural to ask for a similar example for any other regular cardinal κ. Again $V = L$ gives a complete positive answer. It follows from 8.9 and the fact that in L for every regular non-ineffable cardinal κ there is a κ-Kurepa tree with no κ-Aronszajn subtrees (see Remark 8.16(iii)).

8.10. THEOREM. *Assume $V = L$. Then for every regular cardinal κ there is a κ-compact κ-metrizable space of cardinality 2^{κ}.*

Let us now mention some results concerning the consistency of $\neg$KH. An examination of the standard construction of a Kurepa family in L (see JECH [1978; p. 150]) shows that such a family exists in $L[A]$ for every subset A of ω_1. The following theorem is an easy consequence of this fact.

8.11. THEOREM (Solovay). *Kurepa trees exist unless $\aleph_2$ is an inaccessible cardinal in L.*

This shows that the assumption of the existence of an inaccessible cardinal in the next theorem is also necessary.

8.12. THEOREM (Silver). *If there is an inaccessible cardinal κ, then there is a forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ we have $\kappa = \aleph_2$ and there are no Kurepa trees.*

PROOF (sketch). Let $\mathbb{P}$ be the set of all countable partial functions p such that $\mathrm{dom}(p) \subseteq \kappa \times \omega_1$ and $p(\alpha, \xi) < \alpha$ for all $(\alpha, \xi) \in \mathrm{dom}(p)$; $p \leq q$ iff $p \supseteq q$. Thus $\mathbb{P}$ is the standard Lévy collapse of κ to $\aleph_2$ which is a σ-closed κ-cc poset (see JECH

[1978; p. 248] and KUNEN [1980; p. 259]). For $\lambda<\kappa$ we define $\mathbb{P}_\lambda=\{p\in\mathbb{P}\mid p\subseteq\lambda\times\omega_1\}$ and $\mathbb{P}^\lambda=\{p\in\mathbb{P}\mid p\cap(\lambda\times\omega_1)=\emptyset\}$. Then $\mathbb{P}=\mathbb{P}_\lambda\times\mathbb{P}^\lambda$ and $\mathbb{P}^\lambda$ is σ-closed in $V^{\mathbb{P}_\lambda}$. Since every $\aleph_1$-tree from $V^{\mathbb{P}}$ appears in $V^{\mathbb{P}_\lambda}$ for some $\lambda<\kappa$, the crucial point of the proof is to prove that no σ-closed poset adds a new cofinal branch to an $\aleph_1$-tree. The proof of this lemma is given in detail in both of the above mentioned books. Here we just mention the following strengthening of this lemma in case it will have some future use.

8.13. LEMMA. *Let T be an $\aleph_1$-tree and let $\mathbb{P}$ be a σ-closed poset. Then every uncountable initial part of T from $V^{\mathbb{P}}$ contains an uncountable initial part of T which is in V.*

Note that 8.12 shows the consistency of ZFC + GCH + $\neg$KH assuming the consistency of ZFC + "there is an inaccessible cardinal". A similar result holds for any successor of a *regular* uncountable cardinal instead of $\aleph_2$. This leaves (for example) an open problem whether $\mathrm{KH}_{\aleph_\omega}$ is a theorem of ZFC or ZFC + GCH.

A Kurepa tree can also be obtained using a forcing argument. Since such a tree may have some other interesting properties, we shall give some details. So, let κ be a fixed cardinal $\geqslant\aleph_2$, and let us assume that CH holds. The set $\mathbb{P}$ of forcing conditions consists of all pairs $p=\langle T_p, l_p\rangle$ such that

(i) $(T_p,\subseteq)$ is a normal countable subtree of ${}^{\omega_1}2$ of height α_p+1;

(ii) l_p is a 1–1 function from some countable subset of κ onto the last level of T_p;

$p\leqslant q$ iff

(iii) T_p is an end-extension of T_q;

(iv) $\mathrm{dom}(l_p)\supseteq\mathrm{dom}(l_q)$ and $l_q(\xi)\subseteq l_p(\xi)$ for all $\xi\in\mathrm{dom}(l_q)$. It is clear that $\mathbb{P}$ is a σ-closed poset, while a standard Δ-system argument shows that $\mathbb{P}$ satisfies the $\aleph_2$-cc. Thus $\mathbb{P}$ preserves cofinalities. Let G be a generic subset of $\mathbb{P}$, and let us define, in $V[G]$, $T_G=\bigcup\{T_p\mid p\in G\}$. Then T_G is a normal $\aleph_1$-tree. For $\xi<\kappa$, in $V[G]$, we define $b(\xi)=\{t\in T_G\mid t\subseteq l_p(\xi)$ for some $p\in G\}$. Then $b(\xi)$ is an ω_1-branch of T_G. Since $b(\xi)\neq b(\zeta)$ for $\xi\neq\zeta$, we have that T_G is Kurepa tree. The next lemma shows that each ω_1-branch of T_G has the form $b(\xi)$ for some $\xi<\kappa$, and, moreover, that T_G has no Aronszajn subtrees. Hence T_G has exactly κ many ω_1-branches.

8.14. LEMMA ($V[G]$). *For every uncountable initial part A of T_G there is a $\xi<\kappa$ such that $b(\xi)\subseteq A$.*

PROOF. We shall prove this by a forcing argument in V. So, let $p_0\in\mathbb{P}$ and $\boldsymbol{A}\in V^{\mathbb{P}}$ be such that $p_0\Vdash\boldsymbol{A}$ is an initial part of T which contains no $\boldsymbol{b}(\xi)$ for any $\xi<\kappa$. Let λ be a large enough regular cardinal and let N be a countable elementary submodel of $H(\lambda)$ such that $\mathbb{P},\kappa,\boldsymbol{A}\in N$. Let $\langle\xi_n\mid n<\omega\rangle$ be an enumeration of $N\cap\kappa$. By the assumption on p_0 and $\boldsymbol{A}$, we can easily construct a decreasing sequence $\langle p_n \mid n<\omega\rangle$ of elements of $\mathbb{P}\cap N$ such that for every $n<\omega$, $\xi_n \in \mathrm{dom}(l_{p_n})$ and $p_n \Vdash l_{p_n}(\xi_n) \notin \boldsymbol{A}$. Now for $\xi \in N\cap\kappa$, we define $l_p(\xi) = \bigcup\{l_{p_n}(\xi) \mid n<\omega$ and $\xi\in\mathrm{dom}(l_{p_n})\}$. Let $T_p = (\bigcup\{T_{p_n} \mid n<\omega\}) \cup \{l_p(\xi) \mid \xi \in N\cap\kappa\}$ and let $p=\langle T_p, l_p\rangle$. Then $p\in\mathbb{P}$, $p\leqslant p_0$, and $p\Vdash \boldsymbol{A}\subseteq T_p$. Hence $p\Vdash \boldsymbol{A}$ is countable. This completes the proof.

Note that we had a complete freedom in choosing κ, as well as $2^{\aleph_1}$. This has the following interesting application. Working in $V[G]$, let $\mathbb{K}'_G$ be the set of all $s\in {}^{\omega_1+1}2$ such that $\{s\restriction\alpha \mid \alpha<\mathrm{dom}(s)\}$ form a branch of T_G. $\mathbb{K}'_G$ is ordered lexicographically. Then $\mathbb{K}'_G$ is a complete linearly ordered set. Let $\mathbb{K}_G$ be obtained from $\mathbb{K}'_G$ by removing the left hand points of all jumps of $\mathbb{K}'_G$. Then $\mathbb{K}_G$ is a Kurepa continuum (see Section 3). Since CH holds, and since T_G has exactly κ cofinal branches, we have $|\mathbb{K}_G|=\kappa$. Thus, if we choose $\kappa=\aleph_\omega$ in the above forcing construction, then the following result is proved.

8.15. THEOREM. *It is consistent with* ZFC *that there is a Kurepa continuum of cardinality* $\aleph_\omega$.

8.16. REMARKS. (i) The Kurepa Hypothesis was first considered by KUREPA [1935], [1942]. The generalization KH_κ (which is quite often in the literature denoted by $\mathrm{KH}_{\kappa,\kappa}$) was introduced by CHANG [1972]. The statement $\mathrm{KH}^*_{\aleph_1}$ was considered by DEVLIN [1974]. The generalization KH^*_κ appeared in TODORČEVIĆ [1981a] as a statement about trees and subtrees.

(ii) Theorem 8.4 was proved by ERDÖS–HAJNAL–MILNER [1968]. The case cf $\kappa=\kappa$ of 8.6 was proved by JENSEN (see DEVLIN [1973; Ch. 10]). The case cf $\kappa=\aleph_0$ of 8.6 was proved by Jensen and Prikry (unpublished). The proof of Theorem 8.7 is a simple modification of Silver's proof of Jensen's gap-one two cardinal theorem in L (see JENSEN [1972; p. 207]).

(iii) The fact that $\mathrm{KH}^*_{\aleph_1}$ holds in L was proved by Jensen (see DEVLIN [1974] and [1982]). The consistency of KH^*_κ for every regular non-weakly compact cardinal κ (simultaneously) was proved by TODORČEVIĆ [1981a]. The fact that in L for every regular non-ineffable cardinal κ there is a κ-Kurepa tree with no κ-Aronszajn subtrees was proved by DEVLIN [1982].

(iv) Theorem 8.9 was proved by JUHÁSZ–WEISS [1978]. The problem about existence of such a space was asked by SIKORSKI [1950].

(v) The consistency of ZFC + KH using the generic Kurepa tree was proved by STEWART [1966]. For a remark about an earlier result of BUKOVSKY [1966], Lévy and Rowbottom concerning the hypothesis KH we refer the reader to JECH [1971; p. 9]. Theorem 8.11 was proved by Solovay (see DEVLIN [1973; Ch. 18], JECH [1978; p. 252] and KUNEN [1980; p. 240]).

(vi) Theorem 8.12 was proved by SILVER [1971]. The fact that each of the theories ZFC ± CH ± KH ± SH is consistent was proved by DEVLIN [1978a] who also proved the consistency of $\mathrm{ZFC}+\mathrm{MA}_{\aleph_1}+\neg\mathrm{KH}$.

(vii) It is also of interest to consider the following weakening of both CH and

KH denoted by wKH: There is a tree of cardinality $\aleph_1$ with $>\aleph_1$ ω_1-branches. The consistency of ZFC + $\neg$wKH was proved by MITCHELL [1972]. The consistency of ZFC + MA + $\neg$wKH was independently proved by BAUMGARTNER [19··] and TODORČEVIĆ [1981b]. Some applications of MA + $\neg$wKH can be found in DAVIES–KUNEN [19··] and TODORČEVIĆ [1981b].

(viii) Recall that every compact second countable Hausdorff space is either countable or has cardinality $2^{\aleph_0}$. KUNEN [1975] showed that this cannot be generalized to higher cardinals, i.e., that ZFC + CH is consistent with the existence of a compact Hausdorff space X of weight $\aleph_1$ such that $\aleph_1 < |X| < 2^{\aleph_1}$. Note that the Kurepa continuum from 8.15 is also such an example. KUNEN [1975] also proved that such a space cannot be constructed in ZFC or ZFC + CH (see also Juhász's chapter of the Handbook).

9. Nonspecial trees

The first result of this section gives an equivalent definition of speciality not only for trees but also for partially ordered sets in general. It can be regarded as a generalization of a theorem of Cantor which says that every countable linearly ordered set is $\mathbb{Q}$-embeddable.

9.1. THEOREM (Kurepa). *Let E be a partially ordered set. Then E is $\mathbb{Q}$-embeddable iff E is the union of countably many antichains.*

PROOF. We prove only the converse implication, since the direct implication is trivial. It suffices to find a strictly increasing mapping $g: E \to {}^{\omega}2$ such that $|g''E| \leqslant \aleph_0$. (We consider ${}^{\omega}2$ ordered lexicographically.) Let $f: E \to \omega$ be such that each $f^{-1}(n)$ is an antichain of E. For $x \in E$, we define $s = g(x)$ by: $s(n) = 1$ iff $n \leqslant f(x)$ and $\{y \in E \mid y \leqslant x\} \cap f^{-1}(n) \neq \emptyset$. It is easy to check that $g: E \to {}^{\omega}2$ is as required. This completes the proof.

Let $\lessdot$ be a fixed well-ordering of the reals. Then for every $E \subseteq \mathbb{R}$, we define the *Sierpiński's partial ordering* $\leqslant_E$ on E by: $x \leqslant_E y$ iff $x \leqslant y$ and $x \lessdot y$. Then $(E, \leqslant_E)$ is a well-founded partially ordered set with no uncountable chains nor antichains. Thus, if E is uncountable, then $(E, \leqslant_E)$ is not $\mathbb{Q}$-embeddable. This shows that having no uncountable chains is necessary but not sufficient for being $\mathbb{Q}$-embeddable. Note that no uncountable Sierpiński's poset is a tree. However, there is also a tree with no uncountable chains which is not $\mathbb{Q}$-embeddable. Namely, $\sigma\mathbb{Q}$ is such a tree (see Corollary 9.9). Note that the cardinality of $\sigma\mathbb{Q}$ is $2^{\aleph_0}$ but we can have a Sierpiński's poset of cardinality exactly $\aleph_1$. This is an essential difference as the next theorem shows.

For a given partially ordered set E, by $\mathbb{P}(E)$ we denote the set of all finite antichains of E ordered by $\supseteq$.

9.2. LEMMA. *If T is a tree, then $\mathbb{P}(T)$ is a* ccc *poset iff T has no uncountable chains.*

PROOF. Follows easily from Lemma 5.9.

9.3. LEMMA. *If $\mathbb{P}(E)$ is σ-linked, then E is the union of countably many antichains.*

PROOF. If $A \subseteq E$ and if $\{\{x\} \mid x \in A\}$ is a linked subset of $\mathbb{P}(E)$, then A is a antichain of E.

9.4. THEOREM (Baumgartner–Malitz–Reinhardt). *Assume* MA_κ. *Then every tree with no uncountable chains and of cardinality $\leqslant\kappa$ is $\mathbb{Q}$-embeddable.*

PROOF. If MA_κ holds then any ccc poset of cardinality $\leqslant\kappa$ is σ-centered. Now we apply 9.1, 9.2 and 9.3.

9.5. COROLLARY. *If* $\mathrm{MA}_{\aleph_1}$ *holds, then every Aronszajn tree is special.*

9.6. REMARKS. (i) If E is a partially ordered set, then by $\mathbb{P}^\omega(E)$ we denote the set $\{p \in {}^\omega\mathbb{P}(E) \mid \{n < \omega \mid p(n) \neq \emptyset\}$ is finite$\}$ ordered by: $p \leqslant q$ iff $p(n) \supseteq q(n)$ for all $n < \omega$. Then $\mathbb{P}^\omega(E)$ is a standard forcing notion which forces E to be the union of countably many antichains in the extension.

(ii) If T is a tree with no uncountable chains, then as in 9.2 we can prove that $\mathbb{P}^\omega(T)$ is a ccc poset. Hence each power of $\mathbb{P}(T)$ is a ccc poset.

(iii) Assume that $\mathrm{MA}_{\aleph_1}$ holds and that T is a normal Aronszajn tree each level of which is infinite (hence, T has no root). Then, by considering a subposet of $\mathbb{P}^\omega(T)$ and a collection of dense open sets, one can show that there exist antichains A_n, $n < \omega$ of T such that $T = \bigcup\{A_n \mid n < \omega\}$ while each A_n intersects each level of T in exactly one point. Thus, if $\mathrm{MA}_{\aleph_1}$ holds, then *every* Aronszajn tree has a club-antichain.

(iv) Let T be a Suslin tree and let T^* be equal to T with the reverse ordering. Then T^* is a ccc poset. Also by (ii), we have that each finite power of $\mathbb{P}(T)$ is a ccc poset. But $T^* \times \mathbb{P}(T)$ is not a ccc poset, since it contains an uncountable antichain $\{\langle t, \{t\}\rangle \mid t \in T\}$.

Let ξ be an ordinal fixed from now on. Let

$$H_\xi = \{h \in {}^{\omega_\xi}2 \mid \mathrm{supp}(h) \text{ has a maximal element}\},$$

considered as a linearly ordered set under the lexicographical ordering. Then H_ξ is a standard η_ξ-set. For $h \in H_\xi$, we define $\alpha(h) = \max(\mathrm{supp}(h))$. By C_ξ, we denote ${}^{\omega_\xi}2$ under the lexicographical ordering. The next result is a generalization of 9.1 if we restrict ourselves to trees.

9.7. LEMMA. *Assume T is a tree. Then*:

(i) *T is H_ξ-embeddable iff T is the union of $\leqslant\aleph_\xi$ antichains*;

(ii) *T is C_ξ-embeddable iff $T\restriction\{\text{succ}\}$ is H_ξ-embeddable.*

PROOF. We prove only (i). We prove only the direct implication of (i), since the proof of the converse implication of (i) is a trivial generalization of the proof of 9.1. To prove the direct implication, let i: $T\to H_\xi$ be a given strictly increasing mapping. For $\gamma<\omega_\xi$, let $B_\gamma=\{t\in T\mid \alpha(i(t))\leqslant\gamma\}$. Then $T=\bigcup\{B_\gamma\mid\gamma<\omega_\xi\}$. Hence, it suffices to prove that each B_γ is the union of $\leqslant\aleph_\xi$ antichains. So let $\gamma<\omega_\xi$ be fixed. The set $\{f\in H_\xi\mid\alpha(f)\leqslant\gamma\}$ has no well-ordered subsets of cardinality $|\gamma|^+ +\aleph_0$ (see JECH [1978; p. 323]). Thus, B_γ has no chains of cardinality $\aleph_\xi$, hence B_γ is the union of $\leqslant\aleph_\xi$ antichains.

9.8. THEOREM. *σH_ξ is not the union of $\leqslant\aleph_\xi$ antichains.*

PROOF. By Lemma 9.7, it suffices to prove that σH_ξ is not H_ξ-embeddable. Assume, on the contrary, that there exists a strictly increasing mapping i: $\sigma H_\xi\to H_\xi$. By the homogeneity of H_ξ, we may assume that $i''(\sigma H_\xi)$ is bounded in H_ξ. By induction on $\alpha<\omega_{\xi+1}$, we now define increasing sequences $\langle t_\alpha\mid\alpha<\omega_{\xi+1}\rangle$ and $\langle h_\alpha\mid\alpha<\omega_{\xi+1}\rangle$ of elements of σH_ξ and H_ξ, respectively, as follows. Let $t_0=\emptyset$ and $h_0=i(t_0)$. If $\alpha=\beta+1$, let $t_\alpha=t_\beta\cup\{h_\beta\}$ and $h_\alpha=i(t_\alpha)$. If α is limit, let $t_\alpha=\bigcup\{t_\beta\mid\beta<\alpha\}$ and $h_\alpha=i(t_\alpha)$. It is not hard to check that $\langle t_\alpha\mid\alpha<\omega_{\xi+1}\rangle$ and $\langle h_\alpha\mid\alpha<\omega_{\xi+1}\rangle$ are well-defined and strictly increasing. But this gives us a contradiction, since H_ξ contains no well-ordered subsets of cardinality $\aleph_{\xi+1}$. This completes the proof.

A similar proof will show that σC_ξ is not C_ξ-embeddable. Note that σH_ξ is C_ξ-embeddable by Lemma 9.7(ii). If $\xi=0$, then 9.8 gives the following corollary.

9.9. COROLLARY (Kurepa). *$\sigma\mathbb{Q}$ is not the union of countably many antichains.*

In order to give one more application of 9.8, we need to introduce the following definition. We say that a poset $\mathbb{P}$ satisfies the (κ,λ)-cc if there is a partition $\mathbb{P}=\bigcup\{A_\alpha\mid\alpha<\kappa\}$ such that, for every $\alpha<\kappa$, there is no set of λ elements of A_α which are incompatible in $\mathbb{P}$.

9.10. THEOREM (Galvin–Hajnal). *For every infinite cardinal κ there is a poset $\mathbb{P}$ such that*:

(i) $|\mathbb{P}|=2^\kappa$;

(ii) *$\mathbb{P}$ is the union of κ^+ directed sets*;

(iii) *κ^+ is a precaliber for $\mathbb{P}$*,

(iv) *$\mathbb{P}$ does not satisfy the (κ,κ)-chain condition.*

PROOF. Let $\kappa = \aleph_\xi$ and let $\mathbb{P}_\xi$ be the set of all finite antichains of σH_ξ ordered by $\supseteq$. We show that $\mathbb{P}_\xi$ satisfies the theorem.

(ii) Follows easily from the fact that σH_ξ is the union of κ^+ levels and that, for each $\alpha < \kappa^+$, ${}^{(\sigma H_\xi)\alpha}2$ has a dense subset of cardinality κ (see Hodel's chapter of the Handbook, Theorem 11.2).

(iii) Let $\langle p_\alpha \mid \alpha < \kappa^+\rangle$ be a sequence of elements of $\mathbb{P}_\xi$. We have to find $A \subseteq \kappa^+$, $|A| = \kappa^+$ such that $\{p_\kappa \mid \xi \in A\}$ is centered in $\mathbb{P}_\xi$. An easy argument shows that we may assume $p_\alpha \subseteq \sigma H_\xi \restriction \{\text{succ}\}$ for all $\alpha < \kappa^+$. Now the conclusion follows from the fact that $\{p \in \mathbb{P}_\xi \mid p \subseteq \sigma H_\xi \restriction \{\text{succ}\}\}$ is a κ-centered subset of $\mathbb{P}_\xi$. See 9.7(i).

(iv) Follows from 9.8, since if $A \subseteq \sigma H_\xi$ is such that $\{\{t\} \mid t \in A\}$ has no κ pairwise incompatible elements, then A is the union of $\leqslant\kappa$ antichains.

9.11. REMARK. *Let $\mathbb{P}$ be any poset which satisfies* 9.10, *and let X be the Stone space of the Boolean algebra of regular open subsets of $\mathbb{P}$. Then X is an extremely disconnected compact Hausdorff space such that:*

(i) $\kappa^+ = d(X) \leqslant \pi(X) = w(X) = 2^\kappa$;

(ii) *every regular cardinal $>\kappa$ is a caliber for X;*

(iii) *X does not satisfy the (κ, κ)-chain condition.*

One can obtain a similar space using a slightly different method. Namely, let X_ξ be the set of all antichains of σH_ξ considered as a subspace of ${}^{\sigma H_\xi}2$ with the Tychonoff topology (under the usual identification). Then X_ξ is a closed, hence compact, subspace of ${}^{\sigma H_\xi}2$ which also satisfies (i)–(iii) above.

Until the end of this section we consider a class of trees which are, in some case, very far from being special. We also consider the corresponding class of linearly ordered continua and give some applications. So, let A be any subset of ω_1. Then by $U(A)$ we denote the set of all countable and closed in ω_1 subsets of A partially ordered by: $s \leqslant t$ iff s is an initial part of t. Then $U(A)$ is a tree of height $\leqslant\omega_1$ which has no uncountable chains iff $\omega_1 - A$ is stationary.

9.12. LEMMA. (i) *$U(A)$ is special iff A is nonstationary.*

(ii) *$U(A)$ is Baire iff A is stationary.*

(iii) *$U(A) \otimes U(B)$ is special iff $A \cap B$ is nonstationary.*

(iv) *$U(A) \otimes U(B)$ is Baire iff $A \cap B$ is stationary.*

PROOF. We prove here only the converse implication of (ii). So, assume A is a stationary subset of ω_1. Let $\langle D_n \mid n < \omega\rangle$ be a given sequence of dense final parts of $U(A)$, and let $t_0 \in U(A)$ be arbitrary. Let N be a countable elementary submodel of $H(\aleph_2)$ such that $t_0, A, \langle D_n \mid n < \omega\rangle \in N$ and $N \cap \omega_1 = \delta \in A$. Since N is an elementary submodel of $H(\aleph_2)$, by induction on $n < \omega$, we can construct an increasing sequence $\langle t_n \mid n < \omega\rangle$ of elements of $U(A) \cap N$ such that $t_n \in D_n$ and $\sup\{\max(t_n) \mid n < \omega\} = \delta$. Let $s = (\bigcup_n t_n) \cup \{\delta\}$. then $s \in \bigcap_n D_n$ and $t_0 \leqslant s$. This proves that $\bigcap_n D_n$ is dense in $U(A)$. Hence $U(A)$ is Baire.

Let A be a fixed uncountable subset of ω_1. Let $\lhd$ be a linear ordering of A such that:

(i) $(A, \lhd)$ isomorphic to a set of reals;

(ii) $(A, \lhd)$ has no end-points;

(iii) every infinite interval of $(A, \in)$ is dense in $(A, \lhd)$.

The ordering $\lhd$ naturally induces a linear ordering of each node of $U(A)$, hence it also induces a lexicographical ordering $<$ on the set $M(A)$ of all maximal chains of $U(A)$ which can be explicitly defined as follows: $l < m$ iff $\max(s) \lhd \max(t)$, where $s = \min(l - m)$ and $t = \min(m - l)$. Finally, let $\mathbb{K}(A)$ be the Dedekind completion of $M(A)$. Then $\mathbb{K}(A)$ is a linearly ordered continuum. Some properties of $\mathbb{K}(A)$ strongly depend on the nature of the set A. The next theorem lists only a part of results which can be proved about the continuum $\mathbb{K}(A)$.

9.13. THEOREM. *Assume that both A and its complement are stationary subsets of ω_1. Then*:

(i) *$\mathbb{K}(A)$ is first countable.*

(ii) *$\mathbb{K}(A)$ is the union of $\aleph_1$ nowhere dense subsets.*

(iii) *Every first category subset of $\mathbb{K}(A)$ is also nowhere dense.*

(iv) *There is a real function on $\mathbb{K}(A)$ which is not continuous on any dense subset of $\mathbb{K}(A)$.*

PROOF. (i) It suffices to prove that $M(A)$ has no uncountable well-ordered nor conversely well-ordered subsets. This follows easily from the fact that $U(A)$ has no uncountable chains and the properties of the ordering $\lhd$.

(ii) For each $\alpha < \omega_1$, let M_α be the closure of $\{m \in M(A) \mid \mathrm{tp}(m, \leqslant) \leqslant \alpha\}$ in $\mathbb{K}(A)$. Then each M_α is a nowhere dense subset of $\mathbb{K}(A)$ and $\mathbb{K}(A) = \bigcup\{M_\alpha \mid \alpha < \omega_1\}$.

(iii) For $t \in U(A)$, let B_t be the convex closure of $\{m \in M(A) \mid t \in m\}$ in $\mathbb{K}(A)$. It is clear that each open subset of $\mathbb{K}(A)$ contains B_t for some $t \in U(A)$. Hence, if N is a nowhere dense subset of $\mathbb{K}$, then $D_N = \{t \in U(A) \mid B_t \cap N = \emptyset\}$ is a dense final part of $U(A)$. Now the conclusion follows from 9.12(ii).

(iv) Follows from (ii) and (iii) as in the proof of 6.17(iii). This finishes the proof.

One can go further and prove, for example, that if A and B are disjoint stationary subsets of ω_1, then $\mathbb{K}(A) \times \mathbb{K}(A)$ has a dense metrizable subspace, although neither $\mathbb{K}(A)$ nor $\mathbb{K}(B)$ has a dense metrizable subspace.

For technical reasons, we now for each $A \subseteq \omega_1$ define a new tree $U_2(A)$ as follows. It is clear that $U(A)$ isomorphic to an initial part of ${}^{\omega_1}2 \restriction A$, since ht $U(A) \leqslant \omega_1$ and each node of $U(A)$ has cardinality $\leqslant \aleph_1$. Hence, for each $A \subseteq \omega_1$ we can find an initial part $U_2(A)$ of ${}^{\omega_1}2$ such that $U_2(A) \restriction A$ and $U(A)$ are isomorphic. Note that Lemma 9.12 holds also for $U_2(A)$ instead of $U(A)$. Let $Y(A)$ be the set of all paths of $U_2(A)$ considered as a subspace of ${}^{U_2(A)}2$ with the Tychonoff topology. Then it is easily checked that $Y(A)$ is a closed (hence compact) subset of ${}^{U_2(A)}2$.

9.14. THEOREM. *Assume that both A and ω_1-A are stationary subsets of ω_1. Then $Y(A)$ is a compact first countable Corson space with no dense metrizable subspace.*

PROOF. That $Y(A)$ is a first countable Corson space follows from the fact that $U_2(A)$ has no uncountable chains. An easy modification of the proof of 9.13(iii) shows that $Y(A)$ has no dense metrizable subspace.

9.15. REMARKS. (i) The notions of $\mathbb{Q}$-embeddability and $\mathbb{R}$-embeddability were introduced by KUREPA [1937b] (see also [1940] [1941] and [1964]). Theorem 9.1 was proved by KUREPA [1940]. This theorem was rediscovered independently and later by GALVIN (unpublished).

(ii) Theorem 9.4 was proved by BAUMGARTNER–MALITZ–REINHARDT [1970], JENSEN (unpublished) and KUNEN (unpublished). The problem whether every tree T with no uncountable chains and with the property that $|R_\alpha T| < 2^{\aleph_0}$ for all $\alpha < \mathrm{ht}\, T$, is $\mathbb{R}$-embeddable was asked by KUREPA [1940; p. 846]. Let us also mention that SHELAH [1981] constructed a model of ZFC + SH in which not every Aronszajn tree is $\mathbb{Q}$-embeddable. Thus SH does not imply that every Aronszajn tree is special (see BAUMGARTNER–MALITZ–REINHARDT [1970; p. 1749]).

(iii) Corollary 9.9 was proved by KUREPA [1954] who also proved that $\sigma\mathbb{R}$ is not $\mathbb{R}$-embeddable. Note that H_0-embeddable = $\mathbb{Q}$-embeddable and that C_0-embeddable = $\mathbb{R}$-embeddable. Lemma 9.7(ii) (case $\xi = 0$) was proved by Galvin (see BAUMGARTNER [1970a; p. 71]).

(iv) Theorem 9.10 (and Remark 9.11) was proved by GALVIN–HAJNAL [19 · ·]. They used the tree $\{f \in {}^{\kappa^+}\kappa \mid f \text{ is } 1\text{–}1\}$ instead of σH_ξ, where $\kappa = \aleph_\xi$.

(v) $U(A)$ as a forcing notion was apparently first defined by Jensen, who also proved Lemma 9.12(ii) (see also BAUMGARTNER–HARRINGTON–KLEINBERG [1976] and DEVLIN [1978b]). Theorems 9.13 and 9.14 are from TODORČEVIĆ [1981c] where we refer the reader for more detailed proofs.

(vi) Let A and B be two disjoint stationary subset of ω_1. Then by 9.12(iii), $U(A) \otimes U(B)$ is a special tree, but none of $U(A)$ and $U(B)$ is a special tree. Using $\diamondsuit$, JENSEN–JOHNSBRÅTEN [1974] constructed a Suslin tree T such that $\{\langle s, t\rangle \in T \otimes T \mid s \neq t\}$ is a special tree. It is an open problem whether, in ZFC, we can construct a Baire tree U such that $\{\langle t, u\rangle \in U \otimes U \mid t \neq u\}$ is a special tree and $|U| \leq 2^{\aleph_0}$.

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CHAPTER 7

Basic S and L

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

1. Introduction

The problem of the existence of S- and L-spaces has its origins in the 1920's, as a natural outgrowth of Suslin's hypothesis and of investigations of properties of metric, semi-metric, and Fréchet spaces. In its modern form the problem was stated independently in the late 1960's and early 1970's by Countryman and by Hajnal and Juhász. By then enough set theory was in place for a serious attack and for the last ten years it has been one of the most active areas of set theoretic topology.

1.1. *The problem defined.* A space is hereditarily separable iff every subspace has a countable dense subset; it is hereditarily Lindelöf iff every cover of a subspace has a countable subcover. Both properties follow from countable weight. They are dual in the sense that (under a slight restatement) switching 'point' with 'open set' in either definition gives you the other. Do they always coincide?

For non-regular spaces the answer is no. Well-order an uncountable subset of the reals in type ω_1 and refine the topology inherited from the reals by declaring all initial segments open. This space is Hausdorff hereditarily separable non-Lindelöf and not regular. Declare initial segments closed instead of open and the space is Hausdorff hereditarily Lindelöf non-separable and not regular. See e.g. HAJNAL and JUHÁSZ [1968].

The problem is interesting, then, only for regular spaces. Define an S-space as regular Hausdorff hereditarily separable and not hereditarily Lindelöf. Define an L-space as regular Hausdorff hereditarily Lindelöf and not hereditarily separable. Do S- and L-spaces exist?

1.2. *A brief history.* One corollary of KUREPA [1935] is that a Suslin line is an L-space. This fact was widely and independently known in the United States at about the same time. The relation of hereditary separability to Lindelöfness was studied by the Texas school of topology in many classes of spaces. This is the way to prove, for example, that hereditarily separable Moore spaces are metrizable.

In 1967 Jech proved the consistency of the existence of a Suslin line; hence the consistency of the existence of an L-space. In 1972 M.E. Rudin constructed an S-space from a Suslin line.

After that things moved quickly. S- and L-spaces with various properties were constructed from CH (hence are independent of the Suslin hypothesis), from $\diamondsuit$, and by forcing. Every time an S- or L-space was constructed someone showed that the construction had properties ruled out by $\mathrm{MA} + \neg\mathrm{CH}$. The big question became: is it consistent that there are no S- or L-spaces?

The obvious place to look was under $\mathrm{MA} + \neg\mathrm{CH}$ but in 1977 Szentmiklóssy showed that $\mathrm{MA} + \neg\mathrm{CH}$ was consistent with the existence of S-spaces. His proof does not dualize for L-spaces. In 1980 Abraham and Todorčević were even able to get first countable S-spaces consistent with $\mathrm{MA} + \neg\mathrm{CH}$ (see their article in this volume) which was a mild surprise since an earlier theorem by Szentmiklóssy (in SZENTMIKLÓSSY [1980]) ruled out first countable L-spaces under MA + ¬CH. Very recently they noted that their proof did dualize to get the consistency of MA + ¬CH with the existence of L-spaces. It looked as though we were in for a further glut of positive consistency results.

It was not to be. In 1981 TODORČEVIĆ showed the consistency of "there are no S-spaces." His proof does not seem to dualize for L-spaces. So the big questions remaining are: Is "there are no L spaces" consistent? Is there an S-space iff there is an L-space?

1.3. *Uses of* S *and* L. The problem of S and L is made more interesting by the way it pops up in unexpected places. The chapter on Borel measures in this volume is rife with examples. Here is a short sampling (in no way meant to be complete) of other results which use S- and L-space theory.

(HAJNAL and JUHÁSZ [1972b]). The number of open sets of a Hausdorff space can be strictly between 2^{ω} and 2^{ω_1}.

(FEDORCHUK [1976]). A compact space of countable tightness need not have a convergent subsequence.

(JUHÁSZ, KUNEN, RUDIN [1976]). A Dowker space can be first countable.

(OSTASZEWSKI [1976]). Perfectly normal countably compact spaces need not be compact.

(RUDIN and ZENOR [1976]). Perfectly normal non-metrizable manifolds can exist.

(ROITMAN [1979b]). Adding a random or a Cohen real destroys MA_{ω_1}.

(RUDIN [1979]). Perfectly normal non-metrizable manifolds need not exist.

With the exception of (ROITMAN [1976b]) the above are all consistency results.

1.4. *Plan of this paper.* Both M.E. Rudin (RUDIN [1980]) and Juhász (JUHÁSZ [1980a]) have written excellent surveys recently, so there is no need for another survey. Instead, this paper concentrates on basic results. The author bears all responsibility for the application of the word "basic." For pedagogical reasons, easy results are presented in greater detail than harder results. For a detailed difficult S and L argument see the article by Abraham and Todorčević in this volume.

In Section 2 the topological and set-theoretic prerequisites are presented. In Section 3 the basic problem is stripped down to its set-theoretic essentials and duality is explored. Section 4 gives some constructions using CH. Section 5 extends the techniques of Section 4, using $\diamond$ and forcing, to construct counterexamples with applications outside S and L. (Other applications can be found in this volume in e.g. Nyikos' article and the article by GARDNER and PFEFFER.) A sampling of negative MA + ¬CH results is given in Section 6. Finally, Section 7 presents Szentmiklóssy's ccc-indestructible S-space and Todorčević's theorem that "there are no S-spaces" is consistent.

2. Prerequisites

2.1. *Jargon.* The usual set theoretic notation is followed, e.g. $|A|$ denotes the cardinality of A; cardinals are initial ordinals; $2 = \{0, 1\}$; 2^A denotes the set of all functions from A into 2 and also the cardinality of that set; $[\kappa]^\lambda = \{A \subset \kappa : |A| = \lambda\}$; $[\kappa]^{<\lambda} = \{A \subset \kappa : |A| < \lambda\}$; $f|A$ is the restriction of the function f to domain A; and so on. The reader is assumed to be familiar with CH, $\diamond$, and MA + $\neg$CH. The symbol $\mathfrak{c}$ denotes the cardinal 2^ω.

Familiarity with standard topological definitions is assumed. A good reference is ENGELKING [1968]. Less common definitions can be found either there or in JUHÁSZ [1975] or JUHÁSZ [1980b]. Two frequently used concepts are defined forthwith: A space is 0-dimensional iff it has a basis of clopen sets. A discrete subspace is one whose relative topology is the discrete topology (N.B. discrete $\neq$ closed discrete).

Recall that a partial order is ccc iff it has no uncountable pairwise incompatible subsets, and that a space is ccc iff the partial order of its nonempty open sets under reverse inclusion is ccc (i.e. any collection of pairwise disjoint open sets is countable).

Let A, B be arbitrary sets. If σ is a partial function from A to B then $[\sigma] = \{f \in B^A : f > \sigma\}$. (In this context A will usually be the set ω_1; B will usually be the set 2.) Recall that if B is a discrete topological space, then $\{[\sigma]: \operatorname{dom}\sigma$ is a finite subset of A, range $\sigma \subset B\}$ is a basis for the Tychonoff product space B^A.

The boundary of a set Y in a space is written ∂Y and is defined as $\mathrm{cl}(Y) - \mathrm{int}(Y)$.

All spaces are assumed Hausdorff.

2.2. *Δ-system lemma.* Recall the Δ-system lemma (Section 2.3 of Hodel's paper in this volume): If A is an uncountable collection of finite subsets of ω_1, then there is an uncountable $A' \subset A$ and a finite set a so that if b, c are distinct elements of A' then $b \cap c = a$. A' is called a Δ-system and a is its root.

This lemma will be invoked constantly, usually together with a counting argument. Here is a simple illustration of a typical use.

If $f \in 2^{\omega_1}$ define the support of f to be $\{\alpha : f(\alpha) = 1\}$. Let $X = \{f \in 2^{\omega_1}: f$ has finite support$\}$. We'll show that X is a ccc subspace of 2^{ω_1}. (Usually we'll be showing that more complicated subspaces of 2^{ω_1} are hereditarily ccc.)

Suppose by contradiction some uncountable collection $\{[\sigma_\alpha] \cap X : \alpha < \omega_1\}$ were pairwise disjoint, where each $[\sigma_\alpha]$ is basic open as in 2.1 (in particular, $\operatorname{dom}\sigma_\alpha$ is finite) and each $[\sigma_\alpha] \cap X \neq \emptyset$. By a Δ-system argument there is an uncountable set $A_1 \subset \omega_1$, so $\{\operatorname{dom}\sigma_\alpha : \alpha \in A_1\}$ is a Δ-system. Let b be its root. By a counting argument there is an uncountable set $A_2 \subset A_1$ so that if $\alpha, \beta \in A_2$ then $\sigma_\alpha | b = \sigma_\beta | b$. So if α, β are distinct elements of A_2, then $[\sigma_\alpha] \cap [\sigma_\beta] = [\sigma_\alpha \cup \sigma_\beta] \neq \emptyset$. Let $\sigma = \sigma_\alpha | b$ for any $\alpha \in A_2$. Let $X' = X \cap [\sigma]$. For α, $\beta \in A_2$, $\sigma_\alpha \cup \sigma_\beta \cup \{\langle\gamma, 0\rangle : \gamma \notin \operatorname{dom}\sigma_\alpha \cup \operatorname{dom}\sigma_\beta\}$ is a function in $X' \cap [\sigma_\alpha] \cap [\sigma_\beta]$, which is a contradiction.

(Note that for $\alpha, \beta \in A_2$, $X' \cap [\sigma_\alpha] \cap [\sigma_\beta] = \emptyset$ iff $X' \cap [\sigma_\alpha - \sigma] \cap [\sigma_\beta - \sigma] = \emptyset$.)

This cumbersome procedure can be abbreviated as follows: Given the uncountable pairwise disjoint collection $\{[\sigma_\alpha] \cap X : \alpha < \omega_1\}$, by a Δ-system and counting argument we may assume the domains of the σ_α's are pairwise disjoint (i.e. we identify A_2 and ω_1, replace X with X', and replace each σ_α with $\sigma_\alpha - \sigma$). We conclude that

$$\sigma_0 \cup \sigma_1 \cup \{\langle \gamma, 0\rangle : \gamma \notin \operatorname{dom} \sigma_0 \cup \operatorname{dom} \sigma_1\} \in X \cap [\sigma_0] \cap [\sigma_1].$$

In connection with Δ-system and counting arguments the following concepts arise:

2.2.1. DEFINITION. (a) If a, b are sets of ordinals define $a < b$ iff $\sup a < \inf b$.

(b) If A is set of ordinals define the collection of sets of ordinals $\{a_\alpha : \alpha \in A\}$ to be *separated* iff $a_\alpha < a_\beta$ whenever $\alpha < \beta$, for all $\alpha, \beta \in A$.

By a Δ-system and counting argument, if $\{a_\alpha : \alpha \in A\}$ is an uncountable collection of distinct finite subsets of ω_1, $A \subset \omega_1$, then there is some uncountable $A' \subset A$ and finite b so $\{a_\alpha - b : \alpha \in A'\}$ is separated.

2.3. *Simple limit spaces.* If τ is a topology (= the collection of open sets) on X, and $Y \subset X$, we write $\tau|Y$ for the relative topology on Y. If τ is a topology on X, σ a topology on Y, and $Y \subset X$, we say that τ is a conservative extension of σ iff $\sigma = \tau|Y$, i.e. iff the space (Y, σ) is an open subspace of (X, τ).

Suppose $\{X_\alpha : \alpha < \kappa\}$ is an increasing collection of sets, τ_α is a topology on X_α for each α, and τ_α is a conservative extension of τ_β if $\beta < \alpha < \kappa$. We define the simple limit topology $\Sigma_{\alpha<\kappa} \tau_\alpha$ to be the unique conservative extension of each τ_α on $\bigcup_{\alpha<\kappa} X$, i.e. it's the topology on $\bigcup_{\alpha<\kappa} X$ whose basis is $\bigcup_{\alpha<\tau} \tau_\alpha$.

Simple limit constructions will be used in this paper, so it's useful to make a couple of observations:

2.3.1. Observation. Suppose $\tau = \Sigma_{\alpha<\kappa} \tau_\alpha$ as above. If each τ_α is Hausdorff, so is τ. If each τ_α is 0-dimensional, so is τ.

2.3.2. Observation. Suppose $A \subset Y \subset X$, σ is a topology on Y, τ is a topology on X, and τ is a conservative extension of σ. Then

(a) If A is compact in (Y, σ), then A is compact in (X, τ).

(b) If $x \in Y \cap \mathrm{cl}_\sigma(A)$, then $x \in \mathrm{cl}_\tau(A)$.

(c) If $x \in Y$ and $\mathcal{B}$ is a neighborhood basis for x in (Y, σ), then $\mathcal{B}$ is a neighborhood basis for x in (X, τ).

2.4. *A note about forcing.* Forcing arguments appear in several sections. In writing these, I assumed some but not much familiarity with forcing on the part of the reader. Good references for standard facts are JECH [1978], JECH [1971b], and KUNEN [1980]. Since notation is not completely standardized, here is the notation used in this paper.

The phrases 'model' and 'model of set theory' can either refer to countable transitive models of ZFC or to forcing extensions of them. The latter can be viewed either as Boolean-valued models or as two-valued models (= adding the generic object = modding out the Boolean-valued model by the generic ultrafilter). We will use the point of view which is most convenient at the time, e.g. in Section 5.2 the two-valued approach is more natural, while in Section 7.1 the Boolean-valued approach is preferable.

There are several conventions currently in use for terms in the forcing language. The ones used here are as follows: if M is the ground model and N is a forcing extension of M, then terms for objects in $N - M$ are marked by °, while no distinguishing marks are used for terms for objects in M. Thus: "Let $X \in M$ be a topological space, $f \in N - M$ a function and suppose $p \Vdash \mathring{f}: X \to \omega \ldots$" Recall that terms are always in the ground model.

Occasionally we must relativize to a model. Thus, e.g., "ω_1^M" somewhat anthropomorphically means "the ordinal that the model M thinks is ω_1." When we work in the model we dispense with such notation. Thus in writing "In M let $\{A_\alpha: \alpha < \omega_1\}$ be a family of subsets of ω_1" the ω_1's are actually ω_1^M.

3. Canonical S- and L-spaces

The question of S and L reduces to the existence of easily described subspaces of the Tychonoff product 2^{ω_1}. This lays bare the set theoretical nature of the problem. Under these reductions the implicit duality between S and L becomes quite specific, and the limitations of duality become apparent.

Define a space to be right separated iff the points can be well-ordered so that every initial segment is open. It is right separated in type κ if the well-ordering has type κ. Similiarly, a space is left separated iff the points can be well-ordered so that every initial segment is closed, and is left separated in type κ if the well-ordering has type κ. The terminology is due to Hajnal and Juhász who proved

3.1. THEOREM. *A space is hereditarily separable iff it has no uncountable left separated subspace. A space is hereditarily Lindelöf iff it has no uncountable right separated subspace.*

PROOF. An uncountable right separated space is not Lindelöf (look at the first ω_1 points in the enumeration) and an uncountable left separated space is not separable (ditto), so a hereditarily Lindelöf (separable) space has no uncountable right (left) separated subspace.

In the other direction, if X is not hereditarily Lindelöf there is a strictly increasing sequence of open sets $\{u_\alpha: \alpha < \omega_1\}$. Pick $x_\alpha \in u_{\alpha+1} - u_\alpha$ for all $\alpha < \omega_1$ and you have a right separated subspace of type ω_1. If X is not hereditarily separable, let Y be a nonseparable subspace of X. In Y we can define a sequence of points $\{x_\alpha: \alpha < \omega_1\}$ where each $x_\alpha \notin \mathrm{cl}\{x_\beta: \beta < \alpha\}$, so X has a left separated subspace of type ω_1.

3.2. COROLLARY. *Every* S-*space contains a right separated subspace of type* ω_1. *Every* L-*space contains a left separated subspace of type* ω_1.

Right and left separated spaces have the following pleasant property:

3.3. PROPOSITION. *A regular right separated space of type* ω_1 *is an* S-*space iff it has no uncountable discrete subspaces. A regular left separated space of type* ω_1 *is an* L-*space iff it has no uncountable discrete subspaces.*

PROOF. For S-spaces: Let X be right separated in type ω_1. A subspace of X is discrete iff it is left separated by the induced well ordering. So by 3.1 X has an uncountable discrete subspace iff it is not hereditarily separable.

The proof for L-spaces is similar.

HAJNAL and JUHÁSZ [1974] showed that under CH there exist 0-dimensional S- and L-spaces (see Section 4). Juhász noticed that in fact every regular right separated space of type ω_1 is 0-dimensional. Kunen noticed that the same proof showed that under $\neg$CH every regular Lindelöf left separated space of type ω_1 is 0-dimensional.

We sketch the argument for the non-topologist: a regular Lindelöf space is completely regular because normal; any regular locally countable space is completely regular (a homework exercise). So we just have to show that a completely regular space in which every point has a neighborhood of size less than $\mathfrak{c}$ is 0-dimensional. Let X be such a space and pick $x \in X$. Let u be an open neighborhood of x, $|u| < \mathfrak{c}$. By complete regularity there is a continuous function $f: X \to [0, 1]$ where f sends x to 0 and $X - u$ to 1. Since $|u| < \mathfrak{c}$ there is $r \in (0, 1)$ with $r \notin$ range f. Then $f^{-1}[0, r)$ is the desired clopen neighborhood of x contained in u. So X is 0-dimensional.

As corollaries to this argument we have

3.4. CANONICAL FORM. (a) *Every* S-*space contains a* 0-*dimensional subspace which is right separated in type* ω_1 *and therefore also an* S-*space.*

(b) *Assume* $\neg$CH. *Every* L-*space contains a* 0-*dimensional subspace which is left separated in type* ω_1 *and therefore also an* L-*space.*

Putting this together with the CH examples of Hajnal and Juhász we have

3.5. CANONICAL FORM. *If there is an* S-(*respectively* L-) *space, then there is an* 0-*dimensional one which is right* (*respectively left*) *separated in type* ω_1.

Note a certain lack of symmetry: every right-separated space of type ω_1 is 0-dimensional; but we don't know if every left-separated space of type ω_1 even has a 0-dimensional subspace (for 3.5 we had to invoke specific examples under CH).

3.5 sends us in two directions. It immediately tells us that if we want to destroy (or occasionally construct) S- and L-spaces we may without loss of generality identify our spaces with the set ω_1 and assume that we have some sort of 0-dimensional topology in which every initial segment of countable ordinals is respectively open (for S) or closed (for L).

On the other hand, 3.5 firmly connects S- and L-spaces with subspaces of 2^{ω_1} as follows: A 0-dimensional space of weight $\leqslant\kappa$ is homeomorphic to a subspace of 2^κ. Given a 0-dimensional space with no uncountable discrete subspaces which is right or left separated in type ω_1, if you take a weaker 0-dimensional topology which preserves Hausdorff and right or left separation, the original topology is S or L iff the weaker one is. And in a space of size ω_1, to preserve Hausdorff and right or left separation it suffices to use ω_1 many sets. This gives us

3.6. CANONICAL FORM. *There is an* S- (*respectively* L-) *space iff there is a right* (*respectively left*) *separated one embedded in* 2^{ω_1}.

A sharper form of 3.6 is

3.7. CANONICAL FORM. (a) *Every* S-*space contains an uncountable subset which under a possibly weaker topology is homeomorphic to a right separated* S-*subspace of* 2^{ω_1}.

(b) *Assume* $\neg$CH. *Every* L-*space contains an uncountable subset which under a possibly weaker topology is homeomorphic to a left separated* L-*subspace of* 2^{ω_1}.

Guided by 3.6 and 3.7 we define a dual which helps clarify the relation between S and L-spaces.

Suppose $X=\{x_\alpha: \alpha<\omega_1\}\subset 2^{\omega_1}$. We define the dual X^* (dependent on the enumeration of X) by first defining for each $\beta<\omega_1$ the function x^*_β: $x^*_\beta(\alpha)=x_\alpha(\beta)$; and then letting $X^*=\{x^*_\beta: \beta<\omega_1\}$. Note that $X^{**}=X$. We would like to say that X is right separated iff X^* is left separated, but unfortunately this is false. An awkward approximation to the desired state of affairs is

3.8. LEMMA. *Let X be* 0-*dimensional.*

(a) *If X has weight* ω_1 *and is right separated in type* ω_1, *then X can be embedded in* 2^{ω_1} *so its dual is left separated.*

(b) *If X is left separated in type ω_1, then under a possibly weaker topology X can be embedded in 2^{ω_1} so its dual is right separated.*

PROOF. For (a), note that X has a basis of countable clopen sets, so ensure each $[\{\langle\alpha, 1\rangle\}] \cap X$ is countable. For (b), $X = \{x_\alpha : \alpha < \omega_1\}$ where initial segments are closed, so ensure that if $\alpha < \beta$, then $x_\alpha \notin [\{\langle\beta, 1\rangle\}]$.

Let's use the dual to prove a theorem of Zenor: there is a strong S-space iff there is a strong L-space. (Strong S (strong L) is defined to mean every finite product is S (L). ZENOR [1980] actually proved a sharper result: X is strong S (strong L) iff $C(X, H)$ is strong L (strong S) in the topology of pointwise convergence.)

The CH spaces of Hajnal and Juhász can be modified to be strong S and strong L. So by 3.7 and 3.8 all we have to show is

3.9. PROPOSITION (ROITMAN [1979b]). *Suppose $X = \{x_\alpha : \alpha < \omega_1\} \subset 2^{\omega_1}$. Every X^n, $n < \omega$, is hereditarily separable iff every $(X^*)^n$, $n < \omega$, is hereditarily Lindelöf.*

PROOF. To get the idea of the proof across, let's just prove that X^* is hereditarily Lindelöf if every X^n is hereditarily separable. The rest of the proof is an easy generalization with messy notation.

Suppose by contradiction that $Y = \{y_\alpha : \alpha < \omega_1\}$ is a right separated subspace of X^* in type ω_1. Each $y_\alpha = x^*_{\beta_\alpha}$ for some ordinal β_α. For each α let $[\sigma_\alpha]$ be a neighborhood of y_α so $[\sigma_\alpha] \cap Y \subset \{y_\beta : \beta \leq \alpha\}$. For each α let $\{\delta_{\alpha,i} : i < n_\alpha\}$ enumerate dom σ_α, where $|\text{dom}\, \sigma_\alpha| = n_\alpha$. Define $\bar{x}_\alpha \in X^\alpha$ by $\bar{x}_\alpha = \{x_{\delta_{\alpha,i}} : i < n_\alpha\}$. By a series of counting arguments we may assume

(i) for some fixed n, each $n_\alpha = n$

(ii) for each $i < n$ there is $j_i < 2$ so $\sigma_\alpha(\delta_{\alpha,i}) = j_i$ for all α.

Define $\tau_{\alpha,i} = \{\langle\beta_\alpha, j_i\rangle\}$ and define the basic open set u_α in X^n by $u_\alpha = \Pi_{i<n}[\tau_{\alpha,i}]$. By (ii) and duality $\bar{x}_\beta \in u_\alpha$ iff $y_\alpha \in [\sigma_\beta]$ so $\{\bar{x}_\beta : \beta < \omega_1\}$ is a left separated subspace of X^n, which is a contradiction.

Notice that in 3.9 in order to handle just X^* we had to know that X was strong S. "X is an S-space iff X^* is an L-space" is false.

There are many other canonical reductions of S and L, some elegant, few useful. One that has proven to be quite powerful is the canonical form for first countable S-spaces (HAJNAL and JUHÁSZ [1973]). It reduces the question of the existence of first countable S spaces to the existence of a special family of subsets of ω —a considerable improvement over families of functions in 2^{ω_1}.

4. CH constructions

Under CH it is easy to construct S- or L-spaces with various properties. In this section we give two basic inductive techniques and one topological trick. Readers wishing to test their understanding of the inductive constructions are invited to modify them in two ways: to get strong S- or L-spaces (easy); to get S- or L-spaces whose duals are not, respectively, L or S (harder).

Left out are many important and beautiful techniques. Among them are the van Douwen and Kunen constructions in $\mathscr{P}(\omega)$ (VAN DOUWEN and KUNEN [1982]) and the use of direct limits (FEDORCHUK [1976] using $\diamondsuit$, KUNEN [1981] using CH; Kunen informs me that Haydon has a similar construction for a different purpose). Kunen's direct limit construction can be found in Gardner and Pfeffer's article in this volume.

4.1. *The Kunen line.* Appearing in JUHÁSZ, KUNEN, RUDIN [1976] this construction has popularly become known as the KUNEN line (although it is not a linearly ordered space). It gives an example, under CH, of a first countable S-space. HAJNAL and JUHÁSZ [1973b] had already found such a CH example but their construction was difficult. Ostaszewski, motivated by other considerations, had constructed a first countable locally compact countably compact S space using $\diamondsuit$ (OSTASZEWSKI [1976]; his construction appears in 5.3). The Kunen line is an easier version of the Ostaszewski space and also is locally compact.

The idea of the Kunen line is to refine the usual topology on the reals so that the closure of a countable set in the new topology differs from the closure in the usual topology by a countable set. To find a countable dense subset of an uncountable subspace, then, it suffices to take a countable subset dense in the old topology and add at most countably many points to it. So the new space is hereditarily separable. Make it right-separated and 0-dimensional and you're done.

Let's do it. Assume CH and let $\{x_\alpha\colon \alpha<\omega_1\}$ enumerate $\mathbb{R}$. Let $\{A_\alpha\colon \alpha<\omega_1\}$ enumerate all countably infinite subsets of $\mathbb{R}$. Define $X_\alpha=\{x_\beta\colon \beta<\alpha\}$ and let $\mathscr{B}_\alpha=\{A_\beta\colon \beta<\alpha,\ A_\beta\subset X_\alpha, x_\alpha\in \mathrm{cl}_{\mathbb{R}}(A_\beta)\}$. Enumerate each $\mathscr{B}_\alpha$ as $\{C_{\alpha,n}\colon n<\omega\}$ where each $A\in\mathscr{B}_\alpha$ appears infinitely often in the enumeration of $\mathscr{B}_\alpha$.

We will construct a sequence of 0-dimensional topologies $\{\tau_\alpha\colon \alpha<\omega_1\}$ where each τ_α is a topology on X_α and if $\beta<\alpha$ then τ_α is a conservative extension of τ_β. If $\tau=\Sigma_{\alpha<\omega_1}\tau_\alpha$ as in Section 2.3, then each X_α is open under τ, so $(\mathbb{R},\tau)$ is right separated in type ω_1; and since each τ_α is 0-dimensional so is τ.

Constructing the τ_α's. The induction hypothesis at stage $\alpha<\omega_1$ is

(i) if $\beta<\alpha$, then τ_β is a 0-dimensional topology refining the usual topology (as a subspace of $\mathbb{R}$) on X_β

(ii) if $\gamma<\beta<\alpha$, then $\tau_\gamma=\tau_\beta|X_\gamma$

(iii) if $\beta+1<\alpha$ and $A\in\mathscr{B}_\beta$, then $x_\beta\in \mathrm{cl}_{\tau_{\beta+1}}(A)$.

If α is a limit, let $\tau_\alpha = \Sigma_{\beta<\alpha}\tau_\beta$. It is easy to see that the induction hypothesis still holds.

Now suppose $\alpha = \beta + 1$. Pick a sequence $\{y_n: n \in \omega\}$ so each $y_n \in C_{\alpha,n}$ and the distances ε_n between x_β and y_n form a sequence decreasing to zero. Let $\{I_n: n < \omega\}$ be a pairwise disjoint collection of intervals in $\mathbb{R}$ with $x_\beta \notin \mathrm{cl}_{\mathbb{R}}(I_n)$ and $y_n \in I_n$. For each n pick $u_{n,\beta} \in \tau_\beta$, $u_{n,\beta}$ a clopen (in τ_β) neighborhood of y_n contained in I_n. Define

$$v_{n,\beta} = \{x_\beta\} \cup \bigcup_{k>n} u_{k,\beta}$$

and let τ_α be the topology generated by $\tau_\beta \cup \{v_{n,\beta}: n < \omega\}$.

Checking the induction hypothesis we see that (ii) is immediate, (iii) follows from the fact that each $A \in \mathcal{B}_\beta$ appears as infinitely many $C_{\beta,n}$'s, and the second half of (i) is immediate. We have to show that τ_α is 0-dimensional.

Let $\mathcal{U}$ be a clopen basis for τ_β containing no set of the form $\bigcup_{n\in A} v_{n,\beta}$ where A is infinite. $\mathcal{U}$ exists because τ_β refines the usual topology on $\mathbb{R}$. Then $\mathcal{U}' = \mathcal{U} \cup \{u \cap v_{n,\beta}: n < \omega,\ u \in \mathcal{U}$ or $u = X_\alpha\}$ is a basis for τ_α. Suppose $x \in X_\alpha$, $v \in \mathcal{U}'$, and $x \notin v$. We have to find a neighborhood of x avoiding v.

Case 1: *$x \in X_\beta$ and $v \in \mathcal{U}$.* Since τ_β is 0-dimensional we are done.

Case 2: *$x \in X_\beta$ and $v = u \cap v_{n,\beta}$ for some n.* Suppose x lies to the left of x_β in the usual ordering on $\mathbb{R}$. Pick k so that if $s = \inf_{\mathbb{R}} \bigcup_{j>k} I_j$ then $s \neq x$. Since τ_β refines the usual topology on $\mathbb{R}$ we can find w a neighborhood of x in τ_β so $w \cap (u \cap \bigcup_{n<j<k} u_{j,\beta}) = \emptyset$, $w \subset (-\infty, s)$, so we are done. If x lies to the right of x_β the proof is similar.

Case 3: *$x = x_\beta$, $v \in \mathcal{U}$.* By hypothesis, some $v_{n,\beta}$ misses v and we are done.

By construction, each τ_α is first countable, so τ is.

Note that by adding a further induction hypothesis (that each τ_β is locally compact) and requiring the $v_{n,\beta}$'s to be clopen compact, that final space is locally compact.

4.2. *HFD's and HFC's.* HFD's were the second S-spaces to be constructed [HAJNAL and JUHÁSZ 1972b]. They originally were designed for a forcing result about the number of open sets of a Hausdorff space (see Section 5.1). It was quickly realized that in fact they exist under CH (HAJNAL and JUHÁSZ [1974]). The definition of HFD dualizes to a class of spaces known as HFC's (HAJNAL and JUHÁSZ [1972a], HAJNAL and JUHÁSZ [1974]) which give rise to L-spaces. HFD's and HFC's are remarkably flexible constructions. They can be used to provide many counterexamples in topology (e.g. HAJNAL and JUHÁSZ [1972b], HAJNAL and JUHÁSZ [1976], ROITMAN [1980]) and a slight weakening gives further examples (e.g. ROITMAN [1979b], done here in Section 5.2). While HFD's and HFC's do not exist under MA + $\neg$CH, a strengthened HFD gives the ccc-indestructible S-spaces of Szentmiklóssy (see Section 7.1).

4.2.1. *Basic facts about HFD's.* A subset $X \subset 2^{\omega_1}$ is finally dense iff there is some $\alpha < \omega_1$ so that if σ is a finite function from $\omega_1 - \alpha$ into 2, then $[\sigma] \cap X$ is infinite. $X \subset 2^{\omega_1}$ is hereditarily finally dense (HFD) iff every infinite subset of X is finally dense.

Every infinite subspace of an HFD is an HFD.

HFD's are hereditarily separable: suppose by contradiction X is an HFD with a left separated subspace $\{x_\alpha: \alpha < \omega_1\}$, and $\{[\sigma_\alpha]: \alpha < \omega_1\}$ are neighborhoods so each $x_\alpha \in [\sigma_\alpha]$ and $x_\beta \not\in [\sigma_\alpha]$ for $\beta < \alpha$. By a Δ-system and counting argument we may assume the σ_α's have pairwise disjoint domains. But since by left separation $\{x_n: n \in \omega\}$ avoids all but countably many σ_α's and by HFD $\{x_n: n < \omega\}$ meets all but countably many σ_α's we have a contradiction.

Let X be an HFD and for every $x \in X$ assign some $x' \in 2^{\omega_1}$ where $\{\alpha: x'(\alpha) \neq x(\alpha)\}$ is countable. Then $\{x': x \in X\}$ is also an HFD. Thus if there is an HFD there is also a right separated HFD of type ω_1. This is how HFD's give rise to S-spaces.

4.2.2. *Basic facts about HFC's.* An open set is a final cover of an uncountable space X iff it contains all but a countable subset of X. An open set in 2^{ω_1} is nicely split iff it is the countably infinite union of a family $\{[\sigma_i]: i < \omega\}$ where $\{\operatorname{dom} \sigma_i: i < \omega\}$ is pairwise disjoint. $X \subset 2^{\omega_1}$ is defined to be hereditarily finally covered (HFC) iff every nicely split open set in 2^{ω_1} is a final cover of X.

An uncountable subspace of an HFC is an HFC.

By a Δ-system and counting argument similar to the one proving HFD's hereditarily separable, HFC's are hereditarily Lindelöf.

HFC's give rise to L-spaces: suppose $\{\beta_\alpha: \alpha < \omega_1\}$ is an increasing sequence of countable ordinals, $\{x_\alpha: \alpha < \omega_1\}$ is an HFC, and for each $\alpha < \omega_1$ we choose $x'_\alpha \in 2^{\omega_1}$ so $x'_\alpha | \beta_\alpha = x_\alpha | \beta_\alpha$. Then $\{x'_\alpha: \alpha < \omega_1\}$ is an HFC.

4.2.3. *Constructing an HFC.* Assume CH and let $\{u_\alpha: \alpha < \omega_1\}$ enumerate all nicely split open subsets of 2^{ω_1}. For each α pick $\{\sigma_{\alpha,i}: i < \omega\}$ so $u_\alpha = \bigcup\{[\sigma_{\alpha,i}]: i < \omega\}$ and $\{\operatorname{dom} \sigma_{\alpha,i}: i < \omega\}$ is pairwise disjoint. Let $\mathcal{U}_\alpha = \{u_\beta: \beta < \alpha$ and each $\operatorname{dom} \sigma_{\beta,i} \subset \alpha\}$. Enumerate $\mathcal{U}_\alpha$ as $\{v_{\alpha,i}: i < \omega\}$ where each $u \in \mathcal{U}_\alpha$ appears as infinitely many $v_{\alpha,i}$'s. We will construct an HFC $\{x_\alpha: \alpha < \omega_1\}$ by induction, defining $x_\alpha | \alpha$ at stage α and letting $x_\alpha(\beta) = 0$ if $\beta \geqslant \alpha$.

At stage α pick $\{\sigma_i: i < \omega\}$ so that

(i) if $v_{\alpha,i} = u_\beta$, then for some j $\sigma_i = \sigma_{\beta,j}$

(ii) $\{\operatorname{dom} \sigma_i: i < \omega\}$ is pairwise disjoint.

Since the u_β's are nicely split, this is easily done.

Now define $x_\alpha | \alpha$ so $x_\alpha \supset \sigma_i$ for all $i < \omega$. By construction, if δ_β is the first δ so $u_\beta \in \mathcal{U}_\delta$ then $u_\beta \supset \{x_\alpha: \alpha < \delta_\beta\}$ so we are done.

4.2.4. *Constructing an HFD.* Assume CH. We will construct an HFD $\{x_\alpha: \alpha < \omega_1\}$ by induction. At stage α we will determine $x_\beta(\alpha)$ for all $\beta < \alpha$, and declare each

$x_\beta(\gamma)=0$ for $\gamma \leqslant \beta$. (This is dual to the HFC construction where at stage α we decided $x_\alpha(\beta)$ for all $\beta < \alpha$.) Quite a bit of machinery is needed and, unlike the HFC construction, there is a non-vacuous induction hypothesis.

Here's the machinery: Let $\{A_\alpha: \alpha < \omega_1\}$ enumerate all countably infinite subsets of ω_1 and let $B_\alpha = \{\beta < \alpha: A_\beta \subset \alpha\}$. For each $\alpha < \omega_1$ let S_α be the set of all finite functions from α into 2. For each β define $X_\beta = \{x_\gamma: \gamma \in A_\beta\}$. If $\sigma \in S_\alpha$ and $\beta \in B_\alpha$ define $X_{\beta,\sigma} = \{x_\gamma: x_\gamma \in [\sigma] \text{ and } \gamma \in A_\beta\}$. (The $X_{\beta,\sigma}$'s get defined as we go along.) Let $C_\alpha = \{X_{\beta,\sigma}: \beta \in B_\alpha, \alpha \in S_\alpha\} \cup \{X_\beta: \beta \in B_\alpha\}$. The induction hypothesis will guarantee that if $Z \in C_\alpha$, then Z is infinite.

Here's the induction hypothesis at stage α: if $\beta < \alpha$ and $Z \in C_\beta$ then $\{x \in Z: x \in [\{\langle \beta, 0\rangle\}]\}$ and $\{x \in Z: x \in [\{\langle \beta, 1\rangle\}]\}$ are both infinite.

Here's what we do at stage α: Enumerate C_α as $\{Z_i: i < \omega\}$ where each $Z \in C_\alpha$ occurs infinitely often in the list. A quick check of the machinery and the induction hypothesis shows that each $Z \in C_\alpha$ is infinite. Let $\{z_i: i < \omega\}$ be a sequence where

(i) each z_i is some x_β, $\beta < \alpha$;

(ii) if $z_i = x_\beta = z_j$, then $i = j$;

(iii) each $z_i \in Z_i$.

Given $Z \in C_\alpha$ let $\{n_j: j < \omega\}$ enumerate $\{i: Z = Z_i\}$. If $x_\beta = z_{n_j}$ where j is odd, assign $x_\beta(\alpha) = 0$. Otherwise assign, for $\beta < \alpha$, $x_\beta(\alpha) = 1$. The induction hypothesis remains satisfied. By construction, if δ_β is the first δ so $A_\beta \in B_\delta$ then $\{x_\alpha: \alpha \in A_\beta\}$ is dense past δ_β, i.e. $\{x_\alpha: \alpha \in A_\beta\} \cap [\sigma]$ is infinite for every σ a finite function from $\omega_1 - \delta_\beta$ into 2. So we have constructed an HFD.

4.3. *Luzin spaces.* At this point the reader may have the impression that S and L has only the most tenuous relation to topology, and is entirely a matter for combinatorial set theorists. To counteract this impression, here is a sketch of a topological viewpoint which has contributed greatly to L-space theory.

Recall that a nowhere dense set in a topological space is one whose closure has empty interior. Define a Luzin space to be a regular uncountable space with no isolated points in which every nowhere dense set is countable.

Every Luzin space is hereditarily ccc: a discrete subspace of a space with no isolated points is nowhere dense, so a discrete subspace of a Luzin space is countable, and a space is hereditarily ccc iff it has no uncountable discrete subsets.

Every Luzin space is hereditarily Lindelöf [WHITE 1974]: Let X be Luzin, Y an uncountable subset of X, $\mathcal{U}$ a collection of open sets covering Y. Since $\bigcup \mathcal{U}$ is ccc, some countable $\mathcal{V} \subset \mathcal{U}$ has union dense in $\bigcup \mathcal{U}$. Since X is Luzin, $\partial(\bigcup \mathcal{V})$ is countable, so $Y - \bigcup \mathcal{V}$ is countable. So Y is Lindelöf.

Recall that a Baire space is a space in which every countable intersection of dense open sets in dense. We use Baire spaces to construct Luzin spaces (AMIRDZANOV and ŠAPIROVSKIĬ [1974], VAN DOUWEN, TALL, WEISS [1977]).

Assume CH and suppose X is a ccc Baire space with π-weight $\leqslant \omega_1$ and no isolated points. Enumerate some π-basis $\mathcal{B}$ in type ω_1, $\mathcal{B} = \{u_\alpha: \alpha < \omega_1\}$. Note that by ccc every dense open set contains a dense countable union of sets in $\mathscr{B}$, so let $\{V_\alpha: \alpha < \omega_1\}$ enumerate all dense countable unions of sets in $\mathscr{B}$. Construct the subspace $\{x_\alpha: \alpha < \omega_1\}$ so each $x_\alpha \in u_\alpha \cap \bigcap_{\beta<\alpha} V_\beta$ (ensured by X being Baire). Since every nowhere dense subset of X is disjoint from some V_α, this subspace is clearly Luzin.

So under CH the search for L-spaces often ends with the discovery of ccc nonseparable Baire spaces with π-weight ω_1 and no isolated points. Three important examples are the density topology on the real line (TALL [1976]); The Pixley-Roy topology on nonempty compact nowhere dense subsets of $\mathbb{R}$ (VAN DOUWEN, TALL, WEISS [1977]); and the subspace of all functions in 2^{ω_1} with countable support (obvious). The VAN DOUWEN, TALL, WEISS [1977] example gives a first countable L-space.

Non-separable Luzin spaces give L-subspaces, and as a partial converse some L-spaces have Luzin subspaces. The normal Suslin line is a classic example. Recall that a Suslin line is a ccc non-separable linear order; it is called normal if it contains no open set homeomorphic to a subset of reals. (This use of the word normal has nothing to do with the usual topological concept of a normal space.) Given such a line, construct from it a Suslin tree in the standard way (see e.g. JECH [1978] p. 217–218]). Identify branches in the tree with points in the line as usual. A dense set of branches with the property that for each $\alpha < \omega_1$ at most one branch in the set has height α will be a Luzin subspace of the original line, and such a set exists because our line is normal.

Luzin L-spaces are useful for counterexamples. See, e.g., ROITMAN [1978] and ROITMAN [1979b]. The existence of L-spaces does not imply the existence of Luzin spaces: MA + $\neg$CH is consistent with "there are L-spaces" (see Abraham and Todorčević, this volume), but under MA + $\neg$CH there are no Luzin spaces (KUNEN [1976]).

5. Applications

This section presents three applications, chosen mostly for their development of the ideas of §4. In no way are these applications exhaustive. In particular, M.E. Rudin's work on small Dowker spaces (RUDIN [1974a] and JUHÁSZ, KUNEN, RUDIN [1976]; see Rudin's paper in this volume) and the independence of normal nonmetrizable manifolds (RUDIN and ZENOR [1976], RUDIN [1979]; see Nyikos' paper in this volume) are beautiful and important results too long to be included here.

5.1. *The number of open sets.* De Groot's question of how many open sets of Hausdorff space can have (DE GROOT [1965]) focused attention first on spread (= the supremum of the cardinalities of discrete subspaces) and then on S-spaces.

The complete answer under GCH—the number of open sets is either a power of 2 or strongly inaccessible not weakly compact—was essentially given in HAJNAL and JUHÁSZ [1969] and depended on using partition calculus to produce discrete subspaces. To show that radically different answers were possible in other models they constructed [1973a], via forcing, models in which CH held, $2^{\omega_1} > \omega_2$, and there was a hereditarily separable space of size 2^{ω_1}. In these models, if $\kappa \leq 2^{\omega_1}$, there would be a hereditarily separable space X of size κ. If, in addition, $\mathrm{cf}(\kappa) > \omega$, by CH we'd have $\kappa^\omega = \kappa$. So X would have exactly κ many open sets: the number of open sets = the number of closed sets; by hereditary separability the number of closed sets $\leq |X|^\omega$; and by hypothesis $|X|^\omega = \kappa$. Letting $\kappa = \omega_2$, this shows that the number of open sets need not be either a power of 2 or a limit cardinal. The forcing construction used produced, in fact, the first HFD, and the model can be manipulated to make 2^{ω_1} "as large as you want."

Seeing how to force this large HFD after seeing a smaller HFD is a good exercise in the general technique of adapting CH constructions to forcing constructions, so let's sketch the adaptation.

Let λ be a cardinal, $\lambda > \omega_2$, $\mathrm{cf}(\lambda) > \omega_1$. Given a model in which CH holds, we will force an extension which preserves CH, preserves cardinalities, and which contains a space $X = \{x_\alpha : \alpha < \lambda\}$, X an HFD subset of 2^{ω_1}.

The idea is that each forcing condition p should determine a countable set of x_α's on a countable common domain—we let E_p be the indices of the x_α's in the countable set, and D_p be the common domain; and that p should name a countable family M_p of countably infinite subsets of X, each of which is then required to be met by any basic open set whose domain in disjoint from D_p.

The precise definition is: a condition p has the form $\langle F_p, M_p\rangle$, where F_p: $E_p \times D_p \to 2$, $E_p \in [\lambda]^\omega$, $D_p \in [\omega_1]^\omega$, and M_p is a countable subset of $[\lambda]^\omega$. The order is as follows: $p \leq q$ iff $F_p \supseteq F_q$; $M_p \supseteq M_q$; and if $A \in M_q$, $\mathrm{dom}\,\sigma \in [D_p - D_q]^{<\omega}$, and range $\sigma < 2$, then $\{\alpha \in A : \forall \gamma \in \mathrm{dom}\,\sigma\, F_p(\alpha, \gamma) = \sigma(\alpha, \gamma)\}$ is infinite.

The interpretation is that $p \Vdash \mathring{x}_\alpha(\gamma) = F_p(\alpha, \gamma)$, so the last clause in the definition of the partial order translates as: if $p \leq q$ and $A \in M_q$ and σ as above, then $p \Vdash \{\mathring{x}_\alpha : \alpha \in A\} \cap [\sigma]$ is infinite.

This forcing is countably closed: if $p_0 \geq p_1 \geq \cdots \geq p_i \geq \cdots$ for $i < \omega$, then $\langle \bigcup_{i<\omega} F_{p_i}, \bigcup_{i<\omega} M_{p_i}\rangle$ is clearly a condition. Hence this forcing adds no new countable sets of ordinals and, in particular, does not collapse ω_1 and preserves CH. Since CH holds in the ground model, an infinite version of the Δ-system argument shows that ω_2-cc holds, and so cardinals are preserved above ω_1.

A density argument shows that X is an HFD: Every countable subset of λ in the extension is in fact already in the ground model, so if $A \in [\lambda]^\omega$, then $\{p : A \in M_p\}$ is dense. If $A \in M_p$ and $\alpha = \sup D_p$, then for all finite functions σ from $\omega_1 - \alpha$ into 2, $p \Vdash \{\mathring{x}_\alpha : \alpha \in A\} \cap [\sigma]$ is infinite, by definition of the partial order.

5.2. *Adding a Cohen real.* At one time it was conjectured that adding a Cohen real to a model of MA + $\neg$CH preserved MA. This conjecture is false: add one

Cohen real to any model and you get a strong L-space (ROITMAN [1979b]); but a theorem of Kunen (KUNEN [1977]; see Section 6.3 of this paper) says that there are no strong L-spaces under MA + $\neg$CH. Shelah later proved the more difficult theorem that adding a Cohen real adds a Suslin tree but we content ourselves here with the easier construction of strong L-spaces.

Two parenthetical remarks: The strong L-space construction from a Cohen real generalizes easily to get a strong L-space from a random real, but it is not known whether adding a random real adds a Suslin line. Also, to set the record straight yet another time, Jech's article on trees (JECH [1971a]) contains a serious misprint in footnote 5: Tennenbaum's construction of a Suslin tree is not via one Cohen real but via ω_1 Cohen reals (TENNENBAUM [1968]).

Let's proceed with the construction. Recall that the partial order adding one Cohen real consists of all finite functions from ω into 2, where $p \leq q$ iff $p \supseteq q$. The Cohen real is imply the union of the generic filter. Put another way, if $\mathring{x}$ is the term for the generic Cohen real, each $p \Vdash \mathring{x} \supset p$. The salient combinatorial property we want is that if x is Cohen over a model M and $\{\tau_i : i < \omega\} \in V$ is a collection of finite functions from ω into 2 with pairwise disjoint domains, then in the extension $M[x]$, $\{\tau_i : x \in [\tau_i]\}$ is infinite. This is easily proved using a density argument. We also use the fact that every uncountable set of ordinals in an extension by a single Cohen real contains an uncountable set in the ground model. This is because we have only countably many conditions, so if $\mathring{A}$ is a term for an uncountable set of ordinals and G is the generic filter then for some $p \in G\{\alpha : p \Vdash \alpha \in \mathring{A}\}$ is uncountable. But for all p, $\{\alpha : p \Vdash \alpha \in \mathring{A}\}$ is a set in the ground model.

So let M be a ground model and in M let $\{A_\alpha : \alpha < \omega_1\}$ be an almost disjoint family of infinite subsets of ω. For each $\alpha < \omega_1^M$ pick $f_\alpha : \alpha \to A_\alpha$, f_α is 1–1, $f_\alpha \in M$. For convenience, let $\Gamma = \{\alpha < \omega_1^M : \alpha$ a limit but α not a limit of limits$\}$.

Let x be a Cohen real over M, and in $M[x]$ define the functions $\{g_\alpha : \alpha < \omega_1\}$ where each $g_\alpha : \omega_1 \to 2$ by $g_\alpha(\beta) = x \circ f_\alpha(\beta)$ if $\beta < \alpha$; $g_\alpha(\beta) = 0$ otherwise. In $M[x]$ define $X = \{g_\alpha : \alpha \in \Gamma\}$. The claim is that, in $M[x]$, X is a strong L-subspace of 2^{ω_1}.

For the rest of the proof, we remain in $M[x]$.

X is left separated: if $\alpha \in \Gamma$, then $\{n \in A_\alpha : f_\alpha^{-1}(n) > \beta$ for all $\beta \in \Gamma$, $\beta < \alpha\}$ is infinite, so at least one such n has $x(n) = 1$. Call this n_α. The neighborhoods $[\{\langle f_\alpha^{-1}(n_\alpha), 1\rangle\}]$ left separate X.

We must show that each finite product of X is hereditarily Lindelöf. This will follow from the stronger statement:

5.2.1. LEMMA. *Let $n < \omega$ and suppose $U = \{u_i : i < \omega\} \in M$ where each u_i is some $\Pi_{j<n}[\sigma_{i,j}]$ and $\{\bigcup_{j<n} \operatorname{dom} \sigma_{i,j} : i < \omega\}$ is separated (see* Section 2.2 *for the definition of separated). Then if $X_\alpha = \{g_\beta : \alpha < \beta < \omega_1\}$, for some α, $(X_\alpha)^n \subset U$.*

How does Lemma 5.2.1 imply that X^n is an L-space? By a minor variant of 3.3 it suffices to show that no uncountable subset of X^n is discrete. So suppose Y is an uncountable subset of X^n and for each $y \in Y$ let u_y be a basic open set covering y. By a Δ-system and counting argument we may assume that each $y = \langle g_{\beta(y,0)}, \ldots, g_{\beta(y,m)} \rangle$ and each $u_y = \Pi_{i \leq m} (\sigma_{y,i})$ for fixed m where $B = \{\{\beta(y,0), \ldots, \beta(y,m)\} : y \in Y\}$ and $U = \{\bigcup_{i \leq m} \operatorname{dom} \sigma_{i,y} : y \in Y\}$ are separated. (The Δ-system and counting argument may have reduced $m+1<n$ but that's okay.) Since every uncountable set of ordinals in $M[x]$ contains an uncountable set in M, we may further assume that both B and U are in M. But then some countably infinite $U^* \subset U$ is in M. Since U^* and B are separated, by 5.2.1 $\{u_y : \bigcup_{i \leq m} \operatorname{dom} \sigma_{i,y} \in U^*\}$ covers all but countably many elements of Y. So no open covering of Y makes Y discrete.

The only thing left is to prove Lemma 5.2.1.

Let n, U be as in the hypothesis and let $\alpha = \sup \bigcup_{i<\omega} \{\bigcup_{j<\omega} \operatorname{dom} \sigma_{i,j}\}$. Suppose $y = \langle g_{\beta_0}, \ldots, g_{\beta_{n-1}} \rangle \in X^n$ where each $\beta_k > \alpha$. Since the A_β's are pairwise almost disjoint, for all but finitely many $\langle i, j \rangle$

$$(*) \qquad A_{\beta_k} \cap f''_{\beta_{k'}}(\operatorname{dom} \sigma_{i,j}) = \emptyset \quad \text{for all distinct } k, k' < n\,.$$

So assume $(*)$ holds for all $i < \omega$, $j < n$.

Define $c_i = \bigcup_{k<n} f''_{\beta_k}(\operatorname{dom} \sigma_{i,k})$ and define τ_i on c_i by $\tau_i(m) = j$ iff $\sigma_{i,k}(f^{-1}_{\beta,k}(m)) = j$. Each τ_i is well-defined by $(*)$. The τ_i's have pairwise disjoint domains, so by the salient combinatorial fact about Cohen reals, for infinitely many i

$$(**) \qquad \text{if } m \in c_i\,, \quad \text{then } x(m) = \tau_i(m)\,.$$

But $(**)$ holds for i iff $y \in u_i$, so we are done.

A final remark: X is a weak HFC, i.e. every union of uncountably many basic open sets with separated domains contains a countable union of basic open sets which is a final cover. There is a dual construction in $M[x]$ of a strong S-space which is a weak HFD, i.e. every uncountable subset contains a countable finally dense set.

5.3. *The Ostaszewski line.* The Ostaszewski line (again, not a linearly ordered space) is another first countable S-space constructed using the combinatorial principle $\diamondsuit$. From the standpoint of S and L this is not so interesting, since $\diamondsuit$ is a lot stronger than CH. But the Ostaszewski line was not constructed with S-spaces in mind. It is a perfectly normal countably compact space which is not compact. This is interesting to a topologist. For example, such spaces fail to exist under MA + $\neg$CH.

Since the Ostaszewski line is the inspiration for the Kunen line, the construction should be somewhat familiar. Again, a right separated topology of type ω_1 is constructed, so the space will be non-Lindelöf. Again, the final topology is the simple limit of a sequence of topologies. But rather than refining a given topology we have to start from scratch.

The points of our space are identified with the countable ordinals. To give us the extra structure we need without a given topology to refine, $\diamondsuit$ is invoked. We use $\diamondsuit$ in the non-trivially equivalent form $\clubsuit$ + CH, where $\clubsuit$ is the following principle:

($\clubsuit$) There is an increasing sequence $\{\lambda_\alpha: \alpha < \omega_1\}$ of countable limit ordinals and a sequence $\{S_\alpha: \alpha < \omega_1\}$ of countable sets where

(1) each S_α is a cofinal ω-sequence in λ_α;

(2) if $X \in [\omega_1]^{\omega_1}$, then $X \supset S_\alpha$ for some α.

Along with invoking $\clubsuit$ + CH we will continually use the following

Basic Lemma. Suppose X is a locally compact 0-dimensional metrizable space under the topology τ, and A, B are disjoint infinite closed discrete subsets of X. Let Z be a countably infinite set disjoint from X. Then we can put a 0-dimensional locally compact topology τ^* on $Z \cup X$ where τ^* is a conservative extension of τ and every point of Z is a limit point under τ^* of A and B.

PROOF. Let $\{x_n: n < \omega\} \subset X$ be such that $A \supset \{x_n: n \text{ odd}\}$ and $B \supset \{x_n: n \text{ even}\}$. As in the Kunen line use the hypothesis on (X, τ) to construct a pairwise disjoint family $\{u_n: n < \omega\}$ where $x_n \in u_n$ and each u_n is compact clopen. Let $\{P_k: k < \omega\}$ partition ω so each P_k contains infinitely many odd and even integers. Let $Z = \{z_k: k < \omega\}$ and let the basic neighborhoods of z_k have the form $\{z_k\} \cup \bigcup\{u_n: n \in P_k,\ n > m\}$ for each $m < \omega$.

Note that if X is countable the new topology is metrizable.

The actual construction. Assume $\clubsuit$ + CH. Let $\{\lambda_\alpha: \alpha < \omega_1\}$, $\{S_\alpha: \alpha < \omega_1\}$ be as in the definition of $\clubsuit$ and let $\{B_\alpha: \omega \leqslant \alpha < \omega_1\}$ enumerate all countable subsets of ω_1. At stage α, suppose we are given a sequence $\{\tau_\beta: \beta < \alpha\}$ of locally compact 0-dimensional right separated topologies, τ_β is a topology on λ_β, and if $\gamma < \beta < \alpha$ then τ_β is a conservative extension of τ_γ. Let $\alpha^* = \sup\{\lambda_\beta: \beta < \alpha\}$. Let $\tau^* = \sum_{\beta<\alpha} \tau_\beta$.

Case 1. If $\lambda_\alpha = \alpha^*$ let $\tau_\alpha = \tau^*_\alpha$.

Case 2. If α is a limit and $\lambda_\alpha > \alpha^*$, let A be any ω-sequence cofinal in α^* and let $B = B_\gamma$ where γ is the first ordinal indexing a closed discrete infinite subset of α^* under τ^*. By taking subsets if necessary we can assume $A \cap B = \emptyset$. Now invoke the basic lemma to construct τ_α on λ_α so every point of $\lambda_\alpha - \alpha^*$ is a limit of A and of B.

Case 3. If $\alpha = \beta + 1$ for some β, let $A = S_\beta$ and proceed as in the previous case.

The final topology $\tau = \sum_{\alpha<\omega_1} \tau_\alpha$. By construction, the space (ω_1, τ) is locally countable and first countable. An argument similar to 4.1 shows that the space is 0-dimensional. It is countably compact since no countably infinite B can remain closed discrete in all τ_α's. The other properties follow from the

Key Fact. For each α, $\mathrm{cl}(S_\alpha) \supset \{\beta < \omega_1: \beta \geqslant \lambda_\alpha\}$.

PROOF. The induction hypothesis at stage α is that if $\lambda_\gamma \leq \beta < \alpha$ then β is in the closure of S_γ. Assuming this, let $\delta < \alpha$ be the ordinal so that a neighborhood base for α is established at stage δ. Suppose $\lambda_\gamma \leq \alpha$. We must show that α is in the closure of S_γ.

If $\delta = \gamma + 1$ or $\alpha = \lambda_\gamma$ we are in Case 3 and we're done. Otherwise $\lambda_\gamma < \lambda_\delta$ and α is in the closure of an infinite set all but finitely many of whose elements lie above λ_γ. By the induction hypothesis, since each of these is in the closure of λ_γ, so is α, and we're done.

If $X \in [\omega_1]^{\omega_1}$, by ♣ X contains some S_α, so by the key fact X is separable. So (ω_1, τ) is hereditarily separable, and any closed uncountable subset is co-countable.

The space is normal: If two sets H, K are closed disjoint, then one of them, say K, is countable. Let U cover K via compact clopen sets, $\bigcup U \cap H = \emptyset$. By countable compactness there is a finite $V \subset U$, V covers K. But then $\omega_1 - \bigcup V$ is an open cover of H.

Finally, perfect normality is shown by showing that every closed set is a G_δ: a co-countable set in a T_1 space is always a G_δ, and a countable subset of a locally countable T_1 space is a G_δ.

GINSBURG [1977] has a slick variant of this technique where, assuming only ♣, he shows that every regular countably compact space with no non-trivial convergent sequences has a perfect S-subspace. He then applies this construction to countably compact F spaces to get hereditarily extremally disconnected, hereditarily normal, perfect S-spaces. Since his example is so easy, we sketch it here:

Let $\{S_\alpha : \alpha < \omega_1\}$ be a ♣-sequence. Suppose X is regular countably compact with no non-trivial convergent sequence. Suppose we've constructed $\{x_\beta : \beta < \lambda_\alpha\}$ and an open family $\{u_\beta : \beta < \lambda_\alpha\}$ where $x_\beta \in u_\beta - \bigcup_{\gamma<\beta} u_\gamma$ for all $\beta < \lambda_\alpha$. Consider $\{x_\gamma : \gamma \in S_\alpha\} = A_\alpha$. This has uncountably many accumulation points in X, so let $\{x_{\lambda_\alpha + i} : i < \omega\}$ be a discrete subspace of the accumulation points of $A_\alpha - \{x_\beta : \beta < \lambda_\alpha\}$ and let $\{u_{\lambda_\alpha + i} : i < \omega\}$ be an open family where $x_{\lambda_\alpha + i} \in u_{\lambda_\alpha + j}$ iff $i = j$. Continue. Essentially the same proof showing that the Ostaszewski line is hereditarily separable perfect shows that this space is.

6. Destroying S and L using MA + ¬CH

As more and more S- and L-spaces were constructed using combinatorial principles such as CH or ◇ or by forcing, a proof would be found to show that the constructions wouldn't work under MA + ¬CH. In this section we present a brief sampling of these theorems: Silver's lemma (which destroys HFD's and HFC's under MA + ¬CH); Juhász' theorem that under MA + ¬CH there are no compact L-spaces; Kunen's theorem that under MA + ¬CH there are no strong S- or strong L-spaces; and Szentmiklóssy's theorem that under MA + ¬CH there are no compact S-spaces.

6.1. *Silver's Lemma.* Almost as soon as HFD's were invented, Silver observed that they didn't exist under MA + ¬CH. This follows from

6.1.1. LEMMA. *Assume* MA + ¬CH *and suppose* $\{A_n : n < \omega\}$ *is a collection of subsets of* ω_1. *Then for some infinite* $E \subset \omega$ *either* $\bigcap_{n \in E} A_n$ *or* $\bigcap_{n \in E} (\omega_1 - A_n)$ *is uncountable.* (This was also known to Laver.)

PROOF. Define, for each $\alpha < \omega_1$, $B_\alpha = \{n : \alpha \in A_n\}$. By induction construct a sequence $\{X_\alpha : \alpha < \omega_1\}$ of subsets of ω so that if $\alpha < \beta$ then $X_\beta - X_\alpha$ is finite and each X_α is either a subset of B_α or of $\omega - B_\alpha$. By MA + ¬CH (actually by the weaker $\mathrm{MA}_{\sigma\text{-centered}}$—see Weiss' article in this volume) there is some infinite X almost contained in each X_α, i.e. $X - X_\alpha$ is finite for all α. So there is an uncountable $S \subset \omega_1$ and a finite $s \subset \omega$ so that $X - X_\alpha = s$ for all $\alpha \in S$. Without loss of generality assume that $X_\alpha \subset B_\alpha$ for all $\alpha \in S$. Then $S \subset \bigcap\{A_n : n \in X - s\}$. If instead we are forced to assume that $X_\alpha \subset (\omega - B_\alpha)$ for all $\alpha \in S$, then $S \subset \bigcap\{\omega_1 - A_n : n \in X - s\}$.

This lemma destroys HFD's: Let X be a countably infinite subset of 2^{ω_1}. Let F be the set of finite functions from ω_1 into 2 and let $A_x = \{\sigma \in F : x \in [\sigma]\}$. Then there is a countably infinite $Y \subset X$ so either $\bigcap_{x \in Y} A_x$ or $\bigcap_{x \in Y} (F - A_x)$ is uncountable. But then Y is not finally dense.

The lemma destroys HFC's: Suppose X is an HFC and let A be a countably infinite subset of ω_1. By Silver's lemma there is an infinite $B \subset A$ and some $i < 2$ so $X \cap \bigcap_{\alpha \in B} [\{\langle \alpha, i \rangle\}]$ is uncountable. But then $\bigcup_{\alpha \in B} [\{\langle \alpha, 1 - i \rangle\}]$ is not a final cover of X, which is a contradiction.

6.2. *No compact* L-*spaces* (JUHÁSZ [1975]). Assuming MA + ¬CH we'll show that there are no compact L-spaces. Suppose there were a compact L-space. Then there is one which is the closure of a left separated L-space $\{x_\alpha : \alpha < \omega_1\}$. Let X be the compact closure of $\{x_\alpha : \alpha < \omega_1\}$ and for each $\alpha < \omega_1$ let X_α be $\mathrm{cl}_X\{x_\beta : \beta < \alpha\}$. Let $u_\alpha = \mathrm{int}(X_\alpha)$. Since X has no uncountable increasing sequences of open sets, there is an α^* so that $u_\alpha = u_{\alpha^*}$ for all $\alpha \geq \alpha^*$. Let $Y = X - u_{\alpha^*}$. Each $X_\alpha \cap Y$ is nowhere dense in Y. Since X has no uncountable decreasing sequence of closed sets, each point in X is in the closure of some X_α. So $Y = \bigcup_{\alpha < \omega_1} X_\alpha \cap Y$. Thus Y is a compact ccc space which is the union of fewer than 2^ω nowhere dense sets, directly contradicting MA + ¬CH.

6.3. *No strong* S- *or* L-*spaces.* In this section we prove that under MA + ¬CH there are no strong S-spaces [KUNEN 1977]. By Zenor's theorem, this destroys strong L-spaces as well.

First, we define a partial ordering which will also be used in Section 6.4.

6.3.1. DEFINITION. For each $\alpha < \omega_1$ let u_α be a subset of ω_1 with $\alpha \in u_\alpha$. We define a partial order contained in $[\omega_1]^{<\omega}$ which is called the *canonical order* for

$\{u_\alpha: \alpha < \omega_1\}$: its elements are all $a \in [\omega_1]^{<\omega}$ where if α, β are distinct elements of a and $\alpha < \beta$, then $\alpha \notin u_\beta$; the order is reverse inclusion.

Proof of the Theorem. Assume MA + $\neg$CH and suppose X is a strong S space. By Section 3 we may assume X is a 0-dimensional topology on ω_1 where each initial segment of ω_1 is open. For each $\alpha < \omega_1$ choose u_α a clopen neighborhood of α avoiding $\{\beta: \beta > \alpha\}$. Let $\mathbb{P}$ be the canonical order for $\{u_\alpha: \alpha < \omega_1\}$. If $\mathbb{P}$ has ccc, then by MA + $\neg$CH there is an uncountable $D \subset \omega_1$ so that $\{\{\alpha\}: \alpha \in D\}$ is centered in $\mathbb{P}$. But then D is a discrete subspace of X, contradicting that X is an S space.

So we assume that $\mathbb{P}$ is not ccc. Let $\{a_\alpha: \alpha < \omega_1\}$ be an antichain of $\mathbb{P}$. By 2.2 we may assume that a_α's are separated and all have cardinality n. Write $a_\alpha = \{\delta_{i,\alpha}: i < n\}$ and define $v_\alpha = \bigcup_{i<n} u_{\delta_{i,\alpha}}$. Then if $\alpha < \beta$ since a_α is not compatible with a_β there is some i with $\delta_{i,\alpha} \in v_\beta$. Define $\vec{x}_\alpha = \langle \delta_{i,\alpha}: i < n\rangle \in X^n$ and define $w_\alpha = \{x \in X^n: \exists i < n \ (x_i \in v_\alpha)\}$. Since $w_\alpha = \bigcup_{j<n} X^j \times v_\alpha \times X^{n-(j+1)}$, each w_α is clopen. If $\alpha \leq \beta$, then $\vec{x}_\alpha \in w_\beta$ so $\{\vec{x}_\alpha: \alpha < \omega_1\}$ is a left separated subspace of X^n, contradicting that X is a strong S-space.

6.4. *No compact* S-*spaces.* SZENTMIKLÓSSY [1977] proves that under MA + $\neg$CH no compact space of countable tightness has an S-subspace. BALOGH [19··] noticed that SZENTMIKLÓSSY's proof actually gives the following: under MA + $\neg$CH every right separated subspace of a compact space of countable tightness is a countable union of discrete sets (this implies in particular the earlier result that every MA + $\neg$CH every Aronszajn tree is the union of countably many antichains (BAUMGARTNER, MALITZ, REINHARDT [1970]). We are content to prove the simplification.

6.4.1. THEOREM. *Assume* MA + $\neg$CH. *Then there are no compact* S-*spaces.*

Before launching on the proof, some comments:

(1) As a corollary, there are no locally compact S-spaces under MA + $\neg$CH, since the one-point compactification of such a space would be a compact S-space.

(2) SZENTMIKLÓSSY [1977] also proves that under MA + $\neg$CH there are not first countable L spaces. The proof is nearly an exact dual of the proof of 6.4.1. It appears in Abraham and Todorčević's article in this volume.

(3) Szentmiklóssy's 1977 theorems have the corollary that under MA + $\neg$CH there are no compact perfectly normal spaces with S- or L-subspaces. The proof is immediate from the theorem that compact perfectly normal spaces are first countable and the observation that a first countable space has countable tightness.

(4) As important as 6.4.1 is the lemma used to prove it, known as Szentmiklóssy's lemma. This will be a key lemma in our proof of Todorčević's theorem that "there exist no S-spaces" is consistent (TODORČEVIĆ [1981]; see section 7.2 of this paper.)

(5) The proof of the theorem that under MA + ¬CH every compact space of countable tightness contains no S-subspace differs from the proof of 6.4.1 by a single phrase. The reader familiar with countable tightness will immediately see how to adapt the proof.

Proof of 6.4.1. By the comment after 3.5 it suffices to prove the following: if (X, τ) is the compact closure of a space whose points are identified with ω_1, then $(\omega_1, \tau|\omega_1)$ is not an S-space. By 3.1, we may assume that every initial segment of ω_1 is open in $\tau|\omega_1$ (recall this notation from section 2.3).

We begin dually to the proof of 6.2: let $X_\alpha = \text{cl}_X(\omega_1 - \alpha)$ and let $F = \bigcap_{\alpha<\omega_1} X_\alpha$. If $\alpha < \beta$, then $X_\alpha \supset X_\beta$. Any compact $K \subset X - F$ has empty intersection with some X_α. Let $\mathscr{B} = \{K \cap \omega_1 : K \subset X - F$ and K compact$\}$. The proof now proceeds in three steps:

Step 1: to show that $\mathscr{B}$ has the combinatorial property P_l (defined below in 6.4.2)

Step 2: to show that any family with P_l has an uncountable sequence with special properties (to be defined in 6.4.5)

Step 3: to show that the uncountable sequence of Step 2 leads to a contradiction.

Step 1. We define P_l and show that under MA + ¬CH $\mathscr{B}$ satisfies it.

6.4.2. DEFINITION. Let $\mathscr{C}$ be a family of subsets of ω_1. We say that $\mathscr{C}$ has P_l iff there is a finite n and a sequence $\{\langle a_\alpha, B_\alpha\rangle : \alpha < \omega_1\}$ where

(1) each $a_\alpha \in [\omega_1]^n$,

(2) each $B_\alpha \in \mathscr{C}$,

(3) if $\alpha < \beta$, then $a_\alpha < a_\beta$,

(4) each B_α is countable,

(5) if $\alpha < \beta$, then $a_\alpha \cap B_\beta \neq \emptyset$.

Remark. P_l has a dual, P_r, which changes (4) to (4′): each $B_\alpha \subset \omega_1 - \alpha$; and changes (5) to (5′): if $\alpha < \beta$, then $B_\alpha \cap a_\beta \neq \emptyset$. The conclusion of Lemma 6.4.6 below also holds if a family satisfies P_r. This is the key to the proof that under MA + ¬CH there is no first countable L-space.

6.5.3. LEMMA. *Assume* MA + ¬CH *and let ρ be a topology on ω_1 with no uncountable discrete subspaces. Suppose for every $\alpha < \omega_1$ we have assigned a countable closed neighborhood u_α where $\alpha \in u_\alpha$. Let $\mathscr{U} \supset \{u_\alpha : \alpha < \omega_1\}$ where if $u, v \in \mathscr{U}$, then $u \cup v \in \mathscr{U}$. Then $\mathscr{U}$ satisfies P_l.*

PROOF. Let $\mathbb{P}$ be the canonical order for $\{u_\alpha : \alpha < \omega_1\}$ as in Section 6.3. By the proof of 6.3, since ω_1 under ρ has no uncountable discrete subspaces, $\mathbb{P}$ does not have ccc. Let $\{a_\alpha : \alpha < \omega_1\}$ be an uncountable antichain in $\mathbb{P}$. By the usual Δ-system and counting argument we may assume the a_α's all have the same cardinality n and that if $\alpha < \beta$, then $\bigcup\{u_\delta : \delta \in a_\alpha\} < a_\beta$; in particular the a_α's are separated. By incompatibility, if $\alpha < \beta$, then for some $\delta \in a_\alpha$ there is $\gamma \in a_\beta$ with $\delta \in u_\gamma$. Define $B_\alpha = \bigcup\{u_\gamma : \gamma \in a_\alpha\}$ for all α. Then $\{\langle a_\alpha, B_\alpha\rangle : \alpha < \omega_1\}$ satisfies P_l.

6.5.4. COROLLARY. *Assume* MA + ¬CH. $\mathscr{B}$ *satisfies* P_l.

PROOF. Just assign to each α some compact closed countable neighborhood $u_\alpha \in \mathscr{B}$.

In the proof that "there are no S-spaces" is consistent (see Section 7.2) we will use Lemma 6.5.3 in the following restated form:

SZENTMIKLÓSSY'S LEMMA. *For every* $\alpha < \omega_1$ *let* u_α *be a countable set containing* α. *Let* $\mathscr{U} \supset \{u_\alpha : \alpha < \omega_1\}$ *where if* $u, v \in \mathscr{U}$, *then* $u \cup v \in \mathscr{U}$. *Let* $\mathbb{P}$ *be the canonical order for* $\{u_\alpha : \alpha < \omega_1\}$. *If* $\mathbb{P}$ *fails to have* ccc, *then* $\mathscr{U}$ *has* P_l.

Note that in the form Szentmiklóssy's lemma is a theorem of ZFC.

Step 2. Now we have to use P_l to prove a stronger property of $\mathscr{B}$.

6.4.5. DEFINITION. A collection $\{B_\alpha : \alpha < \omega_1\}$ is *cofinally centered* on a set A iff for every uncountable $C \subset A$ there is an α so $\{B_\beta \cap C : \beta \geqslant \alpha\}$ is centered.

6.4.6. LEMMA. *Suppose* $\mathscr{C}$ *is a collection of countable subsets of* ω_1, *every finite union of elements of* $\mathscr{C}$ *is in* $\mathscr{C}$, $\mathscr{C}$ *has* P_l. *Then there is an uncountable* $A \subset \omega_1$ *and a collection* $\{B_\alpha : \alpha < \omega_1\} \subset \mathscr{C}$ *which is cofinally centered on* A.

PROOF. Let n be the smallest integer so some $\{\langle a_\alpha, B_\alpha\rangle : \alpha < \omega_1\}$ with each $|a_\alpha| = n$ and $B_\alpha \in \mathscr{C}$ satisfies the definition of P_l. Let $A = \{\inf a_\alpha : \alpha < \omega_1\}$. Suppose the B_α's are not cofinally centered on A. Let C be a counterexample, i.e. $C \in [A]^{\omega_1}$ and for each $\alpha < \omega_1$ there is some finite $b_\alpha \subset \omega_1 - \alpha$ so

$$(*) \qquad C \cup \bigcap_{\beta \in b_\alpha} B_\beta = \emptyset$$

Let $\{\delta_\alpha : \alpha < \omega_1\}$ be the increasing enumeration of C and define $a^*_\alpha = a_\gamma$ iff $\delta_\alpha = \inf a_\gamma$. Then $a^*_\alpha < a^*_\beta$ for $\alpha < \beta$ (notation as in Section 2.2). By skipping through C we may assume each $b_\alpha < \delta_\alpha$ and that if $\alpha < \beta$, then $a^*_\alpha < b_\beta$.

Define $a'_\alpha = a^*_\alpha - \{\delta_\alpha\}$, $B'_\alpha = \bigcup\{B_\gamma : \gamma \in b_\alpha\}$. If $\alpha < \beta$, then by (*) there is some $\gamma \in b_\beta$ with $\delta_\alpha \notin B_\gamma$. But $\gamma \in B_\beta > a^*_\alpha$ so $\emptyset \neq a'_\alpha \cap B_\gamma \subset a'_\alpha \cap B'_\beta$. Thus $\{\langle a'_\alpha, B'_\alpha\rangle : \alpha < \omega_1\}$ satisfies the definition of P_l. contradicting the minimality of n.

An immediate corollary is that under MA + ¬CH $\mathscr{B}$ contains an uncountable sequence $\{B_\alpha : \alpha < \omega_1\}$ cofinally centered on some $A \in [\omega_1]^{\omega_1}$.

Step 3. The final task is to use the conclusion of Step 2 to derive a contradiction. So assume MA + $\neg$CH and suppose we're given an uncountable $A \subset \omega_1$ and a sequence $\{B_\alpha : \alpha < \omega_1\} \subset \mathscr{B}$ where the sequence is cofinally centered on A. By definition of $\mathscr{B}$ each B_α is some $K_\alpha \cap \omega_1$ where K_α is a compact subset of $X - F$. Let $H_\alpha = \bigcap_{\beta > \alpha} K_\beta$ and let $H = \bigcup_{\alpha < \omega_1} H_\alpha$. Each H_α avoids F, so $H \cap F = \emptyset$. Since X is hereditarily separable, $H = \mathrm{cl}_X(H)$, so H is compact. Let δ be such that $H \cap X_\delta = \emptyset$ and let $C = X_\delta \cap A = A - \delta$. Let α be such that $\{B_\beta \cap C : \beta > \alpha\}$ is centered. Then $\{K_\beta \cap C : \beta > \alpha\}$ is centered, and since $C \subset X_\delta$, $\{K_\beta \cap X_\delta : \beta > \alpha\}$ is centered. By compactness $\emptyset \neq \bigcap_{\beta > \alpha} K_\beta \cap X_\delta = X_\delta \cap \bigcap_{\beta < \alpha} K_\beta = X_\delta \cap H_\alpha \subset X_\delta \cap H$, which is a contradiction.

7. Two theorems about S-spaces

This final section gives proofs of two of the most interesting theorems about S-spaces. In Section 7.1 we prove the consistency of "MA + $\neg$CH + S spaces exist", a theorem of Szentmiklóssy; in Section 7.2 we prove the consistency of "there are no S-spaces" (TODORČEVIĆ [1981]). See the article by Abraham and Todorčević in this volume for the proof of the consistency of "MA + $\neg$CH + L-spaces exist." The consistency of "there are no L-spaces" is not known.

7.1. Ccc-*indestructible spaces.* This section proves Szentmiklóssy's 1978 theorem: "MA + $\neg$CH + S-spaces exist" is consistent. The proof proceeds in three steps:

Step 1. If you want to use ccc forcing to force a right separated S-space to no longer be S, there is only one partial order you need consider. More formally, given an uncountable space X in a model M there is a canonical ordering $\mathbb{P}_X \in M$ (to be defined below) so that if there is some ccc partial ordering which adds an uncountable discrete subset to X, then $\mathbb{P}_X$ has such a sub-order in M. (Recall that by 3.3 an uncountable right separated space is S iff it has no uncountable discrete subspaces.)

Step 2. There is a combinatorial property so tight that if a space X satisfies it then $\mathbb{P}_X$ has no ccc subsets forcing an uncountable discrete subspace of X.

Step 3. A space with this combinatorial property exists under CH.

Szentmiklóssy's theorem immediately follows: Begin with a model of CH and construct the space X of Step 2 in it. Extend to a model of MA + $\neg$CH by ccc forcing. If X is not S in the extension, then by Step 1 $\mathbb{P}_X$ has a ccc subset in the ground model forcing an uncountable discrete subspace of X. But this contradicts the definition of X.

7.1.1. *Step* 1: *The canonical order.* Let's start Step 1 by defining the canonical order $\mathbb{P}_X$. If X is a space, then $\mathbb{P}_X$ is the set of maps from finite subsets of X into the set of its open sets where if $p \in \mathbb{P}_X$, then $x \in p(y)$ iff $x = y$. Order is reverse inclusion. The idea is that each p verifies that its domain is discrete, and that if G is a filter in $\mathbb{P}_X$, then $\bigcup_{p \in G} \operatorname{dom} p$ is discrete.

If $A \subset \mathbb{P}_X$ define $\bar{A}$ to be the algebraic closure of A under finite unions of mutually compatible conditions. i.e. $A \subset \bar{A}$; if p, q, are compatible elements of $\mathbb{P}_X$ in $\bar{A}$, then $p \cup q \in \bar{A}$; and $\bar{A}$ is the smallest such order.

7.1.2. DEFINITION. Let M be a model of set theory, X a space in M. We say that X is ccc-*indestructible* in M iff, for all ccc $\mathbb{Q} \in M$, $M^{\mathbb{Q}} \models X$ has no uncountable discrete subspaces.

The next lemma shows that to verify ccc-indestructibility it suffices to look at the canonical order.

7.1.3. LEMMA. *Let M be a model, X a space in M. X is* ccc-*indestructible in M iff no $\bar{A} \in [\mathbb{P}_X]^{\omega_1} \cap M$ has* ccc.

PROOF. Suppose some $\bar{A} \in [\mathbb{P}_X]^{\omega_1} \cap M$ had ccc. Since A is uncountable ccc, without loss of generality $|\bar{A}| = \omega_1$ and every $p \in \bar{A}$ is compatible with uncountably many elements of $\bar{A}$. Let $\{x_\alpha : \alpha < \omega_1\}$ enumerate $\bigcup_{p \in \bar{A}} \operatorname{dom} p$. Then for each $\alpha < \omega_1$ $\{p \in \bar{A} : \exists \beta > \alpha\ (x_\beta \in \operatorname{dom} p)\}$ is dense.

Now for the other direction. Suppose $\mathbb{Q}$ is a ccc order in M and $M^{\mathbb{Q}} = X$ has an uncountable discrete set. Working in $M^{\mathbb{Q}}$, let Y be this uncountable discrete set and note that $\{u \in M : u$ is an open subset of $X\}$ is a basis for the topology on X. So there is (still in $M^{\mathbb{Q}}$) a function $f: Y \to \{u \in M : u$ is an open subset of $X\}$ where $x \in f(y)$ iff $x = y$. i.e., f verifies that Y is discrete. For the rest of this proof we work in M. Let $A = \{p \in \mathbb{P}_X \cap M : \text{for some } q \in \mathbb{Q},\ q \Vdash p \subset \mathring{f}\}$. A is uncountable because $M^{\mathbb{Q}} \models f$ is uncountable. If $\bar{A}$ is not ccc there is $\{p_\alpha : \alpha < \omega_1\}$ an uncountable antichain of A. For each $\alpha < \omega_1$ pick $q_\alpha \in \mathbb{Q}$ with $q_\alpha \Vdash p_\alpha \subset \mathring{f}$. Then $\{q_\alpha : \alpha < \omega_1\}$ is pairwise incompatible in $\mathbb{Q}$, which is a contradiction, so $\bar{A}$ is ccc.

In practice, it is convenient to dispose of reference to the model M. Thus we say "X is ccc-indestructible" in contexts in which the model is unambiguous. Lemma 7.1.3 then reads: X is ccc-indestructible iff no uncountable $\bar{A} \subset \mathbb{P}_X$ has ccc. This convention is used throughout Step 2, where we assume that we are working in a fixed model.

This completes Step 1.

7.1.4. *Step* 2: *tight HFD's*. Here we define the class of spaces THFD's. Every THFD will be ccc-indestructible, and the existence of a THFD will imply the existence of a right separated THFD S-space.

7.1.5. DEFINITION. Let A be a countable well-ordered set. A subset $B \subset A$ is *tight* in A iff for some $n < \omega$ every set of n consecutive members of A contains a member of B. We say that n *witnesses* the tightness of B in A. Example: the even positive integers are tight in the positive integers ($n = 2$).

We use the following convention: if X is enumerated as $\{x_\alpha: \alpha < \omega_1\}$ and $A \subset X$, then the order type of A is the order type of $\{\alpha: x_\alpha \in A\}$. We say that A is a limit set iff its order type is a limit.

7.1.6. DEFINITION. Let $X = \{x_\alpha: \alpha < \omega_1\} \subset 2^{\omega_1}$. X is a *THFD* iff for every countable limit set $A \subset X$ there is a countable ordinal α so if σ is a finite function from $\omega_1 - \alpha$ into 2, then $A \cap [\sigma]$ is tight in A.

Note that a THFD is an HFD, and that by the same proof as in 4.2.1 if there is a THFD, then there is a THFD which is a right separated S-space.

7.1.7. THEOREM. *If X is a* THFD, *then X is* ccc-*indestructible.*

PROOF. Suppose $X = \{x_\alpha: \alpha < \omega_1\}$ is a THFD and suppose A is an uncountable subset of $\mathbb{P}_X$. We must find an uncountable antichain in $\bar{A}$.

Since $X \subset 2^{\omega_1}$ we may assume without loss of generality that if $p \in \bar{A}$ then each $p(x)$ is some $[\sigma]$. If uncountably many $p \in A$ are each compatible with at most countably many elements of A we're done. So we may assume that each element of A is compatible with uncountably many elements of A. We may also assume that $|A| = \omega_1$. If $p \in A$ define field $p = \{\alpha: x_\alpha \in \operatorname{dom} p$ or $[\sigma] \in \operatorname{range} p$ with $\alpha \in \operatorname{dom} \sigma\}$. Since each field p is finite, by a Δ-system and counting argument we may assume that $A = \{p_\alpha: \alpha < \omega_1\}$ where $\{\text{field } p_\alpha: \alpha < \omega_1\}$ is separated.

Let's say that α works for $C \subset X$ iff $C \cap [\sigma]$ is tight for all finite σ with $\operatorname{dom} \sigma > \alpha$.

Now we construct an uncountable sequence $\{q_\alpha: \alpha < \omega_1\}$ of conditions and an uncountable sequence of ordinals $\{\alpha_\lambda: \lambda$ a countable limit ordinal$\}$. The construction is designed to allow us to extract an uncountable antichain from the q_α's.

Let $q_0 = p_0$. For each $i < \omega$ pick $q_{i+1} \in A$ so $\bigcup_{j \leq i} q_i \in A$ and each field $q_{i+1} >$ field q_i. (We can do this because $\{\text{field } p_\alpha: \alpha < \omega_1\}$ is separated and each p_α is compatible with uncountably many p_β's.) Let α_ω be an ordinal, $\alpha_\omega > \bigcup_{i<\omega}$ field q_i, α_ω works for $\bigcup_{i<\omega} \operatorname{dom} q_i$.

Now suppose we have $\{q_\alpha: \alpha < \lambda\} \subset A$ where λ is a countable limit ordinal. Pick $\alpha_\lambda > \bigcup_{\alpha<\lambda}$ field q_α so α_λ works for $\bigcup_{\alpha<\lambda} \operatorname{dom} q_\alpha$. Pick $\{q_{\lambda+1}: i < \omega\}$ so each field $q_{\lambda+i+1} >$ field $q_{\lambda+i} > \alpha_\lambda$ and $\bigcup_{j \leq i} q_{\lambda+j} \in A$. Continue through all countable limit λ's.

Consider the sequences $\{q_\alpha: \alpha < \omega_1\}$, $\{\alpha_\lambda: \lambda$ a countable limit$\}$ we've just constructed. Pick $[\sigma_\lambda] \in \operatorname{range} q_\lambda$ for each limit $\lambda < \omega_1$ and let n_λ witness the tightness of $[\sigma_\lambda] \cap \bigcup_{\alpha<\lambda} \operatorname{dom} q_\lambda$ in $\bigcup_{\alpha<\lambda} \operatorname{dom} q_\lambda$. Without loss of generality, by passing to an uncountable subset, all $n_\lambda = n$. Define $s_\lambda \in \bar{A}$ by $s_\lambda = \bigcup_{i<n} q_{\lambda+i}$, for λ a countable limit. Then the s_λ's are pairwise incompatible by the definition of the α_λ's, since if $\lambda < \lambda'$, then $\operatorname{dom} s_\lambda \cap [\sigma_{\lambda'}] \neq \emptyset$.

Step 2 is finished.

7.1.8. THFD's *under* CH. The task of Step 3 is to produce a THFD under CH. This will proceed by an adaptation of the construction of an HFD, invoking the following combinatorial lemma:

7.1.9. LEMMA. *If* $\{A_n: n<\omega\}$ *is a collection of countable limit subsets of some ordinal, then there are disjoint* B_1, B_2 *with each* $B_i \cap A_n$ *tight in* A_n.

PROOF. The proof given is due to C. Mills. List each A_n in increasing order as $\{\alpha^n_\gamma: \gamma<\lambda_n\}$ and let $D_{2n}=\{\alpha^n_\gamma: \gamma \text{ is even}\}$ and $D_{2n+1}=\{\alpha^n_\gamma: \gamma \text{ is odd}\}$. It suffices to find pairwise disjoint C_n, $n<\omega$, so each C_n is tight in D_n; for then letting $B_0=\bigcup\{C_n: n \text{ is even}\}$, $B_1=\bigcup\{C_n: n \text{ is odd}\}$ does the trick.

Invoke Hall's theorem: if $\mathscr{A}$ is a family of finite sets so that $|\bigcup\mathscr{F}|\geqslant|\mathscr{F}|$ for each finite $\mathscr{F}\subset\mathscr{A}$, then $\mathscr{A}$ has a transveral (=a 1–1 choice function).

So given D_n break it into consecutive blocks of size 2^{n+1}. There are only countably many of these blocks. List them arbitrarily as $\{D_{n,k}: k<\omega\}$. Let $\mathscr{A}=\{D_{n,k}: n<\omega, k<\omega\}$. We show that $\mathscr{A}$ satisfies the hypothesis of Hall's theorem. If $\mathscr{F}$ is a finite subset of $\mathscr{A}$ let $a=\{n: \exists k<\omega\ (D_{n,k}\in\mathscr{F})$ and for $n\in a$ let $i_n=|\{k: D_{n,k}\in\mathscr{F}\}|$. Then

$$|\mathscr{F}|\leqslant\sum_{n\in a} n\cdot i_n\leqslant\sum_{n\in a}2^n\cdot i_n\leqslant|\bigcup\mathscr{F}|\,.$$

So there is a transversal g of $\mathscr{A}$ where $g(D_{n,k})\in D_{n,k}$ for all n, k. Let $C_n=\bigcup_{k<\omega}g(D_{n,k})$. Then C_n is tight for D_n with 2^{n+2} witnessing the tightness.

7.1.10. THEOREM. *Assume* CH. *Then there is a* THFD.

Sketch of Proof. The proof follows the construction in 4.3.4 so closely that we refer the reader back to 4.3.4 for the (rather cumbersome) notation. Recall that in 4.3.4 at stage α we defined $x_\beta(\alpha)$ for $\beta<\alpha$ so each $[\{\langle\alpha, i\rangle\}]$ intersected each $Z\in C_\alpha$ infinitely often. Now the definition of C_α should be modified slightly to insure that each $Z\in C_\alpha$ is a limit set. At stage α apply Lemma 7.1.9 to C_α to get B_0, B_1 as in the lemma. Require $\{x_\beta: \beta\in B_i\}\subset[\{\langle\alpha, i\rangle\}]$. A routine induction argument left to the reader completes the proof.

7.2. *No* S-*spaces.* Finally, let's prove Todorčević's theorem:

7.2.1. THEOREM. (TODORČEVIĆ [1981]). *It is consistent that there are no* S-*spaces.*

The proof given here is so slick that the reader may well wonder why the question was ever thought difficult. The answer is that it is difficult. Historically, Todorčević proved the consistency of no S-spaces directly, namely forcing each S-space to no longer be S in an iterable fashion (his original write-up gives a nod to the proper forcing axiom, but only a nod[1]). Baumgartner almost immediately found an easier proof by using a combinatorial principle (called G_2) which he had proved consistent just two weeks earlier. Then G_2 got strengthened to TOP which, coincidentally, also strengthens the heart of Todorčević's argument. TOP is so close to what must happen if S-spaces don't exist that we are left with the present short proof. Our proof entirely hides the difficult part of the argument, which is contained in the proof of

7.2.2. THEOREM (Baumgartner). *The following is consistent*: $\mathrm{MA} + \neg\mathrm{CH} +$ *the Thinning-out Principle* (TOP): *if Z, B are uncountable subsets of ω_1 and $\{S_\alpha : \alpha \in B\}$ is a collection cofinally centered on Z with each $S_\alpha \subset \alpha$, then there is an uncountable $Y \subset Z$ where $(Y \cap \alpha)$-S_α is finite for all $\alpha \in B$.* (For a definition of cofinally centered see section 6.4.5.)

The topological implication of TOP we will be interested in is that if each S_α is closed, for $\alpha \in B$, then each $Y \cap \alpha$ is relatively closed, for $\alpha \in B$, since $Y \cap \alpha = (Y \cap S_\alpha) \cup F$ where F is finite hence closed.

Baumgartner's article in this volume proves that PFA (the proper forcing axiom) implies TOP. Since PFA also implies $\mathrm{MA} + \neg\mathrm{CH}$ this would give the proof of 7.2.2 were it not for the awkward fact that PFA also implies the consistency of large cardinals. But PFA can be avoided, as Baumgartner indicates in his Section 9. The reader is invited to check Section 5 of Baumgartner's paper directly to see that the forcing used does not collapse cardinals and can be dovetailed with ccc forcing in an iteration of a length ω_2 in order to get a direct proof of 7.2.2.

We will thus be finished by proving

7.2.3. THEOREM. $\mathrm{MA} + \neg\mathrm{CH} + \mathrm{TOP} \to$ *there are no S-spaces.*

PROOF. Let X be a right-separated 0-dimensional topology on ω_1. We use the terminology of Section 7.1. If X is not ccc-indestructible then some ccc subset $\mathbb{Q}$ of $\mathbb{P}_X$ exists which, via $\mathrm{MA} + \neg\mathrm{CH}$ and judicious use of $\aleph_1$ many dense sets of $\mathbb{Q}$, gives rise to an uncountable discrete subset of X. So X is not an S space.

Now suppose X is ccc-indestructible. Then by Szentmiklóssy's lemma (see Section 6.4) there is an uncountable set $Z \subset X$ and a collection of countable clopen sets $\{u_\alpha : \alpha < \omega_1\}$ cofinally centered on Z. Letting $B = \{\beta_\alpha : \alpha < \omega_1\}$ where each $\beta_\alpha > \sup u_\alpha$ and the β_α's are increasing, and letting each $S_{\beta_\alpha} = u_\alpha$, we have the hypothesis of TOP. So there is an uncountable $Y \subset Z \subset X$ where $(Y \cap \beta_\alpha)$-u_α is finite for each $\alpha < \omega_1$.

By the comment after 7.2.2, Y is the increasing union of closed sets. But then, by 3.1, X is not hereditarily separable, and we are done.

[1]For detailed and exhaustive applications of this approach to a plethora of problems see Todorčević [19··].

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CHAPTER 8

Martin's Axiom and First-Countable S- and L-Spaces

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HANDBOOK OF SET-THEORETIC TOPOLOGY
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Introduction

Let T be a regular Hausdorff topological space of cardinality $\aleph_1$. One can think of two possible properties of the space which make it likely for the space to have an uncountable discrete subspace. The first is that there is some enumeration $T = \{x_i: i \in \omega_1\}$ such that every initial segment $\{x_i: i \leqslant \alpha\}$ is open. In this case every x_α is already separated by an open set from all the points x_β, $\beta > \alpha$; and it seems that to get an uncountable discrete subspace we only have to separate each x_α in some uncountable set from the countably many points in that set enumerated before α. If an uncountable regular Hausdorff space does not possess an uncountable discrete subspace, despite the fact that all initial segments in some ω_1-enumeration of that space are open, then we call that space a right-separated S-space of type ω_1 (or S-space for short).

The second possible property to increase the likelihood of an uncountable discrete subspace is to have an ω_1-enumeration with all final segments $\{x_i: i \geqslant \alpha\}$, $\alpha \in \omega_1$, open. In this case every point has a neighborhood which is disjoint from all points below it, and thus it seems half the way toward acquiring an uncountable discrete subspace is done. If an uncountable, regular, Hausdorff-space does not have an uncountable discrete subspace, despite the fact that all final segments in some ω_1-enumeration of the space are open, then we call the space a left-separated L-space of type ω_1 (or L-space for short).

The reader is referred to the previous chapter in this book for a more general definition of S- and L-spaces, as well as for a proof that we can assume that all the spaces in consideration are 0-dimensional (a basis for the topology exists which consists of clopen sets). It should also be noted that an uncountable subspace of an S-space (L-space) is also an S-space (L-space). An S-space is separable and an L-space is Lindelöf.

We shall deal in this chapter mainly with first countable spaces, i.e. every point in the space has a countable basis for open neighborhoods. When we write "$T = \langle \omega_1, \tau \rangle$ *is a topological space*" we mean of course that the set of all countable ordinals ω_1 is supporting the topology $\tau \subseteq P(\omega_1)$.

Description of the content of the chapter

In Section 1 we give a forcing argument showing the relative consistency (with ZFC) of the existence of a first-countable S-space. (A. HAJNAL and I. JUHÁSZ [1975] used CH to construct a first-countable S-space.) We then define and prove some further properties of the space obtained; these will be used in Section 5.

In Section 3, we prove Z. Szentmiklóssy's theorem that Martin's Axiom + $\neg$CH implies there are no first-countable L-spaces.

A natural question arising, in view of this theorem, is whether Martin's Axiom + $\neg$CH implies the nonexistence of first-countable S-spaces. The next section is a preparation for the negative answer.

In Section 3 we discuss some of the available methods for proving that Martin's Axiom + $\neg$CH does *not* imply a given statement.

In Section 4 we present a combinatorial property (*) such that any S-space with property (*) cannot be destroyed by a ccc poset.

In Section 5 appears the forcing proof of Con(ZFC) $\rightarrow$ Con(ZFC + there is a first-countable S-space satisfying (*)).

Together with Section 4 this gives a proof of

THEOREM[1]. Con(ZFC) $\rightarrow$ Con(ZFC + *Martin's Axiom* + $\neg$CH + *there exists a first countable* S-*space*).

This theorem answers a question of HAJNAL and JUHÁSZ [1975]. Thus, if the axioms of ZFC are consistent, the existence of a first-countable S-space does not imply the existence of a first-countable L-space (which does not exist in a model of Martin's Axiom + $\neg$CH, by SZENTMIKLÓSSY'S theorem in Section 2.)

We don't know, however, if the existence of a first-countable S-space can be deduced from that of a first-countable L-space. (In the model of TODORČEVIĆ [1981] MA + $\neg$CH holds, and so it does not contain a first-countable L-space.)

In Section 6 we outline how our method can give a model of Martin's Axiom + $\neg$CH + there is an L-space (necessarily not first-countable).

1. Consistency of first-countable S-spaces

1.1. THEOREM. *There is a* ccc *poset* $\mathbb{P}$ *such that forcing with* $\mathbb{P}$ *gives a first-countable* S-*space.*

PROOF. We wish to get by forcing a collection $\{U_{\alpha,n}: \alpha \in \omega_1 \text{ and } n \in \omega\}$ where $U_{\alpha,n} \subseteq \alpha + 1$ is the nth clopen basic neighborhood of α, such that the resulting space will be a first-countable S-space. A condition in $\mathbb{P}$ gives a finite information on the $U_{\alpha,n}$'s.

Thus we define, for each $\alpha \in \omega$ and $n \in \omega$, *set constants* $A_{\alpha,n}$ and *ordinal constants* α. Then define:

1.2. DEFINITIONS. (a) A *formula* is one of the following five types:

(i) $\boldsymbol{\beta} \in A_{\alpha,n}$, for $\beta \leq \alpha$.

(ii) $\boldsymbol{\beta} \notin A_{\alpha,n}$, for $\beta \neq \alpha$.

(iii) $A_{\alpha,n} \cap A_{\beta,k} = \emptyset$, for $\alpha \neq \beta$.

(iv) $A_{\beta,k} \subset A_{\alpha,n}$, for $\beta < \alpha$.

(v) $A_{\beta,k} \subset A_{\beta,n}$ for $n \leq k$.

[1]This result was obtained while we were at the Toronto Workshop in set-theoretic topology, July 1980. We wish to thank organizers and participants for the stimulating environment.

(b) Let p be a finite set of formulas, we say $A_{\alpha,n}$ appears in p iff $A_{\alpha,n}$ is part of a formula in p. Similarly, we say $\boldsymbol{\beta}$ appears in p if for some X, $\boldsymbol{\beta} \in X$ is in p, or if $A_{\beta,n}$ appears in p.

1.3. Definition of $\mathbb{P}$. Let $\mathbb{P}$ be the collection of all finite sets, p, of formulas with the following properties:

(a)(1) The formulas $i \in X$ and $i \notin X$ are never both in p.

(2) for any $\boldsymbol{\beta}$ and $A_{\alpha,n}$ appearing in p, either $\boldsymbol{\beta} \in A_{\alpha,n}$ or $\boldsymbol{\beta} \notin A_{\alpha,n}$ is in p. (If $\alpha < \beta$, then $\boldsymbol{\beta} \notin A_{\alpha,n}$ is in p as $\boldsymbol{\beta} \in A_{\alpha,n}$ is not a formula for $\alpha < \beta$.)

(3) If $A_{\alpha,n}$ appears in p, then $\boldsymbol{\alpha} \in A_{\alpha,n}$ is in p; and if $\boldsymbol{\alpha}$ appears in p, then $\boldsymbol{\alpha} \in A_{\alpha,n}$ is in p.

(b) If $A_{\alpha,n}$ appears in p and $m < n$, then $A_{\alpha,n} \subset A_{\alpha,m}$ is in p.

(c) If $\boldsymbol{\beta} \in X$ and $X \subset Y$ are in p, then so is $\boldsymbol{\beta} \in Y$.

(d) $X \cap Y = \emptyset$ is in p iff $Y \cap X = \emptyset$ is in p; and in this case if $\boldsymbol{\beta} \in X$ is in p then $\boldsymbol{\beta} \in Y$ is not.

(e) If $X \subset Y$ and $Y \subset Z$ belong to p, then so does $X \subset Z$.

(f) If $X \subset Y$ and $Y \cap Z = \emptyset$ are in p, then $X \cap Z = \emptyset$ is in p too.

The partial order of $\mathbb{P}$ is inclusion. If $p \subseteq p'$ we say p' extends p.

1.4. Definition. If $A_{\alpha,0}$ appears in $p \in \mathbb{P}$, let n be the maximal number such that $A_{\alpha,n}$ appears in p; we call $A_{\alpha,n}$ *the minimal set of α in p.*

1.5. Lemma. *For $\alpha \in \omega_1$ and $n \in \omega$, every $p \in \mathbb{P}$ has extensions in which $A_{\alpha n}$ appears.*

Proof. If $A_{\alpha,0}$ does not appear in p, then neither does any $A_{\alpha,k}$. In this case just add all formulas $\boldsymbol{\alpha} \in A_{\alpha,l}$ and $A_{\alpha,k} \subset A_{\alpha,l}$ for $l \leqslant k \leqslant n$. For X which appears in p add the formula $\boldsymbol{\alpha} \notin X$. Also, for all $\boldsymbol{\beta}$ which appear in p, add the formulas $\boldsymbol{\beta} \notin A_{\alpha,k}$, $k \leqslant n$. The resulting set of formulas is clearly a condition extending p as required.

If $A_{\alpha,0}$ does appear in p, let $A_{\alpha,m}$ be the minimal set of α in p (thus $A_{\alpha k}$ appears in p for $k \leqslant m$ but not for $k > m$). We can assume $m < n$ of course. For k, $m < k \leqslant n$: Add the formulas $A_{\alpha,k} \subset X$ if $X = A_{\alpha,l}$, $l < k$, or if $A_{\alpha,m} \subset X$ is in p. Add the formulas $A_{\alpha,k} \cap X = \emptyset$ and $X \cap A_{\alpha,k} = \emptyset$ if $A_{\alpha,m} \cap X = \emptyset$ is in p. Also, for $\boldsymbol{\beta}$ appearing in p, add $\boldsymbol{\beta} \in A_{\alpha,k}$ if $\boldsymbol{\beta} \in A_{\alpha,m}$ is in p, and add $\boldsymbol{\beta} \notin A_{\alpha,k}$ otherwise. Again, we get a condition in $\mathbb{P}$ with $A_{\alpha,n}$ appearing in it.

1.6. Definition. Let G be a generic filter over $\mathbb{P}$. Define:

$$U_{\alpha,n} = \{\beta \colon \{\boldsymbol{\beta} \in A_{\alpha,n}\} \in G\}\,.$$

Now Lemma 1.5 gives, by a density argument, that $\alpha \in U_{\alpha,n}$, $U_{\alpha,n} \subseteq \alpha + 1$, $U_{\alpha,n} \supset U_{\alpha,m}$ for $n < m$. Let $\tau = \tau(G)$ be the topology on ω_1 generated by the $U_{\alpha,n}$

as subbase (so unions of finite intersections of the $U_{\alpha,n}$'s give the open sets in τ). Actually we will show that the $U_{\alpha,n}$,s already form a basis of clopen sets. And then we still have to prove that $\mathbb{P}$ satisfies the ccc and that τ is a first countable S-space. But first some simple technical lemmas.

1.7. LEMMA. *If $A_{\alpha,n}$ appears in $p \in \mathbb{P}$ and $\boldsymbol{\beta}$ does not ($\beta < \alpha$), then a condition $p' \supset p$ can be found with p' containing $\boldsymbol{\beta} \in A_{\alpha,n}$.*

PROOF. Let $A_{\alpha,m}$ be the minimal set for α in p. So $n \leqslant m$. Add to p all formulas $\boldsymbol{\beta} \in A_{\alpha,m}$ and $\boldsymbol{\beta} \in X$ whenever $A_{\alpha,m} \subset X$ is in p. For all other X's appearing in p add $\boldsymbol{\beta} \notin X$. Also add $\boldsymbol{\beta} \in A_{\beta,0}$ and $\boldsymbol{\tau} \notin A_{\beta,0}$ for any $\boldsymbol{\tau}$ appearing in p. The resulting set is p'.

1.8. LEMMA. *Let $p \in \mathbb{P}$ and suppose $\boldsymbol{\beta} \notin A_{\alpha,n}$ is in p. Let $A_{\beta,k}$ be the minimal set of β in p. Then some extension of p contains the formula $A_{\alpha,n} \cap A_{\beta,k+1} = \emptyset$.*

PROOF. Add to p the following formulas and check that a condition is obtained:

(1) $A_{\beta,k+1} \subset X$, if X is $A_{\beta,k}$ or if $A_{\beta,k} \subset X$ is in p.

(2) $A_{\alpha,n} \cap A_{\beta,k+1} = \emptyset$, $A_{\beta,k+1} \cap A_{\alpha,n} = \emptyset$, $X \cap A_{\beta,k+1} = \emptyset$ and $A_{\beta,k+1} \cap X = \emptyset$ if $X \subset A_{\alpha,n}$ is in p or if $X \cap A_{\beta,k} = \emptyset$ is in p.

(3) Finally, add $\boldsymbol{\beta} \in A_{\beta,k+1}$ and $\gamma \notin A_{\beta,k+1}$ for $\gamma \neq \boldsymbol{\beta}$ appearing in p.

1.9. LEMMA. *Suppose $\boldsymbol{\beta} \in A_{\alpha,n}$ is in p, let $A_{\beta,k}$ be the minimal set for β in p. Then an extension $p' \supset p$ can be found with $A_{\beta,k+1} \subset A_{\alpha,n}$ in p'.*

PROOF. Add to p, $A_{\beta,k+1} \subset Y$ whenever $Y = A_{\alpha,n}$ or $Y = A_{\beta,k}$ or $A_{\alpha,n} \subset Y$ is in p or $A_{\beta,k} \subset Y$ is in p. Add also $A_{\beta,k+1} \cap X = \emptyset$ and $X \cap A_{\beta,k+1} = \emptyset$ whenever $A_{\beta,k} \cap X = \emptyset$ is in p or $A_{\alpha,n} \cap X = \emptyset$ is in p. Finally add $\boldsymbol{\beta} \in A_{\beta,k+1}$, and $\boldsymbol{\gamma} \notin A_{\beta,k+1}$ whenever $\boldsymbol{\gamma}$ appears in p ($\boldsymbol{\gamma} \neq \boldsymbol{\beta}$).

1.10. LEMMA. *If $p \in \mathbb{P}$ and $\boldsymbol{\beta}$ appears in p, then there are p' extending p and $A_{\beta,k}$ appearing in p' such that for any $A_{\alpha,n}$ appearing in p, if $\boldsymbol{\beta} \in A_{\alpha,n}$ is in p, then $A_{\beta,k} \subset A_{\alpha,n}$ is in p'; and if $\boldsymbol{\beta} \notin A_{\alpha,n}$ is in p, then $A_{\beta,k} \cap A_{\alpha,n} = \emptyset$ is in p'.*

PROOF. Successively apply Lemmas 1.8 and 1.9.

The net result of this lemma is that the topology defined in 1.6 has the collection $\{U_{\alpha,n}: \alpha \in \omega_1,\ n \in \omega\}$ as a basis, and $U_{\alpha,n}$ is cloppen. And $\tau(G)$ is a first-countable, 0-dimensional space.

1.11. DEFINITION. For p a set of sentences, $\mathrm{Ord}(p) = \{\alpha : \boldsymbol{\alpha}$ appears in $p\}$ $(= \{\alpha :$ for some k, $A_{\alpha,k}$ appears in $p\})$.

For $\beta \in \omega_1$ and $p \in \mathbb{P}$, $p|\beta = \{\varphi \in p:$ if $\boldsymbol{\alpha}$ appears in $\{\varphi\}$ then $\alpha < \beta\}$. So $p|\beta$ is the collection of formulas in p which are built only with ordinals in β. The next lemma is trivial.

1.12. LEMMA. *$p|\beta \in \mathbb{P}$ for $p \in \mathbb{P}$.*

The following says that two conditions which are part of a well-behaved Δ-system are compatible.

1.13. LEMMA. *If p_1, $p_2 \in \mathbb{P}$, $\beta \in \omega_1$, $p_1|\beta = p_2|\beta$ and* $\operatorname{Sup}\operatorname{Ord}(p_1) < \inf(\operatorname{Ord}(p_2) - \beta)$, *then $p_1 \cup p_2 \subset p'$ for some $p' \in \mathbb{P}$.*

PROOF. Just add to $p_1 \cup p_2$ all the formulas $\boldsymbol{\alpha} \notin A_{\eta,n}$ for $\alpha \in \operatorname{Ord}(p_1) - \beta$ and $A_{\eta,n}$ appearing in p_2 with $\beta < \eta$. Add also the formulas $\boldsymbol{\alpha} \notin A_{\eta,n}$ for $\eta < \alpha$, where $\boldsymbol{\alpha}$ and $A_{\eta,n}$ appear in $p_1 \cup p_2$. The collection p' thus obtained is a condition.

1.14. CONCLUSION. *$\mathbb{P}$ satisfies the* ccc.

The proof of this is the well known Δ-system argument, using Lemma 1.13 (see Ch. II, Th. 1.5 in KUNEN [1980]).

So $\aleph_1$ is not collapsed in a generic extension $V[G]$ over $\mathbb{P}$ and $\tau(G)$ is thus an uncountable space. At this stage it is possible to analyze the proof of Lemma 1.13 and to see that a minor change gives that $\tau(G)$ does not contain any uncountable discrete subspace—so τ is a first countable S-space. However, for future use in Section 5, we need a stronger statement which follows these definitions.

1.15. DEFINITION. (1) $\bar{F} = \langle F_1, \ldots, F_n \rangle$ is a *disjoint finite sequence* iff F_i is a finite set of countable ordinals and $F_i \cap F_j = \emptyset$ for $i \neq j$. (But, of course, a member of F_i can be between two members of F_j.)

(2) We denote $\bigcup \bar{F} = F_1 \cup \cdots \cup F_n$.

(3) We say $\bar{F}^1 < \bar{F}^2$ iff $\operatorname{Sup}(\bigcup \bar{F}^1) < \inf(\bigcup \bar{F}^2)$.

(4) $\langle \bar{F}^\alpha : \alpha \in \omega_1 \rangle$ is called a *sequence of disjoint finite sequences* (*of length n*) iff
(i) $\bar{F}^\alpha = \langle F_1^\alpha, \ldots, F_n^\alpha \rangle$ is a disjoint finite sequence, for any given $\alpha < \omega_1$, and
(ii) $\bar{F}^\alpha < \bar{F}^\beta$ for $\alpha < \beta$.

Recall from 1.6 that G is a generic filter over $\mathbb{P}$ and $\tau(G)$ is given by the clopen sets $U_{\alpha,n}$.

1.16. LEMMA. *In $V[G]$ the following* 1.17 *is true of $\tau = \tau(G)$.*

1.17. *Let $\langle \bar{F}^\alpha : \alpha < \omega_1 \rangle$ be a sequence of disjoint finite sequences of length $n \in \omega$. Let $\bar{k} = \langle k_1, \ldots, k_n \rangle$ be an n-tuple of natural numbers such that for all $\alpha \in \omega_1$ and $1 \leq i \leq n$, $\bigcap\{U_{\mu,k_i} : \mu \in F_i^\alpha\} \neq \emptyset$. Then there are $\beta < \alpha < \omega_1$ such that $F_i^\beta \subseteq \bigcap\{U_{\mu,k_i} : \mu \in F_i\}$ holds for each $1 \leq i \leq n$.*

PROOF. Before giving the proof we observe that 1.17 implies τ is an S-space: Suppose that $X \subseteq \omega_1$ is uncountable and for every $x \in X$ an open neighborhood V_x is chosen. We must find $x < y$ with $x \in V_y$ thus showing X is not discrete. Now V_X contains a basis member $U_{x,k}$ for some k. By shrinking the uncountable X we can assume that the k is independent of the choice of $x \in X$. X can be regarded as a sequence of disjoint finite sequences of length 1, and 1.17 gives the required inclusion.

Turning to the proof, let $\langle \bar{\boldsymbol{F}}^\alpha : \alpha < \omega_1 \rangle$ be a name in the $\mathbb{P}$-forcing language and $p \in \mathbb{P}$ which forces the premises of 1.17. So, for some n and $\bar{k} = \langle k_1, \ldots, k_n \rangle$,

1.18. $p \Vdash$ "$\langle \bar{\boldsymbol{F}}^\alpha : \alpha \in \omega_1 \rangle$ is a sequence of disjoint finite sequences of length n, and for every $\alpha < \omega_1$ and $1 \leq i \leq n \bigcap \{U_{\mu,k_i} : \mu \in F^\alpha_i\} = \emptyset$".

We seek an extension of p forcing the conclusion of 1.17 for some $\beta < \alpha$. This obviously provides the right dense subset of $\mathbb{P}$ to give the lemma.

For each $\alpha < \omega_1$ pick $p_\alpha \supseteq p$ in P and a disjoint finite sequence $\langle F^\alpha_1, \ldots, F^\alpha_n \rangle$ such that $p_\alpha \Vdash$ "$\bar{\boldsymbol{F}}^\alpha = \langle \check{F}^\alpha_1, \ldots, \check{F}^\alpha_n \rangle$" (where $\check{a}$, for $a \in V$, is the canonical name of a). So p_α "*describes*" $\bar{\boldsymbol{F}}^\alpha$. Extend p_α further, if necessary, to get $F^\alpha_i \subseteq \mathrm{Ord}(p_\alpha)$, $1 \leq i \leq n$, and even that A_{μ,k_i} appears in p_α for each $\mu \in F^\alpha_i$ (Lemma 1.5). We can assume that the sets $\mathrm{Ord}(p_\alpha)$, $\alpha \in \omega_1$, form a Δ-system with root included in $\delta < \omega_1$, and also that $p_\alpha | \delta = p_\beta | \delta$, and that (for $\beta < \alpha$) $\mathrm{Sup}\,\mathrm{Ord}(p_\beta) < \inf(\mathrm{Ord}(p_\alpha) - \delta)$. Lemma 1.13 gives that any two conditions p_α, p_β are compatible. A consequence of this is that $\mathrm{Sup}(F^\beta_1 \cup \cdots \cup F^\beta_n) < \inf(F^\alpha_1 \cup \cdots \cup F^\alpha_n)$, for $\beta < \alpha$. Hence $F^\alpha_1 \cup \cdots \cup F^\alpha_n$ does not belong to the root of the Δ-system and give disjoint sets (for different α's).

Take $\beta < \alpha$ with $\delta < \inf(F^\beta_1 \cup \cdots \cup F^\beta_n)$. We define now $p' \supseteq p_\beta \cup p_\alpha$ as follows. Throw in p' all formulas of $p_\beta \cup p_\alpha$ together with the following additional ones:

(i) $\xi \in A_{\mu,k_i}$ whenever $\xi \in F^\beta_i$ and $\mu \in F^\alpha_i$, $1 \leq i \leq n$.

(ii) If a formula $\xi \in X$ was added in (i), and $X \subset Y$ exists already in p_α, then $\xi \in Y$ is added too.

(iii) $\xi \notin A_{\mu,k}$ whenever ξ and $A_{\mu,k}$ appear in $p_\beta \cup p_\alpha$ and the formula $\xi \in A_{\mu,k}$ is not in $p_\beta \cup p_\alpha$ and was not introduced by (i).

Let us check that property (d) of Definition 1.13 holds for p', letting the reader check the rest. Then, once $p' \in \mathbb{P}$ is established, it is clear by (i) that p' is as required.

Clearly $X \cap Y = \emptyset$ is in p' iff $Y \cap X = \emptyset$ is in p', since no formulas of the kind $X \cap Y = \emptyset$ were added at all. Now assume that the formulas $\xi \in X$ and $\xi \in Y$ are in p', we want to make sure that $X \cap Y = \emptyset$ is not in $p_\beta \cup p_\alpha$. One of these two formulas can be assumed to be new, say $\xi \in X$, thus $\xi \in F^\beta_i$ for a unique $1 \leq i \leq n$, and X appears in p_α but not in p_β. Observe that for μ, $\mu' \in F^\alpha_i$ we cannot have $A_{\mu,k_i} \cap A_{\mu',k_i} = \emptyset$ in $p_\beta \cup p_\alpha$ since the ordinals of F^α_i do not appear in p_β, and by 1.18. Thus, if both formulas were introduced by (i) we are done. Similarly, if both were introduced by (i) or (ii) (use that (f) holds for p_α). Now, if $\xi \in Y$ is not new, then it must belong to p_β (as ξ appears in p_β but not in p_α) hence $X \cap Y = \emptyset$ cannot be in p_α or in p_β.

2. Martin's Axiom and first-countable L-spaces

2.1. THEOREM (SZENTMIKLÓSSY [1980]). *Assume* MA + $\neg$CH. *Then there are no first-countable* L-*spaces.*

PROOF. Assume by way of contradiction that there exists a first countable L-space $T = \langle \omega_1, \tau \rangle$. By 3.4 of the previous chapter, T is 0-dimensional. Hence for each $\alpha < \omega_1$ we can choose a clopen set $V_\alpha \subseteq \omega_1 \backslash \alpha$ which contains α.

Let $\mathbb{P}$ be the set of all finite subsets p of ω_1 such that $\alpha, \beta \in p$ and $\alpha < \beta$ imply $\beta \notin V_\alpha$; the ordering on $\mathbb{P}$ is $\subseteq$.

2.2. LEMMA. $\mathbb{P}$ *satisfies the* ccc.

PROOF. Let $\{p_i : i < \omega_1\}$ be a sequence of elements of $\mathbb{P}$. By the standard Δ-system argument, we may assume that p$_i$'s are disjoint strictly increasing and of the same cardinality n where $1 \leq n < \omega$. For $i < \omega_1$, let $\{p_i(1), \dots, p_i(n)\}$ be the increasing enumeration of p_i. Let $U_i = \bigcup_{\alpha \in p_i} V_\alpha$ for $i < \omega_1$. Then each U_i is a clopen set in T.

Since T is hereditarily Lindelöf we can find an $\aleph_1$-condensation point x_1 of $\{p_i(1): i < \omega_1\}$. Since $\{U_i: x_1 < i < \omega_1\}$ is a family of $\aleph_1$ closed sets in T which do not contain x_1 and since T is first-countable, we can find a neighbourhood W_1 of x_1 such that $A_1 = \{i < \omega_1; W_1 \cap U_i = \emptyset\}$ is uncountable. Let $B_1 = W_1 \cap \{p_i(1): i < \omega_1\}$. Then by the assumption of x_1, B_1 is uncountable. Let x_2 be an $\aleph_1$-condensation point of $\{p_i(2): i \in B_1\}$ and let W_2 be a neighbourhood of x_2 such that $A_2 = \{i \in A_1: W_2 \cap U_i = \emptyset\}$ is uncountable. Let $B_2 = W_2 \cap \{p_i(2): i \in B_1\}$. Then by the assumption on x_2, B_2 is uncountable. Proceeding in this way we construct uncountable sets $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$ and $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n$ such that for every $l \in \{1, \dots, n\}$, every $i \in A_l$ and every $j \in B_l$, $p_j(l) \notin V_i$. Hence if $i \in A_n$ and $j \in B_n$ are such that $i < j$, then p_i and p_j are compatible in $\mathbb{P}$. This proves the lemma.

By MA + $\neg$CH, $\mathbb{P}$ is also a σ-centered poset. Hence $\mathbb{P}$ contains an uncountable centered subset C. Then $\bigcup C$ is an uncountable discrete subspace of T, a contradiction. This completes the proof of Theorem 2.1.

3. Some methods for getting Models of Martin's Axiom

In this section we discuss some methods for proving that certain statements are not resolved by the assumption MA + $\neg$CH, that is, we present some methods for constructing models of Martin's Axiom with some additional properties. It is fair to say that really there are no methods, and that the main difficulties are always of technical nature strongly depending on a particular statement we are considering. However there are some similarities between different proofs and this is what we are presenting.

We split our discussion of proving the consistency of MA + $\neg$CH + P into two parts depending whether the statement P is of the form $\forall x\ Q(x)$ or $\exists x\ Q(x)$. A typical example of the first form is the consistency proof of MA + $\neg$CH + "there are no Kurepa trees" from DEVLIN [1978]. The general idea is to start with a model of SILVER [1971] in which there are no Kurepa trees and which satisfies GCH, and then to iterate ccc posets $\langle \mathbb{P}_\alpha | \alpha \leqslant \omega_2 \rangle$ with finite supports so as to get Martin's Axiom at the end (see SOLOVAY and TENNENBAUM [1971]). The problem is that we must somehow assure that, during the ω_2-long iteration, trees do not acquire more and more branches and become Kurepa trees. So at a given intermediate stage α of the iteration we are presented with a ccc poset Q and would like to use Q as the next step of the iteration, i.e., to put $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * Q$. Suppose, however, that for some tree T in $V^{\mathbb{P}_\alpha}$, forcing with Q adds a new branch to T. Then Q is *dangerous*, and we would not like to use it in our iteration. However, if we want to get MA at the end the only way not to use Q is to kill the ccc property of Q. The arguments of DEVLIN [1978] provide another ccc poset R such that (1) forcing with R introduces an uncountable antichain to Q, but (2) R does not introduce any new branch to an existing tree of height ω_1. So we use R as the next step of the iteration, that is, we let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * R$. A similar approach is in KUNEN [1976a] where he proves the consistency of MA + $\neg$CH + "there are neither (ω_1, c^*) nor (c, c^*) gaps in $\mathscr{P}(\omega)/\text{fin}$".

Suppose now we want to get the consistency of MA + $\neg$CH with the existence of an object satisfying a property $Q(x)$. The most natural way is to construct an object a such that $Q(a)$ is ccc indestructible in the following sense: $Q(a)$ holds in the ground model and in any generic extension done with a ccc poset. Since MA + $\neg$CH can be obtained via ccc posets (see SOLOVAY and TENNENBAUM [1971]), MA + $\neg$CH + $\exists x\ Q(x)$ is consistent. The easiest case is when $Q(a)$ is ccc indestructible for any object a which satisfies $Q(x)$. For example, if $Q(x)$ says that x is maximal almost disjoint family of uncountable subsets of ω_1 such that $|x| = \aleph_2$, then it is easily seen that $Q(\mathscr{F})$ is ccc indestructible for any family $\mathscr{F}$ which satisfies $Q(x)$. However, in general this is not the case, i.e., we may have $Q(x)$ such that $Q(a)$ is ccc destructible for some object a. So we have to construct an object b with a certain stronger property $Q^*(x)$ such that $Q(b)$ is ccc indestructible. Note that $Q^*(b)$ may or may not be ccc indestructible (see for example, KUNEN [1976b] for $Q^*(x)$ which is ccc indestructible). However, in many cases it is not clear what $Q^*(x)$ should be so we have to take a closer look at the behaviour of the property $Q(x)$ during the ccc iterations. A typical example of this approach is the consistency of MA + $\neg$CH + "there exist two nonisomorphic $\aleph_1$-dense sets of reals" proved by Shelah in ABRAHAM and SHELAH [1981] (see also

ABRAHAM, RUBIN and SHELAH [19 · ·] for this and related results). In general terms this approach can be described thus:

Say we want to get the consistency of MA + $\neg$CH with some object satisfying a certain property $Q(x)$; the strategy is to start with some model of ZFC in which there is an object a with $Q(a)$ (a can be constructed using CH, or be obtained by forcing, etc.) and then to try to kill all ccc dangerous posets. In most cases, however, the generic introduction of an uncountable antichain to a dangerous poset might collapse $\aleph_1$ or destroy $Q(a)$ itself (we have to find a non-dangerous poset to kill a given dangerous one). So one looks for a stronger property $Q^*(a)$ which ensures that the killer poset itself is not dangerous. But then there are more dangerous posets: all those which might destroy $Q^*(a)$—and thus we must find a stronger property $Q^{**}(a)$, etc. The problem of course is to find a stable Q^* which will work and will stand the whole iterative process. As the next section shows, Q^* can be quite far from Q ($Q(x)$ is "x is a first countable S-space" and $Q^*(x)$ is "x is a 2-complicated first countable S-space").

Let us give more details now. Suppose $\langle\omega_1, \tau\rangle$ is a first countable S-space. Let $\{S_n^\alpha\colon n \in \omega\}$ be a neighborhood basis for α. So $\alpha \in S_n^\alpha$ and we may assume $S_n^\alpha \subseteq \alpha + 1$ since τ is an S-space (initial segments are open). Call S_n^α the nth neighborhood of α. Suppose $\mathbb{P}$ is a poset such that $\Vdash^{\mathbb{P}}$ "$\langle\omega_1, \tau\rangle$ is not an S-space." This means that the dangerous poset $\mathbb{P}$ introduces an uncountable discrete subspace $\boldsymbol{D} = \{\boldsymbol{a}_i\colon i \in \omega_1\}$. We may well assume that for a fixed n the nth neighborhood of a point in $\boldsymbol{D}$ separates it from all the rest of $\boldsymbol{D}$. For each $i \in \omega_1$ we can pick a condition $p_i \in \mathbb{P}$ which determines the ordinal $\boldsymbol{a}_i$, i.e. $p_i \Vdash$ "$\boldsymbol{a}_i = \check{\alpha}_i$" for some ordinal α_i, and such that p_i forces the fact about $\boldsymbol{D}$ being discrete by virtue of the nth neighborhoods. Now, if $i < j$ and $\alpha_i \in S_n^{\alpha_j}$, then clearly p_i and p_j are incompatible. (Otherwise, if q extends both p_i and p_j, $q \Vdash$ "$\check{\alpha}_j$ and $\check{\alpha}_i$ are in $\boldsymbol{D}$ and hence $\check{\alpha}_i \notin \check{S}_n^{\alpha_j}$".) So, if for uncountably many i's, $i < i' \rightarrow \alpha_i \in S_n^{\alpha_{i'}}$, we would get an uncountable antichain in $\mathbb{P}$. At first, one may believe that this is the way to kill $\mathbb{P}$—by getting uncountably many points no one of which is in the nth neighborhood of the higher ones. But this last property easily implies that $\langle\omega_1, \tau\rangle$ is no longer an S-space. So we must try again. This time choose p_i and two ordinals $\alpha_i > \beta_i$ such that $p_i \Vdash$ "$\check{\alpha}_i$ and $\check{\beta}_i$ are in the discrete subspace $\boldsymbol{D}$". Now, if $i < j$ and

3.1. $\{\alpha_i, \beta_i\} \cap (S_n^{\alpha_j} \cup S_n^{\beta_j}) \neq \emptyset$,

then p_i and p_j are incompatible in $\mathbb{P}$. Here there is more flexibility in introducing an uncountable antichain to $\mathbb{P}$.

Let $\mathbb{R}$ be the poset consisting of all finite sets $a \subseteq \omega_1$ such that for $i, j \in a$, if $i < j$, then 3.1 holds. Clearly $\mathbb{R}$ introduces an antichain to $\mathbb{P}$. If $\mathbb{R}$ would also satisfy the ccc then $\aleph_1$ will not collapse and an iteration could be carried on. The demand that $\mathbb{R}$ satisfies the ccc can be easily formulated as a combinatorial requirement on the topology τ—concerning uncountable sequences of finite sets—and the space obtained in Section 1 satisfies this requirement. However, it is not clear how to keep this requirement undamaged in the successive iterations. This problem lead us to the definition of a 2-complicated space, which enables the process to run. Finally, after all dangerous posets were killed, we discovered that the resulting space satisfies some combinatorial property 4.2 which makes it immune to all ccc posets. So now, starting from the end, we describe first this property and its consequences in Section 4, and in Section 5 we use the notion of a 2-complicated space to get a space satisfying 4.2.

4. S-spaces with strong combinatorial properties

4.1. DEFINITION. Let $T = \langle \omega_1, \tau \rangle$ be a first-countable, 0-dimensional, separable space, and let $\{S^\alpha_n : n \in \omega\}$ be a neighborhood basis for $\alpha \in \omega_1$ consisting of clopen sets, $S^\alpha_n \subseteq \alpha + 1$ and $S^\alpha_{n+1} \subseteq S^\alpha_n$. For $a, b, c, d \in \omega_1$ define:

$$G^n(\langle a, b\rangle, \langle c, d\rangle) \quad \text{iff} \quad a < b < c < d \text{ and } \{a, b\} \cap (S^c_n \cup S^d_n) \neq \emptyset .$$

4.2. DEFINITION. We say t satisfies property $(*)$ iff T is a space as in the assumptions for Definition 4.1 and:

$(*)$ For any uncountable $X \subseteq \omega_1$ and $n < \omega$ with $\bigcap_{x\in X} S^x_n \neq \emptyset$, and for any function $f: X \times \omega_1 \to X$ with $f(x, \alpha) > \alpha$ (for all $x \in X$, $\alpha \in \omega_1$), *there exists* an uncountable sequence $\langle\langle x_i, \alpha_i, y_i\rangle : i \in \omega_1\rangle$ such that $i < j$ implies

4.3. $(x_i < \alpha_i < y_i < x_j)$ & $f(x_i, \alpha_i) = y_i$ & $G_n(\langle x_i, y_i\rangle, \langle x_j, y_j\rangle)$.

Our aim in the next section is to prove: Con(ZFC)$\to$Con(ZFC + there is a topological space satisfying property $(*)$).

In this section we show that any space satisfying property $(*)$ is (a first-countable) S-space and an indestructible one. So we get immediately: Con(ZFC)$\to$Con(ZFC + Martin's Axiom + $\neg$CH + there is a first-countable S-space with property $(*)$). We can have here the continuum as large as asked.

4.4. LEMMA. *A space T satisfying property $(*)$ is a first-countable* S-*space.*

PROOF. That T is first-countable and that all initial segments are open is part of Definition 4.2. So, it suffices to show there are no uncountable discrete subspaces to T. Suppose on the contrary $X \subseteq \omega_1$ is discrete, uncountable and separated by the nth neighborhoods. (This fixed n can be assumed without loss of generality.) Since the space is separable, there is an uncountable $X' \subseteq X$ such that $\bigcap_{\alpha\in X'} S^\alpha_n \neq \emptyset$. Define $f: X' \times \omega_1 \to X'$ by $f(x, \alpha)$ = the first member of X' above α. By $(*)$ we get $x_i < y_i < x_j < y_j$ such that $\beta \in S^\alpha_n$ for some $\beta \in \{x_i, y_i\}$ and $\alpha \in \{x_j, y_j\}$. Contradiction to the assumption that X' is discrete by virtue of the nth neighborhoods.

4.5. LEMMA. If a poset $\mathbb{P}$ satisfies the ccc and T satisfies property $(*)$, then $\Vdash^{\mathbb{P}}$ "T satisfies property $(*)$".

PROOF. It is clear that in a generic extension there might be new open sets, but the basis for the topology remains the same; and, denoting the new topology by T again, it is clear that in the extension T is still first-countable, 0-dimensional, separable space. Since $\mathbb{P}$ satisfies the ccc ω_1 is not collapsed and remains uncountable in an extension with $\mathbb{P}$.

Let $p \in \mathbb{P}$ and let $\boldsymbol{X}$, $\boldsymbol{f}$ be names, $n < \omega$ such that

$$p \Vdash \text{“}f\colon \boldsymbol{X} \times \check{\omega}_1 \to \boldsymbol{X}, \boldsymbol{X} \text{ is uncountable and } \boldsymbol{f}(x, \alpha) > \alpha \text{ for every } \langle x, \alpha\rangle \in \boldsymbol{X} \times \check{\omega}_1, \text{ and } \bigcap\{\check{S}_n^x : x \in \boldsymbol{X}\} \neq \emptyset\text{”}.$$

As $p \Vdash$ "$\exists a \in \bigcap\{\check{S}_n^x : x \in \boldsymbol{X}\}$" we can find $a \in \omega_1$ and some extension of p (which we call p again) such that $p \Vdash \check{a} \in \bigcap\{\check{S}_n^x : x \in \boldsymbol{X}\}$.

Let $Y = \{y \in \omega_1 : \exists q \leq p\ q \Vdash \check{y} \in \boldsymbol{X}\}$. Y is thus the set of all "possible" members of $\boldsymbol{X}$, $Y \subseteq \omega_1$ is uncountable. Pick $p_y \leq p$ for every $y \in Y$, such that $p_y \Vdash$ "$\check{y} \in \boldsymbol{X}$". For every $y \in Y$ and $\alpha \in \omega_1$ choose $p(y, \alpha) \leq p_y$ and $g(y, \alpha) \in \omega_1$ such that $p(y, \alpha) \Vdash$ "$\boldsymbol{f}(\check{y}, \check{\alpha}) = \check{g}(y, \alpha)$". This defines a function $g\colon Y \times \omega_1 \to Y$ such that $g(y, \alpha) > \alpha$. Clearly $a \in \bigcap\{S_n^y : y \in Y\}$.

Using $(*)$ for T, Y, n and g, we can find a sequence $\langle\langle x_i, \alpha_i, y_i\rangle : i \in \omega_1\rangle$ such that $i < j \to x_i < \alpha_i < y_i < x_j$ and $g(x_i, \alpha_i) = y_i$ and $G^n(\langle x_i, y_i\rangle, \langle x_j, y_j\rangle)$. $\mathbb{P}$ satisfies the ccc, hence there is $q \leq p$ such that $q \Vdash$ "$\boldsymbol{U} = \{i \in \omega_1 : p(x_i, \alpha_i) \in G\}$ is uncountable" where G is the name of the generic filter.

Let us prove $q \Vdash$ "$\langle\langle x_i, \alpha_i, y_i\rangle : i \in \boldsymbol{U}\rangle$ satisfies the demands of $(*)$, i.e. $i < j \Rightarrow 4.3$".

So let $i < j$ and $q' \leq q$, $q' \Vdash$ "$\check{\imath}, \check{\jmath} \in \boldsymbol{U}$". As usual, we assume our posets to be separative: $r \Vdash$ "$\check{s} \in G$" implies $r \leq s$. Since $q' \Vdash$ "$\check{p}(x_i, \alpha_i)$, $\check{p}(s_j, \alpha_j) \in G$", we have $q' \leq p(x_i, \alpha_i), p(x_j, \alpha_j)$. This implies $q' \Vdash$ "$\boldsymbol{f}(\check{x}_i, \check{\alpha}_i) = \check{g}(x_i, \alpha_i)$ and $\boldsymbol{f}(\check{x}_j, \check{\alpha}_j) = \check{g}(x_j, \alpha_j)$ and x_i, $x_j \in \boldsymbol{X}$". So $q' \Vdash$ "$\boldsymbol{f}(\check{x}_i, \check{\alpha}_i) = \check{\imath}$; and $\boldsymbol{f}(\check{x}_j, \check{\alpha}_j) = \check{\jmath}$; and $G^n(\langle \check{x}_i, \check{y}_i\rangle, \langle \check{x}_j, \check{y}_j\rangle)$".

In conclusion, for any $p \in \mathbb{P}$ forcing the premises of $(*)$ there is $q \leq p$ forcing the conclusion of $(*)$; a density argument finishes the proof of the lemma.

5. 2-Complicated spaces

In this section we use the condition $(*)$ which was given in Definition 4.2.

5.1. THEOREM. Con(ZFC) $\to$ Con(ZFC + *there is a topological space satisfying* $(*)$).

Section 5 is devoted to the proof of this theorem. Using the result of Section 1 we assume our ground model V to be a model of ZFC + GCH in which there is a first-countable S-space satisfying 1.17. For the reader's convenience we restate the properties of this space $T = \langle \omega_1, \tau \rangle$: There is a neighborhood basis of clopen sets $\alpha \in S^\alpha_{n+1} \subset S^\alpha_n \subset \alpha + 1$ such that the following holds (S^α_n was denoted $U_{\alpha,n}$ in Section 1):

1.17. *Let $\langle \bar{F}^\alpha : \alpha \in \omega_1 \rangle$ be a sequence of disjoint finite sequences of length $n \in \omega$. ($\bar{F}^\alpha = \langle F^\alpha_1, \ldots, F^\alpha_n \rangle$, see Definition 1.15.) Let $\bar{k} = \langle k_1, \ldots, k_n \rangle$ be an n-tuple of natural numbers such that for each $\alpha \in \omega_1$ and $1 \leq i \leq n$, $\bigcap \{S^\mu_{k_i} : \mu \in F^\alpha_i\} \neq \emptyset$. Then there are $\beta < \alpha < \omega_1$ such that $F^\beta_i \subseteq \bigcap \{S^\mu_{k_i} : \mu \in F^\alpha_i\}$ holds for each $1 \leq i \leq n$.*

In Lemma 1.16 it was proved that such a space is an S-space. Hence, the space is separable. Now we will find a generic extension of V in which the space T (with the topology generated by the S^α_n,s) satisfies property (*). This extension is an iteration of length ω_2 which takes care of all possible $X \subseteq \omega_1$ and $f: X \times \omega_1 \to X$ as in Definition 4.2. But first a few definitions.

5.2. DEFINITION. (1) For $D, C \subseteq \omega_1$ we say D is *separated* by C iff for $d_1, d_2 \in D$, $d_1 < d_2 \Rightarrow \exists s \in C (d_1 < s < d_2)$.

Of course, if D is separated by C^* and $C^* \subseteq C$, then D is separated by C.

(2) $\bar{F}$ is an *increasing sequence of pairs* iff

$$\bar{F} = \langle \langle a^i_0, a^i_1 \rangle : i < \mu \leq \omega_1 \rangle \quad \text{where } i < j \Rightarrow a^i_0 < a^i_1 < a^j_0 < a^j_1 .$$

Informally we sometimes view $\bar{F}$ as a *set* of ordered pairs. So we write $\langle a, b \rangle \in \bar{F}$ instead of $\langle a, b \rangle \in \text{range } \bar{F}$. Also,

$$\bigcup \bar{F} = \{x : x = a^i_0 \text{ or } x = a^i_1 \text{ for some } \langle a^i_0, a^i_1 \rangle \in \bar{F}\} .$$

(3) We say $\bar{F}$ (an increasing sequence of pairs) is *separated by C* iff $\bigcup \bar{F}$ is separated by C.

(4) For an increasing sequence of pairs, $\bar{F}$, let

$$C(\bar{F}) = \{\xi \in \omega_1 : \text{if } \langle \alpha, \beta \rangle \in \bar{F} \text{ and } \alpha < \xi, \text{ then also } \beta < \xi\} .$$

Obviously, $C(\bar{F}) \subset \omega_1$ is a club.

(5) $\langle \bar{F}_1, \ldots, \bar{F}_n \rangle$ is a *matrix (of pairs) separated by C* iff $C \subseteq \omega_1$ is a club set; for any $i \leq n$ ($n \in \omega$), $\bar{F}_i$ is a *finite* increasing sequence of pairs; $\bar{F}_1$ is separated by C and $\bar{F}_{i+1}$ is separated by $C \cap C(\bar{F}_1) \cap \cdots \cap C(\bar{F}_i)$.

(6) $\langle \bar{F}_1, \ldots, \bar{F}_n \rangle$ is a *matrix (of pairs)* iff it is a matrix of pairs separated by ω_1.

The next combinatorial key-lemma shows how a separated matrix can be of use.

5.3. LEMMA. *If $\langle \bar{F}_1, \ldots, \bar{F}_n \rangle$ is a matrix of pairs, then there exists a one-to-one choice function on the pairs. In other words, there is a one-to-one function, f, defined on $\bar{F}_1 \cup \cdots \cup \bar{F}_n = \{\langle a, b \rangle : \langle a, b \rangle \in \bar{F}_i \text{ some } i \leq n\}$ such that $f(\langle a, b \rangle) \in \{a, b\}$.*

PROOF. By induction on n, and for a fixed n by induction on the cardinality of $\bar{F}_n$, we show that for each ordinal there is a choice function which does not have this ordinal in its range. For $n = 1$ this is obvious as the pairs in $\bar{F}_1$ are disjoint. Assume it is true for $n-1$, pick any pair in $\bar{F}_n$, say $\langle a, b\rangle$. As $\bar{F}_n$ is separated by $C(\bar{F}_1) \cap C(\bar{F}_2) \cdots \cap C(\bar{F}_{n-1}) = E$ pick $\alpha \in E$, $a < \alpha < b$. Divide $\langle \bar{F}_1, \ldots, \bar{F}_n - \{\langle a, b\rangle\}\rangle$ into two parts: the lower part consisting of those pairs $\langle x, y\rangle$ with $y < \alpha$, and the upper part: those $\langle x, y\rangle$ with $\alpha < x$. Given any ordinal, we have to find a choice function omitting this ordinal. Say the ordinal is below α, use the induction hypothesis to find a choice function for the lower part omitting that ordinal, and pick a choice function for the upper part which omits b. The union of the two functions, extended by sending $\langle a, b\rangle$ to b is as required. Similarly, a choice function is obtained if the ordinal to omit is $\geqslant \alpha$.

5.4. DEFINITION. Let $\langle \omega_1, \tau\rangle$ be a first countable topological space with $\{S_k^\alpha : k \in \omega\}$ as neighborhood basis for $\alpha \in \omega_1$. ($S_{k+1}^\alpha \subseteq S_k^\alpha \subseteq \alpha + 1$.)

(1) $M = \langle \bar{F}_1, \ldots, \bar{F}_n, \bar{k}\rangle$ is a matrix (*of pairs and neighborhoods*) *separated by* C iff $\langle \bar{F}_1, \ldots, \bar{F}_n\rangle$ is a matrix of pairs separated by C and $\bar{k} = \langle k_1, \ldots, k_n\rangle$ is a sequence of natural numbers such that, for $i \leqslant n$, $\bigcap\{S_{k_i}^x : x \in \bigcup \bar{F}_i\} \neq \emptyset$.

(2) For M as above, we say that an ordinal ξ "is in M" iff $\xi \in \bigcup \bar{F}_i$ for some $i \leqslant n$. Two matrices, as above, are *disjoint* iff no ordinal ξ is in both of them. When writing $\langle M_\alpha : \alpha \in \omega_1\rangle$ we assume that the matrices M_α are pairwise disjoint.

(3) For two matrices $M = \langle \bar{F}_1, \ldots, \bar{F}_n, \bar{k}\rangle$ and $M^* = \langle \bar{F}_1^*, \ldots, \bar{F}_n^*, \bar{k}^*\rangle$ define: $G(M, M^*)$ iff (i) $\bar{k} = \bar{k}^*$, (ii) any ordinal in M is below any ordinal in M^*, and (iii) for all $i \leqslant n$ and pairs $u \in \bar{F}_i$, $v \in \bar{F}_i^*$, $G^{k_i}(u, v)$ holds. (See 4.1 for the definition of G^k.)

5.5. DEFINITION. We say that a topological space $\langle \omega_1, \tau\rangle = T$ (with neighborhood basis as in 5.4) is *2-complicated* iff there exists a club $C \subseteq \omega_1$ such that for any sequence $\langle M_\alpha : \alpha \in \omega_1\rangle$ of pairwise disjoint matrices of pairs and neighborhoods separated by C, there are $\alpha < \beta$ such that $G(M_\alpha, M_\beta)$.

We say that the club C as above *shows that T is 2-complicated.*

5.6. LEMMA. *If a space satisfies* 1.17, *then it is* 2-*complicated.*

PROOF. Let $T = \langle \omega_1, \tau\rangle$ with neighborhood basis $\{S_k^\alpha : \alpha \in \omega_1, k \in \omega\}$ satisfy 1.17. We will prove that $C = \omega_1$ shows that T is 2-complicated.

Given a sequence $\langle M_\alpha | \alpha < \omega_1\rangle$ of matrices, assume w.l.o.g. that they all have the same size, $M_\alpha = \langle \bar{F}_1^\alpha, \ldots, \bar{F}_n^\alpha, \bar{k}\rangle$, $\bar{k}$ does not depend on α. Apply Lemma 5.3 to $\langle \bar{F}_1^\alpha, \ldots, \bar{F}_n^\alpha\rangle$ and get $H_1^\alpha, \ldots, H_n^\alpha$ such that $H_i^\alpha \cap H_j^\alpha = \emptyset$ for $i \neq j$ and for any $\langle a, b\rangle \in F_i^\alpha$ $a \in H_i^\alpha$ or $b \in H_i^\alpha$. 1.17 can be applied and we get $\alpha < \beta$ with $H_i^\alpha \subseteq \bigcap_{x \in H_i^\beta} S_{k_i}^x$ for all $i \leqslant n$. Hence, $G(M_\alpha, M_\beta)$.

The net result of Lemma 5.6 and Section 1 (Lemma 1.16) is that we can get the consistency of GCH + there is a first-countable, 0-dimensional, separable, 2-

complicated space. So we fix, for the rest of the proof, a 2-complicated, separable space $T = \langle \omega_1, \tau \rangle$ with clopen neighborhood basis $\{S_n^\alpha : \alpha \in \omega_1, n \in \omega\}$ in a model V of ZFC + GCH. (The space constructed in Section 1 is separable, as it is an S-space.)

5.7. DEFINITION. A poset $\mathbb{P}$ is *good* iff for any $p \in \mathbb{P}$, $p \Vdash$ "T is 2-complicated".

We say that $C \subseteq \omega_1$ *shows that* $\mathbb{P}$ *is good* iff $(\forall p \in \mathbb{P}) p \Vdash$ "$\check{C}$ shows that $\mathbb{P}$ is 2-complicated."

It is well known that if $\mathbb{P}$ satisfies the ccc, then any club subset of ω_1 in the extension contains a club in the ground model. Hence, if $\mathbb{P}$ is good and ccc, then for some C (in the ground model) C shows that $\mathbb{P}$ is good.

5.8. LEMMA. *Assume* $\mathbb{P}$ *is a ccc poset, and* C *is a club in* ω_1. C *shows that* $\mathbb{P}$ *is good iff the following holds*:

5.9. *For any sequence* $\langle\langle p_\alpha, M_\alpha\rangle : \alpha \in \omega_1\rangle$ *where* $p_\alpha \in \mathbb{P}$ *and* M_α *are pairwise disjoint matrices of pairs and neighborhoods separated by* C, there are $\alpha < \beta$ *such that* p_α *and* p_β *are compatible in* $\mathbb{P}$ *and* $G(M_\alpha, M_\beta)$.

PROOF. Assume C shows that $\mathbb{P}$ is good. Let $\langle\langle p_\alpha, M_\alpha\rangle : \alpha \in \omega_1\rangle$ be given as in 5.9. By the ccc of $\mathbb{P}$, there is $p \in \mathbb{P}$ such that $p \Vdash$ "there is an uncountable set $I \subseteq \check{\omega}_1$, such that for $i \in I$, $\check{p}_i \in G$" (where G is the name of the generic filter). Since $p \Vdash$ "$\check{C}$ shows that $\mathbb{P}$ is 2-complicated," and since the matrices M_α are separated by C, there are $q \leq p$ and $\alpha, \beta \in \omega_1$, $\alpha < \beta$ such that $q \Vdash$ "$\alpha, \beta \in I$ and $G(\check{M}_\alpha, \check{M}_\beta)$". But then q extends both p_α and p_β and by absoluteness, $G(M_\alpha, M_\beta)$ holds.

Now for the other direction. We want to prove that C shows that $\mathbb{P}$ is good, assuming (5.9) is true. So suppose

5.10. $p \Vdash$ "$\langle \boldsymbol{M}_\alpha : \alpha \in \check{\omega}_1\rangle$ is a sequence of matrices separated by C".

Then for each $\alpha \in \omega_1$ pick $p_\alpha \in \mathbb{P}$, $p_\alpha \leq p$, and a matrix M_α such that $p_\alpha \Vdash$ "$\boldsymbol{M}_\alpha = \check{M}_\alpha$". We can assume that the matrices M_α are pairwise disjoint. (Recall that 5.10 implies, by a notational convention, that $p \Vdash$ "the $\boldsymbol{M}_\alpha$'s are pairwise disjoint." Form a Δ-system out of the M_α's; now if the root is non-empty, then there are $\aleph_1$ pairwise incompatible p_α's.) Finally, apply 5.9 to get p_α, p_β compatible (by $q \leq p$, say) such that $G(M_\alpha, M_\beta)$. Then, obviously $q \Vdash$ "$G(M_\alpha, M_\beta)$". A density argument finishes the proof of the lemma.

5.11. LEMMA. *An iteration with finite support of length* $\lambda < \omega_2$ *of* ccc *good posets is* ccc *and good.*

PROOF. The fact that ccc is preserved by a finite support iteration (of any length) is well known (SOLOVAY and TENNENBAUM [1971]). The proof of the corresponding fact for 'good' is analogous and its details are left to the reader. At each stage of the iteration pick a club (in the ground model! since the iteration is ccc) which shows that the iteration so far is good. At limit stages take a diagonal intersection of all the clubs so far and use the characterization of Lemma 5.8.

All the technical definitions and lemmas are behind us now, and it is time to show how to get a space satisfying (*). Recall, we assume $T=\langle\omega_1,\tau\rangle$ is a 2-complicated space and the S_n^α are the basic clopen sets.

5.12. LEMMA. *Let $X\subseteq\omega_1$ be uncountable and $n\in\omega$ be such that $\bigcap_{x\in X}S_n^x\neq\emptyset$. Let $f\colon X\times\omega_1\to X$ be given with $f(x,\alpha)>\alpha$. Then there is a forcing poset $\mathbb{P}$ of cardinality $\aleph_1$ which is* ccc *and good such that after forcing with $\mathbb{P}$ the following holds: There is an uncountable sequence $\langle\langle x_i,\alpha_i,y_i\rangle\colon i\in\omega_1\rangle$ such that, for $i<j$,* 4.3 *holds.*

PROOF. Before giving the proof let us see how Lemma 5.12 ends the proof of Theorem 5.1. Since $2^{\aleph_1}=\aleph_2$ in the ground model, there is a bookkeeping device which takes care of all possible $X\subseteq\omega_1$ and $f\colon X\times\omega_1\to X$. So we iterate ccc, good posets of cardinality $\aleph_1$, as given by this Lemma, the iteration is with finite support and of length ω_2. Hence, at each stage $\lambda<\omega_2$, the iterated poset is ccc and good; the bookkeeping device gives us at that stage some X and f, and Lemma 5.12 applies to give the next stage of the iteration. This is a familiar argument, and T satisfies property (*) at the end.

Now we turn to the proof of the lemma. Let C be a club which shows that T is 2-complicated. It is easy to pick an uncountable increasing sequence of pairs $\bar F=\langle\langle a_0^i,a_1^i\rangle\colon i\in\omega_1\rangle$, separated by C, such that for $i\in\omega_1$ $f(a_0^i,\alpha)=a_1^i$ for some $\alpha>a_0^i$ (α is chosen in C, if you want, to cause the separation).

Now define

$$\mathbb{P}=\{p\subset\bar F\colon p \text{ is finite (set of pairs), for any } \langle a_0^\alpha,a_1^\alpha\rangle,\ \langle a_0^\beta,a_1^\beta\rangle \text{ in } p, \text{ if } \alpha<\beta \text{ then } G^n(\langle a_0^\alpha,a_1^\alpha\rangle,\langle a_0^\beta,a_1^\beta\rangle).\}$$

So $\mathbb{P}$ is the natural poset for adding uncountably many pairs $\langle x,f(x,\alpha)\rangle$ such that $G^n(u,v)$ holds for any two pairs (with u below v). Observe, however, that the ordinals in p are separated by C, thus the special properties of T can be used. The partial order on $\mathbb{P}$ is inclusion.

A generic filter over $\mathbb{P}$ can perhaps be countable (or even finite); yet once $\mathbb{P}$ is shown to satisfy the ccc, a condition $p\in\mathbb{P}$ can be found which forces the generic filter to be uncountable. We will then take $\mathbb{P}_p=\{q\in\mathbb{P}\colon q\leqslant p\}$ as the next step of the iteration and thus ensure an uncountable collection of pairs, as the conclusion of (*) requires.

5.13. LEMMA. $\mathbb{P}$ *satisfies the* ccc

PROOF. Let $\langle p_i : i \in \omega_1\rangle$ in $\mathbb{P}$ be given, without loss of generality the p_i are disjoint and increasing. The set p_i can be viewed as a finite increasing sequence of pairs, separated by C (since $\bar{F}$ is separated by C and $p_i \subseteq \bar{F}$). Hence $\langle p_i, n\rangle$ is a matrix (one column) of pairs and neighborhoods, separated by C. (Since $\bigcap\{S_n^x : x \in X\} \neq \emptyset$ is assumed, and our pairs are taken from X.) As C shows that T is 2-complicated, there are $i<j$ with $G(\langle p_i, n\rangle, \langle p_j, n\rangle)$, namely $G^n(u, v)$ holds for any $u \in p_i$ and $v \in p_j$. Hence $p_i \cup p_j \in \mathbb{P}$.

5.14. Lemma. $\mathbb{P}$ *is good and* $C \cap C(\bar{F})$ *shows that.*

PROOF. By Lemma 5.8 we have to consider a sequence $\langle\langle p_\alpha, M_\alpha\rangle : \alpha \in \omega_1\rangle$ where $p_\alpha \in \mathbb{P}$, and M_α is a matrix of pairs and neighborhoods, separated by $C \cap C(\bar{F})$, and the ordinals in the M_α's form pairwise disjoint sets. The p_α can be assumed to form a Δ-system, let p'_α be what remains of p_α after throwing away the root of the Δ-system. It is easy to see that p_α, p_β are compatible iff p'_α, p'_β are compatible. Say $M_\alpha = \langle \bar{F}_1^\alpha, \ldots, \bar{F}_m^\alpha, \bar{k}\rangle$ (m and $\bar{k}$ do not depend on α, without loss of generality). Viewing p'_α as an increasing sequence of pairs separated by C, let $M_\alpha^* = \langle p'_\alpha, \bar{F}_1^\alpha, \ldots, \bar{F}_m^\alpha, \langle n\rangle^\frown \bar{k}\rangle$. Then the M_α's are pairwise disjoint matrices separated by C ($C(\bar{F}) \subseteq C(p'_\alpha)$ as $p'_\alpha \subseteq \bar{F}$, and so $\bar{F}_1^\alpha$ is separated by $C \cap C(p'_\alpha)$ etc.).

Now the assumption of the theorem is that T is 2-complicated, and C shows that. Hence, there are $\alpha < \beta$ such that $G(M_\alpha^*, M_\beta^*)$. This just means that p'_α and p'_β (hence p_α and p_β) are compatible, and that $G(M_\alpha, M_\beta)$ holds. So by Lemma 5.8 $\mathbb{P}$ is good.

6. L-Spaces and Martin's Axiom

By Szentmiklóssy's Theorem 2.1 no first-countable L-space can exist under Martin's Axiom and $\neg$CH. How about L-spaces which are not first-countable?

6.1. THEOREM. Con(ZFC) $\to$ Con(ZFC + *Martin's Axiom* + $\neg$CH + *there is an* L-*space*).

Since the basic ideas in the proof of this theorem are similar to those of the corresponding theorem about first-countable S-spaces, we only outline the proof.

6.2. *The Space.* In a generic extension obtained with finite conditions, we get $V_\alpha \subset \omega_1 - \alpha$ with $\alpha \in V_\alpha$ and use the V_α's as clopen sets to generate a topological space T.

A basic neighborhood of α is determined by a finite set or ordinals B and is equal to

$$\bigcap\{V_\xi : \alpha \in V_\xi \text{ and } \xi \in B\} \cap \bigcap\{\omega_1 - V_\xi : \alpha \notin V_\xi \text{ and } \xi \in B\}.$$

The space thus obtained is an L-space, in fact it is 2-complicated in the following sense.

6.3. *Definition of* 2-*complicated.* First define $M = \langle \bar{F}_1, \ldots, \bar{F}_n, N\rangle$ is a *matrix of pairs and neighborhoods separated by* C as follows: $\langle \bar{F}_1, \ldots, \bar{F}_n\rangle$ is a matrix of pairs separated by C (Definition 5.2), and N is a two place function defined on pairs $(\bar{F}_i, x)$ with $x \in \cap \bar{F}_i$ such that

(1) $N(\bar{F}_i, x)$ is a finite set of ordinals containing x (view this set as a basic neighborhood of x).

(2) If $x, x' \in \bigcup \bar{F}_i$ and $x < x'$, then there is $\alpha \in C \cap C(\bar{F}_1) \cap \cdots \cap (\bar{F}_{i-1})$ such that $x < \alpha < x'$ and $N(\bar{F}_i, x) \subset \alpha$.

(3) For $1 \leqslant i \leqslant n$ there is a such that $a \in \bigcap\{N(\bar{F}_i, x) : x \in \bar{F}_i\}$.

Now we say that a space generated by the clopen sets V_α's is 2-*complicated* iff there exists a club $C \subseteq \omega_1$ such that for any sequence $M_\alpha : \alpha \in \omega_1$ of pairwise disjoint matrices of pairs and neighborhoods separated by C, there are $\alpha < \beta$ such that $G(M_\alpha, M_\beta)$. $G(M_\alpha, M_\beta)$ now means that all the ordinals in M_α are below the ordinals in M_β and for any $1 \leqslant i \leqslant n$, for any $\langle a, b\rangle \in \bar{F}^\beta_i$, $\{a, b\} \cap \bigcap\{N(\bar{F}^\alpha_i, x) : x \in \bigcup \bar{F}^\alpha_i\} \neq \emptyset$.

6.4. *Dangerous posets and how to kill them.* If a ccc poset $\mathbb{P}$ introduces an uncountable discrete subspace to T, or even if after forcing with $\mathbb{P}$, T is no longer 2-complicated, then we call $\mathbb{P}$ dangerous and have to kill it—that is to introduce an uncountable antichain to $\mathbb{P}$ using some non-dangerous poset. If $\mathbb{P}$ is dangerous, we can find a sequence $\langle (p_\alpha, M_\alpha) : \alpha \in \omega_1\rangle$ where $p_\alpha \in \mathbb{P}$ and the M_α form a nice Δ-system; such that if $G(M_\alpha, M_\beta)$, then p_α and p_β are incompatible in $\mathbb{P}$. Then we define a poset Q consisting of finite $q \subseteq \omega_1$ such that, for $\alpha, \beta \in q$, $\alpha < \beta \Rightarrow G(M_\alpha, M_\beta)$. It is obvious that Q introduces an uncountable antichain to $\mathbb{P}$. What remains to be checked is that Q itself is not dangerous.

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CHAPTER 9

Covering Properties

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
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1. Introduction

Paracompact spaces were introduced by J. DIEUDONNÉ in 1944 as a natural generalization of compactness and in 1948 A.H. STONE showed that all metric spaces are paracompact. Since that time paracompactness and its associated tools, such as certain refinement techniques and the locally finite or closure preserving conditions, have played a crucial role in the development of many areas of topology and analysis. As might be expected, several other covering properties have naturally evolved from paracompactness and have proven to be effective in helping to describe the inherent topological structures within a variety of fundamental mathematical systems.

This chapter emphasizes certain characterizations of different covering properties, relationships among them, and invariance under mappings, products, subspaces and sums. Included are covering conditions for pseudocompact or countably compact spaces to be compact and a section on the effect of introducing local compactness and local connectedness. By necessity most of the applications of covering properties could not be included here. It is felt that many of the desired results on applications will be found in the appropriate chapters. The countable versions of covering properties (such as countably paracompact) are not developed here but many important results should be found in the chapters on normal Moore spaces, Dowker spaces, products of normal spaces and others.

The reader should be aware of several recent survey articles and books which contain a substantial amount of material on covering properties and applications. Among these are: R.L. BLAIR [1977], E.K. VAN DOUWEN [1980], R. ENGELKING [1977], H.J.K. JUNNILA [1980], J. NAGATA [1968], and P.J. NYIKOS [1977].

Among other conventions we assume the term 'space' refers to a Hausdorff topological space. A mapping is a continuous onto function and may be denoted by $f: X \to Y$. An ordinal is equal to the set of all smaller ordinals and, as a topological space, will have the order topology. Cardinal numbers are initial ordinals and ω, ω_1 are used to denote the first infinite ordinal and the first uncountable ordinal respectively. For any set A the cardinality of A is denoted by $|A|$. $\mathbb{N}$, $\mathbb{R}$, $\mathbb{P}$ and $\mathbb{Q}$ are used to denote the natural numbers, real numbers, irrational numbers and rational numbers respectively and, unless otherwise stated, are assumed to have the usual topologies inherited from $\mathbb{R}$.

We complete this section with a few of the definitions and concepts used throughout the remainder of the paper.

If $\mathcal{U}$ and $\mathcal{V}$ are covers of a set X, the collection $\mathcal{V}$ is said to be a *refinement* of the collection $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ with $V \subset U$. If the collection $\mathcal{V}$ above is not required to cover X, we say $\mathcal{V}$ is a *partial refinement* of $\mathcal{U}$. A refinement $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ of a cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is said to be a *precise refinement* (relative to the index set Λ) if $V_\alpha \subset U_\alpha$ for every $\alpha \in \Lambda$.

A collection $\mathcal{U}$ of subsets of a space X is said to be *discrete* (resp. *locally finite*, *locally countable*) if for every $x \in X$ there is a neighborhood W of x such that W

has nonempty intersection with at most one (resp. finitely many, countably many) element(s) of $\mathscr{U}$. The collection $\mathscr{U}$ is said to be *point-finite* (*point-countable*) if each point $x \in X$ is an element of at most finitely many (countably many) members of $\mathscr{U}$. For later use we observe the following easily verified properties:

1.1. PROPOSITION. (i) *Any locally finite (locally countable) collection is point-finite (point-countable).*

(ii) *If $\mathscr{U}$ is locally finite (locally countable), then so is $\{\bar{U}: U \in \mathscr{U}\}$.*

(iii) *If $\mathscr{U}$ is locally finite (locally countable) and K is compact, then $\{U \in \mathscr{U} | U \cap K \neq \emptyset\}$ is finite (countable).*

(iv) *If $\mathscr{U}$ is locally finite and $\mathscr{W} \subseteq \mathscr{U}$, then $\bigcup\{\bar{W}: W \in \mathscr{W}\} = \mathrm{cl}(\bigcup\{W: W \in \mathscr{W}\})$.*

In other terminology, statement (iii) says that locally finite collections are *compact-finite* and (iv) says that locally finite collections are *closure-preserving.* Notice that a point-finite closure preserving closed collection is always locally finite. However, if $X = [0, 1]$ and $\mathscr{U} = \{(0, 1/n): n \in \mathbb{N}\}$, then $\mathscr{U}$ is a point-finite closure-preserving open collection which is not locally finite. The reader should be cautioned that if $\mathscr{F} = \{F_\alpha: \alpha \in \Lambda\}$ is a closure preserving collection and $E_\alpha \subset F_\alpha$ for each $\alpha \in \Lambda$, then $\{E_\alpha: \alpha \in \Lambda\}$ need not be closure preserving. The proof of the following proposition is left to the reader.

1.2. PROPOSITION. *If $\mathscr{F}$ is a closure preserving collection of closed sets in X and A is a closed subset of X, then $\{F \cap A: F \in \mathscr{F}\}$ is closure preserving.*

If $\mathscr{U}$ is a collection of subsets of X and $x \in X$ it is convenient to use $\mathrm{ord}(x, \mathscr{U})$ (order of x in $\mathscr{W}$) to denote the cardinality of $\{U \in \mathscr{U}: x \in \mathscr{U}\}$. The following proposition gives a frequently used technique for decomposing a point-finite open collection into a countable number of relatively discrete partial refinements. The proof is left for the interested reader.

1.3. PROPOSITION. *Suppose $\mathscr{U}$ is an open collection in a space X. For each $n < \omega$ let $F_n = \{x \in X: \mathrm{ord}(x, \mathscr{U}) \leq n\}$ and for each $x \in X$ let $W_x = \bigcap\{U \in \mathscr{U}: x \in U\}$.*

(i) *Each F_n is closed.*

(ii) *$\{W_x \cap (F_n - F_{n-1}): x \in F_n - F_{n-1}\}$ is relatively discrete and closed in $F_n - F_{n-1}$.*

(iii) *If V is open in X and $F_{n-1} \subset V$, then $\{W_x \cap (F_n - V): x \in F_n - V\}$ is a discrete closed collection in X covering $F_n - V$.*

A collection $\mathscr{U}$ is said to be *σ-locally finite* (*σ-discrete*, etc.) if $\mathscr{U}$ can be expressed as $\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n$ where each $\mathscr{U}_n$ is locally finite (discrete, etc.). For any set $A \subseteq X$ and collection $\mathscr{U}$, $\mathrm{st}(A, \mathscr{U})$ (the star of $\mathscr{U}$ about A) denotes the set $\bigcup\{U \in \mathscr{U}: U \cap A \neq \emptyset\}$. If $x \in X$, $\mathrm{st}(\{x\}, \mathscr{U})$ is simply denoted by $\mathrm{st}(x, \mathscr{U})$.

2. Characterizations of paracompact spaces

A space X is said to be *paracompact* if every open cover of X has a locally finite open refinement. This section concentrates on a few of the available characterizations of the class of paracompact spaces. Some of this will set the tone and provide motivation for various characterizations of other covering properties in later sections. The results of Theorem 2.1 and Theorem 2.3 were given by E.A. MICHAEL in a series of papers in [1953], [1957], and [1959]. Although not given in proper historical order, A.H. Stone's theorem on the paracompactness of metric spaces will follow easily after Theorem 2.6.

2.1. THEOREM. *For a regular space X the following conditions are equivalent.*

(i) *X is paracompact.*

(ii) *Every open cover of X has a σ-locally finite open refinement.*

(iii) *Every open cover of X has a locally finite refinement.*

(iv) *Every open cover of X has a locally finite closed refinement.*

PROOF. (i)$\rightarrow$(ii). Trivial.

(ii)$\rightarrow$(iii). If $\mathcal{U}$ is an open cover of X and (ii) is assumed, then $\mathcal{U}$ has an open refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ where each $\mathcal{V}_n$ is locally finite. For each $n \in N$ let $H_n = \bigcup \mathcal{V}_n$ and let $\mathcal{G}_n = \{V - \bigcup_{k<n} H_k : V \in \mathcal{V}_n\}$. It is easily verified that $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ is a cover of X refining $\mathcal{U}$. To see that $\mathcal{G}$ is locally finite let $x \in X$ and suppose $m \in N$ such that $x \in H_m$. The collection $\bigcup_{i=1}^{m} \mathcal{V}_i$ is locally finite so there is a neighborhood O of x that O has nonempty intersection with at most finitely many elements of $\bigcup_{i=1}^{m} \mathcal{V}_i$. Since $H_m \cap G = \emptyset$ for all $G \in \bigcup_{k>m} \mathcal{G}_k$ is clear that $W = O \cap H_m$ is a neighborhood of x meeting at most finitely many elements of $\mathcal{G}$. Hence $\mathcal{G}$ is the desired locally finite refinement of $\mathcal{U}$.

(iii)$\rightarrow$(iv). If $\mathcal{U}$ is an open cover of X use regularity of X to find an open cover $\mathcal{H}$ such that $\{\bar{H} : H \in \mathcal{H}\}$ refines $\mathcal{U}$. Now apply (iii) to find a locally finite refinement $\mathcal{G}$ of $\mathcal{H}$. The collection $\{\bar{G} : G \in \mathcal{G}\}$ is the desired locally finite closed refinement of $\mathcal{U}$.

(iv)$\rightarrow$(i). If $\mathcal{U}$ is an open cover of X let $\mathcal{H}$ be a locally finite refinement of $\mathcal{U}$. Now if $\mathcal{V}$ is an open cover witnessing the locally finite condition of $\mathcal{H}$ (i.e., $V \in \mathcal{V}$ implies V intersects at most finitely many elements of $\mathcal{H}$), then $\mathcal{V}$ has a locally finite closed refinement $\mathcal{P}$. For each $H \in \mathcal{H}$ pick $U(H) \in \mathcal{U}$ such that $H \subseteq U(H)$ and let

$$G(H) = \operatorname{int}(\operatorname{st}(H, \mathcal{P})) \cap U(H) .$$

Since $H \subset X - \bigcup\{S : S \in \mathcal{P}, S \cap H = \emptyset\} \subset \operatorname{int}(\operatorname{st}(H, \mathcal{P}))$ it is clear that $\mathcal{G} = \{G(H) : H \in \mathcal{H}\}$ covers X and is an open refinement of $\mathcal{U}$. To see that $\mathcal{G}$ is locally finite, let $x \in X$ and let W be an open neighborhood of x intersecting only finitely many elements of $\mathcal{P}$. Since W intersects only finitely many elements of $\mathcal{P}$ and each $S \in \mathscr{P}$ intersects only finitely many elements of $\mathscr{H}$ it follows that W intersects only finitely many elements of $\{\mathrm{st}(H, \mathscr{P}): H \in \mathscr{H}\}$. This says the collection $\{\mathrm{st}(H, \mathscr{P}): H \in \mathscr{H}\}$ is locally finite; hence $\mathscr{G}$ is locally finite and the theorem is proved.

It should be noted that condition (ii) of Theorem 2.1 says that every regular Lindelöf space is parcompact, originally shown by K. MORITA [1948]. The regular condition is necessary here as there do exist Hausdorff Lindelöf spaces which are not paracompact.

The next theorem shows that paracompactness can actually be obtained using refinements considerably weaker than locally finite. A collection $\mathscr{V}$ is said to be *cushioned* in a collection $\mathscr{U}$ if there exists a function $T: \mathscr{V} \to \mathscr{U}$ such that for any $\mathscr{W} \subset \mathscr{V}$ we have $\mathrm{cl}(\bigcup\{W: W \in \mathscr{W}\}) \subset \bigcup\{T(W): W \in \mathscr{W}\}$. If $\mathscr{V} = \{V_\alpha: \alpha \in \Lambda\}$ and $\mathscr{U} = \{U_\alpha: \alpha \in \Lambda\}$ are both indexed by Λ and the cushioning function T is consistent with this indexing (so $T(V_\alpha) = U_\alpha$) it is convenient to ignore the notation of a formal cushioning function and say that $\mathscr{V}$ is cushioned in $\mathscr{U}$ if

$$\mathrm{cl}(\bigcup\{V_\alpha: \alpha \in \Lambda'\}) \subset \bigcup\{U_\alpha: \alpha \in \Lambda'\}$$

for every $\Lambda' \subset \Lambda$. If $\mathscr{U} = \{U_\alpha: \alpha \in \Lambda\}$ is a cover of X and $\mathscr{U}$ has a cushioned refinement $\mathscr{H}$, via a cushioning function T, it is easy to see that $\mathscr{V} = \{V_\alpha: \alpha \in \Lambda\}$ is a (precise) cushioned refinement of $\mathscr{U}$ if each $V_\alpha = \bigcup\{H \in \mathscr{H}: T(H) = U_\alpha\}$.

Recall that a space X is *collectionwise normal* if whenever $\{F_\alpha: \alpha \in \Lambda\}$ is a discrete collection of closed sets in X there exists a disjoint open collection $\{U_\alpha: \alpha \in \Lambda\}$ such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$. Every paracompact space is collectionwise normal, a fact which can be shown using the original definition of paracompactness. Since this construction is similar to a portion of the argument for Theorem 2.3 we include it in the next lemma.

2.2. LEMMA. (i) *If every open cover of a space X has a cushioned refinement, then X is collectionwise normal.*

(ii) *Every paracompact space is collectionwise normal.*

(iii) *If X is normal, $\{F_\alpha: \alpha \in \Lambda\}$ is a discrete collection of closed sets in X, and $\{U_\alpha: \alpha \in \Lambda\}$ is a disjoint collection of open sets such that $F_\alpha \subset U_\alpha$ there is a discrete collection of open sets $\{V_\alpha: \alpha \in \Lambda\}$ such that $F_\alpha \subset V_\alpha$ for each $\alpha \in \Lambda$. Hence, in a collectionwise normal space, a discrete collection of closed sets can be "separated" by a discrete collection of open sets.*

PROOF. For (i) suppose $\{F_\alpha: \alpha \in \Lambda\}$ is a discrete collection of closed sets where we may assume $F_\alpha \neq F_\beta$ if $\alpha \neq \beta$. For each α let

$$U_\alpha = X - \bigcup_{\beta \neq \alpha} F_\beta .$$

Note that $F_\alpha \subset U_\alpha$ and $U_\alpha \cap F_\beta = \emptyset$ if $\alpha \neq \beta$. The collection $\{U_\alpha: \alpha \in \Lambda\}$ is an open cover of X and, by assumption, has a cushioned refinement $\{C_\alpha: \alpha \in \Lambda\}$. Now let $V_\alpha = X - \mathrm{cl}(\bigcup_{\beta \neq \alpha} C_\beta)$. It is easily verified that $\{V_\alpha: \alpha \in \Lambda\}$ is a disjoint collection of open sets with $F_\alpha \subset V_\alpha$ for each $\alpha \in \Lambda$.

In a regular paracompact space X every open cover of X has a precise locally finite closed refinement (compare with 2.1(iv)). This refinement would actually be a cushioned refinement so part (ii) follows once we show every paracompact space is regular. To this end, suppose F is a closed subset of X and $x \in X - F$. For each $z \in F$ there exists an open neighborhood U_z of z such that $x \notin \bar{U}_z$ (using the Hausdorff condition). If X is paracompact the open cover $\{U_z: z \in F\} \cup \{X - F\}$ has a locally finite open refinement $\mathscr{G}$. The sets $V = \bigcup\{G \in \mathscr{G}: G \cap F \neq \emptyset\}$ and $W = X - \bar{V}$ now form the desired separation of F and x.

To show (iii) assume $\{F_\alpha: \alpha \in \Lambda\}$ and $\{U_\alpha: \alpha \in \Lambda\}$ are as stated. Since $\bigcup \{F_\alpha: \alpha \in \Lambda\}$ is closed we can apply normality to find an open set W such that

$$\bigcup\{F_\alpha: \alpha \in \Lambda\} \subset W \subset \bar{W} \subset \bigcup\{U_\alpha: \alpha \in \Lambda\}.$$

Let $V_\alpha = W \cap U_\alpha$; then V_α is an open set containing F_α. If $x \in X$, then either $x \in X - \bar{W}$ and $X - \bar{W}$ is a neighborhood of x missing every V_α, or $x \in U_\beta$ for some $\beta \in \Lambda$ and U_β is a neighborhood of x interesecting only V_β. In any case, x has a neighborhood intersecting at most one element of $\{V_\alpha: \alpha \in \Lambda\}$. This shows $\{V_\alpha: \alpha \in \Lambda\}$ is a discrete collection so the lemma is proved.

It should be noted that the regularity assumption is given in the hypothesis of the next theorem for the convenience of including (ii). The reader may verify that conditions (i), (iii), (iv), (v) each actually imply X is regular.

2.3. THEOREM. *For any regular space X the following conditions are equivalent.*

(i) *X is paracompact.*
(ii) *Every open cover of X has a σ-closure preserving open refinement.*
(iii) *Every open cover of X has a closure-preserving closed refinement.*
(iv) *Every open cover of X has a σ-cushioned open refinement.*
(v) *Every open cover of X has a cushioned refinement.*

PROOF. It suffices to verify (in some order) that (i)→(ii)→(iii)→(v) and (i)→(iv)→(v)→(i). The implication (v)→(i) is the most difficult and will be left until last.

If $\mathscr{U}$ is an open cover of the paracompact space X use regularity to find an open cover $\mathscr{V}$ such that $\{\bar{V}: V \in \mathscr{V}\}$ refines $\mathscr{U}$ and let $\mathscr{G}$ be a locally finite open refinement of $\mathscr{V}$. It is clear that $\mathscr{G}$ is a σ-closure preserving open refinement of $\mathscr{U}$ and (since $\{\bar{G}: G \in \mathscr{G}\}$ refines $\mathscr{U}$) $\mathscr{G}$ naturally induces a cushioned open refinement as follows. For each $U \in \mathscr{U}$, let

$$H(U) = \bigcup\{G \in \mathscr{G}: \bar{G} \subset U\}.$$

The collection $\mathcal{H}=\{H(U): U\in\mathcal{U}\}$, with the obvious cushioning function, forms a cushioned open refinement of $\mathcal{U}$. This verifies (i)→(ii) and (i)→(iv).

The proofs that (ii)→(iii) and (iv)→(v) are similar; we show only (ii)→(iii) and leave (iv)→(v) to the reader. Assume (ii) and let $\mathcal{U}$ be an open cover of X. $\mathcal{U}$ has an open refinement $\mathcal{G}=\bigcup_{n=1}^{\infty}\mathcal{G}_n$ where each $\mathcal{G}_n$ is closure preserving and, since X is regular, we may assume $\{\bar{G}: G\in\mathcal{G}\}$ refines $\mathcal{U}$. For each n let $H_n=\bigcup\mathcal{G}_n$ and let

$$\mathcal{K}_n=\{\bar{G}-\bigcup_{k<n}H_k: G\in\mathcal{G}_n\}.$$

It is easily verified that $\mathcal{K}=\bigcup_{n=1}^{\infty}\mathcal{K}_n$ is a closed cover of X refining $\mathcal{U}$. To show $\mathcal{K}$ is the desired closure preserving refinement let $\mathcal{S}\subset\mathcal{K}$ and suppose $x\in\mathrm{cl}(\bigcup\mathcal{S})$; we want to show $x\in\bar{S}$ for some $S\in\mathcal{S}$. There is some $m\in\mathbb{N}$ such that $x\in H_m$. Hence H_m is a neighborhood of x such that $S\cap H_m=\emptyset$ for every $S\in\mathcal{S}\cap\mathcal{K}_i$, $i>m$. This says that

$$x\in\mathrm{cl}(\bigcup\{S: S\in\mathcal{S}\cap\mathcal{K}_i, 1\leqslant i\leqslant m\})$$

so there must be some $r\leqslant m$ such that $x\in\mathrm{cl}[\bigcup(\mathcal{S}\cap\mathcal{K}_r)]$. Now it suffices to note that $\mathcal{K}_r$ is closure preserving, a fact assured by Proposition 1.2.

(iii)→(v). If the open cover $\mathcal{U}=\{U_\alpha: \alpha\in\Lambda\}$ has a closure preserving closed refinement $\mathcal{G}$, then $\mathcal{G}$ can be considered to be a cushioned refinement. (For the cushioning function use any $T:\mathcal{G}\to\mathcal{U}$ where $G\subset T(G)$.)

(v)→(i). Assume (v) and let $\mathcal{U}=\{U_\alpha: \alpha\in\Lambda\}$ be an open cover of X where Λ is well ordered. To show X is paracompact we actually show $\mathcal{U}$ has a σ-discrete open refinement and apply 2.1(ii). For notational purposes let U_α be denoted by $U(\alpha,1)$ and $\mathcal{U}(1)=\{U(\alpha,1): \alpha\in\Lambda\}$. Let $\mathcal{C}(1)=\{C(\alpha,1): \alpha\in\Lambda\}$ be a (precise) cushioned refinement of $\mathcal{U}(1)$ and for $\alpha\in\Lambda$ let $U(\alpha,2)=U(\alpha,1)-\mathrm{cl}(\bigcup_{\beta<\alpha}C(\beta,1))$. Notice that $\mathcal{U}(2)=\{U(\alpha,2): \alpha\in\Lambda\}$ is an open cover of X. (Clearly each $U(\alpha,2)$ is open and, for $x\in X$, if γ is the first element of Λ such that $x\in U(\gamma,1)$ then $x\in U(\gamma,2)$ since

$$\mathrm{cl}(\bigcup_{\beta<\gamma}C(\beta,1)\subset\bigcup_{\beta<\gamma}U(\beta,1).)$$

Continuing by induction, for each $n\in\mathbb{N}$, there is an open cover $\mathcal{U}(n)=\{U(\alpha,n): \alpha\in\Lambda\}$ and a corresponding cushioned refinement $\mathcal{C}(n)=\{C(\alpha,n): \alpha\in\Lambda\}$ such that

$$U(\alpha,n+1)=U(\alpha,n)-\mathrm{cl}(\bigcup_{\beta<\alpha}C(\beta,n)).$$

For each $x\in X$, $n\in\mathbb{N}$, let $\delta(x,n)$ be the first $\beta\in\Lambda$ such that $x\in C(\beta,n)$. Note that $\delta(x,1)\geqslant\delta(x,2)\geqslant\cdots$. Since a nonincreasing sequence in a well ordered set must be eventually constant there exists $m(x)\in\mathbb{N}$ such that $\delta(x,m(k))=\delta(x,k)$

for all $k \geq m(x)$. Now, for $\alpha \in \Lambda$, $n \in \mathbb{N}$, let

$$F(\alpha, n) = \{x: x \in C(\alpha, n), \alpha = \delta(x, n), n \geq m(x)\}$$

and $\mathcal{F}(n) = \{F(\alpha, n): \alpha \in \Lambda\}$. Clearly $F(\alpha, n) \subset C(\alpha, n) \subset U(\alpha, n)$ and $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$ covers X. We wish to show that each $\mathcal{F}(n)$ is a discrete collection. Let $x \in X$ and pick $k \in N$ such that $k > \max\{m(x), n\}$. Let

$$W = X - \mathrm{cl}(\bigcup\{C(\beta, k): \beta \neq \delta(x, k)\}).$$

Since $x \in C(\delta(x, k), k-1)$ we see that $x \notin U(\beta, k)$ for all $\beta > \delta(x, k)$. Hence

$$x \notin \mathrm{cl}(\bigcup\{C(\beta, k): \beta > \delta(x, k)\}).$$

Also, $x \notin \mathrm{cl}(\bigcup\{C(\beta, k): \beta < \delta(x, k)\})$ for otherwise x could not be an element of $U(\delta(x, k), k+1)$. This says $x \in W$ (in fact, $F(\delta(x, k), k-1) \subset W$). It is clear from the definition of W that $W \cap C(\beta, k) = \emptyset$ if $\beta \neq \delta(x, k)$, hence $W \cap F(\beta, k) = \emptyset$ if $\beta \neq \delta(x, k)$. Observe that $F(\beta, n) \subset F(\beta, k)$ and it follows that W is a neighborhood of x meeting at most one element of $\mathcal{F}(n)$. Since $\mathcal{F}(n)$ is discrete the family $\{\overline{F(\alpha, n)}: \alpha \in \Lambda\}$ is a discrete closed collection and $\overline{F(\alpha, n)} \subset U(\alpha, n) \subset U(\alpha, 1)$. Lemma 2.2 says that X is collectionwise normal and there is a discrete collection $\mathcal{G}(n) = \{G(\alpha, n): \alpha \in \Lambda\}$ of open sets such that $\overline{F(\alpha, n)} \subset G(\alpha, n)$. We may also assume $G(\alpha, n) \subset U(\alpha, 1)$ (otherwise simply use $G(\alpha, n) \cap U(\alpha, 1)$). Now, $\bigcup_{n=1}^{\infty} \mathcal{G}(n)$ is the desired σ-discrete open refinement of $\mathcal{U}$. That completes the proof of Theorem 2.3.

The proof of the above theorem shows that in a paracompact space X every open cover has a σ-discrete open refinement. With the converse provided by Theorem 2.1(ii) the following may be considered as a strengthening of Theorem 2.1(ii) (MICHAEL [1953]).

2.4. COROLLARY. *A regular space X is paracompact if and only if every open cover of X has a σ-discrete open refinement.*

A cover $\mathcal{V}$ of X is said to be a *barycentric refinement* (also known as *delta* or *point-star* refinements) of a cover $\mathcal{U}$ if $\{\mathrm{st}(x, \mathcal{V}): x \in X\}$ refines $\mathcal{U}$. $\mathcal{V}$ is said to be a *star-refinement* of $\mathcal{U}$ if $\{\mathrm{st}(V, \mathcal{V}): V \in \mathcal{V}\}$ refines $\mathcal{U}$.

2.5. LEMMA. *If $\mathcal{U}_2$, $\mathcal{U}_1$, $\mathcal{U}_0$ are covers of X such that $\mathcal{U}_2$ is a barycentric refinement of $\mathcal{U}_1$ and $\mathcal{U}_1$ is a barycentric refinement of $\mathcal{U}_0$, then $\mathcal{U}_2$ is a star-refinement of $\mathcal{U}_0$.*

PROOF. Let $V \in \mathcal{U}_2$ and pick fixed $z \in V$. Now for every $x \in V$ there is some $W_x \in \mathcal{U}_1$ such that $\mathrm{st}(x, \mathcal{U}_2) \subset W_x$. Notice that $z \in W_x$ for every $x \in V$. Thus

$$\mathrm{st}(V, \mathcal{U}_2) = \bigcup_{x\in V} \mathrm{st}(x, \mathcal{U}_2) \subset \bigcup_{x\in V} W_x \subset \mathrm{st}(z, \mathcal{U}_1) \subset U$$

for some $U \in \mathcal{U}_0$. That completes the proof.

Spaces in which every open cover has an open star-refinement are called *fully normal* and were studied by TUKEY [1940] where it was shown that metric spaces were fully normal. The equivalence of fully normal and paracompact was established by A.H. STONE in [1948]. A related criteria for paracompactness was given by A.V. ARHANGEL'SKIĬ in [1961] using the notion of a locally starring sequence of open covers.

If $\mathcal{U}$ is an open cover of X, a sequence $\{\mathcal{V}_n\}_1^\infty$ of open covers of X is said to be *locally starring* in $\mathcal{U}$ if for any $x \in X$ there is some open neighborhood W of x and some $n \in N$ such that $\mathrm{st}(W, \mathcal{V}) \subset U$ for some $U \in \mathcal{U}$.

2.6. THEOREM. *For any space X the following conditions are equivalent.*

(i) *X is paracompact*

(ii) *Every open cover of X has a barycentric open refinement.*

(iii) *Every open cover of X has an open star-refinement.*

(iv) *For every open cover $\mathcal{U}$ of X there is a sequence $\{\mathcal{V}_n\}_1^\infty$ of open covers which is locally starring in $\mathcal{U}$.*

PROOF. (i)$\to$(ii). Suppose $\mathcal{U}$ is an open cover of the paracompact space X. By Theorem 2.1(iv) $\mathcal{U}$ has a locally finite closed refinement $\mathcal{F}$. For each $F \in \mathcal{F}$ pick $U(F) \in \mathcal{U}$ such that $F \subset U(F)$. For each $x \in X$ define

$$W_x = \Big(\bigcap_{x\in F} U(F)\Big) - \bigcup\{H \in \mathcal{F}: x \notin H\}.$$

Clearly each W_x is an open set containing x so $\mathcal{W} = \{W_x: x \in X\}$ is an open cover of X. To see that $\mathcal{W}$ is a barycentric refinement of $\mathcal{U}$ pick $y \in X$ and some $G \in \mathcal{F}$ with $y \in G$. By the definition of W_x, we see that if $y \in W_x$ it must be true that $x \in G$ so that $W_x \subset U(G)$. Hence $\mathrm{st}(y, \mathcal{W}) = \{W_x: y \in W_x\} \subset U(G)$.

(ii)$\to$(iii). If $\mathcal{U}$ is an open cover of $\mathcal{U}$ and (ii) is true there exist open covers $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V}_1$ is a barycentric refinement of $\mathcal{U}$ and $\mathcal{V}_2$ is a barycentric refinement of $\mathcal{V}_1$. Lemma 2.5 now says that $\mathcal{V}_2$ is a star-refinement of $\mathcal{U}$.

(iii)$\to$(iv). Clear.

(iv)$\to$(i). Suppose $\mathcal{U} = \{U_\alpha: \alpha \in A\}$ is an open cover of X and $\{\mathcal{V}_n\}_1^\infty$ is a sequence of open covers which is locally starring in $\mathcal{V}$. According to Theorem 2.3 and the remark preceding Theorem 2.3 it suffices to show $\mathcal{U}$ has a σ-cushioned open refinement. For each $\alpha \in \Lambda$, $n \in N$, let

$$C(\alpha, n) = \bigcup\{V: V \text{ is open and } \mathrm{st}(V, \mathcal{V}_n) \subset U_\alpha\}.$$

The collection $\{C(\alpha, n): \alpha \in \Lambda, n \in N\}$ is an open refinement of $\mathcal{U}$. Fix $n \in N$. To see that $\{C(\alpha, n): \alpha \in \Lambda\}$ is cushioned in $\mathcal{U}$ let $\Lambda' \subset \Lambda$ and suppose $z \in X - \bigcup\{U_\alpha: \alpha \in \Lambda'\}$. Since $\mathrm{st}(C(\alpha, n), \mathcal{V}_n) \subset U_\alpha$ for every $\alpha \in \Lambda'$ and $z \notin U_\alpha$, $\alpha \in \Lambda'$ we must have $\mathrm{st}(z, \mathcal{V}_n) \cap C(\alpha, n) = \emptyset$ for every $\alpha \in \Lambda'$. Thus $z \notin \mathrm{cl}(\bigcup \{C(\alpha, n): \alpha \in \Lambda'\})$ so that

$$\mathrm{cl}(\bigcup\{C(\alpha, n): \alpha \in \Lambda'\}) \subset \bigcup\{U_\alpha: \alpha \in \Lambda'\}.$$

That completes the proof of Theorem 2.6.

If X is a metric space and $\mathcal{V}_n$ denotes the open cover consisting of open spheres of radius $1/n$ it is easy to see that $\{\mathcal{V}_n\}$ is locally starring in $\mathcal{U}$ for any open cover $\mathcal{U}$ of X.

2.7. COROLLARY. *Every metric space is paracompact.*

An open cover $\mathcal{U}$ of X is said to be a *normal cover* if there exists a sequence $\{\mathcal{V}_n\}_1^\infty$ of open covers such that $\mathcal{V}_1$ star refines $\mathcal{U}$ and $\mathcal{V}_{n+1}$ star refines $\mathcal{V}_n$ for every $n \in \mathbb{N}$. Even in the absence of paracompactness normal covers have locally finite open refinements.

2.8. THEOREM. *If $\mathcal{U}$ is a normal cover of the space X, then $\mathcal{U}$ has a locally finite open refinement which is also σ-discrete.*

PROOF. Suppose $\mathcal{U} = \{U_\alpha: \alpha \in \Lambda\}$ is an open cover of X, with Λ well ordered, and $\{\mathcal{V}_n\}_1^\infty$ is a sequence of open covers where $\mathcal{V}_1$ stars refines $\mathcal{U}$ and $\mathcal{V}_{n+1}$ star refines $\mathcal{V}_n$. For every $x \in X$ let $\delta(x)$ be the smallest element $\beta \in \Lambda$ such that $\mathrm{st}(x, \mathcal{V}_n) \subset U_\beta$ for some $n \in \mathbb{N}$. For $\alpha \in \Lambda$, $n \in \mathbb{N}$, let

$$F(\alpha, n) = \{x: \alpha = \delta(x), \mathrm{st}(x, \mathcal{V}_n) \subset U_\alpha\}.$$

and

$$G(\alpha, n) = \mathrm{st}(F(\alpha, n), \mathcal{V}_{n+3}).$$

It is clear that $\{F(\alpha, n): \alpha \in \Lambda, n \in \mathbb{N}\}$ covers X so $\{G(\alpha, n): \alpha \in \Lambda, n \in \mathbb{N}\}$ is an open refinement of $\mathcal{U}$. We let the reader verify that $\{G(\alpha, n): \alpha \in \Lambda\}$ is a discrete collection for each $n \in \mathbb{N}$. To obtain a locally finite refinement a little more surgery is necessary. If

$$K(n) = \bigcup\{\mathrm{st}(F(\alpha, n), \mathcal{V}_{n+4}): \alpha \in \Lambda\}$$

it is easy to show that

$$\overline{K(n)} \subset \bigcup\{G(\alpha, n): \alpha \in \Lambda\}.$$

Let

$$H(\alpha, n) = G(\alpha, n) - \mathrm{cl}(\bigcup_{k<n} K(k)).$$

It follows that $\{H(\alpha, n)\colon \alpha \in \Lambda,\ n \in \mathbb{N}\}$ is the desired σ-discrete open refinement which is also locally finite.

The next result uses a generalization of the star refinement notion and will be used to prove Tamano's Theorem (2.10). It is convenient to say that a finite collection $\mathcal{H}$ is *centered* if $\bigcap \mathcal{H} \neq \emptyset$ and that $\mathcal{H}$ is *centered at* x if $x \in \bigcap \mathcal{H}$.

2.9. THEOREM. *For any space X the following are equivalent.*

(i) *X is paracompact.*

(ii) *For any open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ such that for any $V \in \mathcal{V}$ there is finite $\mathcal{W} \subset \mathcal{U}$ such that* $\mathrm{st}(V, \mathcal{V}) \subset \bigcup \mathcal{W}$ *and* $V \subset \bigcap \mathcal{W}$.

(iii) *For any open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ such that for any $x \in X$ there is an open set V containing x and finite $\mathcal{W} \subset \mathcal{U}$, centered at x, such that* $\mathrm{st}(V, \mathcal{V}) \subset \bigcup \mathcal{W}$.

PROOF. (i)$\rightarrow$(iii). Let $\mathcal{V}$ be an open star refinement of $\mathcal{U}$.

(iii)$\rightarrow$(ii). Let $\mathcal{V}$ be as given in (iii). For each $x \in X$, pick an open set V_x, with $x \in V_x$ and finite $\mathcal{W}_x \subset \mathcal{U}$, with $x \in \bigcap \mathcal{W}_x$, such that $\mathrm{st}(V_x, \mathcal{V}) \subset \bigcup \mathcal{W}_x$. Let $H_x = V_x \cap (\bigcap \mathcal{W}_x)$ and $\mathcal{H} = \{H_x\colon x \in X\}$. Then $\mathcal{H}$ is an open cover of X, and for $H_x \in \mathcal{H}$ we have

$$\mathrm{st}(H_x, \mathcal{H}) \subset \mathrm{st}(V_x, \mathcal{V}) \subset \bigcup \mathcal{W}_x$$

and $H_x \subset \bigcap \mathcal{W}_x$. Hence (ii) is true.

(ii)$\rightarrow$(i). Let $\mathcal{U}_3$, $\mathcal{U}_2$, $\mathcal{U}_1$ be open covers of X such that $\mathcal{U}_3$ refines $\mathcal{U}_2$ and $\mathcal{U}_2$ refines $\mathcal{U}_1$ in the manner given in (ii). Let $\mathcal{U}_3^{\mathrm{FC}}$, $\mathcal{U}_2^{\mathrm{FC}}$, $\mathcal{U}_1^{\mathrm{FC}}$ denote the covers obtained by taking all unions of finite centered subcollections from $\mathcal{U}_3$, $\mathcal{U}_2$, $\mathcal{U}_1$ respectively. For every $U \in \mathcal{U}_3^{\mathrm{FC}}$ let $\mathcal{G}_U$ denote a finite centered subcollection from $\mathcal{U}_3$ such that $U = \bigcup \mathcal{G}_U$. We show that $\mathcal{U}_3^{\mathrm{FC}}$ star refines $\mathcal{U}_1^{\mathrm{FC}}$. Let $V \in \mathcal{U}_3^{\mathrm{FC}}$. Now, there is a finite collection $\mathcal{H} \subset \mathcal{U}_2$ such that $\mathrm{st}(V, \mathcal{U}_3) \subset \bigcup \mathcal{H}$ and $\bigcap \mathcal{G}_V \subset \bigcap \mathcal{H}$. (To see this, for each $S \in \mathcal{G}_V$ pick finite $\mathcal{W}_S \subset \mathcal{U}_2$ such that $\mathrm{st}(S, \mathcal{U}_3) \subset \bigcup \mathcal{W}_S$ and $S \subset \bigcup \mathcal{W}_S$. Let $\mathcal{H} = \bigcup \{\mathcal{W}_S\colon S \in \mathcal{G}_V\}$.) It follows that

$$\mathrm{st}(V, \mathcal{U}_3^{\mathrm{FC}}) \subset \mathrm{st}(\bigcup \mathcal{H}, \mathcal{U}_3) \subset \mathrm{st}(\bigcup \mathcal{H}, \mathcal{U}_2).$$

For each $H \in \mathcal{H}$ there is a finite $\mathcal{W}_H \subset \mathcal{U}_1$ such that $\mathrm{st}(H, \mathcal{U}_2) \subset \bigcup \mathcal{W}_H$ and $H \subset \bigcap \mathcal{W}_H$. Let $\mathcal{K} = \bigcup \{\mathcal{W}_H\colon H \in \mathcal{H}\}$; then $\mathcal{K}$ is a finite centered subcollection from $\mathcal{U}$ and

$$\mathrm{st}(\bigcup \mathcal{H}, \mathcal{U}_2) \subset \bigcup \mathcal{K} \in \mathcal{U}_1^{\mathrm{FC}}.$$

This shows $\mathcal{U}_3^{FC}$ is a star refinement of $\mathcal{U}_1^{FC}$. Repeated application implies that if $\mathcal{U}$ is any open cover of X, then $\mathcal{U}^{FC}$ is a normal cover so, by Theorem 2.8, must have a locally finite open refinement. It is an easy exercise to show that $\mathcal{U}$ itself must have a locally finite open refinement, completing the proof of 2.9.

The remaining characterizations of paracompactness provide contrast to the open cover characterizations given earlier. Theorem 2.10 is due to TAMANO [1960]. The reader might note that the proof does not depend on specifically using βX; any compactification of X could be used in its place.

2.10. THEOREM. *For a Tychonoff space X the following conditions are equivalent.*
(i) *X is paracompact.*
(ii) *$X \times \beta X$ is normal.*

PROOF. (i)$\to$(ii). Since X is paracompact and βX is compact the product $X \times \beta X$ is actually paracompact (see Theorem 6.1) and hence normal.

(ii)$\to$(i). Suppose $X \times \beta X$ is normal and $\mathcal{U}$ is an open cover of X. We show X satisfies 2.9(iii). For each open set $U \subset X$ assign an open set U^* in βX such that $U^* \cap X = U$ and let $\mathcal{U}^* = \{U^* \colon U \in \mathcal{U}\}$. Apply normality to find an open set $W \subset X \times \beta X$ such that

$$\{(x, x) \colon x \in X\} \subset V \subset \bar{V} \subset W = \bigcup\{U \times U^* \colon U \in \mathcal{U}\}.$$

We may assume V can be expressed as $V = \bigcup\{V_\alpha \times V_\alpha^* \colon \alpha \in \Lambda\}$ where V_α is open in X. Now suppose $x \in X$ and let $E = \beta X - \mathrm{st}(x, \mathcal{U}^*)$. It is clear that $(\{x\} \times E) \cap W = \emptyset$ so $(\{x\} \times E) \cap \bar{V} = \emptyset$ and there exist sets W_1, W_2 open in X and βX respectively such that $\{x\} \times E \subset W_1 \times W_2$ and $(W_1 \times W_2) \cap \bar{V} = \emptyset$. We have $(W_1 \times W_2) \cap (V_\alpha \times V_\alpha^*) = \emptyset$ for every α so $W_2 \cap V_\alpha^* = \emptyset$ whenever $W_1 \cap V_\alpha \neq \emptyset$. This says $W_2 \cap V_\alpha^* = \emptyset$ whenever $W_1^* \cap V_\alpha^* \neq \emptyset$ or

$$W_2 \cap \mathrm{st}(W_1^*, \mathcal{V}^*) = \emptyset .$$

Hence $E \cap \overline{\mathrm{st}(W_1^*, \mathcal{V}^*)} = \emptyset$ so

$$\mathrm{cl}_{\beta X}(\mathrm{st}(W_1^*, \mathcal{V}^*)) \subset \mathrm{st}(x, \mathcal{U}^*) .$$

In fact, there exists a finite subcollection $\mathcal{S} \subset \mathcal{U}^*$, centered at x, such that

$$\mathrm{st}(W_1^*, \mathcal{V}^*) \subset \bigcup \mathcal{S} .$$

If $\mathcal{W} = \{S \cap X \colon S \in \mathcal{S}\}$, then $\mathrm{st}(W_1, \mathcal{V}) \subset \bigcup \mathcal{W}$ and the conditions of 2.9(iii) are satisfied.

A *partition of unity* on a space X is a collection $\Phi = \{f_\alpha \colon \alpha \in \Lambda\}$ of continuous functions from X into $[0, 1]$ such that $\Sigma_{\alpha\in\Lambda} f_\alpha(x) = 1$ for every $x \in X$. A partition of unity Φ is called *locally finite* if $\{f^{-1}(0, 1]: f \in \Phi\}$ is locally finite in X. If $\mathcal{U}$ is a cover, a partition of unity Φ is *subordinated to* $\mathcal{U}$ if the cover $\{f^{-1}((0, 1]): f \in \Phi\}$ is a refinement of $\mathcal{U}$. Our last characterization of paracompactness was given by MICHAEL [1953].

2.11. THEOREM. *For any space X the following conditions are equivalent.*

(i) *X is paracompact.*

(ii) *Every open cover of X has a locally finite partition of unity subordinated to it.*

(iii) *Every open cover of X has a partition of unity subordinated to it.*

The proof can be found in ENGELKING [1977].

3. Definitions and characterizations of other covering properties

A space X is *subparacompact* if every open cover has a σ-discrete closed refinement. This condition was introduced by MCAULEY [1958] as 'F_σ-screenable' and condition (v) in Theorem 3.1 was called 'σ-paracompact' by ARHANGEL'SKIǏ [1966]. The property was given its present name in BURKE [1969] where conditions (i), (ii), (iii) and (v) were shown to be equivalent after (i)→(v) was given by BURKE and STOLTENBERG [1969] and ČOBAN [1969]. The equivalence of (iv) was added by H.J.K. JUNNILA in [1978].

3.1. THEOREM. *For any space X the following are equivalent.*

(i) *X is subparacompact.*

(ii) *Every open cover of X has a σ-locally finite closed refinement.*

(iii) *Every open cover of X has a σ-closure preserving closed refinement.*

(iv) *Every open cover of X has a σ-cushioned refinement.*

(v) *For every open cover $\mathcal{U}$ of X there is a sequence $\{\mathcal{G}_n\}_1^\infty$ of open refinements such that for any $x \in X$ there is some $n \in \mathbb{N}$ with $\mathrm{st}(x, \mathcal{G}_n) \subset U$ for some $U \in \mathcal{U}$.*

(vi) *For any open cover $\mathcal{U}$ of X there is a sequence $\{\mathcal{G}_n\}_1^\infty$ of open refinements such that for any $x \in X$ there is some $n \in \mathbb{N}$ with $\mathrm{ord}(x, \mathcal{G}_n) = 1$.*

PROOF. It is clear that (i)→(ii)→(iii)→(iv). Giving a proof essentially due to H. JUNNILA we show first that (iv)→(i). It will be convenient to use the following notation. For n, $k \in \mathbb{N}$ and any finite sequence $s = (i_1, i_2, \ldots, i_k) \in \mathbb{N}^k$ we let $s \oplus n = (i_1, i_2, \ldots, i_k, n)$. Now suppose (iv) is true and $\mathcal{U} = \{U_\alpha: \alpha \in \Lambda\}$ is an open cover of X where Λ is well-ordered. For every $k \in \mathbb{N}$ and every $s \in \mathbb{N}^k$ we define (by induction on k) an open refinement $\mathcal{G}(s)$ of $\mathcal{U}$ and corresponding σ-cushioned refinement $\mathcal{F}(s)$ as follows: For $t \in N$ (a sequence of length 1) let $V_\alpha(t) = W_\alpha(t) = U_\alpha$ for every $\alpha \in \Lambda$ and let

$$\mathcal{G}(t) = \{V_\alpha(t): \alpha \in \Lambda\} \cup \{W_\alpha(t): \alpha \in \Lambda\}.$$

Assume $\mathscr{G}(s)$, an open refinement of $\mathscr{U}$, has been defined for $s \in \mathbb{N}^k$ where $\mathscr{G}(s)$ has the form

$$\mathscr{G}(s) = \{V_\alpha(s) \colon \alpha \in \Lambda\} \cup \{W_\alpha(s) \colon \alpha \in \Lambda\}.$$

For convenience let $G_\alpha(s) = V_\alpha(s) \cup W_\alpha(s)$. Let $\mathscr{F}(s)$ be a σ-cushioned refinement of $\mathscr{G}(s)$ where we may assume $\mathscr{F}(s)$ has the form

$$\mathscr{F}(s) = \{H_\alpha(s \oplus n) \colon \alpha \in \Lambda, n \in \mathbb{N}\} \cup \{K_\alpha(s \oplus n) \colon \alpha \in \Lambda, n \in \mathbb{N}\},$$

and for each $n \in \mathbb{N}$, $\{H_\alpha(s \oplus n) \colon \alpha \in \Lambda\}$ is cushioned in $\{V_\alpha(s) \colon \alpha \in \Lambda\}$ and $\{K_\alpha(s \oplus n) \colon \alpha \in \Lambda\}$ is cushioned in $\{W_\alpha(s) \colon \alpha \in \Lambda\}$. To complete the induction we assume $\mathscr{G}(s \oplus n)$ is defined by letting

$$V_\alpha(s \oplus n) = G_\alpha(s) - \mathrm{cl}\Big(\bigcup_{\beta \neq \alpha} (H_\beta(s \oplus n) \cup K_\beta(s \oplus n))\Big),$$

$$W_\alpha(s \oplus n) = G_\alpha(s) \cap \Big(\bigcup_{\beta > \alpha} G_\beta(s)\Big) - \mathrm{cl}\Big(\bigcup_{\beta < \alpha} (H_\beta(s \oplus n) \cup K_\beta(s \oplus n))\Big),$$

$$\mathscr{G}(s \oplus n) = \{V_\alpha(s \oplus n) \colon \alpha \in \Lambda\} \cup \{W_\alpha(s \oplus n) \colon \alpha \in \Lambda\}.$$

To see that $\mathscr{G}(s \oplus n)$ covers X suppose $x \in X$ but $x \notin \bigcup_\alpha V_\alpha(s \oplus n)$. Pick the smallest $\gamma \in \Lambda$ such that $x \in G_\gamma(s)$. Then $x \in \mathrm{cl}(\bigcup_{\beta > \gamma}(H_\beta(s \oplus n) \cup K_\beta(s \oplus n)))$ and there must be some $\beta > \gamma$ such that $x \in G_\beta(s)$. It follows that $x \in W_\gamma(s \oplus n)$. Hence $\mathscr{G}(s \oplus n)$ covers X.

Now, for $s \in \mathbb{N}^k$, $n \in \mathbb{N}$, $\alpha \in \Lambda$ let

$$T_\alpha(s \oplus n) = \overline{H_\alpha(s \oplus n)} - \bigcup_{\beta \neq \alpha} V_\beta(s).$$

Since $\{H_\alpha(s \oplus n) \colon \alpha \in \Lambda\}$ is cushioned in $\{V_\alpha(s) \colon \alpha \in \Lambda\}$ it is easy to show that $\{T_\alpha(s \oplus n) \colon \alpha \in \Lambda\}$ is a discrete collection (of closed sets). Obviously $T_\alpha(s \oplus n) \subset U_\alpha$ so we are through if we show

$$\{T_\alpha(s \oplus n) \colon \alpha \in \Lambda, n \in \mathbb{N}, s \in \bigcup_{k=1}^{\infty} \mathbb{N}^k\}$$

covers X. To this end, let $z \in X$ and let δ be the least element of

$$\{\beta \colon z \in H_\beta(r) \cup K_\beta(r), r \in \bigcup_{k=1}^{\infty} \mathbb{N}^k\}.$$

There is some $t \in \bigcup_{k=1}^{\infty} \mathbb{N}^k$ and $n \in \mathbb{N}$ such that $z \in H_\delta(t \oplus n) \cup K_\delta(t \oplus n)$. Let $m \in \mathbb{N}$, $\sigma \in \Lambda$, such that

$$z \in H_\sigma(t \oplus n \oplus m) \cup K_\sigma(t \oplus n \oplus m) \subset V_\sigma(t \oplus n) \cup W_\sigma(t \oplus n).$$

It follows from the definition that $z \notin V_\sigma(t\oplus n)$ and $z \notin W_\sigma(t\oplus n)$ if $\alpha > \delta$ so $\sigma = \delta$ and $z \notin \bigcup_{\beta>\delta} G_\beta(t\oplus n)$ implies $z \notin W_\delta(t\oplus n\oplus m)$. Hence $z \in H_\delta(t\oplus n\oplus m)$. Also, since $z \in H_\delta(t\oplus n) \cup K_\delta(t\oplus n)$ we see that $z \notin V_\beta(t\oplus n)$ if $\beta \neq \delta$. This gives

$$z \in H_\delta(t\oplus n\oplus m) - \bigcup_{\beta\neq\delta} V_\beta(t\oplus n) \subset T_\delta(t\oplus n\oplus m)$$

and the proof that (iv)$\to$(i) is complete.

(i)$\to$(vi). Assume (i), suppose $\mathcal{U}$ is an open cover of X, and let $\bigcup_{n=1}^\infty \mathcal{F}_n$ be a closed refinement of $\mathcal{U}$ where each $\mathcal{F}_n$ is discrete. For each $F \in \mathcal{F}_n$ pick $U(F) \in \mathcal{U}$ such that $F \subset U(F)$. If $E_n = \bigcup \mathcal{F}_n$ and $G(F) = U(F) - (E_n - F)$ let

$$\mathcal{G}_n = \{G(F): F \in \mathcal{F}_n\} \cup \{U - E_n: U \in \mathcal{U}\}.$$

It follows that $\{\mathcal{G}_n\}_1^\infty$ is the desired sequence of open refinements.

(vi)$\to$(v). Clear.

(v)$\to$(iv). Suppose $\mathcal{U} = \{U_\alpha: \alpha \in \Lambda\}$ is an open cover of X and $\{\mathcal{G}_n\}_1^\infty$ is a sequence of open refinements as given in (v). For every $n \in N$, $\alpha \in \Lambda$, let

$$C(\alpha, n) = \{x \in X: \mathrm{st}(x, \mathcal{G}_n) \subset U_\alpha\} \quad \text{and} \quad \mathcal{C}_n = \{C(\alpha, n): \alpha \in \Lambda\}.$$

Clearly $\bigcup_{n=1}^\infty \mathcal{C}_n$ is a refinement of $\mathcal{U}$. To see that each $\mathcal{C}_n$ is cushioned in $\mathcal{U}$ suppose $\Lambda' \subset \Lambda$ and let $z \in X - \bigcup\{U_\alpha: \alpha \in \Lambda'\}$. For any $y \in C(\alpha, n)$, some $\alpha \in \Lambda$; we know that $\mathrm{st}(y, \mathcal{G}_n) \subset C(\alpha, n)$ so $z \notin \mathrm{st}(y, \mathcal{G}_n)$ and $y \notin \mathrm{st}(z, \mathcal{G}_n)$. This says $z \notin \mathrm{cl}(\bigcup\{C(\alpha, n): \alpha \in \Lambda'\})$. That completes the proof of Theorem 3.1.

There are several classes of generalized metric spaces which are naturally subparacompact or which can be shown to be subparacompact using one of the conditions in Theorem 3.1. For example, σ-spaces are seen to be subparacompact by condition (i) and Moore spaces clearly satisfy condition (v). See the chapter on generalized metric spaces for appropriate definitions.

A space X is *metacompact* if every open cover of X has a point finite open refinement. X is *weakly θ-refinable* (BENNETT and LUTZER [1972]) if for any open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{G} = \bigcup_{n=1}^\infty \mathcal{G}_n$ such that if $x \in X$ there is some $n \in \mathbb{N}$ such that $1 \leq \mathrm{ord}(x, \mathcal{G}_n) < \omega$. If this condition is strengthened to require that each $\mathcal{G}_n$ also covers X, then X is said to be *θ-refinable* or *submetacompact*. The class of θ-refinable spaces was introduced by WORRELL and WICKE [1965]; the term "submetacompact" was suggested by H. JUNNILA in [1978]. A sequence $\{\mathcal{G}_n\}_1^\infty$ of covers of X is said to be a θ-sequence of covers if for any $x \in X$ there is some $n \in \mathbb{N}$ such that $\mathrm{ord}(x, \mathcal{G}_n) < \omega$.

We turn our attention to characterizations of metacompactness. The next series of lemmas will be used in the proof of Theorem 3.5.

3.2. LEMMA. *Suppose $\mathscr{U} = \{U_\alpha: \alpha \in \Lambda\}$ is an open cover of X, with Λ well ordered, and $V_\alpha = \bigcup\{U_\beta: \beta \leqslant \alpha\}$ for each $\alpha \in \Lambda$. If $\{V_\alpha: \alpha \in \Lambda\}$ has a precise point finite open refinement $\{W_\alpha: \alpha \in \Lambda\}$ and each $X - \bigcup\{W_\alpha: \gamma > \alpha\}$ has a point finite open cover which is a partial refinement of $\{U_\beta: \beta \leqslant \alpha\}$, then $\mathscr{U}$ has a point-finite open refinement.*

PROOF. We may assume that $W_\alpha \neq \emptyset$ implies $W_\alpha \neq W_\beta$ if $\alpha \neq \beta$. For each $\alpha \in \Lambda$, suppose $\mathscr{S}_\alpha$ is a point-finite open cover of $X - \bigcup\{W_\gamma: \gamma > \alpha\}$ such that $S \in \mathscr{S}_\alpha$ implies $S \subset U_\beta$ for some β. Let

$$\mathscr{P}_\alpha = \{W_\alpha \cap S: S \in \mathscr{S}_\alpha, S \subset U_\beta \text{ for some } \beta \leqslant \alpha\}.$$

It is clear that the family $\mathscr{H} = \bigcup_{\alpha\in\Lambda} \mathscr{P}_\alpha$ is a point-finite open collection such that $H \in \mathscr{H}$ implies $H \subset U_\beta$ for some β. To see that $\mathscr{H}$ covers X, let $x \in X$. The set $\{\alpha \in \Lambda: x \in W_\alpha\}$ is finite so let δ be the largest element. It follows that $x \in X - \bigcup\{W_\gamma: \gamma > \delta\}$ so $x \in T$ for some $T \in \mathscr{S}_\delta$. Hence $x \in W \cap T \in \mathscr{P}_\delta$. That completes the proof.

A cover $\mathscr{V}$ is said to be a *pointwise W-refinement* of a cover $\mathscr{U}$ if for any $x \in X$ there is a finite $\mathscr{K} \subset \mathscr{U}$ such that if $x \in V \in \mathscr{V}$, then $V \subset U$ for some $U \in \mathscr{K}$. This concept was introduced by WORRELL [1966a] in order to characterize metacompactness. The next lemma, essentially due to WORRELL, is a weak form of a result from JUNNILA [1979b].

3.3. LEMMA. *If $\{\mathscr{U}_n\}_1^\infty$ is a sequence of open covers of X such that $\mathscr{U}_{n+1}$ is a pointwise W-refinement of $\mathscr{U}_n$, for each $n \in \mathbb{N}$, then $\mathscr{U}_1$ has a σ-point finite open refinement.*

PROOF. Suppose $\mathscr{U}_1 = \{U_\alpha: \alpha \in \Lambda\}$ with Λ well-ordered. Whenever $U \in \bigcup_{n=1}^\infty \mathscr{U}_n$ let $\delta(U)$ denote the smallest $\beta \in \Lambda$ such that $U \subset U_\beta$. For every $n > 1$ let

$$\mathscr{W}_n = \{W \in \mathscr{U}_n: \delta(W) = \delta(U) \text{ whenever } U \in \mathscr{U}_{n-1}, W \subset U\}.$$

To see that $\bigcup_{n=1}^\infty \mathscr{W}_n$ covers X pick $x \in X$ and for every $n > 1$ let

$$\alpha_n = \sup\{\delta(U): x \in U \in \mathscr{U}_n\}.$$

Note that α_n exists in Λ since $\mathscr{U}_n$ is a pointwise W-refinement of $\mathscr{U}_{n-1}$. Also, $\alpha_2 \geqslant \alpha_3 \geqslant \cdots$ so there is some $\gamma \in \Lambda$ and $m \in \mathbb{N}$ such that $\alpha_k = \gamma$ for all $k \geqslant m$. Now there is a finite $\mathscr{K} \subset \mathscr{U}_{m+1}$ such that $\{U \in \mathscr{U}_{m+2}: x \in U\}$ is a partial refinement of $\mathscr{K}$. Clearly there is $K \in \mathscr{K}$ such that $\delta(K) = \gamma$, for otherwise we would have $\alpha_{m+2} < \alpha_{m+1}$. If $U \in \mathscr{U}_m$ with $K \subset U$ we have $\gamma = \delta(K) \leqslant \delta(U) \leqslant \alpha_m = \gamma$ so $\delta(U) = \delta(K)$. Hence $x \in K \in \mathscr{W}_{m+1}$.

To construct the σ-point finite refinement let

$$V_{n\alpha} = \bigcup\{W \in \mathscr{W}_n : \delta(W) = \alpha\}$$

for any $n > 1$, $\alpha \in \Lambda$. If $\mathscr{V}_n = \{V_{n\alpha} : \alpha \in \Lambda\}$ the collection $\bigcup_{n=1}^{\infty} \mathscr{V}_n$ covers x and refines $\mathscr{U}_1$. To see that $\mathscr{V}_n$ is point finite suppose there is a set $\Lambda' \subset \Lambda$ such that $x \in V_{n\alpha}$, for every $\alpha \in \Lambda'$. Pick corresponding $W_\alpha \in \mathscr{W}_n$ such that $x \in W_\alpha$ and $\delta(W_\alpha) = \alpha$ for every $\alpha \in \Lambda'$. There is a finite $\mathscr{H} \subset \mathscr{U}_{n-1}$ such that $\{W_\alpha : \alpha \in \Lambda'\}$ is a partial refinement of $\mathscr{H}$. By the definition of $\mathscr{W}_n$ we know that $W_\alpha \subset H$ for $H \in \mathscr{H}$ implies $\alpha = \delta(W_\alpha) = \delta(H)$. Since $\mathscr{H}$ is finite we see that Λ' is finite and the lemma is proved.

A collection $\mathscr{U}$ of subsets of a space X is said to be *interior-preserving* if $\operatorname{int}(\bigcap \mathscr{W}) = \bigcap\{\operatorname{int} W : W \in \mathscr{W}\}$ for every $\mathscr{W} \subset \mathscr{U}$. Clearly, an open collection is interior-preserving if and only if $\bigcap \mathscr{W}$ is open for any $\mathscr{W} \subset \mathscr{U}$. A collection $\mathscr{V}$ is said to be *well-monotone* if the subset relation $\subset$ is a well-order on $\mathscr{V}$. $\mathscr{V}$ is *directed* if $U, V \in \mathscr{V}$ implies there exists $W \in \mathscr{V}$ such that $U \cup V \subset W$. Notice that a well-monotone collection of open sets is interior-preserving and directed. The notation $\mathscr{V}^{\mathrm{F}}$ denotes the collection of all unions of finite subcollections from $\mathscr{V}$. The following lemma is due to JUNNILA [1979a], [1979b].

3.4. LEMMA. (i) *If the open cover $\mathscr{U}$ has a point finite refinement $\mathscr{H}$ such that $x \in \operatorname{int}(\operatorname{st}(x, \mathscr{H}))$ for every $x \in X$, then $\mathscr{U}$ has an open pointwise W-refinement.*

(ii) *If $\mathscr{U}$ is an interior-preserving open cover of X, then $\mathscr{U}^{\mathrm{F}}$ has a closure-preserving closed refinement if and only if $\mathscr{U}$ has an interior-preserving open pointwise W-refinement.*

(iii) *If $\mathscr{U}$ is a point finite open cover of X, then $\mathscr{U}^{\mathrm{F}}$ has a closure-preserving closed refinement.*

PROOF. (i) For $H \in \mathscr{H}$ pick $U_H \in \mathscr{U}$ such that $H \subset U_H$. For $x \in X$ let

$$V_x = [\operatorname{int}(\operatorname{st}(x, \mathscr{H}))] \cap [\bigcap\{U_H : x \in H \in \mathscr{H}\}].$$

The collection $\mathscr{V} = \{V_x : x \in X\}$ is the desired open pointwise W-refinement of $\mathscr{U}$.

(ii) If $\mathscr{F}$ is a closure-preserving closed refinement of $\mathscr{U}^{\mathrm{F}}$ and $x \in X$ let

$$V_x = [\bigcap\{U \in \mathscr{U} : x \in \mathscr{U}\}] - \bigcup\{F \in \mathscr{F} : x \notin F\}.$$

Then $\{V_x : x \in X\}$ is an interior-preserving open pointwise W-refinement of $\mathscr{U}$.

Conversely, suppose $\mathscr{V}$ is an interior-preserving open pointwise W-refinement of $\mathscr{U}$. For $G \in \mathscr{U}^{\mathrm{F}}$ let

$$P_G = \{x \in X : \operatorname{st}(x, \mathscr{V}) \subset G\}.$$

The collection $\mathscr{P} = \{P_G : G \in \mathscr{U}^{\mathrm{F}}\}$ is a closure-preserving closed refinement of $\mathscr{U}^{\mathrm{F}}$.

(iii) This follows from (ii) since a point finite open cover of X is an interior-preserving open pointwise W-refinement of itself.

Our next theorem gives several equivalences of metacompactness. The reader is referred to JUNNILA [1978], [1979a], [1979b] for the development of the very powerful and useful conditions (iii), (iv), (v) and (vi). The equivalence of condition (vi) is due to WORRELL [1966a], [1966b]. The portion (i)$\leftrightarrow$(ii) was announced by W.B. SCONYERS [1971]; conditions analogous to (ii) and (iii) were used by J. MACK [1967] to characterize the class of paracompact spaces.

3.5. THEOREM. *For any space X the following are equivalent.*

(i) *X is metacompact.*

(ii) *Every well-monotone open cover of X has a point finite open refinement.*

(iii) *Every directed open cover of X has a closure preserving closed refinement.*

(iv) *For every open cover $\mathcal{U}$ of X, $\mathcal{U}^{\mathrm{F}}$ has a closure preserving closed refinement.*

(v) *Every open cover $\mathcal{U}$ of X has a point finite refinement of $\mathcal{H}$ such that $x \in \operatorname{int}(\operatorname{st}(x, \mathcal{H}))$ for every $x \in X$.*

(vi) *Every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{H}$ such that if $x \in X$ there is finite $\mathcal{K} \subset \mathcal{U}$ where if $x \in H \in \mathcal{H}$, then $H \subset U$ for some $U \in \mathcal{K}$ (i.e., $\mathcal{H}$ is a pointwise W-refinement of $\mathcal{U}$).*

PROOF. (i)$\to$(ii). Clear.

(ii)$\to$(i). Assume (ii) is true. If X is not metacompact there is a smallest cardinal number μ such that there exists an open cover $\mathcal{U}$ of X with no point finite open refinement and $|\mathcal{U}| = \mu$. By this choice of $\mathcal{U}$ we know that whenever $\mathcal{W}$ is an open cover of X with $|\mathcal{W}| < |\mathcal{U}|$, then $\mathcal{W}$ has a point finite open refinement. Express $\mathcal{U}$ as $\mathcal{U} = \{U_\alpha : \alpha < \mu\}$ and for each $\alpha < \mu$, let $V_\alpha = \bigcup_{\beta \leq \alpha} U_\beta$. The collection $\mathcal{V} = \{V_\alpha : \alpha < \mu\}$ is a well-monotone open cover of X so there is a (precise) point finite open refinement $\{W_\alpha : \alpha < \mu\}$ of $\mathcal{V}$. For $\alpha < \mu$ let $F_\alpha = X - \bigcup\{W_\beta : \beta > \alpha\}$. Then $\{X - F_\alpha\} \cup \{U_\beta : \beta \geq \alpha\}$ is an open cover (of X) of cardinality less then μ and, by the minimal condition on μ, must have a point finite open refinement $\mathcal{Q}_\alpha$. If

$$\mathcal{S}_\alpha = \{Q \in \mathcal{Q}_\alpha : Q \cap F_\alpha \neq \emptyset\}$$

the conditions of Lemma 3.2 are satisfied so $\mathcal{U}$ must have a point finite open refinement. This contradiction completes the proof that (ii)$\to$(i).

(iii)$\leftrightarrow$(iv). Clear, since $\mathcal{U}^{\mathrm{F}}$ is always directed and, if $\mathcal{U}$ is directed, then $\mathcal{U}^{\mathrm{F}}$ is a refinement of $\mathcal{U}$.

(i)$\to$(iv). Lemma 3.4(iii).

(iv)$\to$(ii). Assume (iv) is true (we may also use (iii)) and let $\mathcal{U}$ be a well-monotone open cover of X. Clearly $\mathcal{U}$ is an interior-preserving open cover of S. By repeated use of Lemma 3.4(ii) there is a sequence $\{\mathcal{U}_n\}_1^\infty$ of open covers of X such that $\mathcal{U} = \mathcal{U}_1$ and $\mathcal{U}_{n+1}$ is an (interior-preserving) open pointwise W-

refinement of $\mathcal{U}_n$. By Lemma 3.3 $\mathcal{U}$ has an open refinement $\bigcup_{n=1}^{\infty}\mathcal{V}_n$ where each $\mathcal{V}_n$ is point finite. For each n let

$$G_n = \bigcup\{V\colon V \in \mathcal{V}_k, k \leqslant n\}.$$

Now $\{G_n : n \in \mathbb{N}\}$ is a directed open cover of X and must have a closure-preserving closed refinement $\mathcal{F}$ which we may express as $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ where $F_n \subset G_n$. Let

$$\mathcal{H}_n = \{V - \bigcup_{k<n} F_n : V \in \mathcal{V}_n\}.$$

It is easily verified that $\bigcup_{n=1}^{\infty}\mathcal{H}_n$ is a point finite open refinement of $\mathcal{U}$.

(i)→(v). Trivial.

(v)→(vi). Lemma 3.4(i).

(vi)→(i). It is clear from repeated use of (vi) and Lemma 3.3 that a given open cover $\mathcal{U}$ of X has an open refinement $\bigcup_{n=1}^{\infty}\mathcal{V}_n$ where each $\mathcal{V}_n$ is point finite. For each n let $G_n = \bigcup\{V\colon V \in \mathcal{V}_k, k \leqslant n\}$ and let $\mathcal{W}$ be a pointwise W-refinement of $\{G_n : n \in \mathbb{N}\}$. Since $\{G_n : n \in \mathbb{N}\}$ is actually directed, it is clear that $\mathcal{W}$ is a barycentric refinement of $\{G_n : n \in \mathbb{N}\}$. If

$$E_n = \{x : \operatorname{st}(x, \mathcal{W}) \subset G_n\}$$

then

$$\bar{E}_n \subset G_n \quad \text{and} \quad X = \bigcup_{n=1}^{\infty} \bar{E}_n.$$

Let

$$\mathcal{H}_n = \{V - \bigcup_{k<n} \bar{E}_k : n \in \mathbb{N}\},$$

then $\bigcup_{n=1}^{\infty}\mathcal{H}_n$ is a point finite open refinement of $\mathcal{U}$. That completes the proof of the theorem.

Our next theorem gives several characterizations of submetacompactness analogous to those in Theorem 3.5. We do not prove Theorem 3.6 but refer the reader to JUNNILA [1978], [1980] for the nontrivial proofs. The equivalence of (vi) is due to WORRELL [1967].

3.6. THEOREM. *For any space X the following are equivalent.*

(i) *X is submetacompact.*

(ii) *Every well-monotone open cover of X has a θ-sequence of open refinements.*

(iii) *Every directed open cover of X has a σ-closure preserving closed refinement.*

(iv) *For every open cover $\mathcal{U}$ of X, $\mathcal{U}^{\mathrm{F}}$ has a σ-closure preserving closed refinement.*

(v) *Every open cover of X has a θ-sequence of refinements $\{\mathcal{H}_n\}_1^{\infty}$ such that if $x \in X$, $n \in \mathbb{N}$, then $x \in \operatorname{int}(\operatorname{st}(x, \mathcal{H}_n))$.*

(vi) *Every open cover of X has a sequence $\{\mathcal{H}_n\}_1^{\infty}$ of open refinements such that if*

$x \in X$ there is some $n \in \mathbb{N}$ and finite $\mathcal{K} \subset \mathcal{U}$ such that if $x \in H \in \mathcal{H}_n$, then $H \subset U$ for some $U \in \mathcal{K}$.

This list of available equivalences for the class of weakly θ-refinable spaces is not as long as that for the submetacompact spaces. Condition (ii) in the following is especially useful however. Theorem 3.7 appeared in BENNETT and LUTZER [1972].

3.7. THEOREM. *For any space X the following are equivalent.*

(i) *X is weakly θ-refinable.*

(ii) *Any open cover $\mathcal{U}$ of X has an open refinement $\bigcup_{n=1}^{\infty}\mathcal{G}_n$ where for any $x \in X$ there is some $n \in \mathbb{N}$ such that* $\operatorname{ord}(x, \mathcal{G}_n) = 1$.

(iii) *Any open cover $\mathcal{U}$ of X has a refinement $\bigcup_{n=1}^{\infty}\mathcal{F}_n$ such that each collection $\mathcal{F}_n$ is closed and discrete relative to $\bigcup\mathcal{F}_n$.*

PROOF. (ii)$\rightarrow$(i). Trivial.

(i)$\rightarrow$(iii). Suppose $\mathcal{U}$ is an open cover of X and $\bigcup_{n=1}^{\infty}\mathcal{G}_n$ is an open refinement of $\mathcal{U}$ such that for any $x \in X$ there is $n \in \mathbb{N}$ with $0 < \operatorname{ord}(x, \mathcal{G}_n) < \omega$. For each $x \in X$, n, $k \in \mathbb{N}$ let

$$W(x, n) = \bigcap\{G \in \mathcal{G}_n : x \in G\} \quad \text{and} \quad E(n, k) = \{z \in X : \operatorname{ord}(z, \mathcal{G}_n) = k\}.$$

As in Proposition 1.3 the collection $\mathcal{F}_{nk} = \{W(x, n) \cap E(n, k) : x \in X\}$ is closed and discrete relative to $\bigcup\mathcal{F}_{nk}$. Clearly $\bigcup\{\mathcal{F}_{nk} : n, k \in \mathbb{N}\}$ is a refinement of $\mathcal{U}$.

(iii)$\rightarrow$(ii). Suppose $\mathcal{U}$ is an open cover of X and $\bigcup_{n=1}^{\infty}\mathcal{F}_n$ is given as in (iii). For each $F \in \mathcal{F}_n$ there is an open set G_F such that $F \subset G_F \subset U$ for some $U \in \mathcal{U}$ and $F' \cap G_F = \emptyset$ whenever $F, F' \in \mathcal{F}_n$, $F \neq F'$. If $\mathcal{G}_n = \{G_F : F \in \mathcal{F}_n\}$, the open collection $\bigcup_{n=1}^{\infty}\mathcal{G}_n$ is the desired refinement.

A space X is *para-Lindelöf* (*σ-para-Lindelöf*) if every open cover of X has a locally countable (σ-locally countable) open refinement. Recent interest in these spaces and spaces with a σ-locally countable base has generated a number of papers (BURKE [1979], [1980], DAVIS–GRUENHAGE-NYIKOS [1978], FLEISSNER [1979], FLEISSNER–REED [1977], NAVY [1980]). Available techniques for handling and refining locally countable collections are somewhat limited. We give two characterizations from BURKE [1980] in the next two theorems; other results from the above papers will be mentioned in the next section.

3.8. THEOREM. *For any space X the following are equivalent.*

(i) *X is para-Lindelöf.*

(ii) *For any open cover $\mathcal{U}$ of X there is a locally countable refinement $\mathcal{H}$ such that if $x \in X$, then $x \in \operatorname{int}(\operatorname{st}(x, \mathcal{H}))$.*

PROOF. The (i)→(ii) portion is trivial so assume (ii) is true and suppose $\mathscr{U}$ is an open cover of X. Let $\mathscr{H}$ be a locally countable refinement as given in (ii) and let $\mathscr{V}$ be an open cover of X such that every element of $\mathscr{V}$ intersects at most countably many elements of $\mathscr{H}$. Now there is a locally countable refinement of $\mathscr{P}$ of $\mathscr{V}$ such that $x \in \operatorname{int}(\operatorname{st}(x, \mathscr{P}))$ for every $x \in X$. For each $H \in \mathscr{H}$ pick $U(H) \in \mathscr{U}$ such that $H \subset U(H)$ and let

$$G(H) = \operatorname{int}(\operatorname{st}(H, \mathscr{P})) \cap U(H).$$

Since $H \subset \operatorname{int}(\operatorname{st}(H, \mathscr{P}))$ it is clear that $\mathscr{G} = \{G(H): H \in \mathscr{H}\}$ covers X and hence is an open refinement of $\mathscr{U}$. To show $\mathscr{G}$ is locally countable let $x \in X$ and let W be an open neighborhood of x intersecting only countably many elements of $\mathscr{P}$. Now, since W intersects only countably many elements of $\mathscr{P}$ and each $P \in \mathscr{P}$ intersects only countably many elements of $\mathscr{H}$, it follows that W intersects only countably many elements of $\{\operatorname{st}(H, \mathscr{P}): H \in \mathscr{H}\}$. This implies $\mathscr{G}$ is locally countable and the theorem is proved.

The proof of the next result is similar to the proof of Theorem 3.8 and is omitted.

3.9. THEOREM. *For any space X the following are equivalent.*

(i) *X is σ-para-Lindelöf.*

(ii) *Any open cover $\mathscr{U}$ of X has a refinement $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ where each $\mathscr{H}_n$ is locally countable and if $x \in X$ there is some n such that $x \in \operatorname{int}(\operatorname{st}(x, \mathscr{H}_n))$.*

In the space ω_1, with the order topology, every open cover has a locally countable closed refinement. It is easy to show ω_1 is not para-Lindelöf so this shows Theorem 3.8 cannot be strengthened by dropping the "$x \in \operatorname{int}(\operatorname{st}(x, \mathscr{H}))$" condition in (ii).

A family $\mathscr{A}$ of subsets of X is said to be *star-finite* (*star-countable*) if for any $A \in \mathscr{A}$ the collection $\{B \in \mathscr{A}: B \cap A \neq \emptyset\}$ is finite (countable). Clearly a star-finite open cover of X is locally finite. A space X is *strongly paracompact* (also called *hypocompact*) if every open cover of X has a star-finite open refinement. Although it is clear that every strongly paracompact space is paracompact the converse is not true. In fact, we will see that there exist metric spaces which are not strongly paracompact. The next lemma shows that star-countable collections can be decomposed into 'disjoint' countable subcollections.

3.10. LEMMA. *Any star-countable collection $\mathscr{A}$ can be expressed as $\mathscr{A} = \bigcup\{\mathscr{B}_\alpha: \alpha \in \Lambda\}$ where each subcollection $\mathscr{B}_\alpha$ is countable and $(\bigcup\mathscr{B}_\alpha) \cap (\bigcup\mathscr{B}_\beta) = \emptyset$ whenever $\alpha \neq \beta$.*

PROOF. For $A, B \in \mathscr{A}$ let us say that a chain from A to B is a finite collection $\{C_1, C_2, \ldots, C_n\} \subset \mathscr{A}$ where $A = C_1$, $B = C_n$, and $C_i \cap C_{i+1} \neq \emptyset$ for $1 \leq i < n$. Now, if $A \in \mathscr{A}$ let

$$\mathscr{B}(A) = \{B \in \mathscr{A}\colon \text{there is a chain (in } \mathscr{A}) \text{ from } A \text{ to } B\}.$$

It is easy to see that each $\mathscr{B}(A)$ is a countable collection and, for $A_1, A_2 \in \mathscr{A}$, $(\bigcup\mathscr{B}(A_1)) \cap (\bigcup\mathscr{B}(A_2)) \neq \emptyset$ if and only if $\mathscr{B}(A_1) = \mathscr{B}(A_2)$. The conclusion of the lemma now follows.

3.11. LEMMA. *If $\mathscr{U} = \{U_k\colon k \in \mathbb{N}\}$ is a countable open cover of the normal space X and $\mathscr{U}$ has a precise closed refinement $\{F_k\colon k \in N\}$, then $\mathscr{U}$ has a star-finite open refinement.*

PROOF. For any $k \in \mathbb{N}$ we may use normality to find open sets $G(k, n)$, for each $n \in \mathbb{N}$, such that

$$F_k \subset G(k, n) \subset \overline{G(k, n)} \subset G(k, n+1) \subset U_k.$$

For each $n \in \mathbb{N}$ let

$$V_n = \bigcup\{G(i, n)\colon 1 \leq i \leq n\}.$$

Let $H(1, 1) = V_2 \cap G(1, 2)$ and for $n > 1$, $1 \leq k \leq n$ let

$$H(k, n) = (V_{n+1} - \bar{V}_{n-1}) \cap G(k, n+1).$$

It is straightforward (albeit a bit tedious) to show that $\{H(k, n)\colon k,\ n \in \mathbb{N},\ 1 \leq k \leq n\}$ is a star-finite open refinement of $\mathscr{U}$.

The principal characterization of strongly paracompact is due to SMIRNOV [1956].

3.12. THEOREM. *For a regular space X the following conditions are equivalent.*

(i) *X is strongly paracompact.*

(ii) *Every open cover of X has a star-countable open refinement.*

PROOF. (i)$\to$(ii) is trivial so to show (ii)$\to$(i) suppose $\mathscr{U}$ is an open cover of X. If $\mathscr{U}$ has a star-countable open refinement $\mathscr{V}$ then Lemma 3.10 says that $\mathscr{V}$ can be expressed as $\mathscr{V} = \bigcup\{\mathscr{B}_\alpha\colon \alpha \in \Lambda\}$ where each subcollection $\mathscr{B}_\alpha$ is countable and $(\bigcup\mathscr{B}_\alpha) \cap (\bigcup\mathscr{B}_\beta) = \emptyset$ whenever $\alpha \neq \beta$. First we note that (ii) implies X is paracompact for if we express

$$\mathscr{B}_\alpha = \{B_{\alpha n}\colon n \in \mathbb{N}\},$$

then $\mathscr{G}_n = \{B_{\alpha n}\colon \alpha \in \Lambda\}$ is a discrete open collection for each fixed n. This says that $\bigcup_n \mathscr{G}_n$ is a σ-discrete open refinement of the arbitrary open cover $\mathscr{U}$ and we apply Theorem 2.1(ii). Now, for every $\alpha \in \Lambda$, let $Z_\alpha = \bigcup\mathscr{B}_\alpha$. Then Z_α is an open and closed subspace of the paracompact space X so Z_α is paracompact. The countable open cover $\mathscr{B}_\alpha$ of Z_α has a precise closed refinement $\mathscr{F}_\alpha$ so, by Lemma 3.11, $\mathscr{B}_\alpha$ has a star-finite open refinement $\mathscr{H}_\alpha$ (as a cover of Z_α). It is clear that $\bigcup_{\alpha\in\Lambda}\mathscr{H}_\alpha$ is a star-finite open refinement of $\mathscr{U}$ so the theorem is proved.

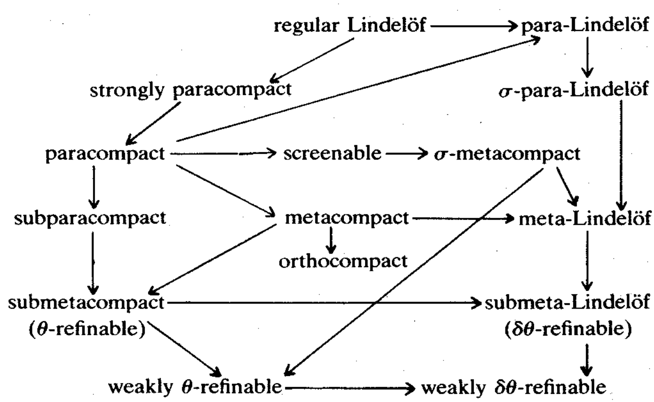
We conclude this section with definitions of a few additional covering properties. X is said to be *screenable* (*σ-metacompact*) if every open cover of X has a σ-disjoint (σ-point-finite) open refinement. Screenable spaces have played an important role in the study of Moore spaces and metrizability (see BING [1951]).

A space X is *meta-Lindelöf* if every open cover of X has a point countable open refinement. X is said to be *weakly $\delta\theta$-refinable* if for any open cover $\mathscr{U}$ of X there is an open refinement $\mathscr{G} = \bigcup_{n=1}^{\infty}\mathscr{G}_n$ of $\mathscr{U}$ such that if $x \in X$ there is some n with $1 \leqslant \operatorname{ord}(x, \mathscr{G}_n) \leqslant \omega$. If the above condition is strengthened to require that each $\mathscr{G}_n$ covers X then X is said to be *$\delta\theta$-refinable* (or *submeta-Lindelöf*).

A space X is *orthocompact* if every open cover of X has an interior-preserving open refinement.

4. Relationships among covering properties

4.1. DIAGRAM.



The diagram above illustrates the general relationships between covering properties mentioned in the last two sections. All of the implications follow easily from the definitions or characterizations given earlier, and it is known that none of the implications are reversible. With few exceptions, it is also known that other implications, not indicated here, are not generally true. One notable exception is in the class of para-Lindelöf spaces.

The reader may be interested in several properties, weaker than 'weakly $\delta\theta$-refinable', which are discussed in Section 6 of Vaughan's article. We continue with a selection of examples which can be used to compare different combinations of covering properties.

4.2. EXAMPLE. A locally compact metacompact space which is not subparacompact.

Let $X = \omega_2 \times \omega_2 - \{(0,0)\}$. For $\alpha \in \omega_2 - \{0\}$ define

$$H_\alpha = \omega_2 \times \{\alpha\} \quad \text{and} \quad V_\alpha = \{\alpha\} \times \omega_2 .$$

Describe a topology on X as follows: For $\alpha \in \omega_2 - \{0\}$ neighborhoods of $(0, \alpha)$ must contain $(0, \alpha)$ and all but finitely many points of H_α. Neighborhoods of $(\alpha, 0)$ must contain $(\alpha, 0)$ and all but finitely many points of V_α. All other points of X are isolated. Since each H_α or V_α is compact it is clear that X is locally compact. Also, any open cover $\mathcal{U}$ of X has a natural open refinement $\mathcal{H}$ where $\operatorname{ord}(x, \mathcal{H}) \leq 2$, for every $x \in X$, so X is metacompact. The following claim will be used to show that X is not subparacompact.

Claim. Suppose A and B are subsets of X such that $A \cap V_\alpha$ and $B \cap H_\alpha$ are countable for every $\alpha \in \omega_2$. Then $X \neq A \cup B$.

PROOF. Suppose $X = A \cup B$. Let

$$\beta_0 = \sup\{\beta : (\alpha, \beta) \in A \cap V_\alpha, \alpha \in \omega_1\} .$$

Then $\beta_0 < \omega_2$ and it follows that $\omega_1 \times (\omega_2 - (\beta_0 + 1))$ is a nonempty subset of X disjoint from A. Thus $\omega_1 \times (\omega_2 - (\beta_0 + 1)) \subset B$. Let $\delta \in \omega_2 - (\beta_0 + 1)$. Then $\omega_1 \times \{\delta\} \subset B \cap H_\delta$ which is in contradiction with the original condition placed on B.

Now, to show X is not subparacompact we let

$$\mathcal{G} = \{V_\alpha : 0 < \alpha < \omega_2\} \cup \{H_\alpha : 0 < \alpha < \omega_2\}$$

and show $\mathcal{G}$ cannot have a σ-discrete refinement. Assume $\mathcal{P} = \bigcup_{n=1}^\infty \mathcal{P}_n$ is a refinement of $\mathcal{G}$ where each $\mathcal{P}_n$ is a discrete collection. Let $\mathcal{V}_n$ and $\mathcal{H}_n$ be all elements of $\mathcal{P}_n$ which are contained in elements of $\{V_\alpha : 0 < \alpha < \omega_2\}$ and $\{H_\alpha : 0 < \alpha < \omega_2\}$ respectively. Note that $\mathcal{P}_n = \mathcal{V}_n \cup \mathcal{H}_n$. For any n, let $A_n = \bigcup \mathcal{H}_n$ and $B_n = \bigcup \mathcal{V}_n$. If $\alpha \in \omega_2$ there is a nieghborhood of $(\alpha, 0)$ which intersects at most one element of $\mathcal{H}_n$. Hence V_α intersects only finitely elements of $\mathcal{H}_n$ and it follows that $A_n \cap V_\alpha$ is finite. Similarly $B_n \cap H_\alpha$ is finite. Let $A = \bigcup_{n=1}^\infty A_n$ and $B = \bigcup_{n=1}^\infty B_n$. Then A and B satisfy the conditions in the above claim and consequently cannot cover X. Thus $\mathcal{P}$ does not cover X.

Notice that the weight of X in this example is ω_2. As we will see in Theorem 4.25, the weight of such a space would have to be greater than ω_1 so the corresponding construction would not work on $\omega_1 \times \omega_1$.

4.3. EXAMPLE. A locally compact metacompact subparacompact space Y which is not paracompact.

Let X be the space of Example 4.2 and let $Y = X \cap (\omega_2 \times \omega)$ with the relative topology inherited from X. Y is clearly metacompact (being a closed subspace of

X) and it is easy to show Y is subparacompact using the fact that Y can be expressed as a union of a countable number of closed paracompact subspaces (see Theorem 7.3). Y is not paracompact since it is not normal (the sets $E = \{(0, \alpha): 0 < \alpha < \omega\}$ and $F = \{(\alpha, 0): 0 < \alpha < \omega_2\}$ are two disjoint closed sets which cannot be separated in Y).

The next two examples use the same basic construction. If D is an infinite set a collection $\mathscr{C}$ of subsets of D is said to be an *almost disjoint* collection if $|A \cap B| < \omega$ whenever $A, B \in \mathscr{C}$, $A \neq B$. Using Zorn's Lemma, there exists an infinite collection $\mathscr{A}$ of countably infinite subsets of D such that $\mathscr{A}$ is an almost disjoint collection and maximal with respect to these properties. The collection $\mathscr{A}$ must be uncountable for if $\mathscr{A} = \{A_1, A_2, A_3, \ldots\}$ there would exist a countably infinite set $B \subset D$ such that $|B \cap A_i| = 1$ for every i. The collection $\mathscr{A} \cup \{B\}$ would now contradict the maximal condition on $\mathscr{A}$. (The cardinality of $\mathscr{A}$ depends on the cardinality of D and the set theory assumed.) Let $\psi(D) = \mathscr{A} \cup D$ and describe a topology on $\psi(D)$ as follows: The points of D are isolated. Basic neighborhoods of a point $A \in \mathscr{A}$ are sets of the form $\{A\} \cup (A - F)$ where F is a finite subset of D. With this topology on $\psi(D)$ the set $\mathscr{A}$ is a closed discrete subset of $\psi(D)$ and D is an open conditionally compact subset (i.e., any infinite subset of D has a limit point in $\mathscr{A}$). We also note that $\psi(D)$ is always locally compact and weakly θ-refinable.

4.4. EXAMPLE. A locally compact Moore space (hence subparacompact) which is not meta-Lindelöf (hence not metacompact).

For this example we use $\psi(N)$. This is the space ψ described in GILLMAN and JERISON [1960], Exercise 5I. It is routine to show that $\psi(N)$ is a Moore space. To see that X is not meta-Lindelöf consider the open cover $\mathscr{U} = \{\{A\} \cup N: A \in \mathscr{A}\}$. Using the fact that $\mathscr{A}$ is uncountable it follows that $\mathscr{U}$ cannot have a point countable open refinement.

4.5. EXAMPLE. A locally compact weakly θ-refinable space which is not submetacompact (not even $\delta\theta$-refinable).

Consider the space $\psi(\omega_1)$. Using the same notation as before, let $\mathscr{U} = \{\{A\} \cup \omega_1: A \in \mathscr{A}\}$ and suppose $\mathscr{U}$ has a $\delta\theta$-sequence of open refinements $\{\mathscr{G}_n\}_1^\infty$. For any $n \in \mathbb{N}$ let $F_n = \{\alpha \in \omega_1: \text{ord}(\alpha, \mathscr{G}_n) \leq \omega\}$. Since $\omega_1 = \bigcup_{n=1}^\infty F_n$ there must be some k where $|F_k| = \omega_1$. Now since the elements of $\mathscr{A}$ are countable, there must be distinct $A_i \in \mathscr{A}$, for $i \in \mathbb{N}$, such that $|A_i \cap F_k| = \omega$. Let $C = \bigcup_{i=1}^\infty (A_i \cap F_i)$. If $\mathscr{B} = \{B \in \mathscr{A}: |B \cap C| = \omega\}$ it follows that $\mathscr{B}$ is uncountable (using the same argument that was used to show $\mathscr{A}$ is uncountable). For every $B \in \mathscr{B}$ pick $G(B) \in \mathscr{G}_k$ such that $B \in G(B)$. The collection $\{G(B): B \in \mathscr{B}\}$ is an uncountable collection of open sets, each of which must intersect the countable set C. It is clear that this collection cannot be point countable on C, hence $\mathscr{G}_k$ is not point countable on C. Since $C \subset F_k$ this contradicts the definition of F_k.

Readers interested in the next example should also see M.E. RUDIN's article (4.2 Example i.2) for an example of a screenable space which is not countably metacompact.

4.6. EXAMPLE. A Tychonoff meta-Lindelöf space which is not submetacompact.

Let $X = \times_{\alpha<\omega_2} X_\alpha$ where $X_\alpha = \{0, 1\}$ for every $\alpha < \omega_2$. For $x \in X$ we use the notation $x(\alpha)$ to denote the α coordinate of x and $C(x) = \{\alpha \in \omega_2 : x(\alpha) = 1\}$. Define

$$Z = \{z \in X : 0 < |C(z)| \leq \omega\}, \qquad F = \{x \in X : 0 < |C(x)| < \omega\}.$$

Our example is the set Z topologized as follows: All points in $Z - F$ are isolated. Points in F have basic neighborhoods in Z inherited from the product topology on X. For $z \in F$ and a finite subset $A \subset \omega_2$ let

$$W(z, A) = \{x \in Z : x(\alpha) = z(\alpha) \text{ for all } \alpha \in A\}.$$

Then the collection $\{W(z, A) : A \subset \omega_1,\ A \text{ finite}\}$ is a neighborhood base (in Z) for the point $z \in F$. To see that Z is meta-Lindelöf notice that any open cover $\mathcal{U}$ can be refined by an open refinement of the form:

$$\mathcal{W} = \{\{z\} : z \in Z - F\} \cup \{W(x, A_x) : x \in F\}$$

where A_x is a finite subset of ω_2 with $C(x) \subset A_x$. It follows from the definition of Z that $\mathcal{W}$ is point countable since $z \in W(x, A_x)$ only if $C(x) \subset C(z)$. Remembering that $C(z)$ is countable and $C(x)$ is finite, this can happen for at most countably many $x \in F$.

For $\alpha < \omega_2$ let y_α denote the element of F such that $y_\alpha(\alpha) = 1$ and $y_\alpha(\beta) = 0$ otherwise. Let $\mathcal{V} = \{W(y_\alpha, \{\alpha\}) : \alpha \in \omega_2\}$ and suppose $\{\mathcal{G}_n\}_1^\infty$ is a sequence of open refinements of $\mathcal{V}$. We show Z is not submetacompact by showing that $\{\mathcal{G}_n\}_1^\infty$ cannot be a θ-sequence for $\mathcal{V}$. Without loss of generality we may assume each $\mathcal{G}_n$ contains a subcollection $\mathcal{W}_n$ of form

$$\mathcal{W}_n = \{W(y_\alpha, B_{n\alpha}) : \alpha < \omega_2\}$$

where $B_{n\alpha}$ is a finite subset of ω_2 with $\{\alpha\} \subset B_{n\alpha}$. Let $B(\alpha) = \bigcup_{n=1}^\infty B_{n\alpha}$. We use the following lemma from BURKE [1970].

LEMMA. *If $B(\alpha)$ is a countable subset of ω_2 for every $\alpha \in \omega_2$ there exists a sequence $\{\alpha_i\}_{i=1}^\infty$ in ω_2 such that $\alpha_i \notin B(\alpha_j)$ if $i \neq j$.*

If $\{\alpha_i\}_1^\infty$ is the sequence as given in the lemma let t be the element of Z such that $t(\alpha_i) = 1$ for every $i \in \mathbb{N}$ and $t(\beta) = 0$ otherwise. It follows that $\operatorname{ord}(t, \mathcal{W}_n) = \omega$

for every n so $\{\mathcal{G}_n\}_1^\infty$ cannot be a θ-sequence. That completes the verification of Example 4.5.

The example below appeared as Example 2.5 in FLEISSNER and REED [1977].

4.7. EXAMPLE. A screenable metacompact Moore space X with a σ-locally countable base (hence σ-para-Lindelöf) which is not para-Lindelöf.

Let

$$X = \omega_1 \cup \{(\alpha, \beta, n): \alpha \leqslant \beta < \omega_1, n < \omega\}.$$

The points in $\{(\alpha, \beta, n): \alpha < \beta < \omega_1, n < \omega\}$ are isolated. For $\beta < \omega_1$ and $n < \omega$ the neighborhoods of the point (β, β, n) are those subsets of X which contain (β, β, n) and all but finitely many elements of $\{(\alpha, \beta, n): \alpha < \beta\}$. For $\alpha < \omega_1$ the neighborhoods of α are those subsets of X which, for some $k < \omega$, contain $\{\alpha\} \cup \{(\alpha, \beta, n): \alpha < \beta < \omega_1, k < n < \omega\}$. The stated properties of X can be verified by the reader.

4.8. EXAMPLE. A collectionwise normal orthocompact space which is not weakly $\delta\theta$-refinable (hence does not have any of the covering properties in Diagram 4.1 except for orthocompactness).

The linearly ordered space ω_1 is collectionwise normal and results from Section 9 (see 9.4) show that ω_1 cannot be weakly $\delta\theta$-refinable. To see that ω_1 is orthocompact let $\mathcal{U}$ be an open cover of X, where we may assume $\mathcal{U}$ contains only basic open sets (intervals). An application of the 'Pressing Down Lemma' shows there is some $\beta \in \omega_1$ such that $[\beta, \omega_1) \subset \mathrm{st}(\beta, \mathcal{U})$. Since $[0, \beta]$ is compact there is a finite subcollection $\mathcal{G}$ of $\mathcal{U}$ covering $[0, \beta]$. Let

$$\mathcal{H} = \mathcal{G} \cup \{U \cap (\beta, \omega_1): \beta \in U \in \mathcal{U}\}.$$

It is straightforward to verify that $\mathcal{H}$ is an interior preserving open refinement of $\mathcal{U}$.

We review the description of the space F (known as Bing's Example G) given by BING [1951]. Let P be an uncountable set, $\mathcal{Q}$ the collection of all subsets of P, and $F = \Pi_{\mathcal{Q}}\{0, 1\}$. For each $p \in P$, let $f_p \in F$ such that for $Q \in \mathcal{Q}$, $f_p(q) = 1$ if and only if $p \in \mathcal{Q}$. Let $E = \{f_p: p \in P\}$. Points in $F - E$ are isolated points of F, and points in E have neighborhoods inherited from the product topology on $\Pi_{\mathcal{Q}}\{0, 1\}$. For a finite subset $\mathcal{R}$ of $\mathcal{Q}$ and $p \in P$, we let

$$U(p, \mathcal{R}) = \{f \in F: f(R) = f_p(R), \text{for each } R \in \mathcal{R}\}.$$

The collection $\{U(p, \mathcal{R}): \mathcal{R} \subset \mathcal{Q}, \mathcal{R} \text{ finite}\}$ is a neighborhood base for the point

$f_p \in F$. F is known to be a completely normal space which is not collectionwise normal. In Example 4.9 below, the assumption that $|P| > c$ is needed in order to show the spaces do not have the indicated covering property. See BURKE [1974] for the details. A more complete outline of the covering properties of various subspaces of F is given in LEWIS [1977].

4.9. EXAMPLE. If P is chosen with $|P| > c$ there are subspaces of F which are:
(i) normal and weakly θ-refinable but not submetacompact,
(ii) normal and metacompact but not subparacompact,
(iii) normal and meta-Lindelöf but not screenable or submetacompact,
(iv) normal and submetacompact but not metacompact or subparacompact.

PROOF. (i) The space F is clearly normal and weakly θ-refinable. It is not submetacompact if $|P| > c$.

(ii) Let $Y = \{f \in F: f(Q) = 1$ for at most two singleton sets $Q \in \mathcal{Q}\}$. Then Y is a normal metacompact space which is not subparacompact.

(iii) Let $K = E \cup \{f \in F: f(Q) = 1$ for at most countably many $Q \in \mathcal{Q}\}$. Then K is a normal meta-Lindelöf space which is not submetacompact.

(iv) Pick any sequence $\{p_n\}_1^\infty$ of distinct points in P. Define $\mathcal{Q}_0 = \{Q: Q$ is a singleton subset of $P\}$ and $\mathcal{Q}_n = \{\{q, p_1, p_2, \ldots, p_n\}: q \in P\}$ for $n \in \mathbb{N}$. Let $X_0 = \{f \in F: f(Q) = 1$ for only finitely many $Q \in \mathcal{Q}_0\}$, $X_n = \{f \in F: f(Q) = 1$ for only finitely many $Q \in \mathcal{Q}_n\} \cup \{f_{p_1}, f_{p_2}, \ldots, f_{p_n}\}$ for $n \in \mathbb{N}$. If $X = \bigcup_{n=0}^\infty X_n$, then X is a normal θ-refinable space that is neither metacompact nor subparacompact.

The remainder of the examples in this section are listed for reference but without verification. The description of each example can be found in the indicated reference.

4.10. EXAMPLE (BING [1951]). Bing's Example H is a normal σ-space (hence perfectly normal and subparacompact) but is not metacompact.

4.11. EXAMPLE (GRUENHAGE and GARDNER [1978]). Assuming (CH) there is a meta-Lindelöf space which is not weakly θ-refinable.

4.12. EXAMPLE (RUDIN [1981]). Assuming $V = L$ there is a collectionwise normal screenable space which is not paracompact (in fact, not submetacompact).

4.13. EXAMPLE (NAVY [1980]). (i) There is a normal para-Lindelöf space which is not collectionwise normal, hence not paracompact.

(ii) Assuming MA + $\neg$CH there is a normal para-Lindelöf Moore space which is not paracompact.

The previous Example 4.13(i) is given in detail in Section 5 of Fleissner's article. M.E. Rudin discusses the next example (4.14) in Section 3.1 of her article.

4.14. EXAMPLE (DE CAUX [1976]). Assuming 'Axiom Club', there is a collectionwise normal weakly θ-refinable space which is not submetacompact.

As seen from the previous examples the presence of normality does not seem to be of much help in upgrading one covering property to a stronger one. In collectionwise normal spaces, however, submetacompact becomes equivalent to paracompact. This result by WORRELL and WICKE [1965] generalized previous results by MICHAEL [1955], NAGAMI [1955], and MCAULEY [1958] for the metacompact and subparacompact cases. The basic construction for Theorem 4.16 is contained in the following lemma.

4.15. LEMMA. *If X is collectionwise normal and $\mathscr{G}$ is an open collection in X such that $\mathscr{G}$ covers a closed subset A and is point finite on A, then there is a σ-discrete open collection in X which covers A and refines $\mathscr{G}$.*

PROOF. For $x \in A$ let $W_x = \bigcap\{G \in \mathscr{G}: x \in G\}$ and for $n \in \mathbb{N}$ let $F_n = \{x \in A: \operatorname{ord}(x, \mathscr{G}) \leq n\}$. The collection $\{W_x \cap F_1: x \in F_1\}$ is a closed discrete collection (see Proposition 1.3) covering F_1 so there is a discrete open collection $\mathscr{U}_1$, covering F_1, such that each element of $\mathscr{U}_1$ is contained in some element of $\mathscr{G}$. Continuing by induction, suppose there exists an open cover $\bigcup_{k=1}^{n}\mathscr{U}_k$ of F_n such that $\bigcup_{k=1}^{n}\mathscr{U}_k$ is a partial refinement of $\mathscr{G}$ and each $\mathscr{U}_k$, $1 \leq k \leq n$, is discrete. Let $V_n = \bigcup\{U: U \in \mathscr{U}_k, 1 \leq k \leq n\}$. Then $F_n \subset V_n$, so $\{W_x \cap (F_{n+1} - V_n): x \in F_{n+1} - F_n\}$ is a closed discrete collection. Using collectionwise normality there exists a discrete open collection $\mathscr{U}_{n+1}$, covering $F_{n+1} - F_n$, which we may assume is a partial refinement of $\mathscr{G}$. We have thus defined a discrete open collection $\mathscr{U}_k$, for every $k \in N$, such that $\bigcup_{k=1}^{\infty}\mathscr{U}_k$ covers A and refines $\mathscr{G}$.

4.16. THEOREM. *For any space X the following conditions are equivalent.*

(i) *X is paracompact.*
(ii) *X is collectionwise normal and subparacompact.*
(iii) *X is collectionwise normal and metacompact.*
(iv) *X is collectionwise normal and submetacompact.*

PROOF. It is clear that (i)$\rightarrow$(ii)$\rightarrow$(iv) and (i)$\rightarrow$(iii)$\rightarrow$(iv) so it suffices to show (iv)$\rightarrow$(i). Assume (iv) is true and let $\mathscr{U}$ be an open cover of X. If $\{\mathscr{G}_n\}_1^\infty$ is a θ-sequence of open refinements of $\mathscr{U}$ and $k \in \mathbb{N}$ let $A_{nk} = \{x \in X: \operatorname{ord}(x, \mathscr{G}_n) \leq k\}$. Then each A_{nk} is closed in X and $\mathscr{G}_n$ is point finite on A_{nk}. Applying Lemma 4.15, there is a σ-discrete open collection $\mathscr{H}_{nk}$ such that $\mathscr{G}_{nk}$ covers A_{nk} and partially refines $\mathscr{G}_n$. It follows that $\bigcup\{\mathscr{H}_{nk}: n, k \in \mathbb{N}\}$ is the desired σ-discrete open refinement of $\mathscr{U}$.

A space X is said to be *perfect* if every closed subset of X is a G_δ-set (or equivalently, every open subset is an F_σ-set). Compare 4.17(ii) below with Example 4.12.

4.17. THEOREM. (i) (BENNETT and LUTZER [1972]). *A perfect weakly θ-refinable space is subparacompact.*

(ii) *A perfectly normal screenable space is paracompact.*

PROOF. (i) Suppose X is a perfect weakly θ-refinable space and $\mathcal{U}$ is an open cover of X. By 3.7(ii) there is an open refinement $\bigcup_{n=1}^{\infty}\mathcal{H}_n$ of $\mathcal{U}$ such that if $x \in X$ then there is some $n \in \mathbb{N}$ with $\operatorname{ord}(x, \mathcal{H}_n) = 1$. For each $n \in \mathbb{N}$ let $H_n = \bigcup \mathcal{H}_n$. Since X is perfect we may express $H_n = \bigcup_{k=1}^{\infty} E_{nk}$, where each E_{nk} is closed. Let

$$\mathcal{G}_{nk} = \mathcal{H}_n \cup \{U \cap (X - E_{nk}) \colon U \in \mathcal{U}\}.$$

Then each $\mathcal{G}_{nk}$ is an open refinement of $\mathcal{U}$ and for any $x \in X$ there is some n, $k \in \mathbb{N}$ such that $\operatorname{ord}(x, \mathcal{G}_{nk}) = 1$. It follows that 3.1(vi) is satisfied so X is subparacompact.

(ii) Since a perfectly normal space is countably paracompact this follows from Theorem 3.2(iii) of M.E. Rudin's article.

4.18. EXAMPLE (MA + $\neg$CH). Any normal metacompact nonmetrizable Moore space is a perfectly normal subparacompact and metacompact space which is not paracompact.

The use of normality and collectionwise normality in the study of paracompact spaces has a parallel theory in the study of subparacompact spaces. A space X is said to be *δ-normal* if whenever A and B are disjoint closed subsets of X there exist disjoint G_δ-sets H, K such that $A \subset H$ and $B \subset K$. X is *collectionwise δ-normal* if whenever $\{A_\alpha \colon \alpha \in \Lambda\}$ is a discrete collection of closed sets in X there is a collection $\{H_\alpha \colon \alpha \in \Lambda\}$ of pairwise disjoint G_δ-sets such that $A_\alpha \subset H_\alpha$ for every $\alpha \in \Lambda$. The class of δ-normal spaces was studied by CHABER [1979] and by KRAMER [1973] (under a different, but equivalent formulation); collectionwise δ-normal spaces were defined by JUNNILA [1980]. We will see that subparacompact spaces are exactly the collectionwise δ-normal submetacompact spaces.

4.19. LEMMA. (i) *Suppose $\{U_\alpha \colon \alpha \in \Lambda\}$ is a faithfully indexed point finite family and for each $\alpha \in \Lambda$ there exists a nonincreasing sequence $\{G_{n\alpha}\}_1^\infty$ of sets such that $G_{1\alpha} \subset U_\alpha$. Then*

$$\bigcup_{\alpha\in\Lambda}\Big(\bigcap_{n=1}^{\infty} G_{n\alpha}\Big) = \bigcap_{n=1}^{\infty}\Big(\bigcup_{\alpha\in\Lambda} G_{n\alpha}\Big).$$

(ii) *Suppose $\mathcal{U}$ is a point finite open cover of a collectionwise δ-normal space X and $F_1 = \{x \in X \colon \operatorname{ord}(x, \mathcal{U}) = 1\}$. There exists a σ-discrete closed collection $\mathcal{F}$ in X and a G_δ-set H such that $\mathcal{F}$ is a partial refinement of $\mathcal{U}$ and $F_1 \subset H \subset \bigcup \mathcal{F}$.*

PROOF. (i) Let $\{U_\alpha \colon \alpha \in \Lambda\}$ and $\{G_{n\alpha} \colon \alpha \in \Lambda, n \in N\}$ be as given. It is clear that

$$\bigcup_{\alpha\in\Lambda}\Big(\bigcap_{n=1}^{\infty} G_{n\alpha}\Big) \subset \bigcap_{n=1}^{\infty}\Big(\bigcup_{\alpha\in\Lambda} G_{n\alpha}\Big).$$

To see the other direction, suppose $x \in \bigcap_{n=1}^{\infty}(\bigcup_{\alpha\in\Lambda} G_{n\alpha})$. For every $n \in \mathbb{N}$, let $\mathcal{G}_n = \{G_{n\alpha} : \alpha \in \Lambda\}$ and let $A_n = \{\alpha \in \Lambda : x \in G_{n\alpha}\}$. Clearly $\{A_n\}_1^{\infty}$ is a nonincreasing sequence of nonempty finite sets so there must be a fixed $\beta \in \Lambda$ such that $\beta \in A_k$ for all $k \in N$. This says $x \in \bigcap_{n=1}^{\infty} G_{n\beta}$ so $x \in \bigcup_{\alpha\in\Lambda}(\bigcap_{n=1}^{\infty} G_{n\alpha})$.

(ii) Suppose $\{U_\alpha : \alpha \in \Lambda\}$ is a faithful indexing for $\mathcal{U}$ and for each $\alpha \in \Lambda$ let $E_\alpha = U_\alpha \cap F_1$. The collection $\{E_\alpha : \alpha \in \Lambda\}$ is a closed discrete collection so there exists a disjoint collection $\{K_\alpha : \alpha \in \Lambda\}$ of G_δ-sets such that $E_\alpha \subset K_\alpha \subset U_\alpha$ for every $\alpha \in \Lambda$. We may assume $K_\alpha = \bigcap_{n=1}^{\infty} K_{n\alpha}$ where $\{K_{n\alpha}\}_1^{\infty}$ is a nonincreasing sequence of open sets with $K_{1\alpha} \subset U_\alpha$. Let $K(n) = \bigcup\{K_{n\alpha} : \alpha \in \Lambda\}$. Apply δ-normality and find disjoint G_δ-sets $H(n)$, $W(n)$ such that $F_1 \subset H(n)$ and $X - K(n) \subset W(n)$. We may express $H(n) = \bigcap_{k=1}^{\infty} H_k(n)$ and $W(n) = \bigcap_{k=1}^{\infty} W_k(n)$ where each $H_k(n)$ and $W_k(n)$ is open. For n, $k \in \mathbb{N}$, let

$$\mathcal{K}_{nk} = \{K_{n\alpha} : \alpha \in \Lambda\} \cup \{W_k(n)\},$$

$$D_{nk} = \{x \in X : \operatorname{ord}(x, \mathcal{K}_{nk}) = 1\}, \quad \text{and} \quad \mathcal{F}_{nk} = \{K_{n\alpha} \cap D_{nk} : \alpha \in \Lambda\}.$$

Notice that $\mathcal{F}_{nk}$ is a closed discrete collection in X partially refining $\mathcal{U}$. Let $\mathcal{F} = \bigcup\{\mathcal{F}_{nk} : n, k \in \mathbb{N}\}$. If $H = \bigcap_{n=1}^{\infty} H(n)$ it is clear that H is a G_δ-set with $F_1 \subset H$. To see that $H \subset \bigcup\mathcal{F}$ pick $z \in H$. Then $z \in H(n)$ for every n implies $z \in K(n)$ for every n. By part (i) there is some $\beta \in \Lambda$ such that $z \in \bigcap_{n=1}^{\infty} K_{n\beta} = K_\beta$. Hence $z \notin K_\alpha$ if $\alpha \neq \beta$ and, in fact, there must be some $k \in \mathbb{N}$ such that $z \notin K_{k\alpha}$ if $\alpha \neq \beta$. We may also assume k is chosen large enough so that $z \notin W_k(n)$ (remember, $z \in H(n)$ and $H(n) \cap W(n) = \emptyset$) and, consequently, $z \in K_{n\alpha} \cap D_{nk} \in \mathcal{F}_{nk}$. We have $F_1 \subset H \subset \bigcup\mathcal{F}$ and that completes the proof of the lemma.

4.20. THEOREM. *If X is collectionwise δ-normal, then every point finite open cover of X has a σ-discrete closed refinement.*

PROOF. Let $\mathcal{U}$ be a point finite open cover of the collectionwise δ-normal space X. For each $n \in \mathbb{N}$ let $F_n = \{x \in X : \operatorname{ord}(x, \mathcal{U}) \leq n\}$. By Lemma 4.19(ii) there is a σ-discrete closed collection $\mathcal{F}_1$ and a G_δ-set H_1 such that $\mathcal{F}_1$ is a partial refinement of $\mathcal{U}$ and $F_1 \subset H_1 \subset \bigcup\mathcal{F}_1$. Continuing by induction, assume we have found a σ-discrete closed collection $\mathcal{F}_n$ and a G_δ-set H_n such that $\mathcal{F}_n$ is a partial refinement of $\mathcal{U}$ and $F_n \subset H_n \subset \bigcup\mathcal{F}_n$. Suppose $H_n = \bigcap_{k=1}^{\infty} H(n, k)$ where each $H(n, k)$ is open and $\{H(n, k)\}_{k=1}^{\infty}$ is nonincreasing. Let $X(n+1, k) = X - H(n, k)$ and let

$$\mathcal{U}(n+1, k) = \{(\bigcap\mathcal{W}) \cap X(n+1, k) : \mathcal{W} \subset \mathcal{U}, |\mathcal{W}| = n+1\}.$$

If $E(n+1, k) = \{x \in X(n+1, k); \operatorname{ord}(x, \mathcal{U}(n+1, k)) = 1\}$ we have $\bigcup_{k=1}^{\infty} \mathrm{E}(n +$

$1, k) = F_{n+1} - H_n$. Apply Lemma 4.19(ii) to the space $X(n+1, k)$ and the point finite open cover $\mathcal{U}(n+1, k)$ to obtain a σ-discrete closed collection $\mathcal{E}(n+1, k)$ and a G_δ-set $K(n+1, k)$ (relative to $X(n+1, k)$) such that $\mathcal{E}(n+1, k)$ is a partial refinement of $\mathcal{U}(n+1, k)$ and

$$E(n+1, k) \subset K(n+1, k) \subset \bigcup \mathcal{E}(n+1, k).$$

Let

$$\mathcal{F}_{n+1} = \mathcal{F}_n \cup \Big(\bigcup_{k=1}^{\infty} \mathcal{E}(n+1, k)\Big) \quad \text{and} \quad H_{n+1} = \bigcap_{k=1}^{\infty} (H(n, k) \cup K(n+1, k)).$$

It is straightforward to verify that $\mathcal{F}_{n+1}$ is a σ-discrete closed partial refinement of $\mathcal{U}$ and H_{n+1} is a G_δ-set such that $F_{n+1} \subset H_{n+1} \subset \mathcal{F}_{n+1}$. It follows that we have $\mathcal{F}_n$ defined for all n so $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is the desired σ-discrete closed refinement of $\mathcal{U}$.

The previous theorem is due to JUNNILA [1980] and is used to prove the following characterization of subparacompactness.

4.21. THEOREM. *A space X is subparacompact if and only if X is collectionwise δ-normal and submetacompact.*

PROOF. If X is subparacompact we know X is submetacompact so to show X is collectionwise δ-normal suppose $\{A_\alpha : \alpha \in \Lambda\}$ is a closed discrete collection in X. For each $\alpha \in \Lambda$ let $U_\alpha = X - \bigcup_{\beta \neq \alpha} A_\beta$. The open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ has a closed refinement $\bigcup_{n=1}^{\infty} \mathcal{D}_n$ where each $\mathcal{D}_n$ is discrete. If $H_{n\alpha} = X - \bigcup\{D \in \mathcal{D}_n : A_\alpha \cap D = \emptyset\}$ and $H_\alpha = \bigcap_{n=1}^{\infty} H_{n\alpha}$, then H_α is a G_δ-set, $A_\alpha \subset H_\alpha$, and $\{H_\alpha : \alpha \in \Lambda\}$ is a disjoint collection.

For the converse, assume X is a collectionwise δ-normal submetacompact space. Theorem 4.20 can now be applied in a manner similar to the way Lemma 4.15 was used to prove Theorem 4.16. The easy details are left to the reader.

It is well known that in a normal space, countable discrete collections of closed sets can be separated by disjoint open sets. It is somewhat surprising that in a δ-normal space, discrete closed collections of cardinality no larger than c can be separated by G_δ-sets (JUNNILA [1980]).

4.22. THEOREM. *If X is δ-normal and $\mathcal{F}$ is a discrete collection of closed sets in X with $|\mathcal{F}| \leq c$, then $\mathcal{F}$ can be separated with disjoint G_δ-sets (i.e., there is a disjoint collection $\{H(F) : F \in \mathcal{F}\}$ of G_δ-sets such that $F \subset H(F)$).*

PROOF. Since $|\mathcal{F}| \leq c$ we may assume there is a subset $T \subset \mathbb{P}$ (the irrational numbers) such that $\mathcal{F}$ may be indexed by T, say $\mathcal{F} = \{F_\alpha : \alpha \in T\}$ where $F_\alpha \neq F_\beta$ if $\alpha \neq \beta$. Now, there exists a sequence $\{S_n\}_1^\infty$ of subsets of $\mathbb{P}$ such that $\{S_n\}_1^\infty$ is point-separating in T, i.e., given distinct $x, y \in T$ there is some $m \in \mathbb{N}$ such that $x \in S_m$ but $y \notin S_m$. For each $n \in \mathbb{N}$ let

$$E_{1n} = \bigcup\{F_\alpha : \alpha \in S_n\} \quad \text{and} \quad E_{2n} = \bigcup\{F_\alpha : \alpha \in T - S_n\}.$$

The sets E_{1n} and E_{2n} are disjoint closed sets, so there exist disjoint G_δ-sets H_{1n} and H_{2n} such that $E_{1n} \subset H_{1n}$ and $E_{2n} \subset H_{2n}$. For each $\alpha \in T$ let

$$K_\alpha = \bigcap\{H_{in} : i \in \{1, 2\}, n \in \mathbb{N}, F_\alpha \subset H_{in}\}.$$

Clearly each K_α is a G_δ-set and if $\beta \in T, \beta \neq \alpha$ there is some S_m such that $\alpha \in S_m$ and $\beta \notin S_m$. It follows that $F_\alpha \subset H_{1m}$ and $F_\beta \subset H_{2m}$, so $K_\alpha \subset H_{1m}$, $K_\beta \subset H_{2m}$, and $H_{1m} \cap H_{2m} = \emptyset$ implies $K_\alpha \cap K_\beta = \emptyset$. That completes the proof of the theorem.

The proof of the next result depends on Theorem 4.22 and the method of proof in 4.20 and 4.21. This theorem, due to JUNNILA [1980], says that δ-normal submetacompact spaces behave like subparacompact spaces for open covers of cardinality $\leqslant c$.

4.23. THEOREM. *If X is δ-normal and submetacompact, then every open cover of cardinality $\leqslant c$ has a σ-discrete closed refinement.*

We turn to a result which allows for σ-closure preserving refinements of locally countable collections under certain conditions (BURKE [1979]). This will give another characterization of subparacompactness in terms of submetacompactness.

4.24. THEOREM. *If $\mathcal{P}$ is a collection of closed subsets of X and $\mathcal{K}$ is a point-finite open cover of X such that each $K \in \mathcal{K}$ intersects at most countably many elements of $\mathcal{P}$, then $\mathcal{P}$ has a σ-closure preserving refinement.*

PROOF. Assume $\mathcal{K} = \{K(\alpha) : \alpha \in \Lambda\}$ where Λ is well-ordered and $K(\alpha) \neq K(\beta)$ if $\alpha \neq \beta$. For each $\alpha \in \Lambda$ the set $\mathcal{H}(\alpha) = \{P \in \mathcal{P} : P \cap K(\alpha) \neq \emptyset\}$ is countable, so express as

$$\mathcal{H}(\alpha) = \{P(1, \alpha), P(2, \alpha), \ldots\}.$$

(Make necessary adjustments in notation if $\mathcal{H}(\alpha)$ is finite or empty.) For each $n \in \mathbb{N}$ let $F_n = \{x \in X : \operatorname{ord}(x, \mathcal{K}) \leqslant n\}$. For each finite sequence $(i_1, i_2, \ldots, i_n)$ of natural numbers and each $\beta \in \Lambda$, let

$$A(i_1, \ldots, i_n, \beta) = \{(\alpha_1, \ldots, \alpha_n) \in \Lambda^n : \alpha_1 < \alpha_2 < \cdots < \alpha_n = \beta$$
$$\text{and } P(i_1, \alpha_1) = P(i_2, \alpha_2) = \cdots = P(i_n, \alpha_n)\}.$$

For each sequence $(i_1, \ldots, i_n) \in \mathbb{N}^n$, we will define a closure preserving collection $\mathcal{D}(i_1, \ldots, i_n)$—this will be done by induction on n.

Let $i \in \mathbb{N}$ (a sequence of length 1), and $\beta \in \Lambda$. Define

$$D(i, \beta) = F_1 \cap P(i, \beta) \cap K(\beta), \quad \text{and} \quad \mathscr{D}(i) = \{D(i, \beta) : \beta \in \Lambda\}.$$

Then $\mathscr{D}(i)$ is a closure preserving collection (in fact, $\mathscr{D}(i)$ is actually discrete). Now let $n \in \mathbb{N}$, $n > 1$ and assume that for any sequence $(j_1, \ldots, j_k) \in \mathbb{N}^k$, with $1 \leqslant k < n$, that $\mathscr{D}(j_1, \ldots, j_k)$ is defined and is a closure preserving collection of subsets of F_k. For any $(i_1, \ldots, i_n) \in \mathbb{N}^n$, $\beta \in \Lambda$, define

$$E(i_1, \ldots, i_n, \beta) =$$

$$\bigcup\{F_n \cap P(i_n, \beta) \cap K(\alpha_1) \cap \cdots \cap K(\alpha_n) : (\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)\},$$

$H(i_1, \ldots, i_n, \beta) = \bigcup\{D(i_{j1}, \ldots, i_{jk}, \alpha_{jk}) : (i_{j1}, \ldots, i_{jk})$ is a subsequence of $(i_1, \ldots, i_n)$, $1 \leqslant k < n$ and $(\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)\}$,

$$D(i_1, \ldots, i_n, \beta) = E(i_1, \ldots, i_n, \beta) \cup H(i_1, \ldots, i_n, \beta),$$

$$\mathscr{D}(i_1, \ldots, i_n) = \{D(i_1, \ldots, i_n, \beta) : \beta \in \Lambda\}.$$

To show $\mathscr{D}(i_1, \ldots, i_n)$ is closure preserving let $\Lambda' \subseteq \Lambda$ and suppose $x \in \mathrm{cl}(\bigcup\{D(i_1, \ldots, i_n, \beta) : \beta \in \Lambda'\})$. If $x \in F_n - F_{n-1}$, then there exists $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Lambda^n$, with $\gamma_1 < \cdots < \gamma_n$ such that

$$x \in W = K(\gamma_1) \cap K(\gamma_2) \cap \cdots \cap K(\gamma_n).$$

Then $W \cap D(i_1, \ldots, i_n, \beta) \neq \emptyset$, for some $\beta \in \Lambda'$, implies $W \cap E(i_1, \ldots, i_n, \beta) \neq \emptyset$ (since $F_n \cap W \subset F_n - F_{n-1}$) which implies $(\gamma_1, \ldots, \gamma_n) \in A(i_1, \ldots, i_n, \beta)$ (so $\gamma_n = \beta$). This says there is only one $\beta \in \Lambda'$ such that $W \cap D(i_1, \ldots, i_n, \beta) \neq \emptyset$; it follows that $x \in \mathrm{cl}(D(i_1, \ldots, i_n, \beta))$, for $\beta = \gamma_n$ and $\gamma_n \in \Lambda'$. Now suppose $\mathrm{ord}(x, \mathscr{K}) = k$, for $1 \leqslant k \leqslant n$; then there exists $(\gamma_1, \ldots, \gamma_k) \in \Lambda^k$, with $\gamma_1 < \cdots < \gamma_k$ such that $x \in V = K(\gamma_1) \cap \cdots \cap K(\gamma_k)$. If $x \in \mathrm{cl}(\bigcup\{H(i_1, \ldots, i_n, \beta) : \beta \in \Lambda'\})$, then $x \in \mathrm{cl}(D(i_{j1}, \ldots, i_{jr}, \alpha_{jr})) \subset \mathrm{cl}(D(i_1, \ldots, i_n, \beta))$ for some subsequence $(i_{j1}, \ldots, i_{jr})$ of $(i_1, \ldots, i_n)$ with $(\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)$, since $\{D(i_{j1}, \ldots, i_{jr}, \alpha_{jr}) : \beta \in \Lambda', (i_{j1}, \ldots, i_{jr})$ is a subsequence of $(i_1, \ldots, i_n)$, $1 \leqslant r < n$ and $(\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)\}$ is closure preserving.

Otherwise we have $x \in \mathrm{cl}(\bigcup\{E(i_1, \ldots, i_n, \beta) : \beta \in \Lambda'\})$. Now note that $V \cap E(i_1, \ldots, i_n, \beta) \neq \emptyset$, for some $\beta \in \Lambda'$, implies there is $(\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)$ and a subsequence $(i_{j1}, \ldots, i_{jk})$ of $(i_1, \ldots, i_n)$ such that $\gamma_1 = \alpha_{j1}$, $\gamma_2 = \alpha_{j2}, \ldots, \gamma_k = \alpha_{jk} \leqslant \beta$. For every subsequence $(i_{j1}, \ldots, i_{jk})$ (of length k) of $(i_1, \ldots, i_n)$ let

$$\Lambda(i_{j1}, \ldots, i_{jk}) = \{\beta \in \Lambda' : \text{there is } (\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)$$

$$\text{such that } \gamma_1 = \alpha_{j1}, \ldots, \gamma_k = \alpha_{jk}\}.$$

Now, since there are only a finite number of such subsequences, there is some subsequence $(i_{j1}, \ldots, i_{jk})$ such that $x \in \mathrm{cl}(\bigcup\{E(i_1, \ldots, i_n, \beta): \beta \in \Lambda(i_{j1}, \ldots, i_{jk})\})$. For each $\beta \in \Lambda(i_{j1}, \ldots, i_{jk})$ we have $E(i_1, \ldots, i_n, \beta) \subset P(i_n, \beta) = P(i_{jk}, \gamma_k)$. So $x \in P(i_{jk}, \gamma_k)$ (since $P(i_{jk}, \gamma_k)$ is closed) and $x \in F_k \cap K(\gamma_1) \cap \cdots \cap K(\gamma_k)$; hence

$$x \in E(i_{j1}, \ldots, i_{jk}, \gamma_k) \subset D(i_{j1}, \ldots, i_{jk}, \gamma_k) \subset D(i_1, \ldots, i_n, \beta)$$

for any $\beta \in \Lambda(i_{j1}, \ldots, i_{jk})$. This shows $\mathscr{D}(i_1, \ldots, i_n)$ is closure preserving and $\mathscr{D} = \bigcup\{\mathscr{D}(i_1, \ldots, i_n): n \in \mathbb{N}, (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is σ-closure preserving.

If $D(i_1, \ldots, i_n, \beta) \in \mathscr{D}$, it follows by construction of $D(i_1, \ldots, i_n, \beta)$ that

$$D(i_1, \ldots, i_n, \beta) \subset P(i_n, \beta) \in \mathscr{P}.$$

To complete the proof we need to show that $\mathscr{D}$ covers $\bigcup\mathscr{P}$. Let $x \in \bigcup\mathscr{P}$ and suppose $\mathrm{ord}(x, \mathscr{K}) = n$. There exist elements $K(\alpha_1), \ldots, K(\alpha_n)$ of $\mathscr{K}$ such that $x \in K(\alpha_1) \cap \cdots \cap K(\alpha_n)$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. For each j, $1 \leqslant j \leqslant n$, there is $i_j \in N$ so that $x \in P(i_j, \alpha_j) \in \mathscr{H}(\alpha_j)$ and $P(i_1, \alpha_1) = P(i_2, \alpha_2) = \cdots = P(i_n, \alpha_n)$. It follows that $x \in D(i_1, \ldots, i_n, \alpha_n) \in \mathscr{D}$ and the theorem is proved.

The next result can now be shown using Theorem 4.24 as the principal tool. We leave the proof for the interested reader (c.f. BURKE [1979]).

4.25. COROLLARY. *A space X is subparacompact if and only if X is submetacompact and every open cover has a σ-locally countable closed refinement.*

If $\mathscr{V}$ is an open cover of X with $|\mathscr{V}| = \omega_1$, say $\mathscr{V} = \{V_\alpha: \alpha < \omega_1\}$, let

$$W_\alpha = V_\alpha - \bigcup\{V_\beta: \beta < \alpha\}.$$

It is easy to see that the collection $\mathscr{W} = \{W_\alpha: \alpha < \omega_1\}$ is a locally countable cover of X. If $\mathscr{U}$ was an open cover such that $\{\bar{V}_\alpha: \alpha < \omega_1\}$ refined $\mathscr{U}$, then $\{\bar{W}_\alpha: \alpha < \omega_1\}$ is a locally countable closed refinement of $\mathscr{U}$. If X is a regular space with the weight of X less than or equal to $\omega_1(w(X) \leqslant \omega_1)$ it follows that every open cover of X has a locally countable closed refinement. Clearly, the condition that $w(X) \leqslant \omega_1$ could be replaced by the condition that X be ω_1-Lindelöf, i.e., every open cover of X has subcover of cardinality $\leqslant\omega_1$. This provides the proof for the next result (JUNNILA [1980]). Notice that Example 4.2 shows that the weight cannot be increased to ω_2.

4.26. COROLLARY. *If X is a regular space with $w(X) \leqslant \omega_1$ (or if X is ω_1-Lindelöf), then X is subparacompact if and only if X is submetacompact.*

A space X is said to be (*strongly*) *collectionwise Hausdorff* if for any closed discrete set A in X the points in A can be separated by a (discrete) disjoint collection of open sets. X is *collectionwise normal with respect to closed Lindelöf (compact) sets* if every discrete closed collection of Lindelöf (compact) subsets of X can be separated by a disjoint collection of open sets. Example 4.13(i) indicates that regular para-Lindelöf spaces need not be collectionwise normal. The following theorem is a weak form of a result from FLEISSNER and REED [1977] giving a closely related separation property.

4.27. THEOREM. *If X is a regular para-Lindelöf space, then X is collectionwise normal with respect to closed Lindelöf sets.*

PROOF. We show first that X is collectionwise Hausdorff. Suppose $A = \{z_\alpha : \alpha \in \Lambda\}$ is a closed discrete set in X (assume $z_\alpha \neq z_\beta$ if $\alpha \neq \beta$). For each $\alpha \in \Lambda$ let U_α be open in X such that $U_\alpha \cap A = \{z_\alpha\}$. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\} \cup \{X - A\}$. Now, $\mathcal{U}$ has a locally countable open refinement $\mathcal{V}$ of the form $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\} \cup \mathcal{G}$ where $z_\alpha \in V_\alpha \subset U_\alpha$. For every $\alpha \in \Lambda$ there is an open neighborhood W_α of z_α such that $\bar{W}_\alpha \subset V_\alpha$ and W_α intersects at most countably many elements of $\mathcal{V}$. This makes the collection $\mathcal{W} = \{W_\alpha : \alpha \in \Lambda\}$ a star-countable collection so $\mathcal{W}$ can be expressed as $\mathcal{W} = \bigcup\{\mathcal{B}_\beta : \beta \in \Gamma\}$ where each $\mathcal{B}_\beta$ is countable and $(\bigcup\mathcal{B}_\beta) \cap (\bigcup\mathcal{B}_\gamma) = \emptyset$ if $\beta \neq \gamma$. Let $A_\beta = A \cap (\bigcup\mathcal{B}_\beta)$. Then A_β is countable so we may express $A_\beta = \{x_{1\beta}, x_{2\beta}, \ldots\}$ and $\mathcal{B}_\beta = \{B_{1\beta}, B_{2\beta}, \ldots\}$ where $x_{i\beta} \in B_{i\beta}$. Define

$$C_{j\beta} = B_{j\beta} - \bigcup_{i<j} \bar{B}_{i\beta}$$

for each $j = 1, 2, \ldots$. Then $\mathcal{C}_\beta = \{C_{1\beta}, C_{2\beta}, \ldots\}$ is a disjoint open collection separating the points in A_β. It follows that $\bigcup\{\mathcal{C}_\beta : \beta \in \Gamma\}$ is a disjoint open collection separating the points of A.

Now suppose $\{E_\alpha : \alpha \in \Lambda\}$ is a discrete collection of closed Lindelöf subsets of X. Consider the quotient map $f: X \to Y$ where Y is the quotient space obtained from X by identifying the points in each E_α. It is easy to see that f is a closed mapping with Lindelöf point inverses, and we see from the later Theorem 5.3 that the para-Lindelöf property is preserved under such a map. Hence Y is para-Lindelöf and must be collectionwise Hausdorff. If $\{y_\alpha : \alpha \in \Lambda\}$ is the set of points in Y where $f^{-1}(y_\alpha) = E_\alpha$ then this set is closed and discrete in Y so there must be a disjoint collection of open sets $\{H_\alpha : \alpha \in \Lambda\}$ where $y_\alpha \in H_\alpha$. The collection $\{f^{-1}(H_\alpha) : \alpha \in \Lambda\}$ gives the desired separation of $\{E_\alpha : \alpha \in \Lambda\}$.

4.28. THEOREM (TRAYLOR [1964] and HODEL [1969]). *A regular meta-Lindelöf space X which is locally separable is strongly paracompact.*

PROOF. Suppose $\mathcal{U}$ is an open cover of X. $\mathcal{U}$ has an open refinement $\mathcal{V}$ consisting of separable subsets and $\mathcal{V}$ has a point countable open refinement $\mathcal{W}$. Now, each $W \in \mathcal{W}$ is also separable so there is a countable set $D \subset W$ such that $W \subset \bar{D}$. If

$H \in \mathscr{W}$, then $H \cap W \neq \emptyset$ if and only if $H \cap D \neq \emptyset$. Since D intersects at most countably many $H \in \mathscr{W}$ we see that W intersects only countably many elements of $\mathscr{W}$. This says that $\mathscr{W}$ is star-countable. Apply Theorem 3.12 and the proof is complete.

5. Mappings and covering properties

Although a few remarks will be made at the end of this section concerning other kinds of mappings, our main emphasis here will be on the role of closed mappings in the study of covering properties. Recall that a mapping $f: X \to Y$ is said to be *perfect* if f is a closed mapping with $f^{-1}(y)$ compact for every $y \in Y$.

The first theorem brings together four very satisfying positive results concerning the invariance of covering properties under closed mappings. These results, not at all obvious using the original definitions, become easy corollaries to the characterizations of the respective covering properties in terms of closure-preserving collections. This is a consequence of the observation that if $f: X \to Y$ is a closed mapping and $\mathscr{S}$ is a closure-preserving collection in X then $\{f(S): S \in \mathscr{S}\}$ is a closure-preserving collection in Y.

5.1. THEOREM. *Suppose $f: X \to Y$ is a closed mapping.*

(i) *If X is paracompact, then so is Y* (MICHAEL [1958]).

(ii) *If X is metacompact, then so is Y* (WORRELL [1966b]).

(iii) *If X is subparacompact, then so is Y* (BURKE [1969]).

(iv) *If X is submetacompact, then so is Y* (JUNNILA [1978]).

PROOF. We prove only (ii). The proofs of (i), (iii), and (iv) follow in a similar manner and are left to the reader. Assume X is metacompact and let $\mathscr{U}$ be an open cover of Y. If $\mathscr{U}^F$ is the collection of all unions of finite subcollections from $\mathscr{U}$ it suffices (by Theorem 3.5.(iv)) to show that $\mathscr{U}^F$ has a closure-preserving closed refinement. Let $\mathscr{W} = \{f^{-1}(U): U \in \mathscr{U}\}$. since $\mathscr{W}$ is an open cover of the metacompact space X the open cover $\mathscr{W}^F$ has a closure-preserving closed refinement $\mathscr{F}$. The collection $\{f(F): F \in \mathscr{F}\}$ is the desired closure-preserving closed refinement of $\mathscr{U}^F$.

In contrast to the situation in the above theorem we see here that the para-Lindelöf property is not preserved under closed mappings.

5.2. EXAMPLE. *If X is a normal (but noncollectionwise normal) para-Lindelöf space there is a closed mapping $\phi: X \to Y$, onto a space Y which is not para-Lindelöf.*

PROOF. Let X be such a space (as in Example 4.13(i)). Since X is not collectionwise normal there is a discrete collection $\{F_\alpha: \alpha \in \Lambda\}$ of closed subsets of X which cannot be separated by open sets. Let Y be the quotient space obtained from X by identifying each F_α with a point p_α and let $\phi: X \to Y$ be the corresponding quotient map. The map ϕ is clearly closed and Y cannot be para-Lindelöf. To show this we recall from Theorem 4.27 that regular para-Lindelöf spaces are collectionwise Hausdorff. The set $\{p_\alpha: \alpha \in \Lambda\}$ is a closed discrete set in Y and a 'separation' by open sets in Y would induce a 'separation' of $\{F_\alpha: \alpha \in \Lambda\}$ by open sets in X. Thus Y is not collectionwise Hausdorff and cannot be para-Lindelöf.

5.3. THEOREM. *If X is para-Lindelöf and $f: X \to Y$ is a closed mapping, with $f^{-1}(y)$ Lindelöf for each $y \in Y$, then Y is para-Lindelöf.*

PROOF. If $\mathscr{U}$ is an open cover of Y let $\mathscr{W}$ be a locally countable open refinement of $\{f^{-1}(U): U \in \mathscr{U}\}$. For any $y \in Y$, the fiber $f^{-1}(y)$ is Lindelöf so there is an open set G_y in X such that $f^{-1}(y) \subset G_y$ and G_y intersects only countably many elements of $\mathscr{W}$. If V_y is the saturated part of G_y (with respect to f), then $f(V_y)$ is an open set about y meeting only countably many elements of $\mathscr{H} = \{f(W): W \in \mathscr{W}\}$. Hence $\mathscr{H}$ is locally countable and it is easy to see that $y \in \operatorname{int}(\operatorname{st}(y, \mathscr{H}))$ for every $y \in Y$. Since $\mathscr{H}$ is a refinement of $\mathscr{U}$ it follows from Theorem 3.8 that Y is para-Lindelöf.

5.4. EXAMPLE. There is a regular σ-para-Lindelöf space X, a space Y which is not σ-para-Lindelöf, and a closed mapping $f: X \to Y$ where $f^{-1}(y)$ is Lindelöf for each $y \in Y$.

PROOF. Let X be the space described in Example 4.7. This space has a σ-locally countable base, so X must be σ-para-Lindelöf. The space Y is obtained as a quotient space of X by identifying the set $\{\beta\} \cup \{(\beta, \beta, n): n \in \omega\}$ as a single point for each $\beta < \omega_1$. If $f: X \to Y$ is the corresponding quotient map it is clear that each $f^{-1}(y)$ is countable and hence Lindelöf. Since the set $D = \omega_1 \cup \{(\beta, \beta, n): \beta < \omega_1, n < \omega\}$ is closed and discrete, and $f^{-1}(y) \subset D$ whenever $|f^{-1}(y)| > 1$ it follows that f is a closed mapping. We let the reader verify that Y is not σ-para-Lindelöf.

5.5. THEOREM. *If X is σ-para-Lindelöf and $f: X \to Y$ is a perfect mapping then Y is σ-para-Lindelöf.*

PROOF. Suppose $\mathscr{U}$ is an open cover of Y and $\mathscr{W} = \bigcup_{n=1}^{\infty} \mathscr{W}_n$ is an open refinement of $\{f^{-1}(U): U \in \mathscr{U}\}$ where each $\mathscr{W}_n$ is locally countable. Because each $f^{-1}(y)$ is compact we may assume, without loss of generality, that $f^{-1}(y) \subset \bigcup \mathscr{W}_k$, for some k. It follows that if $\mathscr{H}_n = \{f(W): W \in \mathscr{W}_n\}$, then $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ is a refinement of $\mathscr{U}$ where each $\mathscr{H}_n$ is locally countable, and if $y \in Y$, then $y \in \operatorname{int}(\operatorname{st}(y, \mathscr{H}_n))$ for some n. The result now follows from Theorem 3.9.

The above results concerning mappings of para-Lindelöf and σ-para-Lindelöf. spaces appeared in BURKE [1980]. The next example, due to G.M. Reed and R.W.

Heath, gives a surprising result. The screenable condition is not preserved under a finite-to-one closed mapping, even with a Moore space domain.

5.6. EXAMPLE. There is a screenable Moore space X and a (finite-to-one) perfect mapping $f: X \to Y$ onto a space Y which is not screenable.

PROOF. Let $H = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y > 0\}$, $X_0 = \mathbb{R} \times \{0\}$, $X_1 = \mathbb{R} \times \{-1\}$, and $X = H \cup X_0 \cup X_1$. Describe a local base for points in X as follows: All points in H are isolated. If $a \in \mathbb{R}$ and $n \in \mathbb{N}$, let

$$U_n(a, 0) = \{(a, 0)\} \cup \{(x, y) \in H: x = y + a, y < 1/n\}$$

and $U_n(a, -1) = \{(a, -1)\} \cup \{(x, y) \in H: x = -y + a, y < 1/n\}$. Using $\{U_n(a, 0)\}_{n=1}^{\infty}$ and $\{U_n(a, -1)\}_{n=1}^{\infty}$ as local bases at $(a, 0)$ and $(a, -1)$ respectively it is easily verified that X is a screenable Moore space. Let $Y = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \geq 0\}$. Define a mapping $f: X \to Y$ by letting $f(x, y) = (x, y)$ if $y > 0$ and $f(a, 0) = f(a, -1) = (a, 0)$ for every $a \in \mathbb{R}$. Give Y the quotient topology induced by this mapping f which is clearly finite-to-one. Using the fact that $X_0 \cup X_1$ is a closed discrete subset of X it follows that X is a closed map. The space Y is homeomorphic to the space described in Example 1 of HEATH [1964], and it is shown there that this space is not screenable. This last fact can also be shown directly by using a 'category argument'.

The next example appeared in BURKE [1980a]. The reader is referred to SCOTT [1975], [1980] for an in-depth discussion of orthocompactness, expecially the product theory. In contrast to this example, JUNNILA [1978a] has shown that orthocompactness is preserved under a closed mapping when the domain is also submetacompact.

5.7. EXAMPLE. There is an orthocompact space X and a perfect mapping $f: X \to Y$ onto a space Y which is not orthocompact.

PROOF. Let $X_0 = \omega_1 \times I \times \{0\}$, $X_1 = \omega_1 \times I \times \{1\}$, and $X = X_0 \cup X_1$ (where I denotes the 'closed unit interval' from $\mathbb{R}$). For $\alpha, \beta \in \omega_1$, α a nonlimit ordinal with $\alpha \leq \beta$, $x \in I$, and $\varepsilon > 0$ define

$$B(\alpha, \beta, x, \varepsilon) = \{(\gamma, z, 0) \in X_0: \alpha \leq \gamma \leq \beta, 0 < |x - z| < \varepsilon\}$$
$$\cup \{(\gamma, z, 1) \in X_1: \alpha \leq \gamma \leq \beta, |x - z| < \varepsilon\}.$$

Topologize X by describing local bases as follows: Points $(\beta, x, 0) \in X_0$ are isolated in X. Points $(\beta, x, 1) \in X_1$ have the set of all $B(\alpha, \beta, x, \varepsilon)$, for nonlimit $\alpha \leq \beta$ and $\varepsilon > 0$, for a local base. It may be revealing to the reader to provide a simple sketch here, and realize that X is similar to the 'Alexandroff double' of $\omega_1 \times I$.

Now let $Y = X_0 \cup \omega_1$ and define a map $f: X \to Y: f(p) = p$ for $p \in X_0$ and $f(\alpha, x, 1) = \alpha$ for $(\alpha, x, 1) \in X_1$. Let Y have the quotient topology induced by f. The details of showing that X is orthocompact, Y is not orthocompact, and f is a perfect mapping are given in BURKE [1980a].

To illustrate the poor behavior of strong paracompactness under closed maps, we state the following result by PONOMAREV [1962].

5.8. THEOREM. *Every paracompact space is the image of a strongly paracompact space under a perfect map.*

This says, of course, that strongly paracompact spaces are not generally preserved under perfect mappings. PONOMAREV [1962a] has shown that strong paracompactness is preserved under open perfect mappings.

We say that a topological property is *inversely preserved* under a class of maps if whenever $f: X \to Y$ is such a map and Y has the property, then X must also have the property. The *fibers* of a mapping $f: X \to Y$ are the sets $f^{-1}(y)$, for $y \in Y$.

5.9. THEOREM. *All of the covering properties in Diagram* 4.1, *except orthocompactness, are inversely preserved under perfect mappings* (*with regular domain*); *in fact, these properties are inversely preserved under closed mappings with Lindelöf fibers* (*and regular domain*).

PROOF. We prove only for the metacompact case. The other cases, each with its own special twist, use similar techniques (some of which do not require a regular domain). Assume $f: X \to Y$ is a closed mapping, with Lindelöf fibers, from the regular space X onto a metacompact space Y. Let $\mathcal{U}$ be an open cover of X and suppose $\mathcal{V}$ is an open cover of X such that $\{\bar{V}: V \in \mathcal{V}\}$ refines $\mathcal{U}$. For every $y \in Y$ pick a countable subcollection $\mathcal{V}_y$ from $\mathcal{V}$, say $\mathcal{V}_y = \{V(y, 1), V(y, 2), \ldots\}$ such that $\mathcal{V}_y$ covers $f^{-1}(y)$ and pick corresponding $U(y, i) \in \mathcal{U}$ such that $\overline{V(y, i)} \subset U(y, i)$. Define $W(y, 1) = U(y, 1) \cap (\bigcup_{i=1}^{\infty} V(y, i))$ and for $n > 1$ let

$$W(y, n) = (U(y, n) - \bigcup_{i<n} \overline{V(y, i)}) \cap (\bigcup_{i=1}^{\infty} V(y, i)).$$

Notice that $\{W(y, n): n \in \mathbb{N}\}$ is a point finite open collection covering $f^{-1}(y)$. Let $G(y)$ be the saturated part of $\bigcup\{W(y, n): n \in \mathbb{N}\}$. Then $\{f(G(y)): y \in Y\}$ is an open cover of Y and must have a precise point finite open refinement $\mathcal{H} = \{H(y): y \in Y\}$. The collection

$$\{W(y,n)\cap f^{-1}(H(y)) \colon y \in Y, n \in \mathbb{N}\}$$

is the desired point finite open refinement of $\mathcal{U}$.

5.10. EXAMPLE. The projection mapping $\pi_1 \colon \omega_1 \times (\omega_1+1) \to \omega_1$ is a perfect mapping from a nonorthocompact space onto an orthocompact space.

PROOF. Since ω_1+1 is compact it is well known that such a projection map is perfect, and we showed ω_1 to be orthocompact in Example 4.8. To see that $\omega_1 \times (\omega_1+1)$ is not orthocompact let

$$\mathcal{U} = \{[0,\alpha] \times (\alpha, \omega_1] \colon \alpha < \omega_1\} \cup \{\omega_1 \times \omega_1\}\,.$$

If $\mathcal{H}$ is any open refinement of $\mathcal{U}$ there is some $\beta < \omega_1$ such that $[\beta, \omega_1) \times \{\omega_1\} \subset \operatorname{st}((\beta, \omega_1), \mathcal{H})$ (treat $\mathcal{H}$ as an open cover of $\omega_1 \times \{\omega_1\}$ and use the 'Pressing Down Lemma'). It follows that $\bigcap\{H \in \mathcal{H} \colon (\beta, \omega_1) \in H\} \subset \omega_1 \times \{\omega_1\}$. Hence $\mathcal{H}$ cannot be interior-preserving and $\omega_1 \times (\omega_1+1)$ is not orthocompact.

A mapping $f \colon X \to Y$ is said to be an *open compact map* if f is an open mapping with compact fibers. We mention here that, except for the Lindelöf property which is preserved under any (continuous) mapping, none of the other properties in Diagram 4.1 are known to be preserved or inversely preserved under open compact mappings. There are examples for most of these properties which illustrate this poor behavior (see GITTINGS [1977] for more details).

If we strengthen the mapping condition to the class of open finite-to-one mappings there are a few positive results. For example, the metacompact and metalindelöf properties are obviously preserved under open finite-to-one mappings, and SCOTT [1980] has shown that the orthocompact property is preserved under open finite-to-one mappings. However, the next example shows that the image of a paracompact space under an open finite-to-one map need not even be subparacompact.

5.11. EXAMPLE. There is an example of a locally compact paracompact space Z and an open finite-to-one map $f \colon Z \to X$ onto a space X which is not subparacompact.

PROOF. Let X be the space described in Example 4.2. Let $Z_1 = \omega_2 \times (\omega_2 - \{0\})$, $Z_2 = (\omega_2 - \{0\}) \times \omega_2$ with topologies given as subspaces of X. Clearly Z_1 and Z_2 are both locally compact and paracompact (in fact, strongly paracompact). If $Z_1 \oplus Z_2$ denotes the free topological sum of Z_1 and Z_2, then $Z_1 \oplus Z_2$ is a locally compact paracompact space and the natural canonical mapping $\phi \colon Z_1 \oplus Z_2 \to X$ is an open finite-to-one map onto the nonsubparacompact space X.

Even though paracompactness is not preserved under open compact mappings it is a sometimes useful (and easy to see) fact that an open compact image of a paracompact space is always metacompact. This observation is superseded by the following result due to JUNNILA [1979b]. Recall that a mapping $f: X \to Y$ is pseudo-open if whenever $f^{-1}(y) \subset U$ where $y \in Y$ and U is open in X, then $y \in \operatorname{int}(f(U))$. The proof of this nontrivial theorem is misleading—the proof is easy only because of JUNNILA's characterization of metacompactness given in 3.5(v).

5.12. THEOREM. *If X is paracompact and $f: X \to Y$ is a pseudo-open compact mapping, then X is metacompact.*

PROOF. Suppose $\mathcal{U}$ is an open cover of Y. Then $\{f^{-1}(u): U \in \mathcal{U}\}$ is an open cover of X and must have a locally finite open refinement $\mathcal{V}$. Let $\mathcal{H} = \{f(V): V \in \mathcal{V}\}$. Since the locally finite collection $\mathcal{V}$ is compact finite every fiber $f^{-1}(y)$ intersects at most finitely many elements of $\mathcal{V}$. It follows that the refinement $\mathcal{H}$ is point finite and (using the pseudo-open condition) it is easy to see that $y \in \operatorname{int}(\operatorname{st}(y, \mathcal{H}))$ for every $y \in Y$. Apply 3.5(v) and the proof is complete.

6. Products

Many of the results on the invariance of covering properties under products are negative, in that covering properties are simply not generally preserved unless one or more of the factors are assumed to satisfy additional conditions. We begin with a useful result which indicates many covering properties are preserved in a product of two spaces if one of the factors is σ-compact.

6.1. THEOREM. *Suppose Y is a regular σ-compact space. If X is a regular space (normal for the para-Lindelöf case) satisfying any of the covering properties in Diagram* 4.1, *except for strongly paracompact or orthocompact, then $X \times Y$ also satisfies the corresponding covering property.*

PROOF. Let property P represent the desired covering property and suppose X is a regular (or normal) space satisfying property P. The projection map $\pi_1: X \times \beta Y \to X$ is a perfect mapping (since βY is compact) so by Theorem 5.9 we know that $X \times \beta Y$ satisfies property P. Express Y as $Y = \bigcup_{n=1}^{\infty} A_n$ where each A_n is compact. Now, $X \times A_n$ is closed in $X \times \beta Y$ so $X \times Y = \bigcup_{n=1}^{\infty}(X \times A_n)$ is an F_σ-subset of $X \times \beta Y$. By Theorem 7.1 we see that property P is preserved to F_σ-subspaces of any space satisfying property P. Hence $X \times Y$ satisfies property P.

Generalizations to certain parts of this theorem have been obtained in KRAMER [1976] and TELGARSKY [1975]. Instead of using a σ-compact space KRAMER uses a space Y which can be covered by a σ-discrete collection of compact spaces. Telgarsky studies paracompact spaces Y which have a σ-closure preserving cover of compact sets and shows that $X \times Y$ must be paracompact when X is any other paracompact space.

The two exceptions to Theorem 6.1 are illustrated in the following examples. Example 6.2(i) was shown in 5.10.

6.2. EXAMPLE. (i) $\omega_1 \times (\omega_1 + 1)$ is not orthocompact even though ω_1 is orthocompact and $\omega_1 + 1$ is compact.

(ii) There is a strongly paracompact metric space X and a σ-compact metric space Y such that $X \times Y$ is not strongly paracompact.

PROOF. (ii) For every $n \in \mathbb{N}$, let $Z_n = \omega_1$ with the discrete topology and let $X = \times_n Z_n$. X is a well known metric space, sometimes called the Baire space of weight ω_1 and denoted by $B(\omega_1)$ (see ENGELKING [1977]). If Y is the open interval $(0, 1)$, with the usual topology, then Y is clearly a σ-compact metric space. We leave the exercise of showing that X is strongly paracompact and $X \times Y$ is not strongly paracompact to the reader.

Within the literature discussing the invariance of covering properties under products there are two examples which have received a considerable amount of deserved exposure. The examples by SORGENFREY [1947] and MICHAEL [1963] emphasize very effectively that one cannot expect invariance of the Lindelöf property or paracompactness in finite products. The *Sorgenfrey Line S* is the set $\mathbb{R}$ endowed with the topology having $\{[a, b): a, b \in \mathbb{R}, a < b\}$ as a basis. It is well known that S is hereditarily Lindelöf but $S \times S$ is not Lindelöf (not even normal). We summarize several properties of S in the following example.

6.3. EXAMPLE. If S is the Sorgenfrey line, then

(i) S is hereditarily Lindelöf but $S \times S$ is not normal, hence not paracompact (SORGENFREY [1947]).

(ii) Assuming MA + $\neg$CH, if $X \subset S$, with $\omega < |X| < c$, then $X \times X$ is normal but not paracompact (PRZYMUSINSKI [1973]).

(iii) $S \times S$ is not metacompact.

(iv) S^n, for $n \leqslant \omega$, is perfect (HEATH and MICHAEL [1971]) and subparacompact (LUTZER [1972]).

We do not prove any of the above. Part (i) is standard material in most topology texts. Part (iii) is an easy exercise and the other parts can be found in the indicated references.

The *Michael Line M* is the set $\mathbb{R}$ topologized by isolating the points of $\mathbb{P}$ and leaving the points of $\mathbb{Q}$ with their usual neighborhoods. We summarize several properties of M.

6.4. EXAMPLE. If M is the Michael Line, then

(i) M is hereditarily paracompact but $M \times \mathbb{P}$ is not normal, hence not paracompact (MICHAEL [1963]).

(ii) $M \times \mathbb{P}$ is hereditarily metacompact (NYIKOS [1978]) and hereditarily subparacompact.

(iii) M^n is paracompact for every $n < \omega$ but M^ω is not normal (MICHAEL [1971]).

PROOF. (i) To see that M is hereditarily paracompact, suppose $X \subset M$ and $\mathcal{U}$ is an open cover of X. There is a countable subcollection $\mathcal{W}$ of $\mathcal{U}$ such that $\mathcal{W}$ covers $Q \cap X$. The collection $\mathcal{W} \cup \{\{x\}: x \in \mathbb{P} \cap X - (\bigcup \mathcal{W})\}$ is a σ-discrete open refinement of $\mathcal{U}$, so X must be paracompact. Now consider the space $M \times \mathbb{P}$. Let $A = \mathbb{Q} \times \mathbb{P}$ and $B = \{(x, x) \in M \times \mathbb{P} : x \in \mathbb{P}\}$. The sets A and B are disjoint closed subsets of $M \times \mathbb{P}$. A 'category argument' shows that if U is open in $M \times \mathbb{P}$ such that $B \subset U$, then $\bar{U} \cap A \neq \emptyset$. Hence $M \times \mathbb{P}$ is not normal.

(ii) Let $\mathcal{U}$ be an open cover of $M \times \mathbb{P}$ and suppose $\mathcal{V}$ is an open cover such that $\{\bar{V}: V \in \mathcal{V}\}$ refines $\mathcal{U}$. We show that $\mathcal{U}$ has a point finite open refinement and a σ-discrete closed refinement. There is a countable subcollection $\{V_i: i \in \mathbb{N}\}$ of $\mathcal{V}$ such that $\{V_i: i \in \mathbb{N}\}$ covers $\mathbb{Q} \times \mathbb{P}$. For each V_i pick $U_i \in \mathcal{U}$ such that $\bar{V}_i \subset U_i$. For each $n \in \mathbb{N}$, define

$$W_n = (U_n - \bigcup_{i<n} \bar{V}_i) \cap (\bigcup_{i=1}^{\infty} V_i).$$

Note that $\{W_n: n \in \mathbb{N}\}$ is a point finite open collection covering $\mathbb{Q} \times \mathbb{P}$. Now, $\mathbb{P} \times \mathbb{P}$ is an open metrizable subset of $M \times \mathbb{P}$ ($\mathbb{P} \times \mathbb{P}$ is a topological sum of open subspaces $\{x\} \times \mathbb{P}$, for $x \in \mathbb{P}$) hence $\mathbb{P} \times \mathbb{P}$ is metacompact. Therefore there is a point finite open cover $\mathcal{W}$ of $\mathbb{P} \times \mathbb{P}$ (open in $\mathbb{P} \times \mathbb{P}$ and also in $M \times \mathbb{P}$) such that $\mathcal{W}$ is a partial refinement of $\mathcal{U}$. The collection $\mathcal{W} \cup \{W_i: i \in \mathbb{N}\}$ is the desired point finite open refinement of $\mathcal{U}$. To construct the σ-discrete closed refinement consider the closed subspace $Z = M \times \mathbb{P} - \bigcup \{V_i: i \in \mathbb{N}\}$. Z is a closed metrizable subspace of $M \times \mathbb{P}$, so there exists a σ-discrete closed collection $\mathcal{F}$ in Z such that $\mathcal{F}$ covers Z and is a partial refinement of $\mathcal{U}$. $\mathcal{F}$ is also a σ-discrete closed collection in $M \times \mathbb{P}$, and $\{\bar{V}_i: i \in \mathbb{N}\}$ is σ-discrete so $\mathcal{F} \cup \{\bar{V}_i: i \in \mathbb{N}\}$ is the desired σ-discrete closed refinement of $\mathcal{U}$. A similar proof shows that $M \times \mathbb{P}$ is hereditarily metacompact and hereditarily subparacompact.

(iii) The proof that M^n, $n \in \mathbb{N}$, is paracompact is left to the reader. To see that M^ω is not normal, pick a countably infinite closed discrete subset $A \subset M$. The space $M \times (\Pi_{n=1}^{\infty} A)$ is homeomorphic to a closed subset of M^ω and is also homeomorphic to $M \times P$ (since $\Pi_{n=1}^{\infty} A$ is homeomorphic to P). Since $M \times \mathbb{P}$ is not normal we see that M^ω is not normal.

When studying the product of paracompact spaces X and Y, special consideration should be given to the situation when one of the factors is metrizable.

Michael's example shows that the product of two paracompact spaces need not be normal even when one of the factors is a complete metric space. This connection between normality of the product and paracompactness of the product is clarified by the following result proved by RUDIN and STARBIRD [1975].

6.5. THEOREM. *If X is metrizable and Y is paracompact the following are equivalent.*

(i) *$X \times Y$ is normal.*

(ii) *$X \times Y$ is paracompact.*

It is enough to assume in the above result that X is a paracompact p-space (see Gruenhage's Chapter) for if $f: X \to Z$ is a perfect map onto a metrizable space Z then $f \times i: X \times Y \to Z \times Y$ is a perfect map (where $i: X \to Y$ is the identity). Hence normality of $X \times Y$ implies normality of $Z \times Y$ and $Z \times Y$ must be paracompact. The paracompact property is then transferred to the preimage of $X \times Y$ via $(f \times i)^{-1}$.

To extend 6.5 to the Lindelöf case we use the following proposition.

6.6. PROPOSITION (WILLARD [1971]). *If the paracompact space Y has a dense Lindelöf subspace, then Y is Lindelöf.*

PROOF. Suppose Z is a dense Lindelöf subspace of the paracompact space Y. If $\mathcal{U}$ is an open cover of Y let $\mathcal{V}$ be an open cover of Y such that $\{\bar{V}: V \in \mathcal{V}\}$ refines $\mathcal{U}$. Now let $\mathcal{W}$ be a locally finite open refinement of $\mathcal{V}$. Since Z is Lindelöf there is a countable subcollection $\{W: i \in \mathbb{N}\}$ from $\mathcal{W}$ covering Z. For each i, pick $U \in \mathcal{U}$ such that $\bar{W}_i \subset U_i$. Using the fact that Z is dense and $\{W_i: i \in \mathbb{N}\}$ is locally finite we see that

$$\bigcup\{\bar{W}_i: i \in \mathbb{N}\} = \mathrm{cl}(\bigcup\{W_i: i \in \mathbb{N}\}) = X.$$

Hence $\{U_i: i \in \mathbb{N}\}$ must cover X.

6.7. COROLLARY. *If X is separable metrizable and Y is regular Lindelöf the following are equivalent.*

(i) *$X \times Y$ is normal.*

(ii) *$X \times Y$ is paracompact.*

(iii) *$X \times Y$ is Lindelöf.*

PROOF. Clearly (iii)$\to$(ii)$\to$(i). To show (i)$\to$(iii) suppose D is a countable dense subset of X. By 6.1 the space $D \times Y$ is a dense Lindelöf subspace of $X \times Y$. If (i) is assumed then $X \times Y$ is paracompact by 6.5. Hence $X \times Y$ is Lindelöf by 6.6.

Working with variations on the Michael line seems to indicate a general construction for building examples is possible. The following procedure, implicit in PRZYMUSINSKI [1980], outlines a technique for building product spaces that suppress certain covering properties.

6.8. GENERAL CONSTRUCTION. Let $A \subset \mathbb{R}$ and let $X = \mathbb{R} - A$ with the usual topology. If ρ is another topology on $\mathbb{R} - A$, stronger than the usual topology, let $Y = \mathbb{R}$ with a topology generated by $\tau \cup \rho$ (where τ is the usual topology on $\mathbb{R}$). Then $\Delta = \{(x, x): x \in X\}$ is a closed subspace of $X \times Y$ homeomorphic to (X, ρ). If A is chosen so that $|A \cap F| = c$ for every closed uncountable subset F of $\mathbb{R}$, then X and Y are Lindelöf but $X \times Y$ will not have any covering property that $Y - A$ (with topology ρ) does not have.

PROOF. Everything given above in 6.8 is easily verified. We discuss only the significance of the condition on A that $|A \cap F| = c$ for every closed uncountable, subset F of $\mathbb{R}$. This condition forces Y to be Lindelöf for if $\mathcal{U}$ is an open cover of Y there is a countable open collection $\mathcal{W}$ (open in $\mathbb{R}$) such that $\mathcal{W}$ covers A and partially refines $\mathcal{U}$. The closed set $F = \mathbb{R} - (\bigcup \mathcal{W})$ is disjoint from A, hence $\mathbb{R} - (\bigcup \mathcal{W})$ is countable. Clearly $\mathbb{R} - (\bigcup \mathcal{W})$ can be covered by a countable subcollection from $\mathcal{U}$ so there must be a countable subcollection from $\mathcal{U}$ covering Y. The next result, from KURATOWSKI [1966], insures that such subsets A exist.

6.9. THEOREM. *There exists a partition $\{A_k: k \in \mathbb{N}\}$ of $\mathbb{R}$ such that $|A_k \cap F| = c$ for any $k \in \mathbb{N}$ and uncountable closed subset F of $\mathbb{R}$.*

PROOF. It is well known that every uncountable closed subset of $\mathbb{R}$ has cardinality c and the collection $\mathcal{F}$ of uncountable closed subsets has cardinality c. Express $\mathcal{F} = \{F_\alpha: \alpha < c\}$ where every element E of $\mathcal{F}$ is repeated c times in this indexing by c. For every $\alpha < c$ it is possible to chose, by induction on α, elements $x_{n\alpha}$, for $n \in \mathbb{N}$ such that

(a) $x_{n\alpha} \in F_\alpha$ and $x_{n\alpha} \neq x_{k\alpha}$ if $n \neq k$.
(b) $x_{k\beta} \neq x_{n\alpha}$, if $\beta < \alpha$ and $n, k \in \mathbb{N}$.
(c) $\{x_{n\alpha}: n \in \mathbb{N}, \alpha < c\} = \mathbb{R}$.

Let $A_k = \{x_{k\alpha}: \alpha < c\}$, for each $k \in \mathbb{N}$; it is easily verified that $\{A_k: k \in \mathbb{N}\}$ has the desired properties.

It is now a simple matter to construct the following example.

6.10. EXAMPLE. There is a separable metric space X and a Lindelöf space Y such that $X \times Y$ is not Lindelöf (hence not normal or paracompact).

PROOF. Let A be a subset of $\mathbb{R}$ such that $|\mathbb{R} - A| = c$ and $|A \cap F| = c$ for every uncountable closed subset F of R. Let $X = \mathbb{R} - A$ with the usual topology. Let Y be the set $\mathbb{R}$ topologized by isolating the points of $R - A$ and letting the points of

A have their usual neighborhoods. Then X is separable metrizable, Y is Lindelöf, and $X \times Y$ is not Lindelöf since the set $\Delta = \{(x, x) \in X \times Y\colon x \in \mathbb{R} - A\}$ is an uncountable closed discrete subset of $X \times Y$.

It is unknown whether the example in 6.10 can be constructed (without extra set-theoretic assumptions) with X a *complete* separable metric space. Such an example was constructed in MICHAEL [1971] using CH, and by BURKE and DAVIS [1981] using an ω_1-scale.

A little more work obtains a separable metric space X and a Lindelöf space Y such that $X \times Y$ is not submetacompact, hence not subparacompact. This example is a modification of a construction given in PRZYMUSIŃSKI [1980].

6.11. EXAMPLE. There is a separable metric space X and a first countable, separable, Lindelöf space Y such that $X \times Y$ is not submetacompact (hence not subparacompact).

PROOF. Suppose $\{A, B, C\}$ is a partition of $\mathbb{R}$ such that $|F \cap A| = |F \cap B| = |F \cap C| = c$ for every closed uncountable subset F of $\mathbb{R}$. We construct a separable, first countable, locally compact topology on $B \cup C$ which is stronger than the usual topology and is not submetacompact. The construction is a modification of the technique given by VAN DOUWEN [1976]. To begin, pick a countable dense subset B_0 of B (with respect to the usual topology) and express $B - B_0 = \{x_\alpha\colon \alpha < c\}$. For every $y \in B - B_0$ find a sequence $\{b_k(y)\}_{k=1}^\infty$ in B_0 such that $|y - b_k(y)| < 1/k$. Let $\{E_\alpha\colon \alpha < c\}$ be an enumeration of all countable subsets E_α of $B - B_0$ such that $|\bar{E}_\alpha| = c$. By induction on α, pick $z_\alpha \in C \cap \bar{E}_\alpha - \{z_\beta\colon \beta < \alpha\}$ for every $\alpha < c$. For each such z_α there is a sequence $\{e_{n\alpha}\}_1^\infty$ in E_α such that $|z_\alpha - e_{n\alpha}| < 1/n$. Without loss of generality, we may assume $C = \{z_\alpha\colon \alpha < c\}$. Define a topology ρ on $B \cup C$ as follows: Points in B_0 are isolated. Points $x_\alpha \in B - B_0$ have

$$U_n(x_\alpha) = \{x_\alpha\} \cup \{b_k(x_\alpha)\colon k \geq n\},$$

for $n \in \mathbb{N}$, as basic neighborhoods. Points $z_\alpha \in C$ have

$$U_n(z_\alpha) = \{z_\alpha\} \cup \{e_{k\alpha}\colon k \geq n\} \cup \{b_r(e_{k\alpha})\colon k \geq n,\ r \geq k\},$$

for $n \in \mathbb{N}$, as basic neighborhoods. It is easy to verify that ρ is a separable, first countable, locally compact topology on $B \cup C$ which is stronger than the usual topology. To check that ρ is not a submetacompact topology consider the open cover $\mathscr{H} = \{B \cup \{z_\alpha\}\colon z_\alpha \in C\}$ and suppose $\{\mathscr{G}_n\}_1^\infty$ is a θ-sequence of open refinements. For each $n \in \mathbb{N}$ let $D_n = \{y \in B - B_0\colon \operatorname{ord}(y, \mathscr{G}_n) < \omega\}$. Since $\bigcup_{n=1}^\infty D_n = B - B_0$ there is some m such that $|D_m| > \omega$. Now there is some $\beta < c$ such that $E_\beta \subset D_m \subset \bar{E}_\beta$ (pick a countable dense subset of D_m) and an uncountable subset $\Lambda \subset c$ such that $E_\alpha \subset E_\beta$ for every $\alpha \in \Lambda$. If for every $\alpha \in \Lambda$ we pick $G_\alpha \in \mathcal{G}_m$ such that $z_\alpha \in G_\alpha$ it follows that $\{G_\alpha : \alpha \in \Lambda\}$ cannot be point finite on E_β, a contradiction to our choice of D_m. Hence $B \cup C$, with the topology ρ, cannot be submetacompact.

To complete the construction of the example let $X = B \cup C$ with the usual topology inherited from $\mathbb{R}$. Let $Y = \mathbb{R}$ with the topology generated by $\tau \cup \rho$, where τ is the usual topology on $\mathbb{R}$. It follows (as in 6.8) that X is separable metrizable, Y is Lindelöf, but $X \times Y$ is not submetacompact.

This now gives the following result, due to PRZYMUSIŃSKI [1980]. ALSTER and ENGELKING [1972] gave the first known example of a paracompact space X such that $X \times X$ is not subparacompact. Their example was not Lindelöf.

6.12. COROLLARY. *There is a first countable, separable, Lindelöf space Z such that $Z \times Z$ is not submetacompact.*

PROOF. If X and Y are as in 6.11 let Z be the topological sum $X \oplus Y$. Clearly Z is a first countable, separable, Lindelöf space and $Z \times Z$ is not submetacompact.

We now turn our attention to the construction of a Lindelöf space X such that X^n is Lindelöf for every $n \in \mathbb{N}$ but X^ω is not Lindelöf. We need the definition of n-cardinality as introduced by PRZYMUSIŃSKI [1978]. For X any set, $n \in \mathbb{N}$, and p, $q \in X^n$ we write $p \cap q = \emptyset$ to mean $p_i \neq q_i$ for $i = 1, 2, \ldots, n$ (this is consistent with the notion that the functions p and q are sets of ordered pairs).

6.13. DEFINITION. For X any set, $n \in \mathbb{N}$, and $A \subset X^n$ the *n-cardinality* of A, denoted by $|A|_n$, is defined by:

$$|A|_n = \max\{|B| : B \subset A \text{ and } p \cap q = \emptyset \text{ for all distinct } p, q \in B\}.$$

The set A is said to be *n-countable* (*n-uncountable*) if $|A|_n \leq \omega$ ($|A|_n > \omega$).

It is clear that '1-cardinality' agrees with 'cardinality' and, in general, $|A|_n \leq |A|$. The following lemma from PRZYMUSIŃSKI [1978] characterizes n-cardinality in the form that we will use. Let $\pi_i : X^n \to X$ be the ith projection map.

6.14. LEMMA. *If $A \subset X^n$ with $|A|_n \geq \omega$, then*

$$|A|_n = \min\{|Y| : Y \subset X \text{ and } A \subset \bigcup_{i=1}^{n} \pi_i^{-1}(Y)\}.$$

We also give, without proof, the following result from PRZYMUSIŃSKI [1978], which generalizes the earlier Theorem 6.9.

6.15. THEOREM. *Let X be a complete separable metric space. There exists a partition*

$\{A_k\colon k \in \mathbb{N}\}$ of X such that for any $n \in \mathbb{N}$ and every n-uncountable Borel subset B of X^n we have $|B \cap A_k^n|_n = c$ for every $k \in \mathbb{N}$.

Our use of this theorem is contained in the following corollary. The proof is a straightforward application of Lemma 6.14 and Theorem 6.15.

6.16. COROLLARY. *With $\{A_k\colon k \in N\}$ as above, if U is any open subset of X^n containing some A_k^n, then $X^n - U$ must be n-countable. Hence there exists a countable set $Y \subset X$ such that $X^n - U \subset \bigcup_{i=1}^n \pi_i^{-1}(Y)$.*

6.17. LEMMA. *Suppose $A \subset \mathbb{R}$ such that whenever U is open in $\mathbb{R}^n$ with $A^n \subset U$ then $|\mathbb{R}^n - U|_n \leqslant \omega$. If X is the space obtained from $\mathbb{R}$ by isolating the points in $\mathbb{R} - A$, then X^n is Lindelöf for every n.*

PROOF. We prove by induction on n. For $n = 1$ the result follows from previous discussions such as the proof of 6.8. Assume true for $k = n - 1$ and consider the space X^n. If $\mathcal{U}$ is an open cover of X^n there is a countable collection $\mathcal{W}$, open in R^n, such that $A^n \subset \bigcup \mathcal{W}$ and $\mathcal{W}$ partially refines $\mathcal{U}$. Hence $\mathbb{R}^n - (\bigcup \mathcal{W})$ is n-countable and there must exist a countable set $Y \subset R$ such that

$$\mathbb{R}^n - (\bigcup \mathcal{W}) \subset \bigcup_{i=1}^{n} \pi_i^{-1}(Y).$$

Now, each $\pi_i^{-1}(Y)$ is homeomorphic to $X^k \times Y$, which by 6.1 is Lindelöf, so $\bigcup_{i=1}^n \pi_i^{-1}(Y)$ is Lindelöf. If $\mathcal{H}$ is a countable subcollection from $\mathcal{U}$ covering $\bigcup_{i=1}^n \pi_i^{-1}(Y)$ then $\mathcal{W} \cup \mathcal{H}$ is a countable refinement of $\mathcal{U}$. It follows that X^n is Lindelöf and the proof is complete.

6.18. REMARK. It is clear that a similar statement holds in 6.17 for any topology σ on X which is stronger than the usual topology on $\mathbb{R}$ and weaker than the given topology on X.

We are now ready to provide the example promised earlier of a first countable space X such that X^n is Lindelöf for every $n \in \mathbb{N}$ but X^ω is not Lindelöf. MICHAEL [1971] has constructed such an example using CH. With a bit more effort PRZYMUSIŃSKI [1980] provided an example of such a space X such that X^ω is not even submetacompact. We describe an example of this type in Remark 6.21.

6.19. THEOREM. *Express $\mathbb{R} = \bigcup_{i=1}^\infty A_i$ as in Theorem 6.15. Let X_i be the space obtained from $\mathbb{R}$ by isolating the points of A_i. If J is any finite subset of N, then $\Pi_{i \in J} X_i$ is Lindelöf but $\Pi_{i \in \mathbb{N}} X_i$ is not Lindelöf. Hence, if $X = \bigoplus_{i \in \mathbb{N}} X_i$ (topological sum), then X^n is a first countable Lindelöf space for every $n \in \mathbb{N}$ but X^ω is not Lindelöf (in fact, not normal).*

PROOF. For every finite $J \subset \mathbb{N}$ let Y_J be the space obtained from $\mathbb{R}$ by isolating the points in $\bigcup_{i\in J} A_i$. If $n = |J|$ it follows from Lemma 6.17 that Y_J^n is Lindelöf; since $\Pi_{i\in J} X_i$ has a weaker topology we see that $\Pi_{i\in J} X_i$ must be Lindelöf. That $\Pi_{i\in\mathbb{N}} X_i$ is not Lindelöf follows by noticing that the diagonal $\Delta = \{(x, x, \ldots): x \in \mathbb{R}\}$ is an uncountable closed discrete subset of $\Pi_{i\in\mathbb{N}} X_i$.

To actually see that X^ω is not normal it suffices to show $\Pi_{i=1}^\infty X_i$ is not normal. For each $i \in \mathbb{N}$, let $B_i = X_i - A_i$ and let $C = B_1 \times B_2 \times \cdots$. It is easy to see that C is a closed set disjoint from Δ. Let U be open in $\Pi_{i=1}^\infty X_i$ such that $\Delta \subset U$; we will show that $\bar{U} \cap C \neq \emptyset$. For each $x \in \mathbb{R}$, let W_x be a basic open set in $\Pi_{i=1}^\infty X_i$ such that $(x, x, x, \ldots) \in W_x \subset U$ and let

$$J_x = \{i \in \mathbb{N}: \pi_i(W_x) \neq X_i\}.$$

Note that J_x is finite. For every finite $M \subset N$ let $Y_M = \{x \in \mathbb{R}: J_x = M\}$. Since $\mathbb{R} = \bigcup\{Y_M: M \subset N, |M| < \omega\}$ there must be some finite $M \subset N$ such that

$$\mathrm{int}_{\mathbb{R}}(\mathrm{cl}_{\mathbb{R}}(Y_M)) \neq \emptyset.$$

Pick some $b \in (\bigcap_{i\in M} B_i) \cap \mathrm{int}_{\mathbb{R}}(\mathrm{cl}_{\mathbb{R}}(Y_M))$ (it is easy to show that $\bigcap_{i\in M} B_i$ is dense in $\mathbb{R}$ since each A_j must be dense). Now let $z \in \Pi_{i\in\mathbb{N}} B_i$ such that $z_i = b$ when $i \in M$. It can be shown that $z \in \bar{U}$. Hence $\bar{U} \cap C \neq \emptyset$ so Δ and C cannot be separated by open sets.

The fact that the example X^ω, in the above theorem, is not normal also says that X^ω is not paracompact. Remark 6.21 shows that such a space X^ω can be constructed so that X^ω is not even submetacompact. This example should be contrasted with the following theorem. The results for the paracompact, Lindelöf, and subparacompact cases are due to OKUYAMA [1967], MICHAEL [1971], and LUTZER [1972] respectively.

6.20. THEOREM. *Let X_i, $i \in \mathbb{N}$, be topological spaces such that each $\Pi_{i=1}^n X_i$ is perfect. Let property P denote any of the covering properties (include normal for the para-Lindelöf case) in Diagram 4.1 except strongly paracompact. If each space $\Pi_{i=1}^n X_i$ satisfies property P, then $\Pi_{i=1}^\infty X_i$ also satisfies property P.*

PROOF. We sketch the proof for the metacompact case. Variations on this argument will give the other results. Assume each $\Pi_{i=1}^n X_i$ is metacompact and suppose $\mathscr{U}$ is an open cover of X consisting of basic open sets. For each $n \in \mathbb{N}$, let $\mathscr{U}_n$ denote the collection of all $U \in \mathscr{U}$ which can be represented as

$$U = U(n) \times \Big(\prod_{i>n} X_i\Big)$$

for some open set $U(n)$ in $\Pi_{i=1}^n X_i$. Clearly $\mathscr{U} = \bigcup_{n=1}^\infty \mathscr{U}_n$. By 7.1 we know that each

$\Pi_{i=1}^n X_i$ is hereditarily metacompact so there is a point-finite open collection $\mathcal{V}_n$ in $\Pi_{i=1}^n X_i$ such that $\mathcal{V}_n$ covers $\bigcup\{U(n): U \in \mathcal{U}_n\}$ and refines $\{U(n): U \in \mathcal{U}_n\}$. Let

$$\mathcal{W}_n = \left\{V \times \left(\prod_{i>n} X_i\right): V \in \mathcal{V}_n\right\}.$$

Then $\bigcup_{n=1}^\infty \mathcal{W}_n$ is a σ-point finite open refinement of $\mathcal{U}$. Since $\Pi_{i=1}^\infty X_i$ is also perfect (HEATH and MICHAEL [1971]) it follows that $\bigcup_{n=1}^\infty \mathcal{W}_n$ has a point-finite open refinement.

6.21. REMARK. If A_i is one of the sets as in Theorem 6.19, we may assume $A_i = B_i \cup C_i$ where $B_i \cap C_i = \emptyset$ and $|F \cap B_i| = |F \cap C_i| = c$ for every uncountable closed subset F of $\mathbb{R}$. (The proof of Theorem 6.9 shows that A_i could be 'split' in this manner). By a construction similar to that in 6.11 there is a locally compact topology ρ_i on A_i, stronger than the usual topology, such that A_i, ρ_i is not submetacompact. If τ is the usual topology on $\mathbb{R}$ let $X_i = \mathbb{R}$ with the topology generated by $\tau \cup \rho_i$ and let $X = \bigoplus_{i\in\mathbb{N}} X_i$. By Remark 6.18 and by modifying the proof of 6.19 we see that X^n is a first countable Lindelöf space for every $n \in \mathbb{N}$ but X^ω is not submetacompact.

A similar construction to the above obtains our last example dealing with finite or countable cartesian products.

6.22. EXAMPLE. For every $k < \omega$ there is a Lindelöf space X such that X^k is Lindelöf but X^{k+1} is not even submetacompact.

PROOF. Let $\{A_1, A_2, \ldots, A_{k+1}\}$ be a partition of $\mathbb{R}$ into $k+1$ sets such that for any $n \in \mathbb{N}$ and every n-uncountable closed set B of $\mathbb{R}^n$ we have $|B \cap A_i^n|_n = c$ for every i, $1 \leq i \leq k+1$. There is a locally compact topology ρ_i on each A_i, stronger than the usual topology, such that A_i, ρ_i is not submetacompact. Let $X_i = \mathbb{R}$ with the topology generated by $\tau \cup \rho_i$ where τ is the usual topology on $\mathbb{R}$. Let $X = \bigoplus_{i=1}^{k+1} X_i$. Arguments similar to those used previously now show that X^k is a Lindelöf space but X^{k+1} is not submetacompact.

An example having the properties of 6.22 was given in PRZYMUSIŃSKI [1980]. Readers interested in the preservation of certain covering properties in finite and countable cartesian products should see the papers by MICHAEL [1971] and PRZYMUSIŃSKI [1980], [1980a].

Much of the study of covering properties in uncountable cartesian products becomes uninteresting (or reduces to the countable case) because of the following example. See POL and PUZIO-POL [1976] or VAN DOUWEN [1980] for a proof. The space $\mathbb{N}$ is assumed to have the discrete topology and $\mathbb{N}^{\omega_1}$ the Tychonoff product topology. The fact that $\mathbb{N}^{\omega_1}$ is not normal (see Theorem 8.2 in Comfort's article for a proof) is a classical result due to STONE [1948]; the space was shown to be not subparacompact by ALSTER and ENGELKING [1972].

6.23. EXAMPLE. The space $\mathbb{N}^{\omega_1}$ is not submetacompact.

A space which is not countably compact always contains a closed copy of $\mathbb{N}$. Hence a product $\Pi_{\alpha\in\Lambda} X_\alpha$ of non-countably compact spaces X_α, where $|\Lambda| \geq \omega_1$, contains a closed copy of $\mathbb{N}^{\omega_1}$. Such a product could never be submetacompact. In fact, since submetacompact countably compact spaces are compact (see 9.3) it follows that if a product space $\Pi_{\alpha\in\Lambda} X_\alpha$ is submetacompact, then all but countably many factor spaces X_α are compact.

We conclude this section with a few positive results on the preservation of covering properties under finite and countable cartesian products. In order to give fairly general results it is necessary to review two technical definitions.

A space X is said to be a *P-space* (MORITA [1964]) if for any open cover $\{U(\alpha_1, \ldots, \alpha_n): \alpha_i \in \Lambda,\ n \geq 1\}$ of X where $U(\alpha_1, \ldots, \alpha_n) \subset U(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ whenever $\alpha_1, \ldots, \alpha_{n+1} \in \Lambda$ there is a closed cover $\{H(\alpha_1, \ldots, \alpha_n): \alpha_i \in \Lambda, n \geq 1\}$ of X such that

(i) $H(\alpha_1, \ldots, \alpha_n) \subset U(\alpha_1, \ldots, \alpha_n)$ whenever $\alpha_1, \ldots, \alpha_n \in \Lambda$;
(ii) if $\{\alpha_i\}_1^\infty$ is any sequence from Λ such that

$$\bigcup_{n=1}^{\infty} U(\alpha_1, \ldots, \alpha_n) = X, \quad \text{then} \quad \bigcup_{n=1}^{\infty} H(\alpha_1, \ldots, \alpha_n) = X.$$

The reader should be warned that the term 'P-space' means something else in GILLMAN–JERISON [1960] and in van Mill's article.

An important special class of P-spaces is given by the following:

6.24. PROPOSITION. *Any perfect space X is a P-space.*

PROOF. Suppose $\{U(\alpha_1, \ldots, \alpha_n): \alpha_i \in \Lambda,\ n \geq 1\}$ is an open cover of the perfect space X where $U(\alpha_1, \ldots, \alpha_n) \subset U(\alpha_1, \ldots, \alpha_{n+1})$ whenever $\alpha_1, \ldots, \alpha_{n+1} \in \Lambda$. Now, each open set $U(\alpha_1, \ldots, \alpha_n)$ can be expressed as $U(\alpha_1, \ldots, \alpha_n) = \bigcup_{k=1}^\infty F_k(\alpha_1, \ldots, \alpha_n)$ where $\{F_k(\alpha_1, \ldots, \alpha_n)\}_{k=1}^\infty$ is a nondecreasing sequence of closed sets. Define $H(\alpha_1, \ldots, \alpha_n) = \bigcup_{k=1}^n F_n(\alpha_1, \ldots, \alpha_k)$. It follows that each $H(\alpha_1, \ldots, \alpha_n)$ is closed, $H(\alpha_1, \ldots, \alpha_n) \subset U(\alpha_1, \ldots, \alpha_n)$, and

$$\bigcup_{n=1}^{\infty} H(\alpha_1, \ldots, \alpha_n) = \bigcup_{n=1}^{\infty} U(\alpha_1, \ldots, \alpha_n)$$

for any sequence $\{\alpha_n\}_1^\infty$ from Λ. That completes the proof.

A space X is a *Σ-space* (NAGAMI [1969] if there is a σ-locally finite closed collection $\mathscr{F}$ in X and a cover $\mathscr{C}$ of closed countably compact sets such that if $C \subset U$, where $C \in \mathscr{C}$ and U is open in X, then $C \subset F \subset U$ for some $F \in \mathscr{F}$. If the elements of C are actually compact then X is said to be a *strong Σ-space*. This will be the case with a Σ-space satisfying any of the covering properties in Diagram 4.1 except orthocompactness (see 9.3).

We remark that it is easy to see that the class of Σ-spaces contains all σ-spaces and M-spaces (see the chapter on Generalized Metric spaces), and the class of Σ-spaces is closed under the action of perfect mappings in both the image and preimage direction.

6.25. THEOREM. *Suppose X is a P-space and Y is a Σ-space. If X and Y are both paracompact (regular Lindelöf, regular subparacompact, submetacompact) then $X \times Y$ is also Lindelöf (paracompact, subparacompact, submetacompact).*

PROOF. We sketch the proof for the submetacompact case. The Lindelöf case and paracompact case are proved in NAGAMI [1969] and the subparacompact case in LUTZER [1972]. Assume X is a submetacompact P-space and Y is a submetacompact Σ-space. Let $\mathscr{F}$ and $\mathscr{C}$ be as given in the definition of Σ-space. Assume $\mathscr{F} = \{F(\alpha)\colon \alpha \in \Lambda\}$. To show $X \times Y$ is submetacompact we let $\mathscr{W}$ be an open cover of $X \times Y$, which is closed under finite unions, and show $\mathscr{W}$ has a σ-closure preserving closed refinement. For $\alpha_1, \ldots, \alpha_n \in \Lambda$ define

$$\mathscr{U}(\alpha_1, \ldots, \alpha_n) = \{U \subset X\colon U \text{ is open and } U \times \Big(\bigcap_{i=1}^{n} F(\alpha_i)\Big) \subset W \text{ for some } W \in \mathscr{W}\}.$$

Let $U(\alpha_1, \ldots, \alpha_n) = \bigcup \mathscr{U}(\alpha_1, \ldots, \alpha_n)$ and let $\mathscr{U} = \{U(\alpha_1, \ldots, \alpha_n)\colon n \geq 1,\ \alpha_i \in \Lambda\}$. Then $\mathscr{U}$ is an open cover of X as in the definition of P-space. Hence there is a closed cover $\{H(\alpha_1, \ldots, \alpha_n)\colon n \geq 1,\ \alpha_i \in \Lambda\}$ of X such that $H(\alpha_1, \ldots, \alpha_n) \subset U(\alpha_1, \ldots, \alpha_n)$ and $\bigcup_{n=1}^{\infty} H(\alpha_1, \ldots, \alpha_n) = X$ whenever $\bigcup_{n=1}^{\infty} U(\alpha_1, \ldots, \alpha_n) = X$. Now each $H(\alpha_1, \ldots, \alpha_n)$ is submetacompact and $\mathscr{U}(\alpha_1, \ldots, \alpha_n)$ is an open cover of $H(\alpha_1, \ldots, \alpha_n)$ so there is a σ-closure preserving closed cover $\mathscr{B}(\alpha_1, \ldots, \alpha_n)$ of $H(\alpha_1, \ldots, \alpha_n)$ refining $\mathscr{U}(\alpha_1, \ldots, \alpha_n)$. Let

$$\mathscr{G} = \{B \times \Big(\bigcap_{i=1}^{n} F(\alpha_i)\Big)\colon n \geq 1, \alpha_1, \ldots, \alpha_n \in \Lambda, B \in \mathscr{B}(\alpha_1, \ldots, \alpha_n)\}.$$

It is clear that $\mathscr{G}$ is a partial refinement of $\mathscr{W}$ and it can be shown that $\mathscr{G}$ is σ-closure preserving. To see that $\mathscr{G}$ covers $X \times Y$ let $(x, y) \in X \times Y$ and pick a compact $C \in \mathscr{C}$ such that $y \in C$. Let $\{\alpha_i\}_1^{\infty}$ be a sequence from Λ such that whenever $C \subset F(\beta) \in \mathscr{F}$, then $F(\beta) = F(\alpha_i)$ for some i. It is easily verified that $\bigcup_{n=1}^{\infty} U(\alpha_1, \ldots, \alpha_n) = X$ so $\bigcup_{n=1}^{\infty} H(\alpha_1, \ldots, \alpha_n) = X$ and there is some m such that $x \in H(\alpha_1, \ldots, \alpha_m)$. If $B \in \mathscr{B}(\alpha_1, \ldots, \alpha_m)$ such that $x \in B$, then

$$(x, y) \in B \times (\bigcap_{i=1}^{m} F(\alpha_i)) \in \mathcal{G}.$$

Hence $\mathcal{G}$ is a σ-closure preserving closed refinement of $\mathcal{W}$. Theorem 3.6(iv) says that $X \times Y$ is submetacompact.

The usefulness of a result such as 6.25 is somewhat diminished by the technical difficulty of the P-space and Σ-space definitions. It is satisfying to have applications in a more familiar setting such as the following corollary. This should be compared to the situation in Examples 6.4, 6.10 and 6.11. The paracompact case was proved by MICHAEL as early as [1953] and the Lindelöf case by MORITA in [1964].

6.26. COROLLARY. *Suppose X is a regular perfect space and Y is a metric space. If X and Y are Lindelöf (paracompact, subparacompact, metacompact, submetacompact), then $X \times Y$ is Lindelöf (paracompact, subparacompact, metacompact, submetacompact).*

The metacompact case was not covered by Theorem 6.25 but a similar construction will work here. Corollary 6.26 is not the strongest available result and does not mention certain other covering properties which could be included (for example, see KRAMER [1976]). The proofs are meant to be indicative of certain situations that may preserve covering properties under finite products. The perfect condition on X and a base condition (related to the covering property) on Y is often the right combination.

To further motivate the use of Σ-spaces in the theory of covering properties of products we make the following observation.

6.27. REMARK. If $\mathcal{E}$ is any class of paracompact spaces which is closed under the formation of countable products and $\mathcal{D}$ is the class of all perfect preimages (preimages under perfect mappings) of elements of $\mathcal{E}$, then $\mathcal{D}$ is a (potentially) larger class of paracompact spaces which is also closed under the formation of countable products.

It is easily verified that if X_i, $i \in \mathbb{N}$, are elements of $\mathcal{D}$ and Y_i, $i \in \mathbb{N}$, are the perfect images, respectively, then $\Pi_{i=1}^{\infty} X_i$ is also a perfect preimage of $\Pi_{i=1}^{\infty} Y_i$ and by Theorem 5.9 $\Pi_{i=1}^{\infty} X_i$ is paracompact. Clearly similar remarks could be made concerning other covering properties.

Since the class of metric spaces is clearly preserved under the formation of countable products we immediately have a much larger class, the perfect preimages of metric spaces, closed under countable products. This class coincides with the paracompact p-spaces of ARHANGEL'SKIĬ [1963] and the paracompact M-spaces of MORITA [1964], and includes all paracompact locally compact spaces and all paracompact Čech complete spaces. A much different class of paracompact spaces, closed under countable products, is the class of paracompact σ-spaces (OKUYAMA [1967]) and its perfect preimages. The class of paracompact Σ-spaces contains all of the above and, as we see below, is also closed under the formation of countable products. See G. GRUENHAGE'S chapter on Generalized Metric Spaces for definitions of the above concepts as well as a proof for a portion of the following result. The paracompact case was proved by NAGAMI [1969] and the metacompact case by NYIKOS [1980].

6.28. THEOREM. *If X_i is a paracompact (regular Lindelöf subparacompact, metacompact, submetacompact) Σ-space for each $i \in \mathbb{N}$, then $\Pi_{i=1}^{\infty} X_i$ is also a paracompact (Lindelöf, subparacompact, metacompact, submetacompact) Σ-space.*

The subparacompact and submetacompact cases of the above come "free" using the fact that strong Σ-spaces are countably productive and that a Σ-space X is a strong Σ-space if and only if X is subparacompact (see Gruenhage's chapter).

7. Subspaces and sums

All of the covering properties listed in Diagram 4.1 are hereditary to closed subspaces and, in a given space, are hereditary to all subspaces if and only if they are hereditary to all open subspaces.

7.1. THEOREM. *If X is a space (normal in the parlindelöf case) satisfying any of the covering properties in Diagram 4.1, except for strongly paracompact, then any F_σ-subspace of X has the corresponding covering property. Hence these properties are all hereditary in such a space which is also perfect.*

PROOF. We prove only the metacompact case. Assume X is metacompact and Y is an F_σ subset of X. Express $Y = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed in X. Let $\mathcal{U}$ be an open collection in X covering Y. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{U \in \mathcal{U}: U \cap F_n \neq \emptyset\}$. The collection $\{X - F_n\} \cup \mathcal{U}_n$ is an open cover of X and must have a point finite open refinement $\mathcal{H}_n$. Let $\mathcal{W}_1 = \{H \cap Y: H \in \mathcal{H}_1,\ H \cap F_1 \neq \emptyset\}$ and for $n > 1$ let

$$\mathcal{W}_n = \{H \cap Y - (\bigcup_{i<n} F_i): H \in \mathcal{H}_n, H \cap F_n \neq \emptyset\}.$$

The collection $\bigcup_{n=1}^{\infty} \mathcal{W}_n$ is easily seen to be a point finite open cover of Y refining $\mathcal{U}$. That completes the proof.

If $B(\omega_1)$ denotes the Baire space of weight ω_1 discussed in Example 6.2(ii) then

$B(\omega_1)\times[0,1]$ is strongly paracompact but the F_σ-subspace $B(\omega_1)\times(0,1)$ is not strongly paracompact. This provides the verification for the following example.

7.2. EXAMPLE. A strongly paracompact metric space with an F_σ subspace which is not strongly paracompact.

A space X is said to be a *countable* (*locally finite*) *closed sum* of subspaces Y_α, $\alpha\in\Lambda$, if $\{Y_\alpha\colon \alpha\in\Lambda\}$ is a countable (locally finite) collection of closed subsets of X such that $X=\bigcup\{Y_\alpha\colon \alpha\in\Lambda\}$. We say that a topological property is preserved under such a sum if every sum X has the property whenever each Y_α has the property.

7.3. THEOREM. (i) *The following covering properties are preserved under countable closed sums: Lindelöf, subparacompact, submetacompact, submeta-Lindelöf, weakly θ-refinable, weakly $\delta\theta$-refinable.*

(ii) *The following covering properties are not generally preserved under countable closed sums: strongly paracompact, paracompact, para-Lindelöf σ-para-Lindelöf, metacompact, σ-meta-Lindelöf, screenable, meta-Lindelöf.*

PROOF. (i) We show the submeta-Lindelöf case. Suppose $X=\bigcup_{n=1}^\infty Y_n$, where each Y_n is a closed submeta-Lindelöf subspace, and let $\mathcal{U}$ be an open cover of X. For each n there is a sequence $\{\mathcal{H}_{nk}\}_1^\infty$ of covers of Y_n such that each $\mathcal{H}_{nk}$ is a partial refinement of $\mathcal{U}$ and $\{\mathcal{H}_{nk}\}_{k=1}^\infty$ is a $\delta\theta$-sequence of open covers with respect to Y_n. For each $H\in\mathcal{H}_{nk}$ pick open $H^*\subset X$ such that $H^*\subset U$ for some $U\in\mathcal{U}$ and $H=Y_n\cap H^*$. Define

$$\mathcal{G}_{nk}=\{H^*\colon H\in\mathcal{H}_{nk}\}\cup\{U\cap(X-Y_n)\colon U\in\mathcal{U}\}\,.$$

Then each $\mathcal{G}_{nk}$ is an open refinement of $\mathcal{U}$ and if $x\in X$ there is some n, $k\in\mathbb{N}$ such that $\operatorname{ord}(x,\mathcal{G}_{nk})\leqslant\omega$. It follows that X must be submeta-Lindelöf.

(ii) The space $\psi(N)$, described in Example 4.4, is a countable closed sum of strongly paracompact subspaces but $\psi(N)$ does not satisfy any of the covering properties listed in (ii).

To prove a locally finite closed sum theorem, suppose X is covered by a locally finite collection $\{A_\alpha\colon \alpha\in\Lambda\}$ of closed sets. Let $\bigoplus_{\alpha\in\Lambda}A_\alpha$ denote the free topological sum of the subspaces A_α and let $\phi\colon\bigoplus_{\alpha\in\Lambda}A_\alpha\to X$ be the natural canonical map. It is easily verified that ϕ is a closed mapping with finite fibers, hence ϕ is a perfect mapping. If the topological sum satisfies a given covering property then X would also satisfy the property in case that property is one generally preserved under perfect mappings. This technique provides the proof of the following theorem, parts of which have appeared in different sources. See HODEL [1969] for related results.

7.4. THEOREM. *The following covering properties are preserved under locally finite closed sums: paracompact, subparacompact, metacompact, submetacompact, para-Lindelöf, σ-para-Lindelöf.*

It is tempting to hope that similar results could be obtained for certain closure-preserving closed sums. The next example shows that such hopes are limited. The first example of this type was given by POTOCZNY [1972] in response to a question by TAMANO [1971], where it was asked whether a closure-preserving sum of compact subsets would give a paracompact space.

7.5. EXAMPLE. There is a locally compact space X which is not subparacompact but is the union of a closure-preserving family of finite subsets.

PROOF. Let $X = \omega_2 \times \omega_2 - \{(0,0)\}$ with the topology as given in Example 4.2. For each $(\alpha, \beta) \in X$ where $\alpha \neq 0$ and $\beta \neq 0$ define

$$F(\alpha, \beta) = \{(\alpha, \beta), (\alpha, 0), (0, \beta)\}.$$

If $\mathscr{F} = \{F(\alpha, \beta): 0 < \alpha < \omega_2, 0 < \beta < \omega_2\}$ it is straightforward to verify that $\mathscr{F}$ is a closure-preserving family (of finite subsets) whose union is X. The space X was previously shown to be a locally compact metacompact space which is not subparacompact.

The metacompactness of X in the above example is expected in view of JUNNILA's characterization (Theorem 3.5(iv)) of metacompactness. It is clear that whenever X is a union of a closure-preserving family of compact subsets and $\mathscr{U}$ is an open cover of X, then $\mathscr{U}^{\mathrm{F}}$ has a closure-preserving closed refinement. This proves the first part of the following result (KATUTA [1974], POTOCZNY and JUNNILA [1975]).

7.6. THEOREM. *If a space X is a closure-preserving sum of compact subspaces, then X is metacompact. If X is locally compact, the converse is true.*

For the proof of the converse suppose X is a locally compact metacompact space. Since X is locally compact there is an open cover $\mathscr{H}$ of X such that $\bar{H}$ is compact for each $H \in \mathscr{H}$ and, since X is metacompact, $\mathscr{H}^{\mathrm{F}}$ has a closure preserving closed refinement $\mathscr{F}$. This collection $\mathscr{F}$ is the desired closure-preserving cover consisting of compact sets.

For additional results concerning spaces which admit a closure-preserving (or σ-closure-preserving) cover by compact sets see POTOCZNY [1973], POTOCZNY and JUNNILA [1975], TELGARSKY [1975], and YAJIMA [1976], [1977].

8. With locally compact or locally connected

The proof that a locally compact locally connected normal Moore space is metrizable (REED and ZENOR [1976]) and recent set-theoretic results on perfectly normal manifolds have generated an interest in the study of covering properties within the context of local compactness or local connectedness. In this section we give a narrow survey of some of this material, starting with results related to F. Tall's question of whether a normal locally compact metacompact space must be paracompact.

Recall that a space X is said to satisfy ccc (*countable chain condition*) if every disjoint open collection in X is countable. Clearly every separable space and every hereditarily Lindelöf space satisfies ccc. Notice that ccc is hereditary to open subspaces, and a dense subspace Z of X satisfies ccc if and only if X satisfies ccc. The next lemma provides an alternate view of the ccc.

8.1. LEMMA. *For any space X the following are equivalent.*

(i) *X satisfies* ccc.

(ii) *Every relatively locally finite open collection in X is countable.*

(iii) *Every relatively locally countable open collection in X is countable.*

PROOF. It is clear that (iii)$\rightarrow$(ii)$\rightarrow$(i). To see that (i)$\rightarrow$(iii) assume (i) is true and suppose $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$ is an open collection of nonempty open sets in X where $\mathcal{H}$ is relatively locally countable, i.e., $\mathcal{H}$ is locally countable relative to the subspace $\bigcup\mathcal{H}$. Assume $H_\alpha \neq H_\beta$ if $\alpha \neq \beta$. For each $\alpha \in \Lambda$ pick a nonempty open subset $G_\alpha \subset H_\alpha$ such that $G_\alpha \cap H_\beta \neq \emptyset$ for at most countably many $\beta \in \Lambda$. Now the collection $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ is a star-countable open collection so it follows from Lemma 3.10 that $\mathcal{G}$ can be expressed as $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$ where each $\mathcal{G}_n$ is a disjoint collection. Since X satisfies ccc we see that each $\mathcal{G}_n$ is countable so $\mathcal{G}$ is countable. The correspondence $H_\alpha \rightarrow G_\alpha$ may not be one-to-one but at least is countable-to-one so $\mathcal{H}$ must also be countable.

The next lemma is a combination of results from MCCOY [1972], and FLETCHER and LINDGREN [1973]. A *Baire space* is a space X where $\bigcap_{n=1}^{\infty} D_n$ is always dense in X whenever D_n, $n \in \mathbb{N}$, are dense open subsets of X.

8.2. LEMMA. *A space X is a Baire space if and only if every point finite open cover of X is locally finite at a dense* (*open*) *set of points.*

PROOF. Suppose X is a Baire space and $\mathcal{U}$ is a point-finite open cover of X. If A is the set of all points at which $\mathcal{U}$ is locally finite it is easy to see that A is open (this does not use the Baire space assumption). To show that A is dense let G be a nonempty open set in X and we will show $A \cap G \neq \emptyset$. Let $E_n = \{x \in G : \operatorname{ord}(x, \mathcal{U}) \leq n\}$. First we see that $\operatorname{int}(E_n) \neq \emptyset$ for some n, for otherwise, each set $D_n = (X - \bar{G}) \cup (G - E_n)$ would be an open dense set and $\bigcap_{n=1}^{\infty} D_n =$

$X-\bar{G}$ which is not dense. If k is the least element of N such that $\operatorname{int}(E_k)\neq\emptyset$ there exists some $z\in\operatorname{int}(E_k)$ such that $\operatorname{ord}(z,\mathcal{U})=k$, say $U_1, U_2, \ldots, U_k$ are the distinct elements of $\mathcal{U}$ such that $z\in\bigcap_{i=1}^k U_i$. Let

$$W=\operatorname{int}(E_k)\cap\Big(\bigcap_{i=1}^{k}U_i\Big).$$

It follows from the definition of E_k that W cannot intersect any elements of $\mathcal{U}$ other than $U_1,\ldots,U_k$. Hence $z\in G\cap A$.

For the converse, assume the condition that every point finite open cover of X is locally finite at a dense set of points, and let $\{D_n\}_1^\infty$ be a sequence of dense open subsets of X. We may assume $\{D_n\}_1^\infty$ is a decreasing sequence of distinct sets. Let $F=\operatorname{cl}(\bigcap_{n=1}^\infty D_n)$. If $F\neq X$ consider the open cover $\{X\}\cup\{D_n-F\colon n\in\mathbb{N}\}$. This cover is point finite and by assumption, there is some $z\in X-F$ such that this cover is locally finite at z. Hence there is an open neighborhood W of Z, $W\subset X-F$, such that $W\cap(D_n-F)\neq\emptyset$ for only finitely many n. This says $W\cap D_n\neq\emptyset$ for only finitely many n, a contradiction since each D_n is dense. Thus $F=X$ and the lemma is proved.

The previous lemmas now give the following result from TALL [1974].

8.3. PROPOSITION. *Every point-finite open collection in a* ccc *Baire space is countable.*

PROOF. If $\mathcal{U}$ is a point-finite open collection in X let $Y=\bigcup\mathcal{U}$. Let A be the set of points in Y at which $\mathcal{U}$ is locally finite. Now, X a Baire space implies the open subspace Y is a Baire space so Lemma 8.2 says that A is dense and open in Y. If $\mathcal{W}=\{U\cap A\colon U\in\mathcal{U}\}$, then $\mathcal{W}$ is a relatively locally finite open collection in X, so Lemma 8.1 says $\mathcal{W}$ is countable. Since $U\cap A\neq\emptyset$ for every $U\in\mathcal{U}$ it follows that $\mathcal{U}$ is also countable.

8.4. COROLLARY (TALL [1974]). *A regular metacompact Baire space X which is locally* ccc *is strongly paracompact.*

PROOF. If $\mathcal{U}$ is an open cover of X it suffices to show $\mathcal{U}$ has a star-countable open refinement (see Theorem 3.12). $\mathcal{U}$ does have a point-finite open refinement $\mathcal{W}$ where we may assume each $W\in\mathcal{W}$ satisfies ccc. Since each $W\in\mathcal{W}$ is also a Baire space it follows from the previous proposition that $\mathcal{W}$ is star-countable.

8.5. COROLLARY (ARHANGEL'SKIĬ [1972]). *A perfect locally compact metacompact space is strongly paracompact.*

PROOF. This follows from 8.4 after verifying that locally compact implies locally Baire (hence Baire), and that perfectly locally compact implies locally ccc.

The space $\psi(N)$ (Example 4.4) shows that 'submetacompact' cannot be substituted for 'metacompact' in Corollaries 8.4 and 8.5, and Example 4.2 shows that the 'perfect' condition in Corollary 8.5 cannot be dropped. It is still unknown whether it can be shown that normal locally compact metacompact spaces are paracompact without extra set-theoretic axioms beyond ZFC. S. WATSON [1982] has shown (see Theorem 8.7 below) that this is the case assuming the axiom '$V = L$'.

P. DANIELS [1981] has recently shown that a normal locally compact space X is paracompact if it is boundedly metacompact, i.e., if for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{H}$ of $\mathcal{U}$ and some $m \in \mathbb{N}$ such that $\operatorname{ord}(x, \mathcal{H}) \leq m$ for all $x \in X$. She relates the study of normal locally compact metacompact spaces to the study of subspaces of certain Pixley–Roy spaces.

In order to give S. WATSON's interesting proof that normal locally compact metacompact spaces are paracompact, assuming $V = L$, we need the following results by W. FLEISSNER which we state without proof. We say a space X is λ-collectionwise Hausdorff (for a cardinal number λ) if whenever D is a closed discrete set in X with $|D| \leq \lambda$, then the points in D can be simultaneously separated by disjoint open sets.

8.6. THEOREM (FLEISSNER [1974], [1977]). (i) ($V = L$) *If κ is a regular cardinal, and X is normal and λ-collectionwise Hausdorff for each $\lambda < \kappa$, then any closed discrete set of cardinality κ, whose points have character less than or equal to κ, can b separated.*

(ii) (GCH) *If κ is singular, and X is normal and λ-collectionwise Hausdorff for each $\lambda < \kappa$, then any closed discrete set of cardinality κ, whose points have character bounded by some $m < \kappa$, can be separated.*

8.7. THEOREM. (*Assume* $V = L$). (i) *Normal locally compact spaces are collectionwise normal with respect to compact sets.*

(ii) *Normal locally compact submetacompact spaces are paracompact.*

PROOF. The proof that collectionwise normal submetacompact spaces are paracompact (Theorem 4.16) can be modified to use only collectionwise normal with respect to compact sets when working in a normal locally compact space. Hence (ii) follows from (i). To show (i), assume otherwise and suppose X is a locally compact normal space and $\{D_\alpha : \alpha < \kappa\}$ is a discrete collection of compact sets which cannot be separated, where κ is minimal with respect to the existence of such a collection taken over all locally compact normal spaces. We show the case for regular κ—the proof can be modified for the singular case. For each $\alpha < \kappa$ there is an open set O_α, with compact closure, such that $D_\alpha \subset O_\alpha$ and $D_\beta \cap \bar{O}_\alpha = \emptyset$ if $\alpha \neq \beta$. It is easy to build an open F_σ-set V_α such that $D_\alpha \subset V_\alpha \subset O_\alpha$ (hence $\bar{V}_\alpha$ is compact). Using normality there is an open set A in X such that

$$\bigcup_{\alpha<\kappa} D_\alpha \subset A \subset \bar{A} \subset \bigcup_{\alpha<\kappa} V_\alpha .$$

Let $Y = \bar{A}$, $U_\alpha = Y \cap V_\alpha$ and

$$E_\alpha = Y - \bigcup_{\beta\neq\alpha} U_\beta .$$

Note that $D_\alpha \subset E_\alpha \subset U_\alpha$, $Y = \bigcup_{\alpha<\kappa} U_\alpha$, and $U_\beta \cap E_\alpha = \emptyset$ if $\alpha \neq \beta$. So $\{E_\alpha : \alpha < \kappa\}$ is a discrete collection of compact sets in Y. If we show there is a collection $\{W_\alpha : \alpha < \kappa\}$ of open sets in Y, which separates $\{E_\alpha : \alpha < \kappa\}$ then $\{W_\alpha \cap A : \alpha < \kappa\}$ would be an open collection in X which separates $\{D_\alpha : \alpha < \kappa\}$. Now, each U_β is an F_σ in Y, so $U_\beta = \bigcup_{n=1}^\infty F_{n\beta}$ for closed $F_{n\beta}$. This gives

$$E_\alpha = \bigcap_{n=1}^{\infty} \bigcap_{\beta\neq\alpha} (Y - F_{n\beta})$$

That is, E_α is the intersection of $\leqslant\kappa$ open sets in Y. If Z is the quotient space obtained from Y, by shrinking each E_α to a point e_α then Z is λ-collectionwise Hausdorff for any $\lambda < \kappa$ (by the minimal condition on κ) and each e_α has character $\leqslant\kappa$. (Use the fact that the pseudo character of e_α is $\leqslant\kappa$ and this agrees with the character in a locally compact space). By Theorem 8.6(i), it follows that the discrete point set $\{e_\alpha : \alpha < \kappa\}$ can be separated (in Z) and this implies $\{E_\alpha : \alpha < \kappa\}$ can be separated in Y. As noted earlier, this induces a separation of $\{D_\alpha : \alpha < \kappa\}$ and this contradiction completes the proof.

Much of the work in this area is directed toward showing that certain spaces satisfy some variation of 'collectionwise separation'. The next result outlines two situations in which a given collectionwise separation property can be strengthened.

8.8. THEOREM. (i) *If X is a normal locally compact space such that X is collectionwise normal with respect to compact sets, then X is collectionwise normal with respect to closed submetacompact subsets.*

(ii) *If $\mathscr{C}$ is a class of λ-collectionwise Haudsorff spaces, for some cardinal λ, such that $\mathscr{C}$ is closed under perfect mappings, then every member of $\mathscr{C}$ is λ-collectionwise normal with respect to compact sets.*

PROOF. (i) Suppose $\{Z_\alpha : \alpha \in \Lambda\}$ is a (faithfully indexed) discrete collection of closed submetacompact subspaces of X. Since X is normal, locally compact and collectionwise normal with respect to compact sets, the proof of Theorem 4.16 can be modified to show each Z_α is paracompact (hence subparacompact). Since Z_α is locally compact and subparacompact there is a σ-discrete cover $\bigcup_{n=1}^\infty \mathscr{B}_{n\alpha}$ of Z_α, where each $\mathscr{B}_{n\alpha}$ is a discrete collection of compact subsets of Z_α. Say $\mathscr{B}_{n\alpha} =$

$\{B(n, \alpha, \beta): \beta \in \Gamma(n, \alpha)\}$. For fixed n, the collection $\bigcup_{\alpha\in\Lambda}\mathcal{B}_{n\alpha}$ is a discrete collection of compact subsets of X so there is a disjoint family $\{U(n, \alpha, \beta): \beta \in \Gamma(n, \alpha), \alpha \in \Lambda\}$ of open subsets of X such that

$$B(n, \alpha, \beta) \subset U(n, \alpha, \beta).$$

By Lemma 2.2(iii) we may assume this family is actually discrete, and we also assume each $U(n, \alpha, \beta)$ has been chosen so that $\overline{U(n, \alpha, \beta)} \cap Z_\gamma = \emptyset$ if $\alpha \neq \gamma$. Now, for $n \in \mathbb{N}$, $\alpha \in \Lambda$ let

$$H(n, \alpha) = \bigcup\{U(n, \alpha, \beta): \beta \in \Gamma(n, \alpha)\}$$

$$-\operatorname{cl}(\bigcup\{U(k, \gamma, \delta): k \leq n, \gamma \in \Lambda, \gamma \neq \alpha, \delta \in \Gamma(k, \gamma)\}.$$

If $H_\alpha = \bigcup_{n=1}^{\infty} H(n, \alpha)$, for each $\alpha \in \Lambda$, it follows that $Z_\alpha \subset H_\alpha$ and $\{H_\alpha: \alpha \in \Lambda\}$ is a disjoint collection of open sets.

We leave the proof of part (ii) as an easy exercise. The argument is similar to that used to finish the proof of Theorem 4.27.

The key to the Reed–Zenor result that a locally compact, locally connected, normal Moore space is metrizable is in showing that a perfectly normal, locally compact, locally connected submetacompact space is paracompact (see Corollary 8.10). Certain generalizations and modifications of this result have been given by various authors. ALSTER and ZENOR [1976] show that a perfectly normal, locally compact, locally connected space is collectionwise normal with respect to closed Lindelöf subsets (c.f., Theorem 8.9). CHABER and ZENOR [1977] weaken the locally compact condition in the Reed–Zenor Theorem to rim compact by showing that a perfectly normal, rim compact, locally connected, subparacompact space is paracompact. GRUENHAGE [1979] extends this result and Corollary 8.10 by using "normal" in place of "perfectly normal" (see Theorem 8.11).

We now prove the theorem due to ALSTER and ZENOR [1976]. The reader might take note of the role played by the 'perfect' condition in this result, in order to compare how the proof of Theorem 8.11 compensates for its absence.

8.9. THEOREM. *A perfectly normal, locally compact, locally connected space X is collectionwise normal with respect to closed submetacompact subsets.*

PROOF. Since the class of perfectly normal, locally compact, locally connected spaces is closed under perfect mappings it suffices, by Theorem 8.8, to show that all such spaces X are collectionwise Hausdorff. To this end, suppose $A = \{a_\alpha: \alpha \in \Lambda\}$ is a closed discrete subset of X (faithfully indexed by Λ). Since X is perfectly normal we may express $A = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \bar{G}_n$ where $\{G_n\}_1^{\infty}$ is a decreasing sequence of open sets. For each $\alpha \in \Lambda$ we can find a decreasing sequence $\{U(n, \alpha)\}_{n=1}^{\infty}$ of connected open sets about a_α such that $U(n, \alpha) \subset G_n$, $\overline{U(1, \alpha)}$ is compact, and $\overline{U(1, \alpha)} \cap A = \{a_\alpha\}$. To finish the proof we show that for each $\alpha \in \Lambda$ there exists $k_\alpha \in \mathbb{N}$ such that

$$U(1, \alpha) \cap U(k_\alpha, \beta) = \emptyset$$

if $\beta \neq \alpha$. The family $\{U(k_\alpha, \alpha): \alpha \in \Lambda\}$ will then be the desired disjoint open collection separating the points of A. To show this, notice that $\partial U(1, \alpha) \cap (\bigcap_{n=1}^{\infty} \bar{G}_n) = \emptyset$ ($\partial U(1, \alpha)$ denotes the boundary of $U(1, \alpha)$). Since $\partial U(1, \alpha)$ is compact there must be some k_α where $\partial U(1, \alpha) \cap \bar{G}_{k_\alpha} = \emptyset$, hence $\partial U(1, \alpha) \cap U(k_\alpha, \beta) = \emptyset$ for every $\beta \neq \alpha$. Since $U(k_\alpha, \beta)$ is connected we must have $U(1, \alpha) \cap U(k_\alpha, \beta) = \emptyset$ as desired. That completes the proof.

8.10. COROLLARY (REED and ZENOR [1976]). *A perfectly normal, locally compact, locally connected, submetacompact space is paracompact.*

The previous corollary is now strengthened in part (i) of the following result by GRUENHAGE [1979]. A space X is *rim compact* if every $x \in X$ has a neighborhood base consisting of open sets with compact boundaries.

8.11. THEOREM. (i) *Every normal, locally compact, locally connected, submetacompact space X is paracompact.*

(ii) *Every normal, rim compact, locally connected subparacompact space X is paracompact.*

PROOF. We prove only (i). The bulk of the proof for (i) is given in two lemmas (due to GRUENHAGE [1979]) which are stated and proved separately.

8.12. LEMMA. (i) *If X is normal, rim compact, locally connected and submetacompact, then X is c-collectionwise Hausdorff.*

(ii) *If X is normal, locally compact, locally connected and submetacompact, then X is c-collectionwise normal with respect to compact sets.*

PROOF. Part (ii) will follow from part (i) and 8.8(ii). To prove (i) let X be a normal, rim compact, locally connected, submetacompact space and suppose $A = \{a_\alpha: \alpha \in \Lambda\}$ is a closed discrete subset of X with $|\Lambda| \leqslant c$. Assume $a_\alpha \neq a_\beta$ if $\alpha \neq \beta$. For each $\alpha \in \Lambda$ pick an open neighborhood W_α of a_α such that the boundary ∂W_α is compact and $\bar{W}_\alpha \cap A = \{a_\alpha\}$. Let $\{\mathcal{U}_n\}_1^{\infty}$ be a θ-sequence of open refinements of $\{W_\alpha: \alpha \in \Lambda\} \cup \{X - A\}$. As in the proof of Theorem 4.22 there exists a countably infinite collection $\{E_{1n}: n \in \mathbb{N}\}$ of subsets of A which is point-separating in A. We may assume $\{E_{1n}: n \in \mathbb{N}\}$ is closed under finite intersections so that whenever $B \subset A$ is finite and $x \in A - B$ there is some m such that $x \in E_{1m}$ and $E_{1m} \cap B = \emptyset$. For each $n \in \mathbb{N}$, let $E_{2n} = A - E_{1n}$ and use normality to find open sets G_{1n} and G_{2n} such that $E_{1n} \subset G_{1n}$, $E_{2n} \subset G_{2n}$, and $\bar{G}_{1n} \cap \bar{G}_{2n} = \emptyset$. By induction on $n \in \mathbb{N}$, we can find connected open sets $V(n, \alpha)$ for each $\alpha \in \Lambda$ such that:

(1) $a_\alpha \in V(n+1, \alpha) \subset \overline{V(n+1, \alpha)} \subset V(n, \alpha) \subset W_\alpha$,
(2) $V(n, \alpha) \subset G_{in}$ if $a_\alpha \in E_{i\alpha}$, $i \in \{1, 2\}$,
(3) $\{V(n, \alpha): \alpha \in \Lambda\}$ is a partial refinement of $\mathcal{U}_n$,
(4) $\mathrm{cl}(\bigcup\{V(n+1, \alpha): \alpha \in \Lambda\}) \subset \bigcup\{V(n, \alpha): \alpha \in \Lambda\}$.

If we show, for each $\alpha \in \Lambda$, there is some $k_\alpha \in \mathbb{N}$ such that $V(k_\alpha, \alpha) \cap V(k_\alpha, \beta) = \emptyset$ if $\beta \in \Lambda$, $\beta \neq \alpha$, then $\{V(k_\alpha, \alpha): \alpha \in \Lambda\}$ would be a collection of open sets separating the points of A.

Assume otherwise and suppose there exist $\alpha \in \Lambda$ and $\beta_n \in \Lambda$, $\beta_n \neq \alpha$, such that $V(n, \alpha) \cap V(n, \beta_n) \neq \emptyset$ for every $n \in \mathbb{N}$. Now $W_\alpha \cap V(n, \beta_n) \neq \emptyset$ and $V(n, \beta_n)$ connected imply there exists some $z_n \in (\partial W_\alpha) \cap V(n, \beta_n)$. Since ∂W_α is compact there is a cluster point $z \in \partial W_\alpha$ of $\{z_n: n \in \mathbb{N}\}$. There is some $m \in \mathbb{N}$ such that $\mathrm{ord}(z, \mathcal{U}_m) < \omega$ and the set $B = \{a_\gamma: z \in V(m, \gamma)\}$ is finite (possibly even empty). Since $a_\alpha \notin B$ there is some $k > m$ such that $a_\alpha \in E_{1k}$ and $B \subset E_{2k}$. For any $n \geqslant k$, $V(n, \alpha) \cap V(n, \beta_n) \neq \emptyset$ implies $V(k, \alpha) \cap V(k, \beta_n) \neq \emptyset$ so conditions (1) and (2) say $z_n \in V(k, \beta_n) \subset G_{1k}$. Hence $z \in \bar{G}_{1k}$ and it follows that $z \notin \bigcup\{V(k, \beta): \beta \in \Lambda\}$. This says $z \notin \mathrm{cl}(\bigcup\{V(k+1, \beta): \beta \in \Lambda\})$, which is impossible since $\{z_i: i \geqslant k+1\} \subset \bigcup\{V(k+1, \beta_i): i \geqslant k+1\}$. This contradiction completes the proof of Lemma 8.12.

8.13. LEMMA. *If X is normal, locally compact, locally connected, submetacompact, and connected, then X is ω_1-Lindelöf (every open cover of X has a subcover of cardinality $\leqslant \omega_1$).*

PROOF. Let $\{\mathcal{U}_n\}_{n=1}^{\infty}$ be a θ-sequence of open covers of X by sets with compact closure. It suffices to show $\bigcup_{n=1}^{\infty}\mathcal{U}_n$ has a subcover of cardinality $\leqslant \omega_1$. For each $n \in \mathbb{N}$, $k < \omega$, let $F(m, k) = \{x \in X: \mathrm{ord}(x, \mathcal{U}_m) \leqslant k\}$, and $E_m = \bigcup_{k<\omega} F(m, k)$. We first observe that whenever $\mathcal{G}$ is a countable subcollection of $\bigcup_{n=1}^{\infty}\mathcal{U}_n$ there is a countable subcollection from $\mathcal{U}_m$ covering $F(m, k) \cap (\overline{\bigcup\mathcal{G}})$. This is clearly true when $k = 0$ since $F(m, 0) = \emptyset$. Continuing by induction, assume that for $k = r$ there is a countable $\mathcal{H} \subset \mathcal{U}_m$ such that

$$F(m, r) \cap (\overline{\bigcup\mathcal{G}}) \subset \bigcup\mathcal{H}.$$

It follows that there is a discrete closed collection $\mathcal{F}$, covering $(F(m, r+1) - \bigcup\mathcal{H}) \cap (\overline{\bigcup\mathcal{G}})$ (see Proposition 1.3(iii)) such that $\mathcal{F}$ is a partial refinement of $\mathcal{U}_m$. Now, $\mathcal{F}$ must be countable, for otherwise we could use Lemma 8.12(ii) and normality to find a discrete collection of ω_1 open sets, each intersecting $\bigcup\mathcal{G}$. Since $\mathcal{G}$ is countable, there would be some $G \in \mathcal{G}$ such that the compact set $\bar{G}$ would intersect uncountably many elements of this discrete collection, a contradiction. The countable collection $\mathcal{F}$ can now be used to find a countable subcollection from $\mathcal{U}_m$ covering $F(m, r+1) \cap (\overline{\bigcup\mathcal{G}})$, completing the induction.

Pick any nonempty $U_0 \in \bigcup_{n=1}^\infty \mathcal{U}_n$ and let $\mathcal{W}_0 = \{U_0\}$. Using the above observation, we can find, for each $\beta < \omega_1$, a countable subcollection $\mathcal{W}_\beta \subset \bigcup_{n=1}^\infty \mathcal{U}_n$ such that:

(1) $\mathcal{W}_\alpha \subset \mathcal{W}_\beta$ if $\alpha < \beta < \omega_1$.

(2) For every $m \in \mathbb{N}$, $\alpha < \omega_1$, $\mathcal{W}_{\alpha+1}$ contains a countable subcollection from $\mathcal{U}_m$ covering $E_m \cap (\overline{\bigcup \mathcal{W}_\alpha})$.

Let $Y = \bigcup\{\bigcup \mathcal{W}_\alpha : \alpha < \omega_1\}$. Clearly Y is open; if we show Y is also closed then $Y = X$ and the result follows. To show this, suppose $z \in \bar{Y} - Y$. Let V be any open neighborhood of z such that $\bar{V}$ is compact, and let $Z = Y \cap \bar{V}$. Since $Z = \bigcup\{(\overline{\bigcup \mathcal{W}_\alpha}) \cap \bar{V} : \alpha < \omega_1\}$ it is easy to show Z is countably compact. It also follows from the construction of Y that $\{Z \cap W : W \in \mathcal{W}_\alpha, \alpha < \omega_1\}$ is a 'weak θ' open cover of Z and, by 9.3 must have a finite subcover. This says there must be some $\beta < \omega_1$ such that $Z \subset \bigcup \mathcal{W}_\beta$, and so $z \in (\overline{\bigcup \mathcal{W}_\beta}) \subset Y$. Hence Y is closed.

To finish the proof of Theorem 8.11(i), it suffices to show each connected component of X is paracompact. This follows using Lemmas 8.12 and 8.13, and techniques from the proofs of Lemma 4.15 and Theorem 4.16.

Any manifold, being locally Euclidean, is a locally compact, locally connected space which is also locally metrizable. An old question, due to ALEXANDROFF [1935] and WILDER [1949], of whether every perfectly normal manifold is metrizable was recently shown to be independent of ZFC. RUDIN and ZENOR [1976] use the continuum hypothesis (CH) to construct a perfectly normal, separable manifold that is not paracompact. KOZLOWSKI and ZENOR [1979] use CH to construct such a manifold which is also real analytic. On the other hand, M.E. RUDIN [1979] shows that every perfectly normal manifold is metrizable assuming (MA + $\neg$CH). D. LANE [1980] extended Rudin's technique from the above result to obtain: If (MA + $\neg$CH), then every perfectly normal, locally compact, locally connected space is paracompact (Theorem 8.15). Rudin's theorem follows from this since locally metrizable paracompact spaces are metrizable.

G. GRUENHAGE [1981] modifies these techniques to obtain the following result which we give without proof. H. JUNNILA has noticed that the proof of this result can be pushed to reduce the condition "collectionwise normal with respect to compact sets" to "collectionwise Hausdorff".

8.14. THEOREM (Assume MA + $\neg$CH). *If X is perfectly normal, locally compact, and collectionwise normal with respect to compact sets, then X is paracompact.*

D. Lane's result now follows from Theorem 8.14 and Theorem 8.9.

8.15. THEOREM (*Assume* MA + $\neg$CH). *If X is perfectly normal, locally compact and locally connected, then X is paracompact.*

An interesting open question, due to G.M. Reed, is whether (MA + ¬CH) implies that every perfectly normal locally compact space is subparacompact. GRUENHAGE [1981] shows that MA implies every perfectly normal locally compact space X, with $|X| \leq \mathfrak{c}$, is subparacompact.

For an additional survey of the connection between set-theoretic topology and nonmetrizable manifolds the reader should see NYIKOS [1981] and the chapter by P. NYIKOS (in this volume) on nonmetrizable manifolds.

9. With countably compact or pseudocompact

In this section we are interested in studying weak covering properties in the presence of a countably compact or pseudocompact condition, and we indicate certain combinations which actually force compactness. We introduce one more covering property which is weaker than all of the properties in Diagram 4.1 except for orthocompactness.

A space X is said to be *weakly* $[\omega_1, \infty)^r$*-refinable* if for any open cover $\mathcal{U}$, of uncountable regular cardinality, there exists an open refinement which can be expressed as $\bigcup\{\mathcal{G}_\alpha : \alpha \in \Gamma\}$ where $|\Gamma| < |\mathcal{U}|$ and if $x \in X$ there is some $\alpha \in \Gamma$ such that $0 < \mathrm{ord}(x, \mathcal{G}_\alpha) < |\mathcal{U}|$. This property was introduced in WORRELL and WICKE [1979] where it was shown that a weakly $[\omega_1, \infty)^r$-refinable, countably compact space is compact (Theorem 9.2 below). A related concept of '$[a, b]$-refinable^r' was previously studied by HODEL and VAUGHAN [1974]. It is clear that weakly $\delta\theta$-refinable spaces are weakly $[\omega_1, \infty)^r$-refinable.

For our purposes, the following lemma takes care of the restriction, in the definition of weakly $[\omega_1, \infty)^r$-refinable, to open covers of regular cardinality.

9.1. LEMMA. *Suppose X is a noncompact space and m is the cardinal number minimal with respect to the condition that there exists an open cover $\mathcal{U}$, $|\mathcal{U}| = m$, but $\mathcal{U}$ has no finite subcover. Then m is regular. Moreover, if $\mathcal{F}$ is any closed filter base in X, $|\mathcal{F}| < m$, then $\bigcap\mathcal{F} \neq \emptyset$.*

PROOF. If m is not regular the open cover $\mathcal{U}$ can be expressed as $\mathcal{U} = \bigcup\{\mathcal{H}_\alpha : \alpha \in \Lambda\}$ where $|\Lambda| < m$ and $|\mathcal{H}_\alpha| < m$ for each $\alpha \in \Lambda$. If $H_\alpha = \bigcup\mathcal{H}_\alpha$ then $\{H_\alpha : \alpha \in \Lambda\}$ is an open cover of cardinality less then m so there is a finite set $A \subset \Lambda$ such that $\{H_\alpha : \alpha \in A\}$ covers X. The cover $\bigcup\{\mathcal{H}_\alpha : \alpha \in A\}$ has cardinality less than m so must have a finite subcover. This finite subcover is also a subcover of $\mathcal{U}$, a contradiction. Hence m is regular. The last statement in the lemma follows easily by considering the open collection of complements of elements from $\mathcal{F}$.

9.2. THEOREM (WORRELL and WICKE [1979]). *If X is countably compact and weakly $[\omega_1, \infty)^r$-refinable, then X is compact.*

PROOF. We first observe that whenever Y is countably compact, $\mathscr{W}$ is an open cover of Y, κ is an infinite cardinal, and $E=\{x\in Y: \operatorname{ord}(x,\mathscr{W})<\kappa\}$, then $\mathscr{W}$ has a subcollection $\mathscr{H}$, covering E, with $|\mathscr{H}|<\kappa$. To see this, pick a sequence $\{x_n\}_1^\infty$, if possible, such that $x_1\in E$ and $x_{n+1}\in E-\bigcup_{i=1}^n \operatorname{St}(x_i,\mathscr{W})$. If z is a cluster point of $\{x_n\}_1^\infty$ there is some $x_k\in \operatorname{St}(z,\mathscr{W})$; then $z\in\operatorname{St}(x_k,\mathscr{W})$ so there must be some $j>k$ such that $x_j\in\operatorname{St}(x_k,\mathscr{W})$. This contradicts the condition placed on the choice of x_j so there does not exist such a sequence. Consequently, there is a finite sequence $x_1, x_2,\ldots,x_n$ from E such that $E\subset\bigcup_{i=1}^n\operatorname{St}(x_i,\mathscr{W})$. The collection $\mathscr{H}=\{H\in\mathscr{W}: x_i\in H, \text{ some } i\leq n\}$ is the desired cover of E.

Now, assuming X is not compact let m and $\mathscr{U}$ be as given in Lemma 9.1. $\mathscr{U}$ has an open refinement $\mathscr{H}$ which we may express as $\mathscr{H}=\bigcup\{\mathscr{G}_\alpha: \alpha<\mu\}$ where μ is a cardinal number less than m, and for any $x\in X$ there is some $\beta<\mu$ such that $0<\operatorname{ord}(x,\mathscr{G}_\beta)<m$. Notice that the conditions on m and $\mathscr{U}$ imply that $\mathscr{H}$ cannot have a subcover of cardinality less than m. For convenience, let $\mathscr{S}$ denote the collection of all closed subspaces of X which cannot be covered by a subcollection from $\mathscr{H}$ of cardinality $<m$. There must be some $\beta<\mu$ such that $X-\bigcup\mathscr{G}_\beta\in\mathscr{S}$, for otherwise the observation above (first paragraph) and regularity of m would imply $X\notin\mathscr{S}$. Let B be a maximal subset of μ such that for any finite $A\subset B$ the set $F(A)=\bigcap_{\alpha\in A}(X-\bigcup\mathscr{G}_\alpha)\in\mathscr{S}$. If $\mathscr{F}$ is the filter base $\{F(A): A\subset B, |A|<\omega\}$ let $Y=\bigcap\mathscr{F}$ and notice that $Y\in\mathscr{S}$. In fact, there must be some $\gamma<\mu$, $\gamma\notin B$ such that $Y-\bigcup\mathscr{G}_\gamma\in\mathscr{S}$. This contradicts the maximal condition placed on B and the theorem is proved.

Theorem 9.2 obviously says that any weakly $\delta\theta$-refinable, countable compact space is compact. The proof of 9.2 actually gives a somewhat stronger result. A cover $\mathscr{H}$ of X is said to be a *weak $\delta\theta$-cover* (*weak θ-cover*) of X if $\mathscr{H}$ can be expressed as $\mathscr{H}=\bigcup_{n=1}^\infty\mathscr{G}_n$ where if $x\in X$ there is some $m\in\mathbb{N}$ such that $0<\operatorname{ord}(x,\mathscr{G}_m)\leq\omega$ ($0<\operatorname{ord}(x,\mathscr{G}_m)<\omega$).

9.3. COROLLARY. *If $\mathscr{H}$ is an open, weak $\delta\theta$-cover of a countably compact space X, then $\mathscr{H}$ has a finite subcover.*

9.4. EXAMPLE. If λ is any limit ordinal, not cofinal with ω, then λ, with the order topology, is not weakly $\delta\theta$-refinable.

PROOF. This follows from 9.3 using the fact that any such λ is countably compact but not compact.

A space X is said to be *isocompact* if every closed countably compact subset is compact. Using this terminology, Theorem 9.2 says that weakly $[\omega_1,\infty)^r$-refinable spaces are isocompact. There are other weak covering properties which imply isocompactness. For example, DAVIS [1979] studied 'property θL' and shows this property generalizes weakly $\delta\theta$-refinable and implies isocompactness. See also J.

VAUGHAN's chapter (in this volume) on countably compact spaces for a discussion of other conditions which force a countably compact space to be compact.

We now turn to conditions which force pseudocompact spaces to be compact. A space X is *pseudocompact* if every continuous real valued function on X has a bounded range. It is well known that a normal space X is pseudocompact if and only if X is countably compact. The space $\psi(N)$, discussed in Example 4.4, is a locally compact, pseudocompact Moore space which is not countably compact. This example also shows that the 'metacompact' condition in Theorem 9.6 cannot be reduced to 'submetacompact'.

Part (i) of the next lemma gives the characterization of pseudocompactness that we will use throughout the rest of this section.

9.5. LEMMA. (i) *A Tychonoff space X is pseudocompact if and only if every countable open filter base in X clusters.*

(ii) *A Tychonoff pseudocompact space is a Baire space.*

(iii) *A space X is a Baire space if and only if every point-finite open cover of X is locally finite at a dense* (*open*) *set of points.*

PROOF. Part (i) is a well known result (see ENGELKING [1977], Theorem 3.10.23(iii)). Part (iii) is a restatement of the earlier Theorem 8.2. Part (ii) can be found in MCCOY [1973]. For a proof, suppose X is a Tychonoff pseudocompact space, and $\{D_n\}_1^\infty$ is a decreasing sequence of dense open subsets of X. To show $\bigcap_{n=1}^\infty D_n$ is dense, let $U \subset X$ be a nonempty open set. Inductively pick nonempty open V_n, $n \in \mathbb{N}$, such that $V_1 = U$ and $\bar{V}_{n+1} \subset U \cap D_n \cap V_n$ for every $n \in \mathbb{N}$. By part (i), $\bigcap_{n=1}^\infty \bar{V}_n \neq \emptyset$ and this implies $U \cap (\bigcap_{n=1}^\infty D_n) \neq \emptyset$.

The next result is due independently to SCOTT [1979] and WATSON [1981].

9.6. THEOREM. *A Tychonoff, pseudocompact, metacompact space X is compact.*

PROOF. Suppose $\mathcal{U}$ is an open cover of X. Using regularity and metacompactness, there is a point-finite open cover $\mathcal{W}$ such that $\{\bar{W}: W \in \mathcal{W}\}$ refines $\mathcal{U}$. It suffices to show there is a finite subcollection $\mathcal{H} \subset \mathcal{W}$ such that $X = \overline{\bigcup \mathcal{H}}$. By 9.5(ii) and 9.5(iii) the set A, of all points at which $\mathcal{W}$ is locally finite, is dense (and open) in X. Let V_1 be a nonempty open subset of A such that V_1 meets only finitely many elements of $\mathcal{W}$. Continuing by induction, choose an infinite sequence $\{V_n\}_1^\infty$, if possible, of nonempty open subsets of A such that each V_k meets only finitely many elements of $\mathcal{W}$, and

$$V_{n+1} \subset A - \bigcup_{i=1}^{n} \mathrm{St}(V_i, \mathcal{W}) .$$

Let $U_n = \bigcup_{k=n}^\infty V_k$; then the filter base $\{U_n: n \in \mathbb{N}\}$ must cluster, so there exists some $z \in \bigcap_{n=1}^{\infty} \bar{U}_n$. Pick some $W \in \mathcal{W}$ such that $z \in W$. Now $W \cap U_n \neq \emptyset$ for every $n \in \mathbb{N}$ implies there exist distinct k, $m \in \mathbb{N}$, say $k < m$, such that $W \cap V_k \neq \emptyset$ and $W \cap V_m \neq \emptyset$. This says that $\mathrm{St}(V_k, \mathcal{W}) \cap V_m \supset W \cap V_m \neq \emptyset$, a contradiction to the original choice of V_m. Hence, in the process for choosing the V_i there must be some n where $A \subset \mathrm{cl}(\bigcup_{i=1}^{n} \mathrm{St}(V_i, \mathcal{W}))$. The collection $\mathcal{H} = \{H \in \mathcal{W}: H \cap V_i \neq \emptyset, \text{ some } i \leq n\}$ is the desired finite subcollection of $\mathcal{W}$.

The next result is due to BURKE and DAVIS [1982]. The proof can be extended to cover the σ-para-Lindelöf case.

9.7. THEOREM. *A Tychonoff, pseudocompact, para-Lindelöf space X is compact.*

PROOF. It is enough to show X is Lindelöf. Let $\mathcal{U}$ be an open cover of X and suppose $\mathcal{W}_1$ is a locally countable open refinement of $\mathcal{U}$. If $\mathcal{V}$ is an open cover such that each $V \in \mathcal{V}$ intersects at most countably many elements of $\mathcal{W}_1$, let $\mathcal{W}_2$ be a locally countable open refinement of $\mathcal{V}$. Then, $W \in \mathcal{W}_2$ implies $\{H \in \mathcal{W}_1: H \cap W \neq \emptyset\}$ is countable. It suffices to show $\mathcal{W}_1$ has a countable subcover. Arguing as in 9.6, it can be shown that there cannot exist an infinite sequence $\{V_n\}_1^{\infty}$ of nonempty open sets such that $V_{n+1} \subset X - \bigcup_{i=1}^{n} \mathrm{St}(V_i, \mathcal{W}_2)$. Hence, there must exist a finite sequence $V_1, V_2, \ldots, V_n$ of nonempty open sets such that each V_i intersects only countably many elements of $\mathcal{W}_2$ and $X = \mathrm{cl}(\bigcup_{i=1}^{n} \mathrm{St}(V_i, \mathcal{W}_2))$. Let $\mathcal{H} = \{H \in \mathcal{W}_2: H \cap V_i \neq \emptyset, \text{ some } i \leq n\}$. Since $\overline{\bigcup \mathcal{H}} = X$ we must have $W \cap (\bigcup \mathcal{H}) \neq \emptyset$ for every nonempty $W \in \mathcal{W}_1$. Since $\mathcal{H}$ is a countable subcollection of $\mathcal{W}_2$ the set $\bigcup \mathcal{H}$ meets at most countably many elements of $\mathcal{W}_1$. Hence $\mathcal{W}_1$ is countable and the theorem is proved.

We conclude by mentioning the existence of an example described in SCOTT [1979] which answers negatively the natural question of whether 9.6 or 9.7 could be extended to meta-Lindelöf spaces.

9.8. EXAMPLE. Assuming CH, there is an example of a Tychonoff, pseudocompact, meta-Lindelöf space which is not compact.

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CHAPTER 10

Generalized Metric Spaces

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
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Introduction

What is a 'generalized metric space'? There is no precise definition. Any class of spaces defined by a property possessed by all metric spaces could technically be called a class of generalized metric spaces. But the term is meant for classes which are 'close' to metrizable spaces in some sense: they usually possess some of the useful properties of metric spaces, and some of the theory or techniques of metric spaces carries over to these wider classes. Sometimes they can be used to characterize the images or pre-images of metric spaces under certain kinds of mappings. They often appear in theorems which characterize metrizability in terms of weaker topological properties. To be most useful, they should be 'stable' under certain topological operations, e.g., finite or countable products, closed subspaces, and perfect (closed, with compact point-inverses) mappings.

There always seem to be many exceptions to any list of criteria one can devise to determine just what classes of spaces deserve to be called generalized metric spaces. So instead of trying to be more precise, let us consider the class of 'p-spaces', which is studied in Section 3, as an illustrative example. This class generalizes both metric spaces and compact spaces, and various theorems which hold for both of these classes can often be generalized and hence unified by showing that they hold for p-spaces. The paracompact p-spaces are precisely the perfect pre-images of metric spaces. A countable product of p-spaces is a p-space, and if each factor is also paracompact, so is the product. (Recall that, in general, the product of two paracompact spaces need not be paracompact.) Also, this class has played an important role in the dimension theory of general spaces. So the class of p-spaces deserves to be called a class of generalized metric spaces.

It would take a large book to discuss in depth all of the classes of generalized metric spaces that have been studied by topologists. Here we outline, with proofs, the basic properties of the most important of these classes, and the relationships among them. We hope the reader can come away with a reasonable working knowledge of each class of spaces discussed in this chapter. For the benefit of the reader who is only interested in knowing the basic properties of one or two of the classes discussed here, we have provided a brief index where, for each class, we give the numbers of the definitions and theorems in which this class appears.

As a starting point for further study, we recommend the general survey articles of BURKE and LUTZER [1976] and AULL [1978]. More specialized surveys of particular classes of generalized metric spaces are mentioned in the appropriate sections. Also, BURKE [1980] contains an excellent survey of the behavior of various classes of generalized metric spaces under closed or perfect mappings.

We have tried to make this chapter self-contained except for basic topological results which appear in most graduate general topology texts (especially ENGELKING [1977]), and some results on covering properties which appear in the Chapter by Burke. It would be helpful for the reader to know a little bit about Moore spaces, studied in the Chapters by Fleissner and Tall. We give a brief review of the most pertinent of these topics in the next section. The reader who is already familiar with this material may skip to Section 2.

All spaces in this chapter are assumed to be regular and T_1, unless stated otherwise. By 'map' we mean a continuous surjection. By 'metric space' we mean 'metrizable space' (a standard abuse of terminology). Letters i, m, and n denote elements of ω, the set of natural numbers. The symbol (a_n) denotes a sequence indexed by ω.

We have provided references for many, but by no means every, result in this chapter. The reader should *not* assume that unreferenced results are due to the author.

1. Review of basic metrization theory, Moore spaces, and weak covering properties

The theory of generalized metric spaces is closely related to what is known as 'metrization theory'. Around 1950, the 'general metrization problem', which asked for necessary and sufficient conditions for a space to be metrizable, was solved independently by BING [1951], NAGATA [1950], and SMIRNOV [1951]:

1.1. THEOREM. *A regular space X is metrizable if and only if X has a σ-locally-finite (Bing, σ-discrete) basis.*

Surprisingly, this was far from the end of the matter. Many other interesting and useful metrization theorems have since appeared. All of the classes of generalized metric spaces discussed in this chapter appear as 'factors' in such theorems. One such theorem is the following metrization theorem for Moore spaces, due to BING [1951].

1.2. THEOREM. *A space X is metrizable if and only if it is a collectionwise-normal Moore space.*

We defer the proof of this theorem until the end of this section. Since Moore spaces are discussed in other chapters, we do not include a section on them in this chapter. However, Moore space theory has greatly influenced the theory of generalized metric spaces, so we will often refer to Moore spaces, and relate them to other classes of generalized metric spaces. Let us recall the definitions. First, if $A \subset X$ and $\mathcal{U}$ is a collection of subsets of X, then $\mathrm{st}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U}: U \cap A \neq \emptyset\}$. For $x \in X$, we write $\mathrm{st}(x, \mathcal{U})$ instead of $\mathrm{st}(\{x\}, \mathcal{U})$.

1.3. DEFINITION. A sequence $(\mathcal{G}_n)$ of open covers of a space X is a *development* for X if for each $x \in X$, the set $\{\mathrm{st}(x, \mathcal{G}_n): n \in \omega\}$ is a base at x. A *developable space* is a space which has a development. A *Moore space* is a regular developable space.

Several other classes of generalized metric spaces also can be characterized in terms of a sequence $(\mathscr{G}_n)$ of open covers satisfying certain conditions. Thus it will be useful to recall now some basic characterizations of metrizability involving sequences of open covers. First we have the Moore metrization theorem (MOORE [1953]).

1.4. THEOREM. *A T_0-space X is metrizable if and only if there exists a sequence $(\mathscr{G}_n)$ of open covers of X such that for each open set U and $x \in U$, there exists an open set V and $n \in \omega$ such that $x \in V$ and* $\operatorname{st}(V, \mathscr{G}_n) \subset U$.

PROOF. The necessity is easy: just let $\mathscr{G}_n$ be an open cover by sets of diameter less than $1/2^n$. We prove the sufficiency. Suppose $(\mathscr{G}_n)$ satisfies the conditions of the theorem. We may assume that $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$ for each n, for if this were not the case, we could replace $\mathscr{G}_n$ by $\mathscr{G}'_n = \{\bigcap_{i \leq n} G_i : G_i \in \mathscr{G}_i\}$. Now observe that, since $(\mathscr{G}_n)$ must be a development, this sequence satisfies the following condition: for each $x \in X$, the set $\{\operatorname{st}^2(x, \mathscr{G}_n) : n \in \omega\}$ is a base at x, where $\operatorname{st}^2(x, \mathscr{G}_n) = \operatorname{st}(\operatorname{st}(x, \mathscr{G}_n), \mathscr{G}_n)$. It is easy to see that $\overline{\operatorname{st}(x, \mathscr{G}_n)} \subset \operatorname{st}^2(x, \mathscr{G}_n)$; it follows that X is regular and T_1, hence a Moore space.

We complete the proof by showing that X is collectionwise-normal. Suppose $\mathscr{H}$ is a discrete collection of closed subsets of X. For each $H \in \mathscr{H}$ and $x \in H$, let $n_x \in \omega$ be such that $\operatorname{st}^2(x, \mathscr{G}_{n_x})$ does not meet any element of $\mathscr{H}$ besides H. Then if $x \in H$ and $x' \in H' \neq H$, it is easy to check that $\operatorname{st}(x, \mathscr{G}_{n_x}) \cap \operatorname{st}(x', \mathscr{G}_{n_{x'}}) = \emptyset$. Thus if we let $U_H = \bigcup_{x \in H} \operatorname{st}(x, \mathscr{G}_{n_x})$, then $H \subset U_H$ and $\{U_H : H \in \mathscr{H}\}$ is disjoint. Thus X is collectionwise-normal, hence metrizable. □

As an easy corollary, we have the Alexandrov–Urysohn metrization theorem.

1.5. THEOREM (ALEXANDROV, URYSOHN [1923]). *A T_0-space X is metrizable if and only if X has a development $(\mathscr{G}_n)$ such that whenever $G, G' \in \mathscr{G}_{n+1}$ and $G \cap G' \neq \emptyset$, then $G \cup G'$ is contained in some member of $\mathscr{G}_n$.*

PROOF. If X is metrizable, let $\mathscr{G}_n = \{B(x, 1/2^n) : x \in X\}$, where $B(x, \varepsilon)$ is the open ball of radius ε centered at x, with respect to some metric on X.

If $(\mathscr{G}_n)$ satisfies the conditions of the theorem, observe that $\operatorname{st}(\operatorname{st}(x, \mathscr{G}_{n+1}), \mathscr{G}_{n+1}) \subset \operatorname{st}(x, \mathscr{G}_n)$, and apply Theorem 1.4. □

Recall that a cover $\mathscr{V}$ of a space X is a *star-refinement* of a cover $\mathscr{U}$ if $\{\operatorname{st}(x, \mathscr{V}) : x \in X\}$ is a refinement of $\mathscr{U}$. Any open cover of a paracompact Hausdorff space has an open star-refinement. Thus we have:

1.6. COROLLARY. *A T_0-space X is metrizable if and only if X has a development $(\mathscr{G}_n)$ such that for each n, $\mathscr{G}_{n+1}$ is a star-refinement of $\mathscr{G}_n$.*

We conclude this section with a brief review of the covering properties known as 'subparacompactness' and 'submetacompactness' (or 'θ-refinability'), which will be useful in the sequel.

1.7. DEFINITION. A space X is *subparacompact* if every open cover of X has a σ-discrete closed refinement. Equivalently, for each open cover $\mathcal{U}$ of X, there exists a sequence $(\mathcal{U}_n)$ of open covers such that, for each $x \in X$, there exists $n \in \omega$ such that $\operatorname{st}(x, \mathcal{U}_n)$ is contained in some member of $\mathcal{U}$.

1.8. DEFINITION. A space X is *submetacompact* (or *θ-refinable*) if for each open cover $\mathcal{U}$ of X, there exists a sequence $(\mathcal{V}_n)$ of open refinements of $\mathcal{U}$ such that for each $x \in X$, there exists $n \in \omega$ such that x is in only finitely many elements of $\mathcal{V}_n$. If we do not require that each $\mathcal{V}_n$ covers X, then X is *weakly submetacompact* (or *weakly θ-refinable*). Such a sequence $(\mathcal{V}_n)$ of refinements is called a (*weak*) *θ-refinement* of $\mathcal{U}$.

All paracompact spaces are subparacompact. All subparacompact and metacompact (= every open cover has a point-finite open refinement) spaces are submetacompact. Collectionwise-normal submetacompact spaces are paracompact. See the Chapter by Burke for proofs and further results.

That Moore spaces are subparacompact follows trivially from the second characterization given in Definition 1.7. In fact, the hard part of Bing's metrization theorem for Moore spaces is essentially showing that Moore spaces satisfy the *first* characterization. Given that these two characterizations are equivalent (which is proven in the Chapter by Burke), we can easily prove Theorem 1.2.

PROOF OF THEOREM **1.2.** Let $(\mathcal{G}_n)$ be a development for X. For each n, let $\mathcal{F}_n = \bigcup_{m\in\omega}\mathcal{F}_{nm}$ be a refinement of $\mathcal{G}_n$ such that each $\mathcal{F}_{nm}$ is closed discrete. Let $\mathcal{B}_{nm} = \{B_F : F \in \mathcal{F}_{nm}\}$ be a discrete collection of open sets such that, for each $F \in \mathcal{F}_{nm}$, $F \subset B_F \subset G$ for some $G \in \mathcal{G}_n$. Then $\bigcup\{\mathcal{B}_{nm} : n, m \in \omega\}$ is a σ-discrete base for X, so X is metrizable by Theorem 1.1. $\square$

2. G_δ-diagonals and submetrizable spaces

The first class of generalized metric spaces we shall consider is the class of spaces having a G_δ-diagonal. The G_δ-diagonal property is a simple property which appears as a factor in many theorems characterizing metrizable or developable spaces. Also, it has a simple characterization in terms of a sequence of open covers (Theorem 2.2), which will facilitate our later study of spaces having a similar but more complicated characterization.

2.1. DEFINITION. The *diagonal* of a set X is the subset $\Delta = \{(x, x): x \in X\}$ of X^2. A space X is said to have a (*regular*) G_δ-*diagonal* if Δ is a (regular) G_δ-set in X^2. (Recall that a set H is a regular G_δ-set in the space Y if $H = \bigcap_n \bar{U}_n$, where each U_n is an open set containing H.)

It is easy to check that a space X is Hausdorff if and only if the diagonal Δ is a closed subset of X^2. Observe that whenever X is metrizable, or more generally, whenever X^2 is perfect (= closed sets are G_δ-sets), then X has a G_δ-diagonal.

The following characterization of spaces having a G_δ-diagonal is very useful for relating them to other classes of generalized metric spaces. It shows that the G_δ-diagonal property is equivalent to a weak form of developability.

2.2. THEOREM. *A space X has a G_δ-diagonal if and only if there exists a sequence $(\mathscr{G}_n)$ of open covers of X such that for each x, $y \in X$ with $x \neq y$, there exists $n \in \omega$ with $y \notin \mathrm{st}(x, \mathscr{G}_n)$ (equivalently, for each $x \in X$, $\{x\} = \bigcap_n \mathrm{st}(x, \mathscr{G}_n)$).*

PROOF. Suppose X has a G_δ-diagonal. Let $\Delta = \bigcap_n U_n$, with U_n open in X^2. For each $x \in X$ and $n \in \omega$, let $g(n, x)$ be an open neighborhood of x such that $g(n, x) \times g(n, x) \subset U_n$. Let $\mathscr{G}_n = \{g(n, x): x \in X\}$. Suppose $\{x, y\} \subset \bigcap_n \mathrm{st}(x, \mathscr{G}_n)$, with $x \neq y$. For each n, choose $z_n \in X$ such that $\{x, y\} \subset g(n, z_n)$. Then $(x, y) \in g(n, z_n) \times g(n, z_n) \subset U_n$, so $(x, y) \in \bigcap_n U_n$, a contradiction.

If $(\mathscr{G}_n)$ satisfies the conditions of the theorem, let $U_n = \bigcup\{G \times G: G \in \mathscr{G}_n\}$. Then $\Delta \subset \bigcap_n U_n$. If $(x, y) \in \bigcap_n U_n$, then for each n, there exists $G_n \in \mathscr{G}_n$ with $(x, y) \in G_n \times G_n$. Thus $y \in \bigcap_n \mathrm{st}(x, \mathscr{G}_n) = \{x\}$, and hence $\Delta = \bigcap_n U_n$. □

A sequence $(\mathscr{G}_n)$ which satisfies the conditions of Theorem 2.2 is called a G_δ-*diagonal sequence* for X.

With this characterization, it is not surprising that the G_δ-diagonal property appears as a factor in many metrization theorems. As a nice application of Theorem 2.2, we prove:

2.3. THEOREM (LUTZER [1969]). *Every linearly ordered space with a G_δ-diagonal is metrizable.*

PROOF. Let $(\mathscr{G}_n)$ be a G_δ-diagonal sequence for the linearly ordered space X. Clearly we may assume that $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$, and that each element of $\mathscr{G}_n$ is convex. We show that $(\mathscr{G}_n)$ is a development for X. Let $x \in X$, and $a < x < b$. There exist integers $n(a)$ and $n(b)$ such that $a \notin \mathrm{st}(\mathscr{G}_{n(a)})$ and $b \notin \mathrm{st}(x, \mathscr{G}_{n(b)})$. Let $n = n(a) + n(b)$. Clearly $\mathrm{st}(x, \mathscr{G}_n) \subset (a, b)$. This shows X is developable, hence a Moore space. Every linearly ordered space is collectionwise-normal (for a proof, see Theorem 5.21), so X is metrizable. □

Examples of non-developable spaces with a G_δ-diagonal are the Sorgenfrey line (the real line with half-open intervals of the form $[a, b)$ as a basis for the topology) and the Michael line (the real line with the irrationals isolated and the rationals having their usual neighborhoods). These examples have something in common—they have a coarser metric topology, in this case the Euclidean topology of the reals.

2.4. DEFINITION. A space (X, τ) is *submetrizable* if there exists a topology τ' on X such that $\tau' \subset \tau$ and (X, τ') is metrizable.

Note that every submetrizable space X has a regular G_δ-diagonal. This is because the square of X with the coarser metric topology is metric, hence the diagonal is a regular G_δ-set (in both topologies).

Just how much stronger is "submetrizable' compared to 'G_δ-diagonal'? In a submetrizable space, one can use the paracompactness of the metric topology to obtain a G_δ-diagonal sequence $(\mathcal{G}_n)$ such that $\mathcal{G}_{n+1}$ is a certain type of refinement of $\mathcal{G}_n$. In particular, we can make $\mathcal{G}_{n+1}$ star-refine $\mathcal{G}_n$ (i.e., $\{\mathrm{st}(x, \mathcal{G}_{n+1}): x \in X\}$ refines $\mathcal{G}_n$). A somewhat weaker condition is that $\mathcal{G}_{n+1}$ is a *regular refinement* of $\mathcal{G}_n$, i.e., that whenever $G, G' \in \mathcal{G}_{n+1}$ and $G \cap G' \neq \emptyset$, then $G \cup G'$ is contained in some element of $\mathcal{G}_n$. This is the same condition as in the Alexandrov–Urysohn metrization theorem (Theorem 1.5). We have the following characterizations of submetrizability:

2.5. THEOREM. *The following are equivalent for a space X:*

(a) *X is submetrizable.*

(b) *X has a G_δ-diagonal sequence $(\mathcal{G}_n)$ such that $\mathcal{G}_{n+1}$ star-refines $\mathcal{G}_n$ for each n.*

(c) *X has a G_δ-diagonal sequence $(\mathcal{G}_n)$ such that $\mathcal{G}_{n+1}$ is a regular refinement of $\mathcal{G}_n$ for each n.*

Before proving Theorem 2.5, we establish a couple of lemmas which will also be used in later sections. The following lemma is useful for building distance functions which satisfy the triangle inequality from certain ones that do not.

2.6. LEMMA (FRINK [1937]). *Suppose d: $X \times X \to \mathbb{R}^+$ (the non-negative reals) satisfies the following condition:*

(*) *For any $\varepsilon > 0$, if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) < 2\varepsilon$.*

Then there is a function ρ: $X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(i) *$\rho(x, z) \leq \rho(x, y) + \rho(y, z)$*

(ii) *$d(x, y)/4 \leq \rho(x, y) \leq d(x, y)$.*

Further, ρ is symmetric (i.e., $\rho(x, y) = \rho(y, x)$) if d is.

PROOF. Define ρ as follows:

$$\rho(a, b) = \inf\left\{\sum_{i=0}^{n} d(x_i, x_{i+1}): n \in \omega, x_i \in X, x_0 = a, \text{ and } x_n = b\right\}.$$

Then ρ clearly satisfies (i), and $\rho(x, y) \leq d(x, y)$. To complete the proof, it will suffice to prove that for each $a, b, x_1, \ldots, x_n \in X$, the following condition holds:

$$(**) \qquad d(a, b) \leq 2d(a, x_1) + 4\sum_{i=1}^{n-1} d(x_i, x_{i+1}) + 2d(x_n, b)\,.$$

With $n = 1$, (**) follows from (*). Suppose (**) holds for all positive integers less than n, and let $a, b, x_1, \ldots, x_n \in X$. By (*), we have $d(a, b) \leq 2d(a, x_i)$ or $d(a, b) \leq 2d(x_i, b)$ for each $i = 1, \ldots, n$. Let k be the least integer such that $d(a, b) \leq 2d(a, x_k)$. If $k = 1$, (**) holds. If $k > 1$, then $d(a, b) \leq 2d(x_{k-1}, b)$, so $d(a, b) \leq d(a, x_k) + d(x_{k-1}, b)$. Now it easily follows from the induction hypothesis, applied to $d(a, x_k)$ and $d(x_{k-1}, b)$, that (**) holds. □

A function $\rho: X \times X \to \mathbb{R}^+$ is a *pseudo-metric* on X if for all $x, y, z \in X$, we have $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$, and $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$. The *topology on X generated by* ρ is the topology τ which has a as a base sets of the form $B(x, \varepsilon) = \{y \in X: \rho(x, y) < \varepsilon\}$. A pseudo-metric is just like a metric, except that $\rho(x, y)$ can be 0 for $x \neq y$. If we define x and y to be equivalent if and only if $\rho(x, y) = 0$, then the quotient space of (X, τ) obtained by identifying the equivalence classes to points is easily seen to be a metrizable space. Observe that the following lemma gives an alternative proof of the Alexandrov–Urysohn metrization theorem (Theorem 1.5).

2.7. LEMMA. *Let $(\mathscr{G}_n)$ be a sequence of open covers of X such that $\mathscr{G}_{n+1}$ is a regular refinement of $\mathscr{G}_n$ for each n. Then there is a pseudo-metric ρ on X such that*

(i) $\rho(x, y) = 0$ *if* $y \in \bigcap_n \operatorname{st}(x, \mathscr{G}_n)$.

(ii) *U is open in the topology generated by ρ iff for each $x \in U$, there exists $n \in \omega$ such that* $\operatorname{st}(x, \mathscr{G}_n) \subset U$.

PROOF. Define $d: X \times X \to \mathbb{R}^+$ by $d(x, y) = 1/2^n$, where n is the least natural number such that $y \notin \operatorname{st}(x, \mathscr{G}_n)$, and $d(x, y) = 0$ if no such n exists. It is easy to see that d is symmetric.

Suppose $d(x, y) < 1/2^{n+1}$ and $d(y, z) < 1/2^{n+1}$. Then $y \in \operatorname{st}(x, \mathscr{G}_{n+1})$ and $z \in \operatorname{st}(y, \mathscr{G}_{n+1})$, so $z \in \operatorname{st}(\operatorname{st}(x, \mathscr{G}_{n+1}), \mathscr{G}_{n+1}) \subset \operatorname{st}(x, \mathscr{G}_n)$. Thus $d(x, z) < 1/2^n$. Hence d satisfies the condition (*) of Lemma 2.6. Let ρ be the function guaranteed by Lemma 2.6, and note that in this case ρ is a pseudo-metric on X. That ρ satisfies property (i) follows easily from 2.6(ii) and the definition of d.

Note that for each $x \in X$, $\operatorname{st}(x, \mathscr{G}_{n+2}) = B_d(x, 1/2^{n+2}) \subset B_\rho(x, 1/2^{n+2}) \subset B_d(x, 1/2^n) = \operatorname{st}(x, \mathscr{G}_n)$. From this it is clear that property (ii) holds. □

Proof of Theorem 2.5. That (b)⇒(c) is trivial, and that (a)⇒(b) has already been noted (see discussion preceding the statement of the theorem).

We prove (c)⇒(a). Let $(\mathscr{G}_n)$ satisfy (c), and let ρ be a pseudo-metric on X

satisfying the properties of Lemma 2.7. Since $\{x\} = \bigcap_n \mathrm{st}(x, \mathcal{G}_n)$, ρ is a metric on X by 2.7(i). By 2.7(ii), the topology generated by ρ is contained in the original topology. □

2.8. REMARK. There is another, much shorter proof of (c)$\Rightarrow$(a) for those who know about uniform spaces. For each n, let $V(n) = \bigcup\{G \times G: G \in \mathcal{G}_n\}$. Then $V(n+1) \circ V(n+1) \subset V(n)$ and $\bigcap_n V(n) = \Delta$, so $\mathcal{V} = \{V(2n): n \in \omega\}$ is a base for a separated uniformity on X whose induced topology τ' is contained in the given topology τ on X. Since $\mathcal{V}$ is countable, τ' is metrizable, and hence X is submetrizable. □

As a corollary to Theorem 2.5, we have the following result:

2.9. COROLLARY (BORGES [1966], OKUYAMA [1964]). *Every paracompact space with a G_δ-diagonal is submetrizable.*

PROOF. Let $(\mathcal{G}_n)$ be a G_δ-diagonal sequence for the paracompact space X. Inductively construct a sequence $(\mathcal{G}'_n)$ of open covers of X such that for each $n \geq 1$, $\mathcal{G}'_n$ star-refines both $\mathcal{G}_n$ and $\mathcal{G}'_{n-1}$. Then $(\mathcal{G}'_n)$ satisfies the conditions of Theorem 2.5(b). □

There are useful notions somewhat stronger than 'G_δ-diagonal' and weaker than 'submetrizable'. We have already defined 'regular G_δ-diagonal' (Definition 2.1). What does the addition of 'regular' mean in terms of a G_δ-diagonal sequence? Following the proof of Theorem 2.2, one can easily see that we can get a G_δ-diagonal sequence $(\mathcal{G}_n)$ satisfying $\{x\} = \bigcap_n \overline{\mathrm{st}(x, \mathcal{G}_n)}$ for each $x \in X$. This is not quite a characterization of the regular G_δ-diagonal property. But this is a simple property and is usually just what one needs, so we make the following definition:

2.10. DEFINITION. A space X has a *G_δ^*-diagonal* if there exists a sequence $(\mathcal{G}_n)$ of open covers such that for each $x \in X$, $\{x\} = \bigcap_n \overline{\mathrm{st}(x, \mathcal{G}_n)}$.

Clearly every Moore space has a G_δ^*-diagonal. As we will see later, not every Moore space has a regular G_δ-diagonal or is submetrizable. In submetacompact spaces, having a G_δ-diagonal is equivalent to having a G_δ^*-diagonal.

2.11. THEOREM. *A submetacompact space with a G_δ-diagonal has a G_δ^*-diagonal.*

This theorem follows immediately from the following lemma, which will be useful later on.

2.12. LEMMA. *Let Y be a submetacompact subspace of X. For each n, let $\mathcal{U}_n$ be a collection of open subsets of X covering Y. Then there exists a sequence $(\mathcal{V}_n)$ of open* *collections covering Y such that for each $y \in Y$,*

$$\bigcap_n \overline{\mathrm{st}(y, \mathcal{V}_n)} = \bigcap_n \mathrm{st}(y, \mathcal{V}_n) \subset \bigcap_n \mathrm{st}(y, \mathcal{U}_n).$$

PROOF. If $\mathcal{U}$ is a collection of subsets of X, let $\mathcal{U} \mid Y = \{U \cap Y: U \in \mathcal{U}\}$. We can use the submetacompactness of Y and the regularity of X to inductively define, for each $m \in \omega$, a sequence $(\mathcal{V}_{mn})$ of open collections covering Y such that

(i) $(\mathcal{V}_{mn} \mid Y)_{n\in\omega}$ is a θ-refinement of each $\mathcal{V}_{ij} \mid Y$, $i<m$, $j<m$, and of each $\mathcal{U}_k \mid Y$, $k \leq m$.

(ii) If $V \in \mathcal{V}_{mn}$, and $i, j < m$, then there exists $W \in \mathcal{V}_{ij}$ such that $\bar{V} \subset W$; also, if $k \leq m$, there exists $U \in \mathcal{U}_k$ with $\bar{V} \subset U$.

Suppose $x \in \bigcap_{i,j} \overline{\mathrm{st}(y, \mathcal{V}_{ij})}$, where $y \in Y$. Fix i and j, and let $m > \max\{i, j\}$. There exists $n \in \omega$ such that y is in only finitely many members of $\mathcal{V}_{mn}$. Then $\overline{\mathrm{st}(y, \mathcal{V}_{mn})} = \bigcup\{\bar{V}: y \in V \in \mathcal{V}_{mn}\}$, which by (ii) is contained in $\bigcup\{W: y \in W \in \mathcal{V}_{ij}\} = \mathrm{st}(y, \mathcal{V}_{ij})$. Thus for each $i, j \in \omega$, $x \in \mathrm{st}(y, \mathcal{V}_{ij})$, so $\bigcap_{i,j} \overline{\mathrm{st}(y, \mathcal{V}_{ij})} = \bigcap_{i,j} \mathrm{st}(y, \mathcal{V}_{ij})$.

By (ii), it is clear that $\bigcap_{i,j} \mathrm{st}(y, \mathcal{V}_{ij}) \subset \bigcap_n \mathrm{st}(y, \mathcal{U}_n)$. Thus if $(\mathcal{V}_n)$ enumerates $\{V_{ij}: i, j \in \omega\}$, then $(\mathcal{V}_n)$ satisfies the desired properties. □

We have as a corollary the following well known result:

2.13. THEOREM (ŠNEĬDER [1945]). *A compact space with a G_δ-diagonal is metrizable.*

PROOF. Let X satisfy the hypotheses. By either Corollary 2.9 or Theorem 2.11, X has a G_δ^*-diagonal sequence $(\mathcal{G}_n)$. (This is also easy to verify directly.) We can assume that each $\mathcal{G}_n$ is finite, and that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. Suppose $x \in U$ with U open in X. Then $\{U\} \cup \{X \backslash \overline{\mathrm{st}(x, \mathcal{G}_n)}: n \in \omega\}$ is an open cover of X, so there exists a finite subcover. From this it follows that there exists $n \in \omega$ with $\mathrm{st}(x, \mathcal{G}_n) \subset U$. Thus $\bigcup_n \mathcal{G}_n$ is a countable base for X. (A slick alternative proof goes as follows. By Corollary 2.9, X has a weaker metric topology. But a topology which is strictly weaker than a compact Hausdorff topology cannot be Hausdorff, so this metric topology is the original topology.) □

The previous theorem also holds for countably compact spaces. If we had a G_δ^*-diagonal, it would be easy. By essentially the same proof as above, we could obtain a development from the G_δ^*-diagonal sequence, and it would follow that the space is compact. However, a countably compact space is not submetacompact unless it is compact (see the Chapter by Burke, Theorem 9.2), so we cannot get the G_δ^*-diagonal as before. We can get compactness, but it takes some work.

2.14. THEOREM (CHABER [1976]). *A countably compact space with a G_δ-diagonal is compact, hence metrizable.*

PROOF. Let X be countably compact, and let $(\mathcal{G}_n)$ be a G_δ-diagonal sequence for X such that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. Suppose X is not compact. Then there exists an open cover $\mathcal{U}$ of X such that $\mathcal{U}$ contains no countable subcover. Pick $x_0 \in X$. Observe that there exists $n(0) \in \omega$ such that $\mathcal{U}$ contains no countable cover of $X \setminus \mathrm{st}(x_0, \mathcal{G}_{n(0)})$. Otherwise, for each n, we could find a countable $\mathcal{U}_n \subset \mathcal{U}$ covering $X \setminus \mathrm{st}(x_0, \mathcal{G}_n)$, and then $\bigcup_n \mathcal{U}_n$ would be a countable cover of $X \setminus \{x\}$, hence $\mathcal{U}$ would contain a countable cover of X.

Now we inductively choose, for each $\alpha < \omega_1$, a point $x_\alpha \in X$ and $n(\alpha) \in \omega$ such that

(i) $x_\alpha \not\in \bigcup_{\beta<\alpha} \mathrm{st}(x_\beta, \mathcal{G}_{n(\beta)})$.

(ii) $\mathcal{U}$ does not contain a countable cover of $X \setminus \bigcup_{\beta \leq \alpha} \mathrm{st}(x_\beta, \mathcal{G}_{n(\beta)})$.

To see that this can be done at a limit stage α, observe that if $\mathcal{U}$ does not contain a countable cover of $X \setminus \bigcup_{\beta \leq \gamma} \mathrm{st}(x_\beta, \mathcal{G}_{n(\beta)})$ for each $\gamma < \alpha$, then $\mathcal{U}$ contains no countable cover of $X \setminus \bigcup_{\beta<\alpha} \mathrm{st}(x_\beta, \mathcal{G}_{n(\beta)})$. Otherwise, $\{\mathrm{st}(x_\beta, \mathcal{G}_{n(\beta)}): \beta < \alpha\} \cup \mathcal{U}$ would contain a finite cover of X. Then if γ is the largest ordinal β such that $\mathrm{st}(x_\beta, \mathcal{G}_{n(\beta)})$ appears in this finite cover, $\mathcal{U}$ would contain a finite cover of $X \setminus \bigcup_{\beta \leq \gamma} \mathrm{st}(x_\beta, \mathcal{G}_{n(\beta)})$, a contradiction.

Now for some $n \in \omega$ and uncountable $A \subset \omega_1$, we have $n(\beta) = n$ for each $\beta \in A$. Then $\{x_\beta : \beta \in A\}$ is an uncountable closed discrete subset of X, for each $G \in \mathcal{G}_n$ meets at most one element of this set. This contradicts the countable compactness of X. Hence X is compact. $\square$

In the next section, we generalize Theorem 2.14 considerably (Corollary 3.8). As another application of these concepts, we have:

2.15. THEOREM. *Let X be locally compact and locally connected. Then*

(a) *if X has a G_δ^*-diagonal, then X is a Moore space;*

(b) *if X has a regular G_δ-diagonal, then X is metrizable.*

PROOF. (a) Let $(\mathcal{G}_n)$ be a G_δ^*-diagonal sequence for X. We may assume as usual that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. By passing to components, we may also assume that each member of $\mathcal{G}_n$ is connected.

We show that $(\mathcal{G}_n)$ is a development for X. Suppose $x \in X$, U is a compact neighborhood of x, and $\mathrm{st}(x, \mathcal{G}_n) \not\subset U$ for each n. Then since $\overline{\mathrm{st}(x, \mathcal{G}_n)}$ is connected, $\overline{\mathrm{st}(x, \mathcal{G}_n)} \cap \partial U \neq \emptyset$ for each n. Since ∂U is compact, $\partial U \cap \bigcap_n \overline{\mathrm{st}(x, \mathcal{G}_n)} \neq \emptyset$, a contradiction.

(b) By (a), X is a Moore space. Let the diagonal $\Delta = \bigcap_n U_n = \bigcap_n \bar{U}_n$, where U_n is open and $U_{n+1} \subset U_n$ for each n. Let $(\mathcal{G}_n)$ be a development for X with $\mathcal{G}_{n+1}$ refining $\mathcal{G}_n$, and such that $G \in \mathcal{G}_n$ implies G is connected and $G \times G \subset U_n$.

We will show that $\{\mathrm{st}^2(x, \mathcal{G}_n): n \in \omega\}$ is a base at each $x \in X$, where $\mathrm{st}^2(x, \mathcal{G}_n) = \mathrm{st}(\mathrm{st}(x, \mathcal{G}_n), \mathcal{G}_n)$; that X is metrizable then follows from Theorem 1.4. Let $x \in V$, where V is open and $\bar{V}$ is compact. Suppose that for each n, $\mathrm{st}^2(x, \mathcal{G}_n) \not\subset V$. Since $\mathrm{st}^2(x, \mathcal{G}_n)$ is connected, $\overline{\mathrm{st}^2(x, \mathcal{G}_n)} \cap \partial V \neq \emptyset$. Let $y \in \bigcap_n (\overline{\mathrm{st}^2(x, \mathcal{G}_n)} \cap \partial V)$. There exists $n \in \omega$ such that $(x, y) \notin \bar{U}_n$, so there exists $m > n$ such that $[\mathrm{st}(x, \mathcal{G}_m) \times \mathrm{st}(y, \mathcal{G}_m)] \cap U_m = \emptyset$. Since $y \in \overline{\mathrm{st}^2(x, \mathcal{G}_m)}$, there exist $G_1, G_2, G_3 \in \mathcal{G}_m$ with $x \in G_1$, $G_1 \cap G_2 \neq \emptyset$, $G_2 \cap G_3 \neq \emptyset$, and $y \in G_3$. Let $z_1 \in G_1 \cap G_2$ and $z_2 \in G_2 \cap G_3$. Then $(z_1, z_2) \in (G_1 \times G_3) \cap (G_2 \times G_2) \subset [\mathrm{st}(x, \mathcal{G}_m) \times \mathrm{st}(y, \mathcal{G}_m)] \cap U_m$, a contradiction. □

2.16. REMARK. As the proof shows, Theorem 2.15 holds if 'locally compact' is replaced by 'locally peripherally compact' (i.e., X has a base consisting of open sets with compact boundaries).

Since there are non-metrizable Moore manifolds (see the Chapter by Nyikos), we see from the above result that not every Moore space has a regular G_δ-diagonal or is submetrizable. It is an open question whether every normal Moore space is submetrizable. REED and ZENOR [1976] show that normal Moore spaces of cardinality $\leqslant 2^\omega$ are submetrizable. They use this and Theorem 2.15(b) to prove that normal, locally compact, locally connected Moore spaces are metrizable.

We conclude this section with an example which shows that 'locally connected' cannot be omitted from Theorem 2.15.

2.17. EXAMPLE. A locally compact submetrizable space with a closed subset which is not a G_δ-set (hence the space is not a Moore space).

PROOF. Let B be a Bernstein subset of $\mathbb{R}$, i.e., every uncountable closed subset of $\mathbb{R}$ meets both B and $\mathbb{R} \setminus B$. Let $\{B_\alpha : \alpha < 2^\omega\}$ be an enumeration of all countable subsets of B having uncountable closure in $\mathbb{R}$. For each $\alpha < 2^\omega$, choose a point $x_\alpha \in \bar{B}_\alpha \setminus (B \cup \{x_\beta : \beta < \alpha\})$, and points $x_{\alpha,n} \in B_\alpha$ with $x_{\alpha,n} \to x_\alpha$. Let $X = B \cup \{x_\alpha : \alpha < 2^\omega\}$. Topologize X by declaring the points of B to be isolated, and $\{x_\alpha\} \cup \{x_{\alpha,n} : n \geqslant m\}$, $m \in \omega$, to be a base at x_α.

Clearly X is locally compact and submetrizable. Let $H = \{x_\alpha : \alpha < 2^\omega\}$. We show that H is not a G_δ-set. Suppose $H = \bigcap_n U_n$, where U_n is open in X. Then $B \setminus U_n$ is countable for each n; otherwise, $B_\alpha \subset B \setminus U_n$ for some $\alpha < 2^\omega$. Since B is uncountable, $B \cap (\bigcap_n U_n) \neq \emptyset$. □

3. Simultaneous generalizations of metric and compact spaces: M-spaces, p-spaces, $w\Delta$-spaces

The classes of spaces suggested by the title of this section have played a major role in the theory of generalized metric spaces. These classes arose in an effort to characterize the perfect pre-images of metric spaces, as well as to find a class of spaces more general than metric spaces such that the countable product of paracompact members of this class is again paracompact. (There are many examples of paracompact spaces X and Y such that $X \times Y$ is not paracompact—see the Chapter by Pryzmusiński.) These classes have also played an important role in dimension theory, and have appeared as a 'factor' in many metrization theorems.

We begin with a class of spaces whose definition looks very similar to that of developable spaces.

3.1. DEFINITION. A space X is a *$w\Delta$-space* if there exists a sequence $(\mathcal{G}_n)$ of open covers of X such that if $x_n \in \mathrm{st}(x, \mathcal{G}_n)$ for each $n \in \omega$, then the set $\{x_n: n \in \omega\}$ has a cluster point in X.

The cluster point referred to in this definition need not be x itself. In fact, if we modified the definition to require x to be the cluster point, then $\{\mathrm{st}(x, \mathcal{G}_n): n \in \omega\}$ would be a base at x (this is easy to check), and we would have re-defined developable spaces.

Besides the developable spaces, all countably compact spaces are $w\Delta$-spaces—simply let $\mathcal{G}_n = \{X\}$ for each n. Also, as we shall see later, every submetacompact locally compact space is a $w\Delta$-space—the submetacompactness being necessary to make $\mathrm{st}(x, \mathcal{G}_n)$ 'get inside' some compact set.

Not surprisingly, if to '$w\Delta$-space' we add a property that enables us to make $\mathrm{st}(x, \mathcal{G}_n)$ 'close down' on x, we get back to developability. To be precise, we need $\{x\} = \bigcap_n \overline{\mathrm{st}(x, \mathcal{G}_n)}$, i.e., the G_δ^*-diagonal property (Definition 2.10). First we prove a simple lemma.

3.2. LEMMA. *Suppose (U_n) is a decreasing sequence of open sets such that $\bigcap_n U_n = \bigcap_n \bar{U}_n$, and $x_n \in U_n$ implies $\{x_n: n \in \omega\}$ has a cluster point. Then (U_n) is a base for the set $\bigcap_n U_n$, i.e., every open set containing $\bigcap_n U_n$ contains some U_n.*

PROOF. Suppose (U_n) satisfies the hypotheses of the lemma but not the conclusion. Then we can find an open set V containing $\bigcap_n U_n$, and points $x_n \in U_n \setminus V$ for each n. Since (U_n) is decreasing, any cluster point of $\{x_n: n \in \omega\}$ must be in $\bigcap_n \bar{U}_n$. But $V \supset \bigcap_n \bar{U}_n$. Thus $\{x_n: n \in \omega\}$ has no cluster point, which is a contradiction. □

3.3. THEOREM (HODEL [1971]). *A space X is developable if and only if it is a $w\Delta$-space with a G_δ^*-diagonal.*

PROOF. The 'only if' part is clear. To see the 'if' part, suppose X is a $w\Delta$-space with a G_δ^*-diagonal. By passing to refinements, we can find a sequence $(\mathcal{G}_n)$ of open covers which is both a $w\Delta$-sequence and a G_δ^*-diagonal sequence, and such that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. By Lemma 3.2, for each $x \in X$, $\{\mathrm{st}(x, \mathcal{G}_n): n \in \omega\}$ is a base at x. Thus $(\mathcal{G}_n)$ is a development. □

Since a submetacompact space with a G_δ-diagonal has a G_δ^*-diagonal (Theorem 2.11), we have:

3.4. COROLLARY. *A space X is a Moore space (resp., a metrizable space) if and only if X is a submetacompact (resp., paracompact) $w\Delta$-space with a G_δ-diagonal.*

Now we define the stronger notion of an M-space, due to MORITA [1964]. Note that, by Theorem 2.5, one could say "M-space is to $w\Delta$-space as submetrizable is to G_δ-diagonal".

3.5. DEFINITION. A space X is an *M-space* if there exists a sequence $(\mathscr{G}_n)$ of open covers of X such that

(i) if $x_n \in \mathrm{st}(x, \mathscr{G}_n)$ for each n, then $\{x_n: n \in \omega\}$ has a cluster point;

(ii) for each n, $\mathscr{G}_{n+1}$ star refines $\mathscr{G}_n$.

We call such a sequence $(\mathscr{G}_n)$ an *M-sequence* for X.

This is an obvious strengthening of the $w\Delta$-space property—we have just added condition (ii). Countably compact spaces are M-spaces; just let $\mathscr{G}_n = \{X\}$ for each n. It is also easy to see that paracompact $w\Delta$-spaces are M-spaces, since, given $(\mathscr{G}_n)$ satisfying (i), we can use paracompactness to modify the sequence to satisfy (ii).

Recall that property (ii) was also a property of the sequence $(\mathscr{G}_n)$ of Theorem 2.5(b) which characterized submetrizability. We could just as well have replaced property (ii) by the weaker condition in 2.5(c), that $\mathscr{G}_{n+1}$ is a regular refinement of $\mathscr{G}_n$. In either case, if $(\mathscr{G}_n)$ is an M-sequence for X, then, by Lemma 2.7, there is a pseudo-metric ρ on X such that $\rho(x, y) = 0$ iff $y \in \bigcap_m \mathrm{st}(x, \mathscr{G}_n)$. Now ρ is not necessarily a metric, since the set

$$C_x = \bigcap_n \mathrm{st}(x, \mathscr{G}_n)$$

may consist of more than just x. But properties (i) and (ii) are exactly what is needed for the C_x's to be a partition of the space such that C_x is countably compact and $\{\mathrm{st}(x, \mathscr{G}_n): n \in \omega\}$ is a base for the set C_x. Then it is not surprising that when we identify the C_x's to points, we get a metrizable space. The quotient map turns out to be *quasi-perfect*, i.e., closed with countably compact point-inverses. Now it is just a matter of checking the details to prove the following characterization of M-spaces.

3.6. THEOREM (MORITA [1964]). *A space X is an M-space if and only if there exists a metric space Y and a quasi-perfect map from X onto Y.*

PROOF. ($\Rightarrow$) Let $(\mathscr{G}_n)$ satisfy the conditions of Definition 3.5. By Lemma 2.7, there is a pseudo-metric ρ on X such that

(i) $\rho(x, y) = 0$ iff $y \in \bigcap_n \mathrm{st}(x, \mathscr{G}_n)$;

(ii) U is open in the topology generated by ρ iff $x \in U$ implies $\mathrm{st}(x, \mathscr{G}_n) \subset U$ for some $n \in \omega$.

Define an equivalence relation on X as follows: $x \sim y$ iff $\rho(x, y) = 0$. Let $Y = X/\sim$, and define $d\colon Y \times Y \to \mathbb{R}^+$ by $d([x], [y]) = \rho(x, y)$. It is easy to check that d is a metric on Y.

We prove that the quotient map $f\colon X \to X/\sim$ is quasi-perfect. Since $f^{-1}(B_d([x], \varepsilon)) = B_\rho(x, \varepsilon)$, and $B_\rho(x, \varepsilon)$ is open in X by (ii) above, f is continuous. Now $f^{-1}([x]) = \{y\colon \rho(x, y) = 0\} = \bigcap_n \operatorname{st}(x, \mathcal{G}_n)$. Thus if $\{x_n\colon n \in \omega\} \subset f^{-1}([x])$, then $x_n \in \operatorname{st}(x, \mathcal{G}_n)$ for each n, so $\{x_n\colon n \in \omega\}$ has a cluster point in X. Since $f^{-1}([x])$ is closed, it follows that each $f^{-1}([x])$ is countably compact.

It remains to prove that f is closed. First observe that since $\mathcal{G}_{n+1}$ star-refines $\mathcal{G}_n$, we have $\operatorname{st}(\operatorname{st}(x, \mathcal{G}_{n+1}), \mathcal{G}_{n+1}) = \operatorname{st}^2(x, \mathcal{G}_{n+1}) \subset \operatorname{st}(x, \mathcal{G}_n)$; in particular, $\overline{\operatorname{st}(x, \mathcal{G}_{n+1})} \subset \operatorname{st}(x, \mathcal{G}_n)$. By Lemma 3.2, $\{\operatorname{st}(x, \mathcal{G}_n)\colon n \in \omega\}$ is a base for the set $\bigcap_n \operatorname{st}(x, \mathcal{G}_n) = [x]$. Now suppose $H \subset X$ is closed, and $x \notin f^{-1}(f(H))$. Then $[x] \cap H = \emptyset$, so $\operatorname{st}(x, \mathcal{G}_n) \cap H = \emptyset$ for some n. If $y \in \operatorname{st}(x, \mathcal{G}_{n+1})$, then $\operatorname{st}(y, \mathcal{G}_{n+1}) \subset \operatorname{st}^2(x, \mathcal{G}_{n+1}) \subset \operatorname{st}(x, \mathcal{G}_n)$, and so $[y] \cap \emptyset$. Hence $\operatorname{st}(x, \mathcal{G}_{n+1}) \cap f^{-1}(f(H)) = \emptyset$. Thus $f^{-1}(f(H))$ is closed, and since f is a quotient map, $f(H)$ is closed.

($\Leftarrow$) Let $f\colon X \to Y$ be quasi-perfect with Y metrizable. Let $(\mathcal{U}_n)$ be a development for Y such that $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$. Let $\mathcal{G}_n = \{f^{-1}(U)\colon U \in \mathcal{U}_n\}$. It is easy to check that $\mathcal{G}_{n+1}$ star-refines $\mathcal{G}_n$. Suppose $x_n \in \operatorname{st}(x, \mathcal{G}_n)$ for each n. We need to show that $\{x_n\colon n \in \omega\}$ has a cluster point. If infinitely many x_n's are in the countably compact set $f^{-1}(f(x))$, we are done. So we may assume that for each n, $x_n \notin f^{-1}(f(x))$. Now $f(x_n) \in \operatorname{st}(f(x), \mathcal{U}_n)$, so $f(x_n) \to f(x)$. Since f is closed, $\{x_n\colon n \in \omega\}$ must have a cluster point in $f^{-1}(f(x))$. $\square$

3.7. COROLLARY. *A space X is a paracompact M-space if and only if X is a perfect pre-image of a metric space.*

PROOF. This follows immediately from Theorem 3.6 and the following facts: countably compact paracompact spaces are compact, and the perfect pre-image of a paracompact space is paracompact (for proofs, see the Chapter by Burke, Theorems 5.9, and 9.2, or ENGELKING [1977], 5.1.20 and 5.1.35). $\square$

We can now see the structure of M-spaces more clearly. An M-space is like a metric space in which points have been 'fattened up' to countably compact sets (compact sets in the case of paracompact M-spaces). If these (countably) compact sets are themselves metrizable, we cannot necessarily conclude that the whole space is metrizable. But, generally speaking, a property which makes countably compact spaces metrizable will also make M-spaces metrizable. The following corollary, which generalizes Theorem 2.14, is an example. (See Corollary 7.10 for another example.)

3.8. COROLLARY. *A space X is metrizable if and only if it is an M-space with a G_δ-diagonal.*

PROOF. Suppose X is an M-space with a G_δ-diagonal. Let $f\colon X \to Y$ be quasi-perfect, with Y metrizable. Each $f^{-1}(y)$ is countable compact and has a G_δ-

diagonal, hence is compact by Theorem 2.14. Thus f is perfect, and so X is paracompact. Then X is metrizable by Corollary 3.4. □

As an application of Corollary 3.7, we can easily show that the class of paracompact M-spaces is countably productive.

3.9. COROLLARY. *If each X_n is a paracompact M-space, so is $\Pi_{n\in\omega}X_n$.*

PROOF. Let $f_n: X_n \to Y_n$, where Y_n is metrizable and f_n is perfect. Then $\Pi f_n: \Pi X_n \to \Pi Y_n$ is perfect (ENGELKING [1977], Theorem 3.7.7) so the result follows from Corollary 3.7. □

Unfortunately, the product of two M-spaces need not be an M-space.

3.10. EXAMPLE. There exist M-spaces X_1 and X_2 such that $X_1 \times X_2$ is not an M-space (or even a $w\Delta$-space).

PROOF. Let Z_1 and Z_2 be any two countably compact spaces such that $Z_1 \times Z_2$ is not countably compact. (For such an example, see the Chapter by Vaughan, Example 3.1, or ENGELKING [1977], Example 3.10.19.) For $i = 1, 2$, let X_i be the ordinal space $\omega_1 + 1$ with each non-limit ordinal replaced by a copy of Z_i. (That is, limit ordinals have their usual interval neighborhoods, except that each non-limit in the interval has been replaced by a copy of Z_i. Points in some copy of Z_i have their usual neighborhoods in Z_i.) Then X_i is countably compact. Suppose $(\mathscr{G}_n)$ is a sequence of open covers of $X_1 \times X_2$. Then $\bigcap_n \mathrm{st}((\omega_1, \omega_1), \mathscr{G}_n)$ contains a closed copy K of $Z_1 \times Z_2$. Since $Z_1 \times Z_2$ is not countably compact, we can choose $x_n \in K \subset \mathrm{st}((\omega_1, \omega_1), \mathscr{G}_n)$ such that $\{x_n : n \in \omega\}$ does not have a cluster point. Thus $(\mathscr{G}_n)$ cannot be an M-sequence, or even a $w\Delta$-sequence, for $X_1 \times X_2$. □

Let us recall Lasnev's 'decomposition theorem' (LASNEV [1965]) for closed maps with metrizable domain: If $f: X \to Y$ is closed and X is metrizable, then $Y = \bigcup_{n\in\omega} Y_n$, where Y_n is a closed discrete subset of Y for each $n \geq 1$, and $f^{-1}(y)$ is compact for each $y \in Y_0$. This useful theorem has been shown to hold for various classes of non-metrizable X. The following theorem shows that it holds for paracompact M-spaces, and even for M-spaces if we are willing to settle for 'countably compact' instead of 'compact'. The proof is deferred until Section 5, where a more general result is proven (Theorem 5.14).

3.11. THEOREM. *Let $f: X \to Y$ be closed, with X an M-space. Then $Y = \bigcup_{n\in\omega} Y_n$, where Y_n is a closed discrete subset of Y for each $n \geq 1$, and $f^{-1}(y)$ is countably compact for each $y \in Y_0$.*

Observe that (by Corollary 3.9 or otherwise) the product of a metric space with a compact space is a paracompact M-space, as well as every closed subset of such a product (both paracompactness and 'M-ness' are closed hereditary properties). It is interesting that *all* paracompact M-spaces can be obtained in this way.

3.12. THEOREM (NAGATA [1969]). *A space X is a paracompact M-space if and only if X is homeomorphic to a closed subset of the product of a metric space and a compact space.*

This theorem follows easily from the following lemma.

3.13. LEMMA. *If $f: X \to Y$ is perfect, and αX is a compactification of X, then the map $h: X \to Y \times \alpha X$ defined by $h(x) = (f(x), x)$ is an embedding of X onto a closed subset of $Y \times \alpha X$.*

PROOF. Clearly h is one-to-one, and h is continuous since the composition of h with each projection map is continuous.

Let K be closed in X. We will show that $h(K)$ is closed in $Y \times \alpha X$; then we will have that $h(X)$ is closed in $Y \times \alpha X$ and $h: X \to h(X)$ is a closed continuous one-to-one map, hence a homeomorphism. Suppose $(y_0, p_0) \in \overline{h(K)} \backslash h(K)$. Clearly, if $p_0 \in X$, then $p_0 \in K$. Thus $p_0 \notin f^{-1}(y_0)$, for otherwise $(y_0, p_0) = h(p_0) \in h(K)$. Since $f^{-1}(y_0)$ is a compact subset of X in αX, there exists V open in αX such that $p_0 \in V$ and $\bar{V}^{\alpha X} \cap f^{-1}(y_0) = \emptyset$. Since f is closed, the set $U = Y \backslash f(\bar{V}^{\alpha X} \cap X)$ is an open subset of Y containing y_0. Then $(y_0, p_0) \in U \times V$, and $U \times V$ does not meet $h(K)$. □

Proof of Theorem 3.12. We have already noted the 'if' part. To see the 'only if' part, let $f: X \to Y$ be perfect, with Y metrizable. By Lemma 3.13, X is homeomorphic to a closed subset of $Y \times \beta X$. □

In view of Theorem 3.12, the following example may be a little surprising.

3.14. EXAMPLE. There is an M-space which is not homeomorphic to a closed subspace of the product of a countably compact space and a metric space.

PROOF. This example is due to KATO [1977]; BURKE and VAN DOUWEN [1977] have also constructed such an example. Let $X = \beta\mathbb{R} \backslash \beta\mathbb{Z}$, where $\beta\mathbb{Z}$ is identified with the closure in $\beta\mathbb{R}$ of the integers $\mathbb{Z}$. First we prove that X is an M-space. Observe that each interval $(i, i+1)$, $i \in \mathbb{Z}$, is a clopen subset of X. Let $(\mathcal{G}_{i,n})_{n\in\omega}$ be a development for $(i, i+1)$ such that $\mathcal{G}_{i,n+1}$ star-refines $\mathcal{G}_{i,n}$. Let

$$\mathcal{G}_n = \{X \backslash (-n, n)\} \cup \Big(\bigcup_{i=-n}^{n-1} \mathcal{G}_{i,n}\Big).$$

Suppose $x_n \in \mathrm{st}(x, \mathcal{G}_n)$ for each n. If $x \in \mathbb{R}$, clearly $x_n \to x$. If $x \notin \mathbb{R}$, then $x_n \in$

$X\backslash(-n, n)$. Then for each n, there exists a closed subset F_n of $\mathbb{R}$ such that $x_n \in \bar{F}_n^{\beta\mathbb{R}} \subset X\backslash(-n, n)$. (Let $F_n = U \cap \mathbb{R}$, where U is a closed neighborhood of x_n in $\beta\mathbb{R}$ missing $\beta\mathbb{Z}$.) Then $\bigcup_n F_n$ is a closed subset of $\mathbb{R}$ missing $\mathbb{Z}$, so

$$\overline{\{x_n : n \in \omega\}}^{\beta\mathbb{R}} \subset \overline{\bigcup_n F_n}^{\beta\mathbb{R}} \subset \beta\mathbb{R}\backslash\beta\mathbb{Z} .$$

(Recall that disjoint closed subsets of a normal space X have disjoint closures in βX (ENGELKING [1977], Corollary 3.6.2.)) Thus $\{x_n : n \in \omega\}$ has a cluster point in X, and so X is an M-space.

Now suppose X is a closed subset of $M \times C$, where M is metrizable and C is countably compact. Then X is a subspace of the countably compact space $\beta M \times C$. Further, if $A \subset X$ is closed and countably compact, then the projection $\pi_1(A)$ on M is countably compact, hence compact, so A is a closed subset of $\pi_1(A) \times C$, which is a closed subset of $\beta M \times C$. Thus X is a dense subspace of a countably compact space $\hat{X}$, namely the closure of X in $\beta M \times C$, and every closed, countably compact subset of X is closed in $\hat{X}$. Also, since X is locally compact, X is an open subset of $\hat{X}$. We will derive our contradiction from this situation.

For each $n \in \mathbb{Z}$, choose $r(n, i) \in \mathbb{R}$ with $|r(n, i) - n| < 1/2^i$, and choose $x_n \in \hat{X}$ such that $x_n \in \overline{\{r(n, i) : i \in \omega\}}$. Since for each n, $r(n, i) \to n \notin X$, we have $x_n \in \hat{X}\backslash X$. Let x be a cluster point in $\hat{X}$ of $\{x_n : n \in \omega\}$; then also $x \in \hat{X}\backslash X$. We aim for a contradiction by showing that x is in the closure of the countably compact subset $X\backslash\mathbb{R}$ of X. Suppose $x \in U$ with U open in $\hat{X}$. Let $A = \{n : x_n \in U\}$, and for each $n \in A$, choose i_n such that $r(n, i_n) \in U$. Then, as in the proof above that X is an M-space, the set $\{r(n, i_n) : n \in A\}$ has a cluster point $p \in X$. Then $p \in (X\backslash\mathbb{R}) \cap \bar{U}$. Hence the closure of every open set containing x meets $X\backslash\mathbb{R}$. By regularity, $x \in \overline{X\backslash\mathbb{R}}$. □

For more information on M-spaces, see the survey article of MORITA [1971].

Now we consider the concept of a 'p-space', due to ARHANGEL'SKII [1963]. While p-spaces turn out to be the same as M-spaces in the presence of paracompactness, they have some advantages in non-paracompact spaces. For example, every locally compact space is a p-space, and the class of p-spaces is countably productive.

The class of p-spaces is also defined in terms of a sequence of open covers, but in some compactification of the space rather than the space itself. For simplicity, we will only define completely regular p-spaces, so we can use the familiar Stone–Čech compactification. Using the Wallman compactification, one can extend this definition to T_1-spaces. Alternately, one can use the characterization Theorems 3.17 and 3.21 as definitions.

3.15. DEFINITION. A completely regular space X is a *p-space* if there exists a sequence $(\mathcal{U}_n)$ of families of open subsets of βX such that

(i) each $\mathcal{U}_n$ covers X;
(ii) for each $x \in X$, $\bigcap_n \mathrm{st}(x, \mathcal{U}_n) \subset X$.
If we also have
(iii) $\bigcap_n \mathrm{st}(x, \mathcal{U}_n) = \bigcap_n \overline{\mathrm{st}(x, \mathcal{U}_n)}$, then X is said to be a *strict p-space*.

As motivation for the definiton, recall that a locally compact space X is an open subset of βX; hence if we let $\mathcal{U}_m = \{X\}$ for each n, $(\mathcal{U}_n)$ trivially satisfies properties (i) and (ii). Thus any locally compact space is a p-space. More generally, any space X which is a G_δ-set in βX is a p-space—let $\mathcal{U}_n = \{U_n\}$, where $X = \bigcap_n U_n$, U_n open in βX. (Such spaces are called *Čech complete.* A metric space is Čech complete if and only if it is completely metrizable (ENGLEKING [1977], Theorem 4.3.26).

We could have used any compactification of X in the definition; for suppose $(\mathcal{U}_n)$ satisfies the conditions of Definition 3.15 in a compactification αX of X. Then there exists a map $f\colon \beta X \to \alpha X$ with $f|X = \mathrm{id}_X$. It is easy to check that $f^{-1}(\mathcal{U}_n) = \{f^{-1}(U)\colon U \in \mathcal{U}_n\}$ satisfies the same conditions in βX.

With the above in mind, it is easy to prove the following result:

3.16. THEOREM. *If each X_n is a p-space, so is $\Pi_{n\in\omega} X_n$.*

PROOF. Suppose $(\mathcal{U}_{nm})_{m\in\omega}$ satisfies conditions (i) and (ii) of Definition 3.15 for X_n. Let $\mathcal{U}_n = \{\Pi_{i\leq n} U_i \times \Pi_{i>n} X_i\colon U_i \in \mathcal{U}_{i,n}, i \leq n\}$. Then $(\mathcal{U}_n)$ is a sequence of open covers of $\Pi_{n\in\omega} X_n$ in $\Pi_{n\in\omega} \beta X_n$ satisfying (i) and (ii). □

Let us see what condition (iii) of Definition 3.15 adds. In this case the set $C_x = \bigcap_n \mathrm{st}(x, \mathcal{U}_n) = \bigcap_n \overline{\mathrm{st}(x, \mathcal{U}_n)}$ is a closed subset of βX contained in X, hence it is a compact subset of X. We can, as usual, assume that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$, for if $(\mathcal{U}_n)$ satisfies conditions (i)–(iii) of Definition 3.15, so does $(\mathcal{U}'_n)$, where $\mathcal{U}'_n = \{\bigcap_{i\leq n} U_i\colon U_i \in \mathcal{U}_i\}$. Now by Lemma 3.1, the set $\{\mathrm{st}(x, \mathcal{U}_n)\colon n \in \omega\}$ is a base in βX for the compact set C_x. This idea leads to the following internal characterization of strict p-spaces.

3.17. THEOREM (BURKE, STOLTENBERG [1969]). *A space X is a strict p-space if and only if there exists a sequence $(\mathcal{G}_n)$ of open covers of X such that*
(i) $C_x = \bigcap_n \mathrm{st}(x, \mathcal{G}_n)$ *is compact;*
(ii) $\{\mathrm{st}(x, \mathcal{G}_n)\colon n \in \omega\}$ *is a base for C_x, i.e., every open set containing C_x contains some* $\mathrm{st}(x, \mathcal{G}_n)$.

REMARK. So that we do not have to include an extra 'completely regular' hypothesis in this and other theorems, we will assume that the Wallman compactification is used in place of βX in this and succeeding proofs, if X is not completely regular.

PROOF. If X is a strict p-space, let $\mathcal{G}_n$ be the trace of $\mathcal{U}_n$ on X, where $(\mathcal{U}_n)$ satisfies Definition 3.15, and $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$. By the above discussion, $(\mathcal{G}_n)$ satisfies properties (i) and (ii).

Now suppose $(\mathscr{G}_n)$ satisfies (i) and (ii). For each $G \in \mathscr{G}_n$, let U_G be open in βX with $U_G \cap X = G$, and let $\mathscr{U}_n = \{U_G : G \in \mathscr{G}_n\}$. Clearly

$$\bigcap_n \operatorname{st}(x, \mathscr{G}_n) \subset \bigcap_n \operatorname{st}(x, \mathscr{U}_n) \subset \bigcap_n \overline{\operatorname{st}(x, \mathscr{U}_n)} \quad \text{for each } x \in X.$$

We complete the proof by showing that $\bigcap_n \overline{\operatorname{st}(x, \mathscr{U}_n)} \subset C_x$. Suppose $y \in \bigcap_n \overline{\operatorname{st}(x, \mathscr{U}_n)}$ but $y \notin C_x$. Then there is a set W open in βX containing y such that $\overline{W} \cap C_x = \emptyset$. Since $\{\operatorname{st}(x, \mathscr{G}_n) : n \in \omega\}$ is a base for C_x, there exists $m \in \omega$ such that $\operatorname{st}(x, \mathscr{G}_m) \cap \overline{W} = \emptyset$; hence $\overline{\operatorname{st}(x, \mathscr{G}_m)}^{\beta X} \cap W = \emptyset$. Since each $G \in \mathscr{G}_m$ is dense in U_G, $\overline{\operatorname{st}(x, \mathscr{U}_m)}^{\beta X} = \overline{\operatorname{st}(x, \mathscr{G}_m)}^{\beta X}$. Hence $y \notin \overline{\operatorname{st}(x, \mathscr{U}_m)}$, a contradiction. □

3.18. Corollary. *Every Moore space is a strict p-space, and every strict p-space is a $w\Delta$-space.*

Proof. A development clearly satisfies properties (i) and (ii) of Theorem 3.17. Also, if $(\mathscr{G}_n)$ satisfies these properties, and $x_n \in \operatorname{st}(x, \mathscr{G}_n)$, then $\{x_n : n \in \omega\}$ must have a cluster point in C_x. Thus such a sequence $(\mathscr{G}_n)$ is also a '*$w\Delta$-sequence*'. □

In the class of submetacompact spaces, the notions of p-space, strict p-space, and $w\Delta$-space are equivalent.

3.19. Theorem. *For a submetacompact space X, the following are equivalent*:

(a) *X is a strict p-space*;

(b) *X is a p-space*;

(c) *X is a $w\Delta$-space.*

Proof. We need to prove (b)⇒(a) and (c)⇒(a). Let X be a submetacompact p-space (resp., $w\Delta$-space), and let $(\mathscr{U}_n)$ satisfy Definition 3.15 (resp., Definition 3.1). Applying Lemma 2.12, we obtain a sequence $(\mathscr{V}_n)$ of open covers of X such that, for each $x \in X$, $\bigcap_n \operatorname{st}(x, \mathscr{V}_n) = \bigcap_n \overline{\operatorname{st}(x, \mathscr{V}_n)} \subset \bigcap_n \operatorname{st}(x, \mathscr{U}_n)$, and $\mathscr{V}_{n+1}$ refines $\mathscr{V}_n$. The result is now clear if X is a p-space. If X is a $w\Delta$-space, then Lemma 3.2 says that $\{\operatorname{st}(x, \mathscr{V}_n) : n \in \omega\}$ is a base for the countably compact set $\bigcap_n \operatorname{st}(x, \mathscr{V}_n)$. But this set is submetacompact (being a closed subset of a submetacompact space), hence is compact (see Theorem 9.2, in the Chapter by Burke). Thus $(\mathscr{V}_n)$ satisfies the conditions of Theorem 3.17. □

Since paracompact $w\Delta$-spaces are M-spaces, we have:

3.20. Corollary. *The following are equivalent.*

(a) *X is a paracompact M-space*;

(b) *X is a paracompact $w\Delta$-space*;

(c) *X is a paracompact p-space.*

Thus one can also characterize the perfect pre-images of metric spaces as the paracompact p-spaces.

From Theorem 3.19 we also see that every submetacompact locally compact space is a $w\Delta$-space. Example 2.17 is a locally compact space, hence a p-space, which is not a $w\Delta$-space. It is not a $w\Delta$-space because otherwise, since it is submetrizable, it would be developable by Theorem 3.3. But it is not developable since it contains a closed subset which is not a G_δ-set. This example also shows that Theorem 3.3 is not valid with '$w\Delta$-space' replaced by 'p-space'.

An interesting unsolved problem is whether every strict p-space is submetacompact, i.e., are strict p-spaces just the submetacompact p-spaces? It is known that this is true for locally hereditarily separable or locally compact spaces (CHABER, JUNILLA [1979]), or more generally, locally ω_1-compact spaces (WAGNER [198 ·]). On the other hand, there is an example of a metacompact locally compact space, hence a strict p-space, which is not subparacompact (see Example 4.2, in the Chapter by Burke).

The following is an internal characterization of p-spaces similar to Theorem 3.17.

3.21. THEOREM (BURKE [1970]). *A space X is a p-space if and only if there exists a sequence $(\mathcal{G}_n)$ of open covers of X satisfying the following condition: If for each n, $x \in G_n \in \mathcal{G}_n$, then*

(i) $\bigcap_n \bar{G}_n$ is compact; and

(ii) $\{\bigcap_{i\leqslant n} \bar{G}_i : n \in \omega\}$ is an outer network for the set $\bigcap_n \bar{G}_n$, i.e., every open set containing $\bigcap_n \bar{G}_n$ contains some $\bigcap_{i\leqslant n} \bar{G}_i$.

PROOF. The proof is in the same spirit as the proof of Theorem 3.17, so we just give the basic outline and leave the details to the reader.

Given $(\mathcal{G}_n)$ satisfying (i) and (ii), expand to $\mathcal{U}_n = \{U_G : G \in \mathcal{G}_n\}$, where U_G is open in βX and $U_G \cap X = G$. Then $(\mathcal{U}_n)$ shows that X is a p-space.

Given $(\mathcal{U}_n)$ satisfying Definition 3.15, let $(\mathcal{G}_n)$ be a sequence of open covers of X such that $\{\bar{G}^{\beta X} : G \in \mathcal{G}_n\}$ refines $\mathcal{U}_n$. Then $(\mathcal{G}_n)$ satisfies (i) and (ii). $\square$

We have not yet given an example of a $w\Delta$-space or an M-space which is not a p-space. We can use this characterization to help find one. Recall that X is a *k-space* if $A \subset X$ is closed whenever $A \cap K$ is closed in K for each compact set $K \subset X$. (The k-spaces are precisely the quotient images of locally compact spaces.)

3.22. THEOREM. *Every p-space is a k-space.*

PROOF. Suppose X is a p-space, and $A \subset X$ is not closed. We need to find a compact subset of X whose intersection with A is not closed.

Let $(\mathcal{G}_n)$ satisfy the conditions of Theorem 3.21. Let $p \in \bar{A} \backslash A$, and choose open sets U_n containing p and $G_n \in \mathcal{G}_n$ with $p \in U_{n+1} \subset \bar{U}_{n+1} \subset U_n \subset \bigcap_{i\leqslant n} G_i$ for each n. It follows from conditions (i) and (ii) of Theorem 3.21 that $K = \bigcap_n U_n$ is compact, and $x_n \in U_n$ implies that $\{x_n : n \in \omega\}$ has a cluster point in X (since every open set containing the compact set $\bigcap_n \bar{G}_n$ contains all but finitely many x_n's, some point of $\bigcap_n \bar{G}_n$ must be a cluster point). Thus by Lemma 3.2, $\{U_n : n \in \omega\}$ is a base for the set K. If $A \cap K$ is not closed, we are done. Suppose $A \cap K$ is closed. Let (V_n) be a sequence of open neighborhoods of p such that $\bar{V}_{n+1} \subset V_n$ and $V_0 \cap A \cap K = \emptyset$. As above, $\{U_n \cap V_n : n \in \omega\}$ is a base for the compact set $K' = \bigcap_n (U_n \cap V_n)$. Note that $K' \cap A = \emptyset$. Choose $x_n \in U_n \cap V_n \cap A$. Then $K'' = K' \cup \{x_n : n \in \omega\}$ is compact, and $K'' \cap A$ is not closed. □

3.23. Example. There is a countably compact subset X of $\beta\mathbb{N}$ which is not a k-space. Hence X is an M-space but not a p-space.

Proof. This construction is due to Frolík [1960]. Since every infinite closed subset of $\beta\mathbb{N}$ has cardinality 2^{2^ω}, and there are 2^{2^ω} closed subsets (Engelking [1977], p. 228), we can, as in the construction of a Bernstein subset of $\mathbb{R}$, inductively define a subset $X \subset \beta\mathbb{N}$ such that both X and $\beta\mathbb{N} \setminus X$ meet every infinite closed set. Then every compact subset of X is finite. But a k-space with this property is easily seen to be discrete, and X is clearly not discrete. Thus X is not a p-space. X is countably compact, because if $A \subset X$ were an infinite closed discrete set in X, then X could not intersect the infinite closed set $\bar{A}^{\beta\mathbb{N}} \setminus A$ of $\beta\mathbb{N}$, a contradiction. □

Finally, we should mention the behavior of M-spaces and p-spaces under certain kinds of mappings. Normal M-spaces (Ishii [1967]) and submetacompact p-spaces (Worrell [198·]) are preserved by perfect maps. On the other hand, if X is the union of countably many convergent sequences, and Y is the space obtained from X by identifying the limit points to a single point, then Y is the closed image of a locally compact separable metric space which is not an M-space or a p-space (since Y is a non-metrizable paracompact space with a G_δ-diagonal). Indeed, it follows from Corollary 4.7 and Corollary 4.12 of the next section that a non-metrizable closed image of a metric space is never an M-space or a p-space.

4. Networks: σ-spaces and Σ-spaces

The concept of a network has been one of the most useful tools in theory of generalized metric spaces. A network is like a base, but its elements are not required to be open.

4.1. Definition. A *network* for a space X is a collection $\mathcal{F}$ of subsets of X such that whenever $x \in U$ with U open, there exists $F \in \mathcal{F}$ with $x \in F \subset U$. The *net weight*, $nw(x)$, of a space X is the least cardinality of a network for X.

Since we are only considering regular spaces, the set of closures of the elements of a network is also a network.

Trivially, $nw(X) \leqslant w(X)$ for any space X. In some cases, a network can be 'fattened up' to a base for the space. For example, it is shown in Chapter 1 that the equality $nw(X) = w(X)$ holds for compact X; in particular, a compact space with a countable network has a countable base. We now show that this equality holds for the much more general class of p-spaces (hence all Moore spaces, locally compact spaces, etc.).

4.2. THEOREM. *If X is a p-space, then $nw(X) = w(X)$.*

PROOF. Let X be a p-space with a closed network $\mathscr{F}$ such that $|\mathscr{F}| \leqslant \kappa$. Let $(\mathscr{G}_n)$ satisfy the conditions of Theorem 3.21. For each $F \in \mathscr{F}$, and $n \in \omega$, pick $G(F, n) \in \mathscr{G}_n$ containing F, if such a set exists. For each $F, F' \in \mathscr{F}$ pick an open set $U(F, F')$ with $F \subset U(F, F') \subset \overline{U(F, F')} \subset X \backslash F'$, if such a set exists.

We complete the proof by showing that the set $\mathscr{B}$ of all finite intersections of the $G(F, n)$'s and $U(F, F')$'s is a base for X. To this end, let $x \in X$ and let V be an open set containing x. For each n, there exists $F_n \in \mathscr{F}$ such that $x \in F_n$ and $G(F_n, n) \in \mathscr{G}_n$ is defined. Let $K = \bigcap_n \overline{G(F_n, n)}$. Then K is compact. For each $y \in K \backslash V$, there exist disjoint elements F_x and F_y of $\mathscr{F}$ containing x and y, respectively, such that $U(F_x, F_y)$ is defined. So for each $y \in K \backslash V$, there exists $B \in \mathscr{B}$ (namely $B = U(F_x, F_y)$) containing x with $y \notin \bar{B}$. Since $K \backslash V$ is compact, there exists a finite $\mathscr{B}' \subset \mathscr{B}$ such that $\{X \backslash \bar{B} \colon B \in \mathscr{B}'\}$ covers $K \backslash V$. By property (ii) of Theorem 3.21, there exists $n \in \omega$ such that $\bigcap_{1 \leqslant n} G(F_i, i) \subset V \cup (\bigcup\{X \backslash \bar{B} \colon B \in \mathscr{B}'\})$. Then

$$x \in \Big(\bigcap_{i \leqslant n} G(F_i, i)\Big) \cap \Big(\bigcap \mathscr{B}'\Big) \subset V. \quad \square$$

We now define a class of generalized metric spaces by replacing 'base' with 'network' in the Bing–Nagata–Smirnov metrization theorem.

4.3. DEFINITION. A space X is a *σ-space* if X has a σ-discrete (equivalently, σ-locally finite) network.

We will prove later (Theorem 4.11) the equivalence suggested in this definition.

What properties do σ-spaces have? We can immediately see that they are subparacompact; hence a collectionwise-normal σ-space must be paracompact. Also, closed sets are G_δ-sets: if $H \subset X$ is closed, let $U_n = X \backslash \bigcup\{F \in \mathscr{F}_n \colon F \cap H = \emptyset\}$, where $\bigcup_n \mathscr{F}_n$ is a σ-discrete closed network for X. Then $H = \bigcap_n U_n$.

The class of σ-spaces is very well behaved in terms of various topological operations. It is easy to check that this class is hereditary and countably productive. Also, a space which is a countable union of closed σ-spaces is a σ-space.

The paracompact σ-spaces are particularly well-behaved. For example, the following is a well-known result for metrizable spaces which carries over to this class.

4.4. THEOREM. *Let X be a paracompact σ-space. Then the following are equivalent.*

(i) *X is (hereditarily) Lindelöf.*
(ii) *X is (hereditarily) separable.*
(iii) *X is (hereditarily)* ccc.
(iv) *X has a countable network.*

PROOF. Any closed discrete collection of subsets of a Lindelöf, separable, or ccc collectionwise-normal space is easily seen to be countable. Thus (i), (ii), and (iii) each imply (iv). On the other hand, any space satisfying (iv) is easily seen to be Lindelöf, separable, and ccc. Since property (iv) is hereditary, the result follows. □

It is also true that a countable product of paracompact σ-spaces is again a paracompact σ-space. We prove this later, as a corollary to the same result for the more general class of Σ-spaces (Definition 4.13).

4.5. THEOREM. *Every Moore space is a* σ-space.

PROOF. This is easy, given that Moore spaces are subparacompact. If $(\mathscr{G}_n)$ is a development for X, let $\mathscr{F}_n$ be a σ-discrete refinement of $\mathscr{G}_n$. Then $\mathscr{F} = \bigcup_n \mathscr{F}_n$ is a σ-discrete network for X. □

Many spaces besides Moore spaces are σ-spaces. Every countable space, more generally every space which is a union of countably many closed metrizable subspaces, is a σ-space. We will see later (Theorem 5.9) that all stratifiable (Definition 5.6) spaces are σ-spaces.

4.6. THEOREM. *Every σ-space has a G^*_δ-diagonal.*

PROOF. Let X be a σ-space. Then X^2 is a σ-space, so the diagonal is a G_δ-set. Since X is subparacompact, X has a G^*_δ-diagonal by Theorem 2.11. □

Putting the last two theorems together with Corollaries 3.4 and 3.8, and Theorem 3.19, we have:

4.7. COROLLARY. (i) *A space X is a Moore space if and only if X is a σ-space and a p-space (or a $w\Delta$-space).*

(ii) *X is metrizable if and only if X is a σ-space and an M-space.*

In particular, a countably compact σ-space is metrizable.

A σ-discrete closed network $\mathcal{F}$ can be used in a natural way to generate a finer metric topology on the space: just let the set $\mathcal{B}$ of all finite intersections of elements of $\mathcal{F}$ be a base for a new topology. It is not difficult to see that $\mathcal{B}$ is also σ-discrete. Since the elements of the network are closed in the original topology, they are clopen in the new topology. So the new topology is regular. Then by the Bing–Nagata–Smirnov metrization theorem, the new topology is metrizable. We have proven:

4.8. THEOREM. *Every σ-space (X, τ) with $nw(X) \leqslant \kappa$ has a finer metric topology (i.e., a topology $\tau' \supset \tau$ with (X, τ') metrizable) of weight $\leqslant \kappa$.*

One nice thing about networks is that, since the sets are not required to be open, the continuous image of a network on the domain is easily seen to be a network on the range. With this in mind, we get the following nice characterization of spaces having a countable network.

4.9. THEOREM. *A space X has a countable network if and only if it is the continuous image of a separable metric space.*

PROOF. The sufficiency is clear from the above discussion. On the other hand, if (X, τ) has a countable network, then by Theorem 4.8, X has a finer separable metric topology τ'. The identity map from (X, τ') to (X, τ) is continuous. $\square$

Since an arbitrary continuous map does not preserve 'discreteness' or 'local finiteness' we cannot expect that the continuous image of a non-separable metric space is a σ-space. The situation is different for closed maps. Closed maps do not preserve discrete or locally finite collections either, but they do preserve 'closure-preserving' collections.

4.10. DEFINITION. A collection $\mathcal{H}$ of subsets of a space X is *closure-preserving* if $\overline{\bigcup \mathcal{H}'} = \bigcup\{\bar{H} : H \in \mathcal{H}'\}$ for each $\mathcal{H}' \subset \mathcal{H}$.

For collections of closed sets, this just means that the union of any subcollection is closed. Observe that locally finite collections are closure-preserving. In many definitions and results in general topology that involve discrete or locally finite collections, one gets an equivalent statement if 'discrete' or 'locally finite' is replaced by 'closure-preserving'. This is the case for σ-spaces, as our next theorem shows. Parts (iv) and (v) of this theorem are other useful characterizations which will help us relate σ-spaces to other classes of generalized metric spaces. The proof of (v)$\Rightarrow$(i) is essentially HEATH's proof [1969] that stratifiable spaces (Definition 5.6) are σ-spaces.

4.11. THEOREM. *The following are equivalent for a space* (X, τ):

(i) *X has a σ-discrete network.*

(ii) *X has a σ-locally finite network.*

(iii) *X has a σ-closure-preserving network.*

(iv) *There exists a function* $g : \omega \times X \to \tau$ *such that*

(a) $x \in g(n, x)$ *for each* $n \in \omega$ *and* $x \in X$;

(b) $y \in g(n, x) \Rightarrow g(n, y) \subset g(n, x)$;

(c) $x \in g(n, x_n) \Rightarrow x_n \to x$.

(v) *There exists a function* $g: \omega \times X \to \tau$ *such that*

(a) $x \in g(n, x)$ *for each* $n \in \omega$ *and* $x \in X$;

(b) $x \in g(n, x_n)$ *and* $x_n \in g(n, y_n)$ *implies* $y_n \to x$.

PROOF. That (i)$\Rightarrow$(ii)$\Rightarrow$(iii) is trivial. Given a function g satisfying (iv), observe that $x \in g(n, x_n)$ and $x_n \in g(n, y_n)$ implies $x \in g(n, x_n) \subset g(n, y_n)$; hence $y_n \to x$. Thus (iv)$\Rightarrow$(v).

To get (iii)$\Rightarrow$(iv), let $\mathcal{F} = \bigcup_n \mathcal{F}_n$ be a network for X such that each $\mathcal{F}_n$ is a closure-preserving collection of closed sets (the set of closures of the members of a closure-preserving collection is closure-preserving). Let $g(n, x) = X \setminus \bigcup\{F \in \mathcal{F}_i : i \leq n, x \notin F\}$. Parts (a) and (b) of (iv) are clearly satisfied. Suppose $x \in g(n, x_n)$ and $x_n \not\to x$. Then some subsequence $S = \{x_{n_k} : k \in \omega\}$ does not contain x in its closure. So there exists $F \in \mathcal{F}$ with $x \in F$ and $F \cap S = \emptyset$. Say $F \in \mathcal{F}_n$. Then if $n_k \geq n$, $g(n, x_{n_k}) \cap F = \emptyset$ by the definition of g. However, $x \in g(n, x_{n_k}) \cap F$, so we have a contradiction.

It remains to prove (v)$\Rightarrow$(i). Let g satisfy (v). We may assume $g(n+1, x) \subset g(n, x)$ for each $n \in \omega$ and $x \in X$. Let $<$ be a well-ordering of X. For $x \in X$ and $i, n \in \omega$, let

$$H(x, i, n) = X \setminus [(\bigcup\{g(i, y) : y < x\}) \cup (\bigcup\{g(n, y) : y \notin g(i, x)\})] .$$

Clearly $H(x, i, n) \subset g(i, x)$. Let $\mathcal{H}(i, n) = \{H(x, i, n) : x \in X\}$. We show $\mathcal{H}(i, n)$ is discrete. Let $z \in X$, and let y be the least element of X such that $z \in g(i, y)$. Clearly $g(i, y) \cap H(x, i, n) = \emptyset$ if $y < x$, and $g(n, z) \cap H(x, i, n) = \emptyset$ if $x < y$. Thus $g(i, y) \cap g(n, z)$ is an open set containing z which meets at most one element (namely $H(y, i, n)$) of $\mathcal{H}(i, n)$. Thus $\mathcal{H}(i, n)$ is discrete.

Now let $F(x, i, n, m) = \{y \in H(x, i, n) : x \in g(m, y)\}$, and let $\mathcal{F}(i, n, m) = \{F(x, i, n, m) : x \in X\}$. Then $\mathcal{F}(i, n, m)$ is discrete because $\mathcal{H}(i, n)$ is discrete.

It remains to prove that $\mathcal{F} = \{\mathcal{F}(i, n, m) : i, n, m \in \omega\}$ is a network for X. Suppose $p \in U$ with U open. For each $i \in \omega$, let x_i be the least element of X such that $p \in g(i, x_i)$. There exists $n(i) \in \omega$ with $p \notin \bigcup\{g(n(i), y) : y \notin g(i, x_i)\}$. (Otherwise, there exists $y_n \notin g(i, x_i)$ with $p \in g(n, y_n)$; but then $y_n \to p$, contradiction.) Note that $p \in H(x_i, i, n(i)) \subset g(i, x_i)$. Thus $x_i \to p$, so for each $m \in \omega$, $x_{i(m)} \in g(m, p)$ for some $i(m) \geq m$. Note that p is in the set $F(x_{i(m)}, i(m), n(i(m)), m)$, which we will denote by F_m.

We complete the proof by showing that $F_m \subset U$ for some m. Suppose not. Choose $y_m \in F_m \setminus U$. Then $p \in g(i(m), x_{i(m)})$ and $x_{i(m)} \in g(m, y_m)$. Since $i(m) \geqslant m$, $p \in g(m, x_{i(m)})$. Thus by (v)(b), $y_m \to p$. But $p \in U$ and $y_m \notin U$, so we have a contradiction. □

4.12. COROLLARY. *The closed image of a σ-space is a σ-space.*

PROOF. We have already noted that the continuous image of a network on the domain is a network on the range. Since the closed image of a closure-preserving collection is closure-preserving (this is not difficult; see the Chapter by Burke for a proof), the result follows from Theorem 4.11 (iii). □

Now we consider the class of Σ-spaces. One reason for the interest in this class is that not only is every σ-space in this class, but every M-space as well. The definition is essentially obtained by replacing points with countably compact sets in the definition of a σ-space.

4.13. DEFINITION. A space X is a (*strong*) *Σ-space* if there exists a σ-discrete collection $\mathscr{F}$, and a cover $\mathscr{C}$ of X by closed countably compact (compact) sets, such that, whenever $C \in \mathscr{C}$ and $C \subset U$ with U open, then $C \subset F \subset U$ for some $F \in \mathscr{F}$.

Of course, any property which makes countably compact subsets compact will make a Σ-space a strong Σ-space. There is such a property which every strong Σ-space has, namely subparacompactness..

4.14. THEOREM. *A space X is a strong Σ-space if and only if it is a subparacompact Σ-space.*

PROOF. Since a subparacompact countably compact space is compact (see the Chapter by Burke, Theorem 9.2, every subparacompact Σ-space is a strong Σ-space.

We prove that every strong Σ-space X is subparacompact. To this end, let $\mathscr{U}$ be an open cover of X, and let $\mathscr{F}$ and $\mathscr{C}$ satisfy the conditions of Definition 4.13. Let $\mathscr{F} = \bigcup_n \mathscr{F}_n$, where each $\mathscr{F}_n$ is discrete. For each $F \in \mathscr{F}$, choose a finite $\mathscr{U}(F) \subset \mathscr{U}$ with $F \subset \bigcup \mathscr{U}_F$, if such a finite set exists. Let $\mathscr{U}(F) = \{U_i(F) : i \leqslant k(F)\}$, and let

$$\mathscr{H}(n, i) = \{F \cap U_i(F) : F \in \mathscr{F}_n \text{ and } U_i(F) \text{ is defined}\}.$$

Each $\mathscr{H}(n, i)$ is discrete since $\mathscr{F}_n$ is. Also, if $x \in C \in \mathscr{C}$, there exists a finite $\mathscr{U}' \subset \mathscr{U}$ with $C \subset \bigcup \mathscr{U}'$, and so $x \in C \subset F \subset \bigcup \mathscr{U}'$ for some $F \in \mathscr{F}$. Thus $\mathscr{H} = \bigcup_{n,i} \mathscr{H}(n, i)$ covers X, and so $\mathscr{H}$ is a σ-discrete refinement of $\mathscr{U}$. □

Trivially, every σ-space is a strong Σ-space. Any quasi-perfect (perfect) preimage of a (strong) Σ-space is a (strong) Σ-space: for if $\mathscr{F}$ and $\mathscr{C}$ satisfy the conditions of Definition 4.13 for Y, and $f: X \to Y$ is quasi-perfect (perfect), it is not difficult to check that $f^{-1}(\mathscr{F})$ and $f^{-1}(\mathscr{C})$ satisfy 4.13 for X. Hence all (paracompact) M-spaces, or more generally, all (perfect) quasi-perfect pre-images of σ-spaces, are (strong) Σ-spaces. NAGAMI [1969] has shown that the class of (strong) Σ-spaces is strictly larger than the class of quasi-perfect (perfect) pre-images of σ-spaces.

Since a quasi-perfect pre-image of a metric space with a G_δ-diagonal is metrizable (Corollary 3.8), perhaps the following is not unexpected in light of the above discussion.

4.15. THEOREM. *A space X is a σ-space if and only if X is a Σ-space with a G_δ-diagonal.*

PROOF. We have already seen that every σ-space and has a G_δ-diagonal. So suppose $\mathscr{F}$ and $\mathscr{C}$ satisfy the conditions of Definition 4.13 for X, and let $(\mathscr{G}_n)$ be a G_δ-diagonal sequence for X such that $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$. Since countably compact spaces having a G_δ-diagonal are compact (Theorem 2.14), the elements of $\mathscr{C}$ are compact, so X is a strong Σ-space, hence subparacompact. Thus by Theorem 2.11, we may assume $(\mathscr{G}_n)$ is a G_δ^*-diagonal sequence, i.e., $\{x\} = \bigcap_n \overline{\mathrm{st}(x, \mathscr{G}_n)}$ for each $x \in X$.

For each n, let $\mathscr{H}_n = \bigcup_{m\in\omega} \mathscr{H}_{nm}$ be a σ-discrete closed refinement of $\mathscr{G}_n$. Let $\mathscr{K}(n, m, p) = \{H \cap F: H \in \mathscr{H}_{nm}, F \in \mathscr{F}_p\}$. Clearly $\mathscr{K}(n, m, p)$ is discrete for each $m, n, p \in \omega$.

We complete the proof by showing that $\mathscr{K} = \bigcup\{\mathscr{K}(n, m, p): n, m, p \in \omega\}$ is a network for X. Pick $x \in X$, an open set U containing x, and $C \in \mathscr{C}$ with $x \in C$. In the proof of Theorem 2.13, we essentially show that a G_δ^*-diagonal sequence for a compact space is a development for that space (provided $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$). By the same argument, there exists $n \in \omega$ such that $\overline{\mathrm{st}(x, \mathscr{G}_n)} \cap C \subset U \cap C$. Then for each $y \in C \setminus U$, there is an open set V_y containing y with $V_y \cap \overline{\mathrm{st}(x, \mathscr{G}_n)} = \emptyset$. There exists $F \in \mathscr{F}$ such that $C \subset F \subset U \cup (\bigcup\{V_y: y \in C\setminus U\})$, and there exists $H \in \mathscr{H}_n$ with $x \in H \subset \mathrm{st}(x, \mathscr{G}_n)$. Then $H \cap F \in \mathscr{K}$, and it is easy to check that $x \in H \cap F \subset U$. $\square$

Strong Σ-spaces, like p-spaces and paracompact M-spaces, behave nicely with respect to countable products.

4.16. *The countable product of (paracompact) strong Σ-spaces is again a (paracompact) strong Σ-space.*

PROOF. Let $\mathscr{F}_i = \bigcup_n \mathscr{F}_{in}$ and $\mathscr{C}_i$ satisfy Definition 4.13 for X_i. Now if a product $\Pi_{i\in\omega} C_i$, where $C_i \in \mathscr{C}_i$, is contained in a set U open in $\Pi_{i\in\omega} X_i$, then there exist open sets $U_i \supset C_i$ and $n \in \omega$ such that $\Pi_{i\in\omega} U_i \subset U$ and $U_i = X_i$ for $i > n$

(ENGELKING [1977], Theorem 3.2.10). With this in mind, it is easy to see that

$$\mathcal{F} = \left\{ \prod_{i \leq n} F_i \times \prod_{i > n} X_i \colon F_i \in \mathcal{F}_i,\ n \in \omega \right\} \quad \text{and} \quad \mathcal{C} = \{\Pi C_i \colon C_i \in \mathcal{C}_i\}$$

satisfy the conditions of Definition 4.13 for ΠX_i. Thus $\Pi_{i\in\omega} X_i$ is a strong Σ-space.

Suppose each X_i is also paracompact. Let $\mathcal{U}$ be an open cover of $X = \Pi_{i\in\omega} X_i$. For each $F \in \mathcal{F}_i$, $i \in \omega$, let $V(F)$ be an open set containing F such that, for each $n \in \omega$, $\{V(F) \colon F \in \mathcal{F}_{in}\}$ is discrete.

For each $F = \Pi_{i\leq n} F_i \times \Pi_{i>n} X_i \in \mathcal{F}$, choose a finite subcollection $\mathcal{U}(F)$ of $\mathcal{U}$, if it exists, such that $F \subset \bigcup \mathcal{U}(F)$. As in the proof of Theorem 4.14, the collection of all F's for which $\mathcal{U}(F)$ is defined covers X. Let $\mathcal{U}(F) = \{U_j(F) \colon j \leq k(F)\}$. Then the collection

$$\mathcal{W}(k_0, k_1, \ldots, k_n, j) = \left\{ U_j(F) \cap V(F) \colon F = \prod_{i \leq n} F_i \times \prod_{i>n} X_i,\ F_i \in F_{ik_1} \right\}$$

is discrete. Thus $\mathcal{W} = \bigcup \{\mathcal{W}(k_0, k_1, \ldots, k_n, j) \colon n, j, k_i \in \omega\}$ is a σ-discrete open refinement of $\mathcal{U}$. $\square$

Since we have already noted that the class of σ-spaces is countably productive, we have:

4.17. COROLLARY. *The countable product of paracompact σ-spaces is a paracompact σ-space.*

In contrast with σ-spaces, the 'discrete' in the definition of a Σ-space cannot be replaced by 'closure-preserving'. (It can, however, be replaced by 'locally finite'.) If this replacement could be made, then the class of Σ-spaces would be closed under closed maps, just as σ-spaces are. The next example shows that even the closed image of a paracompact M-space need not be a Σ-space. On the other hand, the class of Σ-spaces is preserved by quasi-perfect maps (NAGAMI [1969]).

4.18. EXAMPLE. A paracompact locally compact space (hence an M-space) X, and a closed map $f \colon X \to Y$ such that Y is not a Σ-space.

PROOF. X is the topological sum of ω_1 copies of $\omega_1 + 1$, and Y is obtained from X by identifying the point ω_1 in each copy to a single point $\hat{\omega}_1$. Since we have just identified a closed subset of X to a point, the quotient map is closed.

Suppose $\mathcal{F}$ and $\mathcal{C}$ satisfy Definition 4.13 for Y. Since Y is paracompact, each $C \in \mathcal{C}$ is compact. Also, we may assume that the elements of $\mathcal{F}$ are closed. Let $[\alpha(\beta), \hat{\omega}_1)$ denote the image of the interval $[\alpha, \omega_1)$ in the β'th copy of $\omega_1 + 1$ in X. Since $\mathcal{F}$ is σ-discrete, for each $\beta \in \omega$, there exists $\alpha_\beta \in \omega_1$, such that, for each

$F \in \mathscr{F}$, $F \cap [\alpha_\beta(\beta), \hat{\omega}_1) \neq \emptyset$ implies $\hat{\omega}_1 \in F$. Also, $\hat{\omega}_1$ can be in only countably many members of $\mathscr{F}$. For each $F \in \mathscr{F}$, let $\alpha(F) = \{\gamma \in \omega_1 : F \cap [0(\gamma), \hat{\omega}_1) \neq \emptyset\}$. Let $\mathscr{F}_0 = \{F \in \mathscr{F} : \hat{\omega}_1 \in F$ and $\alpha(F)$ is countable$\}$. Choose $\beta < \omega_1$ such that $\beta > \sup \alpha(F)$ for each $F \in \mathscr{F}_0$. Then if $F \in \mathscr{F}$ and $F \cap [\alpha_\beta(\beta)\hat{\omega}_1) \neq \emptyset$, we have $F \cap [0(\gamma), \hat{\omega}_1) \neq \emptyset$ for uncountably many $\gamma \in \omega_1$.

Let (F_n) be an enumeration of $\{F \in \mathscr{F} : F \cap [\alpha_\beta(\beta), \hat{\omega}_1) \neq \emptyset\}$. Let $C \in \mathscr{C}$ be such that $C \cap [\alpha_\beta(\beta), \hat{\omega}_1) \neq \emptyset$. Inductively choose distinct $\gamma_n \in \omega_1$ and $y_n \in Y$ such that $y_n \in (F_n \backslash C) \cap [0(\gamma_n), \hat{\omega}_1)$. This is possible since C, being compact, must be contained in the image of finitely many copies of $\omega_1 + 1$. Then $Y \backslash \{y_n : n \in \omega\}$ is an open subset of Y containing C. Thus $C \subset F \subset Y \backslash \{y_n : n \in \omega\}$ for some $F \in \mathscr{F}$. But $F = F_n$ for some $n \in \omega$, so $y_n \in F$, a contradiction. □

The class of spaces which results when 'discrete' is replaced by 'closure-preserving' has also been studied. These spaces are called $\Sigma^\#$-spaces, and they are preserved under closed maps. Strong $\Sigma^\#$-spaces are submetacompact, and this class is also countably productive (JUNNILA [1978b]). However, it is not known if the product of two paracompact $\Sigma^\#$-spaces is paracompact.

For a survey of the theory of σ-spaces, see OKUYAMA [1971]. For more on Σ-spaces, see NAGAMI [1969], or BURKE and LUTZER [1976]. Some other results concerning Σ-spaces are given in Theorem 6.25 in Burke's article in this Handbook.

5. Stratifiable and related spaces

In the last section, we saw that having a σ-discrete network, a σ-locally finite network, and a σ-closure preserving network are all equivalent notions. If one substitutes 'base' for 'network' in these conditions, the first two are equivalent to metrizability (by the Bing–Nagata–Smirnov metrization theorem), but the third one is more general. We make the following definition.

5.1. DEFINITION. A space X is an *M_1-space* if X has a σ-closure preserving base.

Obviously, M_1-spaces are σ-spaces, so they are subparacompact, perfect (= closed sets are G_δ-sets), and have a G_δ-diagonal. One of Michael's theorems on paracompactness says that, for a regular space, if every open cover has a σ-closure preserving open refinement, the space is paracompact (MICHAEL [1957]; see also the Chapter by Burke, Theorem 2.3). Thus M_1 spaces are paracompact and perfectly normal.

An easy example of a non-metrizable M_1-space is $\mathbb{N} \cup \{p\} \subseteq \beta\mathbb{N}$, where $p \in \beta\mathbb{N} \backslash \mathbb{N}$. The collection of all open sets containing p is clearly closure-preserving, so this collection together with the points of $\mathbb{N}$ form a σ-closure preserving base. For a less trivial example, let X be the quotient image of the closed upper half plane obtained by identifying the x-axis to a point p. It is easy to check that X is not first countable at p. To show that X is an M_1-space, the main task is to show that the point p has a σ-closure preserving base. This is not very difficult, but we will instead show, more generally, that any closed image of a metric space is an M_1-space.

Actually, all that is needed to show that the point $p \in X$ above has a σ-closure preserving base is the following lemma. By a base in X for a subset H of a space X, we mean a collection $\mathcal{U}$ of open sets containing H such that every open set containing H contains a member of $\mathcal{U}$.

5.2. LEMMA. *Every closed subset of a metric space M has a closure-preserving base in M.*

PROOF. Let $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a base for M such that each $\mathcal{B}_n$ is locally finite and $\operatorname{diam}(B) < 1/n$ for each $B \in \mathcal{B}_n$. Let H be a closed subset of M. Let $\mathcal{U}$ be the set of all unions of covers of H by those members of $\mathcal{B}$ which meet H. Clearly $\mathcal{U}$ is a base for H in M.

We prove that $\mathcal{U}$ is closure-preserving. Suppose $\mathcal{U}' \subset \mathcal{U}$ and $x \in \overline{\bigcup \mathcal{U}'}$. We need to show that $x \in \bar{U}$ for some $U \in \mathcal{U}'$. Choose $n \in \omega$ such that $d(x, H) > 1/n$. Then $x \notin \operatorname{Cl}(\bigcup\{B \in \mathcal{B}_k\colon k \geq n, B \subset U \in \mathcal{U}'\})$. Thus $x \in \operatorname{Cl}(\bigcup\{B \in \mathcal{B}_k\colon k < n, B \subset U \in \mathcal{U}'\})$. But $\bigcup_{k<n} \mathcal{B}_k$ is locally finite, so $x \in \bar{B}$ for some $B \in \bigcup_{k<n} \mathcal{B}_k$ with $B \subset U$, $U \in \mathcal{U}'$. Hence $x \in \bar{U}$. □

The notion of an irreducible map will be useful here. A map $f\colon X \to Y$ is *irreducible* if there does not exist a proper closed subset X' of X such that $f(X') = Y$. (Equivalently, every open $U \subset X$ contains $f^{-1}(y)$ for some $y \in Y$.)

If $f\colon X \to Y$, and $U \subset X$, let $f^*(U) = \{y \in Y\colon f^{-1}(y) \subset U\}$. If f is closed and U is open, then $f^*(U)$ is open by the following well known property of closed maps (ENGELKING [1977], 1.4.12): If $f^{-1}(y) \subset U$ with U open, then there is a neighborhood V of y such that $f^{-1}(V) \subset U$.

5.3. LEMMA. *If $\mathcal{U}$ is a closure-preserving collection of open subsets of X, and $f\colon X \to Y$ is a closed and irreducible, then $f^*(\mathcal{U}) = \{f^*(U)\colon U \in \mathcal{U}\}$ is closure-preserving.*

PROOF. Suppose $\mathcal{U}' \subset \mathcal{U}$. We need to show that $\overline{\bigcup f^*(\mathcal{U}')} = \bigcup\{\overline{f^*(U)}\colon U \in \mathcal{U}'\}$. Suppose $y \in \overline{\bigcup f^*(\mathcal{U}')}$. Then $y \in \overline{f(\bigcup \mathcal{U}')} = f(\overline{\bigcup \mathcal{U}'})$, so there exists a point $x \in f^{-1}(y) \cap (\overline{\bigcup \mathcal{U}'})$. Since $\mathcal{U}$ is closure-preserving, $x \in \bar{U}$ for some $U \in \mathcal{U}'$. Let W be any open set containing y. Then, since f is irreducible, $f^{-1}(z) \subset f^{-1}(W) \cap U$ for some $z \in Y$. Thus $z \in W \cap f^*(U)$. Hence $y \in \overline{f^*(U)}$, and so $\overline{\bigcup f^*(\mathcal{U}')} \subset \bigcup\{\overline{f^*(U)}\colon U \in \mathcal{U}'\}$. Containment the other direction is clear. □

Next we want to show that any closed image of a metric space is a closed irreducible image of a metric space. Our next lemma accomplishes this task. Recall that a space X is a *Fréchet space* if whenever $A \subset X$ and $x \in \bar{A}$, there exist $a_n \in A$ with $a_n \to x$. First countable spaces are Fréchet, and the Fréchet property is easily seen to be preserved under closed maps.

5.4. LEMMA. *If $f: X \to Y$ is a closed map, with X paracompact and Y Fréchet, then there is a closed $X_0 \subset X$ such that $f \mid X_0: X_0 \to Y$ is irreducible.*

PROOF. For each isolated point $y \in Y$, pick a point $x(y) \in f^{-1}(y)$. Let $I = \{x(y): y$ isolated in $Y\}$. Then $X' = I \cup (\bigcup\{f^{-1}(y): y$ not isolated in $Y\}$ is closed in X, and $f(X') = Y$.

Suppose $\{X_\alpha: \alpha < \kappa\}$ is a decreasing collection of closed subsets of X' such that $f(X_\alpha) = Y$ for each $\alpha < \kappa$. We will show $f(\bigcap_{\alpha<\kappa} X_\alpha) = Y$. Then one can apply Zorn's lemma to obtain a minimal closed subset X_0 of X such that $f(X_0) = Y$, and so $f \mid X_0$ is irreducible.

Clearly $\bigcap_{\alpha<\kappa} X_\alpha \supset I$. So we need to show that if $y \in Y$ is not isolated, then $f^{-1}(y) \cap (\bigcap_{\alpha<\kappa} X_\alpha) \neq \emptyset$. Since Y is Fréchet, there is a sequence $y_n \to y$, with $y_n \neq y$ for each n. We show that $K = \{x \in f^{-1}(y): x \in \overline{\bigcup_n f^{-1}(y_n)}\}$ is compact. If not, then since X is paracompact, K is not countably compact. Let $\{x_n: n \in \omega\} \subset K$ be closed discrete, and let $\{U_n: n \in \omega\}$ be a discrete collection of open sets with $x_n \in U_n$. Pick $z_n \in U_n \cap f^{-1}(y_k)$ for some $k \geqslant n$. Then $Z = \{z_n: n \in \omega\}$ is closed in X, but $f(z_n) \to y \notin f(Z)$. Thus $f(Z)$ is not closed, a contradiction which proves that K is compact.

Let $\alpha < \kappa$. Since $f(X_\alpha) = Y$, one can find $w_n \in X_\alpha \cap f^{-1}(y_n)$. Then $y \in \overline{f(\{w_n: n \in \omega\})} = f(\overline{\{w_n: n \in w\}})$. Thus $\emptyset \neq f^{-1}(y) \cap \overline{\{w_n: n \in \omega\}} \subset K \cap X_\alpha$. Since K is compact, $K \cap (\bigcap_{\alpha<\kappa} X_\alpha) \neq \emptyset$, and so $f(\bigcap_{\alpha<\kappa} X_\alpha) = Y$. $\square$

Finally we have enough to prove our theorem.

5.5. THEOREM (SLAUGHTER [1973]). *The closed image of a metric space is an M_1-space.*

PROOF. Let $f: M \to X$ be a closed map, with M metrizable. By Corollary 4.12, X has a closed network $\mathscr{F} = \bigcup_n \mathscr{F}_n$ such that each $\mathscr{F}_n$ is discrete. Now X is collectionwise-normal (since it is the closed image of a collectionwise-normal space), so to each $F \in \mathscr{F}_n$, one can assign an open set $U_F \supset F$ such that $\{U_F: F \in \mathscr{F}_n\}$ is discrete. By Lemma 5.2, each $f^{-1}(F)$ has a closure-preserving base $\mathscr{B}_F$; we may assume $\bigcup \mathscr{B}_F \subset f^{-1}(U_F)$. By Lemma 5.4, we may assume that f is irreducible. Then $f^*(\mathscr{B}_F)$ is closure-preserving by Lemma 5.3, and since $\bigcup f^*(\mathscr{B}_F) \subset U_F$, $\{f^*(\mathscr{B}_F): F \in \mathscr{F}_n\}$ is also closure-preserving. It is now easy to check that $\bigcup_n \{f^*(\mathscr{B}_F): F \in \mathscr{F}_n\}$ is a σ-closure preserving base for X. $\square$

Closed images of metric spaces are often called *Lasnev spaces*, after N. Lasnev for his work on this class of spaces. He obtained an internal characterization of this class, as well as his 'decomposition theorem' mentioned earlier (see Theorem 3.11). The class of Lasnev spaces furnishes us with a reasonably wide and important class of spaces which are M_1-spaces. Another important subclass of M_1-spaces is Whitehead's CW-complexes (see CEDER [1961] for the proof).

However, the M_1-spaces have serious failings in terms of preservation under various topological operations. While the obvious proof shows that a countable product of M_1-spaces is M_1, it is not known if every closed or even perfect image of an M_1-space is M_1, or if every subspace or even closed subspace of an M_1-space is M_1. The answers to all these questions would be affirmative if it could be shown that M_1-spaces are the same as stratifiable spaces, which we define below. Stratifiable spaces have almost all the preservation properties one could hope for, and have proven to be one of the most useful classes of generalized metric spaces. Yet it is still not known if they are really the same as M_1-spaces.

Stratifiable spaces and M_1-spaces were introduced by CEDER [1961], who used the term 'M_3-space' for 'stratifiable space'. The latter term was proposed by BORGES [1966]. The class of semi-stratifiable spaces was introduced and studied by CREEDE [1970].

5.6. DEFINITION. A space X is *semi-stratifiable* if there is a function G which assigns to each $n \in \omega$ and closed set $H \subset X$, an open set $G(n, H)$ containing H such that

(i) $H = \bigcap_n G(n, H)$;

(ii) $H \subset K \Rightarrow G(n, H) \subset G(n, K)$.

If also

(iii) $H = \bigcap_n \overline{G(n, H)}$, then X is *stratifiable.*

REMARK. We can always assume the following additional condition holds:

(iv) $G(n+1, H) \subset G(n, H)$ for each $n \in \omega$;

for if G stratifies (i), (ii) and/or (iii), so does $G'(n, H) = \bigcap_{i \leq n} G(i, H)$.

For a fixed closed set H, the open sets $G(n, H)$ show that H is a G_δ-set (regular G_δ-set in the case of stratifiable spaces), and property (ii), whose power is yet to be seen, says that this is done in a monotone fashion—the bigger H is, the bigger $G(n, H)$ is. It is easy to see how to define such a G for a metrizable space X: just let $G(n, H)$ be the set of all points whose distance from H is less than $1/2^n$. Thus metric spaces are stratifiable. Also note that every developable space is semi-stratifiable: let $G(n, H) = \mathrm{st}(H, \mathcal{G}_n)$, where $(\mathcal{G}_n)$ is a development.

To see that M_1-spaces are stratifiable, suppose $\mathcal{B} = \bigcup_n \mathcal{B}_n$ is a σ-closure preserving base for X. If H is a closed subset of X, let

$$G(n, H) = X \setminus \bigcup \{\bar{B} : B \in \mathcal{B}_n, \bar{B} \cap H = \emptyset\} .$$

Then G is easily seen to satisfy properties (i)–(iii) above.

The following dual characterization of stratifiable spaces is sometimes useful: X

is stratifiable if for each open set U and $n \in \omega$, one can assign an open set U_n such that $\bar{U}_n \subset U$, $U = \bigcup_n U_n$, and $U \subset V$ implies $U_n \subset V_n$. To get this characterization from a function G satisfying Definition 5.6, let $U_n = X \setminus \overline{G(n, X \setminus U)}$. On the other hand, to get G from the U_n's, let $G(n, H) = X \setminus \overline{(X \setminus H)_n}$.

The above dual characterization, together with another of Michael's characterizations of paracompactness, will easily give us:

5.7. THEOREM. *Stratifiable spaces are paracompact.*

PROOF. Let $\mathcal{U}$ be an open cover of the stratifiable space X. For fixed $n \in \omega$, the collection $\{(U_n, U): U \in \mathcal{U}\}$, where U_n is as in the dual characterization given above, has the following property: if $\mathcal{U}' \subset \mathcal{U}$, then $\overline{\bigcup\{U_n: U \in \mathcal{U}'\}} \subset \overline{(\bigcup \mathcal{U}')_n} \subset \bigcup \mathcal{U}'$. This means that $\{U_n: U \in \mathcal{U}\}$ is *cushioned* in $\mathcal{U}$, and so $\mathcal{U}$ has a σ-cushioned refinement. Thus X is paracompact (MICHAEL [1959]; see also the Chapter by Burke, Theorem 2.3). $\square$

The above result also follows from Theorems 5.11 and 5.18.

The following is another very useful characterization of stratifiable and semi-stratifiable spaces.

5.8. THEOREM. *A space (X, τ) is semi-stratifiable iff there exists a function $g: \omega \times X \to \tau$ such that*

(i) $\{x\} = \bigcap_n g(n, x)$

(ii) $y \in g(n, x_n) \Rightarrow x_n \to y$.

X is stratifiable is one can also obtain

(iii) *if $y \notin H$, where H is closed, then $y \notin \overline{\bigcup\{g(n, x): x \in H\}}$ for some $n \in \omega$.*

PROOF. ($\Rightarrow$) The first thing that comes to mind works: let $g(n, x) = G(n, \{x\})$, where G satisfies Definition 5.6, with $G(n+1, H) \subset G(n, H)$ for each n. Property (i) is clearly satisfied. To see (ii), suppose $y \in g(n, x_n)$ but $x_n \not\to y$. Then there is an infinite subset $A \subset \omega$ such that $y \notin \overline{\{x_n: n \in A\}}$. Then $y \notin G(m, \overline{\{x_n: n \in A\}})$ for some $m \in \omega$. Choose $n \in A$ with $n \geq m$. Then $y \in g(n, x_n) = G(n, \{x_n\}) \subset G(n, \overline{\{x_n: n \in A\}})$, a contradiction. Finally, to check (iii), assume that y is not in the closed set H. There exists $n \in \omega$ with $y \notin \overline{G(n, H)}$. But $G(n, H) \supset \bigcup_{x \in H} G(n, \{x\})$, so $y \notin \overline{\bigcup\{g(n, x): x \in H\}}$.

($\Leftarrow$) Let $G(n, H) = \bigcup_{x \in H} g(n, x)$. One can easily verify that G satisfies the conditions of Definition 5.6. $\square$

Note the similarity of this characterization with the characterizations of σ-spaces given in (iv) and (v) of Theorem 4.11. It turns out that σ-spaces fit properly between stratifiable and semi-stratifiable spaces.

5.9. THEOREM. *Every stratifiable space is a σ-space, and every σ-space is semi-stratifiable.*

PROOF. If $g: \omega \times X \to \tau$ satisfies (a) and (c) of Theorem 4.11 (iv), then $\{x\} = \bigcap_n g(n, x)$, so g satisfies (i) and (ii) of Theorem 5.8. Thus σ-spaces are semi-stratifiable.

We show that if g satisfies (i)–(iii) of Theorem 5.8, and $g(n+1, x) \subset g(n, x)$ (which can always be assumed), then g satisfies Theorem 4.11 (v). Part (a) of 4.11 (v) is obvious. Suppose $x \in g(n, x_n)$ and $x_n \in g(n, y_n)$. We need to show, given any subsequence $S = \{y_{n(k)}: k \in \omega\}$ of the y_n's, that $x \in \bar{S}$. Suppose $x \notin \bar{S}$. By 5.8 (iii), $x \notin \overline{\bigcup\{g(n, z): z \in \bar{S}\}}$ for some $n \in \omega$. Since $x_n \in g(n, y_n)$, $x \notin \overline{\{x_{n(k)}: n(k) \geq n\}}$. This contradicts $x_{n(k)} \to x$. □

For an example of a non-stratifiable σ-space, one can take any non-metrizable Moore space. Example 9.10 is an example of a semi-metrizable (Definition 9.5), hence semi-stratifiable, space which is not a σ-space.

The following is another easy corollary to Theorem 5.8.

5.10. THEOREM. *The classes of semi-stratifiable and stratifiable spaces are hereditary and countably productive.*

PROOF. That these classes are hereditary is clear. To see that they are countably productive, let $g_i: \omega \times X_i \to \tau_i$ satisfy the conditions of Theorem 5.8 for X_i, and let $x = (x_n) \in \Pi_{n\in\omega} X_n$. Let $g(n, x) = \Pi_{i\leq n} g_i(n, x_i) \times \Pi_{i>n} X_i$. It is easy to check that g satisfies the conditions of Theorem 5.8 for $\Pi_{n\in\omega} X_n$. □

5.11. THEOREM. *Every semi-stratifiable space is subparacompact and has a G_δ^*-diagonal.*

PROOF. Let X be semi-stratifiable. Then X^2 is semi-stratifiable, so the diagonal is a G_δ-set. That X has a G_δ^*-diagonal will follow after we prove that X is subparacompact (by Theorem 2.11).

Let $\mathcal{U}$ be an open cover of X well-ordered by "$<$". Let g satisfy conditions (i) and (ii) of Theorem 5.8. For each $U \in \mathcal{U}$ and $n \in \omega$, let

$$U_n = U \setminus [(\bigcup\{U' \in \mathcal{U}: U' < U\}) \cup (\bigcup\{g(n, y): y \notin U\})\,.$$

Let $\mathcal{F}_n = \{U_n: U \in \mathcal{U}\}$, and let $\mathcal{F} = \bigcup_n \mathcal{F}_n$. To see that $\mathcal{F}$ covers X, pick $x \in X$ and let $U(x)$ be the least element of $\mathcal{U}$ which contains x. By property (ii) of 5.8, there exists $n \in \omega$ with $x \notin \bigcup\{g(n, y): y \notin U(x)\}$. Hence $x \in U(x)_n$, and so $\mathcal{F}$ covers X and is a refinement of $\mathcal{U}$.

We complete the proof by showing that each $\mathcal{F}_n$ is discrete. Fix n, and let $x \in X$. Let $U(x) \in \mathcal{U}$ be as above. Then $U(x) \cap U_n = \emptyset$ if $U > U(x)$. Also, $g(n, x) \cap U_n = \emptyset$ if $U < U(x)$. Thus $g(n, x) \cap U(x)$ meets at most one member of $\mathcal{F}_n$, namely $U(x)_n$. Hence $\mathcal{F}_n$ is discrete. □

We now can see the relationship of (semi)-stratifiable spaces with the spaces of Section 3.

5.12. COROLLARY. (i) *A space X is a Moore space if and only if X is semi-stratifiable, and a $w\Delta$-space or a p-space.*

(ii) *A space X is metrizable if and only if X is stratifiable, and a $w\Delta$-space or a p-space.*

PROOF. Since stratifiable spaces are paracompact, (ii) follows from (i). Part (i) follows from the following: semi-stratifiable spaces are subparacompact and have a G_δ^*-diagonal, p-spaces and $w\Delta$-spaces are the same for submetacompact spaces (Theorem 3.19), and a $w\Delta$-space with a G_δ^*-diagonal is developable (Theorem 3.3). □

By Corollary 5.12, we have in particular that locally compact stratifiable spaces are metrizable. We should also mention that a semi-stratifiable space which is countably compact, or more generally an M-space, is also metrizable by Theorem 3.8.

We will see later that stratifiable spaces are preserved by closed maps. For now we have:

5.13. THEOREM. *The closed image of a semi-stratifiable space is semi-stratifiable.*

PROOF. Let $f: X \to Y$ be closed, and suppose G satisfies conditions (i) and (ii) of Definition 5.6 for X. For each closed $H \subset Y$, let $G'(n, H) = f^*(G(n, f^{-1}(H)))$, where $f^*(U) = \{y: f^{-1}(y) \subset U\}$. As noted prior to Lemma 5.3, $f^*(U)$ is open in Y if U is open in X. Hence $G'(n, H)$ is an open set containing H. It is easy to check that G' satisfies 5.6(i) and (ii) for Y. □

Now we show that Lasnev's "decomposition theorem" (see discussion preceding Theorem 3.11) holds for stratifiable, or more generally, normal semi-stratifiable spaces. We have formulated the following result so that it implies Theorem 3.11 as well.

5.14. THEOREM. *Suppose X is a quasi-perfect pre-image of a normal semi-stratifiable space, and $f: X \to Y$ is a closed map. Then $Y = \bigcup_{n\in\omega} Y_n$, where $f^{-1}(y)$ is countably compact for each $y \in Y_0$, and for each $n \geqslant 1$, Y_n is a closed discrete subset of Y.*

PROOF. Let $h: X \to Z$ be quasi-perfect, where Z is semi-stratifiable. Let G satisfy the conditions of Definition 5.6 for Z, with $G(n+1, H) \subset G(n, H)$. For a closed set $K \subset X$, let $U_n(K) = h^{-1}(G(n, h(K)))$.

Since f is closed, for each $y \in Y$ we can find an open set $O_n(y)$ containing y such that $f^{-1}(O_n(y)) \subset U_n(f^{-1}(y))$. For $n \geqslant 1$, let $Y_n = \{y \in Y: y' \neq y \Rightarrow y \notin O_n(y')\}$, and let $Y_0 = Y \backslash \bigcup_{n=1}^{\infty} Y_n$. It is easy to check that each Y_n for $n \geqslant 1$ is a closed discrete subset of Y.

Let $y \in Y_0$. We need to show that $f^{-1}(y)$ is countably compact. Since $y \notin \bigcup_{n=1}^{\infty} Y_n$, for each $n \geqslant 1$, there exists $y_n \neq y$ such that $y \in O_n(y_n)$. Let S be any infinite subset of $\{n \in \omega: n \geq 1\}$. Then

$$f^{-1}(y) \subset \bigcap_{n\in S} f^{-1}(O_n(y_n)) \subset \bigcap_{n\in S} U_n(f^{-1}(y_n)) = \bigcap_{n\in S} h^{-1}(G(n, h(f^{-1}(y_n)))$$

$$= h^{-1}(\bigcap_{n\in S} G(n, h(f^{-1}(y_n))) \subset h^{-1}(\bigcap_{n\in S} G(n, \overline{\bigcup_{n\in S} h(f^{-1}(y_n))})$$

$$= h^{-1}(\overline{\bigcup_{n\in S} h(f^{-1}(y_n))}) = h^{-1}(h(\overline{\bigcup_{n\in S} f^{-1}(y_n)})) .$$

Thus for each $x \in f^{-1}(y)$, $h(x) \in h(\overline{\bigcup_{n\in S} f^{-1}(y_n)})$. From this it follows that no infinite subset of $\{f^{-1}(y_n): n \geq 1\}$ is a closed discrete in X.

Now suppose $f^{-1}(y)$ is not countably compact. Then there is an infinite discrete set $\{x_n: n \in \omega\} \subset f^{-1}(y)$. Since each $h^{-1}(z)$ is countably compact, we may assume $h(x_n) \neq h(x_m)$ for $n \neq m$. Then since h is closed, $\{h(x_n): n \in \omega\}$ is closed discrete in Z. Since Z is normal, there exists a discrete collection $\{V_n: n \in \omega\}$ of open subsets of Z such that $h(x_n) \in V_n$ for each n. By the above paragraph, $V_n \cap h(\bigcup_{k\geq n} f^{-1}(y_k)) \neq \emptyset$, so there exists $x'_n \in X$ and $k(n) \geq n$ with $x'_n \in f^{-1}(y_{k(n)})$ and $h(x'_n) \in V_n$. Since $\{V_n: n \in \omega\}$ is discrete, so is $\{x'_n: n \in \omega\}$, and since f is closed, so is $\{f^{-1}(f(x'_n)): n \in \omega\} = \{f^{-1}(y_{k(n)}): n \in \omega\}$, a contradiction. □

What exactly is the difference between stratifiable and semi-stratifiable spaces? At first glance, one might suspect 'normality', since then one could modify the $G(n, H)$'s to get $H = \bigcap_n \overline{G(n, H)}$. But an arbitrary modification like this may destroy the 'monotone' condition. What is needed is the following concept due to P. Zenor:

5.15. DEFINITION. A space X is *monotonically normal* if to each pair (H, K) of disjoint closed subsets of X, one can assign an open set $D(H, K)$ such that

(i) $H \subset D(H, K) \subset \overline{D(H, K)} \subset X \setminus K$;

(ii) if $H \subset H'$ and $K \supset K'$, then $D(H, K) \subset D(H', K')$.

The function D is called a *monotone normality operator* for X. Observe that one can always modify D so that $D(H, K) \cap D(K, H) = \emptyset$: if D does not satisfy this condition, just let $D'(H, K) = D(H, K) \cap X \setminus \overline{D(K, H)}$.

We now show that monotone normality can be thought of as the 'difference' between stratifiable and semi-stratifiable spaces.

5.16. THEOREM. *A space X is stratifiable if and only if X is semi-stratifiable and monotonically normal.*

PROOF. ($\Rightarrow$) Let X be stratifiable, with G satisfying the conditions of Definition 5.6, and $G(n+1, H) \subset G(n, H)$. To prove that X is monotonically normal, we simply mimic the proof that if every closed set is a regular G_δ-set, then the space is normal. That is, we define

$$D(H, K) = \bigcup_n G(n, H) \setminus \overline{G(n, K)}\,.$$

Clearly $D(H, K)$ is an open set containing H. Suppose $y \in K$. Then $y \notin \overline{G(m, H)}$ for some $m \in \omega$. Hence $(X \setminus \overline{G(m, H)}) \cap G(m, K)$ is an open set containing y missing $D(H, K)$. Thus $\overline{D(H, K)} \subset X \setminus K$. Finally, the monotonicity of D (property (ii) of Definition 5.15) follows from the monotonicity of G.

($\Leftarrow$) Suppose X is semi-stratifiable and monotonically normal, with G and D satisfying the conditions of Definitions 5.6 and 5.15, respectively. Let $G'(n, H) = D(H, X \setminus G(n, H))$. It is easy to check that G' satisfies (i)–(iii) of Definition 5.6, hence X is stratifiable. $\square$

5.17. THEOREM. *Monotonically normal spaces and stratifiable spaces are preserved under closed maps.*

PROOF. We have already shown that semi-stratifiable spaces are preserved under closed maps, so by Theorem 5.16, all we have to show is that monotonically normally spaces are preserved under closed maps.

Suppose $f: X \to Y$ is closed, and D_X is a monotone normality operator for X. Let H and K be disjoint closed subsets of Y. Define

$$D_Y(H, K) = f^*(D_X(f^{-1}(H), f^{-1}(K)))\,,$$

where $f^*(U) = \{y: f^{-1}(y) \subset U\}$. As noted prior to Lemma 5.3, $f^*(U)$ is open if U is, so $D_Y(H, K)$ is an open set containing H. It is easy to check that D_Y satisfies condition (ii) of Definition 5.15. Thus it remains to prove $\overline{D_Y(H, K)} \subset Y \setminus K$. Suppose $y \in K$. Then $y \in Y \setminus \overline{f(D_X(f^{-1}(H), f^{-1}(K))}$, and this set misses $D_Y(H, K)$. Thus $\overline{D_Y(H, K)} \subset Y \setminus K$. $\square$

Monotonically normal spaces need not be paracompact: any non-paracompact linearly ordered space such as ω_1 is an example, as we shall see later. But we have the following result.

5.18. THEOREM. *Monotonically normal spaces are collectionwise-normal.*

PROOF. Let $\mathscr{H}$ be a discrete collection of closed subsets of X, and let D be a monotone normality operator on X such that $D(H, K) \cap D(K, H) = \emptyset$. For each $H \in \mathscr{H}$, let $H^* = \bigcup\{H' \in \mathscr{H}: H' \neq H\}$, and let $U_H = D(H, H^*)$. Then $H \subset U_H$, and $U_{H_0} \cap U_{H_1} = D(H_0, H_0^*) \cap D(H_1, H_1^*) \subset D(H_0, H_1) \cap D(H_1, H_0) = \emptyset$ whenever $H_0 \neq H_1 \in \mathscr{H}$. $\square$

Since collectionwise-normal subparacompact spaces are paracompact, from Theorems 5.11 and 5.18 we have another proof that stratifiable spaces are paracompact. We should also mention that monotonically normal spaces are countably paracompact (see the Chapter by Rudin, Theorem 2.3 for the proof), and monotonically normal spaces having a G_δ-diagonal are paracompact (PALENZ [1982]).

We now prove a very useful characterization of monotonically normal spaces.

5.19. THEOREM. *A space X is monotonically normal if and only if for each open set $U \subset X$ and $x \in U$, one can assign an open set U_x containing x satisfying the following condition*:

$$U_x \cap V_y \neq \emptyset \quad \textit{implies} \quad x \in V \textit{ or } y \in U .$$

PROOF. ($\Rightarrow$) Let D be a monotone normality operator for X such that $D(H, K) \cap D(K, H) = \emptyset$. Let $U_x = D(\{x\}, X \setminus U)$. Suppose $x \notin V$ and $y \notin U$. Then $U_x \subset D(\{x\}, \{y\})$ and $V_y \subset D(\{y\}, \{x\})$. Hence $U_x \cap V_y = \emptyset$, so the desired condition holds.

($\Leftarrow$) Suppose an assignment satisfying the conditions of the theorem can be made. Given disjoint closed sets H and K, let

$$D(H, K) = \bigcup \{U_x : x \in H \text{ and } U \cap K = \emptyset\} .$$

Clearly $D(H, K)$ is open and contains H. Suppose $y \in K$. Then $(X \setminus H)_y \cap U_x = \emptyset$ whenever $x \in H$ and $U \cap K = \emptyset$, for otherwise either x must be in $X \setminus H$ or y in U. Thus $y \notin \overline{D(H, K)}$, and so $\overline{D(H, K)} \subset X \setminus K$. That D is 'monotone' is clear from the definition. $\square$

It is now easy to prove the following result.

5.20. *Every subspace of a monotonically normal space is monotonically normal.*

PROOF. Suppose X is monotonically normal, and for each open $U \subset X$ and $x \in U$, we have U_x satisfying the conditions of Theorem 5.19. If V is open in some subspace Y of X, let $V_y = V'_y$, where V' is an open subset of X with $V' \cap Y = V$. It is easy to check that this assignment satisfies the conditions of Theorem 5.19 for Y. $\square$

One reason for the interest in monotonically normal spaces is that, besides stratifiable spaces, all suborderable spaces (= subspaces of linearly ordered spaces) are monotonically normal, as we now show.

5.21. *Every suborderable space is monotonically normal.*

PROOF. By the previous result, we need only prove that if X is a linearly ordered space, then X is monotonically normal. Let "$<$" be the linear order on X, and let

"$<$" be a well-ordering of X. If $a \lneqq x \lneqq b$, let a_x and b_x be the $<$-least elements of X such that $a \leqslant a_x < x < b_x \leqslant b$. For $x \in U$, U open, pick $a < x < b$ with $(a, b) \subset U$, and let $U_x = (a_x, b_x)$. It is not difficult to check that the U_x's satisfy the conditions of Theorem 5.19. □

Note that from the above result, together with Theorem 5.18, we get the well-known fact that linearly ordered spaces are (hereditarily) collectionwise-normal.

The following theorem demonstrates how poorly behaved monotonically normal spaces are in terms of products. The product of a monotonically normal space X with a convergent sequence is not monotonically normal unless X is stratifiable.

5.22. THEOREM. *A space X is stratifiable if and only if $X \times (\omega + 1)$ is monotonically normal.*

PROOF. If X is stratifiable, then $X \times (\omega + 1)$ is also stratifiable, hence monotonically normal. Now suppose $X \times (\omega + 1)$ is monotonically normal. Then X is monotonically normal, so it suffices to show that X is semi-stratifiable. The proof of this is similar to the proof that the hereditary normality of $X \times (\omega + 1)$ implies the perfect normality of X (KATETOV [1948]; see also the Chapter by Przymusiński, Theorem 5.1).

Suppose that for the space $X \times (\omega + 1)$, we have U_x's satisfying the conditions of Theorem 5.19. For a closed set $H \subset X$, let

$$G(n, H) = \pi_X\big(\bigcup_{x \in H} \{U_{(x,n)}\colon U \subset X \times \{n\},\ U \text{ open}\}\big),$$

where π_X is the projection onto X.

If $H \subset K$, clearly $G(n, H) \subset G(n, K)$. It remains to prove $H = \bigcap_n G(n, H)$. Suppose $p \in (\bigcap_n G(n, H)) \setminus H$. Then for each n, $(p, n) \in U(n)_{(x_n, n)}$ for some $U(n) \subset X \times \{n\}$ and $x_n \in H$. Hence $(p, \omega) \in \overline{\bigcup_n U(n)_{(x_n,n)}}$. Thus

$$[(X \setminus H) \times (\omega + 1)]_{(p,\omega)} \cap U(n)_{(x_n,n)} \neq \emptyset \quad \text{for some } n.$$

But this contradicts 5.19, since $(p, \omega) \not\in U(n)$ and $(x_n, n) \not\in (X \setminus H) \times (\omega + 1)$. □

Extension of continuous functions

We now briefly discuss the problem of the extension of continuous functions defined on a closed subspace A of a stratifiable or related type of space X. If X is normal, the Tietze–Urysohn theorem says that a continuous function $f\colon A \to \mathbb{R}$ can be extended to a continuous function defined on all of X. If X is metrizable one can do much more: if $\mathscr{C}(A)$ and $\mathscr{C}(X)$ are the collections of real-valued continuous functions defined on A and X, respectively, then there exists a "linear extender", i.e., a map ϕ which takes a function $f \in \mathscr{C}(A)$ to an extension $\Phi(f) \in \mathscr{C}(X)$ in such a way that $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$ for all $f, g \in \mathscr{C}(A)$. Further, the range of $\Phi(f)$ can be made to be contained in the convex hull of the range of f. The above result is known as the Dugundji extension theorem (DUGUNDJI [1966]). We now show that his theorem holds for stratifiable spaces. This will illustrate but one of many results concerning extension properties of stratifiable, monotonically normal, and other spaces.

5.23. THEOREM (BORGES [1966]). *Let A be a closed subspace of the stratifiable space X. Then there exists a function Φ: $\mathscr{C}(A) \to \mathscr{C}(X)$ such that*

(i) *for each $f \in \mathscr{C}(A)$, $\Phi(f)$ extends f;*

(ii) *Φ is linear;*

(iii) *the range of $\Phi(f)$ is contained in the convex hull of the range of f.*

PROOF. We follow Borges' original proof. Let G satisfy 5.6(i)–(iii) for X, and assume $G(n+1, H) \subset G(n, H)$. First we define, for each open set U and $x \in U$, an open set U_x containing x and an integer $n(x, U)$ such that the following condition holds:

(*) $\qquad U_x \cap V_y \neq \emptyset$ and $n(x, U) \leq n(y, V)$ implies $y \in U$.

Let $n(x, U)$ be the least natural number such that $x \notin \overline{G(n, X \backslash U)}$, and let

$$U_x = (X \backslash \overline{G(n(x, U), X \backslash U)}) \cap G(n(x, U), \{x\}).$$

Suppose $n(x, U) \leq n(y, V)$, and $y \notin U$. Then $G(n(y, V), \{y\}) \subset G(n(x, U), X \backslash U)$, so

$$U_x \cap V_y \subset [X \backslash G(n(x, U), X \backslash U)] \cap G(n(y, V), \{y\}) = \emptyset.$$

Thus (*) holds.

Let $B = X \backslash A$, and let $B' = \{x \in B: x \in U_y$ for some $y \in A$ and open U containing $y\}$. For each $x \in B'$, let

$$m(x) = \max\{n(y, U): y \in A \text{ and } x \in U_y\}.$$

By condition (*), $m(x) < n(x, B)$.

Let $\mathscr{V}$ be a locally finite open refinement of $\{B_x: x \in B\}$ in the subspace B. For each $V \in \mathscr{V}$, pick $x(V) \in B$ with $V \subset B_{x(V)}$. If $x(V) \in B'$, pick $a(V) \in A$ and open $U(V)$ containing $a(V)$ such that $x(V) \in U(V)_{a(V)}$ and $n(a(V), U(V)) = m(x(V))$. If $x(V) \notin B'$, let $a(V)$ be a fixed point $a_0 \in A$.

Let $\{\rho_V: V \in \mathscr{V}\}$ be a partition of unity subordinated to $\mathscr{V}$ (see ENGELKING [1977], pp. 374–375). Suppose $f \in \mathscr{C}(A)$. We define an extension $\Phi(f) = g$ as follows:

(i) $g(x) = f(x)$ if $x \in A$;

(ii) $g(x) = \Sigma_{V \in \mathscr{V}} \rho_V(x) f(a_v)$ if $x \notin A$.

It is easy to see that the range of g is contained in the convex hull of the range of f, that g is continuous on $X \backslash A$, and that Φ is linear.

It remains to prove that g is continuous at an arbitrary point $a \in A$. Let O be an open interval containing $f(a)$, and let N be an open neighborhood of a such that $f(A \cap N) \subset O$. We complete the proof by showing that $g((N_a)_a) \subset O$. If $x \in (N_a)_a \cap A \subset N \cap A$, then $g(x) = f(x) \in O$. Suppose $x \in (N_a)_a \backslash A$, and let $V \in \mathscr{V}$ with $x \in V$. Since $x \in (N_a)_a \cap B_{x(V)}$, and $a \notin B$, we have by (∗) that $x(V) \in N_a$. Hence $x(V) \in B'$ and $n(N, a) \leq m(x(V)) = n(U(V), a(V))$. Since $x(V) \in N_a \cap U(V)_{a(V)}$, again by (∗) we have that $a(V) \in N$. Thus $f(a(V)) \in O$, and by the convexity of O, $g(x) \in O$. This completes the proof. □

It is possible to strengthen the above result by making the linear extender be continuous whenever both $\mathscr{C}(A)$ and $\mathscr{C}(X)$ are given the compact-open topology, the topology of pointwise convergence, or the topology of uniform convergence.

In suborderable spaces, one can obtain a linear extender from $\mathscr{C}^*(A)$ to $\mathscr{C}^*(X)$, the bounded real-valued functions on A and X, respectively (HEATH, LUTZER [1974]). In monotonically normal spaces, one can obtain an extender Φ such that if $f(x) \leq g(x)$ for each $x \in A$, then $\Phi(f)(x) \leq \Phi(g)(x)$ for each $x \in X$ (HEATH, LUTZER, ZENOR [1973]). We should point that Theorem 5.23 does not hold for paracompact σ-spaces (VAN DOUWEN [1975]).

Stratifiable vs. M_1

We mentioned earlier that it is not known if every stratifiable space is an M_1-space, i.e., if every stratifiable space has a σ-closure preserving open base. We conclude this section with a brief discussion of this problem, and a proof that stratifiable spaces are the same as M_2-spaces, which we now define.

5.24. DEFINITION. A collection $\mathscr{B}$ of subsets of X is a *quasi-base* for X if whenever $x \in U$, U open, then $x \in B^0 \subset B \subset U$ for some $B \in \mathscr{B}$. An *M_2-space* is a space with a σ-closure-preserving quasi-base.

We observe that every M_2-space X is a closed subset of some M_1-space. Let $\mathscr{B} = \bigcup_n \mathscr{B}_n$ be a quasi-base for X, with each $\mathscr{B}_n$ closure-preserving. Let Z be the space obtained from $X \times (\omega + 1)$ by isolating the points of $X \times \omega$. For each $B \in \mathscr{B}$ and $m \in \omega$, let $B(m) = (B \times [m, \omega)) \cup (B^0 \times \{\omega\})$. It is not difficult to check that

$$\mathscr{B}(m, n) = \{B(m): B \in \mathscr{B}_n\}$$

is closure-preserving in Z, and so

$$\Big(\bigcup_{m,n} \mathcal{B}(m,n)\Big) \cup \Big(\bigcup_{n} X \times \{n\}\Big)$$

is a σ-closure preserving base for Z. Thus Z is M_1, and X is homeomorphic to the closed subspace $X \times \{\omega\}$.

The above argument is due to HEATH and JUNNILA [1981], who also showed that there is a subspace $Z' \subset Z$ containing $X \times \{\omega\}$ such that the projection onto $X \times \{\omega\}$ is a perfect map. Then from the fact that stratifiable spaces are M_2-spaces, which we shall presently prove, it follows that the question of whether stratifiable spaces are M_1 is equivalent to the question of whether M_1-spaces are closed hereditary or preserved under perfect mappings.

First we need a characterization of M_2-spaces similar to the characterization of stratifiable spaces given by Theorem 5.8.

5.25. THEOREM. *A space X is an M_2-space if and only if there is a function $g: \omega \times X \to \tau$ such that*

(i) $\{x\} = \bigcap_n g(n, x)$;

(ii) $x \in g(n, x_n) \Rightarrow x_n \to x$;

(iii) *if H is closed and $y \notin H$, then $y \notin \overline{\bigcup\{g(n, x): x \in H\}}$ for some $n \in \omega$.*

(iv) $y \in g(n, x) \Rightarrow g(n, y) \subset g(n, x)$.

PROOF. ($\Rightarrow$) Let $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a quasi-base for X. Let

$$g(n, x) = X \setminus \bigcup\{\bar{B}: B \in \mathcal{B}_i,\ i \leq n,\ x \notin \bar{B}\}.$$

The proof that (i), (ii), and (iv) are satisfied is the same as the proof of (iii)$\Rightarrow$(iv) of Theorem 4.11. Property (iii) follows easily from the fact that $\mathcal{B}$ is a quasi-base.

($\Leftarrow$) Suppose g satisfies conditions (i)–(iv) for X. Let $\mathcal{G}_n = \{g(n, x): x \in X\}$ and let $\mathcal{B}_n = \{X \setminus \bigcup \mathcal{G}'_n: \mathcal{G}'_n \subset \mathcal{G}_n\}$. That $\mathcal{B}_n$ is a quasi-base follows easily from the fact that g satisfies (iii).

Let $\mathcal{G}'_n \subset \mathcal{G}_n$. By (iv), if $y \in \bigcap \mathcal{G}'_n$ then $g(n, y) \subset \bigcap \mathcal{G}'_n$; hence $\bigcap \mathcal{G}'_n$ is open. Suppose $\mathcal{G}_n(\alpha) \subset \mathcal{G}_n$ for each $\alpha \in \Lambda$. Then

$$\bigcup_{\alpha \in \Lambda} \big(X \setminus \bigcup \mathcal{G}_n(\alpha)\big) = X \setminus \bigcap \Big(\bigcup_{\alpha \in \Lambda} \mathcal{G}_n(\alpha)\Big)$$

is closed. It follows that $\mathcal{B}_n$ is closure-preserving. $\square$

Note that in the above proof, all we need is that g satisfy (i)–(iii), and that $\mathcal{G}_n = \{g(n, x): x \in X\}$ be 'interior-preserving', i.e., that $\mathcal{G}'_n \subset \mathcal{G}_n$ implies $\bigcap \mathcal{G}'_n$ is open. The idea of the proof that stratifiable spaces are M_2 is to take a function g satisfying (i)–(iii) and modify it so that $\mathcal{G}_n$ is point-finite and hence interior-preserving. In order to do this, we need to 'fatten up' the $g(n, x)$'s a bit. If

$N: X \to \tau$ assigns an open neighborhood $N(x)$ of x to each $x \in X$, let

$$N^2(x) = \bigcup\{N(y): y \in N(x)\} \quad \text{and} \quad N^3(x) = \bigcup\{N(z): z \in N^2(x)\}.$$

It turns out that if $g(n, x)$ satisfies (i)–(iii) above (with $g(n+1, x) \subset g(n, x)$), so does $g^3(n, x)$. The following is then the key lemma.

5.26. LEMMA (JUNNILA [1978]). *Let (X, τ) be a metacompact semi-stratifiable space, and $N: X \to \tau$ such that $x \in N(x)$ for each x. Then there is a point-finite open cover $\mathscr{V}$ of X such that, for each $x \in X$,*

$$\bigcap\{V \in \mathscr{V}: x \in V\} \subset N^3(x).$$

PROOF. Let $g: \omega \times X \to \tau$ satisfy conditions (i) and (ii) of Theorem 5.8 (i.e., g witnesses that X is semi-stratifiable). Clearly we may assume that $g(n+1, x) \subset g(n, x)$ and $g(0, x) \subset N(x)$.

Let $\mathscr{G}_k = \{g(k, x): x \in X\}$, and let $\mathscr{Q}_k$ be a point-finite refinement of $\mathscr{G}_k$. Let

$$H_k = \{x \in X: x \in g(k, y) \Rightarrow y \in N(x)\}.$$

By property 5.8(ii), each $x \in X$ is in some H_k; let $k(x)$ be the least $n \in \omega$ such that $x \in \bar{H}_n$. Let

$$Q_n(x) = \bigcap\{Q \in \mathscr{Q}_i: i \leq n, x \in Q\}, \quad \text{and} \quad V(x) = Q_{k(x)}(x) \setminus \bigcup_{i<k(x)} \bar{H}_i.$$

Then $\mathscr{V} = \{V(x): x \in X\}$ is easily seen to be a point-finite open cover of X, since $p \in V(x) \Rightarrow k(p) \leq k(x)$.

To complete the proof, we show that for each $x \in X$, there exist $y, z \in X$ such that $\bigcap\{V \in \mathscr{V}: x \in V\} \subset N(y)$, $y \in N(z)$, and $z \in N(x)$. Let $m = k(x)$. Since $x \in \bar{H}_m$, there exists $z \in H_m \cap g(m, x) \cap Q_m(x)$. We have $z \in g(m, x) \subset N(x)$. Since $\mathscr{Q}_m$ refines $\mathscr{G}_m$, $Q_m(x) \subset g(m, y) \subset N(y)$ for some $y \in X$. Thus

$$\bigcap\{V \in \mathscr{V}: x \in V\} \subset V(x) \subset Q_{k(x)}(x) = Q_m(x) \subset N(y).$$

It remains to show $y \in N(z)$. We have $z \in Q_m(x) \subset g(m, y)$. Since $z \in H_m$, then, $y \in N(z)$. $\square$

5.27. THEOREM (GRUENHAGE [1976]; JUNNILA [1978]). *A space X is stratifiable if and only if X is an M_2-space.*

PROOF. The proof that M_2-spaces are stratifiable is essentially the same as the proof for M_1-spaces. So assume X is stratifiable, and g satisfies conditions (i)–(iii) of Theorem 5.8, with $g(n+1, x) \subset g(n, x)$. Let $g^3(n, x)$ be as defined prior to Lemma 5.26.

We show that g^3 also satisfies (i)–(iii). To show that g^3 satisfies 5.8(ii), we need to prove that if $x \in g(n, x_n)$, $x_n \in g(n, y_n)$, and $y_n \in g(n, z_n)$, then $z_n \to x$. But we already showed that $y_n \to x$ in the proof of Theorem 4.11, (iv)$\Rightarrow$(v). Showing $z_n \to x$ is just a matter of repeating a similar argument. Now 5.8(i) follows easily from (ii) and the fact that $x \in g(n\, x)$. We leave the straightforward verification that g^3 satisfies 5.8(iii) to the reader.

Now for each n, let $\mathcal{V}_n$ be the point-finite open cover obtained by applying Lemma 5.16 to $N(x) = g(n, x)$. Let

$$g'(n, x) = \bigcap\{V \in \mathcal{V}_n : x \in V\}.$$

Since $g'(n, x) \subset g^3(n, x)$, it easily follows that g' also satisfies 5.8(i)–(iii), which are the same conditions as 5.25(i)–(iii). Now if $y \in g'(n, x)$, clearly $g'(n, y) \subset g'(n, x)$, so g' also satisfies 5.25(iv). Thus X is an M_2-space. □

6. Base of countable order and other monotonic properties

Recall that a sequence $(\mathcal{G}_n)$ of open covers of a space X is a development for X if the following holds:

(*) $\{\mathrm{st}(x, \mathcal{G}_n) : n \in \omega\}$ is a base at x for each $x \in X$.

It is easy to check that it would be the same to say:

(**) Whenever $x \in G_n \in \mathcal{G}_n$, $\{G_n : n \in \omega\}$ is a base at x.

Similarly, X has a G_δ-diagonal if there is a sequence $(\mathcal{G}_n)$ of open covers such that $x \in G_n \in \mathcal{G}_n$ implies $\{x\} = \bigcap_n G_n$. This leads to the following 'monotonic generalization' of these classes.

6.1. DEFINITION. A space X is said to have a *base of countable order* (resp., *W_δ-diagonal*) if there is a sequence $(\mathcal{B}_n)$ of bases for X such that:

(***) Whenever $x \in b_n \in \mathcal{B}_n$, and (b_n) is decreasing (by set inclusion), then $\{b_n : n \in \omega\}$ is a base at x (resp., $\{x\} = \bigcap b_n$)).

We use 'BCO' to abbreviate 'base of countable order'. Spaces having a BCO have also been called *monotonically developable* spaces.

To see that developable spaces and spaces having a G_δ-diagonal have a BCO and a W_δ-diagonal, respectively, simply let $\mathscr{B}_n$ be a base which is also a refinement of $\mathscr{G}_n$, where $(\mathscr{G}_n)$ is a development (resp., G_δ-diagonal sequence) for X.

The reader may have noticed that 'base of countable order' does not quite fit the definition. It turns out that if X has a base of countable order according to Definition 7.1, then there is a single base $\mathscr{B}$ such that whenever $x \in b_n \in \mathscr{B}$ and (b_n) is strictly decreasing, then $\{b_n : n \in \omega\}$ is a base at x. We never use this fact, so we omit the proof; but this is where the terminology arose. The term 'monotonically developable' is more descriptive, but not as common for historical reasons and perhaps because it suggests a property stronger than developability (e.g., recall that 'monotonically normal' is stronger than 'normal').

Spaces having a BCO were introduced by Arhangel'skii, and studied in depth by WICKE and WORRELL [1965]. While we restrict our attention here to BCO's and W_δ-diagonals, one can define similar 'monotonic generalizations' of some other classes of generalized metric spaces. For example, if in Definition 6.1, one replaces

"then $\{b_n : n \in \omega\}$ is a base at x"

with

"then $\{\bar{b}_n : n \in \omega\}$ is an outer network at the compact set $\bigcap_n \bar{b}_n$"

one has a 'monotonic generalization' of p-spaces (cf. Theorem 3.21).

A few special characteristics of these monotonic generalizations make them particularly useful. One is the "local implies global" characteristic (Theorem 6.5): If X possesses one of these properties locally, then the whole space has the property. In particular, any locally metrizable space has a BCO—the countable ordinals, for example, Another important characteristic is that if a space has one of these 'monotonic' properties and is submetacompact, then the space has the 'non-monotonic' property (Theorem 6.6). For example, a submetacompact space having a BCO is developable. Finally, these properties are preserved by open compact maps and by perfect maps. Since perfect maps also preserve submetacompactness, this can be combined with the above result to prove that certain classes of generalized metric spaces are preserved under perfect maps (e.g., developable spaces and submetacompact p-spaces).

The definition is not particularly good for actually working with spaces having a BCO or a W_δ-diagonal. For example, try proving that these properties are hereditary. WICKE and WORRELL [1965], [1971], [1972] used the ideas contained in their theory of *primitive sequences* to prove most of the results mentioned above. (See BURKE and LUTZER [1976] for a brief survey of this theory.) Later CHABER, ČOBAN, and NAGAMI [1974] introduced the notion of a 'sieve', which provided another convenient framework for the study of these monotonic properties. A sieve is essentially a tree of open sets of height ω with the following properties: the first level is just an open cover of the space, and each element of the tree is equal to the union of its immediate successors. Thus a branch (= a maximal chain) of the tree corresponds to a decreasing sequence of open sets.

6.2. DEFINITION. A *sieve* for a space (X, τ) is a pair (G, T), where T is a tree of height ω and $G: T \to \tau$ is a function such that

(i) $G(T_0) = \{G(t): t \in T_0\}$ covers X, where T_0 is the least level of T.

(ii) For each $t \in T$, $G(t) = \bigcup\{G(t'): t'$ is an immediate successor of $t\}$.

Note that, by (i) and (ii), $G(T_n)$ covers X for each n, where T_n is the nth level of T, and $t \leq t'$ implies $G(t) \supset G(t')$.

The following result essentially says that instead of having to consider all decreasing sequences taken from a sequence of bases, we only have to worry about the branches of a tree of open sets.

6.3. THEOREM. *The following are equivalent*:

(i) *X has a* BCO (*resp.*, *W_δ-diagonal*).

(ii) *X has a decreasing sequence $(\mathscr{B}_n)$ of bases satisfying the conditions of Definition* 6.1.

(iii) *X has a sieve (G, T) such that if b is a branch of T, and $x \in \bigcap_{t \in b} G(t)$, then $\{G(t): t \in b\}$ is a base at x (resp., $\{x\} = \bigcap_{t \in b} G(t)$).*

PROOF. We prove this for the case that X has a BCO; the other case is analogous. That (ii)$\Rightarrow$(i) is trivial. Let us prove (i)$\Rightarrow$(iii). Suppose $(\mathscr{B}_n)$ is a sequence of bases satisfying the conditions of Definition 7.1. Let $T_0 = \mathscr{B}_0$, and let $G \mid T_0$ be the identity. If T_n and $G \mid T_n$ has been defined, then define T_{n+1} by letting the followers of $t \in T_n$ be copies of those elements of $\mathscr{B}_{n+1}$ contained in $G(t)$, and define $G \mid T_{n+1}$ in the obvious way. Clearly (G, T) satisfies (iii).

It remains to prove (iii)$\Rightarrow$(ii). Let (G, T) satisfy (iii). Well-order each level of T so that if $s < t$ in T_n, then in T_{n+1}, the successors of s precede the successors of t. For each $x \in X$, let $t_n(x)$ be the least element t of T_n such that $x \in G(t)$. Note that $\{t_n(x): n \in \omega\}$ is a branch of T. Let $\mathscr{B}_n(x)$ be a base for the point x contained in $G(t_n(x))$, and let

$$\mathscr{B}_n = \bigcup\{\mathscr{B}_i(x): x \in X, i \geq n\}.$$

Then $(\mathscr{B}_n)$ is a decreasing sequence of bases for X.

Suppose $p \in b_n \in \mathscr{B}_n$, and (b_n) is decreasing. Now $b_n \in \mathscr{B}_{i(n)}(x_n)$ for some $x_n \in X$ and $i(n) \geq n$, so $b_n \subset G(t_{i(n)}(x_n)) \subset G(t_n(x_n))$. For each $k \in \omega$, let s_k be the least element s of T_k such that $G(s)$ contains some b_n. We will be done if we show that $\{s_k: k \in \omega\}$ is a branch of T, for then $\{G(s_k): k \in \omega\}$, and hence also $\{b_k: k \in \omega\}$, is a base at p. To this end, fix k and choose $m > k$ such that $b_m \subset G(s_k) \cap G(s_{k+1})$. Now $b_m \in \mathscr{B}_i(x)$ for some $x \in X$ and $i \geq m$. Thus $x \in b_m \subset G(t_i(x)) \subset G(t_{k+1}(x))$. Since $x \in G(s_{k+1})$, we must have $t_{k+1}(x) \leq s_{k+1}$. But $s_{k+1} \leq t_{k+1}(x)$ since $b_m \subset$

$G(t_{k+1}(x)) \cap G(s_{k+1})$. Thus $t_{k+1}(x) = s_{k+1}$; similarly, $t_k(x) = s_k$, so s_{k+1} is a successor of s_k. □

6.4. THEOREM. *The property of having a* BCO *(W_δ-diagonal) is hereditary and countably productive.*

PROOF. Let (G, T) be a sieve for X satisfying the conditions of Theorem 6.3(iii). Suppose $Y \subset X$. Then (G', T), where $G'(t) = G(t) \cap Y$, satisfies 6.3(iii) for Y.

To see that these properties are countably productive, suppose $(\mathscr{B}_{in})_{n\in\omega}$ satisfies Definition 6.1 for X_i. It is easy to check that $(\mathscr{B}_n)$ satisfies 6.1 for $\Pi_{i\in\omega} X_i$, where

$$\mathscr{B}_n = \left\{\prod_{i\leq n} B_i \times \prod_{j>n} X_j \colon B_i \in \mathscr{B}_{in},\ i \leq n\right\}. \qquad \square$$

With the sieve characterization, it is easy to prove the "local ⇒ global" property.

6.5. THEOREM. *If every point of X has a neighborhood which has a* BCO *(W_δ-diagonal), then X has a* BCO *(W_δ-diagonal).*

PROOF. Let $\mathscr{U}$ be an open cover of X such that each $U \in \mathscr{U}$ has a sieve $(G(U), T(U))$ satisfying 6.3(iii). We may assume the $T(U)$'s are pairwise disjoint. Let $T = \bigcup_{U\in\mathscr{U}} T(U)$, and $G = \bigcup_{U\in\mathscr{U}} G(U)$. Then (G, T) is a sieve for X satisfying 6.3(iii). □

For the two cases we have been considering, we now show that in a submetacompact space, having a monotonic property is equivalent to having the property itself. As a corollary, we have that a paracompact (submetacompact) locally developable space is metrizable (developable).

6.6. THEOREM. *Let X be a submetacompact space. If X has a* BCO *(W_δ-diagonal), then X is developable (has a G_δ-diagonal).*

PROOF. We prove this for the case that X has a BCO, the other case being entirely analogous. We present a modification of the original proof of WICKE and WORRELL [1965], as outlined in CHABER, COBAN, NAGAMI [1974].

Suppose $(\mathscr{B}_n)$ satisfies the conditions of Definition 6.1 for X. We define, by induction, an open cover $\mathscr{U}(s)$ of X for each finite sequence s of natural numbers. Assume $\mathscr{B}_0 = \{X\}$, and let $\mathscr{U}(\emptyset) = X$. If $s \in \omega^n$ and $\mathscr{U}(s)$ has been defined, let $\bigcup_{k\in\omega} \mathscr{U}_k(s)\}$ be a θ-refinement of

$$\{B \in \mathscr{B}_{n+1} \colon B \subset U \text{ for some } U \in \mathscr{U}(s)\},$$

and if $t \in \omega^{n+1}$ extends s, let $\mathscr{U}(t) = \mathscr{U}_{t(n)}(s)$.

We claim that $\{\mathcal{U}(s): s \in \omega^n, n \in \omega\}$ is a development for X. To see this, let $x \in X$. Let $s \in \omega^\omega$ be such that x is in only finitely many members of $\mathcal{U}(s \mid n))$ for each n. Let V be a neighborhood of X, and suppose that, for each n, $\mathrm{st}(x, \mathcal{U}(s \mid n)) \not\subset V$. Let $\mathcal{W}_n$ be all elements of $\mathcal{U}(s \mid n))$ which are not contained in V. For each $W \in \mathcal{W}_{n+1}$, there exist $B(W) \in \mathcal{B}_{n+1}$ and $U \in \mathcal{U}(s \mid n)$ with $W \subset B(W) \subset U$. (Note that $U \in \mathcal{W}_n$.) Let $\mathcal{B}'_n = \{B(W): W \in \mathcal{W}_n\}$, $n \geq 1$, and let $\mathcal{B}'_0 = \{X\}$. Then $\mathcal{B}'_n$ is finite, and it is easy to see that for each n, there is a sequence B_i, $i \leq n$, such that $B_i \in \mathcal{B}'_i$ and $B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_n$. By Konig's lemma, there is an infinite decreasing sequence (B_n) with $B_n \in \mathcal{B}'_n$. But $\{B_n: n \in \omega\}$ is then a base at x, so some B_n, hence some $W \in \mathcal{W}_n$, is contained in V, which is a contradiction. $\square$

7. Spaces with a point-countable base

One can obtain many generalizations of metric spaces by weakening the 'σ-locally finite base' condition in the Bing–Nagata–Smirnov theorem in various ways. We have already considered two such classes—spaces having a σ-locally-finite network (σ-spaces), and spaces having a σ-closure preserving base (M_1-spaces).

Other generalizations that come to mind are 'σ-point finite base', 'σ-locally countable base', and 'point-countable base'. Of these three classes, the latter class, which is also the most general, has played the biggest role in the theory of generalized metric spaces and metrization theory.

We begin with a classic characterization theorem. Recall that a map f is an *s-map* provided each point-inverse is separable.

7.1. THEOREM (PONOMAREV [1960]). *The following are equivalent*:

(i) *X has a point-countable base*;

(ii) *X is an open s-image of a metric space*;

(iii) *X is an open s-image of a space with a point-countable base.*

PROOF. That (ii)$\Rightarrow$(iii) is obvious. To see (iii)$\Rightarrow$(i), let $f: Y \to X$ be an open s-map, and let $\mathcal{B}$ be a point-countable base for Y. Since f is open, $f(\mathcal{B}) = \{f(B): B \in \mathcal{B}\}$ is a base for X. Since each $f^{-1}(y)$ is separable, only countably many members of $\mathcal{B}$ meet $f^{-1}(y)$. Hence $f(\mathcal{B})$ is point-countable.

It remains to prove (i)$\Rightarrow$(ii). Let $\mathcal{B}$ be a point-countable base for X, and let Y be the following subspace of $\mathcal{B}^\omega$, where the set $\mathcal{B}$ is given the discrete topology:

$$Y = \{(B_n) \in \mathcal{B}^\omega: \{B_n: n \in \omega\} \text{ is a base at some } x \in X\}.$$

Define $f: Y \to X$ by declaring $f((B_n))$ to be the point x at which the B_n's form a base.

Clearly Y is metrizable. Since $\mathscr{B}$ is point-countable, each $f^{-1}(x)$ is a subspace of a countable product of countable discrete spaces, hence is separable. If $\sigma: n \to \mathscr{B}$, then $[\sigma] = \{y \in Y: y \text{ extends } \sigma\}$ is a typical basic open set in Y. We see that $f([\sigma]) = \bigcap_{i<n} \sigma(i)$. Thus f is open. From the fact that the coordinates of each point of $f^{-1}(x)$ form a base at x, it is easy to see that f is continuous. $\square$

REMARK. We should mention the following in connection with Theorem 7.1:

(1) The images of metric spaces under open *compact* mappings (i.e. point-inverses are compact) are precisely the metacompact developable spaces (HANAI [1961], ARHANGEL'SKIĬ [1966]).

(2) The perfect image, or more generally, the bi-quotient s-image of a space with a point-countable base has a point-countable base (FILIPPOV [1968]; see also BURKE and MICHAEL [1972]). (A map $f: X \to Y$ is *bi-quotient* if whenever $y \in Y$ and $\mathscr{U}$ is an open cover of $f^{-1}(y)$, then there is a finite $\mathscr{U}' \subset \mathscr{U}$ such that $y \in f(\bigcup \mathscr{U}')^{\circ}$.)

Let us briefly consider some examples of non-metrizable spaces with a point-countable base. It is easy to see directly, or via the above remark, that any non-metrizable metacompact Moore space, e.g., the Pixley–Roy topology on the reals, is such an example. The Michael line (p. 430) is easily seen to be a non-developable space with a point-countable base. One can generate suborderable spaces with a point-countable base from trees of height $\leqslant \omega_1$. If T is a tree, let $B(T)$ denote the 'branch space' of T, i.e., $B(T)$ is the set of all branches (= maximal chains) of T, and for $t \in T$,

$$[t] = \{b \in B(T): t \in b\}$$

is a basic open set. (The class of all subspaces of these branch spaces coincides with the so-called *non-archimedean* spaces (cf. NYIKOS [1975]).) If every chain in T is countable, it is easy to see that the set of all such basic open sets is point-countable. If T is a Souslin tree, then $B(T)$ is hereditarily Lindelöf non-separable space with a point-countable base.

An interesting unsolved problem is whether there is a model of set theory in which every perfectly normal paracompact space with a point-countable base is metrizable. As noted above, if T is a Souslin tree, then $B(T)$ is an L-space (hereditarily Lindelöf, non-separable) example. Other L-space examples have been constructed assuming the continuum hypothesis (VAN DOUWEN, TALL, WEISS [1977]). But the problem is not completely solved even for non-archimedean spaces or linearly ordered spaces.

Even if 'paracompact' is omitted from the hypothesis, the problem remains unsolved. But in this case, non-metrizable normal metacompact Moore spaces are examples. By Fleissner's Moore space construction (see the Chapter by Fleissner), then, one sees that if there is a model of set theory in which every perfectly normal space with a point-countable base is metrizable, then there is an inner model with a measurable cardinal.

Metrization

Now we consider conditions under which spaces having a point-countable base are metrizable. First, it is easy to see that in a separable space, a point-countable base must be countable. In fact:

7.2. THEOREM. *If a locally separable space X has a point-countable base, then X is the disjoint union of clopen subspaces, each of which has a countable base.*

PROOF. Let $\mathcal{B}$ be a point-countable base for X; we may assume each $B \in \mathcal{B}$ is separable. Then each member of $\mathcal{B}$ meets only countably many others. Define $B \equiv B'$ if there is a sequence B_i, $i \leq n$, of elements of $\mathcal{B}$ such that $B = B_0$, $B_n = B'$, and $B_i \cap B_{i+1} \neq \emptyset$ for each $i < n$. It is easy to check that the following hold:

(i) "$\equiv$" is an equivalence relation;

(ii) the equivalence class $[B]$ of each $B \in \mathcal{B}$ is countable;

(iii) $\{\bigcup[B]: B \in \mathcal{B}\}$ is a partition of X.

It follows that each $\bigcup[B]$ is clopen and separable, hence has a countable base. □

It is a now classic result of MIŠČENKO [1962] that a compact space with a point-countable base is metrizable. But 'base' is not necessary in this result; all we need is a point-countable collection which separates points in the T_1-sense.

7.3. DEFINITION. A cover $\mathcal{U}$ of a space X is *T_1-separating* if whenever x and y are two points of X, we have $x \in U \subset X \setminus \{y\}$ for some $U \in \mathcal{U}$.

Note that any space which admits a weaker topology having a point-countable base (e.g., a submetrizable space) has a point-countable T_1-separating open cover.

We aim to prove that every countably compact space with a point-countable T_1-separating open cover is metrizable. We will need Miščenko's lemma; for a proof of this lemma, see the Chapter by Hodel.

7.4. LEMMA (MIŠČENKO [1962]). *Let $\mathcal{U}$ be a point-countable cover of a set X. Then $\mathcal{U}$ contains only countably many minimal finite subcovers of X.* (A cover $\mathcal{V}$ of X is *minimal* if no proper subset of $\mathcal{V}$ covers X.)

We will also use the following simple but useful fact:

7.5. LEMMA. *Every point-countable open cover of an ω_1-compact (= every closed discrete subset has cardinality $<\omega_1$) space has a countable subcover.*

PROOF. Let $\mathcal{U}$ be a point-countable open cover of the ω_1-compact space X. Inductively define $\{x_\alpha: \alpha < \kappa\}$ such that $x_\alpha \notin \bigcup_{\beta<\alpha} \operatorname{st}(x_\beta, \mathcal{U})$ and $X = \bigcup_{\alpha<\kappa} \operatorname{st}(x_\alpha, \mathcal{U})$. Then $\{x_\alpha: \alpha < \kappa\}$ is closed discrete, and so $|\kappa| = \omega$. Thus $\{U \in \mathcal{U}$: there exists $\alpha < \kappa$ with $x_\alpha \in U\}$ is a countable subcover. □

7.6. THEOREM. *Suppose X is a countably compact space having a point-countable T_1-separating open cover. Then X is a compact metric space.*

PROOF. We show that X^2 has a G_δ-diagonal. The result then follows from Theorem 2.14. First observe that X^2 has a point-countable T_1-separating open cover; call it $\mathcal{U}$. Let Δ denote the diagonal of X^2, and let

$$\mathcal{V} = \{\bigcup \mathcal{U}' : \mathcal{U}' \subset \mathcal{U} \text{ is a minimal finite cover of } \Delta\}.$$

By Miščenko's lemma, $\mathcal{V}$ is countable. So it remains to prove that $\Delta = \bigcap \mathcal{V}$. Suppose $p \in X^2 \backslash \Delta$. For each $x \in \Delta$, choose $U_x \in \mathcal{U}$ with $x \in U_x \subset X^2 \backslash \{p\}$. Then $\{U_x : x \in \Delta\}$ is a point-countable open cover of Δ. Now Δ is homeomorphic to X, hence countably compact, so by Lemma 7.5, $\{U_x : x \in \Delta\}$ contains a countable, and thus a minimal finite, subcover $\mathcal{U}'$. Then $\Delta \subset \bigcup \mathcal{U}' \subset X^2 \backslash \{p\}$. □

There are many theorems of the kind which say that if a member of some class of generalized metric spaces has a point-countable base or a T_1-separating open cover, then it is metrizable or at least developable. We can obtain most of these results easily from a couple of results involving the class of 'β-spaces', introduced by HODEL [1972].

7.7. DEFINITION. A space (X, τ) is a *β-space* if there is a function $g : \omega \times X \to \tau$ such that

(i) $x \in g(n, x)$;

(ii) if $x \in g(n, x_n)$, then the set $\{x_n : n \in \omega\}$ has a cluster point in X.

The β-spaces are quite general: almost every class of spaces introduced in Sections 3, 4 and 5 is a subclass of the β-spaces. Note that if we had required $x_n \to x$ in part (ii) of Definition 7.7, then by Theorem 5.8 we would have defined semi-stratifiable spaces. In fact, by a proof similar to that of Theorem 3.3, one can show that semi-stratifiable spaces are precisely those β-spaces which have a G^*_δ-diagonal. In any case, we see that all semi-stratifiable spaces, hence all stratifiable spaces, Moore spaces, and σ-spaces, are β-spaces.

Besides the above classes, all $w\Delta$-spaces, hence all countably compact spaces, M-spaces, and strict p-spaces, are β-spaces. To see this, let $g(n, x)$ be some member of $\mathcal{G}_n$ containing x, where $(\mathcal{G}_n)$ is a "$w\Delta$-sequence" for X. Then $x \in g(n, x_n)$ implies $x_n \in \mathrm{st}(x, \mathcal{G}_n)$, so $\{x_n : n \in \omega\}$ has a cluster point.

We can also show that every Σ-space, in fact every $\Sigma^\#$-space, is a β-space. Define $g(n, x) = X \backslash \bigcup \{F \in \mathcal{F}_i : x \notin F, i \leq n\}$, where $\mathcal{F} = \bigcup_n \mathcal{F}_n$ satisfies the $\Sigma^\#$-space property with respect to a cover $\mathcal{C}$ of X by closed countably compact subsets. If $x \in C \in \mathcal{C}$, and $x \in g(n, x_n)$, one can show $\{x_n : n \in \omega\}$ clusters at some point of C. The proof is the same as the analogous result for σ-spaces: see Theorem 4.11, (iii)$\Rightarrow$(iv).

It should also be mentioned that every β-space is countably metacompact (i.e., every countable open cover has a point-finite open refinement; equivalently, whenever (H_n) is a decreasing sequence of closed sets with empty intersection, there exist open sets $U_n \supset H_n$ with $\bigcap_n U_n = \emptyset$). To see this, if (H_n) is a decreasing sequence of closed subsets of X with empty intersection, and $g: \omega \times X \to \tau$ satisfies 7.7, let $U_n = \bigcup\{g(n, x): x \in H_n\}$. It is not difficult to check that $\bigcap_n U_n = \emptyset$.

The following theorem summarizes what we have so far.

7.8. THEOREM. (i) *The following are β-spaces: $w\Delta$-spaces (in particular, Moore spaces, countably compact spaces, M-spaces, and strict p-spaces), semi-stratifiable and stratifiable spaces, σ-spaces and Σ-spaces.*

(ii) *A space X is semi-stratifiable if and only if X is a β-space with a G_δ^*-diagonal.*

(iii) *Every β-space is countably metacompact.*

We now prove a quite general and very useful result on β-spaces having a point-countable base or a point-countable T_1-separating open cover. The proof, due to CHABER [1977], is a nice application of the monotonic generalizations of developable spaces and spaces having a G_δ-diagonal studied in the previous section.

7.9. THEOREM. *Let X be a submetacompact β-space.*

(i) *If X has a point-countable T_1-separating open cover, then X has a G_δ-diagonal.*

(ii) *If X has a point-countable base, then X is developable.*

PROOF. We prove (ii); part (i) is analogous. Let $\mathcal{B}$ be a point-countable base for X. We will show that X has a BCO; then since X is submetacompact, X is developable by Theorem 6.6. (This is the only place submetacompactness is used.)

For each $x \in X$, let $(B_n(x))$ be a fixed enumeration of $\{B \in \mathcal{B}: x \in B\}$. Let $g: \omega \times X \to \tau$ satisfy Definition 7.7. Let $T_0 = X$, and for each $x \in T_0$, let $G(x) = g(0, x) \cap B_0(x)$.

Inductively define $T_n \subset X^{n+1}$ and $G: T_n \to \tau$ such that for each $n \geq 1$,

(i) $T_n = \{(x_0, \ldots, x_n): t = (x_0, \ldots, x_{n-1}) \in T_{n-1} \quad \text{and} \quad x_n \in G(t)\}$;

(ii) $G((x_0, \ldots, x_n)) = G((x_0, \ldots, x_{n-1})) \cap g(n, x_n) \cap (\bigcap\{B_i(x_j): i, j \leq n$ and $x_n \in B_i(x_j)\})$.

Let $T = \bigcup_n T_n$, and define $s \leq t$ if s is an initial segment of t. Clearly (G, T) is a sieve for X.

Suppose $b = \{t_n: n \in \omega\}$ is a branch of T with $t_n \in T_n$, and $p \in \bigcap_n G(t_n)$. Then there is a sequence (x_n) such that $t_n = (x_0, \ldots, x_n)$. Since $p \in G(t_n) \subset g(n, x_n)$, the set $\{x_n: n \in \omega\}$ has a cluster point, say x. Let $x \in B \in \mathcal{B}$. There exist $k, i \in \omega$ such that $x_k \in B$ and $B = B_i(x_k)$. Choose $n \geq i+k$ such that $x_n \in B$. Then $p \in G((x_0, \ldots, x_n)) \subset B_i(x_k) = B$. Since B was an arbitrary element of $\mathscr{B}$ containing x, we must have $p = x$ and $\{G(t_n): n \in \omega\}$ is a base at p. Thus X has a BCO by Theorem 6.3. □

7.10. COROLLARY. *Suppose X has a point-countable T_1-separating open cover. Then*:

(i) *X is developable if and only if X is submetacompact and a $w\Delta$-space or a p-space.*

(ii) *X is metrizable if and only if X is an M-space.*

(iii) *X is a σ-space if and only if X is a Σ-space.*

PROOF. By Theorem 7.9, in each case X has a G_δ-diagonal. The corollary now follows from previous results: Corollary 3.4 and Theorem 3.19 for (i), Corollary 3.8 for (ii), and Theorem 4.15 for (iii). □

Note that 7.10(ii) generalizes Theorem 7.6.

7.11. COROLLARY. *Suppose X has a point-countable base. Then*:

(i) *X is developable if and only if X is semi-stratifiable space or a Σ-space.*

(ii) *X is metrizable if and only if X is a stratifiable space or an M-space.*

PROOF. (i) We need to show that a semi-stratifiable space or a Σ-space with a point-countable base is submetacompact, for then Theorem 7.9 applies. But every semi-stratifiable space is subparacompact (Theorem 5.11), and a Σ-space with a point-countable base is subparacompact by Corollary 7.10(iii).

(ii) For M-spaces, this follows from 7.10(ii). For stratifiable spaces, it follows from 7.11(i) and the fact that stratifiable spaces are paracompact. □

In the next section, we show that in Theorem 7.9(ii) and Corollary 7.11, 'point-countable base' can be replaced by '$\delta\theta$-base', a notion more general than both point-countable base and developability.

8. Quasi-developable spaces, θ-bases, and $\delta\theta$-bases

8.1. DEFINITION. A collection $\mathscr{B} = \bigcup_n \mathscr{B}_n$ of open subsets of a space X is a *θ-base* (resp., *$\delta\theta$-base*) if whenever $x \in U$ with U open, there exist $n \in \omega$ and $B \in \mathscr{B}_n$ such that

(i) $1 \leq \operatorname{ord}(x, \mathscr{B}_n) < \omega$ (resp., $1 \leq \operatorname{ord}(x, \mathscr{B}_n) \leq \omega$);

(ii) $x \in B \subset U$.

The notion of a θ-base is due to WICKE and WORRELL [1965], while AULL [1974] was the first to consider $\delta\theta$-bases. One might want to consider $\delta\theta$-bases because many theorems concerning point-countable bases are valid for this wider class, and, while not every Moore space has a point-countable base (consider any separable non-metrizable Moore space), every Moore space has a $\delta\theta$-base, in fact, a θ-base (just take a θ-refinement of each member of a development).

We begin by showing that Theorem 7.9(ii) and Corollary 7.11 are valid with 'point-countable base' replaced by '$\delta\theta$-base'.

8.2. THEOREM (CHABER [1977]). *A submetacompact β-space with a $\delta\theta$-base is developable.*

PROOF. We show that a β-space with a $\delta\theta$-base has a BCO. The result then follows from Theorem 6.6. Let $\mathscr{B} = \bigcup_n \mathscr{B}_n$ be a $\delta\theta$-base for X. Note that X is first-countable. Let

$$F_n = \{x \in X : \operatorname{ord}(x, \mathscr{B}_n) \leq \omega\}.$$

Then F_n is closed, because by the first-countability, if $\operatorname{ord}(x, \mathscr{B}_n) > \omega$, there exists an open set U containing x such that

$$|\{B \in \mathscr{B}_n : B \supset U\}| > \omega.$$

For each $x \in X$, let $N(x) = \{j \in \omega : 1 \leq \operatorname{ord}(x, \mathscr{B}_j) \leq \omega\}$, and let $(B_n(x))$ be an enumeration of $\{B \in \mathscr{B}_j : j \in N(x), x \in B\}$. Construct a sieve (G, T) with $T_n \subset X^{n+1}$ satisfying the same conditions (i) and (ii) as in the proof of Theorem 7.9, as well as

(iii) $\overline{G((x_0, \ldots, x_n))} \subset \bigcap\{X \setminus F_j : j \leq n \text{ and } x_n \notin F_j\}$.

Suppose $b = \{t_n : n \in \omega\}$ is a branch of T with $t_n = (x_0, \ldots, x_n)$ and $p \in \bigcap G(t_n)$. The proof that $\{G(t_n) : n \in \omega\}$ is a base at p is similar to the proof of 7.9. Let x be a cluster point of $\{x_n : n \in \omega\}$. Note that $x \in \bigcap_n \overline{G(t_n)}$. Let $x \in B \in \mathscr{B}_j$, where $j \in N(x)$. By (iii), $x_n \in F_j$ for $n \geq j$. By the same argument as in 7.9, there exists $n \geq j$ such that $G(t_n) \subset B$, and so $p = x$ and $\{G(t_n) : n \in \omega\}$ is a base at p. Hence X has a BCO. □

The following corollary generalizes Corollary 7.11. The proof follows from Theorem 8.2 in the same way that 7.11 follows from Theorem 7.9.

8.3. COROLLARY. *Suppose X has a $\delta\theta$-base. Then*

(i) *X is developable if and only if X is a semi-stratifiable space or a Σ-space.*

(ii) *X is metrizable if and only if X is stratifiable or an M-space.*

Spaces having a θ-base take on further interest in light of a characterization which we now consider. The following definition is due to BENNETT [1971].

8.4. DEFINITION. A space X is *quasi-developable* if there exists a sequence $(\mathcal{G}_n)$ of families of open subsets of X such that for each $x \in X$, $\{\mathrm{st}(x, \mathcal{G}_n): n \in \omega\}$ is a base at x.

This definition is the same as that of developable spaces, except that the $\mathcal{G}_n$'s do not have to cover the space. It turns out that quasi-developable spaces are precisely the same as spaces having a θ-base.

8.5. THEOREM (BENNET, LUTZER [1972]). *The following are equivalent for a space X:*

(i) *X has a θ-base;*

(ii) *X is quasi-developable.*

PROOF. (i)$\Rightarrow$(ii). Let $\mathcal{B} = \bigcup_n \mathcal{B}_n$ be a θ-base for X. Let $\mathcal{B}_{nm}$ be the set of all intersections of m distinct elements of $\mathcal{B}_n$. Let $\{\mathcal{B}_{nm}: n, m \in \omega\} = \{\mathcal{G}_n: n \in \omega\}$. We claim that $(\mathcal{G}_n)$ is a quasi-development for X. Suppose $p \in U$ with U open. Let $n \in \omega$ and $B \in \mathcal{B}_n$ be such that $p \in B \subset U$ and $\mathrm{ord}(p, \mathcal{B}_n) = k < \omega$. Then p is in precisely one element W of $\mathcal{B}_{nk}$. Thus $\mathrm{st}(p, \mathcal{B}_{nk}) = W \subset B \subset U$.

(ii)$\Rightarrow$(i). Suppose $(\mathcal{G}_n)$ is a quasi-development for X. It is easy to see that every subspace of X is quasi-developable. Thus if we can prove that quasi-developable spaces are weakly submetacompact, we can obtain a θ-base for X by taking a weak θ-refinement of each $\mathcal{G}_n$.

We first prove that X has a network $\mathcal{F} = \bigcup_n \mathcal{F}_n$ such that each $\mathcal{F}_n$ is relatively discrete, i.e., $\mathcal{F}_n$ is closed discrete in $\bigcup \mathcal{F}_n$. To this end, well-order each $\mathcal{G}_n$, and for each $n, m \in \omega$ and $G \in \mathcal{G}_n$, let

$$G_m = G \cap (\bigcup \mathcal{G}_m) \setminus [(\bigcup\{G' \in \mathcal{G}_n: G' < G\}) \cup (\bigcup\{G'' \in \mathcal{G}_m: G'' \not\subset G\})].$$

Let $\mathcal{F}_{nm} = \{G_m: G \in \mathcal{G}_n\}$ and $\mathcal{F} = \bigcup_{n,m} \mathcal{F}_{nm}$.

We show that $\mathcal{F}$ is a network for X. Suppose $x \in U$ with U open. Choose n such that $\mathrm{st}(x, \mathcal{G}_n) \subset U$, and let G be the least element of $\mathcal{G}_n$ containing x. Choose m such that $\mathrm{st}(x, \mathcal{G}_m) \subset G$. It is easy to check that $x \in G_m \subset U$.

We now prove that $\mathcal{F}_{nm}$ is relatively discrete. Suppose $G \in \mathcal{G}_n$ and $x \in G_m \in \mathcal{F}_{nm}$. Clearly G is the least member of $\mathcal{G}_n$ containing x, and $G \cap G'_m = \emptyset$ if $G < G' \in \mathcal{G}_n$. Since $x \in G_m$, there exists $H \in \mathcal{G}_m$ with $x \in H$. Then $H \not\subset G''$ if $G > G'' \in \mathcal{G}_n$, and so $H \cap G''_m = \emptyset$. Thus $H \cap G$ is an open set containing x which meets only one member of $\mathcal{F}_{nm}$, namely G_m. Hence $\mathcal{F}_{nm}$ is relatively discrete.

Finally, we can see that X is weakly submetacompact. Let $\mathcal{U}$ be an open cover of X, and for each $F \in \mathcal{F}_{nm}$ with $F \subset U$ for some $U \in \mathcal{U}$, choose an open set V_F with $F \subset V_F \subset U$, and such that $V_F \cap (\bigcup \mathcal{F}_{nm}) = F$. Let $\mathcal{V}_{nm} = \{V_F: F \in \mathcal{F}_{nm}\}$. Then $\{\mathcal{V}_{nm}: n, m \in \omega\}$ is a weak θ-refinement of $\mathcal{U}$. $\square$

Quasi-developments are often easier to work with than θ-bases, particularly if one is used to working with developable spaces. For example, the following result is easy to prove.

8.6. THEOREM. *A space X is developable if and only if X is perfect (= closed sets are G_δ-sets) and quasi-developable.*

PROOF. Suppose $(\mathscr{G}_n)$ is a quasi-development for X. For each n, let $\bigcup\mathscr{G}_n = \bigcup_m F_{nm}$, where each F_{nm} is closed. Let $\mathscr{G}_{nm} = \mathscr{G}_n \cup \{X \setminus F_{nm}\}$, and let $(\mathscr{G}'_n)$ be an enumeration of $\{\mathscr{G}_{nm}: n, m \in \omega\}$. Then $(\mathscr{G}'_n)$ is easily seen to be a development for X. □

9. Semi-metrizable and symmetrizable spaces

In this section and the next, we study generalizations of metric spaces whose topology can be described in terms of a "distance function" which is not quite a metric. Distance functions of various kinds arise naturally in many settings. For example, there is a natural distance function on a quotient space X of a metric space using the distance between point-inverses in the metric space as a measure of the distance between points of X. One can also use a sequence $(\mathscr{G}_n)$ of open covers satisfying certain conditions to define distance functions on developable and other spaces (see Lemma 2.7, for example).

In this section, we look at two classes of spaces which are defined in terms of a distance function which has the usual properties of a metric except for the triangle inequality.

9.1. DEFINITION. A function $d: X \times X \to \mathbb{R}^+$ (the non-negative reals) is a *symmetric* on the set X if, for each $x, y \in X$,

(i) $d(x, y) = 0 \Rightarrow x = y$;
(ii) $d(x, y) = d(y, x)$.

Because we do not have the triangle inequality, the usual ε-balls

$$B(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\}$$

do not necessarily form a base for a topology: if $y \in B(x, \varepsilon)$, then without the triangle inequality or some other conditions (such as continuity of d), we cannot show that there exists $\delta > 0$ with $B(y, \delta) \subset B(x, \varepsilon)$. But there are other ways to relate the ε-balls to a topology.

9.2. DEFINITION. A space X is said to be *symmetrizable* if there is a symmetric d on X satisfying the following condition: $U \subset X$ is open if and only if for each $x \in U$, there exists $\varepsilon > 0$ with $B(x, \varepsilon) \subset U$.

Note that by the above discussion, the ε-balls themselves are not necessarily open, so this is *not* the same as saying: "U is open if and only if U is a union of ε-balls".

It is useful to note the following complementary version of 9.2. A space X is symmetrizable if there is a symmetric d on X satisfying the following condition: $H \subset X$ is closed if and only if $d(x, H) > 0$ for each $x \notin H$.

The following lemma further clarifies the relation of the ε-balls and the symmetric d to the topology of a symmetrizable space.

9.3. LEMMA. *Suppose X is symmetrizable with respect to the symmetric d. Then the following are equivalent*:

(i) $x_n \to x$;

(ii) $d(x_n, x) \to 0$;

(iii) $\{x_n\colon n \in \omega)\backslash B(x, \varepsilon)$ *is finite for each* $\varepsilon > 0$.

PROOF. It is easy to see (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i). Assume (i), and suppose (iii) does not hold. Then the set $A = \{x_n\colon n \in \omega\}\backslash B(x, \delta)$ is infinite for some $\delta > 0$. Since $\{x\} \cup A$ is closed, for each point $p \notin \{x\} \cup A$, we have $B(p, \varepsilon(p)) \subset X\backslash A$ for some $\varepsilon(p) > 0$. Since $B(x, \delta) \subset X\backslash A$, the set $X\backslash A$ is open by Definition 9.2, a contradiction. Thus we have (i)$\Rightarrow$(iii). $\square$

Suppose X is symmetrizable with respect to d, and $H \subset X$ is not closed. The for some $x \notin H$, we have $d(x, H) = 0$. From the above lemma, we see that there exists $x_n \in H$ with $x_n \to x \notin H$. It follows that every symmetrizable space is *sequential*, i.e., a set A is closed if and only if A contains the limit point of every convergent sequence in A.

An example is in order. Let X be the quotient space of the real line $\mathbb{R}$ obtained by identifying n with $1/n$ for each positive integer n. Let f be the quotient map. Then for each n, the images under f of sets of the form

$$(1/n - \varepsilon, 1/n + \varepsilon) \cup (n - \varepsilon, n + \varepsilon), \quad \varepsilon > 0,$$

are a base at the point $\{1/n, n\}$. To get a base at 0, we can take images under f of sets of the form

$$(-\varepsilon, \varepsilon) \cup \Big(\bigcup_{n > 1/\varepsilon} (n - \varepsilon(n), n + \varepsilon(n))\Big).$$

Since the choice of the $\varepsilon(n)$'s is arbitrary, we see that X is not first-countable at 0.

The space X turns out to be symmetrizable. To see this, for each $x, y \in X$ define $d(x, y) = \rho(f^{-1}(x), f^{-1}(y))$, where ρ is the usual metric on $\mathbb{R}$. Then an ε-ball about $\{1/n, n\}$ is just the typical basic neighborhood of $\{1/n, n\}$ described above, and $B(0, \varepsilon)$ is the image under f of $(-\varepsilon, \varepsilon)$. With this and the description of the basic open sets given above, it is easy to check that X is symmetrizable with respect to d. Note that $B(0, \varepsilon)$ does not contain 0 in its interior, since every point of the form $\{1/n, n\}$, $n > 1/\varepsilon$, is in $B(0, \varepsilon)$ but not in $B(0, \varepsilon)^{\circ}$.

The following theorem shows that the above example is just a special case of a more general construction.

9.4. THEOREM. *Let M be a metric space, with metric ρ, and let $f: M \to X$ be a quotient map. Then X is symmetrizable with respect to*

$$d(x, y) = \rho(f^{-1}(x), f^{-1}(y)) \quad \textit{if and only if} \quad \rho(f^{-1}(x), f^{-1}(H)) > 0$$

for each closed set $H \subset X$ and $x \notin H$. (In particular, the condition holds if each point-inverse is compact.)

PROOF. The condition $\rho(f^{-1}(x), f^{-1}(H)) > 0$ is equivalent to $d(x, H) > 0$. Thus if X is symmetrizable with respect to d, then by the dual version of Definition 9.2, it follows that $\rho(f^{-1}(x), f^{-1}(H)) > 0$ for each closed set H and $x \notin H$.

Suppose the condition on ρ holds. If U is open in X and $x \in U$, then $d(x, X \backslash U) > 0$, so $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$. Suppose $V \subset X$ is such that whenever $y \in V$, $B(y, \varepsilon) \subset V$ for some $\varepsilon > 0$. We need to show V is open in X, or, equivalently, $f^{-1}(V)$ is open in M. Let $z \in f^{-1}(V)$, and let ε be such that $B(f(z), \varepsilon) \subset V$. Then $\rho(z, X \backslash f^{-1}(V)) \geqslant \varepsilon$. Hence $z \in f^{-1}(V)^{\circ}$, so $f^{-1}(V)$ is open. □

We saw that our example of a symmetrizable space given above was not first countable. An interesting thing happens when first-countability is added: the point x is always in the interior of $B(x, \varepsilon)$. To see this, suppose $x \notin B(x, \varepsilon)^{\circ}$. Then there exist $x_n \to x$ with $x_n \notin B(x, \varepsilon)$. (Choose x_n in the nth member of a decreasing countable base for x.) But $d(x_n, x) \not\to 0$, contradicting Lemma 9.3. The proof goes through just the same if X is assumed to be a *Fréchet* space, i.e., $x \in \bar{A}$ iff there exist $x_n \in A$ with $x_n \to x$.

Now if $x \in B(x, \varepsilon)^{\circ}$, then the $B(x, \varepsilon)$'s form a neighborhood base at x.

9.5. DEFINITION. A space X is said to be *semi-metrizable* if there is a symmetric d on X such that for each $x \in X$, $\{B(x, \varepsilon): \varepsilon > 0\}$ forms a (not necessarily open) neighborhood base at x.

By the above discussion, we have:

9.6. THEOREM. *The following are equivalent*:

(i) *X is semi-metrizable*;

(ii) *X is symmetrizable and first countable*;

(iii) *X is symmetrizable and Fréchet.*

There is another useful way of illustrating the difference between symmetrizable and semi-metrizable spaces. In a semi-metric space, a point p is in the closure of a set H if and only if $B(p, \varepsilon) \cap H \neq \emptyset$ for each $\varepsilon > 0$; i.e., $d(p, H) = 0$. This is not necessarily the case in a symmetrizable space. In the example preceding

Theorem 9.4, 0 is in the clousre of $[1,\infty)\backslash\mathbb{N}$, but $d(0,[1,\infty)\backslash\mathbb{N})=1$. The situation is precisely as follows:

9.7. THEOREM. *Let X be a space, and let d be a symmetric on the set X. Then*:

(i) *X is semi-metrizable with respect to d if and only if for each $p\in X$ and $H\subset X$,*

$$d(p,H)=0 \quad\Leftrightarrow\quad p\in\bar{H}\,;$$

(ii) *X is symmetrizable with respect to d if and only if for each $H\subset X$,*

$$H \text{ is closed} \quad\Leftrightarrow\quad d(p,H)>0 \text{ for each } p\notin H.$$

PROOF. Part (ii) is just the dual version of Definition 9.2, and we have already noted the 'only if' part of (i). So suppose d satisfies the conditions of part (i). Then for each $p\in X$ and $\varepsilon>0$, $d(p,X\backslash B(p,\varepsilon))>0$, so $p\notin\overline{X\backslash B(p,\varepsilon)}$; hence $p\in B(p,\varepsilon)^\circ$. □

Theorem 9.6 factors semi-metrizability into first countability and symmetrizability. Curiously, there is another quite different factorization of semi-metrizability involving first countability. It is easy to see that each semi-metrizable space is semi-stratifiable (Definition 5.6): let $G(n,H)=\{y\colon d(y,H)<1/2^n\}^\circ$. Since $x\in B(x,1/n)^\circ$, $H\subset G(n,H)$. Clearly $H=\bigcap_n G(n,H)$, and $H\subset K$ implies $G(n,H)\subset G(n,K)$. Thus the function G satisfies the necessary conditions. So semi-metrizable spaces are first countable and semi-stratifiable. The reverse is also true:

9.8. THEOREM (CREEDE [1970]). *A space X is semi-metrizable if and only if X is semi-stratifiable and first countable.*

PROOF. By the above discussion, we only need to prove the sufficiency. Suppose (X,τ) is first countable and semi-stratifiable, with $g\colon\omega\times X\to\tau$ satisfying the conditions of Theorem 5.8. For each $x\in X$, let $\{b(n,x)\colon n\in\omega\}$ be a decreasing neighborhood base at x. Let $h(n,x)=b(n,x)\cap g(n,x)$. Define $d(x,x)=0$, and for $x\neq y$, define

$$d(x,y)=\sup\{1/2^n\colon x\notin h(n,y) \text{ and } y\notin h(n,x)\}\,.$$

Clearly d is a symmetric on the set X.

Note that $y\in h(n,x)$ implies $d(x,y)<1/2^n$; hence $h(n,x)\subset B(x,1/2^n)^\circ$. It remains to prove that $\{B(x,1/2^n)\colon n\in\omega\}$ is a base at x. If not, there exists an open set U containing x and $y_n\in B(x,1/2^n)\backslash U$. Since $d(x,y_n)<1/2^n$, either $y_n\in h(n,x)\subset b(n,x)$ or $x\in h(n,y_n)\subset g(n,y_n)$. No matter which case occurs infinitely often, some subsequences of the y_n's converges to x, which is a contradiction. □

Recalling that Moore spaces are semi-stratifiable, we see from the above result that the class of semi-metrizable spaces fits between Moore spaces and semi-stratifiable spaces, and that semi-metrizable spaces are therefore subparacompact, perfect, and have a G_δ-diagonal. The following example shows that symmetrizable spaces need not have any of these properties.

9.9. EXAMPLE. A symmetrizable space X which does not have a G_δ-diagonal, and is not perfect or submetacompact.

PROOF. Let $\mathfrak{c} = 2^\omega$. The set X is $\mathfrak{c} \cup B(\mathfrak{c})$, where $B(\mathfrak{c}) = \mathfrak{c}^\omega$ is the Baire space of weight $\mathfrak{c}$. Points of $B(\mathfrak{c})$ have their usual neighborhoods, i.e., basic open sets are of the form

$$[\sigma] = \{f \in B(\mathfrak{c}) \colon \sigma \subset f\} \quad \text{where } \sigma \colon n \to \mathfrak{c}$$

The set $\mathfrak{c}$ is going to be a closed discrete subset of X which is not a G_δ-set. We inductively define basic neighborhoods of the elements of $\mathfrak{c}$. First, for $n \geq 1$, let $\mathcal{S}(n)$ be the set of all countable subsets S of $\mathfrak{c}^n$ such that $s, s' \in S$ implies $s|_1 \neq s'|_1$. Let $\mathcal{S} = \bigcup_{n=1}^\infty \mathcal{S}(n) = \{S_\alpha \colon \alpha < \mathfrak{c}\}$, and let $k(\alpha)$ be such that $S_\alpha \subset \mathfrak{c}^{k(\alpha)}$.

We inductively define, for each $\alpha < \mathfrak{c}$, points $f_{\alpha,n} \in B(\mathfrak{c})$ and functions $\sigma_{\alpha,n} \in \bigcup_k \mathfrak{c}^k$ such that

(i) $f_{\alpha,n} \in [\sigma_{\alpha,n}] \subset [\sigma]$ for some $\sigma \in S_\alpha$;
(ii) if $i \neq j$, $f_{\alpha,i}|_1 \neq f_{\alpha,j}|_1$;
(iii) $\sigma_{\alpha,n} \in \bigcup_{i=n+k(\alpha)}^\infty \mathfrak{c}^i$; and
(iv) if $\beta < \alpha$, and $m, n \in \omega$, then $f_{\beta,m} \notin [\sigma_{\alpha,n}]$, and if $\sigma_{\beta,m} \in \bigcup_{i=k(\alpha)+1}^\infty \mathfrak{c}^i$, then $[\sigma_{\alpha,n}] \cap [\sigma_{\beta,m}] = \emptyset$.

That this construction can be carried out follows easily from the fact that at stage α, fewer than $\mathfrak{c}$ $f_{\beta,m}$'s and $\sigma_{\beta,m}$'s have been chosen.

Now if $f, g \in B(\mathfrak{c})$, define $d(f, g) = \Sigma_{n=0}^\infty \delta(n)/2^n$, where $\delta(n) = 0$ if $f(n) = g(n)$ and $\delta(n) = 1$ otherwise (this is a standard metric on $B(\mathfrak{c})$.) Now extent d to all of X as follows:

$$d(\alpha, f_{\alpha,n}) = d(f_{\alpha,n}, \alpha) = 1/2^n\,; \qquad d(\alpha, \alpha) = 0\,;$$

$$d(\alpha, y) = d(y, \alpha) = 2 \text{ otherwise}\,.$$

The topology on X is then the topology generated by this symmetric according to Definition 9.2. It is easy to see that points of $B(\mathfrak{c})$ have their usual neighborhoods, and a neighborhood of a point $\alpha \in \mathfrak{c}$ is a tail of the sequence $(f_{\alpha,n})$, together with arbitrarily chosen neighborhoods in $B(\mathfrak{c})$ of each point of this tail. Then the regularity of X follows from the fact that

$$\{f_{\alpha,n} \colon n \in \omega\} \cap (\bigcup\{[\sigma_{\beta,m}] \colon m \in \omega\})$$

is finite for $\alpha \neq \beta$, which in turn follows from properties (i)–(iv) above.

Let $F_n = \{\alpha < \mathfrak{c}\colon k(\alpha) \geq n\}$. Then (F_n) is a decreasing sequence of closed sets with empty intersection. Suppose $F_n \subset U_n$ with U_n open. We will show $\bigcap_n U_n \neq \emptyset$. From this it clearly follows that X is not perfect, and also not submetacompact, for suppose $(\mathcal{V}_m)$ is a θ-refinement of $\{X \backslash F_n\colon n \in \omega\}$. Let $W_n = \bigcup_{i \leq m} \operatorname{st}(F_n, \mathcal{V}_i)$; one easily checks that $\bigcap_n W_n = \emptyset$.

Let $A_n = \{\sigma \in \bigcup_{k \geq n} \mathfrak{c}^k\colon [\sigma] \cap U_n = \emptyset$. Suppose $\{\sigma|_1\colon \sigma \in A_n\}$ is uncountable. Then A_n contains some $S_\alpha \in \mathcal{S}$ with $k(\alpha) \geq n$. But by the construction, α is a limit point of $\bigcup\{[\sigma]\colon \sigma \in S_\alpha\}$, so $[\sigma] \cap U_n \neq \emptyset$ for some $\sigma \in S_\alpha$, a contradiction. Hence $\{\sigma|_1\colon \sigma \in \bigcup_n A_n\} = C$ is countable.

From the above paragraph, it follows that each U_n is dense in $B = \{f \in B(\mathfrak{c})\colon f|_1 \notin C\}$. Since B is easily seen to be a Baire space, there is some point $x \in B \cap (\bigcap_n U_n)$. Thus X is not perfect or submetacompact. A similar argument involving $B^2 \backslash \Delta$, where Δ is the diagonal, shows that X does not have a G_δ-diagonal. □

The proof shows that the above example is not countably metacompact, hence not a β-space. (See Theorem 7.8 and the discussion preceding it.) The example is also not normal. It is not known if every normal or even collectionwise normal symmetrizable space is subparacompact or perfect. It is also not known if every point of a regular symmetrizable space must be G_δ; however, the above example can be used to construct a Hausdorff counterexample.

Since σ-spaces are semi-stratifiable, Theorem 9.8 shows, in particular, that first-countable σ-spaces are semi-metrizable. The following example shows that not every semi-metrizable space is a σ-space; this is also the first example we have seen of a semi-stratifiable space which is not a σ-space.

9.10. EXAMPLE. A semi-metrizable space which is not a σ-space.

PROOF. Let $X = \mathbb{R}^2$ with the 'bowtie' topology; that is, a neighborhood of a point $(s, t) \in X$ is the 'bowtie':

$$\{(s, t)\} \cup \{(s', t')\colon 0 < |s - s'| < \varepsilon \text{ and } |(t' - t)/(s' - s)| < \delta\},$$

where $\varepsilon > 0$ and $\delta > 0$ can vary. Define

$$d((s, t), (s', t')) = \begin{cases} 0 & \text{if } s = s' \text{ and } t = t' \\ 1 & \text{if } s = s',\ t \neq t' \\ |s' - s| + |(t' - t)/(s' - s)| & \text{otherwise}. \end{cases}$$

It is easy to check that d satisfies the conditions of Theorem 9.7(i); thus X is semi-metrizable.

Suppose X has a network $\mathcal{F} = \bigcup_n \mathcal{F}_n$, where each $\mathcal{F}_n$ is discrete. For each $x \in X$, let $B(x)$ be the bowtie neighborhood of x with $\varepsilon = \delta = 1$, and let $n(x)$ and F_x be such that $x \in F_x \subset B(x)$ and $F_x \in F_{n(x)}$. There exists $m \in \omega$ and $X' \subset X$ such that $n(x) = m$ for each $x \in X'$ and X' is dense in some Euclidean open set U. Let $p \in U$. We will obtain a contradiction by showing that if W is open in X and contains p, then there exists $F \in \mathscr{F}_m$ such that $p \notin F$ and $W \cap F \neq \emptyset$. Since $W \cap U$ contains a Euclidean open set, it is easy to see that there are two points x_1, x_2 in $X' \cap W \cap U$ such that $x_1 \notin B(x_2)$ and $x_2 \notin B(x_1)$. Then $F_{x_1} \neq F_{x_2} \in \mathscr{F}_m$, so one of F_{x_1} and F_{x_2} does not contain p. □

REMARK. Using CH, or more generally $b = \omega_1$ (see the Chapter by van Douwen for the definition of b), it is possible to construct a hereditarily Lindelöf subspace of X which is not a σ-space (BURKE, DAVIS [198 ·]).

It is not known if there is a real example of a hereditarily Lindelöf semi-metrizable space which is not a σ-space.

From Theorem 9.8 and previous results on semi-stratifiable spaces, it is clear that the class of semi-metrizable spaces is hereditary and countably productive. Perhaps a case can be made for ignoring semi-metrizable spaces in favor of semi-stratifiable spaces. Most theorems involving semi-metrizable spaces seem to carry over to semi-stratifiable spaces, and semi-stratifiable spaces have the following advantage: they are preserved under closed maps (Theorem 5.13), but semi-metrizable spaces are not even preserved by perfect maps.

9.11. EXAMPLE. A semi-metric space X and a perfect $f: X \to Y$ such that Y is not first-countable. Thus Y is not semi-metrizable, or even symmetrizable, since perfect maps preserve the Fréchet property.

PROOF. The example X is the subspace $[0,1] \times [0,1]$ of Example 9.10. Let $K = [0,1] \times \{0\}$ and $Y = X/K$. Since K is compact, the quotient map f is perfect. It is easily seen that the point $f(K)$ of Y does not have a countable neighborhood base. □

Metrization

Now we turn to the metrization theory of these spaces. Since semi-metrizable spaces are a semi-stratifiable, we have from previous results that a semi-metrizable space is a Moore space if it is a p-space or a $w\Delta$-space (Theorem 5.12), or has a point-countable base or a $\delta\theta$-base (Corollary 8.3), and is metrizable if it is an M-space (Theorem 5.12). In particular, a semi-metrizable countably compact space is metrizable. It turns out that all these results are true for symmetrizable spaces as well. The key step is to prove that a symmetrizable countably compact space is hereditarily Lindelöf, hence compact and first countable. The following lemma does a bit more. Recall that a space X is κ-compact, κ a cardinal, if every closed discrete subset of X has cardinality less than κ.

9.12. LEMMA. *Let X be an ω_1-compact symmetrizable space. Then X is hereditarily Lindelöf.*

PROOF. Suppose $Y \subset X$ is not Lindelöf, and let $\mathcal{U}$ be an open cover of Y which contains no countable subcover. Inductively choose $x_\alpha \in Y$ and $U_\alpha \in \mathcal{U}$ containing x_α such that $x_\alpha \notin \bigcup_{\beta<\alpha} U_\beta$. Let $X' = \{x_\alpha : \alpha < \omega_1\}$, and note that for each α, $x_\alpha \notin F_\alpha = \overline{\{x_\beta : \beta > \alpha\}}$. (So far, this is just the standard construction of an uncountable right-separated subspace of a non-hereditary Lindelöf space).

Since $d(x_\alpha, F_\alpha) > 0$, by passing to a subsequence we may assume that there exists $\varepsilon > 0$ such that $d(x_\alpha, F_\alpha) > \varepsilon$ for each $\alpha < \omega_1$. Now X' cannot be closed in X, for otherwise it would follow from 9.7(ii) that X' is discrete. Hence $d(x, X') = 0$ for some $x \notin X'$. Let $x_{\alpha_n} \in X'$ such that $d(x_{\alpha_n}, x) \to 0$. Then $x_{\alpha_n} \to x$. We may assume that (α_n) is an increasing sequence of ordinals. Then for each n, $x \in F_{\alpha_n}$. But $d(x_{\alpha_n}, x) \to 0$, while $d(x_{\alpha_n}, F_{\alpha_n}) > \varepsilon$, a contradiction. □

9.13. THEOREM. *Let X be a symmetrizable space. Then:*

(i) *X is a Moore space if and only if X is a p-space, a $w\Delta$-space, or has a $\delta\theta$-base;*

(ii) *X is metrizable if and only if X is an M-space.*

PROOF. By the remarks preceding Lemma 9.12, all we need to show is that in each case X is first-countable. Certainly every space with a $\delta\theta$-base is first countable.

If X is a $w\Delta$-space or a p-space, we will first prove that every point of X is contained in a compact set of countable character in X. For p-spaces, this has already been done in the proof of Theorem 3.22. Suppose $(\mathcal{G}_n)$ is a "$w\Delta$-sequence" for X. Let $x \in X$, and choose a sequence (U_n) of open sets such that

$$x \in U_{n+1} \subset \bar{U}_{n+1} \subset U_n \subset G_n \in \mathcal{G}_n$$

for each n and some $G_n \in \mathcal{G}_n$. Let $K = \bigcap_n U_n$. By Lemma 3.2, the U_n's are a base for K. Also, if $\{x_n : n \in \omega\} \subset K$, then $x_n \in \mathrm{st}(x, \mathcal{G}_n)$, so $\{x_n : n \in \omega\}$ has a cluster point which must be in the closed set K. Thus K is countably compact. By Lemma 9.12, K is hereditarily Lindelöf, hence compact.

Also by Lemma 9.12, compact subsets of X are first-countable. That X is first-countable now follows from the following general fact: if x has countable character in the compact subspace K, and K has countable character in X, then x has countable character in X. The obvious works: if $\{U_n : n \in \omega\}$ is a base for K and the trace of $\{V_n : n \in \omega\}$ on K is a base for x in K, and $W_n = U_n \cap V_n$, then $\{W_n : n \in \omega\}$ is a base for x in X. The details are left to the reader. □

We conclude this section with a metrization theorem for symmetrizable spaces involving properties of the distance function.

9.14. THEOREM (ARHANGEL'SKIĬ [1966]). *Let X be symmetrizable with respect to the symmetric d. Then X is metrizable if either of the following (equivalent) conditions hold:*

(i) *$d(K, H)>0$ whenever K is compact, H closed, and $K\cap H=\emptyset$;*
(ii) *$d(p, x_n)\to 0$ and $d(x_n, y_n)\to 0$ implies $d(p, y_n)\to 0$.*

PROOF. First we prove condition (i) implies condition (ii). Suppose d satisfies (i), and p, x_n, and y_n satisfy the hypotheses of (ii). Recall that $d(z, z_n)\to 0$ is equivalent to $z_n\to z$; hence we must prove that $y_n\to p$. If $y_n\not\to p$, there is an open neighborhood U of p such that $A=\{y_n: n\in\omega\}\setminus U$ is infinite. There exists $k\in\omega$ such that $x_n\in U$ for $n\geqslant k$. Then $K=\{p\}\cup\{x_n: n\geqslant k\}$ is a compact subset of U, and $d(K, K\setminus U)=0$, a contradiction. Hence $y_n\to p$. We leave the straightforward proof that (ii)$\Rightarrow$(i) to the reader.

Now we prove that (ii) implies that d is a semi-metric for X. Let $H\subset X$ and let $H'=\{x: d(x, H)=0\}$. If we prove that H' is closed, then d is a semi-metric for X by Theorem 9.7(i).

Suppose H' is not closed. Then $d(p, H')=0$ for some $p\not\in H'$. There exist $x_n\in H'$ with $d(p, x_n)\to 0$, and $y_n\in H$ with $d(x_n, y_n)\to 0$. Then $d(p, y_n)\to 0$, so $p\in H'$, a contradiction. So H' is closed and d is a semi-metric for X.

Now let $\mathscr{G}_n=\{B(x, 1/2^n)^{\circ}: x\in X\}$. We complete the proof by showing that for each $x\in X$, $\{\mathrm{st}^2(x, \mathscr{G}_n): n\in\omega\}$ is a base at x, where $\mathrm{st}^2(x, \mathscr{G}_n)=\mathrm{st}(\mathrm{st}(x, \mathscr{G}_n), \mathscr{G}_n)$. (We can then apply Theorem 1.4.) Suppose this is not the case. Then there exists x, an open set U containing x, and x_n, y_n such that $x\in B(x_n, 1/2^n)$, $B(x_n, 1/2^n)\cap B(y_n, 1/2^n)\neq\emptyset$, and $B(y_n, 1/2^n)\not\subset U$. Let $z_n\in B(x_n, 1/2^n)\cap B(y_n, 1/2^n)$, and $q_n\in B(y_n, 1/2^n)\setminus U$. Then $d(x, x_n)\to 0$, $d(x_n, z_n)\to 0$, $d(z_n, y_n)\to 0$, and $d(y_n, q_n)\to 0$. By induction, $d(x, q_n)\to 0$, contradicting $q_n\not\in U$. □

10. Quasi-metrizable and related spaces

In the previous section, we studied spaces admitting a 'distance function' satisfying the usual properties of a metric except for the triangle inequality. Now we omit the symmetric property instead.

10.1. DEFINITION. A function $d: X\times X\to\mathbb{R}^+$ is called a (*non-archimedean*) *quasi-metric* on the set X if for each $x, y, z\in X$,

(i) $d(x, y)=0$ iff $x=y$;
(ii) $d(x, z)\leqslant d(x, y)+d(y, z)$ $(d(x, z)\leqslant\max\{d(x, y), d(y, z)\})$.

A topological space X is said to be (*non-archimedean*) *quasi-metrizable* if there is a (non-archimedean) quasi-metric on X such that $\{B(x, \varepsilon): \varepsilon>0\}$ forms a base at each $x\in X$.

To see how quasi-metrizable spaces arise and how they might be characterized topologically, let us consider the properties of the $1/2^n$-balls $\{B(x, 1/2^n): n \in \omega, x \in X\}$. If $y \in B(x, 1/2^n)$, $n \geq 1$, then by the triangle inequality, we have $B(y, 1/2^n) \subset B(x, 1/2^{n-1})$. If the quasi-metric were non-archimedian, we would have $B(y, 1/2^n) \subset B(x, 1/2^n)$. The 'neighborhood assignment' $g(n, x) = B(x, 1/2^n)$ provides us with a convenient way of characterizing (non-archimedean) quasi-metrizable spaces and relating them to other classes.

10.2. THEOREM. *A space (X, τ) is [non-archimedean] quasi-metrizable if and only if there is a function $g: \omega \times X \to \tau$ such that*
(i) *$\{g(n, x): n \in \omega\}$ is a base at x;*
(ii) *$y \in g(n+1, x) \Rightarrow g(n+1, y) \subset g(n, x)$*
[(ii)′ *$y \in g(n, x) \Rightarrow g(n, y) \subset g(n, x)$*.]

PROOF. We have already observed that such a function g exists if X is (non-archimedean) quasi-metrizable. Suppose g satisfies (i) and (ii)′. Define $d(x, y) = 1/2^k$, where k is the largest $n \in \omega$ such that $y \in g(n, x)$. Then $B(x, \varepsilon) = g(m, x)$, where m is the least integer n such that $1/2^n < \varepsilon$, so the $B(x, \varepsilon)$'s form a base at x.

Suppose $d(x, y) = 1/2^j$, $d(y, z) = 1/2^k$, and $m = \min\{j, k\}$. Then $y \in g(m, x)$ and $z \in g(m, y) \subset g(m, x)$, so $d(x, z) \leq 1/2^m = \max\{d(x, y), d(y, z)\}$. Thus d is a non-archimedean quasi-metric for X.

Now suppose d satisfies (i) and (ii). Define $d(x, y)$ as above. In the same way, the $B(x, \varepsilon)$'s form a base at x. Unfortunately d does not necessarily satisfy the triangle inequality. However it does satisfy the conditions of Frink's Lemma (Lemma 2.6). To see this, suppose $d(x, y) < 1/2^k$ and $d(y, z) < 1/2^k$. Then $y \in g(k+1, x)$ and $z \in g(k+1, y)$. By (ii), $z \in g(k, x)$, so $d(x, z) < 1/2^{k-1}$.

Now let ρ be the function guaranteed by Frink's Lemma; then ρ satisfies the triangle inequality and

$$\tfrac{1}{4}d(x, y) \leq \rho(x, y) \leq d(x, y)\,.$$

Then ρ is certainly a quasi-metric on the set X, and

$$B_d(x, \tfrac{1}{4}\varepsilon) \subset B_\rho(x, \tfrac{1}{4}\varepsilon) \subset B_d(x, \varepsilon)\,.$$

From this it easily follows that (X, τ) is quasi-metrizable with respect to ρ. □

Now we can see better what spaces are quasi-metrizable. Suppose X has a σ-point-finite base $\mathscr{B} = \bigcup_n \mathscr{B}_n$. For each $x \in X$, let $g(n, x) = \bigcap\{B \in \mathscr{B}_i: i \leq n, x \in B\}$. This function g is easily seen to satisfy (i) and (ii)′ above. Hence every space with a σ-point-finite base is non-archimedean quasi-metrizable. Actually, all that we need in the above proof is that the collections $\mathscr{B}_n$ be *interior-preserving*, i.e., that $\bigcap\{B \in \mathscr{B}_n: x \in B\}$ be open for each $x \in X$. This leads to the following topological characterization of non-archimedean quasi-metrizable spaces.

10.3. THEOREM (NEDEV [1967]; FLETCHER and LINDGREN [1972]). *A space X is non-archimedean quasi-metrizable if and only if X has a σ-interior-preserving base.*

PROOF. Given the remarks above, it remains to prove that every non-archimedean quasi-metrizable space has a σ-interior-preserving base. But it is easily seen that $\{g(n, x): x \in X\}$ is interior-preserving, where g satisfies 10.2(i), (ii)'. □

We see from the above result that, while not every metrizable space is non-archimean metrizable (only the strongly 0-dimensional ones are (HAUSDORFF [1934], DE GROOT [1956])), every metric space is non-archimedean quasi-metrizable. Metacompact Moore spaces, as well as the Michael line, are easily seen to have σ-point-finite bases, hence are non-archimedean quasi-metrizable. So is the Sorgenfry line, for $\mathscr{B} = \{[x, r): r \in \mathbb{Q}\}$ is a σ-interior preserving base (consider fixed r's). The Sorgenfrey line does not have a σ-point-finite base: if it did, it would be a perfect space with a θ-base (Definition 8.1), hence developable by Theorem 8.7.

There does not seem to be such a nice topological characterization of quasi-metrizable spaces. From a practical point of view, perhaps this is not too serious, since quasi-metrizable spaces which are not non-archimedean quasi-metrizable seem to be difficult to find. Essentially the only known example is a modification of the plane, where a neighborhood of a point $p = (x_0, y_0)$ is p together with the interior of a disk tangent to $y = y_0$ at p, lying above this line. If $g(n, p)$ is such a neighborhood with the disk having radius $1/2^n$, then g satisfies 10.2(i), (ii), so the space is quasi-metrizable. See KOFNER [1973] or KOFNER [1980] for a proof that this space is not non-archimedean quasi-metrizable. This space is not a Moore space (each vertical line looks like the Sorgenfrey line); it is not known if every quasi-metrizable Moore space is non-archimedean quasi-metrizable.

10.4. EXAMPLE (KOFNER [1973]; HEATH [1972]). A Moore space which is not quasi-metrizable.

PROOF. Let X be the irrationals $\mathscr{I}$ on the x-axis, together with the double rationals D in the upper half-plane. Points of D have their usual Euclidean neighborhoods. A basic neighborhood of a point $p \in \mathscr{I}$ is p together with the interior of an equilateral triangle above the x-axis with one vertex at p and one side parallel to the x-axis. X is clearly a Moore space.

Suppose $g: \omega \times X \to \tau$ satisfies conditions (i) and (ii) of Theorem 10.2. For $x \in \mathscr{I}$, let $T(n, x)$ be the triangular neighborhood of x described above, where the altitude of this triangle is $1/2^n$. By the Baire property, the set

$$A(j, n, k) = \{x \in \mathscr{I}: T(j, x) \subset g(n+1, x) \subset g(n, x) \subset T(k, x)\}$$

is dense in some interval (a, b) on the x-axis. It is not hard to see that if

$y=(r_0, r_1)\in D$, with $r_0\in(a, b)$ and $r_1<\min\{1/2^j, \frac{1}{2}(b-a)\}$, then for some $x\in A(j, n, k)$, we have $y\in g(n+1, x)$, but $g(n+1, y)\not\subset g(n, x)$. $\square$

Now we define a class of spaces slightly more general than quasi-metrizable spaces.

10.5. DEFINITION. A space (X, τ) is a *γ-space* if there exists a function $g\colon \omega\times X\to\tau$ such that

(i) $\{g(n, x)\colon n\in\omega\}$ is a base at x;

(ii) for each $n\in\omega$ and $x\in X$, there exists $m\in\omega$ such that $y\in g(m, x)$ implies $g(m, y)\subset g(n, x)$.

Recall that if we can always take $m=n+1$ in (ii) above, then we have a quasi-metrizable space. Only recently has a regular example of a γ-space which is not quasi-metrizable been found (FOX [198·]). In developable spaces (JUNNILA [1978]) and suborderable spaces (KOFNER [1981]), the γ-space property is equivalent to quasi-metrizability. Example 10.4 is a Moore space which is not a γ-space; this can be seen by Junnila's result above, or by a slight modification of the proof that the space is not quasi-metrizable.

The γ-spaces have appeared independently in several disguises, which has been part of their interest for topologists. Let us give some of these characterizations. In the following theorem, a *distance function on* X is a function $d\colon X\times X\to\mathbb{R}^+$ such that $\{B(x, \varepsilon)\colon \varepsilon>0\}$ is a base at each $x\in X$.

10.6. THEOREM. *The following are equivalent for a space* (X, τ):

(i) *X is a γ-space;*

(ii) *there exists a function $g\colon \omega\times X\to\tau$ such that $y_n\to p$ whenever $x_n\in g(n, p)$ and $y_n\in g(n, x_n)$;*

(ii)′ *there exists a distance function d on X such that $d(p, y_n)\to 0$ whenever $d(p, x_n)\to 0$ and $d(x_n, y_n)\to 0$;*

(iii) *there exists a function $g\colon \omega\times X\to\tau$ such that if K is compact, and U is an open set containing K, then $\bigcup_{x\in K} g(n, x)\subset U$ for some $n\in\omega$;*

(iii)′ *there exists a distance function d on X such that $d(K, H)>0$ whenever K is compact, H closed, and $K\cap H=\emptyset$.*

PROOF. Properties (ii)′ and (iii)′ are just the 'distance function equivalents' of properties (ii) and (iii). If d is a distance function, let $g(n, x)=B(x, 1/2^n)$; if $g\colon \omega\times X\to\tau$, and $g(n+1, x)\subset g(n, x)$, let $d(x, y)=1/2^n$, where n is the least $k\in\omega$ such that $y\notin g(k, x)$. These definitions easily get one back and forth between (ii) and (ii)′, and (iii) and (iii)′.

Now the conditions on d in (ii)′ and (iii)′ are the same as the conditions of Theorem 9.14, (ii) and (i) respectively, and these conditions were seen to be equivalent there. By the same proof, (ii)′ and (iii)′ are equivalent. Hence (ii) and (iii) are equivalent, so it remains to prove (ii)$\Leftrightarrow$(i).

It is easily checked that a function g satisfying Definition 10.5, with $g(n+1, x) \subset g(n, x)$, also satisfies (ii). So suppose g satisfies (ii). Clearly $x_n \in g(n, p)$ implies $x_n \to p$, and so $\{g(n, p): n \in \omega\}$ is a base at p. Fix $n \in \omega$, and suppose that for each $m \in \omega$, there exists $x_m \in g(m, p)$ such that $g(m, x_m) \not\subset g(n, p)$. Pick $y_m \in g(m, x_m) \setminus g(n, p)$. By (ii), $y_m \to p$, a contradiction. Hence g satisfies Definition 10.5. □

Observe that by Theorem 9.14, if X admits a symmetric distance function satisfying one of the equivalent conditions (ii)′ and (iii)′, then X is metrizable. And of course, any distance function which is both a symmetric and a quasi-metric is a metric. But if X is symmetrizable with respect to one distance function, and quasi-metrizable with respect to another, one probably would not expect to get metrizability. But we do get developability. Recall that quasi-metrizable space is first-countable, so a symmetrizable, quasi-metrizable space is semi-metrizable (Theorem 9.6), hence semi-stratifiable (Theorem 9.8), hence a β-space (Theorem 7.8). The following is a quite general result from which we can derive various metrization and 'developability' theorems.

10.7. THEOREM (HODEL [1972]). *If X is a β-space and a γ-space, then X is developable.*

PROOF. Let $g_1: \omega \times X \to \tau$ satisfy Definition 7.7, and let $g_2: \omega \times X \to \tau$ satisfy Theorem 10.6(ii), with $g_i(n+1, x) \subset g_i(n, x)$ in both cases. Let

$$g(n, x) = g_1(n, x) \cap g_2(n, x),$$

and let $\mathcal{G}_n = \{g(n, x): x \in X\}$. We show that $(\mathcal{G}_n)$ is a development for X. Suppose it is not. Then there exists $p \in X$, an open set U containing p, and points x_n, y_n in X such that

$$p \in g(n, x_n) \quad \text{and} \quad y_n \in g(n, x_n) \setminus U.$$

By the β-space condition, $\{x_n: n \in \omega\}$ must have a cluster point x. Then for each $k \in \omega$, we can choose $n(k) \geq k$ with $x_{n(k)} \in g(k, x)$, and since $p \in g(n(k), x_{n(k)}) \subset g(k, x_{n(k)})$, this contradicts 10.6(ii) unless $p = x$. It follows that p is a cluster point of the x_n's. Now by a similar trick, choosing $x_{n(k)} \in g(k, p)$, we can use 10.6(ii) to show that p is a limit point of the y_n's, which is a contradiction. □

10.8. COROLLARY. *Let X be a γ-space. Then:*

(i) *X is developable if and only if X is a Σ-space, a $w\Delta$-space, semi-stratifiable, or symmetrizable;*

(ii) *X is metrizable if and only if X is an M-space (in particular, countably compact), a stratifiable space, or a collectionwise-normal symmetrizable space.*

PROOF. (i) All spaces mentioned, except for symmetrizable spaces, are β-spaces (Theorem 7.8). But a first-countable symmetrizable space is a semi-metrizable space, hence semi-stratifiable, hence a β-space. The result now follows from Theorem 10.7.

(ii) Except for the case where X is an M-space, this is immediate from (i) (recalling that stratifiable spaces are paracompact). If X is an M-space, then X is a quasi-perfect pre-image of a metric space under a mapping f. By (i), a countably compact γ-space is developable, hence has a G_δ-diagonal, and thus is compact by Theorem 2.14. Therefore f is a perfect map, and so X is paracompact. The result now follows from (i). □

See KOFNER [1980b] for an excellent survey of quasi-metrizable spaces.

11. k-networks: $\aleph$-spaces and $\aleph_0$-spaces

There are several reasons for the interest in the classes of spaces discussed in this section. One is that the class of $\aleph_0$-spaces can be used to characterize the quotient images of separable metric spaces (Theorem 11.3). Another is a useful application to the space $\mathscr{C}(X, Y)$ of all continuous functions from X into Y with the compact-open topology (see Theorem 11.5 and preceding comments). These classes are also very well-behaved under various topological operations.

11.1. DEFINITION. A collection $\mathscr{F}$ of subsets of a space X is a *k-network* if whenever K is a compact subset of an open set U, there exists a finite $\mathscr{F}' \subset \mathscr{F}$ such that $K \subset \bigcup \mathscr{F}' \subset U$. A regular space with σ-locally-finite (countable) k-network is called an *$\aleph$-space* (*$\aleph_0$-space*).

A base for a space is clearly a k-network, so metrizable spaces are $\aleph$-spaces. Certainly every k-network is a network, so $\aleph$-spaces are σ-spaces. In particular, $\aleph$-spaces are subparacompact, perfect, and have a G_δ-diagonal.

The following theorem begins to show how well-behaved these classes are.

11.2. THEOREM. *The classes of $\aleph$-spaces, $\aleph_0$-spaces, and paracompact $\aleph$-spaces are hereditary and countably productive.*

PROOF. That $\aleph$-spaces and $\aleph_0$-spaces are hereditary is straightforward and left to the reader. If X is a paracompact $\aleph$-space, then every open subspace of X, being an F_σ-set, is paracompact. Thus X hereditarily paracompact. The obvious k-networks show that $\aleph$-spaces and $\aleph_0$-spaces are countably productive. Finally, since $\aleph$-spaces are σ-spaces, the countable product of paracompact $\aleph$-spaces is paracompact by Corollary 4.17. □

Since $\aleph_0$-spaces have a countable network, they are hereditarily Lindelöf and hereditarily separable. We saw that spaces having a countable network characterize the continuous images of separable metric spaces. It turns out that k- and -$\aleph_0$-spaces (i.e., $\aleph_0$-spaces which are k-spaces) characterize the quotient images of separable metric spaces.

11.3. THEOREM (MICHAEL [1966]). *The following are equivalent for a space X:*
(i) *X is a quotient space of a separable metric space;*
(ii) *X is a k- and -$\aleph_0$-space.*

PROOF. (i)$\Rightarrow$(ii). Let $f: M \to X$ be a quotient map, with M a separable metric space. Let $\mathscr{B}$ be a countable base for M; we show $f(\mathscr{B}) = \{f(B): B \in \mathscr{B}\}$ is a k-network for X. Suppose $K \subset U \subset X$, K compact, U open. Suppose there does not exist a finite cover of K by $\mathscr{C} = \{C \in f(\mathscr{B}): C \subset U\}$. Let (C_n) enumerate $\mathscr{C}$, and pick $x_n \in K \setminus \bigcup_{i \leq n} C_i$.

Now K has a countable network, hence is metrizable (Theorem 4.2), so by passing to a subsequence if necessary, we can assume $x_n \to x \in K$, and $x \neq x_n$ for all n. Since f is quotient and $A = \{x_n: n \in \omega\}$ is not closed in X, $f^{-1}(A)$ is not closed in M. Let $z \in \overline{f^{-1}(A)} \setminus f^{-1}(A)$. Then $z \in f^{-1}(K) \subset f^{-1}(U)$. Choose $B \in \mathscr{B}$ with $z \in B \subset f^{-1}(U)$. Then $f(B) \in \mathscr{C}$, and $f(B)$ contains infinitely many x_n's. This contradicts the way the x_n's were chosen.

(ii)$\Rightarrow$(i). Let $\mathscr{F}$ be a countable closed k-network for X, closed under finite unions and intersections. Let $M = \mathscr{F}^\omega$, where the set $\mathscr{F}$ is given the discrete topology. Then M is a separable metric space. Let N be the set of all sequences (F_n) in M such that $\{F_n: n \in \omega\}$ is a network at some point $x \in X$, i.e., $x \in \bigcap_n F_n$, and every neighborhood of x contains some F_n. Note that $\{x\} = \bigcap_n F_n$.

Define $f: N \to X$ by declaring $f((F_n))$ to be the unique element x of $\bigcap_n F_n$. Since (F_n) is a network at x, this map is clearly continuous.

It remains to prove that f is a quotient map. Suppose $A \subset X$ is not closed, but $f^{-1}(A)$ is closed in N. Since X is a k-space, $K \cap A$ is not closed for some compact set $K \subset X$. Now K is a compact σ-space, hence metrizable, so there exists $x \in K \setminus A$ and $a_n \in K \cap A$ with $a_n \to x$. Let $\{G_n: n \in \omega\} = \{F \in \mathscr{F}: x \in F$ and $F \supset \{a_n: n \geq m\}$ for some $m \in \omega\}$. This set is non-empty since $\mathscr{F}$ is a k-network closed under finite unions. That $\{G_n: n \in \omega\}$ is a network at x follows easily from the fact that $\mathscr{F}$ is a k-network. Thus $(G_n) \in N$. But since each G_n contains a tail of the sequence (a_n), it follows that $(G_n) \in \overline{f^{-1}(A)}$, a contradiction. $\square$

REMARK. Recall that a map $f: N \to X$ is compact-covering if every compact subset of X is the image of a compact subset of N. One can show that if X is an $\aleph_0$-space, then the map $f: N \to X$ defined in the proof of (ii)$\Rightarrow$(i) of the above result is compact-covering, whether X is a k-space or not. For suppose $K \subset X$ is compact. Let $(\mathscr{F}_n)$ enumerate all finite covers of K by elements of $\mathscr{F}$. Then $\Pi_n \mathscr{F}_n$ is a compact subset of $\mathscr{F}^\omega$. Suppose $x \in K$, and $x \in F_n \in \mathscr{F}_n$ for each n. If U is a neighborhood of x, one can easily find a finite subset $\mathscr{F}_x \subset \mathscr{F}$ such that $K \subset \bigcup \mathscr{F}_x$ and $\mathrm{st}(x, \mathscr{F}_x) \subset U$. Since

$F_x = F_n$ for some n, we have $F_n \subset U$ for some n. Hence the set

$$L = \{(F_n) \in \prod_n \mathscr{F}_n : (\bigcap_n F_n) \cap K \neq \emptyset\}$$

is a subset of N, and $f(L) = K$. Since the elements of $\mathscr{F}$ are closed and K is compact, L is evidently closed in $\Pi_n \mathscr{F}_n$, hence compact. Thus f is compact-covering, and so any $\aleph_0$-space is a compact-covering image of a separable metric space.

Recall that a space X is a Fréchet space if $x \in \bar{A}$ implies $a_n \to x$ for some sequence (a_n) of points of A. The following result shows that imposing the Fréchet condition or first-countability on an $\aleph$-space has surprisingly strong consequences.

11.4. THEOREM. (i) *A Fréchet $\aleph$-space is stratifiable* (FOGED [198·]);
(ii) *A first-countable $\aleph$-space is metrizable* (O'MEARA [1971]).

PROOF. (i) Since $\aleph$-spaces are σ-spaces, hence semi-stratifiable, it follows from Theorem 5.16 that we need only show that Fréchet $\aleph$-spaces are monotonically normal. Suppose X is Fréchet, and $\mathscr{F} = \bigcup_n \mathscr{F}_n$ is a σ-locally-finite closed k-network for X.

Let H and K be disjoint closed subsets of X. Let

$$U_n = \bigcup\{F \in \mathscr{F}_i : i \leq n, F \cap K = \emptyset\} \setminus \bigcup\{F \in \mathscr{F}_i : i \leq n, F \cap H = \emptyset\},$$

and let $D(H, K) = (\bigcup_n U_n)^\circ$. Clearly D is 'monotone', i.e., if $H \subset H'$ and $K \supset K'$, then $D(H, K) \subset D(H', K')$. So we must prove that $H \subset D(H, K)$ and $\overline{D(H, K)} \subset X \setminus K$. If $H \not\subset D(H, K)$, then $x \notin (\bigcup_n U_n)^\circ$ for some $x \in H$. Since X is Fréchet, there exists $x_n \in X \setminus \bigcup_n U_n$ with $x_n \to x$. Since $\mathscr{F}$ is a k-network, some $F \in \mathscr{F}_m$ contains infinitely many x_n's, and does not intersect K. Since $x_n \to x \in H$, then U_m contains infinitely many x_n's, a contradiction. Thus $H \subset D(H, K)$. A similar type of argument shows $\overline{D(H, K)} \subset X \setminus K$. It follows that X is monotonically normal, hence stratifiable.

(ii) Let X be a first countable $\aleph$-space, with σ-locally-finite closed k-network $\mathscr{F} = \bigcup_n \mathscr{F}_n$. We first prove that for each $x \in X$ and open U containing x, the point x is contained in the interior of a finite union of elements of $\mathscr{F}_{x,U} = \{F \in \mathscr{F} : x \in F \subset U\}$. Suppose not. Let $\mathscr{F}_{x,U} = \{F_n : n \in \omega\}$, and let (U_n) be a decreasing local base at x with $U_0 \subset U$. Choose $x_n \in U_n \setminus \bigcup_{i \leq n} F_i$. Then $K = \{x\} \cup \{x_n : n \in \omega\}$ is a compact subset of U. Since there is a finite cover of K by elements of $\mathscr{F}_{x,U}$, some F_k must contain infinitely many x_n's, which is impossible.

Let $\mathscr{B}_n = \{(\bigcup \mathscr{F}')^\circ : \mathscr{F}' \subset \bigcup_{i \leq n} \mathscr{F}_i, |\mathscr{F}'| < \omega\}$. It is clear from the above paragraph that $\mathscr{B} = \bigcup_n \mathscr{B}_n$ is a base for X. We prove that each $\mathscr{B}_n$ is interior-preserving. Let $x \in X$; we need to show that $x \in (\bigcap\{B \in \mathscr{B}_n : x \in B\})^\circ$. If $x \in (\bigcup \mathscr{F}')^\circ$, $\mathscr{F}' \subset \bigcup_{i \leq n} \mathscr{F}_i$, then since each $F \in \mathscr{F}$ is closed, $x \in (\bigcup \mathscr{F}'_x)^\circ$, where $\mathscr{F}'_x = \{F \in \mathscr{F}' : x \in F\}$.

Since x is in only finitely many elements of $\bigcup_{i\leq n}\mathscr{F}_i$, the set

$$\mathscr{B}_{n,x}=\left\{(\bigcup\mathscr{F}'_x)^\circ\colon F'\subset\bigcup_{i\leq n}\mathscr{F}_i,\ |\mathscr{F}'|<\omega,\ x\in(\bigcup\mathscr{F}')^\circ\right\}$$

is finite, and $x\in\bigcap\mathscr{B}_{n,x}\subset(\bigcap\{B\in\mathscr{B}_n\colon x\in B\})^\circ$. Hence $\mathscr{B}_n$ is interior-preserving. Now by Theorem 10.3, X is non-archimedean quasi-metrizable. By (i), X is stratifiable, and it now follows from Corollary 10.8(ii) that X is metrizable. □

One of the most interesting applications of $\aleph$-spaces is in the area of function spaces. Given spaces X and Y, let $\mathscr{C}(X,Y)$ denote the space of continuous functions from X into Y with the compact-open topology. If X is a discrete space, then $\mathscr{C}(X,Y)$ is just the product space $Y^{|X|}$. Thus some restrictions must be made if we are to expect $\mathscr{C}(X,Y)$ to have good topological properties. It turns out that if X is an $\aleph_0$-space, then $\mathscr{C}(X,Y)$ is an $\aleph_0$-space, an $\aleph$-space, or a paracompact $\aleph$-space whenever Y is. We will prove this result for the case that Y is an $\aleph_0$-space. The proofs of the remaining cases are due to O'MEARA [1971] for Y a paracompact $\aleph$-space and FOGED [198·] for Y an $\aleph$-space.

11.5. THEOREM (MICHAEL [1966]). *If X and Y are $\aleph_0$-spaces, so is $\mathscr{C}(X,Y)$.*

PROOF. We prove the theorem by establishing a series of claims.

Claim 1. We may assume that X is a separable metric space. If X is an $\aleph_0$-space, then by the remark following Theorem 11.3, there is a separable metric space N and a map $f\colon N\to X$ such that every compact subset of X is the image of some compact subset of N. Define a map

$$\Phi\colon\mathscr{C}(X,Y)\to\mathscr{C}(N,Y)$$

by $\Phi(g)=g\circ f$. It is not difficult to check that Φ is a homeomorphism onto its range. Hence if $\mathscr{C}(N,Y)$ is an $\aleph_0$-space, so is $\mathscr{C}(X,Y)$ by Theorem 11.2. This proves Claim 1.

If $A\subset X$ and $B\subset Y$, let $W(A,B)=\{f\in\mathscr{C}(X,Y)\colon f(A)\subset B\}$. Recall that all sets of the form $W(C,U)$, where C is compact in X and U is open in Y, form a sub-base for $\mathscr{C}(X,Y)$.

Now let $\mathscr{B}$ be a countable base for X, and let $\mathscr{Q}$ be a countable k-network for Y, both closed under finite unions. Let

$$\mathscr{P}=\{W(B,Q)\colon B\in\mathscr{B},\ Q\in\mathscr{Q}\}.$$

Claim 2. Suppose $K\subset W(C,U)$, with K compact in $\mathscr{C}(X,Y)$, C compact in X, and U open in Y. Then there exists $P\in\mathscr{P}$ with $K\subset P\subset W(C,U)$.

Let us see why this claim is true. First, recall the following general fact concerning function spaces (see e.g., DUGUNDJI [1966], 2.4, pg. 260):

(i) If $C \subset X$ is compact, then the evaluation map $(f, x) \to f(x)$ from $\mathscr{C}(X, Y) \times C \to Y$ is continuous.

Now for $\mathscr{F} \subset \mathscr{C}(X, Y)$ and $A \subset X$, let $\mathscr{F}(A) = \{f(x): f \in \mathscr{F} \text{ and } x \in A\}$. From (i), one can easily obtain:

(ii) If $x_n \to x$ and $K(\{x\}) \subset U$, where U is open in Y, then $K(\{x_n\}) \subset U$ for sufficiently large n.

(To see how (ii) is obtained from (i), consider $f_n \in K$ with $f_n(x_n) \notin U$, and a cluster point $f \in K$ of the f_n's. Let $C = \{x_n: n \in \omega\} \cup \{x\}$. Since $f(x) \in U$, the continuity of the evaluation map is violated at (f, x).)

To prove Claim 2, it will suffice to find an open set V containing C and $Q \in \mathscr{Q}$ with $K(V) \subset Q \subset U$; for then $K \subset W(B, Q) \subset W(C, U)$ for any $B \in \mathscr{B}$ with $C \subset B \subset V$. Let (Q_n) enumerate all $Q \in \mathscr{Q}$ with $Q \subset U$. If no such V and Q exist, then there exist $x_n \in X$ with $d(x_n, C) < 1/2^n$, where d is some metric on X, and $f_n \in K$ with $f_n(x_n) \notin \bigcup_{i \leq n} Q_i$. (Recall that $\mathscr{Q}$ is closed under finite unions.) By passing to a subsequence, we may assume $x_n \to x$ for some $x \in C$.

Since $K \subset W(C, U)$, we have $K(\{x\}) \subset U$. By (ii) above, $K(\{x_n\}) \subset U$ for sufficiently large n. Let $A = \{x\} \cup \{x_n: K(\{x_n\}) \subset U\}$. Then A is compact and $K(A) \subset U$. By (i) above, $K(A)$ is compact. Therefore $K(A) \subset Q \subset U$ for some $Q \in \mathscr{Q}$. But $Q = Q_m$ for some $m \in \omega$, and so $K(A) \subset Q$ contradicts $f_n(x_n) \notin Q$ for $n \geq m$. This proves Claim 2.

The next claim completes the proof of the theorem.

Claim 3. Let $\mathscr{P}$ be as in Claim 2. Then the set $\hat{\mathscr{P}}$ of all finite intersections of elements of $\mathscr{P}$ is a k-network for $\mathscr{C}(X, Y)$.

Let $\mathscr{W}$ be the base for $\mathscr{C}(X, Y)$ consisting of all finite intersections of the $W(C, U)$'s. From Claim 2, it is easy to see that if $K \subset W \subset \mathscr{W}$, where $K \subset \mathscr{C}(X, Y)$ is compact, then $K \subset P \subset W$ for some $P \in \hat{\mathscr{P}}$. Claim 3 now follows from the following easily observed fact:

If $H \subset U$, with H compact and U open in $\mathscr{C}(X, Y)$, then H has a finite cover by closed, hence compact, subsets, each of which is contained in some $W \in \mathscr{W}$ with $W \subset U$. $\square$

We should mention that the analogue of Theorem 11.5 for spaces having a countable network (instead of k-network) is not valid. In fact, there is a separable stratifiable space Y (hence Y has a countable network) such that $\mathscr{C}(I, Y)$ is not normal, where I is the unit interval (MICHAEL [1966], Example 12.1). In particular, $\mathscr{C}(I, Y)$ cannot have a countable network by Theorem 4.4.

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CHAPTER 11

An Introduction to $\beta\omega$

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

Introduction

The aim of this paper is to give an introduction to the space $\beta\omega$, i.e. the Stone space of the Boolean algebra $\mathcal{P}(\omega)$ of subsets of ω. There are several arguments in favour of writing such a paper. Firstly, in the last five years several important questions concerning the structure of $\beta\omega$ were solved. We have a good picture of $\beta\omega$ now. Secondly, results about $\beta\omega$ usually have wide applications in various parts of mathematics. The space $\beta\omega$ is an exciting place where topologists, set theorists, infinite combinatorists, Boolean algebraists, and sometimes even number theorists and analysts, meet.

Since this is a chapter in the Handbook of Set Theoretic Topology, I have written this paper from the perspective of a topologist. Our language is topological but at several places it was more natural to use Boolean algebras instead of their Stone spaces, so we freely did this. We mention our perspective at this early stage of the introduction since this gives the reader an idea about what types of results are to be expected in this paper. In addition, we do not aim to be complete. Several important results will not be proven in detail, or will not even be mentioned. For this reason we have called this paper "An introduction to $\beta\omega$". Also, we will not give lengthy historical comments giving proper credit to everybody, but we will usually only refer to the paper giving the final solution of the problem we are discussing.

It is probably true that the following facts are the most important results obtained in $\beta\omega$ in recent years:

(1) it is consistent that P-points do not exist in $\beta\omega\backslash\omega$ (Shelah; see MILLS [1978] or WIMMERS [1978]).

(2) some but not all points in $\beta\omega\backslash\omega$ are weak P-points (KUNEN [1978]),

(3) every point in $\beta\omega\backslash\omega$ is a $\mathfrak{c}$-point (BALCAR & VOJTÁŠ [1980]).

(1) will interest set theorists most, (2) fascinates topologists and (3) is connected with and important in Boolean algebras as well as topology. Due to our perspective, we will discuss (2), but we leave (1) and (3) untouched.

The space $\beta\omega$ is a monster having three heads. If one works in a model in which the Continuum Hypothesis (abbreviated CH) holds, then one will see only the first head. This head is smiling, friendly, and makes you feel comfortable working with $\beta\omega$. I do not know many open problems on $\beta\omega$ the answers of which are unknown under CH. In fact, one usually does not work with $\beta\omega$ or with $\beta\omega\backslash\omega$, but with a Boolean algebra satisfying a certain completeness property which characterizes the Boolean algebra $\mathcal{P}(\omega)$/fin under CH. This is the theme in Section 1. Here we discuss the spaces $\beta\omega$ and $\beta\omega\backslash\omega$ under CH. We begin by identifying the completeness property which characterizes $\mathcal{P}(\omega)$/fin and then work in Boolean algebras satisfying this completeness property. Because of the presence of the CH, transfinite inductions have length ω_1 and because of the special properties of the Boolean algebras under consideration, we can always continue the transfinite inductions until stage ω_1. The reader should observe that nowhere in Section 1 do we use the special structure of $\beta\omega$, with the exception, of course, of the completeness property of $\mathcal{P}(\omega)/\text{fin}$. If one works in a model in which CH does not hold, then one will see the second head of $\beta\omega$. This head constantly tries to confuse you and you will never be able to decide whether it speaks the truth. This head of $\beta\omega$ will be discussed in Section 2. It turns out that all but three of the CH results derived in Section 1 are consistently false. After reading the first two sections, the reader might feel that $\beta\omega$ is a horrible creature since it seems that all statements about it depend on special set theoretic assumptions. What can there 'really' (=in ZFC) be said about $\beta\omega$? The answer to this question is: quite a bit. The third head of $\beta\omega$ is its head in ZFC. Because of the first two heads, this head is rather vague, but some parts of it are very clear. If one wants to see the clear part, one will have to work like a slave, inventing ingeneous combinatorial arguments. One will have to use special properties of $\beta\omega$ and not only global properties. Some ZFC results on $\beta\omega$ are discussed in Sections 3 and 4.

I am indebted to W.W. Comfort, F. van Engelen, I. Juhász, P. Nyikos and J.E. Vaughan for many helpful comments.

0. Preliminaries

In order to be able to understand the arguments in this paper, one should know some elementary facts about Boolean algebras and Čech–Stone compactifications. All one needs to know can be found in COMFORT & NEGREPONTIS [1974, §2]. A Boolean algebra is usually denoted by $\mathcal{B}$, its universal bounds by 0 and 1, concepts such as homomorphism, embedding, isomorphism, etc., should be familiar. Cardinals are initial (von Neumann) ordinals, and get the discrete topology. If α is an ordinal, then $W(\alpha)$ denotes the topological space with underlying set α equipped with the order topology. What should one know about Čech–Stone compactifications? Well, one should know that βX is the unique compactification of the (completely regular Hausdorff) space X with the property that disjoint zero-sets in X have disjoint closures. This easily implies that given a map $f: X \to K$, where K is compact, there exists a unique map $\beta f: \beta X \to K$ extending f. This map is called the *Stone extension* of f. I often hear the remark that $\beta\omega$ is clear, since it is the Stone space of $\mathcal{P}(\omega)$, but βX, for arbitrary X, is not clear, partly because it is not the Stone space of a Boolean algebra. For this reason in this paper we almost exclusively work with *strongly zero-dimensional* spaces, i.e. those spaces X for which βX is zero-dimensional, or equivalently, those spaces X for which βX is equivalent to the Stone space of the Boolean algebra $\mathcal{B}(X)$ consisting of all clopen (=both closed and open) subsets of X. Observe that in this case the existence of the Stone extension βf discussed above is clear, since the existence of f implies that $\mathcal{B}(K)$ can be embedded in $\mathcal{B}(X)$. *Henceforth, all topological spaces under discussion are assumed to be completely regular and Hausdorff.* The *Stone space* of a Boolean algebra $\mathcal{B}$ is denoted by $\text{st}(\mathcal{B})$. Recall that a subset U of a space X is called *regular open* provided that $U = \operatorname{int}_X \operatorname{cl}_X U$. Let $\mathrm{RO}(X) = \{U \subseteq X\colon U \text{ is regular open}\}$. Then $\mathrm{RO}(X)$ becomes a complete Boolean algebra under the following operations:

$$U \wedge V = U \cap V,$$
$$U \vee V = \operatorname{int}_X \operatorname{cl}_X(U \cup V),$$
$$U' = \operatorname{int}_X(X \backslash U).$$

If X is compact, then the Stone space of $\mathrm{RO}(X)$ will be denoted by EX. Since $\mathrm{RO}(X)$ is complete, EX is *extremally disconnected* (=closure of an open set is open). It is easily seen that topologically, EX is characterized as follows: EX is the unique extremally disconnected space which admits an *irreducible* (a continuous surjection $f\colon S \to T$ is called irreducible provided that $f(A) \neq T$ for all closed $A \subseteq S$ with $A \neq S$) perfect map $\pi\colon EX \to X$. The space EX is called the *projective cover* of X. For a recent survey on projective covers, see WOODS [1979]. A space X is called *basically disconnected* if the closure of each open F_σ is again open. Observe that, trivially, each extremally disconnected space is basically disconnected, but not conversely. As usual, fin denotes the ideal of finite subsets of ω, and $\mathscr{P}(\omega)/\text{fin}$ is the Boolean algebra we obtain from $\mathscr{P}(\omega)$ by calling A, $B \in \mathscr{P}(\omega)$ equivalent iff $A \,\Delta\, B \in \text{fin}$ $(A \,\Delta\, B = (A \backslash B) \cup (B \backslash A))$. As remarked above, $\beta\omega$ denotes $\mathrm{st}(\mathscr{P}(\omega))$. If $n > \omega$, then we identify n with the point

$$\{A \in \mathscr{P}(\omega)\colon n \in A\}$$

from $\mathrm{st}(\mathscr{P}(\omega))$. Points from $\beta\omega \backslash \omega$ are called *free ultrafilters*. Obviously, $\beta\omega \backslash \omega \approx \mathrm{st}(\mathscr{P}(\omega)/\text{fin})$. If $A \subseteq \omega$, we put

$$A^* = \{x \in \beta\omega \backslash \omega : A \in x\}.$$

It is clear that the collection $\{A^* : A \in \mathscr{P}(\omega)\}$ is a base for $\beta\omega \backslash \omega$. Also observe that $\beta\omega \backslash \omega = \omega^*$.

0.1. LEMMA. (a) *If $V \subseteq \omega$ is infinite, then $\bar{V}$ is homeomorphic to $\beta\omega$.*
(b) *If V, $W \subseteq \omega$ are infinite, then $V^* \cap W^* = \emptyset$ iff $|V \cap W| < \omega$.*

PROOF. (a) Is clear since $\mathscr{P}(V)$ is isomorphic to $\mathscr{P}(\omega)$. We leave the proof of (b) as an exercise to the reader. □

A point x of a space X is called a *P-point* if the intersection of countably many neighborhoods of x is again a neighborhood of x.

Whenever X is a set and κ is a cardinal we define (as usual)

$$[X]^\kappa = \{A \subseteq X : |A| = \kappa\},$$

$$[X]^{\leqslant\kappa} = \{A \subseteq X : |A| \leqslant \kappa\},$$

$$[X]^{<\kappa} = \{A \subseteq X : |A| < \kappa\},$$

respectively. We also let $\subset$ denote *proper* inclusion.

If X is a space, then X^* denotes $\beta X \backslash X$ and if $U \subseteq X$ is open, then

$$\operatorname{Ex}(U) = \beta X \backslash \mathrm{cl}_{\beta X}(X \backslash U).$$

Observe that $\operatorname{Ex}(U)$ is open and that $\operatorname{Ex}(U) \cap X = U$. The reader can easily verify that the collection

$$\{\operatorname{Ex}(U) : U \subseteq X \text{ is open}\}$$

is a base for the topology of βX. If $U \subseteq X$ is open, let

$$U' = \operatorname{Ex}(U) \cap X^*.$$

If X is normal and Y is closed in X, then $\mathrm{cl}_{\beta X} Y = \beta Y$. We identify $\mathrm{cl}_{\beta X} Y \backslash Y$ and Y^* in this case. For definitions such as character, π-weight, cellularity etc., see JUHÁSZ [1980], or HODEL [1983]. By "X is ccc" we mean that X satisfies the countable chain condition. We say that a family of sets $\mathcal{F}$ has the *n-intersection property* $(n < \omega)$ provided that $\bigcap \mathcal{G} \neq \emptyset$ for all $\mathcal{G} \in [\mathcal{F}]^{\leq n}$. To indicate that two spaces X and Y are homeomorphic, we write $X \approx Y$.

A *zero-set* of a space X is any set of the form $f^{-1}(\{0\})$, where $f: X \to I$ is continuous. A *cozero-set* is the complement of a zero-set.

Let α and κ be cardinals. We define, as usual,

$$\alpha^{\underline{\kappa}} = \sum \{\alpha^\lambda : (\lambda \text{ is a cardinal and}) \ \lambda < \kappa\}.$$

1. The spaces $\beta\omega$ and $\beta\omega\backslash\omega$ under CH

In this section we will see how $\beta\omega$ and ω^* behave under CH.

1.1. A characterization of $\mathcal{P}(\omega)$/fin

Let $\mathcal{B}$ be a Boolean algebra and let $F, G \subseteq \mathcal{B}$. We say that $F < G$ provided that for all $F' \in [F]^{<\omega}$, $G' \in [G]^{<\omega}$ we have that $\vee F' < \wedge G'$.

1.1.1. DEFINITION. Let $\mathcal{B}$ be a Boolean algebra. We say that $\mathcal{B}$ satisfies *condition* H_ω provided that for all $F \in [\mathcal{B}\backslash\{1\}]^{\leq\omega}$ and $G \in [\mathcal{B}\backslash\{0\}]^{\leq\omega}$ such that $F < G$ there is an element $x \in \mathcal{B}$ such that $F < \{x\} < G$.

1.1.2. LEMMA. *$\mathcal{P}(\omega)$/fin satisfies condition H_ω.*

PROOF. We will begin by proving the following assertion: if $A \in [\mathcal{P}(\omega)/\mathrm{fin}]^{\leq\omega}$ and $\{0\} < A$, then there is a $y \in \mathcal{P}(\omega)$/fin such that $\{0\} < \{y\} < A$. Indeed, enumerate A

as $\{a_n : n < \omega\}$, and for all $n < \omega$, let $C_n \in [\omega]^\omega$ be a representative of a_n. By induction, pick points y_n for all $n < \omega$, such that

$$y_n \in \bigcap_{0 \le i \le n} C_i \backslash \{y_0, y_1, \ldots, y_{n-1}\}$$

Let $Y = \{y_n : n < \omega\}$ and let y be the element of $\mathcal{P}(\omega)/\text{fin}$ corresponding to y. It is a good exercise to show that $\{0\} < \{y\} < A$.

Now let us return to the proof of the lemma. Suppose that $F, G \in [\mathcal{P}(\omega)/\text{fin}]^{\le\omega}$, $1 \notin F$, $0 \notin G$ and $F < G$. If $\vee F$ or $\wedge G$ exists, then using the above assertion, it is easy to find the required x. So assume that this is not the case. Enumerate F as $\{f_n : n < \omega\}$ and G as $\{g_n : n < \omega\}$. It is clear that without loss of generality we may assume that $f_0 < f_1 < \cdots$ and $g_0 > g_1 > \cdots$. For each $n < \omega$ take representatives $A_n, B_n \in [\omega]^\omega$ of f_n, resp. g_n. By induction on $k < \omega$, pick a point $d_k < \omega$ such that

(1) $d_k \in \bigcap_{0 \le i \le k} B_i \backslash (\bigcup_{0 \le i \le k} A_i \cup \{d_0, \ldots, d_{k-1}\})$

and put $D = \{d_k : k < \omega\}$. In addition, define

$$A' = \bigcup_{k<\omega} (A_k \cap \bigcap_{0 \le i \le k} B_i) .$$

Put $C = A' \cup D$. Then $C \in [\omega]^\omega$ while moreover

(2) if $n < \omega$, then $|A_n \backslash C| < \omega$, and

(3) if $m < \omega$, then $|C \backslash B_m| < \omega$.

Let x be the element of $\mathcal{P}(\omega)/\text{fin}$ corresponding to C. It is easy to see that (2) and (3) imply that $F < \{x\} < G$. □

1.1.3. DEFINITION. Let $\mathcal{B}$ be a Boolean algebra. We say that $\mathcal{B}$ satisfies *condition* R_ω provided that for all nonempty $F \in [\mathcal{B}\backslash\{1\}]^{\le\omega}$, $G \in [\mathcal{B}\backslash\{0\}]^{\le\omega}$ and $H \in [\mathcal{B}]^{\le\omega}$ such that

(1) $F < G$, and

(2) $\forall \tilde{F} \in [F]^{<\omega} \forall \tilde{G} \in [G]^{<\omega} \forall h \in H : h \not\le \vee \tilde{F}$ and $\wedge \tilde{G} \not\le h$,

there is an element $x \in \mathcal{B}$ such that

(3) $F < \{x\} < G$, and

(4) $\forall h \in H$: $h \not\le x$ and $x \not\le h$.

The main reason that ω^* is relatively easy to deal with under CH is, as we will see later, because of the following lemma.

1.1.4. LEMMA. *If a Boolean algebra $\mathcal{B}$ satisfies condition H_ω, then it satisfies condition R_ω.*

PROOF. Let $F \in [\mathcal{B}\backslash\{1\}]^{\leq\omega}$, $G \in [\mathcal{B}\backslash\{0\}]^{\leq\omega}$ and $H \in [\mathcal{B}]^{\leq\omega}$ be as in 1.1.3 (1) and (2). Enumerate F as $\{f_n : n < \omega\}$, G as $\{g_n : n < \omega\}$ and H as $\{h_n : n < \omega\}$. For each $h \in H$ and finite $\tilde{F} \in [F]^{<\omega}$ we have that $(\vee\tilde{F})' \wedge h \neq 0$, consequently there exists, by applying condition H_ω for all $n < \omega$, an element $d_n \in \mathcal{B}\backslash\{0\}$ such that

$$(1)\ d_n < h_n \quad \text{and} \quad \forall f \in F, f \wedge d_n = 0\,.$$

Similarly, we can find $e_n \in \mathcal{B}\backslash\{0\}$ such that

$$(2)\ \{e_n\} < G \quad \text{and} \quad e_n \wedge h_n = 0\,.$$

If the d_n's and e_n's are chosen with a little extra care, we can assure that $e_n \wedge d_m = 0$ for all n, $m < \omega$. Now define for all $n < \omega$,

$$\tilde{f}_n = f_n \vee e_n, \quad \text{and} \quad \tilde{g}_n = g_n \wedge d'_n\,.$$

Notice if n, $m < \omega$ then $\bigvee_{0 \leq i \leq n} \tilde{f}_i \leq \bigwedge_{0 \leq j \leq m} \tilde{g}_j$. By H_ω, we can therefore find an element $x \in \mathcal{B}$ such that for all n, $m < \omega$,

$$\bigvee_{0 \leq i \leq n} \tilde{f}_i \leq x \leq \bigwedge_{0 \leq j \leq m} \tilde{g}_j\,.$$

An easy check shows that x is as required. □

1.1.5. COROLLARY. *$\mathcal{P}(\omega)$/fin satisfies condition R_ω.*

We now come to the main result of this section. The proof we give is slightly incomplete. The reader is encouraged to fill in all missing details (in case of problems, see COMFORT & NEGREPONTIS [1974, Lemma 6.10]). If $\mathcal{B}$ is a Boolean algebra (abbreviated: BA) and if $A \subseteq \mathcal{B}$, then $\langle\langle A\rangle\rangle \subseteq \mathcal{B}$ denotes the subalgebra of $\mathcal{B}$ generated by A.

1.1.6. THEOREM (CH). *If $\mathcal{B}$ is a Boolean algebra of cardinality at most $\mathfrak{c}$ satisfying condition H_ω, then $\mathcal{B}$ is isomorphic to $\mathcal{P}(\omega)$/fin.*

PROOF. Let $\mathcal{B}$ and $\mathcal{E}$ be BA's satisfying condition H_ω such that $|\mathcal{B}|$, $|\mathcal{E}| \leq \mathfrak{c}$. By CH list $\mathcal{B}$ as $\{b_\alpha : \alpha < \omega_1\}$ and $\mathcal{E}$ as $\{e_\alpha : \alpha < \omega_1\}$.

Without loss of generality we may assume that $e_0 = 0$ and $b_0 = 0$. By transfinite induction, for $\alpha < \omega_1$ we will construct countable subalgebras $\mathcal{B}_\alpha \subseteq \mathcal{B}$ and $\mathcal{E}_\alpha \subseteq \mathcal{E}$ and an isomorphism $\sigma_\alpha : \mathcal{B}_\alpha \to \mathcal{E}_\alpha$ such that

(1) $b_\alpha \in \mathcal{B}_\alpha$ and $e_\alpha \in \mathcal{E}_\alpha$,

(2) if $\beta < \alpha$, then $\mathcal{B}_\beta \subseteq \mathcal{B}_\alpha$, $\mathcal{E}_\beta \subseteq \mathcal{E}_\alpha$ and $\sigma_\alpha \restriction \mathcal{B}_\beta = \sigma_\beta$.

Let $\mathcal{B}_0 = \{0, 1\}$ and $\mathcal{E}_0 = \{0, 1\}$ and let $\sigma_0 : \mathcal{B}_0 \to \mathcal{E}_0$ be defined in the obvious way. Suppose that $\mathcal{B}_\beta$, $\mathcal{E}_\beta$ and σ_β are defined for all $\beta < \alpha < \omega_1$ satisfying (1) and (2). If $b_\alpha \in \bigcup_{\beta<\alpha} \mathcal{B}_\beta$ and $e_\alpha \in \bigcup_{\beta<\alpha} \mathcal{E}_\beta$, then define $\mathcal{B}_\alpha = \bigcup_{\beta<\alpha} \mathcal{B}_\beta$, $\mathcal{E}_\alpha = \bigcup_{\beta<\alpha} \mathcal{E}_\beta$ and

$\sigma_\alpha = \bigcup_{\beta<\alpha} \sigma_\beta$. Suppose next that e.g. $b_\alpha \notin \bigcup_{\beta<\alpha} \mathscr{B}_\beta = \mathscr{F}$. Let $\sigma = \bigcup_{\beta<\alpha} \sigma_\beta$. Put

$$\mathscr{F}_0 = \{f \in \mathscr{F} : f < b_\alpha\}, \qquad \mathscr{F}_1 = \{f \in \mathscr{F} : b_\alpha < f\}, \quad \text{and} \quad \mathscr{F}_2 = \mathscr{F}\backslash(\mathscr{F}_0 \cup \mathscr{F}_1).$$

By Lemma 1.1.4 there is an element $e \in \mathscr{E}$ such that $\sigma(\mathscr{F}_0) < \{e\}$, $\{e\} < \sigma(\mathscr{F}_1)$, and for all $\tilde{e} \in \sigma(\mathscr{F}_2)$, $\tilde{e} \not\leq e$ and $e \not\leq \tilde{e}$. If we put $\sigma(b_\alpha) = e$ and $\sigma(b'_\alpha) = e'$, then σ can be extended to an isomorphism $\tilde{\sigma}: \langle\langle \mathscr{F} \cup \{b_\alpha\}\rangle\rangle \to \langle\langle \sigma(\mathscr{F}) \cup \{e\}\rangle\rangle$. If $e_\alpha \notin \langle\langle \sigma(\mathscr{F}) \cup \{e\}\rangle\rangle$, then we do the same thing as above with σ replaced by σ^{-1}. This shows how to construct $\mathscr{B}_\alpha$ and $\mathscr{E}_\alpha$.

We conclude that $\mathscr{B}$ and $\mathscr{E}$ are isomorphic and by Corollary 1.1.5, this also shows that both $\mathscr{B}$ and $\mathscr{E}$ are isomorphic to $\mathscr{P}(\omega)/\text{fin}$. □

1.1.7. REMARK. Observe that each BA satisfying condition H_ω has cardinality at least $\mathfrak{c}$.

1.2. A topological translation

In this section we will translate the results of Section 1.1 in topological language.

Let X be a space. A subset $A \subseteq X$ is called *C^*-embedded* in X provided that each map $f: A \to [0, 1]$ can be extended to a map $f: X \to [0, 1]$.

1.2.1. DEFINITION. A space X is called an *F-space* if each cozero-set in X is C^*-embedded in X.

The following lemma summarizes some relevant information on F-spaces.

1.2.2. LEMMA. (a) *X is an F-space iff βX is F-space.*

(b) *A normal space X is an F-space iff any two disjoint open F_σ subsets of X have disjoint closures in X.*

(c) *Each basically disconnected space is an F-space.*

(d) *Any closed subspace of a normal F-space is again an F-space.*

(e) *If an F-space X satisfies the countable chain condition, then it is extremally disconnected.*

PROOF. For (a), use that X is C^*-embedded in βX. The proof of (b) is routine and (c) is trivial. The proof of (d) is easy if one uses the characterization of normal F-spaces stated in (b). For (e), first observe that it suffices to show that disjoint open subsets of X have disjoint closures. Let U, $V \subseteq X$ be open and disjoint. Use the fact that X is ccc to find dense cozero-sets $U' \subseteq U$ and $V' \subseteq V$. The function $f: U' \cup V' \to [0, 1]$ defined by $f(x) = 0$ if $x \in U'$ and $f(x) = 1$ if $x \in V'$ can be extended to a map $\bar{f}: X \to [0, 1]$. Since $U \subseteq \bar{f}^{-1}(\{0\})$ and $V \subseteq \bar{f}^{-1}(\{1\})$, we conclude that $\bar{U} \cap \bar{V} = \emptyset$. □

The following result gives a topological translation of condition H_ω.

1.2.3. LEMMA. *Let X be a compact zero-dimensional space. The following statements are equivalent*:

(1) *$\mathscr{B}(X)$ satisfies condition H_ω*,

(2) *X is an F-space and each nonempty G_δ in X has infinite interior.*

PROOF. (1)$\Rightarrow$(2) follows directly from Lemma 1.2.2(b) and the fact that in a compact zero-dimensional space every open F_σ is a countable union of clopen sets. That (2) implies (1) is routine. $\square$

1.2.4. COROLLARY (CH). *Let X be a space. The following statements are equivalent*:

(1) *$X \approx \omega^*$*,

(2) *X is a compact zero-dimensional F-space of weight $\mathfrak{c}$ in which each nonempty G_δ has infinite interior.*

PROOF. Follows directly from Theorem 1.1.6 and Lemma 1.2.3. $\square$

A compact zero-dimensional F-space of weight $\mathfrak{c}$ in which each non-empty G_δ has infinite interior, will be called a *Parovičenko space* from now on. Corollary 1.2.4 says that, under CH, ω^* is, up to homeomorphism, the only Parovičenko space.

Corollary 1.2.4 is a very useful result since it turns out that the class of Parovičenko spaces is quite large. The following result, which is of independent interest, is the key in finding more Parovičenko spaces.

1.2.5. THEOREM. *Let X be a locally compact, σ-compact and noncompact space. Then X^* is an F-space and each nonempty G_δ in X^* has infinite interior.*

PROOF. Let $F \subseteq X^*$ be any F_σ and let $f: F \to [0,1]$ be continuous. Since $Y = X \cup F$ is σ-compact, it is normal and therefore, since F is closed in Y, the Tietze Extension Theorem implies that f can be extended to a map $\bar{f}: Y \to [0,1]$. Put $g = \bar{f} \restriction X$. Then g can be extended to a map $\bar{g}: \beta X \to [0,1]$. Since clearly $\bar{g} \restriction F = f$, we see that $\tilde{f} = \bar{g} \restriction X^*$ is the required extension of f.

Let $S \subseteq X^*$ be a nonempty G_δ. Since the set $\{U' : U \text{ open in } X\}$ is a basis for X^*, it is clear that we can find open sets $U_n \subseteq X$ for all $n < \omega$, such that

$$\bar{U}_{n+1} \subseteq U_n \quad \text{and} \quad \emptyset \neq \bigcap_{n<\omega} U'_n \subseteq S$$

(since $U_n \subseteq X$ for all n, the bar means closure in X of course). Since X is locally compact and σ-compact, we can write X as $\bigcup_{n<\omega} K_n$, where each K_n is compact and moreover each compact $K \subseteq X$ is contained in some K_n. For each $n < \omega$ choose a nonempty open set $V_n \subseteq U_n$ such that

$$\bar{V}_n \text{ is compact and misses } K_n .$$

Put $V = \bigcup_{n<\omega} V_n$. If $n < \omega$, then $V\backslash U_n$ has compact closure in X, whence

$$V' \subseteq \bigcap_{n<\omega} U'_n \subseteq S .$$

In addition, $V' \neq \emptyset$ since V does not have compact closure in X. The easy proof that V' is infinite is left to the reader. □

We now present an interesting topological consequence of Theorem 1.1.6.

1.2.6. THEOREM (CH). *Let X be a zero-dimensional, locally compact, σ-compact, noncompact space of weight at most $\mathfrak{c}$. Then X^* and ω^* are homeomorphic.*

PROOF. Since X is a zero-dimensional Lindelöf space, X is strongly zero-dimensional and has at most $\mathfrak{c}^\omega = \mathfrak{c}$ clopen sets. It follows that X^* is a zero-dimensional compact space of weight at most $\mathfrak{c}$. By Theorem 1.2.5 and Lemma 1.2.3 $\mathscr{B}(X^*)$ satisfies condition H_ω. This implies that $\mathscr{B}(X^*)$ and $\mathscr{P}(\omega)/\text{fin}$ are isomorphic (Theorem 1.1.6), and consequently, by Stone duality, that X^* and ω^* are homeomorphic. □

1.3. Continuous images of ω^*

In this section we characterize the continuous images of ω^*.

1.3.1. THEOREM. *Let $\mathscr{B}$ be a Boolean algebra of cardinality at most ω_1. Then $\mathscr{B}$ can be embedded in $\mathscr{P}(\omega)/\text{fin}$.*

PROOF. Use the same technique as in the proof of Theorem 1.1.6. □

By Stone duality, Theorem 1.3.1 is equivalent to the statement that each compact and zero-dimensional space of weight at most ω_1 is a continuous image of ω^*. This result suggests the question whether the same result holds without the assumption on zero-dimensionality. This is indeed the case, see Theorem 1.3.3 below.

1.3.2. LEMMA. *Let X be a compact space of weight κ. Then there is a compact zero-dimensional space Y of weight κ which can be mapped onto X.*

PROOF. Let $\mathscr{B} \in [\text{RO}(X)]^\kappa$ be such that $\mathscr{B}$ is a basis and put $\mathscr{E} = \langle\langle\mathscr{B}\rangle\rangle \subseteq \text{RO}(X)$. Observe that $|\mathscr{E}| = \kappa$. Let Y be the Stone space of $\mathscr{E}$. □

1.3.3. THEOREM. *Each compact space of weight at most ω_1 is a continuous image of ω^*.*

PROOF. Let X be a compact space of weight ω_1 and let Y be as in Lemma 1.3.2. By Theorem 1.3.1, $\mathscr{B}(Y)$ embeds in $\mathscr{P}(\omega)/\text{fin}$. Consequently, by Stone duality, ω^* can be mapped onto Y. $\square$

1.3.4. COROLLARY (CH). *Each compact space of weight at most $\mathfrak{c}$ is a continuous image of ω^*.*

1.4. Closed subspaces of $\beta\omega$

In this section we characterize topologically the closed subspaces of $\beta\omega$. If X is a closed subspace of $\beta\omega$, then X must clearly be of weight at most $\mathfrak{c}$ and X must be a zero-dimensional compact F-space by Lemma 1.2.2(d). It turns out that, under CH, these conditions are not only necessary but also sufficient.

Let X be a space. A subset of $B \subseteq X$ is called a *P-set* provided that the intersection of countably many neighborhoods of B is again a neighborhood of B.

Let X and Y be compact spaces. Let $A \subseteq X$ be closed and let $f: A \to Y$ be a continuous surjection. It is easily seen that the collection

$$\mathscr{B} = \{f^{-1}(y): y \in Y\} \cup \{\{x\}: x \in X \backslash A\}$$

is an upper-semicontinuous decomposition of X; the decomposition space $X/\mathscr{B}$ will be denoted by $X \cup_f Y$. If $\pi: X \to X \cup_f Y$ is the decomposition map, then we identify Y and $\pi(A)$.

1.4.1. LEMMA. *Let X and Y be compact F-spaces, let $A \subseteq X$ be a closed P-set, and let $f: A \to Y$ be a continuous surjection. Then $X \cup_f Y$ is an F-space.*

PROOF. Let U and V be disjoint open F_σ subsets of $X \cup_f Y$. Since Y is an F-space, $(U \cap Y)^- \cap (V \cap Y)^- = \emptyset$. Let E and F be closed G_δ neighborhoods of $(U \cap Y)^-$ and $(V \cap Y)^-$ such that $E \cap F = \emptyset$. Then $U \backslash E$ and $V \backslash F$ are disjoint open F_σ subsets of X which both do not meet A. Since X is an F-space and A is a P-set,

(1) $(U \backslash E)^- \cap (V \backslash F)^- = \emptyset$, and

(2) $((U \backslash E)^- \cup (V \backslash F)^-) \cap A = \emptyset$,

This easily implies that $\bar{U} \cap \bar{V} = \emptyset$.

By Lemma 1.2.2(b), we may now conclude that $X \cup_f Y$ is an F-space. $\square$

1.4.2. LEMMA. *Let X be a compact space with the property that each nonempty G_δ has infinite interior. If $A \subseteq X$ is closed and nowhere dense and if $f: A \to Y$ is a continuous surjection, then the space $X \cup_f Y$ has also the property that each nonempty G_δ has infinite interior.*

PROOF. Obvious. $\square$

1.4.3. LEMMA (CH). *ω^* contains a nowhere dense closed P-set A which is homeomorphic to ω^*.*

PROOF. By Theorem 1.2.6, we can represent ω^* by

$$Z = (\omega \times W(\omega_1 + 1))^* .$$

Let $A = (\omega \times \{\omega_1\})^*$. Trivially, $A \approx \omega^*$ and that A is a P-set follows easily from the fact that ω_1 is a P-point in $W(\omega_1 + 1)$. That A is nowhere dense is clear. □

We now come to the main result of this section.

1.4.4. THEOREM (CH). *Let X be a space. The following statements are equivalent:*

(1) *X is a compact zero-dimensional F-space of weight at most $\mathfrak{c}$,*
(2) *X can be embedded in $\beta\omega$ as a closed subspace,*
(3) *X can be embedded as a nowhere dense closed P-set in ω^*.*

PROOF. The implications (2)$\Rightarrow$(1) and (3)$\Rightarrow$(2) are trivial, so it suffices to prove that (1)$\Rightarrow$(3). To this end, let X be a compact zero-dimensional F-space of weight at most $\mathfrak{c}$. By Lemma 1.4.3 we can find a closed nowhere dense P-set A of ω^* such that $A \approx \omega^*$. In addition, by Corollary 1.3.4, there is a continuous surjection $f: A \to X$. Put $Z = \omega^* \cup_f X$. It is routine to verify that

(a) Z is zero-dimensional,
(b) Z is of weight $\mathfrak{c}$,
(c) X is a nowhere dense closed P-set of Z.

Lemma 1.4.1 followed by Lemma 1.4.2 imply that Z is a compact F-space in which each nonempty G_δ has infinite interior. Consequently, by Corollary 1.2.4, $Z \approx \omega^*$. □

Let $\mathscr{B}$ be a Boolean algebra. We say that $\mathscr{B}$ is *weakly countably complete* (abbreviated: WCC) iff the Stone space of $\mathscr{B}$ is an F-space. In Boolean algebraic language, $\mathscr{B}$ is a WCC BA iff $\mathscr{B}$ is a BA and

$$\forall B, C \in [\mathscr{B}]^{\leq\omega} \text{ such that } \forall b \in B\ \forall c \in C: b \wedge c = 0 ,$$

$$\text{there is an } a \in \mathscr{B} \text{ with } b \leq a \leq c' \text{ for all } b \in B,\ c \in C .$$

The following result is a purely Boolean algebraic consequence of Theorem 1.4.4.

1.4.5. THEOREM (CH). *Let $\mathscr{B}$ be a Boolean algebra. The following statements are equivalent:*

(1) *$\mathscr{B}$ is WCC and $|\mathscr{B}| \leq \mathfrak{c}$,*
(2) *$\mathscr{B}$ is a homomorphic image of $\mathscr{P}(\omega)$.*

1.4.6. COROLLARY (CH). *Each WCC Boolean algebra of cardinality at most* $\mathfrak{c}$ *is a homomorphic image of a complete Boolean algebra.*

We will now prove an interesting result without the aid of the CH.

1.4.7. THEOREM. *Let* X *be a compact extremally disconnected space of weight at most* $\mathfrak{c}$. *Then* X *can be embedded in* $\beta\omega$.

PROOF. We may assume that $X \subseteq I^{\mathfrak{c}}$, where as usual, $I = [0, 1]$. Since $I^{\mathfrak{c}}$ is separable, ENGELKING [1977, 2.3.16], there is a continuous surjection $f : \beta\omega \to I^{\mathfrak{c}}$. Let $g = f \restriction f^{-1}(X)$ and take a closed $Z \subseteq f^{-1}(X)$ such that $g \restriction Z : Z \to X$ is irreducible. The existence of Z easily follows from Zorn's Lemma (order all closed sets of $f^{-1}(X)$ that map onto X by reverse inclusion). We claim that $h = g \restriction Z$ is a homeomorphism. For this it suffices to show that h is one to one. To this end, take distinct points $x, y \in Z$. Find disjoint clopen neighborhoods U and V of, respectively, x and y (in Z). Since h is irreducible,

$$h(x) \in \overline{\operatorname{int} h(U)}, \qquad h(y) \in \overline{\operatorname{int} h(V)}, \quad \text{and} \quad \operatorname{int} h(U) \cap \operatorname{int} h(V) = \emptyset .$$

Consequently, by the extremal disconnectivity of X,

$$\overline{\operatorname{int} h(U)} \cap \overline{\operatorname{int} h(V)} = \emptyset ,$$

and we conclude that $h(x) \neq h(y)$. $\square$

1.5. C^*-embedded subspaces of $\beta\omega$

In this section we will characterize those subspaces of $\beta\omega$ that are C^*-embedded in $\beta\omega$. It is interesting that being C^*-embedded in $\beta\omega$ turns out to be a topological property and does not depend on how a given set is placed in $\beta\omega$.

1.5.1. DEFINITION. A space X is called *weakly Lindelöf* provided that for any open cover $\mathcal{U}$ of X there is a countable subfamily $\mathcal{E} \subseteq \mathcal{U}$ such that $(\bigcup \mathcal{E})^- = X$.

Observe that each space satisfying the countable chain condition is weakly Lindelöf.

The following important result shows that $\beta\omega$ has 'many' C^*-embedded subspaces.

1.5.2. THEOREM. *Let* $X \subseteq \beta\omega$ *be weakly Lindelöf. Then* X *is* C^**-embedded in* $\beta\omega$.

PROOF. It clearly suffices to show that disjoint zero-sets in X have disjoint closures in $\beta\omega$, COMFORT & NEGREPONTIS [1974, Theorem 2.6]. To prove this, let Z_0,

$Z_1 \subseteq X$ be a disjoint zero-sets. There are disjoint open neighborhoods U and V of Z_0 and Z_1 such that $\mathrm{cl}_X U \cap \mathrm{cl}_X V = \emptyset$. For each $x \in X$ let $C_x \subseteq \beta\omega$ be a clopen neighborhood of x in $\beta\omega$ such that

(1) if $x \in \mathrm{cl}_X U$, then $C_x \cap \mathrm{cl}_X V = \emptyset$,
(2) if $x \in \mathrm{cl}_X V$, then $C_x \cap \mathrm{cl}_X U = \emptyset$,
(3) if $x \notin (\mathrm{cl}_X U \cup \mathrm{cl}_X V)$, then $C_x \cap (\mathrm{cl}_X U \cup \mathrm{cl}_X V) = \emptyset$.

Since X is weakly Lindelöf, there is a sequence x_n $(n < \omega)$ in X such that $\bigcup_{n<\omega} (C_{x_n} \cap X)$ is dense in X. For each $n < \omega$, put

$$E_n = C_{x_n} \backslash \bigcup_{i<n} C_{x_i}.$$

Then the family $\{E_n : n < \omega\}$ is a pairwise disjoint collection clopen subsets of $\beta\omega$ such that

(1) $\bigcup_{n<\omega} (E_n \cap X)$ is dense in X, and
(2) each E_n meets at most one of the $\mathrm{cl}_X U$ and $\mathrm{cl}_X V$.

Put $E = \bigcup\{E_n : n < \omega \ \& \ E_n \cap U \neq \emptyset\}$ and $F = \bigcup\{E_n : n < \omega \ \& \ E_n \cap V \neq \emptyset\}$. Then $E \cap F = \emptyset$ and therefore, since $\beta\omega$ is an F-space, $\bar{E} \cap \bar{F} = \emptyset$. Since obviously $U \subseteq \bar{E}$ and $V \subseteq \bar{F}$, we conclude that Z_0 and Z_1 have disjoint closures in $\beta\omega$. □

We will now show that, under CH, the converse of Theorem 1.5.2 is true, thus giving a *topological* characterization of those subspaces of $\beta\omega$ that are C^*-embedded in $\beta\omega$.

1.5.3. THEOREM (CH). *Let $X \subseteq \beta\omega$. The following statements are equivalent*:

(1) *X is weakly Lindelöf*,
(2) *X is C^*-embedded in $\beta\omega$*,
(3) $|C^*(X)| = \mathfrak{c}$.

PROOF. (1)$\Rightarrow$(2) is shown in Theorem 1.5.2 and (2)$\Rightarrow$(3) is clear since $|C^*(\beta\omega)| = \mathfrak{c}$. It therefore suffices to prove that the negation of (1) imples the negation of (3).

If X is not weaky Lindelöf then, by CH, there is a family $\{C_\alpha : \alpha < \omega_1\}$ of clopen subsets of $\beta\omega$ such that

(1) $X \subseteq \bigcup_{\alpha<\omega_1} C_\alpha$,
(2) for each $\alpha < \omega_1$, $X \backslash (\bigcup_{\beta<\alpha} C_\beta \cap X)^- \neq \emptyset$.

We can therefore find a strictly increasing sequence of ordinals $\kappa_\alpha < \omega_1$ $(\alpha < \omega_1)$ and for each $\alpha < \omega_1$ a clopen set $D_\alpha \subseteq C_{\kappa_\alpha}$ such that

(3) $D_\alpha \cap X \neq \emptyset$,
(4) $D_\alpha \cap (\bigcup_{\beta<\alpha} (C_\beta \cap X) \cup \bigcup_{\beta<\alpha} (D_{\kappa_\beta} \cap X))^- = \emptyset$.

For each $\alpha < \omega_1$, put $\tilde{C}_\alpha = C_\alpha \cap \bar{X}$ and let $D = \bigcup_{\alpha<\omega_1} D_\alpha \cap \bar{X}$.

Claim. D is C^*-embedded in $\bigcup_{\alpha<\omega_1} \tilde{C}_\alpha$.

Let $f: D \to [0, 1]$ be given. For each $\alpha < \omega_1$ put

$$f_\alpha = f \restriction D \cap \bigcup_{\beta \leqslant \alpha} \tilde{C}_\beta .$$

Observe that (4) implies that $\mathrm{dom}(f_\alpha)$ is an open F_σ-subset of $\bar{X}$ for all $\alpha < \omega_1$. We will construct, for each $\alpha < \omega_1$, an extension $g_\alpha : \bigcup_{\beta \leqslant \alpha} \tilde{C}_\beta \to I$ of f_α such that for all $\beta < \alpha$,

$$g_\beta \subseteq g_\alpha .$$

Suppose this is done for all $\beta < \alpha$. The function $\bigcup_{\beta<\alpha} g_\beta \cup f_\alpha$ is continuous on $\bigcup_{\beta<\alpha} \tilde{C}_\beta \cup (D \cap \tilde{C}_\alpha)$, and this set is an open F_σ-subset of $\bar{X}$. Therefore, since $\bar{X}$ is an F-space (Lemma 1.2.2(d)), this function can be extended to get the required g_α.

Finally put $g = \bigcup_{\alpha<\omega_1} g_\alpha$. It is clear that g is as required.

Since D is a union of ω_1 pairwise disjoint nonempty clopen sets, $|C^*(D)| \leqslant 2^{\omega_1}$ and consequently, by the Claim, $|C^*(\bigcup_{\alpha<\omega_1} \tilde{C}_\alpha)| \geqslant 2^{\omega_1}$. Since X is dense in $\bigcup_{\alpha<\omega_1} \tilde{C}_\alpha$ this imples that

$$|C^*(X)| \geqslant 2^{\omega_1} > \mathfrak{c} ,$$

which is a contradiction. $\square$

1.5.4. COROLLARY (CH). *If $x \in \omega^*$, then $\omega^* \backslash \{x\}$ is not C^*-embedded in ω^*.*

PROOF. If $\omega^* \backslash \{x\}$ is C^*-embedded in ω^*, then $\omega^* \backslash \{x\}$ is C^*-embedded in $\beta\omega$ since ω^* is C^*-embedded in $\beta\omega$. By Theorem 1.5.3 it therefore suffices to prove the following easy

Fact. If $x \in \omega^*$, then $\omega^* \backslash \{x\}$ is not weakly Lindelöf.

Assume, to the contrary, that $\omega^* \backslash \{x\}$ is weakly Lindelöf. Put $\mathcal{U} = \{C \subseteq \omega^* : C$ is clopen and $x \notin C\}$. Since by assumption $\omega^* \backslash \{x\}$ is weakly Lindelöf, there are $C_n \in \mathcal{U}$ $(n < \omega)$ such that $\bigcup_{n<\omega} C_n$ is dense in ω^*. By Lemma 1.1.2 or by Theorem 1.2.5, there is a nonempty clopen $E \subseteq \omega^*$ such that $E \cap (\bigcup_{n<\omega} C_n) = \emptyset$. It is clear that without loss of generality we may assume that $x \notin E$. Then E must meet $\bigcup_{n<\omega} C_n$, which is not the case and therefore we have obtained the desired contradiction. $\square$

1.6. Autohomeomorphisms of ω^*

In this section we will concentrate on autohomeomorphisms of ω^*. Our main results are Theorems 1.6.4 and 1.6.5.

If π: $\omega \to \omega$ is a permutation, then $\beta\pi \restriction \omega^*$ is an autohomeomorphism of ω^*.

Let π_0 and π_1 be two permutations of ω. We claim that if $\beta\pi_0\restriction\omega^* = \beta\pi_1\restriction\omega^*$, then $|\{n<\omega : \pi_0(n) \neq \pi_1(n)\}| < \omega$. If not, then we can find an infinite set $E \subseteq \omega$ such that $\pi_0(E) \cap \pi_1(E) = \emptyset$. Take $x \in \omega^*$ such that $E \in x$. Since $\pi_i(E) \in \beta\pi_i(x)$ for $i<2$, we conclude that $\beta\pi_0(x) \neq \beta\pi_1(x)$, which contradicts our assumptions. Since it is clear that we can find a family $\{\pi_\xi : \xi < \mathfrak{c}\}$ of permutations of ω such that for all $\eta < \xi < \mathfrak{c}$ we have that $\{n : \pi_\eta(n) \neq \pi_\xi(n)\}$ is infinite, this shows that ω^* has at least $\mathfrak{c}$ autohomeomorphisms which are induced from permutations on ω. Are there others? Under CH, there are.

1.6.1. LEMMA (CH). *ω^* has precisely $2^{\mathfrak{c}}$ autohomeomorphisms.*

PROOF. Since by Theorem 1.2.6, $\omega^* \approx (\omega \times 2^{\mathfrak{c}})^*$ (here $2^{\mathfrak{c}}$ denotes the Cantor cube of weight $\mathfrak{c}$) and since $2^{\mathfrak{c}}$ has $2^{\mathfrak{c}}$ autohomeomorphisms, being a topological group of cardinality $2^{\mathfrak{c}}$, it easily follows that ω^* has at least $2^{\mathfrak{c}}$ autohomeomorphisms. Since ω^* has weight $\mathfrak{c}$, it cannot have more than $2^{\mathfrak{c}}$ autohomeomorphisms. $\square$

We will now prove two results which are steps in the proof of Theorem 1.6.4.

1.6.2. LEMMA. *Let U and V be noncompact open F_σ-subsets of ω^*. Then there is an autohomeomorphism $h:\omega^* \to \omega^*$ with $h(U) = V$.*

PROOF. Find partitions $\{A_n : n<\omega\}$ and $\{B_n : n<\omega\}$ of ω in infinite sets such that

$$U = \bigcup_{n<\omega} A_n^* \quad \text{and} \quad V = \bigcup_{n<\omega} B_n^* .$$

Let $\pi:\omega\to\omega$ be a permutation such that $\pi(A_n) = B_n$ for all $n<\omega$. Then $h = \beta\pi\restriction\omega^*$ is clearly as required. $\square$

1.6.3. COROLLARY (CH). *Let S and T be nowhere dense P-sets in ω^* such that $S \approx T \approx \omega^*$. Then there is an autohomeomorphism*

$$h:\omega^*\to\omega^* \quad \textit{with } h(S) = T,$$

PROOF. Let X be a homeomorph of $\omega\times\omega^*$ disjoint from ω^*. Since, by Theorem 1.2.6, $X^* \approx \omega^* \approx S$, we can identify X^* and S. In other words, we assume that $\beta X \cap \omega^* = S$. We topologize $Z_0 = \omega^* \cup X$ by pasting βX and ω^* together. Formally,

$$U \subseteq Z_0 \text{ is open} \quad \text{iff} \quad \begin{array}{l} U\cap\beta X \text{ is open in } \beta X \\ \text{and } U\cap\omega^* \text{ is open in } \omega^* . \end{array}$$

By using similar arguments as in the proofs of Lemmas 1.4.1 and 1.4.2, the reader can easily verify that Z_0 is a Parovičenko space, consequently $Z_0 \approx \omega^*$. Similarly, take a homeomorph Y of $\omega \times \omega^*$ disjoint from ω^* such that $\beta Y \cap \omega^* = T$ and form the Parovičenko space $Z_1 = \omega^* \cup Y$. By Lemma 1.6.2 there is a homeomorphism $\tilde{h}: Z_0 \to Z_1$ with $\tilde{h}(X) = Y$. Then $h = \tilde{h} \restriction \omega^*$ is clearly as required. $\square$

1.6.4. THEOREM (CH). *Let S, $T \subseteq \omega^*$ be nowhere dense P-sets such that $S \approx T \approx \omega^*$ and let $h: T \to S$ be a homeomorphism. Then h can be extended to a homeomorphism $\bar{h}: \omega^* \to \omega^*$.*

PROOF. Let $f: \omega^* \to \omega^*$ be a homeomorphism such that $f(S) \cap T = \emptyset$. It is clear that such homeomorphism exists since all clopen subsets of ω^* are homeomorphic to ω^* and $S \cup T$ is nowhere dense. Put $Z = f(S) \cup T$ and define $\varphi: Z \to Z$ by

$$\begin{cases} \varphi(t) = f(h(t)) & \text{if } t \in T, \\ \varphi(t) = h^{-1}(f^{-1}(t)) & \text{if } t \in f(S). \end{cases}$$

Now if we can extend $\varphi: Z \to Z$ to a homeomorphism $\bar{\varphi}: \omega^* \to \omega^*$ then $\bar{h} = f^{-1} \circ \bar{\varphi}$ is a homeomorphism of ω^* extending h. Since $Z \approx \omega^*$, in view of Corollary 1.6.3, it therefore suffices to prove the following

Fact. There is a nowhere dense P-set $A \subseteq \omega^*$ such that $A \approx \omega^*$ and each autohomeomorphism of A extends to an autohomeomorphism of ω^*.

Put $X = \omega \times W(\omega_1 + 1) \times \omega^*$ and $Y = \omega \times \{\omega_1\} \times \omega^*$. It is easy to see that $Y^* \subseteq X^*$ is a nowhere dense P-set. The projection $\pi: Y \to \omega^*$ extends to a map $\beta\pi: \beta Y \to \omega^*$. Let $f = \beta\pi \restriction Y^*$ and define $B = X^* \cup_f Y^*$. By Lemmas 1.4.1 and 1.4.2, B is a Parovičenko space. Obviously, $A = \omega^*$ is a nowhere dense P-set in B. Let $h: A \to A$ be any homeomorphism. The homeomorphism $\tilde{h} = \mathrm{id} \times \mathrm{id} \times h$ of X extends to a homeomorphism $\beta\tilde{h}: \beta X \to \beta X$. Define $\bar{h}: B \to B$ by

$$\begin{cases} \bar{h}(x) = h(x) & \text{if } x \in A, \\ \bar{h}(x) = \beta\tilde{h}(x) & \text{if } x \notin A. \end{cases}$$

An easy check shows that $\bar{h}$ is an autohomeomorphism of B extending h. Since $B \approx \omega^*$ (Corollary 1.2.4) this is as required. $\square$

The following result is, in a sense, also a result on extending homeomorphisms.

1.6.5. THEOREM (CH). *Let p, $q \in \omega^*$ be P-points. Then there is an autohomeomorphism $h: \omega^* \to \omega^*$ with $h(p) = q$.*

PROOF. Adapt the proof of Theorem 1.1.6. $\square$

Observe that Theorem 1.4.4 implies that P-points in ω^* exist.

1.6.6. REMARK. Theorem 1.6.4 and 1.6.5 have a common generalization. In VAN DOUWEN & VAN MILL [1981b] it will be shown that any homeomorphism between (arbitrary) nowhere dense closed P-sets extends to an autohomeomorphism of ω^*. The proof which van Douwen and I have of this result is conceptually simple, but technically complicated. Since we believe that the proof is not in its final form yet, in this section we have only worked out some special cases which have simpler proofs.

1.7. P-points and nonhomogeneity of ω^*

Since ω is homogeneous, ω^* looks homogeneous and the question naturally arises whether ω^* is homogeneous. We will prove that, under CH, this is not the case. We will show later that ω^* is not homogeneous in ZFC. Observe that Theorem 1.4.4 implies that, under CH, ω^* contains a P-point. If all points of ω^* are P-points, then the compactness of ω^* implies that ω^* is finite, which is clearly not the case. Therefore, ω^* contains both P-points and non P-points and we conclude that ω^* is not homogeneous under CH. The proof, just given here that there are P-points in ω^*, is not very economical. We will therefore give an easier proof of this fact.

1.7.1. LEMMA. *ω^* cannot be covered by ω_1 nowhere dense sets.*

PROOF. Let $\{D_\alpha : \alpha < \omega_1\}$ be a family of ω_1 nowhere dense subsets of ω^*. By using Lemma 1.1.2 or Theorem 1.2.5, find a family $\{C_\alpha : \alpha < \omega_1\}$ of nonempty clopen subsets of ω^* such that for all $\alpha < \omega_1$,

(1) $C_\alpha \cap D_\alpha = \emptyset$,

(2) if $\beta < \alpha$, then $C_\alpha \subseteq C_\beta$.

Consequently, any point of $\bigcap_{\alpha<\omega_1} C_\alpha$ misses $\bigcup_{\alpha<\omega_1} D_\alpha$. $\square$

1.7.2. COROLLARY (CH). *ω^* contains P-points.*

PROOF. Let $\mathscr{A} = \{\bar{U}\backslash U : U \subseteq \omega^* \text{ is an open } F_\sigma\}$. By CH, $|\mathscr{A}| \leq \omega_1$. By Lemma 1.7.1, $\omega^*\backslash \bigcup \mathscr{A} \neq \emptyset$ and each point of this set is a P-point. $\square$

Since by Theorem 1.6.5, for any two P-points $x, y \in \omega^*$, under CH there is an autohomeomorphism $h : \omega^* \to \omega^*$ with $h(x) = y$, all P-points are topologically the same. In view of the above results, one therefore naturally wonders whether P-points and non P-points are the only types of points in ω^*. This is not true, as the next result shows.

A point x of a space X is called a *weak P-point* provided that $x \notin \bar{F}$ for all countable $F \subseteq X\backslash\{x\}$.

1.7.3. THEOREM (CH). (1) *There is a weak P-point in ω^* which is not a P-point,*
(2) *There is a point $x \in \omega^*$ such that*
(a) *for some countable $F \subseteq \omega^* \backslash \{x\}$ we have that $x \in \bar{F}$,*
(b) *for all countable discrete $D \subseteq \omega^* \backslash \{x\}$ we have that $x \notin \bar{D}$.*

PROOF. Let $\mathscr{M}$ be the BA of Lebesgue measurable subsets of $[0, 1]$ and let $\mathscr{N}$ be the ideal of null-sets. We put $\mathscr{B} = \mathscr{M}/\mathscr{N}$. It is well-known, and easy to prove, that $|\mathscr{B}| = \mathfrak{c}$ and that $\mathscr{B}$ is complete. Consequently, $X = \mathrm{st}(\mathscr{B})$ is an extremally disconnected compactum of weight $\mathfrak{c}$. If $M \in \mathscr{M}$, then the $\mathscr{N}$-equivalence class of M is denoted by $[M]$. λ denotes Lebesgue measure.

Fact 1. *If $D \subseteq X$ is countable, then D is nowhere dense.*

Take $M \in \mathscr{M}$ and list D as $\{d_n : n < \omega\}$. Since d_n is an ultrafilter in the BA $\mathscr{B}$, there exists an element $M_n \in \mathscr{B}$ such that
(1) $[M_n] \in d_n$, and
(2) $\lambda(M_n) < 2^{-2-n} \cdot \lambda(M)$.
Then $\{x \in X : [M \backslash \bigcup_{n<\omega} M_n] \in x\}$ is a nonempty open subset of $[M]$ which misses $\{d_n : n < \omega\}$.

Fact 2. *X is* ccc.

Let $\mathscr{A} \subseteq \mathscr{M}$ be uncountable such that $\lambda(A) > 0$ for all $A \in \mathscr{A}$ while moreover the family

$$\{\{x \in X : [A] \in x\} : A \in \mathscr{A}\}$$

is pairwise disjoint. Let $\mathscr{U}$ be a countable open basis for $[0, 1]$ which is closed under finite unions and for all $U \in \mathscr{U}$, put

$$\mathscr{A}(U) = \{A \in \mathscr{A} : \lambda(A \cap U) > \tfrac{1}{2}\lambda(U)\}.$$

If $A \in \mathscr{A}$, then there is a compact $K \subseteq A$ with $\lambda(K) > 0$. For this K there is an element $U \in \mathscr{U}$ with $K \subseteq U$ and $\lambda(K) > \frac{1}{2}\lambda(U)$. We conclude that $A \in \mathscr{A}(U)$ and since A is arbitrarily chosen, this implies that

$$\bigcup_{U \in \mathscr{U}} \mathscr{A}(U) = \mathscr{A}.$$

Hence there must be an element $U \in \mathscr{U}$ such that $\mathscr{A}(U)$ is uncountable. But this contradicts the definition of $\mathscr{A}(U)$.

Fact 3. *There is a family $\mathscr{D}$ of $\mathfrak{c}$ nowhere dense subsets of X such that each nowhere dense subset of X is contained in an element of $\mathscr{D}$.*

Since X has weight $\mathfrak{c}$, by Fact 2 we can take $\mathscr{D}$ to be the collection of all nowhere dense closed G_δ's.

Fact 4 (CH). *There is a point $x \in Y^*$, where $Y = \omega \times X$, such that*
(1) *x is a P-point of Y^*,*
(2) *if $D \subseteq \omega \times X$ is any nowhere dense set, then $x \notin \bar{D}$.*

By Fact 3 and by CH there is a family $\mathscr{E}$ *of* ω_1 nowhere dense subsets of Y such that each nowhere dense subset of Y is contained in an element of $\mathscr{E}$. By Theorem 1.2.5, $Y^* \approx \omega^*$, and therefore by Lemma 1.7.1, there is a point

$$x \in Y^*\backslash(\bigcup\{E^* : E \in \mathscr{E}\} \cup \bigcup\{\bar{U}\backslash U : U \subseteq Y^* \text{ is an open } F_\sigma\})$$

(it is easily seen that if $E \in \mathscr{E}$, then $E^* \subseteq Y^*$ is nowhere dense). It is clear that x is as required.

Now, since βY is an extremally disconnected compactum of weight $\mathfrak{c}$, by Theorem 1.4.4 (3), βY can be embedded in ω^* as a closed P-set. If we take $x \in \beta Y$ such as in Fact 4, then Facts 1 and 2 imply (if we identify βY with a P-set in ω^*) that x is a weak P-point which is not a P-point (in ω^* as well as in βY). This proves (1). To prove (2), substitute X by the projective cover of the Cantor set and proceed similarly. The details of checking this out are left to the reader. □

1.7.4. REMARK. As we will see later, Theorem 1.7.3 is true in ZFC by a more complicated argument. We have included the above proof since this way of constructing points will be used frequently in the remaining part of this paper. If the reader understands the proof of Theorem 1.7.3, she or he will have less trouble understanding the more complicated forthcoming constructions.

1.8. Retracts of $\beta\omega$ and ω^*

In this section we study subspaces of $\beta\omega$ or ω^* on which $\beta\omega$ or ω^* can be retracted.

1.8.1. THEOREM (CH). *Let X be a closed P-set of ω^*. Then X is a retract of ω^*.*

PROOF. By CH, we can enumerate $\mathscr{B}(X)$ by $\{C_\alpha : \alpha < \omega_1\}$. It is easy, using the fact that X is a P-set, to construct for each $\alpha < \omega_1$ a countable subalgebra $\mathscr{B}_\alpha \subseteq \mathscr{B}(X)$ and an embedding $\rho_\alpha : \mathscr{B}_\alpha \to \mathscr{B}(\omega^*)$ such that

(1) if $\beta < \alpha$, then $\mathscr{B}_\beta \subseteq \mathscr{B}_\alpha$,
(2) if $\beta < \alpha$, then $\rho_\alpha \restriction \mathscr{B}_\beta = \rho_\beta$,
(3) $C_\alpha \in \mathscr{B}_\alpha$,
(4) if $C \in \mathscr{B}_\alpha$, then $\rho_\alpha(C) \cap X = C$.

Define $\rho : \mathscr{B}(X) \to \mathscr{B}(\omega^*)$ by $\rho(C) = \rho_\alpha(C)$ if $C \in \mathscr{B}_\alpha$.

Now define $r : \omega^* \to X$ by

$$\{r(x)\} = \bigcap\{C \in \mathscr{B}(X) : x \in \rho(C)\}.$$

An easy check shows that r is a retraction. □

We will now prove a result on retracts of $\beta\omega$ which does not use CH.

1.8.2. THEOREM. *Let $X \subseteq \beta\omega$ be a closed subspace of countable π-weight. Then X is a retract of $\beta\omega$.*

PROOF. Let $\mathscr{C}$ be a countable subalgebra of $\mathscr{B}(X)$ which forms a π-basis for X (for the definition of π-basis, see JUHÁSZ [1980] or HODEL [1983]). It is trivial to find a function $\rho : \mathscr{C} \to \mathscr{B}(\beta\omega)$ such that

(1) ρ is an embedding,

(2) if $C \in \mathscr{C}$, then $\rho(C) \cap X = C$.

Define a function $\kappa : \mathscr{B}(X) \to \mathscr{B}(\beta\omega)$ by

$$\kappa(A) = (\bigcup\{\rho(C) : C \in \mathscr{C}, C \subseteq A\})^{-} .$$

Since $\mathscr{P}(\omega)$ is complete or, in topological language, since $\beta\omega$ is extremally disconnected, κ is well-defined.

Fact 1. *If $A \in \mathscr{B}(X)$, then $\kappa(A) \cap X = A$.*

Since $\bigcup\{\rho(C) : C \in \mathscr{C},\ C \subseteq A\} \cap \bigcup\{\rho(C) : C \in \mathscr{C},\ C \subseteq X \backslash A\} = \emptyset$, this is immediate.

Fact 2. *κ is an embedding.*

This follows easily from the fact that ρ is an embedding.

Now, as in the proof of the previous theorem, define $r : \beta\omega \to X$ by

$$\{r(x)\} = \bigcap\{A \in \mathscr{B}(X) : x \in \kappa(A)\} .$$

By Stone dualty, r is continuous, and by Fact 1, $r \restriction X =$ identity. $\square$

Observe that the above theorem is interesting since it shows that a certain class of subspaces of $\beta\omega$ is always a retract of $\beta\omega$ no matter how these sets are placed in $\beta\omega$. However, topologically, there are not many closed subspaces of $\beta\omega$ which have countable π-weight, so in this sense the theorem is quite restrictive. M. TALAGRAND [1981] has given recently a quite complicated example of a separable closed subspace of $\beta\omega$ which is not a retract of $\beta\omega$ (his construction is under CH; it is desirable to find such an example in ZFC only). Therefore, the above theorem cannot be generalized. The following result shows precisly how far one can go, and it also illustrates the complexity of Talagrand's Example.

1.8.3. THEOREM. *Any separable, extremally disconnected compact space can be embedded in ω^* in such a way that it is a retract of $\beta\omega$.*

PROOF. Let X be a separable, extremally disconnected compact space. Since $\beta\omega$ maps onto each separable compact space, there is a continuous surjection $f : \beta\omega \to X$. Let $Z \subseteq \beta\omega$ be such that $f \restriction Z$ is an irreducible surjection from Z onto X. Since X is extremally disconnected, as in the proof of Theorem 1.4.7, $f \restriction Z$ is a homeomorphism.

Since ω^* is infinite, it contains a countable relatively discrete subspace D. By Theorem 1.5.2, $\bar{D} \approx \beta\omega$, and by Theorem 1.8.2, $\bar{D}$ is a retract of $\beta\omega$. Since, as was shown above, X embeds in $\bar{D}$ as a retract, the desired result follows. $\square$

1.9. Nowhere dense P-sets in ω^*

Nowhere dense P-sets have played an important role in this section. The question therefore naturally arises whether each point $x \in \omega^*$ is contained in a nowhere dense P-set. Under CH, we will show that this is not the case.

1.9.1. DEFINITION. Let $\kappa \geq \omega$. A subset A of a space X is called a *P_κ-set* provided that the intersection of fewer than κ neighborhoods of A is again a neighborhood of A.

P_{ω_1}-sets are precisely the P-sets of course, and any subset of any space is a P_ω-set.

1.9.2. LEMMA. *Let X be a space of π-weight $\leq\kappa$, where κ is regular and $\kappa \geq \omega$. For each $1 \leq n < \omega$ there is a family $\mathscr{F}_n$ of closed subsets of X such that*

(1) $\mathscr{F}_n$ has the n-intersection property,

(2) if $K \subseteq X$ is any nowhere dense P_κ-set, then for some $F \in \mathscr{F}_n$ we have that $F \cap K = \emptyset$.

PROOF. Let $\mathscr{U} = \{U_\alpha : \alpha < \kappa\}$ be a π-basis for X and let $\mathscr{D}$ be the family of all nowhere dense subsets of X. For each $D \in \mathscr{D}$, put

$$H(D) = \{\alpha < \kappa : \bar{U}_\alpha \cap D = \emptyset\}.$$

Define ordinals $\mu(D, m) < \kappa$ $(1 \leq m < \omega)$ as follows

$$\mu(D, 1) = \min H(D),$$

$$\mu(D, m) = \min\{\alpha < \kappa : \forall\beta \leq \mu(D, m-1)\, \exists\xi \leq \alpha\ \bar{U}_\xi \subseteq U_\beta\backslash\bar{D}\}$$

(observe that $\mu(D, 2)$ need not be defined if κ is singular).
Define $F(D, n) = \bigcup\{\bar{U}_\alpha : \alpha \in H(D) \text{ and } \alpha \leq \mu(D, n)\}$.

Fact. $\{F(D, n) : D \in \mathscr{D}\}$ *has the n-intersection property. In fact, if $\mathscr{E} \in [\mathscr{D}]^n$ then there is an $\alpha \leq \max\{\mu(D, n) : D \in \mathscr{E}\}$ such that $U_\alpha \subseteq \bigcap\{F(D, n) : D \in \mathscr{E}\}$.*

Induction on n. The case $n = 1$ is trivial. Suppose the fact to be true for all $i \leq n$ and take $D_1, D_2, \ldots, D_{n+1} \in \mathscr{D}$. We may assume that for all $i \leq n+1$ we have that $\mu(D_i, n) \leq \mu(D_{n+1}, n)$.

By induction hypothesis, there is an $\alpha \leq \max\{\mu(D_i, n) : 1 \leq i \leq n\}$ such that $U_\alpha \subseteq \bigcap\{F(D_i, n) : 1 \leq i \leq n\} \subseteq \bigcap\{F(D_i, n+1) : 1 \leq i \leq n\}$. Since $\alpha \leq \mu(D_{n+1}, n)$, there is a $\beta \leq \mu(D_{n+1}, n+1)$ with $\bar{U}_\beta \subseteq U_\alpha\backslash\bar{D}_{n+1}$.

Now let $\mathcal{F}_n = \{\overline{F(K,n)} : K \subseteq X$ is a nowhere dense P_κ-set$\}$. Since $F(K, n)$ is a union of less than κ closed sets each of which do not intersect K, clearly $\overline{F(K,n)} \cap K = \emptyset$. Consequently, $\mathcal{F}_n$ is as required. □

We now come to the main result of this section.

1.9.3. THEOREM. *Let X be a compact space of π-weight $\leqslant\kappa$ $(\kappa > \omega)$. Then there is an $x \in X$ such that $x \notin K$ for all nowhere dense P_κ-sets $K \subseteq X$.*

PROOF. Suppose first that κ is regular. Let $\{V_n : n < \omega\}$ be a sequence of countably many nonempty pairwise disjoint open subsets of X. By Lemma 1.9.2, there is a family $\mathcal{F}_n$ of closed subsets of $\bar{V}_n$ such that

(1) $\mathcal{F}_n$ has the n-intersection property,

(2) if $K \subseteq X$ is a nowhere dense P_κ-set, then there is an $F \in \mathcal{F}_n$ with $F \cap K = \emptyset$.

(Observe that if $K \subseteq X$ is a nowhere dense P_κ-set, then $K \cap \bar{V}_n$ is a nowhere dense P_κ-set of $\bar{V}_n$). Take any point x in the intersection

$$\bigcap\Big\{\overline{\bigcup_{n<\omega} g(n)} : g \in \prod_{n<\omega} \mathcal{F}_n\Big\}.$$

Since $\kappa > \omega$, x is as required.

Now observe that if κ is singular, then any P_κ-set of X is a P_{κ^+}-set. Therefore, the theorem for singular κ follows from the theorem for regular κ. □

1.9.4. COROLLARY (CH). *There is a point $x \in \omega^*$ such that $x \notin K$ for all nowhere dense P-sets $K \subseteq \omega^*$.*

Notes for Section 1

Theorem 1.1.6 is due to PAROVIČENKO [1963]. Lemma 1.2.2 is well known. For other results of this type see COMFORT & NEGREPONTIS [1974]. Corollary 1.2.4 is due to PAROVIČENKO [1963]. Theorem 1.2.5 can be found in GILLMAN & JERISON [1960]. The argument given here is due to NEGREPONTIS [1967]. That Theorem 1.2.6 is a consequence of Parovičenko's characterization of ω^*, was observed by many people. Theorem 1.3.1 is due to PAROVIČENKO [1963]. For another proof of this result see BLASZCZYK & SZYMAŃSKI [1980b]. Lemmas 1.4.1 and 1.4.2 are implicit in BALCAR, FRANKIEWICZ & MILLS [1980]. Theorem 1.4.4 (1)$\Leftrightarrow$(2) is due to LOUVEAU [1973]; the equivalence (1)$\Leftrightarrow$(3) can be found in BALCAR, FRANKIEWICZ & MILLS [1980]. Theorem 1.4.7 is due to EFIMOV [1970]. A result stronger than Theorem 1.5.2 is due to COMFORT, HINDMAN & NEGREPONTIS [1969]. A result stronger than Theorem 1.5.3 is due to WOODS [1976a]. For a related result, see WOODS [1976b]. An important step in the proof of Theorem 1.5.3 is due to FINE &

GILLMAN [1960]. Corollary 1.5.4 is due to GILLMAN [1966]. Lemma 1.6.1 is due to W. RUDIN [1956] but the proof we give is due to van DOUWEN & VAN MILL [1981d]. Lemma 1.6.2 is well-known and Theorem 1.6.4 is a special case of a result in VAN DOUWEN & VAN MILL [1981b]. Theorem 1.6.5 is due to W. RUDIN [1956]. Corollary 1.7.2 is also due to W. RUDIN [1956]. Theorem 1.7.3 is due to KUNEN [1976]. Theorem 1.8.1 can be found in VAN DOUWEN & VAN MILL [1981b]. The easy Theorem 1.8.2 seems to be new. The proof is in the spirit of VAN MILL [1979b]. For a related result see VAN DOUWEN & VAN MILL [1980]. Theorem 1.8.3 is well-known. I don't know who proved this for the first time. Corollary 1.9.4 is due to Kunen and Theorem 1.9.3 is due to KUNEN, VAN MILL & MILLS [1980]. The proof presented here, which was suggested to me by Alan Dow, is different from the proof given in KUNEN, VAN MILL & MILLS [1980]. It is in the spirit of CHAE & SMITH [1980] and VAN DOUWEN [1981].

2. The spaces $\beta\omega$ and $\beta\omega\backslash\omega$ under $\neg$CH

In this section we will see how $\beta\omega$ and ω^* behave in various models in which CH is not true. All the CH results derived in Section 1 are consistently false, except for Theorem 1.7.3, which is true in ZFC (see Theorems 4.3.3 and 4.4.1), and Lemma 1.4.3 and Theorem 1.7.3 of which we do not know whether they can be false.

2.1. A characterization of $\mathcal{P}(\omega)$/fin, II

The main result in Section 1.1, namely Theorem 1.1.6, is false under $\neg$CH. In fact, Theorem 1.1.6 is equivalent to CH.

2.1.1. THEOREM. CH *is equivalent to the statement that all Boolean algebras of cardinality $\mathfrak{c}$ which satisfy condition H_ω are isomorphic.*

PROOF. Our proof is in topological language, since this is more convenient here. The Boolean algebraic reader can easily translate this proof in Boolean algebraic language.

We will construct two Parovičenko spaces that cannot be homeomorphic under $\neg$CH.

Example 1. A Parovičenko space S having a point p such that $\chi(p, X) = \omega_1$.

By Lemma 1.2.3 there is an ω_1-sequence $\langle C_\alpha : \alpha < \omega_1\rangle$ of clopen subsets in ω^* with $C_\alpha \subset C_\beta$ if $\beta < \alpha < \omega_1$. Let $P = \bigcap_{\alpha<\omega_1} C_\alpha$, and let $S = X/P$, the quotient space obtained from X by collapsing P to one point. By Lemma 1.4.1, S is an F-space. The other properties of S required in the definition of a Parovičenko space are easily checked. If we put $p = \{P\}$, then, obviously, $\chi(p, S) = \omega_1$.

Example 2. A Parovičenko space T with $\pi(x, T) = \mathfrak{c}$ for all $x \in T$.

Put $T = (\omega \times 2^{\mathfrak{c}})^*$. Since $\omega \times 2^{\mathfrak{c}}$ is a zero-dimensional Lindelöf space of weight $\mathfrak{c}$, T is zero-dimensional and has weight $\mathfrak{c}$. Consequently, Theorem 1.2.5 implies that T is a Parovičenko space.

For $\alpha < \mathfrak{c}$ denote the α-th projection $2^{\mathfrak{c}} \to 2$ by π_α. For $\alpha < \mathfrak{c}$ and $i < 2$ define

$$K(\alpha, i) = T \cap (\omega \times \pi_\alpha^{-1}(\{i\}))^- .$$

Note that $K(\alpha, i)$ is a nonempty clopen subset of T and that $K(\alpha, i) = K(\alpha', i')$ iff $\alpha = \alpha'$ and $i = i'$. Define

$$\mathcal{K} = \{K(\alpha, i) : \alpha < \mathfrak{c}, i < 2\} .$$

Claim. Any intersection of ω_1 distinct members of $\mathcal{K}$ has empty interior.

For symmetry reasons it suffices to prove that $I = \bigcap_{\alpha<\omega_1} K(\alpha, 0)$ has empty interior. Suppose that this is not true. Then there is a clopen $U \subseteq \beta(\omega \times 2^{\mathfrak{c}})$ such that $\emptyset \neq U \cap T \subseteq I$. For every $\alpha < \omega_1$ the set $U \backslash (\omega \times \pi_\alpha^{-1}(\{0\}))$ is a compact subset of $\omega \times 2^{\mathfrak{c}}$, and since $U \cap (\omega \times 2^{\mathfrak{c}})$ is not compact, there is an integer n_α such that $\emptyset \neq U \cap (\{n_\alpha\} \times 2^{\mathfrak{c}}) \subseteq \{n_\alpha\} \times \pi_\alpha^{-1}(\{0\})$. There is an integer n such that $A = \{\alpha < \omega_1 : n_\alpha = n\}$ is infinite. But then $\{n\} \times \bigcap_{\alpha \in A} \pi_\alpha^{-1}(\{0\})$ is a subset of $\{n\} \times 2^{\mathfrak{c}}$ with nonempty interior, which is absurd.

Let $x \in T$ be arbitrary, and let $\mathcal{U}$ be a π-base for x. The family $\mathcal{F} = \{K \in \mathcal{K} : x \in K\}$ has cardinality $\mathfrak{c}$. For each $K \in \mathcal{F}$ there is a $U(K) \in \mathcal{U}$ with $U(K) \subseteq K$, hence $|\mathcal{U}| \geq |\mathcal{F}| = \mathfrak{c}$ since the Claim implies that $\{|K \in \mathcal{K} : U(K) = U|\} \leq \omega$ for all $U \in \mathcal{U}$. It follows that $\pi(x, T) = \mathfrak{c}$ since we know already that T has weight at most $\mathfrak{c}$. □

2.1.2. REMARK. The above result suggests the following interesting question: is CH equivalent to the statement that (*) all Boolean algebras of cardinality $\mathfrak{c}$ which satisfy condition H_ω have isomorphic completions? This question was first considered by Broverman & Weiss [1981], who showed that (*) is not a theorem of ZFC. Subsequently, VAN MILL & WILLIAMS [1983] proved that (*) implies that $\mathfrak{c} < 2^{\omega_1}$. Whether (*) iff CH is still unknown. (See the remarks on p. 564.)

2.2. *A topological translation, II*

Section 2.1 of course implies that Corollary 1.2.4 is equivalent to CH. Whether Theorem 1.2.6 is equivalent to CH is unknown, although it is easy to show it is not a Theorem of ZFC. In the proof of Theorem 2.1.1 we showed that, among others, $(\omega \times 2^{\mathfrak{c}})^*$ contains a nonempty intersection of ω_1 clopen sets with empty interior. However, MA + $\neg$CH implies that each nonempty intersection of ω_1 clopen subsets of ω^* has nonempty interior, see 2.3. Consequently, MA + $\neg$CH implies that $(\omega \times 2^{\mathfrak{c}})^*$ is not homeomorphic to ω^*. This argument does not apply to prove that a space such as $(\omega \times 2^{\omega})^*$ is not homeomorphic to ω^*, since MA + $\neg$CH easily implies that if X is a locally compact, σ-compact, noncompact space of countable π-weight, then each nonempty intersection of fewer than $\mathfrak{c}$ open subsets of X^* has nonempty interior. One might therefore hope that if X is topologically very 'close' to ω, then X^* and ω^* are homeomorphic. Even this is not true.

2.2.1. THEOREM. *It is consistent that ω^* and $(\omega \times W(\omega+1))^*$ are not homeomorphic.*

PROOF. SHELAH [1978] has recently shown that it is consistent that all homeomorphisms of ω^* are induced, i.e. that for each autohomeomorphism $h:\omega^* \to \omega^*$ there is a permutation $\pi:\omega \to \omega$ such that $h = \beta\pi \restriction \omega^*$. We will show that for any permutation π of ω, the set of fixed points of $\beta\pi$ is a clopen subset of $\beta\omega$, consequently, the set of fixed points of $\beta\pi \restriction \omega^*$ is a clopen subset of ω^*. Let $\pi:\omega \to \omega$ be a permutation, and let $p \in \omega^*$ be a fixed point of $\beta\pi$. Let $E = \{n < \omega : \pi(n) = n\}$. If $E \in p$, then p has clearly a clopen neighborhood consisting of fixed points of $\beta\pi$, namely, the closure of E in $\beta\omega$. So assume that $E \notin p$. Define $F = \omega\backslash E$. Since for all $n \in F$ we have that $\pi(n) \neq n$, it is easy to split F in two sets F_0 and F_1 such that $\pi(F_0) \cap F_0 = \emptyset$ and $\pi(F_1) \cap F_1 = \emptyset$. Without loss of generality, $F_0 \in p$. Then $\pi(F_0) \in \beta\pi(p)$, whence $p \neq \beta\pi(p)$, a contradiction. We conclude that the set of fixed points of $\beta\pi$ is open, whence clopen. To prove that in Shelah's model, ω^* and $(\omega \times W(\omega+1))^*$ are not homeomorphic, it therefore suffices to produce an autohomeomorphism h of $(\omega \times W(\omega+1))^*$ such that the set $\mathrm{Fix}(h)$ of fixed points of h is not clopen. To this end, let E, $F \subseteq \omega$ be two complementary infinite sets and let $\pi:\omega \to \omega$ be a permutation such that $\pi(E) = F$ (which implies that $\pi(F) = E$). Define $f:\omega \times W(\omega+1) \to \omega \times W(\omega+1)$ by

$$\begin{cases} f(\langle n, m\rangle) = \langle n, \pi(m)\rangle \quad (m \in \omega), \\ f(\langle n, \omega\rangle) = \langle n, \omega\rangle . \end{cases}$$

Put $h = \beta f \restriction (\omega \times W(\omega+1))^*$. It is easily seen that

$$\mathrm{Fix}(h) = (\omega \times \{\omega\})^* ,$$

which implies that $\mathrm{Fix}(h)$ is not clopen. □

2.2.2. REMARK. Observe that in the proof of Theorem 2.2.1 we found an easily described topological property that distinguishes between ω^* and $(\omega \times W(\omega+1))^*$.

2.3. Continuous images of ω^*, II

It is well known that Corollary 1.3.4 is not a result of ZFC. KUNEN [1968, 12.7 and 12.3] proved that in a model formed by adding ω_2 Cohen reals to a model of CH, there is no ω_2 sequence of subsets of ω which is strictly decreasing (mod fin).

Therefore, in this model ω^* cannot be mapped onto $W(\mathfrak{c}+1)$. VAN DOUWEN & PRZYMUSIŃSKI [1980] have used results of ROTHBERGER [1952] and PRZYMUSIŃSKI [1978] to show that Corollary 1.3.4 is not true under the following hypothesis:

$$(*) \qquad \omega_2 \leq \mathfrak{c} < 2^{\omega_1} = \omega_{\omega_2}.$$

This is interesting since (*) only involves cardinals.

For a discussion of Martin's Axiom (abbreviated MA), see M.E. RUDIN [1977]. The following statements are consequences of MA:

$P(\mathfrak{c})$: *If $\mathcal{A}$ is a family of less than $\mathfrak{c}$ subsets of ω such that for all $\mathcal{B} \in [\mathcal{A}]^{<\omega}$ we have that $|\bigcap \mathcal{B}| = \omega$, then there is an infinite $B \subseteq \omega$ such that $|B \backslash A| < \omega$ for all $A \in \mathcal{A}$,*

$S(\mathfrak{c})$: *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are families of less than $\mathfrak{c}$ subsets of ω such that for all $A \in \mathcal{A}$ and $\mathcal{F} \in [\mathcal{B}]^{<\omega}$ we have that $|A \cap \bigcap \mathcal{F}| = \omega$. Then there is an infinite $C \subseteq \omega$ such that $|C \cap A| = \omega$ for each $A \in \mathcal{A}$, and $|C \backslash B| < \omega$ for all $B \in \mathcal{B}$.*

We are now in a position to generalize Theorem 1.3.3.

2.3.1. THEOREM (MA). *Each compact space of weight less than $\mathfrak{c}$ is a continuous image of ω^*.*

PROOF. Let Y be a compact space of weight κ, where $\kappa < \mathfrak{c}$. We may assume that Y is a nowhere dense subspace of $[0, 1]^\kappa$. Since $\kappa \leq \mathfrak{c}$, $[0, 1]^\kappa$ is separable, and we can therefore find a countable dense set D of $[0, 1]^\kappa$ which misses Y. Let $\mathcal{E}$ be an open base for $[0, 1]^\kappa$ which is closed under finite unions and such that $|\mathcal{E}| = \kappa$. Put

$$\mathcal{A} = \{E \cap D : E \in \mathcal{E} \ \&\ E \cap Y \neq \emptyset\},$$

and

$$\mathcal{B} = \{E \cap D : E \in \mathcal{E} \ \&\ Y \subseteq E\},$$

respectively. It is easily seen that $\mathcal{A}$ and $\mathcal{B}$ satisfy the hypotheses of $S(\mathfrak{c})$. Consequently, we can find a subset $J \subseteq D$ such that

(1) $|A \cap J| = \omega$ for all $A \in \mathcal{A}$,

(2) $|J \backslash B| < \omega$ for all $B \in \mathcal{B}$.

Let $Z = Y \cup J$. We claim that Z is compact and that J is a dense set of isolated points of Z. If this is true, then Z is a compactification of ω, which implies that Y is a continuous image of ω^*.

If $E \in \mathcal{E}$ and $E \cap Y \neq \emptyset$, then $E \cap D \in \mathcal{A}$ which implies that $E \cap J$ is infinite. Hence J is dense in Z. Let x be a limit point of J which does not belong to Y. Since $\mathcal{E}$ is closed under finite unions and since Y is compact, there are disjoint E_0, $E_1 \in \mathcal{E}$ with $Y \subseteq E_0$ and $x \in E_1$. Then $E_0 \cap D \in \mathcal{B}$ which implies, by (2), that $J \backslash E_0$

is finite. But E_1 contains infinitely many points of J, contradiction. We conclude that Z is compact and that J is relatively discrete. $\square$

Of course, the above theorem suggests the question, due to VAN DOUWEN & PRZYMUSIŃSKI [1980, 2.8], whether MA implies that each compact space of weight $\mathfrak{c}$ is a continuous image of ω^*. In the remaining part of this section we will show that this is not the case.

Let κ and λ be infinite cardinals and consider the following statement:

$G(\kappa,\lambda)$: there are a κ-sequence $\langle U_\xi : \xi<\kappa\rangle$ of clopen sets in ω^* and a λ-sequence $\langle V_\xi : \xi<\lambda\rangle$ of clopen sets in ω^* such that

(1) $U_\xi \subset U_\eta$ if $\xi<\eta<\kappa$,
(2) $V_\xi \subset V_\eta$ if $\xi<\eta<\lambda$,
(3) $(\bigcup_{\xi<\kappa} U_\xi)\cap(\bigcup_{\xi<\lambda} V_\xi)=\emptyset$, but
(4) $(\bigcup_{\xi<\kappa} U_\xi)^-\cap(\bigcup_{\xi<\lambda} V_\xi)^-\neq\emptyset$.

This has a straightforward translation in terms of the existence of certain families of subsets of ω which we leave to the reader.

By Lemma 1.1.2, $G(\omega,\omega)$ is false, but interestingly, $G(\omega_1,\omega_1)$ is true, HAUSDORFF [1936].

2.3.2. THEOREM. *There is a compact space X and a continuous surjection $f: X\to\omega^*$ such that, under* MA $+\neg$CH $+\neg G(\mathfrak{c},\mathfrak{c})$, *$X$ has weight $\mathfrak{c}$, f is irreducible, and ω^* cannot be mapped onto X.*

PROOF. Let $Y=\omega^*$ with the $G_{<\mathfrak{c}}$-topology, i.e the underlying set of Y is ω^* and the intersections of fewer than $\mathfrak{c}$ clopen subsets of ω^* form an open basis for Y. Let $\mathscr{E}$ be a basis for Y of cardinality $w(Y)$ consisting of clopen sets. By transfinite induction, for each $\alpha<\mathfrak{c}$ we will construct subalgebras $\mathscr{E}_\alpha\subseteq\mathscr{B}(Y)$ such that

(1) $\mathscr{E}_0=\langle\langle\mathscr{E}\rangle\rangle$,
(2) $|\mathscr{E}_\alpha|\leq|\bigcup_{\beta<\alpha}\mathscr{E}_\beta|^{\mathfrak{c}}$,
(3) if $\beta<\alpha<\mathfrak{c}$ and if $\mathscr{F}\in[\mathscr{E}_\beta]^{<\mathfrak{c}}$, then $\bigcup\mathscr{F}\in\mathscr{E}_\alpha$.

It is straightforward to construct these algebras since the union of fewer than $\mathfrak{c}$ clopen subsets of Y is clopen.

Put $\mathscr{B}=\bigcup_{\alpha<\mathfrak{c}}\mathscr{E}_\alpha$ and observe that if $\mathfrak{c}$ is regular, then $\mathscr{B}$ is a $<\mathfrak{c}$-closed subalgebra of $\mathscr{B}(Y)$. Let $X=\mathrm{st}(\mathscr{B})$. We claim that X is as required. It is clear that the function $f: X\to\omega^*$ defined by

$$\{f(x)\}=\bigcap\{\bar{B}: B\in x\}$$

is a continuous surjection.

From now on, assume MA $+\neg$CH $+\neg G(\mathfrak{c},\mathfrak{c})$. We also identify Y and the subspace of X consisting of the fixed ultrafilters on $\mathscr{B}$.

First observe that (2) implies that X has weight $\mathfrak{c}$, since MA implies that $2^\kappa = \mathfrak{c}$ for all $\omega \leq \kappa < \mathfrak{c}$, see M.E. RUDIN [1977]. We first claim that f is irreducible. To this end, let $A \subseteq X$ be a proper closed subset. Since Y is dense in X, we can find a point $y \in Y \setminus A$. Choose $E \in \mathscr{E}$ such that $\bar{E} \cap A = \emptyset$ and $y \in \bar{E}$ (the closure is taken in X). Since E is an intersection of fewer than $\mathfrak{c}$ clopen subsets of ω^*, by $P(\mathfrak{c})$ we can find a nonempty clopen $C \subseteq \omega^*$ such that $C \subseteq E$. It is clear that $C \cap f(A) = \emptyset$.

We will now show that ω^* cannot be mapped onto X. Fix $y \in Y$. We will construct a family $\{B(y, \alpha) : \alpha < \mathfrak{c}\}$ and a family $\{E(y, \alpha) : \alpha < \mathfrak{c}\}$ of clopen subsets of X such that

(4) $\alpha < \beta < \mathfrak{c} \to B(y, \alpha) \subset B(y, \beta) \subset X \setminus \{y\}$,

(5) $\alpha < \beta < \mathfrak{c} \to E(y, \alpha) \subset E(y, \beta) \subset X \setminus \{y\}$,

(6) $\alpha < \beta < \mathfrak{c} \to B(y, \alpha) \cap E(y, \beta) = \emptyset$,

(7) $(\bigcup\{B(y, \alpha) : \alpha < \mathfrak{c}\})^- \cap (\bigcup\{E(y, \alpha) : \alpha < \mathfrak{c}\})^- = \{y\}$,

(8) $\bigcup\{B(y, \alpha) : \alpha < \mathfrak{c}\} \cup \bigcup\{E(y, \alpha) : \alpha < \mathfrak{c}\} = X \setminus \{y\}$.

(This construction is a triviality of course). Let $\{Z_\alpha : \alpha < \mathfrak{c}\}$ enumerate the family of all clopen subsets of X containing y. To achieve (7) and (8), we will make the required families of clopen sets such that

(9) $Z_\alpha \cap B(y, \alpha) \neq \emptyset$, $Z_\alpha \cap E(y, \alpha) \neq \emptyset$ and $X \setminus Z_\alpha \subseteq B(y, \alpha) \cup E(y, \alpha)$.

So our induction hypotheses are (4), (5), (6) and (9). Suppose that we have completed the construction for all $\alpha < \beta < \mathfrak{c}$. Put

$$B = \bigcup_{\alpha<\beta} B(y, \alpha) \quad \text{and} \quad E = \bigcup_{\alpha<\beta} E(y, \alpha).$$

Then $\bar{B}$ and $\bar{E}$ are both open, since $\mathscr{B}$ is $<\mathfrak{c}$-closed, which implies, by (6), that $\bar{B} \cap \bar{E} = \emptyset$. Since y is a $P_\mathfrak{c}$-point of X, i.e. the intersection of fewer than $\mathfrak{c}$ neighborhoods of y is again a neighborhood of y, $y \notin \bar{B} \cup \bar{E}$. Let $F \subseteq Z_\alpha$ be a clopen neighborhood of y which misses $\bar{B} \cup \bar{E}$. Take two disjoint clopen nonempty subsets G, $H \subseteq F$ which do not contain y. Define

$$B(y, \alpha) = \bar{B} \cup G \cup (X \setminus (\bar{E} \cup Z_\alpha)) \quad \text{and} \quad E(y, \alpha) = \bar{E} \cup H.$$

It is clear that our inductive hypotheses are satisfied.

Now suppose that there is a continuous surjection $g : \omega^* \to X$. Put

$$B_y = (\bigcup\{g^{-1}(B(y, \alpha)) : \alpha < \mathfrak{c}\})^- \quad \text{and} \quad E_y = (\bigcup\{g^{-1}(E(y, \alpha)) : \alpha < \mathfrak{c}\})^-.$$

By $\neg G(\mathfrak{c}, \mathfrak{c})$, $B_y \cap E_y = \emptyset$. Observe that y is the unique point of Y with the property that $g^{-1}(y)$ meets both B_y and E_y.

Let $Y_0 = \{y \in Y : B_y \cup E_y = \omega^*\}$. Then B_y and E_y are both clopen and since if $y_0, y_1 \in Y$ are distinct, then $B_{y_0} \neq B_{y_1}$, we have that $|Y_0| \leq \mathfrak{c}$.

Let $Y_1 = \{y \in Y : B_y \cup E_y \neq \omega^*\}$. If $y \in Y_1$, then $g^{-1}(y)$ has nonempty interior in ω^*, which implies that $|Y_1| \leq \mathfrak{c}$.

Since by 3.1.2 (c), $|Y| = 2^{\mathfrak{c}}$ and $Y = Y_0 \cup Y_1$, we have the desired contradiction. □

2.3.3. REMARK. An inspection of the proof of Theorem 2.3.2 will show that we 'only' need the following hypotheses:

(1) $2^\kappa = \mathfrak{c}$ if $\omega \leqslant \kappa < \mathfrak{c}$, and

(2) $\neg G(\mathfrak{c}, \mathfrak{c})$.

2.3.4. REMARK. If $\mathscr{B}$ is the BA of clopen subsets of the space of Theorem 2.3.2, then $\mathscr{P}(\omega)$/fin can be embedded in $\mathscr{B}$, $\mathscr{P}(\omega)$/fin and $\mathscr{B}$ have isomorphic completions, but $|\mathscr{B}| = \mathfrak{c}$ and $\mathscr{B}$ cannot be embedded in $\mathscr{P}(\omega)$/fin.

2.3.5. REMARK. Let X be the space of Theorem 2.3.2. Observe that $Z = (\omega \times X)^*$ is a Parovičenko space which is not a continuous image of ω^*, since Z can be mapped onto X.

Until now it is not clear yet that Theorem 2.3.2 has some use, for it is not obvious at all that MA + $\neg$CH + $\neg G(\mathfrak{c}, \mathfrak{c})$ can be true. Fortunately, KUNEN [1981] has shown the following.

2.3.6. THEOREM. (A) *It is consistent with* MA + $\neg$CH *that* $G(\omega_1, \mathfrak{c})$ *and* $G(\mathfrak{c}, \mathfrak{c})$ *both are false,*

(B) *it is consistent with* MA + $\neg$CH *that* $G(\omega_1, \mathfrak{c})$ *and* $G(\mathfrak{c}, \mathfrak{c})$ *both are true.*

Let X be the space constructed in the proof of Theorem 1.7.3, i.e. X is the Stone space of the reduced measure algebra of $[0, 1]$. It is unknown whether it is consistent that X is not a continuous image of ω^*. It will not be possible to deduce this from rather global properties of X, since BELL [1980] has constructed in ZFC examples of spaces which are very similar to X and which are continuous images of ω^*.

Let us finally notice that PRZYMUSIŃSKI [1982] has shown that each perfectly normal compact space is a continuous image of ω^*. The big open question in this area is whether every first countable compactum is a continuous image of ω^*.

2.4. *Closed subspaces of $\beta\omega$, II*

In Section 1.4 we showed that every compact zero-dimensional F-space of weight $\mathfrak{c}$ embeds, under CH, in $\beta\omega$. This suggests to consider the following statement:

FE: *Every compact zero-dimensional F-space can be embedded in an Extremally disconnected space.*

Observe that in Boolean algebraic language FE is the statement that each WCC BA is a homomorphic image of some complete BA.

It is convenient to factor FE as FB + BE, where

FB: *Every compact zero-dimensional F-space can be embedded in a Basically disconnected space,*

and

BE: *Every Basically disconnected compact space can be embedded in an Extremally disconnected space.*

Of course, both FB and BE have straightforward Boolean algebraic translations. In Section 1.4 we showed, in particular, that the restriction of FE to spaces of weight $\mathfrak{c}$ holds under CH. VAN DOUWEN & VAN MILL [1980] construct, under $\mathrm{MA}+\mathfrak{c}=\omega_2$, an example of a compact zero-dimensional F-space V of weight $\mathfrak{c}$ that cannot be embedded in any basically disconnected space. As a consequence, neither FE nor FB are theorems of ZFC. Very little is known about BE, it is known however that the Čech–Stone compactification of any P-space embeds in an extremally disconnected space, see Section 4.4. We conclude that Theorems 1.4.4, 1.4.5 and Corollary 1.4.6 are false under $\mathrm{MA}+\mathfrak{c}=\omega_2$.

2.5. C^*-embedded subspaces of $\beta\omega$, II

It is easy to see that Theorem 1.5.3 need not be true. If $2^{\omega_1}=\mathfrak{c}$, then by Theorem 1.4.7, $\beta\omega_1$ embeds in ω^*, say by the embedding h. It is clear that $h(\omega_1)$ is C^*-embedded in $\beta\omega$, but $h(\omega_1)$ is not weakly Lindelöf. It is not so clear that Corollary 1.5.4 need not be true.

2.5.1. LEMMA $[\forall\kappa<\mathfrak{c},\ \neg G(\kappa,\omega)+\neg G(\mathfrak{c},\mathfrak{c})]$. *If $A\subseteq\omega^*$ is a closed $P_{\mathfrak{c}}$-set, then $\omega^*\backslash A$ is C^*-embedded in ω^*.*

PROOF. Striving for a contradiction, assume there are disjoint, nonempty, closed G_δ-subsets Z_0, $Z_1\subseteq\omega^*\backslash A$ such that $\bar{Z}_0\cap\bar{Z}_1\neq\emptyset$. Pick a point $a\in\bar{Z}_0\cap\bar{Z}_1$ and let $\{C_\alpha:\alpha<\mathfrak{c}\}$ enumerate the family of all clopen subsets of ω^* containing a. By transfinite induction on $\alpha<\mathfrak{c}$, we will construct clopen subsets G^i_α $(i<2)$ of ω^* such that

(1) $G^i_\alpha\subseteq Z_i$ and $G^i_\alpha\cap C_\alpha\neq\emptyset$,

(2) if $\beta<\alpha$, then $G^i_\beta\subset G^i_\alpha$.

If we can complete the induction, then we contradict $\neg G(\mathfrak{c},\mathfrak{c})$. Suppose that the sets G^i_β are defined for all $\beta<\alpha$, $i<2$. Since A is a $P_{\mathfrak{c}}$-set, there is a clopen $C\subseteq\omega^*\backslash A$ such that

$$\bigcup_{\beta<\alpha}G^0_\beta\cup\bigcup_{\beta<\alpha}G^1_\beta\subseteq C.$$

By $\neg G(\alpha,\omega)$, we can find clopen sets $C_i\subseteq C\cap Z_i$ such that

$$\bigcup_{\beta<\alpha}G^i_\beta\subseteq C_i.$$

Put $C'_\alpha = C_\alpha \cap (\omega^*\backslash C)$. Take $x_i \in Z_i \cap C'_\alpha$. Let $E_i \subseteq \omega^*$ be clopen neighborhoods of x_i not meeting A. By Theorem 1.2.5, $E_i \cap Z_i$ contains a non-empty clopen set, say F_i. Define $G^i_\alpha = C_i \cup F_i$. This completes the induction, which gives us the required contradiction. □

2.5.2. COROLLARY [$\forall\kappa < \mathfrak{c}$, $\neg G(\kappa, \omega) + \neg G(\mathfrak{c}, \mathfrak{c})$]. *Let $A = \{x \in \omega^* : \exists$ closed nowhere dense $P_\mathfrak{c}$-set $B \subseteq \omega^*$ containing $x\}$. If $x \in A$, then $\omega^*\backslash\{x\}$ is C^*-embedded in ω^*.*

PROOF. By Lemma 2.5.1, if $x \in A$ and if B is a nowhere dense closed $P_\mathfrak{c}$-set containing x, then $\beta(\omega^*\backslash B) = \omega^*$. But this easily implies that $\beta(\omega^*\backslash\{x\}) = \omega^*$. □

The question arises of course whether Corollary 2.5.2 is of any use, i.e. is it possible that the set A is nonempty, while moreover the combinatorial hypotheses required for the proof of Corollary 2.5.2 hold. The answer is yes of course. By Theorem 2.3.6, there is a model in which MA + $\neg$CH + $\neg G(\mathfrak{c}, \mathfrak{c})$ is true. It is easy to show that MA implies there are $P_\mathfrak{c}$-points and that MA implies $\neg G(\omega, \kappa)$ for all $\kappa < \mathfrak{c}$.

Consequently, we obtain

2.5.3. COROLLARY TO COROLLARY. *It is consistent that for some $x \in \omega^*$ we have that $\beta(\omega^*\backslash\{x\}) = \omega^*$.*

It is unpleasant that the point x of Corollary 2.5.3 does not 'really' exist, since it is a $P_\mathfrak{c}$-point. We will show that there are many points which 'really' exist for which it is consistent that their complements in ω^* are C^*-embedded in ω^*.

2.5.4. THEOREM [$\forall\kappa < \mathfrak{c}$, $\neg G(\kappa, \omega) + \neg G(\mathfrak{c}, \mathfrak{c})$]. *If $x \in \omega^*$ is not a P-point, then $\omega^*\backslash\{x\}$ is C^*-embedded in ω^*.*

PROOF. Let $x \in \omega^*$ be not a P-point and let $U \subseteq \omega^*$ be an open F_σ such that $x \in \bar{U}\backslash U$. Let $f : \omega^*\backslash\{x\} \to [0, 1]$ be continuous. Let $f_0 = f \restriction \bar{U}\backslash\{x\}$ and $f_1 = f \restriction \omega^*\backslash\bar{U}$. By $\forall\kappa < \mathfrak{c}$, $\neg G(\kappa, \omega)$, we have that $\bar{U}$ is a $P_\mathfrak{c}$-set in ω^*. Consequently, by Lemma 2.5.1, we can extend f_1 to a continuous map $g_1 : \omega^*\backslash\bar{U} \to [0, 1]$. By Theorem 1.2.5, $\omega^*\backslash\bar{U} = (\omega^*\backslash\bar{U}) \cup (\bar{U}\backslash U)$. This implies that $g_1(t) = f(t)$ for all $i \in \bar{U}\backslash(U \cup \{x\})$. By Theorem 1.5.2, U is C^*-embedded in $\bar{U}$, consequently, $\bar{U}\backslash\{x\}$ is C^*-embedded in $\bar{U}$. We therefore conclude that we can extend f_0 to a continuous map $g_0 : \bar{U} \to [0, 1]$. Since $g_0 \restriction \bar{U}\backslash(U \cup \{x\}) = g_1 \restriction \bar{U}\backslash(U \cup \{x\})$, we conclude that $g_0 \restriction \bar{U}\backslash U = g_1 \restriction \bar{U}\backslash U$ since x is not isolated (Theorem 1.2.5). Define $g : \omega^* \to [0, 1]$ by

$$\begin{cases} g(t) = g_0(t) & \text{if } t \in \bar{U}, \\ g(t) = g_1(t) & \text{if } t \notin \bar{U}. \end{cases}$$

It is obvious that g is continuous and that g extends f. □

Since P-points need not exist in ω^*, one might hope that the following statements can be simultaneously true:

(1) $\forall\kappa<\mathfrak{c}$, $\neg G(\kappa,\omega)$,

(2) $\neg G(\mathfrak{c},\mathfrak{c})$,

(3) no P-points,

for then, there would be a model in which $\beta(\omega^*\backslash\{x\})=\omega^*$ for *all* $x\in\omega^*$. Unfortunately, as was pointed out to me by Ken Kunen, (1) implies $\neg$(3). Define an order $<^*$ on ω^ω by

$$f<^*g \quad \text{iff} \quad |\{n<\omega : f(n)\geq g(n)\}|<\omega\,.$$

A subset $A\subseteq\omega^\omega$ is called *dominating* if for each $f\in\omega^\omega$ there is a $g\in A$ with $f<^*g$.

2.5.5. LEMMA. *Suppose that no subset of ω^ω of cardinality less than $\mathfrak{c}$ dominates. Then ω^* contains a P-point.*

PROOF. Let $\{f_\alpha:\alpha<\mathfrak{c}\}$ enumerate ω^ω. By transfinite induction on $\alpha<\mathfrak{c}$ we will construct a filter $\mathscr{F}_\alpha\subseteq\mathscr{P}(\omega)$ such that

(1) finite intersections of elements of $\mathscr{F}_\alpha$ have infinite intersections,

(2) there is an element $F\in\mathscr{F}_\alpha$ such that either $|f_\alpha^{-1}(n)\cap F|<\omega$ for all $n<\omega$, or $F\subseteq f_\alpha^{-1}(\{0,1,\ldots,n\})$ for certain $n<\omega$,

(3) if $\kappa<\alpha$ then $\mathscr{F}_\kappa\subseteq\mathscr{F}_\alpha$ and $|\mathscr{F}_\alpha|\leq|\alpha|\cdot\omega$.

Suppose we have constructed everything for all $\kappa<\alpha$ and define $\mathscr{F}=\bigcup_{\kappa<\alpha}\mathscr{F}_\kappa$. Observe that $|\mathscr{F}|\leq|\alpha|\cdot\omega<\mathfrak{c}$. For each $F\in\mathscr{F}$ define a function $g(F):\omega\to\omega$ by

$$g(F)(n)=\begin{cases}\min(F\cap f_\alpha^{-1}(n)) & \text{if } F\cap f_\alpha^{-1}(n)\neq\emptyset\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Since $|\mathscr{F}|<\mathfrak{c}$, we can find a function $f\in\omega^\omega$ such that $f\not<^* g(F)$ for all $F\in\mathscr{F}$. Define

$$X=\bigcup_{n<\omega} f_\alpha^{-1}(n)\cap\{j<\omega : j\leq f(n)\}$$

and define $\mathscr{F}_\alpha$ to be the filter generated by $\mathscr{F}\cup\{X\}$ if $|F\cap X|=\omega$ for all $F\in\mathscr{F}$. Otherwise, define $\mathscr{F}_\alpha=\mathscr{F}$.

It is clear that any ultrafilter extending $\bigcup_{\alpha<\mathfrak{c}}\mathscr{F}_\alpha$ is a P-point. $\square$

In view of the above Lemma it therefore suffices to prove the following.

2.5.6. LEMMA. $(\forall\kappa<\mathfrak{c},\ \neg G(\kappa,\omega))\to$ (*no subset of ω^ω of cardinality less than $\mathfrak{c}$ dominates*).

PROOF. Let $\kappa = \min\{\lambda : \exists F \subseteq \omega^\omega$ such that F dominates and $|F| = \lambda\}$. Choose $F \subseteq \omega^\omega$ of cardinality κ such that F dominates. We may assume that $F = \{f_\alpha : \alpha < \kappa\}$, where $\alpha < \beta$ implies that $f_\alpha <^* f_\beta$. For each $\alpha < \kappa$, let

$$S_\alpha = \{\langle m, n\rangle : n < f(m)\}.$$

If $T_n = (\omega\backslash\{0, 1, \ldots, n\}) \times \omega$ for all $n < \omega$, then the families $\{S_\alpha : \alpha < \kappa\}$ and $\{T_n : n < \omega\}$ form a (κ, ω) gap (defined on $\omega \times \omega$). $\square$

2.6. *Autohomeomorphisms of ω^*. II*

As remarked in Section 2.2, SHELAH [1978] has shown it to be consistent that all autohomeomorphisms of ω^* are induced by a permutation of ω. Consequently, in this model ω^* has precisely $\mathfrak{c}$ autohomeomorphisms and we conclude that Lemma 1.6.1 can be false. I do not know whether Theorem 1.6.4 is a result of ZFC. This is caused by the fact that I do not know whether in ZFC there is a nowhere dense P-set in ω^* which is homeomorphic to ω^*. Theorem 1.6.5 is false under MA + $\neg$CH, since this axiom easily implies that there are $P_\mathfrak{c}$-points in ω^* and P-points which are not $P_\mathfrak{c}$-points.

2.7. *P-points and nonhomogeneity of ω^*, II*

It was an open problem for many years whether P-points in ω^* could be constructed without using additional set theoretic hypotheses. Finally, Shelah, see MILLS [1980] or WIMMERS [1980], proved it to be consistent that P-points in ω^* do not exist. Therefore, Corollary 1.7.2 cannot be established in ZFC. In Sections 3 and 4 we will give several proofs that ω^* is not homogeneous.

Theorem 1.7.3 is true in ZFC, see 4.3.3 and 4.4.1.

2.8. *Retracts of $\beta\omega$ and ω^*, II*

Theorem 1.8.1 has to be consistently false of course. This can be seen in various ways, one of which we give below. As usual,

$$U(\omega_1) = \{p \in \beta\omega_1 : \forall P \in p,\ |P| = \omega_1\}.$$

2.8.1. THEOREM (MA + $\neg$CH). *There is a nowhere dense closed P-set X in ω^* which is not a retract of ω^*.*

PROOF. By KUNEN [1976, 1.2], the space $\beta\omega_1$ embeds in ω^* as a $P_\mathfrak{c}$-set under MA + $\neg$CH. Let $h : \beta\omega_1 \to \omega^*$ be an embedding such that $h(\beta\omega_1)$ is a P-set and put $X = h(U(\omega_1))$. Since $U(\omega_1)$ is a P-set in $\beta\omega_1$, X is a P-set in ω^*. By COMFORT & NEGREPONTIS [1974, 12.2], it follows that there is a family of $\mathscr{A}$ of ω_2 uncountable subsets of ω_1 such that for distinct $A, B \in \mathscr{A}$ the set $A \cap B$ is countable. This immediately implies that the cellularity of $U(\omega_1)$ is at least ω_2. Since ω_1 is dense in $\beta\omega_1$, there cannot be a retraction from $\beta\omega_1$ onto $U(\omega_1)$, which immediately implies that there cannot be a retraction from ω^* onto X. □

2.9. Nowhere dense P-sets in ω^*, II

BALCAR, FRANKIEWICZ & MILLS [1980] prove it to be consistent that ω^* can be covered by nowhere dense closed P-sets. Consequently, Corollary 1.9.4 is not a result of ZFC.

DOW & VAN MILL [1981] show that no compact space can be covered by nowhere dense ccc P-sets i.e. P-sets satisfying the countable chain condition. It is not known whether there is a compact space that can be covered by nowhere dense P-sets of cellularity at most ω_1, however it is known that ω^* is not a consistent example.

2.9.1. PROPOSITION. *There is a point $x \in \omega^*$ such that $x \notin K$ for any nowhere dense P-set $K \subseteq \omega^*$ of cellularity at most ω_1.*

PROOF. Under CH, this is a consequence of Theorem 1.9.3. So assume $\neg$CH. It is left to the reader to prove that the R-points constructed in the proof of Lemma 3.3.4 have the required property. □

It is unknown whether in ZFC there is a nowhere dense P-set in ω^* of cellularity at most ω_1.

The cover of ω^* constructed by BALCAR, FRANKIEWICZ and MILLS consists of P-sets of different 'cofinalities'. Interestingly, NYIKOS [1982] has recently shown that it is consistent that ω^* can be covered by nowhere dense closed P-sets which are all an intersection of a chain of ω_1 clopen subsets of ω^*.

Notes for Section 2

Theorem 2.1.1 is due to VAN DOUWEN & VAN MILL [1978]. Theorem 2.2.1 is due to VAN DOUWEN & VAN MILL [1981d]. That Theorem 2.3.1 holds was established in VAN DOUWEN & PRZYMUSIŃSKI [1980]. It was known from the work of Baumgartner that MA does not imply that each compact space of weight $\mathfrak{c}$ is a continuous image of ω^*. Theorem 2.3.2, which is due to the author, gives another proof of this result. As noticed in Section 2.3, the interesting Theorem 2.3.6 is due to KUNEN [1980].

Section 2.5, with the exception of 2.5.5 and 2.5.6, was taken from VAN DOUWEN & VAN MILL [1981c]. Lemma 2.5.5 is due to KETONEN [1976]. That $\forall\kappa < \mathfrak{c}$, $\neg G(\kappa, \omega)$ implies that P-points in ω^* exist, was pointed out to me by Ken Kunen. Lemma 2.5.6 is due to Kunen and HECHLER [1975]. Theorem 2.8.1 is well-known. Proposition 2.9.1 is due to DOW & VAN MILL [1980].

3. Partial orderings on $\beta\omega$

In this section we will concentrate on various partial orderings on $\beta\omega$, which can be used to prove that certain spaces are not homogeneous.

3.1. The Rudin–Keisler order on $\beta\omega$

Let $f:\omega\to\omega$ be a function and let $\beta f:\beta\omega\to\beta\omega$ be its Stone extension. It is easily verified that

$$(*)\qquad \beta f(p)=q \quad\text{iff}\quad \forall P\in p: f(P)\in q \quad\text{iff}\quad \forall Q\in q: f^{-1}(Q)\in p\,.$$

Define an equivalence relation $\sim$ on $\beta\omega$ by

$$p\sim q \quad\text{iff}\quad \exists \text{ permutation } \pi:\omega\to\omega \text{ with } \beta\pi(p)=q\,.$$

It is clear that $\sim$ is indeed an equivalence relation.

Let p, $q\in\beta\omega$ and write

$$p\leqslant q \quad\text{iff}\quad \exists f\in\omega^\omega \text{ with } \beta f(q)=p\,.$$

The following theorem, which we will not prove in detail, summarizes relevant information about $\sim$ and $\leqslant$.

3.1.1. THEOREM. *Let p, q, $r\in\beta\omega$. Then*
(a) $p\leqslant p$,
(b) *if $p\leqslant q$ and $q\leqslant r$, then $p\leqslant r$,*
(c) *if $p\leqslant q$ and $q\leqslant p$, then $p\sim q$.*

Observe that only 3.1.1(c) requires proof. For information concerning the proof of Theorem 3.1.1(c) and many related things, see COMFORT & NEGREPONTIS [1974, section 9].

Observe that Theorem 3.1.1 shows that the quotient relation defined by $\leqslant$ on $\beta\omega/\sim$ is a partial ordering.

The relation $\leqslant$ on $\beta\omega$ is called the *Rudin–Keisler* order on $\beta\omega$. If $p\in\beta\omega$, then the set $\{q\in\beta\omega: q\leqslant p\}$ is equal to

$$\{\beta f(p): f\in\omega^\omega\}$$

and therefore has cardinality at most $\mathfrak{c}$, since $|\omega^\omega|=\mathfrak{c}$. Is there for all $p\in\beta\omega$ a point $q\in\beta\omega$ such that $q\not\leqslant p$? It seems strange, but at this moment we do not have the tools yet to answer this question, since we have almost not deduced any results about $\beta\omega$ in ZFC alone. In fact, we did not even find the cardinality of $\beta\omega$. Let us quickly compute $|\beta\omega|$, in order to answer the above question.

3.1.2. LEMMA. (a) *There is a family* $\{A_\alpha : \alpha < \mathfrak{c}\}$ *of infinite subsets of* ω *such that if* $\alpha < \beta$, *then* $A_\alpha \cap A_\beta$ *is finite.*

(b) *There is a family* $\{\langle A_\alpha^0, A_\alpha^1\rangle : \alpha < \mathfrak{c}\}$ *of pairs of disjoint subsets of* ω *such that for all finite* $F \subseteq \mathfrak{c}$ *and for each* $f : F \to 2$ *we have that* $\bigcap_{\alpha \in F} A_\alpha^{f(\alpha)}$ *is infinite.*

(c) $|\beta\omega| = 2^{\mathfrak{c}}$; *in fact, if* $A \subseteq \beta\omega$ *is countably infinite, then* $|\bar{A}| = 2^{\mathfrak{c}}$.

PROOF. For each irrational number $r \in \mathbb{R}$ choose a sequence $S(r)$ of rational numbers converging to r. The family $\{S(r) : r \text{ irrational}\}$ is obviously as required in (a), except that it does not consist of subsets of ω, but of the countable set $\mathbb{Q}$. But this causes no problems of course.

Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be a family of subsets of ω as in (a). For each $\alpha < \mathfrak{c}$, define

$$B_\alpha^0 = \{F \in [\omega]^{<\omega} : F \cap A_\alpha \neq \emptyset\}, \quad \text{and} \quad B_\alpha^1 = [\omega]^{<\omega} \setminus B_\alpha^0 .$$

An easy check shows that the family $\{\langle B_\alpha^0, B_\alpha^1\rangle : \alpha < \mathfrak{c}\}$ has the properties of the required family in (b), except that it is not defined on ω, but on the countable set $[\omega]^{<\omega}$. But this again causes no problems of course.

Let $\{\langle A_\alpha^0, A_\alpha^1\rangle : \alpha < \mathfrak{c}\}$ be a family as in (b).

If $f \in 2^{\mathfrak{c}}$, take a point $p_f \in \beta\omega$ such that $\{A_\alpha^{f(\alpha)} : \alpha < \mathfrak{c}\} \subseteq p_f$. It is clear that $|\beta\omega| \geqslant |\{p_f : f \in 2^{\mathfrak{c}}\}| = 2^{\mathfrak{c}}$. Since $|\mathscr{P}(\omega)| = \mathfrak{c}$, $|\beta\omega| \leqslant 2^{\mathfrak{c}}$ which proves that $|\beta\omega| = 2^{\mathfrak{c}}$. Statement (c) now follows from Theorem 1.5.2 and from the fact that each countably infinite space contains a countably infinite relatively discrete subspace. $\square$

The proof of the above lemma tells us two important facts, namely that combinatorial arguments are important if one wishes to study $\beta\omega$ without extra hypotheses, and that for obtaining certain families of subsets of ω, one should not try to define them directly on ω but rather on a suitable countable set which is, in the given situation, easier to handle than ω.

Let us now return to our question: given $p \in \beta\omega$, is there a point $q \in \beta\omega$ such that $q \not\leqslant p$? The answer is now easy of course, since $|\{q \in \beta\omega : q \leqslant p\}| \leqslant \mathfrak{c}$ and $|\beta\omega| = 2^{\mathfrak{c}}$, by Lemma 3.1.2(c). Let us specify the question a little bit: given $p \in \omega^*$, is there a point $q \in \beta\omega$ such that $p \not\leqslant q$ and $q \not\leqslant p$? It may come as a shock, but the answer to this question is not known. Under CH it is easy to show that the answer is yes, but in ZFC the answer is unknown. It is known, however, that at least there are points $p, q \in \beta\omega$ with $p \not\leqslant q$ and $q \not\leqslant p$ and in the remaining part of this section we will construct such points.

3.1.3. DEFINITION. Let $\mathscr{F} \subseteq \mathscr{P}(\omega)$ be a filter no element of which is finite. An indexed family $\{\langle A_i^0, A_i^1\rangle : i \in I\}$ of pairs of disjoint subsets of ω is called an *independent family with respect to* $\mathscr{F}$ provided that for all $\sigma \in [I]^{<\omega}$, $f \in 2^\sigma$ and $F \in \mathscr{F}$ the set $F \cap \bigcap_{i \in \sigma} A_i^{f(i)}$ is infinite.

Let $\mathscr{C}\mathscr{F}$ denote the filter of cofinite subset of ω.

3.1.4. LEMMA. *There is an independent family* $\{\langle A^0_\alpha, A^1_\alpha\rangle : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ *w.r.t.* $\mathcal{CF}$.

PROOF. Lemma 3.1.2 (b). $\square$

If $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we denote by $\langle \mathcal{A}\rangle$ the (possibly improper) filter on ω generated by $\mathcal{A}$.

3.1.5. LEMMA. *Let* $\mathcal{F}$, $\mathcal{G} \subseteq \mathcal{P}(\omega)$ *be filters and assume that* $\{\langle A^0_i, A^1_i\rangle : i \in I\}$ *is independent w.r.t.* $\mathcal{F}$ *as well as* $\mathcal{G}$. *For each* $f \in \omega^\omega$ *there is a finite* $J \subseteq I$ *and a subset* $A \subseteq \omega$ *such that* $\{\langle A^0_i, A^1_i\rangle : i \in I\backslash J\}$ *is independent w.r.t.* $\langle \mathcal{F} \cup \{A\}\rangle$ *as well as* $\langle \mathcal{G} \cup \{\omega\backslash f^{-1}(A)\}\rangle$.

PROOF. Fix $a \in I$ arbitrarily.

Case 1: $\{\langle A^0_i, A^1_i\rangle : i \in I\backslash\{a\}\}$ *is independent w.r.t.* $\langle \mathcal{G} \cup \{\omega\backslash f^{-1}(A^0_a)\}\rangle$.

We then put $A = A^0_a$ and $J = \{a\}$. An easy check shows that A and J are as required.

Case 2: *Not Case* 1.

Then there are a finite $K \subseteq I\backslash\{a\}$ and a function $\tilde{f} \in 2^K$ and an element $G \in \mathcal{G}$ such that

$$(*) \qquad \left| \bigcap_{i\in K} A^{\tilde{f}(i)}_i \cap G \cap (\omega\backslash f^{-1}(A^0_a))\right| < \omega .$$

Now put $A = \omega\backslash A^0_a$ and $J = K \cup \{a\}$. It is clear that $\{\langle A^0_i, A^1_i\rangle : i \in I\backslash J\}$ is independent w.r.t. $\langle \mathcal{F} \cup \{A\}\rangle$, so it remains to be shown that $\{\langle A^0_i, A^1_i\rangle : i \in I\backslash J\}$ is independent w.r.t. $\langle \mathcal{G} \cup \{\omega\backslash f^{-1}(A)\}\rangle$. To this end, let $L \subseteq I\backslash J$ be finite and take $g \in 2^L$. Choose $G_0 \in \mathcal{G}$ arbitrarily. Then

$$\bigcap_{i\in L} A^{g(i)}_i \cap G_0 \cap (\omega\backslash f^{-1}(A)) = \bigcap_{i\in L} A^{g(i)}_i \cap G_0 \cap f^{-1}(A^0_\alpha)$$

$$\supseteq \bigcap_{i\in L} A^{g(i)}_i \cap \bigcap_{i\in K} A^{\tilde{f}(i)}_i \cap (G_0 \cap G) \cap f^{-1}(A^0_a),$$

which is infinite by $(*)$ and by our assumption that $\{\langle A^0_i, A^1_i\rangle : i \in I\}$ is independent w.r.t. $\mathcal{G}$. $\square$

We now come to the main result of this section.

3.1.6. THEOREM. *There are points* $p, q \in \beta\omega$ *such that* $p \not\leqslant q$ *and* $q \not\leqslant p$.

PROOF. By Lemma 3.1.4 there is an independent family $\{\langle A^0_\alpha, A^1_\alpha\rangle : \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ w.r.t. $\mathcal{CF}$. Let $\{f_\alpha : 1 \leqslant \alpha < \mathfrak{c}\}$ enumerate ω^ω. By transfinite induction on α we will construct $\mathcal{F}_\alpha$, $\mathcal{G}_\alpha$ and $K_\alpha \subseteq 2^\omega$ so that

(1) $\mathscr{F}_\alpha$ and $\mathscr{G}_\alpha$ are filters on ω and $\{\langle A^0_\xi, A^1_\xi\rangle : \xi \in K_\alpha\}$ is independent w.r.t. $\mathscr{F}_\alpha$ as well as $\mathscr{G}_\alpha$,

(2) $K_0 = 2^\omega$ and $\mathscr{F}_0 = \mathscr{G}_0 = \mathscr{C}\mathscr{F}$,

(3) $\kappa < \alpha$ implies $\mathscr{F}_\kappa \subseteq \mathscr{F}_\alpha$, $\mathscr{G}_\kappa \subseteq \mathscr{G}_\alpha$ and $K_\alpha \subseteq K_\kappa$,

(4) for each α, $|2^\alpha \backslash K_\alpha| \leq |\alpha| \cdot \omega$,

(5) for each $\alpha \geq 1$ there are sets A, $B \subseteq \omega$ with $\{A, \omega\backslash f_\alpha^{-1}(B)\} \subseteq \mathscr{F}_\alpha$ and $\{B, \omega\backslash f_\alpha^{-1}(A)\} \subseteq \mathscr{G}_\alpha$.

Suppose that we have completed the construction for all $\kappa < \alpha$, $\alpha < \mathfrak{c}$. Put $K = \bigcap_{\kappa<\alpha} K_\kappa$, $\mathscr{G} = \bigcup_{\kappa<\alpha} \mathscr{G}_\kappa$ and $\mathscr{F} = \bigcup_{\kappa<\alpha} \mathscr{F}_\kappa$. Observe that, by (4), $|K| = \mathfrak{c}$ and that, by (1), $\{\langle A^0_\xi, A^1_\xi\rangle : \xi \in K\}$ is independent w.r.t. $\mathscr{F}$ as well as $\mathscr{G}$. Using Lemma 3.1.5 twice, it is easy to construct $\mathscr{F}_\alpha$, $\mathscr{G}_\alpha$ and K_α satisfying (1) through (5).

Now let $p \in \beta\omega$ extend $\bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ and let $q \in \beta\omega$ extend $\bigcup_{\alpha<\mathfrak{c}} \mathscr{G}_\alpha$. By (5) it easily follows that $p \not\leq q$ and $q \not\leq p$. □

3.1.7. REMARK. Points p and q as in the above Theorem 3.1.6 are called *$\leq$-incomparable.* Observe that if p, $q \in \beta\omega$ are $\leq$-incomparable, then both p and q belong to ω^*.

3.1.8. REMARK. The technique of proof used in Theorem 3.1.6 is quite important. At each stage of the construction we give up a negligible number of the initial independent family in order to obtain in return a required property of the filter(s) we wish to construct. Under CH (or MA) such a delicate process is not necessary, since then at each stage of the construction we are at a countable level and one can then construct by hand enough sets to continue the induction. Under $\neg$CH, in a transfinite induction of length $\mathfrak{c}$ one has to pass level ω_1, and if one then for example in the previous steps constructed a family of sets which constitute a Hausdorff gap (i.e. a family of sets which witnesses the fact that $G(\omega_1, \omega_1)$ holds), then there is usually no way to continue the induction. In the proof of Theorem 3.1.6 this cannot happen, since before starting the induction enough sets were identified which ensure that one can always pick new sets to continue the induction.

3.1.9. REMARK. If $\mathfrak{c}^+ = 2^\mathfrak{c}$, then ω^* has a $\leq$-cofinal well-ordered subset (of cardinality $2^\mathfrak{c}$); and the condition $\mathfrak{c}^+ < 2^\mathfrak{c}$ is equivalent to the statement that any subset of ω^* of cardinality $2^\mathfrak{c}$ has a pairwise $\leq$-incomparable subset of cardinality $2^\mathfrak{c}$. For details, see COMFORT & NEGREPONTIS [1974, Corollaries 10.11 and 10.15].

3.2. The Rudin–Frolík order on ω^*

The *Rudin–Frolík (pre-)order $\sqsubseteq$ on ω^** is defined as follows:

$$p \sqsubseteq q \quad \text{iff} \quad \text{there is an embedding } h : \beta\omega \to \omega^* \text{ with } h(p) = q\,.$$

This order and the Rudin–Keisler order $\leq$ of Section 3.1 are related by the following lemma.

3.2.1. LEMMA. *If p, $q \in \omega^*$ and if $p \sqsubseteq q$, then $p \leqslant q$.*

PROOF. Let $h : \beta\omega \to \omega^*$ be an embedding with $h(p) = q$. Since $h(\omega)$ is a relatively discrete, there is a sequence C_n of subsets of ω such that for all $n < \omega$,

(1) $h(n) \in \bar{C}_n$, and

(2) if $n < m$, then $C_n \cap C_m = \emptyset$.

By adding $\omega \backslash \bigcup_{n<\omega} C_n$ to C_0 we may assume that the sequence $\{C_n : n < \omega\}$ is a partition of ω. Define $g : \omega \to \omega$ by

$$g(k) = n \quad \text{if } k \in C_n .$$

and let $\beta g : \beta\omega \to \beta\omega$ be its Stone extension. We claim that $\beta g(q) = p$. Take $Q \in q$ arbitrarily. The set $\{n : h(n) \in \bar{Q}\}$ must belong to p, since $h(p) = q$, but

$$\{n : h(n) \in \bar{Q}\} \subseteq g(Q) ,$$

and we therefore may conclude that $g(Q) \in p$. Consequently, $\beta g(q) = p$, which is as required. $\square$

3.2.2. COROLLARY. *If p, $q \in \omega^*$ and if $p \sqsubseteq q$ and $q \sqsubseteq p$, then $p \sim q$.*

PROOF. Apply Lemma 3.2.1 and Theorem 3.1.1(c). $\square$

3.2.3. LEMMA. *If p, q, $r \in \omega^*$ and if $p \sqsubseteq q$ and $q \sqsubseteq r$, then $p \sqsubseteq r$.*

PROOF. Let $f : \beta\omega \to \omega^*$ be an embedding with $f(p) = q$ and, similarly, let $g : \beta\omega \to \omega^*$ be an embedding with $g(q) = r$. Let $h = (g \restriction f(\beta\omega)) \circ f$. Then h is an embedding with $h(p) = r$. $\square$

Observe that Corollary 3.2.2, Lemma 3.2.3 and Theorem 3.1.1(c) show that the quotient relation defined by $\sqsubseteq$ on $\beta\omega/\sim$ is a partial ordering.

We will now show that the orders $\leqslant$ and $\sqsubseteq$ are powerful tools if one wishes to study $\beta\omega$. First a preliminary lemma.

3.2.4. LEMMA. *Let $f : \omega^* \to \omega^*$ be a homeomorphism and let $q \in \omega^*$. Then*

$$\{p \in \omega^* : p \sqsubseteq q\} = \{p \in \omega^* : p \sqsubseteq f(q)\} .$$

PROOF. Obvious. $\square$

This enables us to give our first 'real' proof that ω^* is not homogeneous.

3.2.5. THEOREM *ω^* is not homogeneous.*

PROOF. Let $D=\{d_n : n<\omega\}$ be a relatively discrete subset of ω^* and take a point $x\in \bar{D}\backslash D$. Observe that D is C^*-embedded in ω^* (Theorem 1.5.2), hence $\bar{D}=\beta D$. Put $A=\{y\in \bar{D}\backslash D : \exists$ homeomorphism $f:\omega^*\to\omega^*$ with $f(x)=y\}$. Let $h:\omega\to D$ be defined by $h(n)=d_n$ and let βh be its Stone extension. If $y\in A$, then clearly $(\beta h)^{-1}(y)\sqsubseteq y$ and consequently, by Lemma 3.2.3, $(\beta h)^{-1}(y)\sqsubseteq x$. Since βh is one to one, and since by Lemma 3.2.1 $|\{q\in\beta\omega : q\sqsubseteq x\}|\leqslant\mathfrak{c}$, we conclude that $|A|\leqslant\mathfrak{c}$. In Lemma 3.1.2 (c) we proved that $|\beta\omega|=2^{\mathfrak{c}}$. Since $\beta D\approx\beta\omega$, we therefore can find $2^{\mathfrak{c}}$ points in $\bar{D}\backslash A$. $\square$

3.3. *Another order on $\beta\omega$*

Define an order $\leqslant$ on $\beta\omega$ by

$$p\leqslant q \quad \text{if} \quad \text{there is a finite to one } f\in\omega^\omega \text{ with } \beta f(q)=p\,.$$

This order is obviously quite similar to the Rudin–Keisler order. One might hope that at least on ω^*, the orders $\leqslant$ and $\preccurlyeq$ are the same. The aim of this section is to show that this is not true. We will construct points $p, q\in\omega^*$ such that $p\preccurlyeq q$, but p and q are $\leqslant$-incomparable. We first need a generalization of the concept of an independent family of subsets of ω. We will directly translate the new concept in terms of clopen subsets of ω^*.

3.3.1. DEFINITION. An indexed family $\{A^i_j : I\in I,\ j\in J\}$ of clopen subsets of ω^* is called a *J by I independent matrix* if

(1) for all distinct $j_0,\ j_1\in J$ and $i\in I$ we have that $A^i_{j_0}\cap A^i_{j_1}=\emptyset$,
(2) if $F\in[I]^{<\omega}$ and $f\in J^F$, then

$$\bigcap\{A^\alpha_{f(\alpha)} : \alpha\in F\}\neq\emptyset\,.$$

We will first show that large families of this type exist.

3.3.2. LEMMA. *There is a $\mathfrak{c}$ by $\mathfrak{c}$ independent matrix of clopen subsets of ω^*.*

PROOF. Let $S=\{\langle k,f\rangle : k<\omega\ \&\ f\in\mathcal{P}(k)^{\mathcal{P}(k)}\}$. For each $X,\ Y\in\mathcal{P}(\omega)$, put

$$A^Y_X=\{\langle k,f\rangle\in S : f(Y\cap k)=X\cap k\}\,.$$

An easy check shows that the family $\{A^Y_X : X,\ Y\in\mathcal{P}(\omega)\}$, defined on the countable set S, gives us a $\mathfrak{c}$ by $\mathfrak{c}$ independent matrix of clopen subsets of ω^*. $\square$

3.3.3. DEFINITION. A closed subset $A\subseteq\omega^*$ is called an *R-set* if there is an open F_σ $U\subseteq\omega^*$ such that $A\subseteq\bar{U}\backslash U$ and $A\cap\bar{F}=\emptyset$ for all $F\cap[U]^{<\mathfrak{c}}$. An R-set consisting of precisely one point is called an *R-point.*

3.3.4. LEMMA. *There exists an R-point in ω^*.*

PROOF. Let $\{C_n : n<\omega\}$ be a sequence of pairwise disjoint nonempty clopen subsets of ω^*. Put $C=\bigcup_{n<\omega} C_n$. For each $n<\omega$, let $\{A^i_\alpha(n) : i<\omega,\ \alpha<\mathfrak{c}\}$ be a $\mathfrak{c}$ by ω independent matrix of clopen subsets of C_n (Lemma 3.3.2). Put

$$\mathscr{F}=\{F\subseteq C : \forall n<\omega\ \forall i\leqslant n\ \exists\alpha<\mathfrak{c} \text{ such that } A^i_\alpha(n)\subseteq F\}.$$

Notice that if $\mathscr{G}\in[\mathscr{F}]^n$, then $\bigcap\mathscr{G}\cap C_i\neq\emptyset$ for all $i\leqslant n-1$. Let $D\in[C]^{<\mathfrak{c}}$. For each $n<\omega$ and $i\leqslant n$ choose $\alpha(n,i)<\mathfrak{c}$ such that $A^i_{\alpha(n,i)}\cap D=\emptyset$ and put

$$F=\bigcup_{n<\omega}\bigcup_{i\leqslant n} A^i_{\alpha(n,i)}(n).$$

Then $F\in\mathscr{F}$ and $F\cap D=\emptyset$. Since F is clopen (in C) and since disjoint clopen subsets of C have disjoint closures in ω^* (recall that ω^* is an F-space), we conclude that $\bar{F}\cap\bar{D}=\emptyset$. (Observe that $D\subseteq C\backslash F$). Consequently, each point of $\bigcap_{F\in\mathscr{F}}\bar{F}$ is an R-point of ω^*. $\square$

The following result is the key in deriving our main result of this section.

3.3.5. THEOREM. *Let $\mathscr{A}$ be a family of $\mathfrak{c}$ R-sets in ω^*. If $\{C_n : n<\omega\}$ is a family of countably many nonempty clopen subsets of ω^*, then for each $n<\omega$ there is a point $x_n\in C_n$ such that*

$$\bigcup\mathscr{A}\cap\{x_n : n<\omega\}^-=\emptyset.$$

PROOF. List $\mathscr{A}$ as $\{A_\alpha : \alpha<\mathfrak{c}\}$. By induction, for each $\alpha<\mathfrak{c}$ we will construct for each $n<\omega$ a nonempty closed subset $F^n_\alpha\subseteq C_n$ such that

(1) $(\bigcup_{n<\omega} F^n_\alpha)^-\cap A_\alpha=\emptyset$.

(2) $\chi(F^n_\alpha,\omega^*)\leqslant|\alpha|\cdot\omega$ for each $n<\omega$,

(3) if $\kappa<\alpha$ and $n<\omega$, then $F^n_\alpha\subseteq F^n_\kappa$.

Let $U\subseteq\omega^*$ be an open F_σ which witnesses the fact that A_0 is an R-set. Define $E=\{n<\omega : C_n\backslash\bar{U}\neq\emptyset\}$ and for each $n\in E$ choose a nonempty clopen $E_n\subseteq C_n\backslash\bar{U}$. For all $n\notin E$, pick a point $t_n\in C_n\cap U$. Since A_0 is an R-set, $A_0\cap\{t_n : n\notin E\}^-=\emptyset$. Consequently, we can find for any $n\notin E$ a clopen neighborhood E_n of t_n such that $E_n\subseteq C_n$ and $(\bigcup_{n\notin E} E_n)^-\cap A_0=\emptyset$. For each $n<\omega$ define $F^n_0=E_n$. By construction, $(\bigcup_{n\notin E} E_n)^-\cap A_0=\emptyset$ and since ω^* is an F-space, $(\bigcup_{n\in E} E_n)^-\cap A_0\subseteq(\bigcup_{n\in E} E_n)^-\cap\bar{U}=\emptyset$. Consequently, the F^n_0's are as required.

Suppose that we have completed the construction for all $\mu<\alpha<\mathfrak{c}$. Put $G_n=\bigcap_{\mu<\alpha} F^n_\mu$ for all $n<\omega$ and observe that $\chi(G_n,\omega^*)\leqslant|\alpha|\cdot\omega$. Let $U\subseteq\omega^*$ be an open F_σ which witnesses the fact that A_α is an R-set. Put $E=\{n<\omega : G_n\cap\bar{U}\neq\emptyset\}$ and for each $n\in E$ let $\{V^n_\rho : \rho<|\alpha|\cdot\omega\}$ be a neighborhood basis for G_n. For each $n \in E$ and $\rho < |\alpha| \cdot \omega$ pick a point in $V^n_\rho \cap U$ and let Z be the set of points obtained in this way. Then $|Z| < \mathfrak{c}$ and therefore there is a clopen neighborhood C of A_α which misses $\bar{Z}$. Define $F^n_\alpha = G_n$ if $n \notin E$ and $F^n_\alpha = G_n \backslash C$ if $n \in E$. An easy check, again using the fact that ω^* is an F-space, shows that everything is defined properly.

For each $n < \omega$ take a point $x_n \in \bigcap_{\alpha < \mathfrak{c}} F^n_\alpha$. Then $\{x_n : n < \omega\}$ is as required. □

We need one more lemma.

3.3.6. LEMMA. *Let $f \in \omega^\omega$ be finite to one. If $p \in \omega^*$ is an R-point, then $\beta f^{-1}(\{p\})$ is an R-set.*

PROOF. Observe that $\beta f : \beta\omega \to \beta\omega$ is open and that $\beta f(\omega^*) \subseteq \omega^*$. If $p \notin \beta f(\beta\omega)$, then there is nothing to prove, so, without loss of generality, f is onto. Let $U \subseteq \omega^*$ be an open F_σ which witnesses the fact that p is an R-point. Since βf is open, $Z = \beta f^{-1}(\{p\}) \subseteq \bar{V} \backslash V$, where $V = \beta f^{-1}(U)$. Notice that V is an open F_σ of ω^* since $\beta f^{-1}(\omega^*) = \omega^*$. It is clear that if $F \in [V]^{<\mathfrak{c}}$, then $\bar{F} \cap Z = \emptyset$, whence Z is an R-set. □

We now come to the main result of this section.

3.3.7. THEOREM. *For each R-point $x \in \omega^*$ there is a point $y \in \omega^*$ such that $x \leqslant y$ but x and y are $\leqslant$-incomparable.*

PROOF. Let $f : \omega \to \omega$ be such that $|f^{-1}(\{n\})| = \omega$ for all $n < \omega$. For each $n < \omega$, let $\{E^i_n : i < \omega\}$ be a family of countably many pairwise disjoint (faithfully indexed) nonempty clopen subsets of $\beta f^{-1}(\{n\}) \cap \omega^*$. By Theorem 3.3.5 and Lemma 3.3.6 we may pick for all i, $n < \omega$ a point $x^i_n \in E^i_n$ such that $\{x^i_n : i, n < \omega\}^- \cap \bigcup\{\beta g^{-1}(\{x\}) : g \in \omega^\omega$ is finite to one$\} = \emptyset$. For each $i < \omega$, let $S_i = \{x^i_n : n < \omega\}$. Observe that $\beta f(S_i) = \omega$ which implies that $\bar{S}_i \cap \beta f^{-1}(\{x\}) \neq \emptyset$. If $i \neq j$ then, since ω^* is an F-space, $\bar{S}_i \cap \bar{S}_j = \emptyset$ and this implies that

$$|\{x^i_n : i, n < \omega\}^- \cap \beta f^{-1}(\{x\})| \geqslant \omega ,$$

Therefore $\{x^i_n : i, n < \omega\}^- \cap \beta f^{-1}(\{x\})$ contains a countably infinite relatively discrete set, which is C^*-embedded in ω^* by Theorem 1.5.2, and which therefore has the property that its closure has cardinality $2^\mathfrak{c}$ (Lemma 3.1.2(c)). Since $|\{p \in \omega^* : p \leqslant x\}| \leqslant \mathfrak{c}$, we can therefore find a point $y \in \beta f^{-1}(\{x\}) \backslash (\{p \in \omega^* : p \leqslant x\} \cup \bigcup\{\beta g^{-1}(\{x\}) : g \in \omega^\omega$ is finite to one$\})$. It is clear that y is as required. □

Since, by Lemma 3.3.4, R-points in ω^* exist, we have therefore obtained, the following.

3.3.8. COROLLARY. *There are points p, $q \in \omega^*$ such that $p \preccurlyeq q$ but p and q are $\leqslant$-incomparable.*

3.3.9. REMARK. It might come as a surprise that the proof in this section has not very much in common with the proof in Section 3.1 that $\leqslant$-incomparable points in $\beta\omega$ exist. I do not know whether Corollary 3.3.8 can also be obtained by the method of Section 3.1. Notice however that both methods have an important fact in common, namely that beforehand things have been arranged so that a transfinite induction of length $\mathfrak{c}$ was possible.

3.3.10. REMARK. In Section 4.5 we will compare the orders $\leqslant$, $\sqsubseteq$ and $\preccurlyeq$ with one another.

3.4. Applications of the Rudin–Keisler order

In this section we will give a surprisingly general nonhomogeneity result. This will allow us to give another proof that ω^* is not homogenous.

3.4.1. THEOREM. *Let X be an infinite compact space in which all countable discrete subspaces are C^*-embedded. Then X is not homogeneous.*

PROOF. Since X contains a countable discrete subspace, for convenience assume that $\omega \subseteq X$. The assumptions on X then imply that $\bar{\omega} = \beta\omega$. By Theorem 3.1.6 there are points p, $q \in \beta\omega$ which are $\leqslant$-incomparable. We claim there is no homeomorphism $h: X \to X$ with $h(p) = q$. Striving for a contradiction, assume that such an h exists.

Let $\{U_n : n < \omega\}$ be a family of open subsets of X such that

(1) $n \in U_n \subseteq \bar{U}_n \subseteq X \backslash \omega^*$,

(2) if $n \neq m$, then $\bar{U}_n \cap \bar{U}_m = \emptyset$.

Put $E = \{n < \omega : h(n) \notin \bigcup_{m<\omega} U_m \cup \bar{\omega}\}$.

Case 1: $q \in \overline{h(E)}$. Since $h(E) \cup \omega$ is clearly a discrete subset of X, and since $h(E) \cap \omega = \emptyset$, the assumptions on X imply that $\overline{h(E)} \cap \bar{\omega} = \emptyset$, which is impossible since $q \in \bar{\omega}$.

Put $F = \{n < \omega : h(n) \in \omega^*\}$.

Case 2: $q \in \overline{h(F)}$. Then $p \in \bar{F}$ and we may conclude that $p \sqsubseteq q$ and consequently, by Lemma 3.2.1, $p \leqslant q$. This is a contradiction.

Put $G = \{n < \omega : h(n) \in \bigcup_{m<\omega} U_m\}$.

Case 3: $q \in \overline{h(G)}$. Define a function $f: \omega \to \omega$ by

$$\begin{cases} f(k) = 0 & \text{if } k \notin G, \\ f(k) = n & \text{if } k \in G \text{ and } h(k) \in U_n . \end{cases}$$

An easy check shows that $\beta f(p) = q$, i.e. $q \leqslant p$, which is also a contradiction.

Since $E \cup F \cup G = \omega$ and $p \in \bar{\omega}$, $q = h(p) \in \overline{h(E)} \cup \overline{h(F)} \cup \overline{h(G)}$. We therefore have derived a contradiction. □

3.4.2. COROLLARY. *No compact infinite F-space is homogeneous. In particular, ω^* is not homogeneous.*

PROOF. By using the same technique as in the proof of Theorem 1.5.2, the reader can easily check that every countable subspace of an F-space is C^*-embedded. □

Notes for Section 3

The Rudin–Keisler order on $\beta\omega$ was defined by KATĚTOV [1961] and independently, by M.E. RUDIN [1966] and KEISLER [1967]. Theorem 3.1.1 is due to Katětov, M.E. Rudin and Keisler. Lemma 3.1.2(a) and (b) are well-known. The proof of Lemma 3.1.2(a) is due to SIERPIŃSKI [1928]. The proof of Lemma 3.1.2(b) from 3.1.2(a) is new and was suggested to me by Charley Mills. Lemma 3.1.2(c) is due to HAUSDORFF [1936]. Lemma 3.1.4 is well-known and also follows from the fact that $2^{\mathfrak{c}}$ is separable. Lemma 3.1.5 and Theorem 3.1.6 are due to KUNEN [1972]. In fact, Kunen proves that there are $\mathfrak{c}$ pairwise $\leqslant$-incomparable points. Recently, SHELAH and R.E. RUDIN [1978] showed that there even exist $2^{\mathfrak{c}}$ pairwise $\leqslant$-incomparable points. The Rudin–Frolík order on ω^* was defined by M.E. RUDIN [1966] and FROLÍK [1967a]. See also M.E. RUDIN [1971]. Theorem 3.2.3 is due to FROLÍK [1967a]. Under CH, it was earlier shown by W. RUDIN [1956]. Lemma 3.3.2 is due to KUNEN [1978]. All other results in section 3.3 were taken from VAN MILL [1981a]. Theorem 3.4.1 was formulated in COMFORT [1977]. The method of proof used in Theorem 3.4.1 is due to FROLÍK [1967b]. Much of the material presented in this chapter can also be found in COMFORT and NEGREPONTIS [1974].

For some recent information concerning the Rudin–Frolík order, see BUKOVSKÝ & BUTKOVICOVÁ [1981].

4. Weak P-points and other points in ω^*

We have seen that, under CH, there are P-points and non P-points in ω^*, whence ω^* is not homogeneous, see section 1.7. However, in section 2.7 we saw that this nonhomogeneity proof in ZFC did not work. In sections 3.2 and 3.4 we gave proofs in ZFC that ω^* is not homogeneous, but these proofs 'only' showed *that* ω^* is not homogeneous but not *why* it is not homogeneous. The aim of this section is to present several 'special' points in ω^*, thus giving a 'real' proof that ω^* is not homogeneous.

4.1. A technical result

The aim of this section is to prove a technical result which enables us later to construct several special points in ω^*.

4.1.1. DEFINITION. Let X be a compact extremally disconnected space, and let $\mathscr{C} = \{C_n : n < \omega\}$ be a sequence of nonempty, faithfully indexed, pairwise disjoint, clopen subsets of X and put $Z = X \setminus \bigcup \mathscr{C}$. In addition, let $f : Z \to Y$ be a continuous surjection and let $B \subseteq Z$ be closed.

If $1 \leq n < \omega$, an indexed family $\{A_i : i \in I\}$ of clopen subsets of X is *precisely n-linked* w.r.t. $\langle B, f\rangle$ if for all $\sigma \in [I]^n$,

$$f\left(\bigcap_{i \in \sigma} A_i \cap B\right) = Y,$$

but for all $\sigma \in [I]^{n+1}$, $\bigcap_{i \in \sigma} A_i \cap Z = \emptyset$.

An indexed family $\{A_{in} : i \in I,\ 1 \leq n < \omega\}$ of clopen subsets of X is a *linked system* w.r.t. $\langle B, f\rangle$, if for each n, $\{A_{in} : i \in I\}$ is precisely n-linked w.r.t. $\langle B, f\rangle$, and for each n and i, $A_{in} \subseteq A_{i,n+1}$. An indexed family $\{A^j_{in} : i \in I,\ 1 \leq n < \omega,\ j \in J\}$ is an *I by J independent linked family* w.r.t. $\langle B, f\rangle$ if for each $j \in J$, $\{A^j_{in} : i \in I,\ 1 \leq n < \omega\}$ is a linked system w.r.t. $\langle B, f\rangle$, and:

$$f\left(\bigcap_{j \in \tau}\left(\bigcap_{i \in \sigma_j} A^j_{in_j}\right) \cap B\right) = Y,$$

whenever $\tau \in [J]^{<\omega}$, and for each $j \in \tau$, $1 \leq n_j < \omega$ and $\sigma_j \in [I]^{n_j}$.

If $f : \omega \to \omega$ is a function, let $\bar{f} = \beta f \restriction \omega^*$. In Definition 4.1.1, let $X = \beta\omega$ and, for all $n < \omega$, let $C_n = \{n\}$. We then have the following important lemma.

4.1.2. LEMMA. *There is a finite to one function $\pi : \omega \to \omega$ and a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family of clopen subsets of $\beta\omega$* w.r.t. $\langle \omega^*, \bar{\pi}\rangle$.

PROOF. Let $S = \{\langle k, g\rangle : k \in \omega\ \&\ g \in \mathscr{P}\mathscr{P}(k)^{\mathscr{P}(k)}\}$. Identify S and ω and define $\pi : S \to \omega$ by $\pi(\langle k, g\rangle) = k$. It is clear that π is finite to one. For all $X, Y \in \mathscr{P}(\omega)$ and $n < \omega$, define

$$A^Y_{Xn} = \{\langle k, g\rangle \in S : |g(Y \cap k)| \leq n\ \&\ X \cap k \in g(Y \cap k)\}.$$

It is easily seen that the family

$$\{E^Y_{Xn} : X \in \mathscr{P}(\omega),\ 1 \leq n < \omega,\ Y \in \mathscr{P}(\omega)\},$$

where E^Y_{Xn} is the closure of A^Y_{Xn} in βS, is as required. $\square$

4.1.3. DEFINITION. Let X be a space. A closed subspace $A \subseteq X$ is called κ-OK provided that for each sequence $\{U_n : n < \omega\}$ of neighborhoods of A, there is a sequence $\{V_\alpha : \alpha < \kappa\}$ of neighborhoods of A such that for each $n \geq 1$ and $a_1 < \alpha_2 < \cdots < \alpha_n < \kappa$,

$$\bigcap_{1 \leq i \leq n} V_{\alpha_i} \subseteq U_n .$$

Observe that the property of being κ-OK gets stronger as κ gets bigger. A point $x \in X$ is called a κ-OK *point* if $\{x\}$ is a κ-OK set of X.

4.1.4. DEFINITION. Let X be a space and let $\mathscr{U} = \{U_n : n < \omega\}$ be a family of open subsets of X. A closed subset $Z \subseteq \overline{\bigcup \mathscr{U}} \backslash \bigcup \mathscr{U}$ is called *nice* w.r.t. $\mathscr{U}$ provided that for each neighborhood V of Z the set $\{n < \omega : V \cap U_n = \emptyset\}$ is finite.

We now come to the main result of this section.

4.1.5. THEOREM. *Let X be a compact extremally disconnected space of weight $\mathfrak{c}$ and let $\mathscr{C} = \{C_n : n < \omega\}$ be a sequence of nonempty, faithfully indexed, pairwise disjoint, clopen subsets of X and put $Z = \overline{\bigcup \mathscr{C}} \backslash \bigcup \mathscr{C}$. If $A \subseteq Z$ is nice w.r.t. $\mathscr{C}$ and if Y is a continuous image of ω^*, then there is a closed set $B \subseteq A$ which is a $\mathfrak{c}$-OK set of Z and which admits an irreducible surjection on Y.*

PROOF. Since $\overline{\bigcup \mathscr{C}}$ is clopen in X, without loss of generality, $\overline{\bigcup \mathscr{C}} = X$. Define $f : \bigcup \mathscr{C} \to \omega$ by $f(x) = n$ iff $x \in C_n$ and let $\beta f : X \to \beta\omega$ be its Stone extension. Since X is an F-space, $\overline{\bigcup \mathscr{C}} = \beta(\bigcup \mathscr{C})$. Let $\pi : \omega \to \omega$ be the finite to one function of Lemma 4.1.2 and let $\{E^\beta_{\alpha n} : \alpha < \mathfrak{c},\ 1 \leq n < \omega,\ \beta < \mathfrak{c}\}$ be the $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family of clopen subsets of $\beta\omega$ w.r.t. $\langle \omega^*, \bar{\pi} \rangle$ of Lemma 4.1.2. In addition, let $g : \omega^* \to Y$ be a continuous surjection.

Define $h : Z \to Y$ by $h = g \circ \bar{\pi} \circ (\beta f \restriction Z)$ and observe that the family

$$\{A^\beta_{\alpha n} : \alpha < \mathfrak{c},\ 1 \leq n < \omega,\ \beta < \mathfrak{c}\},$$

where $A^\beta_{\alpha n} = \beta f^{-1}(E^\beta_{\alpha n})$, is an independent linked family w.r.t. $\langle A, h \rangle$. For this one only needs to verify that $\beta f(A) = \omega^*$, and this is easy. Let $\{Z_\mu : \mu < \mathfrak{c}\ \&\ \mu \text{ is even}\}$ enumerate the family of all clopen subsets of X and let $\{\langle S_{\mu n} : n < \omega \rangle : \mu < \mathfrak{c}\ \&\ \mu$ is odd$\}$ enumerate all sequences of nonempty clopen subsets of X satisfying

$$S_{\mu, n+1} \subseteq S_{\mu n} \backslash \bigcup_{i \leq n} C_i .$$

Furthermore, assume that each sequence is listed cofinally often. By induction on μ we construct F_μ and K_μ so that:

(1) $F_\mu \subseteq A$ is closed, $K_\mu \subseteq \mathfrak{c}$, and $\{A^\beta_{\alpha n} : \alpha < \mathfrak{c},\ 1 \leq n < \omega,\ \beta \in K_\mu\}$ is an independent linked family w.r.t. $\langle F_\mu, h\rangle$,

(2) $K_0 = 2^\omega$ and $F_0 = A$;

(3) $\nu < \mu$ implies $F_\nu \supseteq F_\mu$ and $K_\mu \subseteq K_\nu$,

(4) if μ is a limit ordinal, $F_\mu = \bigcap_{\nu<\mu} F_\nu$ and $K_\mu = \bigcap_{\nu<\mu} K_\nu$,

(5) for each μ, $K_\mu \backslash K_{\mu+1}$ is finite,

(6) if μ is even, either $F_{\mu+1} \subseteq Z_\mu$ or $h(F_{\mu+1} \cap Z_\mu) \neq Y$,

(7) if μ is odd and $F_\mu \subseteq \bigcap_{n<\omega} S_{\mu n}$, then there are clopen neighborhoods $D_{\mu\alpha}$ of $F_{\mu+1}$ for $\alpha < \mathfrak{c}$ such that for all $n \geq 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \mathfrak{c}$, there is an $m < \omega$ such that

$$(D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}) \backslash S_{\mu n} \subseteq \bigcup_{i\leq m} C_i .$$

Fix $\mu < \mathfrak{c}$ and assume that F_ν, K_r have been constructed for all $\nu \leq \mu$. We will construct $F_{\mu+1}$ and $K_{\mu+1}$.

Suppose first that μ is even and define $T = F_\mu \cap Z_\mu$. If

$$\{A^\beta_{\alpha n} : \alpha < \mathfrak{c},\ 1 \leq n < \omega,\ \beta \in K_\mu\}$$

is an independent linked family w.r.t. $\langle T, h\rangle$, we put $F_{\mu+1} = T$ and $K_{\mu+1} = K_\mu$. If not, then

$$h\Big(F_\mu \cap Z_\mu \cap \bigcap_{\beta\in\tau} \Big(\bigcap_{\alpha\in\sigma_\beta} A^\beta_{\alpha n_\beta}\Big)\Big) \neq Y$$

for some $\tau \in [K_\mu]^{<\omega}$, $n_\beta < \omega$ and $\sigma_\beta \in [\mathfrak{c}]^{n_\beta}$. Then let $K_{\mu+1} = K_\mu \backslash \tau$, and let

$$F_{\mu+1} = F_\mu \cap \bigcap_{\beta\in\tau} \Big(\bigcap_{\alpha\in\sigma_\beta} A^\beta_{\alpha n_\beta}\Big) .$$

Clearly, $F_{\mu+1}$ and $K_{\mu+1}$ are as required.

If μ is odd and there is an $n < \omega$ such that $F_\mu \backslash S_{\mu n} \neq \emptyset$, put $F_{\mu+1} = F_\mu$ and $K_{\mu+1} = K_\mu$. In case $F_\mu \subseteq \bigcap_{\eta<\omega} S_{\mu n}$, fix $\beta \in K_\mu$ and let $K_{\mu+1} = K_\mu \backslash \{\beta\}$. For each $\alpha < \mathfrak{c}$, define

$$D_{\mu\alpha} = \Big(\bigcup_{1\leq n<\omega} A^\beta_{\alpha n} \cap S_{\mu n}\Big)^- ,$$

and put $F_{\mu+1} = \bigcap_{\alpha<\mathfrak{c}} D_{\mu\alpha} \cap F_\mu$. We claim that $F_{\mu+1}$ and the sequence $\langle D_{\mu\alpha} : \alpha < \mathfrak{c}\rangle$ are as required.

First observe that each $D_{\mu\alpha}$ is clopen since it is the closure of an open set in X. To verify condition (7), let $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \mathfrak{c}$ and put

$$T = (D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}) \setminus S_{\mu n}.$$

If $n = 1$, then clearly $T = \emptyset$, since $D_{\mu\alpha_1} \subseteq S_{\mu 1}$. Therefore, assume that $n > 1$.

Claim. $T \subseteq A^{\beta}_{\alpha_1, n-1} \cap \cdots \cap A^{\beta}_{\alpha_n, n-1}$.

Take $x \in D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}$ and assume that

$$x \in \bigcap_{1 \leq i \leq n} A^{\beta}_{\alpha_i k_i} \cap S_{\mu k_i},$$

where $k_{i_0} \geq n$ for some $1 \leq i_0 \leq n$. Since $S_{\mu k_{i_0}} \subseteq S_{\mu n}$ it follows that $x \notin T$. Next suppose that

$$x \in \bigcap_{1 \leq i \leq n} A^{\beta}_{\alpha_i k_i} \cap S_{\mu k_i}$$

where $k_i < n$ for all $1 \leq i \leq n$. Since $A^{\beta}_{\alpha_i k_i} \subseteq A^{\beta}_{\alpha_i, n-1}$ for all $1 \leq i \leq n$, this implies that $x \in \bigcap_{1 \leq i \leq n} A^{\beta}_{\alpha_i, n-1}$. We therefore conclude that

$$\begin{aligned} T &= (D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}) \cap (X \setminus S_{\mu n}) \\ &= \Big(\bigcap_{1 \leq i \leq n} \Big(\bigcup_{1 \leq k < \omega} A^{\beta}_{\alpha_i k} \cap S_{\mu k} \Big) \cap (X \setminus S_{\mu n}) \Big)^{-} \\ &\subseteq \bigcap_{1 \leq i \leq n} A^{\beta}_{\alpha_i, n-1}. \end{aligned}$$

This implies that for some $m < \omega$ we have that $T \subseteq \bigcup_{i \leq m} C_i$, since the $\{A^{\beta}_{\alpha_i, n-1} : 1 \leq i \leq n\}$ are precisely $(n-1)$-linked.

Finally, to verify condition (1), observe that $D_{\mu\alpha} \supseteq S_{\mu n} \cap A^{\beta}_{\alpha n}$ for each $n < \omega$.

Now put $B = \bigcap_{\mu < \mathfrak{c}} F_\mu$. We claim that B is as required. By (1), $h(F_\mu) = Y$ for all $\mu < \mathfrak{c}$ and therefore, by compactness and by (3), $h(B) = Y$. By (7), B is a $\mathfrak{c}$-OK set of Z and it therefore suffices to prove that $h \restriction B$ is irreducible. If $B' \subseteq B$ is a proper closed set, then for some even $\mu < \mathfrak{c}$, $B' \subseteq Z_\mu$ and $B \setminus Z_\mu \neq \emptyset$. Then, by (6), $h(F_{\mu+1} \cap Z_\mu) \neq Y$. Since $B' \subseteq F_{\mu+1} \cap Z_\mu$, we conclude that $h(B') \neq Y$, consequently, $h \restriction B$ is irreducible. $\square$

4.2. *A compactification of ω*

We will show that there is a compactification $\gamma\omega$ of ω such that $\gamma\omega \setminus \omega$ is not separable but yet satisfies the countable chain condition. This compactification we need in the next section to construct several special points in ω^*.

Let $P = \{f \in \omega^\omega : 0 \leq f(n) \leq n+1$ for each $n < \omega\}$ and $N = \{f \restriction n : f \in P$ and $n < \omega\}$. Define $T = \{\pi \in N^\omega : \mathrm{dom}(\pi(n)) = n+1$ for each $n < \omega\}$. For each $s \in N$, let $C_s = \{t \in N : s \subseteq t\}$ and for each $\pi \in T$ put

$$C_\pi = \bigcup_{n<\omega} C_{\pi(n)} .$$

Observe that $N \backslash C_\pi$ is infinite for each π. Let $\mathscr{B}$ be the smallest Boolean subalgebra of $\mathscr{P}(N)$ containing $\mathscr{A} = \{C_\pi : \pi \in T\} \cup \{N \backslash C_\pi : \pi \in T\}$. Notice that $\{\{s\} : s \in N\} \cup \{C_s : s \in N\} \subseteq \mathscr{B}$. Let $\gamma\omega$ denote the Stone space of $\mathscr{B}$. Clearly, $\gamma\omega$ is a compactification of the countable discrete space $\{\{B \in \mathscr{B} : s \in B\} : s \in N\}$ which we identify with ω, Put $X = \gamma\omega \backslash \omega$.

4.2.1. LEMMA. *X is not separable.*

PROOF. Let $\{p_n : n < \omega\}$ be countably many free ultrafilters on $\mathscr{B}$. For each $n < \omega$, there exists $\pi(n)$ with $\mathrm{dom}(\pi(n)) = n+1$ such that $C_{\pi(n)} \in p_n$. Simply observe that $N = \{s \in N : \mathrm{dom}(s) \leq n\} \cup \bigcup\{C_s : \mathrm{dom}(s) = n+1\}$ for each $n < \omega$. Consequently, $\{p \in X : N \backslash C_\pi \in p\}$ is a nonempty open set of X disjoint from $\{p_n : n < \omega\}$. $\square$

A family of sets is called *linked* provided that each subfamily of cardinality at most 2 has nonempty intersection. Call a family of sets *σ-linked* provided that it is the union of countably many linked subfamilies. It is obvious that a space having a σ-linked base is ccc.

4.2.2. LEMMA. *X has a σ-linked base.*

PROOF. It suffices to show that $\{B \in \mathscr{B} : |B| = \omega\} = \bigcup_{n \in \omega} \mathscr{B}_n$ such that for each n every two members of $\mathscr{B}_n$ have infinite intersection. To this end, for each $j \in \omega$ and for each $s \in N$ with $2j - 1 \leq \mathrm{dom}\, s$, define

$$\mathscr{B}(j, s) = \Big\{B \in \mathscr{B} : \exists K \in [T]^{<\omega} \text{ and } L \in [T]^j \text{ with } s \in \bigcap_{\pi \in K} C_\pi \cap \bigcap_{\pi \in L} N \backslash C_\pi \in [B]^\omega\Big\} .$$

Since for each $B \in \mathscr{B}$ with $|B| = \omega$, there exists a set D which is a finite intersection of elements of $\mathscr{A}$, with $D \in [B]^\omega$ and since any infinite subset of N contains elements of arbitrarily large domain, it follows that

$$\{B \in \mathscr{B} : |B| = \omega\} = \bigcup\{\mathscr{B}(j, s) : j \in \omega,\ s \in N, \text{ and } 2j - 1 \leq \mathrm{dom}\, s\} .$$

Fix an index j and $s \in N$ with $2j - 1 \leq \mathrm{dom}\, s$. If $\{B_0, B_1\} \subseteq \mathscr{B}(j, s)$, then there exist

$K_i \in [T]^{<\omega}$ and $L_i \in [T]^j$ such that for each $i < 2$,

$$s \in D_i = \bigcap_{\pi \in K_i} C_\pi \cap \bigcap_{\pi \in L_i} N \backslash C_\pi \in [B_i]^\omega .$$

We now define, by induction on dom $s \leqslant n$, an $h \in P$ such that

$$\{h \restriction n : \operatorname{dom} s \leqslant n\} \subseteq D_0 \cap D_1 .$$

Stage dom s: Let $h \restriction \operatorname{dom} s = s$. Then $h \restriction \operatorname{dom} s \in D_0 \cap D_1$. Assume we have defined $h \restriction n$ for some dom $s \leqslant n$ such that $h \restriction n \in D_0 \cap D_1$.

Stage $n+1$: Define $h \restriction n+1$ to be some sequence in N of domain $n+1$ that extends $h \restriction n$ and such that $h \restriction n+1 \notin \{\pi(n) : \pi \in L_0 \cup L_1\}$. This is possible because there are $n+2$ sequences in N of domain $n+1$ that extend $h \restriction n$ and $|L_0 \cup L_1| \leqslant 2j < \operatorname{dom} s + 2 \leqslant n+2$. Then $h \restriction n+1 \in D_0 \cap D_1$. □

4.3. *Weak P-points in ω^**

In this section we will show that ω^* contains at least two types of weak P-points. Let X be a space. A subset $F \subseteq X$ is called a *weak P-set* provided that $F \cap \bar{D} = \emptyset$ for any countable $D \subseteq X \backslash F$.

4.3.1. LEMMA. *Let X be a space and let $S \subseteq X$. Then*
(a) *if S is ω_1-OK then S is a weak P-set of X, and*
(b) *if S is κ-OK, where $\operatorname{cf}(\kappa) \geqslant \omega_1$, and S is not a P-set, then $c(X) \geqslant \kappa$.*

PROOF. Let $F = \{t_n : n < \omega\} \subseteq X \backslash S$ be any sequence. Since S is ω_1-OK, we can find a collection $\{U_\xi : \xi < \omega_1\}$ of neighborhoods of S such that for all $\xi_1 < \xi_2 < \cdots < \xi_n < \omega_1$, we have that

$$(*) \qquad \bigcap_{1 \leqslant i \leqslant n} U_{\xi_i} \subseteq X \backslash \{t_n\} .$$

If $U_\xi \cap F \neq \emptyset$ for all $\xi < \omega_1$, then there are an uncountable $A \subseteq \omega_1$ and an $n < \omega$ such that $t_n \in \bigcap_{\xi \in A} U_\xi$. But this obviously contradicts (*). For (b), let $F_n \subseteq \bar{F}_n \subseteq X \backslash S$ $(n < \omega)$ be a sequence of open sets in X such that

$$S \cap \overline{\left(\bigcup_{n<\omega} \bar{F}_n \backslash \bigcup_{n<\omega} \bar{F}_n \right)} \neq \emptyset .$$

Choose a family $\{U_\xi : \xi < \kappa\}$ of neighborhoods of S such that for all $\xi_1 < \xi_2 < \cdots < \xi_n < \kappa$, $\bigcap_{1 \leqslant i \leqslant n} U_{\xi_i} \subseteq X \backslash \bar{F}_n$. Since $\operatorname{cf}(\kappa) \geqslant \omega_1$, there have to be a set $A \in [\kappa]^\kappa$

and an $n<\omega$ such that $U_\xi \cap F_n \neq \emptyset$ for all $\xi \in A$. Then

$$\mathcal{B} = \{U_\xi \cap F_n : \xi \in A\}$$

is a family of κ open subsets of X such that any intersection of n of them has empty intersection. By transfinite induction, for each $\xi<\kappa$ we will define a maximal subfamily $\mathcal{G}_\xi \subseteq \mathcal{B}$ such that $\bigcap \mathcal{G}_\xi \neq \emptyset$ and $\mathcal{G}_\xi \neq \mathcal{G}_\eta$ for all $\eta<\xi<\kappa$. If $\mathcal{G}_\eta$ has been defined for all $\eta<\xi<\kappa$, then take

$$B \in \mathcal{B} \setminus \bigcup_{\eta<\xi} \mathcal{G}_\eta$$

Such a B exists since $|\mathcal{G}_\eta| \leqslant n-1$ for all $\eta<\xi$. Then let $\mathcal{G}_\xi$ be any maximal subfamily of $\mathcal{B}$ which contains B and has nonempty intersection. The family $\{\bigcap \mathcal{G}_\xi : \xi<\kappa\}$ consists of κ pairwise disjoint nonempty open subsets of X, whence $c(X) \geqslant \kappa$. $\square$

4.3.2. COROLLARY. *Let $x \in X$ be ω_1-OK. Then x is a weak P-point of X.*

We now come to the main result of this section.

4.3.3. THEOREM. *Let $A = \{x \in \omega^* : x$ is $\mathfrak{c}$-OK$\}$ and $B = \{x \in \omega^* : x$ is a weak P-point and $x \in \bar{C} \setminus C$ for some $C \subseteq \omega^*$ satisfying the* ccc$\}$. *Then $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$.*

PROOF. That $A \neq \emptyset$ follows directly from Theorem 4.1.4 and that $A \cap B = \emptyset$ is a consequence of Lemma 4.3.1(b). It remains to show that $B \neq \emptyset$. To this end, let X be the ccc nonseparable remainder of ω constructed in section 4.2. Observe that the one point compactification $a(\omega \times X)$ of $\omega \times X$, is a continuous image of ω^*, whence, by Theorem 4.1.5, there is a closed $\mathfrak{c}$-OK set $Z \subseteq \omega^*$ and irreducible surjection $f: Z \to a(\omega \times X)$. For each $n<\omega$, put $Z_n = f^{-1}(\{n\} \times X)$. Observe that by irreducibility of f, we have that $(\bigcup_{n<\omega} Z_n)^- = Z$. Let $\pi : \omega \times X \to X$ be the projection and let $\mathcal{U}$ be a maximal disjoint family of separable clopen subsets of X. Since X is ccc but not separable, $(\bigcup \mathcal{U})^- \neq X$ and consequently we can pick a nonempty clopen $C \subseteq X$ which misses $\bigcup \mathcal{U}$. Observe that C is nowhere separable. For each countable $D \subseteq \bigcup_{n<\omega} Z_n$, let $\{C_n(D) : n<\omega\}$ be a maximal disjoint family of nonempty clopen subsets of C such that

$$\bigcup_{n<\omega} C_n(D) \cup (\overline{\pi(f(D))} \cap C) = \emptyset$$

and define

$$F(D) = \bigcup_{n<\omega} f^{-1}\Big(\bigcup_{i\leqslant n} \{n\} \times C_i(D)\Big).$$

Define $F = \bigcap\{\overline{F(D)}: D \in [f^{-1}(\omega \times X)]^{\leq\omega}\}$. It is easily seen that F is nice w.r.t. $\{f^{-1}(\{n\} \times X): n < \omega\}$. Since X is an F-space, being closed in ω^*, and since, by irreducibility of f, Z is ccc, we may conclude that Z is extremally disconnected (Lemma 1.2.2). By Theorem 4.1.4, there is a point $x \in S = Z \backslash \bigcup_{n<\omega} Z_n$ which belongs to F and which is a $\mathfrak{c}$-OK point of S.

Claim. If $D \in [\bigcup_{n<\omega} Z_n]^{\leq\omega}$, then $x \notin \bar{D}$.

By construction, $F(D)$ is a clopen subspace of $\bigcup_{n<\omega} Z_n$ which misses D. Since Z is extremally disconnected,

$$\overline{F(D)} \cap \left(\overline{\bigcup_{n<\omega} Z_n \backslash F(D)}\right) = \emptyset .$$

Since $x \in \overline{F(D)}$, we conclude that $x \notin \bar{D}$.

The claim now can be used to prove quite easily that x is a weak P-point of ω^*. □

4.3.4. COROLLARY. *ω^* contains weak P-points.*

Notice that Theorem 4.3.3 proves once again that ω^* is not homogeneous. This proof is totally different from the previous ones, since we found an easy to state topological property shared by some but not all points in ω^*.

A space X is called *first order homogeneous* provided that no property which can be expressed in first order language distinguishes points of X. It is clear that any homogeneous space is first order homogeneous, but not conversely. It was shown by VAN DOUWEN & VAN MILL [1981a] that $\mathfrak{c}$-OK points can be used to show that ω^* is not first order homogeneous.

4.4. Some other points of interest

In this section we will continue our search for 'special' points in ω^*. In Theorem 4.3.3 we constructed two types of weak P-points in ω^*. It is natural to ask whether every point $x \in \omega^*$ which is a limit point of some countable subset of $\omega^* \backslash \{x\}$ is also a limit point of some countable discrete subspace of $\omega^* \backslash \{x\}$. Our first result is that the answer to this question is in the negative.

4.4.1. THEOREM. *There is a point $x \in \omega^*$ such that x is a limit point of some countable subset of $\omega^* \backslash \{x\}$, but not of any countable discrete subset of $\omega^* \backslash \{x\}$.*

PROOF. It is clear that the one point compactification S of $\omega \times [0, 1]$ is a continuous image of ω^*. Therefore, by Theorem 4.1.4, there is a $\mathfrak{c}$-OK set $T \subseteq \omega^*$ which can be mapped by an irreducible map, say f, onto S. For all $n < \omega$, put $T_n = f^{-1}(\{n\} \times [0, 1])$.

Claim. For each $n \geq 1$ there is a family $\mathcal{F}_n$ of closed subsets of $[0, 1]$ such that

(1) *$\mathscr{F}_n$ has the n-intersection property,*
(2) *if $D \subseteq [0, 1]$ is nowhere dense, then there is an $F \in \mathscr{F}_n$ with $F \cap D = \emptyset$.*
This was shown in Lemma 1.9.2.

For each $n < \omega$, let $\mathscr{G}_n = \{f^{-1}(\{n\} \times F) : F \in \mathscr{F}_{n+1}\}$. Observe that G_n has the $(n+1)$-intersection property and that for each nowhere dense set $D \subseteq T_n$ there is a $G \in \mathscr{G}_n$ with $G \cap D = \emptyset$. (This is obvious since $f(D)$ is nowhere dense in $\{n\} \times [0, 1]$.) Put

$$G = \bigcap\{\bar{A} : A \subseteq f^{-1}(\omega \times [0, 1]) \text{ and } A \cap T_n \in \mathscr{G}_n \text{ for all } n < \omega\}.$$

It is easily seen that $G \subseteq T \backslash \bigcup_{n<\omega} T_n$ and that G is nice w.r.t. the sequence $\{T_n : n < \omega\}$.

By similar arguments as in the proof of Theorem 4.3.3 we may conclude that T is extremally disconnected and by an appeal to Theorem 4.1.5 we can find a point $x \in G$ which is a $\mathfrak{c}$-OK point of $T \backslash \bigcup_{n<\omega} T_n$. It is easily seen that the point x is as required. The details of checking this out are left to the reader. □

Let us now pose a rather innocent question. Does every point in ω^* have character $\mathfrak{c}$ in ω^*? Under CH, this is obviously true. However, under $\neg$CH there can be points in ω^* which have character ω_1, KUNEN [1972]. The following now directly comes to mind: can all points in ω^* be of character less than $\mathfrak{c}$? The answer to this question is in the negative. In Lemma 3.3.4 we showed that there is an R-point in ω^*, and R-points obviously have character $\mathfrak{c}$ in ω^*. Let us give a somewhat easier proof than the one in Lemma 3.3.4, that points of character $\mathfrak{c}$ in ω^* exist.

4.4.2. THEOREM. *There is a point $x \in \omega^*$ such that $\chi(x, \omega^*) = \mathfrak{c}$.*

PROOF. Let $\{\langle A^0_\alpha, A^1_\alpha\rangle : \alpha < \mathfrak{c}\}$ be a family of pairs of disjoint subsets of ω^* such as in Lemma 3.1.2(b). Take any point x in the intersection

$$\bigcap\{\overline{A^0_\alpha} \cap \omega^* : \alpha < \mathfrak{c}\} \cap \bigcap\Big\{C : C \subseteq \omega^* \text{ is clopen and } \exists D \in [\mathfrak{c}]^\omega \text{ such that } \omega^* \backslash C \subseteq \bigcap_{\alpha \in D} \overline{A^0_\alpha}\Big\}.$$

By using similar arguments as in the proof of Theorem 2.1.1, it follows that x has character $\mathfrak{c}$ in ω^*. □

In the proof of Theorem 2.1.1, we constructed a Parovičenko space T such that $\pi(x, T) = \mathfrak{c}$ for all $x \in T$. In view of the above result, is therefore quite natural to ask whether the above result can be strengthened to the statement that there is a point $x \in \omega^*$ such that $\pi(x, \omega^*) = \mathfrak{c}$. This is impossible. BELL & KUNEN [1980] show it to be consistent with $\mathfrak{c} = \omega_{\omega_1}$ that each point $x \in \omega^*$ has π-character $\omega_1 (< \mathfrak{c})$. However, the following is true in ZFC.

4.4.3. THEOREM. *There is a point $x \in \omega^*$ with $\pi(x, \omega^*) \geq \mathrm{cf}(\mathfrak{c})$.*

PROOF. Let $\{A^\alpha_\beta : \alpha, \beta < \mathfrak{c}\}$ be a $\mathfrak{c}$ by $\mathfrak{c}$ independent matrix of clopen subsets of ω^* (Lemma 3.3.2) and let $\{C_\alpha : \alpha < \mathfrak{c}\}$ enumerate the family of all clopen subsets of ω^*. For all $\alpha, \kappa < \mathfrak{c}$ there is at most one $\beta < \mathfrak{c}$ such that $C_\kappa \subseteq A^\alpha_\beta$. It is therefore easy to pick for each $\alpha < \mathfrak{c}$ an element $f(\alpha) < \mathfrak{c}$ such that for all $\beta < \alpha$,

$$C_\beta \not\subseteq A^\alpha_{f(\alpha)} .$$

Take any point $x \in \bigcap_{\alpha < \mathfrak{c}} A^\alpha_{f(\alpha)}$. We claim that $\pi(x, \omega^*) \geq \mathrm{cf}(\mathfrak{c})$. If $\pi(x, \omega^*) < \mathrm{cf}(\mathfrak{c})$, then there is an $\alpha < \mathfrak{c}$ such that the family $\{C_\beta : \beta < \alpha\}$ constitutes a π-basis for x. But $A^\alpha_{f(\alpha)}$ is a neighborhood of x which does not contain any C_β for all $\beta < \alpha$. $\square$

Since ω^* is an F-space, each countable subspace of ω^* is C^*-embedded in ω^* (Theorem 1.5.2). If $2^{\omega_1} = \mathfrak{c}$, then $\beta\omega_1$ can be embedded in ω^* since $\beta\omega_1$ is extremally disconnected and has weight $\mathfrak{c}$ (Theorem 1.4.7). Therefore, under $2^{\omega_1} = \mathfrak{c}$, ω^* contains subspaces of cardinality ω_1 that are C^*-embedded. Having this in mind, it is quite natural to ask whether all subspaces of ω^* of cardinality ω_1 can be C^*-embedded. We will show that this is not the case. As usual, a P-space is a space in which all G_δ's are open.

4.4.4. THEOREM. *Let X be a P-space of weight at most $\mathfrak{c}$. Then X can be embedded in ω^*.*

PROOF. We may assume that $X \subseteq 2^\mathfrak{c}$ (here $2^\mathfrak{c}$ denotes the Cantor cube of weight $\mathfrak{c}$). Take $p \in 2^\mathfrak{c}$. The map $g_p : 2^\mathfrak{c} \to 2^\mathfrak{c}$ defined by $g_p(x) = x + p$ lifts to a map $eg_p : E(2^\mathfrak{c}) \to E(2^\mathfrak{c})$ ($E(2^\mathfrak{c})$ is the projective cover of $2^\mathfrak{c}$, see Section 0). The homeomorphism eg_p will be called h_p for short. Let π be the canonical irreducible surjection from $E(2^\mathfrak{c})$ onto $2^\mathfrak{c}$, i.e. π is defined by

$$\{\pi(u)\} = \bigcap \{\bar{U} : U \in u\} .$$

Take a point $u_0 \in \pi^{-1}(\mathbf{0})$, where $\mathbf{0}$ denotes the identity of $2^\mathfrak{c}$. If $p \in X$, let $u_p = h_p(u_0)$. Observe that

$$\pi(u_p) = \pi(h_p(u_0)) = g_p(\pi(u_0)) = g_p(\mathbf{0}) = p ,$$

whence $u_p \in \pi^{-1}(p)$.

Let U be a regular open subset of $2^\mathfrak{c}$. We can find a countable subset $D \subseteq \mathfrak{c}$ and a regular open subset $U' \subseteq 2^D$ such that if $\pi_D : 2^{\mathfrak{c}} \to 2^D$ denotes the projection, then $\pi_D^{-1}(U') = U$ (uses the fact that $2^{\mathfrak{c}}$ is ccc, JUHÁSZ [1980]).

Claim. If $p \restriction D = q \restriction D$ for $p, q \in X$, then $U \in u_p$ iff $U \in u_q$.

Indeed, simply observe that $U \in u_p$ iff $U + p \in u_0$ iff $U + p + q \in u_q$ iff $U \in u_q$.

Now, let $P = \{u_p : p \in X\}$. We claim that $\pi \restriction P : P \to X$ is a homeomorphism. For convenience, put $f = \pi \restriction P$. Then f is clearly one to one, onto and continuous. It therefore suffices to show that f is open. Let U be a regular open subspace of $2^{\mathfrak{c}}$ and let U' and D be as above. The set $\tilde{U} = \{u \in E(2^{\mathfrak{c}}) : U \in u\}$ is a basic open subset of $E(2^{\mathfrak{c}})$, so we only need to show that $f(\tilde{U})$ is open in X. Take $p \in f(\tilde{U})$ arbitrarily. Define $Z = \{q \in X : p \restriction D = q \restriction D\}$. By the claim, $Z \subseteq f(\tilde{U})$. Observe that $Z = \pi_D^{-1}(p \restriction D) \cap X$, whence Z is a G_δ in X. Since G_δ's in X are open, and since $p \in Z$, we conclude that $f(\tilde{U})$ is a neighborhood of p.

We conclude that X can be embedded in $E(2^{\mathfrak{c}})$. Since $E(2^{\mathfrak{c}})$ is separable, it has weight $\mathfrak{c}$, and therefore, by Theorem 1.4.7, it embeds in $\beta\omega$. Since $\beta\omega$ embeds in ω^*, we are done. □

4.4.5. COROLLARY. *There is a point $x \in \omega^*$ and a (relatively) discrete sequence $\{x_\alpha : \alpha < \omega_1\} \subseteq \omega^* \backslash \{x\}$, such that each neighborhood of x contains all but countably many of the x_α's.*

PROOF. There is clearly a P-space of cardinality ω_1 and containing precisely one nonisolated point. Now apply Theorem 4.4.4. □

4.4.6. REMARK. Observe that the proof of Theorem 4.4.4 actually shows that if X is a P-space, then βX can be embedded in the Čech–Stone compactification of some discrete space.

4.5. Partial orderings on $\beta\omega$, II

In Section 3 we defined three 'partial' orders on $\beta\omega$, namely $\preccurlyeq$, $\sqsubseteq$ and $\leqslant$. We observed that the following relations hold:

$$\begin{array}{ccc} p \sqsubseteq q & \searrow & \\ & & p \preccurlyeq q \\ p \leqslant q & \nearrow & \end{array}$$

(see Lemma 3.2.1 and the definition of $\leqslant$). In 3.3.8 we showed that there are points $p, q \in \omega^*$ with $p \preccurlyeq q$ but p and q are $\leqslant$-incomparable. We begin by establishing a similar result for the order $\sqsubseteq$.

4.5.1. THEOREM. *There is a finite to one function $\pi : \omega \to \omega$ such that for all $x \in \omega^*$ there is a $\mathfrak{c}$-OK point $y \in \omega^*$ with $\bar{\pi}(y) = x$.*

PROOF. We will be brief. Let $\pi : \omega \to \omega$ be the finite to one function of Lemma

4.1.2 and let $\mathscr{A} = \{A^\beta_{\alpha n} : \alpha < \mathfrak{c},\ 1 \leq n < \omega,\ \beta < \mathfrak{c}\}$ be the $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family of clopen subsets of $\beta\omega$ w.r.t. $\langle \omega^*, \bar{\pi}\rangle$ given by Lemma 4.1.2. Take $x \in \omega^*$ arbitrarily. Since $\mathscr{A}$ is independent w.r.t. $\langle \omega^*, \bar{\pi}\rangle$, $\mathscr{A}$ is also an independent linked family w.r.t. $\langle \bar{f}^{-1}(x), g\rangle$, where $g : \omega^* \to \{0\}$ maps ω^* onto 0. By using precisely the same technique as in the proof of Theorem 4.1.4, we can construct a $\mathfrak{c}$-OK point $y \in \omega^*$ which belongs to $\bar{f}^{-1}(x)$. □

4.5.2. COROLLARY. *ω^* contains $2^\mathfrak{c}$ $\mathfrak{c}$-OK points.*

PROOF. This is clear since $|\omega^*| = 2^\mathfrak{c}$, Lemma 31.2(c). □

4.5.3. COROLLARY. *There are points $p, q \in \omega^*$ with $p \leq q$, and consequently $p \preccurlyeq q$, but p and q are $\sqsubseteq$-incomparable.*

PROOF. By Theorem 4.3.3, there is a $\mathfrak{c}$-OK point $p \in \omega^*$. An application of Theorem 4.5.1 gives us a $\mathfrak{c}$-OK point $q \in \omega^*$ with $p \leq q$. Since p and q are weak P-points (Corollary 4.3.2), p and q are obviously $\sqsubseteq$-incomparable. □

Since $p \preccurlyeq q$ whenever $p \sqsubseteq q$, the question naturally arises whether $p \sqsubseteq q$ implies that $p \leq q$. We will show that this is not the case.

4.5.4. THEOREM. *There are points $p, q \in \omega^*$ with $p \sqsubseteq q$, and consequently $p \preccurlyeq q$, but p and a are $\leq$-incomparable.*

PROOF. Let $\{C_n : n < \omega\}$ be a sequence of pairwise disjoint nonempty clopen subsets of ω^*. For each $n < \omega$, let $\{E^n_m : m < \omega\}$ be a sequence of pairwise disjoint nonempty clopen subspaces of C_n. In addition, let $p \in \omega^*$ be an arbitrarily chosen R-point (Lemma 3.3.4). Let $G = \{f \in \omega^\omega : f \text{ is finite to one}\}$. For all $f \in G$ put $A_f = \bar{f}^{-1}(\{p\})$. By Lemma 3.3.6, each A_f is an R-set of ω^*. By Theorem 3.3.5 for all n, $m < \omega$ we can pick a point $x^n_m \in E^n_m$ such that

$$\{x^n_m : n, m < \omega\}^- \cap \bigcup_{f \in G} A_f = \emptyset\,.$$

For each $n < \omega$, put $Z_n = \{x^n_m : n, m < \omega\}^-$. Observe that $Z_n \approx \beta\omega$ since ω^* is an F-space and that consequently $|Z_n| = 2^\mathfrak{c}$, Lemma 3.1.2(c). For each $n < \omega$, let $\langle q^n_\alpha : \alpha < 2^\mathfrak{c}\rangle$ enumerate Z_n. We choose the enumeration to be most economical, i.e. each point of Z_n occurs precisely once in the sequence $\langle q^n_\alpha : \alpha < 2^\mathfrak{c}\rangle$. Observe that this implies that if $\alpha < \beta < 2^\mathfrak{c}$, then

(*) $\{q^n_\alpha : n < \omega\}^- \cap \{q^n_\beta : n < \omega\}^- = \emptyset$

(use that $\bigcup_{n<\omega} C_n$ is C^*-embedded in ω^*, Theorem 1.5.2). For each $\alpha < 2^\mathfrak{c}$, define

$g_\alpha : \omega \to \omega^*$ by

$$g_\alpha(n) = q_\alpha^n .$$

Then (*) implies that the set $\{\beta g_\alpha(p) : \alpha < 2^{\mathfrak{c}}\}$ has cardinality $2^{\mathfrak{c}}$. Since $|\{x \in \omega^* : x \leqslant p\}| \leqslant \mathfrak{c}$, we can therefore find an $\alpha < 2^{\mathfrak{c}}$ such that $q = \beta g_\alpha(p) \not\in \{x \in \omega^* : x \leqslant p\}$. Then q is as required. □

Notes for Section 4

The notion of a κ-OK point is due to KUNEN [1978]. Theorem 4.1.5 for the special case $X = \beta\omega$, $Z = \omega^*$, $A = \omega^*$ and $Y = \{0\}$ is due to KUNEN [1978]. Theorem 4.1.5 is implicit in VAN MILL [1981b] and was subsequently partly generalized in VAN MILL [1982]. The ccc nowhere separable remainder of ω described in Section 4.2 is due to BELL [1981]. Interestingly, this compactification is also an important step in the proof of the main result of VAN MILL [1982]. Lemma 4.3.1 is due to KUNEN [1978]. That the set A of Theorem 4.3.3 is nonempty is due to Kunen and that the set B of Theorem 4.3.3 is nonempty is due to VAN MILL [1981b]. Corollary 4.3.4 is due to KUNEN [1978] and for generalizations see VAN MILL [1979a], [1981b], [1982] and DOW [1982]. Theorem 4.4.1 is due to VAN MILL [1981b] and Theorem 4.4.2 to POSPÍŠIL [1939]. The proof of Theorem 4.4.2 presented here was taken from KUNEN [1974]. Theorem 4.4.3 is due to BELL & KUNEN [1981]. Corollary 4.4.5 is due independently to BALCAR, SIMON & VOJTÁS [1981], KUNEN and SHELAH. Theorem 4.4.4 is due to VAN DOUWEN (unpublished), but the proof presented here is due to DOW & VAN MILL [1982]. Our proof of Theorem 4.4.4 differs from van Douwen's proof, but both proofs have in common that they are based on the technique of Balcar, Simon, Vojtás, Kunen and Shelah. All other results in this chapter are new.

We have seen that there are many 'special' points in ω^*. In VAN MILL [1981b] it is shown that there are at least 16 definable types in ω^*. Call a space π-*homogeneous* provided that all nonempty open subspaces have the same π-weight. Define

$A_1 = \{x \in \omega^* : \exists$ countable discrete $D \subseteq \omega^* \backslash \{x\}$ with $x \in \bar{D}\}$.

$A_2 = \{x \in \omega^* : \exists$ countable, dense in itself, π-homogeneous subset $D \subseteq \omega^* \backslash \{x\}$ of countable π-weight such that $x \in \bar{D}\}$,

$A_3 = \{x \in \omega^* : \exists$ countable, dense in itself, π-homogeneous subset $D \subseteq \omega^* \backslash \{x\}$ of π-weight ω_1 such that $x \in \bar{D}\}$.

$A_4 = \{x \in \omega^* : \exists$ locally compact, ccc, nowhere separable $D \subseteq \omega^* \backslash \{x\}$ with $x \in \bar{D}\}$.

By using similar ideas as developed in this section it can be shown that for all subsets $F \subseteq \{1, 2, 3, 4\}$ the set

$$\bigcap_{i\in F} A_i \setminus \bigcup_{i\notin F} A_i$$

is nonempty. This gives 16 definable types of points in ω^*. For details, see VAN MILL [1981b].

5. Remarks

The reader will undoubtedly have noticed that we did not discuss several important facts about $\beta\omega$. For example, we did not say anything about normality in $\beta\omega$. There are several simple proofs that for any $p \in \omega^*$, the spaces $\beta\omega\setminus\{p\}$ and $\omega^*\setminus\{p\}$ are not normal under CH. However, for years there has not been made significant progress in this area of $\beta\omega$. We don't know that $\omega^*\setminus\{p\}$ is not normal for any $p \in \omega^*$ without the help of some set theoretic hypothesis. It is known however, that for some $x \in \omega^*$ the space $\omega^*\setminus\{x\}$ is not normal. The best result of this type is, as far as I know, due to BLASZCZYK & SZYMAŃSKI [1980a]. They showed that if $x \in \omega^*$ is a limit point of some countable discrete subset of ω^*, then $\omega^*\setminus\{x\}$ is not normal.

What else is there to say about $\beta\omega$? Consider the following question: is $\beta\omega$ homeomorphic to $(\beta\omega)^2$? The answer is of course: NO! It is easy to see that $(\beta\omega)^2$ is not extremally disconnected. Make the question a little bit less trivial: is $(\beta\omega)^2$ homeomorphic to $(\beta\omega)^3$? This question is easy to state, but the answer to the question is not simple at all. VAN DOUWEN [1982] showed that $(\beta\omega)^n \approx (\beta\omega)^m$ iff $n = m$, for all $n, m \geq 1$. The list of interesting results about $\beta\omega$ seems endless.

Let X be a space which is dense in itself. Define

$$n(X) = \min\{\kappa : X \text{ can be covered by } \kappa \text{ nowhere dense sets}\}.$$

This number is called the Novák number of X. It is clear that if $n(\omega^*) > \mathfrak{c}$, then ω^* contains P-points. Therefore, in Shelah's model in which there are no P-points, $n(\omega^*) \leq \mathfrak{c}$. Observe that $n(\omega^*) \geq \omega_2$. The Novák number $n(\omega^*)$ can be almost anything you want, for details, see BALCAR, PELANT & SIMON [1980] and also HECHLER [1978].

In this paper we have restricted our attention to the space $\beta\omega$. The reader should however realize that many of the results we obtained can be generalized to higher cardinals with proofs that are essentially identical. For example the Rudin–Keiler order can without any problem be defined for higher cardinals and the proof we gave for the existence of $\leq$-incomparable points in $\beta\omega$ can be copied to prove without extra difficulty that there are $\leqslant$-incomparable uniform ultrafilters on any infinite cardinal κ. However, there are also results that exclusively only work for ω. For example, the Rudin–Frolík order on ω^* cannot even be defined for higher cardinals.

Open problems

The following problems are unsolved as far as I know. It is recognized that a few of the problems listed below may be inadequately worded, be trivial or be known. Of many of the problems it is unknown who asked the problem. For that reason we do not credit anybody for posing a certain problem. The following mathematicians (with addresses listed in the AMS-MAA Combined Membership List) are sources of continuing information on many of the problems and their background: B. Balcar, W.W. Comfort, E.K. van Douwen, N. Hindman, K. Kunen, J. van Mill, M.E. Rudin and R.G. Woods.

1. Is ω^* homeomorphic to ω_1^*? (No if MA.)
2. Are there points $p, q \in \omega^*$ such that if $f\colon \omega \to \omega$ is any finite to one map, then $\beta f(p) \neq \beta f(q)$? (Yes if MA.)
3. Are $\omega^* \setminus \{p\}$ and $\beta\omega \setminus \{p\}$ nonnormal for any $p \in \omega^*$? (Yes if MA.)
4. Is there a model in which there are no P-points *and* no Q-points in ω^*?
5. Is there a model in which every point in ω^* is an R-point?
6. Is there a ccc closed P-set in ω^*? (Yes if CH.)
7. Let X be a compact space that can be mapped onto ω^*. Is X non-homogeneous? (Yes if X has weight at most $\mathfrak{c}$.)
8. Is the autohomeomorphism group of ω^* algebraically simple? (Yes is consistent.)
9. Is there an extremally disconnected, normal, locally compact space that is not paracompact? (Yes if MA + $\neg$CH or if there is a weakly compact cardinal.)
10. Is every first countable compactum a continuous image of ω^*?
11. Which spaces can be embedded in $\beta\omega$?
12. Is there a separable closed subspace of ω^* which is not a retract of $\beta\omega$? (Yes if CH.)
13. Let $(*)$ denote the statement that every Parovičenko space is coabsolute with ω^*. Is $(*)$ equivalent to CH? (It is known that $\mathfrak{c} < 2^{\omega_1}$ implies $\neg(*)$.)

14. Let X be the Stone space of the reduced measure algebra of $[0, 1]$. Is it consistent that X is not a continuous image of ω^*?

15. Let (**) denote the statement that every compact zero-dimensional F-space of weight $\mathfrak{c}$ can be embedded in ω^*. Is (**) equivalent to CH? (It is known that CH implies (**) but $\mathrm{MA} + \mathfrak{c} = \omega_2$ implies $\neg$(**).)

16. Is it consistent that there is a compact basically disconnected space of weight $\mathfrak{c}$ that cannot be embedded in $\beta\omega$? (Such an example cannot be the Čech-Stone compactification of a P-space.)

17. Is there a $p \in \omega^*$ such that $\omega^* \backslash \{p\}$ is not C^*-embedded in ω^*? (Yes if CH.)

18. Assume MA. Are there $P_\mathfrak{c}$-points p, $q \in \omega^*$ which are not of the same type, i.e. for which $h(p) \neq q$ for any autohomeomorphism h of ω^*?

19. Is it true that for all $p \in \omega^*$ there is a $q \in \omega^*$ such that p and q are $\leqslant$-incomparable?

20. Is every subspace of ω^* strongly zero-dimensional?

21. Is there a point $p \in \omega^*$ such that every compactification of $\omega \cup \{p\}$ contains a copy of $\beta\omega$?

22. Is there a point $p \in \omega^*$ for which there is a compactification of $\omega \cup \{p\}$ that does not contain a copy of $\beta\omega$? (Yes if MA.)

23. Let D be any nowhere dense subset of ω^*. Is D a $\mathfrak{c}$-set, i.e. is there a disjoint family $\mathscr{A}$ of $\mathfrak{c}$ open sets in ω^* such that $D \subseteq \bar{A}$ for all $A \in \mathscr{A}$? (Yes if $|D| = 1$.)

24. Is there a point $p \in \omega^*$ such that if $f: \omega \to \omega$ is any map, then either $\beta f(p) \in \omega$ or $\beta f(p)$ has character $\mathfrak{c}$ in $\beta\omega$? (Yes if MA.)

Remarks added in August 1982. Murray Bell has constructed a consistent example of a compact space X of weight $\mathfrak{c}$ which is first countable in all but one point and which in addition is not a continuous image of ω^*. This gives a partial answer to Question 10. Alan Dow showed that if $\mathrm{cf}(\mathfrak{c}) = \omega_1$, then all Parovičenko spaces are coabsolute. This solves Question 13 in the negative.

Remark added in May 1983. Andrzej Szymański has recently constructed, under MA, a separable closed subspace of ω^* which is not a retract of $\beta\omega$ (this concerns Question 12).

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CHAPTER 12

Countably Compact and Sequentially Compact Spaces

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
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1. Introduction

The notions of countable compactness and sequential compactness, which concern the fundamental properties of convergence and clustering of sequences, are among the oldest in topology. These concepts occur in the early work of M. Fréchet and F. Hausdorff in the settings of convergence and metric spaces. Countable compactness was defined as part of the general scheme of $[a, b]$-compactness by ALEXANDROFF and URYSOHN [1929] (see Stephenson's article in the Handbook for more details about this).

The purpose of this article is to present the basic theory, examples, and techniques of countable compactness and sequential compactness, and to discuss the current status of the questions asked in the titles of Sections 3, 5, and 6. Different techniques are needed to answer these questions; so there is not much overlap among these sections.

Recent results in set theory have had a strong influence in this area, and many results in this article are of recent vintage. The cardinal numbers p, s, and t, which concern subsets of the set ω of natural numbers, are important in this study. These three cardinals are discussed in van Douwen's article, and we recall the definitions here for convenience.

Let $[\omega]^\omega$ denote the set of all infinite subsets of ω. For $A, B \in [\omega]^\omega$ we write $A \subset^* B$ provided $|A - B| < \omega$ and we write $A < B$ provided $A \subset^* B$ and $|B - A| = \omega$. A family $\mathcal{F} \subset [\omega]^\omega$ has the *strong finite intersection property* (s.f.i.p.) provided every intersection of finitely many elements of $\mathcal{F}$ is an infinite set. Such a family $\mathcal{F}$ is called *maximal* provided for no $A \in [\omega]^\omega$ is $A \subset^* F$ for all $F \in \mathcal{F}$. Then the cardinal $\mathfrak{p}$ is defined to be the smallest cardinality of a maximal family with the s.f.i.p. A family $\mathcal{T}$ is called a decreasing *tower* provided there exists an ordinal γ and a function $f: \gamma \to \mathcal{T}$ onto $\mathcal{T}$ such that if $\alpha < \beta < \gamma$, then $f(\beta) \subset^* f(\alpha)$. Obviously every tower has the s.f.i.p. The cardinal $\mathfrak{t}$ is defined to be the smallest cardinality of a maximal tower.

A family $\mathcal{S} \subset [\omega]^\omega$ is called a *splitting family* provided for every $N \in [\omega]^\omega$ there exists $S \in \mathcal{S}$ such that $|N \cap S| = |N - S| = \omega$. The cardinal $\mathfrak{s}$ is defined to be the smallest cardinality of a splitting family.

Most of the theory of countable compactness can be given in the class of T_3-spaces (= regular and T_2), and much of it goes through for T_2-spaces (= Hausdorff spaces). Some parts of the theory require no separation axioms at all. We will assume that all spaces are T_3-spaces unless otherwise stated. In a few cases we will mention separation axioms in order to draw attention to their use for the interested reader. While on the subject of separation axioms, we mention that a countably compact, first countable T_2-space is always regular (the proof is routine) but need not be completely regular ($T_{7/2}$); see VAUGHAN [1979].

2. Basic properties and examples

For completeness, we recall the basic definitions.

2.1. A *sequence* in a set X is a function $f\colon \omega \to X$. Usually one can identify a sequence with its range and write $\langle f_n \rangle$ for the sequence, but when set-theoretic considerations become crucial, it is better to treat a sequence as a function. If A is an infinite subset of ω and f a sequence, we call the restricted function $f \mid A$ a *subsequence* of f. If there is a point x in X such that $f(n) = x$ for all n in A, we say that f is *constant* on A, and if $f(n) = x$ for all but finitely many n in A we say that f is *eventually constant* on A.

2.2. A sequence $f\colon \omega \to X$ in a topological space X is said to *converge* to a point x (resp. *cluster* at x) provided for every open set U in X with x in U we have $f^{-1}(U)$ is a final segment (resp. is infinite) in ω. We often write $\{n \in \omega : f(n) \in U\}$ instead of $f^{-1}(U)$.

We now come to the two basic properties of this chapter.

2.3. A space X is called *sequentially compact* provided that every sequence in X has a convergent subsequence.

2.4. A space X is called *countably compact* provided every sequence in X has a cluster point.

Since the range of a convergent sequence taken together with its limit point is a compact set, the next definition is an obvious generalization of sequential compactness.

2.5. A space X is called *totally countably compact* provided that every sequence f in X has a subsequence $f \mid A$ whose range is contained in a compact subset of X.

The next property is clearly stronger than that in 2.5.

2.6. A space X is called *ω-bounded* provided for every sequence f in X the range of f is contained in a compact subset of X.

The concepts in 2.5 and 2.6 are of use mainly in the study of countably compact products, and are given here because they are naturally related to 2.3 and 2.4. We can summarize the relations by the following diagram:

$$\begin{tikzcd}
\text{sequentially compact} \arrow[dr] & & \omega\text{-bounded} \arrow[dl] \\
& \text{totally countably compact} \arrow[d] & \\
& \text{countably compact} &
\end{tikzcd}$$

In many cases it is just as easy to work with infinite subsets of a space X as it is to work with sequences in X, and we give these versions too.

2.7. A point x is a *complete accumulation point* of an infinite set E in a space X provided for every open set U with x in U, $|U \cap E| = |E|$. This is different in general from the concept of an accumulation point. A point x is an *accumulation point* of a set E provided for every open set U with x in U, $U \cap (E - \{x\}) \neq \emptyset$. In a T_1-space, if x is an accumulation point of E, then for every neighborhood U of x, $U \cap E$ is infinite. Thus, in T_1-spaces, the concepts 'accumulation point' and 'complete accumulation point' are equivalent *for countably infinite sets E*. We can, therefore, use the shorter term 'accumulation point' in this article.

We restate Definitions 2.3–2.6 as follows. *Sequentially compact*: every countably infinite set in X has an infinite subset having exactly one (complete) accumulation point. *Countably compact*: every countably infinite set in X has a (complete) accumulation point. *ω-bounded*: every countably infinite set in X has compact closure (here we are using the assumption that all spaces are T_2; so that compact sets are closed). *Totally countably compact*: every countably infinite set in X contains an infinite subset with compact closure.

2.8. REMARK. The work 'totally' comes from the concept of a total filter base, or total net. A *filter base* $\mathscr{F}$ is a family of subsets of a set X which is not empty and has the property that every intersection of finitely many members of $\mathscr{F}$ is non-empty. A filter base $\mathscr{F}$ on a space X *converges* to a point x in X provided for every neighborhood U of x, there is an F in $\mathscr{F}$ with $F \subset U$. A filter base F *clusters* at a point x (or has x as an *adherent point*) provided $x \in \bigcap\{\bar{F} : F \in \mathscr{F}\}$. A filter base $\mathscr{F}$ is *total* if every filter base $\mathscr{G}$ which is finer than $\mathscr{F}$ (i.e., each F in $\mathscr{F}$ contains some G in $\mathscr{G}$) has an adherent point. Note that every convergent filter base is total, and in a compact space, every filter base is total, but not necessarily convergent. From the point of view of filter bases the four properties can be defined as follows: *Sequentially compact*: every countably filter base has a finer countable filter base which is convergent. *Countably compact*: every countable filter base has an adherent point. ω-bounded: Every filter base on a countable set has an adherent point (even if the filter base is not countable). *Totally countably compact*: every countable filter base has a finer countable filter base which is total.

Only countable compactness and ω-boundedness have convenient characterizations using open covers, so we do not use open covers very much in this article (one exception is in Section 6). Since the basic idea in the theory of convergence is the filter base of neighborhoods of a point, we believe that filter bases are the most natural way to develop the theory of countable compactness. On the other hand, the treatment using sequences is probably the one which is best known to most mathematicians, and further, the theory of r-limits uses sequences (see Section 4). For these reasons we will use sequences or infinite subsets as the main approach to countable compactness. We leave the proofs of the equivalences of the various "definitions" of countable compactness to the reader. In some cases, separation axioms are needed to prove the equivalences.

In Section 3, we consider Frolík's class $\mathfrak{C}$ and in Section 4 we consider the concept of r-compactness.

The remainder of this section is devoted to constructing examples (2.10–2.14) which show that the four properties (2.3–2.6) are distinct. Except for 2.11, these examples are subspaces of $\beta(\omega)$: the Stone–Čech compactification of the natural numbers ω, and 2.11 can be described as a quotient of a subspace of $\beta(\omega)$.

2.9. *Facts about* $\beta(\omega)$. (i) We consider $\beta(\omega)$ as the set of all ultrafilters on ω with the topology which is generated by taking as a base all sets of the form $\bar{V}=\{u\in\beta(\omega)\colon V\in u\}$ where $V\in[\omega]^{\omega}$. The integers are identified with the fixed (= principal) ultrafilters, and ω is dense in $\beta(\omega)$.

(ii) Every infinite set in $\beta(\omega)$ has $2^{\mathfrak{c}}$ cluster points, hence the only convergent sequences in $\beta(\omega)$ are those which are eventually constant; therefore if X is a subspace of $\beta(\omega)$ and X is sequentially compact, then X is finite.

(iii) For every $V\in[\omega]^{\omega}$, $\bar{V}$ is homeomorphic to $\beta(\omega)$, hence $|\bar{V}|=2^{\mathfrak{c}}$. Hence if X is a totally countably compact subspace of $\beta(\omega)$ and $\omega\subset X\subset\beta(\omega)$, then $|X|=2^{\mathfrak{c}}$ (because the infinite set ω has an infinite subset V with compact closure in X, i.e., $\bar{V}\subset X$).

(iv) If V, $W\in[\omega]^{\omega}$, then $V^*\cap W^*=\emptyset$ if and only if $V\cap W$ is finite, where $V^*=\bar{V}-\omega$.

For details about 2.9, see van Mill's article.

2.10. EXAMPLE. A compact space which is not sequentially compact. The space $\beta(\omega)$ is a compact (hence ω-bounded and totally countably compact) space which is not sequentially compact by 2.9(ii).

2.11. EXAMPLE. A sequentially compact space which is not ω-bounded.

Let k be any cardinal for which there exists a family $\{H_\alpha\colon\alpha<k\}$ of infinite subsets of ω such that (i) $\beta<\alpha$ implies $H_\beta\subset^* H_\alpha$, and (ii) for every $H\in[\omega]^{\omega}$ there exists $\alpha<k$ such that $H\not\subset^* H_\alpha$. By definition, the cardinal $\mathfrak{t}$ is the smallest such cardinal number. The underlying set for the desired space is the set $k\cup\omega$ where we consider k and ω to be disjoint. The topology is defined as follows: For each $\alpha<k$ we define a basic neighborhood of α for each $\beta<\alpha$ and each $F\in[\omega]^{<\omega}$ by

$$N(\alpha\colon\beta,F)=\{\gamma<k\colon\beta<\gamma\leqslant\alpha\}\cup((H_\beta-H_\alpha)-F)\,.$$

Points in ω are declared to be isolated. With this topology on $X=k\cup\omega$ it is clear that ω is a countable dense set of isolated points, and k is a closed subset of X whose subspace topology is the usual order topology on k. Since the cofinality of k is uncountable, the ordered space k is sequentially compact, but not compact.

Thus, the countable set ω does not have compact closure in X; so X is not ω-bounded. It is easy to prove that X is sequentially compact. Property (ii) of the tower $\{H_\alpha : \alpha < k\}$ is used to show that every sequence in ω has a subsequence which converges to a point of k. The details of this are given in van Douwen's article in this Handbook.

Note also that if $\mathfrak{t} = \omega_1$, then the above space can be taken to be first countable. It is an open problem to find in ZFC a first countable, countably compact, separable, T_3-space which is not compact.

2.12. EXAMPLE. A countably compact space which is not totally countably compact. For this example we can take any countably compact X such that $\omega \subset X \subset \beta(\omega)$ and $|X| \leq \mathfrak{c}$. It follows from 2.9(iii) that X is not totally countably compact and moreover, every compact subset of X is finite. Such a space X can be easily constructed as follows (we give this construction a number because it is of independent interest):

2.13. *There exists a countably compact space X with $\omega \subset X \subset \beta(\omega)$ and $|X| \leq \mathfrak{c}$.*

PROOF. We proceed by transfinite induction on ω_1. Let $X_0 = \omega$. Assume for all $\alpha < \gamma$ where $\gamma < \omega_1$ that we have constructed X_α such that:

(i) If $\alpha < \beta < \gamma$, then $X_\alpha \subset X_\beta$.

(ii) If $\alpha + 1 < \gamma$, then every sequence in X_α has a cluster point in $X_{\alpha+1}$.

(iii) $|X_\alpha| \leq \mathfrak{c}$.

(iv) If α is a limit ordinal, then $X_\alpha = \bigcup\{X_\beta : \beta < \alpha\}$.

To complete the construction, we show how to construct X_γ. Since (iv) determines the definition of X_γ in case γ is a limit ordinal, we need only show what to do in the case when γ is a successor ordinal, say $\gamma = \alpha + 1$. By (iii) there are $|X_\alpha|^\omega \leq \mathfrak{c}^\omega = \mathfrak{c}$ sequences in X_α. Since $\beta(\omega)$ is compact, we may select one cluster point for each sequence in X_α. Let $X_{\alpha+1}$ be the union of X_α and the set of cluster points so selected. Thus $|X_{\alpha+1}| \leq \mathfrak{c} + \mathfrak{c} = \mathfrak{c}$; and clearly (i)–(iii) hold for $X_{\alpha+1}$. This completes the induction. Define $X = \bigcup\{X_\alpha : \alpha < \omega_1\}$. Now $|X| \leq \mathfrak{c} \cdot \omega_1 = \mathfrak{c}$, and X is countably compact since every sequence in X has its range in some X_α and therefore has a cluster point in $X_{\alpha+1} \subset X$.

This is a fundamental construction in the theory of countably compact spaces and has been subjected to many variations. We will use it again in Section 3.

In accordance with 2.9(iv), we let $\omega^* = \beta(\omega) - \omega$.

2.14. The spaces K_x for x in ω^*. For each x in ω^*, define a subspace of $\beta(\omega)$ by $K_x = \beta(\omega) - \{x\}$. Thus $\omega \subset K_x \subset \beta(\omega)$. We note that each K_x is totally countably compact. This holds because any infinite set E in K_x has a cluster point y in $\beta(\omega)$ with $y \neq x$. Thus any closed neighborhood of y which does not contain x will be a subset of K_x and will be a compact set containing an infinite subset of E. Thus K_x is totally countably compact, but not ω-bounded since the countable set $\omega \subset K_x$ is dense in $\beta(\omega)$ and thus does not have compact closure in K_x.

The last result in this section is a generalization of 2.14. Since we do not use this result in this article, we have omitted the proof (see Theorem 3.7 in FROLÍK [1960a]).

2.15. THEOREM. *If X is a countably compact $T_{7/2}$-space, and has a compactification K such that $|K - X| < c$, then X is totally countably compact.*

3. When is a product of countably compact spaces countably compact?

We will be concerned with only one topology on product spaces: the product (or Tychonoff) topology. If $\{X_\alpha : \alpha \in I\}$ is a family of topological spaces (and I is an index set which we often take to be a cardinal number) the product topology on the Cartesian product set $X = \Pi\{X_\alpha : \alpha \in I\}$ is defined to be the topology on X having as a base all sets of the form $\bigcap\{\pi_\alpha^{-1}(U_\alpha) : \alpha \in F\}$, where $\pi_\alpha : X \to X_\alpha$ is the usual projection map, U_α is an open set in X_α, and F is a finite subset of I. This is, of course, the topology for which Tychonoff's product theorem holds (every product of compact spaces is compact).

It is easy to check that the product of a compact space Y and a countably compact space X is countably compact. (For any sequence $f(n) = \langle x_n, y_n\rangle$ in $X \times Y$, let x be a cluster point of the sequence $\langle x_n\rangle$ in X. If $y \in Y$ but $\langle x, y\rangle$ is not a cluster point of f, we can find open sets U_y and V_y such that $x \in U_y$, $y \in V_y$ and $\{n \in \omega : f(n) \in U_y \times V_y\}$ is finite. If no $y \in Y$ gives a cluster point $\langle x, y\rangle$, then there is a finite set $Z \subset Y$ such that $\{V_y : y \in Z\}$ covers Y. But this is impossible since $U = \bigcap\{U_y : y \in Z\}$ is a neighborhood of x such that $\{n \in \omega : x_n \in U\}$ is finite).

It is not true, however, that the product of every two countably compact spaces is countably compact. The first thing to do in this section is to construct two countably compact spaces X and Y such that their product $X \times Y$ is not countably compact. This can easily be done by a slight extension of the inductive construction in 2.13.

3.1. (J. NOVÁK [1953] and H. TERASAKA [1952]). *There exist countably compact spaces X and Y such that*

(i) *$\omega \subset X \subset \beta(\omega)$ and $\omega \subset Y \subset \beta(\omega)$,*

(ii) *$X \cap Y = \omega$.*

PROOF. First we consider the method to be used to show that $X \times Y$ is not countably compact.

LEMMA. *If X and Y are subsets of $\beta(\omega)$ such that*
(i) $\omega \subset X \subset \beta(\omega)$ *and* $\omega \subset Y \subset \beta(\omega)$, *and*
(ii) $\omega = X \cap Y$,
then $X \times Y$ is not countably compact.

PROOF. The infinite set $H = \{\langle n, n\rangle : n \in \omega\}$ (of isolated points) has no cluster point in $X \times Y$. This can be seen as follows: let $\Delta = \{\langle u, u\rangle : u \in \beta(\omega)\}$ denote the diagonal in $\beta(\omega) \times \beta(\omega)$. Since in T_2-spaces the diagonal is closed, we have $\bar{H} \subset \Delta$.

Thus, if $\langle p, q\rangle$ is a cluster point of H in $X \times Y \subset \beta(\omega) \times \beta(\omega)$, then $p = q$, but this is impossible by (ii).

The construction. Let $X_0 = Y_0 = \omega$, and assume that we have defined X_α and Y_α for all $\alpha < \gamma$ where $\gamma < \omega_1$ such that

(i) If $\alpha < \beta < \gamma$, then $X_\alpha \subset X_\beta$ and $Y_\alpha \subset Y_\beta$,

(ii) If $\alpha + 1 < \gamma$, then every sequence in X_α (resp. Y_α) has a cluster point in $X_{\alpha+1}$ (resp. $Y_{\alpha+1}$).

(iii) $X_\alpha \cap Y_\alpha = \omega$ and $|X_\alpha| \leq \mathfrak{c}$ and $|Y_\alpha| \leq \mathfrak{c}$

(iv) If α is a limit ordinal, then $X_\alpha = \bigcup\{X_\beta : \beta < \alpha\}$ and $Y_\alpha = \bigcup\{Y_\beta : \beta < \alpha\}$.

We need to construct X_γ and Y_γ for the case $\gamma = \alpha + 1$. We note as in 2.13 that there are at most $\mathfrak{c}$ sequences in X_α and in Y_α. Pick for each sequence in X_α a cluster point in $\beta(\omega) - Y_\alpha$ (this is possible because every sequence in X_α is either eventually constant or has $2^{\mathfrak{c}}$ cluster points by 2.9(ii)). Let $X_{\alpha+1}$ be the union of X_α and this set of $\leq\mathfrak{c}$ cluster points. Thus $|X_{\alpha+1}| \leq \mathfrak{c}$. Next, for each sequence in Y_α pick a cluster point in $\beta(\omega) - X_{\alpha+1}$. Let $Y_{\alpha+1}$ be the union of Y_α and these cluster points. Clearly $X_{\alpha+1} \cap Y_{\alpha+1} = \omega$, and this completes the induction. Put $X = \bigcup\{X_\alpha : \alpha < \omega_1\}$ and $Y = \bigcup\{Y_\alpha : \alpha < \omega_1\}$. This completes the proof of 3.1.

There are many interesting examples concerning products of two or more countably compact spaces. We list some of them here without proofs. The first one can be proved by using a slight extension of the construction in 3.1.

3.2. *Further examples.* (i) There exist spaces X and Y such that every sequence in X and in Y has $2^{\mathfrak{c}}$ cluster points, and $X \times Y$ is not countably compact (see NOVÁK [1953]).

(ii) There exist countably compact spaces X and Y such that $X \times Y$ is not countably compact, but is countably paracompact (see PARSONS [1977]) or is normal (assuming MA; see VAN DOUWEN [1980]) or is pseudocompact (see GINSBURG and SAKS [1975]) or an M-space (see STEINER [1971]).

(iii) Assuming MA there exist two countably compact topological groups G and H such that $G \times H$ is not countably compact (see VAN DOUWEN [1980]).

(iv) There exists a countably compact space X such that for all $n < \omega$ X^n is countably compact (the product of n copies of X), but X^ω is not countably compact (see FROLÍK [1967]).

Recall that the cardinal $\mathfrak{t}$ is the smallest cardinal for which there is a maximal decreasing tower of infinite subsets of ω in the mod finite order.

3.3. THEOREM. (A) *If X is totally countably compact, and Y is countably compact, then $X \times Y$ is countably compact.*

(B) *If $\{X_\alpha : \alpha < k\}$ is a family of totally countably compact spaces and $k < \mathfrak{t}$, then the product $X = \Pi\{X_\alpha : \alpha < k\}$ is totally countably compact. In particular, the product of countably many totally countably compact spaces is totally countably compact.*

(C) *If $\{X_\alpha : \alpha < \mathfrak{t}\}$ is a family of totally countably compact spaces, then $X = \Pi\{X_\alpha : \alpha < \mathfrak{t}\}$ is countably compact. In particular, the product of no more than ω_1 totally countably compact spaces is countably compact.*

PROOF. (A) For an infinite set $H \subset X \times Y$, project H into X. Since X is totally countably compact, there is a compact set $K \subset X$ such that infinitely many points of H project into K (we have to say it this way since H might project onto a finite subset of X, and the definition does not apply to finite sets). Then $K \times Y$ is countably compact (as is any product of a compact space and a countably compact space) and contains infinitely many points of H. Thus H has a cluster point.

(B) and (C). These have such similar proofs they can be done together. Let H be an infinite subset of the product $X = \Pi\{X_\alpha : \alpha < k\}$. Identify H with ω. Assume we have defined infinite sets $A_\alpha \subset H$ for all $\alpha < \gamma$ where $\gamma < k$ such that

(i) $\pi_\alpha(A_\alpha)$ has compact closure in X_α.

(ii) if $\alpha < \beta < \gamma$, then $A_\beta \subset^* A_\alpha$.

Since $\gamma < k$ and $k \leqslant \mathfrak{t}$ (depending on (B) or (C)) there exists an infinite set A such that $A \subset^* A_\alpha$ for all $\alpha < \gamma$. Since X is totally countably compact, there is an infinite subset $A_\gamma \subset A$ such that $\pi_\gamma(A_\gamma)$ has compact closure in X_γ (this follows as in (A)). This completes the induction. In case (B), since $k < \mathfrak{t}$, we may pick an infinite set $A \subset H$ such that $A \subset^* A_\alpha$ for all $\alpha < k$. We show that the closure of A in X is compact. Let Y_α be the closure of $\pi_\alpha(A)$ in X_α. Since $A \subset^* A_\alpha$ we have that $\pi_\alpha(A) \subset \pi_\alpha(A_\alpha) \cup F_\alpha$ where F_α is a finite subset of X_α. Thus $\pi_\alpha(A)$ has compact closure in X_α for all $\alpha < k$, i.e., Y_α is compact for all $\alpha < k$. By the Tychonoff product theorem $Y = \Pi\{Y_\alpha : \alpha < k\}$ is compact, and clearly $A \subset Y$. Thus the infinite set H has an infinite subset A with compact closure, and this proves (B). To prove (C) requires a little more work because it is possible that the tower $\{A_\alpha : \alpha < \mathfrak{t}\}$ is maximal, but we only have to show that H has a cluster point. As before, let Y_α denote the closure of $\pi_\alpha(A_\alpha)$ in X_α and $Y = \Pi\{Y_\alpha : \alpha < \mathfrak{t}\}$. If no point of Y is an accumulation point of H, we can get an open set U in the product space X such that $Y \subset U$ and $U \cap H = \emptyset$. Since Y is compact, there is a finite set $\{W_i : i < n\}$ of basic open sets such that $Y \subset \bigcup\{W_i : i < n\} \subset U$. For each $i < n$, let $F_i = \{\alpha < \mathfrak{t} : \pi_\alpha(W_i) \neq X_\alpha\}$, and put $F = \bigcup\{F_i : i < n\}$. By definition of the basic open sets, as defined at the beginning of this section, each F_i is finite, and thus F is finite. Pick any $x \in \bigcap\{A_\alpha : \alpha \in F\}$. Hence, $\pi_\alpha(x) \in \pi_\alpha(A_\alpha) \subset Y_\alpha$ for all

$\alpha \in F$, which implies that there exists $y \in Y$ such that $\pi_\alpha(y) = \pi_\alpha(x)$ for all $\alpha \in F$. There exists $i < n$ such that $y \in W_i$; so by definition of F, $x \in W_i$, and thus $x \in U \cap H$, a contradiction. This shows that H has an accumulation point in Y, and completes the proof of 3.3.

The theorem above was motivated by the following results.

3.4. PROPOSITION. (A) *The product of a sequentially compact space and a countably compact space is countably compact.*

(B) *The product of countably many sequentially compact spaces is sequentially compact.*

(C) (SCARBOROUGH–STONE [1966]). *The product of no more than ω_1 sequentially compact spaces is countably compact.*

PROOF. Parts (A) and (C) are direct corollaries of 3.3, but (B) requires a separate proof (a straightforward 'Cantor diagonal argument'). Further, (B) can be improved to show that if $k < \mathfrak{t}$, then the product of k sequentially compact spaces is sequentially compact. The proof of this is analogous to the proof of Theorem 3.3(B).

For a cardinal number k (finite or infinite) let X^k denote the product space obtained by taking the product of k copies of X. The next result gives another interesting property of totally countably compact spaces.

3.5. PROPOSITION. *Assume* $\mathfrak{t} = \mathfrak{c}$. *If X is totally countably compact, and $|X| \leqslant \mathfrak{c}$, then X^k is countably compact for every cardinal k.*

PROOF. Since $|X| \leqslant \mathfrak{c}$ there are at most $\mathfrak{c}$ sequences in X. We list them as $\{f_\alpha : \alpha < \mathfrak{c}\}$. Define a sequence $g : \omega \to X^\mathfrak{c}$ by the rule that the αth coordinate of $g(n)$ is $f_\alpha(n)$, i.e., let π_α denote the usual projection map from X^k onto X and define g by

$$\pi_\alpha(g(n)) = f_\alpha(n) \quad \text{for all } \alpha < \mathfrak{c}, n < \omega .$$

By Theorem 3.3(C), the product $X^\mathfrak{t}$ is countably compact; so by our assumption $X^\mathfrak{c}$ is countably compact. Let x be a cluster point of g in $X^\mathfrak{c}$. We sketch the remainder of the proof. We have to show that X^k is countably compact for $k > \mathfrak{c}$; so we consider a sequence $h : \omega \to X^k$. For each $\alpha < k$, $\pi_\alpha \circ h$ is a sequence in X, hence appears in our list. Define a point y in X^k as follows: If $\pi_\alpha \circ h = f_\beta$, then put the αth coordinate of y equal to the βth coordinate of x. The fact that x is a cluster point of g implies that y is a cluster point of h. Thus X^k is countably compact.

It is known that Proposition 3.5 is not provable in ZFC (see Section 7). In Section 4 we get further insight into why Theorem 3.5 is true (it is because the hypotheses imply the space X is r-compact for some r in ω^*). The next result gives the connection between countably compact k-spaces and totally countably compact spaces. Since we have to work with two topologies on the same set, we now give the notation to do this.

3.6. DEFINITION. Let (X, T) be a topological space, where X is a set and T is the topology on X, and let $\mathscr{C}(X, T)$ denote the family of all compact subsets of (X, T). Let $\mathscr{K}(X, T)$ denote the family of all subsets F of X which have the property that $F \cap K$ is open in K (with respect to the subspace topology on K) for all K in $\mathscr{C}(X, T)$. Clearly $T \subset \mathscr{K}(X, T)$, and $\mathscr{K}(X, T)$ is a topology on the set X. A space (X, T) is called a *k-space* (or is said to be *compactly generated*) provided $T = \mathscr{K}(X, T)$.

3.7. THEOREM. *A space (X, T) is totally countably compact if and only if the space $(X, \mathscr{K}(X, T))$ is countably compact. In particular, every countably compact k-space is totally countably compact.*

PROOF. First we assume that $(X, \mathscr{K}(X, T))$ is countably compact and show that (X, T) is totally countably compact. Let H be an infinite subset of X, and let x be an accumulation point of H in $(X, \mathscr{K}(X, T))$. Then $H - \{x\}$ is not closed in $(X, \mathscr{K}(X, T))$; so there exists a compact set K of (X, T) such that $K \cap (H - \{x\})$ is not closed in K. Thus $K \cap (H - \{x\})$ is an infinite subset of H contained in a compact subset of (X, T). Thus (X, T) is totally countably compact. Next, we assume that (X, T) is totally countably compact, and show that $(X, \mathscr{K}(X, T))$ is countably compact. Let H be an infinite subset of X. There is a compact set K such that $K \cap H$ is infinite. Thus $K \cap H$ has an accumulation point in K, which is obviously an accumulation point of H in $(X, \mathscr{K}(X, T))$ by definition of that topology. This completes the proof.

One often sees in books the result that if X is countably compact, and Y is a countably compact k-space, then $X \times Y$ is countably compact. This is a corollary of Theorem 3.3(A) and Theorem 3.7. In addition we get the following result which may be new: The product of no more than ω_1 countably compact k-spaces is countably compact. In connection with this, recall that the product of two k-spaces need not be a k-space.

We have seen that some countably compact spaces X have the property that $X \times Y$ is countably compact for every countably compact space Y. It is natural to consider the class of all such spaces. This class, however, seems to be a bit awkward to handle (at least as far as we know now). The only known characterizations of this class involve the manner in which the spaces are situated in their compactifications. For this reason this class is usually defined only for $T_{7/2}$-spaces.

3.9. DEFINITION. Let $\mathfrak{C}$ denote the class of all $T_{7/2}$-spaces X with the property that for every $T_{7/2}$-space Y, $X \times Y$ is countably compact.

Since every space is homeomorphic to the product of itself and a single point, it follows that all the spaces in $\mathfrak{C}$ are countably compact. We also know that every totally countably compact space is in class $\mathfrak{C}$ (by 3.3(A)). The only use we have for the following characterization of class $\mathfrak{C}$ in this paper is in proving Corollary 3.12 which is used to verify Example 3.13. The following easy result is used in the characterization.

3.10. LEMMA. *Let $F: X \to Y$ be a closed, continuous function into a countably compact space Y such that $f^{-1}(y)$ is countably compact for all y in Y. Then X is countably compact.*

PROOF. Let H be an infinite subset of X. If $f(H)$ is finite, then there is a point y in Y such that $H \cap f^{-1}(y)$ is infinite. Since $f^{-1}(y)$ is countably compact, we see that H has an accumulation point. If $f(H)$ is an infinite subset of Y, then it has an accumulation point y in Y. Since f is a closed map, it is easy to see that H must have an accumulation point in $f^{-1}(y)$.

3.11. THEOREM. *The following are equivalent for a $T_{7/2}$-space X.*

(i) *X belongs to class $\mathfrak{C}$.*

(ii) *For every infinite discrete subset N of X, there is a compactification K of X such that for every subset $S \subset K - X$, the subspace $N \cup S$ of K is not countably compact.*

(iii) *For every infinite discrete subset N of X, and for every subset $S \subset \beta(X) - X$, the subspace $N \cup S$ of $\beta(X)$ is not countably compact.*

(iv) *For every infinite discrete subset N of X, for every compactification K of X, and for every subset $S \subset K - X$, the subspace $N \cup S$ of K is not countably compact.*

PROOF. (i)$\to$(ii). We prove this by contradiction. Suppose (ii) fails. Then there is an infinite discrete subset N of X such that for every compactification of X (so pick one and call it K) there is a subset $S \subset K - X$ such that the subspace $N \cup S$ of K is countably compact. But the set $H = \{\langle n, n\rangle : n \in N\}$ is a closed discrete subset of $X \times (N \cup S) \subset K \times K$ (this can be seen in a manner similar to the Lemma in 3.1) which contradicts (i).

(ii)$\to$(iii). Let N be an infinite discrete subset of X. By (ii) there is a compactification K of X which satisfies the condition of (ii). Let S be a subset of $\beta(X) - X$. We must show that the subspace $N \cup S$ of $\beta(X)$ is not countably compact. Let $f: \beta(X) \to K$ be a continuous function onto K which leaves X pointwise fixed (i.e., $f(x) = x$ for all x in X) and $f(\beta(X) - X) = K - X$ (for details about this step see Chapter 3 in ENGELKING [1977]). Clearly $f(N \cup S) = N \cup f(S)$

and $f(S)$ is a subset of $K-X$. Thus if $N \cup S$ is countably compact; so is $N \cup f(S)$, but that is impossible by (ii).

(iii)$\rightarrow$(iv). Let N be an infinite discrete subset of X, and K any compactification of X, and let S be a subset of $K-X$. We must show that the subspace $N \cup S$ of K is not countably compact. Again, let $f: \beta(X) \rightarrow K$ be a continuous map onto K which leaves X pointwise fixed and such that $f(\beta(X)-X)=K-X$. Clearly $f^{-1}(N \cup S)=N \cup f^{-1}(S)$ and $f^{-1}(S)$ is a subset of $\beta(X)-X$. By (iii) we know that $N \cup f^{-1}(S)$ is not countably compact. Since $g=f \mid f^{-1}(N \cup S)$ is a closed map from $N \cup f^{-1}(S)$ onto $N \cup S$ such that $g^{-1}(y)$ is compact for all y in $N \cup S$ $(g^{-1}(y)=f^{-1}(y))$, it follows from Lemma 3.10 that $N \cup S$ is not countably compact.

(iv)$\rightarrow$(i). We prove this by contradiction. Suppose that X is a T_{712}-space which satisfies (iv) and X does not belong to class $\mathfrak{C}$. Then there is a countably compact space Y (no separation axioms are needed for Y) such that $X \times Y$ is not countably compact, and therefore an infinite subset H of $X \times Y$ such that H has no complete accumulation points in $X \times Y$. Thus if C is a countably compact subset of $X \times Y$, then $H \cap C$ is finite. In particular, for finite sets $F \subset X$ and $G \subset Y$, $H \cap ((F \times Y) \cup (X \times G))$ is finite. From this we can construct by induction a sequence $(x_i, y_i) \in H$ such that the function $h(y_i)=x_i$ is one-to-one. Since every infinite set in a T_2-space contains an infinite discrete subset, we may assume that $N=\{x_i: i<\omega\}$ is a discrete subset of X. Let K be any compactification of X. In order to get a contradiction (to (iv)) we will define a set $S \subset K-X$ such that $N \cup S$ is countably compact. Let A denote the set of all accumulation points of $M=\{y_i: i<\omega\}$ in Y. For each $a \in A$, let $\mathcal{N}_a$ denote the neighborhood filter of a in Y. Then $\mathcal{F}_a=\{h(U \cap M): U \in \mathcal{N}_a\}$ is a filter base on $N \subset K$, hence

$$\alpha(a)=\overline{\{h(U \cap m)^K: U \in \mathcal{N}_a\}}$$

is a compact, non-empty subset of K. Put $S=\bigcup\{\alpha(a): a \in A\}$. First note that $S \subset K-X$, for otherwise there exist $a \in A$ and $x \in \alpha(a)$ which implies that (x, a) is an accumulation point if H in $X \times Y$. Next we show that every infinite subset $E \subset N$ has an accumulation point in S. Since $h^{-1}(E)$ is an infinite subset of M, there is an $a \in A$ such that a is an accumulation point of $h^{-1}(E)$. Thus the filter base $\{h(U \cap h^{-1}(E)): U \in \mathcal{N}_a\}$ is finer than $\mathcal{F}_a$, and therefore has an adherent point x in $\alpha(a)$. It follows that x is an accumulation point of E and $x \in S$. The last step is to show that every infinite subset of S has an accumulation point in S. Let E be an infinite subset of S. If $E \cap \alpha(a)$ is infinite for some $a \in A$, then we are done since $\alpha(a)$ is compact. Otherwise we can construct a sequence of distinct points $z_n \in E$ and $a_n \in A$ $(n<\omega)$ such that $z_n \in E \cap \alpha(a_n)$. Let y be an accumulation point of $\{a_n: n<\omega\}$ in Y; so $y \in A$. It suffices to show that $\bar{E}^K \cap \alpha(y) \neq \emptyset$. If this fails to hold, then there exists an open set W, such that $\alpha(y) \subset W$, and $W \cap E=\emptyset$. By the compactness of K, the filter base $\mathcal{F}_y$ "converges to its set of adherent points $\alpha(y)$," that is, there exists $F \in \mathcal{F}_y$ such that $F \subset W$. By definition of $\mathcal{F}_y$ there is $U \in \mathcal{N}_y$ such that $\overline{h(U \cap M)}^K \subset W$. Since $U \in \mathcal{N}_y$, there is an $n \in \omega$ such that $a_n \in U$ (i.e., $U \in \mathcal{N}_{a_n}$). Thus $z_n \in \overline{h(U \cap M)}^K \subset W$ which contradicts that $W \cap E = \emptyset$. This completes the proof.

Theorem 3.11 is not always easy to apply to a given space X, but works nicely in the following situation. Let RF denote the set of all points x in ω^* such that x is not an accumulation point of any countably discrete subset of ω^*. The 'RF' comes from the fact that these points are minimal in the Rudin–Frolík order. A point is called a weak P-point in ω^* if it is not an accumulation point of any countable subset of ω^*. Thus every weak P-point is in RF. It is known that there are $2^{\mathfrak{c}}$ weak P-points in ω^*, and that they form a dense subset of ω^* (see van Mill's article for more on these concepts).

3.12. COROLLARY. *If $\omega \subset X \subset \beta(\omega)$ and $\beta(\omega) - X \subset \mathrm{RF}$, then X belongs to class $\mathfrak{C}$.*

PROOF. First note that such spaces X are countably compact by definition of RF. If X does not belong to class $\mathfrak{C}$, then since $\beta(\omega)$ is a compactification of X, there exists (by Theorem 3.11(iii)) an infinite discrete subset N of X and a subset S of $\beta(\omega) - X$ such that the subspace $N \cup S$ of $\beta(\omega)$ is countably compact. Now $N - \omega$ is finite, for otherwise it would have an accumulation point in S, which is impossible since $S \subset \mathrm{RF}$. Thus $N \cap \omega$ is infinite, and can be partitioned into infinitely many mutually disjoint infinite sets $\{N_i \colon i \in \omega\}$. Since $N \cup S$ is countably compact, each N_i has an accumulation point in S, and these accumulation points are distinct by 2.9(iv). Thus $S \cap \omega^*$ is infinite, and hence contains an infinite, discrete subset H. Thus H must have an accumulation point in S, but as before this contradicts that $S \subset \mathrm{RF}$.

3.13. EXAMPLE. A space X in class $\mathfrak{C}$ which is not totally countably compact.

PROOF. Take $X = \beta(\omega) - \mathrm{RF}$. It follows from Corollary 3.12 that X belongs to class $\mathfrak{C}$. To see that X is not totally countably compact, one shows that the infinite subset ω of X has no infinite subset with compact closure in X (this uses the denseness of RF in ω^*, and is similar to the analogous step in 2.14).

We conclude this section with two simple, but useful examples of FROLÍK [1960a] concerning products of the spaces K_x of Example 2.14.

3.14. EXAMPLE. The product of $2^{\mathfrak{c}}$ totally countably compact spaces need not be totally countably compact. For this example we take $K = \Pi\{K_x \colon x \in \omega^*\}$. For each x let $X_x = \beta(\omega)$, and put $X = \Pi\{X_x \colon x \in \omega^*\}$ and $\Delta = \{f \in X \colon f \text{ is constant}\}$. Let $D = \{f \in X \colon f \text{ is constant and the constant is in } \omega\}$. Since $\omega \subset K_x$ for all x, we see that $D = \Delta \cap K \subset X$. Thus D is a countable subset of K all of whose accumulation points are in Δ, hence not in K. Thus K is not countably compact.

3.15. EXAMPLE. The product of $\mathfrak{c}$ totally countably compact spaces need not be totally countably compact. Let X be the space of example 2.12, i.e., X is countably compact, $|X| = \mathfrak{c}$ and $\omega \subset X \subset \beta(\omega)$. Let $Y = \Pi\{K_x : x \in X\}$. Then Y is the product of $\mathfrak{c}$ totally countably compact spaces but $Y \times X$ is not countably compact (the proof is similar to 3.14); so Y is not totally countably compact nor even in class $\mathfrak{C}$.

If we consider these two examples and assume the first two steps of the generalized continuum hypothesis, (i.e., $\mathfrak{c} = 2^{\omega} = \omega_1$ and $2^{\omega_1} = \omega_2$), we see that it is consistent that the product of ω_1 totally countably compact spaces need not be totally countably compact (but the product will be countably compact by Theorem 3.3) and the product of ω_2 totally countably compact spaces need not be countably compact. For another use of these examples see the discussion following Theorem 4.10.

4. Products and r-limits

If $f: \omega \to X$ is a sequence in a space X, and x is a cluster point of f then the neighborhood filter of x in X can be reflected back to a filter on ω in a natural way: Let $\mathcal{N}$ denote the filter of all neighborhoods of x. For each U in $\mathcal{N}$, $f^{-1}(U) = \{n \in \omega : f(n) \in U\}$ is an infinite subset of ω, and the family $\mathcal{F} = \{f^{-1}(U) : U \in \mathcal{N}\}$ is a filter on ω, hence is contained in at least one ultrafilter and possibly in more than one. Indeed, it is easy to get examples where the two extreme cases obtain: (i) an $\mathcal{F}$ such that $\mathcal{F}$ is contained in every ultrafilter on ω, and (ii) an $\mathcal{F}$ such that $\mathcal{F}$ is contained in exactly one ultrafilter. The first example is trivial: if f converges to x, then each $f^{-1}(U)$ is a final subset of ω (i.e., $\omega - f^{-1}(U)$ is finite) and therefore $\mathcal{F} \subset r$ for every $r \in \omega^*$. For the other example, let $f: \omega \to \beta(\omega)$ be the identity sequence defined by $f(n) = n$ for all $n \in \omega$; then every $x \in \omega^*$ is a cluster point of f. Pick one $x \in \omega^*$. Since basic neighborhoods of x in $\beta(\omega)$ are of the form $U = \{y \in \beta(\omega) : V \in y\}$ where $V \in x$, it follows that $\{n \in \omega : f(n) \in U\} = V$. Thus $\mathcal{F} = x$; so in this case $\mathcal{F}$ is contained in exactly one ultrafilter. The ultrafilters which contain $\mathcal{F}$ describe in some sense the manner in which f clusters at the point x. This is the idea of r-limits.

4.1. DEFINITION. Let $r \in \omega^*$ and let $f: \omega \to X$ be a sequence in a space X. We say that the point $x \in X$ is the r-limit of f (denoted $x = r\text{-}\lim f$) provided for every neighborhood U of x, $\{n \in \omega : f(n) \in U\} \in r$.

Since an ultrafilter cannot contain a pair of disjoint sets, a sequence (in T_2-spaces) can have at most one r-limit point. Thus, the r-limit of a sequence is unique if it exists. Also note that an r-limit point of a sequence is a cluster point of the sequence since the ultrafilters in ω^* contain only infinite sets. We summarize these remarks in the following result.

4.2. A point x is a cluster point of a sequence f if and only if there exists $r \in \omega^*$ such that $x = r\text{-}\lim f$.

The notion of r-limits plays an intrinsic role in the theory of products of countably compact spaces. This is obvious from 4.4, which we prove using the following easy result.

4.3. LEMMA. *Continuity preserves r-limits, i.e., if $f: \omega \to X$ is a sequence in X and $x = r\text{-}\lim f$, and $g: X \to Y$ is continuous, then $g(x) = r\text{-}\lim g \circ f$.*

4.4. THEOREM. *Let $\{X_\alpha : \alpha \in I\}$ be a family of spaces. Then the product space $X = \Pi\{X_\alpha : \alpha \in I\}$ is countably compact if and only if for every sequence $f: \omega \to X$, there exists $r \in \omega^*$ such that $\pi_\alpha \circ f$ has an r-limit in X_α for all $\alpha \in I$.*

PROOF. If X is countably compact, then f has a cluster point $x \in X$, so by 4.2 there is an ultrafilter $r \in \omega^*$ such that $x = r\text{-}\lim f$. Since the projection maps are continuous, we have by 4.3 that $\pi_\alpha \circ f(x) = r\text{-}\lim \pi_\alpha \circ f$ for all $\alpha \in I$. To prove the converse, let x_α be the r-limit of $\pi_\alpha \circ f$ for all $\alpha \in I$. We show that the point $x \in X$ whose αth coordinate is x_α is the r-limit of f in X. Let $B = \bigcap\{\pi_\alpha^{-1}(U_\alpha): \alpha \in F\}$ be a basic neighborhood of x in X (so F is a finite subset of I and U_α is a neighborhood of x_α in X_α for all $\alpha \in F$). We have to show that $\{n \in \omega : f(n) \in B\} \in r$. For each $\alpha \in F$ we have that $\{n \in \omega : \pi_\alpha \circ f(n) \in U_\alpha\} \in r$ and thus the intersection of these finitely many sets is in r. But this completes the proof since

$$\{n \in \omega : f(n) \in B\} = \bigcap_{\alpha \in F} \{n \in \omega : \pi_\alpha \circ f(n) \in U_\alpha\}.$$

In a countably compact space, every sequence f has a r-limit for some $r \in \omega^*$, but different sequences may require different r's. This suggests the next definition.

4.5. DEFINITION. Let $r \in \omega^*$. A space X is called *r-compact* provided every sequence in X has an r-limit point in X.

4.6. LEMMA. *Every r-compact space is countably compact.*

The converse of this lemma is not true. In fact, Example 3.1 gives us a simple way to construct an example of a countably compact space which is not r-compact for any $r \in \omega^*$. To see this, we need the next important result.

4.7. THEOREM. *Let $r \in \omega^*$. Every product of r-compact spaces is r-compact.*

PROOF. Let $\{X_\alpha : \alpha \in I\}$ be a family of r-compact spaces, and let $X = \Pi\{X_\alpha : \alpha \in I\}$. Suppose that $f: \omega \to X$ is a sequence in X. For each $\alpha \in I$, $\pi_\alpha \circ f: \omega \to X_\alpha$ is a sequence in X_α, hence has an r-limit in X_α by hypothesis. It follows from 4.4 that f has an r-limit in X.

4.8. EXAMPLE. There exists a countably compact space Z which is not r-compact for any $r \in \omega^*$.

PROOF. By Lemma 4.6 and Theorem 4.7, it suffices to find a countably compact space Z such that $Z \times Z$ is not countably compact. Let X and Y be the spaces in Example 3.1. Thus X and Y are countably compact and $X \times Y$ is not countably compact. Take Z to be the disjoint union of X and Y (so X and Y are clopen subsets of Z). Then Z is countably compact, and $Z \times Z$ is not countably compact since $Z \times Z$ contains $X \times Y$ as a closed subspace.

For a stronger example, see 4.13.

What kind of space has the property that it is r-compact for all $r \in \omega^*$? In T_3-spaces the answer is given by the next result.

4.9. THEOREM. *A T_3-space X is ω-bounded if and only if it is r-compact for all $r \in \omega^*$.*

PROOF. If X is ω-bounded and $f\colon \omega \to X$ is a sequence in X, then there is a compact set $K \subset X$ such that $f\colon \omega \to K$. Thus, f gives a correspondence of each ultrafilter $r \in \omega^*$ to an ultrafilter base on K, which converges since K is compact. This point of convergence is easily seen to be the r-limit of f. To prove the converse, suppose X is r-compact for all $r \in \omega^*$, and H is a countable subset of X. Let $f\colon \omega \to H$ be a bijection. We show that every filter base on $\bar{H}$ has an adherent point in $\bar{H}$ (so $\bar{H}$ is compact). Let $\mathcal{F}$ be a filter on $\bar{H}$. If $\mathcal{F}$ has no adherent point in $\bar{H}$, then since X is T_3, the filter base $\mathcal{F}' = \{U\colon U \text{ is open in } X \text{ and there exists } F \text{ in } \mathcal{F} \text{ with } F \subset U\}$ has no adherent point in $\bar{H}$. Since H is dense in $\bar{H}$, $U \cap H$ is not empty for all U in $\mathcal{F}'$, and $\mathcal{F}'' = \{U \cap H\colon U \in \mathcal{F}'\}$ has no adherent point in $\bar{H}$. Let r be an ultrafilter containing $\mathcal{F}''$ (we are identifying ω and H). By hypothesis, the sequence f has an r-limit point x in X. But then x is an adherent point of $\mathcal{F}''$, hence of $\mathcal{F}$. This contradiction completes the proof.

We now consider two more theorems which use the notion of r-limits.

4.10. THEOREM. *Let $\{X_\alpha\colon \alpha \in I\}$ be a family of spaces. Then $X = \Pi\{X_\alpha\colon \alpha \in I\}$ is countably compact if and only if for every $J \in [I]^{2^{\mathfrak{c}}}$, $\Pi\{X_\alpha\colon \alpha \in J\}$ is countably compact.*

PROOF. Let $X_J = \Pi\{X_\alpha\colon \alpha \in J\}$ for any subset J of I. Since X_J is a closed subset of X_I for all $J \subset I$, half of the theorem is obvious. We now assume that if $J \subset I$ and

$|J| \leqslant 2^{\mathfrak{c}}$, then X_J is countably compact, and prove that X_I (the full product) is countably compact. If X_I is not countably compact, then by Theorem 4.4, there is a sequence $f: \omega \to X_I$ such that for every $r \in \omega^*$ there is a coordinate $\alpha(r) \in I$ such that $\pi_{\alpha(r)} \circ f$ has no r-limit in $X_{\alpha(r)}$. Put $J = \{\alpha(r); r \in \omega^*\}$. Then $|J| \leqslant |\omega^*| = 2^{\mathfrak{c}}$; so X_J is countably compact by hypothesis. This is impossible because the sequence $\pi_J \circ f: \omega \to X_J$, where π_J is the projection of X_I onto X_J, has no cluster point in X_I (if it did have a cluster point, it would have an r-limit for some $r \in \omega^*$ by 4.2, and then by continuity, $\pi_{\alpha(r)} \circ f$ would have an r-limit in $X_{\alpha(r)}$).

QUESTION. Can the number '$2^{\mathfrak{c}}$' in Theorem 4.10 be replaced by a smaller number? If we assume the first two steps of the generalized continuum hypothesis, then the answer is "No" by Example 3.14. The question, however, is still unanswered in ZFC. VICTOR SAKS [1979] has shown that the answer is "No" under MA and several other set-theoretic assumptions.

4.11. THEOREM. *The following are equivalent for a space X:*

(i) *X^k is countably compact for every cardinal k.*
(ii) *$X^{2^{\mathfrak{c}}}$ is countably compact.*
(iii) *$X^{|X|^\omega}$ is countably compact.*
(iv) *There is an r in ω^* such that X is r-compact.*

PROOF. It is trivial that (i)→(ii), and (ii)→(iii) follows at once from Theorem 4.10. We next prove (iii)→(iv). Let $\{f_\alpha: \alpha < |X|^\omega\}$ list the set of all sequences in X, and define $g: \omega \to X^{|X|^\omega}$ by the rule $\pi_\alpha \circ g(n) = f_\alpha(n)$ (compare with the proof of 3.5). Thus g is a sequence in $X^{|X|^\omega}$ whose projection on the αth coordinate is f_α. Since $X^{|X|^\omega}$ is countably compact, there exists $r \in \omega^*$ such that g has an r-limit. Since the projection maps are continuous, the sequences $\pi_\alpha \circ g = f$ have r-limits for all $\alpha < |X|^\omega$. Thus X is r-compact. The implication (iv)→(i) follows at once from Theorem 4.7. This completes the proof.

We now expand the diagram following 2.6 to include class $\mathfrak{C}$ and r-compactness for an arbitrary, but fixed, $r \in \omega^*$.

$$\begin{tikzcd}
\text{sequentially compact} \arrow[d] & \\
\text{totally countably compact} \arrow[d] & \omega\text{-bounded} \arrow[l] \arrow[dd] \\
\text{class } \mathfrak{C} \arrow[d] & \\
\text{countably compact} & r\text{-compact} \arrow[l]
\end{tikzcd}$$

We need several more examples in order to show that all the properties in the above diagram are distinct. For a sequentially compact space which is not

r-compact for any r in ω^* we have to resort to using an extra set-theoretic assumption, but the other examples are given in ZFC.

4.12. EXAMPLE. *For every r in ω^* there exist spaces X and Y such that X is r-compact, Y is countably compact, and $X \times Y$ is not countably compact. Thus, X is an r-compact space which does not belong to class $\mathfrak{C}$.*

PROOF. Let $r \in \omega^*$, and construct X along the lines of the inductive construction given in 2.13, keeping $|X_\alpha| \leq \mathfrak{c}$, but adding the r-limit for each sequence in X_α instead of adding an arbitrary accumulation point. The resulting space X is r-compact, $\omega \subset X \subset \beta(\omega)$ and $|X| \leq \mathfrak{c}$. (Note that at this point we can see that X is not totally countably compact by 2.9(iii).) Now construct Y along the lines of the inductive construction given in 3.1, except at the successor ordinal step pick the needed accumulation points in $\beta(\omega) - X$. Thus, Y is countably compact, $\omega \subset Y \subset \beta(\omega)$ and $X \cap Y = \omega$. It follows from the Lemma in 3.1 that $X \times Y$ is not countably compact. We note that by Theorem 4.7 the space Y is a countably compact space which is not r-compact, but the next example gives a much stronger space of this kind.

4.13. EXAMPLE. A totally countably compact space X which is not r-compact for any r in ω^*. We use the spaces K_x $(x \in \omega^*)$ defined in 2.14, and consider them as pairwise disjoint. Let ∞ be a point not in any of the K_x. We define the space X as the 'one-point countably-compactification' of the K_x. To be precise, let $\bigoplus\{K_x \colon x \in \omega^*\}$ denote the disjoint union of the spaces K_x, and put $X = \bigoplus\{K_x \colon x \in \omega^*\} \cup \{\infty\}$. Define the topology on X so that each K_x with its usual topology is a clopen subset of X, and such that every neighborhood of ∞ contains all but finitely many of the K_x. Since for each r in ω^*, K_r is a closed subset of X, X is not r-compact for any r. To see that X is totally countably compact, let H be a countably infinite subset of X. If there exists an r in ω^* such that $H \cap K_r$ is infinite, then $H \cap K_r$ contains an infinite subset with compact closure (because K_r is totally countably compact). If for every r in ω^*, $H \cap K_r$ is finite, then $H \cup \{\infty\}$ is compact, in fact a convergent sequence.

4.14. EXAMPLE. Assume $\mathfrak{b} = \mathfrak{c}$. A sequentially compact space which is not r-compact for any $r \in \omega^*$.

For each $r \in \omega^*$, let S_r be a sequentially compact space which is not r-compact. Such spaces have been constructed, assuming $\mathfrak{b} = \mathfrak{c}$ (see van Douwen's article). Then (as in 4.13) the space $X = \bigoplus\{S_r \colon r \in \omega^*\} \cup \{\infty\}$ works.

Recall that there do exist sequentially compact spaces which are not ω-bounded (Example 2.11).

4.15. EXAMPLE. Assume $\mathfrak{b} = \mathfrak{c}$. There exists a family of sequentially compact spaces whose product is not sequentially compact. Let S be the space of Example

4.14. Then the product of 2^{c} copies of S is a product of sequentially compact spaces which is not countably compact by Theorem 4.11.

It is an open problem to determine whether 4.14 and 4.15 can be proved in ZFC. Some of the work on this is discussed in the notes at the end of this article.

4.16. REMARK. In $T_{7/2}$-spaces, it is sometimes convenient to work with countable compactness and r-compactness in the following forms. Let X be a $T_{7/2}$-space. X is countably compact if and only if for every sequence $f\colon \omega \to X$, one has $f^{\beta}(\omega^*) \cap X \neq \emptyset$, where $f^{\beta}\colon \beta(\omega) \to \beta(X)$ denotes the Stone extension of f. X is r-compact if and only if for every sequence $f\colon \omega \to X$, one has $f^{\beta}(r) \in X$. For instance, see W.W. COMFORT [1977].

5. When is a countably compact or compact space sequentially compact?

We have already discussed several aspects of sequentially compact spaces, such as the relations this class has with the other classes discussed. We now consider both topological and set-theoretic conditions under which countably compact or compact spaces are sequentially compact.

We begin with three fairly general classes of spaces where countable compactness and sequential compactness are equivalent. These are the classes of sequential spaces, suborderable (= generalized orderable) spaces and scattered (= dispersed = right separated) spaces.

5.1. DEFINITION. A subset A of a space X is called *sequentially closed* provided that for every sequence $f\colon \omega \to A$, if f converges to a point $x \in X$, then $x \in A$. Clearly, every closed set in every space is sequentially closed, but the converse is false (ω_1 is sequentially closed but not closed in the space $X = (\omega_1 + 1)$ with the order topology). A space X is called a *sequential space* provided every sequentially closed set in X is closed in X.

The following diagram indicates the general nature of this property,

$$\begin{tikzcd}
\text{first countable} \arrow[r] \arrow[d] & \text{semimetrizable} \arrow[d] \\
\text{Fréchet} \arrow[d] & \text{symmetrizable} \arrow[d] \\
\text{sequential} & \text{weakly first countable} \arrow[l]
\end{tikzcd}$$

5.2. REMARK. In order to show that a space is sequentially compact, it suffices to show that every one-to-one sequence has a convergent subsequence.

5.3. *Every countably compact, sequential space X is sequentially compact.*

PROOF. Let $f\colon \omega \to X$ be a one-to-one sequence in X, and let x be a cluster point of f. The set $A = \text{Range}(f) - \{x\}$ is not closed in X, hence not sequentially closed. Thus, there is a sequence $g\colon \omega \to A$ which converges to a point $y \notin A$. Since g cannot be eventually constant, by passing to a subsequence we may assume that g is one-to-one. Pick $n_i \in \omega$ such that $f(n_i) = g(i)$ for $i \in \omega$. The set $\{n_i\colon i \in \omega\}$ is infinite and hence determines a convergent subsequence of f. This completes the proof.

This proof can be modified slightly to show that countable compactness and sequential compactness are equivalent in the class of spaces which can be embedded into a sequential space, and more generally in the class of spaces which can be embedded into a chain net space (see DEVI, MEYRE and RAJAGOPALAN [1976]).

5.4. DEFINITION. A space is called *suborderable* provided it can be embedded into an ordered space (see Todorčević's article).

5.5. *Every countably compact, suborderable space X is sequentially compact.*

PROOF. Let $f\colon \omega \to X$ be a one-to-one sequence in X, and $(Z, \leqslant)$ a (linearly) ordered space containing X as a subspace, and let x be a cluster point of f in X. We will show that f has a subsequence converging to x. Working with the order in Z, we note that either there is a subsequence of f increasing up to x, or a subsequence of f decreasing down to x (otherwise we could find $a, b \in Z$ with $a < x < b$ and $\{n \in \omega\colon f(n) \in (a, b)\} = \emptyset$).

An alternate proof can be given by showing that every suborderable space is a chain net space.

5.6. DEFINITION. A space X is called scattered provided every non-empty subspace $Y \subset X$ has an isolated point (with respect to the subspace topology on Y).

5.7. THEOREM. *Every countably compact, scattered T_3-space is sequentially compact.*

PROOF. It suffices to show that every point in X has a neighborhood (not necessarily open) which is sequentially compact. Claim: $Y = \{x \in X\colon$ no neighborhood of x is sequentially compact$\}$ is empty. If not, there is $y \in Y$ and an open set U in X such that $U \cap Y = \{y\}$. Let V be an open neighborhood of y with $\bar{V} \subset U$. Then $\bar{V}$ is sequentially compact because every sequence in $\bar{V}$ either converges to y or (since $\bar{V}$ is countably compact) has a cluster point $x \neq y$. Since

$x \notin Y$, x has a sequentially compact neighborhood; so f has a convergent subsequence. This contradicts that $y \in Y$.

We next turn to cardinality conditions. The general theme is that if a countably compact space is small enough then it must be sequentially compact. We have seen how to construct countably compact spaces of cardinality $\mathfrak{c}$ having no non-trivial convergent subsequences (2.12), and which are therefore not sequentially compact. It is not possible to construct such examples of smaller cardinality.

5.8. THEOREM. *Every countably compact space of cardinality $<2^\omega$ is sequentially compact.*

PROOF. We prove the contrapositive. Let f be a sequence in X having no convergent subsequence. Thus every subsequence of f has at least two cluster points. We will show that $|X| \geqslant \mathfrak{c}$ by constructing a "Cantor set worth of points in X." By mathematical induction, construct for each $\sigma \in \bigcup\{{}^n2: n \in \omega\}$ an infinite subset $T_\sigma \subset \omega$. Assume we have constructed these for all $\sigma \in \bigcup\{{}^n2: n < N\}$ satisfying

(i) If $\sigma \subset \mu$, then $T_\mu \subset T_\sigma$.

(ii) If σ, $\mu \in {}^n2$ and $\sigma \neq \mu$, then $\overline{\text{Range}(f \mid T_\sigma)}$ and $\overline{\text{Range}(f \mid T_\mu)}$ are disjoint.

In order to construct T_μ for $\mu \in {}^N2$, it suffices to consider each $\sigma \in {}^{(N-1)}2$ and define T_{σ^0} and T_{σ^1} where $\sigma^0 = \sigma \cup \{\langle N-1, 0\rangle\}$ and $\sigma^1 = \sigma \cup \{\langle N-1, 1\rangle\}$. So let $\sigma \in {}^{(N-1)}2$, and pick two cluster points $a \neq b$ of $f \mid T_\sigma$, and (since X is T_3) open sets U and V in X such that $a \in U$, $b \in V$, and $\bar{U} \cap \bar{V} = \emptyset$. Put $T_{\sigma^0} = \{n \in T_\sigma: f(n) \in U\}$ and $T_{\sigma^1} = \{n \in T_\sigma: f(n) \in V\}$. Clearly (i) and (ii) are satisfied, and this completes the induction. To get the points in X, form, for each $\phi \in {}^\omega2$, the filter base $\mathscr{F}_\phi = \{\text{Range}(f \mid T_\sigma): \sigma = \phi \mid n \text{ for } n \in \omega\}$. This is the image under the function f of the countable tower $\{T_\sigma: \sigma = \phi \mid n \text{ for } n \in \omega\}$. Since X is countably compact, we may select an adherent point $x(\phi)$ of each $\mathscr{F}_\phi$. To complete the proof we have to show that if $\phi \neq \psi$, then $x(\phi) \neq x(\psi)$.

If $\phi \neq \psi$, then there exists $n \in \omega$ such that $\sigma = \phi|n \neq \psi|n = \mu$, and then we have that $x(\phi) \in \overline{\text{Range}(f \mid T_\sigma)}$ and $x(\psi) \in \overline{\text{Range}(f \mid T_\mu)}$, which shows $x(\phi) \neq x(\psi)$ since by (ii) these are disjoint sets.

This result suggests the question: Does there exist a compact space of cardinality $\mathfrak{c}$ which is not sequentially compact? We will see from two interesting results below that this question cannot be answered within ZFC. The next result gives us half of it.

5.9. THEOREM. *Every compact space of cardinality $< 2^{\mathfrak{t}}$ is sequentially compact. In particular, every compact space of cardinality $< 2^{\omega_1}$ is sequentially compact.*

PROOF. The proof is an extension of the construction in 5.8. Let f be a sequence in X having no convergent subsequence. By transfinite induction on $\mathfrak{t}$ construct for each $\sigma \in \{2^\alpha : \alpha < \mathfrak{t}\}$ an infinite set $T_\sigma \subset \omega$ satisfying

(i) If $\sigma \subset \mu$, then $T_\mu \subset^* T_\sigma$

(ii) If $\sigma, \mu \in {}^\alpha 2$ and $\sigma \neq \mu$, then $f(T_\sigma)$ and $f(T_\mu)$ have no accumulation points in common (they may have a finite number of points in common).

The remainder of the proof is similar to that of Theorem 5.8. This result is also given in Theorem 6.3 in van Douwen's article in this Handbook.

5.10. COROLLARY. *Assume* $\mathfrak{t} = \mathfrak{c}$. *Every compact space of cardinality* $< 2^{\mathfrak{c}}$ *is sequentially compact. In particular, every compact space of cardinality* $\leqslant \mathfrak{c}$ *is sequentially compact.*

The next result shows that there exist models of ZFC in which there are compact spaces of cardinality $\mathfrak{c}$ which are not sequentially compact. In fact, these spaces can be of a very simple kind.

5.11. DEFINITION. Let D denote the two point discrete space ($D = 2$) and for an infinite cardinal k, let D^k denote the product of k copies of D with the product topology. Thus D^ω is the Cantor space, a compact metric space, hence sequentially compact.

We now use the splitting number $\mathfrak{s}$ (see Section 1).

5.12. THEOREM. *The following are equivalent for a cardinal* k.

(i) D^k *is sequentially compact.*

(ii) $k < \mathfrak{s}$.

(iii) *Every compact space of weight* $\leqslant k$ *is sequentially compact.*

In particular, $D^{\mathfrak{c}}$ *is not sequentially compact.*

For the proof, see Theorem 6.1 in van Douwen's article.

5.13. COROLLARY [$2^\omega = 2^{\omega_1}$ and $\omega_1 = \mathfrak{s}$]. *There exists a compact space* X *of cardinality* $\mathfrak{c}$ *which is not sequentially compact.*

PROOF. Take $X = D^{\omega_1}$. The set-theoretic hypothesis is known to be consistent (for example, if one adds ω_2 Cohen reals to a model of GCH).

5.14. THEOREM. *The following two statements are consistent with and independent of* ZFC:

(i) *Every compact space of cardinality* $\leqslant \mathfrak{c}$ *is sequentially compact.*

(ii) D^{ω_1} *is sequentially compact.*

PROOF. Part (i) follows from 5.10 and 5.13. To prove (ii), note that by Theorem 5.12, the space D^{ω_1} is sequentially compact if and only if $\omega_1 < \mathfrak{s}$.

Even more is possible under the assumption of Corollary 5.13. V. FEDORČUK [1977] proved that under the assumption ($2^{\omega} = 2^{\omega_1}$ and $\mathfrak{s} = \omega_1$) there exists a compact space of cardinality $\mathfrak{c}$ having no non-trivial convergent sequences.

The next result gives an interesting connection between topology and measure theory via set theory.

5.15. *If D^k is sequentially compact, then every subset of the real line having cardinality k has Lebesgue measure zero.*

PROOF (outline). Define a map $d: [\omega]^{\omega} \to [0, 1]$ by the rule $d(T) = \Sigma\{T(n)2^{-n-1}: n \in \omega\}$ where T is the characteristic function of T. The range of d is $(0, 1]$. Let S be a subset of $(0, 1]$, of cardinality $\leqslant k$ and for each x in S let T_x be an infinite subset of ω such that $d(T_x) = x$. By Theorem 5.12, $\{T_x: x \in S\}$ is not a splitting family; so there exists $N \subset \omega$ such that $N \subset^* T_x$ or $N \subset^* (\omega - T_x)$ for all $x \in S$. Thus $S \subset \{d(T): N \subset^* T\} \cup \{d(T): N \subset^* \omega - T\}$. Next show for $M \in [\omega]^{\omega}$ that $\{d(R): M \subset R\}$ has measure zero. Then $\{d(T): N \subset^* T\}$ is a countable union of such sets.

6. When is a countably compact space compact?

We are interested in this section in properties $\mathscr{P}$ which satisfy the implication

6.1. *Countably compact* + $\mathscr{P} \to$ *compact*, that is, properties $\mathscr{P}$ such that every countably compact space with property $\mathscr{P}$ is compact. We have compiled the following partial list of such properties, most of which are discussed in this section. Covering properties are listed first.

6.2. List of properties which satisfy 6.1. Lindelöf, paracompact, subparacompact, metacompact, submetacompact ($=\theta$-refinable), meta-Lindelöf, submeta-Lindelöf ($=\delta\theta$-refinable), weakly submetacompact ($=$ weakly θ-refinable), weakly submeta-Lindelöf ($=$ weakly $\delta\theta$-refinable), ultrapure, astral, pure, G_δ-diagonal, quasi-G_δ-diagonal, realcompact, closed-complete ($= a$-realcompact), weakly $[\omega_1, \infty]^r$-refinable, $[\mathfrak{p}, \infty]$-compact and separable, and perfect (assuming $\mathfrak{p} > \omega_1$).

Since many of the properties in this list are related, we do not have to prove a separate result for each one. One of the weakest properties in the list is the property of being a pure space in the sense of ARHANGEL'SKII [1980]. This can be considered a weak covering property since every weakly submeta-Lindelöf space is pure, and also a weak global property since every space with a quasi-G_δ-diagonal is pure. Most of these covering properties are discussed in Burke's article in this Handbook, but we will define them here so the reader can see how these properties evolved from 'Lindelöf' to 'pure'.

In order to prove that every property in the list 6.2 is a property $\mathcal{P}$ which satisfies 6.1, we proceed as follows. First we call on the result (see 4.1 in Burke's article) that every subparacompact space is submetacompact. This, combined with an inspection of the following definitions, shows that each of the first 12 properties in 6.2 implies the property 'pure.' We prove in Theorem 6.3 that "countably compact + pure → compact"; so this takes care of the first 12 properties. The next four properties in 6.2 (i.e., 'G_δ-diagonal' to 'closed complete') also imply 'pure,' but we refer to the literature for the proofs:

(1) Every space with a (quasi)-G_δ-diagonal is pure (see ARHANGEL'SKIĬ [1980]).

(2) Every realcompact space is closed-complete (this is obvious; see N. DYKES [1970]).

(3) Every closed-complete space is pure (this follows easily from Theorem 4.1 of R.L. BLAIR [1977]).

For the property $[\omega_1, \infty]^r$-refinable, see Theorem 9.2 in Burke's article. For the last two properties in 6.2, see Theorems 6.5 and 6.8 in this section.

Recall that a space X is countably compact if and only if every open cover of X has a finite subcollection which covers X (called a finite subcover). A Lindelöf space is one in which every open cover has a countable subcover; so it is trivial that every countably compact, Lindelöf space is compact. A space is called *paracompact* provided every open cover $\mathcal{U}$ has a locally finite open refinement $\mathcal{V}$ (i.e., $\mathcal{V}$ is an open cover such that each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$ and each point has a neighborhood which intersects at most finitely many members of $\mathcal{V}$). It is easy to see that in a countably compact space every locally finite collection is finite (else pick a sequence of distinct points, one from each member of the locally finite family and note that a cluster point of that sequence contradicts the locally finiteness). Hence, every countably compact, paracompact space is compact. This is the start of the evolution of covering properties which satisfy 6.1. The proofs that the remaining properties in the list satisfy 6.1 are non-trivial, and become more involved as the properties get weaker.

A family $\mathcal{V}$ is called a *weak refinement* of a cover $\mathcal{U}$ provided for each $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ with $V \subset U$. If in addition $\mathcal{V}$ is also a cover, V is called a *refinement* of $\mathcal{U}$. For a family $\mathcal{V}$ of subsets of a set X, and a point x in X, we define the *order of x with respect to $\mathcal{V}$* as $\mathrm{o}(x, \mathcal{V}) = |\{V \in \mathcal{V}: x \in V\}|$. A family $\mathcal{V}$ is called point-finite provided for every x in X, $\mathrm{o}(x, \mathcal{V}) < \omega$. A space is called *metacompact* (resp. *meta-Lindelöf*) provided every open cover of X has an open refinement $\mathcal{V}$ with $\mathcal{V}$ point-finite (resp. point-countable (i.e., for all $x \in X$, $\mathrm{o}(x, \mathcal{V}) \leq \omega$)). A space is called *submetacompact* (resp. *submeta-Lindelöf*) provided for every open cover $\mathcal{U}$ of X there exists a sequence $\{\mathcal{V}_n: n < \omega\}$ of open refinements of $\mathcal{U}$ such that for every $x \in X$ there exists $n < \omega$ such that $\mathrm{o}(x, \mathcal{V}_n) < \omega$ (resp. $\mathrm{o}(x, \mathcal{V}_n) \leq \omega$). These terms are also known under the names θ-refinable (resp. $\delta\theta$-refinable). A space is called *weakly submetacompact* (resp. *weakly submeta-Lindelöf*) provided for every open cover $\mathcal{V}$ of X there exists a sequence $\{\mathcal{V}_n: n < \omega\}$ of open weak refinements of $\mathcal{U}$ such that for every x in X, there is $n < \omega$ such that $0 < \mathrm{o}(x, \mathcal{V}_n) < \omega$ (resp. $0 < \mathrm{o}(x\mathcal{V}_n) \leq \omega$).

For 6.1 and other things, the properties of the sequence $\{\mathscr{V}_n: n \in \omega\}$ may be further weakened. First of all, we do not need the full strength of "each V in $\mathscr{V}_n$ is open in X," but can get by with "each V in $\mathscr{V}_n$ is open in $\bigcup\mathscr{V}_n$." Second, we do not need the full strength of $\mathscr{V}$ refines $\mathscr{U}$ and of

$$\text{“}(\forall x \in X)(\exists n \in \omega)(0 < \mathrm{o}(x, \mathscr{V}_n) \leq \omega)\text{”}$$

but can get by with

$$\text{“}(\forall x \in \bigcup\mathscr{V}_n)(\exists \mathscr{U}' \in [\mathscr{U}]^{\omega}) \text{ such that } \mathrm{St}(x, \mathscr{V}_n) \subset \bigcup\mathscr{U}'.\text{”}$$

If we make these modifications to the definitions of weakly submeta-Lindelöf, we arrive at the definition of an ultrapure space. Further generalizations, however, are more naturally stated in terms of closed families $\mathscr{F}$ such that $\bigcap\mathscr{F} = \emptyset$ than in terms of open covers; so we will give the definition of ultrapure in terms of closed families too.

A countable family $\mathscr{V} = \{\mathscr{V}_n: n \in \omega\}$ of collections of subsets of a space X is called an *interlacing* on X provided $\bigcup\mathscr{V}$ is a cover of X and for each $n \in \omega$, each $V \in \mathscr{V}_n$ is open in $\bigcup\mathscr{V}_n$. A space X is called *ultrapure* provided for each family $\mathscr{F}$ of closed sets with $\bigcap\mathscr{F} = \emptyset$, there is an interlacing $\mathscr{V} = \{\mathscr{V}_n: n \in \omega\}$ on X which is δ-suspended from $\mathscr{F}$ (i.e., for each $n \in \omega$ and for each $x \in \bigcup\mathscr{V}_n$ there is a countable family $\mathscr{F}' \in [\mathscr{F}]^{\omega}$ such that $\mathrm{St}(x, \mathscr{V}_n) \cap (\bigcap\mathscr{F}) = \emptyset$. By restricting the class of families $\mathscr{F}$ for which we require δ-suspensions, we can get the concepts of astral and pure (we omit discussion of astral). A space X is called a *pure* space provided for each maximal centered family $\mathscr{F}$ of closed sets with $\bigcap\mathscr{F} = \emptyset$, there exists an interlacing on X which is δ-suspended from $\mathscr{F}$.

6.3. THEOREM. *Every countably compact, pure space is compact.*

PROOF. If false, there exists a countably compact, pure space X which is not compact, hence has a maximal centered family $\mathscr{F}$ of closed sets such that $\bigcap\mathscr{F} = \emptyset$. Since X is countably compact, $\mathscr{F}$ is countably complete (i.e., for each $\mathscr{F}' \in [\mathscr{F}]^{\omega}, \bigcap\mathscr{F}' \in \mathscr{F}$)). Let $\mathscr{V} = \{\mathscr{V}_n: n \in \omega\}$ be an interlacing on X which is δ-suspended from $\mathscr{F}$. Since $\{\bigcup\mathscr{V}_n: n \in \omega\}$ is a countable cover of X, and $\mathscr{F}$ is countably complete, there exists $n \in \omega$ such that $\mathscr{F} \cap (\bigcup\mathscr{V}_n) \neq \emptyset$ for every $F \in \mathscr{F}$. For each $V \in \mathscr{V}_n$, let $W(V)$ be an open set in X such that $W(V) \cap (\bigcup\mathscr{V}_n) = V$, and put $\mathscr{W} = \{W(V): V \in \mathscr{V}_n\}$.

(∗) There exists an $F \in \mathscr{F}$ such that $F \subset \bigcup\mathscr{W}$.

(If not, by the maximality of $\mathscr{F}$, for every closed set H in X either $H \in \mathscr{F}$ or there exists $F \in \mathscr{F}$ such that $F \cap H = \emptyset$. Since the closed set $(X - \bigcup\mathscr{W})$ misses $\bigcup\mathscr{V}_n$, it is not an element of $\mathscr{F}$; so the first possibility does not obtain). We now work with the set F given by (∗) and look at $F \cap (\bigcup\mathscr{V}_n)$. Pick any point $y \in F \cap (\bigcup\mathscr{V}_n)$. Since $\mathscr{V}$ is δ-suspended from $\mathscr{F}$, there exists $\mathscr{F}_y \in [\mathscr{F}]^{\omega}$ such that $\mathrm{St}(y, \mathscr{V}_n) \cap \bigcap\mathscr{F}_y = \emptyset$. Pick any point

$$y' \in (F \cap (\bigcup \mathcal{V}_n)) - \mathrm{St}(y, \mathcal{V}_n)$$

and get $\mathcal{F}_{y'} \in [\mathcal{F}]^{\omega}$ such that $\mathrm{St}(y', \mathcal{V}_n) \cap \bigcap \mathcal{F}_{y'} = \emptyset$. Proceed to construct (either by transfinite induction, or Zorn's lemma) a set $Y \subset F \cap (\bigcup \mathcal{V}_n)$ such that (a) each $V \in \mathcal{V}_n$ contains at most one $y \in Y$, and (b) $\bigcup\{\mathrm{St}(y, \mathcal{V}_n) \colon y \in Y\} \supset F \cap (\bigcup \mathcal{V}_n)$. By (a), Y is a discrete subset since each $V \in \mathcal{V}_n$ is open in $\bigcup \mathcal{V}_n$. By (b) Y is closed in X (if x were an accumulation point of Y, then since $x \in F \subset \bigcup \mathcal{W}$, there is $V \in \mathcal{V}_n$ such that $x \in W(V)$. Since $W(V)$ is open in X, it must contain infinitely many points of Y, but $W(V) \cap Y = V \cap Y$ which contradicts (b)). Since X is countably compact, the closed discrete set Y must be finite. Thus $\mathcal{F}' = \bigcup\{\mathcal{F}_y \colon y \in Y\}$ is countable. But then the countable, centered family $\mathcal{F}' \cup \{F\}$ of closed set has empty intersection, which is impossible in a countably compact space. This contradiction completes the proof.

Another direction in which generalizations have been considered: a restriction to regular cardinality. Given any open cover property, one can get a formally weaker one by requiring the definition to apply only to covers of regular cardinality. The property weakly $[\omega_1, \infty]^r$-refinable of WORRELL and WICKE [1980] is such a property and is discussed in Section 9 of Burke's article.

List 6.2 is not intended to be complete. One property not mentioned there which could go in the list is the property of having a point countable T_1-separating open cover. Countably compact spaces with this property are not only compact but also metrizable. See Section 7 in Gruenhage's article for the proof.

We next consider properties which satisfy 6.1 under the assumption of extra set-theoretic conditions. By using the cardinal $\mathfrak{p}$ (see Section 1) we can state some of these results in ZFC.

6.4. LEMMA. *Every countably compact, separable space has the following property: For every open cover $\mathcal{U}$ of X with $|\mathcal{U}| < \mathfrak{p}$, there exists a finite subfamily $\mathcal{U}' \subset \mathcal{U}$ such that $X = \bigcup\{\bar{U} \colon U \in \mathcal{U}'\}$.*

PROOF. Suppose this is false, and $\mathcal{U}$ is an open cover of X such that $|\mathcal{U}| < \mathfrak{p}$ and no finite subcollection of $\mathcal{U}$ has dense union in X. Let D be a countable dense subset of X (we identify D with ω) and put $\mathcal{F} = \{D - U \colon U \in \mathcal{U}\}$. Since D cannot be covered by finitely many members of $\mathcal{U}$, we see that $\mathcal{F}$ has the strong finite intersection property (i.e., the intersection of finitely many members of $\mathcal{F}$ is infinite). Since $|\mathcal{F}| < \mathfrak{p}$, there exists an infinite set $A \subset D$ such that $A \subset^* F$ for all $F \in \mathcal{F}$. Since X is countably compact, A has an accumulation point x. But there exists $U \in \mathcal{U}$ with $x \in U$; so $U \cap A$ is infinite, which contradicts that $A \subset^* D - U \in \mathcal{F}$.

Our first use of this lemma involves a generalization of the Lindelöf property (every open cover has a countable subcover). A space X is called $[\mathfrak{p}, \infty]$-*compact* provided every open cover of X has a subcover of cardinality strictly less than $\mathfrak{p}$.

6.5. THEOREM. *Every countably compact, separable* $[\mathfrak{p}, \infty]$*-compact space X is compact.*

PROOF. Let X be such a space, and $\mathscr{W}$ any open cover of X. Since X is a T_3-space, there exists an open cover $\mathscr{V}$ of X such that for each $V \in \mathscr{V}$ there exists $W \in \mathscr{W}$ such that $\bar{V} \subset W$. In order to show that $\mathscr{W}$ has a finite subcover, it suffices to show that there is a finite $\mathscr{V}' \subset \mathscr{V}$ such that $X = \{\bar{V}: V \in \mathscr{V}'\}$. Since X is $[\mathfrak{p}, \infty]$-compact, $\mathscr{V}$ has a subcover of cardinality strictly less than $\mathfrak{p}$, and the result follows from Lemma 6.4.

6.6. COROLLARY. *Every countably compact,* $[\mathfrak{p}, \infty]$*-compact T_3-space is ω-bounded.*

In connection with this result we mention that the spaces K_x of 2.14 are countably compact and separable, but neither ω-bounded nor compact.

Recall that a space is called *perfect* provided that every open set in X is an F_σ-set (i.e., a countable union of closed sets).

6.7. LEMMA. *Every countably compact, perfect space X has countable spread (i.e. $s(X) = \omega$).*

PROOF. We have to show that every discrete subset of X is countable. Suppose this is false, and X has a discrete subspace of cardinality ω_1, say $S = \{x_\alpha: \alpha < \omega_1\}$. Let $\{U_\alpha: \alpha < \omega_1\}$ be open sets in X such that $U_\alpha \cap S = \{x_\alpha\}$. Since X is perfect, there are closed sets F_i for $i < \omega$ such that $\bigcup\{U_\alpha: \alpha < \omega_1\} = \bigcup\{F_i: i < \omega\}$; so some F_i contains uncountably many points of S. Let x be an accumulation point of $F_i \cap S$. Since $F_i \subset \bigcup\{U_\alpha: \alpha < \omega_1\}$, there is $\alpha < \omega_1$ such that $x \in U_\alpha$. This implies that infinitely many x_β are in U_α, which is a contradiction.

The following two results (6.8 and 6.9) combine to show that perfect, countably compact spaces may or may not be compact, depending on set- theoretic assumptions. This is one of the major results in set-theoretic topology.

6.8. THEOREM (WEISS [1978]). *Assume $\mathfrak{p} > \omega_1$. Every countably compact, perfect T_3-space is compact.*

PROOF. Suppose this is false and that X is a countably compact, perfect T_3-space which is not compact, hence not Lindelöf. Let $\mathscr{U}$ be an open cover of X having no countable subcover. Pick points $\{x_\alpha: \alpha < \omega_1\} \subset X$ and open sets $\{U_\alpha: \alpha < \omega_1\}$ from $\mathscr{U}$ such that

$$(*) \qquad \text{for } \alpha < \omega_1, \qquad x_\alpha \in U_\alpha \quad \text{and} \quad U_\alpha \cap \{x_\beta: \alpha < \beta < \omega_1\} = \emptyset.$$

This is possible since the countable subfamilies $\{U_\beta: \beta < \alpha\}$ of $\mathscr{U}$ do not cover X. Put $Y = \{x_\alpha: \alpha < \omega_1\}$ (Y is a right separated subspace of X).

Claim 1. *Y is separable.* If Y is not separable, we can select points $x_{\alpha_\tau} \in Y$

such that

$$(**) \qquad x_{\alpha_\tau} \notin \overline{\{x_{\alpha_\tau}: \gamma < \tau\}} \quad \text{for all } \tau < \omega_1 .$$

This is possible because the closures of the countable sets $\{x_{\alpha_\gamma}: \gamma < \tau\}$ do not contain Y. Put $Z = \{x_{\alpha_\tau}: \tau < \omega_1\}$. Now Z is a discrete subspace of X since for each $\tau < \omega_1$

$$V = U_{\alpha_\tau} \cap (X - \overline{\{x_{\alpha_\gamma}: \gamma < \tau\}})$$

is an open set such that $V \cap Z = \{x_{\alpha_\tau}\}$. Thus the assumption that Y is not separable implies that Y (and therefore X) has an uncountable discrete subspace, i.e., $s(X) > \omega$, but this contradicts Lemma 6.7.

Claim 2. *Y is hereditarily separable.* It suffices to show that every uncountable subset $Y' \subset Y$ is separable, and the proof of this is very similar to the proof of Claim 1. We obviously have the needed analogue of (*) for Y' and we can get a subspace $Z' \subset Y'$ satisfying (**).

We finish the proof by contradiction. For each $\alpha < \omega_1$ let V_α be an open set such that $x_\alpha \in V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$. Let $W_\alpha = V_\alpha \cap \bar{Y}$ for all $\alpha < \omega_1$ and put $\mathscr{W} = \{W_\alpha: \alpha < \omega_1\}$. Note that by definition of U_α, $W_\alpha \cap Y$ is countable for $\alpha < \omega_1$. Put $W = \bigcup\{W_\alpha: \alpha < \omega_1\}$. Since $\bar{Y}$ is a perfect subspace of X, and W is open in $\bar{Y}$, there exist closed sets F_i for $i < \omega$ such that $W = \bigcup\{F_i: i < \omega\}$. Let i be such that $F_i \cap \{x_\alpha: \alpha < \omega_1\}$ is uncountable. Since Y is hereditarily separable, $F_i \cap Y$ is separable; so $X' = \overline{F_i \cap Y}$ is both separable and countably compact, and since $F_i = \bar{F}_i \subset W$, X' is covered by $\{W_\alpha: \alpha < \omega_1\}$. By hypothesis, $\omega_1 < \mathfrak{p}$; so by Lemma 6.4, there is a finite set $A \subset \omega_1$ such that $X' \subset \bigcup\{\bar{W}_\alpha: \alpha \in A\}$, but this is impossible because each $\bar{W}_\alpha \subset \bar{V}_\alpha \subset U_\alpha$ and each U_α contains only countably many of the points x_β. This contradiction completes the proof.

The next result shows that Theorem 6.8 can not be proved in ZFC.

6.9. EXAMPLE (OSTASZEWSKI [1976]). Assume $\diamondsuit$. There exists a countably compact, perfect T_3-space which is not compact. Moreover, there is such a space which is hereditarily separable (hence an S-space), first countable, normal (hence perfectly normal), and zero dimensional.

For the construction of this example, see Section 3 in Roitman's article in this Handbook.

7. Notes

Total countable compactness was introduced by Z. FROLÍK [1960b] under a different name, and has been rediscovered in several forms since then (for a discussion of this see J.E. VAUGHAN [1978a, p. 212]). The concept of ω-boundedness was introduced by S.L. GULDEN, W.F. FLEISCHMAN, and J.H. WESTON [1970]. Total filter bases were defined by B.J. PETTIS [1969] in a slightly different form, and studied in a topology by VAUGHAN [1976], [1978a]. Example 2.11 is due to S.P. FRANKLIN and M. RAJAGOPALAN [1971]. The spaces K_x in Example 2.14 are due to FROLÍK [1960a].

YU. SMIRNOV [1951] proved that the product of a countably compact space and a compact space is countably compact. C.T. SCARBOROUGH and A.H. STONE [1966] proved that the product of no more than ω_1 sequentially compact spaces is countably compact. (Proposition 3.4(C)). This was generalized to strongly ω-compact spaces by V. SAKS and R.M. STEPHENSON, Jr. [1971]. In turn, this was extended (from ω_1 to the cardinal $\mathfrak{t}$) by S.H. HECHLER [1975], and VAUGHAN [1978] gave a general setting for these results using total filter bases. Proposition 3.5 is due to SCARBOROUGH and STONE [1966] (they assumed CH). Theorem 3.7 is due to N. NOBLE [1969]. Theorem 3.11 (i)$\leftrightarrow$(ii) is due to FROLÍK [1960a] and T. ISIWATA [1964] proved (ii)$\leftrightarrow$(iv) via (iii). Corollary 3.12 and Example 3.13 are essentially from VAUGHAN [1978b] and extend results in ISIWATA [1964].

The notion of r-limits was introduced by A. BERNSTEIN [1970] using the approach of non-standard analysis, and studied by J. GINSBURG and V. SAKS [1975], and also by M. RAJAGOPALAN and R.G. WOODS [1977], V. SAKS [1978] [1979] and VAUGHAN [1975]. The results 4.2–4.9 can be found for the most part in GINSBURG and SAKS [1975] and some are in BERNSTEIN [1970]. The first reduction theorem (like 4.10) was proved by SCARBOROUGH and STONE [1966] for products of $2^{2^{\mathfrak{c}}}$ spaces. GINSBURG and SAKS [1975] improved this to $2^{\mathfrak{c}}$ products for powers of one space. W.W. COMFORT (see Math. Reviews 52 #1633) and V. SAKS [1978] noticed that the idea of the Ginsburg-Saks proof gives Theorem 4.10. Theorem 4.11 is in GINSBURG and SAKS [1975]. Example 4.12 may be new. Example 4.13 is due to FROLÍK [1967].

The first example of a family of sequentially compact spaces whose product is not countably compact (and Example 4.14) were given by RAJAGOPALAN and WOODS [1977] assuming CH and later using the same construction by RAJAGOPALAN [1976] assuming "$\mathfrak{p} = \mathfrak{c}$". An alternate construction, which is an extension of Ostaszewski's technique, was given by VAUGHAN [1975] (assuming $\diamondsuit$ or CH). E. VAN DOUWEN extended this construction to the assumption "$\mathfrak{b} = \mathfrak{c}$" (see his article in this volume). Similar spaces were constructed independently by I. JUHÁSZ, ZA. NAGY, and W. WEISS [1979] (assuming MA). A weak assumption which gives a family of sequentially compact spaces whose product is not countably compact is "$\mathfrak{b} = \mathfrak{c}$". When the stronger assumption $\diamondsuit$ is assumed, the sequentially compact spaces can be made perfectly normal. This cannot be done in ZFC, by Theorem 6.8 of Weiss (see VAUGHAN [1975]). A refinement of this problem is to find a single space X which is first countable and countably compact, and for which a product of copies of X is not countably compact. This has been done by K. KUNEN (unpublished) assuming the first two steps of GCH.

PETER NYIKOS [1982] has shown that under the set-theoretic assumption that there exist two unbounded chains in ${}^{\omega}\omega$ (mod finite order) of different cofinalities (see 6.1 in William's article in this Handbook) there exists a family $\mathscr{F}$ of sequentially compact spaces (each in the form of the space in Example 2.11) whose product is not countably compact. This set-theoretic assumption, which contradicts "$\mathfrak{b} = \mathfrak{c}$", allows the family $\mathscr{F}$ to be constructed with fewer than $\mathfrak{c}$ spaces in it. This construction can be combined with the one in Example 4.13 to give a sequentially compact space X with $|X| = \aleph_2 < \mathfrak{c}$ (and $\mathfrak{c}$ can be arbitrarily large) such that the product space X^{ω_2} is not countably compact (contrast this to Theorem 3.5 in this article). These and related results will appear in NYIKOS and VAUGHAN [19··].

Theorem 5.3 is due to S.P. FRANKLIN [1967]. Theorem 5.5 is due to S. PURISH [1973] and Theorem 5.7 to J.W. BAKER [1973]. For a generalization, see K.M. DEVI, P.R. MEYER, and M. RAJAGOPALAN [1976]. We do not know who first proved Theorem 5.8. The special case (for 2^{ω_1}) of 5.9 is due to FRANKLIN [1969] (with a different proof), and a related result is given by A. SZMANSKI and M. TURZANSKI [1976]. Corollary 5.10 is due to V. MALYHIN and B. ŠAPIROVSKIĬ [1973]. Theorems 5.12, 5.13, and 5.15 are due to D. BOOTH [1974].

Theorem 6.3 is due to A.V. ARHANGEL'SKII [1980], but our proof is more along the lines of J.M. WORRELL, Jr., and H.H. WICKE [1980]. In any case, Theorem 6.3 is the result of many hands. For more about the restriction to regular cardinality, see WICKE and WORRELL [1979]. For another variation see BLAIR [1980]. Lemma 6.4 is due to MALYHIN and ŠAPIROVSKIĬ [1973] and independently to HECHLER [1975a]. Theorem 6.5 is a version of 6.4. Lemma 6.7 is due to R.M. STEPHENSON, Jr. [1972].

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CHAPTER 13

Initially κ-Compact and Related Spaces

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
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1. Introduction

A main reason for studying initial κ- and, more generally, interval compactness (defined below) is that compactness, countable compactness, and the Lindelöf property are special cases of one or both of these concepts. Another is that the theory of initially κ-compact and related spaces sometimes provides a means for answering fundamental questions which arise in other areas of topology. A third reason is that results in this area illustrate the usefulness of the close relationship that exists between the set theory of uncountable cardinals and properties of topological spaces.

In this section our main purpose is to give most of the conventions, notation, and terminology that will be used throughout this chapter.

CH, GCH, MA, and ZFC will denote, respectively, the Continuum Hypothesis, the Generalized Continuum Hypothesis, Martin's Axiom, and the usual axioms of set theory.

ω_0 will be denoted by ω, and θ, κ, and λ will always denote infinite cardinal numbers (although occasionally, for emphasis, we may also write, "Let κ be an infinite ..."); other Greek letters will denote ordinal numbers. An ordinal number is taken to be the set of smaller ordinal numbers, and a cardinal number α is the smallest ordinal number equipotent with some fixed set; if a set X has cardinal number α, we write $\alpha = |X|$ and $2^\alpha = |\{Y\colon Y \subset X\}|$. Recall that a cardinal number is called *regular* if it is not the sum of fewer, smaller cardinal numbers; otherwise, it is said to be *singular*. The *cofinality* of a limit ordinal β, denoted $\mathrm{cf}(\beta)$, is the smallest ordinal number γ such that there exist ordinal numbers $\delta_\alpha < \beta$, $\alpha \in \gamma$, with $\beta = \sup\{\delta_\alpha\colon \alpha < \gamma\}$. $\mathrm{cf}(\beta)$ is always a regular cardinal number, and an infinite cardinal number κ is regular iff $\mathrm{cf}(\kappa) = \kappa$. The smallest cardinal number greater than a cardinal number α is denoted α^+, and the smallest ordinal number greater than an ordinal number α is denoted $\alpha + 1$.

By a *filter base* $\mathscr{F}$ on a set X one means a nonempty family $\mathscr{F}$ of nonempty subsets of X such that the intersection of any two members of $\mathscr{F}$ contains a member of $\mathscr{F}$. If Y and Z are sets, $f\colon Y \to X$, and $g\colon X \to Z$, then $f^{-1}(\mathscr{F}) = \{f^{-1}(F)\colon F \in \mathscr{F})$ and $g(\mathscr{F}) = \{g(F)\colon F \in \mathscr{F}\}$. If $|\mathscr{F}| \leq \kappa$ and $|F| = \kappa$ for every $F \in \mathscr{F}$, we shall say that $\mathscr{F}$ is of *type* κ. If $A \subset X$ and $F \cap A \neq \emptyset$ for every $F \in \mathscr{F}$, the filter base $\{F \cap A\colon F \in \mathscr{F}\}$ is denoted $\mathscr{F}|A$ and called the *restriction* of $\mathscr{F}$ to A. If $\mathscr{F}$ and $\mathscr{G}$ are filter bases on some set, $\mathscr{G}$ is said to be *finer than* $\mathscr{F}$, written $\mathscr{F} < \mathscr{G}$, if each member of $\mathscr{F}$ contains some member of $\mathscr{G}$. If $\mathscr{H}$ is a family of subsets of a set, $[\mathscr{H}] = \{\bigcap \mathscr{F}\colon \mathscr{F} \subset \mathscr{H}$ and $|\mathscr{F}| < \omega\}$.

In a topological space X, the closure of a set A will be denoted $\mathrm{Cl}_X A$, $\mathrm{Cl}\, A$, $\bar{A}$, or A^-: if a subspace Y of X and a subset B of Y are under consideration, $\bar{B}$ may be written for $\mathrm{Cl}_X B$ but not for $\mathrm{Cl}_Y B$ unless $\mathrm{Cl}_X B = \mathrm{Cl}_Y B$. By an *adherent point* of a filter base $\mathscr{F}$ *in* X one means any point in $\bigcap\{\bar{F}\colon F \in \mathscr{F}\}$. The set of all adherent points of $\mathscr{F}$ in X is called the *adherence of* $\mathscr{F}$ *in* X and is denoted $\mathrm{ad}_X \mathscr{F}$ or $\mathrm{ad}\, \mathscr{F}$. If $x \in X$ and $\mathscr{V} < \mathscr{F}$ for some neighborhood base $\mathscr{V}$ for x, then x is called a *convergent point of* $\mathcal{F}$ *in* X. If γ is an ordinal number, $f: \gamma \to X$ is a mapping, and $x \in X$, then x is said to be a *cluster point of the sequence* f if for each ordinal number $\alpha < \gamma$, $x \in \{f(\beta): \alpha \leqslant \beta\}^-$. If $E \subset X$, a point $x \in X$ is called a *complete accumulation point of* E if $|V \cap E| = |E|$ for every neighborhood V of x in X.

Given sets A and B, the set of all mappings of A into B will be denoted B^A. Thus, for an ordinal number λ, $X^\lambda = \{f: f: \lambda \to X\}$.

In many of the results to be presented, no separation axioms are needed. Thus, unless some axiom is mentioned, the reader should not assume it is required.

Whenever reference is made to a numbered section or result without further identification, it should be understood that the section or result referred to is in this article.

2. Initial κ- and $[\theta, \kappa]$-compactness defined

A topological space X is called $[\theta, \kappa]$-*compact* if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leqslant \kappa$ has a subcover of cardinality $<\theta$. If $\theta = \omega$ then X is called *initially* κ-*compact,* and if $\kappa \geqslant |X|$ then X is called *finally* θ-*compact.*

Thus, a space is countably compact if and only if it is initially ω-compact, and it is Lindelöf if and only if it is finally ω_1-compact. Because a space X is finally ω-compact if and only if it is initially κ-compact for all κ, a finally ω-compact space used to be called 'bicompact' some years ago, but now most mathematicians (and we shall) use the term 'compact' when referring to a space in which every open cover has a finite subcover.

In the case of $[\omega, \kappa]$-, that is, initial κ-compactness, a number of useful conditions other than the one in the definition above can be used to test a space for the presence of compactness-like properties. Before giving them, we need to prove the following technical result.

2.1. LEMMA. *Let* $\mathcal{F}$ *be a filter base of type* κ *on a set* X. *Then there exists a filter base* $\mathcal{H}$ *of type* κ *on* X *such that* $\mathcal{H}$ *is finer than* $\mathcal{F}$, *and for every subset* V *of* X, *if* $|V| < \kappa$ *then there is a set* $H \in \mathcal{H}$ *with* $H \cap V = \phi$.

PROOF. Let $\{\lambda_\alpha | \alpha \in \mathrm{cf}(\kappa)\}$ be a collection of regular cardinals such that $\sup\{\lambda_\alpha | \alpha \in \mathrm{cf}(\kappa)\} = \kappa$. Then there exists a family of filter bases $\{\mathcal{F}(\alpha) | \alpha \in \mathrm{cf}(\kappa)\}$ on X such that

(1) if $V \subset X$ and $|V| < \lambda_\alpha$, then there is a set $H \in \mathcal{F}(\alpha)$ with $H \cap V = \emptyset$; and

(2) each $[\mathcal{F} \cup (\bigcup\{\mathcal{F}(\delta) | \delta \leqslant \alpha\})]$ is of type κ.

For suppose that $\gamma \in \mathrm{cf}(\kappa)$ and filter bases $\mathcal{F}(\alpha)$, $\alpha \in \gamma$, on X of type κ have already been defined so that (1) and (2) hold for all $\alpha \in \gamma$. We wish to define $\mathcal{F}(\gamma)$.

By (2), $\mathcal{G} = [\mathcal{F} \cup (\bigcup\{\mathcal{F}(\beta) | \beta \in \gamma\})]$ is a filter base of type κ, so by a theorem of SIERPIŃSKI [1958, p. 455], there is a family $\mathcal{P}$ of pairwise disjoint subsets of X such that $|\mathscr{P}| = \lambda_\gamma$ and $|P \cap G| = \kappa$ for all $P \in \mathscr{P}$ and $G \in \mathscr{G}$. Let $\{P_\varepsilon | \varepsilon \in \lambda_\gamma\}$ be a one–one indexing of the members of $\mathscr{P}$, and define

$$\mathscr{F}(\gamma) = \{G \cap (\bigcup\{P_\varepsilon | \varepsilon \geqslant \eta\}) | G \in \mathscr{G} \text{ and } \eta \in \lambda_\gamma\}.$$

Then, clearly, (2) holds for all $\alpha \leqslant \gamma$. Furthermore, if $V \subset X$ and $|V| < \lambda_\gamma$, it follows from the regularity of the cardinal λ_γ that for some $\eta \in \lambda_\gamma$, $V \cap (\bigcup\{P_\varepsilon | \varepsilon \geqslant \eta\}) = \emptyset$. Thus (1) holds for all $\alpha \leqslant \gamma$.

Set $\mathscr{H} = [\mathscr{F} \cup (\bigcup\{\mathscr{F}(\alpha) | \alpha \in \mathrm{cf}(\kappa)\})]$. Then by (2), $\mathscr{H}$ is a filter base on X of type κ that is finer than $\mathscr{F}$. Furthermore, if $V \subset X$ and $|V| < \kappa$, with, say, $|V| < \lambda_\alpha$, then it follows from (1) that there is a set $H \in \mathscr{F}(\alpha) \subset \mathscr{H}$ such that $H \cap V = \emptyset$.

In case the cardinal κ in Lemma 2.1 is regular, the proof can be simplified, since then any set F well ordered as $\{x_\alpha : \alpha < \kappa\}$ has the property that for every subset V of F, $|V| < \kappa$ if and only if there exists $\beta < \kappa$ such that $v \leqslant x_\beta$ for all $v \in V$.

2.2. THEOREM. *The following are equivalent for a topological space X and infinite cardinal number κ.*

(i) *Every sequence $\{x_\alpha : \alpha < \delta\}$, $\delta \leqslant \kappa$, of points of X has a cluster point in X.*

(ii) *Every decreasing sequence of nonempty closed subsets of X, well ordered by inclusion as*

$$F_0 \supset F_1 \supset \cdots \supset F_\alpha \supset \cdots, \qquad \alpha < \delta,$$

where $\delta \leqslant \kappa$, has nonempty intersection.

(iii) *For every filter base $\mathscr{H}$ on X, if $|\mathscr{H}| \leqslant \kappa$ then* ad $\mathscr{H} \neq \emptyset$.

(iv) *For every infinite subset E of X, if $|E| \leqslant \kappa$ then E has a complete accumulation point in X.*

(v) *x is initially κ-compact.*

PROOF. (i)$\Rightarrow$(ii). Let $\{F_\alpha : \alpha < \delta\}$ be as in the hypothesis of (ii), and for each $\alpha < \delta$ choose a point $x_\alpha \in F_\alpha$. By (i), the sequence $\{x_\alpha : \alpha < \delta\}$ has a cluster point x. Then for any $\beta < \delta$ and neighborhood V of x, there exists $\gamma \geqslant \beta$ with $x_\gamma \in V$, so $\emptyset \neq V \cap F_\gamma \subset V \cap F_\beta$, that is, $x \in \mathrm{Cl} F_\beta = F_\beta$, and $\bigcap\{F_\alpha : \alpha < \delta\} \neq \emptyset$.

(ii)$\Rightarrow$(iii). Suppose some filter base on X fails to have an adherent point. Let μ be the smallest cardinal for which there exists a filter base $\mathscr{H}$ on X with ad $\mathscr{H} = \emptyset$ and $|\mathscr{H}| = \mu$. We shall prove $\mu > \kappa$.

Well order $\mathscr{H}$ as $\{H_\beta : \beta < \mu\}$. For each $\alpha < \mu$, $F_\alpha \equiv \mathrm{ad}[\{H_\beta : \beta \leqslant \alpha\}] \neq \emptyset$ by the choice of μ. Then by (ii), $\emptyset \neq \bigcap\{F_\alpha : \alpha < \mu\}$ if $\mu \leqslant \kappa$, but since $\bigcap\{F_\alpha : \alpha < \mu\} =$ ad $\mathscr{H} = \emptyset$, one must have $\mu > \kappa$.

(iii)$\Rightarrow$(iv). Let E be an infinite subset of X with $|E| \leqslant \kappa$. Let $\mathscr{H}$ be the filter base on E produced by applying Lemma 2.1 to $\mathscr{F} = \{E\}$. By (iii), $\mathscr{H}$ has an adherent point $p \in H$. Since each subset of E having cardinality $<|E|$ is disjoint with some member of $\mathscr{H}$, it follows from $p \in \bigcap\{\bar{H} : H \in \mathscr{H}\}$ that there does not exist a neighborhood V of p with $|V \cap E| < |E|$, i.e., p is a complete accumulation point of E.

(iv)$\Rightarrow$(v). Suppose X fails to be initially θ-compact for some cardinal member θ. Let μ be the smallest cardinal for which X fails to be initially μ-compact. We shall prove that $\mu > \kappa$.

Let $\mathcal{V}$ be an open cover of X such that $|\mathcal{V}| = \mu$ and $\mathcal{V}$ has no finite subcover. Well order $\mathcal{V}$ as $\{V_\alpha : \alpha < \mu\}$, and note that since $\mathcal{V}$ has no finite subcover and X is initially θ-compact for all $\theta < \mu$, it must be the case that for each $\beta < \mu$, one has $|X \setminus \bigcup\{V_\alpha : \alpha < \beta\}| \geq \mu$. Thus, by transfinite induction there exist points $e_\beta \in X \setminus \bigcup\{V_\alpha : \alpha < \beta\}$, $\beta < \mu$, so that whenever $\delta < \gamma < \mu$, $e_\gamma \neq e_\delta$. Thus, the set $E = \{e_\beta : \beta < \mu\}$ has cardinality μ and since $|E \cap V| < \mu$ for each set V in the open cover $\mathcal{V}$, no point of X is a complete accumulation point of E. By (iv), it follows that $\mu > \kappa$.

(v)$\Rightarrow$(i). let $\{x_\alpha : \alpha < \delta\}$, $\delta \leq \kappa$, be a sequence of points of X. For each $\gamma < \delta$ let $F_\gamma = \mathrm{Cl}\{x_\alpha : \alpha \geq \gamma\}$. Then $\mathcal{V} = \{X \setminus F_\alpha : \alpha < \delta\}$ is a family of open subsets of X with $|\mathcal{V}| \leq \kappa$, and so by (v), if $\bigcup \mathcal{V} = X$, one must also have some $X \setminus F_\alpha = X$, i.e., some $F_\alpha = \emptyset$. Since the latter cannot occur, it follows that $X \setminus \bigcup \mathcal{V} \neq \emptyset$, but any point in $X \setminus \bigcup \mathcal{V}$ is clearly a cluster point of $\{x_\alpha : \alpha < \delta\}$.

In case κ is a singular cardinal, a space can be tested for the presence of initial κ-compactness with the following.

2.3. THEOREM. *Suppose that κ is a singular cardinal number and X is a topological space that is initially θ-compact for every cardinal number $\omega \leq \theta < \kappa$. Then X is initially κ-compact.*

PROOF. Let $\mathcal{V}$ be an arbitrary open cover of X with $|\mathcal{V}| \leq \kappa$. We wish to prove that $\mathcal{V}$ has a finite subcover.

If $|\mathcal{V}| < \kappa$, then we are done, so suppose that $\mathcal{V}$ is well ordered as $\{V_\alpha : \alpha < \kappa\}$, and choose cardinals $\kappa_\beta < \kappa$, $\beta \in \mathrm{cf}(\kappa)$, for which $\sup\{\kappa_\beta\} = \kappa$. For each $\beta \in \mathrm{cf}(\kappa)$ let $W_\beta = \bigcup\{V_\alpha : \alpha < \kappa_\beta\}$, and let $\mathcal{W} = \{W_\beta : \beta \in \mathrm{cf}(\kappa)\}$. Then $\bigcup \mathcal{W} = \bigcup \mathcal{V} = X$ and $|\mathcal{W}| \leq \mathrm{cf}(\kappa) < \kappa$, so $\mathcal{W}$ must have a finite subcover, that is, for some $\beta \in \mathrm{cf}(\kappa)$, $X = W_\beta = \bigcup\{V_\alpha : \alpha < \kappa_\beta\}$. But then it follows from the initial κ_β-compactness of X that $\{V_\alpha : \alpha < \kappa_\beta\}$ and, hence, $\mathcal{V}$ have a finite subcover.

2.4. REMARK. By Theorem 2.3, the smallest cardinal number κ for which a given noncompact space fails to be initially κ-compact must be regular.

3. Basic properties of initially κ-compact and related spaces

Like compactness, initial κ-compactness is preserved by continuous mappings, perfect pre-images, and closed subets. We state these results without proofs.

3.1. THEOREM. *Suppose that $\theta \leqslant \kappa$ and X is a $[\theta, \kappa]$-compact space. Then every closed subset and every continuous image of X is* $[\theta, \kappa]$-compact.

3.2. THEOREM. *Let X be a topological space, and suppose that there exist an initially κ-compact space Y and a closed continuous mapping f of X onto Y such that each $f^{-1}(y)$, $y \in Y$, is initially κ-compact. Then X is initially κ-compact.*

If one considers the shrinking of an infinite discrete space X to a point by a mapping f, it is clear that in Theorem 3.2 the requirement on point inverses cannot be deleted. If there exist initially κ-compact spaces E_0 and E_1 such that the product space $X = E_0 \times E_1$ fails to be initially κ-compact, and if one takes $f = \pi_{E_0}$, it is evident that the word 'closed' cannot be left out of Theorem 3.2; for $\kappa = \omega$, such spaces are constructed in the article by J.E. Vaughan, and for $\kappa > \omega$, their existence is discussed in Sections 5 and 6. Simple examples, e.g., see Example 4.2, show that, in general, one cannot conclude every $[\theta, \kappa]$-compact subspace of a $[\theta, \kappa]$-compact space is a closed subset, but under suitable restrictions, one can do so.

A Hausdorff space X is said to be of *H-pseudocharacter* κ if κ is the smallest cardinal number having the following property: for each point $x \in X$ there exists a family $\mathscr{B}$ of neighborhoods of x such that $|\mathscr{B}| \leqslant \kappa$ and $\{x\} = \bigcap\{\bar{B}: B \in \mathscr{B}\}$.

3.3. THEOREM. *Let X be a space of H-pseudocharacter λ and suppose that F is an initially κ-compact subset of X for some $\lambda \leqslant \kappa$. Then F is a closed subset of X.*

PROOF. Consider any point $x \in \bar{F}$, and choose a family $\mathscr{B}$ of neighborhoods of x with $\{x\} = \bigcap\{\bar{B}: B \in \mathscr{B}\}$ and $|\mathscr{B}| \leqslant \lambda$. By the initial κ-compactness of F, the filter base $[\mathscr{B}]|F$ must have an adherent point $y \in F$. Then $y \in \bigcap\{\bar{B}: B \in \mathscr{B}\} = \{x\}$ and so $x = y \in F$.

The next result is also of interest.

3.4. THEOREM. *Every initially κ-compact space of H-pseudocharacter κ is a regular space of character κ.*

PROOF. Consider any point $x \in X$ and family $\mathscr{B}$ of neighborhoods of x such that $|\mathscr{B}| \leqslant \kappa$ and $\{x\} = \bigcap\{\bar{B}: B \in \mathscr{B}\}$. Let V be an open neighborhood of x. If $\bar{C} \cap (X \setminus V) \neq \emptyset$ for every $C \in \mathscr{C} \equiv [\mathscr{B}]$, then by the initial κ-compactness of $X \setminus V$, $(X \setminus V) \cap \bigcap\{\bar{C}: C \in \mathscr{C}\} \neq \emptyset$. But the latter could not occur since $\bigcap\{\bar{C}: C \in \mathscr{C}\} = \{x\}$ and $x \in V$. Thus, for some $C \in \mathscr{C}$, $V \supset \bar{C}$.

In a search to identify initially κ-compact subspaces of a given initially κ-compact space, the construction below is sometimes quite useful. It extends to higher cardinals the construction 2.13 in Vaughan's article.

3.5. CONSTRUCTION. *Let X be an initially κ-compact space and A any subset of X of cardinality $\leq 2^\kappa$. Then there exists an initially κ-compact subset G of X such that $A \subset G$ and $|G| \leq 2^\kappa$. In case X is a topological group, then G may be taken to be a subgroup of X.*

PROOF. For any subset B of X, we shall denote by B' a set obtained as follows: for each infinite subset I of B such that $|I| \leq \kappa$ choose one complete accumulation point $p_I \in X$ of I and let B' be the union of B and all of the points p_I obtained. Note that if $|B| \leq 2^\kappa$, then $|B'| \leq (2^\kappa)^\kappa \cdot \kappa = 2^\kappa$.

Let $G_0 = A$ and for each ordinal number $\alpha < \kappa^+$ let $G_\alpha = (\bigcup\{G_\beta : \beta < \alpha\})'$. Define $G = \bigcup\{G_\alpha : \alpha < \kappa^+\}$. Then $A \subset G$ and $|G| \leq 2^\kappa \cdot \kappa^+ = 2^\kappa$. To see that G is initially κ-compact, note that if $I \subset G$ and $|I| \leq \kappa$, then by the regularity of κ^+, there must exist $\alpha < \kappa^+$ such that $I \subset G_\alpha$. Thus I has a complete accumulation point in $G_{\alpha+1}$ and hence in G.

In case X is a topological group, one can modify the construction by taking each G_α, $0 < \alpha < \kappa^+$, to be the group generated by $(\bigcup\{G_\beta : \beta < \alpha\})'$, and then the space G will be a subgroup of X of cardinality $\leq 2^\kappa$.

In the study of initially κ-compact spaces, several properties somewhat stronger than initial κ-compactness have proved to be quite useful, particularly when one is trying to determine if a product space is initially κ-compact. Several of these will be defined next, and relationships that hold among them will be examined.

A filter base on a space X is called *total* if every finer filter base has an adherent point in X. Likewise, a subset Y of a space X is called *total* if every filter base on Y has an adherent point in X, or, equivalently, if the filter base $\{Y\}$ is a total filter base on X.

A space X is said to be *totally initially κ-compact* provided that for every filter base $\mathcal{F}$ on X, if $|\mathcal{F}| \leq \kappa$, then there exists a total filter base $\mathcal{G}$ on X such that $\mathcal{F} < \mathcal{G}$ and $|\mathcal{G}| \leq \kappa$. If a space X has the property that every subset of X of cardinality $\leq \kappa$ is a total subset, then X is called *κ-total*, and if every subset of X of cardinality $\leq \kappa$ is contained in a compact subset of X, then X is said to be *κ-bounded*.

One more concept needs to be defined. For any infinite cardinal number θ, $\mu(\theta)$ will denote the set of all ultrafilters $\mathcal{U}$ on θ such that $|U| = \theta$ for all $U \in \mathcal{U}$. The members of $\mu(\theta)$ are called the *uniform ultrafilters* on θ. $\beta(\theta)$ will denote the set of all ultrafilters on θ, and θ^* will be the set of all *free ultrafilters* on $\theta (= \{\mathcal{U} \in \beta(\theta) : \bigcap \mathcal{U} = \emptyset\})$. For a topological space X, ultrafilter $\mathcal{U} \in \theta^*$ and transfinite sequence $f: \theta \to X$, a point $x \in X$ is called a *$\mathcal{U}$-limit of f* provided that for every neighborhood V of x, $\{\alpha < \theta : f(\alpha) \in V\} \in \mathcal{U}$.

Suppose that for each cardinal number $\omega \leq \theta \leq \kappa$, one has selected a uniform ultrafilter $\mathcal{U}_\theta \in \mu(\theta)$. Then a topological space X is called *$\{\mathcal{U}_\theta : \theta \leq \kappa\}$-compact* if for every cardinal number $\omega \leq \theta \leq \kappa$, every transfinite sequence $f: \theta \to X$ has a $\mathcal{U}_\theta$-limit in X.

3.6. DIAGRAM. This diagram indicates implications holding among these concepts for an arbitrary space X and cardinal number κ. Implications denoted with ---- hold if X is a regular space.

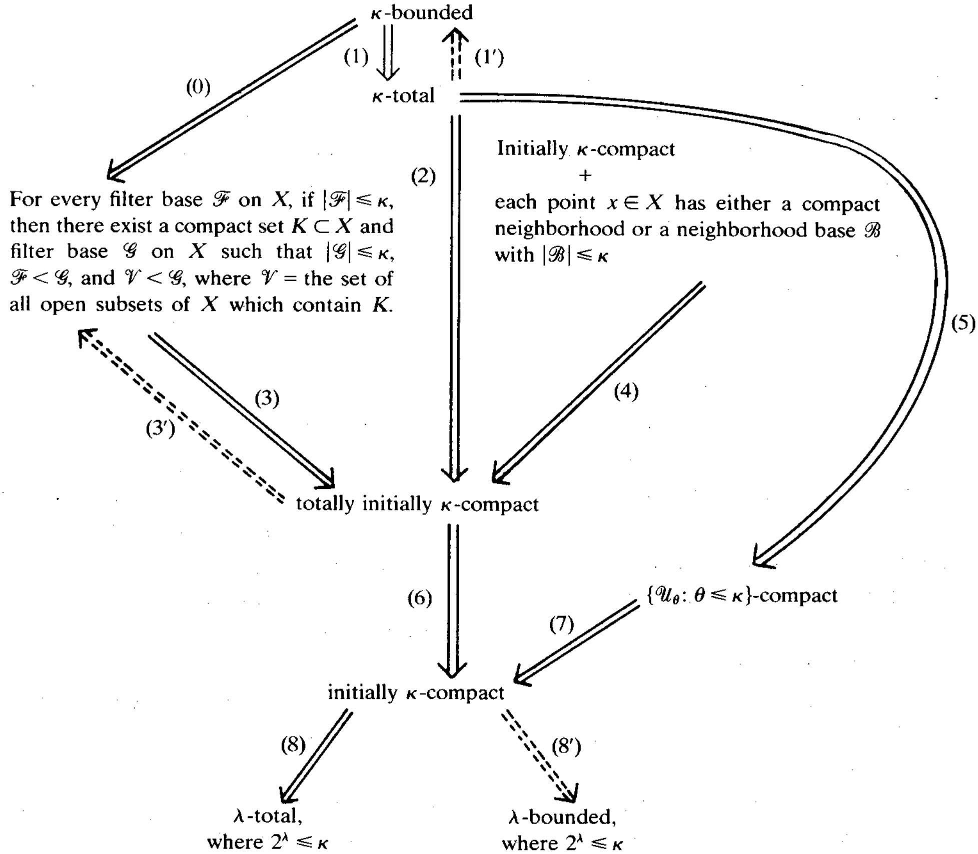

Diagram 3.6.

PROOF. To obtain (2), note that if $\mathscr{F}$ is a filter base on X with $|\mathscr{F}| \leqslant \kappa$ and if one chooses a point $x_F \in F$ for each $F \in \mathscr{F}$, then $\mathscr{F}|\{x_F\colon F \in \mathscr{F}\}$ is a total filter base finer than $\mathscr{F}$, by the κ-totalness of X. For (3), observe that the filter base $\mathscr{G}$ in the statement above (3) is total, because one can show easily that any filter base finer than $\mathscr{G}$ has an adherent point in the compact set K.

To prove (4), consider any filter base $\mathscr{F}$ on X such that $|\mathscr{F}| \leqslant \kappa$ and adherent point $x \in X$ of $\mathscr{F}$. If x has a compact neighborhood K, then $\mathscr{G} = \mathscr{F}|K$ is a finer total filter base with $|\mathscr{G}| \leqslant \kappa$. And if x has a neighborhood base $\mathscr{V}$ with $|\mathscr{V}| \leqslant \kappa$, then $\mathscr{G} = [\mathscr{F} \cup \mathscr{V}]$ is a finer, convergent (hence, total) filter base satisfying $|\mathscr{G}| \leqslant \kappa$.

For (5), note that for any cardinal number $\omega \leqslant \theta \leqslant \kappa$ and transfinite sequence $f\colon \theta \to X$, the set $f(\theta)$ is total and so $f(\mathscr{U}_\theta)$, like any base for an ultrafilter, converges to any of its adherent points, and hence f has a $\mathscr{U}_\theta$-limit in X.

To obtain (7), assume X is $\{\mathscr{U}_\theta\colon \theta \leqslant \kappa\}$-compact. We prove that (iv) of Theorem

2.2 is satisfied. Let $E \subset X$ with $\omega \leqslant \theta = |E| \leqslant \kappa$, and let $f\colon \theta \to E$ be one-to-one. Then any point $x \in X$ which is a $\mathcal{U}_\theta$-limit of f is a complete accumulation point of E, for each $U \in \mathcal{U}_\theta$ has cardinality θ (since $\mathcal{U}_\theta \in \mu(\theta)$), and thus for every neighborhood V of x, $V \cap E \supset f(W)$ for some $W \in \mathcal{U}_\theta$, and so $|V \cap E| = |f(W)| = \theta$.

(0), (1), (6), and (8) are obvious. To prove (1′) and thus (8′), note that by the regularity of X, if $A \subset X$ is total then $\bar{A}$ is compact, for given any filter base $\mathcal{H}$ on $\bar{A}$, if one lets $\mathcal{V} = \{V\colon V$ is open in $\bar{A}$ and $V \supset H$ for some $H \in \mathcal{H}\}$, then by the totalness of A and the regularity of X, we have $\emptyset \neq \operatorname{ad}(\mathcal{V}|A) \subset \operatorname{ad} \mathcal{V} \subset \operatorname{ad} \mathcal{H}$.

Finally, to prove (3′), suppose X is regular and totally initially κ-compact, and let $\mathcal{F}$ be a filter base on x such that $|\mathcal{F}| \leqslant \kappa$. Then choose any finer, total filter base $\mathcal{G}$ on X with $|\mathcal{G}| \leqslant \kappa$, and define $K = \operatorname{ad} \mathcal{G}$. To prove K is compact, it suffices by the regularity of X to prove that if $\mathcal{U}$ is any filter base of open subsets of X such that $U \cap K \neq \emptyset$ for each $U \in \mathcal{U}$, then $\emptyset \neq K \cap \operatorname{ad} \mathcal{U}$. But for any such $\mathcal{U}$,

$$\mathcal{C} = \{U \cap G\colon G \in \mathcal{G} \text{ and } U \in \mathcal{U}\}$$

is a filter base finer than the total filter base $\mathcal{G}$, and so $\emptyset \neq \operatorname{ad} \mathcal{C} \subset \operatorname{ad} \mathcal{U} \cap K$. Thus the statement above (3) holds.

In the next and later sections constructions will be given which show most of the implications in Diagram 3.6 cannot be reversed. The following summarizes what will be proved.

Result, assumptions	show	Implication is *not* reversible
4.8, $\kappa > \omega$ and regular		(0), (2)
4.12, $\kappa > \omega$ and regular		(1)
4.1		(4)
4.5		(5)
4.11		(6)
6.16, $\kappa > \omega$ and regular, GCH		(7)
4.2, GCH		(8), (8′)

Examples 4.1 and 4.11 will also show that for each $\kappa \geqslant \omega$ there exist spaces which are $\{\mathcal{U}_\theta\colon \theta \leqslant \kappa\}$-compact but not totally initially κ-compact and spaces which are κ-bounded but neither initially κ^+-compact, locally compact, nor of character $\leqslant \kappa$.

For additional, related results concerning the case $\kappa = \omega$, we refer the reader to the article by Vaughan.

We shall conclude this section by giving an application of the theory of initial κ-compactness to the theory of cardinal invariants. Before doing so, a definition and lemma are needed.

A T_1-space X is said to be of *pseudocharacter* κ if κ is the smallest cardinal such that each point of X is an intersection of κ or fewer open sets.

3.7. LEMMA. *Let X be a T_1-space of pseudocharacter κ and suppose that Y is a total subset of X. Then every initially κ-compact subspace Z of Y is compact.*

PROOF. Let $\mathcal{F}$ be a maximal filter base of closed subsets of Z. We prove $\bigcap \mathcal{F} \neq \emptyset$. Choose an adherent point $x \in X$ of $\mathcal{F}$, and let $\mathcal{B}$ be a family of open neighborhoods of x in X such that $\bigcap \mathcal{B} = \{x\}$ and $|\mathcal{B}| \leqslant \kappa$. For each $C \in \mathcal{C} = [\mathcal{B}]$, the family of all finite intersections of members of $\mathcal{B}$, it follows from the maximality of $\mathcal{F}$ that there exists a set $F_C \in \mathcal{F}$ with $F_C \subset C$. Then by the initial κ-compactness of Z, $\emptyset \neq \bigcap\{F_C: C \in \mathcal{C}\}$, and since $\bigcap\{F_C: C \in \mathcal{C}\} \subset \bigcap \mathcal{B} = \{x\}$, one has $x \in Z$ and $\emptyset \neq Z \cap \operatorname{ad} \mathcal{F} = \bigcap \mathcal{F}$.

3.8. THEOREM. *If X is a 2^κ-total space of pseudocharacter $\leqslant\kappa$, then $|X| \leqslant 2^\kappa$ and hence X is compact.*

PROOF. For each point $x \in X$ let $\mathcal{B}_x$ be a family of open neighborhoods of x with $|\mathcal{B}_x| \leqslant \kappa$ and $\bigcap \mathcal{B}_x = \{x\}$, and let c be a mapping into X such that $c(A) \in A$ for every nonempty subset A of X. We construct a sequence $\{F_\alpha: \alpha < \kappa^+\}$ of compact subsets of X in the following way.

Choose a point $x \in X$ and let $F_0 = \{x\}$. Suppose β is an ordinal number $<\kappa^+$ and F_α has been defined for every $\alpha < \beta$ so that each F_α is compact and of cardinality $\leqslant 2^\kappa$. If β is a limit ordinal, then $A = \bigcup\{F_\alpha: \alpha < \beta\}$ has cardinality $\leqslant 2^\kappa$, so by Construction 3.5 there exists an initially κ-compact subspace G of X such that $A \subset G$ and $|G| \leqslant 2^\kappa$. Take $F_\beta = G$ and note that F_β is compact by Lemma 3.7 and the 2^κ-totalness of X. Suppose next that $\beta = \gamma + 1$. Let $\mathcal{C}_\gamma = \bigcup\{\mathcal{B}_x: x \in F_\gamma\}$ and $\mathcal{G}_\gamma = \{\mathcal{F}: |\mathcal{F}| < \omega,\ \mathcal{F} \subset \mathcal{C}_\gamma,\ F_\gamma \subset \bigcup \mathcal{F},$ and $\bigcup \mathcal{F} \neq X\}$. Take $A = F_\gamma \cup \{c(X \backslash \bigcup \mathcal{F}): \mathcal{F} \in \mathcal{G}_\gamma\}$. Since $|A| \leqslant 2^\kappa$, we may, as above, find a compact subset G of X such that $A \subset G$ and $|G| \leqslant 2^\kappa$. Then define $F_\beta = G$.

Finally, let $D = \bigcup\{F_\alpha: \alpha < \kappa^+\}$. Then $|D| \leqslant 2^\kappa \cdot \kappa^+ = 2^\kappa$ and D is initially κ-compact, so D is, in fact, compact. To complete the proof, it suffices to show that $D = X$.

Suppose, on the contrary, that there is a point $y \in X \backslash D$. Then there exists a finite collection $\mathcal{F} \subset \bigcup\{\mathcal{B}_x: x \in D\}$ such that $y \notin \bigcup \mathcal{F}$ and $D \subset \bigcup \mathcal{F}$. But then for some $\gamma < \kappa^+$, $\mathcal{F} \in \mathcal{G}_\gamma$, and hence we would have $c(X \backslash \bigcup \mathcal{F}) \notin \bigcup \mathcal{F}$, in contradiction of the fact that $D \subset \bigcup \mathcal{F}$ and $c(X \backslash \bigcup \mathcal{F}) \in F_{\gamma+1} \subset D$.

4. Examples of initially κ-compact spaces

A number of interesting examples, some quite simple, show that most of the concepts of the previous section are distinct. Several also answer questions which have been raised over the years and indicate both limitations and applications of the theorems given in this Handbook. All except the last of the examples given in this section are completely regular and Hausdorff, and several are normal.

4.1. EXAMPLE. Let X be the set of ordinal numbers $<\kappa^+$, with the order topology, and let $P = X^\lambda$, where λ is a cardinal number. Then it is easily shown that P is κ-bounded but not initially κ^+-compact. P is locally compact if and only if $\lambda < \omega$, and P is of character $\leqslant\kappa$ if and only if $\lambda \leqslant \kappa$.

4.2. EXAMPLE. For each ordinal number $\alpha < \kappa^+$ choose a compact space X_α with $|X_\alpha| \geqslant 2$ and fix a point $p_\alpha \in X_\alpha$. Let $X = \{x \in \Pi\{X_\alpha\}: |\{\alpha: x_\alpha \neq p_\alpha\}| \leqslant \kappa\}$. Then X is κ-bounded, and if each X_α is a topological group with identity p_α, the space X is also a topological group. If each X_α is a T_1-space, then X is not initially κ^+-compact.

According to Theorem 3.8, if a 2^κ-total, T_1-space has pseudocharacter $\leqslant\kappa$, then it is compact and has cardinality $\leqslant 2^\kappa$. The next example shows that '2^κ-total' cannot be weakened to 'κ-total'.

4.3. EXAMPLE. For any two infinite cardinal numbers κ and λ, there exists a κ-bounded space X of character $\leqslant\kappa$ which is of cardinality λ. Take X to be the set of all ordinal numbers $\alpha \leqslant \lambda$ such that $\mathrm{cf}(\alpha) \leqslant \kappa$, with the subspace topology X inherits from $[0, \lambda]$, where the latter has the order topology.

PROOF. Since every successor ordinal is in X, $|X| = \lambda$. To see that X has character $\leqslant\kappa$, note that if $\alpha \in X$ is a limit ordinal, there exists $B \subset [0, \alpha)$ with $|B| = \mathrm{cf}(\alpha)$ and $\sup B = \alpha$, so that $\mathscr{B} = \{X \cap (\beta, \alpha]: \beta \in B\}$ is a neighborhood base for α in X of cardinality $\leqslant\kappa$. Other points of X are isolated.

To prove that X is κ-bounded, consider any subset A of X, and let $K = \mathrm{Cl}_X A$. If K fails to be compact, then it cannot be a closed subset of $[0, \lambda]$, so there must exist $\beta \leqslant \lambda$ such that $\mathrm{cf}(\beta) > \kappa$ but $A \cap (\alpha, \beta] \neq \emptyset$ for all ordinals $\alpha < \beta$. But from the latter it would follow that $|A| \geqslant \mathrm{cf}(\beta) > \kappa$.

Note that the space X need not be locally compact, for if there exist $\beta \in X$ and $B \subset [0, \beta) \cap ([0, \lambda] \backslash X)$ such that $\sup B = \beta$, then X fails to be locally compact at β.

The construction given in the previous section, 3.5, can be used to generate lots of examples which show that for every cardinal number κ there exist initially κ-compact spaces (and groups) that are not κ-total.

4.4. EXAMPLE. Let $\{X_\alpha: \alpha < 2^\kappa\}$ be compact Hausdorff spaces such that each $|X_\alpha| \geqslant 2$ and each X_α has a dense subset of cardinality $\leqslant\kappa$. Let $X = \Pi\{X_\alpha: \alpha < 2^\kappa\}$. Then X is a compact Hausdorff space and $|X| \geqslant 2^{2^\kappa}$. It is known (see ENGELKING [1978]) that since each X_α has a dense subset of cardinality $\leqslant\kappa$ there exists a dense subset A of X such that $|A| \leqslant \kappa$. Let G be the initially κ-compact space produced by Construction 3.5. Then $A \subset G$ and $|G| \leqslant 2^\kappa < 2^{2^\kappa}$, so $\mathrm{Cl}_G A$ is a dense, proper subset of X. Thus $\mathrm{Cl}_G A$ is not compact, and G fails to be κ-total.

A construction similar to 3.5 shows that for each cardinal number κ, if

$\mathscr{U}_\theta \in \mu(\theta)$ for each cardinal number $\omega \leqslant \theta \leqslant \kappa$, then by an approach like the one above, one can prove there exist $\{\mathscr{U}_\theta\colon \theta \leqslant \kappa\}$-compact spaces that are not κ-total.

4.5. CONSTRUCTION. *Let κ be a cardinal number and for each cardinal number $\omega \leqslant \theta \leqslant \kappa$, $\mathscr{U}_\theta$ an ultrafilter in $\mu(\theta)$. Let X be a $\{\mathscr{U}_\theta\colon \theta \leqslant \kappa\}$-compact space and A any subset of cardinality $\leqslant 2^\kappa$. Then there exists a $\{\mathscr{U}_\theta\colon \theta \leqslant \kappa\}$-compact subset G of X such that $A \subset G$ and $|G| \leqslant 2^\kappa$. In case X is a topological group, then G may be taken to be a topological group.*

PROOF. For any subset B of X, denote by B' a set obtained as follows: for each cardinal number $\omega \leqslant \theta \leqslant \kappa$ and transfinite sequence $f\colon \theta \to B$, choose one $\mathscr{U}_\theta$-limit point $p_f \in X$ of f, and let B' be the union of B and all of the points p_f obtained. Note that if $|B| \leqslant 2^\kappa$ then $|B'| \leqslant (2^\kappa)^\kappa \cdot \kappa = 2^\kappa$.

Now, as in Construction 3.5, let $G_0 = A$ and for each ordinal number $\alpha < \kappa^+$, let $G_\alpha = (\bigcup\{G_\beta\colon \beta < \alpha\})'$, define $G = \bigcup\{G_\alpha\colon \alpha < \kappa^+\}$. The remainder of the proof is similar to that of 3.5.

The next few examples we need to study are subspaces of the Stone–Čech compactification of a discrete space. It will be convenient to have the following conventions. The cardinal number κ will be topologized with the discrete topology. $\beta(\kappa)$ will denote its Stone–Čech compactification and will be taken to be the set $\kappa \cup \kappa^*$. Thus $x \in \beta(\kappa)$ if and only if x is an ordinal number $<\kappa$ or x is a free ultrafilter on κ. Recall that a base for the topology on $\beta(\kappa)$ is $\{E(B)\colon B \subset \kappa\}$, where for $B \subset \kappa$, $E(B) \equiv B \cup \{\mathscr{U} \in \kappa^*\colon B \in \mathscr{U}\}$. Then for $\alpha < \kappa$, $E(\{\alpha\}) = \{\alpha\}$, i.e., α is an isolated point of $\beta(\kappa)$, and for $\mathscr{U} \in \kappa^*$, a neighborhood base for the point $\mathscr{U}$ is $\{E(B)\colon B \in \mathscr{U}\}$. Because each $\mathscr{U} \in \kappa^*$ is a filter, each $E(B \cap C) = E(B) \cap E(C)$, and because each $\mathscr{U} \in \kappa^*$ is an ultrafilter, one easily sees that $E(B) \cup E(C) = E(B \cup C)$, $E(\kappa \backslash B) = \beta(\kappa) \backslash E(B)$, and $E(B) = \bar{B}$ for all $B, C \subset \kappa$. Note that a point $\mathscr{U} \in \kappa^*$ is in $\mu(\kappa)$ iff for every subset B of κ, $\mathscr{U} \in \bar{B} \Rightarrow |B| = \kappa$.

4.6. LEMMA. *$\beta(\kappa)\backslash\mu(\kappa)$ is an open subspace of $\beta(\kappa)$ which is θ-bounded for each $\theta < \operatorname{cf}(\kappa)$ but fails to be initially κ-compact.*

PROOF. Let $Y \subset \beta(\kappa)\backslash\mu(\kappa)$, where $|Y| = \theta < \operatorname{cf}(\kappa)$. For each point $y \in Y$, choose a subset B_y of κ such that $y \in B_y^-$ and $|B_y| < \kappa$. Let $B = \bigcup\{B_y\colon y \in Y\}$. Then $|B| < \kappa$, so $\bar{B} \subset \beta(\kappa)\backslash\mu(\kappa)$ and $\bar{B}$ is a compact subset of $\beta(\kappa)\backslash\mu(\kappa)$ which contains Y. To see that $\beta(\kappa)\backslash\mu(\kappa)$ is not initially κ-compact, let $\mathscr{H}$ be the filter base of cofinal subsets in Lemma 2.1, whrere $\mathscr{F} = \{\kappa\}$. Then for any point $p \in \beta(\kappa)\backslash\mu(\kappa)$ there exists $V \subset \kappa$ with $p \in \bar{V}$ and $|V| < \kappa$, and so $p \in \bar{V} \Rightarrow p \notin \operatorname{ad} \mathscr{H}$.

REMARK. Whenever $\mathscr{J}$ is a filter base on κ and $\mathscr{U} \in \kappa^*$, the point $\mathscr{U}$ is in the adherence of $\mathscr{J}$ iff $\mathscr{J} \subset \mathscr{U}$.

Our next example will show that *for every regular cardinal number $\kappa > \omega$ there are totally initially κ-compact spaces that are not κ-total.* To present it, some information about certain subsets of κ will be needed.

A subset F of κ is called a *cub set* (short for closed, unbounded set) if $\sup F = \kappa$ and $\sup B \in F \cup \{\kappa\}$ for every subset B of F (or equivalently, if F is a closed, unbounded subset of κ, when κ is given the order topology).

The next lemma is well known to set-theorists. Its proof may be found in K. KUNEN [1980], p. 78 and p. 80.

4.7. LEMMA. *Let $\mathscr{C} = \mathscr{C}_\kappa$ be the family of all cub subsets of κ, where κ is a regular cardinal number $>\omega$. Then the following hold.*

(i) *For any subset I of κ, if $|I| < \kappa$, then $I \cap C = \emptyset$ for some $C \in \mathscr{C}$.*

(ii) *$\mathscr{C}$ is a filter base, and if $\mathscr{D} \subset \mathscr{C}$ and $|\mathscr{D}| < \kappa$, then $\bigcap\mathscr{D} \in \mathscr{C}$.*

(iii) *If $C_\alpha \in \mathscr{C}$ for each ordinal number $\alpha < \kappa$, then the set $\Delta\{C_\alpha : \alpha < \kappa\} \equiv \{\beta \in \kappa : \beta \in C_\alpha$ for all $\alpha < \beta\}$ is in $\mathscr{C}$.*

4.8. THEOREM. *Let κ be a regular, uncountable cardinal number and $\mathscr{C}$ the family of all cub subsets of κ. Denote by V the subspace of $\beta(\kappa)$ defined by*

$$V = \bigcup\{\bar{I} : I \subset \kappa \text{ and } I \cap C = \emptyset \text{ for some } C \in \mathscr{C}\}\,.$$

Then the following hold.

(i) *$\beta(\kappa) \backslash \mu(\kappa) \subset V$.*

(ii) *$V \cap \mu(\kappa)$ is κ-bounded and V is θ-bounded for every $\omega \leqslant \theta < \kappa$.*

(iii) *V is locally compact and initially κ-compact (hence totally initially κ-compact) but not κ-total.*

PROOF. (i) If $x \in \beta(\kappa)\backslash\mu(\kappa)$, either $x \in \kappa$, or $x \in \kappa^*$ and there is a set $I \subset \kappa$ such that $I \in x$ and $|I| < \kappa$. In either case, there exists $I \subset \kappa$ with $|I| < \kappa$ and $x \in \bar{I}$. By (i) of Lemma 4.7, there exists $C \in \mathscr{C}$ with $I \cap C = \emptyset$. Thus $\bar{I} \subset V$ and $x \in V$.

(ii) Suppose $Z \subset V \cup \mu(\kappa)$ and $Z = \{x_\alpha : \alpha < \theta\}$, where $\theta \leqslant \kappa$. For each $\alpha < \theta$ choose $C_\alpha \in \mathscr{C}$ such that $x_\alpha \notin \bar{C}_\alpha$. By Lemma 4.7, if $\theta < \kappa$, then $C = \bigcap\{C_\alpha : \alpha < \theta\} \in \mathscr{C}$, and if $\theta = \kappa$, then $C = \{\beta < \kappa : \beta \in C_\alpha$ for all $\alpha < \beta\} \in \mathscr{C}$. In either case, $C \subset C_\alpha \cup [0, \alpha]$ and $x_\alpha \in \beta(\kappa)\backslash(C_\alpha \cup [0, \alpha])^-$ for each $\alpha < \theta$, so $Z \subset \beta(\kappa)\backslash\bar{C} = (\kappa\backslash C)^-$, a compact set contained in V. The second statement in (ii) follows from the first one and Lemma 4.6.

(iii) Like any open subset of a locally compact space, V is locally compact. Since the filter base $\mathscr{C}$ has an adherent point in the compact space $\beta(\kappa)$, and any ultrafilter $x \in \operatorname{ad}\mathscr{C}$ is a filter on the set κ that has no adherent point in V, the space V fails to be κ-total. Now consider any $E \subset V$ with $\omega \leqslant |E| \leqslant \kappa$. We wish to prove that E has a complete accumulation point in V. If $|E| < \kappa$, then by the $|E|$-boundedness of V, E has a complete accumulation point in V. Suppose $|E| = \kappa$. If $|E \cap \mu(\kappa)| = \kappa$ then E has a complete accumulation point in $V \cap \mu(\kappa)$

by the κ-boundedness of $V \cap \mu(\kappa)$. Likewise, if there exists $I \subset \kappa$ with $|I| < \kappa$ and $|\bar{I} \cap E| = \kappa$, then the compact set $\bar{I} \subset V$ and E has a complete accumulation point in $\bar{I}$. So, to complete the proof, it suffices for us to prove that for any subset E of V, if $|E| = \kappa$, $E \subset V \backslash \mu(\kappa)$, and $|E \cap \bar{I}| < \kappa$ whenever $I \subset \kappa$ and $|I| < \kappa$, then E has a complete accumulation point in V. For such a set E, one can define by transfinite induction a set $C \in \mathscr{C}$ such that $|E \cap (\kappa \backslash C)^-| = \kappa$, as follows. Let $\gamma_0 = 0$. Suppose $0 < \beta < \kappa$ and $\gamma_\alpha < \kappa$ has been defined for each $0 < \alpha < \beta$. If β is a limit ordinal, define $\gamma_\beta = \sup\{\gamma_\alpha : \alpha < \beta\}$ and note that by the regularity of κ, $\gamma_\beta < \kappa$. If not and $\beta = \delta + 1$, choose $x \in E$ such that $x \notin [0, \gamma_\delta]^-$ and define γ_β to be an ordinal number $< \kappa$ such that $x \in (\gamma_\delta, \gamma_\beta)^-$. The set $C = \{\gamma_\alpha : \alpha < \kappa\}$ so obtained satisfies $C \in \mathscr{C}$ and $|E \cap (\kappa \backslash C)^-| = \kappa$. Since $(\kappa \backslash C)^-$ is a compact subset of V, E has a complete accumulation point in V.

In Section 6 some results will be given concerning the relationship between κ-bounded and totally initially κ-compact, where κ is a singular cardinal.

We next study in further detail properties of initially κ-compact subspaces G of $\beta(\kappa)$. The results obtained will show that *for every cardinal number κ, there exist initially κ-compact (in fact, $\{\mathscr{U}_\theta : \theta \leqslant \kappa\}$-compact, for any $\mathscr{U}_\theta \in \mu(\theta)$) subspaces of $\beta(\kappa)$ that fail to be totally initially κ-compact.*

A subset D of a topological space X will be called a Stone–Čech discrete subset of X if $\bar{A} \cap \bar{B} = \emptyset$ whenever A and B are disjoint subsets of D, and D will be called a *strongly discrete subset of* X if there exist pairwise disjoint open subsets $\{V_d : d \in D\}$ of X such that for each $d \in D$, $d \in V_d$. For any subset D of a space X, we extend our use of the symbol 'μ' by defining $\mu(D) = \{p \in X : p$ is a complete accumulation point of $D\}$.

4.9. LEMMA. *Let D be an infinite subset of a space X with $|D| = \theta$.*

(i) *If X is Hausdorff, $|\bar{D}| \leqslant 2^{2^\theta}$.*

(ii) *If $X = \beta(\kappa)$ and D is strongly discrete, then D is Stone–Čech discrete, and if, in addition, $\theta = \kappa$, then $\mu(D) \subset \mu(\kappa)$.*

(iii) *If D is Stone–Čech discrete, and X is compact, $|\mu(D)| \geqslant 2^{2^\theta}$.*

PROOF. For (i), see the article by Hodel. To obtain (ii), first note that if A and B are disjoint subsets of D, there exist disjoint open subsets U and V of $\beta(\kappa)$ containing A and B, and thus $(U \cap \kappa)^-$ and $(V \cap \kappa)^-$ are disjoint closed-and-open subsets of $\beta(\kappa)$ containing A and B. Next, if $p \in \mu(D)$ and $V \subset \kappa$ with $p \in \bar{V}$, then $E = D \cap \bar{V}$ has cardinality κ and $V = \bar{V} \cap \kappa$ has a subset of cardinality κ, namely $\bigcup\{V_d \cap V : d \in E\}$.

To prove (iii), recall that by a result of HAUSDORFF [1936], there exists a sequence of subsets $\{X_\varepsilon : \varepsilon < 2^\theta\}$ of D such that for all disjoint finite subsets F and G of 2^θ,

$$|(\bigcap\{X_\varepsilon : \varepsilon \in F\}) \cap (\bigcap\{D \backslash X_\varepsilon : \varepsilon \in G\})| = \theta .$$

For each function $f: 2^\theta \to \{0, 1\}$ let p_f be an adherent point of

$$[\{X_\varepsilon : f(\varepsilon) = 1\} \cup \{D \backslash X_\varepsilon : f(\varepsilon) = 0\} \cup \{D \backslash G : G \subset D \text{ and } |G| < \theta\}]$$

in X. Then each $p_f \in \mu(D)$ and since the rule $g(f) = p_f$ defines a one-to-one mapping, $|\mu(D)| \geq 2^{2^\theta}$.

4.10. THEOREM. *For any cardinal number $\kappa \geq \omega$ and totally initially κ-compact subspace $\kappa \subset G \subset \beta(\kappa)$ of $\beta(\kappa)$, if $\mathscr{F}$ is a filter base on κ of type κ, then* $|\mathrm{ad}_G \mathscr{F} \cap \mu(\kappa)| = 2^{2^\kappa}$.

PROOF. By Lemma 2.1 there exists a finer filter base $\mathscr{H}$ on κ such that $|\mathscr{H}| \leq \kappa$ and for every subset U of κ, if $|U| < \kappa$, then $U \cap H = \emptyset$ for some $H \in \mathscr{H}$. By hypothesis there exist a compact subset K of G and a filter base $\mathscr{I}$ on κ such that $\mathscr{H} < \mathscr{I}$, $|\mathscr{I}| \leq \kappa$, and $\mathscr{V} < \mathscr{I}$, where $\mathscr{V}$ is the family of all open subsets of G containing K. Since $K = \bigcap\{\bar{W}: W$ is an open subset of $\beta(\kappa)$ containing $K\}$ and $\mathscr{V} < \mathscr{I}$, $\mathrm{ad}_{\beta(\kappa)}\mathscr{I} \subset K \subset G$, and thus $\mathrm{ad}_{\beta(\kappa)}\mathscr{I}$ is a compact subset of G. Since also $\mathscr{H} < \mathscr{I}$, we have $\mathrm{ad}_{\beta(\kappa)}\mathscr{I} \subset \mu(\kappa)$, so that $\mathrm{ad}_{\beta(\kappa)}\mathscr{I} \subset G \cap \mu(\kappa)$. By SIERPINSKI's theorem [1958, p. 455], there exists a family $\mathscr{P}$ of pairwise disjoint subsets of κ such that $|\mathscr{P}| = \kappa$ and $|P \cap I| = \kappa$ for all $I \in \mathscr{I}$ and $P \in \mathscr{P}$. For each $P \in \mathscr{P}$, choose an adherent point p_P of $\mathscr{I}|P$. Then the set $D \equiv \{p_P : P \in \mathscr{P}\}$ has the following properties: $|D| = \kappa$; $D \subset \mathrm{ad}_{\beta(\kappa)}\mathscr{I}$; and D is strongly discrete. Thus $\bar{D} \subset \mathrm{ad}_{\beta(\kappa)}\mathscr{I} \subset (G) \cap \mu(\kappa)$, and by Lemma 4.9, $|\bar{D}| = 2^{2^\kappa}$. So, it follows from $\mathrm{ad}_{\beta(\kappa)}\mathscr{I} = \mathrm{ad}_G \mathscr{I} \subset \mathrm{ad}_G \mathscr{F}$ that $|\mathrm{ad}_G \mathscr{F} \cap \mu(\kappa)| = 2^{2^\kappa}$.

4.11. COROLLARY. *For any cardinal number $\kappa \geq \omega$, if $\mathscr{U}_\theta \in \mu(\theta)$ for each cardinal number $\omega \leq \theta \leq \kappa$, then there exists a $\{\mathscr{U}_\theta: \theta \leq \kappa\}$-compact space G such that G fails to be totally initially κ-compact.*

PROOF. Apply Construction 4.5 to κ and $\beta(\kappa)$.

We conclude this section by giving examples which show that for $\kappa > \omega$ and regular, κ-total does not imply κ-bounded.

4.12. EXAMPLE. Let κ be a regular uncountable cardinal number and let V be the space in Theorem 4.8. Denote by X the Hausdorff space whose points are the same as those of $\beta(\kappa)$ but whose topology is

$$\{S \cup (T \backslash (V \cap \mu(\kappa))): S \text{ and } T \text{ are open subsets of } \beta(\kappa)\}.$$

Because $V \cap \mu(\kappa)$ is κ-bounded and $\beta(\kappa)$ is compact, it is immediate that X is κ-total. Since κ is dense in X and X is not compact, (it has a closed, noncompact subset, namely, $V \cap \mu(\kappa)$), X fails to be κ-bounded.

5. Products of initially κ-compact spaces

As discussed in Vaughan's article, it was discovered in the early 50's that the product of two countably compact spaces can fail to be countably compact. This fact shows, at least for the case $\kappa = \omega$, that when one is forming product spaces, initial κ-compactness may behave quite differently than compactness and fail to be productive. In this section and the next, we focus on the question: When is a product of initially κ-compact spaces initially κ-compact? Special attention will be given to properties defined in Sections 2 and 3 which are only mildly stronger than initial κ-compactness, and which force certain product spaces to be initially κ-compact.

Our first result shows that for cofinally many singular cardinals κ, *every* product of initially κ-compact spaces is initially κ-compact!

Recall that a cardinal number κ is said to be a *strong limit cardinal number* if $2^\lambda < \kappa$ whenever $\lambda < \kappa$. It is well known that there are cofinally many singular, strong limit cardinal numbers (given $\lambda \geqslant \omega$, define $\kappa_0 = \lambda$, $\kappa_{i+1} = 2^{\kappa_i}$, and $\kappa = \sup\{\kappa_i: i \in \omega\}$).

5.1. THEOREM. *Let $\{X_a: a \in A\}$ be a family of initially κ-compact spaces, where κ is a singular, strong limit cardinal number. Then $X = \Pi\{X_a: a \in A\}$ is initially κ-compact.*

PROOF. It suffices by Theorem 2.3 to prove that for each cardinal number $\theta < \kappa$, X is initially θ-compact. Let $\mathscr{F}$ be a filter base on X such that $|\mathscr{F}| \leqslant \theta$. We will prove that ad $\mathscr{F} \neq \emptyset$. For each $F \in \mathscr{F}$ choose a point $p_F \in F$, and set $Z = \{p_F: F \in \mathscr{F}\}$. The filter base $\mathscr{F}|Z$ is contained in an ultrafilter $\mathscr{U}$ on Z, and $|\mathscr{U}| \leqslant 2^{|Z|} < \kappa$. Then for each $a \in A$, $\pi_a(\mathscr{U})$ (the projection of $\mathscr{U}$ onto X_a) is a base for an ultrafilter on the initially κ-compact space X_a, and $|\pi_a(\mathscr{U})| < \kappa$, so $\pi_a(\mathscr{U})$ converges to a point of X_a. Thus $\mathscr{U}$ converges to a point of X and hence $\emptyset \neq \mathrm{ad}\,\mathscr{U} \subset \mathrm{ad}\,\mathscr{F}$.

Since every infinite singular cardinal number is a strong limit cardinal under the assumption of GCH, one can conclude that it is consistent with ZFC for initial κ-compactness to be productive whenever κ is singular (however, see Section 6).

In Section 6 a construction will be given which shows that if GCH holds, then for every regular cardinal $\kappa > \omega$ there exist two initially κ-compact spaces whose product fails to be initially κ-compact.

The next two theorems, when combined, show that if the cardinality of the indexing set is not too large, initially κ-compact spaces that are locally compact or of character $\leqslant\kappa$ have initially κ-compact products.

5.2. THEOREM. *Let X be a totally initially κ-compact space, and suppose that Y is an initially κ-compact space. Then $X \times Y$ is initially κ-compact.*

PROOF. Let $\mathscr{F}$ be a filter base on $X \times Y$ such that $|\mathscr{F}| \leqslant \kappa$. We wish to show that $\operatorname{ad} \mathscr{F} \neq \emptyset$.

By hypothesis there is a total filter base $\mathscr{G}$ on X such that $|\mathscr{G}| \leqslant \kappa$ and $\mathscr{G}$ is finer than $\pi_X(\mathscr{F})$. Then

$$\mathscr{H} = \{F \cap (G \times Y) \colon F \in \mathscr{F} \text{ and } G \in \mathscr{G}\}$$

is a filter base on $X \times Y$ with $|\mathscr{H}| \leqslant \kappa$, and since Y is initially κ-compact, $\pi_Y(\mathscr{H})$ has an adherent point y. Let $\mathscr{W}$ be the family of all neighborhoods of y. Then for all $H \in \mathscr{H}$ and $W \in \mathscr{W}$, we have $H \cap (X \times W) \neq \emptyset$, so for all $F \in \mathscr{F}$, $G \in \mathscr{G}$, and $W \in \mathscr{W}$,

$$\emptyset \neq F \cap (G \times Y) \cap (X \times W) = F \cap (G \times W).$$

Thus,

$$\mathscr{I} = \{F \cap (G \times W) \colon F \in \mathscr{F}, G \in \mathscr{G}, \text{ and } W \in \mathscr{W}\}$$

is a filter base, and since $\pi_X(\mathscr{I})$ is finer than the total filter base $\mathscr{G}$, the filter base $\pi_X(\mathscr{I})$ has an adherent point x. Then for every neighborhood V of x, W of y, $F \in \mathscr{F}$, and $G \in \mathscr{G}$, one has

$$\emptyset \neq F \cap (G \times W) \cap (V \times Y) \subset F \cap (V \times W),$$

which shows that $(x, y) \in \operatorname{ad} \mathscr{F}$.

5.3. LEMMA. *Let $\mathscr{F}$ be a filter base on the product space $X = \Pi\{X_a \colon a \in A\}$. Then $\mathscr{F}$ is total if and only if $\pi_a(\mathscr{F})$ is total on X_a for all $a \in A$.*

PROOF. Let $a \in A$ and suppose $\mathscr{G}$ is a filter base which is finer than $\pi_a(\mathscr{F})$. If $\mathscr{F}$ is total, the finer filter base

$$\mathscr{H} = \{F \cap \pi_a^{-1}(G) \colon G \in \mathscr{G}\}$$

must have an adherent point x, and hence $x_a \in \operatorname{ad} \pi_a(\mathscr{H}) \subset \operatorname{ad} \mathscr{G}$, which shows that $\pi_a(\mathscr{F})$ is total.

Conversely, suppose that $\mathscr{J}$ is a filter base on X and $\pi_a(\mathscr{J})$ is total for each $a \in A$. Let $\mathscr{I}$ be a filter base finer than $\mathscr{J}$. We need to show that $\mathscr{I}$ has an adherent point. There exists an ultrafilter $\mathscr{U}$ on X finer than $\mathscr{I}$, and since each $\pi_a(\mathscr{U})$ is finer than $\pi_a(\mathscr{J})$, each $\pi_a(\mathscr{U})$ converges to some point x_a. Then $x = (x_a) \in \operatorname{ad} \mathscr{U} \subset \operatorname{ad} \mathscr{I}$ and $\mathscr{J}$ is total.

5.4. THEOREM. *Let $X = \Pi\{X_a \colon a \in A\}$ where each X_a is totally initially κ-compact.*

(i) *If $|A| \leqslant \kappa$, then x is totally initially κ-compact.*

(ii) If $|A| \leqslant \kappa^+$, then X is initially κ-compact.

PROOF. We combine the proofs of (i) and (ii) and use transfinite induction. Let $\lambda = |A|$ and $\mathscr{F}$ be a filter base on X such that $|\mathscr{F}| \leq \kappa$. We may assume that $\lambda = A$. Suppose $\beta < \lambda$ and for all $\alpha < \beta$ we have defined filter bases $\mathscr{F}_\alpha$ on X such that

(a) $\mathscr{F} < \mathscr{F}_\alpha$,

(b) $|\mathscr{F}_\alpha| \leq \kappa$ and $\pi_\alpha(\mathscr{F}_\alpha)$ is total on X_α,

(c) if $\gamma < \delta < \beta$, then $\mathscr{F}_\gamma < \mathscr{F}_\delta$.

Then by (a) and (c), $\mathscr{G} = [\mathscr{F} \cup \{\mathscr{F}_\alpha : \alpha < \beta\}]$ is a filter base on X, and by (b), $|\mathscr{G}| \leq \kappa \cdot (|\beta| + 1) \leq \kappa$. Since X_β is totally initially κ-compact, there exists a total filter base $\mathscr{H}$ on X_β such that $|\mathscr{H}| \leq \kappa$ and $\mathscr{H}$ is finer than $\pi_\beta(\mathscr{G})$. Now define $\mathscr{F}_\beta = [\mathscr{G} \cup \pi_\beta^{-1}(\mathscr{H})]$, and note that $\mathscr{F} < \mathscr{F}_\beta$, $|\mathscr{F}_\beta| \leq \kappa$, and $\mathscr{F}_\alpha < \mathscr{F}_\beta$ for all $\alpha < \beta$. Since $\mathscr{H} < \pi_\beta(\mathscr{F}_\beta)$, the latter is total on X_β. This completes the inductive step.

Now define $\mathscr{F}^* = [\bigcup\{\mathscr{F}_\alpha : \alpha < \lambda\}]$. By (c), $\mathscr{F}^*$ is a filter base, and $\pi_\alpha(\mathscr{F}_\alpha) < \pi_\alpha(\mathscr{F}^*)$ for each $\alpha < \lambda$, so by (b) and Lemma 5.3, $\mathscr{F}^*$ is total. Thus $\emptyset \neq \operatorname{ad} \mathscr{F}^* \subset \operatorname{ad} \mathscr{F}$, which proves (ii). In case (i), we have $|\lambda| \leq \kappa$ and hence by (a) and (b), $\mathscr{F}^*$ is a total filter base with $\mathscr{F} < \mathscr{F}^*$ and $|\mathscr{F}^*| \leq \kappa \cdot |\lambda| \leq \kappa$.

Using Diagram 3.6, Lemma 5.3, and Theorems 5.2 and 5.4, one immediately obtains such theorems as the following.

5.5. COROLLARY. *Every product of κ or fewer initially κ-compact spaces, all but one of which are locally compact or of character $\leq\kappa$, is initially κ-compact.*

5.6. COROLLARY. *Every product of κ^+ or fewer initially κ-compact spaces, all of which are locally compact or of character $\leq\kappa$, is initially κ-compact.*

If one replaces 'totally initially κ-compact' with 'κ-total', the restriction on the indexing set's cardinality can be removed.

5.7. THEOREM. *Let $X = \Pi\{X_a : a \in A\}$, where each X_a is κ-total. Then X is κ-total (and hence initially κ-compact).*

PROOF. Let $\emptyset \neq Y \subset X$ with $|Y| \leq \kappa$, and consider any filter base $\mathscr{F}$ on Y. For each $a \in A$, $|\pi_a(Y)| \leq \kappa$, so $\pi_a(Y)$ is total, and $\pi_a(\mathscr{F})$, like any filter base on a total set, is total. Then by Lemma 5.3, $\mathscr{F}$ is total, and hence, Y is also.

As will be seen below, if all of the factors X_a are the same space, κ-total can be replaced by the much weaker condition $\{\mathscr{U}_\theta : \theta \leq \kappa\}$-compact. The usefulness of the latter concept in the theory of initially κ-compact product spaces is illustrated by the next few results.

5.8. LEMMA. *Let $g: X \to Y$ be a continuous mapping of a space X into a space Y, and suppose that $\mathscr{U} \in \theta^*$, $f: \theta \to X$ and $x \in X$ is a $\mathscr{U}$-limit of f in X. Then $g(x)$ is a $\mathscr{U}$-limit of $g \circ f$ in Y.*

5.9. LEMMA. *For a space X and cardinal number κ, the following are equivalent.*

(i) *X is initially κ-compact.*

(ii) *For each cardinal number $\omega \leq \theta \leq \kappa$ and transfinite sequence $f: \theta \to X$, there exists an ultrafilter $\mathcal{U} \in \mu(\theta)$ such that f has a $\mathcal{U}$-limit in X.*

PROOF. The proof that (ii)⇒(i) is similar to the proof of (7) in Diagram 3.6. To see that (i)⇒(ii), assume X is initially κ-compact and $f: \theta \to X$, where $\omega \leq \theta \leq \kappa$. We need to prove that f has a $\mathcal{U}$-limit in X for some $\mathcal{U} \in \mu(\theta)$. According to Lemma 2.1, there exists a filter base $\mathcal{H}$ on θ such that $|\mathcal{H}| \leq \theta$ and for every subset V of θ if $|V| < \theta$, then there is a set $H \in \mathcal{H}$ with $H \cap V = \emptyset$. The filter base $f(\mathcal{H})$ has an adherent point $x \in X$ since $|f(\mathcal{H})| \leq \theta \leq \kappa$ and X is initially κ-compact. Let $\mathcal{W}$ be a neighborhood base at x. For each $W \in \mathcal{W}$, $|f^{-1}(W)| = \theta$, since, otherwise, some $H \in \mathcal{H}$ would have to satisfy $\emptyset = H \cap f^{-1}(W)$, whereas $f(H) \cap W \neq \emptyset$. Thus, $|f^{-1}(W)| = \theta$ for all $W \in \mathcal{W}$ and we may use Zorn's lemma to obtain an ultrafilter $\mathcal{U} \in \mu(\theta)$ which is finer than $f^{-1}(\mathcal{W})$. Then the point x is a $\mathcal{U}$-limit of f.

5.10. LEMMA. *Let $X = \Pi\{X_a: a \in A\}$ be a product space and θ a cardinal number. Suppose $x \in X$, $f: \theta \to X$, and $\mathcal{U} \in \theta^*$. Then x is a $\mathcal{U}$-limit of f if and only if for each $a \in A$, x_a is a $\mathcal{U}$-limit of $\pi_a \circ f$.*

PROOF. Lemma 5.8 establishes the 'only if'. Suppose each x_a is a $\mathcal{U}$-limit of $\pi_a \circ f$, and consider any basic neighborhood of x, say $B = \bigcap\{\pi_{a_i}^{-1}(U_i): i = 1, \ldots, n\}$. Since $f^{-1}(B) = \bigcap\{(\pi_{a_i} \circ f)^{-1}(U_i): i = 1, \ldots, n\}$ is the intersection of a finite number of members of $\mathcal{U}$, $f^{-1}(B) \in \mathcal{U}$ and x is a $\mathcal{U}$-limit of f.

5.11. LEMMA. *For a product space $X = \Pi\{X_a: a \in A\}$, the following are equivalent.*

(i) *X is initially κ-compact.*

(ii) *For every cardinal number $\omega \leq \theta \leq \kappa$ and transfinite sequence $f: \theta \to X$, there exists an ultrafilter $\mathcal{U} \in \mu(\theta)$ such that for every $a \in A$, $\pi_a \circ f$ has a $\mathcal{U}$-limit in X_a.*

Using Lemmas 5.10 and 5.11, several product theorems can be obtained.

5.12. THEOREM. *Let κ be a cardinal number and suppose that for each cardinal number $\omega \leq \theta \leq \kappa$, $\mathcal{U}_\theta \in \mu(\theta)$. Then every product of $\{\mathcal{U}_\theta: \theta \leq \kappa\}$-compact spaces is $\{\mathcal{U}_\theta: \theta \leq \kappa\}$-compact (and hence initially κ-compact).*

5.13. THEOREM. *For a space X and cardinal number κ, the following are equivalent.*

(i) *X^A is initially κ-compact for every nonempty set A.*

(ii) *There exists a transfinite sequence $\{\mathcal{U}_\theta: \theta \leq \kappa\}$ such that X is $\{\mathcal{U}_\theta: \theta \leq \kappa\}$-compact.*

PROOF. By Theorem 5.12, (ii)⇒(i). For the converse, suppose (ii) is false. Then there exists a cardinal number $\omega \leqslant \theta \leqslant \kappa$ such that for every ultrafilter $\mathcal{U} \in \mu(\theta)$, there exists a transfinite sequence $f_{\mathcal{U}}: \theta \to X$ having no $\mathcal{U}$-limit in X. Let $A = \mu(\theta)$ and define $f: \theta \to X^A$ by the rule $(\pi_{\mathcal{U}} \circ f)(\alpha) = f_{\mathcal{U}}(\alpha)$. Then by Lemma 5.11, X^A fails to be initially κ-compact.

We conclude this section with a reduction theorem.

5.14. THEOREM. *For a product space $X = \Pi\{X_a: a \in A\}$ and cardinal number κ, the following are equivalent.*

(i) *X is initially κ-compact.*

(ii) *For every nonempty subset B of A such that $|B| \leqslant 2^{2^\kappa}$, the product space $X_B = \Pi\{X_a: a \in B)$ is initially κ-compact.*

PROOF. By Theorem 3.1, (i)$\Rightarrow$(ii). Conversely, suppose (i) is false. By Lemma 5.11 there exist a cardinal number $\omega \leqslant \theta \leqslant \kappa$ and a transfinite sequence $f: \theta \to X$ such that for every ultrafilter $\mathcal{U} \in \mu(\theta)$ there exists $a_{\mathcal{U}} \in A$ for which $\pi_{a_{\mathcal{U}}} \circ f$ has no $\mathcal{U}$-limit. Choose $B = \{a_{\mathcal{U}}: \mathcal{U} \in \mu(\theta)\}$. Then $|B| \leqslant 2^{2^\theta} \leqslant 2^{2^\kappa}$ by Lemma 4.9. Define $g: \theta \to X_B$ by $(\pi_{a_{\mathcal{U}}} \circ g)(\alpha) = (\pi_{a_{\mathcal{U}}} \circ f)(\alpha)$. By applying Lemma 5.11 to g, θ, and X_B, one can conclude that X_B fails to be initially κ-compact.

6. Further examples, product theorems, and open problems

In this section we study examples and theorems which fill or provide further information about gaps in the theory of initially κ-compact spaces. Some of the results to be developed will require additional axioms such as GCH, and others can be obtained within ZFC.

According to Theorem 5.4, a product space $X = \Pi\{X_a: a \in A\}$ of totally initially κ-compact spaces X_a is (i) totally initially κ-compact if $|A| \leqslant \kappa$ and (ii) initially κ-compact if $|A| \leqslant \kappa^+$. As discussed in Vaughan's article, for the case $\kappa = \omega$, simple examples show that X can fail to be initially ω-compact if $|A| = 2^{2^\omega}$ and X can fail to be totally initially ω-compact if $|A| = 2^\omega$. The next few results indicate some ways in which Theorem 5.4 cannot be improved for $\kappa > \omega$.

In Section 4 an example (Theorem 4.8) was given to show that for every uncountable regular cardinal number κ there exists a locally compact, initially κ-compact subspace V of $\beta(\kappa)$ such that V fails to be κ-total. It will be shown below that V^{2^κ} fails to be totally initially κ-compact, thereby showing that *for $\kappa > \omega$ and regular, the restriction on $|A|$ in Theorem 5.4(i) cannot be weakened to 2^κ.*

In what follows, $h(\kappa)$ denotes the smallest cardinal number which is the cardinality of a family $\mathcal{F} \subset \kappa^\kappa$ with the property that for every filter base $\mathcal{H}$ of type κ on κ there exists $f \in \mathcal{F}$ such that for every $H \in \mathcal{H}$, $f(H)$ contains a final segment of κ.

6.1. LEMMA. $\kappa < h(\kappa) \leqslant 2^{\kappa}$.

PROOF. The first inequality follows from the fact that for each cardinal number $\theta \leqslant \kappa$, V^{θ} is totally initially κ-compact (by Theorems 5.4 and 4.8) but $V^{h(\kappa)}$ is not (by Theorem 6.2). To prove the second inequality, it suffices to show that κ^{κ} satisfies the condition in the definition of $h(\kappa)$. Consider any filter base $\mathscr{H}$ of type κ on κ. By SIERPIŃSKI's theorem [1958, p. 455] there exists a family $\{B_H: H \in \mathscr{H}\}$ of pairwise disjoint subsets of κ such that $B_H \subset H$ and $|B_H| = \kappa$ for all $H \in \mathscr{H}$. By mapping each B_H onto κ, we can construct a function $f \in \kappa^{\kappa}$ such that $f(H) = \kappa$ for all $H \in \mathscr{H}$.

6.2. THEOREM. *If $X = \Pi\{X_\alpha: \alpha < h(\kappa)\}$ is a product space which is totally initially κ-compact, then there exists $\alpha < h(\kappa)$ such that X_α is κ-total.*

PROOF. The proof is by contradiction. Assume for each $\alpha < \kappa$ there exists a set $S_\alpha \subset X_\alpha$ such that $|S_\alpha| \leqslant \kappa$ and S_α is not total. Let $\{x^\alpha_\beta: \beta < \kappa\}$ be a listing of the members of S_α in such a way that each point in S_α appears κ times in the list. Let $\mathscr{F} = \{f_\alpha: \alpha < h(\kappa)\}$ be a subset of κ^{κ} which satisfies the condition in the definition of $h(\kappa)$. Define a sequence $y: \kappa \to X$ by the rule $\pi_\alpha \circ y(\delta) = x^\alpha_{f_\alpha(\delta)}$. By Lemma 2.1, there exists a filter base $\mathscr{S}$ on κ such that $|\mathscr{S}| \leqslant \kappa$ and for every subset V of κ, if $|V| < \kappa$, then $V \cap S = \emptyset$ for some $S \in \mathscr{S}$. Then $y(\mathscr{S})$ is a filter base on X of cardinality $\leqslant \kappa$, so by hypothesis there exists a filter base $\mathscr{G}$ of cardinality $\leqslant \kappa$ which is total and finer than $y(\mathscr{S})$. $y^{-1}(\mathscr{G})$ is a filter base on κ of cardinality $\leqslant \kappa$, and since each subset of κ of cardinality $< \kappa$ misses some member of $\mathscr{S}$, it follows that $y^{-1}(\mathscr{G})$ is of type κ. Hence for some α every $f_\alpha(y^{-1}(G))$ contains a final segment, say (τ_G, κ). By definition of y, one then has $\pi_\alpha(G) \supset \{x^\alpha_\beta: \tau_G < \beta < \kappa\}$ for each $G \in \mathscr{G}$. Thus $\pi_\alpha(G) \supset S_\alpha$ (by the redundant listing of S_α) for all $G \in \mathscr{G}$. But $\pi_\alpha(\mathscr{G})$ is total by Lemma 5.3, and since $\{S_\alpha\}$ is a filter base finer than $\pi_\alpha(\mathscr{G})$, it and thus the set S_α would have to be total, in contradiction of our assumption at the beginning of the proof.

6.3. EXAMPLE. Let κ be regular and uncountable and let V be the space in Theorem 4.8. Then V is locally compact and (also totally) initially κ-compact, but $V^{h(\kappa)}$ fails to be totally initially κ-compact.

For κ regular and uncountable, it is not known if the restriction $|A| \leqslant \kappa^+$ in Theorem 5.4 (ii) can be weakened. In fact, the following problems are unsolved.

6.4. QUESTION. *If κ is an uncountable cardinal number, is every product of totally initially κ-compact spaces initially κ-compact?*

6.5. QUESTION. *If κ is a singular cardinal number, is total initial κ-compactness productive?*

By Theorem 6.2, Question 6.5 has a negative answer iff the following question has one also.

6.6. QUESTION. *If κ is a singular cardinal number, is every totally initially κ-compact space κ-total?*

We understand that an article now being prepared (see P.J. NYIKOS and J.E. VAUGHAN (to appear)) will establish the following result: *It is consistent with ZFC that c be arbitrarily large and for every cardinal number κ with $\kappa^{++} \leqslant c$, there exists a family $\mathcal{P}$ of totally initially κ-compact, sequentially compact, separable, spaces of character $\leqslant\kappa$ such that $|\mathcal{P}| = \kappa^{++}$ and $\Pi\mathcal{P}$ fails to be countably compact.*

We turn now to developing a method for constructing subspaces E_0 and E_1 of $\beta(\kappa)$, where κ has the discrete topology, such that E_0 and E_1 are initially κ-compact but $E_0 \times E_1$ is not. It will be shown that: *for κ regular and uncountable,* GCH *implies such spaces exist*; *and for $\kappa = \omega$,* MA *implies there exist such spaces* E_0 and E_1 *so that, in addition,* E_0 and E_1 *are initially θ-compact for every* $\omega \leqslant \theta < 2^{\omega}$.

The lemmas below show how the spaces E_0 and E_1 will be constructed.

6.7. LEMMA. *Suppose that κ is a cardinal number $\geqslant\omega$ and E_0 and E_1 are subspaces of $\beta(\kappa)$ such that $\kappa \subset E_0 \cap E_1 \subset \beta(\kappa)\backslash\mu(\kappa)$. Then $E_0 \times E_1$ fails to be initially κ-compact.*

PROOF. If $E_0 \times E_1$ were initially κ-compact, then the set $\{(\alpha, \alpha): \alpha < \kappa\}$ would have a complete accumulation point $p \in E_0 \times E_1$, but p would have to be of the form (x, x) for some $x \in \mu(\kappa) \cap E_0 \cap E_1$.

Note that for the case $\kappa = \omega$, the condition in Lemma 6.7 is the same as the requirement that $\omega = E_0 \cap E_1$.

6.8. LEMMA. *Assume κ is a cardinal number, and let E be a subset of $\beta(\omega)$ such that* (i) $\omega \subset E$, (ii) *$E\backslash\omega$ is κ-bounded, and* (iii) *for every infinite subset I of ω, $\bar{I} \cap (E\backslash\omega) \neq \emptyset$. Then E is initially κ-compact.*

PROOF. Let I be an infinite subset of E with $|I| \leqslant \kappa$. If $|I \cap \omega| = |I|$, then $|I| = \omega$, and by (iii), I has a complete accumulation point in E. If $|I \cap \omega| < |I|$ then $|I \cap (E\backslash\omega)| = |I|$ and I has a complete accumulation point in E by (ii).

Notation. For $\kappa \geqslant \omega$, $SD(\kappa)$ will denote the set of all subsets I of $\beta(\kappa)\backslash\mu(\kappa)$ such that $|I| = \kappa$ and I is a strongly discrete subset of $\beta(\kappa)$.

6.9. LEMMA. *Assume κ is a regular cardinal number and let E be a subset of $\beta(\kappa)$ such that* (i) $\beta(\kappa)\backslash\mu(\kappa) \subset E$, (ii) *$E \cap \mu(\kappa)$ is κ-bounded, and* (iii) *for every $I \in SD(\kappa)$, $\bar{I} \cap E \cap \mu(\kappa) \neq \emptyset$. Then E is initially κ-compact.*

PROOF. Let I be an infinite subset of E with $|I| \leqslant \kappa$. We prove that I has a complete accumulation point in E. By (ii), $(I \cap \mu(\kappa))^- \subset E$, so if $|I \cap \mu(\kappa)| = |I|$, then $I \cap \mu(\kappa)$ and hence I have a complete accumulation point in the compact set $(I \cap \mu(\kappa))^-$. Suppose $|I \cap \mu(\kappa)| < |I|$. Let $J = I \backslash \mu(\kappa)$ and note that $|J| = |I|$. If there exists a subset S of $\beta(\kappa) \backslash \mu(\kappa)$ such that $|S| < \kappa$ and $|\bar{S} \cap J| = |J|$, then by the θ-boundedness of $\beta(\kappa) \backslash \mu(\kappa)$ for every $\theta < \kappa$ (use Lemma 4.6), $\bar{S} \subset \beta(\kappa) \backslash \mu(\kappa) \subset E$, and J and hence I have a complete accumulation point in $\bar{S}$. Thus, to complete the proof, it suffices to prove that if K is any subset of $\beta(\kappa) \backslash \mu(\kappa)$ such that $|K| = \kappa$ and $|\bar{S} \cap K| < \kappa$ whenever $S \subset \beta(\kappa) \backslash \mu(\kappa)$ and $|S| < \kappa$, then K has a complete accumulation point in E. By (iii), it is enough to prove that such a K must contain a member Y of $SD(\kappa)$, for any point of $\bar{Y} \cap \mu(\kappa)$ would be a complete accumulation point of K since $K \subset \beta(\kappa) \backslash \mu(\kappa)$. Let $\beta < \kappa$ and suppose we have chosen points $\{k_\alpha : \alpha < \beta\} \subset K$ and pairwise disjoint subsets $\{V_\alpha : \alpha < \beta\}$ of κ such that each $|V_\alpha| < \kappa$ and $k_\alpha \in V_\alpha^-$. There must exist a point $x \in K \backslash \bar{S}$, where $S = \bigcup\{V_\alpha : \alpha < \beta\}$ and also a subset V of $\kappa \backslash \bar{S}$ with $|V| < \kappa$ and $x \in \bar{V}$. Define $k_\beta = x$ and $V_\beta = V$. Thus, by transfinite induction there exist points $\{k_\alpha : \alpha < \kappa\} \subset K$ and pairwise disjoint subsets $\{V_\alpha : \alpha < \kappa\}$ of κ such that $k_\alpha \in V_\alpha^-$ for $\alpha < \kappa$, so K contains a member of $SD(\kappa)$.

6.10. LEMMA. *Let $\kappa > \omega$ and assume* GCH. *Then* $|\beta(\kappa) \backslash \mu(\kappa)| \leqslant \kappa^+$.

PROOF. If $V \subset \kappa$ and $\omega \leqslant |V| = \lambda < \kappa$, then by Lemma 4.9, $|\bar{V}| = 2^{2^\lambda} = 2^{\lambda^+} \leqslant 2^\kappa$. Since $\beta(\kappa) \backslash \mu(\kappa) = \bigcup\{\bar{V} : V \subset \kappa,\ |V| < \kappa\}$, it follows that $|\beta(\kappa) \backslash \mu(\kappa)| \leqslant 2^\kappa \cdot 2^\kappa = 2^\kappa = \kappa^+$.

6.11. LEMMA. (i) $|SD(\omega)| = 2^\omega$. (ii) *Assume* GCH *and $\kappa > \omega$. Then* $|SD(\kappa)| = 2^\kappa$.

PROOF. Since $\beta(\omega) \backslash \mu(\omega) = \omega$, $SD(\omega)$ is the set of all infinite subsets of ω. Thus $|SD(\omega)| = 2^\omega$.

To prove (ii), note first that by Lemma 6.10, $|SD(\kappa)| \leqslant (\kappa^+)^\kappa \leqslant (2^\kappa)^\kappa = 2^\kappa$. Since there are 2^κ subsets of κ of cardinality κ, $|SD(\kappa)| \geqslant 2^\kappa$.

6.12. LEMMA. *Suppose that X is compact, $I \subset X$ is infinite, and $\mathscr{F}$ is a filter base on X such that $|\bar{F} \cap I| = |I|$ for each $F \in \mathscr{F}$. Then I has a complete accumulation point in* ad $\mathscr{F}$.

PROOF. If no point in the compact set ad $\mathscr{F}$ were a complete accumulation point of I, there would exist an open set $V \supset \text{ad}\,\mathscr{F}$ with $|V \cap I| < |I|$, but then it would follow from the compactness of X that $V \supset \bar{F}$ for some $F \in \mathscr{F}$, and we would have $|\bar{F} \cap I| < |I|$.

6.13. LEMMA. *Assume 2^κ is a regular cardinal number, and suppose $\{X_\varepsilon : \varepsilon < 2^\kappa\}$ is a sequence of subsets of κ. Let $B_0 = \{\mathscr{U} \in \mu(\kappa)$: for some $\beta < 2^\kappa$, $\kappa \backslash X_\varepsilon \in \mathscr{U}$ for all*

$\varepsilon \geqslant \beta\}$ and $B_1 = \{\mathcal{U} \in \mu(\kappa)$: for some $\beta < 2^\kappa$, $X_\varepsilon \in \mathcal{U}$ for all $\varepsilon \geqslant \beta\}$. Then B_0 and B_1 are θ-bounded for every $\theta < 2^\kappa$.

PROOF. We prove B_0 has the desired property. Suppose $\emptyset \neq Y \subset B_0$, and $|Y| = \theta < 2^\kappa$. For each $y \in Y$ choose $\beta_y < 2^\kappa$ such that $\kappa \backslash X_\varepsilon \in y$ for all $\varepsilon \geqslant \beta_y$. Since 2^κ is regular, $\beta \equiv \sup\{\beta_y: y \in Y\} < 2^\kappa$. Let $K = \mathrm{ad}[\{\kappa \backslash X_\varepsilon: \varepsilon \geqslant \beta\}]$. Then K is a closed subset of $\beta(\kappa)$, so $K \cap \mu(\kappa)$ is a compact subset of B_0 which contains Y.

REMARK. Since MA implies 2^ω is regular and GCH implies $2^\kappa = \kappa^+$, which is regular, Lemma 6.13 can be used if one assumes MA and $\kappa = \omega$ or GCH.

Notation. Given an ordinal α, $H(\alpha)$ will denote the set of all mappings f into $\{0, 1\}$ such that the domain of f is a finite subset of α. Given a sequence of subsets of κ, $\{X_\varepsilon: \varepsilon < \alpha\}$, and a mapping $f \in H(\alpha)$, $X(f)$ denotes $(\bigcap\{X_\varepsilon: f(\varepsilon) = 1\}) \cap \{\bigcap\{\kappa \backslash X_\varepsilon: f(\varepsilon) = 0\})$. For $i = 0$ or 1, we define $\Sigma_i = \{x: 2^\kappa \to \{0, 1\}$: for some $\beta < 2^\kappa$, $x(\alpha) = i$ for all $\alpha \geqslant \beta\}$.

6.14. LEMMA. *Let κ be a regular cardinal number, and assume* MA *if $\kappa = \omega$ or* GCH *if $\kappa > \omega$. Let $SD(\kappa) = \{I_\varepsilon: \varepsilon < 2^\kappa\}$. Then there exist points $\{x_{i,\varepsilon}: \varepsilon < 2^\kappa\} \subset \Sigma_i$, $i < 2$, and subsets $\{X_\varepsilon: \varepsilon < 2^\kappa\}$ of κ such that for each $I_\varepsilon \in SD(\kappa)$, $i < 2$, and $f \in H(2^\kappa)$, if $f \subset x_{i,\varepsilon}$ then $|X(f)^- \cap I_\varepsilon| = \kappa$.*

PROOF. The proof is by transfinite induction on 2^κ. Let $\beta < 2^\kappa$ be an ordinal number, and suppose that subsets $\{X_\varepsilon: \varepsilon < \beta\}$ of κ and points $\{x_{i,\varepsilon}: \varepsilon < \beta\} \subset \Sigma_i$, $i < 2$, have been chosen so that for each $\alpha < \beta$, the statement

(S_α) for each $\varepsilon \leqslant \alpha$, $i < 2$, and $f \in H(\alpha + 1)$, if $f \subset x_{i,\varepsilon}$ then $|X(f)^- \cap I_\varepsilon| = \kappa$

holds. We wish to construct $X_\beta \subset \kappa$ and $x_{i,\beta} \in \Sigma_i$, $i < 2$, so that (S_β) is true.

Suppose $\beta = 0$. In this case, for $i < 2$, define $x_{i,0} \in \Sigma_i$ to be a constant function, i.e., $x_{i,0}(\alpha) = i$ for all $\alpha < 2^\kappa$. By Lemma 4.9, $|\mu(I_0)| \geqslant 2$, so there exist disjoint subsets A and B of κ such that $|\bar{A} \cap I_0| = \kappa = |\bar{B} \cap I_0|$. Define $X_0 = A$. Then (s_0) is true.

Suppose $\beta > 0$. Take and fix some point $p \in \mu(I_\beta)$. Since p is an ultrafilter, for each $\alpha < \beta$, either $X_\alpha \in p$ or $\kappa \backslash X_\alpha \in p$. Define $x: \beta \to \{0, 1\}$ by the rule $x(\alpha) = 1$ iff $X_\alpha \in p$. For $i < 2$, define $x_{i,\beta} \in \Sigma_i$ by

$$x_{i,\beta}(\alpha) = \begin{cases} x(\alpha) & \text{if } \alpha < \beta, \\ i & \text{if } \alpha \geqslant \beta. \end{cases}$$

Next, we define the set X_β, using different approaches according to whether $\kappa = \omega$ (Case 1) or $\kappa > \omega$ (Case 2). For either case, if $i < 2$, $\varepsilon \leqslant \beta$, and $f \in H(\beta)$ with $f \subset x_{i,\varepsilon}$, define $J(i, \varepsilon, f) = X(f)^- \cap I_\varepsilon$, and let $\mathscr{C}$ be the family of all such sets. It follows from the definition of $x_{i,\beta}$, $i < 2$, and the assumption, (S_α) is true for $\alpha < \beta$, that for each $C \in \mathscr{C}$, $|C| = \kappa$ and $C \subset \beta(\kappa) \backslash \mu(\kappa)$.

Case 1. $\kappa = \omega$. $\mathscr{C}$ is a family of infinite subsets of ω and $|\mathscr{C}| \leq \max\{\omega, |\beta|\} < 2^\omega$, so by a well known consequence of MA, there exist disjoint subsets A and B of ω such that $|C \cap A| = \omega = |C \cap B|$ for every $C \in \mathscr{C}$. We define $X_\beta = A$.

Case 2: $\kappa > \omega$. In this case, GCH is assumed, and X_β will be defined by transfinite induction. Since $|\mathscr{C}| \leq \max\{\omega, |\beta|\} < 2^\kappa = \kappa^+$, the members of $\mathscr{C}$ may be listed as $\{C_\alpha: \alpha < \kappa\}$, where each member of $\mathscr{C}$ appears κ times in the listing.

Let $\gamma < \kappa$ and suppose that we have defined subsets A_α and B_α of κ, $\alpha < \gamma$, so that whenever $\rho < \gamma$ and $\delta < \gamma$,

(a) $A_\rho \cap B_\delta = \emptyset$;

(b) $|A_\rho| < \kappa$ and $|B_\rho| < \kappa$;

(c) $A_\rho \cap A_\delta = \emptyset = B_\rho \cap B_\delta$ if $\rho \neq \delta$; and

(d) there exist points p_ρ and q_ρ in C_ρ such that $p_\rho \in A_\rho^-$ and $q_\rho \in B_\rho^-$.

We wish to define A_γ and B_γ so that (a)–(d) hold for all $\rho \leq \gamma$ and $\delta \leq \gamma$. Let $A' = \bigcup\{A_\alpha: \alpha < \gamma\}$ and $B' = \bigcup\{B_\alpha: \alpha < \gamma\}$, and note that by the regularity of κ, $|A' \cup B'| < \kappa$. Since $C_\gamma \in SD(\kappa)$, $\mu(C_\gamma) \subset \mu(\kappa)$ by Lemma 4.9. Let $x \in \mu(C_\gamma)$. Then $x \notin (A' \cup B')^-$, so there exists $V \subset \kappa$ with $x \in \bar{V}$ and $V \cap (A' \cup B') = \emptyset$. Take and fix two points p_γ and q_γ in $\bar{V} \cap C_\gamma$. There exist disjoint subsets A_γ and B_γ of V such that $p_\gamma \in A_\gamma^-$, $q_\gamma \in B_\gamma^-$, $|A_\gamma| < \kappa$, and $|B_\gamma| < \kappa$. With p_γ, q_γ, A_γ, and B_γ so chosen, the conditions (a)–(d) hold as desired. Thus, there exist subsets $\{A_\alpha: \alpha < \kappa\}$ and $\{B_\alpha: \alpha < \kappa\}$ of κ such that for all ρ, $\delta < \kappa$, (a)–(d) hold.

Let $A = \bigcup\{A_\alpha: \alpha < \kappa\}$ and $B = \bigcup\{B_\alpha: \alpha < \kappa\}$. By (a)–(d) and the redundant listing of the members of $\mathscr{C}$, A and B are disjoint subsets of κ such that for each $C \in \mathscr{C}$, $|\bar{A} \cap C| = \kappa = |\bar{B} \cap C|$. Define $X_\beta = A$.

Then, whether $\kappa = \omega$ or $\kappa > \omega$, it follows from the above (and the equality of the closure of the intersection of finitely many subsets of κ with the intersection of their closures) that (S_β) holds.

Thus, by transfinite induction there exist points $\{x_{i,\varepsilon}: \varepsilon < 2^\kappa\} \subset \Sigma_i$, $i < 2$, and subsets $\{X_\varepsilon: \varepsilon < 2^\kappa\}$ of κ such that for each $\alpha < 2^\kappa$, $\varepsilon \leq \alpha$, $i < 2$, and $f \in H(\alpha + 1)$, if $f \subset x_{i,\varepsilon}$ than $|X(f)^- \cap I_\varepsilon| = \kappa$, and that completes the proof of Lemma 6.14.

6.15. THEOREM. *Let κ be a regular cardinal number, and assume* MA *if $\kappa = \omega$ or* GCH *if $\kappa > \omega$. Then there exist initially κ-compact subspaces E_0 and E_1 of $\beta(\kappa)$ such that $E_0 \times E_1$ fails to be initially κ-compact, and in case $\kappa = \omega$, E_0 and E_1 can be constructed so that they are initially θ-compact for every $\theta < 2^\omega$.*

PROOF. Let $SD(\kappa) = \{I_\varepsilon: \varepsilon < 2^\kappa\}$ and $\{x_{i,\varepsilon}: \varepsilon < 2^\kappa\} \subset \Sigma_i$, $i < 2$, and $\{X_\varepsilon: \varepsilon < 2^\kappa\}$ be as in Lemma 6.14. With respect to these sets, let B_0 and B_1 be as in Lemma 6.13, and define $E_i = (\beta(\kappa) \backslash \mu(\kappa)) \cup B_i$, $i < 2$. Since $B_0 \cap B_1 = \emptyset$, $E_0 \times E_1$ fails to be initially κ-compact, by Lemma 6.7. To verify that E_0 and E_1 have the other properties stated above, it suffices by Lemmas 6.8, 6.9, and 6.13 to prove that for each $\varepsilon < 2^\kappa$ and $i < 2$, $I_\varepsilon^- \cap B_i \neq \emptyset$. Let $\mathscr{F} = [\{X(f): f \in H(2^\kappa) \text{ and } f \subset x_{i,\varepsilon}\}]$. Then $\mathscr{F}$ is a filter base on $\beta(\kappa)$ such that $|\bar{F} \cap I_\varepsilon| = \kappa$ for every $F \in \mathscr{F}$. By Lemma 6.12, I_ε has a complete accumulation point p in ad $\mathscr{F}$, and since $I_\varepsilon \in \mathrm{SD}(\kappa)$, $p \in \mu(\kappa)$. Because there exists $\beta < 2^\kappa$ such that $x_{i,\varepsilon}(\alpha) = i$ for all $\alpha \geqslant \beta$, ad $\mathcal{F} \subset (\beta(\kappa)\backslash\mu(\kappa)) \cup B_i$. Thus $p \in I_\varepsilon^- \cap B_i$.

6.16. COROLLARY. *Let κ be a regular cardinal number and assume* MA *if $\kappa = \omega$ or* GCH *if $\kappa > \omega$. There exists a space X which is initially κ-compact but not $\{\mathcal{U}_\theta\colon \theta \leqslant \kappa\}$-compact for any choice $\mathcal{U}_\theta \in \mu(\theta)$, $\theta \leqslant \kappa$, and in case $\kappa = \omega$, X can be constructed so that it is initially θ-compact for every $\theta < 2^\omega$.*

PROOF. Let X be the direct sum of E_0 and E_1 and use Theorem 5.13.

6.17. COROLLARY. *Let κ be a singular cardinal number $<\mathfrak{c}$. The following question cannot be answered within* ZFC. *Is initial κ-compactness finitely productive?*

PROOF. Use Theorems 5.1 and the MA part of 6.15.

Results of Eric van Douwen and Theorem 5.15 also establish a similar result for certain singular cardinal numbers $\kappa > \mathfrak{c}$ (see VAN DOUWEN [1982]).

6.18. QUESTION. *Is it true that the only cardinal numbers for which initial κ-compactness is productive are singular, strong limit cardinals?*

7. Notes

The concept of $[\theta, \kappa]$-compactness dates back to the work of P. ALEXANDROFF and P. URYSOHN [1929]; the definition of $[\theta, \kappa]$-compactness given in this article is due to Y.M. SMIRNOV [1950]. In addition to the above articles of Alexandroff and Urysohn and Smirnov, the articles in the References, I.S. GAAL [1957] and [1958] and J.E. VAUGHAN [1974], [1975], and [1978] provide considerable information about $[\theta, \kappa]$-compactness. Lemma 2.1 is due to R.M. STEPHENSON, JR. and J.E. VAUGHAN [1974]. The equivalence of statements (ii), (iv), and (v) in Theorem 2.2 is due to ALEXANDROFF and URYSOHN [1929], p. 20. Theorem 2.3 is noted on p. 177 of STEPHENSON and VAUGHAN [1974]. Theorem 3.1 is implicit in ALEXANDROFF and URYSOHN [1929]. An earlier version of Theorem 3.2 (given for 'perfect mappings') is noted in Z. FROLÍK [1960a]. For the case $\lambda = \omega$, Theorem 3.3 is due to C.E. AULL [1967]. Theorem 3.4 is an extension of results obtained, for $\kappa = \omega$, in ALEXANDROFF and URYSOHN [1929], p. 28, and for $\kappa > \omega$, in VICTOR SAKS [1978], p. 93. Construction 3.5 is due to Saks (see SAKS and STEPHENSON [1971]) and was discovered while Saks was a graduate student working under W.W. Comfort. The concept, κ-bounded, is due to S.L. GULDEN, W.M. FLEISHMAN, and J.H. WESTON [1970], and the concepts, κ-total and totally initially κ-compact, are due to VAUGHAN [1972], [1976], and [1983]. Total filter bases are due to B.J. PETTIS [1969]. $\{\mathcal{U}_\theta\colon \theta \leqslant \kappa\}$-compactness is defined in SAKS [1978]. The implications in Diagram 3.6 give or strengthen results which have appeared in the above mentioned papers of Gulden et al., Saks, Saks and Stephenson, Stephenson and Vaughan, and Vaughan. Lemma 3.7 extends a result of A.A. GRYZLOV [1980], and Theorem 3.8 is a formal extension of GRYZLOV'S result [1980] that every compact T_1-space of pseudocharacter $\leqslant\kappa$ is of cardinality $\leqslant 2^\kappa$. Example 4.1 is mentioned in N. NOBLE [1971], Example 4.2, which is based on examples due to H.H. CORSON [1959], I. GLICKSBERG [1959], and others, is used in SAKS and STEPHENSON [1971]. Example 4.4 appears in Saks and Stephenson. Construction 4.5 is due to Saks and appears in SAKS [1978]. Lemma 4.6 is an extension obtained by ERIC VAN DOUWEN [1983a] of Lemma 4.6 in SAKS and STEPHENSON [1971]. Theorem 4.8 is due to VAN DOUWEN [1983b]. Lemma 4.9(ii) and (iii) strengthen Lemma 3.1 of STEPHENSON and VAUGHAN [1974], and Theorem 4.10 follows from results of, and Corollary 4.11 improves Example 3.9 of the same article. Theorem 5.1 appears in STEPHENSON and VAUGHAN [1974] and improves an earlier result obtained by Stephenson which was published in SAKS and STEPHENSON [1971]. Theorem 5.2 is a special case of Theorem 2 of VAUGHAN [1978] and an extension of an earlier result of STEPHENSON (Theorem 2.2 of STEPHENSON and VAUGHAN [1974]). Lemma 5.3 and Theorem 5.4 are due to VAUGHAN [1976] and [1978]. Corollaries 5.5 and 5.6 generalize: Stone's Theorem 5.5 of C.T. SCARBOROUGH and A.H. STONE [1966]; pp. 379–380 of I. GLICKSBERG [1959]; p. 347 of FROLÍK [1960b]; and Theorem 2.4 of Noble [1971]. Theorems 5.12, 5.13, and 5.14 and most of Lemmas 5.8, 5.9, 5.10, and 5.11 are results or special cases of results due to Saks which appear in SAKS [1978]; for $\kappa = \omega$, Theorem 5.14 improves Stone's reduction theorem in SCARBOROUGH and STONE [1966]. Lemma 6.1, Theorem 6.2, and Example 6.3 are due to VAUGHAN [1983]. Most of Lemmas 6.7 through 6.14, Theorem 6.15, and Corollary 6.17 are results or slight modifications of results due to VAN DOUWEN in [1983a], where additional results of van Douwen are used to show (under the same hypotheses as those of Theorem 6.15) that there exist *normal* initially κ-compact spaces E_0 and E_1 such that $E_0 \times E_1$ fails to be initially κ-compact. A few of the results of this article have not appeared in print previously.

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CHAPTER 14

The Theory of Nonmetrizable Manifolds

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Contents

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HANDBOOK OF SET-THEORETIC TOPOLOGY
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1. Introduction

Manifolds have long occupied center stage in algebraic, differential, and geometric topology. On the other hand, point-set topology has had very little to do with them since its earliest days. Manifolds have been considered too specialized, and the distinctions between them too fine and subtle for the techniques of point-set topology to give us interesting results about them. As far as metrizable manifolds are concerned, this may still be a fair assessment of the situation. However, there is no longer any reason to feel this way about nonmetrizable manifolds, nor about the 'metrization theory' of manifolds, i.e. the problem of what hypotheses are necessary and sufficient for a manifold to be metrizable. Already we have many examples (some of which will be presented in Section 3) of manifolds constructed by methods that are minor variations on constructions found elsewhere in this book. Moreover, these manifolds share many basic properties with the spaces they mimic, making it possible to bring the tools of set theory into play. The structure theory of manifolds in general (see especially Section 5) also provides a fertile ground for the application of set-theoretic concepts and techniques.

Manifolds also provide us with a convenient central category from which to branch out in our study of topological spaces. With a few exceptions (all of which will be pointed out) the results of this article extend to spaces that are locally compact, locally connected, and locally metrizable. However, most results do not use all three hypotheses, at least not fully. Manifolds are: locally compact, locally separable, locally (arcwise) connected, locally Lindelöf, locally ccc, first countable, locally second countable, locally metrizable, and completely regular; they also have bases of countable order. Of course, some of these conditions imply others, but the point is that a weaker one may be 'more right' than a stronger one.

Now it is obviously impractical to master all the different classes of spaces that one might encounter through such a generalization process, but if one has a good understanding of the theory of nonmetrizable manifolds, one can pick up more easily on one class or the other if the need arises. Moreover, many 'proofs' can take on the following format: "The proof of Theorem ______ in [ ] only uses the fact that manifolds are ______, ______, and ______". By the same token, to obtain a manifold with a certain property is obviously more useful than to obtain a space that is merely, say, locally compact and first countable, and has the same property.

A fringe benefit of studying nonmetrizable manifolds is that one is sometimes made aware of unsolved problems concerning more general sorts of spaces. Here are three such problems:

1.1. PROBLEM. Is every normal, locally compact, locally connected space collectionwise normal?

1.2. PROBLEM. In a normal first countable space, is every closed discrete subspace a G_δ?

1.3. PROBLEM. Does there exist a first countable, countably compact, noncompact space which does not contain a copy of ω_1?

The first problem was a natural outgrowth of the Reed–Zenor Theorem 2.21 and is still untouched; even consistency results are lacking. The second was inspired by the result (Corollary 2.16) that a normal manifold is collectionwise Hausdorff (cwH) iff every closed discrete subspace is a G_δ, coupled with the usefulness of the cwH property in Sections 2 and 4. It has an affirmative answer if $V = L$ (becuase then every normal first countable space is collectionwise Hausdorff: see the article by Tall) but its status under MA + $\neg$CH, where we would really like to have it, is unknown. Problem 1.3 will be discussed in Section 6.

In this article we will adopt the convention that manifolds are Hausdorff and connected. We also adopt the convention that 'space' means 'regular T_2 space', so that we say:

1.4. DEFINITION. A *manifold* is a connected space for which there is a positive integer n such that each point has a neighborhood which is homeomorphic to $\mathbb{R}^n$.

'Manifold-with-boundary' is defined similarly; for it, we also allow some points to have neighborhoods homeomorphic to closed half-space $\{\langle x_1, \ldots, x_n\rangle \in \mathbb{R}^n : x_1 \geqq 0\}$.

Section 2 gives some simple ZFC results on cardinal invariants and covering properties, and several variations on the theme of normality and the cwH property. It ends with the result that every locally compact, locally connected normal Moore space is metrizable. Section 3 gives several techniques for building and analyzing manifolds, beginning with the long line and ending in M.E. Rudin's original perfectly normal nonmetrizable manifold assuming the axiom $\diamondsuit$. In Section 4 we give an assortment of applications of Martin's axiom and the negation of the continuum hypothesis, all related to the consistency (also shown by M.E. Rudin) of every perfectly normal manifold being metrizable. Section 5 studies the structure of countably compact surfaces, climaxing with a theorem that every ω-bounded 2-manifold is a connected sum of a metrizable 'bag' and finitely many 'long pipes' (Theorem 5.14). Section 6 has to do with what kinds of long pipes are possible.

For the most part, the various sections can be read independently of each other. In Sections 4 and 5 we do need 2.10–2.12 as well as the techniques in the proof of Theorem 2.9. In Section 4 we also use 2.17–2.20. For Sections 5 and 6 we also need 3.7, 4.3, 4.4, and 4.9. Section 6 uses Lemmas 3.4 and 3.12, the statement of Corollary 5.15, and the discussion following Corollary 5.16. It also has a passing reference to 3.8.

Among the abbreviations employed is 'bd A' for 'the boundary of A,' 'cub' for 'closed unbounded' [always in reference to subsets of ω_1], 'cwH' for 'collectionwise Hausdorff,' and '$x_n \nearrow x$' for '$x_n \to x$ and $x_n < x_{n+1}$ for all n'. We will use $\langle\ ,\ \rangle$

for ordered pairs and finite sequences, reserving parentheses for open intervals.

Owing to lack of space, several results promised for this article in NYIKOS [1982] will appear elsewhere. Both NYIKOS [1982] and SPIVAK [1970], Appendix A, are good supplementary sources of information about nonmetrizable manifolds.

2. Cardinal invariants, covering and separation properties

Some of the most useful information on manifolds comes from simple ZFC results on the different classes of spaces mentioned in the first section. Since every point of a manifold has a metrizable neighborhood, there follows [for the definition of BCO see 6.1 in Gruenhage's article]:

2.1. LEMMA. *Every manifold has a base of countable order* (BCO).

Indeed, every space which is locally a BCO space has a BCO. This was shown by WICKE and WORRELL (1965), along with the results that metrizable and Moore spaces have BCO's, and that a BCO space is developable if and only if it is submetacompact ('θ-refinable') if and only if it is subparacompact ('F_σ-screenable'.)

2.2. COROLLARY. *A manifold is a Moore space if and only if it is submetacompact.*

2.3. COROLLARY. *Every submetacompact manifold is subparacompact.*

2.4. COROLLARY. *A manifold is metrizable if, and only if, it is paracompact.*

Bringing connectedness, local compactness, and local separability into the picture takes us further.

2.5. THEOREM. *Let M be a manifold. The following are equivalent.*

(i) *M is metrizable.*
(ii) *M is paracompact.*
(iii) *M is Lindelöf.*
(iv) *M is meta-Lindelöf.*

PROOF. To show (ii)$\Rightarrow$(iii) we use connectedness and the fact (cf. ENGELKING (1977) p. 382) that every locally compact, paracompact space is the disjoint union of clopen Lindelöf subspaces. (iv)$\Rightarrow$(ii) follows from the fact that every locally separable, meta-Lindelöf space is paracompact: every point-countable cover of a space by separable open subsets is star-countable, and a space in which every open cover has a star-countable open refinement is paracompact (ibid., pp. 404–405).

The equivalence of 'metrizable' and 'Lindelöf' in manifolds is behind the stock assumption of second countability in texts and papers dealing only with metrizable manifolds. A manifold is second countable if and only if it is Lindelöf; more generally:

2.6. LEMMA. *Let M be a manifold. The following cardinal numbers are equal:*

(1) *The weight of M, i.e. the least cardinality of base for the topology on M;*

(2) *The Lindelöf number of M, i.e. the least infinite cardinal κ such that every open cover has a subcover of cardinality $\leqq\kappa$.*

(3) *The least infinite cardinal κ such that M has a cover by $\leqq\kappa$ relatively compact open subsets.*

The proof is routine, but it is perhaps worth noting that a base for M can be found by taking the union of all members of bases for members of an open cover. This is what allows manifold theorists to specify a manifold merely by giving a cover by Euclidean (this word will always mean 'homeomorphic to $\mathbb{R}^n$ for some n') open sets.

Lemma 2.6 extends to subspaces of manifolds, provided one substitutes 'second countable' for 'relatively compact'. A related result is:

2.7. LEMMA. *Let X be a subspace of a manifold. The spread of X is equal to its hereditary density.*

PROOF. We show that if X has a subspace Y with no dense subspace of cardinality $\leqq\kappa$, then X has a discrete subspace of cardinality κ. By induction, choose points y_α of Y for each $\alpha<\kappa$, such that $y_\alpha\notin \mathrm{cl}\{y_\beta\colon \beta<\alpha\}$. For each y_α choose a second countable neighborhood U_α in Y which does not meet $\{y_\beta\colon \beta<\alpha\}$.

Suppose κ is regular. Then, for each $\beta<\kappa$ there exists $\beta'>\beta$ such that for all $\alpha\geqq\beta'$, $y_\alpha\notin\bigcup\{U_\gamma\colon \gamma\leqq\beta\}$. Indeed, this union has weight $|\beta|$ while $\{y_\beta\colon \beta<\kappa\}$ has weight κ since its density and cardinality both equal κ, and density $\leqq$ weight $\leqq$ cardinality for any locally second countable space. It is now a simple matter to choose a discrete "cofinal" subspace of $\{y_\beta\colon \beta<\kappa\}$.

If κ is singular, we take an increasing sequence of regular cardinals $\{k_\alpha\colon \alpha<\lambda\}$ where $\lambda=\mathrm{cof}\,\kappa$. For each $\alpha<\lambda$ let Y_α be a discrete subspace of $\{y_\gamma\colon \kappa_\gamma\leqq\gamma<\kappa_{\gamma+1}\}$ of cardinality $\kappa_{\gamma+1}$. All but $\leqq\kappa_\gamma$ of these miss the neighborhoods U_β of points in the 'earlier' Y_β's so again we have a discrete subspace of cardinality κ.

We have almost finished establishing:

2.8. THEOREM. *Let X be a subspace of a manifold. The following inequalities apply to X:*

$$\textit{cellularity} = \textit{density} \leqq \textit{hereditary density} = \textit{spread}$$
$$\leqq \textit{Lindelöf number} = \textit{weight} \leqq \textit{cardinality}\,.$$

PROOF. All that remains to be shown is cellularity $\geqq$ density. And this is easy: if $\mathcal{U}$ is any infinite maximal disjoint collection of separable open sets, then X has a dense subspace of cardinality $|\mathcal{U}|$.

Local second countability of X was all that was really needed in this theorem. There is a ZFC manifold counterexample to the reversal of the first inequality (example 2). Pertaining to the second, there is the theorem of SZENTMIKLÓSSY [1978], proven in the article by Roitman, that if MA + $\neg$CH then every locally compact space of countable spread is Lindelöf; on the other hand, there is the $\diamondsuit$ example 3.14 in the following section, of a hereditarily separable, non-Lindelöf manifold. A similar example using only CH was given by RUDIN and ZENOR [1976].

As for the third inequality, Euclidean n-space itself shows it does not reverse. Moreover, cardinality is an extremely simple matter where manifolds are concerned.

2.9. THEOREM. *Every manifold is of cardinality* $2^{\aleph_0} = \mathfrak{c}$.

PROOF. Given a manifold M, let U_0 be a Euclidean open subspace of M. With U_β defined for all $\beta < \lambda$ where λ is a limit ordinal, let $U_\lambda = \bigcup\{U_\beta : \beta < \lambda\}$. With U_α defined, let $U_{\alpha+1}$ be an open set containing the closure of U_α, obtained by defining a cover of bd U_α by the least possible number of second countable open sets and taking their union with U_α.

Now $|U_0| = \mathfrak{c}$. If $|U_\alpha| = \mathfrak{c}$, then $|\bar{U}_\alpha| = \mathfrak{c}$ because each point of $\bar{U}_\alpha$ has a sequence from U_α converging to it. Hence $U_{\alpha+1} = \mathfrak{c}$ also. We finish by showing U_{ω_1} is clopen, hence equal to M.

Clearly U_{ω_1} is open. If p is a point in the closure of U_{ω_1}, then p is in the closure of countably many points of U_{ω_1}. But then there exists U_α ($\alpha < \omega_1$) containing these points, whence $p \in U_{\alpha+1} \subset U_{\omega_1}$.

Manifolds which admit a family of open subspaces U_α as in the proof such that $\bar{U}_\alpha$ (hence U_α itself) is Lindelöf for all α have an especially nice structure theory. For the sake of convenience we will refer to them as 'Type I' manifolds. More generally:

2.10. DEFINITION. A space X is *of Type* I if it is the union of an ω_1-sequence $\{U_\alpha : \alpha < \omega_1\}$ of open subspaces such that $\bar{U}_\alpha \subset U_\beta$ whenever $\beta < \alpha$, and such that $\bar{U}_\alpha$ is Lindelöf for all α.

Although 'Type I' is not a hereditary concept in general, it is if the U_α's are hereditarily Lindelöf, as in the case of manifolds. It is easy to see that every Type I manifold is of weight $\leqq \aleph_1$, and also:

2.11. LEMMA. *If X is a subspace of a Type I manifold, then its density equals its weight.*

PROOF. It is easy to see that a Type I manifold M is metrizable (equivalently, second countable) if, and only if, for any sequence $\{U_\alpha : \alpha < \omega_1\}$ as in the definition, $U_\alpha = M$ for some α, and that a subspace X of M is second countable if, and only if, $X \subset U_\alpha$ for some α. This is also easily seen to be equivalent to separability. Since the only possible values for density and weight are $\aleph_0$ and $\aleph_1$, the rest is obvious.

2.12. COROLLARY. *A Type I manifold is metrizable if, and only if, it is separable.*

The remainder of this section is devoted to properties related to normality. Not every manifold is normal even under the conventions we have adopted: see Example 3.7, which is not even pseudonormal. However, it is easy to see that every Type I manifold is pseudonormal, i.e. if we are given a closed subset X and a countable closed subset Y disjoint from it, there are disjoint open subsets containing X and Y, respectively. All we have to do is find α such that $Y \subset U_\alpha$, put $X \cap U_{\alpha+1}$ and Y into disjoint open subsets U and V of $U_{\alpha+1}$ (hence of X), then let $W = U \cup \bar{U}_\alpha^c$, whence V and W are disjoint open sets containing Y and X respectively.

In checking for hereditary normality, etc. we can be guided by the rules of thumb that "if a normality-like property is inherited by open subspaces, it is inherited by all subspaces" and "if every member of a disjoint collection of clopen sets has a normality-like property, so does their union." Thus we have:

2.13. LEMMA. (i) [(ii)]. *A manifold is hereditarily* [*collectionwise*] *normal if, and only if, every open submanifold is* [*collectionwise*] *normal.*

(iii) [(iv)] *A manifold is hereditarily* [*strongly*] *collectionwise Hausdorff if, and only if, every open submanifold is* [*strongly*] *collectionwise Hausdorff.*

Outline of the Proof. See ENGELKING [1977] p. 97 for a proof that a space is normal if, and only if, every open subspace is normal. The proofs of the analogous result for the other three properties are similar. In each case one takes the closure in the original space M of the union of the sets one is trying to separate in the subspace X. The 'bad' points (the ones in the closure of more than one member, or in the closure of the union but not in the closures of the individual members) form a closed subspace K of M that is disjoint from X, and we separate the sets in the open subspace $M - K$.

[*Definition.* A space is [strongly] *collectionwise Hausdorff* if for every closed discrete subspace D there is a disjoint [resp. discrete] collection of open sets covering D, each of which meets D in only one point.]

To go from 'subspace' to 'submanifold' we use the local connectedness of manifolds ("components of open subsets are open") plus the second rule of thumb above, which obviously holds for all four properties.

Apropos of 'open submanifold': it is a fact, often going under the name 'invariance of domain', that a submanifold of an n-manifold is open if, and only if, it is of dimension n. But, while this is occasionally useful even in set-theoretic contexts, we will have no need of this result in this article.

2.14. LEMMA. *A space X is hereditarily collectionwise Hausdorff if, and only if, for each discrete subspace D of X, there exists a family $\{U_d: d \in D\}$ of disjoint open subsets of X such that $U_d \cap D = \{d\}$ for all $d \in D$.*

Indeed, if X is hereditarily collectionwise Hausdorff, and we are given D, we can separate the points of D in the open subspace $(X - \bar{D}) \cup D$ since D is closed in the relative topology. The other implication is even more trivial.

There is a similar result for hereditary collectionwise normality, with a similar proof. Call a collection $\{F_\alpha: \alpha \in A\}$ of subsets of a space X *relatively discrete* if no member has a point in the closure of the union of the other members. Then X is hereditarily collectionwise normal if, and only if, every relatively discrete collection of subsets "can be put into disjoint open sets," i.e. there are open sets $U_\alpha \supset F_\alpha$ such that $U_\alpha \cap U_\beta \neq \emptyset$ if, and only if, $F_\alpha = F_\beta$.

For manifolds we have some interesting equivalents of 'collectionwise Hausdorff', which we will often abbreviate as 'cwH'.

2.15. LEMMA. *If a discrete subspace D of a manifold M is a zero-set, there exists a family $\{U_d: d \in D\}$ of disjoint open subsets of M such that $U_d \cap D = \{d\}$ for all $d \in D$.*

PROOF. Let D be a discrete zero-set and let $f: M \to [0, 1]$ be a continuous function satisfying $D = f^{-1}\{0\}$. For each point $d \in D$ pick a compact, connected neighborhood N_d that does not meet the rest of D. If B_d is the boundary of N_d, then $f(B_d)$ is bounded away from 0.

For each positive integer n, let

$$D_n = \{d \in D: f(B_d) \subset [1/n, 1]\}\,.$$

Pick, for each $d \in D$, and each positive integer m, a connected neighborhood $U_m(d) \subset f^{-1}[0, 1/m)$. If $d \in D_n$, $U_n(d) \subset N_d$ by connectedness of $U_n(d)$. Moreover, $U_n(d) \cap U_m(d') = \emptyset$ for any other $d' \in D$ and $m \geqq n$. For each $d \in D$, let $U_d = U_n(d)$ for the least integer n such that $d \in D_n$. Then the sets U_d are as required.

Later we will see examples (cf. Example 3.7) that show 'zero-set' cannot be weakened to 'closed G_δ' in Lemma 2.5. Of course, the concepts are equivalent in a normal space (cf. ENGELKING [1977], p. 64), so:

2.16. COROLLARY. *Let M be a normal manifold. The following are equivalent.*

(i) *M is collectionwise Hausdorff.*

(ii) *M is strongly collectionwise Hausdorff.*

(iii) *Every closed discrete subspace of M is a G_δ.*

PROOF. It is an elementary exercise to show that every normal cwH space is strongly cwH, and (iii)$\Rightarrow$(i) follows immediately from Lemma 2.15 and the above observation. Finally, (i)$\Rightarrow$(iii) is a routine exercise in any normal space in which each point is a G_δ: first put the points of the closed discrete subspace D into disjoint open sets, and inside each one pick a sequence whose intersection is the unique point of D inside it. Let U_n be the union of the nth terms of all the sequences.

2.17. COROLLARY. *Every subspace of a perfectly normal manifold is strongly* cwH.

PROOF. Perfect normality is a hereditary property (cf. ENGELKING [1977], p. 96), so in the light of Lemma 2.13 it is enough to show every perfectly normal manifold is strongly cwH, and that is immediate from 2.16.

2.18. COROLLARY. *Every separable, perfectly normal manifold is hereditarily separable.*

PROOF. Spread equals cellularity for hereditarily cwH spaces, so this follows from Lemma 2.7.

The proofs of the results beginning with Lemma 2.13 (except for Corollaries 2.16 and 2.18) can be applied to all locally compact, locally connected spaces in place of manifolds. These are also the only results thus far which use local connectedness. A big advance in this area would result from a solution to Problem 1.1, or even:

2.19. PROBLEM. Is every normal manifold collectionwise normal?

Additional information on these problems may be found in NYIKOS [1982] and in the articles by Burke and Tall, where one can find proofs of:

2.20. THEOREM. *Every perfectly normal, locally compact, locally connected space is collectionwise normal with respect to closed submetacompact subsets.*

2.21. THEOREM (REED and ZENOR [1976]). *Every locally compact, locally connected, normal Moore space is metrizable.*

3. Examples

The examples in this section are meant to illustrate, especially towards the end, what a wide variety of manifolds are possible despite the results of Section 2; and to illustrate some of the techniques for constructing manifolds and distinguishing between them.

Often a manifold can best be defined by first defining a related manifold-with-boundary and then modifying that.

3.1. DEFINITION. A *manifold-with-boundary* is a connected space N in which every point has a neighborhood homeomorphic to a Euclidean half-space

$$\{\langle x_1, \ldots, x_n\rangle : x_i \in \mathbb{R} \text{ for all } i, x_1 \geqq 0\}.$$

The set of all points that do not have neighborhoods homeomorphic to $\mathbb{R}^n$ is called *the boundary of* N.

Manifolds are the special case where the boundary is empty. Some texts use 'manifolds' to include manifolds-with-boundary.

It is easy to show that the boundary of an n-manifold is a disjoint union of submanifolds of dimension $n-1$, each clopen in the relative topology of the boundary. [Of course when $n = 1$ we must stretch the definition of 'manifold' to include 'one-point space'.]

3.2. EXAMPLES. Since we are assuming the Hausdorff axiom and connectedness, there are 'really' only four 1-manifolds: the real line, the circle, the (open) long ray, and the long line. If we add manifolds with boundary we only get three more: the closed interval, the half-open interval, and the closed long ray.

The two long rays are the best-known examples of nonmetrizable manifolds-with-boundary. The closed long ray, which can safely be attributed to CANTOR [1883], is formed by adding arcs to join successive countable ordinals. We define, for each countable ordinal α and each real number $r \in (0, 1)$, an element $\alpha + r$ such that $\alpha < \alpha + r < \alpha + s < \alpha + 1$ whenever $r < s < 1$. The closed long ray is the set of all such elements $\alpha + r$ with the order topology. If we remove 0, the remaining subspace is called the (open) long ray and is denoted L^+. The long line L is formed by identifying the zero points of two disjoint copies of the closed long ray.

It is not hard to see that this set of seven is exhaustive. A manifold-with-boundary must contain a subspace homeomorphic to $(0, 1)$, and this can only be continued in one or both of two directions. Unless the two join up to form a circle, the space is linearly orderable. Using local compactness and connectedness, one easily shows that the interval $[x, y]$ between any two points is homeomorphic to the closed unit interval. If the space has an uncountable subset that is either well-ordered or dual to a well order, it must contain a copy of ω_1 by Dedekind completeness. If it contains a copy of ω_1, that copy must either be cofinal or coinitial—there "can be nothing beyond it" without violating first countability. It is now an easy exercise to complete the job of classification.

There are easy 'internal' ways of distinguishing between these seven spaces; for example, L is the only countably compact, noncompact 1-manifold. In higher dimensions, classification is a much more difficult job even for metrizable manifolds. [Recall the Poincaré conjecture!] Many of the criteria we will use are foreshadowed by:

3.3. LEMMA. *Every closed, non-Lindelöf subset of L^+ contains a closed unbounded subset of ω_1.*

PROOF. A subset of L^+ is Lindelöf if, and only if, it is bounded above by some ordinal α; and the closure of any unbounded subset S of L^+ contains an unbounded (and closed) subset of ω_1. Like almost all results on closed unbounded subsets of ω_1, this is shown by a leapfrog argument: given $\alpha_0 \in \omega_1$ choose $s_0 \in S$ such that $s_0 > \alpha_0$, then pick $\alpha_1 \in \omega_1$ such that $\alpha_1 > s_0$, and so forth; the supremum of the α_n is in the closure of S.

This lemma makes one aware of a 'granularity' of L^+ that is only apparent on the global level. There is no way to distinguish ordinals, or even countable sets of ordinals, topologically from other points or even countable subsets of the space. Some might even argue that the ordinals are mere artifacts and have no place in an 'intrinsic' description of the space. Yet the filter of closed unbounded sets ('the cub filter') on ω_1 is really indispensible. On the one hand, ω_1 is itself closed in L^+ [it is a countably compact subspace of a first countable space], so any cub subset of ω_1 is closed and 'unbounded' in L^+; on the other hand, we have Lemma 3.3. These two facts reduce many questions about L^+ to questions about the cub filter on ω_1. For instance, the fact that any countable family of closed, nonmetrizable subsets of L^+ has nonempty intersection is immediate from the same fact for ω_1.

One might think that we have much more freedom in higher dimensions. But consider, for example:

3.4. LEMMA. *In the 2-manifold $L^+ \times \mathbb{R}$:*

(i) *Every closed nonmetrizable subspace contains a (closed) copy of ω_1.*

(ii) *Every copy of ω_1 is, with the exception of at most countably many points, a subset of $L^+ \times \{r\}$ for some $r \in \mathbb{R}$.*

(iii) *For every copy M of L^+ in $L^+ \times \mathbb{R}$, there exists α such that $M \cap [\alpha, \omega_1) \times \mathbb{R} = [\alpha, \omega_1) \times \{r\}$ for some $r \in \mathbb{R}$.*

PROOF. Let X be any non-Lindelöf subset of $L^+ \times \mathbb{R}$. There exists $r \in \mathbb{R}$ such that for some uncountable $Y \subset X$, Y is a subset of $L^+ \times [r, +\infty)$ but Y is not contained in any subspace of the form $(0, \alpha) \times \mathbb{R}$. With Y fixed, let $r = \sup\{q \in \mathbb{Q}: Y \setminus (L^+ \times [q, +\infty)) \text{ is countable}\}$. Then the closure of Y contains $C \times \{r\}$ for some closed unbounded subset C of ω_1. This is a copy of ω_1, hence (i) follows. To show (ii), let X be a copy of ω_1 and use the fact that ω_1 does not contain two disjoint copies of itself. To show (iii), let $X = M$, show by contradiction that all but a Lindelöf subset of M must be contained in $L^+ \times \{r\}$, and use the connectedness of M.

3.5. EXAMPLES. Similar 'unavoidable' subspaces occur in other 2-manifolds that one can construct from 1-manifolds by elementary operations, but there are some important distinctions. In $L^+ \times L^+$, any copy of ω_1 which does not meet the diagonal in a closed unbounded set must, with the exception of a countable subset, be contained in some 'vertical line' $\{x\} \times L^+$ or some 'horizontal line'. However, a copy of ω_1 which 'runs along the diagonal' can deviate from it on an uncountable set. For example,

$$\{\langle\alpha, \alpha+1\rangle : \alpha \in \omega_1\} \cup \{\langle\alpha, \alpha\rangle : \alpha \in \omega_1\}$$

is a (closed) copy of ω_1. We can even connect the points in a 'long stairway' by adding the line segments $\{\alpha\} \times (\alpha, \alpha+1)$ and $(\alpha, \alpha+1) \times \{\alpha+1\}$; this is a copy of L^+.

The special status of the diagonal provides one way of distinguishing between $L^+ \times L^+$ and $L^+ \times L$: the latter contains a pair of disjoint copies of L^+ from which copies of ω_1 can deviate in an uncountable set. The 'state of Utah' formed by removing the closed first quadrant from $L \times L$ contains a set of three such disjoint subspaces, while $L \times L$ contains four. It is easy to manufacture similar-looking spaces with any prescribed finite number of disjoint 'diagonals'. In subsection 3.8 we will use examples with $\aleph_0$ disjoint 'diagonals', while in Section 6 we will take the number up to $\aleph_1$.

3.6. EXAMPLE. Nonmetrizable manifolds quite different from the foregoing can be constructed by using a manifold-with-boundary whose construction is a variation on the well-known tangent disk space. In the closed upper half-plane of $\mathbb{R}^2$ we replace each point $\langle x, 0\rangle$ on the x-axis by a line segment whose underlying set is $\{\langle x, 0\rangle\} \times \mathbb{R}$. Points in the open upper half-plane have the usual base of neighborhoods, while a local base at $\langle x, 0, a\rangle$ is a system of wedges and segments centered on the line of slope $1/a$ emanating from $\langle x, 0\rangle$. [Of course, if $a = 0$, we center on the vertical line.] Define $U_\varepsilon(x, 0, a)$ to be the union of $\langle x, 0\rangle \times (a - \varepsilon, a + \varepsilon)$ with the set of points within ε of $\langle x, 0\rangle$ in the open sector of the upper half-plane bounded by the lines of slope $1/(a - \varepsilon)$ and $1/(a + \varepsilon)$ emanating from $\langle x, 0\rangle$.

It is easy to show that if we replace any point $\langle x, 0\rangle$ by a line segment in this way, what results is a manifold-with-boundary, the boundary being the line segment. The 'narrowing of the wedges to a point' which one might raise as an objection is an illusion which can be dispelled by considering that a closed triangular region in $\mathbb{R}^2$ with the apex removed is homeomorphic to a closed square region with one side removed. [This observation will serve us well in later examples.] What is initially surprising to many is that we can replace all points on the x-axis simultaneously in this way and still have a manifold-with-boundary. Crudely put, the individual attachments do not interfere with one another. The boundary is a discrete collection of $\mathfrak{c}$ closed copies of $\mathbb{R}$, and the space is not metrizable, nor even normal, because of the Jones lemma (cf. 3.5 in Hodel's article, or NYIKOS [1981]). It is a Moore space, because if we let $\mathcal{U}_n$ be the collection of all open disks of radius $1/n$ in the plane, together with all wedge neighborhoods $U_\varepsilon(x, 0, a)$ with $\varepsilon = 1/n$, then $\{\mathcal{U}_n : n \in N\}$ is a development.

Recalling the nonmetrizable normal Moore spaces that exist under MA + $\neg$ CH, one might think that we could obtain another by throwing away all but $\aleph_1$ components of the boundary; but that is defeated by Theorem 2.21. A direct proof can be had in this case by noting that whatever is left of $\mathbb{R} \times \{0\} \times \{0\}$ and $\mathbb{R} \times \{0\} \times \{1\}$ is a pair of disjoint closed sets that cannot be put into disjoint open sets.

3.7. EXAMPLES. There are two elementary ways of obtaining a manifold from a manifold-with-boundary which we can use now. One is the method of collaring: identify each component M_α of the boundary with the subspace $M_\alpha \times \{0\}$ of $M_\alpha \times [0, 1)$, while $M_\alpha \times (0, 1)$ 'stands away' from the rest of the original space. [To be precise: it is an open subspace whose closure is $M_\alpha \times [0, 1)$.] For example, collaring the closed long ray gives the open long ray. If we attach a collar to each $\{\langle x, 0\rangle\} \times \mathbb{R}$ in Example 3.6, we obtain the *Prüfer manifold*, first described in print by RADÓ [1925]. This is a Moore manifold of cellularity $\mathfrak{c}$. Unlike the manifolds thus far considered, it is not of Type I, because it contains a separable nonmetrizable subset.

We can construct a separable, nonmetrizable Moore manifold by taking two disjoint copies of Example 3.6 and identifying the corresponding portions of the boundaries in the two copies. Of course, this technique too can be extended to all manifolds-with-boundary. We have already used it in obtaining the long line from the closed long ray.

3.8. EXAMPLES. We now use Example 3.6 and the 'closed first octant'

$$\{\langle x, y\rangle \in L \times L : 0 \leqq y \leqq x\}$$

to construct, for each uncountable cardinal $\kappa \leqq \mathfrak{c}$, a collection of 2^κ topologically distinct manifolds of weight κ. HODEL [198·] has shown that there are no more than 2^κ topologically distinct Čech complete spaces of weight x altogether, and every locally compact space is Čech complete, so this is optimal. Our examples will all be simply connected 2-manifolds. In themselves they are not especially interesting, and they will not be used later on in this article, except briefly in Section 6. Our main reason for including them is to illustrate how special subspaces like the diagonal of $L^+ \times L^+$ can be used to show that two manifolds are topologically distinct.

We begin with the following observation. Given any two octants, we can either join them by identifying their 'x-axes' ('front to front') or their diagonals ('back to back') or the x-axis of one to the diagonal of the other ('front to back'). The copy A of L^+ that is now common to the two spaces behaves differently in each case. In the first, any copy of ω_1 that meets it in an uncountable set must be 'almost' (i.e.

with the exception of a countable subset) contained in it. This is not true in the other two identification spaces. In the first of these, removal of A leaves two components, each of which can contain uncountably many points of the same copy of ω_1 in the identification space; in this case we call the common copy of L^+ a 'two-sided diagonal'. In the second, every copy of ω_1 in the identification space that 'runs along' the common copy of L^+ can meet one of the components only in a countable set; in this case we call the common copy a 'one-sided diagonal'.

Next, suppose we are given a sequence σ: ω-$\{0\} \to \{1, 2\}$. We construct a manifold M_σ by joining together countably many octants, as follows. We fix one octant as 'the first' and remove its x-axis. If $\sigma(1) = 1$, we attach the first octant to the second 'back to front', identifying the diagonal of the first with the x-axis of the second, and calling this common, one-sided diagonal Δ_1. If $\sigma(1) = 2$, we attach the octants 'back to back' and call the common two-sided diagonal Δ_1. We call the whole manifold-with-boundary that results $M_{\langle\sigma(1)\rangle}$ in either case and fix one octant as its 'last'. (In the first case, it is the one whose diagonal is still 'free'.)

If $M_{\langle\sigma(1),\ldots,\sigma(n)\rangle}$ has been formed, and the 'free' side of its last octant is a diagonal, we proceed as follows. If $\sigma(n+1) = 1$, attach a new octant to this one 'front to back', calling this new manifold $M_{\langle\sigma(1),\ldots,\sigma(n+1)\rangle}$, calling this new octant the last, and labeling as Δ_{n+1} the copy of L^+ which is common to this last octant and $M_{\langle\sigma(1),\ldots,\sigma(n)\rangle}$. If $\sigma(n+1) = 2$, attach a new octant to this one 'back to back', calling the common two-sided diagonal Δ_{n+1}; then attach another octant to the new one 'front to front', calling *it* the last octant of the resulting manifold-with-boundary $M_{\langle\sigma(1),\ldots,\sigma(n+1)\rangle}$.

If the 'free' side of the last octant of $M_{\langle\sigma(1),\ldots,\sigma(n)\rangle}$ is an x-axis, and $\sigma(n+1) = 1$, attach a new octant to this one 'back to front' and proceed as in case $\sigma(n+1) = 1$ above. If $\sigma(n+1) = 2$, attach a new octant to this one 'front to front' and then attach another octant to this new one 'back to back,' calling the common two-sided diagonal Δ_{n+1}, calling the resulting manifold $M_{\langle\sigma(1),\ldots,\sigma(n+1)\rangle}$ and calling this 'other' octant its last octant.

We let M_σ be the union of all these manifolds-with-boundary with the direct limit topology; that is, a subspace is open if, and only if, it is a union of sets each of which is an open subset of $M_{\langle\sigma(1),\ldots,\sigma(n)\rangle}$ for some n. Clearly, this is a simply connected 2-manifold.

Given any M_σ constructed in this way, we can recover the sequence σ that gave rise to it as follows. Call a copy D of the long ray a *diagonal* if it is possible for a copy of ω_1 to both meet it and deviate from it in an uncountable number of elements. If we remove a diagonal (or indeed any copy of L^+) from M_σ, we can remove a metrizable subspace of what is left in such a way that what now remains are two components, exactly one of which contains an infinite set of mutually disjoint diagonals. Also, there will be a diagonal D_0 whose removal divides the space into two components, one of which has no diagonals at all. There is a copy of ω_1 common to D_0 and Δ_0. In the component that does contain diagonals, there will be a diagonal D_1 whose removal divides the component into two components, one of which contains no diagonals. In this way we define D_n for all n and can routinely show that it meets Δ_n in a non-Lindelöf subset.

Let $\tau(i) = 1$ if there is a 'one-sided diagonal' D_1' (defined as above: if a copy W of ω_1 meets it in an uncountable set, there is a component of $M - D_1'$ whose intersection with W is Lindelöf) which meets D_1 in a non-Lindelöf set; otherwise let $\tau(i) = 2$. It is routine to show that $\tau = \sigma$: for example, if $\sigma(n) = 2$, then every copy of L^+ that meets Δ_n in a non-Lindelöf subset is a two-sided diagonal.

Thus it is that distinct sequences give rise to non-homeomorphic manifolds. Now take any subset S of ${}^{\omega}\{1, 2\}$ of cardinality κ. Removing all but κ components of the boundary of Example 3.6, set up a bijection from S to the set of remaining components. For each $\sigma \in S$ we attach the manifold M_σ to the corresponding component so that M_σ 'stands away' from the rest of Example 3.6 and the other 'leaves' M_τ. One method of performing the attachment is to first reattach the interval $(\langle 0, 0\rangle, \langle 1, 0\rangle)$ of the x-axis of the first octant to M_σ, then identify it with the appropriate boundary component of Example 3.6.

Claim. Given any two distinct uncountable subsets S and T of ${}^{\omega}\{1, 2\}$, the resulting manifolds M_S and M_T are not homeomorphic. It follows from this claim that there are 2^κ pairwise non-homeomorphic manifolds of weight κ.

To prove the claim, pick any $\sigma \in S$, and suppose that there is a homeomorphism ψ from M_S to M_T. The removal of Example 3.6 from M_T splits the image of M_σ into the portions in common with each 'leaf' M_τ. Now, it is impossible to produce two nonmetrizable components of M_σ by removing a Lindelöf subspace, and in fact all but a Lindelöf subset of $\psi(M_\sigma)$ must actually be contained in one leaf M_τ. Similarly, the pre-image of M_τ must be contained (except for a Lindelöf subspace) in M_σ. Thus $\psi(M_\sigma) \cap M_\tau$ is a co-Lindelöf subset of M_τ and is homeomorphic to one of M_σ. But the analysis that recaptured σ from M_σ is good enough to recapture it from any co-Lindelöf subset, whence $\sigma = \tau$, and $S \subset T$. And now one sees that S must equal T.

3.9. EXAMPLE. Besides the two techniques mentioned in 3.7, there is another way of producing a Moore manifold from Example 3.6, and that is by identifying each point $\langle x, 0, a\rangle$ with $\langle x, 0, -a\rangle$ in each component of the boundary. This example, due to R.L. Moore himself, was originally described in an entirely different way [1942], [1962]; the description was preceded in both printings by a similar description of the Prüfer manifold.

Unlike the first example in 3.7, this manifold is separable; unlike the second, it is simply connected. But its true significance lies in a way of looking at it that is the key to constructing manifold analogues of many locally compact, locally countable counterexamples, both in ZFC and under various set-theoretic hypotheses.

The manifold can be thought of as the end result of pushing $\mathfrak{c}$ copies of $[0, 1)$ into the open upper half-plane P without in any way affecting the topology of P, which remains as an open subspace of the resulting space. One might imagine the following Fig. 1 to accompany the addition of one copy of $[0, 1)$.

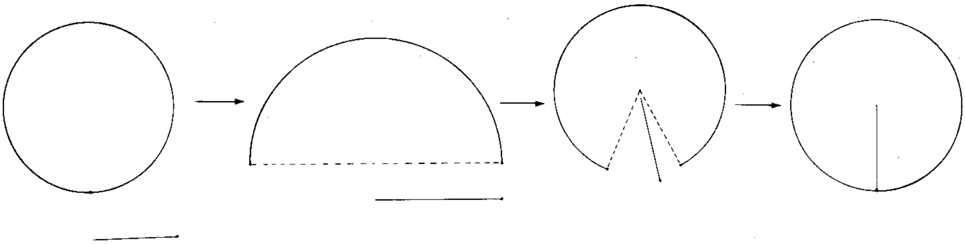

Fig. 1.

There are many details in going from the first picture to the last that can be varied according to the effect one is after. If we are given an infinite closed discrete subspace D of a 'nicked disk' [copy of the closed Euclidean n-ball $\bar{B}^n$, for $n = 2$, with one point removed from the boundary] and D does not meet the boundary, then we can add a copy I of $[0, 1)$ to it as illustrated, and we can make D display "any behaviour with respect to I that is topologically compatible with the last picture." The next few paragraphs will explain what is meant by this.

Let e_1 denote the 'rightmost' point $\langle 1, 0, \ldots, 0\rangle$ of $\bar{B}^n = \{p \in \mathbb{R}^n : d(p, \mathbf{0}) \leqq 1\}$ [we are rotating each picture above to make the notation easier]. If D is any closed discrete subspace of $\bar{B}^n - \{e_1\}$, it converges to e_1, and we can use this fact to set up a homeomorphism of $\bar{B}^n - \{e_1\}$ with

$$B'_n = \bar{B}^n - \{\langle r, 0, \ldots, 0\rangle : 0 \leqq r \leqq 1\}$$

that carries D bijectively onto any closed discrete subspace D' of B'_n and that leaves the boundary [in the sense of Definition 3.1] of $\bar{B}^n - \{e_1\}$ pointwise fixed. The key to demonstrating this is that only finitely many points of D' lie outside any given ε-neighborhood of the positive x-axis. If we think of $\bar{B}^n - \{e_1\}$ and B'_n as onions in which the kth layer is the part between the $1/k$ and $1/k+1$ neighborhoods of e_1 and the non-negative x-axis respectively, then we can construct the desired homeomorphism by a back-and-forth argument involving finitely many layers at a time, each time having to contend with only finitely many points of D and D'. The details are elementary but tedious, and belong more properly to the study of metrizable manifolds, so they are omitted here.

For more sophisticated constructions, one may have to deal with a disjoint family of countably many closed discrete subspaces D_i of $\bar{B}^n - \{e_1\}$ simultaneously, sending each D_i to a given $D'_i \subset B'_n$. We can send all but finitely many elements of D_i to a cofinite subset of D'_i by taking care of the first j members of each family in the jth step of the back-and-forth construction. In this paper we will only need to deal with a special case where it does not matter exactly what D'_i is; all that is required is that the image of each D_i have all of $[0, 1) \times \{0\} \times \cdots \times \{0\}$ in its closure. Moreover, we will only be concerned with the *open* unit n-ball and the homeomorphism we choose will not even have to be extendible to the boundary. This case, for $n = 2$, was essentially handled in the RUDIN–ZENOR paper [1976] and was adapted to smooth manifolds by KOZLOWSKI and ZENOR [1979].

Our next example includes a special case of a variation on this theme: given a closed copy J of $[0, 1)$ in $\bar{B}^n - \{e_1\}$ that does not meet the boundary, it is possible to map $\bar{B}^n - \{e_1\}$ homeomorphically onto B'_n leaving the boundary fixed and sending J onto any given closed copy of $[0, 1)$ in B'_n that does not meet the boundary.

3.10. EXAMPLE (An 'almost Dowker' manifold). B.M. Scott has called a space 'almost Dowker' if it is a regular space that is not countably metacompact. This is equivalent to saying that there is a decreasing sequence $\langle F_n : n \in \omega\rangle$ of closed sets whose intersection is empty and 'which cannot be followed down by open sets,' i.e. given any choice of $U_n \supset F_n$ with U_n open, $\bigcap_{n\in\omega} U_n \neq \emptyset$. We can construct such a manifold with the open first octant of $L \times L$ as our starting point:

$$A = \{\langle x, y\rangle \in L \times L : 0 < y < x\}.$$

The example, paralleling a construction of a zero-dimensional 'almost Dowker' space by DAVIES [1979], is a good illustration of how to 'translate' zero-dimensional constructions to manifolds.

We will add, at each limit ordinal λ, a copy I_λ of $[0, 1)$ in such a way that (1) the vertical line $\{\lambda\} \times [0, \lambda)$ 'converges to the copy of 0 in I_λ'; that is, the one endpoint of I_λ has the property that every neighborhood contains some final segment $\{\lambda\} \times [\alpha, \lambda)$ of this vertical line; and (2) each I_λ in the resulting manifold has a neighborhood whose closure does not meet I_γ for any $\gamma \neq \lambda$. Of course, we also arrange it so that A is an open subset of the resulting manifold M.

Once this is done, it follows that the I_λ are a discrete collection of closed subsets of M. If we take countably many disjoint stationary sets S_n of limit ordinals in ω_1, let $F'_n = \bigcup\{I_\lambda : \lambda \in S_n\}$, and let $F_n = \bigcup_{i=n}^{\infty} F'_i$, then the set of all F_n behaves as described above. To show this, we use the following result, also known as Fodor's lemma. [See the article by Baumgartner].

The pressing-down lemma. If S is stationary subset of ω_1 and $f : S \to \omega_1$ is regressive (i.e. $f(\alpha) < \alpha$ for all $\alpha \in S$), then there exists $\beta \in \omega_1$ such that $f(S') = \{\beta\}$ for some stationary set $S' \subset S$.

Suppose $F'_n \subseteq U_n$, where U_n is open. For each $\lambda \in S_n$ we pick an ordinal $f(\lambda) < \lambda$ such that $\{\lambda\} \times [f(\lambda), \lambda) \subset U_n$. By the pressing-down lemma, there is a stationary set S'_n and an ordinal β_n such that $\{\lambda\} \times [\beta_n, \lambda) \subset U_n$ for all $\lambda \in S'_n$. Let $\beta = \sup\{\beta_n : n \in \omega\}$, then $\langle \lambda, \beta\rangle \in U_n$, for any $\lambda \in S'_n$ such that $\beta < \lambda$, and any n.

We use the pressing-down lemma a second time: given $\langle \lambda, \beta\rangle \in U_n$, choose $g(\lambda) < \lambda$ such that $[g(\lambda), \lambda] \times \{\beta\} \subset U_n$. There is a stationary set $S''_n \subset S'_n$ and an ordinal γ_n such that $[\gamma_n, \lambda] \times \{\beta\} \subset U_n$ for all $\lambda \in S''_n$. Let $\gamma = \sup\{\gamma_n : n \in \omega\}$, then $\langle \gamma, \beta\rangle \in U_n$ for all n.

To actually construct M so that (1) and (2) are satisfied, we choose for each limit ordinal λ a sequence $\langle \lambda_n : n \in \omega\rangle$ of ordinals such that $\lambda_n \nearrow \lambda$, meaning that the λ_n form an increasing sequence whose supremum is λ. We join the point $\langle\lambda_{n+1}, \lambda_n\rangle$ of A to $\langle\lambda_{n+2}, \lambda_{n+1}\rangle$ by a path $f_n: [0,1] \to A$ so that if $0 \leqq r < s \leqq 1$, then both coordinates of $f_n(r)$ are less than both coordinates of $f_n(s)$. We join the point $\langle\lambda + 1/(n+1), \lambda_n\rangle$ to $\langle\lambda + 1/(n+2), \lambda_{n+1}\rangle$ by a path g_n so that if $r < s$ then the first coordinate of $g_n(r)$ is greater than the first of $g_n(s)$, the second smaller than the second of $g_n(s)$. Together with $[\lambda_0, \lambda+1] \times \{\lambda_0\}$, the union of the images of all these paths forms a 'triangle' T_0 with the apex $\langle\lambda, \lambda\rangle$ missing, forming the boundary of a 'nicked disk' R_0 which is closed in A. By similar techniques we can construct, for each $n \in \omega$, a 'triangle' T_n whose bottom is part of the line $[\lambda_n, \lambda + 1/(n+1)] \times \{\lambda_n\}$, whose other sides are closed copies of $[0,1)$ inside the interior of the nicked disk R_{n-1} whose boundary is T_{n-1}, staying on opposite sides of the line $\{\lambda\} \times [0, \lambda)$.

We can now arrange a homeomorphism ψ from R_0 onto $\bar{B}^2 - \{e_1\}$, sending T_n to the 'nicked circle' of radius $1/n+1$ tangent to e_1, and the vertical line $\{\lambda\} \times [\lambda_0, \lambda)$ onto the portion of the x-axis inside $\bar{B}^2 - \{e_1\}$. Let ϕ be a homeomorphism from $B^2 - \{e_1\}$ onto $B'_{\frac{1}{2}}$ leaving the boundary fixed and sending $[0,1) \times \{0\}$ to $[-\frac{1}{2}, 0) \times \{0\}$. Let θ: $B^2 - \{e_1\} \to R_0 \cup I_\lambda$ be the function such that $\theta(p) = \psi^{-1}(\phi^{-1}(p))$ for all $p \in B'_{\frac{1}{2}}$, and $\theta(\langle r, 0\rangle) = r_\lambda$ ('the copy of r in I_λ') for all $r \geqq 0$. Then the relative topology in M of $R_0 \cup I_\lambda$ is one making θ a homeomorphism. See Fig. 2.

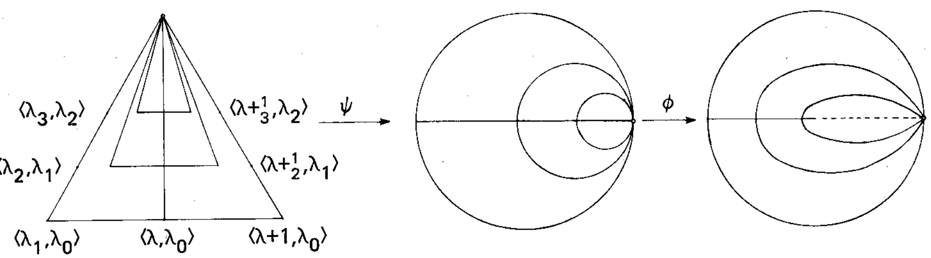

Fig. 2.

After we have done this for all λ, (1) will be satisfied because of the way each θ has been defined, while (2) is satisfied because any subset of A whose closure in M meets I_γ would have $\langle\gamma, \gamma\rangle$ in its closure in the topology on $L^+ \times L^+$, and the only point on the diagonal in the closure of R_0 in $L^+ \times L^+$ is the point $\langle\lambda, \lambda\rangle$ at its 'apex'.

Both this example and Example 3.9 had the property that the added intervals I_λ were a discrete collection of closed subsets of the resulting manifold. In order to be able to imitate most of the standard constructions of locally compact, locally countable spaces, we usually need to add the I_λ one at a time by transfinite induction, and allow later I_λ's to be in the closure of earlier ones. That is what we do in our remaining examples.

3.11. EXAMPLE [CH]. Let us recall one of the simplest uses of the continuum hypothesis to construct a space that has not yet been obtained 'from ZFC': a separable, first countable, countably compact, noncompact space.

Begin with a copy of ω and add points p_α indexed by the countable ordinals. Using CH, arrange the countably infinite subsets of $\omega \cup \{p_\alpha : \alpha < \omega_1\}$ in an ω_1-sequence $\{A_\alpha : \alpha < \omega_1\}$. At the αth stage, assume $X_\alpha = \omega \cup \{p_\beta : \beta < \alpha\}$ has been given a topology making it locally compact and noncompact; that ω is dense in X_α; and that X_β is a locally compact subspace of X_α for all $\beta < \alpha$. Pick the first γ such that $A_\gamma \subset X_\alpha$ and A_γ is closed discrete; such a set exists because X_α is metrizable and noncompact. Let $B_\gamma \subset A_\gamma$ be such that $|B_\gamma| = |A_\gamma \backslash B_\gamma| = \omega$. Since X_α is locally compact and zero-dimensional, there is a discrete collection of compact open sets U_γ^n each of which meets B_γ in one point and misses the rest of A_γ. Let a local base at p_α be the collection of all sets of the form $\bigcup\{U_\gamma^n : n \geqq k\} \cup \{p_\alpha\}$ $(k \in \omega)$. This puts p_α in the closure of ω and gives it a local base of compact clopen sets; thus $X_{\alpha+1}$ satisfies all the induction hypotheses.

At a limit ordinal λ, we let X_λ have the direct limit topology (see Example 3.8 for the definition). The induction hypotheses are satisfied at each $\lambda \leqq \omega_1$; in particular, if $\alpha_n \nearrow \lambda$ then $\{p_{\alpha_n} : n \in \omega\}$ is closed discrete. Now it follows from the way the induction is done that each A_γ has a limit point in X_α for some countable α; so X_{ω_1} is countably compact in addition to being locally compact and locally countable (hence first countable and zero-dimensional) and having ω as a dense subspace. Moreover $\{X_\alpha : \alpha < \omega_1\}$ is an open cover with no countable subcover.

To construct a manifold along these lines, we begin with the n-dimensional open ball B^n instead of ω. Instead of a point p_α we add a copy I_α of $[0, 1)$ at the αth stage. We let X_α be the union of B^n with $\bigcup\{i_\beta : \beta < \alpha\}$ for each $\alpha \leqq \omega_1$. The underlying set for the space is X_{ω_1}.

Use CH as before, to index the countable subsets of X_{ω_1} as $\{A_\alpha : \alpha < \omega_1\}$. The induction hypothesis at α is that the topology has been defined on X_β for all $\beta < \alpha$ in such a way that X_β is homeomorphic to B^n itself and contains X_γ as a dense open subspace for all $\gamma < \beta$. ['Open' is automatic from density and local compactness.] If α is a limit ordinal, we let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ with the direct limit topology. All the induction hypotheses are easy to verify except homeomorphism with B^n. For that we can use the following result of BROWN [1961]: the union of an ascending chain of spaces homeomorphic to B^n is itself homeomorphic to B^n. A differentiable analogue of this was proven by KOZLOWSKI and ZENOR [1979]: if a differentiable manifold M has an atlas $\{(U_i, \phi_i) : i \in \omega\}$ such that $U_i \subset U_{i+1}$ and $\phi_i(U_i) = \mathbb{R}^n$ for all $i \in \omega$, then M is diffeomorphic to $\mathbb{R}^n$.

If α is any ordinal and the topology on X_α has been defined, we define the topology on $X_{\alpha+1} = X_\alpha \cup I_\alpha$ as follows. Pick the least γ such that $A_\gamma \subset X_\alpha$ and A_γ is closed discrete in X_α. Let $\psi_\alpha : X_\alpha \to B^n$ be a homeomorphism, and let $\phi_\alpha : B^n \to B^n$ be an embedding whose range is $B^n - [0, 1) \times \{0\} \times \cdots \times \{0\}$, and which sends $\psi_\alpha(A_\gamma)$ to a (discrete) subspace D' whose closure in $\mathbb{R}^n$ is $D' \cup [0, 1] \times \{0\} \times \cdots \times \{0\}$. Let θ be the function from B^n to $X_\alpha \cup I_\alpha$ which takes $\langle r, 0, \ldots, 0\rangle$ to the copy of r in I_α and which takes any other point p to $\psi_\beta^{-1}\phi_\beta^{-1}(p)$. Let the topology on $X_{\alpha+1}$ be that which makes θ a homeomorphism. Clearly the induction hypotheses are satisfied.

Now $M = X_{\omega_1}$ is an n-manifold because each point is contained in some open subspace X_α ($\alpha < \omega_1$). B^n is a dense subspace, so X_{ω_1} is separable. And countable compactness is proven exactly as for the locally compact, locally countable space whose construction we have been imitating.

Many properties of M depend upon the order in which the subsets A_α are chosen and the way ψ_α and ϕ_α are chosen at each stage. It is possible to rig the induction so that X_{ω_1} is normal, and also so that it is not normal. Our last example, based on the famous countably compact S-space of OSTASZEWSKI [1976] and due to M.E. RUDIN, is a special case of Example 3.11 which is perfectly normal. To pave the way for it, we state a result which is very useful in constructing manifolds under CH:

3.12. LEMMA. *Let $\{A_\alpha : \alpha < \omega_1\}$ be a collection of sets such that $|A_\alpha| \leqq \omega_1$. Let $X_\alpha = \bigcup_{\beta<\alpha} A_\alpha$. Let Y be a subset of X_{ω_1} such that $Y \cap X_\alpha$ is countable for all α. If $\rho: X_{\omega_1} \to \omega_1$ is injective, and C is the set of ordinals λ such that $\rho^{-1}(\lambda) \cap Y = Y \cap X_\lambda$, then C is a closed unbounded subset of λ.*

PROOF. That C is closed is immediate from the identities $\rho^{-1}(\lambda) = \bigcup_{\alpha<\lambda} \rho^{-1}(\lambda)$ and $Y \cap X_\lambda = \bigcup_{\alpha<\lambda} Y \cap X_\alpha$ whenever λ is a limit ordinal. As usual, unboundedness is shown by a leapfrog argument. For any α, let $Y_\alpha = Y \cap X_\alpha$. Let $\alpha_0 < \omega_1$ be arbitrary. With α_n defined, let β_n be the supremum of $\rho(Y_\alpha)$. With β_n defined, let α_{n+1} be the least ordinal $\geqq \alpha_n$ such that $\rho^{-1}(\beta_n) \subset X_{\alpha+1}$. Clearly the supremum of the α_n is in C. Our construction will employ:

3.13. AXIOM. Let Λ be the set of all countable limit ordinals. There exists a collection $\{S_\lambda : \lambda \in \Lambda\}$ of subsets of ω_1 such that $S_\lambda \subset \lambda$ and λ is the only limit point of λ, with the following property: if A is an uncountable subset of ω_1, then $S_\lambda \subset A$ for a stationary set of λ's.

This axiom is formally stronger than (but can be shown equivalent to) the axiom ♣ of OSTASZEWSKI [1976]. The argument used to derive ♣ from the axiom $\diamondsuit$ on p. 32 of M.E. RUDIN's lecture notes [1975] can be used to derive Axiom 3.13. We will not need the statement of either ♣ or $\diamondsuit$ here but merely pause to recall the theorem that ♣ + CH is equivalent to $\diamondsuit$, hence so is Axiom 3.13 + CH.

3.14. EXAMPLE [$\diamondsuit$] (A perfectly normal, countably compact, (hereditarily) separable, nonmetrizable manifold). We continue to use the notation of Example 3.11, of which this is a special case. Let $\rho: X_{\omega_1} \to \omega_1$ be a bijection, let S_λ be as in Axiom 3.13 and let $T_\lambda = \rho^{-1}(S_\lambda)$.

Along with the usual induction hypotheses, we carry the following: (1) if T_λ is a closed discrete subspace of X_λ, for any limit ordinal $\lambda < \alpha$, then I_λ is in the closure of T_λ; and (2) if $\delta < \beta < \alpha$, then I_β is in the closure of I_δ.

Suppose this has been done for all $\beta < \alpha$. If α is a limit ordinal, the topology on X_α is the direct limit topology as always. Then, if T_α is closed discrete in X_α, we let $D_\alpha = T_\alpha$ and $D_{\alpha+1} = A_\gamma$ for the least γ such that A_γ is closed discrete in X_α. Now if $\delta < \alpha$, the closure of I_δ in X_α is metrizable, but not compact: if we let $\alpha_n \nearrow \alpha$ and pick $x_n \in I_{\alpha_n}$, then by the hypothesis (2) all but finitely many x_n are in the closure of I_δ; but $\{x_n : n \in \omega\}$ is a closed discrete subspace of X_α since we are operating under the constraints of Example 3.11. Therefore, there exists an infinite discrete subspace D_δ of X_δ that is closed in X_α.

Now when we define ψ_α and ϕ_α we make sure that the image of each D_δ ($\delta \leqq \alpha + 1$) has all of $[0, 1) \times \{0\} \times \cdots \times \{0\}$ in its closure. Then obviously (1) and (2) are satisfied.

Finally, if $\alpha = \beta + 1$ and the topology on X_α has been defined, we define D_δ for each $\delta \leqq \alpha$ just as in the case where α is a limit ordinal. This time D_δ exists because the closure of I_δ includes I_β which is a closed copy of $[0, 1)$ in X_α. We follow the same instructions in defining ψ_α and ϕ_α except that $D_{\alpha+1}$ is not defined.

Claim. If F is a closed non-Lindelöf subset of $M = X_{\omega_1}$, then there exists α such that $I_\beta \subset F$ for all $\beta > \alpha$.

Perfect normality follows easily from this claim. If F_1 and F_2 are disjoint closed sets, then one is Lindelöf, being contained in X_α for some α; but then it is compact because M is countably compact. Hence F_1 and F_2 can be put into disjoint open sets. If F is a closed subset of X_1 and F is compact, F is a subset of some X_α, which is metrizable, so F is a G_δ in X_α, hence in M. If F is not compact, then its complement is an open subset of X_α for some α, hence is an F_σ—indeed, σ-compact.

Proof of Claim. If F is a closed non-Lindelöf subset of X, then $F \cap I_\alpha$ is nonempty for uncountable many α, so there is an uncountable $Y \subset F$ such that $Y \cap I_\alpha$ is finite, and $Y \cap X_\alpha$ is countable, for all α. By Lemma 3.12, the set C of all ordinals λ such that $\rho^{-1}(\lambda) \cap Y = Y \cap X_\lambda$ is closed and unbounded. Let C' be the derived set of C. By Axiom 3.13, there exists $\lambda \in C'$ such that $S_\lambda \subset \rho(Y)$, hence $T_\lambda \subset Y$. Pick $\lambda_n \nearrow \lambda$ with $\lambda_n \in C$. Now $\rho^{-1}(\lambda_n) \cap Y = Y \cap X_\lambda$, and $(0, \lambda_n)$ meets S_λ in a finite set, so $Y \cap X_\lambda$ meets T_λ in a finite set. Since $X_\lambda = \bigcup_{n\in\omega} X_{\lambda_n}$, T_λ is a closed discrete subspace of X_λ. Then $I_\lambda \subset F$, and $I_\beta \subset F$ for all $\beta > \lambda$.

Of course, M is separable because B^n is dense in M, so M is hereditarily separable by Corollary 2.18.

It is instructive to make a four-way comparison between Example 3.14, OSTASZEWSKI's [1976] example, the RUDIN–ZENOR [1976] perfectly normal manifold, and the Kunen 'line' (the first example in the JUHÁSZ, KUNEN, and RUDIN [1976] paper)—or, more precisely, the Kunen construction that uses $(-1, 1) \times (0, 1)$ rather than the real line as the underlying set. The use of CH in the former two is entirely different from that in the latter two: Rudin and Zenor, like Kunen, use CH in their example to obtain perfect normality; in Ostaszewski's space it is ♣ that gives perfect normality and CH that gives countable compactness; while in Example 3.14, CH was used for both purposes; in particular, its role in insuring perfect normality was by way of Lemma 3.12. The fact that manifolds have the cardinality of the continuum in all models sometimes has this effect of translating a ♣ construction into a ♣ + CH (= $\diamondsuit$) construction.

With this, we hope that we have given the interested reader enough keys to the translation of many familiar locally compact, locally countable spaces to manifolds. In some cases, particularly where a construction runs beyond ω_1, a translation may be impossible; but there are many examples where it is possible besides the ones given here.

The set of techniques presented in this section is by no means exhaustive, either. There is, for example, the technique of adding nicked disks B_α instead of half-open intervals I_α at each stage of an induction. By sacrificing countable compactness, we can use this technique as in Example 3.14 to produce a perfectly normal manifold of uncountable cellularity. [Of course, every countably compact, perfectly normal manifold is of countable spread, hence hereditarily separable: in a perfectly normal space, every discrete subspace is an F_σ.] There are some messy details that need to be thrashed out if the union of the interiors of the B_α's is to be an F_σ, so we omit the actual construction. The same difficulty arises in the RUDIN–ZENOR [1976] example, 'Corollary 1,' which is also done using nicked disks (referred to as 'wedges' in that paper).

In Section 6 we will see another method of constructing manifolds by induction, using ascending ω_1-sequences of disks. Other transfinite inductions can be used to attach long rays or long lines to an n-sphere to produce separable n-manifolds with pathological properties.

Except for a passing reference, nothing has been said so far about the construction of smooth manifolds. An article at least the length of this one could be written on the subject of 'smoothing' the various constructions we have given and then using them to construct vector and fiber bundles by standard methods. Even so 'ordinary' a manifold as the long ray can be used to produce a wealth of examples in this way (cf. NYIKOS [1979]), all quite different from any in this section.

4. Martin's Axiom and Type I spaces

The focal point of this section is M.E. RUDIN's result [1979] that every perfectly normal manifold is metrizable if one assumes MA + $\neg$CH. Together with Example 3.14 this laid to rest a problem posed by ALEXANDROFF [1935] and WILDER [1949], whether there exists a perfectly normal nonmetrizable manifold.

This result has been generalized in several directions, and can now be seen as part of a general theory (to which it gave birth) of when a perfectly regular space (i.e. a regular space in which every closed subset is a G_δ) is subparacompact or paracompact. One of the standard procedures in this theory is to reduce the problem to one involving Type I spaces, usually by way of SZENTMIKLÓSSY's [1978] MA + $\neg$CH theorem that every locally compact space of countable spread is hereditarily Lindelöf. Additional information on this theorem may be found in the article by Roitman. For our purpose the starting point is:

4.1. LEMMA [MA + $\neg$CH]. *Every perfectly regular, collectionwise Hausdorff manifold is of Type I.*

PROOF (Similar to that of Theorem 2.9). Let U_0 be a relatively compact, nonempty open subset of the manifold X. Then U_0 is hereditarily Lindelöf. Assume that α is a countable ordinal and the U_β has been defined for all $\beta < \alpha$, to be a hereditarily Lindelöf, open subset of X, with $\bar{U}_\gamma \subset U_\beta$ whenever $\gamma < \beta$. If α is a limit ordinal, let $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$. Suppose $\alpha = \beta + 1$. In a perfectly regular space, every discrete subspace is a countable union of closed subspaces [routine exercise] and $\bar{U}_\beta$ cannot have an uncountable closed discrete subspace, because U_β is hereditarily Lindelöf and $\bar{U}_\beta$ is cwH. Therefore, $\bar{U}_\beta$ is of countable spread, hence hereditarily Lindelöf by Szentmiklóssy's theorem. Cover the boundary with countably many relatively compact open subsets and let Y_α be the union of these with U_β. As in the proof of Theorem 2.9, $\bigcup_{\alpha<\omega_1} U_\alpha = X$.

Of course, Example 3.7 shows that 'collectionwise Hausdorff' cannot be omitted from Lemma 4.1.

The above proof was worded to also make it a proof of:

4.2. LEMMA [MA + $\neg$CH]. *Every perfectly regular collectionwise Hausdorff, locally compact, connected space is of Type I.*

Indeed, every perfectly regular, Lindelöf space is hereditarily Lindelöf (cf. Engelking's text, exercise 3.8A, part (b), which should read "Show that a regular Lindelöf space is a hereditarily Lindelöf space if and only if it is perfectly normal."), and every locally compact, perfectly regular space is first countable. This is all that is needed to make the proof of Lemma 4.1 go through here.

Lemma 4.2 was further generalized by GRUENHAGE [1980] to show that every perfectly regular, locally compact, collectionwise Hausdorff space is the disjoint union of clopen Type I subspaces. See Lemma 3.6 in Tall's article.

Now it is worth going a little more deeply into the structure of Type I spaces.

4.3. DEFINITION. Let X be a Type I space. A *canonical sequence for* X is a sequence $\Sigma = \langle X_\alpha : \alpha < \omega_1 \rangle$ of open subspaces of X such that $\bar{X}_\alpha$ is Lindelöf for all α, $\bar{X}_\alpha \subset X_{\alpha+1}$ for all α, $X_\lambda = \bigcup_{\beta<\lambda} X_\beta$ for any limit ordinal λ, and $X = \bigcup_{\alpha<\omega_1} X_\alpha$.

Any two canonical sequences agree on a cub set of indices:

4.4. LEMMA. *Let $\Sigma = \langle X_\alpha : \alpha < \omega_1 \rangle$ and $\Sigma^1 = \langle X'_\alpha : \alpha < \omega_1 \rangle$ be canonical sequences for the Type I space X. There exists a closed unbounded subset C of ω_1 such that $X_\alpha = X'_\alpha$ for all $\alpha \in C$.*

In fact, the set of all indices α such that that $X_\alpha = X'_\alpha$ *is* a closed unbounded set. 'Closed' is trivial, while 'unbounded' is proved in a leapfrog fashion as in Lemma 3.12. The key here is that since $\bar{X}_\alpha$ is Lindelöf, it is contained in the open subspace X'_β for some $\beta < \omega_1$; similarly, $\bar{X}'_\beta \subset X_\gamma$ for some $\gamma < \omega_1$.

Because of this close correspondence, we will sometimes write X_α, given X (or M_α, given M, etc.) without bothering to specify Σ. We will also abuse language in talking about properties of Σ as though they were properties of X.

4.5. THEOREM [MA + $\neg$CH]. *Every perfectly normal, locally compact, Type I space is subparacompact.*

PROOF. Let X be a perfectly normal, locally compact Type I space. We will show that X can be covered by countably many discrete collections of compact sets. This is equivalent to subparacompactness in a locally compact space: if we take an open cover by relatively compact sets, then any σ-discrete closed refinement is a cover of X by compact sets as desired; conversely, suppose X has a cover as desired, let $\mathscr{U}$ be an open cover of X and let $\mathscr{K}$ be one of the countably many discrete collections of compact sets. Then each $K \in \mathscr{K}$ is the union of finitely many closed sets, each of which is contained in some member of $\mathscr{U}$ (Engelking's text, Theorem 1.5.18). By the axiom of choice, $\mathscr{U}$ has a σ-discrete closed refinement.

Our proof will use the following consequence of Martin's Axiom, proven in the article by Weiss: *every* ccc *poset of cardinality* $<\mathfrak{c}$ *is* σ*-linked.*

Let X be a locally compact, perfectly normal, Type I space X and let $\Sigma = \langle X_\alpha : \alpha < \omega_1 \rangle$ be canonical for X. For each successor ordinal $\alpha + 1 < \omega_1$, let $X^\#_{\alpha+1} = X_{\alpha+1} \backslash X_\alpha$. Let $\mathscr{H}_{\alpha+1}$ be a countable collection of open sets with compact closures such that (i) $X^\#_{\alpha+1} \subset \bigcup \mathscr{H}_{\alpha+1}$ (ii) $\bar{H} \subset X_{\alpha+1}$ for all $H \in \mathscr{H}_{\alpha+1}$ and (iii) $H \cap X^\#_{\alpha+1} \neq \emptyset$ for all $H \in \mathscr{H}_{\alpha+1}$. Let $\mathscr{H} = \bigcup \{\mathscr{H}_{\alpha+1} : \alpha < \omega_1\}$. For each $H \in \mathscr{H}$, let $\gamma(H)$ be the greatest ordinal γ such that $H \cap X^\#_\gamma \neq \emptyset$; this is also the unique ordinal $\alpha + 1$ such that $H \in \mathscr{H}_{\alpha+1}$.

Let $(P, \leqq)$ be the partially ordered set defined as follows. Members of P are finite subsets $p = \{H_1, \ldots, H_n\}$ of $\mathscr{H}$ such that if $\gamma(H_i) < \gamma(H_j)$, then $H_j \cap H_i \cap X^\#_{\gamma(H_i)} = \emptyset$. We define $p \leqq q$ if $q \subset p$.

Suppose $Q = \{p_\alpha : \alpha < \omega_1\}$ is an uncountable collection of pairwise incompatible members of P. We may assume all members of Q are of the same cardinality n and that every incompatible collection whose members are of cardinality $<n$ is countable. We may also assume that Q forms a Δ-system. And now, since subtracting off the common root from each member of Q leaves the members incompatible, any two members of Q are disjoint. Because each $\mathscr{H}_\alpha$ is countable, we may assume that if $\alpha < \beta$ then $\gamma(H) < \gamma(K)$ for all $H \in p_\alpha$, $K \in p_\beta$. Finally, we may assume the set of all ordinals of the form $\gamma(H)$ for some $H \in p_\alpha \in Q$ is a discrete subspace of ω_1. Let C be the derived set of this subspace.

Of course, C is a closed unbounded subset of ω_1, and $F = \bigcup \{\bar{X}_\alpha \backslash X_\alpha : \alpha \in C\}$ is a closed subset of X. Let $\{U_n : n \in \omega\}$ be a decreasing sequence of open subsets of X

satisfying $F = \bigcap_{n\in\omega}\bar{U}_n$. Since $\bar{H} \cap X^{\#}_{\gamma(H)}$ is compact and misses F for each $H \in \bigcup Q$ there exists m such that, for uncountably many indices α, $\bar{U}_m \cap \bar{H} \cap X^{\#}_{\gamma(H)} = \emptyset$ for all $H \in p_\alpha$. Let α_i be the ith such index ($i \in \omega$) and let $\lambda = \sup\{\gamma(H)$: $H \in p_{\alpha_i}$ for some $i\}$. If $p_\alpha \in Q$ and $\alpha > \alpha_i$ for all i, the incompatibility of Q requires that there be $H \in p_\alpha$ and infinitely many α_i such that $H \cap \bar{H}' \cap X^{\#}_{\gamma(H')} \neq \emptyset$ for some $H' \in p_{\alpha_i}$. But since $\bar{H}$ is (countably) compact, the set of all such intersections must have a limit point in $\bar{X}_\lambda \backslash X_\lambda$. But since $\lambda \in C$ this contradicts the claim that U_m is open and $F \subset U_m$.

Now by the above consequence of Martin's Axiom, $P = \bigcup_{n\in\omega}P_n$ where each P_n is linked in P. This means that if $p_\alpha, p_\beta \in P_n$ then $p_\alpha \cup p_\beta \in P$. Hence for any $H, H' \in \bigcup P_n$ either $\gamma(H) = \gamma(H')$ or $H' \cap H \cap X^{\#}_{\gamma(H)} = \emptyset$. By the axiom of choice and countability of each $\mathscr{H}_\alpha$, we can split each $\bigcup P_n$ and hence $\bigcup P = \mathscr{H}$ into countably many collections $\mathscr{H}^n$ such that $\gamma(H) \neq \gamma(H')$ and $H' \cap H \cap X^{\#}_{\gamma(H)} = \emptyset$ for distinct $H, H' \in \mathscr{H}^n$.

Now $\bigcup \mathscr{H}^n$ is the union of countably many closed sets K_i, and for each $H \in \mathscr{H}^n$ we can express $H \cap X^{\#}_{\gamma(H)}$, a locally compact Lindelöf space, as the union of compact sets $K^H_{im} \subset K_i$. Then $\{K^H_{im}: H \in \mathscr{H}^n\}$ is a discrete collection of compact sets, and X is easily seen to be the union of all such collections.

Local compactness cannot be removed from the hypotheses: R. POL's [1976] example of a perfectly normal, collectionwise normal, nonparacompact space is of Type I.

4.6. THEOREM [MA + $\neg$CH]. *Every perfectly normal, locally compact, locally connected space is paracompact.*

PROOF. Any locally connected space is the disjoint union of clopen connected subspaces, and so it suffices to prove the theorem for these components of X. By the extension of Corollary 2.17 to locally compact, locally connected perfectly normal spaces, each component of X is cwH and hence (by Lemma 4.2) of Type I. Let $\mathscr{U}$ be a cover of the component by relatively compact open sets and (by Theorem 4.5) let $\mathscr{K} = \bigcup_{n\in\omega}\mathscr{K}_n$ be a refinement of $\mathscr{U}$ with closed sets, with each $\mathscr{K}_n$ discrete. By Theorem 2.20, the members of $\mathscr{K}_n$ can be put into a discrete collection $\mathscr{V}_n$ of open sets each of which is contained in some member of $\mathscr{U}$. Then $\bigcup_{n\in\omega}\mathscr{V}_n$ is a σ-discrete open refinement of $\mathscr{U}$.

4.7. COROLLARY [MA + $\neg$CH]. *Every perfectly normal manifold is metrizable.*

This is the theorem of Rudin alluded to at the beginning of this section, originally proven in a manner very similar to the proof of Theorem 4.5 and using Theorem 2.21 (which can be used to derive Corollary 4.7 immediately from Theorem 4.5 and Corollary 2.2). Theorem 4.6 is D. LANE's [1980] generalization, further extended by G. GRUENHAGE [1980]. If we assume Gruenhage's lemma that every perfectly regular, locally compact, collectionwise Hausdorff space is the disjoint union of clopen Type I subspaces [see the article by Tall, Lemma 3.6], we will in due course arrive at Gruenhage's theorem:

4.8. THEOREM [MA + $-$CH]. *Every perfectly normal, locally compact,* cwH *space is paracompact.*

First, though, it is convenient to make a few more observations about Type I spaces. The usefulness of the following concept has already been hinted at by Lemma 3.2 and the definition of F in the proof of Theorem 4.5.

4.9. DEFINITION. Let X be a Type I space, and let $\Sigma = \langle X_\alpha : \alpha < \omega_1 \rangle$ be canonical for X. The *skeleton of X associated with Σ* is the sequence $\langle \bar{X}_\alpha \backslash X_\alpha : \alpha < \omega_1 \rangle$. The set $\bar{X}_\alpha \backslash X_\alpha$ is *the αth bone of X associated with Σ.*

Of course, we will often leave off the words 'associated with Σ,' and it is to be understood that there is a fixed Σ being used as a reference.

An immediate consequence of Lemma 4.4 is that any two skeletons of X agree on a cub of bones (i.e. on the bones indexed by some cub subset of ω_1). Moreover, if $\Sigma = \langle X_\alpha : \alpha < \omega_1 \rangle$ is canonical for X and $C \subset \omega_1$ is a cub set, then a listing of $\{X_\alpha : \alpha \in C\}$ in numerical order is canonical for X: thus the bones indexed by any cub set in any skeleton will themselves form a skeleton.

4.10. LEMMA. *Let X be a locally compact Type I space. The following are equivalent.*

(i) *X is paracompact.*
(ii) *Some skeleton of X is empty.*
(iii) *Every skeleton of X has a cub of empty bones.*

PROOF. The equivalence of (ii) and (iii) is clear from what has just been said. If X is paracompact, then by local compactness, X is the union of ω_1 disjoint clopen Lindelöf subspaces (cf. Engelking's text). If we arrange these in an ω_1-sequence and let X_α be the union of the ones indexed by an ordinal $< \alpha_1$, then $\Sigma = \langle X_\alpha : \alpha < \omega_1 \rangle$ produces an empty skeleton. Hence (i) implies (ii), and the converse is routine.

4.11. LEMMA. *Let X be a locally hereditarily Lindelöf Type I space. Then X is [hereditarily] cwH if, and only if every closed discrete [resp. discrete] subspace misses a cub set of bones.*

PROOF. If $\{\bar{X}_\alpha \backslash X_\alpha : \alpha \in C\} = F$ misses D, and C is closed unbounded in ω_1, then F is closed in X, and for each $d \in D$ there are unique ordinals α_1 and α_2 such that

$d \in X_\alpha \backslash \bar{X}_\alpha$ and α_2 immediately follows α_1 in C. Thus $\{X_\alpha \backslash \bar{X}_{\alpha'}$: α' immediately precedes α in $C\}$ is a disjoint collection of hereditarily Lindelöf open sets whose union contains D. If D is discrete, it meets each such open set in a countable discrete set and so its elements can be put into disjoint open sets.

On the other hand, suppose D meets a stationary set of bones in some (hence any) skeleton of X. Then from the pressing-down lemma it follows that the elements of D cannot be put into disjoint open sets: some X_α would have to contain a collection of uncountably many disjoint open sets.

Now Theorem 4.8 follows from Gruenhage's lemma and:

4.12. LEMMA [MA + $\neg$CH]. *Every perfectly normal, locally compact, collectionwise Hausdorff Type I space is paracompact.*

PROOF. By Lemma 4.5, X is the union of countably many sets each of which is the union of a discrete collection $\mathcal{K}_n$ of compact sets. Suppose X has no empty skeleton; then for each skeleton there is a stationary subset S of ω_1 and an n such that $\mathcal{K}_n$ meets every bone indexed by S. By the usual leapfrog method we can find a closed unbounded set $C \subset \omega_1$ such that no two bones indexed by members of $C \cap S$ will meet the same member of $\mathcal{K}_n$ and thus find a closed discrete subspace D of X that meets a stationary set of bones. Now use Lemma 4.11 and the fact that in a perfectly regular Type I space, X_α is hereditarily Lindelöf for all α [see comment following Lemma 4.2].

Example 3.6 shows that 'normal' cannot be weakened to 'regular' in Theorem 4.6. However, Theorem 4.8 has been extended to perfectly regular spaces in Z. BALOGH's [198 ·] paper. We will now sketch his proof, which does not use Lemma 4.5 but uses the consequence of Martin's Axiom mentioned there is a different way.

As can be seen from Roitman's article, Szentmiklóssy used a slightly weaker consequence of MA + $\neg$CH ("in a ccc poset P, every uncountable subset X has an uncountable subset Y which is linked in P") in proving the following result: in a compact space of countable tightness, every locally countable subspace of cardinality ω_1 has an uncountable discrete subspace. Using essentially the same argument and the stronger consequence that P is σ-linked (the countable union of collections linked in P) if it is ccc and of cardinality $<\mathfrak{c}$, Balogh showed that:

4.13. THEOREM [MA + $\neg$CH]. *In a compact space of countable tightness, every locally countable subset of cardinality $<\mathfrak{c}$ is the countable union of discrete subspaces.*

By Gruenhage's lemma, we need only show the version of Lemma 4.12 with 'regular' in place of 'normal' to show:

4.14. THEOREM [MA + ¬CH]. *Every perfectly regular,* cwH, *locally compact space is paracompact.*

Now, the one-point compactification of a perfectly regular space has countable tightness, because: (1) if a compact space X does not have countable tightness, it contains an ω-bounded noncompact subspace, and so does every cofinite subspace of X (a space is *ω-bounded* if every countable subset has compact closure) (2) [GRUENHAGE, 1977] an ω-bounded noncompact space contains a perfect preimage of ω_1; but (3) no perfect preimage of ω_1 can be perfectly regular.

Next, suppose X is perfectly regular, locally compact, and of Type I. If we take one point apiece from each bone of a skeleton, the resulting subspace Y is locally countable, hence (by Theorem 4.13) the countable union of discrete subspaces. By perfect regularity, it will therefore be σ-discrete (the countable union of *closed* discrete subspaces D_n). By cwH, there must be a closed unbounded subset C_n of ω_1 such that the bones indexed by C_n miss D_n. Hence the bones indexed by $\bigcap_{n\in\omega} C_n$ miss Y; but then these bones are empty, and so X is paracompact by Lemma 4.10.

Incidentally, the arguments for Lemma 4.1 and Theorem 4.14 are enough to establish the following result of BALOGH [198 ·]:

4.15. THEOREM [MA + ¬CH]. *Every hereditarily* cwH, *nonmetrizable manifold must contain a perfect preimage of ω_1.*

For Type I manifolds, Balogh has been able to drop 'hereditarily' here; more generally, he has shown:

4.16. THEOREM [MA + ¬CH]. *Every* cwH *nonmetrizable manifold of weight $<\mathfrak{c}$ contains a perfect preimage of ω_1.*

One ingredient in the proof is his result that, for manifolds of weight $<\mathfrak{c}$, 'perfectly regular' may be dropped from Lemma 4.1.

4.17. COROLLARY [MA + ¬CH]. *Every ω_1-compact manifold of weight $<\mathfrak{c}$ is either metrizable or contains a perfect preimage of ω_1.*

It may be a long time before the last word on generalizations of M.E. Rudin's theorem is in. Three of the most ambitious unanswered questions along these lines are:

4.18. PROBLEM. It is consistent that every hereditarily normal manifold of dim >1 is metrizable?

4.19 [4.20]. PROBLEMS. Is it consistent with ¬CH that every manifold of weight $<\mathfrak{c}$ is either developable or contains [a perfect preimage of] ω_1?

The best candidate we have for affirmative answers is PFA (see the article by Baumgartner), although MA + $\neg$CH has not yet been ruled out.

5. The structure of ω-bounded manifolds

As is well known, compact 2-manifolds with boundary were completely classified in a very simple way early in this century. Those that are manifolds are a connected sum of finitely many tori and projective planes, and the basic relationship is succinctly stated on p. 8 of LANG's [1965] algebra textbook. Those that are not manifolds (what MASSEY [1967] and others refer to as 'bordered surfaces') can be constructed even more simply by pasting rectangular strips to the edge of a disc (MASSEY [1967], pp. 43–46).

In this section we will extend these ideas to ω-bounded 2-manifolds with boundary.

5.1. DEFINITION. A space is *ω-bounded* if every countable subset has compact closure.

Of course, every ω-bounded space is countably compact. 'Conversely' all known examples of 'real' countably compact manifolds are ω-bounded. Under CH, there are counterexamples, e.g. Example 3.11.

The main theorem of this section will be called the 'bagpipe theorem' because it says that any ω-bounded 2-manifold can be constructed by attaching n 'long pipes' ($0 \leqq n < \omega$) to a metrizable 'bag' (compact 2-manifold with boundary). For convenience, we adopt the following jargon.

5.2. DEFINITION. A space is *a long pipe* if it is the union of a chain $\{U_\alpha : \alpha < \omega_1\}$ of open subspaces U_α homeomorphic to $S^1 \times \mathbb{R}$, such that $\bar{U}_\alpha \subset U_\beta$ and such that the boundary of U_α in U_β is homeomorphic to S^1 whenever $\alpha < \beta$.

It is easy to see that any long pipe can be obtained by removing a single point from an ω-bounded 2-manifold which is the union of an increasing sequence of subspaces homeomorphic to $\mathbb{R}^2$, and that it does not matter which point we remove. Two topologically distinct examples of long pipes are $S^1 \times L^+$ and the space obtained by removing one point from $L \times L$; distinctness can be shown as in Example 3.5. In Section 6 we will see many more.

Unlike in the preceding section, we will not be straining towards the greatest generality, so most results will use the word "manifold" in the hypotheses. However, these results can be extended at least to locally compact, connected, locally connected, locally metrizable spaces. Those which cannot will use the words 'surface' (meaning '2-manifold with boundary') or '2-manifold' instead of 'manifold'.

We omit the elementary proof of:

5.3. LEMMA. *A Type I, locally separable space is ω-bounded if, and only if, it is countably compact.*

Of course, any such space is locally compact.

5.4. COROLLARY. *A manifold is ω-bounded if, and only if, it is countably compact and of Type I.*

5.5. LEMMA. *Let X be a Type I manifold, and let $\langle X_\alpha\colon \alpha<\omega_1\rangle$ be canonical for X. Then $X\backslash\bar{X}_\alpha$ has at most countably many components. Moreover, if X is countably compact, $X\backslash\bar{X}_\alpha$ has at most finitely many nonmetrizable components.*

PROOF. Since X is locally connected, the components of $X\backslash\bar{X}_\alpha$ are open in X. Since X is connected, each component meets $X_{\alpha+1}\backslash\bar{X}_\alpha$. But $X_{\alpha+1}$ has countable cellularity, so there can be only countably many components altogether.

Of these components, only finitely many can meet $X\backslash\bar{X}_{\alpha+1}$ if X is countably compact. If there were infinitely many, let $\{M_n\colon n\in\omega\}$ be the set of all that do; then $\{X_{\alpha+1}\}\cup\{M_n\colon n\in\omega\}$ is an infinite open cover of X with no proper subcover. However, $X_{\alpha+1}$ is metrizable and so cannot contain any nonmetrizable components. So there are only finitely many of these.

To analyze the nonmetrizable components further, we introduce the tree Υ ("upsilon").

5.6. DEFINITION. Let X be a Type I manifold, and let $\Sigma=\langle X_\alpha\colon \alpha<\omega_1\rangle$ be canonical for X. *The tree of nonmetrizable-component boundaries associated with* Σ, denoted $\Upsilon(\Sigma)$, is the collection of all sets of the form *bd* C such that C is a nonmetrizable component of $X\backslash\bar{X}_\alpha$ for some α, with the following order: if $A, B\in\Upsilon(\Sigma)$, then $A\leqq B$ if, and only if, B is a subset of a component whose boundary is A.

If Σ or X is clear from the context, we will write $\Upsilon(X)$ or simply Υ.

It is easy to see that $\leqq$ is a partial order, and that Υ is a tree; that is, the members of Υ that are $\leqq$ a given member of Υ form a well-ordered set. Like all trees, Υ can be divided up into levels, with the lowest ('zeroth') level the set of all minimal members, which are the boundaries of components of $X\backslash\bar{X}_0$. In general, the αth level is the set of boundaries of components of $X\backslash\bar{X}_\alpha$.

An important feature of Υ is that each element has successors at all levels: if C is a nonmetrizable component of $X\backslash\bar{X}_\alpha$, then by connectedness it must meet $X_{\beta+1}\backslash\bar{X}_\beta$ for all $\beta\geqq\alpha$, and any component of $C\backslash\bar{X}_\beta$ is a component of $X\backslash\bar{X}_\beta$.

The following concept has already proven useful (Examples 3.6):

5.7. DEFINITION. A locally connected space X is *trunklike* if, given any closed Lindelöf subset C, $X\backslash C$ has at most one non-Lindelöf component.

5.8. LEMMA. *A Type I manifold is trunklike* $\Leftrightarrow$ $Y(\Sigma)$ *is a (possibly empty) chain for some* $\Sigma \Leftrightarrow Y(\Sigma)$ *is a (possibly empty) chain for all* Σ.

PROOF. The first condition clearly implies the third, since $\bar{X}_\alpha$ is Lindelöf for each α and every component of $X \backslash \bar{X}_\alpha$ is a manifold and thus is metrizable if, and only if, it is Lindelöf. So we need only show the second statement implies the first.

If X is not trunklike, $X \backslash C$ has two non-Lindelöf components K_1 and K_2 for some closed Lindelöf subset C. Pick $X_\beta \in \Sigma$ such that $C \subset X_\beta$. Then $K_1 \backslash \bar{X}_\beta$ and $K_2 \backslash \bar{X}_\beta$ are both nonmetrizable and disjoint and open in X, so each has a nonmetrizable component which is also a component of $X \backslash \bar{X}_\beta$. But the boundaries of these components are both nonempty; so they are incomparable members of $Y(\Sigma)$, contradiction.

The structure of ω-bounded manifolds really begins to emerge with:

5.9. THEOREM ('The Bagpipe Lemma'). *Every ω-bounded manifold X has an open Lindelöf subset U such that $X \backslash \bar{U}$ is the free union of finitely many trunklike manifolds.*

PROOF. Pick a canonical Σ and let n_α be the cardinality of the αth level of $Y(\Sigma)$. By Lemma 5.5, each n_α is finite, and the sequence of n_α's is nondecreasing since each element has successors at all levels. Hence there is an $\alpha < \omega_1$ and $n \in \omega$ such that $n_\beta = n$ for all $\beta \geqq \alpha$. In other words, the portion of Y above the αth level is the union of n disjoint chains. Let $\{M_1, \dots, M_n\}$ be the set of nonmetrizable components of $X \backslash X_{\alpha+1}$. The trace of Σ on M_i is a canonical sequence, and the resulting tree Y_i is a chain. Lemma 5.8 implies each M_i is trunklike, and $U = X \backslash \bigcup_{i=1}^n M_i$ is the desired subset.

Of course 'trunklike' is still a long way from 'long pipe' even for 2-manifolds. There may be 'holes' (like the 'hole' in a torus) in a trunklike 2-manifold as well as cross-caps attached to it. On a trunklike 2-manifold with boundary there may even be holes of the sort one has on the pipes of a bagpipe. But now we will start bringing in machinery from the theory of metrizable manifolds which implies this 'garbage' must cease above some α.

Our first step is to show that every ω-bounded surface is the union of a chain of compact manifolds-with-boundary. This is complicated by the fact that the $\bar{X}_\alpha$ (which seem like a natural choice) are not necessarily manifolds-with-boundary. In fact, as will be explained in Section 6, it may be impossible to choose a canonical sequence for even a long pipe such that $\bar{X}_\alpha$ is a surface for each α. However, it *can* be done so that every $\bar{X}_{\alpha+1}$ is a compact surface. For this we use a theorem from regular neighborhood theory:

5.10. THEOREM (RUSHING [1973], p. 20). *In a piecewise linear manifold-with-boundary, each compact subset has a polyhedral neighborhood which is a compact manifold-with-boundary.*

To use this theorem, it is not necessary to know what a piecewise linear (PL) manifold is, only that every metrizable surface can be given a PL structure and hence every compact subset has a neighborhood which is a compact surface.

5.11. COROLLARY. *If M is an ω-bounded surface, then there is a canonical $\Sigma = \langle M_\alpha : \alpha < \omega_1 \rangle$ for M such that $\bar{M}_{\alpha+1}$ is a compact surface for each α.*

PROOF. Let M_0 be an open disc in M. With M_α defined as in the definition of 'canonical', pick a metrizable connected open subspace N of M containing $\bar{M}_\alpha$. This can be done since $\bar{M}_\alpha$ is Lindelöf and hence compact. Inside this pick a compact surface whose interior (in N, hence in M) contains $\bar{M}_\alpha$. Let $M_{\alpha+1}$ be this interior. [Note that if it is the interior of a polyhedral surface, it is connected.] Of course, we let $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for any limit ordinal λ. Then M_λ is separable and hence $\bar{M}_\lambda$ is compact.

Already we have a result whose proof cannot be extended to dimensions greater than 3: there exist compact 4-manifolds which cannot be given a PL structure.

The boundary [collection of points without Euclidean neighborhoods] of a compact surface is a disjoint union of compact 1-manifolds. Hobson's choice applies: they are circles! As such they make it possible to bring the machinery of homology or homotopy theory directly to bear.

5.12. DEFINITION. If M is a manifold-with-boundary and A and B are closed subsets of M, then A and B are *cobordant in M* if $A \cup B$ is the boundary (in M) of an open subspace of M.

The following lemma is an important part of the proof that 'the garbage must cease above some α'.

5.13. LEMMA. *If M is an ω-bounded surface, then M cannot contain an infinite set of disjoint pairwise non-cobordant circles.*

PROOF. First suppose M is compact. For any finite set $\mathscr{A}$ of disjoint circles (copies of S^1) in M, there is a triangulation of M so that each circle in $\mathscr{A}$ is the union of edges of the triangulation. Then each $A \in \mathscr{A}$ is represented by a 1-cycle in the 1-dimensional simplicial chain group with coefficients in the 2-element group (cf. HOCKING and YOUNG [1961] Section 6-4). Two members of $\mathscr{A}$ are cobordant if, and only if, the sum of the corresponding cycles mod 2 is a boundary; in other words, the cycles will have the same image in the 1-dimensional homology group. But this group is a topological invariant (SPANIER [1966] pp. 155–6, 191) and M has a finite triangulation (RADÓ [1925], AHLFORS and SARIO [1960], Chapter 1) so the group is finite, and its cardinality is a bound on the size of pairwise non-cobordant sets of circles.

In the noncompact case, we use Corollary 5.11. Suppose $\mathscr{A}$ is a countably infinite set of circles in M. Let α be such that $\bar{M}_{\alpha+1}$ is a compact surface whose interior contains each member of $\mathscr{A}$. By adding the finitely many metrizable components of $X - \bar{M}_{\alpha+1}$ to $\bar{M}_{\alpha+1}$, we may assume every component of $X - \bar{M}_{\alpha+1}$ is nonmetrizable. Replace each of these nonmetrizable components M_i by a metrizable one in the following way. The boundary of M_i is a union of k_i $(<\omega)$ circles; we cut k_i disjoint open disks out of a sphere, and attach the resulting boundary circles in 1–1 fashion to these k_i circles resulting from the removal of M_i. It is easy to see that two circles in $\bar{M}_{\alpha+1}$ are cobordant in M if, and only if, they are cobordant in the new space; the key is that if $A \cup B$ (where A and B are circles in $\bar{M}_{\alpha+1}$) is the boundary of some open set U that meets M_i, then U must contain all of M_i; the same is true in the new space, *mutatis mutandis*. But the new space is a compact surface, and this contradicts the claim that $\mathscr{A}$ is infinite.

5.14. THEOREM ('The Bagpipe Theorem'). *Every ω-bounded 2-manifold M has a compact subsurface K such that $M\backslash K$ is the union of finitely many disjoint open long pipes* [Definition 5.2].

PROOF. By the bagpipe lemma (5.9) M has a compact subset $\bar{U}$ such that $X\backslash\bar{U}$ is the union of finitely many disjoint open trunklike nonmetrizable subsurfaces, and we will be done as soon as we show each of these has a compact subsurface whose complement is a long pipe. So we may assume without loss of generality that M is trunklike.

Let $\Sigma = \langle M_\alpha : \alpha < \omega_1\rangle$ be a canonical sequence for M as in Corollary 5.11. Then $Y(\Sigma) = Y$ is a chain and each member of Y is a finite union of disjoint circles. First we will show that uncountably many members of Y are single circles.

Let $\mathscr{C}$ be the collection of all components of successor members of Y. We will use the partition relation $\omega_1 \to (\omega_1, \omega)^2$, proven e.g. in RUDIN [1975], p. 8. This relation states that if E is a set of cardinality ω_1, and $[E]^2 = P_1 \cup P_2$, then either there is $A \subset E$ such that $[A]^2 \subset P_1$ and $|A| = \omega_1$, or there is $B \subset E$ such that $[B]^2 \subset P_2$ and $|B| = \omega$. Since $\mathscr{C}$ is uncountable and contains no infinite pairwise non-cobordant subset, this relation implies that $\mathscr{C}$ contains an uncountable subcollection $\mathscr{C}'$ such that any two are cobordant, and by cutting this down we may assume any two members of $\mathscr{C}'$ are subsets of different members of Y.

Let C_{α_1}, C_{α_2}, and C_{α_3} be members of $\mathscr{C}'$, with C_{α_i} part of the boundary of $M - \bar{M}_{\alpha_i+1}$, etc. and $\alpha_1 < \alpha_2 < \alpha_3$. Since $\bar{M}_{\alpha_i+1}$ is connected, it is easy to see that there must be a pair $i<j$ such that $C_{\alpha_i} \cup C_{\alpha_j}$ is the boundary of a subset of $\bar{M}_{\alpha_j+1}\backslash M_{\alpha_i+1}$. So another application of $\omega_1 \to (\omega_1, \omega)^2$ shows that $\mathscr{C}'$ has an uncountable subcollection $\mathscr{D}$ so that any two members C_α, C_β $(\alpha < \beta)$ form the boundary of a subset of $\bar{M}_{\beta+1}\backslash M_{\alpha+1}$. The union of the members of $\mathscr{D}$ and these subsets they bound is a nonmetrizable closed subsurface of M whose boundary in M is the 'lowest' member of $\mathscr{D}$, call it C_α. Because the removal of C_α disconnects M, C_α is all of the $\alpha + 1$st member of Y. Similarly, each member of $\mathscr{D}$ is a member of Y.

Now suppose the portion $\bar{M}_{\beta+1}\backslash M_{\alpha+1}$ bounded by two members C_α, C_β, of $\mathcal{D}$ is not homeomorphic to $S^1\times[0,1]$. Then the space obtained by identifying C_α and C_β each to a point is not a 2-sphere. So by the converse of the Jordan Curve theorem, there is a circle C which does not separate it, and this circle (which can be assumed to miss both identification points) is not cobordant in M with C_α or C_β, nor with any circle in $M\backslash\bar{M}_{\beta+1}$ or in $\bar{M}_{\alpha+1}$. Indeed, if C were cobordant with a circle 'above' C_β, then the subset they bound would have to contain C_β, which disconnects it and thus is cobordant with C in M, contradition. Similarly for a circle 'below' C_α. Thus there cannot be, in $\mathcal{D}$, an infinite sequence of circles so that the portion bounded by any successive pairs is *not* homeomorphic to $S^1\times[0,1]$. Hence there must be an α such that $\bar{M}_{\beta+1}\backslash M_{\alpha+1}$ is homeomorphic to $S^1\times[0,1]$ for all $\beta>\alpha$, $C_\beta\in\mathcal{D}$, and so $M\backslash\bar{M}_{\alpha+1}$ is a long pipe.

5.15. COROLLARY. *Every ω-bounded 2-manifold is the connected sum of a sphere and finitely many tori, projective planes, and long pipes.*

Note that a kind of converse of Corollary 5.12 is also true: every connected sum of a sphere and finitely many tori, projective planes, and long pipes is an ω-bounded 2-manifold, as long as in connecting the long pipe U to the rest of the manifold we do it so the boundary of each U_α is a union of two circles, one of which is its interface with the complement of the long pipe; and so that the closure of some (hence any) U_α is compact.

What about other ω-bounded surfaces? their boundary components, being countably compact, are of two kinds: circles and long lines. If these are only circles, then by 5.13 there are only finitely many of them, and so there is an α such that M_α contains them all. From this and the bagpipe theorem there follows:

5.16. COROLLARY. *Every ω-bounded surface M with compact boundary has a compact subsurface K such that $M\backslash K$ is the union of finitely many disjoint open long pipes.*

If the boundary includes one or more long lines, we can employ the trick in the second paragraph of 3.7. To simplify matters, let us first sew a disk onto every boundary circle of our manifold, then make two copies of what results, and join them by their boundaries. The result is an ω-bounded manifold. Each long line in our original surface has two 'ends' each of which runs up a long pipe of the new manifold. If we cut this long pipe at one of the circles of Y, we see that there must be exactly two long rays from the boundary of the original surface that 'run its length', i.e. meet it in a copy of L^+. So half of this long pipe belongs to the original surface, and it is what might be called a 'long strip': the union of a chain open subspaces U_α each homeomorphic to $[0,1]\times\mathbb{R}$, with the boundary of U_α in U_β a copy of the closed unit interval whenever $\alpha<\beta$.

So now, if we picture compact bordered surfaces like on p. 45 of MASSEY's

[1967] text, we can amend the pictures to include all ω-bounded surfaces. All we need do is glue finitely many long strips and long pipes to the boundary of the bordered surface. The long strips we attach by their 'lower ends' so their interface with the rest of the surface is a closed arc; while to each long pipe we first add a circle at the 'lower end' to countably-compactify it, then identify a closed arc of this circle with an arc on the boundary of the bordered surface.

The following question is natural, and difficult:

5.17. PROBLEM. How can these results be generalized to higher dimensions? For example, is it true that an ω-bounded n-manifold has a compact submanifold-with-boundary K such that $M\backslash K$ is the union of finitely many subspaces, each of which is the union of a chain of subsets homeomorphic to $N \times \mathbb{R}$ for the same compact $(n-1)$-manifold N?

Difficulties arise in three places: Corollary 5.11 with its reliance on PL structures, Lemma 5.13 with its reliance on circles and their special role in homology, and the last paragraph in the proof of Theorem 5.14.

6. A variety of ω-bounded 2-manifolds

With the Bagpipe Theorem and the classical results on compact surfaces, the study of ω-bounded 2-manifolds reduces to the study of long pipes. This in turn is essentially equivalent to the study of simply connected, ω-bounded, nonmetrizable 2-manifolds. [We can take 'simply connected' to mean that every Jordan curve (copy of S^1) separates the surface into exactly two components, one of which has compact closure.] This is because of the comments following Definition 5.2 and the following:

6.1. LEMMA. *A noncompact 2-manifold is simply connected and ω-bounded if, and only if, it is the union of a chain $\{U_\alpha : \alpha < \omega_1\}$ of open subspaces U_α, each homeomorphic to $\mathbb{R}^2$, such that $\bar{U}_\alpha \subset U_\beta$ whenever $\alpha < \beta$.*

PROOF. If a 2-manifold M is the union of such a chain, and S is a copy of S^1, then there exists α such that $S \subset U_\alpha$. Then by the Jordan Curve Theorem, S separates U_α into two components; $U_\alpha \backslash S = V_1 \oplus V_2$. One component V_i has the property that $S \cup V_i$ is compact, the other does not. Then it is easy to show that $M\backslash S$ is the disjoint union of the open subspaces V_i and $M\backslash(S \cup V_i)$ and that both are connected.

Conversely, suppose M is a simply connected, ω-bounded 2-manifold. By the bagpipe theorem (5.14), M has a compact subsurface K such that $M\backslash K$ is a finite union of long pipes. There is a copy S of S^1 in one long pipe that separates a nonmetrizable part of it from the rest of the space N, which must be metrizable.

So there is only one long pipe. Moreover, S can be chosen so that it has a closed neighborhood homeomorphic to $S^1 \times [0, 1]$, oriented so the copy of $S^1 \times \{0\}$ is a subset of N. It is routine (using simple connectedness of M) to show that the union of N with the copy of $S^1 \times [0, 1)$ is simply connected and metrizable, hence homeomorphic to $\mathbb{R}^2$. If we let $\{U'_\alpha: \alpha < \omega_1\}$ be so that U'_α is like U_α in Definition 5.2, then the same conclusion holds for the boundary of U'_α in $U'_{\alpha+1}$ in place of S. So we can let $U_\alpha = U'_\alpha \cup N$.

The sequence $\langle U_\alpha: \alpha < \omega_1\rangle$ given by this proof is *not* necessarily canonical. The definition of 'long pipe' does not say that $U'_\lambda = \bigcup_{\alpha<\lambda} U'_\alpha$ whenever λ is a limit ordinal, and it may well be impossible to add this and still satisfy the definition of 'long pipe'. It is true that $\bigcup_{\alpha<\lambda} U_\alpha$ as defined just now will be homeomorphic to $\mathbb{R}^2$, because of the converse of the Jordan curve theorem, but its boundary need not be a circle. It could (for example) be a Warsaw circle or even the shoreline of one of the Lakes of Wada. These were first described by YONEYAMA [1917] as three sharing a single shoreline, an indecomposable continuum. A related construction is illustrated between (!) pages 424 and 425 of BROUWER [1910]. See also pp. 119–124 and 143–145 of HOCKING and YOUNG [1961]. [*Caution.* The description of the pseudo-arc there is incorrect; correct descriptions may be found in MOISE [1948] and BING [1948].] At any rate, there is an enormous variety of possibilities, and there may be no way of avoiding them in a given example. The boundaries are the bones of a skeleton, [Definition 4.9], and if a pathological boundary occurs on a stationary set of indices, it will show up in any skeleton.

This last fact can be used in any elementary way to construct $2^{\aleph_1}$ topologically distinct ω-bounded, simply connected 2-manifolds. This is the greatest number possible, because there are only $2^{\aleph_1}$ topologically distinct Čech complete spaces of weight $\aleph_1$ (HODEL [198·]) and every ω-bounded manifold is of weight $\leqq \aleph_1$.

6.2. EXAMPLES. It is possible to divide up ω_1 into $\aleph_1$ disjoint stationary sets, $\{A_\xi: \xi < \omega_1\}$, cf. KUNEN [1980], p. 79. Suppose $\sigma: \omega_1 \to \{0, 1\}$ given, we will construct an ω-bounded simply connected 2-manifold M^σ with a canonical sequence $\{M^\sigma_\alpha: \alpha < \omega_1\rangle$ such that $M^\sigma_\alpha \approx \mathbb{R}^2$ for each α, and the boundary of M^σ_α in M^σ ('the αth bone of M') is a circle iff $\sigma(\xi) = 0$ for the unique ξ satisfying $\alpha \in A_\xi$.

If $\sigma \neq \tau$, the resulting manifolds cannot be homeomorphic. Indeed, suppose $\sigma(\xi) \neq \tau(\xi)$, and suppose there is a homeomorphism $\psi: M^\sigma \to M^\tau$. Then by Lemma 4.4, there is a cub C of indices α such that $\psi(M^\sigma_\alpha) = M^\tau_\alpha$. But if $\alpha \in A_\xi \cap C$, then the boundary of M^σ_α in M^σ is not homeomorphic to the boundary of M^τ_α in M^τ, contradiction. Hence there are $2^{\aleph_1}$ topologically distinct ω-bounded, simply connected 2-manifolds.

Begin by putting the discrete topology on ω_1 and forming the product $B^2 \times \omega_1$, where B^2 is the open unit disc in R^2, Let $M^\sigma_0 = B^2 \times \{0\}$. From now on we suppress the superscript σ since it will be fixed throughout.

Suppose M_α has been defined along with a homeomorphism $\psi_\alpha: M_\alpha \to B^2$. If $\alpha \in A_\xi$ and $\sigma(\xi) = 0$, let $\phi_\alpha: B^2 \to B^2 \times \{\alpha + 1\}$ be defined by $\phi_\alpha(x, y) =$

$\langle \frac{1}{2}x, \frac{1}{2}y, \alpha+1\rangle$. If $\sigma(\xi)=1$, let ϕ_α embed B^2 in $B^2 \times \{\alpha+1\}$ in such a way that the image has compact closure and a boundary that is not homeomorphic to S^1. [For the sake of definiteness, we could let ϕ_α be defined as in Example 3.11, except with the range contracted by a factor of 2. Then the boundary can be pictured as a circle of radius $\frac{1}{2}$ with a line segment connecting its center to a point on its boundary.] Let the underlying set for $M_{\alpha+1}$ be $[(B^2 \times \{\alpha+1\}) \setminus \phi_\alpha(\psi_\alpha(M_\alpha))] \cup M_\alpha$. Let $f_{\alpha+1}\colon M_{\alpha+1} \to B^2 \times \{\alpha+1\}$ be defined by letting it agree with $\phi_\alpha \circ \psi_\alpha$ on M_α and otherwise defining $f_{\alpha+1}(p)=p$. Let the topology on $M_{\alpha+1}$ be the one making $f_{\alpha+1}$ a homeomorphism. Of course, M_α is an open subspace of $M_{\alpha+1}$ and the relative topology is the original one; also, $M_{\alpha+1}$ is homeomorphic to $\mathbb{R}^2$.

If λ is a limit ordinal, we let $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$, with the direct limit topology. In other words, we decree that each M_α is an open subspace of M_λ (see the comments following Lemma 2.6). By the converse of the Jordan curve theorem, M_λ is homeomorphic to $\mathbb{R}^2$. (See also BROWN [1961].)

Of course, there are many ways of generalizing this example. We can easily arrange it so that distinct indices ξ satisfying $\sigma(\xi)=1$ have associated with them topologically distinct boundaries. For example, a boundary could be a circle with a discrete sequence of +'s and |'s attached to it, converging to a point on the boundary; two such sets are homeomorphic if, and only if, the associated sequences agree on every term, so there are $\mathfrak{c}$ such topologically distinct boundaries. Of course, as hinted before the above construction, there is a much greater variety of possibilities; just how great can be gleaned from two papers of BING: [1952], especially Theorem 6 and [1962], especially Theorem 9. A good reference to the early history of this theme of boundaries of planar regions is KURATOWSKI's [1924] paper.

Even if the 'bones' are all circles, there is no need for two simply connected ω-bounded 2-manifolds to be homeomorphic. Consider $L \times L$, which has four 'diagonal rays' (Examples 3.5), and a sphere with a disc cut out of it and a long pipe homeomorphic to $L^+ \times S^1$ attached to the boundary. In the resulting space, any two copies of L^+ must either become disjoint or coincide outside some Lindelöf subset. With a little more effort we can construct $2^{\aleph_1}$ distinct examples, each with a skeleton of circles. Such a construction might run along the lines of the examples in 3.8, with the sequences σ running up to ω_1 instead of stopping at ω. Of course, the M_σ's in 3.8 are not ω-bounded, but we could start the construction with four homeomorphic octants oriented with respect to each other like the first, fourth, fifth, and eighth octants of $L \times L$, then ever after proceeding with fourfold symmetry. The details of what to do at limit ordinal stages are left as an exercise for the reader.

More important than these purely quantitative considerations are certain qualitative problems. For example: how strong a normality condition can a simply connected, ω-bounded 2-manifold satisfy? What sorts of zero-dimensional and one-dimensional subspaces can/must one have? In particular, does every such manifold contain a copy of ω_1?

These are difficult problems which apparently depend upon extra set-theoretic axioms, at least to some extent. It is true that no ω-bounded space can be perfectly normal and noncompact [see comment (2) following the statement of Theorem 4.14]. It is also true that there are examples of non-normal, simply connected, ω-bounded 2-manifolds in ZFC (NYIKOS [1979]). But what happens in between is still open. The search for a hereditarily normal example is made difficult by:

6.3. LEMMA. *If M is a hereditarily normal ω-bounded trunklike manifold, then M cannot contain a pair of disjoint closed noncompact subsets.*

PROOF. As usual, let $\Sigma = \langle M_\alpha : \alpha < \omega_1 \rangle$ be canonical for M. Any closed noncompact subset E must meet the bones indexed by some cub subset of ω_1. This is because F cannot be contained in any M_α, and if it meets $M_{\alpha_n+1} \backslash M_{\alpha_n}$ and $\langle \alpha_n : n \in \omega \rangle$ is an increasing sequence, it must meet $\bar{M}_\alpha \backslash M_\alpha$ (where $\alpha = \sup\{\alpha_n\}_{n=0}^{\infty}$) by countable compactness.

Suppose disjoint noncompact closed subsets F_1 and F_2 are given. By Urysohn's Lemma there is a continuous function $f: M \to [0, 1]$ such that $f(F_1) = \{0\}$ and $f(F_2) = \{1\}$. Now for each $r \in [0, 1]$, $f^{-1}\{r\}$ is noncompact, otherwise its removal would disconnect the space, violating 'trunklike'. For each rational $q \in [0, 1]$ let C_q be a cub subset of ω_1 such that $f^{-1}\{q\}$ meets every bone indexed by C_q. Let C' be the set of all limit points of $C = \bigcap\{C_q : q \in \mathbb{Q} \cap [0, 1]\}$. By continuity of f, and compactness of each bone, every $f^{-1}\{r\}$ ($r \in [0, 1]$) meets every bone indexed by a member of C.

The remainder of the proof is reminiscent of the argument that $\omega_1 \times (\omega + 1)$ is not hereditarily normal (see the article by Przymusinski). [Compare also the proof in NYIKOS [1982] that under MA + $\neg$CH, every hereditarily normal, hereditarily cwH vector bundle is metrizable.] Let

$$A_0 = f^{-1}\{0\} \backslash \bigcup\{\bar{M}_\alpha \backslash M_\alpha : \alpha \in C'\},$$

$$A_1 = \bigcup\{\bar{M}_\alpha \backslash M_\alpha : \alpha \in C'\} \backslash f^{-1}\{0\}.$$

Then A_0 and A_1 are separated (that is, $\bar{A}_0 \cap A_1 = \emptyset$ and $\bar{A}_1 \backslash A_0 = \emptyset$) but, as we will now show, they cannot be put into disjoint open subsets of M; this will show M is not hereditarily normal.

Suppose U is an open set such that $A_0 \subset U$. For each $\alpha \in C \backslash C'$ there exists $\varepsilon_\alpha > 0$ such that $(\bar{M}_\alpha \backslash M_\alpha) \cap f^{-1}[0, \varepsilon_\alpha) \subset U$. Hence there is some $\varepsilon > 0$ such that $f^{-1}\{\varepsilon\} \cap U$ meets uncountably many distinct bones $\bar{M}_\alpha \backslash M_\alpha$. But $f^{-1}\{\varepsilon\}$ is closed and so there is $\gamma \in C'$ such that some point of $f^{-1}\{\varepsilon\} \cap (\bar{M}_\alpha \backslash M_\alpha)$ is in the closure of U. But $f^{-1}\{\varepsilon\} \cap (\bar{M}_\alpha \backslash M_\alpha) \subset A_1$, and we are done.

The proof we have just given clearly extends to locally compact, locally connected spaces (in place of manifolds). In particular, we have:

6.4. COROLLARY. *Every hereditarily normal ω-bounded surface has a compact boundary.*

PROOF. Suppose the boundary contained a long line. Then part of this line would form part of the boundary of a 'long strip', while disjoint from it would be a long ray forming 'the other edge' of the long strip. But the closure of the long strip is an ω-bounded trunklike manifold-with-boundary with a pair of disjoint closed long rays, contradicting the aforementioned extension of Lemma 6.3.

It may be that, in some model of set theory, Corollary 6.4 can be strengthened to read "Every hereditarily normal, ω-bounded surface is compact." On the other hand, the last example in this article is a noncompact ω-bounded surface constructed using $\diamond$ which is hereditarily collectionwise normal. This will follow from the following stronger property: *every closed noncompact subset contains all the bones indexed by some cub subset of* ω_1. This also shows that the example does not contain a copy of ω_1 or, indeed, any noncompact zero-dimensional subset.

Just how far we are from a ZFC example may be seen from this, that at present we do not have any 'real' example of a countably compact, first countable, noncompact space which does not contain a copy of ω_1. We do not even have one assuming only CH! In fact, the author was made aware of this deficiency in our knowledge through trying to construct an ω-bounded surface with this property.

Our $\diamond$ example will have certain other properties which we will introduce now. For the most part we will be following the notation and conventions of HUSEMOLLER [1966]. (Some authors reserve 'bundle' for 'locally trivial bundle' or 'locally trivial fiber bundle'.)

6.5. DEFINITION. A *bundle* is a triple $\langle E, p, M\rangle$ where E and M are topological spaces and $p\colon E \to M$ is continuous. The space E is called the etale or *total* space, M is called the *base* space, and p the *projection* of the bundle. For each $x \in M$, the set $p^{-1}(x)$ is called the *fiber* [var. sp. 'fibre'] *of* $\langle E, p, M\rangle$ *over* x. If $p^{-1}(x)$ is homeomorphic to a fixed space F for each x, then $\langle E, p, M\rangle$ is *a bundle with fiber* F.

We will often identify a bundle with its total space and say 'E is a bundle over M' if $\langle E, p, M\rangle$ is a bundle.

6.6. LEMMA. *A long pipe can be made into a bundle over L^+ with fiber S^1 if, and only if, it has a skeleton in which every bone is homeomorphic to S^1.*

PROOF. If the pipe E has such a skeleton, take the αth bone to α and define the rest of p by composing, for each $\alpha < \omega_1$, any homeomorphism of $\overline{E_{\alpha+1}}\backslash E_\alpha$ with $(0, 1] \times S^1$ that sends the αth bone to $\{1\} \times S^1$ with the first projection map and the obvious map from $(0, 1]$ to $(\alpha, \alpha + 1]$. Conversely, if $\langle E, p, L^+\rangle$ is a bundle with fiber S^1, we can let $E_\alpha = p^{-1}(0, \alpha)$ and $\langle E_\alpha\colon \alpha < \omega_1\rangle$ will be a canonical sequence.

6.7. DEFINITION. A bundle $\langle E, p, M\rangle$ with fiber F is *trivial* if there is a homeomorphism ϕ of E with $M \times F$ which preserves fibers, i.e. $\psi(p^{-1}(x)) = \pi_1^{-1}(x)$ for all $x \in M$. We say ϕ is a *bundle isomorphism of* $\langle E, p, M\rangle$ *with* $\langle M \times F, \pi_1, M\rangle$. The bundle is *locally trivial* if for each $x \in M$ there is a neighborhood U of x and a bundle isomorphism ϕ of $p^{-1}(U)$ with $U \times F$.

In the second part of this definition, it is understood that the projection of 'the bundle $p^{-1}(U)$' is the restriction of p to $p^{-1}(U)$. The terminology is a bit misleading because most of the examples that interest researchers are 'locally trivial'. This means much more than just the condition that $p^{-1}(U)$ is homeomorphic to $U \times F$. For example, one might begin with an ordinary open cyclinder with the axis horizontal and a sequence of copies of S^1 with bights in them that circle the axis of the cylinder, converging to a circle without a bight, as illustrated in Fig. 3.

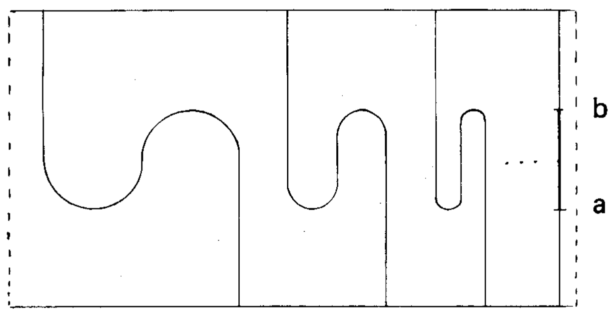

Fig. 3.

This is to be done so that each point between a and b on the last circle will have a base of neighborhoods, each of which meets all but finitely many of the circles in three disjoint intervals. It is easy to make the cyclinder into a bundle over $(0, 1)$ with fiber S^1 so that each of these circles are fibers over various points; such a bundle cannot be locally trivial.

Assuming ♣, one can use this bundle as a prototype for a bundle over L^+ with fiber S^1 whose total space is a long pipe (which is therefore locally homeomorphic to $(0, 1) \times S^1$) which *cannot be the total space of a locally trivial bundle over* S^1. In fact, it is just as easy to use the following weakening of ♣:

6.8. AXIOM. It is possible to choose, for each countable limit ordinal λ, a sequence $S_\lambda \subset \lambda$ whose only limit point is λ, in such a way that if C is a cub subset of ω_1, there exists a λ such that $S_\lambda \subset C$.

Of course, any such λ is itself in C, and C contains a stationary set of such λ's.

6.9. EXAMPLE [Assuming Axiom 6.8]. One proceeds as in the examples of 6.2, building up E_{ω_1} by maps ψ_α, ϕ_α and subspaces E_α. If α is a successor, one defines

ϕ_α as it was defined when $\sigma(\xi)=0$. As before, E_λ is the direct limit if λ is a limit ordinal, and then one defines $\phi_\lambda : B^2 \to B^2 \times \{\lambda + 1\}$ as follows. Let $\langle \alpha_n : n \in \omega \rangle$ list S_λ in increasing order. The image of B^2 is to have as its boundary the copy of the circle of radius $\frac{1}{2}$ centered on the copy of the origin; the image under $\phi_\lambda \circ \psi_\lambda$ of the Jordan curve that forms the boundary of M_{α_n} is to have a bight in it; and the bights are to converge to an interval on the boundary of $\phi_\lambda(B^2)$ [see Fig. 4].

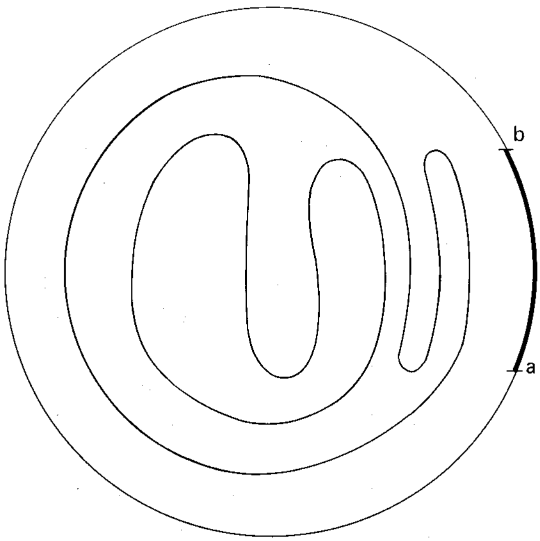

Fig. 4.

Finally, we remove a point from E_{ω_1} to form a long pipe E. Consider now any bundle $\langle E, p, L^+\rangle$ with fiber S^1. The fibers over the ordinals α of L^+ form a skeleton of E, which will agree with the set of boundaries of E_α for a cub C of indices. Pick $S_\lambda \subset C$; the bundle fails to be locally trivial at λ.

This example illustrates how easily a non-topological concept (in this case, a bundle construction) can be 'writ large' in a nonmetrizable manifold as a topological property with the help of Axiom 6.8 and Lemma 4.4. In this case, local triviality becomes transmuted into a *global* property in an essential way: the initial segments of E *do* admit locally trivial (in fact, trivial) bundle maps into initial segments of L^+, but although the initial segments cover E, none of the bundle maps can be extended to all of E.

In some constructions which begin with the use of Axiom 6.8, it is possible to do something similar in ZFC, but if so the constructions usually become much less natural and more intricate. Anyway, Axiom 6.8 is reasonably general, being implied by ♣ and *a fortiori* $\diamond$, and is also compatible with MA + $\neg$CH. For if one obtains model of MA by the usual method of iterated ccc forcing, then every cub subset of ω_1 contains a cub in the ground model [set-theoretic folklore]. On the other hand, PFA (see the article by Baumgartner) negates Axiom 6.8, and so does Shelah's axiom ✡ which is compatible with CH. These axioms even negate the modification of 6.8 which concludes with '$S_\lambda \cap C$ is infinite' rather than '$S_\lambda \subset C$', which is adequate for Example 6.9.

Much has been done on the question of when a bundle is locally trivial, cf. McAULEY [1966] and later papers in the same volume, and DYER and HAMSTROM [1957]. For our purpose, the following discussion will suffice.

Suppose, in Example 6.2, the maps $\psi_\alpha: M_\alpha \to B^2$ can be chosen so that the composition $\psi_\gamma \circ \psi_\alpha^{-1}: B^2 \to B^2$ takes circles centered on the origin to other circles centered on the origin whenever $\alpha < \gamma$; in particular, then, $\psi_\alpha^{-1}(\langle 0, 0\rangle)$ is the same point for all α. Then the manifold E that results when the preimage of the origin is removed 'is' a locally trivial bundle over L^+ in a canonical way. The construction begins with the natural map $k_1: B^2 - \{\langle 0, 0\rangle\}$ to L^+ sending the circle of radius r centered on the origin to r. The general idea is to define for each $\alpha < \omega_1$, a continuous map k_γ from $B^2 - \{\langle 0, 0\rangle\}$ onto $(0, \gamma)$ so that:

(i) The preimage of each $x \in (0, \gamma)$ is a circle centered on the origin;

(ii) $k_\gamma \circ \psi_\gamma \restriction E_\alpha = k_\alpha \circ \psi_\alpha$ for all $\alpha < \gamma$; in particular, k_γ sends the image of E_α under ψ_γ to $(0, \alpha)$.

In this way, $p = \bigcup\{k_\gamma \circ \psi_\gamma: \gamma < \omega_1\}$ becomes a well-defined continuous function from E to L^+; the fibers are all homeomorphic to S^1; and each ψ_γ is an isomorphism of the bundle $\langle E_\gamma, k_\gamma \circ \psi_\gamma, (0, \gamma)\rangle$ with $\langle B^2 - \{\langle 0, 0\rangle\}, k_\gamma, (0, \gamma)\rangle$, which is obviously locally trivial (even trivial). A definition of k_γ would be complicated if written out in full, but the construction poses no real problem. If γ is a limit ordinal, k_γ is unique, while if $\gamma = \alpha + 1$, condition (ii) determines $k_\gamma \restriction \psi_\gamma(E_\alpha)$, and k_γ itself can be any extension compatible with the other constraints.

Conversely, any locally trivial bundle over L^+ with fiber S^1 can be constructed in exactly the same way, using maps $\psi_\alpha: E_\alpha \to B^2 - \{\langle 0, 0\rangle\}$ as described. One begins with maps guaranteed by the following lemma, and composes them with maps from $(0, \alpha) \times S^1$ onto $B^2 - \{\langle 0, 0\rangle\}$ sending π_1-fibers to circles concentric to the origin.

6.10. LEMMA. *Let α be a countable ordinal. Every locally trivial bundle over $(0, \alpha)$ with fiber S^1 is trivial.*

PROOF. We can let $\alpha = 1$ without loss of generality. Let $\langle E, p, (0, 1)\rangle$ be locally trivial, and for each $x \in (0, 1)$, pick an open interval (a, b) about x so the restriction of p there is trivial. By paracompactness, etc. we may pick, for each $n \in \mathbb{Z}$, points x_n so the intervals (a_n, b_n) cover $(0, 1)$ and overlap only with the immediately adjacent intervals. By local triviality there are homeomorphisms $f_n: p^{-1}(a_n, b_n) \to (a_n, b_n) \times S^1$ which send the fiber over x to $\{x\} \times S^1$ for each $x \in (a_n, b_n)$. Thus $f_{n+1} \circ f_n^{-1}$ is a homeomorphism of $(a_{n+1}, b_n) \times S^1$ which sends each $\{x\} \times S^1$ onto itself. By induction we can, if necessary, alter each f_n to make $f_{n+1} \circ f_n^{-1}$ orientation-preserving.

For each interval (a_{n+1}, b_n) pick a nondegenerate closed subinterval $[c_n, d_n]$. By elementary homotopy theory, there is a map g_n from $[c_n, d_n] \times S^1$ onto itself which equals $f_{n+1} \circ f_n^{-1}$ on $\{d_n\} \times S^1$ and is the identity on $\{c_n\} \times S^1$. Then for each n, the function f'_n which equals f_n on $[d_{n-1}, c_n]$ and $g_n \circ f_n$ on $[c_n, d_n]$ is a homeomorphism, and $\bigcup\{f'_n\colon n \in \mathbb{Z}\}$ is a well-defined homeomorphism from E to $(0, 1) \times S^1$ sending $p^{-1}(x)$ to $\{x\} \times S^1$, as desired.

The correspondence we have just outlined is generalized in many ways: see AUSLANDER and MACKENZIE [1977] Section 9-4 or Chapter 5 of HUSEMOLLER [1966]. Once one has established local triviality, it is traditional, in the theory of bundles, to have a map like ψ_α go from E_α directly to $(0, \alpha) \times S^1$ rather than $B^2 - \{\langle 0, 0\rangle\}$ or (which amounts to the same thing, if one imposes the usual polar coordinate system) $(0, 1) \times S^1$ as we have done, and then k_γ becomes simply the projection π_1. [The set-up of 6.2 and the discussion preceding Lemma 6.10 is more appropriate to differential geometry.]

Of course, we cannot replace α by ω_1 in the above lemma: removing $\langle 0, 0\rangle$ from $L \times L$ gives a space which can be made very naturally into a locally trivial bundle over L^+ (of course it is most natural to think of the fibers as squares rather than circles!) which is not trivial because the total space is not even homeomorphic to $L^+ \times S^1$. [Topology alone is not always enough to tell whether a locally trivial bundle is trivial: see the comments following Theorem 6.15].

Many of the basic concepts of differential geometry and the theory of fiber bundles incorporate transition maps like $\psi_\gamma \circ \psi_\alpha^{-1}$ and $f_{n+1} \circ f_n^{-1}$ in their definitions. In the case of locally trivial bundles over L^+ with fiber S^1, we may, as we have just seen, assume these compositions take circles concentric to the origin to other such circles. To bring this more in line with the traditional picture, we can follow this composition up with a map $k_{\alpha\gamma}\colon B^2 \to B^2$ which moves points along the rays joining then to the origin in a generalized 'dilation-contraction' so that the map $k_{\alpha\gamma} \circ \psi_\gamma \circ \psi_\alpha^{-1}$ sends each circle concentric to the origin to itself. Then we can classify locally trivial bundles over S^1 according to how each such map moves the points of each circle.

If this motion is a rotation each time and for each circle, the resulting object is, in a natural way, what is known as a 'principal $\mathbb{T}$-bundle'. Here $\mathbb{T}$, misleadingly called 'the torus group', is the multiplicative group of all complex numbers of absolute value 1. (Recall that each such number $e^{i\theta}$, which we will write as θ, can be identified with a rotation of any circle through θ radians.) Applying Ockham's Razor (at least to the extent appropriate to the present article) to pp. 39–41 of HUSEMOLLER [1966], we arrive at:

6.11. DEFINITION. Let G be a topological group. A bundle $\langle E, p, M\rangle$ together with a continuous map $\psi\colon E \times G \to E$ is *an effective G-bundle* if:

(i) For all $x \in E$, $s, t \in G$, $x(st) = (xs)t$ where xs is the image of $\langle x, s\rangle$ under ψ and st is the product taken in G.

(ii) $xs = x$ iff $s = 1$, the identity of G.

(iii) The orbit of x under G, $\{xs\colon s \in G\}$, is the fiber $p^{-1}(p(x))$.

One generally omits mention of the map ψ, and thinks of G as a group of homeomorphisms of E leaving fibers fixed.

6.12. DEFINITION. Let E, G, and M be as above, then $\langle E, p, M\rangle$ is a *principal G-bundle* if it is an effective G-bundle and the map $\langle x, xs\rangle \to s$ is (jointly) continuous on the subspace of E^2 where it is defined.

In other words, if G, E, and M are first countable, then E is a principal G-bundle if it is effective and whenever $x_n \to x$ and $y_n \to y$ and x_n, y_n are in the same fiber, then the homeomorphism (element of G) that takes x_n to y_n will converge to the homeomorphism that takes x to y. Moreover, for each $x \in E$, the map $\theta \mapsto x\theta$ is a homeomorphism from G onto the fiber to which x belongs.

Let us see how this applies to the case of a principal $\mathbb{T}$-bundle E over L^+. There is a natural temptation to identify each fiber with $\mathbb{T}$, but the whole point is that there is no canonical way of doing this. However, one can and does speak of the member θ of $\mathbb{T}$ sending x to y (assuming x and y are on the same fiber) and write $y = x\theta$.

If E is locally trivial, we can choose the maps ψ_α discussed earlier so that they not only take fibers to circles concentric to the origin, but also are determined by the image of a single point y on each fiber. We simply let $\psi_\alpha(y\theta) = [\psi_\alpha(y)]\theta$, where as before we identify θ with a rotation of any circle through an angle of θ. To define $\psi_\alpha(y)$ itself, we use the method of local cross sections.

6.13. DEFINITION. Let $\langle E, p, M\rangle$ be a bundle. A *global cross section* (also called a *cross section* or simply a *section*) is a map $s: M \to E$ whose composition with p is the identity on M. A *local cross section* is a map $S: U \to E$ from an open subset U of M whose composition with p is the identity on U.

It is easy to see that a section is an embedding of M [resp. U] in E sending each point into the fiber over it. If $M = (0, \alpha)$ $(\alpha < \omega_1)$ and E_α is locally trivial, then there is no problem in finding such an embedding s_α because all we need is to construct an arc that meets each fiber in a single point, and we can use Lemma 6.10. If we think like fiber bundle theorists, we will now take E_α to $(0, \alpha) \times S^1$ by sending $s_\alpha(x)$ to $\langle x, 0\rangle$ and then taking $[s_\alpha(x)]\theta$ to $\langle x, \theta\rangle$. If we think like differential geometers, we will take a nice map f_α from $(0, \alpha)$ onto the unit interval of the positive real axis in the complex plane, then let ψ_α take $[s_\alpha(x)]\theta$ to $\langle f_\alpha(x), \theta\rangle$ for each θ [polar notation]. In particular, then, $\psi_\alpha(s_\alpha(x)) = \langle f_\alpha(x), 0\rangle$. One cannot expect to be able to prolong s_α to a global cross section, so we can expect there to be some γ and some $x \in (0, \alpha)$ such that $s_\gamma(x) \neq s_\alpha(x)$. What happens in such a case is that ψ_γ takes $s_\gamma(x)$ to $\langle f_\gamma(x), 0\rangle$; as for $s_\alpha(x)$, we must find the θ_x that satisfies $s_\alpha(x) = s_\gamma(x)\theta_x$ and then ψ_γ takes $s_\alpha(x)$ to $\langle f_\gamma(x), \theta_x\rangle$; also $\psi_\gamma(s_\alpha(x)\theta) = \langle f_\gamma(x), \theta_x + \theta\rangle$ in general. Thus $\psi_\gamma\psi_\alpha^{-1}\langle f_\alpha(x), \theta\rangle = \langle f_\gamma(x), \theta_x + \theta\rangle$; the map $\psi_\gamma\psi_\alpha^{-1}$, after a dilation or contraction, turns the fiber over x through an angle of θ_x radians. (This angle will, of course, be a continuous function of x.) Had we followed the bundle theorists, there would be no dilation or contraction, and θ_x would stand alone.

Of course, rotations are a very special class of objects, and so are principal $\mathbb{T}$-bundles: it is not easy to find a nontrivial example, because of the following lemma and clauses (i) and (iii) of Theorem 6.15.

6.14. LEMMA. *A principal G-bundle is trivial if, and only if, it has a global cross section.*

PROOF. If $\langle E, p, M\rangle$ is trivial, the inverse image of $M \times \{s\}$ under a bundle isomorphism from E to $M \times G$ is a global cross section for any $s \in G$. Conversely, if $f: M \to E$ is a cross section, it is routine to show that $\phi: [f(x)]s \mapsto \langle x, s\rangle$ is a homeomorphism (see HUSEMOLLER [1966], Theorem 3.2, p. 42) and it clearly preserves fibers.

6.15. THEOREM. *Let $\langle E, p, L^+\rangle$ be a locally trivial bundle with fiber S^1. The following are equivalent.*

(i) *$\langle E, p, L^+\rangle$ is trivial.*

(ii) *There is a homeomorphism ψ from E to $L^+ \times S^1$ and there exists $\alpha < \omega_1$ such that $\psi^{-1}[\alpha, \omega_1) \times \{\theta\}$ is the range of a local cross section for each $\theta \in S^1$.*

(iii) *There exists a subspace of E that is homeomorphic to L^+, and $\langle E, p, L^+\rangle$ can be made into a principal $\mathbb{T}$-bundle.*

Moreover, any homeomorphism ψ from E to $L^+ \times S^1$ must, in such an event, have the property described in (ii) *for some $\alpha < \omega_1$.*

PROOF. (i)$\to$(ii) and (i)$\to$(iii) are obvious. [The obvious way of making $L^+ \times S^1$ into a principal $\mathbb{T}$-bundle works.]

(ii)$\to$(i). By Lemma 4.4, any homeomorphism ψ from E to $L^+ \times S^1$ must have the property that $\pi_1 \circ \psi$ and p agree on the inverse image of cub C. Let β be the least member of C such that $\alpha \leq \beta$. Below β, we can adjust ψ using Lemma 6.10 to make its composition with π_1 equal p. And above β, we can compose it with the map f that takes $\psi(x) = \langle \gamma, \theta\rangle$ to $\langle p(x), \theta\rangle$. The restriction of f to cylinders $[c_1, c_2] \times S^1$ between successive members of C is a homeomorphism because of the triviality of p over $[c_1, c_2]$, and so f is an auto-homeomorphism of $[\beta, \omega_1) \times S^1$. It is easy to see that the composition of ψ with f is an isomorphism of $\langle E\backslash E_\beta, p, [\beta, \omega_1)\rangle$ with $\langle [\beta, \omega_1) \times S^1, \pi, [\beta, \omega_1)\rangle$.

(iii)$\to$(i). By Lemma 6.14, it is enough to show that $\langle E, p, L^+\rangle$ has a global cross section—in other words, we must find a copy of L^+ in E that meets each fiber in a singleton. And by Lemma 6.10, it is enough to find $\alpha < \omega_1$ and a copy of L^+ in $p^{-1}[\alpha, \omega_1)$ that meets each fiber in a singleton.

Suppose it is impossible to do this, and Y is a copy of L^+ in E. For each $\alpha < \omega_1$, there exists $x_\alpha \in (\alpha, \omega_1)$ such that the fiber over x_α meets Y in at least two points,

y_α and $y_\alpha\theta_\alpha$. We may assume there exists $\varepsilon > 0$ such that (the radian measure of) $\theta_\alpha \geqq \varepsilon$ for all α.

Fix a homeomorphism $\phi: Y \to L^+$. By Lemma 4.4, $p^{-1}(0, \alpha) \cap Y$ and $\phi^{-1}\pi_1^{-1}(0, \alpha)$ must agree for all α in some cub C. So we may pick an increasing sequence $\{\alpha_n : n \in \omega\}$ from C so that $x_{\alpha_n} \in (\alpha_n, \alpha_{n+1}]$. Then $\alpha = \sup_n \alpha_n$ is in C and $\langle y_{\alpha_n} \rangle$ must converge to $\phi^{-1}\pi_1^{-1}(\alpha)$. But $y_{\alpha_n}\theta_{\alpha_n}$ would also have to converge to the same point, which is impossible.

To prove the 'moreover' portion, let $\phi: \langle E, p, L^+ \rangle \to \langle L^+ \times S^1, \pi_1, L^+ \rangle$ be a bundle isomorphism, and consider the homeomorphism $\psi \circ \phi^{-1}$ of $L^+ \times S^1$. The image of each $\omega_1 \times \{\theta\}$ is a copy of L^+, so by Lemma 3.4, there must exist α_θ such that $\psi \circ \phi^{-1}([\alpha_\theta, \omega_1) \times \{\theta\}) = [\beta_\theta, \omega_1) \times \{\theta'\}$ for some θ'. Let Q be a countable dense subset of S^1, and let $\alpha = \sup\{\alpha_\theta : \theta \in Q\}$, $\beta = \sup\{\beta_\theta : \theta \in Q\}$. Then $\psi \circ \phi^{-1}([\alpha, \omega_1) \times \{\theta\}) = [\beta, \omega_1) \times \{\theta'\}$ for all $\theta \in S^1$ and $[\beta, \omega_1) \times \{\theta'\}$ meets the image of each fiber under ψ in the singleton $\{\psi(\phi^{-1}(\theta))\}$.

With all this behind us, it is a routine exercise to give examples of locally trivial, nontrivial bundle maps from $L^+ \times S^1$ itself to L^+ with fiber S^1. For example, we can pick a θ and make sure the fiber over $\lambda + 1$ runs along an interval of $\omega_1 \times \{\theta\}$ for each limit ordinal $\lambda < \omega_1$, and make this fiber meet every other $\omega_1 \times \{\theta'\}$ in a singleton. That way, Theorem 6.15(ii) insures that the resulting bundle cannot be trivial, but it can be made locally trivial by a transfinite induction. It is routine to define the fibers between λ and the next limit ordinal to 'straighten' the bundle out, and in fact we can arrange it so the fiber over x is $\{x\} \times S^1$ whenever $x \notin (\lambda, \lambda + 2)$ for some limit ordinal λ. Details are routine but lengthy, and we omit them here.

It is much less routine to define a nontrivial principal $\mathbb{T}$-bundle over L^+ because we cannot allow it to contain a copy of L^+ by Theorem 6.15(iii). [It can be shown that every principal $\mathbb{T}$-bundle over L^+ is locally trivial.] It can be done in ZFC, but the available constructions all take longer to describe than our final example, which uses $\diamondsuit$ and does not even contain a copy of ω_1. It becomes a principal $\mathbb{T}$-bundle upon the removal of its 'origin', and it is a simply connected, ω-bounded surface with the property that *every noncompact closed subset contains the bones indexed by a cub subset of* ω_1. Most of the topological properties follow quickly from this: the bones are 'concentric' circles and so there cannot be a (necessarily closed) copy of ω_1; the space is normal because in any regular space, a compact set and a closed set disjoint from it can be put into disjoint open sets (and of course any two noncompact closed sets meet in a whole skeleton); it is collectionwise normal because every collection of disjoint closed subsets is countable (and at most one can be noncompact); and it is totally normal, and therefore (HODEL [1966]) it is hereditarily collectionwise normal:

6.16. DEFINITION. A space X is *totally normal* if every open subset U of X has a cover by open F_σ subsets of X that is locally finite in U.

For if U has a compact complement, it *is* an F_σ because the complement is contained in some M_α which is metrizable. If not, then U is broken up by its complement into a disjoint family of open subsets, each contained in some M_α and therefore an F_σ.

It is also easy to prove hereditary collectionwise normally directly.

6.17. EXAMPLE [$\Diamond$]. We will construct a principal $\mathbb{T}$-bundle E over L^+ and augment it with 'the point at 0' to obtain the ω-bounded manifold M just described.

The underlying *set* for E will be $L^+ \times S^1$, but the *topology* will only coincide with the product topology on subsets of the form $(0, \omega) \times S^1$ and $(\lambda, \lambda + \omega) \times S^1$ where ω is a limit ordinal.

To define the topology we will, for once, emulate the fiber bundle theorists and use maps ψ_λ from $(0, \lambda) \times S^1$ into $(0, \lambda + \omega) \times S^1$. It is elementary to modify them to maps into $B^2 - \{\langle 0, 0 \rangle\}$ taking $(0, \lambda) \times S^1$ to the open annulus of radius $\frac{1}{2}$.

By CH, there is a bijection $\zeta\colon L^+ \times S^1 \to \omega_1$. With $\{S_\lambda\colon \lambda \in \Lambda\}$ as in Axiom 3.13, let $T_\lambda = \zeta^{-1}(S_\lambda)$. Fix, once for all, a countable dense subset $Q = \{q_n\colon n \in \omega\}$ of S^1.

Our first map is $\psi_\omega\colon (0, \omega) \times S^1 \to (0, \omega + \omega) \times S^1$. If $T_\omega \subset (0, n) \times S^1$ for some n, we let ψ_ω be the identity embedding. If not, we will define ψ_ω so that

(i) Each fiber is invariant under ψ_ω; that is, $\psi_\omega(\{x\} \times S^1) = \{x\} \times S^1$ for all $x \in (0, \omega)$.

(ii) The restriction of ψ_ω to each fiber $\{x\} \times S^1$ is a rotation through a fixed angle $-\theta_\omega(x)$.

(iii) Every point of $\{\omega\} \times S^1$ is in the closure of $\psi_\omega(T_\omega)$.

To accomplish this, we let $\{\langle t_n, s_n \rangle\colon n \in \omega\}$ be a subset of T_ω such that $n < t_n < t_{n+1}$ for all n. Let A_ω be an arc which spirals outward from the origin, meeting each fiber $\{x\} \times S^1$ in one point, and crossing the fiber $\{t_n\} \times S^1$ at the point $\langle t_n, s_n + q_n \rangle$. See Fig. 5.

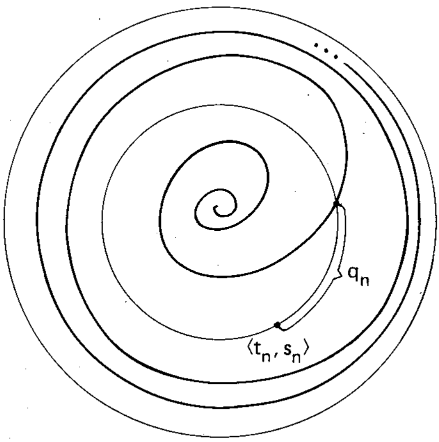

Fig. 5.

Let ψ_ω be a map which 'straightens out A_ω by a rotation on each fiber': if $\langle x, \theta_\omega(x)\rangle \in A_\omega$ we let $\psi_\omega(\langle x, \theta_\omega(x)\rangle) = \langle x, 0\rangle$ and in general $\psi_\omega(\langle x, \theta\rangle) = \langle x, \theta - \theta_\omega(x)\rangle$. Since A_ω is an arc, this function is continuous in the usual topology of $(0, \omega) \times S^1$. [Aside: by choosing A_ω 'smoothly' we can make ψ_ω a C^∞ function.] It is easy to see that ψ_ω satisfies (i) and (ii), and since $\psi_\omega(\langle t_n, s_n\rangle) = \langle t_n, -q_n\rangle$, condition (iii) is also satisfied.

Let $h_\omega(\langle x, \theta\rangle) = \psi_\omega^{-1}(\langle x, \theta\rangle)$ for all $x \in (0, \omega)$, and let $h_\omega(\langle x, \theta\rangle) = \langle x, \theta\rangle$ if $x \in [\omega, \omega + \omega)$. Let the E-topology on $(0, \omega + \omega) \times S^1$ be the one which makes h_ω a homeomorphism. [Note, in particular, that the arc A_ω, which probably 'looks like' it has every point of $\{\omega\} \times S^1$ in its closure, actually converges to the point $\langle \omega, 0\rangle$ in the E-topology.]

Suppose ψ_γ and A_γ have been defined for all limit ordinals $\gamma < \lambda$, so that A_γ is an arc in the E-topology meeting each fiber over $(0, \gamma)$ in exactly one point, and converging to the point $\langle \gamma, 0\rangle$. We also define $A_{\gamma+n}$ for each $n \in \omega$ as $A_\gamma \cup ([\gamma, \gamma + n) \times \{0\})$.

Pick an increasing sequence of ordinals α_n whose limit is λ; if possible, let α_n be such that there is a point $\langle \alpha_n, s_n\rangle \in T_\lambda$. Let A_λ be an arc in E meeting each fiber over $(0, \lambda)$ in exactly one point and agreeing with $A_{\alpha_n} + s_n + q_n$ [by which is meant the arc which passes through $\langle x, \theta + s_n + q_n\rangle$ iff A_{α_n} passes through $\langle x, \theta\rangle$] on the annulus $(\alpha_{n-1} + \frac{1}{2}, \alpha_n) \times S^1$ for each $n > 0$, and with $A_{\alpha_0} + s_0 + q_0$ on $(0, \alpha_0) \times S^1$. There is no difficulty about all this since each segment $A_{\alpha_n} + s_n + q_n$ is an arc by induction, and we have the annuli $(\alpha_{n-1}, \alpha_{n-1} + \frac{1}{2}) \times S^1$, where the topology agrees with the usual product topology, in which to define arcs which will link segments up.

As with ψ_ω, let ψ_λ be the map which 'straightens out A_λ' by taking each $\langle x, \theta_\lambda(x)\rangle \in A_\lambda$ to $\langle x, 0\rangle$ and also taking each $\langle x, \theta\rangle$ to $\langle x, \theta - \theta_\lambda(x)\rangle$. Let $h_\lambda \colon (0, \lambda + \omega) \times S^1 \to (0, \lambda + \omega) \times S^1$ agree with ψ_λ^{-1} on $(0, \lambda) \times S^1$ and be the identity on $[\lambda, \lambda + \omega) \times S^1$. Let the E-topology on $(0, \lambda + \omega) \times S^1$ be the one that makes h_λ a homeomorphism. (The domain of h_λ is taken to be $(0, \lambda + \omega) \times S^1$ with the product topology.)

Here too, $\psi_\lambda(\langle \alpha_n, s_n\rangle) = \langle \alpha_n, -q_n\rangle$ and so the closure of $\{\langle \alpha_n, s_n\rangle : n \in \omega\}$ includes all of $\{\lambda\} \times S^1$, while A_λ converges to $\langle 0, \lambda\rangle$ in the E-topology. On the annulus $[\alpha_{n-1} + \frac{1}{2}, \alpha_n] \times S^1$ with the usual topology, $\psi_\lambda \circ \psi_{\alpha_n}^{-1}$ rotates each fiber through $s_n + q_n$ radians. On the 'transitional annuli' $(\alpha_{n-1}, \alpha_{n-1} + \frac{1}{2})$ this composition does not have such a neat formula, but it still is a rotation on each fiber.

Let E be the space that results when we use the open cover $\{(0, \lambda) \times S^1 : \lambda \in \Lambda\}$ to define the topology (see the comment following Lemma 2.6). We will now show that every subset Y of E which is not contained in some $E_\lambda = (0, \lambda) \times S^1$ has all of $C_1 \times S^1$ in its closure for some cub $C_1 \subset \omega_1$.

It is enough to prove this when $Y \cap E_\lambda$ is countable for each λ. In that case, Lemma 3.12 applies and there is a cub C such that $\zeta^{-1}(\lambda) \cap Y = Y \cap E_\lambda$ for all $\lambda \in C$. Let C' be the set of all limit points of C, and select $\lambda \in C'$ such that $S_\lambda \subset \zeta(Y)$. By the same proof as in Example 3.14, T_λ is a closed discrete subspace of E_λ and so it was possible to let $\langle \alpha_n, s_n \rangle \in T_\lambda$, and all of $\{x\} \times S^1$ is in the closure of Y. We can repeat the argument for $Y \backslash E_\lambda$ in place of Y; repeating the argument uncountably many times gives an uncountable collection $A \subset \omega_1$ such that $\{\alpha\} \times S^1$ is in the closure of Y for all $\alpha \in A$; finally, let C_1 be the closure of A in ω_1. By the connectedness of E, $C_1 \times S^1$ is the closure of $A \times S^1$.

To construct M, we simply add $\langle 0, 0 \rangle$ to E by declaring its base of neighborhoods to be all sets of the form $\{\langle 0, 0 \rangle\} \cup (0, 1/n) \times S^1$ in E, $n \in \omega - \{0\}$.

With somewhat more care it is possible to construct spaces with all the properties of E and M using $\clubsuit$ alone. Details will appear in NYIKOS [198 ·], where it is also shown how to construct an assortment of principal $\mathbb{R}$-bundles over L^+, including a Moore example in ZFC and a Dowker example assuming $\diamondsuit$.

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CHAPTER 15

Normality versus Collectionwise Normality

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Contents

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HANDBOOK OF SET-THEORETIC TOPOLOGY
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Introduction

A topological space is *normal* if given disjoint closed sets A and B, there exist open sets U_A, U_B which *separate* them, i.e. $A \subset U_A$, $B \subset U_B$, and $U_A \cap U_B = \emptyset$. A trivial induction establishes that normality implies any finite number of disjoint closed sets can be simultaneously separated. A convergent sequence with its limit point shows, however, that 'finite' cannot be extended to 'countable'. To exclude this sort of difficulty, define a disjoint collection of closed sets to be *discrete* if the union of any subcollection is closed. Intuitively, the closed sets are 'far apart'. Define a space to be *collectionwise normal* if any discrete collection of closed sets can be separated. The natural question—which is the subject of this article and the one of Fleissner in this volume as well—is, *under what circumstances does normal imply collectionwise normal*? Since collectionwise normal Moore spaces are metrizable, the conjecture that normal Moore spaces are metrizable is a special case of this question.

The organization of this paper is as follows. Section I is implicitly divided by theorem numbering into four sections. In the first we prove theorems of ZFC which partially answer the question above. This is good old-fashioned general topology. In the second and third sections we enlist the aid of additional axioms and set-theoretic machinery to attain stronger positive results. The second section is essentially concerned with points while the third deals with arbitrary closed sets. The short fourth section collects the results of the previous sections and directly applies them to normal Moore spaces. Section II provides a host of normal non-collectionwise normal spaces, some constructed with the aid of additional axioms. Section III lists open problems. To avoid cluttering earlier chapters, we have added Section IV which provides historical notes and references. (My practice will usually be to give references in the text if and only if the quoted result is not proved in the text.) In contrast to my earlier survey (TALL [1979]), our organization here is thematic rather than historical. We have aimed at comprehensiveness—all significant results are mentioned, most are proved. The solution of the normal Moore space problem has stimulated—rather than killed—activity in this area. This will be evident from the large number of recent as yet unpublished results we have included, as well as the long list of interesting open questions.

There is a natural dividing line in the subject between the examples that involve discrete collections of points, and those that involve discrete collections of closed sets more complicated than points. The latter are much more difficult and lengthy to describe. Therefore, in this article we shall enumerate their properties, but the details will be handled by Fleissner in his article. He will also give the proof of the consistency of the Product Measure Extension Axiom relative to a strongly compact cardinal. The two articles together, then, will give a reasonably complete picture of the area, an area which has for fifteen years been at the cutting edge of set-theoretic topology, very frequently being the first topological consumer of a new set-theoretic technique.

Our topological prerequisites are minimal and should not deter any passing set-theorist. Undefined terms may be found in ENGELKING [1977]. Our set-theoretic prerequisites are commensurate with the material under discussion; e.g. when we force, we assume elementary knowledge of forcing. Our basic set-theoretic reference is KUNEN [1980].

Our notation is—with minor exceptions—standard. ${}^{\kappa}\lambda = \{f: f$ is a function from κ into $\lambda\}$, ${}^{\underline{\kappa}}\lambda = \{f: f$ is a function from some $\alpha \in \kappa$ into $\lambda\}$. A collection $\mathcal{Y}$ of sets is *separated* if there exist pairwise disjoint open sets about each member of the collection; it is *normalized* if given any subcollection $\mathcal{Z}$, there exist disjoint open sets, one about $\bigcup\mathcal{Z}$, the other about $\bigcup(\mathcal{Y} - \mathcal{Z})$. A collection $\mathcal{Y}$ is *discrete* if about each point of the space there is an open set intersecting at most one element of the collection. A space is *κ-collectionwise Hausdorff* (*$<\kappa$-collectionwise Hausdorff*) if each discrete collection of points of cardinality $\leqslant\kappa$ ($<\kappa$) is separated. Given a discrete collection $\mathcal{Y}$ in a space X, *the canonical cover* is $\{X - \overline{\bigcup\mathcal{Y}}\} \cup \{X - \overline{\bigcup\{Y' \in \mathcal{Y}: Y' \neq Y\}}: Y \in \mathcal{Y}\}$.

Theorems, lemmas, and corollaries are referred to via "section · subsection · number", but within a section, the first numeral is omitted. We also omit the section reference when confusion is unlikely.

Our bibliography is liberally padded—we have attempted to list every significant paper bearing upon the subject, whether or not we refer to it specifically.

I would like to thank Steve Watson for catching numerous errors in previous versions of this chapter, and for improving many of the proofs.

I. Theorems

One can get from normality to collectionwise normality (with respect to discrete collections of some sort) in ZFC if one or both of two circumstances hold: the canonical cover has a 'nice' refinement or the elements of the collection are 'nice'. 'Nice' in the first case means some variant of *closure-preserving* (a collection is closure-preserving if for any subcollection, 'the union of the closures equals the closure of the union'); in the second, it means 'point', 'compact', Lindelöf, etc. In fact if the elements of the collection are nice, one proves that the canonical cover has a nice refinement. In general, normality allows one to weaken nice to 'σ-nice', as we shall see. Now for the results.

1.1. THEOREM. *Suppose the canonical cover of $\mathcal{Y}$ has an open refinement $\mathcal{U} = \bigcup_{n<\omega} \mathcal{U}_n$ such that each $\mathcal{U}_n$ is closure-preserving, and the closure of each member of $\mathcal{U}$ meets at most one member of $\mathcal{Y}$. Then $\mathcal{Y}$ can be separated.*

PROOF. If $\mathcal{U}$ were closure-preserving, the proof would be obvious. The 'σ' is handled by the standard technique used in proving e.g. that Lindelöf regular spaces are normal. For each $n \in \omega$ and each $Y \in \mathscr{Y}$, let $U_{Y,n} = \bigcup\{U \in \mathscr{U}_n : U \cap Y \neq \emptyset\}$. Let $U'_{Y,n} = \bigcup\{U \in \mathscr{U}_n$: for some $Y' \in \mathscr{Y}$, $Y' \neq Y$, $U \cap Y' \neq 0\}$. Then the desired separation is composed of sets $V_Y = \bigcup_{n<\omega}(U_{Y,n} - \bigcup_{j=0}^{n} \overline{U'_{Y,j}})$.

1.2. COROLLARY. *Paracompact Hausdorff spaces are collectionwise normal.*

PROOF. Such spaces are normal; locally finite collections are closure-preserving.

Although our next result could in most cases be thought of as a corollary to 1.1, it is more natural to explicate it in the following fashion—if we can break up a discrete collection into subcollections simultaneously separated from each other and if we can separate the elements of each subcollection, then we can separate the whole collection.

1.3. THEOREM. *Suppose every countable subcollection of $\mathscr{Y}$ is separated, and suppose there is a collection $\mathscr{S}$ of open sets covering $\mathscr{Y}$ such that each $S \in \mathscr{S}$ intersects at most one element of $\mathscr{Y}$ and at most countably other members of $\mathscr{S}$. Then $\mathscr{Y}$ is separated.*

PROOF. For each $S \in \mathscr{S}$, let

$$\mathscr{S}(S, 0) = \{S\},$$

$$\mathscr{S}(S, n+1) = \{S' \in \mathscr{S}: \text{for some } S'' \in \mathscr{S}(S, n), S'' \cap S' \neq \emptyset\},$$

$$\mathscr{S}(S, \omega) = \bigcup_{n<\omega} \mathscr{S}(S, n).$$

Note that each $\mathscr{S}(S, \omega)$ is countable and that the relation '$S' \in \mathscr{S}(S, \omega)$' is an equivalence relation which partitions $\mathscr{S}$ into countable subcollections with disjoint open sets as unions. Separating inside the disjoint open sets by countable separation yields a separation of $\mathscr{Y}$.

Countable separations are easy to achieve:

1.4. THEOREM. *In a normal space, countable discrete collections are separated.*

PROOF. Let $\{Y_n\}_{n<\omega}$ be discrete. For each n, let $U_n \supset Y_n$, $U'_n \supset \bigcup_{n' \neq n} Y_{n'}$ be disjoint. Let $V_n = U_n \cap \bigcap_{k<n} U'_n$. Then $\{V_n\}_{n<\omega}$ is the desired separation.

Since in a regular space, two Lindelöf sets with disjoint closures can be separated, the same proof yields that in a regular space, countable discrete collections of Lindelöf sets are separated. It is then not difficult to obtain from Theorem 1.3:

1.5. COROLLARY. *Paralindelöf regular spaces are collectionwise normal with respect to Lindelöf sets.*

A space is *screenable* if every open cover has a refinement $\mathscr{S} = \bigcup_{n<\omega} \mathscr{S}_n$, where each $\mathscr{S}_n$ is disjoint. One expects to get some collectionwise normality from screenability plus normality and one does:

1.6. THEOREM. *Screenable normal spaces are collectionwise normal with respect to countably metacompact closed sets.*

PROOF. Let $\mathscr{M}$ be a discrete collection of countably metacompact (every countable open cover has a point-finite refinement) closed sets. Let $\mathscr{S} = \bigcup_{n<\omega} \mathscr{S}_n$ be a σ-disjoint refinement of the canonical cover. The closed subspace $\bigcup \mathscr{M}$ is normal and countably metacompact. It follows that the cover $\{\bigcup \mathscr{S}_n \cap \bigcup \mathscr{M} : n < \omega\}$ of it can be shrunk to $\{T_n\}_{n<\omega}$, $\bar{T}_n \subseteq \bigcup \mathscr{S}_n \cap \bigcup \mathscr{M}$. $\bigcup \mathscr{M}$ is closed so the closure sign is unambiguous. $\{M \cap \bar{T}_n : M \in \mathscr{M}\}$ for each n is a discrete collection separated by the open sets $S_{n,M} = \bigcup\{S \in \mathscr{S}_n : S \cap M \neq \emptyset\}$. By normality, the $S_{n,M}$'s may be shrunk to a discrete separation $\{S'_{n,M} : M \in \mathscr{M}\}$. Then $\bigcup\{S'_{n,M} : M \in \mathscr{M}\}$ yields a separation by 1.1.

Thus screenable normal spaces which are not collectionwise normal are not only *Dowker spaces* (see RUDIN [1983b], this volume), but they fail to be collectionwise normal only with respect to Dowker spaces!

It is often useful to be able to conclude that a space is collectionwise normal, knowing only that it's normal and collectionwise normal with respect to closed sets satisfying some nice property P. What is needed is that the space locally satisfies property P, plus a covering condition sufficient to produce a discrete unseparated collection of sets with property P, if there were any discrete unseparated collection. The weakest reasonable covering condition that works is *submetacompactness* (formerly θ-refinability) which is a common generalization of metacompactness and subparacompactness (see BURKE [1983], this volume). A space is submetacompact if every open cover has an open refinement consisting of countably many open covers, such that each point has finite order with respect to one of the covers. Thus one has for example

1.7. THEOREM. *Locally compact normal submetacompact spaces which are collectionwise normal with respect to compact sets are paracompact* (*and hence collectionwise normal*).

We omit the proof, which consists of noting that in the proof (see e.g. Burke op. cit.) that submetacompact plus collectionwise normal yields paracompact, if one refines the original cover by (in this case) open sets with compact closure, all that the proof requires is collectionwise normality with respect to compact sets.

A recent result worth mentioning is that locally compact normal spaces such that for each open cover there is an n such that the cover has a point-n refinement are paracompact (DANIELS [19··]). We discuss later the question of whether all metacompact locally compact normal spaces are paracompact, but let us note here

1.8. THEOREM. *Metacompact locally compact perfectly normal spaces are paracompact.*

PROOF. See BURKE, Theorem 3.12 in this volume.

A priori, it seems unlikely that local connectedness has anything to do with collctionwise normality. However the longstanding absence of even a consistent example of a locally compact, locally connected normal nonmetrizable Moore space led to the correct conjecture that none existed. The proof was later simplified and now yields

1.9. THEOREM. *Locally compact, locally connected perfectly normal spaces are collectionwise normal with respect to submetacompact closed sets.*

PROOF. It suffices to prove collectionwise normality with respect to compact sets. For assume this and then note that 'locally compact', 'normal' and 'collectionwise normal with respect to compact sets' are closed hereditary, so submetacompact closed sets are paracompact. Locally compact paracompact spaces are free unions of Lindelöf spaces and Lindelöf locally compact spaces are σ-compact. It is an easy exercise to use normality, Theorem 1.1, and collectionwise normality with respect to compact sets to separate σ-compact sets. Suppose therefore that $\{Y_\alpha: \alpha \in A\}$ is a discrete collection of compact sets in a perfectly normal, locally compact, locally connected space X. Let $Y = \bigcup\{Y_\alpha: \alpha \in A\}$. Then there are open sets $\{U_n: n < \omega\}$ such that $Y = \bigcap_{n<\omega} U_n = \bigcap_{n<\omega} \bar{U}_n$. For each n let $U_{\alpha,n}$ be the union of those components of U_n which meet Y_α. Then for each α there is an $N(\alpha)$ such that $U_{\alpha,N(\alpha)} \cap U_{\beta,N(\alpha)} = \emptyset$ for all $\beta \neq \alpha$ in A. To see this, take a compact neighbourhood V of Y_α disjoint from $Y - Y_\alpha$. Then by compactness, for some N, U_N misses the boundary of V. It follows that each component of U_N is either included in V or is disjoint from $\bar{U}_N$, so N is the desired N_α. Now for each n, let $Y_n = \bigcup\{Y_\alpha: N(\alpha) < n\}$. By normality let $\{W_n\}_{n<\omega}$ separate the H_n's. Let $D_\alpha = W_{N(\alpha)} \cap U_{\alpha,N(\alpha)}$. Then $\{D_\alpha: \alpha \in A\}$ is the required separation.

By a somewhat more complicated argument (GRUENHAGE [1979]) one can prove that normal, locally connected, locally compact, submetacompact spaces are paracompact.

In view of 1.1, one might conjecture that normal collectionwise Hausdorff spaces are collectionwise normal with respect to Lindelöf sets. Partial results in this direction were achieved by Steve Watson after he had read the first draft of this article. We shall give the simplest case of his results since they are not yet in final form.

1.10. THEOREM. *If X is collectionwise Hausdorff and collectionwise normal with respect to discrete collections of $\leqslant\aleph_1$ closed sets, then X is collectionwise normal with respect to discrete collections of hereditarily Lindelöf sets.*

PROOF. Let $\{Y_\gamma\}_{\gamma<\lambda}$ be a discrete collection of hereditarily Lindelöf sets. Using normality and collectionwise Hausdorffness, by induction on $\beta\in\omega_1$ define $S_\beta\subseteq\lambda$ and U_γ^β, $\gamma\in S_\beta$ as follows: $\{U_\gamma^0\colon \gamma<\lambda\}$ is a discrete collection of open sets such that $U_\gamma^0\cap Y_\gamma\neq\emptyset$ and $\overline{U_\gamma^0}\cap Y_{\gamma'}=\emptyset$ for $\gamma\neq\gamma'$. If S_α and U_γ^α have been defined for all $\alpha<\beta$ and $\gamma\in S_\alpha$, let $S_\beta=\{\gamma\colon \bigcup_{\alpha<\beta}U_\gamma^\alpha$ does not cover $Y_\gamma\}$ and define U_γ^β for $\gamma\in S_\beta$ as before, requiring that $U_\gamma^\beta\cap(Y_\gamma-\bigcup_{\alpha<\beta}U_\gamma^\alpha)\neq\emptyset$. Because each Y_γ is hereditarily Lindelöf, for each γ there is a least $\beta(\gamma)\in\omega_1$ such that $\gamma\notin S_{\beta(\gamma)}$. For each $\beta\in\omega_1$ let $Z_\beta=\bigcup\{Y_\gamma\colon \beta(\gamma)=\beta\}$. We can separate the Z_β's from each other by hypothesis; within each Z_β we can separate the Y_γ's by 1.1, so we are done.

Venturing beyond ZFC, we shall show that simple relations among cardinals have implications for normality versus collectionwise normality. For example,

2.1. THEOREM. $2^{\aleph_0}<2^{\aleph_1}$ *implies separable normal spaces have no uncountable closed discrete subspaces and hence* (*by* 1.4) *are collectionwise normal.*

We shall prove a generalization of this result shortly, so let us just indicate a proof here. Given an uncountable closed discrete subspace Y and a countable dense set D, correspond to each $Z\subseteq Y$, $U_Z\cap D$, where $U_Z\supseteq Z$ is open, $\bar{U}_Z\cap(Y-Z)=\emptyset$. Then $2^{\aleph_1}=2^{\aleph_0}$. Alternatively, use Urysohn's Lemma to separate the $2^{\aleph_1}$ subsets of Y, but note that by separability there are only $2^{\aleph_0}$ real-valued continuous functions. These proofs obviously generalize to other cardinals, but there is also a nontrivial generalization:

DEFINITION. A space is *weakly collectionwise Hausdorff* if for each cardinal κ and each closed discrete subspace Y of cardinality κ, there is a separated $Z\subseteq Y$ of cardinality κ.

2.2. THEOREM. *Suppose for every cardinal κ that $2^\kappa<2^{\kappa^+}$. Then every normal space of character $\leqslant 2^{\aleph_0}$ is weakly collectionwise Hausdorff.*

A typical useful consequence is

2.3. COROLLARY. $2^{\aleph_0}<2^{\aleph_1}$ *implies that normal spaces of character $\leqslant 2^{\aleph_0}$ satisfying the countable chain condition have no uncountable closed discrete subspaces.*

To prove 2.2, suppose Y is a closed discrete subspace of X. For $|Y| \leq \aleph_0$, 1.4 does it, so let us first consider the case when $|Y|$ is a successor, κ^+. Fix for each $y \in Y$ a neighbourhood base $\mathcal{B}_y$ for y, $|\mathcal{B}_y| \leq 2^{\aleph_0}$. By normality, for each $Z \subseteq Y$ there is an open $U_Z \supseteq Z$, $\bar{U}_Z \cap (Y - Z) = \emptyset$. For each Z pick a maximal disjoint collection $\mathcal{S}_Z$ of elements of $\mathcal{B}_y$'s, $y \in Z$. $Z_1 \neq Z$ implies $\mathcal{S}_{Z_1} \neq \mathcal{S}_{Z_2}$. If no subspace of Y of cardinality κ^+ is separated, each $|\mathcal{S}_Z| \leq \kappa$, so the map $Z \to \mathcal{S}_Z$ establishes that $2^{\kappa^+} \leq (2^{\aleph_0} \cdot \kappa^+)^\kappa = 2^\kappa$.

For κ a limit, the cofinality of $2^{\underline{\kappa}}$ is κ, so by König's Lemma, $2^{\underline{\kappa}} < 2^\kappa$. For κ regular limit then, the same proof as for the successor case works. For κ singular, again take maximal disjoint collections $\mathcal{S}_Z$ for each $Z \subseteq Y$. We may as well assume each $|\mathcal{S}_Z| < \kappa$, else we are done. There are $\leq (2^{\aleph_0})^\kappa$ such collections. It is not difficult to construct a family of 2^κ subsets of κ such that any two have a difference of power κ (e.g. an independent family). It suffices to observe that on this family, the map $Z \to \mathcal{S}_Z$ is one–one, contradicting $2^{\underline{\kappa}} < 2^\kappa$. To see this, note that $|Z - \overline{\bigcup \mathcal{S}_Z}| < \kappa$, for if not, since we may assume X is $|\mathcal{S}_Z|^+$-weakly collectionwise Hausdorff, $\mathcal{S}_Z$ could not be maximal. Thus if $Z_1 \neq Z_2$, there is a $z \in Z_2 - Z_1$ such that $z \in \bigcup \mathcal{S}_{Z_2}$. But $z \notin \bar{U}_{Z_1} \supseteq \bigcup \mathcal{S}_{Z_1}$, so $\mathcal{S}_{Z_1} \neq \mathcal{S}_{Z_2}$.

There are several variants of the above results. One worth mentioning is

2.4. THEOREM. *Suppose X is normal and has character $\leq 2^{\aleph_0}$. Then any closed discrete subspace of X of cardinality $2^{\aleph_0}$ includes a separated subspace of cardinality $\aleph_1$.*

There are also versions of these results in which countable paracompactness replaces normality. See TALL [1976a].

An important strengthening of weak collectionwise Hausdorff results is in TAYLOR [1981]:

2.5. THEOREM. *Suppose X is normal and has character $\leq 2^{\aleph_0}$. If $2^{\aleph_0} < 2^{\aleph_1}$ and $\{y_\alpha\}_{\alpha < \omega_1}$ is a closed discrete subspace of X, then there is a stationary $S \subseteq \omega_1$ such that $\{y_\alpha : \alpha \in S\}$ is separated.*

We omit the proof but apply the result later to show special Aronszajn trees not normal.

By using more powerful methods and assumptions we can improve our weakly collectionwise Hausdorff results to get collectionwise Hausdorffness.

2.6. THEOREM. *$V = L$ implies every normal space of character $\leq 2^{\aleph_0}$ is collectionwise Hausdorff.*

The conclusion also holds in other models, some allowing more freedom in cardinal arithmetic. The Theorem is proved by induction on the cardinality of closed discrete subspaces. We state the cases in a sequence of lemmas. For future use we shall prove these lemmas in greater generality than needed at this point. First, a

DEFINITION. $\mathcal{A} = \{A_f : f \in {}^\kappa\kappa\}$ is a *stationary system* (for κ) if each A_f is a stationary subset of κ, and whenever $\alpha \in \kappa$ and $f, g \in {}^\kappa\kappa$, if $f|\alpha = g|\alpha$, then $A_f \cap (\alpha+1) = A_g \cap (\alpha+1)$. $\diamondsuit$ *for stationary systems* (at κ) is the assertion that for each stationary system $\mathcal{A}$ for κ, there is a sequence $\{f_\alpha\}_{\alpha<\kappa}$, such that $f_\alpha \in {}^\alpha\alpha$ and for each $f \in {}^\kappa\kappa$ there is a stationary $S \subseteq A_f$ such that $\beta \in S$ implies $f|\beta = f_\beta$.

2.7. LEMMA. *Suppose κ is regular, X is normal and $<\kappa$-collectionwise Hausdorff, and $\chi(X) \leq \kappa$. Then $\diamondsuit$ for stationary systems at κ implies X is κ-collectionwise Hausdorff.*

2.8. LEMMA. *Suppose* $\mathrm{cf}(\kappa) = \aleph_0$, *and X is normal and $<\kappa$-collectionwise Hausdorff. Then X is κ-collectionwise Hausdorff.*

2.9. LEMMA. *Suppose $\aleph_0 \leq \mathrm{cf}(\kappa) < \kappa$ and x is normal and $<\kappa$-collectionwise Hausdorff. If $\chi(X) < \kappa$ and for all λ, $\chi(X) \leq \lambda < \kappa$, $2^\lambda = \lambda^+$, then X is κ-collectionwise Hausdorff.*

The second lemma is immediate by the same argument as in 1.4. The other two are non-trivial. In order to prove the first, we require another lemma. Some notation will prove useful. '$N(y, \beta)$' will denote the βth basic neighbourhood of a point y, in some fixed enumeration of a local basis. When we are dealing with a discrete subspace, we without loss of generality take these basic neighbourhoods to contain no other point of the discrete subspace. If Y is a discrete subspace of cardinality κ, to simplify notation we often identify Y with κ. If each point of Y has character $\leq\lambda$, we identify neighbourhood assignments to all members of Y with functions from κ into λ. We then for $f \in {}^\kappa\lambda$ define

$$W_f(\alpha) = \textstyle\bigcup_{\beta<\alpha} N(\beta, f(\beta))\,.$$

2.10. LEMMA. *Let X be a space. Let κ be a regular cardinal. Suppose the character of each point in the closed discrete set $Y \subseteq X$ is $\leq\kappa$. Suppose $|Y| = \kappa$ and each subset of Y of smaller cardinality is separated. If there is an $f \in {}^\kappa\kappa$ such that $A_f = \{\alpha : \overline{W_f(\alpha)} \cap (\kappa - \alpha) \neq \emptyset\}$ is nonstationary, then Y is separated.*

PROOF. Let C be a closed unbounded subset of κ disjoint from A_f. For every $\alpha \in \kappa$, define $C^*(\alpha)$ = the greatest $\gamma \in C \cup \{0\}$ such that $\gamma \leq \alpha$. Note C^* is always defined and that it partitions κ into equivalence classes of size $<\kappa$. We can define a function $g \in {}^\kappa\kappa$ so that for each α and for all $\beta < C^*(\alpha)$, $N(\alpha, g(\alpha)) \cap$

$N(\beta, f(\beta)) = \emptyset$. On the other hand, by hypothesis there is an $h \in {}^\kappa\kappa$ such that whenever $C^*(\alpha) = C^*(\alpha')$, $N(\alpha, h(\alpha)) \cap N(\alpha', h(\alpha')) = \emptyset$. Then $\{N(\alpha, g(\alpha)) \cap N(\alpha, f(\alpha)) \cap N(\alpha, h(\alpha)): \alpha \in \kappa\}$ is the required separation.

The proof of 2.7 is a straightforward $\diamond$ argument—we construct a partition of κ by recursion so as to prevent any member of a $\diamond$ sequence of functions from being an initial segment of a neighbourhood assignment normalizing the partition. The usual argument doesn't quite show that this suffices to take care of all functions, because A_f is in general stationary rather than closed unbounded. That is why $\diamond$ for stationary systems is needed. In special cases, e.g. for special Aronszajn trees, we may assume without loss of generality that A_f is closed unbounded and then $\diamond$ does imply such trees are not normal, since they are not collectionwise Hausdorff. See Section II.

Let X be a space, Y a closed discrete unseparated subspace of power κ. The $\{A_f: f \in {}^\kappa\kappa\}$ obtained in the previous lemma form a stationary system. Let $\{f_\alpha\}_{\alpha<\kappa}$ be a $\diamond$ sequence for the A_f's. Define a partial function $\nu: \kappa \to \kappa$ by $\nu(\alpha) =$ the least $\beta \in \kappa$ (if any) such that $\beta \in \overline{W_{f_\alpha}(\alpha)} \cap (\kappa - \alpha)$. (We extend the W_f notation to partial functions.) Next, define a partition H, K of κ by induction on $\alpha \in \kappa$ as follows:

(a) If α limit, let $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, $K_\alpha = \bigcup_{\beta<\alpha} K_\beta$.

(b) If $\nu(\alpha)$ is defined and is in $\overline{W_{f_\alpha}(K_\alpha)} - (H_\alpha \cup K_\alpha)$, then $H_{\alpha+1} = H_\alpha \cup \{\nu(\alpha)\}$, $K_{\alpha+1} = K_\alpha \cup \{\alpha + 1 - H_{\alpha+1}\}$.

(c) If $\nu(\alpha)$ is defined and is in $\overline{W_{f_\alpha}(H_\alpha)} - (\overline{W_{f_\alpha}(K_\alpha)} \cup H_\alpha \cup K_\alpha)$, then $K_{\alpha+1} = K_\alpha \cup \{\nu(\alpha)\}$, $H_{\alpha+1} = H_\alpha \cup \{\alpha + 1 - K_{\alpha+1}\}$.

(d) If $\nu(\alpha)$ is not defined or $\nu(\alpha) \in H_\alpha \cup K_\alpha$, then $H_{\alpha+1} = H_\alpha$, $K_{\alpha+1} = K_\alpha \cup \{\alpha + 1 - H_{\alpha+1}\}$.

Let $H = \bigcup_{\alpha<\kappa} H_\alpha$, $K = \bigcup_{\alpha<\kappa} K_\alpha$.

Suppose some $f \in {}^\kappa\kappa$ normalizes H, K. Note that $C = \{\alpha \in \kappa: \beta < \alpha$ implies $\nu(\beta) < \alpha$ or is undefined$\}$ is closed unbounded. Hence there is an $\alpha \in C \cap A_f$ such that $f|\alpha = f_\alpha$. By construction, for any γ, $\gamma \subseteq H_\gamma \cup K_\gamma \subseteq \gamma \cup \{\nu(\beta): \beta < \gamma$ & $\nu(\beta)$ defined$\}$. Therefore, since $\alpha \in C$, $\alpha = H_\alpha \cup K_\alpha$. Since $\alpha \in A_f$, $\nu(\alpha)$ is defined and is in $\overline{W_f(\alpha)} = \overline{W_{f_\alpha}(\alpha)} = \overline{W_{f_\alpha}(H_\alpha)} \cup \overline{W_{f_\alpha}(K_\alpha)}$. Since $\nu(\alpha) \notin H_\alpha \cup K_\alpha$, in the construction of $H_{\alpha+1}$ and $K_{\alpha+1}$, clause (b) or (c) must have applied. But then f couldn't have normalized H, K.

The case of singular cardinals of uncountable cofinality is handled similarly to that of regular cardinals in L. However, instead of using $\diamond$ for stationary systems to ensure that a small number of functions defined on initial segments trap all functions, we use a $\diamond$-like principle for singular cardinals which is implied by a segment of GCH. Using this principle we deduce that no matter how we order our points, there is a neighbourhood assignment f which at each point doesn't stick out too far. We then systematically push down the excess until—by the nonexistence of infinite descending sequences of ordinals—it disappears. Now for the details.

2.11. LEMMA. *Suppose κ is a limit cardinal, $\chi < \kappa$, and for all μ, $\chi \leq \mu < \kappa$, $2^\mu = \mu^+$. Then there is a sequence $\{g_\alpha : \chi \leq \alpha < \kappa\}$, where g_α is a partial function from κ into χ, such that for any $g \in {}^\kappa\chi$ and any β, $\chi \leq \beta < \kappa$, there is an $\alpha < \kappa$ such that $g|\beta = g_\alpha$ and $|\beta| = |\alpha|$.*

PROOF. $\{[\mu, \mu^+): \mu \in [\chi, \kappa)$ a cardinal$\}$ partitions $[\chi, \kappa)$. For each cardinal $\mu \in [\chi, \kappa)$ and each ordinal $\eta \in [\mu, \mu^+)$, $|{}^\eta\chi| \leq \mu^+$. Let $\{g_\alpha : \chi \leq \alpha < \kappa\}$ be such that for each $\mu \in [\chi, \kappa)$, $\{g_\alpha : \mu \leq \alpha < \mu^+\}$ enumerates $\bigcup\{{}^\eta\chi : \eta \in [\mu, \mu^+)\}$. For any $g \in {}^\kappa\chi$ and any $\beta \in [\delta, \kappa)$, there is a $\mu \in [\chi, \kappa)$ such that $\beta \in [\mu, \mu^+)$. $g|\beta$ is then some g_α, $\mu \leq \alpha < \mu^+$, and $|\beta| = |\alpha|$, which was to be proved.

DEFINITION. Let $S \subseteq \kappa$. Let ρ be a one–one function from S into κ. Let f be a partial function from κ into χ. Let $\alpha \in \kappa$. Let $W_{\alpha,f}(S, \rho) = \bigcup\{N(\beta, f(\beta)) : \beta \in S \cap \operatorname{dom} f \ \&\ \rho(\beta) < \alpha\}$. We say f is *thin* (with respect to S, ρ) if for every $\alpha \geq \chi$, the cardinality of $E_{\alpha,f}(S, \rho) = \overline{W_{\alpha,f}} \cap \{\beta \in S : \rho(\beta) \geq \alpha\}$ is $\leq |\alpha|$.

2.12. LEMMA. *Suppose $\chi < \kappa$ and for all μ, $\chi \leq \mu < \kappa$, $2^\mu = \mu^+$, and κ is a limit. Then if X is normal, for every $S \subseteq \kappa$ and every reordering ρ of S, there is an f which is thin with respect to S, ρ.*

PROOF. We define H, $K \subseteq S$ by recursion. Suppose for each $\beta < \alpha < \kappa$ we have defined H_β, $K_\beta \subseteq S$ so that $\bigcup_{\beta<\omega} H_\beta$ and $\bigcup_{\beta<\alpha} K_\beta$ are disjoint. If $E_{\alpha,g_\alpha}(S, \rho) \subseteq \bigcup_{\beta<\alpha} H_\beta \cup \bigcup_{\beta<\alpha} K_\beta$, let $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, $K_\alpha = \bigcup_{\beta<\alpha} K_\beta$. If not, let δ be the least element of $E_{\alpha,g_\alpha}(S, \rho) - (\bigcup_{\beta<\alpha} H_\beta \cup \bigcup_{\beta<\alpha} K_\beta)$. If

$$\delta \in \overline{\bigcup\{N(\gamma, g_\alpha(\gamma)) : \rho(\gamma) < \alpha \ \&\ \gamma \in \operatorname{dom} g_\alpha \cap \bigcup_{\beta<\alpha} H_\beta\}}\,,$$

let $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, $K_\alpha = \{\delta\} \cup \bigcup_{\beta<\alpha} K_\beta$. Alternatively, if

$$\delta \in \overline{\bigcup\{N(\gamma, g_\alpha(\gamma)) : \rho(\gamma) < \alpha \ \&\ \gamma \in \operatorname{dom} g_\alpha \cap \bigcup_{\beta<\alpha} K_\beta\}}\,,$$

let $K_\alpha = \bigcup_{\beta<\alpha} K_\beta$, $H_\alpha = \{\delta\} \cup \bigcup_{\beta<\alpha} H_\beta$. Now take some $g : \kappa \to \chi$ which normalizes H and K. Claim g is thin. Suppose not. Then there is a β, $\chi \leq \beta < \kappa$, $|E_{\beta,g}(S, \rho)| > |\beta|$. $g|\beta = g_\alpha$ for some α such that $|\beta| = |\alpha|$. Then at stage α we must have assigned a point which prevented g from separating H and K, contradiction.

We now can finish the singular cardinal of uncountable cofinality case (2.9) and hence the Theorem. Let $C = \{c_\alpha : \alpha < \operatorname{cf}(\kappa)\}$ be a set of cardinals closed and unbounded in κ such that each $c_\alpha > \operatorname{cf}(\kappa)$. Let $B_\alpha = \{\gamma : \gamma = \operatorname{cf}(\kappa) \cdot \delta + \alpha$ for some $\delta < c_\alpha\}$, each $\alpha < \operatorname{cf}(\kappa)$. The B_α's are disjoint subsets of κ and each B_α is a subset of c_α of cardinality c_α.

Define by recursion for each $i \in \omega$, S_i, ρ_i, f_i. Let $S_0 = \kappa$, ρ_0 the identity function on κ. Let f_i be thin with respect to S_i, ρ_i. Let $S_{i+1} = \bigcup\{E_{c_\alpha, f_i}(S_i, \rho_i) : c_\alpha \in C\}$. Let

$\rho_{i+i}: S_{i+1} \to \kappa$ be one–one such that $\rho_{i+1}E_{c_\alpha,f_i}(S_i, \rho_i) \subseteq B_\alpha$. This can be done since $|B_\alpha| = c_\alpha$ and f_i is thin with respect to ρ_i, S_i. Claim $\kappa = \bigcup_{i<\omega} (S_i - S_{i+1})$. Equivalently—since $S_{i+1} \subseteq S_i$—we show $\bigcap_{j<\omega} S_i = \emptyset$. Note if $\gamma \in E_{c_\alpha,f_i}(S_i, \rho_i) \subseteq S_{i+1}$, then $\rho_i(\gamma) \geqslant c_\alpha > \rho_{i+1}(\gamma)$, since $\rho_{i+1}(\gamma) \in B_\alpha \subseteq c_\alpha$. Thus if some $\gamma \in \bigcap_{i<\omega} S_i$, we have $\rho_0(\gamma) > \rho_1(\gamma) > \cdots > \rho_i(\gamma) > \cdots$, contradiction.

By normality, since the $S_i - S_{i+1}$ are disjoint, it suffices to show each $S_i - S_{i+1}$ can be separated. Let $D = \{\alpha: \beta \in \overline{W_{\alpha,f_i}(S_i, \rho_i)} \cap (S_i - S_{i+1})$ implies $\rho_i(\beta) < \alpha\}$. Then $C \subseteq D$. Proceeding as in the proof of Lemma 2.10, for each β such that $\rho_i(\beta) < c_\alpha$, there is a $g \in {}^\kappa\chi$ such that $N(c_\alpha, g(c_\alpha)) \cap N(\rho_i(\beta), f_i(\rho_i(\beta))) = \emptyset$. Thus $\{N(\rho_i(\beta), f_i(\rho_i(\beta))) \cap N(\rho_i(\beta), g(\rho_i(\beta))): \beta \in S_i - S_{i+1}\}$ separates all $\beta \in S_i - S_{i+1}$ which are in different $\sim_C$ classes. But each $\sim_C$ class has cardinality $<\kappa$ so can be separated by hypothesis. Thus we obtain a separation of κ.

REMARK. In 2.6, normality can be replaced by countable paracompactness, indeed by the common weakening of the two that asserts that the canonical cover of any countable partition of a closed discrete set has a locally finite refinement. This requires a rather delicate argument, however (Watson [1982]).

We can also establish the consistency of the conclusion of 2.7 by forcing. This enables us to work in more general situations and to deal with collections of certain types of closed sets, not just points. However, for pedagogical reasons, we shall only do the case of points here.

2.13. THEOREM. *Let $\mathcal{M}$ be a model of "κ, λ are regular cardinals, $2^{\underline{\kappa}} = \kappa$, $\lambda \geqslant \kappa^+$". Let $\mathcal{P}$ be* $\mathrm{Fn}(\lambda, 2, \kappa)$ *i.e. the set of partial functions from λ into 2 with domain of cardinality less than κ, ordered by inclusion. Let G be $\mathcal{P}$-generic over $\mathcal{M}$. In $\mathcal{M}[G]$ every normal space of character less than λ is κ-collectionwise Hausdorff if it is $<\kappa$-collectionwise Hausdorff.*

PROOF. $\mathcal{P}$ is the partial order for adding λ Cohen subsets of κ. The proof proceeds in several steps. We intend to show that if the space is not κ-collectionwise Hausdorff, it is not normal. We first claim that we may restrict our attention to spaces of cardinality and weight less than λ. Such spaces 'appear at some initial stage', which we may as well assume is the 'next to last' stage. We then show that the adjunction of a Cohen subset (e.g., the 'Final' one) of κ forces normality to fall.

We state the first lemma but defer the proof until the examples section, since it is a variation of a standard technique for splitting points.

2.14. LEMMA. *Let Y be a closed discrete subset of a normal space X such that the sup of the characters of the points of Y is μ which is less than λ. There is a normal space X' including Y as a closed discrete subspace such that both the* weight (*least cardinal of a base*) *and the cardinality of X' are $\leqslant |Y| \cdot \mu$.*

The second lemma is the meat of the matter.

2.15. LEMMA. *Let $\mathcal{P}$ be the partial order for adding one Cohen subset of κ to a model $\mathcal{M}$ of $2^{\underset{\smile}{\kappa}} = \kappa$. Let $\langle X, \mathcal{T}\rangle$ be a topological space in $\mathcal{M}$. Let Y be a closed discrete subset of X of power κ. Suppose Y is not separated but is $<\kappa$-separated in X. Let G be $\mathcal{P}$-generic over $\mathcal{M}$. Let $\mathcal{T}(G)$ be the topology $\mathcal{T}$ generates on X in $\mathcal{M}[G]$. Then in $\mathcal{M}[G]$, Y is not normalized.*

PROOF. Identify Y with κ. G partitions κ into the pieces $G_\varepsilon = \{\alpha: (\exists p \in G)(p(\alpha) = \varepsilon\}$, $\varepsilon = 0, 1$. We shall show that there do not exist disjoint open sets about G_0 and G_1. It suffices to show that for each assignment of a member of $\mathcal{T}$ to each element of κ, there are $\alpha_0 \in G_0$ and $\alpha_1 \in G_1$ such that their assigned open sets intersect. I.e. it suffices to show that if $p \Vdash f: \check{\kappa} \to \check{\mathcal{T}}$, then

$$D_f = \{q: q \Vdash \bigcup\{f(\alpha): \alpha \in \mathbf{G}_0\} \cap \bigcup\{f(\alpha): \alpha \in \mathbf{G}_1\} \neq \emptyset\}$$

is dense below p. Let q be arbitrary below p. Since $\mathcal{P}$ is κ-closed, we can as usual construct a descending sequence of conditions $\{p_\alpha: \alpha < \kappa\}$ below q such that $p_\alpha \Vdash f(\alpha)$ (P_α 'decides' $f(\alpha)$), i.e. $(\exists U \in \mathcal{T})(p_\alpha \Vdash f(\check{\alpha}) = \check{U})$. In fact we can do more; we can additionally require that $\alpha \in \text{domain } p_\alpha$ and that when p_α is 'flipped' at $\alpha \notin \text{domain } p$, i.e. the pair $\langle \alpha, \varepsilon\rangle$ is replaced by $\langle \alpha, 1-\varepsilon\rangle$ to obtain what we shall call '$\hat{p}_\alpha$', $\hat{p}_\alpha$ also decides f at α. These additional requirements are ensured by an easy density argument we leave to the reader. We now define a function h in the ground model by $h(\alpha) = U_\alpha \cap V_\alpha$, where $p_\alpha \Vdash f(\check{\alpha}) = \check{U}_\alpha$ and for $\alpha \in \text{domain } p$, $V_\alpha = U_\alpha$, while for $\alpha \notin \text{domain } p$, $\hat{p}_\alpha \Vdash f(\check{\alpha}) = \check{V}_\alpha$. The idea is that since Y is not separated, the open sets given by h intersect sufficiently often so that the open sets about G_0 and G_1 specified by f—whatever they may be—must also intersect. By the same argument as in 1.3, there is an α_0 and there is an α_1 greater than the sup of the domain of p_{α_0} such that $h(\alpha_0) \cap h(\alpha_1) \neq \emptyset$. Both p_{α_1} and $\hat{p}_{\alpha_1}$ are $\leqslant p_{\alpha_0}$ and hence $\leqslant q$. Let the one which has different values at α_0 and α_1 be called 'p^*'. $p^* \Vdash (\forall \beta \leqslant \alpha_1)(h(\beta)) \subseteq f(\beta))$, so

$$p^* \Vdash (\bigcup\{f(\alpha): \check{p}^*(\alpha) = \check{0}\} \cap \bigcup\{f(\alpha): \check{p}^*(\alpha) = \check{1}\} \neq \emptyset)$$

so

$$p^* \Vdash (\bigcup\{f(\alpha): \alpha \in \mathbf{G}_0\} \cap \bigcup\{f(\alpha): \alpha \in \mathbf{G}_1\} \neq \emptyset)\,. \qquad \square$$

Now to prove the Theorem, let $\langle X, \mathcal{T}\rangle$ be a normal topological space in $\mathcal{M}[G]$ with Y a closed discrete subspace of power κ which is $<\kappa$-separated. By Lemma 2.14, without loss of generality assume $\mathcal{T}$ has a basis $\mathcal{B}$ of cardinality $<\lambda$ and $|X| < \lambda$. Therefore, letting $\mathcal{M}[G] = \mathcal{M}[G_\alpha: \alpha < \lambda]$ where G_α are Cohen subsets of κ, we have X, and Y appear in some $\mathcal{M}[G_\alpha: \alpha < \lambda, \alpha \neq \alpha_0]$, $\alpha_0 < \lambda$. If Y is not separated in $\langle X, \mathcal{T}\rangle$ in $\mathcal{M}[G]$, it is not separated in the topology $\bar{\mathcal{B}}$ on X generated by $\mathcal{B}$ in $\mathcal{M}[G_\alpha: \alpha < \lambda, \alpha \neq \alpha_0]$. By 2.15, Y is not normalized in $\langle X, \bar{\mathcal{B}}(G_{\alpha_0})\rangle = \langle X, \mathcal{T}\rangle$ in $\mathcal{M}[G_\alpha: \alpha < \lambda, \alpha \neq \alpha_0][G_{\alpha_0}] = \mathcal{M}[G]$, contradiction.

We have done one step toward proving the consistency of normal spaces of character $2^{\aleph_0}$ being collectionwise Hausdorff by forcing. The rest of the proof would be to apply 2.13 at regular cardinals and 2.9 at singular ones. The technical details are messy since to formalize this one has to do 'reverse Easton forcing' over a class of cardinals. See TALL [1969] and MENAS [1973].

There is yet another way of getting normal spaces of character $\leq\aleph_1$ to be $\aleph_1$-collectionwise Hausdorff, and that is to use $\diamond^*$. Recall $\diamond^*$ says there is a sequence $\{\mathcal{S}_\alpha : \alpha \in \omega_1\}$, $\mathcal{S}_\alpha \in [\mathcal{P}(\alpha)]^\omega$, such that for every $A \subseteq \omega_1$, $\{\alpha : A \cap \alpha \in \mathcal{S}_\alpha\}$ is closed unbounded. $\diamond^*$ is incomparable with $\diamond$ for stationary systems (SHELAH [1979a]). There are several different routes (SHELAH [1979a], TAYLOR [1981], WATSON [1983]) from $\diamond^*$ which lead to the result, and it is not yet clear which is the best one. The theory is also somewhat unsatisfactory in that it does not generalize conveniently to $\aleph_2$-collectionwise Hausdorff although one has the interesting fact that, assuming the first two levels of GCH, first countable spaces which are collectionwise normal with respect to discrete collections of power $\leq\aleph_1$ are $\aleph_2$-collectionwise Hausdorff (FLEISSNER [1978b]). We shall content ourselves here with defining yet another combinatorial principle Y, which is a common generalization of $\diamond^*$ and $\diamond$ for stationary systems and implies normal spaces of character $\leq\aleph_1$ are $\aleph_1$-collectionwise Hausdorff.

DEFINITION. A partial function $F: {}^{\omega_1}2 \to 2$ is *defined most of the time* if the σ-ideal $\mathcal{S}_F$ generated by sets of the form $\{\alpha : F(f|\alpha) \text{ is undefined}\}$, for each $f: \omega_1 \to 2$, does not include a closed unbounded set.

Y: For every $F: {}^{\omega_1}2 \to 2$ which is defined most of the time, there is a $g: \omega_1 \to 2$, such that for every $f: \omega_1 \to 2$, there is an $\alpha \in \omega_1$ such that $F(f|\alpha) = g(\alpha)$.

2.16. THEOREM. *$\diamond^*$ and $\diamond$ for stationary systems at $\aleph_1$ both imply Y.*

2.17. THEOREM. *Y implies normal spaces of character $\leq\aleph_1$ are $\aleph_1$-collectionwise Hausdorff.*

Proofs. See Watson cited above.

An essential ingredient in Theorem 2.6 is the restriction on character. It is therefore quite surprising that for locally compact T_2 spaces, no such restriction is needed. For convenience in this section, we take 'compact' to mean 'compact Hausdorff'.

2.18. THEOREM. *$V = L$ implies locally compact normal spaces are collectionwise Hausdorff.*

Actually, the Theorem *is* a corollary of the earlier collectionwise Hausdorff results; the key step, as we shall see, is to observe that when attempting to separate $\leqslant\kappa$ points, one may without loss of generality assume that each has character $\leqslant\kappa$.

Theorem 2.18 has a number of interesting corollaries. First note that we get collectionwise normality for compact sets, since the quotient map which collapses a discrete collection of compact sets to points preserves local compactness and normality. In fact one can even obtain collectionwise normality for paracompact sets by observing they are discrete unions of σ-compact sets. One may also weaken the hypothesis of Theorem 2.18 to merely require the space to be of *point-countable type*, i.e. every point is included in a compact set of countable character. Besides first countable and locally compact spaces, Čech-complete spaces (or, more generally, p-spaces) satisfy this criterion. More important is the following result which follows immediately by 1.7.

2.19. THEOREM. *$V = L$ implies locally compact normal submetacompact spaces are paracompact.*

Stating the most general results, we have

2.20. THEOREM. (a) *$V = L$ implies locally compact normal spaces are collectionwise normal with respect to discrete collections of closed submetacompact subsets.*

(b) *$V = L$ implies normal spaces of countable type are collectionwise normal with respect to compact sets.*

(c) *$V = L$ implies normal spaces of point-countable type are collectionwise Hausdorff.*

A space is of *countable type* if every compact set is included in a compact set of countable character. Locally compact spaces are of countable type: given a compact set K, enclose it in an open set U with compact closure and use regularity inductively to get a compact G_δ about K included in U and then observe that closed G_δ's in compact T_2 spaces have countable character.

The main ingredient in our results is

2.21. LEMMA. *Suppose X is a normal space and every discrete collection of fewer than κ compact sets which are included in compact sets of countable character, is separated. Then if X admits a discrete unseparated collection of κ compact sets which are included in compact sets of countable character, it admits one in which each set has character $\leqslant\mathrm{cf}(\kappa)$.*

PROOF. First suppose κ is regular. Let $\{G_\alpha : \alpha < \kappa\}$ be a discrete collection of compact sets. By repeatedly applying normality there exist open F_σ-sets $O_\alpha \supseteq G_\alpha$, $\bar{O}_\alpha \cap \bigcup\{G_\beta : \beta \neq \alpha\} = \emptyset$. By normality there is an open A such that $\bigcup\{G_\alpha : \alpha < \kappa\} \subseteq A \subseteq \bar{A} \subseteq \bigcup\{O_\alpha : \alpha < \kappa\}$.

For each α, let K'_α be a compact set of countable character containing it. Let K_α

be a closed G_δ in K'_α containing G_α and included in $K'_\alpha \cap O_\alpha \cap A$. Then as before, K_α has countable character in K'_α and hence in X.

For each α let $L_\alpha = K_\alpha - \bigcup\{O_\beta : \beta \neq \alpha\}$. $\{L_\alpha : \alpha < \kappa\}$ is a discrete collection of compact sets: the O_α's prove discreteness for points in $\bigcup\{O_\alpha : \alpha < \kappa\}$, while $X - \bar{A}$ takes care of the rest of the space. The L_α's are unseparated since $L_\alpha \supseteq G_\alpha$. The complement of L_α in K_α is the union of $\leqslant\kappa$ closed subsets of K_α (since the O_β's are F_σ's), so in the compact space K_α, L_α is the intersection of $\leqslant\kappa$ open sets and hence has character $\leqslant\kappa$ there. But then the character of L_α in X is $\leqslant\kappa$ since K_α has countable character.

The case of singular κ proceeds essentially the same way. We partition κ into disjoint subsets $\{S_\gamma : \gamma < \mathrm{cf}(\kappa)\}$ and assume by normality and hypothesis that for each γ, $\{O_\alpha : \alpha \in S_\gamma\}$ are discrete. Proceeding as before, we now claim $K_\alpha - L_\alpha$ is the union of $\leqslant\mathrm{cf}(\kappa)$ closed subsets of K_α. The point is that by compactness K_α intersects at most finitely many elements of each $\{O_\beta : \beta \in S_\gamma\}$, and so at most $\mathrm{cf}(\kappa)$ many elements of $\{O_\beta : \beta \neq \alpha\}$.

We now can prove 2.18(b) by induction on the number of compact sets. Normality gets us started, 2.7 gets us past regular cardinals while 2.8 and 2.9 take care of singular ones. To prove (a), note that as mentioned, locally compact spaces are of countable type and so are collectionwise normal with respect to compact sets. Then proceed as in 1.1.9.

Note that we have actually proved that $V = L$ implies that normal spaces are collectionwise normal with respect to compact sets which are included in compact sets of countable character. Thus (c) follows. It is not clear whether we can improve (c) to get collectionwise normality for compact sets. Incidentally, W.S. Watson has recently constructed examples of normal spaces which are collectionwise Hausdorff but are not collectionwise normal with respect to compact sets.

It is worth pointing out that by using the techniques of 2.21 together with the ones at the beginning of this section, we can prove such results as

2.22. THEOREM. *Suppose for every κ that $2^\kappa < 2^{\kappa^+}$. Then normal spaces of point-countable type are weakly collectionwise Hausdorff.*

So far none of the set-theoretic techniques we have exhibited has enabled us to separate discrete collections of arbitrary closed sets. We shall now consider *measure extension axioms* which will do this. The strongest (for our purposes) is what is known as ***Fisher's Axiom*** or, more descriptively, the

Product Measure Extension Axiom. For any cardinal λ, the usual product measure on ${}^\lambda 2$ can be extended to a c-additive measure defined on all subsets of ${}^\lambda 2$.

A weaker axiom is the

$\mathfrak{c}$ *Measure Extension Axiom*. Given $\mathfrak{c}$ subsets of ${}^{\mathfrak{c}}2$, there is a $\mathfrak{c}$-additive measure extending the usual product measure and defined on those sets.

Finally, we shall consider the

Weak Measure Extension Axiom. For every $\kappa < \mathfrak{c}$, for any collection of κ subsets of ${}^{\kappa}2$, the usual product measure on ${}^{\kappa}2$ can be extended to a κ-additive measure measuring these sets.

3.1. THEOREM. *If there is a model of set theory with a strongly compact* (*weakly compact*) *cardinal, there is a model of the Product Measure Extension Axiom* (*the* $\mathfrak{c}$ *Measure Extension Axiom plus* $\mathfrak{c} = 2^{\underline{\mathfrak{c}}}$). *If there is a model of set theory, there is a model in which* $\mathfrak{c}$ *is anything reasonable,* $2^{\underline{\mathfrak{c}}} = \mathfrak{c}$, *and* WMEA *holds.*

PROOF. See Fleissner's article (FLEISSNER [1983b]) in this volume. $\mathfrak{c}$MEA is equiconsistent with the existence of a weakly compact cardinal. The Product Measure Extension Axiom has strength at least the existence of 'many' measurable cardinals and implies $2^{\underline{\mathfrak{c}}} = \mathfrak{c}$.

All three axioms are obtained by adjoining random reals, say to a model of GCH. In particular WMEA may be obtained by adjoining $\aleph_2$ random reals.

3.2. THEOREM. *If the usual product measure on* ${}^{\lambda}2$ *can be extended by* κ-*many sets to a* ρ-*additive, measure* $\rho \leqslant \kappa$, *then normal spaces of character* $<\rho$ *are collectionwise normal with respect to discrete collections of power* $\leqslant\lambda$ *and union of cardinality* $\leqslant\kappa$. *In particular, the Product Measure Extension Axiom implies normal spaces of character* $<\mathfrak{c}$ *are collectionwise normal.*

PROOF. Let $\{Y_\alpha\}_{\alpha<\lambda}$ be a discrete collection, $|\bigcup\{Y_\alpha\colon \alpha<\lambda\}| \leqslant \kappa$. For each $f \in {}^{\lambda}2$ there exist disjoint open sets $U_i(f)$ including $\bigcup\{Y_\alpha\colon f(\alpha) = i\}$, $i = 0, 1$. For each $y \in \bigcup\{Y_\alpha\colon \alpha < \lambda\}$ let $\{N_\beta(y)\colon \beta < \chi_y\}$ be a neighbourhood base for y, χ_y a cardinal less than ρ. Then for each y and for each $f \in {}^{\lambda}2$ there is a $\gamma(f, y) < \chi_y$ such that $N_{\gamma(f,y)}(y) \subseteq U_0(f)$ or $U_1(f)$ depending on the value of $f(\alpha)$, where $y \in Y_\alpha$. Let $M(\beta, y) = \{f \in {}^{\lambda}2\colon \gamma(f, y) = \beta\}$. Extend the product measure on ${}^{\lambda}2$ by the κ many $M(\beta, y)$'s to a ρ-additive measure μ. By ρ-additivity and since the $N_\beta(y)$'s are a basis, for each y there is an $F_y \in [\rho]^{<\omega}$ such that $\mu(\bigcup_{\beta\in F_y} M(\beta, y)) > \frac{7}{8}$. For any y, $z \in \bigcup\{Y_\alpha\colon \alpha < \lambda\}$ then, we have

$$\mu\Big(\bigcup_{\beta\in F_y} M(\beta, y) \cap \bigcup_{\beta\in F_z} M(\beta, z)\Big) > \tfrac{3}{4}\cdot$$

Suppose $y \in Y_\alpha$ and $z \in Y_\beta$ for different α, β. Then since $\mu(\{f \in {}^{\lambda}2\colon f(\alpha) = 0$ and

$f(\beta)=1\})=\frac{1}{4}$, there is an $f\in\bigcup_{\beta\in F_y}M(\beta,y)\cap\bigcup_{\beta\in F_z}M(\beta,z)$ such that $f(\alpha)=0$ and $f(\beta)=1$. But then $\bigcap_{\beta\in F_y}N_\beta(y)\subseteq U_0(f)$ and $\bigcap_{\beta\in F_z}N_\beta(z)\subseteq U_1(f)$ and these are disjoint. Thus the required separation is $\{\bigcup\{\bigcap_{\beta\in F_y}N_\beta(y)\colon y\in Y_\alpha\}\colon \alpha<\lambda\}$.

3.3. THEOREM. $\mathfrak{c}$MEA *implies normal, locally compact, locally connected, first countable spaces (e.g. normal manifolds) are collectionwise normal.*

PROOF. It suffices to show each component has cardinality $\leqslant\mathfrak{c}$. Working in a component, take an open U_0 with compact closure. By ARHANGEL'SKIĬ [1969] (for a proof, see HODEL [1983], this volume), compact first countable spaces have cardinality $\leqslant\mathfrak{c}$ so $\bar{U}_0$ may be covered by $\leqslant\mathfrak{c}$ open sets, each with compact closure. Let U_1 be the union of these open sets. Inductively construct $\{U_\alpha\}_{\alpha\leqslant\omega_1}$ such that $\bar{U}_\alpha\subseteq U_{\alpha+1}$ and $|U_\alpha|\leqslant\mathfrak{c}$ by using the successor step indicated above and by taking unions at limits, noting that in a first countable T_2 space, the closure of a countable union of sets of size $\leqslant\mathfrak{c}$ has size $\leqslant\mathfrak{c}$. By first countability, U_{ω_1} is closed so we are done.

3.4. THEOREM. *Assume* WMEA. *Suppose $\chi(X)\leqslant w(X)<2^{\aleph_0}$, where '$\chi$' and '$w$' stand for 'character' and 'weight' respectively. Then if X is normal, it is collectionwise normal.*

PROOF. Examine the proof of 3.2 and observe that in this case there are $\leqslant w$ many distinct sets by which we need to extend the measure to a χ^+-additive measure.

The L-theoretic and measure-theoretic methods we have exhibited above may be combined to prove the following

3.5. THEOREM. *Adjoin $\aleph_2$ random reals to a model of $V=L$. Then every locally compact perfectly normal space is collectionwise normal.*

First we shall prove a technical

3.6. LEMMA. *If X is perfectly normal, locally compact, and collectionwise normal with respect to compact sets, then X is the free union of spaces, each of which is the union of $\leqslant\aleph_1$ compact sets.*

PROOF. For $\alpha\in\omega_1$, inductively define closed sets F_α and collections $\mathcal{U}_\alpha$ of F_α-open sets as follows:

Let $F_0=X$. Let $\mathcal{U}_0$ be a maximal collection of open sets with disjoint compact closures. Suppose F_β, $\mathcal{U}_\beta$, $\beta<\alpha$ have been defined. Let $F_\alpha=X-\bigcup\{\bigcup\mathcal{U}_B\colon\beta<\alpha\}$ and let $\mathcal{U}_\alpha$ be a maximal collection of F_α-open sets with disjoint compact closures. Claim $X=\bigcup\{\bigcup\mathcal{U}_\alpha\colon\alpha<\omega_1\}$. If not, there is an $x\in\bigcap\{F_\alpha\colon\alpha<\omega_1\}$. Let U be an open neighbourhood of x with compact closure. By perfection, U is hereditarily

Lindelöf so there is a $\alpha < \omega_1$ such that $F_\alpha \cap U = F_{\alpha+1} \cap U$. But then $\bigcup \mathcal{U}_\alpha \cap U$ is empty, so $F_\alpha \cap U$ must be empty, because $\bigcup \mathcal{U}_\alpha$ is dense in F_α. But that is a contradiction.

Again by perfection, for each $\alpha < \omega_1$ let $\bigcup \mathcal{U}_\alpha = \bigcup \{F_{\alpha,n}: n < \omega\}$, $F_{\alpha,n}$ closed. Then $\mathcal{F}_{\alpha,n} = \{\overline{F_{\alpha,n} \cap U}: U \in \mathcal{U}_\alpha\}$ is a discrete collection of compact subsets of X. By collectionwise normality for compact sets and local compactness let $\mathcal{W}_{\alpha,n} = \{W(U, n): U \in \mathcal{U}_\alpha\}$ be a discrete collection of open sets with compact closures such that $F_{\alpha,n} \cap U \subset W(U, n)$. Then $\mathcal{W} = \bigcup \{\mathcal{W}_{\alpha,n}: \alpha < \omega_1,\ n < \omega\}$ is a cover of X by open sets with compact closures, which can be decomposed as the union of $\aleph_1$ discrete subcollections. Proceeding as in the proof of Theorem 1.3 above, we can then partition x into $\aleph_1$ clopen pieces, each of which is the union of $\leq \aleph_1$ of the $\mathcal{W}_{\alpha,n}$'s.

REMARK. The final hypothesis of 3.6 can be weakened to 'collectionwise Hausdorff' by an application of the technique of 1.10, but we don't need it to prove 3.5.

Now for the proof of the Theorem. By Lemma 3.2, in our model first countable normal spaces are $\aleph_1$-collectionwise Hausdorff. $\diamond$ for stationary systems holds for regular $\kappa \geq \aleph_2$, in $L[A]$, $A \subseteq \omega_2$, so by Lemma 2.7 locally compact perfectly normal spaces are collectionwise normal with respect to compact sets since the character of compact sets is $\leq \aleph_0$. By Lemma 3.6 then, we just have to consider locally compact perfectly normal spaces which are unions of $\aleph_1$ compact sets. Discrete collections then have cardinality at most $\aleph_1$, and each element of such a collection is the union of $\aleph_1$ compact sets of countable character, say $Y_\gamma = \bigcup_{\alpha<\omega_1} \dot{K}_{\gamma\alpha}$. Look back at the proofs of 3.2 and 3.4 and note that we never used the fact that distinct points are disjoint. It follows that we may obtain the desired separation via WMEA, extending the measure on $\{0, 1\}^{\aleph_1}$ by the $\aleph_1$ sets corresponding to the open sets forming neighbourhood bases about the $\aleph_1$ compact sets involved.

A useful result in delimiting the kinds of normal non-collectionwise normal spaces that exist absolutely is

3.7. THEOREM. *It is consistent with* ZFC *that first countable normal spaces are collectionwise normal with respect to discrete collections of copies of* ω_1.

This was originally proved (FLEISSNER [1977a]) with the aid of an inaccessible cardinal: mimic the proof that collapsing such a cardinal kills Kurepa trees and get that the character of any copy of ω_1 (consider it as a point) in a first countable space is $\aleph_1$. Having forced over L, $\diamond$ for stationary systems and GCH still hold everywhere, so one can apply 2.7–2.9 to get the result. A more delicate argument (TALL [1980b]) keeps GCH but avoids the inaccessible at the (necessary) cost of failing to keep the character of ω_1 so small. Giving up CH, however, we can get a stronger result by the same method as for 3.5.

3.8. THEOREM. *In the model obtained by adjoining $\aleph_2$ random reals to a model of $V = L$, normal first countable spaces are collectionwise normal with respect to arbitrary discrete collections of sets of power $\leqslant\aleph_1$.*

PROOF. Measure extension takes care of collections of size $\leqslant\aleph_1$. Collapse the sets to points and they have character $\leqslant\aleph_2$. GCH holds at $\aleph_2$ and above, as does $\diamondsuit$ for stationary systems at regular cardinals, so we may as usual proceed by induction to get the result.

The new proof (see the end of this chapter) that "normality implies collectionwise normality" results may be obtained via random real forcing without first proving measure extension axioms, arrived too late to be considered in the foregoing. It enables 3.7 and 3.8 to be improved to allow $\chi \leqslant \aleph_1$. This is discussed further in TALL [19··b].

I have recently proved (TALL [19··c]):

3.9. THEOREM. *If the existence of a weakly compact cardinal is consistent with* ZFC, *so is "every locally compact normal space of weight $\leqslant\mathfrak{c}$ is collectionwise normal".*

The proof is rather complicated so I won't get into it here. By the same method I prove

3.10. THEOREM. *It's consistent with* ZFC *that locally compact normal spaces are collectionwise normal with respect to discrete collections of copies of ω_1.*

In addition to measure extension axioms and Fleissner's normal non-metrizable Moore space (see his chapter), large cardinals enter into the normality versus collectionwise normality in more-or-less standard set-theoretic fashion. By that I mean that there are many properties which if true below a weakly compact cardinal, hold at that cardinal, and if true below a strong compact or perhaps a supercompact, hold at and above that cardinal. To be concrete, FLEISSNER [1977b] proved

3.11. THEOREM. *If κ is weakly (strongly) compact, $\chi(X) < \kappa$, and X is $<\kappa$-collectionwise Hausdorff, then X is κ-collectionwise Hausdorff (λ-collectionwise Hausdorff for all $\lambda \geqslant \kappa$).*

I proved the analogous results for collectionwise normality in 1980 (unpublished). These results illuminate the relevance of strongly and weakly compact cardinals for the normal Moore space problem. More interesting than these results is the situation that occurs when these large cardinals are Lévy-collapsed to $\aleph_2$. Again in standard set-theoretic fashion, one wonders whether, if a property holds below $\aleph_2$, it now holds at $\aleph_2$ or at all $\lambda \geqslant \aleph_2$. The results here are fragmentary; two worth mentioning are

3.12. THEOREM. (SHELAH [1977a]. *In the model obtained by Lévy-collapsing a supercompact, locally countable $\aleph_1$-collectionwise Hausdorff spaces are collectionwise Hausdorff.*

3.13. COROLLARY. (TALL [19··b]). *In the same model, locally countable spaces which hereditarily are collectionwise normal with respect to discrete collections of $\leq\aleph_1$ sets, each of cardinality $\leq\aleph_1$, are hereditarily collectionwise normal.*

Some restriction on the spaces is necessary since there is an absolute example of a normal $\aleph_1$-collectionwise Hausdorff space which is not $\aleph_2$-collectionwise Hausdorff (PRZYMUSIŃSKI [1975]). An interesting conjecture of FLEISSNER [1979] is that in the above model, first countable $\aleph_1$-collectionwise Hausdorff spaces are collectionwise Hausdorff. I conjecture a similar result holds for $\aleph_1$-collectionwise normality.

Shelah's method actually works for a much less restricted class of spaces, but I do not yet have the 'right' formulation, so I won't pursue the subject further here.

The major impetus toward studying normality versus collectionwise normality has of course been the normal Moore space problem. A *Moore space* X is a regular space with a sequence $\{\mathcal{G}_n\}_{n<\omega}$ of open covers such that for each $p \in X$ and each open U containing p, there is an $n \in \omega$ such that $\bigcup\{G \in \mathcal{G}_n : p \in G\} \subseteq U$. The normal Moore space problem is whether normal Moore spaces are metrizable. Collectionwise normal ones are—indeed that is a characterization of metrizability— (BING [1951], see GRUENHAGE [1983], this volume), so the problem we have been expounding arises. The most useful properties of Moore spaces in connection with collectionwise normality are that they are first countable, closed sets are G_δ's, and they are subparacompact. The first is obvious, the second is easy, and a proof of the third can be found in the references just mentioned. Using these we easily obtain the following corollaries of our earlier results.

4.1. THEOREM. (a) *Locally compact, locally connected, normal Moore spaces are metrizable.*

(b) $2^{\aleph_0} < 2^{\aleph_1}$ *implies countable chain condition normal Moore spaces are metrizable.*

(c) $V = L$ *implies locally compact normal Moore spaces are metrizable.*

(d) PMEA ($\mathfrak{c}$MEA) (WMEA) *implies normal Moore spaces* (*of weight* $\leq\mathfrak{c}$) (*of weight* $<\mathfrak{c}$) *are metrizable.*

If κ^{++} random reals are adjoined to a model of $V = L$, normal Moore spaces of *local* weight $\leq\kappa$ are metrizable; if weakly compact many are adjoined, normal Moore spaces of local weight $<\mathfrak{c}$ are metrizable. See TALL [19··b] for a proof.

II. Examples

There is a wealth of examples of normal non-collectionwise normal spaces—and in particular, normal non-metrizable Moore spaces. In this section we shall discuss extensively the principal ones of interest that are not given by Fleissner in his chapter.

The standard example of a normal non-collectionwise normal space is Example G of BING [1951]. We shall call it

EXAMPLE A. *A normal non-$\aleph_1$-collectionwise Hausdorff space of character* $2^{\aleph_1}$.

The construction is given in Fleissner's article. To see that the space has character $2^{\aleph_1}$, observe that the character of a non-isolated point is equal to its character in the product space. Standard arguments (see e.g. TALL [1969]) prove that the character of a point in $\{0, 1\}^{2^{\aleph_1}}$ is $2^{\aleph_1}$.

EXAMPLE B. *A perfectly normal non-$\aleph_1$-collectionwise Hausdorff space.*

This is Example H of BING [1951]. Again we refer to Fleissner's chapter. There are many variations of these two examples of Bing—see LEWIS [1977]. The most notable is Michael's subspace [1955]:

EXAMPLE C. *A metacompact normal non-$\aleph_1$-collectionwise Hausdorff space of cardinality* $\aleph_1$.

Throw out all isolated points from Example A except those with finite support. The remaining subspace M has cardinality $\aleph_1$ and is normal and not collectionwise Hausdorff as before. Given an open cover $\mathcal{U}$, pick open sets U_α about each non-isolated point y_α. Let $V_\alpha = U_\alpha \cap \Pi^{-1}_{\{y_\alpha\}}(1)$. The V_α's plus the isolated points form a point-finite refinement of $\mathcal{U}$.

Bing's examples can be generalized to provide for every regular uncountable κ a perfectly normal metacompact space which is $<\kappa$ collectionwise normal but is not κ-collectionwise Hausdorff. See PRZYMUSIŃSKI [1975].

None of the above examples—or anything resembling them—is locally compact. The point is that they are 0-dimensional and so, if U is a compact open neighbourhood of some non-isolated point P, take a smaller compact open neighbourhood V of p by fixing more coordinates. Then $U - V$ is a compact infinite discrete space, which is absurd.

Another example (KUNEN [1977b]) of a normal non-$\aleph_1$-collectionwise Hausdorff space is not widely known and is no more difficult than Bing's. It was the first absolute example of an extremally disconnected normal space which was not paracompact.

EXAMPLE D. *An extremally disconnected normal non-$\aleph_1$-collectionwise Hausdorff space.*

Let $\{A_\alpha: \alpha \in 2^{\omega_1}\}$ be a maximal family of independent subsets of ω_1. (Recall 'independent' means any finite combination of A_α's and $\omega_1 - A_\alpha$'s has non-empty intersection if the α's are distinct. Independent families of size $2^{\aleph_1}$ are easily constructed using the traces of basic open subsets of $\{0, 1\}^{2^{\aleph_1}}$ on a dense set of power $\aleph_1$.) Let $\{H_\alpha: \alpha \in 2^{\omega_1}\}$ enumerate $\mathcal{P}(\omega_1)$. Define $U'_\xi \subseteq \mathcal{P}(\omega_1)$, $\xi \in \omega_1$, by $A_\alpha \in U'_\xi$ if $\xi \in H_\alpha$; $\omega_1 - A_\alpha \in U'_\xi$ if $\xi \notin H_\alpha$. Then U'_ξ is a filter subbase. Let U'_ξ generate a filter on ω_1 and extend it to an ultrafilter U_ξ. The U_ξ's are distinct. Let ω_1 be the discrete space of power $\aleph_1$ and consider the U_ξ's as points in $\beta\omega_1 - \omega_1$. Then $X = \omega_1 \cup \{U_\xi: \xi \in \omega_1\}$ is a dense subspace of the extremally disconnected space $\beta\omega_1$ and so is extremally disconnected. $\{U_\xi: \xi \in \omega_1\}$ is a closed discrete subspace of X. X is normal, for if $H \subseteq \omega_1$, $H \cup \{U_\xi: \xi \in H\}$, $(\omega_1 - H) \cup \{U_\xi: \xi \in \omega_1 - H\}$ are disjoint and open. Suppose the U_ξ's were separated. Then there would exist pairwise disjoint subsets $F_\xi \subseteq \omega_1$, $F_\xi \in U_\xi$. At most one F_ξ could be an A_α so without loss of generality, assume none is. By maximality, for every ξ there is a finite partial function s_ξ from 2^{ω_1} into 2 such that, letting $A^0_\alpha = A_\alpha$ and $A^1_\alpha = \omega - A_\alpha$, $F_\xi \supseteq \bigcap\{A^{s_\xi(\alpha)}_\alpha: \alpha \in \operatorname{dom} s_\xi\}$. Without loss of generality, assume the domains of the s_ξ's form a Δ-system with root r, and that $s_\xi|r = s_{\xi'}|r$ for all ξ, ξ'. But then, by independence, the F_ξ's have non-empty intersection, contradiction.

Steve Watson maintains (in work in progress) that all (known) normal non-collectionwise normal spaces are essentially just variants of Bing's original example. We leave it as an exercise for the reader to put Kunen's example into this form.

By assuming extra set-theoretic hypotheses, spaces with 'nicer' properties than Bing's (and its variants) or Kunen's may be obtained. The three nice properties Bing's and Kunen's spaces fail to have are separability, first countability, and local compactness. This is no accident, since these properties are excluded by Theorems 2.1, 2.6, and 2.19 respectively. To obtain separability, we prove a converse of 2.1:

EXAMPLE E. Assume $2^{\aleph_0} = 2^{\aleph_1}$. Then there is *a separable normal T_1 space with an uncountable closed discrete subspace.*

PROOF. Let L be a set of cardinality $\aleph_1$ disjoint from ω. Let $\{F_\alpha\}_{\alpha<\mathfrak{c}}$ be an independent family of subsets of ω. Using $2^{\aleph_0} = 2^{\aleph_1}$, construct a complement-preserving map f from $\mathcal{P}(L)$ onto $\{F_\alpha\}_{\alpha<\mathfrak{c}} \cup \{\omega - F_\alpha\}_{\alpha<\mathfrak{c}}$. Let $X = L \cup \omega$ with a subbasis $\mathcal{S}$ for a topology defined by

(a) if $M \subseteq L$, then $M \cup f(M) \in \mathcal{S}$,

(b) if $n \in \omega$, then $\{n\} \in \mathcal{S}$,

(c) if $p \in X$, then $X - \{p\} \in \mathcal{S}$.

(c) assures T_1; (b) makes L closed; (a) makes it discrete. Separability is clear. If Y, Z are disjoint closed subsets of X,

$$U_Y = (Y \cup f(Y \cap L)) \cap (X - Z),$$
$$U_Z = ((Z - L) \cup ((L - Y) \cup f(L - Y)) \cap (X - Y)$$

are disjoint open sets about Y, Z respectively, so X is normal.

A slick way of getting this result is to note that the non-collectionwise normality of Bing's example depends only on the set of isolated points being dense in $\{0, 1\}^{2^{\aleph_1}}$. By hypothesis $2^{\aleph_1} = 2^{\aleph_0}$ and we know $\{0, 1\}^{2^{\aleph_0}}$ is separable so that does it.

To obtain a *first countable* separable normal non-collectionwise normal space, we must use a hypothesis strictly stronger than $2^{\aleph_0} = 2^{\aleph_1}$, since WMEA and $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ are consistent. What we need is a *Q-set.*

DEFINITION. A *Q-set* is an uncountable set of reals such that in the subspace topology, every subset of it is an F_σ.

It is not difficult to show that, assuming Martin's Axiom plus $2^{\aleph_0} > \aleph_1$, every uncountable set of reals of power less than continuum is a Q-set (TALL [1969]). It follows from 1.2.1 plus the next example that $2^{\aleph_0} < 2^{\aleph_1}$ implies there are no uncountable Q-sets. For more on Q-sets, see MILLER [1983] this volume. The first and most famous example of a normal non-metrizable Moore space can easily be obtained from a Q-set:

EXAMPLE F. Let B be an uncountable set of real numbers. $M(B)$, *the Moore space derived from* B, is defined to be the Euclidean upper half-plane plus the points of B, topologized by giving the points above the axis their usual neighbourhoods, while a point x in B has as neighbourhoods $\{x\}$ together with the interior of a disc in the upper half plane, tangent at x to the axis. $M(B)$ is a *separable non-metrizable Moore space.* $M(B)$ *is normal* if (and only if) B is a Q-set.

PROOF. We shall omit the less interesting 'only if'. See TALL [1969]. The points above the axis with both coordinates rational are dense in $M(B)$, so it's separable. $M(B)$ is not metrizable since B is an uncountable closed discrete subspace which cannot be separated. More elementarily, $M(B)$ is not metrizable since any countable collection of open sets fails to be a basis at some points of B although $M(B)$ is separable. A development can easily be constructed by taking as the n'th cover, Euclidean $1/n$ discs $(\bigcap M(B))$ about points above the axis plus basic neighbourhoods with $1/n$ discs for points on the axis. Points are closed since $M(B)$ strengthens the Euclidean topology. It is not difficult to show $M(B)$ is regular; indeed given $p \in U$ open, there is an open V containing p with even the closure of V in the Euclidean topology included in U.

Suppose B is a Q-set. To show $M(B)$ is normal it suffices to show that if $Y \subseteq B$, then there exist disjoint open sets about Y and $B - Y$. Arbitrary pairs of disjoint closed sets can be handled using the fact that above the axis, $M(B)$ has the

Euclidean topology. There exist closed sets of reals $\{F_n\}_{n<\omega}$ and $\{G_n\}_{n<\omega}$ such that $Y = B \cap \bigcup_{n<\omega} F_n$, $B - Y = B \cap \bigcup_{n<\omega} G_n$. For any n, $Y \cap G_n = 0$, so points in Y are at positive distance from G_n. We can put bubbles about the points in $G_n \cap B$ and then, with a little computation, find other bubbles disjoint from them about the points in Y. But that means that, letting U_n be the union of the first set of bubbles, $\bar{U}_n \cap Y = 0$. Similarly, for any n construct $V_n \supset F_n \cap B$ such that $\bar{V}_n \cap (B - Y) = \emptyset$. But then by 1.1.1 we can separate Y and $B - Y$.

The existence of a Q-set is actually equivalent to the existence of a separable normal non-metrizable Moore space (HEATH [1964a]). There are a large number of other equivalents (TALL [1969], [1974b]), in particular the existence of a separable normal first countable space which is not collectionwise normal.

$M(B)$, B a Q-set, is often called 'the bubble space'. There are a large number of variations on it. For example one can isolate the points with rational coordinates above the axis and throw out all the other points above the axis without changing the essential properties. Having made these modifications, to get *a locally compact separable normal non-metrizable Moore space*, take for each point x in B a sequence of points with rational coordinates above the axis which converges to x in the Euclidean topology. Let the basic neighbourhoods of x be tails of the sequence chosen for x. It is not difficult to check that the resulting space has the desired properties. We shall call it *Example* G.

An essentially equivalent space is

EXAMPLE G′. The *Cantor tree* is the subspace of the full binary tree ${}^{\underline{\omega}}2 \cup {}^{\omega}2$ with the tree topology, obtained by omitting the points in the complement of a Q-set. It is easy to see the Cantor tree is *locally compact, separable, normal, first countable, and not* $\aleph_1$*-collectionwise Hausdorff.* If there is a Q-set, there is one in the Cantor space ${}^{\omega}2$, since sets of reals of power less than continuum are 0-dimensional, while Cantor space is universal for 0-dimensional separable metric spaces.

Another variation of the bubble space is

EXAMPLE H. Assuming the existence of a Q-set, *a metacompact normal non-metrizable Moore space.*

This is 'Heath's V-space'. Metacompactness is attained at the cost of losing separability (separable metacompact spaces are Lindelöf and hence, if Moore, are metrizable). Isolate all the points above the axis. The nth neighbourhood of a point on the axis is a 90° angle with vertex at the point, each side of length $1/n$, and each side at a 45° angle with the axis. The isolated points together with all the nth neighbourhoods form the nth cover of a development. It is easy to show that every open cover has a point-2 refinement. Regularity and normality proceed as before. The space is still not collectionwise Hausdorff, since disjoint V's have disjoint discs nestled inside them.

A Q-set also yields a much more pathological normal non-metrizable Moore space:

EXAMPLE I. If there is a Q-set, there is *a countable chain condition normal metacompact Moore space which is not separable* (and hence not metrizable).

We in fact will need an apparent strengthening of the Q-set condition:

DEFINITION. A Q-set S is *strong* if for all $n \in \omega$, S^n is a Q-set in $\mathbb{R}^n$.

Under Martin's Axiom every uncountable set of reals of power less than continuum is a strong Q-set (PRZYMUSIŃSKI and TALL [1974]) but Q-sets (at least those of cardinality $\aleph_2$) need not be strong (FLEISSNER [19··a]). However, in so far as existence is concerned, there is no difficulty.

1. LEMMA. *If there is a Q-set, there is a strong Q-set.*

See Miller's chapter (MILLER [1983]) in this volume for a proof. Our space will be constructed using finite subsets of a strong Q-set. The correspondence of $\mathbb{R}^n$ with $[\mathbb{R}]^n$ is the reason that the strength of the Q-set will be the key toward proving normality.

DEFINITION. Given a T_1 space Y the *Pixley–Roy* topology on the collection X of non-empty finite subsets of Y is defined by taking as a basis all sets of the form

$$U(x, V) = \{y \in X : x \subseteq y \subseteq V\}$$

where $x \in X$ and V is open in Y.

This topology is well-defined, since if $z \in U(x_1, V_1) \cap U(x_2, V_2)$, then $z \in U(z, V_1 \cap V_2) \subseteq U(x_1, V_1) \cap U(x_2, V_2)$. It's Hausdorff, for given $x \neq y \in X$, say e.g. $t \in x - y$, take V open in Y, $y \subset V$, $t \notin V$. Then $U(y, V) \cap U(x, Y) = 0$. In fact the space is zero-dimensional but we won't bother proving this. X is metacompact, for if $\mathscr{W}$ is an open cover of X, take $x \in W_x \in \mathscr{W}$, all $x \in X$ and consider $\{U(x, Y) \cap W_x : x \in X\}$. This is a point-finite open cover refining $\mathscr{W}$ since $y \in U(x, Y)$ if and only if y is one of the finitely many subsets of x. If y is first countable, X is a Moore space. To see this, choose for each $x \in X$ a decreasing neighbourhood base $\{W_n(x) : n \in \omega\}$ in Y for x. We can do this since x is finite and Y is first countable T_1. For each $n \in \omega$ define $\mathscr{G}_n = \{U(x, W_n(x)) : x \in X\}$. Claim $\{\mathscr{G}_n\}_{n \in \omega}$ is a development. For let V be an open set of X containing x. For n sufficiently large, $U(x, W_n(x)) \subseteq U$. For each nonempty proper subset y of x, there is an $n(y) \in \omega$ such that $x \not\subseteq W_{n(y)}(y)$. Let $n = \max\{n(y) : \emptyset \neq y \subseteq x\}$. Then $x \not\subseteq W_n(y)$ if y is a proper subset of x, so

$$\bigcup\{G \in \mathscr{G}_n : x \in G\} = U(x, W_n(x)) \subseteq V.$$

We now leave the general theory of Pixley–Roy spaces and take Y to be a strong Q-set S with the Euclidean topology. In this case (in fact in all but trivial cases) X is not separable, for if D were countable dense in X, take $x \in X$ which is not a subset of any member of D. Then $D \cap U(x, S) = \emptyset$. We next show X satisfies the countable chain condition. It more than suffices to show the basis defining the topology on X is the union of countably many centred families. Let $\mathscr{B}$ be a countable basis for S. For $B \in \mathscr{B}$, let $\mathscr{H}(B) = \{U(x, V) : x \subseteq B \subseteq V\}$. Then each $\mathscr{H}(B)$ is centred.

Note that the collection of singletons of S is a closed discrete subspace of X, so X is not collectionwise Hausdorff, since it satisfies the countable chain condition.

We consider a member of $[\mathbb{R}]^n$ to be ordered by magnitude and hence to be a member of $\mathbb{R}^n$. We denote the ith component of x by 'x_i'.

To show normality—indeed perfect normality—it suffices to find for each open subset U of X, open sets U_n, $n < \omega$ such that $U = \bigcup_{n<\omega} U_n$ and $\bar{U}_n \subseteq U$. Given U open, for each $x \in U$ take $\mu(x)$ such that $U(x, 1/\mu(x)) \subseteq U$. Let $\rho(x) = \min_{i \neq j} |x_i - x_j|$. Let

$$A_{n,m} = \{x \in \bar{U} : |x| = n, \mu(x) \leqslant m, \rho(x) \geqslant 1/m\}.$$

Then $U = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m}$. We consider $A_{n,m}$ as a subset of S^n, as indicated above. Since S is strong Q, there exist closed subsets $F_{n,m,k}$ of S^n such that $A_{n,m} = \bigcup_{k=1}^{\infty} F_{n,m,k}$. Let $V_{n,m,k} = \bigcup\{U(x, 1/2m) : x \in F_{n,m,k}\}$. Then

$$U = \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} V_{n,m,k}.$$

It suffices to show $\bar{V}_{n,m,k} \subseteq U$. Let $z = \{z_1, \ldots, z_r\} \in \bar{V}_{n,m,k}$. For each integer $p > 0$, there is a $t^p \in X$ and an $x^p \in F_{n,m,k}$ such that $t^p \in U(z, 1/p) \cap U(x^p, 1/2m)$. Therefore $t^p \subseteq B(z, 1/p) \cap B(x^p, 1/2m)$ (where $B(x, \varepsilon)$ is the ε-ball about the point x in Euclidean $|x|$-space) and $t^p \supseteq z \cup x^p$, so

(i) $z \cup x^p \subseteq B(z, 1/p) \cap B(x^p, 1/2m)$.

For each p there is an $i_1(p)$ such that $|x_1^p - z_{i_1(p)}| < 1/p$. Since z is finite, there is an i_1 and an infinite set P_1 of positive integers such that for each $p \in P_1$, $|x_1^p - z_{i_1}| < 1/p$. Again, there is an infinite $P_2 \subseteq P_1$ and i_2 such that for each $p \in P_2$, $|x_2^p - z_{i_2}| < 1/p$. Continuing, we obtain an infinite set P_n of positive integers, and positive integers $i_1, \ldots, i_n$ such that for each $p \in P_n$ and for every j, $1 \leqslant j \leqslant n$,

(ii) $|x_j^p - z_{i_j}| < 1/p$.

By (ii) and the triangle inequality, we get $\min_{j \neq j'} |z_{i_j} - z_{i_{j'}}| \geqslant 1/m$ and then $z_{i_1} <$

$\cdots < z_{i_n}$. Let $z' = \{z_{i_1}, \ldots, z_{i_n}\} \subseteq z$. Then $z' \in F_{n,m,k'}$, for if not, there would be an $\varepsilon > 0$ such that $(\Pi_{j=1}^n B(\{z_{i_j}\}, \varepsilon)) \cap F_{n,m,k} = \emptyset$. Take $p \in P_n$ such that $1/p < \varepsilon$. By (ii), for each j we have $x_j^p \in B(\{z_{i_j}\}, \varepsilon)$, which is impossible. Since $z' \in F_{n,m,k}$, $U(z', 1/m) \subseteq U$. Take any $y \in z$ and $p \in P_n$ such that $1/p < 1/2m$. By (i), $y \in B(x^p, 1/2m)$ so $|y - z_{i_j}| < 1/m$. But y was an arbitrary member of z so $z \subseteq B(z', 1/m)$ and hence $z \in U(z', 1/m) \subseteq U$, completing the proof.

One can actually obtain all the significant properties of Example I except first countability without special axioms. This construction is due to JUNNILA [1979]. Verification of the properties is lengthy so we confine ourselves to defining the space. The basic idea is to iterate Bing's construction of Example A (or rather Michael's subspace (Example C)) countably many times. At the first stage of the construction, open sets about the points in the closed discrete subspace satisfy the countable chain condition, but there are too many isolated points. At the $(n+1)$th stage, we add a new set of isolated points which insures that the open sets about the previously isolated points satisfy the countable chain condition. After ω steps, the resulting space satisfies the countable chain condition.

EXAMPLE J. *A perfectly normal countable chain condition metacompact space which is not $\aleph_1$-collectionwise Hausdorff.*

Let A_0 be an uncountable set. Let $A_{n+1} = [\mathcal{P}(A_n)]^{<\omega}$. Let $X = \bigcup_{n\in\omega} A_n$. For $x \in X$, let $n(x)$ be that $n \in \omega$ such that $x \in A_n$. For $x \in X$ and $z \in A_{n(x)+1}$, let $z_x = \{a \in z : x \in a\}$. Let $P_z(x) = \{u \in A_{n(x)+1} : u \cap z = z_x\}$. Define $U \subseteq X$ to be open if and only if for each $x \in U$, $P_z(x) \subseteq U$ for some $z \in A_{n(x)+1}$.

In addition to the topological properties besides normality and collectionwise normality that we have been singling out in our descriptions of examples so far, there are others of lesser interest that have been considered by various mathematicians so we shall survey them briefly.

Many of our examples have dense metrizable subspaces because they are separable or have a dense set of isolated points; on the other hand, the Pixley–Roy space does not have a dense metrizable subspace since countable chain condition metrizable spaces are separable. Similarly it is not locally metrizable, while the bubble space or Heath's V-space, for example, are. The bubble space is locally connected; the Cantor tree is not. Under MA plus not CH, there is a normal Moore space whose square is not normal (COOK [19··]); under the same hypotheses, locally compact separable normal Moore spaces do have normal squares (ALSTER and PRZYMUSIŃSKI [1976]).

There is a general method for proving spaces to be normal, assuming Martin's Axiom, which gives unified short proofs of the normality of Examples F–I. We did not use this method for these examples, since it is of interest that only the existence of a Q-set is needed. For our next two however, there is no proof known which uses anything weaker than the general method, so we introduce it here. Define an open cover to be T_2 *point-separating* if any two distinct points can be separated by disjoint elements of the cover.

2. THEOREM. *Assume Martin's Axiom ($P(\mathfrak{c})$ will do). Suppose $\mathcal{P}$ is a countable T_2 point-separating open cover of a space $\langle X, \mathcal{T}\rangle$ and suppose every point of X has a $\mathcal{T}$-neighbourhood base consisting of sets which are compact in the topology generated by $\mathcal{P}$ (say X is* locally $\mathcal{P}$-compact). *Then any pair of disjoint closed sets of cardinality $<\mathfrak{c}$ is normalized.*

PROOF. See Weiss' chapter in this volume.

EXAMPLE K. Assuming Martin's Axiom plus not CH, there is *a homogeneous normal non-$\aleph_1$-collectionwise Hausdorff space of character $<\mathfrak{c}$.*

Our previous spaces were unions of two metrizable spaces and had essentially two kinds of points—those in the closed discrete subspace and those outside. Moreover they 'lived' on sets of power $\aleph_1$, i.e. each had a normal non-collectionwise Hausdorff subspace of power $\aleph_1$. The example we shall now sketch is quite different—subspaces of size $<\mathfrak{c}$ are discrete and the space is homogeneous. Recall that the *density topology* (see e.g. TALL [1976a]) on the real line is defined by taking as open those Lebesgue measurable sets which have density 1 at each of their points, where

$$d(x, E) = \lim_{h\to 0} \frac{m(E \cap [x-h, x+h])}{2h}.$$

Using Martin's Axiom, it is not difficult to inductively construct a *generalized Sierpiński set S*, i.e. a subset of $\mathbb{R}$ of power $\mathfrak{c}$ such that its intersection with every null set has cardinality $<\mathfrak{c}$, with the additional property that it is an additive subgroup of $\mathbb{R}$. Since the density topology refines the Euclidean topology, S remains homogeneous as a subspace of that topology. (Closed) discrete subspaces of the density topology coincide with sets of measure zero, so by Martin's Axiom, subsets of S of power $<\mathfrak{c}$ are closed discrete. Any subset of the density topology is the union of a countable chain condition piece and a closed discrete piece. The countable chain condition subset must be dense in S, else S would have isolated points and so by homogeneity be closed discrete. But S is not measurable. Hence S satisfies the countable chain condition and so is not collectionwise Hausdorff. To prove S is normal, note that it suffices to separate the boundaries of any two disjoint closed sets. Since S is generalized Sierpiński, as a subspace of the density topology it is *generalized Luzin*, i.e. nowhere dense subsets of it have cardinality $<\mathfrak{c}$. Boundaries thus have cardinality $<\mathfrak{c}$, so by Theorem 2, to prove S is normal it suffices to find a countable T_2 point-separating open cover $\mathcal{P}$ for the density topology for which that topology is locally $\mathcal{P}$-compact. We will then be able to normalize pairs of disjoint closed sets of power $<\mathfrak{c}$ in the density topology and hence in S. Any basis for the Euclidean topology is T_2 point-separating, so it remains to show that the density topology is locally $\mathcal{P}$-compact for some countable basis $\mathcal{P}$ for the Euclidean topology. By the Lusin–Menchoff Theorem (see e.g. ZAHORSKI [1950]), given a point x in the density topology interior of a Euclidean Borel set B, there is a perfect set P, such that $P \subseteq B$ and x is in the density topology interior of P. The density topology is regular and regular open subsets of it are Euclidean $F_{\sigma\delta}$, so we are done. Since there are only $2^{\aleph_0}$ Euclidean $F_{\sigma\delta}$'s while subsets of size $<\mathfrak{c}$ are closed discrete, it is clear S has character $\mathfrak{c}$.

Steve Watson has recently announced that he has a 'homogenizing machine' which yields, among other things, an absolute example of a normal homogeneous non-collectionwise Hausdorff space, a separable example under $2^{\aleph_0} = 2^{\aleph_1}$ and a Moore space under Martin's Axiom plus not CH.

One of the more interesting examples of a normal non-metrizable Moore space (under MA plus $2^{\aleph_0} > \aleph_1$) is a *special Aronszajn tree.*

EXAMPLE L. *A locally compact weakly collectionwise Hausdorff non-$\aleph_1$ collectionwise Hausdorff Moore space* which is normal under MA plus $2^{\aleph_0} > \aleph_1$, but not normal assuming $\diamondsuit$, or even $2^{\aleph_0} < 2^{\aleph_1}$.

DEFINITION. A *tree* is a partially ordered set $\langle T, \leqslant \rangle$ such that for each $t \in T$, $\{x : x < t\}$ is well-ordered by $<$. The *height* of t is the ordinal isomorphic to $\{x : x < t\}$. The height of T is the sup of the heights of its elements. T_α, the αth *level* of T is the collection of all $t \in T$ of height α. $A \subseteq T$ is an *antichain* if it is pairwise unordered. $S \subseteq T$ is a *cofinal branch* if S is totally ordered and $\sup\{\text{height } s : s \in S\} = \text{height } T$. An ω_1*-tree* is a tree of height ω_1 with countable levels. An Aronszajn tree is an ω_1-tree with no cofinal branch. An ω_1-tree is *special* if it is the union of countably many antichains. The *tree topology* on a tree $\langle T, \leqslant \rangle$ is generated by taking the collection of all open intervals as a basis. A tree is Hausdorff if distinct elements have distinct sets of predecessors Special Aronszajn Hausdorff trees exist—see e.g. Todorčević, this volume.

3. LEMMA. *A tree is special if and only if it is a Moore space.*

PROOF. We shall only prove the forward implication. Let $T = \bigcup_{n\in\omega} A_n$, where each A_n is a maximal antichain. Given an $x \in T$ with height (x) a limit ordinal α, let $\{\alpha_m\}_{m\in\omega}$ be a sequence of ordinals strictly increasing to α as limit. Let x_m be that element of T_{α_m} which is below x. Let $U_m(x) = \{z : x_m < z \leqslant x\}$. Let $V_m(x) = \{z : z \leqslant x_m\} \cup \{z : x < z\}$. For x of successor height, define for all $m \in \omega$, $U_m(x) = \{x\}$, $V_m(x) = \{z : z < x\} \cup \{z : z > x\}$. Then $U_m(x)$ and $V_m(x)$ are open for all $x \in T$. Let $\mathcal{G}_{n,m} = \{U_m(x) : x \in A_n\} \cup \{V_m(x) : x \in A_n\}$. $\mathcal{G}_{n,m}$ is an open cover of X. Claim $\{\mathcal{G}_{n,m}\}_{n,m\in\omega}$ is a development of T. (Strictly speaking the covers should be indexed by ω, but the obvious bijection takes care of this.) Let $x \in T$, say $x \in A_n$, and let V

be an arbitrary open set about x. If x is of successor height, then $\bigcup\{G \in \mathcal{G}_{n,0}\colon x \in G\} = \{x\} \subseteq V$ and we are done. If x is of limit height, suppose $x \in G \in \mathcal{G}_{n,m}$. G cannot be $U_m(x')$ for $x \neq x' \in A_n$. So either $G = U_m(x)$ or G is $V_m(x')$, $x' \in A_n$. The second alternative is impossible since x is not incomparable with x' and its height is greater than m. But for m sufficiently large, $U_m(x) \subseteq V$, so we are done.

To see that a special Aronszajn tree is not $\aleph_1$-collectionwise Hausdorff, first observe that antichains are closed discrete in the tree topology. Therefore, some closed discrete subspace A must intersect stationarily many levels. Without loss of generality assume A has at most one point on each level. Since basic open sets 'look back', a separation of A would give rise to a regressive function on a stationary set, whence uncountably many members of the separation would trace on the same level and so couldn't be disjoint.

On the other hand, to see that T is weakly collectionwise Hausdorff, let $\{t_\alpha\}_{\alpha\in\omega_1}$ be distinct elements of T. Define $\{s_\beta\}_{\beta\in\omega_1}$ by letting s_β be the first t_α such that height $(t_\alpha) < \sup\{\text{height}(s_\gamma)\colon \gamma < \beta)$. Define $g(\beta) = \sup\{\text{height}(s_\gamma)\colon \gamma < \beta\}$. Let r_β be that predecessor of s_β of height $g(\beta)$. Let $U_\beta = \{t\colon r_\beta < t \leqslant s_\beta\}$. Then the U_β's separate the s_β's.

The proof that $\diamondsuit$ implies special Aronszajn trees are not normal is exactly like the proof of 1.2.10. To avoid $\diamondsuit$ for stationary systems, well-order the antichain A with type ω_1. The same regressive function argument as above shows that this time the A_f's are actually closed unbounded. The more delicate argument that $2^{\aleph_0} < 2^{\aleph_1}$ suffices uses the result of Taylor alluded to earlier (1.2.5). That hypothesis would imply a stationary subset of an antichain could be separated, but the pressing down argument prevents that. (The original Devlin–Shelah argument is given in Todorčević's chapter.)

To prove that Martin's Axiom plus not CH implies special Aronszajn trees are normal, since the trees are locally compact, it suffices by Theorem 2 to prove.

4. LEMMA. *A special ω_1-tree has a countable T_2 point-separating open cover.*

PROOF. Let $T = \bigcup_{n\in\omega} A_n$, where the A_n's are antichains. For each $x \in T$, let $x\uparrow = \{y\colon y > x\}$. Note that for $x, y \in A_n$, $x\uparrow$ and $y\uparrow$ are disjoint open sets. Let I be the set of isolated points of T. Let

$$U_n = T - (\bigcup\{x\uparrow : x \in A_n\} \cup (A_n \cap I)).$$

U_n is open since $y \leqslant x \in U_n$ implies $y \in U_n$. Let

$$\mathcal{U}_n = \{x\uparrow : x \in A_n\} \cup \{U_n\} \cup \{\{x\}\colon x \in A_n \cap I\}.$$

Then $\mathcal{U}_n$ is a disjoint open cover of T. Since $|\mathcal{U}_n| \leqslant c$, for each n there is a continuous function $f_n\colon T \to \mathbb{R}$ defined by sending the members of $\mathcal{U}_n$ to distinct reals. Let $\mathscr{B}$ be a countable base for $\mathbb{R}$. Claim

$$\mathscr{P} = \{f_n^{-1}(B): n \in \omega, B \in \mathscr{B}\}$$

is a T_2 point-separating open cover for T. For let $x \neq y \in T$. If x, y are comparable, say $x < y$, take $z < x$ such that $z \nless y$, $z \in A_n$. Then $x \in z\uparrow$ and $y \not\subset z\uparrow$, so $f_n(x) \neq f_n(y)$ and so P separates x and y. If on the other hand say $x \nleq y$, if x is not isolated, argue as before. If $x \in A_n$ is isolated, $\{x\} \in \mathscr{U}_n$ so again $f_n(x) \neq f_n(y)$.

All the first countable examples we have seen so far do not exist if the continuum hypothesis is assumed, or even $2^{\aleph_0} < 2^{\aleph_1}$. Nonetheless, it is possible to modify a special Aronszajn tree and then force without adding new countable sets to obtain a normal first countable non-$\aleph_1$-collectionwise Hausdorff space. Although we won't do the rather difficult forcing argument here, we will define the space. It is remarkable that the concepts involved were introduced in Shelah's work on the Whitehead problem in group theory (Shelah [1977b], [1980]).

EXAMPLE L. *A consistent-with* GCH *normal Moore space which is not $\aleph_1$-collectionwise Hausdorff.*

For an ordinal $\delta \in \omega_1$, a *ladder* on δ is a strictly increasing ω-sequence cofinal in δ. A *2-colouring* of a ladder η on δ is an $f \in {}^{\omega}2$. (Think of f colouring $\eta(n)$ with $f(n)$.) If $E \subseteq \omega_1$, a *ladder system* on E is a sequence $\eta = \langle \eta_\delta : \delta \in E\rangle$ such that η_δ is a ladder on δ for each $\delta \in E$. A 2-colouring of η is a sequence $f = \langle f_\delta : \delta \in E\rangle$ such that f_δ is a 2-colouring of η_δ for each $\delta \in E$. A *uniformization* of f is a $g \in {}^{\omega_1}2$ such that for each $\delta \in E$ there is an $n \in \omega$ such that for $m \geq n$, $f_\delta(m) = g(\eta_\delta(m))$.

If you think about what it means for Example G to be normal, you will see it is quite close to the idea of 2-colourings possessing uniformizations. Think of the sequences as being ladders, 2-colourings as being partitions of the closed discrete subspace, and a uniformization as picking out a cofinite piece of each sequence, and hence determining open sets about each point. Separability, however, is absent in the present context. Nonetheless, we define a topology in much the same fashion: given a ladder system η on $E \subseteq \omega_1$, isolate the points in $\omega_1 - E$ and let a neighbourhood of $\delta \in E$ be $\{\delta\}$ together with a tail of η_δ. It is routine to verify that such ladder system topologies are first countable and T_3.

Let us now start with a special Aronszajn tree T and construct a particular ladder system topology from it. As before we may take an antichain E which meets stationarily many levels and moreover meets each such level in exactly one point. We can then split E into disjoint stationary sets E_1 and E_2 with these properties. For each $\delta \in E_1$, take an ω-sequence η_δ converging to it in the tree topology and—except for an indexing problem—we have a ladder system. To get the system to sit on ω_1 as it's supposed to, we note that without loss of generality we could have identified the tree with ω_1 so that, letting $<$ be the tree ordering,

(i) $\alpha < \beta \to \alpha \in \beta$,
(ii) $\alpha \in \omega_1 \to \text{height}(\alpha) = \alpha$,
(iii) $\alpha \in \omega_1 \to \sup\{\beta \in \alpha : \beta < \alpha\} = \alpha$.

Now let X be the topology defined by this ladder system. Since X is obtained by isolating some points of a Moore space, it is clear that it is a Moore space. The same proof as before establishes that X is not $\aleph_1$-collectionwise Hausdorff. If every 2-colouring of η is uniformizable, it is not difficult to prove X is normal. We leave that as an exercise. The point is that naturally defined open sets are disjoint, else some point would be coloured in two different ways. To complete the proof we thus—modulo standard consistency proof handwaving—only need the following

5. THEOREM. *Let $\mathcal{M}$ be a countable transitive model of* ZFC + GCH. *In $\mathcal{M}$, let E be a stationary costationary subset of ω_1, and let η be a ladder system on E. Then there is a generic extension $\mathcal{N}$ of $\mathcal{M}$ such that*:
(i) *$\mathcal{M}$ and $\mathcal{N}$ have the same cardinals and cofinality function,*
(ii) ${}^{\omega}\mathcal{M} \cap \mathcal{M} = {}^{\omega}\mathcal{M} \cap \mathcal{N}$,
(iii) $\mathcal{N} \models$ GCH,
(iv) $\mathcal{N} \models$ "*E and $\omega_1 - E$ are stationary*",
(v) $\mathcal{N} \models$ "*Every 2-colouring of η is uniformizable*".

Given this theorem, define T, E, η, and X as above inside $\mathcal{M}$. Extend to $\mathcal{N}$. Since $\mathcal{M}$ and $\mathcal{N}$ have the same cardinals and E remains stationary, in $\mathcal{N}$ X is a non-$\aleph_1$-collectionwise Hausdorff Moore space, which by (v) is normal.

Let us call a metrizable space which is not the union of countably many closed discrete subspaces but in which each subset is F_σ, a *Q-space*. REED [1979] showed that the existence of a Q-space of weight $\aleph_1$ is equivalent to a version of the conclusion of the Theorem, more specifically to the existence of a collection S of almost disjoint ω-sequences of elements of ω_1 such that no collection of tails of all the sequences is pairwise disjoint, and any 2-colouring of S is uniformizable.

SHELAH [1977a] constructed another interesting example, taking an $\aleph_1$-collectionwise Hausdorff not $\aleph_2$-collectionwise Hausdorff Moore space of FLEISSNER [1978a] (obtained from $E(\omega_2)$) and forcing it to be normal. We omit the proof but state the result:

EXAMPLE M. It is consistent with CH (or with MA plus not CH) that there is *a locally countable, locally compact normal Moore space which is $\aleph_1$-collectionwise Hausdorff* but not $\aleph_2$-collectionwise Hausdorff.

Compare with Theorem 1.3.12.

One of the fine arts of point-set topology is to manipulate a counter-example to keep some desired properties and introduce new ones. In the normal non-collectionwise normal area, perhaps the most useful technique is that of 'splitting points'. For simplicity's sake, think first of a first countable normal non-collectionwide Hausdorff space in which the complement of the unseparated closed discrete subspace is composed of isolated points. We split each of those isolated points into several new ones, e.g. one for each pair of basic open sets about non-isolated points the original point is in. Neighbourhoods of a point in the closed discrete subspace are then defined by taking former neighbourhoods and replacing former isolated points by their offspring. The point of the exercise is to preserve the intersection properties of basic open sets, and hence normality and non-collectionwise Hausdorffness. The splitting can be done in a large variety of ways; for example we can vary the number of basic open sets new isolated points are associated with. There are several delicate applications of this technique, see e.g. TALL [1973], CHABER [1976], DAVIS, REED, and WAGE [1976], and DAVIES [1979]. One application of even the simplest splitting is to make the new space metacompact or a Moore space. The method may also be applied to normal non-collectionwise normal spaces, but naturally account must then be taken of the structure of the members of the unseparated collection. As an example of the technique, we shall prove

6. THEOREM. *If there is a normal Moore space which is not collectionwise normal with respect to a discrete collection of metacompact Moore spaces, then there is a metacompact normal non-metrizable Moore space.*

7. COROLLARY. *There is a normal non-metrizable metacompact Moore space if there is either*

(1) *a normal locally metrizable non-metrizable Moore space, or*

(2) *a normal first countable T_1 space which is not collectionwise Hausdorff.*

To prove the theorem, let $\{M_\alpha\}_{\alpha<\lambda}$ be the discrete unseparated collection of metacompact spaces in the normal Moore space X_0. Let $\bigcup_{\alpha<\lambda} M_\alpha = M$. Without loss of generality we assume $X_0 - M$ is composed of isolated points. The new space X will consist of the points of M plus the members of $(X_0 - M)\times[M]^2$. We will take the latter points to be isolated. Let us fix for each point p in M a neighbourhood base $\{B(p, n)\}_{n\in\omega}$ in X_0 such that $\{\{B(p, n)\colon p\in M\}\cup\{\{d\}\colon d\in X_0 - M\}$ is a development of X_0. Define for $p\in M$,

$$N(p, n) = (B(p, n)\cap M)\cup\{\langle d, \{x, y\}\rangle\in X - M\colon d\in B(p, n) \text{ and } \{x, y\}\cap B(p, n)\neq\emptyset\}\,.$$

The isolated points plus the $N(p, n)$'s, $p\in M$, $n\in\omega$, form a basis for a topology on X, for if $q\in M\cap N(p, n)\cap N(p', n')$, then $q\in B(p, n)\cap B(p', n')$, so there is a k such that $q\in B(q, k)\subseteq B(p, n)\cap B(p', n')$. Then $q\in N(q, k)\subseteq N(p, n)\cap N(p', n')$. In fact, if $\mathscr{G}_n = \{N(p, n)\colon p\in M\}\cup\{\{z\}\colon z\in X - M\}$, then $\{\mathscr{G}_n\}_{n<\omega}$ is clearly a development of X.

Since $N(p, n) \cap N(p', n') \neq \emptyset$ if and only if $B(p, n) \cap B(p', n') \neq \emptyset$, X is normal and not collectionwise normal. X is T_1, for if $d \notin B(p, n)$, $\langle d, \{x, y\}\rangle \notin N(p, n)$. Finally, to see that X is metacompact, let $\mathcal{U}$ be an open cover. Let $\mathcal{V}$ be a refinement of $\mathcal{U}$ which refines the canonical cover and is point-finite on M. Claim $\mathcal{V}$ is point-finite. For if $\langle d, \{x, y\}\rangle$ is a member of infinitely many members of $\mathcal{V}$, say $\{V_n\}_{n<\omega}$, then for each n, either $x \in V_n$ or $y \in V_n$. But then either for infinitely many $n \in \omega$, $x \in V_n$, or for infinitely many n, $y \in V_n$, contradiction.

The first half of the corollary is because such a space is not collectionwide normal with respect to some collection of metrizable—and therefore metacompact—sets. The second half is a corollary of the proof rather than the result: we may without loss of generality assume for each $p \in M$ that the $B(p, n)$'s are descending; there is a k such that $d \notin B(x, k) \cup B(y, k)$, so $\langle d, \{x, y\}\rangle \notin N(x, k) \cup N(y, k)$. Then $\{\langle d, \{x, y\}\rangle\}$ is the only element of the kth open cover containing $\langle d, \{x, y\}\rangle$.

Note that if we identify $\langle d, \{x, y\}\rangle$, $\langle d', \{x, y\}\rangle$ if they are in exactly the same neighbourhoods of x and y, then the cardinality of the whole space equals $|M| \cdot \aleph_0$. This observation leads immediately to the promised proof of 1.2.14, for we first isolate the complement of the closed discrete subspace, then split the isolated points, then perform the identification. The cardinality of the final space is then the cardinality of the closed discrete subspace times the character of its points.

We now move on to discuss normal collectionwise Hausdorff non-collectionwise normal spaces. Since these are dealt with extensively in Fleissner's chapter, we shall confine ourselves to listing their properties and making some remarks.

EXAMPLE N. *A normal collectionwise Hausdorff non-collectionwise normal space of character* $\mathfrak{c}$.

This is 'George' (FLEISSNER [1976]). The closed sets that cannot be separated are copies of ω_1 with the order topology. That is the motivation for Theorems 2.3.7 and 2.3.10 of the previous chapter. George is not locally compact by much the same argument as for Bing's example. It is not metacompact since it includes ω_1 as a closed subspace.

EXAMPLE O. *A normal metacompact paralindelöf (and hence collectionwise Hausdorff) space of character* $2^{\aleph_1}$ *which is not collectionwise normal.*

EXAMPLE F. Assuming MA plus not CH, *a paralindelöf metacompact normal non-metrizable Moore space.*

Both examples are due to NAVY [1982]. The first one is expounded in Fleissner's chapter; the second is obtained by applying Navy's machine to the bubble space on a Q-set or any other 'nice' first countable normal non-collectionwise normal space. For a clarification of 'nice', see WATSON [19· ·b].

EXAMPLE Q. Under CH, *a paralindelöf metacompact normal non-metrizable Moore space of cardinality* $\mathfrak{c}$.

EXAMPLE R. Assuming a combinatorial proposition whose failure implies the existence of an inner model with 'many' measurable cardinals, *a metacompact normal non-metrizable Moore space.*

See FLEISSNER's chapter and [1982a], [1982b] for details of these. It is not known whether the paralindelöfness of the first example can be obtained from the weaker hypothesis of the second.

EXAMPLE S. Assuming $\diamondsuit^{++}$ (a technical apparent strengthening of $\diamondsuit^{+}$), *a first countable normal space which is not collectionwise normal with respect to a discrete collection of copies of a topology refining the usual topology on* ω_1.

This space was christened 'Son of George and $V = L$' by FLEISSNER [1983a]. The construction of it is lengthy and rather difficult and will therefore be omitted. $\diamondsuit^{++}$ was later used by RUDIN [1983a] to construct a screenable normal non-countably paracompact space. Assuming the existence of such a space, she modified it to produce

EXAMPLE T. *A screenable normal non-collectionwise normal space* (assuming the existence of a screenable normal non-countably paracompact space, e.g. under $\diamondsuit^{++}$).

For details, see RUDIN [1983b] or her chapter in this volume.

III. Problems

The prejudices of the author are sometimes reflected in the way the problems are stated, but sometimes it is merely neater to state them the way they are.

1. Locally compact spaces

(a) Is it consistent that every locally compact normal space (every normal space of (point-)countable type?) is collectionwise normal? I conjecture the answer is yes, assuming large cardinals. I also conjecture that one can prove, à la Example R, that large cardinals are necessary. MA plus not CH of course provides a counterexample.

(b) Is it consistent that every locally compact, locally connected normal space is collectionwise normal? There are no independence results known here.

(c) Is it consistent that every normal manifold is collectionwise normal? The question here is whether the result is a theorem, a consistency result, or whether it requires a large cardinal.

(d) Is it consistent that there be a locally compact normal metacompact space which is not paracompact?

(e) Is there a locally compact (perfectly) normal non-collectionwise normal space in L?

2. Normal Moore spaces

(a) Is it consistent, say assuming a supercompact cardinal, that $2^{\aleph_0} = \aleph_2$ and all normal Moore spaces are metrizable? An opposing conjecture is that if $2^{\aleph_0}$ is not a regular limit cardinal, then there is a normal non-metrizable Moore space. This would generalize Fleissner's CH example and provide a more direct proof of the necessity of large cardinal for proving the consistency of the normal Moore space conjecture.

(b) Is it consistent without large cardinals that

(i) every paralindelöf normal Moore space is metrizable?

(ii) every normal Moore space of cardinality $\leqslant c$ is metrizable?

(c) If there is a normal non-metrizable Moore space, is there a metacompact one?

(d) Does the existence of a countable chain condition (metacompact?) normal non-metrizable Moore space imply the existence of a Q-set?

3. Collectionwise Hausdorffness

(a) Is it consistent, assuming the existence of a supercompact cardinal, that every first countable $\aleph_1$-collectionwise Hausdorff space is collectionwise Hausdorff?

(b) When does $<\kappa$-collectionwise Hausdorff imply κ-collectionwise Hausdorff, say for spaces of character $<\kappa$, κ limit or successor of a singular?

4. Screenable spaces

(a) Is it consistent that every normal screenable space is collectionwise normal? Example S is a counterexample from $\diamondsuit^{++}$.

(b) Is every normal space with a σ-disjoint base collectionwise normal? PMEA implies yes.

(c) Is every locally compact normal screenable space collectionwise normal? Paracompact?

5. Countably paracompact spaces

(a) Is it consistent that countably paracompact Moore spaces are

(i) normal?

(ii) metrizable?

Large cardinals are needed since there is a machine (WAGE [1976]) for transforming normal non-metrizable Moore spaces into countably paracompact non-normal Moore spaces.

(b) Does $2^{\aleph_0} < 2^{\aleph_1}$ imply countably paracompact separable (first countable?) spaces are collectionwise normal? $2^{\aleph_0} < 2^{\aleph_1} \leq \aleph_{\omega_1}$ does (J. STEPRĀNS and) WATSON [19··b]). $2^{\aleph_0} = 2^{\aleph_1}$ (MA plus not CH) provides (first countable) counterexamples since Examples E and F are easily seen to be countably paracompact.

6. *What's so special about* $\mathfrak{c}$?

Nyikos' result does not obviously generalize to higher cardinals. The following two questions go to opposite extremes.

(a) Is it consistent, presumably assuming large cardinals, that CH holds and spaces of character $<2^{\mathfrak{c}}$ which are collectionwise normal with respect to discrete collections of power $\aleph_1$ are collectionwise normal?

(b) Does CH imply there is a normal non-collectionwise Hausdorff space of character $\aleph_2$ in which G_δ's are open?

IV. Historical notes

The proofs of 1.1 and 1.2 appear in the fundamental BING [1951]. 1.3 appears in TALL [1969] although the idea goes back to SMIRNOV [1956]. 1.4 is folklore which first appears in BING [1965]. 1.5 appears in TALL [1969]. 1.6 was proved first for points in TALL [1969], then in this form in TALL [19··b]. The idea of 1.7 is due to FITZPATRICK and TRAYLOR [1966] but first appears explicitly in TALL [1969]. 1.8 is due to ARHANGEL'SKIĬ [1971]. 1.11 is due to ALSTER and ZENOR [1976] simplifying REED and ZENOR [1976]. 2.1 is from the old paper JONES [1937] that first stated the normal Moore space conjecture. 2.2 is in TALL [1976]; 2.3 is in ŠAPIROVSKIĬ [1972]. 2.6 is due to FLEISSNER [1974] which followed the forcing proof for regular cardinals (2.13) in TALL [1969]. Our treatment of the singular cardinal case incorporates an improvement (2.11) due to WATSON [19··b]. The forcing proof was shortened by Watson as well. 2.18–2.21 appear in WATSON [19··a] and incorporate improvements due to PRZYMUSIŃSKI and GRUENHAGE. 2.22 is due to the author.

Measure extension problems have a long history we won't enter into here. Axioms close to PMEA were considered in FISHER [1970]. Kunen proved the consistency of PMEA in the unpublished note KUNEN [19··]. Nyikos christened the axiom and popularized it in NYIKOS [1980] where he proved 3.2. $\mathfrak{c}$MEA and WMEA are neologisms first introduced here, although the basic results on consistency appear in CARLSON [19··]. For cMEA the result is due jointly to Prikry and Carlson. 3.3 is due to NYIKOS [19··b]. 3.4 appears in TALL [19··b], 3.5 in TALL [1982]. 3.6 is in GRUENHAGE [1980]. The improvement of 3.6 is an unpublished result of Junnila. Watson rediscovered the idea which he used to prove 1.10. 3.7 and 3.8

appear in TALL [19··b]. 4.1(a) is in REED and ZENOR [1976], (b) in ŠAPIROVSKII [1972]. The consistency of the conclusion of (c) was an unpublished result of the author, using the reverse Easton model. (c) itself is due to FLEISSNER [1974]. (d) is NYIKOS [1980].

Now we turn to the examples. For most of these we have already given credit in the text. The proof given that Examples A, B, and C are not locally compact is due to Watson. Example E is due to HEATH [1964b]; the present proof is in TALL [1969]; another proof is in CHARLESWORTH, HODEL, TALL [1975]. The 'slick way' is due to Watson and no doubt others. The bubble space is in BING [1951], G in FITZPATRICK and TRAYLOR [1966], G' is folklore, H is in HEATH [1964a]. Pixley–Roy space over all the reals was introduced in PIXLEY and ROY [1969]; its normal subspace appeared in PRZYMUSIŃSKI and TALL [1974]. The proof given that the space satisfies the countable chain condition is due to VAN DOUWEN [1977] and improves the original one of Pixley and Roy. Lemma 1 is due to PRZYMUSIŃSKI [1980]. Example K is due to TALL [1978]. The special Aronszajn tree example is due to JONES [1966], who proved it was a non-metrizable Moore space. The first proofs of its normality and non-normality appear in FLEISSNER [1975]. We give another proof of normality, due to JUHÁSZ and WEISS [1978]. The fact that the tree is not normal under $2^{\aleph_0} < 2^{\aleph_1}$ is due to DEVLIN and SHELAH [1979]; another proof is in TAYLOR [1981]. Lemma 5.1 first appears explicitly in Juhász and Weiss op. cit. Ladder systems first appear explicitly in Devlin and Shelah op. cit., where they are credited to SHELAH [1977b], [1980] in which Theorem 5 in effect appears. The technique of splitting points appears implicitly in BING [1965] and explicitly in TALL [1973], [1974a]. Corollary 2.7 for the case of points is TALL [1974a]; the more general case is due to Watson.

It is hard to credit problems. I hope the following is more or less accurate. Problem 1(a) is due to Watson, 1(b) and (c) to Nyikos, (d) to Arhangel'skiĭ and myself, (e) to Watson, and (perfectly) to Fleissner. 2(a) is due to myself and Fleissner; the opposing conjecture is due to Watson. 2(b) (i) (without the 'without large cardinals') is due to Heath, (ii) to folklore. 2(c) is probably due to Traylor; 2(d) is due to Przymusiński and myself. 3(a) is due to Fleissner; (b) to him and myself. 4(a) and (b) are due to M.E. Rudin and myself, (c) to myself. 5(a) is due to Younglove; (b) is due to Fleissner, (first countable) to Przymusiński or myself. 6(a) is due to myself; 6(b) is due to Watson.

Added in proof, July 1983

There have been a number of exciting developments since this article was written a year ago. Since so many people have seen earlier versions, I decided it would be most useful to gather the new material into one place, rather than inserting it throughout the text. We shall proceed in more-or-less the same order as the main body of the article.

WATSON [1983] has proved:

5.1. THEOREM. *Normal collectionwise Hausdorff spaces are collectionwise normal with respect to scattered paracompact sets.*

Balogh has proved:

5.2. THEOREM. *Normal collectionwise Hausdorff locally Lindelöf spaces are collectionwise normal with respect to closed Lindelöf sets.*

FLEISSNER [19· ·b] observed that the last use of normality in the proof of 2.9 is superfluous. This enabled him to improve results of Watson to obtain:

5.3. THEOREM. *$V = L$ implies if X is a Hausdorff space of character $\leq \aleph_2$ in which every open cover of cardinality $\aleph_1$ has a locally countable open refinement, then X is collectionwise Hausdorff.*

I have proved:

5.4. THEOREM. *If the existence of infinitely many supercompact cardinals is consistent with* ZFC, *so is "every normal space of countable type in which each point has character $< \beth_\omega$ is collectionwise normal".*

Recall $\beth_\omega$ is the limit of the $\beth_n$'s, $n < \omega$, where $\beth_0 = 2^{\aleph_0}$ and $\beth_{n+1} = 2^{\beth_n}$. The proof is very long and uses a lot of set-theoretic technology so I shall not attempt to sketch it here. I suspect 'countable type' can be weakened to 'point-countable type'. The obvious question is whether the character restriction can be omitted at the cost of increasing the strength of the large cardinal hypotheses. Even for locally compact spaces, I do not know the answer. The proof (which is by induction on character) runs into technical difficulties at $\beth_\omega$ and at this point in time it is unclear whether or not these difficulties are essential.

The measure extension proofs of Chapter 3 leave unanswered the question of whether measure theory has an accidental or necessary relationship with 'normality versus collectionwise normality'. It is not at all clear how to generalize measure to arbitrary regular cardinals. Such a generalization would seem to be needed to obtain even such straightforward generalizations of known results as e.g. the consistency modulo large cardinals of CH plus "every normal space of character $<2^{\aleph_1}$ in which G_δ's are open, is collectionwise normal". In fact the most significant advance in the past year has been the revelation that measure theory is superfluous. The author and W. Weiss (TALL and WEISS [19· ·]) proved

5.5. THEOREM. *Adjoin to a model of set theory supercompact many Cohen reals. Then every normal space of character $<2^{\aleph_0}$ is collectionwise normal.*

In terms of consistency strength, the Kunen–Nyikos result is better (although FLEISSNER [19· ·c] later weakened our 'supercompact' to 'strongly compact' by a different argument). The significance of our result, however, is that in contrast to random real forcing, Cohen real forcing *can* be generalized to higher cardinals. Thus I was able to obtain:

5.6. THEOREM. *If* ZFC *plus the existence of a supercompact cardinal is consistent, so is* ZFC *plus* CH *plus "every normal space of character* $<2^{\aleph_1}$ *in which* G_δ*'s are open, is collectionwise normal".*

Variations of this result, and the results about spaces of countable type were alluded to earlier. Because of the importance of 5.5 I want to sketch a proof here. The reader should review the proof of 2.13, and particularly 2.15. Why were we not able to obtain collectionwise normality in that situation? Think of the proof for $\aleph_1$-collectionwise Hausdorff. We went from 'not separated' to 'not locally countable' to 'not normalized'. Another way of looking at the proof would be to say that a normalization of the generic partition gave rise to a locally countable neighbourhood assignment 'downstairs', which yielded a separation. This wouldn't work for arbitrary closed sets for the same reason that paralindelöf doesn't imply collectionwise normal. Paracompact does of course imply collectionwise normal, so forcing with conditions with *finite* support – e.g. Cohen reals – is indicated.

The key to the proof of 2.15, was the 'representation' of a function f in the extension by a function h in the ground model. Using κ-closure we could represent a function with domain κ. To obtain full collectionwise normality we must somehow represent functions with arbitrary domains. This can be done via an important combinatorial lemma due to Alan Dow [1983]. We say that a partial order is *endowed* (Dow should not be blamed for this terminology) if there is a function F which picks from each maximal antichain a finite subset of it, so that for every pair A, A' of maximal antichains, there is a $p \in F(A)$ and $p' \in F(A')$ such that p and p' are compatible. The partial order for adding any number of random reals is endowed since, taking the measure to be in the interval [0, 1], one can always take a finite subset of a maximal antichain to have union of measure $>\frac{1}{2}$.

5.7. THEOREM (DOW). *The partial orders for adding any number of Cohen reals are endowed.*

The essential point of Dow's proof is to show that given a maximal antichain A, and an $n \in \omega$, there is a finite subset $F_n(A)$ such that any condition with domain of size $\leqslant n$ is compatible with some member of $F_n(A)$. Given an endowed partial order and a function f in the extension, – say $f(\alpha)$ is a neighbourhood of α – one can represent f in the ground model by

$$h(\alpha) = \bigcap\{b\colon (\exists p \in F(A_\alpha))\ p \Vdash f(\alpha) = b\}\,,$$

where A_α is a maximal antichain deciding f at α. More particularly, to make 5.4 go through, if $\alpha \in Y_\gamma$, define

$$h(\alpha) = \bigcap\{b : (\exists p \in F(A_\alpha))(p \Vdash f(\alpha) = b) \vee (\exists p \in F(A_\alpha))(p^\gamma \Vdash f(\alpha) = b)\}.$$

With these ideas in mind, the reader should be able to fill in the details of a proof for

5.8. THEOREM. *Add κ many Cohen reals. Then normal spaces of character $<2^{\aleph_0}$ are collectionwise normal with respect to collections $\mathcal{Y}$ such that $(\bigcup \mathcal{Y}) < \kappa$.*

The problem of dealing with collections with union of cardinality $\geqslant\kappa$ leads us to large cardinals. We need a 'reflection principle' that says roughly speaking, that if there is a counterexample, there is one of size $<\kappa$. Supercompactness of κ is such a reflection principle. When κ Cohen reals are adjoined however, κ loses its supercompactness, so one must use reflection on the appropriate forcing statement in the ground model. The details of this sort of argument are well known to set-theorists. I unfortunately know of no exposition in print which would be reasonably comprehensible to the typical reader of this book, although the technique is extremely useful in proving the consistency of 'for all' statements – e.g. PFA – via large cardinals plus iterated or product forcing. The point is that one can take care of all objects of all cardinalities by iterating only κ many times, taking care of all objects of size $<\kappa$. Weiss and I will attempt to give a suitable exposition of this method when we write up the proof of 5.3 for publication. For the reader who is familiar with reflection arguments, I should mention two particulars of the present proof; these together with the general framework should enable that reader to prove 5.3. First of all, note that $\chi < \kappa$ implies $j''X$ is a sub*space* of $j(X)$; secondly, note Dow's lemma can be used to show that unseparated collections cannot be separated by adjoining Cohen reals.

Let me also mention that by using a slightly stronger form of Dow's lemma, one can avoid all references to finite support and paracompactness, and directly get that h is a separation. This gives a unified proof that either supercompact many Cohen reals or supercompact many random reals suffice and enables us to prove 5.7 for random reals as well. There are also '$\leqslant\kappa$' versions for weakly compact many reals. Since 5.7 and these versions enable us to prove stronger results, there is now really no point in considering cMEA or WMEA unless one is allergic to forcing.

Moving on to the Problems section, Fleissner in a private communication pointed out that if one avoids the use of CH, the construction used for Example Q still yields a metacompact paralindelöf normal non-collectionwise normal space of character c.

WATSON [1983] has constructed a normal collectionwise Hausdorff space which is not collectionwise normal with respect to a discrete collection of copies of the unit interval.

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CHAPTER 16

The Normal Moore Space Conjecture and Large Cardinals

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

1. Introduction, notation, definitions

The normal Moore space conjecture (NMSC) has already been introduced earlier in this volume in Tall's article. There this topological problem is connected with basic set theoretic tools such as cardinal arithmetic, transfinite induction, and special axioms, for example, MA + $\neg$CH and $V = L$. The goal of this article is to present the connection with large cardinals. The connection was first revealed in Nyikos' theorem that PMEA implies NMSC (Section 2). Kunen showed that PMEA was consistent relative to a strongly compact cardinal (Section 3). It was known immediately that a large cardinal is necessary for PMEA; it was not so clear that a large cardinal is necessary for NMSC.

Progress in the other direction, constructing normal nonmetrizable Moore spaces, proceeded by developing better and better techniques of constructing normal, not collectionwise normal spaces (Section 5). We will pay particular attention to Navy's space (Section 6), and Fleissner's space from CH (Section 7). The generalization from ω to strong limit cardinals of cofinality ω (Section 8) gives a connection to large cardinals. Because this last construction is possible unless the covering lemma fails for the core model, large cardinals really are needed to construct a model of NMSC.

1.1. *Set theoretic notation.* We use the usual notations. See [KUNEN 1979]. For example, an ordinal is the set of its predecessors, and cardinals are initial ordinals. We denote the cardinality of a set X by $|X|$. The class of limit ordinals is called LIM. We use $[\lambda]^{<\rho}$ to denote $\{A \subset \lambda : |A| < \rho\}$. We call κ a strong limit cardinal if $\lambda < \kappa$ implies $2^\lambda < \kappa$. We use $\mathfrak{c}$ for 2^ω, the cardinal of the continuum.

We will use the Pressing Down Lemma [KUNEN 1979, p. 80], [JECH 1978, p. 59]: Let $\kappa > \omega$ be regular, S a stationary subset of κ, and $f: S \to \kappa$ such that for all $\delta \in S$, $f(\delta) < \delta$; then for some stationary subset S_1 of S, $f|S_1$ is constant.

1.2. *Topological definitions.* A family, $\mathcal{Y} = \{Y_i : i \in I\}$ of subsets of a space X is called *discrete* if every $x \in X$ has a neighborhood meeting at most one element of $\mathcal{Y}$. If $\mathcal{U} = \{U_J : J \subset I\}$ is a family of open sets such that for every $J \subset I$, $\bigcup\{Y_i : i \in J\} \subset U_J$ and $\bigcup\{Y_i : i \notin J\} \subset X - \overline{U_J}$, we say that $\mathcal{U}$ *normalizes* $\mathcal{Y}$ and that $\mathcal{Y}$ is normalized.

Let $\mathcal{W} = \{W_i : i \in I\}$ be a family of open sets with $Y_i \subset W_i$ for all $i \in I$. If $\mathcal{W}$ is disjoint, $\mathcal{W}$ *separates* $\mathcal{Y}$ and $\mathcal{Y}$ is *separated.* If for some countable $C \subset I$, $\{W_i : i \in I - C\}$ is disjoint, then $\mathcal{W}$ *almost separates* $\mathcal{Y}$. Note that normalized, almost separated collections are separated.

Of course the prototypes are that a space is normal iff every discrete collection of closed sets is normalized and that a space is collectionwise normal iff every discrete family is separated. In view of Bing's theorem [BING 1951] (see Gruenhage's article), the normal Moore space conjecture is a consequence of the statement that normalized collections can be separated in Moore spaces.

While the motivating question is NMSC, the theorems usually apply in more generality. Then following uncommon variations of weight and character will be useful in expressing results in the strongest form. We define $\hat{\chi}(X)$ to be the least cardinal strictly greater than $\chi(x, X)$ for all $x \in X$. We say that a space X is *of character below* ρ iff $\hat{\chi}(X) \leq \rho$. For Z a subset of a space, X, we define $\hat{\chi}(Z, X)$ to be the least cardinal strictly greater than $\chi(z, X)$ for all $z \in Z$. Also, we define a *base for* Z *in* X to be a collection, $\mathcal{B}$, of open subsets of X such that whenever U is open in X and $z \in Z \cap U$, then there is $B \in \mathcal{B}$ such that $z \in B \subset U$. We define the *weight of* Z *in* X, denoted $w(Z, X)$ to be $\min\{|\mathcal{B}|: \mathcal{B}$ is a base for Z in $X\}$.

Note that $w(Z) \leq w(Z, X) \leq |Z| \cdot \sup\{\chi(z, X): z \in Z\}$. Examples where the inequalities are strict are easy to construct. Note also that if $\mathcal{Y}$ is discrete family in X, then $|\mathcal{Y}| \leq w(\bigcup\mathcal{Y}, X)$.

1.3. *Measure theory.* In this article, we will use the unmodified term measure to mean σ-additive probability measure. Precisely, a *measure*, m, on a set, I, is a function such that

(i) dom m is a σ-algebra of subsets of I and ran $m \subset [0, 1]$,

(ii) $m(\emptyset) = 0$ and $m(I) = 1$,

(iii) if $A \subset A' \subset I$, then $m(A) \leq m(A')$,

(iv) if $\{A_n: n \in \omega\}$ is a disjoint family of subsets of I, then $m(\bigcup_{n\in\omega} A_n) = \Sigma_{n\in\omega} m(A_n)$.

We want to discuss strengthenings of (iv). For κ, a cardinal, we say that a measure, m, is κ-*additive* if it satisfies (i), (ii), (iii) and

(iv)$_\kappa$ If $\{A_\alpha: \alpha < \beta\}$, where $\beta < \kappa$, is a disjoint family of subsets of I, then $m(\bigcup_{\alpha<\beta} A_\alpha) = \Sigma_{\alpha<\beta}\, m(A_\alpha)$.

Thus σ-additive means ω_1-additive and finitely additive means ω-additive.

We will find the following variant of (iv)$_\kappa$ convenient.

(iv)$'_\kappa$ If $\{A_\alpha: \alpha < \beta\}$, where $\beta < \kappa$, is a family of subset of I closed under finite union, and δ is a positive real number, then there is $\gamma < \beta$ so that $m(\bigcup_{\alpha<\beta} A_\alpha) < m(A_\gamma) + \delta$.

We say that a filter, $\mathcal{F}$, of subsets of a set, I, is κ-*complete* if the characteristic function $\chi_{\mathcal{F}}$, defined by $\chi_{\mathcal{F}}(A) = 1$ if $A \in \mathcal{F}$ and $\chi_{\mathcal{F}}(A) = 0$ if $(I - A) \in \mathcal{F}$, is a κ-additive measure.

When $\mathcal{F}$ is a filter on a set I, and f and g are functions with domain I, we let the phase "$f < g$ mod $\mathcal{F}$" abbreviate "there is $F \in \mathcal{F}$ such that for all $i \in F$, $f(i) < g(i)$". Phrases like "f is constant mod $\mathcal{F}$" should be interpreted similarly.

One particular measure will be used frequently in this article, the product measure on 2^I, the set of functions from I to $\{0, 1\}$. Let H_I be the set of functions from a finite subset of I to $\{0, 1\}$. For each $\eta \in H_I$, we define $[\eta]_I = \{f \in 2^I: \eta \subset f\}$. The *product measure*, m_I, is defined on B_I, the σ-algebra generated by $\{[\eta]_I: \eta \in H_I\}$, so that $m_I([\eta]_I) = 2^{-k}$, where $|\eta| = k$. As usual, we will omit subscripts when no confusion will result.

1.4. *Axioms. The Product Measure Extension Axiom* (PMEA) asserts that for every set I there is a $\mathfrak{c}$-additive measure $\bar{m}_I$ defined on *all* of $\mathscr{P}(2^I)$ extending the product measure. In symbols, $\operatorname{dom}(\bar{m}_I) = \mathscr{P}(2^I)$ and $m_I \subset \bar{m}_I$. Of course, it is sufficient to prove the existence of $\bar{m}_\lambda$ for arbitrarily large cardinals.

Let PMEA (κ, λ, ρ) be the axiom: For any family, $\mathscr{S}$, of λ-many subsets of 2^κ, the product measure on 2^κ can be extended to a ρ-additive measure, $m_{\mathscr{S}}$, with $\mathscr{S} \subset \operatorname{dom} m_{\mathscr{S}}$.

Let Sep(∞) be the assertion: If $\mathscr{Y}$ is a normalized collection in a space X and $\hat{\chi}(\bigcup\mathscr{Y}, X) \leq \mathfrak{c}$, then $\mathscr{Y}$ is separated. Let Sep$(\mathfrak{c})$ be the assertion Sep(∞) altered by adding "$w(\bigcup\mathscr{Y}, X) \leq \mathfrak{c}$" to the hypothesis. Let Sep$(<\mathfrak{c})$ be the assertion Sep(∞) altered by adding "$w(\bigcup\mathscr{Y}, X) < \mathfrak{c}$ and $|\bigcup\mathscr{Y}| < \mathfrak{c}$" to the hypothesis.

Some axioms which are consequences of $V = L$ are defined in 8.1.

2. Nyikos' theorem

The idea of Nyikos' theorem is amazingly simple. Given a family, $\mathscr{Y} = \{Y_i : i \in I\}$ of subsets of a space X normalized by $\mathscr{U} = \{U_J : J \subset I\}$, we choose for each point y of $\bigcup\mathscr{Y}$, a neighborhood which is contained either in U_J or $X - \bar{U}_J$ for 'most' J. PMEA is exactly what it needed to make the vague 'most' precise.

2.1. THEOREM [NYIKOS 1978]. *Assume* PMEA. *Every normalized collection in a space of character below* $\mathfrak{c}$ *is separated. In particular, normal Moore spaces are metrizable.*

PROOF. Let $\mathscr{Y} = \{Y_i : i \in I\}$ be a family of subsets of a space X of character below $\mathfrak{c}$, normalized by $\mathscr{U} = \{U_J : J \subset I\}$. For each $y \in \bigcup\mathscr{Y}$, fix a neighborhood base $\mathscr{N}_y$, where $|\mathscr{N}_y| < \mathfrak{c}$. For each $y \in \bigcup\mathscr{Y}$ and $J \subset I$, choose $N(J, y) \in \mathscr{N}_y$ such that either $N(J, y) \subset U_J$ or $N(J, y) \subset X - \bar{U}_J$.

We identify $\mathscr{P}(I) = \{J : J \subset I\}$ with $2^I = \{f : f \text{ is a function from } I \text{ to } \{0, 1\}\}$ via characteristic functions; let $m = \bar{m}_I$, the product measure extension on 2^I given by PMEA. For each $y \in \bigcup\mathscr{Y}$ and each $N \in \mathscr{N}_y$, set $J(N, y) = \{J \in 2^I : N(J, y) = J\}$. Now, for each $y \in \bigcup\mathscr{Y}$, 2^I is the union of the disjoint family $\mathscr{J}_y = \{J(N, y) : N \in \mathscr{N}_y\}$. Since m is $\mathfrak{c}$-additive, there is a finite $\mathscr{K}_y \subset \mathscr{J}_y$ such that $m(\bigcup\mathscr{K}_y) > \frac{7}{8}$. Set $W_y = \bigcap\{N \in \mathscr{N}_y : J(N, y) \in \mathscr{K}_y\}$. Set $W_i = \bigcup\{W_y : y \in Y_i\}$.

We claim that $\{W_i : i \in I\}$ separates $\mathscr{Y}$. If not, there are $i \neq j$, $y \in Y_i$, and $z \in Y_j$ so that $W_y \cap W_z \neq \emptyset$. Set $[\eta] = \{f \in 2^I : f(i) = 1 \text{ and } f(j) = 0\}$. Since $m(\bigcup\mathscr{K}_y) > \frac{7}{8}$, $m(\bigcup\mathscr{K}_z) > \frac{7}{8}$, and $m([\eta]) = \frac{1}{4}$, there is $J \in [\eta] \cap (\bigcup\mathscr{K}_y) \cap (\bigcup\mathscr{K}_z)$. Then $U_J \cap (X - \bar{U}_J) \supset W_y \cap W_z \neq \emptyset$. Contradiction. $\square$

Nyikos' theorem opened a door in this area of topology. Of course, people immediately tried to get stronger conclusions or to use a weaker assumption. For example, Junnila made the paradoxical observation when it is possible to choose W_y, then it is unnecessary to choose a specify W_y for each y. For each $i \in I$ and $x \in X$, set

$$\mathscr{D}(i, x) = \{J \subset I: x \in U_j \text{ if } i \in J \text{ and } x \in X - \bar{U}_J \text{ if } i \notin J\}.$$

For each $r \in [0, 1]$, the set $\{x \in X: m(\mathscr{D}(i, x)) > r\}$ is open. (This observation uses $\hat{\chi}(X) \leq \mathfrak{c}$, where Nyikos' proof uses only $\hat{\chi}(\bigcup \mathscr{Y}, X) \leq \mathfrak{c}$.) This enabled Junnila to extend Nyikos' theorem to spaces with weak character less than $\mathfrak{c}$. A space X is said to be of *weak character below* $\mathfrak{c}$ if for each $x \in X$, there is a filter base $\mathscr{F}_x$, $|\mathscr{F}_x| < \mathfrak{c}$, such that a subset U of X is open iff for every $x \in U$ there is $F \in \mathscr{F}_x$ with $x \in F \subset U$. Junnila also played the leading role in proving Proposition P (an axiom useful in Borel theory of nonseparable metric spaces) from PMEA [FLEISSNER, HANSELL and JUNNILA 1982].

The main effort in modifying Nyikos' theorem was towards weakening the assumption of PMEA. Kunen's proof of Con(ZFC + PMEA) starts with Con(ZFC+$\exists$ strongly compact cardinal). (See the next section.) Earlier work of Solovay [SOLOVAY 1971] showed that at least a measurable cardinal is needed to get PMEA. Nyikos' proof seems to use so little of PMEA. For example, we really don't care about the exact value of $m(\bigcup \mathscr{K}_y)$. All we need is $(\bigcup \mathscr{K}_y) \cap (\bigcup \mathscr{K}_z) \cap [\eta] \neq \emptyset$. Hence, people tried to use Nyikos' proof assuming weak versions of PMEA which didn't require large cardinals.

The following restatement of Nyikos' theorem is useful in describing such efforts.

2.2. THEOREM [NYIKOS 1978]. *Assume* PMEA(κ, λ, ρ) *where* $\rho \leq \lambda$. *Whenever* $\mathscr{Y} = \{Y_\alpha: \alpha < \kappa\}$ *is a normalized collection with* $|\bigcup \mathscr{Y}| \leq \lambda$, (*alternatively* $w(\bigcup \mathscr{Y}, X) \leq \lambda$) *in a space* X, *and* $\hat{\chi}(\bigcup \mathscr{Y}, X) \leq \rho$, *then* $\mathscr{Y}$ *is separated.*

PROOF. Repeat the proof of Theorem 2.1. Here we only point out where the hypotheses above are used. In the primary case $\mathscr{S} = \{J(N, y): y \in \bigcup \mathscr{Y}, N \in \mathscr{N}_y\}$ is a family of $\leq\lambda$-many subsets of 2^κ, so by PMEA(κ, λ, ρ) we get $m_{\mathscr{S}}$. In the alternative case, let $\mathscr{B}$ be a base for $\bigcup \mathscr{Y}$ in X (see 1.2). For $B \in \mathscr{B}$, set $J(B) = \{J \in 2^I: B \subset U_J \text{ or } B \subset X - \overline{U_J}\}$ and set $\mathscr{S} = \{J(B): B \in \mathscr{B}\}$.

Now for each $y \in \bigcup \mathscr{Y}$, $2^I = \bigcup \mathscr{J}_y$ and $|\mathscr{J}_y| < \rho$, so ρ-additivity applies. In the alternative case, for each $y \in \bigcup \mathscr{Y}$, choose a neighborhood base $\mathscr{J}_y \subset \mathscr{B}$, with $|\mathscr{J}_y| < \rho$, and use (iv)$'_\rho$. $\square$

The use of $w(\bigcup \mathscr{Y}, X)$ in the above proof is due to Tall.

Note that the fact that dom $m_{\mathscr{S}}$ is a σ-algebra was not fully used. We needed only closure under finite insections. We used (iv)$'_\rho$ only when $\bigcup \mathscr{J}_y = 2^I$.

3. Strongly compact cardinals, random reals, Con(PMEA)

We need only one of the many equivalent definitions of strongly compact cardinals [KEISLER and TARSKI 1964], [JECH 1978, Section 33]. We say that a cardinal, κ, is *strongly compact* if every κ-complete filter on a field of sets can be extended to a κ-complete ultrafilter. We illustrate how to use this definition by proving the exact result we will need.

3.1. LEMMA. *Let κ be strongly compact. For every cardinal λ, there are a set I, a κ-complete free ultrafilter $\mathcal{U}$ on I, and functions f_α, $\alpha < \lambda$, such that*

(i) *for all $\alpha < \lambda$, $f_\alpha: I \to \kappa$,*

(ii) *if $\alpha < \beta < \lambda$, then $\{i \in I: f_\alpha(i) < f_\beta(i)\} \in \mathcal{U}$,*

(iii) *for all $\alpha < \lambda$ and $\delta < \kappa$, $\{i \in I: f_\alpha(i) \neq \delta\} \in \mathcal{U}$.*

PROOF. Let $I = \{x \subset \lambda: |x| < \kappa\}$. For each $\alpha \in \lambda$, set $X_\alpha = \{x \in I: \alpha \in x\}$. Since $\{X_\alpha: \alpha \in \lambda\}$ generates a free κ-complete filter $\mathcal{F}$ on I, we can use the assumption that κ is strongly compact to get a free κ-complete ultrafilter, $\mathcal{U}$, on I extending $\mathcal{F}$.

For each $\alpha \in \lambda$ we define $f_\alpha: I \to \kappa$ so that if $\alpha \in x$, then α is the $f_\alpha(x)$th element of x in the order inherited from λ. Let $f_\alpha(x) = 0$ if $\alpha \notin x$. Let $\alpha < \beta < \lambda$. For all $x \in X_\alpha \cap X_\beta \in \mathcal{F} \subset \mathcal{U}$, then $f_\alpha(x) < f_\beta(x)$. $\square$

There is an easy version of Lemma 3.1 that holds for all κ.

3.2. LEMMA. *Let κ be an infinite cardinal. There are functions f_α, $\alpha < \kappa$, and a κ-complete free filter $\mathcal{F}$ on κ such that*

(i) *for all $\alpha < \kappa$, $f_\alpha: \kappa \to \kappa$,*

(ii) *if $\alpha < \beta < \kappa$, then $\{i \in \kappa: f_\alpha(i) \neq f_\beta(i)\} \in \mathcal{F}$,*

(iii) *for all $\alpha < \kappa$ and $\delta < \kappa$, $\{i \in \kappa: f_\alpha \neq \delta\} \in \mathcal{F}$.*

PROOF. Let $\{P_\alpha: \alpha < \kappa\}$ be a partition of κ into κ disjoint sets each of cardinality κ. Let $f_\alpha: \kappa \to P_\alpha$ be one to one. Set $\mathcal{F} = \{z \subset \kappa: |\kappa - z| < \kappa\}$. $\square$

We present Kunen's proof that Con($\exists$ strongly compact)$\to$Con(PMEA). The main idea is to use the functions f_α from Lemma 3.1 to pull the Solovay measure on 2^I back to 2^λ.

For κ a infinite cardinal, set $B_\kappa^* = B_\kappa - \{h \in B_\kappa: m_\kappa(b) \neq 0\}$. By the phrase "add κ random reals" we mean "force with $\langle B_\kappa^*, \supseteq\rangle$". For the basic properties of this forcing the reader is referred to Kunen's article in this volume.

(When forcing, there is always the question of whether to use posets or complete Boolean algebras. For random reals, the quotient Boolean algebra, $B_\kappa \bmod \{b: m(b) = 0\}$, is often the better way. In our context, the two methods are about the same, and I prefer posets. Using the Boolean algebra avoids the arbitrary choice in defining $[[\sigma]]$, but in exchange requires an arbitrary choice in defining supp(b)).

Note that by our definition (see 1.3), an element b of B_κ is a subset of 2^κ. We can show by transfinite induction on the construction of the σ-algebra that every $b \in B_\kappa$ has countable support. Precisely, for every $b \in B_\kappa$, there is a countable subset, $\mathrm{supp}(b)$, of κ so that whether $f \in b$ is completely determined by $f|\mathrm{supp}(b)$; i.e. for some $b' \subset 2^{\mathrm{supp}(b)}$, $f \in b$ iff $f|\mathrm{supp}(b) \in b'$.

For this discussion, we fix κ and omit subscripts. Let σ be a statement of the forcing language for B^*. We want to define $[[\sigma]]$ to be the 'Boolean value' of σ. We define $[[\sigma]]$ to be an element, b, of B^* with $m(b)$ maximal. First, we show that the maximum is attained. Let $W \subset B^*$ be a maximal pairwise disjoint family of elements forcing σ. Because B^* is ccc, W is countable, and hence $\bigcup W \in B^*$, and of course $m(\bigcup W)$ is maximal. Next we must justify the use of the notation $[[\sigma]]$ despite the fact that the definition does not define a unique element of B^*. It is O.K. because what we ultimately use are $m([[\sigma]])$ and $m([[\sigma]] \cap b)$, and these values are the same for all elements of B^* satisfying the definition of $[[\sigma]]$.

3.3. THEOREM. [SOLOVAY 1971]. *Let M be a model of set theory. In M, let $\mathcal{U}$ be a κ-complete ultrafilter on a set I. Let m be the product measure, defined on $B = B_I$. Let G be an M-generic filter on B^*. Then in $M[G]$, there is a κ-additive real-valued measure, ν, (the Solovay measure), extending $\chi_{\mathcal{U}}$ and defined for every subset of I.*

PROOF. We will use $\Vdash \sigma$ to mean that every condition forces σ. Let $\hat{\emptyset}$ and $\hat{I}$ be names for $\emptyset$ and I, respectively.

For expository purposes, we begin with an argument which almost works. We work in M. Let A be a name for a subset of I (i.e. $\Vdash A \subset \hat{I}$). For $r \in [0, 1]$, let $J(A, r) = \{i \in I: m([[i \in A]]) = r\}$. Since $\{J(A, r): r \in [0, 1]\}$ partitions I into $\mathfrak{c}$-many pieces, for some unique r, $J(A, r) \in \mathcal{U}$. (Recall that $\mathfrak{c} < \kappa$). Define $\nu_1(A)$ to be that r. It is straightforward to check that

(i) $\nu_1(\hat{I}) = 1$, $\nu_1(\hat{\emptyset}) = 0$;

(ii) if $\Vdash A \subset A'$, then $\nu_1(A) \leq \nu_1(A')$; and

(iii) if $\Vdash (A = \{A_\alpha: \alpha < \beta\}$, where $\beta < \kappa$, is disjoint), then $\nu_1(\bigcup \mathcal{A}) = \Sigma_{\alpha<\beta}\, \nu_1(A_\alpha)$.

The flaw in the above argument is that ν_1 is defined on names in M, and is not well defined on elements of $M[G]$. To illustrate this, choose $b \in B$, $0 < m(b) < 1$ and let A be a name for a subset of I defined by $b \Vdash i \in A$, $(2^\kappa - b) \Vdash i \notin A$, for all $i \in I$. Then $\nu_1(A) = m(b)$, but $i_G(A) = I$ if $b \in G$ and $i_G(A) = \emptyset$ if $b \notin G$. Hence the name, τ, for $\nu(A)$ should be defined by $b \Vdash \tau = 1$, $(2^\kappa - b) \Vdash \tau = 0$.

Towards the correct definition of ν, for $b \in B^*$ and A a name for a subset of I, define $\nu_b(A)$ to be the unique $r \in [0, 1]$ such that

$$\left\{i \in I : \frac{m([[i \in A]] \cap b)}{m(b)} = r\right\} \in \mathcal{U}.$$

It is easy to verify that (i), (ii), and (iii) above hold with ν_b replacing ν_1 and $b \Vdash$ replacing $\Vdash$.

Now, we define

$$\nu(A) = \lim_{b \in G} \nu_b(A).$$

To show that the limit exists, we prove that for any finite partition $\mathcal{P} = \{P_t : t < n\}$ of $[0, 1]$ into intervals there is a partition $\mathcal{C} = \{c_t : t < n\}$ of 2^I into measurable sets such that $b \subseteq c_t$ implies $\nu_b(A) \in \overline{P_t}$. This suffices because G contains exactly one c_t and the partitions can be made arbitrarily fine.

For each name A and $b \in B$, let

$$\lambda_A(b) = \nu_b(A) \cdot m(b).$$

We see that λ_A is a measure on B such that $m(b) = 0$ implies that $\lambda_A(b) = 0$. By the Radon–Nikodym theorem, there is a measurable function, h_A, such that for all $b \in B$, $\lambda_A(b) = \int_b h_A \, \mathrm{d}m$. Because h_A is measurable, for every interval, P_t, the set $c_t = \{x \in 2^I : h_A(x) \in P_t\}$ is measurable. If $b \subseteq c_t$, then

$$\inf P_t \leqslant \nu_b(A) = \frac{\lambda_A(b)}{m(b)} = \frac{\int_b h_A \, \mathrm{d}m}{m(b)} \leqslant \sup P_t.$$

Now we can see that ν is well defined on elements of $M[G]$. For if $i_G(A) = i_G(A')$, then there is $b \in G$, $b \Vdash A = A'$. For all $c \leqslant b$, $\nu_c(A) = \nu_c(A')$; hence $\nu(A) = \nu(A')$.

It is straightforward to verify that ν is a finitely additive measure. Towards κ-additivity, let $\mathcal{A}$, $(A_\alpha)_{\alpha<\beta}$, and A be names, $\beta < \kappa$, and $b \in G$ be such that $b \Vdash \mathcal{A} = \{A_\alpha : \alpha < \beta\}$ is disjoint and $A = \bigcup \mathcal{A}$.

For each n, $\{\alpha < \beta : (\exists b)(m(b) > 1/n \,\&\, \nu_b(A_\alpha) > 1/n)\}$ is finite, so we may assume that $\beta = \omega$. For $k < \omega = \beta$, let h_k represent $\bigcup_{\alpha<k} A_\alpha$. Let $\delta > 0$ be arbitrary; we will find $d \in G$ and $k < \omega$ such that for all $x \in d$, $h_k(x) + \delta > \nu(A)$. First we pick $c \in G$ so that for all $x \in c$, $h_A(x) + \delta > \nu(A)$. Note that $N = \{x \in c : h_A(x) \neq \lim h_k(x)\}$ is a null set. For each $x \in c - N$, $\lim h_k(x) = h_A(x) > \nu(A) - \delta$. Set $d_k = \{x \in c - N : h_k(x) > \nu(A) - \delta\}$, a measurable set. Since $\bigcup d_k = c - N \in G$, for some k, $d_k \in G$. $\square$

The Radon–Nikodym theorem was a convenient way to get the measurable sets c_t and d_k. For a proof not using the theorem, see [JECH 1978, p. 423].

We have gathered the pieces and we are ready to assemble them into Kunen's theorem.

3.4. THEOREM. *Let $M[G]$ be constructed by adding κ random reals to M, where κ is strongly compact in M. Then* PMEA *holds in $M[G]$.*

PROOF. In M, let λ be an arbitrary cardinal greater than κ. By Lemma 3.1, there is a κ-complete free ultrafilter $\mathcal{U}$ on a set I and functions $f_\alpha: I \to \kappa$, $\alpha < \lambda$, such that if $\alpha < \beta < \lambda$, then $f_\alpha \neq f_\beta$ mod $\mathcal{U}$.

In $M[G]$, define $\phi_\alpha: I \to 2$ by $\phi_\alpha(i) = 1$ iff $\{g \in 2^\kappa: g(f_\alpha(i)) = 1\} \in G$, and define $\phi: I \to 2^\lambda$ by $\phi(i)(\alpha) = \phi_\alpha(i)$. Let ν be the Solovay measure constructed from I and $\mathcal{U}$ in Theorem 3.3. Let σ be the measure on 2^λ defined by pulling ν back via ϕ. Explicitly

$$\sigma(Y) = \nu(\phi^{-1} Y).$$

Because inverse functions preserve the basic set theoretic properties, σ is a $\mathfrak{c}$-additive real-valued measure.

It remains to check that σ extends the product measure on 2^λ. Let

$$A = \{g \in 2^\lambda: (g(\alpha_1) = e_1) \& \cdots \& (g(\alpha_n) = e_n)\}$$

be a basic clopen set in 2^λ. We must show that $\sigma(A) = 2^{-n}$. By definition $\sigma(A) = \nu(D)$ where $D = \{i \in I: (\phi_{\alpha_1}(i) = e_1) \& \cdots \& (\phi_{\alpha_n}(i) = e_n)\}$. By our choice of the f_α's, there is a set $J \in \mathcal{U}$ such that for all $i \in J$, $f_{\alpha_i}(i), \ldots, f_{\alpha_n}(i)$ are all distinct. For $i \in J$, $m([[i \in D]]) = 2^{-n}$.

Recall that every $b \in B$ is a cylinder over some countable set, supp(b), of coordinates. If for all k, $1 \leq k \leq n$, $f_{\alpha_k}(i) \notin \text{supp}(b)$, then $m([[i \in D]] \cap b) = (2^{-n})m(b)$. Hence for all $b \in \mathcal{B}$, $\nu_b(D) = 2^{-n}$, and finally $\sigma(A) = \nu(D) = 2^{-n}$. $\square$

If in Theorem 3.3, we let κ be measurable and $\mathcal{U}$ a κ-complete ulterafilter on κ, then $\nu: \mathcal{P}(\kappa) \to [0, 1]$. In such a case, κ is called a *real-valued measurable* cardinal. In the notation of this paper, PMEA$(\kappa, 2^\kappa, \kappa)$ holds in $M[G]$.

Suppose we lower our sights to PMEA(κ, κ, κ). Then we have to define $\nu(A)$ not for all names for subsets of 2^κ, but only for $Y \in \mathcal{Y}$, an arbitrary family of κ-many names for subsets of 2^κ.

Using Lemma 3.2, we can get function f_α, $\alpha < \kappa$, and in $M[G]$, define ϕ_α and ϕ. Then in M there is $\mathcal{S}$, a family of κ-many names for subsets of κ such that for each $Y \in \mathcal{Y}$ there is $A \in \mathcal{S}$ such that $\Vdash A = \phi^{-1}(Y)$. To continue as in the proofs of 3.3 and 3.4, we need a free κ-complete ultrafilter $\mathcal{U}$ on the κ-field of sets, $\mathcal{R}$, generated by

$$\{T(A, b, r): A \in S, b \in B^*, r \in [0, 1]\}$$

where

$$T(A, b, r) = \left\{\alpha \in \kappa : \frac{m([[\alpha \in A]] \cap b)}{m(b)} = r\right\}.$$

(A κ-field of set is closed under complements, unions, and intersections of subsets of cardinality less than κ. $\mathcal{U}$ is an ultrafilter on $\mathcal{R}$ means that $\mathcal{U} \subset \mathcal{R}$ and for all $R \in \mathcal{R}$, either $R \in \mathcal{U}$ or $(\kappa - R) \in \mathcal{U}$}.

Since $|\mathcal{R}| = \kappa$, we can get such an ultrafilter if κ is weakly compact. (We call κ *weakly compact* if κ is Π^1_1-indescribable; equivalently, if κ is strongly inaccessible and every κ-complete filter on a κ-field of sets $\mathcal{R}$ where $|\mathcal{R}| = \kappa$ can be extended to a κ-complete ultrafilter. See [KEISLER and TARSKI 1964] or [JECH 1978, §22]).

What can be done with weaker large cardinal assumptions, or ever no large cardinal assumption at all? We have just shown that we can slightly modify the proof of Theorems 3.3 and 3.4 to show that when κ is weakly compact, adding κ random reals gives a model of PMEA(κ, κ, κ). The results without large cardinal assumption are due to Carlson. Some combinatorial preparation is needed to define the measure.

The following cardinal arithmetic hypothesis is a bit cumbersome. Assuming GCH, it includes three cases: (i) $\rho = \lambda < \lambda^+ = \kappa$, where λ is regular; (ii) $\rho = \mathrm{cf}(\nu) < \lambda < \nu < \nu^+ = \kappa$; and (iii) $\rho \leq \lambda < \kappa$, where κ is inaccessible.

3.5. *Cardinal Arithmetic Hypothesis.* (a) λ and κ are regular;
(b) $\omega < \lambda$, $2^\omega < \kappa$;
(c) $\rho \leq \lambda < \kappa$; and
(d) if $\nu < \kappa$, then $\nu^{<\rho} < \kappa$.

3.6. LEMMA. *Assume ρ, λ, κ to satisfy* 3.5. *Set $S = \{\delta < \kappa : \mathrm{cf}(\delta) \geq \lambda\}$. When for each $\alpha \in \lambda$, $f_\alpha : S \to \kappa$ satisfies $f_\alpha(\delta) < \delta$ for all $\delta \in S$, then there is a ρ-complete filter $\mathcal{F}$ on κ extending the club filter, such that for each $\alpha \in \lambda$ there is $F_\alpha \in \mathcal{F}$ such that $f_\alpha | F_\alpha$ is constant.*

PROOF. For each $\delta \in S$ and each $A \in [\lambda]^{<\rho}$, define $Z(A, \delta) = \{\beta \in \kappa$: for all $\alpha \in A$, $f_\alpha(\beta) = f_\alpha(\delta)\}$. We establish the following by contradiction.

Claim. For some $\delta \in \lambda$, $Z(A, \delta)$ is stationary for every $A \in [\lambda]^{<\rho}$.

Proof. If not, for each $\delta \in S$ choose $A_\delta \in [\lambda]^{<\rho}$ so that $Z(A_\delta, \delta)$ is not stationary. Define $g(\delta) = \sup\{f_\alpha(\delta) : \alpha \in A_\delta\} < \delta$. By the Pressing Down Lemma, there are a stationary subset $S_1 \subset S$ and $\gamma < \kappa$ so that for all $\delta \in S_1$, $g(\delta) = \gamma$. Now $|[\lambda]^{<\rho}| < \kappa$, and for each $A \in [\lambda]^{<\rho}$ there are $|\gamma|^{<\rho} < \kappa$ many functions from A to γ. Hence there is a stationary subset $S_2 \subset S_1$ for which δ, $\beta \in S_2$ implies $A_\delta = A_\beta$ and for all $\alpha \in A_\delta$, $f_\alpha(\beta) = f_\alpha(\delta)$. In other words, $Z(A_\delta, \delta) \supset S_2$ stationary; contradiction, which proves the Claim.

Having established the claim, let δ be as asserted and let $\mathcal{F}$ be generated by the club filter and $\{Z(A, \delta) : A \in [\lambda]^{<\rho}\}$. $\mathcal{F}$ is ρ-complete, and $f_\alpha(\beta) = f_\alpha(\delta)$ for all $\beta \in Z(\{\alpha\}, \delta)$. $\square$

3.7. THEOREM [CARLSON 198 ·]. *Assume* ρ, λ, κ *to satisfy* 3.5. *Adding* κ *random reals yields a model of* PMEA(κ, λ, ρ) & $\mathfrak{c} = \kappa$.

PROOF. That $\mathfrak{c} = \kappa$ in $m[G]$ is, of course, the first thing learned about adding random reals. We work in M. Towards verifying PMEA(κ, λ, ρ), let Z_α, $\alpha \in \lambda$, be names for subsets of 2^κ. We can and do assume that $\mathscr{X} = \{i_G(Z_\alpha)\colon \alpha \in \lambda\}$ is closed under complementation, union, and intersection, but because $\lambda < \kappa = \mathfrak{c}$ in $m[G]$, we cannot assume either that $\mathscr{X}$ includes B_κ or that $\mathscr{X}$ is a σ-algebra. Carlson proved PMEA(κ, λ, ρ), but here we prove a weaker result: The existence of a function σ, with dom $\sigma \supset \mathscr{X} \cup \{[\eta]_\kappa\colon \eta \in H_\kappa\}$, which fails to be a measure only in that dom σ is not a σ-algebra and satisfying (iv)$_\rho$ only when $\bigcup_{\alpha<\beta} A_\alpha \in$ dom σ. This weaker result is enough for Theorem 2.2.

As we did in the weakly compact case, we use Lemma 3.2 to define f_α, $\alpha < \kappa$, ϕ_α, $\alpha < \kappa$, and ϕ. Below, we let Y_α be a name for the subset of κ, $\phi^{-1}(Z_\alpha)$. For $\alpha \in \lambda$ and $\delta \in \kappa$, set $s(\delta, \alpha) = \text{supp}[[\delta \in Y_\alpha]]$, and recall that it is a countable subset of κ. Let $\{\beta^\alpha_\gamma\colon \gamma < \xi_\alpha\}$ list $s(\delta, \alpha)$ in natural order, where $\xi_\alpha =$ order type $s(\delta, \alpha)$. For each $\xi < \omega_1$, there are $\mathfrak{c}$ recipes by which $b \in B_\kappa$ can be cooked from the subbasic sets $\{[\langle \beta_\gamma, e\rangle]\colon \gamma < \xi,\ e \in \{0, 1\}\}$. Via a bijection between $\mathfrak{c}$ and $\omega_1 \times \mathfrak{c}$, let ζ^δ_α code both ξ_α and the recipe for $[[\delta \in Y_\alpha]]$. Then $C = \{\gamma < \kappa\colon$ if $\delta < \gamma$ then $\mathfrak{c} \cdot \delta < \gamma\}$ is club. Set $S_0 = C \cap \{\delta \in \kappa\colon \text{cf}(\delta) \geq \lambda\}$.

For each $\alpha \in \lambda$ and $\delta \in S$, define

$$f_\alpha(\delta) = \mathfrak{c} \cdot \sup(s(\delta, \alpha) \cap \delta) + \zeta^\delta_\alpha, \quad \text{if } \delta \in S_0,$$

$$f_\alpha(\delta) = 0, \qquad\qquad \text{if } \delta \in S - S_0.$$

Since $f_\alpha(\delta) < \delta$, we can apply Lemma 3.6 to get a ρ-complete filter $\mathscr{F}$, where each f_α is constant on $F_\alpha \in \mathscr{F}$.

Next we verify that for each $\alpha \in \lambda$ and $b \in B^*_\kappa$, there is a (unique) $r \in [0, 1]$ so that $\{\delta \in \kappa\colon m([[\delta \in Y_\alpha]] \cap b)/m(b) = r\} \in \mathscr{F}$, and we will define $\nu_b(Y_\alpha)$ to be that r. We first note that $\{s(\delta, \alpha)\colon \delta \in F_\alpha\}$ is a Δ-system with root R. (i.e. there is a fixed set R such that for any two distinct δ, $\delta' \in F_\alpha$, $s(\delta, \alpha) \cap s(\delta', \alpha) = R$.) Thus, for each $b \in B^*_\kappa$, the set $V(b, \alpha) = \{\delta \in F_\alpha\colon \text{supp}(b) \cap (s(\delta, \alpha) - R) \neq \emptyset\}$ is countable, and $F_\alpha - V(b, \alpha) \in \mathscr{F}$. On this set of δ's, $m([[\delta \in Y_\alpha]] \cap b)$ will have a constant value because each $[[\delta \in Y_\alpha]] \cap b$ is cooked from the same recipe.

Having defined $\nu_b(Y_\alpha)$, we may continue exactly as in 3.3 and 3.4. $\square$

4. Summary and an alternative proof

We begin by combining the results of the previous two sections. (The Sep axioms are defined in 1.4.)

4.1. THEOREM. *Let $M[G]$ be constructed by adding κ random reals to a model, M.*
(a) *If κ is strongly compact in M, then* Sep(∞) *holds in $M[G]$.*
(b) *If κ is weakly compact in M, the* Sep($\mathfrak{c}$) *hold in $M[G]$.*
(c) *If in M, $\kappa \geqslant \mathfrak{c}$ and $\nu < \kappa$ implies $\nu^{<\nu} < \kappa$, then* Sep($<\mathfrak{c}$) *holds in $M[G]$.*

PROOF. Combine Theorem 2.2 with Theorem 3.4, the remarks after Theorem 3.3, and Theorem 3.8. □

Next, we rearrange the ideas of the previous theorems to prove a theorem very similar to Theorem 4.1. The advantages of Theorem 4.1 are (i) it is the original version, (ii) it proves not only Sep(∞), but the stronger PMEA, which has other applications, and (iii) a strongly compact cardinal (rather than supercompact) is used in part (a). The advantages of Theorem 4.2 are (i) the proof is more direct, and in my opinion, much clearer, (ii) only $\mathrm{cf}(\kappa) = \omega$ is excluded from part (c).

4.2. THEOREM. *Let $M[G]$ be constructed by adding κ random reals to a model, M.*
(a) *If κ is supercompact in M, then* Sep(∞) *holds in $M[G]$.*
(b) *If κ is weakly compact in M, then* Sep($\mathfrak{c}$) *holds in $M[G]$.*
(c) *If in M, $\kappa \geqslant \mathfrak{c}$ and* $\mathrm{cf}(\kappa) \neq \omega$, then Sep($<\mathfrak{c}$) *holds in $M[G]$.*

Before beginning the proof, let us record some facts about random real forcing. (See Kunen's article). For $I \subset \kappa$ and $G \subset B_\kappa$, we define $G|I = \{b \in G: \mathrm{supp}(b) \subset I\}$. If G is an M-generic filter on B_κ and $I \subset \kappa$, then $M[G] = M[G|I][G|(\kappa - I)]$, $G|I$ is M-generic on B_I, and $G|(\kappa - I)$ is $M[G|I]$-generic on $B_{\kappa - I}$. Because B_κ is ccc and elements of B_κ have countable support, whenever $A \in M[G]$ is a set of ordinals, then there is $I \subset \kappa$, $|I| = |A| \cdot \omega$ such that $A \in M[G|I]$.

We also need a lemma showing that if a counterexample exists, a small, nice counterexample exists.

4.3. LEMMA. *If $\mathscr{Y}$ is a discrete family in a space X, and $\mathscr{B}$ is a base for $\bigcup\mathscr{Y}$ in X, then there are X', $\mathscr{Y}'$, and $\mathscr{B}'$ such that the point set of X' is $\nu = \max\{|\mathscr{B}|, |\bigcup\mathscr{Y}|\}$, (i.e. the set of ordinals of lesser cardinality) and $\mathscr{Y}$ is separated or normalized iff $\mathscr{Y}'$ is.*

PROOF. Choose $X'' \subset X$, such that (i) $|X''| = \nu$, (ii) if B, $B' \in \mathscr{B}$ and $B \cap B' \neq \emptyset$, then $B \cap B' \cap X'' \neq \emptyset$, and (iii) $\bigcup\mathscr{Y} \subset X''$. Define $\mathscr{B}'' = \{B \cap X'': B \in \mathscr{B}\}$. There is a space X' with point set ν homeomorphic to X''. Define $\mathscr{B}'$ and $\mathscr{Y}'$ accordingly. □

Proof of Theorem 4.2. We first prove part(c). Let $\mathscr{Y}$ be a discrete family in a space X, $\mathscr{B}$ a base for $\bigcup\mathscr{Y}$ in X, and $|\mathscr{B}| = \nu < \mathfrak{c}$. We will show that $\mathscr{Y}$ is not normalized.

By Lemma 4.3, we can assume that the point set of X is ν. By the facts about random real forcing, there is $I \subset \kappa$, $|I| = \nu$ such that X, $\mathcal{Y}$, $\mathcal{B} \in M[G|I]$. Index $\mathcal{Y}$ with ordinals in $\kappa - I$ so that $\mathcal{Y} = \{Y_\alpha : \alpha \in I'\}$, where $I' \subset \kappa - I$. Let J be the "generic" subset of I'; precisely, we define $J = \{\alpha \in I' : [\langle \alpha, 1\rangle]_I \in G\}$. We will show that if $\bigcup\{Y_\alpha : \alpha \in J\}$ and $\bigcup\{Y_\alpha : \alpha \notin J\}$ can be separated by disjoint open sets in $M[G]$, then $\mathcal{Y}$ is almost separated in $M[G|I]$, a fortiori, in $M[G]$. (Hence if $\mathcal{Y}$ is not almost separated, then $\mathcal{Y}$ is not normalized.)

Assume that $b_0 \in B^*_{\kappa-I}$ and $\dot{U}$ is a name such that

$$b_0 \Vdash \dot{U} \text{ is open in } X, \bigcup\{Y_\alpha : \alpha \in \dot{J}\} \subset \dot{U}, \text{ and}$$

$$\bigcup\{Y_\alpha : \alpha \in \dot{J}\} \subset X \dot{-} \bar{U}$$

where we work in $M[G|I]$ and force with $B_{\kappa-I}$. For each $y \in \bigcup\mathcal{Y}$, choose a neighborhood $N_y \in \mathcal{B}$ and a condition $b_y \leq b_0$ such that $m(b_y) > \frac{7}{8}\, m(b_0)$ and

$$b_y \Vdash N_y \subset \dot{U} \quad \text{or} \quad N_y \subset X \dot{-} \bar{U}. \tag{$*$}$$

(We justify. Fix y. In M, let D be a maximal set of pairs $(b_i, \dot{N}_i)$ such that (i) $b_i \in B_\kappa$, (ii) $\dot{N}_i$ is a name for a neighborhood of y, (iii) b_i, $\dot{N}_i$ satisfy $(*)$; and (iv) the b_i's are disjoint. Let $E = \bigcup\{b_i : i \in \omega\}$; then $m_\kappa(E) = 1$. By Fubini's Theorem

$$m_I(\{h \in 2^I : m_{\kappa-I}(\{k \in 2^{\kappa-I} : (h \cup k) \in E\}) = 1\}) = 1\,.$$

By density, $(\bigcap G)|I$ is such an h. Hence in $M[G|I]$ there is $j \in \omega$ such that

$$m_{\kappa-I}(\{k \in 2^{\kappa-I} : ((\bigcap G)|I \cup k) \in \bigcup\{b_i : i < j\}\} \cap b_0) > \tfrac{7}{8} m(b_0)\,.$$

Let $N_y = \bigcap\{i_{G|I}(\dot{N}_i) : i < j\}$).

For each $\alpha \in I'$, set $W_\alpha = \bigcup\{N_y : y \in Y_\alpha\}$. Note that $\{W_\alpha : \alpha \in I'\} \in M[G|I]$. We claim that $\{W_\alpha : \alpha \in I - \operatorname{supp}(b_0)\}$ is disjoint. Let α, β be distinct elements of $I - \operatorname{supp}(b_0)$ and let $y \in Y_\alpha$, $z \in Y_\beta$. Set $b = b_y \cap b_z \cap [\langle \alpha, 1\rangle, \langle \beta, 0\rangle]$; $m(b) > 0$ and

$$b \Vdash (N_y \cap N_z) \subset (\dot{U} \cap X \dot{-} \bar{U}) = \emptyset\,.$$

We have shown that $\mathcal{Y}$ is almost separated.

For parts (a) and (b) we assume that there is a counterexample, use a reflection principle to get a small counterexample, and proceed as above. Suppose that in $M[G]$, $\mathcal{Y}$ is a not almost separated, discrete family in a space X, whose point set is an ordinal, ($\leq \kappa$ in case b)), $\mathcal{B}$ is a base for $\bigcup\mathcal{Y}$ in X, and $\hat{\chi}(\bigcup\mathcal{Y}, X) \leq \mathfrak{c}$. Let $\mathcal{N} : \bigcup\mathcal{Y} \times \kappa \to \mathcal{B}$ and $\gamma : \bigcup\mathcal{Y} \to \kappa$ be such that for all $y \in \bigcup\mathcal{Y}$, $\{\mathcal{N}(y, \alpha) : \alpha < \kappa\}$ is a neighborhood base for y and for all $\alpha > \gamma(y)$, $\mathcal{N}(y, \alpha) = X$ (which we may assume is in $\mathscr{B}$). In M, there are B_κ-names for these, namely $\dot{\mathscr{Y}}$, $\dot{X}$, $\dot{\mathscr{B}}$, $\dot{\mathscr{N}}$, $\dot{\gamma}$. Consider the structure $\langle V_\mu, \in, B_\kappa, \dot{\mathscr{Y}}, \dot{X}, \dot{\mathscr{B}}, \dot{\mathscr{N}}, \dot{\gamma}\rangle$, where in case (b) $\mu = \kappa$.

By reflection, we obtain a substructure $\langle V_{\nu'}, \in, B_\nu, \dot{\mathscr{Y}}', \dot{X}', \dot{\mathscr{B}}', \dot{\mathscr{N}}', \dot{\gamma}'\rangle$.

In case (a) we simply take a second order submodel see [KANAMORI and MAGIDOR 1978, pp. 185–186], but in case (b) we must specify the Π_1^1 statement to be preserved. It is $\forall \dot{\mathscr{W}}\, \forall b \in \mathscr{B}_\kappa$ not $b \Vdash \dot{\mathscr{W}}$ almost separates $\dot{\mathscr{Y}}$. Now let's look at all these terms interpreted in $M[G|\nu]$. $\mathscr{Y}'$ is a discrete, not almost separated family in X'. Because of $\mathscr{N}$ and γ, $\mathscr{B}'$ includes a neighborhood base for each $y \in \bigcup \mathscr{Y}'$. To complete the proof, we must show that in $M[G] = M[G|\nu][G|(\kappa - \nu)]$, $\mathscr{Y}'$ is not normalized in X', hence not in X. This is done exactly as in case (c). □

Many of the ideas of the above proof are in Tall's thesis. There he added ω_2 Cohen subsets of ω_1. Spaces of cardinality ω_1 appear in some $M[G|I]$, and if a discrete collection is 'starry', then a generic subset shows that it is not normalized. It is probable that the subject would have developed differently had Tall added ω_2 random reals rather than indolently choosing Cohen subsets of ω_1 [TALL 1977, p. 25].

The techniques of this section can be used with Cohen reals in place of random reals. To choose N_y satisfying (*) in the proof of Theorem 4.2(c), we use Dow's Lemma in place of measure.

4.4. LEMMA [DOW 1983]. *For every set I and every $n \in \omega$, there is a family L of clopen sets such that* (i) *whenever $\{c_i: i \in \omega\}$ is a maximal pairwise disjoint family of basic open subsets of 2^I, then there is a finite set a such that $\bigcup\{c_i: i \in a\} \in L$; and* (ii) *whenever $\eta \in H_I$, $|\eta| = n$, and $b_j \in L$ for $j < n$, then $[\eta] \cap (\bigcap\{b_j: j < h\}) \neq \emptyset$.*

For the reflection argument in Theorem 4.2(a), a 'strong' cardinal (defined in [DODD 1983] suffices. It seems plausible to me that Mitchell forcing or iterated Sacks forcing might create a model of NMSC + $2^\omega = \omega_2$. As above, take a potential counter example, reflect almost separated, and a generic subset makes it not normalized. The problem is to show that the remaining forcing does not introduce separations.

The proof of Theorem 4.2 can be presented without the notion almost separated. Because B_κ is so homogeneous, if in $M[G]$ there is a counterexample, its existence is forced with Boolean value 2^κ. We get a specific counterexample and the open set U with Boolean value 2^κ by using the maximal principle. Hence $b_0 = 2^\kappa$, $\text{supp}(b_0) = \emptyset$, and $\{W_\alpha: \alpha \in I\}$ is a separation.

5. Normal, not collectionwise normal spaces

Having said all that is known about making NMSC true, let us turn to constructing normal, nonmetrizable Moore spaces. In view of Bing's Theorem, such spaces must be normal and not collectionwise normal. We will begin with describing Bing's examples G and H [BING 1951], and follow with the problem of lowering the character.

5.1. *Bing's example G.* The point set of G is $2^{\mathcal{P}(\omega_1)} = \{f : f$ is a function from $\mathcal{P}(\omega_1)$ to $\{0, 1\}\}$. For $\alpha \in \omega_1$, define $y_\alpha \in 2^{\mathcal{P}(\omega_1)}$ by $y_\alpha(A) = 1$ iff $\alpha \in A$. Points y_α have the usual product topology; all other points are isolated. First, let's verify that G is normal. Let H and K be disjoint closed sets, and set $Y = \{y_\alpha : \alpha \in \omega_1\}$. Let $A = \{\alpha : y_\alpha \in H\}$ and $U = \{f \in G : f(A) = 1\}$. Then $(U \cup H)$-K is an clopen set separating H and K.

This argument illustrates a 'standard trick'. In order to show that a space is normal, it suffices to consider disjoint closed subsets of the nonisolated points.

Towards showing that G is not collectionwise normal, first note that $\mathcal{Y} = \{\{y_\alpha\} : \alpha \in \omega_1\}$ is discrete. If $y_\alpha \in U_\alpha$, then U_α contains an open set in the usual product topology, a set with positive product measure in $2^{\mathcal{P}(\omega_1)}$. So any family of disjoint U_α's is countable, and $\mathcal{Y}$ can't be separated.

Bing's example H is $G \times (\omega + 1)$ where points of the form (g, n), $n \in \omega$, are isolated and points of the form (g, ω) have the product topology. H is normal, not collectionwise normal, and perfect. In fact every subset (not only closed subsets) is G_δ.

The reason that G is not a Moore space is that the character is too large—$\chi(G) = 2^{\omega_1} = |\mathcal{P}(\omega_1)|$. For over twenty years there were only minor variations on G and H—for example taking subspaces, using $\mathcal{P}(I)$ in place of $\mathcal{P}(\omega_1)$ where I has various cardinalities, and changing usual (finite support) product topology to countable support product topology. None of these variations lowered the character or made the space collectionwise Hausdorff. See [LEWIS 1977] for a catalogue of these spaces.

The reader is referred to Tall's article for the results between Bing's article [BING 1951] and Fleissner's Thesis [FLEISSNER 1974]. We will not dwell on the latter, but only describe how it directed research towards proving NMSC consistent. It focused too much attention on $V = L$, and away from PMEA, which contradicts $V = L$ in several ways very strongly. Towards constructing an absolute counterexample, it showed that the discrete family $\mathcal{Y}$ cannot consist of singletons, or even Lindelöf sets. In this context, it reemphasized Tall's question of the existence of a normal, collectionwise Hausdorff, not collectionwise normal space.

A space is called *collectionwise Hausdorff* if every discrete family of singletons can be separated. Talking about families of singletons is rather pedantic, so it is usual in this context to talk about discrete sets. We call D a discrete *set* if $\mathcal{Y} = \{\{d\} : d \in D\}$ is a discrete *family*.

Tall's question was answered by the space George. The idea of the construction is to replace Bing's G with an ω_1-sequence of approximations of G. In [FLEISSNER 1975] it is described as $\bigcup_{\beta<\omega_1} G_\beta$ where $G_\beta = 2^{\mathcal{P}(\beta)}$, containing special points $\{y_\alpha^\beta : \alpha < \beta\}$, with the example G topology. The special points are 'tied together' so that $Y_\alpha = \{y_\alpha^\beta : \alpha < \beta < \omega_1\}$ is homeomorphic to $\{\beta : \alpha < \beta < \omega_1\}$ with the order topology. This line of thinking leads directly to a collectionwise Hausdorff space of character 2^ω with a normalized not separated family, but the space itself is not normal. George was defined by making some apparently ad hoc modifications.

Przymusiński presented a version of George in which these modifications seem natural [PRZYMUSIŃSKI 1976]. The idea is that the point set should be $2^{\mathcal{T}}$, where $\mathcal{T}$ is the family of clopen, or even open, subsets of a space. This idea includes Bing's Example G as a special case because the family of clopen sets of ω_1 with the discrete topology is $\mathcal{P}(\omega_1)$.

5.2. *Definition of George.* Let

$$F = \{(\alpha, \beta) : \alpha \leq \beta < \omega_1\},$$

$$Y_\alpha = \{\alpha\} \times (\omega_1 - \alpha) \subset F, \qquad Z_\beta = \{(\alpha, \gamma) \in F : \gamma \leq \beta\}.$$

Topologize F so that $\mathcal{Y} = \{Y_\alpha : \alpha \in \omega_1\}$ is a discrete collection and for each α, $Y_\alpha \cong (\omega_1 - \alpha)$ via $(\alpha, \beta) \mapsto \beta$. In other words, F is the "upper triangle" of ω_1 with the discrete topology cross ω_1 with the order topology.

Let $\mathcal{U}$ be the family of clopen subsets of F; set $\mathcal{U}_\beta = \{U \in \mathcal{U} : U \subset Z_\beta\}$; set $\mathcal{W}^\beta = \{U \in \mathcal{U} : U \cap Z_\beta = \emptyset\}$; and let $J = 2^{\mathcal{U}}$. We identify $(\alpha, \beta) \in F$ with the point $y_{\alpha,\beta} \in J$ defined by $y_{\alpha,\beta}(U) = 1$ iff $(\alpha, \beta) \in U$. Points of $J - F$ are isolated. Neighborhoods of $y_{\alpha,\beta} \in J$ are indexed by finite subsets of $\mathcal{U}_\beta$. For $(\alpha, \beta) \in F$ and $a \in [\mathcal{U}_\beta]^{<\omega}$, we define

$$B(\alpha, \beta, a) = \{g \in J : (\forall U \in \mathcal{W}_\beta)(g(U) = 0) \,\&\, (\forall U \in a)(g(U) = y_{\alpha,\beta}(U))\}.$$

It is routine to verify that we have presented a base for a T_1, regular topology. Moreover, the subspace topology on F is exactly the topology on F originally given.

5.3. *George is normal.* Let H and K be disjoint closed sets. Using the standard trick (see 4.1), we may assume that $H \cup K \subset F$. Let A be a clopen subset of F, $H \subset A \subset F - K$. Set $U = \bigcup\{B(\alpha, \beta, \{A\}) : (\alpha, \beta) \in A\}$. Now in J, U is clopen and $H \subset U \subset J - K$.

5.4. *George is not collectionwise normal.* We will show that the discrete family $\mathcal{Y} = \{Y_\alpha : \alpha < \omega_1\}$ can not be separated. For each $\alpha < \omega_1$, let W_α be an open subset of J containing Y_α. For each $(\alpha, \beta) \in F$, choose $a(\alpha, \beta)$ so that $y_{\alpha,\beta} \in B(\alpha, \beta, a(\alpha, \beta)) \subset W_\alpha$. Let $\gamma(\alpha, \beta) = \min\{\gamma : y_{\alpha,\gamma} \in B(\alpha, \beta, a(\alpha, \beta))\}$. By the Pressing Down Lemma, for each $\alpha \in \omega_1$, we can find $n_\alpha \in \omega$ and $\gamma_\alpha \in \omega_1$ so that $\{\beta : |a(\alpha, \beta)| = n_\alpha \,\&\, \gamma(\alpha, \beta) = \gamma_\alpha\}$ is uncountable. Choose $m \in \omega$, $\zeta \in \omega_1$, and $r \subset$

ω_1 so that $|r| = 2^m + 1$, and for $\alpha \in r$, $n_\alpha = m$, $\gamma_\alpha \leqslant \zeta$. Further, for $\alpha \in r$, choose $\beta_\alpha > \zeta$ so $n(\alpha, \beta_\alpha) = m$ and $\gamma(\alpha, \beta_\alpha) = \gamma_\alpha$. We abbreviate $B(\alpha, \beta_\alpha, a(\alpha, \beta_\alpha))$ by B_α.

Set $\mathscr{V} = \mathscr{U} - \mathscr{W}^\zeta$. The map $\theta(g) = g|\mathscr{V}$ from J to $2^{\mathscr{V}}$ takes each B_α, $\alpha \in r$, to a set, C_α, of measure 2^{-m} in the product measure on $2^{\mathscr{V}}$ because $a(\alpha, \beta_\alpha) \subset \mathscr{V}$. Since $|r| > 2^m$, there are distinct $\alpha, \alpha' \in r$ and $g_0 \in C_\alpha \cap C'_\alpha$. Define $g \in J$ by $g(U) = g_0(U)$ if $U \in \mathscr{V}$ and $g(U) = 0$ if $U \in \mathscr{W}^\zeta$. Then $g \in B_\alpha \cap B_{\alpha'}$. Hence the U_α's are not disjoint and J is not collectionwise normal.

5.5. *J is collectionwise Hausdorff.* Let $D \subset J$ be a discrete set. As usual we may assume that $D \subset F$. Because Y_α is countably compact, $|D \cap Y_\alpha| < \omega$; hence

$$C = \{\gamma \in \omega_1: \text{if } y_{\alpha,\beta} \in D \text{ and } \alpha < \gamma, \text{ then } \beta < \gamma\}$$

is club. For $\gamma \in C$, let $\gamma^\#$ be the successor of γ in C; i.e. $\gamma^\# = \min(C - (\gamma + 1))$. For $\gamma \in C$, set $F_\gamma = \{(\alpha, \beta) \in F: \gamma \leqslant \beta < \gamma^\#\}$ and set $D_\gamma = F_\gamma \cap D$. In regular, T_1, spaces, countable discrete sets can be separated. So for each $\gamma \in C$, there is a collection $\{U_{\alpha,\beta}: y_{\alpha,\beta} \in D_\gamma\}$, of open sets separating the discrete set D_γ. Finally, $\bigcup_{\gamma \in C}\{U_{\alpha,\beta} \cap B(\alpha, \beta, \{F_\gamma\}: y_{\alpha,\beta} \in F_\gamma\}$ separates D.

6. Navy's space

To use the idea of approximating Bing's example G to get a first countable space, the approximating spaces must be of the form 2^A, where A is a countable set. A linear series of such approximations seems impossible under mild set theoretic assumptions, but is suggested by $\diamondsuit^+$. These ideas were developed in [FLEISSNER 1982], where $\diamondsuit^{++}$ is formulated, proved from $V = L$, and used to construct a first countable, normal, not collectionwise normal space. The base space F is a discrete union of stationary subsets, and hence not a Moore space.

The ideas that finally worked came from an apparently unrelated problem—the problem of whether regular, paraLindelöf spaces are paracompact. (A space is called paraLindelöf if every open cover has a locally countable refinement.)

Fleissner suggested using the metric space F (defined below) with the entwined pairs $\langle \rho, \tau \rangle$ as isolated points to define a paraLindelöf not paracompact space [FLEISSNER 1978]. The space defined there failed to be paraLindelöf because there wasn't "enough normality". Navy saw that replacing the single isolated point, $\langle \rho, \tau \rangle$, with a copy of Bing's G would create the desired space. In this section we present Navy's example [NAVY 1981].

6.1. *Navy's space defined.* Let F be product of ω copies of ω_1 with the discrete topology. Thus the point set is $\{f: f \text{ is a function from } \omega \text{ to } \omega_1\}$, and a metric d can be defined by $d(f, g) = 2^{-n}$ where n is the least integer such that $f(n) \neq g(n)$. For our purposes, it is more useful to look at a σ-discrete basis $\mathscr{B} = \bigcup_{n \in \omega} \mathscr{B}_n$. Set $\Sigma_n = \{f|n: f \in F\}$, set $\Sigma = \bigcup_{n \in \omega} \Sigma_n$; for $\sigma \in \Sigma$, set $[\sigma] = \{f \in F: \sigma \subset F\}$, and for

$n \in \omega$, set $\mathscr{B}_n = \{[\sigma]: \sigma \in \Sigma_n\}$. Let $\mathscr{T}$ be the family of open subsets of F, and set $G = 2^{\mathscr{T}}$.

We say that $\rho, \tau \in \Sigma_n$ are *entwined* if,

$$\text{for all } i < n-1, \quad \rho(i) < \tau(i) < \rho(i+1) < \tau(i+1)\,.$$

Let $Q = \{\langle g, \rho, \tau\rangle : g \in G, \text{ and } \rho, \tau \text{ are entwined}\}$.

The point set of Navy's space, N, is $F \cup Q$. Points of Q are isolated. Neighborhoods of points of F have the form $B(\sigma, a)$, where $\sigma \in \Sigma$ and a is a finite subset of $\mathscr{T}$.

$$\begin{aligned} B(\sigma, a) = [\sigma] \cup \{\langle g, \rho, \tau\rangle :\ & \text{(i) either } \sigma \subset \rho \text{ or } \sigma \subset \tau, \text{ and} \\ & \text{(ii) } (\forall U \in a) \text{ if } [\sigma] \subset U \text{ then } g(U) = 1, \text{ and} \\ & \text{(iii) } (\forall U \in a) \text{ if } [\sigma] \cap U = \emptyset \text{ then } g(U) = 0\}\,. \end{aligned}$$

We pause to describe a mental picture. The base space, F, sits underneath a whole swarm of copies of $G = 2^{\mathscr{T}}$, indexed by entwined pairs $\langle \rho, \tau\rangle$. A basic open set N is a basic open set of F with two arms. One arm consists of "corners" of copies G indexed by $\langle \rho, \tau\rangle$ where $\sigma \subset \rho$, the other arm is the same with $\sigma \subset \tau$.

6.2. *N is normal.* Let us show that N is normal. Let H and K be disjoint closed subsets of N. We may assume that $H \cup K \subset F$. Since H and K are disjoint closed subsets in the metric subspace F, there are disjoint open in F sets U, V with $H \subset U$ and $K \subset V$. Then $B(U, \{U\})$ and $B(V, \{U\})$ are disjoint open subsets of N containing H and K, respectively.

6.3. *N is not collectionwise normal.* The first step in showing that N is not collectionwise normal is to note that $\mathscr{Y} = \{Y_\alpha : \alpha < \omega_1\}$ where $Y_\alpha = \{f \in F: f(0) = \alpha\}$ is a discrete family. For $\alpha \in \omega_1$, let U_α be an open set containing Y_α. We would like to argue as in Bing's G or as in George, that each U_α meets G in a set of positive measure, hence the U_α's are not disjoint. The problem is that there are many copies of G, not just one. Moreover, it is easy to choose the U_α's so that each G meets only two U_α's.

The key idea is that of a full set. We call $S \subset \Sigma_n$ *full* if for all $\sigma \in S$ and $j < n$, $\{\upsilon(j): \sigma | j \subset \upsilon \in S\}$ is uncountable.

In case $n = 1$, full just means uncountable. The following combinatorial lemma will show that we have full sets and can use them to focus on a particular copy of G.

6.4. LEMMA. (a) *If* $\bigcup\{[\sigma]: \sigma \in T\} = F$, *then for some* $n \in \omega$, $\Sigma_n \cap T$ *has a full subset.*

(b) *Suppose that* $S \subset \Sigma_n$ *is full and that* $h: S \to \omega$. *Then there is a full* $S' \subset S$ *on which h is constant.*

(c) *If* $S' \subset \Sigma_n$ *is full, then there is* $W = \{\sigma_k : k \in \omega\}$ *such that each pair* (σ_k, σ_m) *of elements of W is entwined.*

PROOF. (a) Towards a contradiction, assume that for all $n \in \omega$, $T \cap \Sigma_n$ does not have a full subset. We will construct $f \notin \bigcup\{[\sigma]: \sigma \in T\}$.

We start with an informal argument. Well, since $T \cap \Sigma_1$ is not full, all but a countable set of choices of $f(0)$ will satisfy $f \notin \bigcup\{[\sigma]: \sigma \in T \cap \Sigma_1\}$. However, we might have chosen $f(0)$ carelessly so that all choices of $f(1)$ place f in $\bigcup\{[\sigma]: T \cap \Sigma_2\}$. But since $T \cap \Sigma_2$ is not full, only a countable set of choices could be so unfortunate. We must choose $f(0)$ not only to avoid $T \cap \Sigma_1$, but also to avoid being 'trapped' by $T \cap \Sigma_n$, any $n \in \omega$.

Formally, by induction on $i \in \omega$ define $f|i$ so that for each $n > i$ and $T' \subset T \cap \Sigma_n$ there is $\upsilon \in T'$ such that either $f|i \not\subset \upsilon$ or there is j, $i \leq j < n$ such that $\{\sigma(j): \upsilon|j \subset \sigma \in T'\}$ is countable.

(b) We have a function $h = h_0$ defined on S, the top level of a tree. We define a function h_1 on the next lower level, i.e., on $S_1 = \{\upsilon|n-1: \upsilon \in S\}$. Well, for each $\sigma \in S_1$, there are uncountably many $\upsilon \in S$ such that $\sigma = \upsilon|n-1$. So we can define $h_1(\sigma)$ so that $\{\upsilon(n-1): \sigma \subset \upsilon \in S \ \&\ h(\upsilon) = h_1(\sigma)\}$ is uncountable. In the same way, we continue to define by induction on $i \leq n$, a function h_i defined on $S_i = \{\upsilon|n-i: \upsilon \in S\}$ so that for each $\sigma \in S_i$, $\{\upsilon(n-i): \sigma \subset \upsilon \in S \ \&\ h_{i-1}(\upsilon|(n-i)+1) = h_i(\sigma)\}$ is uncountable. Now $S' = \{\sigma \in S: \text{ for all } i < n,\ h_i(\sigma|n-1) = h_n(\emptyset)\}$ satisfies the conclusion of the lemma.

(c) By induction on $(i, k) \in \omega \times \omega$ with the lexicographic order, define $\sigma_k(i)$ so that $\sigma_k \in S'$ and $\sigma_k(i) < \sigma_{k'}(i')$ iff $(i, k) < (i', k')$ in the lexicographic order. $\square$

We return to the proof that N is not collectionwise normal. Let $T = \{\sigma: \text{for some } a,\ B(\sigma, a) \subset U_{\sigma(0)}\}$. Since $\bigcup \mathcal{Y} = F$, we can apply part (a) of the lemma to get $n \in \omega$ and S, a full subset of $T \cap \Sigma_n$. For $\sigma \in S$, define $a(\sigma)$ so that $B(\sigma, a(\sigma)) \subset U_{\sigma(0)}$ and set $h(\sigma) = |a(\sigma)|$. Apply part (*b*) to get a full subset $S' \subset S$ and $n \in \omega$ so that for all $\sigma \in S'$, $|a(\sigma)| = n$. Finally, we apply part (c) to get a subset $W = \{\sigma_k: k \in \omega\}$ of S'.

Let us look at the structure of $B(\sigma, a(\sigma))$ for $\sigma \in W$. It consists of $[\sigma]$ together with "two arms of the same corner of G". More precisely, there is a basic open subset P_σ of G of measure 2^{-n} such that $B(\sigma, a(\sigma)) = [\sigma] \cup \{(g, \rho, \tau): g \in P_\sigma \text{ and either } \sigma \subset \rho \text{ or } \sigma \subset \tau\}$. Since there are infinitely many P_σ's and G has measure 1, we can find $\rho, \tau \in W$ and $\bar{g} \in G$ so that $\bar{g} \in P_\rho \cap P_\tau$ and $\rho(0) < \tau(0)$. Since any two elements of W are entwined

$$(\bar{g}, \rho, \tau) \in B(\rho, a(\rho)) \cap B(\tau, a(\tau)) \subset U_{\rho(0)} \cap U_{\tau(0)}\,.$$

We conclude that $\mathcal{Y}$ can't be separated and N is not collectionwise normal.

6.5. *N is paraLindelöf.* Let $\mathcal{U}$ be an open cover of N; we must find $\mathcal{R}$, a locally countable open refinement of $\mathcal{U}$. First, we look at $\mathcal{U}|F = \{U \cap F: U \in \mathcal{U}\}$, an open cover of F. Now F is such a nice space that there is a disjoint open cover $\mathcal{W}$ of F refining $\mathcal{U}|F$. (The name of this property is ultraparacompactness.) We may assume that $\mathcal{W}$ consists of basic open sets; i.e. $\mathcal{W} \subset \mathcal{B}$. Set $\mathcal{W}_n = \mathcal{W} \cap \mathcal{B}_n$ and set $H_n = \bigcup \mathcal{W}_n$. $\{H_n : n \in \omega\}$ is a countable disjoint family of closed subsets of N. Since we have already shown that N is normal, we can find a disjoint family $\mathcal{V} = \{V_n : n \in \omega\}$ of open subsets of N satisfying $H_n \subset V_n$. For each $n \in \omega$, and each $[\sigma] \in \mathcal{W}_n$, find a_σ and $U \in \mathcal{U}$ so that $B(\sigma, a_\sigma) \subset U$. Set $\mathcal{R}_n = \{B(\sigma, a_\sigma) \cap V_n : [\sigma] \in \mathcal{W}_n\}$.

It is routine to show that

$$\mathcal{R} = \bigcup_{n\in\omega} \mathcal{R}_n \cup \{\{q\} : q \notin \bigcup_{n\in\omega} (\bigcup \mathcal{R}_n)\}.$$

is an open family covering N and refining $\mathcal{U}$. We show that $\mathcal{R}$ is locally countable. If $q \in N - F$, then $\{q\}$ is open and meets at most two members of $\mathcal{R}$. If $f \in F$, then for some $n \in \omega$, $f \in H_n$. Set $N_f = B(f|n+1, \emptyset) \cap V_n$. Because $N_f \subset V_n$, $\mathcal{R}_f = \{R \in \mathcal{R} : N_f \cap R \neq \emptyset\} \subset \mathcal{R}_n$. We claim that $\mathcal{R}_f$ is countable. When $[\sigma] \in \mathcal{W}_n$ and $B(\sigma, a_\sigma) \cap N_f \neq \emptyset$, then for all $i < n$, $\sigma(i) < f(n) < \omega_1$. Hence there are at most countably many such σ's.

N is collectionwise Hausdorff because regular, paraLindelöf spaces are collectionwise Hausdorff [TALL 1979].

7. The CH example

We have presented two examples of normal, collectionwise Hausdorff, not collectionwise normal spaces. Which is closer to being a Moore space? In the sense of character, George is, and son of George [FLEISSNER 1982], being first countable, is closer still, for $2^{\omega_1} \geqslant 2^{\omega} > \omega$. However, if we judge by the subspace of nonisolated points, we see that Navy's space is built from a metrizable space, where George contains copies of ω_1. Son of George contains stationary sets, and this 'non-Mooreness' seems essential to the construction. We focus our efforts, therefore, on trying to lower the character of Navy's space.

One thing we soon notice is that using g's defined on all of $\mathcal{T}$, (recall that $\mathcal{T}$ is the family of open subsets of the metric space F) both made the character large and made the proof of normality almost trivial. Can we trade a more difficult proof of normality for a lower character?

Note that for each $\sigma \in \Sigma_{n+1}$ there are uncountably many $\upsilon \in \Sigma_{n+1}$ which are entwined with but there are only countably many $\upsilon|n$ so that σ and υ are entwined. (This oblique view is exactly what is needed for a paraLindelöf, not paracompact space.) This observation allows us to replace $\mathcal{T}$, the family of arbitrary unions of elements from $\mathcal{B}$, with $\mathcal{U}$, the family of countable unions of elements from $\mathcal{B}$.

Mimicking the proof of normality from the previous section, let H and K be disjoint closed subsets of F, and let U, V, be disjoint open subsets of F, $H \subset U$

and $K \subset V$. For $\sigma \in \Sigma$, define $U|\sigma = \bigcup\{[\upsilon]: \upsilon \in \Sigma$, max range $\upsilon \leqslant$ max range σ, and $[\upsilon] \subset U\}$. Let $[\rho] \subset U$, $[\tau] \subset V$, and, without loss of generality $\rho(0) < \tau(0)$. We want to show that $B(\rho, \{U|\rho\}) \cap B(\tau, \{U|\tau\}) = \emptyset$. However, if, as is likely, there is $[\sigma] \subset U$ with max range $\rho <$ max range $\sigma <$ max range τ, there are g with $g(U|\rho) = 1$ and $g(U|\tau) = 0$, i.e. $B(\rho, \{U|\rho\}) \cap B(\tau, \{U|\tau\}) \neq \emptyset$.

We solve this problem by defining $B(\sigma, a_\sigma)$ to cut the cube indexed by (ρ, τ) on the coordinate $V|\rho$, for $V \in a_\sigma$, both when $\sigma \subset \rho$ and when $\sigma \subset \tau$. The proof of not collectionwise normal is slightly complicated by the fact that if $\rho \neq \rho'$, then the cube attached to $\langle \rho, \tau \rangle$ is not the same as that attached to $\langle \rho', \tau' \rangle$. We will solve this problem as we solved the similar problem with George (see Section 5.4).

Rather than dwell on the above modification, we continue to lower the character. Assuming CH we can enumerate $\mathscr{U}$ as $\{U_\alpha: \alpha < \omega_1\}$. To make the space first countable we restrict the parameter a in $B(\sigma, a)$ to the finite subsets of $\mathscr{U}^\sigma = \{U_\alpha: \alpha <$ max range $\sigma)$. Well, if $U|\sigma \in \mathscr{U}^\sigma$, things go fine, but if $U|\sigma \notin \mathscr{U}^\sigma$, we need a new idea. Moreover, unless we enumerate $\mathscr{U}$ carefully, we may lose regularity.

It is convenient introduce another method to separate basic open sets. We modify F be requiring that $f(0)$ be a limit ordinal, δ. For each limit $\delta < \omega_1$, we fix an increasing sequence δ_i, $i \in \omega$, cofinal in δ. For basic open sets $B(\rho, a)$ and $B(\tau, a')$ to intersect, we require that for $i < k$, $\rho(0)_i = \tau(0)_i$, where k depends on ρ and τ. We make a few other inessential changes to clean up the presentation of the space. For example, we deal with the countable sets, S, of basic open sets, rather than the open set, $\bigcup S$. We have made so many changes that it is time to define the space from scratch.

7.1. *Construction of M.* Let F be the set of functions, f, from ω to ω_1 such that $f(0)$ is a limit ordinal. For $n \in \omega$, set $\Sigma_n = \{f|n: f \in F\}$ and set $\Sigma = \bigcup_{n\in\omega}\Sigma_n$. For $\sigma \in \Sigma$, set $[\sigma] = \{f \in F: \sigma \subset f\}$. For each limit ordinal $\delta < \omega_1$, we fix an increasing sequence δ_i, $i \in \omega$, cofinal in δ.

Let $\mathscr{S}_n$ be the set of countable subsets of Σ_n, set $\mathscr{S} = \bigcup_{n\in\omega}\mathscr{S}_n$. By CH we can list $\mathscr{S}$ as $\{S_\alpha: \alpha < \omega_1\}$. For $\sigma \in \Sigma$, we define σ^* to be max range σ, and set $\Sigma^\sigma = \{\upsilon \in \Sigma: \upsilon^* < \sigma^*\}$. For $\sigma \in \Sigma$, list $\{S_\alpha: \alpha < \sigma^*\}$ as $\{R(\sigma, n): n \in \omega\}$. For $\sigma \in \Sigma_n$ set $A^\sigma = \{R(\sigma|j, k): j, k \leqslant n\}$. Note that if $\upsilon \subset \sigma$, then $A^\upsilon \subset A^\sigma$, and that $|A^\sigma| \leqslant (n+1)^2$.

Let Q_k be the set of triples (g, ρ, τ) which satisfy

(i) $\rho, \tau \in \Sigma_k$;

(ii) for all $i < k-1$, $\rho(i) < \tau(i) < \rho(i+1) < \tau(i+1)$; and

(iii) g is a function from $\{S_\alpha: \alpha < \rho^*\}$ to $\{0, 1\}$.

Set $Q = \bigcup_{k\in\omega}Q_k$. For $\sigma \in \Sigma_n$, define $G(\sigma)$ to be the set of triples $(g, \rho, \tau) \in Q$ satisfying

(i) either $\sigma \subset \rho$ or $\sigma \subset \tau$;

(ii) for all $i \leqslant n$, $\rho(0)_i = \tau(0)_i$; and

(iii) if $S \in A(\sigma)$ and $S \subset \Sigma_m$, then $g(S \cap \Sigma^{\rho|m+1}) = 1$ iff $\sigma|m \in s$.

The point set of space M, is $F \cup Q$. Points of Q are isolated. The other basic open sets have the form $B(\sigma) = [\sigma] \cup G(\sigma)$. It is easy to verify that if $B(\sigma) \cap B(\upsilon) \cap F \neq \emptyset$, then $\sigma \subset \upsilon$ and $B(\sigma) \supset B(\upsilon)$ (or vice versa). Hence we have given a basis.

Clearly, M is T_1. To show that M is regular, we show that $B(\sigma)$ is closed. If $x \in Q - B(\sigma)$, then $\{x\}$ is open. If $f \in F$, and $f(0) \neq \sigma(0)$ choose i so that $f(0)_i \neq \sigma(0)_i$; then $B(f|i) \cap B(\sigma) = \emptyset$. If $f \in [\sigma|1] - [\sigma]$, choose $[f|i] \cap [\sigma] = \emptyset$; then $B(f|i) \cap B(\sigma) = \emptyset$.

For $n \in \omega$, set $\mathcal{G}_n = \{B(\sigma) : \sigma \in \Sigma_n\} \cup \{\{g\} : g \in Q\}$. Then $\mathcal{G}_n$, $n \in \omega$, is a development for M. Hence M is a Moore space.

7.2. *M is normal.* Now we turn to showing that M is normal. Let H and K be disjoint subsets of M. As usual, it suffices to consider the case where $H \cup K \subset F$. Since the subspace F is a very nice space, we can expand H and K to disjoint clopen sets, $H^+ \cup K^+ = F$. (The name of this property is strongly zero dimensional). Let

$$H_n = \bigcup\{[\sigma] \in \Sigma_n : [\sigma] \subset H^+\} \quad \text{and} \quad K_n = \bigcup\{[\sigma] \in \Sigma_n : [\sigma] \subset K^+\}.$$

Let U_n be open in M and satisfy $H_n \subset U_n \subset \bar{U}_n \subset H_n \cup Q$; let V_n be open in M and satisfy $K_n \subset V_n \subset \bar{V}_n \subset K_n \cup Q$. Then

$$U = \bigcup_{n \in \omega} (U_n - \bigcup_{i \leq n} \bar{V}_n) \quad \text{and} \quad V = \bigcup_{n \in \omega} (V_n - \bigcup_{i \leq n} \bar{U}_n)$$

are disjoint open sets in M containing H and K respectively. (We call this the shoelace trick.)

In light of the above argument, we see that to show that M is normal, it suffices to separate for each $n \in \omega$ and each $S \subset \Sigma_n$, the sets $H_S = \bigcup\{[\sigma] : \sigma \in S\}$ and $K_S = F - H_S$. Fix n, S, H_S, K_S.

Define $C = \{\gamma \in \omega_1 :$ if $\sigma^* < \gamma$ then $S \cap \Sigma^\sigma = S_\alpha$ for some $\alpha < \gamma\}$. C is closed and unbounded in ω_1. For $\beta \in \omega_1$ define $\gamma(\beta)$ to be the least element of C greater than β. For $\gamma \in C$, define m_γ from $\{\delta \in \gamma : \delta \in \omega_1 \cap \mathrm{LIM}\}$ to ω so that if $\delta \neq \eta$, then for some $i < \max\{m_\gamma(\delta), m_\gamma(\eta)\}$, $\delta_i \neq \eta_i$.

Our plan for separating H_S and K_S uses two tactics. We will use C to divide F into 'fast' functions and 'slow' functions. To separate two fast functions we place $S \cap \Sigma^\sigma$ in $A(\sigma')$ where σ' extends σ. To separate two slow functions we use the m_γ's. What about separating a fast function from a slow function? Because of the entwining requirement, we need only worry about slow fast functions and fast slow functions. Both the fast function argument and the slow function arguments work in such a case.

For $\sigma \in \Sigma_{n+3}$, define $j(\sigma) \geq n+3$ to satisfy $j(\sigma) \geq m_{\gamma(\sigma(n+1))}(\sigma(0))$, and if $\gamma(\sigma(n)) < \sigma(n+2)$, then $S \cap \Sigma^{\sigma(n)} = R(\sigma, k)$ for some $k < j(\sigma)$. Set $W_\sigma = \bigcup\{B(\rho) : \sigma \leq \rho \in \Sigma_{j(\sigma)}\}$. To separate H_S and K_S, it will suffice to show that if σ, $\upsilon \in \Sigma_{n+3}$ where $\sigma \in S$ and $\upsilon \notin S$, then $W_\sigma \cap W_\upsilon = \emptyset$. Aiming for a contradiction, assume that $(g, \rho, \tau) \in B_{\sigma'} \cap B_{\upsilon'}$, where $\sigma \subset \sigma' \in \Sigma_{j(\sigma)}$ and $\upsilon \subset \upsilon' \in \Sigma_{j(\upsilon)}$.

We have either $\sigma \subset \rho$ or $\sigma \subset \tau$. Without loss of generality, we assume $\sigma \subset \rho$ and hence also $\upsilon \subset \tau$.

Case 1. $\gamma(\upsilon(n)) < \sigma(n+2)$ (i.e. σ and υ are 'fast'). Then $S \cap \Sigma^{\sigma|n+1} \in A_{\sigma'}$ and $S \cap \Sigma^{\upsilon|n+1} \in A_{\upsilon'}$. Since $\sigma|n \in S$ and $\upsilon|n \notin S$, we get $1 = g(S \cap \Sigma^{\rho|n+1}) = 0$.

Case 2. $\sigma(n+2) \leq \gamma(\upsilon(n))$ (i.e. σ and υ are 'slow'). Then $\gamma(\sigma(n+1)) = \gamma(\upsilon(n+1))$. From the definition of $j(\sigma)$, we get that for some $i < \max\{j(\sigma), j(\upsilon)\}$, $\rho(0)_i \neq \tau(0)_i$. Hence $(g, \rho, \tau) \notin B(\sigma') \cap B(\upsilon')$.

7.3. *M is not collectionwise normal.* While the proof of normality for M above was rather more complex than for N, the proof of not collectionwise normal for M is basically the same as for N. We begin by replacing the notion of full with the notion stafull. We say that a subset, S of Σ_n is *stafull* if for all $\sigma \in S$ and $j < n$, the set $\{\tau(j): \sigma|j \subset \tau \in S\}$ is stationary.

7.4. LEMMA. (a) *If* $\bigcup\{[\sigma]: \sigma \in T\} = F$, *then for some* n, $T \cap \Sigma_n$ *has a stafull subset.*

(b) *Suppose that* $S \subset \Sigma_n$ *is stafull, and that* $h: S \to \omega_1$ *satisfies* $h(\sigma) < \sigma(0)$ *for all* $\sigma \in S$. *Then h is constant on a stafull subset* S' *of S.*

(c) *If* $S' \subset \Sigma_n$ *is stafull, then there is* $W = \{\sigma_k: k \in \omega\}$ *such that each pair of elements of W is entwined.*

PROOF. The same as Lemma 6.4. □

Following the proof in 6.3, for each limit $\delta < \omega_1$, we set $Y_\delta = \{f \in F: f(0) = \delta\}$. Now, $\mathcal{Y} = \{Y_\delta: \delta \in \omega_1 \cap \mathrm{LIM}\}$ is discrete. We will assume that $\mathcal{U} = \{U_\delta: \delta \in \omega_1 \cap \mathrm{LIM}\}$ separates $\mathcal{Y}$ and reach a contradiction. Let $T = \{\sigma \in \Sigma: B(\sigma) \subset U_{\sigma(0)}\}$. By part (a) of Lemma 7.4, for some $n \in \omega$, $T \cap \Sigma_n$ contains a stafull set, S. Apply part (b) $n+1$ times to get a stafull $S' \subset S$ such that for all σ, $\upsilon \in S'$ and $i \leq n$, $\sigma(0)_i = \upsilon(0)_i$. Next, apply part (c) to get $W = \{\sigma_m: m \in \omega\}$ where every pair is entwined.

Let us set $\rho = \sigma_0$, $l = (n+1)^2$, and $\Delta = \{g: g$ is a function from Σ^ρ to $\{0, 1\}\}$. For $0 < m < \omega$, let $D_m = \{g \in \Delta: (g, \rho, \sigma_m) \in B(\sigma_m)\}$. At this point, we must verify that condition (iii) of the definition of $G(\sigma)$ cannot make $G(\sigma)$ empty by requiring contradictory conditions. This verification is done by noting that if ρ, σ are entwined; z_1, $z_2 \in A(\sigma) \cap \Sigma_m$; and $z_1 \cap \Sigma^{\rho|m+1} = z_2 \cap \Sigma^{\rho|m+1}$; then $\sigma|m \in z_1$ iff $\sigma|m \in z_2$ because $(\sigma|m)^* = \sigma(m-1) < \rho(m) = (\rho|m+1)^*$. Since D_m is defined by fixing at most l of the values of g, D_m is a subset of Δ of measure $\geq 2^{-l}$. Choose m, m', and g satisfying $m < m'$, and $g \in D_m \cap D_{m'}$. Define g', a function from Σ^{σ_m} to $\{0, 1\}$, so that if $S \in \Sigma_k$ then $g'(S \cap \Sigma^{\sigma_m|k}) = g(S \cap \Sigma^{\rho|k})$. Then $(g', \sigma_m, \sigma_{m'}) \in B(\sigma_m) \cap B(\sigma_{m'})$. This contradicts the assumption that $\mathcal{U}$ is disjoint, and establishes that M is not collectionwise normal.

8. Large cardinals again

Let us review the situation. We can construct a normal, nonmetrizable Moore space assuming either MA + $\neg$CH or CH. We can prove NMSC assuming PMEA. Thus settling NMSC assuming ZFC is impossible. This situation is unfortunate in the opinion of many mathematicians. The 'next best thing' would be that NMSC is consistent with and independent of ZFC. The problem is that the consistency of at least a measurable cardinal is needed to show the consistency of PMEA. The attempts to prove NMSC with non large cardinal versions of PMEA were unsuccessful. Theorem 8.2, below, explains why. The contrapositive of (a)$\rightarrow$(e) of Theorem 8.2 is "if all normal Moore spaces are metrizable, then there is an inner model with a measurable cardinal". Since the existence of an inner model with a measurable cardinal implies Con(ZFC + $\exists$ a measurable cardinal), we can apply Gödel's Second Incompleteness Theorem. Either the existence of a measurable cardinal is inconsistent with the axioms of set theory or any proof of the Normal Moore Space Conjecture must start with a large cardinal assumption. While many hope for the first alternative, it seems to me that this hope is mostly a wish for large cardinal consistency results go away and let the nonlogician mathematicians prove or disprove outright their conjectures. I believe we are stuck with the second alternative, even though it is aesthetically displeasing.

There is no avoiding the fact that the proof of Theorem 8.2 is deep, long, and very technical. Some will be satisfied with the statement of the theorem; others will be restless and want to know something about the proof. For the latter, the implication from (a) to (e) has been split into four steps. Implication (a) to (b) is beyond the scope of this article, but the other implications are accessible if one accepts that K, the core model, satisfies many of the same combinatorial properties as L, the class of constructible sets; in particular, the axioms GCH and $\forall\kappa\ \square_\kappa$.

8.1. *Definitions.* (a) K, *the core model*, is a class model of set theory, satisfying $L \subseteq K \subseteq V$. If $V = L$, then of course, $L = K = V$. If there is a measurable cardinal, then $L \neq K \neq V$. In this case, L can be built up to K by unioning the mice (this is more general than iterating the sharps); and V can be cut down to K by intersecting all transitive models, M, of set theory such that M contains all ordinals and M is a model of $\exists$ a measurable cardinal. ([KANAMORI and MAGIDOR 1978, §§0, 1, 29] and [DODD 1983] are good places to start learning about all this.)

(b) $\square_\kappa$: there is a sequence $(C_\lambda : \lambda \in \kappa^+ \cap \text{LIM})$ such that

(i) C_λ is a closed unbounded subset of λ;

(ii) if $\text{cf}(\lambda) < \kappa$, then order type $C_\lambda < \kappa$;

(iii) if γ is a limit point of C_λ, then $C_\gamma = \gamma \cap C_\gamma$.

(c) $E(\lambda)$: there is a set E, consisting only of ordinals of cofinality ω, such that E is stationary in λ, but $E \cap \beta$ is not stationary in β for all $\beta < \lambda$.

Let us remark here that $E(\omega_1)$ is always true; let $E = \omega_1 \cap \text{LIM}$. Also, ω is a strong limit cardinal of cofinality ω. Thus CH implies condition (d) of Theorem 8.2.

Section 8 is summarized in Theorem 8.2.

8.2. THEOREM. *Each of the following statements implies the next.*

(a) *No inner model of set theory has a measurable cardinal.*

(b) *Covering(V, K): For every uncountable set X of ordinals there exists a core model set $Y \supset X$ such that $|Y| = |X|$ and $Y \in K$.*

(c) *For all singular strong limit cardinals κ, $2^\kappa = \kappa^+$ and there is an $E(\kappa^+)$ set.*

(d) *For some strong limit cardinal κ of cofinality ω, $2^\kappa = \kappa^+$ and there is an $E(\kappa^+)$ set.*

(e) *There is a normal nonmetrizable Moore space.*

PROOF. (a)$\to$(b) This is a very hard and technical theorem. See [DODD and JENSEN 1982].

(b)$\to$(c). The first application of covering was that Covering(V, L) implies for all strong limit cardinals κ, $2^\kappa = \kappa^+$. Any model of GCH, for example K, can be used in place of L.

Let κ be a singular strong limit cardinal, and set $\mathcal{P} = [\kappa]^\lambda = \{B \subset \kappa : |B| = \lambda\}$. The first task is to show that $|\mathcal{P}| = 2^\kappa$, (see [JECH 1978, Lemma 6.5]), and then it will suffice to show that $|\mathcal{P}| = \kappa^+$. Let (κ_α), $\alpha < \kappa$, be a sequence of cardinals cofinal in κ such that if $\alpha < \beta$, then $2^{\kappa_\alpha} < \kappa_\beta$. For $\alpha < \lambda$, list $2^{\kappa_\alpha} = \{A^\alpha_\beta : \beta < \kappa_{\alpha+1}\}$. We code each subset $A \subset \kappa$ by the function $f_A : \lambda \to \kappa$ defined so that, if $\beta = f_A(\alpha)$, then $A^\alpha_\beta = A \cap \kappa_\alpha$. Now f_A is a subset of $\lambda \times \kappa$, and via a fixed bijection between $\lambda \times \kappa$ and κ, we have a one to one map from $\mathcal{P}(\kappa)$ to $\mathcal{P}$.

By Covering(V, K), for each $B \in \mathcal{P}$ there is $C \in \mathcal{P}$, $B \subset C$ and $C \in K$ (the core model). Since for each $C \in \mathcal{P}$, $|\{B \in \mathcal{P} : B \subset C\}| < \kappa$, it suffices to show that $|\mathcal{P} \cap K| = \kappa^+$. $\mathcal{P} \cap K$ is just "K's idea of $\{B < \kappa : |B| = \lambda\}$", so using GCH in K, we obtain $|\mathcal{P} \cap K| = (\kappa^+)^K$, "$K$'s ideas of the successor of κ". Now any bijection between κ and α in K is a bijection between κ and α in V (the class of all sets). So $|\mathcal{P} \cap K| = (\kappa^+)^K \leq \kappa^+$.

Next, we show that in fact $(\kappa^+)^K = \kappa^+$. Let $\kappa \leq \alpha < \kappa^+$. Since κ is singular, there is a subset A cofinal in α, $|A| < \kappa$. Applying Covering(V, K), get B in K, $|B| < \kappa$, B cofinal in α. So α is singular in K, hence $\alpha \neq (\kappa^+)^K$.

Having shown that $2^\kappa = \kappa^+$, we now show $E(\kappa^+)$. The proof of $\square_\kappa$ from $V = L$ [DEVLIN 1977, p. 474], and probably also the proof from $V = K$, constructs an E satisfying $E(\kappa^+)$ (see 8.1) at the same time. E is stationary in κ^+, contains only ordinals of cofinality ω, and the conclusion of 8.1(b) (iii) is strengthened to include $\gamma \notin E$.

Let $\mathcal{C} = (C_\lambda : \lambda \in (\kappa^+)^K \cap \mathrm{LIM})$ satisfy $\square_\kappa$ in K. We ask whether $\mathcal{C}$ satisfies $\square_\kappa$ in V. Conditions (i), (ii), (iii), are absolute. We checked above that $(\kappa^+)^K = \kappa^+$, so $\mathcal{C}$ satisfies $\square_\kappa$ in V. What we really want is the set E satisfying $E(\kappa^+)$ in V, but 'E stationary in κ^+' in K does not imply 'E stationary in κ^+' in V, i.e. there might be $C \in V - K$ that is closed, unbounded in κ^+ and disjoint from E.

Fortunately, we can prove the existence of the desired E from $\square_\kappa$. For $\delta \in \{\alpha: \kappa < \alpha < \kappa^+ \ \&\ \mathrm{cf}(\alpha) = \omega\}$, define $f(\delta) =$ order type C_δ. By (ii) $f(\delta) < \kappa < \delta$, so there is a stationary set E on which f is constant. It is easy to see that $E \cap \beta$ is not stationary in β for $\beta < \kappa^+$ because $C_\beta \cap E$ contains at most one limit of C_β.

(c)→(d) Trivial.

(d)→(e) As we pointed out before stating Theorem 8.2, CH implies (d). In fact, (d) is just an abstraction from CH of what is needed for the construction of Section 7. We sketch below the slight modifications needed to use 8.2(d) in place of CH. We replace ω_1 with κ^+ and $\omega_1 \cap \mathrm{LIM}$ with E. For example, we start by setting $F = \{f: f$ is a function from ω to κ^+ such that $f(0) \in E\}$. We use $2^\kappa = \kappa^+$ to enumerate $\mathcal{S}$, the collection of sets of $<\kappa^+$ basic open sets of F, as $\{S_\alpha: \alpha < \kappa^+\}$. Then for each $\sigma \in \Sigma$, $|\{S_\alpha: \alpha < \sigma^*\}| \leq \kappa$ and so $\{S_\alpha: \alpha < \sigma^*\}$ is a union of countably many sets of cardinality $<\kappa$. We use the fact that $E \cap \beta$ is not stationary in β, for $\beta < \kappa^+$, to deal with the slow functions in the proof of normality. In place of the measure argument used in showing not collectionwise normal, we use

8.4. LEMMA. *Let $\rho < 2^\rho < \lambda < \kappa$, and, for $\alpha < \lambda$, let D_α be a subset of the functions from a set of cardinality κ to $\{0, 1\}$ defined by fixing $\leq\rho$ values. Then $\{D_\alpha: \alpha < \lambda\}$ is not disjoint.*

Lemma 8.4 is an easy consequence of the infinitary Δ-system lemma. See [KUNEN 1979, Theorem 1.6], or [JECH 1978, Lemma 24.4].

If the above concepts are unfamiliar to the reader, it is easy to be distracted by them. We reemphasize that essentially the same construction as in Section 7 works assuming (d). For details, see [FLEISSNER 1983]. □

Let κ be a singular strong limit cardinal of cofinality ω. Mitchell has constructed inner models with type-1 measurables (hypermeasurables, respectively) from $2^\kappa > \kappa^+$ (not $E(\kappa^+)$, respectively). We don't describe these results here for several reasons. First, the large cardinal definitions are rather technical. Second, the results haven't been published. Third, progress in this area has been rapid, and current results are likely to be soon out of data.

In my opinion, the 'cleanest' resolution still possible is for ZFC + NMSC to be equiconsistent with ZFC + ∃ strongly compact cardinal. It seems that this result will have to wait for the development of the theory of inner models with strongly compact cardinals.

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CHAPTER 17

Dowker Spaces

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
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By a *space* we mean a Hausdorff topological space. A *Dowker space* is a normal space whose Cartesian product with the ordinary closed unit interval I is not normal.

The property of a space X, called binormal, that $X \times I$ be normal, had long been a standard hypothesis for certain homotopy extension theorems; one, of course, wondered whether this said more than that X is normal. HUGH DOWKER [1951] proved that for normal spaces X, $X \times I$ is normal if and only if X is countably paracompact. Dowker's proof gave a useful combinatorial structure for spaces which are normal but not countably paracompact and again raised the issue of the existence of such spaces; thus any normal but not countably paracompact space soon became known as a Dowker space. M.E. RUDIN [1955] showed that the existence of a Souslin line implied the existence of a Dowker space; nothing was known at this time about the existence of either class of spaces. However in the mid 1960's, (SOLOVAY, TENNENBAUM [1971]) the existence of a Souslin line was shown to be independent of Zermelo-Franekl Set Theory; thus the existence of a Dowker space was known to be consistent with the usual axioms for set theory. In RUDIN [1970] a Dowker space is constructed without using any special set theoretic assumptions. This space is still the only known 'real' Dowker space; however it is neither separable nor 1st-countable, and it has cardinality $(\omega_\omega)^\omega$; it is pathological in many ways. Knowledge of the existence of Dowker spaces having additional properties, especially those having various cardinal functions (such as density, character, and weight) countable, is often basic to the solution of other, sometimes apparently unrelated, problems. So the search for 'nice' Dowker spaces continues and many types have been constructed under a wide variety of set theoretic assumptions.

In spite of the existence of Dowker spaces, MORITA [1974] and STARBIRD [1974] showed that the '$X \times I$ normal' assumption in the old homotopy extension theorem is unnecessary. Dowker spaces are clearly part of the broader question of normality in products [see Chapter 18]; but they are sufficiently important and have such a large literature that we devote a special section to them.

We begin by proving Dowker's old theorem for it gives us the basic structure needed throughout. We then present a few theorems which tell us some of the circumstances under which there are *no* Dowker spaces. These theorems are mostly very simple ones: either we do not yet have an effective technique for killing Dowker spaces, or they exist with more additional properties than we would wish! Next, we present a variety of examples of Dowker spaces constructed under a variety of set theoretic assumptions. We close with the Morita–Starbird theorem and some related topics.

1. Dowker's theorem

1.1. THEOREM. *For a normal space X, the following are equivalent*:

(i) *If* $D_1 \supset D_2 \supset \cdots$ *is a nested sequence of closed sets in* X *with* $\bigcap\{D_n \mid n \in \omega\} = \emptyset$, *then, for each* $n \in \omega$, *there is an open* $U_n \supset D_n$ *with* $\bigcap\{U_n \mid n \in \omega\} = \emptyset$.

(ii) X *is countably paracompact*; *in fact, if* $\{V_n \mid n \in \omega\}$ *is an open cover of* X, *then there is an open cover* $\{W_n \mid n \in \omega\}$ *of* X *such that* $\bar{W}_n \subset V_n$ *for each* n.

(iii) $X \times Y$ *is normal for all infinite, compact, metric* Y.

(iv) $X \times Y$ *is normal for some infinite, compact, metric* Y.

PROOF. (i) *implies* (ii). Suppose that $\{V_n \mid n \in \omega\}$ is an open cover of X. If $D_n = X - \bigcup\{V_i \mid i \leq n\}$, then D_n is closed and $\bigcap\{D_n \mid n \in \omega\} = \emptyset$; so, for each n, there is an open $U_n \supset D_n$ with $\bigcap\{U_n \mid n \in \omega\} = \emptyset$. Since X is normal we can find an open Y_n with $D_n \subset Y_n \subset \bar{Y}_n \subset U_n$; then $\bigcap\{\bar{Y}_n \mid n \in \omega\} = \emptyset$. Let $Z_n = X - \bigcap\{\bar{Y}_i \mid i \leq n\}$; $\{Z_n \mid n \in \omega\}$ is an open cover of X and $\bar{Z}_n \subset \bigcup\{V_i \mid i \leq n\}$.

Since $\{V_n - \bigcup\{\bar{Z}_i \mid i < n\} \mid n \in \omega\}$ is a locally finite open refinement of $\{V_n \mid n \in \omega\}$ which covers X, we have proved that X is countably paracompact.

We want to prove that there exist W_n's as in (ii) if (i) or if X is countably compact.

If X is countably paracompact and $\{V_n \mid n \in \omega\}$ is an open cover of X, there is a locally finite open refinement $\mathscr{Z}$ of $\{V_n \mid n \in \omega\}$ and if $Z_n = X - \bigcup\{\bar{Z} \mid Z \in \mathscr{Z}$ and $Z \cap (X - V_n) \neq \emptyset\}$, then $\{Z_n \mid n \in \omega\}$ is an open cover of X and $\bar{Z}_n \subset \bigcup\{V_i \mid i \leq n\}$.

Thus we can assume in any case that $\{V_n \mid n \in \omega\}$ is an open cover of a countably paracompact, normal space X and that $\{Z_n \mid n \in \omega\}$ is an open cover of X with $\bar{Z}_n \subset \bigcup\{V_i \mid i \leq n\}$.

If M is a finite subset of ω, let $\mathscr{A}_M = \{A \mid$ there is an $n \in \omega$ such that $A = \{x \in Z_n \mid x \in V_m$ iff $m \in M\}\}$; $\mathscr{A}_M$ is a countable family of open sets. By induction on $|M|$, for each finite $M \subset \omega$ and $A \in \mathscr{A}_M$, choose an open set $S_A \subset \bar{S}_A \subset \bigcap\{V_m \mid m \in M\}$ such that the closed set $(A - \bigcup\{S_B \mid B \in \mathscr{A}_N$ for some $N \subset \omega$ with $|N| < |M|\}) \subset S_A$. Since X is normal and $A \subset \bigcap\{V_n \mid m \in M\}$ this is possible. Since $\{S_A \mid A \in \mathscr{A}_M$ for some finite $M \subset \omega\}$ is a countable open cover of the countably paracompact space X, it has a locally finite refinement $\mathscr{W}$. If we let $W_n = \bigcup\{W \in \mathscr{W} \mid n = \min\{m \in \omega \mid \bar{W} \subset V_n\}\}$, then $\{W_n \mid n \in \omega\}$ is a locally finite refinement of $\{V_n \mid n \in \omega\}$ with $\bar{W}_n \subset V_n$ for all n, as desired.

(ii) *implies* (iii). Suppose that Y is an infinite compact metric space and that H and K are disjoint closed subsets of $X \times Y$. Our goal is to find an open set containing H whose closure misses K. Let $\{B_1, B_2, \ldots\}$ be the set of all finite unions of members of some countable basis for Y.

For each $n \in \omega$, let $V_n = \{x \in X \mid y \in B_n$ if $\langle x, y\rangle \in K$ and $y \in (Y - \bar{B}_n)$ if $\langle x, y\rangle \in H\}$. If $x \in X$, $x \in V_n$ for some n. Since Y is compact, if $x \in V_n$ there is an open set containing x in V_n. So $\{V_n \mid n \in \omega\}$ is an open cover of X, and, by (2), there is a locally finite open refinement $\{W_n \mid n \in \omega\}$ of $\{V_n \mid n \in \omega\}$ with $\bar{W}_n \subset V_n$ for each n. Since $\{W_n \mid n \in \omega\}$ is locally finite, if $U = \bigcup\{W_n \times B_n \mid n \in \omega\}$, $\bar{U} \subset \bigcup\{V_n \times \bar{B}_n \mid n \in \omega\}$. So U is an open set containing H whose closure misses K, as desired.

(iii) implies (iv) trivially; so we only need the proof of:

(iv) *implies* (i). Suppose that $D_0 \supset D_1 \supset \cdots$ is a nested sequence of closed sets from X with $\bigcap\{D_n \mid n \in \omega\} = \emptyset$.

If there is an infinite, compact metric space Y for which $X \times Y$ is normal, choose a limit point y from Y and a sequence $y_0, y_1, \ldots$ of distinct points from Y converging to y. The sets $H = \bigcup\{D_n \times \{y_n\} \mid n \in \omega\}$ and $K = (X \times \{y\})$ are closed and disjoint in $X \times Y$; so, by the normality of $X \times Y$, there is an open set U in $X \times Y$ with $H \subset U$ and $K \cap \bar{U} = \emptyset$. Since the projection map is open, $U_n = \{x \in X \mid \langle x, y_n\rangle \in U\}$ is open in X. Observe that $U_n \supset D_n$. And, if $x \in X$, since $\langle x, y\rangle \in K$, there is an open neighborhood B of y in Y with $(\{x\} \times B) \cap U = \emptyset$. There is an n with $y_n \in B$; thus $x \notin U_n$; hence $\bigcap\{U_n \mid n \in \omega\} = \emptyset$, as desired.

An immediate consequence of Theorem 1.1 is:

1.2. THEOREM. *A space X is a Dowker space iff X is normal and*:

($*$) *X has a sequence $D_0 \supset D_1 \supset \cdots$ of closed sets with $\bigcap\{D_n \mid n \in \omega\} = \emptyset$ and, whenever $\{U_n \mid n \in \omega\}$ is a family of open sets with $D_n \subset U_n$, then $\bigcap\{U_n \mid n \in \omega\} \neq \emptyset$.*

(*Or equivalently, X is normal and not countably paracompact.*)

2. Classes of spaces which cannot be Dowker

There are, of course, no metric or compact or T_3-Lindelöf Dowker spaces, for all such spaces are paracompact. There is no countably compact or pseudocompact Dowker space since a set $\{x_n \mid n \in \omega\}$ with $x_n \in D_n$ of ($*$) has no limit point. But there are consistency examples of Dowker spaces which are separable, even hereditarily separable, 1st countable, even locally compact and locally countable, or cardinality and weight ω_1, screenable, hereditarily normal, collectionwise normal, extremely disconnected .... Are there any nontrivial classes of spaces which can never be Dowker?

The fact that there are no perfect Dowker spaces (Theorem 2.1) is an important theorem even though it is easy to prove. This tells us, for instance, that there are no Moore Dowker spaces. That there are no linearly ordered Dowker spaces (Theorem 2.2) is not so easy to prove or important; but its proof, partitioning the space into equivalence classes, is typical and usually a feature of every theorem and counterexample in this area. That there are no monotonically normal Dowker spaces (Theorem 2.3) is a harder, more modern, theorem which implies Theorem 2.2 and again uses an equivalence class proof. It is used primarily as an illustration of when such theorems might be useful. DIANA PALENZ [1980] was trying to prove that certain classes of monotonically normal spaces were paracompact; she could prove that her spaces were screenable; normal, screenable spaces are paracompact if they are countably paracompact; so she needed to prove her spaces were not Dowker; Theorem 2.3 does the trick.

2.1. THEOREM. *There is no perfect Dowker space.*

PROOF. Recall that a space is perfect if every closed set is a G_δ-set.

Suppose that X is a normal space satisfying (*) and that each D_n is a G_δ-set. Then we can find a family $\{V_{ni} \mid i \in \omega\}$ of open sets whose intersection is D_n.

If $U_n = \bigcap\{V_{mi} \mid i \leqslant n,\ m \leqslant n\}$, $\{U_n \mid n \in \omega\}$ is a family of open sets with $D_n \subset U_n$ and $\bigcap\{U_n \mid n \in \omega\} = \bigcap\{D_n \mid n \in \omega\} = \emptyset$; which contradicts (*).

2.2. THEOREM. *There is no linearly ordered Dowker space.*

PROOF. Suppose that X is a Dowker space, that $(X, \leqslant)$ is a totally ordered set, and that there is a basis for X consisting of intervals from $(X, \leqslant)$. (U is an interval in $(X, \leqslant)$ if $U \subset X$ and there are no x and y in U and z in $X - U$ with $x < z < y$).

If $\{D_n \mid n \in \omega\}$ satisfies (*), for each $n \in \omega$ let $\mathscr{J}_n$ be the set of all maximal intervals of $(X, \leqslant)$ which do not intersect D_n. Let $\mathscr{S}$ be the set of all sequences $J_0 \subset J_1 \subset \cdots$ where $J_N \in \mathscr{J}_n$ for each n. If $J_0, J_1, \ldots$ and $K_0, K_1, \ldots$ are in $\mathscr{S}$, we say they are 'equivalent' if there is an n such that $J_m = K_m$ for all $m \geqslant n$. Let $\mathscr{L}$ be a maximal family of nonequivalent members of $\mathscr{S}$.

If $x \in X$ let $i_x = \min\{j \mid x \notin D_j\}$ and, for $n \geqslant i_x$, let $J_n(x)$ be the term of $\mathscr{J}_n$ containing x. There is a unique $K_0, K_1, \ldots, \in \mathscr{L}$ and an $m > i_x$ in ω such that $J_n(x) = K_n$ for all $n \geqslant m$. Choose an open interval U_x in $(X, \leqslant)$ such that $x \in U_x \subset J_{i_x}(x)$ and, if $n < i_x$ and $x \notin \bar{K}_n$, make sure that $U_x \cap \bar{K}_n = \emptyset$.

If $U_n = \bigcup\{U_x \mid x \in D_n\}$ then U_n is certainly an open set containing D_n; we claim $\bigcap\{U_n \mid n \in \omega\} = \emptyset$ contrary to (*).

To see this suppose that $x \in \bigcap\{U_n \mid n \in \omega\}$. If $K_0, K_1, \ldots$ and m are as above, the interval K_m of $(X, \leqslant)$ can have at most two end points and $K_m \subset (X - D_m)$; so there is $k > m$ in ω such that $\bar{K}_m \subset (X - D_k)$. By assumption, $x \in U_y$ for some $y \in D_k$. Since $m < i_y$, and $U_y \subset J_{i_y}(y)$ and $x \in U_j$, $J_{i_y}(y) = J_{i_y}(x) = K_{i_y}$. And, since and K's and J's are nested, $J_n(y) = K_n$ for all $n \geqslant i_y$. Thus, by the definition of U_y, since $y \notin \bar{K}_m$, $U_y \cap \bar{K}_m = \emptyset$. But this contradicts $x \in U_y$ since $x \in J_m(x) = K_m$.

2.3. THEOREM. *There is no monotonically normal Dowker space.*

PROOF. A space X is monotonically normal if associated with every $x \in X$ and open neighborhood U of x there is an open neighborhood $H(x, U)$ of x contained in U such that:

(a) if $x \neq y$ in X, then $H(x, X - \{y\}) \cap H(y, X - \{x\}) = \emptyset$, and

(b) if $x \in U \subset V$, then $H(x, U) \subset H(x, V)$.

Observe that X is normal and that $\overline{H(x, U)} \subset U$.

In order to prove the contrary, we assume that (*) holds. For $x \in X$, let $i_x = \min\{j \mid x \notin D_j\}$. Define $V_0(x) = H(x, X - D_{i_x})$. And, by induction, for $n > 0$, define $V_n(x) = H(x, V_{n-1}(x))$.

Let $\mathscr{S}$ be the set of all subsets $\{x_n \mid n \in M\}$ of X where M is an infinite subset of

ω, $i_{x_n} = n$ for all $n \in M$, and $x_n \in V_{m-n}(x_m)$ for all $m > n$ in M. For $S = \{x_n \mid n \in M\} \in \mathcal{S}$, define

$$S' = \{x \in X \mid x \in V_{n-i_x}(x_n) \text{ for all } n > i_x, n \in M\}.$$

Index $\mathcal{S}$ as $\{S_\alpha \mid \alpha < \lambda\}$ for some ordinal λ; we now choose a subset Λ of λ by transfinite induction: If $\beta \cap \Lambda$ has been chosen for some $\beta \in \lambda$, we put β into Λ if and only if $S_\beta \cap S'_\alpha = \emptyset$ for all $\alpha < \beta$, $\alpha \in \Lambda$.

If $x \in X - \bigcup\{S'_\alpha \mid \alpha \in \Lambda\}$, define $W(x) = H(x, V_{i_x}(x))$.

If $x \in S'_\alpha$ and α is minimal for $x \in S'_\alpha$ and $S_\alpha = \{x_n \mid n \in \omega\}$, define $W(x) = H(x, (V_{i_x}(x) - \bigcup\{\overline{V_0(x_n)} \mid n < i_x\}))$. Since $\overline{V_0(x_n)} \subset (X - D_n)$, and $x \in D_n$ for all $n < i_x$, $W(x)$ is well defined.

If $U(x) = H(x, W(x))$ and $U_n = \bigcup\{U(x) \mid x \in D_n\}$, then U_n is an open set containing D_n and we claim that $\bigcap\{U_n \mid n \in \omega\} = \emptyset$ contrary to (*).

Suppose on the contrary that $x \in \bigcap\{U_n \mid n \in \omega\}$. Then for every n, $x \in U(p_k)$ for some $p_k \in D_n$. Since $\bigcap\{D_n \mid n \in \omega\} = \emptyset$, there is an infinite $K \subset \omega$ with $i_{p_k} = k$ and $x \in U(p_k)$ for all $k \in K$.

If $k < j$ in K, $p_j \in D_k$ and $U(p_k) \subset H(p_k, X - D_k)$ while $U(p_j) \subset H(p_j, W(p_j))$. Since $x \in U(p_j) \cap U(p_k)$, $p_k \in W(p_k)$. Since $V_{j-k}(p_j) \subset V_j(p_j) \subset W(p_j)$, $\{p_k \mid k \in K\} \in \mathcal{S}$.

Let α be minimal in Λ for $S'_\alpha \cap \{p_k \mid k \in K\} \neq \emptyset$; choose $k \in K$ with $p_k \in S'_\alpha$. If $S_\alpha = \{x_n \mid n \in M\}$, $p_k \in V_{n-k}(x_n)$ for $n \in M$, $n > k$.

If $k < j$ in K and $n \in M$ with $n > j$, then $x_n \in D_j$ and $V_{j-k}(p_j) \subset H(p_j, X - D_j)$ while $V_{n-k}(x_n) \subset H(x_n, V_{n-j}(x_n))$. Since $p_k \in V_{j-k}(p_j) \cap V_{n-k}(x_n)$, $p_j \in V_{n-j}(x_n)$. Thus, for all $j > k$ in K, $p_j \in s'_\alpha$ and α is minimal in Λ for $p_j \in S'_\alpha$.

Choose $m \in M$ with $m > k$ and $j \in K$ with $j > m$. We have shown that $p_k \in W(p_j)$; and $p_k \in V_{m-k}(x_m) \subset V_0(x_m)$. But by the definition of $W(p_j)$, $W(p_j) \cap V_0(x_m) = \emptyset$ which is a contradiction.

3. Some Dowker spaces

To construct a Dowker space one must find a normal space X with D_n's which satisfy (*).

If $n \in \omega$, let $E_n = \{x \in X \mid n \text{ is minimal for } x \notin D_n\}$. The E_n's partition X into layers. Since $(X - D_n)$ is an open set containing E_n, basic neighborhoods 'look back'. In order to achieve (*), i.e., to guarantee that for all open $U_n \supset D_n$, $\bigcap\{U_n \mid n \in \omega\} \neq \emptyset$, every open U_n must contain all of $\bigcup\{E_i \mid i \leq n\}$ except for a 'trivial' set; and trivial must be defined in such a way that the union of countably many trivial sets is trivial.

The most elementary idea is to take each E_n to have cardinality ω_1 and to define 'trivial' to be countable. This is done in RUDIN [1955] with each E_n being a Souslin tree, and again in RUDIN [1973] where a hereditarily separable, 1st countable X is constructed. In JUHÁSZ, KUNEN, RUDIN [1976] CH is used to construct a Dowker space where each E_n is a Lusin subset of the real line. But in DE CAUX [1976], using $\clubsuit$ and the OSTASZEWSKI [1976] technique and taking each E_n to be a simple copy of ω_1, the technique is clarified and made easy to use.

Since Souslin trees, CH, and $\clubsuit$ are all consequences of the set theoretic assumption $\diamondsuit$ whose consequences are often denied by MA + $\neg$CH, one might hope that MA + $\neg$CH would kill 'small' Dowker spaces. However WEISS [1981] and BELL [1981] have killed this hope by constructing 1st countable (even separable and locally compact in Weiss's case) Dowker spaces whose existence is consistent with MA + $\neg$CH (and implied by it in Bell's case). As one should expect, the cardinality of each E_n is 2^ω here, $2^\omega > \omega_1$, and 'trivial' means of cardinality less than 2^ω. But let us actually look at some constructions.

3.1. *Some de Caux type Dowker spaces*

In each case we make an assumption which is consistent with ZFC.

(i) Assume $\clubsuit$. *For every limit α in ω_1 there is a simple sequence S_α (i.e. order isomorphic with ω), cofinal with α, such that, if S is an uncountable subset of ω_1, then there is a limit α with $S_\alpha \subset S$.*

For each α, partition S_α into infinite, disjoint subsets $\{S_{\alpha ijn} \mid i, j, n \in \omega\}$. Let $X = \omega_1 \times \omega$.

Topologize X by declaring a set $U \subset X$ to be open if and only if, for every $\langle \alpha + j, n \rangle \in U$ where α is a limit in ω_1 and n and j are in ω, and every $i \leqslant n$, there is a cofinite subset S^i of $S_{\alpha ijn}$ such that $\{\langle \beta, i \rangle \mid \beta \in S^i\} \subset U$.

It is trivial to check that this is indeed a T_1 topology on X. To check other Dowker properties, observe:

LEMMA. *X has no two disjoint uncountable closed subsets.*

PROOF. Suppose that H is an uncountable subset of X. There is a minimal $i \in \omega$ such that $H \cap (\omega_1 \times \{i\})$ is uncountable. If $S = \{\beta \in \omega_1 \mid \langle \beta, i \rangle \in H\}$, then, by $\clubsuit$, there is a limit α with $S_\alpha \subset S$. Clearly $\{\langle \alpha + j, n \rangle \mid n \geqslant i, j \in \omega\} \subset \overline{S \times \{i\}}$, thus $\{\langle \gamma, n \rangle \mid \gamma \geqslant \alpha,\ n \geqslant i\} \subset \overline{S \times \{i\}}$. Since $(S + \{i\}) \subset H$, our lemma is proved; and we have also shown that X is hereditarily separable.

Define $E_n = \omega_1 \times \{n\}$ and $D_n = \bigcup\{E_i \mid i > n\}$. If $U_n \supset D_n$ and U_n is open, $X - U_n$ is a closed set disjoint from D_n. By the lemma, $X - U_n$ is countable. Thus $\bigcap\{U_n \mid n \in \omega\} \neq \emptyset$; and $(*)$ holds.

If H and K are disjoint closed sets in X, one of them is countable, say $H \subset (\beta \times \omega)$ for some $\beta \in \omega_1$. By simple induction on its points, one can cover $(\beta + \omega) \times \omega$ by disjoint open sets U and V so that

$$H \subset U \quad \text{and} \quad (K \cap (\beta \times \omega)) \cup ((\beta + \omega) - \beta) \times \omega) \subset V.$$

Then U and $V \cup ((\omega_1 - (\beta+\omega)) \times \omega)$ are disjoint open sets containing H and K respectively; and X is proved normal and thus a Dowker space.

It is easy to check that X is also hereditarily normal, 0-dimensional, collectionwise normal, ... but not real compact or irreducible. Also X is not first countable or locally compact, properties we would especially like to have.

(ii) We construct a 1st countable locally compact Dowker space, by defining our topology by induction. We assume CH and ♣ and define $S_{\alpha ijn}$ as before. Let $\{K_\alpha \mid \alpha < \omega_1\}$ be an indexing of the countable subsets of $(\omega_1 \times \omega)$.

By transfinite induction, for each $\alpha \in \omega_1$ and $n \in \omega$ we define a countable set $\mathcal{B}(\alpha, n)$ of subsets of $(\alpha+1)\times\omega$ such that $\bigcup\{\mathcal{B}(\beta, n) \mid \beta \leq \alpha, n \in \omega\}$ is a compact open basis for a T_3 topology on $(\alpha+1)\times\omega$ with $\mathcal{B}(\beta, n)$ being a local basis for (β, n) for every $\beta \leq \alpha$ and $n \in \omega$.

Assume that α is a limit ordinal and that $\mathcal{B}(\beta, n)$ has been defined for all $\beta < \alpha$ and $n \in \omega$. Let τ_α be the topology on $\alpha \times \omega$ having $\bigcup\{\mathcal{B}(\beta, n) \mid \beta < \alpha \text{ and } n \in \omega\}$ as a basis.

For $\beta \in S_{\alpha ijn}$ choose $B_\beta \in \mathcal{B}(\beta, i)$ so that $\{B_\beta \mid \beta \in S_\alpha\}$ are disjoint and their union is closed in τ_α. Also, if $\gamma < \alpha$ and $\delta < \alpha$ and K_γ is a closed-in-τ_α subset of $(\delta \times \omega)$, choose B_β so at most finitely many B_β's intersect K_γ. If n and j are in ω, let

$$\begin{aligned}\mathcal{B}(\alpha+j, n) = \{B \mid &\text{for some cofinite subset } S^i \text{ of } S_{\alpha ijn} \text{ for each } i \leq n,\\ &B = \bigcup\{B_\beta \mid \beta \in \bigcup\{S^i \mid i \leq n\}\}\}.\end{aligned}$$

Our space X is again $\omega_1 \times \omega$, this time with the topology induced by having $\{B(\alpha, n) \mid \alpha \in \omega_1, n \in \omega\}$ as a basis. It has all of the properties of the previous example for the same reasons; and is trivially 1st countable and locally compact in addition. For other Ostaszewski type constructions see Chapter 7.

(iii) WEISS [1981] points out that if ZFC is consistent, then so is ZFC + MA + $2^\omega = \omega_2 + \diamondsuit_{\omega_2}(E)$ where $E = \{\alpha \in \omega_2 \mid \text{cofinality of } \alpha = \omega\}$. We assume MA + $2^\omega = \omega_2$ + a ♣ -like consequences of $\diamondsuit_{\omega_2}(E)$:

For every $\alpha \in E$ and $i \in \omega$ there is a simple, cofinal with α, sequence $S_{\alpha i}$ such that, if $i \leq j$ in ω and S and T are cardinality ω_2 subsets of ω_2, then there is an $\alpha \in E$ with $S_{\alpha i} \subset S$ and $S_{\alpha j} \subset T$.

Without loss of generality $\{S_{\alpha i} \mid i \in \omega\}$ are disjoint and their union is a simple sequence S_α cofinal with α. Partition each $S_{\alpha i}$ into disjoint infinite subsets $\{S_{\alpha in} \mid n \in \omega\}$.

Define $\mathcal{B}(\alpha, n) = \{\{\langle\alpha, n\rangle\}\}$ if $\alpha \notin E$ or $n = 0$.

Otherwise choose $\mathcal{B}(\alpha, n)$ by induction in such a way that $\bigcup\{\mathcal{B}(\beta, n) \mid \beta \leq \alpha, i \in \omega\}$ is a locally compact open base for a T_3 topology τ_α on $(\alpha+1)\times\omega$ with $\mathcal{B}(\beta, i)$ being a local base for $\langle\beta, i\rangle$ for every $\beta \leq \alpha$ and $i \in \omega$. Also, if $\gamma < \beta$ in ω_2, there is a member of $\mathcal{B}(\beta, i)$ contained in $((\beta - \gamma) \times i) \cup \{\langle\beta, i\rangle\}$.

Suppose that $\alpha \in E$ and $n > 0$ and $\mathcal{B}(\beta, i)$ has been chosen for all $\beta < \alpha$ and $i \in \omega$. If γ is the immediate predecessor of β in S_α and $\beta \in S_{\alpha in}$, choose

$B_\beta \in \mathscr{B}(\beta, i)$ contained in $((\beta-\gamma)\times i)\cup\{\langle\beta, i\rangle\}$ and let $\mathscr{B}(\alpha, n) = \{\langle\alpha, n\rangle\}\cup\{\{\bigcup\{B_\beta \mid \beta\in S\} \mid S$ is some cofinite subset of $\bigcup\{S_{\alpha in} \mid i < n\}\}$.

Our space X is $\omega_2\times\omega$ topologized by using $\mathscr{B} = \bigcup\{\mathscr{B}(\alpha, n) \mid \alpha\in\omega_2, n\in\omega\}$ as a basis. Clearly X is T_3, locally compact, 1st countable, locally countable.

LEMMA. *X has no two disjoint closed sets of cardinality ω_2.*

Suppose H and K are such sets. There are $i \leqslant j$ in ω and S and T in ω_2 such that S and T have cardinality ω_2 and $(S\times\{i\})\subset H$ and $(T\times\{j\})\subset K$; so there is an $\alpha\in E$ such that $S_{\alpha i}\times\{i\}\subset H$ and $S_{\alpha j}\times\{j\}\subset K$; thus $H\cap K\neq\emptyset$.

Also, if U_n is an open set containing $D_n = \omega_2\times(\omega - n)$, then $X - U_n$ has cardinality at most ω_1, so $\bigcap\{U_n \mid n\in\omega\}\neq\emptyset$. So (*) holds.

Since $(\omega_2 - (\alpha+1))\times\omega$ is both open and closed for all $\alpha\in\omega_2$, the above lemma together with the following one suffice to prove that X is normal and thus a Dowker space.

LEMMA. *Suppose H and K are disjoint closed subsets of X of cardinality less than ω_2. Then* MA + ¬CH *implies there are disjoint open sets containing H and K, respectively.*

PROOF. For each $x\in H\cup K$, let $\mathscr{B}^*(x) = \{B\in\mathscr{B}(x) \mid \bar{B}$ meets only one of H and $K\}$. Let

$$P = \{f\colon A\to\mathscr{B} \mid A \text{ is a finite subset of } H\cup K,\ f(x)\in\mathscr{B}^*(x) \text{ for } x\in A, \\ \text{and } f(x)\cap f(y)=\emptyset \text{ if } x\in A\cap H \text{ and } y\in A\cap K\}.$$

If f and g belong to P, define $f\leqslant g$ if f extends g;$\langle P, \leqslant\rangle$ is a partially ordered set. For each $x\in H\cup K$, let $F_x = \{f\in P \mid x\in \text{domain } f\}$. Since each F_x is dense in $\langle P, \leqslant\rangle$, if $\langle P, \leqslant\rangle$ is ccc, there is a generic subset G of P meeting every F_x, i.e. every pair of functions in G has a common extension in P and every $x\in H\cup K$ is in the domain of some function in G. Thus $\bigcup\{f(x) \mid f\in G$ and $x\in H\}$ and $\bigcup\{f(y) \mid f\in G$ and $y\in K\}$ are disjoint open sets containing H and K, respectively.

In order to prove that $\langle P, \leqslant\rangle$ is ccc, assume that $\{f_\delta \mid \delta<\omega_1\}\subset P$. We can assume without loss of generality that for all δ, the domain of f_δ is $\{x_{0\delta}, \ldots, x_{h\delta}, y_{0\delta}, \ldots, y_{k\delta}\}$ where for $i\leqslant h$ and $j\leqslant k$ $x_{i\delta}\in H$ and $y_{j\delta}\in K$.

Fix $i\leqslant h$ and $j\leqslant k$ and let $B_\delta = f_\delta(x_{i\delta})$ and $B^*_\delta = f(y_{j\delta})$. We claim there is an uncountable $\Delta\subset\omega_1$ with $B_\delta\cap B^*_{\delta'}=\emptyset$ for all $\delta\neq\delta'$ in Δ. When we prove this claim, our proof that $\langle P, \leqslant\rangle$ is ccc will be complete, for by repeating the argument for every pair $\langle i, j\rangle$ with $i\leqslant h$ and $j\leqslant k$, we will arrive at an uncountable subset Δ^* of ω_1 with $f_\delta(x)\cap f_{\delta'}(y)=\emptyset$ for all $x\in(H\cap\text{domain } f_\delta)$ and $y\in(K\cap\text{domain } f_{\delta'})$ and $\delta\neq\delta'$ in Δ^*. Thus f_δ and $f_{\delta'}$ are compatible.

To find Δ, we let $\Sigma_\delta = \{\gamma\in\omega_2 \mid \langle\gamma, m\rangle\in B_\delta$ for some $m\}$. By our inductive definition of $\mathscr{B}(\alpha, n)$, Σ_δ is a closed subset of ω_2, order isomorphic with some countable ordinal σ less than the product of countably many copies of ω. Because of this bound, we can assume that σ is the same for all δ's.

If $\gamma \leqslant \sigma$, let γ_δ be the γth member of Σ_δ; if $m \in \omega$, we can assume $\langle \gamma_\delta, m\rangle \in B_\delta$ if and only if $\langle \gamma_{\delta'}, m\rangle \in B_{\delta'}$, for all δ' in ω_1.

Let $\rho = \min\{\gamma \leqslant \sigma \mid$ there is an uncountable $\Gamma_\gamma \subset \omega_1$ such that, if $\gamma \leqslant \mu \leqslant \sigma$, either all μ_δ's are the same or all μ_δ's are different for δ's in $\Gamma_\gamma\}$. Since σ satisfies the condition required of γ, ρ is well defined. We prove $\rho = 0$ by showing it can't be anything else:

Suppose $\rho = \eta + 1$. We can choose an uncountable $\Gamma \subset \Gamma_\rho$ with all η_δ's the same or all η_δ's different for δ's in Γ_ρ. Which contradicts the minimality of ρ.

Suppose ρ is a limit and all ρ_δ's are the same for δ's in Γ_ρ. Then $X_\delta = \{\langle \rho_\delta, m\rangle \in B_\delta \mid m \in \omega\}$ is the same for all $\delta \in \Gamma_\rho$. By our inductive definition of $\mathscr{B}(\alpha, n)$, there is $\gamma < \rho$ and, for each $x \in X_\delta$, $B_x \in \mathscr{B}(x)$ such that

$$(((\rho_\delta - \gamma_\delta) \times \omega) \cap B_\delta) \subset \bigcup\{B_x \mid x \in X_\delta\}.$$

We can assume that γ is the same for all $\delta \in \Gamma_\rho$, and that the B_x's are the same, and that all $(((\rho_\delta - \gamma_\delta) \times \omega) \cap B_\delta)$ are the same. Thus $(\gamma + 1)$ denies that ρ is minimal.

Suppose ρ is a limit and all ρ_δ's are different for $\delta \in \Gamma_\rho$. We can choose an uncountable $\Gamma \subset \Gamma_\rho$ with $\{\rho_\delta \mid \delta \in \Gamma\}$ discrete in ω_2. Then choose an uncountable $\Gamma' \subset \Gamma$ and $\gamma < \rho$ with

$$(\rho_\delta - \gamma_\delta) \cap \{\rho_{\delta'} \mid \delta' \neq \delta \text{ in } \Gamma\} = \emptyset \quad \text{for all } \delta \in \Gamma'.$$

Then $(\gamma + 1)$ denies that ρ is minimal.

So there is an uncountable $\Gamma_0 \subset \omega_1$ such that, for all $\delta \in \Gamma_0$ and $\mu \leqslant \sigma$, all μ_δ's are the same or all μ_δ's different. Let $M = \{\mu \leqslant \sigma \mid$ all μ_δ's are the same for δ's in $\Gamma_0\}$. We can choose $\Delta = \{\delta_\alpha \mid \alpha < \omega_1\} \subset \Gamma_0$ by induction, simply requiring that

$$\{\mu_{\delta_\alpha} \mid \mu \not\in M\} \cap \{\mu_{\delta_\beta} \mid \beta < \alpha \text{ and } \mu \not\in M\} = \emptyset \quad \text{if } \beta < \alpha.$$

This Δ has the property that $B_\delta \cap B^*_{\delta'} = \emptyset$ for all $\delta \neq \delta'$ in Δ; for if $B_\delta \cap B^*_{\delta'} \neq \emptyset$, $B_\delta \cap B^*_\delta \neq \emptyset$; which contradicts our original definition of P. This completes the proof of our lemma.

Unlike the Dowker spaces constructed in (i) and (ii), the Dowker space X constructed in (iii) is not separable. A theorem of SZENMITKLÓSSY [1979] tells us that every locally compact, hereditarily separable space is Lindelöf if MA + $\neg$CH; so we know X cannot be made hereditarily separable. However it can be made separable by just 'gluing' on a countable dense set:

Let T be the Cantor tree with the usual topology, see RUDIN [1974]; T is a nice locally compact, 1st countable, T_3 space which is the union of a discrete subset C

of cardinality 2^ω and a countable discrete subset S which is dense in T. Using MA + $\neg$CH we can construct a subset K of C of cardinality 2^ω such that if H is a subset of K of cardinality ω_1, then there are disjoint open sets containing H and $K - H$, respectively. If $e: X \to K$ is a 1–1 function, the quotient space of the disjoint union of X and $S \cup K$ gotten by identifying x with $e(x)$ is a separable, locally compact, 1st countable, Dowker space.

3.2. *More exotic Dowker spaces*

(i) *The 'real' Dowker space.* The reader should see Chapter 4 for more details on box products.

Let X be the subspace of the box product $\square_{n\in\mathbb{N}}\omega_n$ consisting of those functions f for which there is a $k \in \omega$ with $\omega_1 \leqslant$ (cofinality of $f(i)$) $< \omega_k$ for all $i \in \mathbb{N}$. Define $D_n = \{f \in X \mid f(i) = \omega_i$ for all $i \leqslant n+1\}$. Clearly D_n is closed and $\bigcap_{n\in\omega} D_n = \emptyset$. If U_n is open and $U_n \supset D_n$ we can find a function $f_n: \mathbb{N} \to \omega_\omega$ such that $f_n(i) < \omega_i$ for all $i \in \mathbb{N}$ and $U_n \supset \{g \in X \mid f_n(i) \leqslant g(i)$ for all $i \in \mathbb{N}\}$. Clearly $\bigcap_{n\in\omega} U_n \neq \emptyset$. So (*) holds.

The proof that such f_n's exist and that X is normal can be found in RUDIN [1971]; especially the latter is a now standard box product proof. The reader should observe that unlike the example in 3.1 where 'trivial' sets were just of small cardinality, a trivial set of functions in this space is a set bounded below by a function $f: \mathbb{N} \to \omega_\omega$ with $f(i) < \omega_i$ for all i.

HART [1981] and [1982] gives a number of properties of this space: in particular he proves that it is strongly collectionwise normal, orthocompact, and finitely-fully-normal but not ω_0-full-normal.

(ii) *An extremally disconnected Dowker space.* The example X in (i) is not as useful as one would like because its cardinal functions are too large. However one of its 'too large' properties has proved useful: X is a P-space; i.e. every G_δ set in X is open. The question had been raised by HARDY and JUHÁSZ [1976] as to whether there was an extremally disconnected Dowker space, i.e. one in which the closure of every open set is open. (See WOODS [1979].)

DOW and VAN MILL [1982] have answered this question in the affirmative by proving that every P-space can be embedded in $\beta\kappa$ for some discrete space κ. Since $\beta\kappa$ embeds in $(\beta\kappa - \kappa)$, assume $X \subset (\beta\kappa - \kappa)$. Then the subspace $X \cup \kappa$ of $\beta\kappa$ is an extremally disconnected Dowker space.

(iii) *A screenable Dowker space.* A space is screenable if every open cover has a σ-disjoint refinement. K. NAGAMI [1955] proved:

THEOREM. *Every screenable, normal, countably paracompact space is paracompact.*

PROOF. If $\mathcal{G}$ is an open cover of such a space X, for each $n \in \omega$, let $\mathcal{G}_n$ be a family of disjoint open sets with $\bigcup\{\mathcal{G}_n \mid n \in \omega\}$ refining $\mathcal{G}$. If $V_n = \bigcup \mathcal{G}_n$, then $\{V_n \mid n \in \omega\}$ is an open cover of X; and by Theorem 1.1(ii) there is an open cover $\{W_n \mid n \in \omega\}$

of X with $\overline{W_n} \subset V_n$ for each n. Let $\{Z_n \mid n \in \omega\}$ be an open refinement of $\{W_n \mid n \in \omega\}$ with $\overline{Z_n} \subset W_n$. Then $\{U \subset X \mid$ there is $G \in \mathscr{G}_n$ for some $n \in \omega$ such that $U = (W_n \cap G) - \bigcup\{\overline{Z_i} \mid i < n\}\}$ is a locally finite refinement of $\mathscr{G}$.

In RUDIN [1982] a complicated example of a screenable Dowker space is constructed using the very strong set-theoretic assumption $\diamondsuit^{++}$; the space is not 1st countable. At this time there is no clue as to the existence of a Dowker space with a σ-disjoint base under any assumption; the situation is similar for para Lindelöf Dowker spaces or symmetrizable Dowker spaces. The basic open questions in this area seem to be of two kinds: Can we find 'real' Dowker spaces with small cardinal functions? And can we construct Dowker Spaces with strong global countable structures under any assumptions?

4. Related topics

4.1. Homotopy extension theorem

THEOREM (STARBIRD [1975] and MORITA [1975]. *Suppose that A is a closed subset of a normal space X, that Y is either a compact Hausdorff space or an absolute neighborhood retract, and that $H\colon (A \times I) \cup (X \times \{0\}) \to Y$ is continuous. Then H can be continuously extended to $(X \times I)$.*

The BORSUK [1937] theorem had the additional hypothesis that $X \times I$ be normal; even though there are Dowker spaces, the additional hypothesis is unnecessary. We prove this by proving:

LEMMA. *Suppose that A is a closed subset of a normal space X and that $f\colon (A \times I) \to I$ is continuous. Then*

(i) *There is a continuous $g\colon (X \times I) \to I$ with $f^{-1}(0) \subset g^{-1}(0)$ and $f^{-1}(1) \subset g^{-1}(1)$.*

(ii) *There is a continuous $h\colon (X \times I) \to I$ extending f.*

(iii) *If $F\colon (A \times I) \cup (X \times \{0\}) \to I$ is a continuous extension of f, there is a continuous $G\colon (X \times I) \to I$ extending F.*

The homotopy extension theorem standardly follows from (iii).

To prove (iii) from (ii) just observe that if $H(x, t) = h(x, t) + F(x, 0) - h(x, 0)$, we can define G by $G(x, t) = H(x, t)$ if $0 \leq H(x, t) \leq 1$, $G(x, t) = 0$ if $H(x, t) \leq 0$, and $G(x, t) = 1$ if $H(x, t) \geq 1$.

By the standard proof of Tietze's extension theorem, (i) for all possible f is exactly what is required to prove (ii). So the theorem follows from:

PROOF OF (i). Let $\mathscr{B}$ be the set of all unions of finitely many members from some countable open basis for I. Then let $\{\langle B_{1n}, B_{2n}, B_{3n}, B_{4n}\rangle \mid n \in \omega\}$ be an indexing of

$\{\langle B_1, B_2, B_3, B_4\rangle \mid \text{each } B_i \in \mathscr{B} \text{ and, for } i = 1, 2, 3, \overline{B_i} \subset B_{i+1}\}$. Choose a continuous $f_n : I \to I$ with $f_n(B_{2n}) = 0$ and $f_n(X - B_{3n}) = 1$. Define

$$K_n = \{x \in A \mid t \in \bar{B}_{1n} \text{ if } f(x, t) = 0 \text{ and } t \in \overline{(X - B_{4n})} \text{ if } f(x, t) = 1\},$$

and

$$V_n = (X - A) \cup \{x \in A \mid t \in B_{2n} \text{ if } f(x, t) = 0 \text{ and } t \in (x - B_{3n}) \text{ if } f(x, t) = 1\}.$$

Since K_n is closed and $K_n \subset V_n$ which is open in the normal space X, we can choose a continuous $g_n; X \to I$ with $g_n(K_n) = 1$ and $g_n(X - V_n) = 0$. Let

$$U_n = g_n^{-1}((0, 1]) - \bigcup\{g_k^{-1}([1/n, 1] \mid k < n\}.$$

Finally let $U = \bigcup\{U_n \mid n \in \omega\}$ and choose a continuous $p : X \to I$ with $p(A) = 1$ and $p(X - U) = 0$.

If $g : X \times I \to I$ is defined by $g(x, t) = 0$ if $x \notin U$ and

$$g(x, t) = p(x) \cdot \frac{\sum\{g_n(x) \cdot f_n(t) \mid x \in U_n\}}{\sum\{g_n(x) \mid x \in U_n\}},$$

g has the desired properties. If $x \in A$, $p(x) = 1$ and $g(x, t)$ is just the average of the $f_n(t)$'s with $x \in U_n$ each of which has $f_n(t) = 0$ if $f(x, t) = 0$ and $f_n(t) = 1$ if $f(x, t) = 1$. To see that g is continuous observe that the p factor guarantees this at points (x, t) where $x \notin U$. If $x \in U$, $x \in U_{n_0}$ for some $n_0 \in \omega$ and $\{U_n \mid n \in \omega\}$ is locally finite in U. So the g_n being small near the boundary of U_n with $x \in \overline{U_n} - U_n$ guarantees continuity at these points also.

4.2. *Almost Dowker spaces and anti-Dowker spaces*

There are text book examples of regular but not normal spaces: the Cantor tree, the Sorgenfrey plane, the 'bubble space', the product of the two Michael lines. All of these spaces are *very* nice locally and are perfect (closed sets are G_δ sets); but, almost as a surprise, none of them are countably paracompact. This fact has led us to search for examples of the following types of spaces:

(i) *Almost Dowker spaces* are defined by SCOTT [1978] to be regular spaces which are not countably metacompact. A space is defined to be countably metacompact if every countable open cover has a point-finite refinement. It is easy to check that a space is countably metacompact if and only if (∗) fails. By Theorem 1.1, then, a normal space is countably paracompact if and only if it is countably metacompact. But the proof of Theorem 2.1 shows that every perfect space satisfies (∗). So all of the standard examples of regular, not normal spaces are also examples of countably metacompact but not countably paracompact spaces. However there *are* examples of almost Dowker spaces which are not Dowker; we will give two and others can be found in SCOTT [1978], CHABER [1976], GRUENHAGE [1978], DAVIS [1979], VAN DOUWEN [1978].

But there is also an interesting theory for almost Dowker spaces. A space X is said to be *orthocompact* if every open cover of X has a refinement with the property that the intersection of any subset of the refinement is open. In SCOTT [1975] we find the nice theorem (whose proof parallels that of Theorem 1.1) that $X \times I$ is orthocompact if and only if X is countably metacompact.

EXAMPLE (i.1) (CHABER [1976]). If $\mathbb{Q}$ denotes the rationals and J the irrationals in the line $\mathbb{R}$, for all $r \in \mathbb{Q}$ choose a maximal family M_r of almost disjoint sequences from J converging to r.

Let $X = J \cup (\bigcup\{M_r \mid r \in \mathbb{Q}\})$. A $U \subset X$ is a basic open set if either $U = \{x\}$ for some $x \in J$ or $U = \{S\} \cup T$ for some $S \in M_r$ for some $r \in \mathbb{Q}$ and $T \subset J$ cofinite with S. One easily checks that this yields a regular topology on X. Indexing $\mathbb{Q} = \{r_n \mid n \in \omega\}$, define $D_n = \bigcup\{M_{r_m} \mid m \geqslant n\}$; each D_n is then closed and $\bigcap\{D_n \mid n \in \omega\} = \emptyset$. If $D_n \subset U_n$ which is open, then $J - U_n$ is countable; hence $\bigcap\{U_n \mid n \in \omega\} \neq \emptyset$ and (*) holds.

Thus X is almost Dowker; CHABER [1976] also gives a σ-disjoint base example.

EXAMPLE (i.2). For all $n \in \omega$, let $F_n = \{f\colon (\omega - n) \to \omega_1 \mid : f \text{ is increasing}\}$. If $n \in \omega$, $f \in F_n$, $\beta < f(n)$ in ω_1, and $k \in \omega$, then let

$$B_{\beta k}(f) = \{g \in F_i \mid i \leqslant n,\ \beta < g(i), \text{ and } g(j) = f(j) \text{ for } n \leqslant j \leqslant n + k\}.$$

Let $X = \bigcup\{F_n \mid n \in \omega\}$. And a set $U \subset X$ is basic open if for some n and $f \in F_n$, U is either $\{f\}$ if $f(n) = 0$ or some $B_{\beta k}(f)$ if $f(n) > 0$. It is easy to check that a regular topology is induced.

If $D_n = \bigcup\{F_m \mid m \geqslant n\}$, then each D_n is closed and $\bigcap\{D_n \mid n \in \omega\} = \emptyset$. Suppose that U_n is open and $U_n \supset D_n$. Without loss of generality $U_0 \supset U_1 \supset \cdots$. For each n, choose $G_n \subset F_n$ so that for each $\alpha \in \omega_1$, there is precisely one $f \in G_n$ with $f(n) = \alpha$. There are $k_n \in \omega$ and $\beta_n \in \omega_1$ and an uncountable $H_n \subset G_n$ such that $B_{\beta_n k_n}(f) \subset U_n$ for all $f \in H_n$. For each $i \in \omega$ we now inductively choose an $n_i \in \omega$ and $f_i \in F_{n_i}$.

Let $n_0 = 0$ and choose $f_0 \in F_0$ so that $f_0(0) > \sup\{\beta_n \mid n \in \omega\}$.

Having chosen n_i and $f_i \in F_{n_i}$ define $n_{i+1} = n_i + k_{n_i} + 1$. Then choose $f_{i+1} \in F_{n_{i+1}}$ so that $f_{i+1}(n_{i+1}) > f_i(n_i + k_{n_i})$.

If we define $f \in F_0$ by $f(n_i + j) = f_i(n_i + j)$ whenever $n_i \leqslant j \leqslant (n_i + k_{n_i})$, then

$$f \in \bigcap\{B_{\beta_{n_i} k_{n_i}}(f_i) \mid i \in \omega\} \subset \bigcap\{U_n \mid n \in \omega\}.$$

Hence X satisfies (*).

This X is thus an almost Dowker space with a σ-disjoint base.

(ii) *Anti-Dowker spaces.* Observing that not normal (but regular) spaces are 'usually' also not countably paracompact, and that Dowker spaces, i.e., normal but not countably paracompact spaces, are both important and hard to find, one is led to wonder about the existence of anti-Dowker spaces, i.e., not normal (but regular) and countably paracompact ones. Anti-Dowker spaces were first studied because of their relationship to Moore spaces:

THEOREM (ii.1) (WAGE [1976]). *If there is a normal, nonmetrizable, Moore space, there is an anti-Dowker Moore space.*

PROOF. Let M be a normal, nonmetrizable, Moore space. Let $\{C_\alpha \mid \alpha \in A\}$ be a discrete family of closed sets which cannot be separated by disjoint open sets; such must exist since M is not metrizable. Let $C = \bigcup\{C_\alpha \mid \alpha \in A\}$. Define

$$X = [C \times \{0, 1\}] \cup [(M - C) \times A^2].$$

If $\alpha \in A$ and U is open in M, let

$$U_0(\alpha) = [(U \cap C) \times \{0\}] \cup [(U - C) \times \{\alpha\} \times (A - \{\alpha\})]$$

and

$$U_1(\alpha) = [(U \cap C) \times \{1\}] \cup [(U - C) \times (A - \{\alpha\}) \times \{\alpha\}].$$

Our topology on X is the one having $\{U_i(\alpha) \mid i \in \{0, 1\},\ \alpha \in A,\ U \text{ open in } M\}$ as a basis.

Since X is clearly regular and, if $\mathcal{G}$ is a development for M, $\{U_i(\alpha) \mid i \in \{0, 1\},\ \alpha \in A, \text{ and } U \in \mathcal{G}\}$ is a development for X, X is Moore. Also $C \times \{0\}$ and $C \times \{1\}$ are disjoint closed sets which cannot be separated by disjoint open sets in M since $\{C_\alpha \mid \alpha \in A\}$ cannot be so separated in X, so X is not normal.

Suppose $\mathcal{V}$ is a countable open cover of X. For $i \in \{0, 1\}$, and $V \in \mathcal{V}$, define $V_i = (M - C) \cup \{x \in C \mid \langle x, i\rangle \in V\}$. For each i, $\{V_i \mid V \in \mathcal{V}\}$ is a countable open cover of X. By Theorem 2.1, since M is a normal space in which closed sets are G_δ sets, M is countably paracompact and, for each i, there is a locally finite open refinement $\mathcal{W}_i$ of $\{V_i \mid V \in \mathcal{V}\}$. Then $\{U_i(\alpha) \mid i \in \{0, 1\},\ \alpha \in A, \text{ and } U \in \mathcal{W}_i\}$ is a locally finite open refinement of $\mathcal{V}$.

So X is an anti-Dowker Moore space.

The construction we have just described is actually a machine (WAGE [1977]) for turning any not-collectionwise normal but normal and perfect space into an anti-Dowker space. Other properties such as being Moore depend upon the space you put into the machine. If one starts with a space that is not-collectionwise Hausdorff, the resulting space is screenable and if one starts with a perfect, not-collectionwise Hausdorff space, the result has a σ-disjoint base. Thus BING's [1951] G or H yield screenable or σ-disjoint base anti-Dowker spaces.

Another interesting connection with Moore spaces is found in the well known theorem that the existence of a *separable* normal nonmetrizable Moore space is undecidable in ZFC: Martin's Axiom plus the negation of the Continuum Hypothesis implies the existence of such a space while the Continuum Hypothesis denies the existence of any such. The reader may wish to refer to FLEISSNER [1982] or to RUDIN [1975]. By Theorem (ii.1), Martin's Axiom implies the existence of an anti-Dowker Moore space and FLEISSNER, REED, WAGE [1976] proved that the Continuum Hypothesis implies there is no separable anti-Dowker Moore space. So, for the same reasons, the existence of a separable anti-Dowker Moore space is undecidable in ZFC. Example (ii.2) shows that this is an occasion when Moore spaces do not behave like 1st countable ones. There are many examples of 'real' very strong anti-Dowker spaces now in the literature, WAGE's [1976] example is separable and submetrizable, VAN DOUWEN's [1976] example is separable and locally compact, VAUGHAN's [1979] example is locally compact and countably compact, WEISS' [1980] example is locally compact, 1st countable, submetrizable, and separable. They illustrate the usefulness of the OSTASZEWSKI [1975], JUHÁSZ, KUNEN, RUDIN [1976] technique even in the construction of 'real' examples.

EXAMPLE (ii.2). *A spearable, 1st countable anti-Dowker space.* Let $\mathbb{Z}$ be the set of all integers and $H=\{f: \omega \rightarrow \mathbb{Z}\}$. We say $f<g$ in H if $f(n)<g(n)$ for all but finitely many n in ω. It is not difficult to prove that one can choose (Hausdorff [1936]) subsets $F=\{f_\alpha \mid \alpha<\omega_1\}$ and $G=\{g_\alpha \mid \alpha<\omega_1\}$ of H such that:

(a) $f_\alpha<f_\beta$ and $g_\alpha>g_\beta$ for all $\alpha<\beta$ in ω_1,

(b) $f_\alpha<g_\alpha$ for all α in ω_1,

(c) there is no $h \in H$ with $f_\alpha<h<g_\alpha$ for all α in ω_1,

If $\alpha<\omega_1$ has cofinality ω, choose $\alpha_0<\alpha_1<\cdots$ converging to α.

Let $X=F \cup G \cup \omega^2$; basic open sets for X are $\{B_n(x) \mid n \in \omega, x \in X\}$ where:

(a) $B_n(x)=\{x\}$ if $x \in \omega^2$,

(b) $B_n(f_\alpha)=\{f_\alpha\} \cup\{\langle i, j\rangle \in \omega^2 \mid i \geqslant n$ and $f_\alpha(i)=j\}$ if α does not have cofinality ω (same interchanging f and g),

(c) $B_n(f_n)=\{f_\beta \mid \alpha_n \leqslant \beta \leqslant \alpha\} \cup\{\langle i, j\rangle \in \omega^2 \mid i \geqslant n$ and $f_{\alpha_n}(i) \leqslant j \leqslant f_\alpha(i)\}$ if α has cofinality ω (same interchanging f and g).

The induced topology is easily seen to be regular and 1st countable. Since ω^2 is dense in X, X is separable. Since F and G cannot be separated by disjoint open sets, X is not normal.

Yet X is countably paracompact. For suppose that $\mathscr{V}$ is a countable open cover of X. By our choice of $B_n(f_\alpha)$ for α of cofinality ω, finitely many members of $\mathscr{V}$ cover F; and similarly for G. Thus $\mathscr{V}$ has the desired locally finite refinement.

4.3. κ-Dowker spaces

See ATSUJI [1976]; RUDIN [1976].

Suppose κ is an infinite cardinal. Thinking of a Dowker space as an ω-Dowker space, we would like to generalize to a κ-Dowker space. The following ideas are pertinent.

Let Y_κ be $(\kappa+1)$ with the topology induced by having each $\alpha<\kappa$ be isolated and neighborhoods of κ be those sets containing κ and having complements of cardinality less than κ.

If Y is a space, let the minimal tightness $\mathrm{mt}(Y)$ be the minimal cardinal κ for which there is a subset of Y having cardinality κ which has a limit point; if Y is discrete, let $\mathrm{mt}(Y)=0$. Observe that $\mathrm{mt}(Y_\kappa)=\kappa$ for infinite κ.

We say a space has property $(\kappa *)$ provided: There is a decreasing family $\{D_\alpha \mid \alpha<\kappa\}$ of closed sets with $\bigcap\{D_\alpha \mid \alpha<\kappa\}=\emptyset$ such that, if $U_\alpha \supseteq D_\alpha$ and U_α is open for each α, then $\bigcap\{U_\alpha \mid \alpha<\kappa\}\neq\emptyset$.

For a T_4 space X consider the following possible properties:

(1) X is κ-paracompact and κ-Lindelöf.

(2) $X\times Y_\kappa$ is normal.

(3) $X\times Y$ is normal for some Y with $\mathrm{mt}(Y)=\kappa$.

(4) For every open cover $(U_\alpha \mid \alpha<\kappa\}$ of X, there is an open cover $\{V_\alpha \mid \alpha<\kappa\}$ of X with $\overline{V_\alpha}\subset U_\alpha$ for all α.

(5) For every nested open cover $\{U_\alpha \mid \alpha<\kappa\}$ of X, there is an open cover $\{V_\alpha \mid \alpha<\kappa\}$ of X with $\overline{V_\alpha}\subset U_\alpha$ for all α.

(6) For every nested open cover $\{U_\alpha \mid \alpha<\kappa\}$ of X, there is a nested open cover $\{V_\alpha \mid \alpha<\kappa\}$ of X with $\overline{V_\alpha}\subset U_\alpha$ for all α.

By the same techniques used in the proof of Theorem 1.1, we see that (1) implies all of the other properties; (6) = (2) which implies (3). All of the other properties imply (5) and other relationships between (3), (4), and (5) are unknown. There are no other implications since ω_1 with the usual order topology satisfies (6) with $\kappa=\omega_1$ (and thus (1) and (2) also) but not (3), (4), or (5).

The important and difficult-to-deny property here is clearly (5). Although in RUDIN [1976] a κ-Dowker space is defined to be a T_4 space not satisfying (2), I would *like* to define a κ-Dowker space to be a T_4 space satisfying $(\kappa *)$ (and thus not (5)).

The construction of the real Dowker space in 3.2(i) generalizes almost word for word: Choose an increasing family $\{\lambda_\alpha \mid \alpha<\kappa\}$ of regular cardinals with $\lambda_\alpha^\kappa=\lambda_\alpha$.

Let X be the subspace of the box product $\square_{\alpha<\kappa}\lambda_\alpha$ consisting of exactly those functions f for which there is a $\beta<\kappa$ such that each $\alpha<\kappa$ has (cofinality $f(\alpha))=\lambda_\gamma$ for some $\gamma\leqslant\beta$.

The space X so constructed is normal and satisfies $(\kappa *)$ for precisely the same reasons the real Dowker space was normal and satisfied $(*)$. Thus X is a κ-Dowker space. Since every space Y has $\mathrm{mt}(Y)=\kappa$ for some k:

THEOREM (ii). *If $X\times Y$ is normal for all normal spaces X, Y is discrete.*

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CHAPTER 18

Products of Normal Spaces

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

1. Introduction

Theory of product spaces constitutes a very interesting and complex part of set-theoretic topology. In this chapter, we intend to present its small, but—in the author's opinion—very beautiful fragment, dealing with the preservation of separation and covering properties under the product operation. We will be especially interested in the preservation of normality, paracompactness and the Lindelöf property.

Unlike all of the weaker separation properties, normality, paracompactness and the Lindelöf property, in general, are not preserved by products. We substantiate this statement in Section 2, providing a variety of examples. There exists however a rich and well-developed theory, which determines conditions under which the above mentioned properties *are* actually preserved. We present its reasonably self-contained outline in Sections 3–6. Although, no survey of this kind can claim completeness, we strived to illustrate all of the most interesting topics. Sections 3, 4 and 5 deal with products of finitely many factors, whereas Section 6 is devoted to the study of infinite (countable and uncountable) products. Normality of important subspaces of product spaces, called Σ-spaces, is investigated in Section 7. Section 8 contains a selection of problems that remain open.

Most of the results are presented with complete proofs. Those whose proofs are either omitted or only sketched are denoted by an *asterisk* (*).

All spaces are assumed to be completely regular and all mappings are continuous. By $\mathbb{R}$, $\mathbb{I}$, $\mathbb{Q}$, $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{D}$ we denote, unless otherwise indicated, the real line, unit interval $[0, 1]$, rationals, irrationals, natural numbers and the two-point discrete space, respectively. Ordinals and cardinals are considered as sets of smaller ordinals and, unless explicitly stated to the contrary, are equipped with the order topology and therefore are collectionwise normal. For a limit ordinal α by $\bar{\alpha}$ we denote the set $\{\beta: \beta \leqslant \alpha\} = \alpha + 1$, equipped with the order topology. A space is said to be *κ-paracompact* if its every open covering of cardinality $\leqslant\kappa$ has a locally finite open refinement and a space is *κ-collectionwise normal* if for every discrete family $\{F_\alpha\}_{\alpha<\kappa}$ of its closed subsets there is a family $\{U_\alpha\}_{\alpha<\kappa}$ of mutually disjoint open sets such that $F_\alpha \subset U_\alpha$. A space is *perfect* if all of its open subsets are F_σ sets and a space is *perfectly normal* (*paracompact, Lindelöf*) if it is perfect and normal (paracompact, Lindelöf). For undefined notions and symbols and also for facts that are used without mention, the reader is referred to ENGELKING's book [1977].

For information about related subjects, the reader is referred to M.E. Rudin's chapter on Dowker spaces, S. Williams' chapter on box products, Burke's chapter on covering properties and also to my survey article [1980b].

The author wishes to express his thanks to Professors E. Michael, P. Nyikos and R. Telgársky for their comments.

We finish this section with a lemma which will be used several times in the sequel.

1.1. JONES' LEMMA. *If a separable normal space X contains a discrete closed subspace F of cardinality κ, then $2^\kappa \leqslant 2^\omega$.*

PROOF. For every subset K of F there must exist an open set U_K such that $K \subset U_K$ and $\overline{U_K} \cap (F \setminus K) = \emptyset$. Since X is separable, it has only continuum many different open subsets, which obviously implies $2^\kappa \leqslant 2^\omega$. □

2. Products of normal spaces need not be normal

Products of Hausdorff, regular or completely regular spaces are always Hausdorff, regular or completely regular, respectively. However, products of *normal, paracompact or Lindelöf* spaces often fail to be normal, paracompact or Lindelöf.

2.1. EXAMPLE (SORGENFREY [1947]). *The square of a Lindelöf space does not have to be normal: the square of the Sorgenfrey line is not normal.*

PROOF. The Sorgenfrey line S is the real line $\mathbb{R}$ with the topology generated by the half-open intervals of the form $[a, b)$, where $a < b$. The space S is hereditarily Lindelöf: indeed, if $\{[a_s, b_s)\}_{s \in S}$ is a family of half-open intervals, then we can find a countable subset $S_0 \subset S$ such that $\bigcup\{(a_s, b_s): s \in S\} = \bigcup\{(a_s, b_s): s \in S_0\}$ and it remains only to observe that the set

$$\bigcup\{[a_s, b_s): s \in S\} \setminus \bigcup\{(a_s, b_s): s \in S\}$$

is countable. Also note that the antidiagonal $\nabla = \{\langle x, -x\rangle : x \in S\}$ is a discrete closed subset of S^2.

The non-normality of S^2 can now be proved in two ways, either applying the Jones Lemma 1.1 or the Baire Category Theorem.

(A) *Applying the Jones Lemma.* The space S^2 is separable and contains a closed discrete subset of cardinality continuum, hence it cannot be normal.

(B) *Applying the Baire Category Theorem.* Let $K_0 = \{\langle q, -q\rangle : q \text{ is rational}\}$ and $K_1 = \{\langle p, -p\rangle : p \text{ is irrational}\}$. The sets K_0 and K_1 are disjoint closed subsets of S^2. Suppose that U_0 and U_1 are disjoint open subsets of S^2 such that $K_i \subset U_i$, $i = 0, 1$. For every irrational number p let us choose an $\varepsilon(p) > 0$ such that

$$[p, p + \varepsilon(p)) \times [-p, -p + \varepsilon(p)) \subset U_1\,,$$

and let $P_n = \{p \in \mathbb{P} : \varepsilon(p) \geqslant 1/n\}$. Clearly, $P = \bigcup_{n<\omega} P_n$ and since $\mathbb{P}$ is not an F_σ subset of $\mathbb{R}$, there exists an n and a $q \in \mathbb{Q}$ such that q belongs to the euclidean closure of the set P_n. One easily checks that the point $\langle q, -q\rangle$ belongs to the closure of U_1 in S^2 and therefore U_0 and U_1 are not disjoint. □

Observe, that the category argument actually produces a pair of disjoint closed sets that cannot be separated, whereas the Jones Lemma only asserts their existence.

Products of compact spaces are of course compact and therefore normal, but a product of a normal space and a compact space need not be normal.

2.2. EXAMPLE (DIEUDONNÉ [1939]). *The product of a normal space and a compact space need not be normal: the product space $\omega_1 \times \overline{\omega_1}$ is not normal.*

PROOF. Let $K_0 = \{\langle\alpha, \alpha\rangle : \alpha < \omega_1\}$ and $K_1 = \{\langle\alpha, \omega_1\rangle : \alpha < \omega_1\}$. The sets K_0 and K_1 are closed and disjoint in $\omega_1 \times \overline{\omega_1}$. Suppose that U_0 and U_1 are open subsets of the product such that $K_i \subset U_i$ for $i = 0, 1$. For every $\alpha < \omega_1$ there exists an $f(\alpha) < \alpha$ such that $(f(\alpha), \alpha] \times (f(\alpha), \alpha] \subset U_0$ and therefore by the Pressing Down Lemma there is an $\alpha_0 < \omega_1$ such that $f(\alpha) = \alpha_0$ for uncountably many α's in ω_1. One can easily verify that the point $\langle\alpha_0 + 1, \omega_1\rangle$ belongs to the closure of U_0 and since it also belongs to K_1, the sets U_0 and U_1 are not disjoint. □

Actually, essentially the same proof shows that the space $\kappa \times \bar{\kappa}$ is not normal for any cardinal κ of uncountable cofinality.

Products of countably many metrizable spaces are of course metrizable and therefore normal, but a product of a metric space and a normal space may be non-normal.

2.3. EXAMPLE (MICHAEL [1963]). *The product of a normal space and a metric space may be non-normal:*

(1) *There exists a paracompact space X such that the product space $X \times \mathbb{P}$ of X with the space $\mathbb{P}$ of irrationals is non-normal.*

(2) *There exists a Lindelöf space X and a metric space M such that the product space $X \times M$ is not normal.*

PROOF. (1) Let X be the real line $\mathbb{R}$ with a new topology generated by the base $\mathcal{B} = \{U : U \text{ is open in } \mathbb{R} \text{ or } U \text{ is a subset of } \mathbb{P}\}$. This space is called the *Michael line.* Let us observe first that the space X is paracompact. Indeed, if $\mathcal{U}$ is a covering of X by basic open sets and $\mathcal{V}$ is the subfamily of $\mathcal{U}$ consisting of all euclidean-open sets, then we can choose a countable $\mathcal{V}_0 \subset \mathcal{V}$ so that $\bigcup\mathcal{V} = \bigcup\mathcal{V}_0$ and observe that $F = X \setminus \bigcup\mathcal{V}$ is a discrete closed-and-open subset of X.

The sets $K_0 = \mathbb{Q} \times \mathbb{P}$ and $K_1 = \{\langle p, p\rangle : p \in \mathbb{P}\}$ are easily seen to be disjoint and closed in $X \times \mathbb{P}$. Suppose that U_0 and U_1 are disjoint open sets such that $K_i \subset U_i$ for $i = 0, 1$. Let us choose a countable dense subset T of $\mathbb{P}$. For every $p \in \mathbb{P}$ there exists a point $t(p) \in T$ such that $\langle p, t(p)\rangle \in U_1$. Define $P_t = \{p \in \mathbb{P} : t(p) = t\}$ for $t \in T$. Since $\mathbb{P} = \bigcup_{t\in T} P_t$ and $\mathbb{P}$ is not an F_σ subset of $\mathbb{R}$, there exists a rational q and a $t \in T$ such that q belongs to the euclidean closure of P_t. One easily sees that the point $\langle q, t\rangle$ belongs to the closure of U_1 and since it also belongs to K_0, the sets U_0 and U_1 are not disjoint.

(2) This is similar to part (1) except that instead of $\mathbb{P}$ we use a *totally imperfect* (see KURATOWSKI [1966], Chapter III, §40) subset P' of the real line, i.e. a subset such that no uncountable closed subset of the real line is contained either in P' or in $\mathbb{R}\setminus P'$. The space X is the real line $\mathbb{R}$ with the topology generated by the base $\mathcal{B}' = \{U: U$ is open in $\mathbb{R}$ or U is contained in $P'\}$ and M is the subspace P' of the real line.

The proof of the non-normality of $X \times M$ is completely analogous (of course, P' must be uncountable, and hence is not an F_σ in $\mathbb{R}$). To see that the space X is Lindelöf observe that in the argument given above the set $F = X\setminus\bigcup\mathcal{V}$ is a closed subset of P' and therefore must be countable. □

An example with even more pathological properties was constructed by PRZYMUSIŃSKI [1980a].

The metric space M is not complete; using CH, Michael constructed a Lindelöf space X such that $X \times \mathbb{P}$ is not normal (MICHAEL [1971]; see Problem 7).

Perhaps the most striking example of a non-normal product space is the following famous example due to M.E. RUDIN.

2.4*. EXAMPLE (RUDIN [1971]). *The product of a normal space and a compact metric space does not have to be normal: there exists a normal space X such that its product $X \times \mathbb{I}$ with the unit interval is not normal.* □

In fact Example 2.4 can be even generalized.

2.5*. EXAMPLE (RUDIN [1975], ATSUJI [1977]). *For every non-discrete space Y there is a normal space X such that $X \times Y$ is not normal.* □

Normal spaces whose products with the unit interval are not normal are called *Dowker spaces* and are hard to come by. For more information about Dowker spaces as well as for the description of Examples 2.4 and 2.5 the reader is referred to M.E. Rudin's article (this volume), which is entirely devoted to this class of spaces. Here we only point out that a normal space is Dowker if and only if it is *not* countably paracompact (see Corollary 3.9).

The above examples shed some light on the behavior of normality under finite products. The example below shows that a countable power of a space may fail to be normal even though all finite subproducts are Lindelöf. A consistent example of such a space has been first obtained by MICHAEL [1971].

2.6*. EXAMPLE (PRZYMUSIŃSKI [1980a]). *There exists a space X such that X^n is Lindelöf for all $n < \omega$ but X^ω is not normal.*

Sketch of Proof[1]. Let us define the notion of n-cardinality first. A subset A of a product space X^n, where n is a natural number, is said to have *n-cardinality* equal to κ (briefly, $|A|_n = \kappa$) if κ is the smallest cardinal such that

$$A \subset \bigcup_{i=1}^{n} (X^{(i-1)} \times Y \times X^{(n-i)}),$$

with $|Y| \leqslant \kappa$. The notion of n-cardinality generalizes the notion of cardinality in a way which is suitable for product space considerations (see PRZYMUSIŃSKI [1978]).

We can represent the plane as the union of a disjoint family $\{T_m\}_{m<\omega}$ of sets having the following property (see PRZYMUSIŃSKI [1978] for details):

(*) For every n, $m < \omega$ and for every closed subset F of $(\mathbb{R}^2)^n$, if $F \cap (T_m)^n = \emptyset$, then $|F|_n \leqslant \omega$.

For every $m < \omega$ we define a topology τ_m on the plane $\mathbb{R}^2$ and denote the new space $\langle \mathbb{R}^2, \tau_m \rangle$ by X_m. The topology τ_m is generated by the base $\mathcal{B} = \mathcal{E} \cup \{B_k(\langle r, 0\rangle): \langle r, 0\rangle \in T_m, k = 1, 2, \ldots\}$, where $\mathcal{E}$ is the family of open subsets of $\mathbb{R}^2$ and $B_k(\langle r, 0\rangle)$ is the union of the one point set $\{\langle r, 0\rangle\}$ and the interior of a disc of radius $1/k$ lying in the upper half-plane and tangent to the x-axis at the point $\langle r, 0\rangle$. In other words, all points of the plane have standard euclidean neighborhoods except for points belonging to T_m and lying on the x-axis, which have Niemytzki-type neighborhoods.

The space X is defined as the discrete union $X = \bigoplus_{m<\omega} X_m$ of spaces X_m. It is easy to see that X^ω is not normal. Indeed, the subset $\Delta = \{\langle x, x, x \ldots\rangle: x = \langle r, 0\rangle, r \in R\}$ of $\Pi_{m<\omega} X_m$ is closed and discrete in X^ω and since X^ω is separable its non-normality follows from the Jones Lemma 1.1. It remains to show that all finite powers of X are Lindelöf or—equivalently—that for every $n < \omega$ and every sequence $m_1, m_2, \ldots, m_n$ of natural numbers the space $Z = X_{m_1} \times X_{m_2} \times \cdots \times X_{m_n}$ is Lindelöf.

Suppose that X^k was already proved to be Lindelöf for all $k < n$, let $\mathcal{U}$ be an open covering of Z and let $m = \max(m_1, m_2, \ldots, m_n) + 1$. Since all points of T_m^n in Z have standard euclidean neighborhoods, there exists a countable family $\mathcal{V}$ of euclidean-open sets that covers T_m^n and refines $\mathcal{U}$. The set $F = (\mathbb{R}^2)^n \setminus \bigcup \mathcal{V}$ is closed in $(\mathbb{R}^2)^n$ and $F \cap T_m^n = \emptyset$, therefore, by (*), F is n-countable. Using the inductive assumption and the definition of n-cardinality it is easy to show that there is a countable subfamily $\mathcal{W}$ of $\mathcal{U}$ that covers F. Now, $\mathcal{W} \cup \mathcal{V}$ is a countable open refinement of $\mathcal{U}$. □

A space with even more pathological properties was constructed by ALSTER [1982].

When we pass to uncountable products, normality is very seldom preserved.

[1]A complete proof can be found in Burke's article (this volume).

Naturally, any product of compact spaces is normal, but, as the following example shows, uncountable products of metric spaces do not have to be normal. In fact, a product of an uncountable family of noncompact metric spaces is never normal (see Corollary 6.5).

2.7. EXAMPLE (STONE [1948]). *The product of uncountably many metric spaces may be non-normal: the product space* $\mathbb{N}^{\omega_1}$ *of* ω_1 *copies of natural numbers is not normal.*

PROOF. We define two disjoint closed subsets K_0 and K_1 in $\mathbb{N}^{\omega_1}$ as follows. Let K_i for $i = 0, 1$ be the set of all those elements $(n_\alpha)_{\alpha<\omega_1}$ of $\mathbb{N}^{\omega_1}$ such that for every natural $j \neq i$ there is at most one $\alpha < \omega_1$ with $n_\alpha = j$. It is easy to verify that they are indeed closed and disjoint. Suppose that U is an open subset of $\mathbb{N}^{\omega_1}$ such that $K_0 \subset U \subset \bar{U} \subset \mathbb{N}^{\omega_1} \backslash K_1$. Let $\mathcal{V}$ be a maximal family of mutually disjoint basic open subsets of $\mathbb{N}^{\omega_1}$ contained in U; i.e. every element $V \in \mathcal{V}$ is of the form $\pi_A^{-1}(W)$, where $A = A_V$ is a finite subset of ω_1, W is a subset of $\Pi_{\alpha\in A} N_\alpha$ and $\pi_A: \mathbb{N}^{\omega_1} \to \Pi_{\alpha\in A} N_\alpha$ is a projection. Since the space $\mathbb{N}^{\omega_1}$ satisfies the countable chain condition (see ENGELKING [1977], 2.3.18), the family $\mathcal{V}$ is countable and the set $L = \bigcup\{A_V : V \in \mathcal{V}\}$ is countable. One easily checks that:

$$(**) \qquad \bar{U} = \pi_L^{-1}(\pi_L(\bar{U}))\,.$$

We will now show that this is impossible. Let us choose a point $t = (t_\alpha)_{\alpha\in L}$ such that for all $j < \omega$ there is at most one α with $t_\alpha = j$ and let $x = (x_\alpha)_{\alpha<\omega_1}$, $y = (y_\alpha)_{\alpha<\omega_1}$, where

$$x_\alpha = \begin{cases} t_\alpha, & \alpha \in L\,, \\ 0, & \text{otherwise}\,, \end{cases} \qquad y_\alpha = \begin{cases} t_\alpha, & \alpha \in L\,, \\ 1, & \text{otherwise}\,. \end{cases}$$

Now it is enough to observe that $x \in K_0$, $y \in K_1$ and $x, y \in \pi_L^{-1}(t)$, which contradicts (**). □

If normality is so easily destroyed by products, then maybe hereditary normality is better preserved? Unfortunately, this is not the case. The product of two hereditarily normal spaces does not have to be normal as Example 2.1 shows. Here is another example.

2.8. EXAMPLE. *The product of two hereditarily normal compact spaces or even hereditarily Lindelöf compact spaces does not have to be hereditarily normal:*

(1) *The product space* $\mathbb{I} \times \overline{\omega_1}$ *is not hereditarily normal.*

(2) *The square of the double arrow space is not hereditarily normal.*

PROOF. (1) (TYCHONOFF [1930]). We will show that the subspace $X = (\mathbb{I} \times \bar{\omega}_1)\backslash\{\langle 0, \omega_1\rangle\}$ of $\mathbb{I} \times \bar{\omega}_1$ is not normal. Indeed, consider disjoint closed subsets

$$K = (0, 1] \times \{\omega_1\} \quad \text{and} \quad L = \{0\} \times \omega_1$$

of X. Suppose that U is open in X and contains L. For every $\alpha < \omega_1$ there is an $n(\alpha) < \omega$ such that $[0, 1/n(\alpha)) \times \{\alpha\} \subset U$ and therefore there is a cofinal subset $T \subset \omega_1$ and an $n < \omega$ with $n(\alpha) = n$ for all $\alpha \in T$. Then $\langle 1/2n, \omega_1 \rangle \in \bar{U} \cap K$.

(2) The double arrow space X (ALEXANDROFF and URYSOHN [1929]) is the set $\mathbb{I} \times \{0, 1\}$ with the following topology: every point of type $\langle t, 0 \rangle$ has a base of neighborhoods of the form

$$[t, t + 1/n) \times \{0\} \cup (t, t + 1/n) \times \{1\} \quad \text{for } n = 1, 2, \ldots$$

and every point of type $\langle t, 1 \rangle$ has a base of neighborhoods consisting of sets of the form

$$(t - 1/n, t) \times \{0\} \cup (t - 1/n, t] \times \{1\} \quad \text{for } n = 1, 2, \ldots.$$

It is easy to observe that X is compact and since the subspaces $(0, 1) \times \{0\}$ and $(0, 1) \times \{1\}$ are clearly homeomorphic to the Sorgenfrey line, X is also hereditarily Lindelöf. Because of the same reason, X^2 is not hereditarily normal. $\square$

As is seen from the example below, hereditary normality of all finite subproducts of X^ω does not even imply normality of X^ω.

2.9*. EXAMPLE (VAUGHAN [1975]). *There exists a space X such that X^n is hereditarily paracompact for all $n < \omega$ but X^ω is not normal.*

Sketch of Proof. Let $\hat{D}(\omega_1)$ denote the set $\tilde{\omega}_1$ in which all points $\alpha < \omega_1$ are isolated and basic neighborhoods of the point ω_1 have the form $(\alpha, \omega_1]$, for $\alpha < \omega_1$. Let X denote the box product $\square \hat{D}(\omega_1)^\omega$ of countably many copies of $\hat{D}(\omega_1)$. (See S. Williams' article for information about *box products.*) One easily sees that X is homeomorphic with any finite power of itself and it can be proved without difficulty that X is hereditarily paracompact (in fact, X is ω_1-metrizable). Consequently, X^n is hereditarily paracompact for all $n < \omega$.

To show that X^ω is not normal it is enough to show that $X \times D(\omega_1)^\omega$ is not normal, where $D(\omega_1)$ is a discrete space of cardinality ω_1. Indeed, X contains a closed subset homeomorphic to $D(\omega_1)$ and therefore X^ω contains a closed subset homeomorphic to $D(\omega_1)^\omega$ which would imply that $X \times X^\omega \simeq X^\omega$ is not normal.

The space X also contains an open subset $W = \{(x_n)_{n<\omega}$: for every $n < \omega$, $x_n < \omega_1\}$, which we can of course identify with $D(\omega_1)^\omega$ (with discrete topology).

Using an argument resembling the argument used in Michael's example 2.3, one proves that the following two closed and discrete subsets K_0 and K_1 of $X \times D(\omega_1)^\omega$ cannot be separated by open sets:

$$K_0 = (X \setminus D(\omega_1)^\omega) \times D(\omega_1)^\omega ,$$

$$K_1 = \{\langle x, x\rangle \in X \times D(\omega_1)^\omega : x \in D(\omega_1)^\omega\} . \qquad \square$$

Finally, the following theorem completely describes the behavior of hereditary normality under uncountable products:

2.10. THEOREM (POSPIŠIL [1937]). *The product of uncountably many spaces containing at least two points is never hereditarily normal.*

PROOF. Let $Z = \Pi_{s\in S} X_s$, where $|S| \geq \omega_1$ and $|X_s| > 1$. For all $s \in S$ let Y_s be a two-point subspace of X_s and let S be decomposed into $|S|$ disjoint sets S_t, $t \in S$, of cardinality ω. For every $t \in S$ the space $C_t = \Pi_{s\in S_t} Y_s$ is homeomorphic with the Cantor set and therefore contains a subspace N_t homeomorphic to N. By 2.7 $\Pi_{t\in S} N_t$ is not normal and, obviously, $\Pi_{t\in S} N_t \subset \Pi_{t\in S} C_t = \Pi_{s\in S} Y_s \subset \Pi_{s\in S} X_s$, which proves the non-normality of Z. $\square$

In the following sections we will present several new examples showing how normality can be destroyed by the product operation, but we will be primarily interested in proving *positive* theorems about the preservation of normality and related properties under products.

3. Products with a compact factor

In this section we investigate products of the form $X \times C$ with C being compact; C always denotes a *compact space* and $\mathcal{B}$ denotes any *base of* C closed with respect to finite unions and finite intersections.

3.1. DEFINITION. Let $\mathcal{B}$ be a base of a compact space C. An open covering $\mathcal{G} = \{G(B, D)\colon B, D \in \mathcal{B}$ and $\bar{B} \cap \bar{D} = \emptyset\}$ of a space X is called a *$\mathcal{B}$-covering* if $G(B, D) \cap G(B', D') = G(B \cap B', D \cap D')$, for all $B, B', D, D' \in \mathcal{B}$.

The following theorem gives a complete characterization of normality of products with a compact factor.

3.2. CHARACTERIZATION THEOREM. *Let $\mathcal{B}$ be a base of a compact space C. The product space $X \times C$ is normal if and only if X is normal and every $\mathcal{B}$-covering of X has a locally finite open refinement.*

PROOF. ($\Leftarrow$) Let K, L be disjoint closed subsets of $X \times C$. For B, $D \in \mathcal{B}$ with $\bar{B} \cap \bar{D} = \emptyset$ define

$$G(B, D) = \{x \in X\colon K_x \subset B \text{ and } L_x \subset D\} ,$$

where

$$K_x = \{y \in C: \langle x, y\rangle \in K\}, \qquad L_x = \{y \in C: \langle x, y\rangle \in L\}.$$

It is easy to check that $\mathscr{G} = \{G(B, D): B, D \in \mathscr{B}$ and $\bar{B} \cap \bar{D} = \emptyset\}$ is a $\mathscr{B}$-covering of X. Let $\mathscr{V} = \{V(B, D): B, D \in \mathscr{B}$ and $\bar{B} \cap \bar{D} = \emptyset\}$ be a locally finite open covering of X such that $V(B, D) \subset G(B, D)$. By the normality of X there exist a closed covering $\{F(B, D): B, D \in \mathscr{B},\ \bar{B} \cap \bar{D} = \emptyset\}$ of X such that $F(B, D) \subset V(B, D)$ and continuous functions $f_{B,D}: X \to I$ such that $f_{B,D}(F(B, D)) \subset \{1\}$ and $f_{B,D}(X \setminus V(B, D)) \subset \{0\}$. By the normality of C there exist continuous functions $g_{B,D}: C \to I$ such that $g_{B,D}(\bar{B}) \subset \{0\}$ and $g_{B,D}(\bar{D}) \subset \{1\}$. Define $h: X \times C \to R$ as follows:

$$h(x, y) = \sum \{f_{B,D}(x) \cdot g_{B,D}(y): B, D \in \mathscr{B} \text{ and } \bar{B} \cap \bar{D} = 0\}.$$

Since the family $\mathscr{V}$ is locally finite, h is continuous and it is routine to verify that $h(K) \subset \{0\}$ and $h(L) \subset [1, \infty)$, which proves the normality of $X \times C$.

($\Rightarrow$) Suppose that $\mathscr{G} = \{G(B, D): B, D \in \mathscr{B},\ \bar{B} \cap \bar{D} = \emptyset\}$ is a $\mathscr{B}$-covering of X. Define:

$$K = X \times C \setminus \bigcup \{G(B, D) \times (C \setminus \bar{B}): B, D \in \mathscr{B}, \bar{B} \cap \bar{D} = \emptyset\},$$

and

$$L = X \times C \setminus \bigcup \{G(B, D) \times (C \setminus \bar{D}): B, D \in \mathscr{B}, \bar{B} \cap \bar{D} = \emptyset\}.$$

It is obvious that the sets K and L are closed. We will show that they are disjoint. Indeed, let $\langle x, y\rangle \in X \times C$. There exist $B, D \in \mathscr{B}$, $\bar{B} \cap \bar{D} = \emptyset$ such that $x \in G(B, D)$ and either $y \notin \bar{B}$ or $y \notin \bar{D}$, therefore either $\langle x, y\rangle \notin K$ or $\langle x, y\rangle \notin L$.

Let $f: X \times C \to I$ be a continuous function such that $f(K) \subset \{0\}$ and $f(L) \subset \{1\}$ and define a continuous pseudometric ρ on X as follows:

$$\rho(x, x') = \sup_{y \in C} |f(x, y) - f(x', y)|.$$

In order to prove that $\mathscr{G}$ has a locally finite refinement it suffices to show that the family $\{B(x, \frac{1}{3}): x \in X\}$ of open balls in ρ of radius $\frac{1}{3}$ refines $\mathscr{G}$.

Fix $x_0 \in X$. Since the base B is closed w.r.t. finite unions, we can find $B, D \in \mathscr{B}$ such that $\{y \in C: f(x_0, y) \leqslant \frac{1}{3}\} \subset B$, $\{y \in C: f(x_0, y) \geqslant \frac{2}{3}\} \subset D$ and $\bar{B} \cap \bar{D} = \emptyset$. We will show that $B(x_0, \frac{1}{3}) \subset G(B, D)$. Observe that if $x \in B(x_0, \frac{1}{3})$, then $K_x \subset B$ and $L_x \subset D$. Indeed, otherwise there would exist, for example, a point $y \in K_x \setminus B$ and then $f(x, y) = 0$, $f(x_0, y) > \frac{1}{3}$ and $\rho(x_0, x) < \frac{1}{3}$, which is impossible. We have to show that $x \in G(B, D)$.

By the definition of K and L we have

$$\{x\} \times (C \setminus B) \subset \bigcup \{G(B', D') \times (C \setminus \bar{B}'): B', D' \in \mathscr{B} \text{ and } \bar{B}' \cap \bar{D}' = \emptyset\},$$

and

$$\{x\} \times (C \setminus D) \subset \bigcup \{G(B'', D'') \times (C \setminus \bar{D}''): B'', D'' \in \mathscr{B}, \bar{B}'' \cap \bar{D}'' = \emptyset\}.$$

There exist therefore finite families $\{\langle B'_1, D'_1\rangle, \ldots, \langle B'_n, D'_n\rangle\}$ and $\{\langle B''_1, B''_1\rangle, \ldots, \langle B''_m, D''_m\rangle\}$ such that

$$C\backslash B \subset \bigcup_{i=1}^{n}(C\backslash \bar{B}'_i),\ x \in \bigcap_{i=1}^{n} G(B'_i, D'_i),$$

$$C\backslash D \subset \bigcup_{j=1}^{m}(C\backslash \bar{D}''_j),\ x \in \bigcap_{j=1}^{m} G(B''_j, D''_j)$$

and therefore

$$B \supset \bigcap_{i=1}^{n} B'_i, \qquad D \supset \bigcap_{j=1}^{m} D''_j,$$

and

$$x \in G\Big(\bigcap_{i=1}^{n} B'_i \cap \bigcap_{j=1}^{m} B''_j,\ \bigcap_{i=1}^{n} D'_i \cap \bigcap_{j=1}^{m} D''_j\Big) \subset G(B, D). \quad \square$$

REMARK. Theorem 3.2 has the following Boolean-algebraic counterpart:

3.2.A. THEOREM. *The product space $X \times S(\mathscr{B})$ of a normal space X and the Stone space $S(\mathscr{B})$ of a Boolean algebra $\mathscr{B}$ is normal* iff *every open covering $\mathscr{G} = \{G(B, D): B, D \in \mathscr{B},\ B \cdot D = 0\}$ of X, with the property $G(B, D) \cap G(B', D') = G(B \cdot B', D \cdot D')$, has a locally finite open refinement* iff *every open covering $\mathscr{G} = \{G(B): B \in \mathscr{B}\}$ of X with the property $G(B \cdot B' \cdot B'') \cap G(B + B' + B'') = G(B) \cap G(B') \cap G(B'')$, has a locally finite open refinement.* $\square$

Many theorems involving products with a compact factor can be derived from Theorem 3.2. As usual, $w(X)$ denotes the *weight* of a space X.

3.3. COROLLARY (MORITA [1961/2]). *If X is normal and $w(C)$-paracompact, then $X \times C$ is normal.*

PROOF. Choose a base $\mathscr{B}$ of C of cardinality $w(C)$ and apply 3.2. $\square$

3.4. COROLLARY (MORITA [1961/2], TAMANO [1962]). *The following conditions are equivalent for a space X:*
(i) *X is paracompact.*
(ii) *$X \times \beta X$ is normal.*
(iii) *$X \times \alpha X$ is normal for some (every) compactification αX of X.*

PROOF. By 3.3, (i) implies (ii) and (iii). Suppose that αX is a compactification of X, $X \times \alpha X$ is normal and $\mathscr{U}$ is an open covering of X by sets open in αX. Choose $\mathscr{B}$ to be the family of all open subsets of αX and define

$$G(B, D) = \begin{cases} B, & \text{if } \alpha X\backslash D \subset \bigcup \mathscr{U}, \\ \emptyset, & \text{otherwise}. \end{cases}$$

Then $\mathcal{G} = \{G(B, D): \bar{B} \cap \bar{D} = \emptyset\}$ is a $\mathcal{B}$-covering of X and therefore has a locally finite open refinement. Observe however, that the closure in αX of every element of that refinement is contained in $\bigcup \mathcal{U}$ and therefore is covered by finitely many elements of $\mathcal{U}$, which proves paracompactness of X. $\square$

3.5. COROLLARY (DIEUDONNÉ [1944]). *The product $X \times C$ of a paracompact space X and a compact space C is paracompact.*

PROOF. By 3.3 $X \times C \times \beta(X \times C)$ is normal and therefore, by 3.4, $X \times C$ is paracompact. $\square$

3.5.R. REMARK. The last corollary can be easily proved without referring to 3.2. Indeed, if $\mathcal{U}$ is an open covering of $X \times C$ let $\mathcal{V}$ denote the family of all finite unions of elements of $\mathcal{U}$. For every $V \in \mathcal{V}$ let $W(V) = X \setminus \pi(X \times C \setminus V)$ where $\pi: X \times C \to X$ is a projection. Since π is closed the family $\{W(V): V \in \mathcal{V}\}$ is an open covering of X and therefore it has a locally finite open refinement $\mathcal{H}$. Now, the family $\{H \times C: H \in \mathcal{H}\}$ is a locally finite covering of $X \times C$ every element of which is contained in finitely many elements of $\mathcal{U}$, which proves paracompactness of $X \times C$.

Different variants of this argument can be used to prove many analogous theorems, e.g. that the product of a Lindelöf and a compact space is Lindelöf, that the preimage of a paracompact space under a perfect mapping is paracompact, etc. $\square$

3.6. COROLLARY (TAMANO [1962]). *If $X \times Y$ is normal for every paracompact space Y, then $X \times Y$ is paracompact for every space Y.*

PROOF. Let Y be paracompact. By 3.5, $Y \times \beta(X \times Y)$ is paracompact and therefore $X \times Y \times \beta(X \times Y)$ is normal, which by 3.4 implies that $X \times Y$ is paracompact. $\square$

3.7. COROLLARY (KUNEN). *Let α be an ordinal. The product space $X \times \bar{\alpha}$ is normal if and only if X is normal and $|\alpha|$-paracompact.*

PROOF. ($\Leftarrow$) This implication follows immediately from 3.3.

($\Rightarrow$) Let λ be the smallest ordinal for which the theorem fails. Observe that λ has to be a cardinal. Let $\mathcal{U} = \{U_\alpha: \alpha < \lambda\}$ be an open covering of X. By the inductive assumption it suffices to show that the open covering $\mathcal{V} = \{V_\alpha: \alpha < \lambda\}$, where $V_\alpha = \bigcup\{U_\beta: \beta < \alpha\}$ has a locally finite open refinement. Let $\mathcal{B}$ denote the set of all open subsets of $\bar{\lambda}$ and define:

$$G(B, D) = \begin{cases} V_\alpha, & \text{where } \alpha \text{ is the smallest ordinal such that } (\alpha, \lambda] \subset B, \\ & \text{if such an ordinal exists,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\mathcal{G} = \{G(B, D): B, D \in \mathcal{B}, \bar{B} \cap \bar{D} = \emptyset\}$ is a $\mathcal{B}$-covering of $\bar{\lambda}$ refining $\mathcal{V}$. $\square$

3.8. COROLLARY (MORITA [1961/2]). *The following conditions are equivalent for a space X:*

(i) *X is normal and κ-paracompact.*
(ii) *$X \times I^{\kappa}$ is normal.*
(iii) *$X \times D^{\kappa}$ is normal.*

PROOF. Clearly (i) implies (ii) by 3.3 and obviously (ii) implies (iii). The space D^{κ} contains a closed copy of $\bar{\kappa}$ and therefore by 3.7, (iii) implies (i). □

As we mentioned in Section 2, a normal space X for which $X \times \mathbb{I}$ is *not* normal is called a *Dowker* space. The following corollary shows that a normal space X is Dowker if and only if it is not countably paracompact.

3.9. COROLLARY (DOWKER [1951]). *The following conditions are equivalent for a space X:*

(i) *X is normal and countably paracompact.*
(ii) *$X \times \mathbb{I}$ is normal.*
(iii) *$X \times C$ is normal, for some (every) infinite compact metric space C.*

PROOF. By 3.3, (i) implies both (ii) and (iii). On the other hand, since every infinite compact metric space contains a closed subspace homeomorphic to $\bar{\omega}$, Corollary 3.7 shows that both (ii) and (iii) imply (i). □

Let us recall that a compact space is *dyadic* if it is a continuous image of $\mathbb{D}^{\kappa}$ for some cardinal κ. A space is *extremally disconnected* if any two disjoint open sets have disjoint closures (see ENGELKING [1977]). The following two theorems are based on fairly recent and deep results obtained by EFIMOV [1977], GERLITS [1976], BALCAR and FRANEK [1982].

3.10*. THEOREM. *If C is dyadic, then $X \times C$ is normal if and only if X is normal and $w(C)$-paracompact.*

Sketch of Proof. The sufficiency follows immediately from 3.3. We prove necessity. Let $\lambda = w(C)$. If $\lambda = \omega$, then the necessity follows from 3.9. If $\mathrm{cf}(\lambda) > \omega$, then by EFIMOV [1977] or GERLITS [1976] the space C contains a homeomorph of $\mathbb{D}^{\lambda}$, hence also $X \times \mathbb{D}^{\lambda}$ is normal and thus, by 3.8, X is λ-paracompact. If $\lambda > \omega$ and $\mathrm{cf}(\lambda) = \omega$, then there exist λ_n such that $\lambda = \sup\{\lambda_n : n \in N\}$ and $\omega < \lambda_n < \lambda$. By EFIMOV [1977]; Remark 3.12, for every n the space C contains a homeomorph of $\mathbb{D}^{\lambda_n}$ and thus X is λ_n-paracompact for all $n < \omega$. This easily implies λ-paracompactness of X. □

Notice, that Theorem 3.10 generalizes Corollaries 3.3, 3.6, 3.8 and 3.9.

3.11*. THEOREM. *If C is compact and extremally disconnected, then $X \times C$ is normal if and only if X is normal and $w(C)$-paracompact.*

Sketch of Proof. By BALCAR and FRANEK [1982], every compact extremally disconnected space of weight λ can be continuously mapped onto $\mathbb{D}^\lambda$. If $X \times C$ is normal, then also $X \times \mathbb{D}^\lambda$ is normal and consequently X is λ-paracompact. □

It is not true that in order for the product space $X \times C$ to be normal the space X *has to be* $w(C)$-paracompact (see 3.15). It must however be $w(C)$-collectionwise normal and countably paracompact.

3.12. THEOREM. (RUDIN [1975], DOWKER [1951]). *If C is an infinite compact space and $X \times C$ is normal, then X is countably paracompact and $w(C)$-collectionwise normal.*

PROOF. Countable paracompactness of X follows immediately from 3.9 (or 3.10) and the fact that every infinite compact space C contains a closed subspace that can be mapped onto $\bar{\omega}$ (for example the closure of an infinite and discrete-in-itself subspace of C).

Let $\lambda = w(C)$. We now prove that X is λ-collectionwise normal. Let $\{F_\alpha : \alpha < \lambda\}$ be a discrete collection of closed subsets of X, and $F = \bigcup\{F_\alpha : \alpha < \lambda\}$. It is easy to see that the space $C(C)$ of continuous real functions on C with the sup norm has weight λ and since it is a normed space it contains a family $\{f_\alpha : \alpha < \lambda\}$ of functions such that $\|f_\alpha - f_\beta\| \geq 1$ for all $\alpha \neq \beta$. Define a function $\phi : F \times C \to R$ as follows:

$$\phi(x, y) = f_\alpha(y), \quad \text{for } x \in F_\alpha \text{ and } \alpha < \lambda\,,$$

let Φ be a continuous extension of ϕ onto $X \times C$ and let ρ be a continuous pseudo-metric on X defined by:

$$\rho(x, x') = \sup\{|\Phi(x, y) - \Phi(x', y)| : y \in C\}\,.$$

It is immediate that the sets $V_\alpha = \{x \in X : \rho(x, F_\alpha) < \frac{1}{2}\}$ are open and disjoint and obviously $F_\alpha \subset V_\alpha$. □

Countable paracompactness and $w(C)$-collectionwise normality of X do not in general ensure the normality of $X \times C$.

3.13. EXAMPLE. *Countable paracompactness and $w(C)$-collectionwise normality of X do not imply the normality of $X \times C$: the spaces $\omega_1 \times \overline{\omega_1}$ and $\omega_1 \times I^{\omega_1}$ are not normal.*

PROOF. Example 2.2 or Corollary 3.7 and Corollary 3.8. □

For a cardinal number κ we shall denote by $A(\kappa)$ the one-point compactification of the discrete space of cardinality κ. We shall identify points of $A(\kappa)$ with ordinals $\alpha \leqslant \kappa$ with κ being the unique non-isolated point.

3.14. THEOREM (ALAS [1971]). *The space $X \times A(\kappa)$ is normal if and only if X is countably paracompact and κ-collectionwise normal.*

PROOF. Necessity follows immediately from 3.13. We prove sufficiency. Let us first observe, that it is sufficient to prove that for every closed subset L of $X \times A(\kappa)$ disjoint from $X \times \{\kappa\}$ there exists an open neighborhood W of L such that $\bar{W} \cap (X \times \{\kappa\}) = \emptyset$.

Indeed, let K_0 and K_1 be disjoint closed subsets of $X \times A(\kappa)$ and let $K_0^1 = K_0 \cap (X \times \{\kappa\})$ and $K_1^1 = K_1 \cap (X \times \{\kappa\})$. By the normality of X there exist disjoint open sets U_0 and U_1 in $X \times A(\kappa)$ such that $K_0^1 \subset U_0$ and $K_1^1 \subset U_1$ and also disjoint open sets V_0 and V_1 in $X \times A(\kappa)$ such that $K_0 \backslash K_0^1 \subset V_0$ and $K_1 \backslash K_1^1 \subset V_1$. Since the closed sets $L_0 = K_0 \backslash U_0$ and $L_1 = K_1 \backslash U_1$ do not intersect $X \times \{\kappa\}$, there exist open sets W_0 and W_1 in $X \times A(\kappa)$ such that $L_i \subset W_i$ and $\overline{W_i} \cap (X \times \{\kappa\}) = \emptyset$, for $i = 0$, 1. We can of course assume that $W_i \subset V_i$. Now, the sets:

$$G_0 = U_0 \backslash \bar{W}_1 \cup W_0 \quad \text{and} \quad G_1 = U_1 \backslash \bar{W}_0 \cup W_1$$

are open and disjoint, $K_0 \subset G_0$ and $K_1 \subset G_1$.

Suppose now that L is closed in $X \times A(\kappa)$ and disjoint from $X \times \{\kappa\}$. One easily checks that the family $\{F_\alpha : \alpha < \kappa\}$, where $F_\alpha = \{x \in X : \langle x, \alpha \rangle \in L\}$ is locally finite and consists of closed sets. Therefore, by collectionwise normality and countable paracompactness of X there exists a locally finite family $\{U_\alpha : \alpha < \kappa\}$ of open sets in X such that $F_\alpha \subset U_\alpha$ (see ENGELKING [1977], Problem 5.5.18). The set $W = \bigcup \{U_\alpha \times \{\alpha\} : \alpha < \kappa\}$ is open, contains L and $\bar{W} \cap (X \times \{\kappa\}) = \emptyset$, which completes the proof. $\square$

We have seen already that $w(C)$-collectionwise normality and countable paracompactness of X do not guarantee the normality of $X \times C$. On the other hand we have:

3.15. EXAMPLE. *$w(C)$ paracompactness of X is not necessary for the product $X \times C$ to be normal: $\omega_1 \times A(\omega_1)$ is normal.*

PROOF. By 3.14 (see also 3.19). $\square$

As the following theorem shows, some degree of paracompactness is however necessary.

3.16. THEOREM (KATUTA [1971]). *If $X \times C$ is normal and C compact, then X is λ-paracompact for every $\lambda < t(C)$, where $t(C)$ denotes the tightness of C. Moreover, if $t(C)$ is not weakly inaccessible, then X is $t(C)$-paracompact.*

We recall that the *tightness* $t(X)$ of X is the smallest cardinal number κ such that for every point $x_0 \in X$ and $S \subset X$, if, $x_0 \in \bar{S}$ then $x_0 \in \bar{S}_0$, for some $S_0 \subset S$ with $|S_0| \leqslant \kappa$ (See ARHANGEL'SKIĬ and PONOMARIEV [1968]; also see Hodel's article on cardinal functions in this volume).

The following lemma will be used in the proof of Theorem 3.16 and seems also to be interesting in itself.

3.17. LEMMA. *Let C be compact. The tightness $t(C)$ of C is equal to* $\sup\{\lambda \in \text{Card}:$ *there exists a closed subset F of C and a continuous mapping $f: F \to \bar{\lambda}$ of F onto $\bar{\lambda}$*$\}$.

PROOF. By ARHANGEL'SKIĬ's theorem [1971] $t(C) = \sup\{\lambda:$ there exists a free sequence $\{x_\alpha\}_{\alpha<\lambda}$ in $C\}$, where $\{x_\alpha\}_{\alpha<\lambda}$ is a *free sequence* if $\overline{\{x_\beta: \beta < \alpha\}} \cap \overline{\{x_\gamma: \gamma \geqslant \alpha\}} = \emptyset$ for all $\alpha < \lambda$. For a fixed λ, let $C_\alpha = \overline{\{x_\gamma: \gamma \geqslant \alpha\}}$ for all $\alpha < \lambda$, $C_\lambda = \bigcap\{C_\alpha: \alpha < \lambda\}$, $C_{\lambda+1} = \emptyset$ and $F = C_0$. The sets C_α are closed and open in F. The function $f: F \to \bar{\lambda}$ defined by

$$f(t) = \sup\{\alpha: t \in C_\alpha\}$$

is easily verified to be continuous and surjective.

Since the tightness of a compact space cannot be increased by a continuous mapping, no closed subset of a compact space can be mapped onto $\bar{\lambda}$ with $\lambda > t(C)$, which completes the proof of the Lemma. □

Proof of Theorem 3.16. Suppose that $X \times C$ is normal and $\kappa = t(C)$. If there exists a continuous mapping $f: F \to \bar{\lambda}$ of a closed $F \subset C$ onto $\bar{\lambda}$, then $X \times \bar{\lambda}$ is normal as a perfect image of a normal space $X \times F$ and, by 3.7, the space X is λ-paracompact, which shows, by 3.17, that $t(C) \leqslant \sup\{\lambda: X$ is λ-paracompact$\}$.

If κ is not a limit cardinal, then this obviously implies that X is $t(X)$-paracompact. If κ is not regular, then it is easy to note that being λ-paracompact for all $\lambda < \kappa$ implies being κ-paracompact. □

REMARK. KATUTA [1977] showed that the assumption that $t(C)$ is not weakly inaccessible is essential.

The above quoted results seem to suggest that for every compact space C there may exist a cardinal κ, $\omega < \kappa \leqslant w(C)^+$, such that for any space X, $X \times C$ is normal if and only if X is $w(C)$-collectionwise normal and λ-paracompact for every $\lambda < \kappa$. The following example (suggested by G. Gruenhage) refutes this conjecture (a consistent example has been given earlier by K. Kunen).

3.18*. EXAMPLE. *There exists a compact space C and two spaces X_1 and X_2 such that $X_1 \times C$ is normal, $X_2 \times C$ is not normal, but X_1 and X_2 have the same degree of paracompactness.*

Sketch of Proof. Let X_2 be the VAN DOUWEN *line* [1976], C its one point compactification and $X_1 = \omega_1$. Since X_2 is separable and non-Lindelöf an easy modification of Michael's proof (see Example 2.3) shows that $X_2 \times C$ is not normal. Moreover, it is known that X_2 is countably paracompact and is not ω_1-paracompact. On the other hand, one easily checks that C has countable tightness and therefore $X_1 \times C$ is normal by Theorem 3.19 below. Clearly, ω_1 is not ω_1-paracompact, either. □

Preceding results described restrictions imposed on X by the normality of $X \times C$. The two theorems below show, vice versa, that for some special spaces X the normality of $X \times C$ imposes strong restrictions on C.

3.19. THEOREM. (NOGURA [1976]). *For a cardinal number κ and a compact C the space $\kappa^+ \times C$ is normal if and only if $t(C) \leqslant \kappa$.*

PROOF. Since κ^+ is not κ^+-paracompact, necessity follows immediately from 3.16. We prove sufficiency. Let K_0, K_1 be disjoint closed subsets of $\kappa^+ \times C$.

Since C is compact, it is enough to show that for every $y \in C$ there exists a neighborhood V_y such that the sets $L_0 = K_0 \cap (\kappa^+ \times \bar{V}_y)$ and $L_1 = K_1 \cap (\kappa^+ \times \bar{V}_y)$ can be separated by open sets in $\kappa^+ \times \bar{V}_y$ (cf. Lemma 4.2). Take $y \in C$.

There exists an $\alpha < \kappa^+$ such that either $K_{0,y} \subset \bar{\alpha}$ or $K_{1,y} \subset \bar{\alpha}$, where $K_{i,y} = \{\beta \leqslant \kappa^+ : \langle \beta, y \rangle \in K_i\}$ for $i = 0, 1$. Suppose that e.g. $K_{0,y} \subset \bar{\alpha}$. There exists a neighborhood V_y of y in C such that $K_0 \cap (\kappa^+ \times \bar{V}_y) \subset \bar{\alpha} \times \bar{V}_y$. Indeed, suppose not. Then $y \in \bar{T}$, where $T = \{t \in C$: there exists a $\beta_t > \alpha$ with $\langle \beta_t, t \rangle \in K_0\}$ and by $t(C) \leqslant \kappa$, there exists a $T_0 \subset T$ such that $y \in \bar{T}_0$ and $|T_0| \leqslant \kappa$. Then the set $\{\beta_t : t \in T_0\}$ has an upper limit γ in κ^+ and since the space $\bar{\gamma}$ is compact, $y \notin \bar{T}_0$, which is a contradiction.

Since the space $\bar{\alpha}$ is compact and open in κ^+, the sets $L_0 \cap (\bar{\alpha} \times \bar{V}_y)$ and $L_1 \cap (\bar{\alpha} \times \bar{V}_y)$ can be separated by open sets in $\kappa^+ \times \bar{V}_y$ and therefore the sets L_0 and L_1 can be separated. □

REMARK. Observe that the above proof essentially shows that if Y is a paracompact space with $t(Y) \leqslant \kappa$, then $\kappa^+ \times Y$ is normal (KOMBAROV [1978]). Also as an application of 3.19 observe that the product space of ω_1 with the double arrow space (Example 2.8) is normal. □

3.20*. THEOREM (PRZYMUSIŃSKI [1975]). *For every cardinal κ there exists a perfectly normal space X_κ such that $X_\kappa \times C$ is normal if and only if $w(C) \leqslant \kappa$, for a compact space C.*

Sketch of Proof. By Theorems 3.3 and 3.12 it suffices to construct a perfectly normal space X_κ such that X_κ is κ-paracompact but not κ^+-collectionwise normal. □

It is well-known that a closed image of a normal space is normal (see

ENGELKING [1977], Theorem 1.5.20) and that the product $f \times g$ of two perfect mappings is again perfect. However, the product of two *closed* mappings, in general, is not closed. Thus the following theorem is particularly interesting and powerful. Its proof, however, is beyond the scope of this paper and it is still an open problem to find a reasonably short proof of it (Problem 10).

3.21*. THEOREM (RUDIN [1975]). *Suppose that C is compact and $f: X \twoheadrightarrow X'$ and $g: C \twoheadrightarrow C'$ are closed continuous mappings. If $X \times C$ is normal, then so is $X' \times C'$.* □

We complete this section with the following theorem which generalizes Corollary 3.6 and will be needed in the next section.

3.22. THEOREM. (MORITA [1963b]). *If X is paracompact and Y is paracompact and σ-locally compact, then the product space $X \times Y$ is paracompact.*

PROOF. Let $\mathscr{G}$ be an open covering of $X \times Y$ and let $Y = \bigcup_{n<\omega} \mathscr{F}_n$, where $\mathscr{F}_n$ is a locally finite family of compact subsets of Y. By paracompactness of Y there exist locally finite families $\mathscr{U}_n = \{U(F): F \in \mathscr{F}_n\}$ of open sets in Y such that $F \subset U(F)$. Since the sets F are compact, for any fixed $F \in \mathscr{F} = \bigcup_{n<\omega} \mathscr{F}_n$ and every $x \in X$ there exists an open neighborhood $W(F, x)$ of x in X and a finite family $\mathscr{V}(F, x)$ of open subsets of Y such that

$$F \subset \bigcup \mathscr{V}(F, x) \subset U(F)$$

and for every $V \in \mathscr{V}(F, x)$ there exists a $G \in \mathscr{G}$ with $W(F, x) \times V \subset G$.

By paracompactness of X there is a locally finite open covering $\{G(F, x): x \in X\}$ of X such that $G(F, x) \subset W(F, x)$. The family

$$\mathscr{U} = \{G(F, x) \times V: F \in \mathscr{F}, x \in X, V \in \mathscr{V}(F, x)\}$$

is σ-locally finite and refines $\mathscr{G}$, which completes the proof. □

REMARK. In a similar way one can prove that $X \times Y$ is countably paracompact if X is countably paracompact and Y is metrizable and σ-locally compact. □

Extensive research has been done to investigate the class of all spaces Y such that $X \times Y$ is paracompact for every paracompact space X. As is shown by Theorem 3.22 this class contains all locally compact paracompact spaces. In fact, it is much broader and, in particular, contains all closed images of locally compact paracompact spaces. As yet, however, no satisfactory characterization of this class is known (cf. however KATUTA [1971]). Theory of games plays an interesting role in this problem. For more information, the reader is referred to TELGÀRSKY's papers [1969], [1971], [1975] and to my survey article PRZYMUSIŃSKI [1980b]; also see Problems 5 and 6.

4. Products with a metric factor

In this section we investigate products of the form $X \times M$ where M is metrizable; M always denotes a *metrizable space* and $\mathcal{B}$ denotes any *base of* M such that $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ is the union of locally finite coverings $\mathcal{B}_n$ of M by sets of diameter $<1/n$ (with respect to some metric on M).

The following theorem gives a complete characterization of normality of products with a metric factor. We call a family $\{F_B: B \in \mathcal{B}\}$ of sets *monotone* if $F_B \subset F_{B'}$ provided $B \subset B'$.

4.1. CHARACTERIZATION THEOREM. Let $\mathcal{B}$ be a base of a metrizable space M. The product space $X \times M$ is normal if and only if X is normal and for every monotone family $\{F_B: B \in \mathcal{B}\}$ of closed subsets of X, such that $\bigcap\{F_B: z \in B\} = \emptyset$ for all $z \in M$, there exists a family $\{U_B: B \in \mathcal{B}\}$ of open subsets of X such that $F_B \subset U_B$ and $\bigcap\{U_B: z \in B\} = \emptyset$ for all $z \in M$.

In order to prove Theorem 4.1 we shall need the following lemma which is often used in proofs of normality.

4.2. LEMMA. *A space Z is normal if and only if for every pair K_0, K_1 of disjoint closed subsets of Z there exists a σ-locally finite open covering $\mathcal{U}$ of Z such that either $\bar{U} \cap K_0 = \emptyset$ or $\bar{U} \cap K_1 = \emptyset$ for every $U \in \mathcal{U}$.*

PROOF. Let $\mathcal{U} = \bigcup_{n<\omega} \mathcal{U}_n$, where the families $\mathcal{U}_n$ are locally finite. Define

$$U_i(n) = \bigcup\{U \in \mathcal{U}_n : \bar{U} \cap K_{1-i} = \emptyset\}$$

for all $n < \omega$ and $i = 0, 1$. Clearly, $\bigcup_{i=1}^{2} \bigcup_{n<\omega} U_i(n) = Z$ and $\overline{U_i(n)} \cap K_{1-i} = \emptyset$. Let $V_i(n) = U_i(n) \setminus \bigcup_{j \leq n} \overline{U_{1-i}(j)}$, for $i = 0, 1$ and $n < \omega$ and let

$$V_i = \bigcup_{n<\omega} V_i(n) .$$

One easily checks that $K_i \subset V_i$ and $V_0 \cap V_1 = \emptyset$. □

Proof of Theorem 4.1. (⇒) Let K_0 and K_1 be disjoint closed subsets of $X \times M$. For every $B \in \mathcal{B}$ and $i = 0, 1$ let

$$F_B^i = \overline{\pi(K_i \cap (X \times \bar{B}))} \quad \text{and} \quad F_B = F_B^0 \cap F_B^1 ,$$

where $\pi: X \times M \to X$ is the projection.

Clearly, the family $\{F_B: B \in \mathscr{B}\}$ is monotone and closed. Let $z \in M$ and $x \in X$. There is $i = 0, 1$ such that $\langle x, z\rangle \notin K_i$. Find an open set $U \times V$ containing $\langle x, z\rangle$ such that $(U \times V) \cap K_i = \emptyset$. There exists a $D \in \mathscr{B}$ such that $z \in D$ and $\bar{D} \subset V$. Then $(U \times \bar{D}) \cap K_i = \emptyset$ and therefore $U \cap F_D = \emptyset$, $x \notin F_D$ and $\bigcap\{F_B: z \in B\} = \emptyset$.

Let $\{U_B: B \in \mathscr{B}\}$ be a family of open sets in X such that $F_B \subset U_B$ and:

$$(*) \qquad \bigcap\{U_B: z \in B\} = \emptyset \quad \text{for all } z \in M.$$

Since $U_B \supset F_B^0 \cap F_B^1$, $X \backslash U_B \subset X \backslash F_B^0 \cup X \backslash F_B^1$ and by normality of X there exist open sets V_B^0 and V_B^1 such that

$$X \backslash U_B \subset V_B^0 \cup V_B^1, \qquad \overline{V_B^0} \cap F_B^0 = \emptyset \quad \text{and} \quad \overline{V_B^1} \cap F_B^1 = \emptyset.$$

Let $\mathscr{V} = \{V_B^0 \times B: B \in \mathscr{B}\} \cup \{V_B^1 \times B: B \in \mathscr{B}\}$. The family $\mathscr{V}$ is open and since $\mathscr{B}$ is σ-locally finite, so is $\mathscr{V}$. Moreover,

$$\overline{V_B^i \times B} \cap K_i = \emptyset \quad \text{for all } B \in \mathscr{B} \text{ and } i = 0, 1.$$

Therefore, by Lemma 4.1 to prove the normality of $X \times M$ it suffices to show that $\mathscr{V}$ is a covering of $X \times M$. Let $\langle x, z\rangle \in X \times M$ and let $B \in \mathscr{B}$ be such that $z \in B$ and $x \notin U_B$. Consequently, $x \in V_B^0 \cup V_B^1$ and therefore $\langle x, z\rangle \in V_B^0 \times B \cup V_B^1 \times B \subset \bigcup \mathscr{V}$.

($\Leftarrow$) Suppose now that the space $X \times M$ is normal and let $\{F_B: B \in \mathscr{B}\}$ be an arbitrary monotone family of closed sets such that:

$$(**) \qquad \bigcap\{F_B: z \in B\} = \emptyset \quad \text{for every } z \in M.$$

Let $\mathscr{B}^* = \{B \in \mathscr{B}: B \text{ is } \textit{not} \text{ discrete-in-itself}\}$. By transfinite induction we can easily choose points p_B^0, $p_B^1 \in B$, for $B \in \mathscr{B}^*$, so that the family $\{p_B^0\}_{B \in \mathscr{B}^*} \cup \{p_B^1\}_{B \in \mathscr{B}^*}$ consists of different points. Let $K_i = \bigcup_{B \in \mathscr{B}^*}(F_B \times \{p_B^i\})$, for $i = 0, 1$. The sets K_i are clearly disjoint. Moreover, they are closed. Indeed, letting

$$K_i^n = \bigcup\{F_B \times \{p_B^i\}: B \in \mathscr{B}^* \cap \mathscr{B}_n\}$$

we have $K_i = \bigcup_{n<\omega} K_i^n$. The sets K_i^n are closed by the local finiteness of $\mathscr{B}_n$. It is enough to prove that the family $\{K_i^n\}_{n<\omega}$ is locally finite, for $i = 0, 1$. Let $\langle x, z\rangle \in X \times M$ and let $B \in \mathscr{B}$ be such that $z \in B$ and $x \notin F_B$. There is an n and a $B' \in \mathscr{B}_n$ such that $z \in B'$ and

$$\operatorname{St}(B', \mathscr{B}_n) = \bigcup\{B \in \mathscr{B}_n: B \cap B' \neq \emptyset\} \subset B.$$

If $m \geqslant n$, then $((X \backslash F_B) \times B') \cap K_i^m = \emptyset$, because otherwise we would have

$$\emptyset \neq ((X \backslash F_B) \times B') \cap (F_D \times \{p_D^i\})$$

for some $D \in \mathscr{B}^* \cap \mathscr{B}_m$ and $B' \cap D \neq \emptyset$, which would imply $D \subset B$ and $F_D \subset F_B$, which is impossible.

Now, let U_0 and U_1 be open sets such that $\bar{U}_0 \cap \bar{U}_1 = \emptyset$ and $K_i \subset U_i$, for $i = 0, 1$. For $B \in \mathscr{B}^*$ define

$$U_B = \{x \in X : \langle x, p_B^i \rangle \in U_i, \text{ for } i = 0, 1\},$$

and for $B \in \mathscr{B} \backslash \mathscr{B}^*$ define

$$U_B = \begin{cases} \emptyset, & \text{if } B \text{ consists of one point}, \\ X, & \text{otherwise}. \end{cases}$$

Clearly $F_B \subset U_B$. Let z be a non-isolated point of M and $x \in X$. There is an i such that $\langle x, z \rangle \notin \bar{U}_i$, a neighborhood V of x in X and a $B \in \mathscr{B}^*$ containing z such that $(V \times B) \cap \bar{U}_i = \emptyset$. Then $V \cap U_B = \emptyset$ and therefore $x \notin U_B$.

For an isolated point $z \in M$ the proof is trivial. □

Most of the theorems involving products with a metric factor can be derived from Theorem 4.1. The following beautiful result underscores the essential difference existing between products with a metric and with a compact factor.

4.3. COROLLARY (MORITA [1966], TAMANO [1960], RUDIN and STARBIRD [1975]). *The product $X \times M$ of a normal space X and a non-discrete metric space M is normal if and only if it is countably paracompact.*

PROOF. ($\Rightarrow$) Suppose that $X \times M$ is normal. By Corollary 3.9 it suffices to show that $X \times \bar{\omega} \times M$ is normal. Since $X \times M$ is normal and M is not discrete, $X \times \bar{\omega}$ is normal. Let $\{F_B : B \in \mathscr{B}\}$ be a monotone family of closed subsets of $X \times \bar{\omega}$ satisfying (**). Define $F'_B = p(F_B)$, where $p: X \times \bar{\omega} \to X$ is the projection. Since p is closed one easily sees that the family $\{F'_B : B \in \mathscr{B}\}$ is monotone and satisfies (**). Therefore, by Theorem 4.1, there exists a family $\{U'_B : B \in \mathscr{B}\}$ of open subsets of X such that $F'_B \subset U'_B$ and (*) is satisfied. Clearly the family $\{U_B : B \in \mathscr{B}\}$, where $U_B = p^{-1}(U'_B)$ satisfies (*) and $F_B \subset U_B$.

($\Leftarrow$) Suppose now that $X \times M$ is countably paracompact and let $\{F_B : B \in \mathscr{B}\}$ be a monotone family of closed subsets of X satisfying (**), where we choose $\mathscr{B}$ so that $\mathscr{B}_{n+1}$ is a refinement of $\mathscr{B}_n$, for all $n < \omega$ (recall that Theorem 4.1 holds for any base $\mathscr{B}$). Define

$$F_n = \bigcup \{F_B \times \bar{B} : B \in \mathscr{B}_n\}.$$

Observe, the sets F_n are closed and also $F_{n+1} \subset F_n$ for all $n < \omega$. Indeed, if $\langle x, z \rangle \in F_{n+1}$, then $\langle x, z \rangle \in F_B \times \bar{B}$ for some $B \in \mathscr{B}_{n+1}$, and there exists a $B' \in \mathscr{B}_n$ such that $B \subset B'$ and, consequently, $\langle x, z \rangle \in F_{B'} \times \overline{B'} \subset F_n$.

Moreover, $\bigcap_{n<\omega} F_n = \emptyset$. Indeed, if $\langle x, z\rangle \in X \times M$, then there exists a $B \in \mathscr{B}$ such that $z \in B$ and $x \notin F_B$. There exists an n and a $B' \in \mathscr{B}$ containing z and such that

$$\mathrm{St}(B', \mathscr{B}_n) = \bigcup\{D \in \mathscr{B}_n \colon D \cap B' \neq \emptyset\} \subset B .$$

If $D \in \mathscr{B}_n$ and $\langle x, z\rangle \in F_D \times D$, then $D \cap B' = \emptyset$ and therefore $D \subset B$. Consequently, $F_D \subset F_B$ and $x \notin F_D$, which is impossible.

Let $U_n \supset F_n$ be decreasing open subsets of $X \times M$ such that $\bigcap_{n<\omega} \bar{U}_n = \emptyset$ and for $B \in \mathscr{B}$ define $U_B = \pi(U_n \cap (X \times \bar{B}))$, if B is not discrete and $n = n(B)$ is the smallest integer for which $B \in \mathscr{B}_n$, and $U_B = \emptyset$, if $|B| = 1$ and $U_B = X$, in the remaining case.

Clearly, $U_B \supset F_B$. Let $z \in M$ and $x \in X$. There is an $n < \omega$ and open sets V, W such that $\langle x, z\rangle \in V \times W$ and $(V \times W) \cap U_n = \emptyset$. Let B be such that $z \in B$, $\bar{B} \subset W$, and $m = n(B) \geqslant n$. Then $(V \times \bar{B}) \cap U_m = \emptyset$ and therefore $x \notin U_B$, which completes the proof. $\square$

4.4. COROLLARY (RUDIN and STARBIRD [1975]). *Suppose that C is compact and M is metric. If the product spaces $X \times M$ and $X \times C$ are normal, then the product space $X \times M \times C$ is normal.*

PROOF. This is trivial if M is discrete. By 4.3 the space $X \times M$ is countably paracompact and therefore also $X \times M \times C$ is countably paracompact (see Remark 3.5.R) and consequently, again by 4.3, $X \times M \times C$ is normal. $\square$

It is not clear to what extent the assumptions that C is compact and M is metric in Corollary 4.4 can be relaxed (see Problem 1). There exist however 'nice' spaces such that X^2 is normal but X^3 is not (see MICHAEL [1971], PRZYMUSIŃSKI [1980c], [1980a] or Example 4.14), so some assumptions on C and M are necessary.

4.5. COROLLARY (RUDIN and STARBIRD [1975]). *Suppose that X is paracompact and M metrizable. The following conditions are equivalent.*

(i) *$X \times M$ is normal,*

(ii) *$X \times M$ is countably paracompact.*

(iii) *$X \times M$ is paracompact.*

Moreover, if M is separable and X Lindelöf, then each of the above conditions is equivalent to:

(iv) *$X \times M$ is Lindelöf.*

PROOF. This is trivial if M is discrete. Clearly (iii) implies (ii) and (i) and (ii) are equivalent by 4.3. It suffices to prove that (i) implies (iii). But $X \times \beta(X \times M)$ is normal by 3.5 and therefore $X \times M \times \beta(X \times M)$ is normal by 4.4, which in view of 3.4 implies that $X \times M$ is paracompact.

Suppose that X is Lindelöf and M is separable. It remains to prove that (iii) implies (iv). Let S be a countable dense subset of M; clearly $X \times S$ is dense in $X \times M$ and Lindelöf. Suppose now that $\mathcal{U}$ is an open covering of $X \times M$. Let $\mathcal{V}$ be a locally finite open refinement of $\mathcal{U}$. Since $X \times S$ is Lindelöf, it can intersect only countably many elements of $\mathcal{V}$ and since $X \times S$ is dense, the family $\mathcal{V}$ itself must be countable. □

The following result corresponds to Theorem 3.21 for compact spaces.

4.6. COROLLARY (RUDIN and STARBIRD [1975]). *Suppose that $f: X \twoheadrightarrow X'$ and $g: M \twoheadrightarrow M'$ are closed continuous mappings and M and M' are metrizable. If $X \times M$ is normal (paracompact), then so is $X' \times M'$.*

PROOF. First observe that $X \times M'$ is normal. Indeed, since g is closed there exists a closed subspace M_0 of M such that $g|M_0: M_0 \twoheadrightarrow M'$ is perfect (see ENGELKING [1977], Problem 5.5.11(c)). Therefore, since $X \times M_0$ is normal, the space $X \times M'$ is also normal as a perfect image of $X \times M_0$.

Now, we prove that $X' \times M'$ is normal. Let $\mathcal{B}$ be an appropriate base for M' and let $\{F'_B: B \in \mathcal{B}\}$ be a monotone collection of closed subsets of X' satisfying (**). The family $\{F_B: B \in \mathcal{B}\}$, where $F_B = f^{-1}(F'_B)$, is also a monotone family of closed subsets of X satisfying (**) and therefore there exists a family $\{U_B\}$ of open subsets of X satisfying (*). Define $U'_B = X' \setminus f(X \setminus U_B)$, for $B \in \mathcal{B}$. One easily checks that $F'_B \subset U'_B$ and that the family $\{U'_B: B \in \mathcal{B}\}$ satisfies (*). Therefore $X' \times M'$ is normal by 4.1. □

The following result clearly does not have an exact compact counterpart. However, it is unknown whether its restricted version, with $M = M'$ holds for compact spaces (see Problem 4).

4.7. COROLLARY. *Let $f: X \twoheadrightarrow X'$ and $g: M \twoheadrightarrow M'$ be perfect mappings and let M and M' be metric non-discrete. If spaces $X' \times M'$ and X are normal, then so is $X \times M$.*

PROOF. The mapping $f \times g$ is perfect and $X' \times M'$ is countably paracompact by 4.3. Therefore $X \times M$ is countably paracompact (see Remark 3.5.R) and consequently, again by 4.3, $X' \times M'$ is normal. □

Before proving the next corollary we shall need the following lemma.

4.8. LEMMA (MICHAEL [1953]). *The product of a metric and a perfect space is perfect.*

PROOF. Let X be perfect, M-metrizable, $\mathcal{B}$ a base for M and U an open subset of $X \times M$. For every $B \in \mathcal{B}$ let $V_B = \bigcup\{V: V \text{ is open in } X \text{ and } V \times \bar{B} \subset U\}$. Let

$V_B = \bigcup_{n<\omega} F_B^n$, where the sets F_B^n are closed in X. Clearly, $U = \bigcup\{V_B \times \bar{B}: B \in \mathcal{B}\} = \bigcup_{n<\omega} \bigcup\{F_B^n \times \bar{B}: B \in \mathcal{B}\}$ and therefore, U is a σ-locally finite union of closed sets. □

4.9. COROLLARY (MORITA [1964]). *The product of a perfectly normal space and a metric space is perfectly normal.*

PROOF. Let X be a perfectly normal space and let M be metrizable. Let $\mathcal{B}$ be a base of M which additionally satisfies the condition that for all $n < \omega$ and every $B \in \mathcal{B}_{n+1}$, the set B intersects only finitely many members of $\mathcal{B}_n$. Let $\{F_B: B \in \mathcal{B}\}$ be a monotone collection of closed subsets of X satisfying (**). Choose open sets $G_B^n \supset F_B$ so that $G_B^{n+1} \subset G_B^n$ and $\bigcap_{n<\omega} G_B^n = F_B$. For every $B \in \mathcal{B}_n$ let $U_B = \emptyset$ if $|B| = 1$ and $U_B = \bigcap\{G_D^n: D \supset B$ and $D \in \bigcup_{i\leq n} \mathcal{B}_i\}$, otherwise.

Let $z \in M$ and $x \in X$. There is a $B \in \mathcal{B}_k$, $B \ni z$ such that $x \notin F_B$. There is an $n \geq k$ such that $x \notin G_B^n$. Choose $B' \in \mathcal{B}_m$, $B' \ni z$ so that $B' \subset B$ and $m \geq n$. Then $x \notin U_{B'}$, which, in view of 4.1, proves normality of $X \times M$. Perfect normality now follows from 4.8. □

It follows from Corollary 4.5 that the product $X \times M$ of a perfectly paracompact (Lindelöf) space X and a metric (separable) space M is again perfectly paracompact (Lindelöf).

4.10. COROLLARY. (STONE [1948]). *If X is countably compact and normal and M is metric, then $X \times M$ is normal.*

PROOF. Let $\{F_B: B \in \mathcal{B}\}$ be a monotone closed collection of subsets of X satisfying (**). Since every decreasing collection of non-empty closed subsets of X has a non-empty intersection, for every $z \in M$ there is a $B \in \mathcal{B}$, $B \ni z$ such that $F_B = \emptyset$. It suffices to define

$$U_B = \begin{cases} X, & \text{if } F_B \neq \emptyset \\ \emptyset, & \text{otherwise}. \end{cases} \quad \square$$

As is seen from Corollaries 4.9 and 4.10 the class of spaces whose products with all metric spaces are normal contains all perfectly normal spaces and all normal countably compact spaces. Morita gave a complete characterization of this class, which we present below. First we give a definition of a P-space; it differs slightly from the original definition given by MORITA [1964], but is obviously equivalent.

For a cardinal number κ let us denote by $\kappa^{<\omega}$ the family of all functions $\sigma: n \to \kappa$, $n < \omega$, ordered by inverse inclusion: $\sigma < \delta$ iff $\sigma \supset \delta$.

DEFINITION. A space X is a *P-space* if for every cardinal κ and every monotone family $\{F_\sigma: \sigma \in \kappa^{<\omega}\}$ of closed subsets of X there exists a family $\{U_\sigma: \sigma \in \kappa^{<\omega}\}$ of open subsets of X such that for every $f \in \kappa^\omega$, if $\bigcap_{n<\omega} F_{f|n} = \emptyset$, then $\bigcap_{n<\omega} U_{f|n} = \emptyset$.

4.11. COROLLARY (MORITA [1964]). *The product space $X \times M$ is normal (paracompact) for every metric space M if and only if X is a normal (paracompact) P-space.*

PROOF. ($\Rightarrow$) Let $\{F_\sigma : \sigma \in \kappa^{<\omega}\}$ be as in the definition of a P-space and let $M = \{f \in \kappa^\omega : \bigcap_{n<\omega} F_{f|n} = \emptyset\}$ be a subspace of κ^ω, considered as a countable product of discrete spaces of cardinality κ (i.e. κ^ω is the *Baire space* of weight κ). Clearly $\mathcal{B} = \{[\sigma] \cap M : \sigma \in \kappa^{<\omega}\}$, where $[\sigma] = \{f \in \kappa^\omega : f \supset \sigma\}$ is a base of M and therefore from normality of $X \times M$ it follows that there exist open subsets U_σ of X such that $F_\sigma \subset U_\sigma$ and $\bigcap\{U_\sigma : f \in [\sigma]\} = \emptyset$ for every $f \in M$, and therefore X is a P-space.

($\Leftarrow$) Let X be a normal P-space and M be metric. There exists a subspace M_0 of κ^ω, where $\kappa = w(X)$, and a perfect mapping $f: M_0 \twoheadrightarrow M$ (see ENGELKING [1977]; Exercise 4.4.3). Since a perfect image of a normal space is normal, it suffices to prove that $X \times M_0$ is normal and therefore we will simply assume that M already is a subspace of κ^ω.

Let $\mathcal{B} = \{[\sigma] \cap M : \sigma \in \kappa^{<\omega}\}$ be the base of M and let $\{F_\sigma : \sigma \in \kappa^{<\omega}\}$ be a monotone family of closed subsets of X such that $\bigcap\{F_\sigma : f \in [\sigma]\} = \emptyset$ for every $f \in M$. Then, by the definition of a P-space, there exists a family $\{U_\sigma : \sigma \in \kappa^\omega\}$ of open subsets of X such that $F_\sigma \subset U_\sigma$ and $\cap\{U_\sigma : f \in [\sigma]\} = \emptyset$ for every $f \in M$, which proves that $X \times M$ is normal. If X is paracompact, then paracompactness of $X \times M$ follows from Corollary 4.5. $\square$

The class of normal P-spaces is closed with respect to closed mappings (4.6) and contains all perfectly normal spaces (4.9), all normal countably compact spaces (4.10) and all σ-locally compact paracompact spaces (3.22). The last two classes can be significantly broadened:

4.12. COROLLARY. (YAJIMA [1979]). *Every space X having a σ-conservative closed covering by countably compact sets is a P-space.*

PROOF. Recall, that a closed covering $\mathcal{A}$ is *conservative* if $\bigcup\mathcal{B}$ is closed for every $\mathcal{B} \subset \mathcal{A}$. Suppose that $\{F_\sigma : \sigma \in \kappa^{<\omega}\}$ is as in the definition of a P-space and let $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$ be a σ-conservative closed covering of X by countably compact sets.

For $\sigma \in \kappa^n$ define

$$U_\sigma = X \setminus \bigcup\{F : F \in \bigcup_{i=1}^{n} \mathcal{F}_i \text{ and } F \cap F_\sigma = \emptyset\}.$$

Clearly the sets U_σ are open and $F_\sigma \subset U_\sigma$. Suppose that $f \in \kappa^\omega$ and $\bigcap_{n<\omega} F_{f|n} = \emptyset$. Take $x_0 \in X$ and $n < \omega$ such that $x_0 \in F_0 \in \mathcal{F}_n$. Since F_0 is countably compact there is an m such that $F_{f|m} \cap F_0 = \emptyset$. For $k = \max\{n, m\}$, $x_0 \notin U_{f|k}$ and therefore $\bigcap_{k<\omega} U_{f|k} = \emptyset$. $\square$

Theorem 4.11 gave a complete characterization of the class of those spaces X whose products $X \times M$ with metric spaces M are normal (paracompact). The following theorem settles the converse problem.

4.13*. THEOREM (MORITA [1963a, b]). *Let M be metric. The space $X \times M$ is normal (paracompact) for every normal and countably paracompact (paracompact) space X if and only if M is σ-locally compact.*

Sketch of Proof. The sufficiency follows from Theorem 3.22, the Remark following 3.22 and Corollary 4.3.

It remains to prove that if $X \times M$ is normal for every paracompact space, then M is σ-locally compact.

Suppose that M is not σ-locally compact. By a classical result of Stone [1962] there exists a metric space M_0 containing M as a dense non F_σ subset. Now, we follow Michael's technique (Example 2.3) to construct a paracompact space X such that $X \times M$ is not normal.

Let X be the set M_0 with the topology generated by the base $\mathcal{U} = \{U: U$ is open in M_0 or $U \subset M\}$. Similarly as in 2.3 we observe that X is paracompact. The sets $K_0 = (X \backslash M) \times M$ and $K_1 = \{\langle z, z\rangle: z \in M\}$ are easily seen to be closed and disjoint. Suppose that U_0 and U_1 are disjoint open subsets of $X \times M$ such that $K_i \subset U_i$, for $i = 0, 1$, and let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ be a σ-discrete base of M_0. For every $z \in M$ choose an $n = n(z)$ and a $B \in \mathcal{B}_n$ such that $z \in B$ and $\{z\} \times (\bar{B} \cap M) \subset U_1$ and let $P_n = \{z \in M: n(z) = n\}$. There exists an n such that P_n is not closed in M_0 and since $\mathcal{B}_n$ is discrete, there exists a $B \in \mathcal{B}_n$ such that $\bar{B} \cap P_n$ is not closed in M_0. Choose $z_0 \in X \backslash M$ such that $z_0 \in \overline{B \cap P_n}^{M_0}$ and an arbitrary $z_1 \in \bar{B}$. Clearly $\langle z_0, z_1\rangle \in \bar{U}_1 \cap K_0$, which is a contradiction. □

None of the results stated in this section is valid if the assumption that M is metrizable is dropped. The reader can easily provide counterexamples for some of the results, using the material from Sections 2 and 3. Below we present the non-trivial counterexamples.

The first example shows that normality of a product of two paracompact spaces does not necessarily imply its paracompactness (cf. 4.5 and 4.18).

4.14*. EXAMPLE (PRZYMUSIŃSKI [1980a]). *There exists a Lindelöf space X such that X^2 is normal but not paracompact.* □

REMARKS. The construction of this example is beyond the scope of this article. It is possible, however, to construct such examples—with less effort—under various set-theoretic assumptions. In particular, one can prove, using Martin's Axiom, that the square X^2 of every subspace X of the Sorgenfrey line S (Example 2.1) of cardinality less than continuum is normal (see PRZYMUSIŃSKI [1976]; see also ALSTER and PRZYMUSIŃSKI [1976] for a more general result). Assuming additionally the negation of CH and choosing X, so that $|X| = \omega_1$ and $X = \{-x : x \in X\}$ we obtain a Lindelöf space X, whose square X^2 is normal but not paracompact, because X^2 is separable and contains an uncountable closed discrete subset $\{\langle x, -x\rangle : x \in X\}$, (see PRZYMUSIŃSKI [1976]).

Another consistent example of this kind was constructed under CH by ALSTER and ZENOR [1977].

It is also worth pointing out that for every $n < \omega$ the space X in 4.14 can be constructed so that X^n is Lindelöf and X^{n+1} is normal, but X^{n+1} is not paracompact. □

The next example refers to Corollary 4.6 (see also 4.19).

4.15*. EXAMPLE (CHIBA and CHIBA [1974], RUDIN and STARBIRD [1975]). *There exists a space X and a closed mapping $f: M \twoheadrightarrow Y$ of a metric space M onto Y such that $X \times M$ is normal, but $X \times Y$ is not.*

Sketch of Proof. The space X is Bing's example of a normal non-collectionwise normal space (see ENGELKING [1977]; Example 5.1.23), i.e. X is the set $2^{\mathcal{P}(\omega_1)}$, with all points isolated, except for points $f_\alpha \in X$, for $\alpha < \omega_1$, which have standard product neighborhoods, where f_α is defined by:

$$f_\alpha(A) = \begin{cases} 0, & \text{if } \alpha \notin A \\ 1, & \text{otherwise}. \end{cases}$$

The space M is the discrete sum of ω_1 copies of $\bar{\omega}$, i.e. $M = \omega_1 \times \bar{\omega}$, with ω_1 equipped with discrete topology, and Y is the quotient space of M obtained by identifying all limit points of M to a single point θ, i.e. $Y = \{\theta\} \cup (\omega_1 \times \omega)$, with all points of $\omega_1 \times \omega$ isolated. Let f be the corresponding quotient mapping.

We denote by F the closed set $\{f_\alpha : \alpha < \omega\}$ and for each $\alpha < \omega_1$ we renumber $\{f_\beta\}_{\beta<\alpha}$ as $\{f_i^\alpha\}_{i<\omega}$. The sets

$$K = F \times \{\theta\} \quad \text{and} \quad L = \{\langle f_i^\alpha, \langle \alpha, i\rangle\rangle \in X \times Y : i < \omega, \alpha < \omega_1\}$$

are closed and disjoint in $X \times Y$ and one can prove that they cannot be separated by disjoint open sets. □

The last example shows that metrizability of M is essential in 4.13 (cf. 4.21).

4.16*. EXAMPLE (VAN DOUWEN [1976]). *There exists a normal, locally compact and countably paracompact space X and a metric space M such that the product space $X \times M$ is not normal.*

Sketch of Proof. The space X is the so called VAN DOUWEN *line* [1976], i.e. X is the real line $\mathbb{R}$ with a new, locally compact topology, which is stronger than the usual topology on $\mathbb{R}$. The space X contains an open and discrete-in-itself subset A which is not an F_σ in X. Define M to be the set A with the topology of the subspace of $\mathbb{R}$. Essentially, repeating Michael's argument from Example 2.3, one can show that the sets

$$K_0 = (X\backslash A) \times M \quad \text{and} \quad K_1 = \{\langle z, z\rangle \in X \times M : z \in A\}$$

are disjoint closed subsets of $X \times M$ that cannot be separated by disjoint open sets. □

Some of the results proved in this section can be extended however onto a broader class of spaces that contains both metrizable and compact spaces.

DEFINITION. Closed subspaces of products $M \times C$ of a metric space M and a compact space C are called *paracompact p-spaces* (ARHANGEL'SKIĬ [1965], MORITA [1963a]). (MORITA calls them *paracompact M-spaces.*)

Obviously, countable products of paracompact p-spaces are paracompact p-spaces again, and hence—in particular—paracompact.

4.17. THEOREM. *A space T is a paracompact p-space if and only if T is an inverse image of a metrizable space under a perfect mapping.*

PROOF. If T is a closed subset of $M \times C$ then the mapping $\pi|T\colon T \twoheadrightarrow M'$, where $\pi\colon M \times C \to M$ is the projection, is a perfect mapping of T onto a metric space $M' = \pi(T)$.

Conversely, if $f\colon T \twoheadrightarrow M$ is perfect, and g is an embedding of T into any compact space C, then the diagonal mapping $h = f \Delta g\colon T \to M \times C$, defined by $h(t) = (f(t), g(t))$, is a homeomorphic embedding of T onto a closed subspace of $M \times C$. □

The following result generalizes Corollary 4.5.

4.18. COROLLARY. *Let X be paracompact and T a paracompact p-space. The following are equivalent.*

(i) $X \times T$ is normal.

(ii) $X \times T$ is countably paracompact.

(iii) $X \times T$ is paracompact.

Moreover, if both X and T are Lindelöf, then the above conditions are equivalent to

(iv) $X \times T$ *is Lindelöf.*

PROOF. Clearly (iv) implies (iii) and (iii) implies both (ii) and (i). Let f be a perfect mapping of T onto a metric space M. If $X \times T$ is countably paracompact, then so is $X \times M$ as a perfect image of $X \times T$ under $i \times f$, where $i: X \twoheadrightarrow X$ is the identity. Therefore $X \times M$ is paracompact (Lindelöf) by 4.5 and, consequently, $X \times T$ is paracompact (or Lindelöf) as a perfect preimage of a paracompact (Lindelöf) space. This shows that (ii) implies (iii) (or (iv)). Analogous proof shows that (i) implies (iii) (or (iv)). □

The next result generalizes half of Corollary 4.6. It is not known whether the other half has a similar generalization (see Problem 2).

4.19. COROLLARY (RUDIN and STARBIRD [1975]). *Suppose that $f: X \twoheadrightarrow X'$ and $g: T \to T'$ are closed mappings and T and T' are paracompact p-spaces. If $X \times T$ is paracompact, then so is $X' \times T'$.*

PROOF. First of all, observe that $X \times T'$ is paracompact. Indeed, since $g: T \twoheadrightarrow T'$ is closed, there exists a closed subspace T_0 of T such that $g|T_0: T_0 \twoheadrightarrow T'$ is perfect, and therefore $X \times T'$ is paracompact as a perfect image of a paracompact space $X \times T_0$ (see ENGELKING [1977], Theorem 5.1.33).

We can therefore assume that $T = T'$. Let $h: T \twoheadrightarrow M$ be a perfect mapping of T onto a metric space M. Since $X \times T$ is paracompact, also the space $X \times M$ is paracompact as a perfect image of $X \times T$. By 4.6 the space $X' \times M$ is paracompact and therefore also $X' \times T$ is paracompact as a perfect preimage of a paracompact space. □

The next corollary is a counterpart of 4.11 for paracompact p-spaces.

4.20. COROLLARY. (MORITA [1964]). *The product space $X \times T$ is normal (paracompact) for every paracompact ρ-space T if and only if X is a paracompact P-space.*

PROOF. The necessity follows immediately from 4.11 and 3.4.

Suppose that X is a paracompact P-space, T is a paracompact p-space and let $f: T \twoheadrightarrow M$ be a perfect mapping onto a metric space M. By 4.11 and 4.5 the space $X \times M$ is paracompact and therefore $X \times T$ is paracompact as a perfect preimage of a paracompact space. □

Finally, the following result is a counterpart of 4.13.

4.21. COROLLARY (MORITA [1964]). *Let T be a paracompact p-space. The product space $X \times T$ is normal (paracompact) for every paracompact space X if and only if T is σ-locally compact.*

PROOF. Let $f: T \twoheadrightarrow M$ be a perfect mapping onto a metric space. Since the product mapping $i \times f$, where $i: X \to X$ is the identity, is perfect, the product space $X \times T$ is paracompact for every paracompact space X if and only if $X \times M$ is paracompact, which by 4.13 is equivalent to M being σ-locally compact. This, together with 3.6, completes the proof. □

REMARK. RUDIN and STARBIRD [1975] proved a counterpart of Theorem 3.21 and Corollary 4.6 for the class of ordinals; namely, if α is an ordinal and $f: X \twoheadrightarrow Y$ is a closed mapping, then normality of $X \times \alpha$ implies the normality of $Y \times \alpha$. □

5. Hereditarily normal products

In this section we investigate hereditary normality of finite products. Hereditary normality of a product space $X \times Y$ imposes rather strong restrictions on the factors:

5.1. THEOREM (KATĚTOV [1948]). *If $X \times Y$ is hereditarily normal, then either X is perfectly normal or every countable subset of Y is closed and discrete.*

PROOF. Suppose that F is a non-G_δ closed subset of X and $D = \{d_n: n < \omega\}$ is a countable subset of Y with an accumulation point y_0 (i.e. $y_0 \in \overline{D \setminus \{y_0\}}$). We may of course assume that $y_0 \notin D$. We will show that the open subset $G = (X \times Y) \setminus (F \times \{y_0\})$ of $X \times Y$ is not normal. Indeed, the sets $K = F \times (Y \setminus \{y_0\})$ and $L = (X \setminus F) \times \{y_0\}$ are disjoint closed subsets of G. Suppose that U and V are disjoint open subsets of $X \times Y$ such that $K \subset U$ and $L \subset V$. For every $n < \omega$ let $U_n = \{x \in X: \langle x, d_n\rangle \in U\}$. To get a contradiction it suffices to show that $\bigcap_{n<\omega} U_n = F$. Clearly $F \subset \bigcup_{n<\omega} U_n$. If $x \in \bigcup_{n<\omega} U_n$, then $\langle x, y_0\rangle \in \bar{U}$ and therefore $x \in F$. □

5.2. COROLLARY (KATĚTOV [1948]). *If the product $C \times C'$ of two compact spaces C and C' is hereditarily normal, then both C and C' are perfectly normal.*

PROOF. A compact space cannot contain infinite closed and discrete subsets. □

Clearly, in 5.2 it is enough to assume that the spaces C and C' are countably compact. Notice, that—as Example 2.8(b) shows—the converse of Corollary 5.2 does not hold.

The following theorem implies in particular that the square C^2 of a compact space C is perfectly normal if and only if C is metrizable. Notice, however, that there exists a non-metrizable compact space C such that $C \times \mathbb{I}$ is perfectly normal: for example take C to be the double arrow space (Example 2.8) and use

Corollary 4.9. There also exist consistent examples of non-metrizable compact spaces C and C' such that $C \times C'$ is perfectly normal (RUDIN [1979]; also see Problem 9).

5.3. THEOREM. (ŠNEIDER [1945]). *The diagonal of the square of a compact space C is a G_δ set if and only if C is metrizable.*

PROOF. The sufficiency is obvious. Let Δ be the diagonal of C^2 and let $\{G_n\}_{n<\omega}$ be a family of open subsets of C^2 such that $\Delta \subset G_n$ and $\Delta = \bigcap_{n<\omega} \bar{G}_n$. For every $n < \omega$ let $f_n\colon C^2 \to I$ be a continuous function such that $f_n(\Delta) \subset \{0\}$ and $f_n(C^2 \backslash G_n) \subset \{1\}$, let ρ_n be a continuous pseudometric on C defined for $x, x' \in C$ by

$$\rho_n(x, x') = \sup_{y \in C} |f_n(x, y) - f_n(x', y)| .$$

and let ρ be the continuous pseudometric on C defined by

$$\rho(x, x') = \sum_{n=0}^{\infty} \frac{1}{2^n} \rho_n(x, x') .$$

It suffices to show that ρ generates the topology of C. Let $x_0 \in C$ and let U be an open neighborhood of x_0. There exists an n such that $G_n \cap ((C \backslash U) \times \{x_0\}) = \emptyset$ and therefore $f_n((C \backslash U) \times \{x_0\}) \subset \{1\}$ and, consequently, $x_0 \in \{x \in C\colon \rho(x, x_0) < 1/2^n\} \subset U$, which completes the proof. □

CHABER [1976] showed that the assumption of compactness in 5.3 can be relaxed to countable compactness.

5.4. COROLLARY (KATĚTOV [1948]). *The cube C^3 of a compact space C is hereditarily normal if and only if C is metrizable.*

PROOF. The sufficiency is obvious. If C^3 is hereditarily normal, then by 5.2 the square C^2 is perfectly normal and therefore, by 5.3, C is metrizable. □

The following example obtained under the assumption of Martin's Axiom and the negation of the Continuum Hypothesis shows that C^3 in Corollary 5.4 cannot be replaced by C^2. On the other hand, G. GRUENHAGE [1982] proved that C is metrizable, if C^2 is hereditarily paracompact.

5.5*. EXAMPLE (NYIKOS [1977]) (MA + $\neg$CH). *There exists a compact non-metrizable space C, whose square C^2 is hereditarily normal.*

Sketch of Proof. Let C^* be the double arrow space, i.e. $C^* = \mathbb{I} \times \{0, 1\}$, with the topology defined in Example 2.8(b). Choose a subset A of $\mathbb{I}$ of cardinality ω_1 and let C be the quotient space of C^* obtained by identifying pairs $\{\langle x, 0\rangle, \langle x, 1\rangle\}$ of points, for all $x \notin A$. Using the fact that under MA$+\neg$CH the square of any subspace of the Sorgenfrey line of cardinality ω_1 is (perfectly) normal (see PRZYMUSIŃSKI [1973] and the Remarks following Example 4.14) one proves that the square of C is hereditarily normal. $\square$

ZENOR [1971] proved that a compact space C is metrizable if and only if C^3 is hereditarily countably paracompact.

We finish this section with the following interesting characterization of separable metric spaces.

5.6. THEOREM. (TAMANO [1962]). *A space X is separable and metrizable if and only if there exists a compactification αX of X such that the product space $X \times \alpha X$ is perfectly normal.*

PROOF. Suppose that $X \times \alpha X$ is perfectly normal. The diagonal $\Delta = \{\langle x, x\rangle : x \in X\}$ is a closed subset of $X \times \alpha X$ and therefore there exists a function $f: X \times \alpha X \to \mathbb{I}$ such that $f^{-1}(0) = \Delta$. Define a pseudometric ρ on X by

$$\rho(x, x') = \sup_{y \in \alpha X} |f(x, y) - f(x', y)|$$

for $x, x' \in X$. Clearly ρ is continuous and since the range of f is separable, $\langle X, \rho\rangle$ is separable. It suffices to show that ρ generates the topology of X. Let $x_0 \in X$ and let U be a neighborhood of x_0 in X. Choose an open V in αX so that $V \cap X = U$. There exists an $\varepsilon > 0$ such that $f(\{x_0\} \times (\alpha X \setminus V)) \subset [\varepsilon, 1]$ and therefore $x_0 \in \{x \in X : \rho(x, x_0) < \varepsilon\} \subset U$, which shows that ρ is a metric for X.

The necessity is obvious. $\square$

6. Infinite products

We begin our discussion of infinite products with *countable* products, i.e. products of countably many factors.

Naturally, a necessary condition for a product space $X = \Pi_{n<\omega} X_n$ to be normal (paracompact, Lindelöf; respectively) is that all finite subproducts of X, i.e. spaces $X^{(m)} = \Pi_{n \leq m} X_n$, must be normal (paracompact, Lindelöf; respectively). Example 2.6 shows, however, that this condition is *not* sufficient. Below, we present a theorem which gives necessary *and* sufficient conditions. Observe, that this theorem is quite analogous to Corollaries 4.5 and 4.18.

Throughout this section we will assume that *spaces X_n contain at least two points.*

6.1. THEOREM. (ZENOR [1969], [1971], NAGAMI [1972]). *Suppose that all finite subproducts of a product space* $X = \Pi_{n<\omega} X_n$ *are normal (paracompact, Lindelöf; respectively). Then, the following conditions are equivalent:*

(i) *X is normal.*

(ii) *X is countably paracompact.*

(iii) *X is normal (paracompact, Lindelöf; respectively).*

PROOF. The implication (iii)→(i) is obvious.

(i)→(ii). Suppose that X is normal and $\{F_n\}_{n<\omega}$ is a decreasing sequence of closed subsets of X such that $\bigcap_{n<\omega} F_n = \emptyset$. We have to find a sequence $\{U_n\}_{n<\omega}$ of open subsets of X such that $F_n \subset U_n$ and $\bigcap_{n<\omega} \bar{U}_n = \emptyset$. For every $m < \omega$ let π_m denote the projection $\pi_m : X \twoheadrightarrow X^{(m)}$, where $X^{(m)} = \Pi_{n\leq m} X_n$.

Let the space X_n contain two different points p_n^0 and p_n^1. For every $m < \omega$ and $i = 0, 1$ define

$$K_m^i = \overline{\pi_m(F_m)} \times \{q_m^i\},$$

where $q_m^i \in \Pi_{n>m} X_n$ are defined by

$$q_m^i(n) = p_n^i, \quad \text{for } n > m.$$

Clearly, the sets K_m^i are closed and the sets $K_i = \bigcup_{m<\omega} K_m^i$, $i = 0, 1$, are disjoint. We shall show that the sets K_i are closed. Indeed, suppose that $x \in X$. There exists a k such that $x \notin F_k$ and an open subset G of some $X^{(m)}$ such that $x \in \pi_m^{-1}(G)$ and $\pi_m^{-1}(G) \cap F_k = \emptyset$. Then $\pi_m^{-1}(G) \cap F_l = \emptyset$ for every $l \geq k$ and consequently $\pi_m^{-1}(G)$ is disjoint from all sets K_l^i, for $l \geq \max\{k, m\}$, which proves that K_i is closed.

By the normality of X there exist open sets V_i, $i = 0, 1$, such that $\bar{V}_0 \cap \bar{V}_1 = \emptyset$ and $K_i \subset V_i$. Define

$$U_n = \pi_n^{-1}(\pi_n(V_0) \cap \pi_n(V_1)),$$

for $n < \omega$. Clearly, $F_n \subset U_n$. We shall show that $\bigcap_{n<\omega} \bar{U}_n = \emptyset$.

Let $x \in X$. There exists an $i = 0, 1$ such that $x \notin \bar{V}_i$ and an open subset G of some $X^{(m)}$ such that $x \in \pi_m^{-1}(G)$ and $\pi_m^{-1}(G) \cap V_i = \emptyset$. Therefore, $G \cap \pi_m(V_i) = \emptyset$, which implies that $\pi_m^{-1}(G) \cap U_m = \emptyset$.

(ii)→(iii). Suppose now that X is countably paracompact and let U be an open covering of X. (In case of finite subproducts being only *normal* we will additionally assume that $\mathcal{U}$ consists of *two* elements.) It suffices to show that there exists a σ-locally finite (countable, if all subproducts are *Lindelöf*) open covering $\mathcal{V}$ of X such that for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $\bar{V} \subset U$ (cf. Lemma 4.2).

For every $m < \omega$ and $U \in \mathcal{U}$ let $G_m(U) = \bigcup\{G \subset X^{(m)} : \pi_m^{-1}(G) \subset U\}$. Clearly, $\pi_m^{-1}(G_m(U)) \subset U$ and $U = \bigcup_{m<\omega} G_m(U)$. Let

$$\mathscr{G}_m = \{G_m(U) : U \in \mathscr{U}\}, \qquad G_m = \bigcup \mathscr{G}_m \quad \text{and} \quad F_m = X^{(m)} \setminus G_m .$$

Clearly, $\pi_{m+1}^{-1}(F_{m+1}) \subset \pi_m^{-1}(F_m)$ and $\bigcap_{m<\omega} \pi_m^{-1}(F_m) = \emptyset$. Therefore, there exists a decreasing family $\{U_n\}_{n<\omega}$ of open subsets of X such that $\pi_m^{-1}(F_m) \subset U_m$ and $\bigcap_{m<\omega} \bar{U}_m = \emptyset$. Define

$$K_m = X^{(m)} \setminus \pi_m(U_m) \quad \text{and} \quad H_m = \operatorname{int}_{X^{(m)}} K_m .$$

The sets K_m are closed in $X^{(m)}$ and contained in G_m. Consequently, there exist locally finite (countable, if $X^{(m)}$ is Lindelöf) open coverings $\mathscr{W}_m$ of K_m such that for every $W \in \mathscr{W}_m$ there exists a $U \in \mathscr{U}$ such that $\bar{W} \subset G_m(U)$. Let

$$\mathscr{V} = \{\pi_m^{-1}(W \cap H_m) : W \in \mathscr{W}_m, m < \omega\} .$$

Clearly, $\mathscr{V}$ is a σ-locally finite (countable) family of open sets in X such that for every $V \in \mathscr{V}$ there is a $U \in \mathscr{U}$ with $\bar{V} \subset U$. It remains to prove that $\bigcup \mathscr{V} = X$.

To this end it suffices to show that $X = \bigcup_{m<\omega} \pi_m^{-1}(H_m)$. Let $x \in X$. There is a k such that $x \notin \bar{U}_k$, an $m \geqslant k$ and an open subset G of $X^{(m)}$ such that $x \in \pi_m^{-1}(G)$ and $\pi_m^{-1}(G) \cap U_k = \emptyset$. Then $G \cap \pi_m(U_m) = \emptyset$, $G \subset K_m$ and therefore $x \in \pi_m^{-1}(H_m)$, which completes the proof. □

Naturally, a necessary condition for a product space $X = \Pi_{n<\omega} X_n$ to be hereditarily normal (resp. paracompact) is that all finite subproducts $X^{(m)} = \Pi_{n \leqslant m} X_n$ must be hereditarily normal (resp. paracompact). Example 2.9 shows, however, that this condition is *not* sufficient. Our next theorem gives necessary *and* sufficient conditions. It also implies (Corollary 6.3), that X is hereditarily Lindelöf if and only if all finite subproducts have this property, so Example 2.9 cannot be improved.

6.2. THEOREM (KATĚTOV [1948], OKUYAMA [1968]). *For a product space $X = \Pi_{n<\omega} X_n$ the following conditions are equivalent*:

(i) *X is hereditarily normal* (*resp. paracompact*).

(ii) *X is perfectly normal* (*resp. paracompact*).

(iii) *All finite subproducts of X are perfectly normal* (*resp. paracompact*).

6.3. COROLLARY (MICHAEL [1971]). *A product space $X = \Pi_{n<\omega} X_n$ is hereditarily Lindelöf if and only if all finite subproducts of X are hereditarily Lindelöf.*

Proof of Corollary 6.3. Recall that a space is hereditarily Lindelöf, if and only if it is perfectly Lindelöf, therefore if all finite subproducts of X are hereditarily Lindelöf, then by 6.2 the space X is perfectly normal and by 6.1 it is also Lindelöf, so it is hereditarily Lindelöf. □

Proof of Theorem 6.2. Clearly, (ii) implies (i).

(i)$\rightarrow$(iii). Let $m < \omega$ and $X^{(m)} = \Pi_{n\leq m} X_n$. Since all spaces X_n contain at least two points, the space $\Pi_{n>m} X_n$ contains a homeomorphic copy of the Cantor set C. Therefore, the space $X^{(m)} \times C$ is hereditarily normal (paracompact), which in view of Theorem 5.1 implies that $X^{(m)}$ is perfectly normal (paracompact).

(iii)$\rightarrow$(ii). Suppose that all finite subproducts of X are perfectly normal (paracompact). In view of Theorem 6.1 it suffices to show that X is perfectly normal. Let F be a closed subset of X. For every $m < \omega$ let $F_m = \overline{\pi_m(F)}$. Since $X^{(m)}$ is perfectly normal, there exists a continuous function $f_m: X \rightarrow I$ such that $f_m^{-1}(0) = \pi_m^{-1}(F_m)$ (see ENGELKING [1977], Theorem 1.5.19). Define a continuous function $f: X \rightarrow R$ by

$$f(x) = \sum_{m<\omega} \frac{1}{2^m} f_m(x) .$$

It suffices to show that $f^{-1}(0) = F$. Clearly, $f(F) \subset \{0\}$. On the other hand, suppose that $x \notin F$. Then, there exists an m and an open subset G of $X^{(m)}$ such that $x \in \pi_m^{-1}(G)$ and $\pi_m^{-1}(G) \cap F = \emptyset$. Therefore $G \cap \pi_m(F) = \emptyset$, $x \notin \pi_m^{-1}(F_m)$ and $f_m(x) > 0$. Consequently, $f(x) > 0$, which completes the proof. □

REMARKS. ZENOR [1971] proved that $X = \Pi_{n<\omega} X_n$ is hereditarily countably paracompact if and only if X is perfectly normal. R. POL [1978] constructed an example of a non-paracompact, locally metrizable space Z such that Z^ω is perfectly normal. It can be easily proved that $X = \Pi_{n<\omega} X_n$ is perfect if and only if all finite subproducts of X are perfect (HEATH and MICHAEL [1971]). □

We now proceed to a discussion of *uncountable* product spaces, i.e. products of uncountably many factors. Such products are rarely normal.

6.4. THEOREM (STONE [1948]). *If the product space $X = \Pi_{\alpha<\kappa} X_\alpha$ is normal, then all spaces, with the exception of at most countably many, are countably compact.*

PROOF. Suppose that uncountably many spaces X_α for, say, $\alpha < \omega_1$, are not countably compact. Therefore each of the spaces X_α, for $\alpha < \omega_1$, contains a closed copy of the discrete space $\mathbb{N}$ of natural numbers. This implies that $\mathbb{N}^{\omega_1}$ is normal, which is impossible by Example 2.7. □

6.5. COROLLARY (for metrizable spaces, STONE [1948]). *For a family $\{T_\alpha\}_{\alpha<\kappa}$ of paracompact p-spaces (e.g. for metrizable spaces T_α) the following conditions are equivalent*:

(i) $\Pi_{\alpha<\kappa} T_\alpha$ *is normal.*

(ii) $\Pi_{\alpha<\kappa} T_\alpha$ *is paracompact.*

(iii) *All spaces T_α, with the exception of at most countably many, are compact.*

PROOF. Clearly (ii) implies (i).

(i)→(iii). By 6.4 all spaces T_α, with the exception of at most countably many, are countably compact, but countably compact paracompact spaces are compact.

(iii)→(ii). This is an immediate consequence of Corollary 3.5 and the fact that the product of countably many paracompact p-spaces is paracompact. □

REMARK. Notice that in Corollary 6.5 the assumption that the spaces T_α are paracompact p-spaces can be replaced by a much weaker assumption that all countable subproducts of $T = \Pi_{\alpha<\kappa} T_\alpha$ are paracompact. *Paracompact Σ-spaces* (NAGAMI [1969]) constitute a fairly broad class of spaces whose countable products are paracompact (see G. Gruenhage's chapter on generalized metric spaces).

6.6. COROLLARY (NOBLE [1971]). *For a space X the following conditions are equivalent*:

(i) *X is compact.*

(ii) *X^κ is normal for every cardinal κ.*

(iii) *X^κ is normal for some uncountable cardinal $\kappa \geqslant w(X)$.*

PROOF. The implications (i)→(ii) and (ii)→(iii) are obvious.

(iii)→(i). By 6.5, the space X is countably compact. If X contains at least two points, then X^κ contains $\mathbb{D}^\kappa$, the Cantor cube of weight κ, as a closed subspace and therefore the space $X \times \mathbb{D}^\kappa \subset X \times X^\kappa \cong X^\kappa$ is normal. By Corollary 3.8, the space X is κ-paracompact and since $\kappa \geqslant w(X)$ the space X is paracompact and therefore compact. □

The next theorem illustrates the preservation of normality under products by investigating the normality of product spaces κ^λ, where κ and λ are cardinals and κ is equipped with the order topology. As a corollary to this theorem we obtain an example (6.10) showing that the assumption in 6.6(iii) that κ be greater than or equal to $w(X)$ is essential. It also shows that the assumptions in Corollary 6.5 are essential and that countable compactness in 6.4 cannot be replaced by compactness.

Naturally, κ^λ is always normal when $\lambda \leqslant 1$ or $\kappa < \omega$. Moreover, from Example 2.7 it follows that if $\kappa = \omega$, then κ^λ is normal if and only if $\lambda \leqslant \omega$. Therefore, we will assume that $\lambda > 1$ and $\kappa > \omega$.

6.7. THEOREM. *For cardinals $\kappa > \omega$ and $\lambda > 1$ the following conditions are equivalent*:

(i) *κ^λ is normal.*

(ii) *κ is regular and $\lambda < \kappa$.*

In order to prove Theorem 6.7 we shall need the following lemma.

6.8. LEMMA. *Let κ be a regular cardinal and λ, $\tau < \kappa$. Every cover of κ^λ by τ open sets has a finite subcover. In particular, κ^λ is τ-paracompact for every $\tau < \kappa$.*

Proof of Lemma 6.8. Let τ be the first cardinal for which this is false and let $\{U_\alpha: \alpha < \tau\}$ be an open cover of κ^λ which does not have a finite subcover. Define $V_\beta = \bigcup\{U_\alpha: \alpha < \beta\}$, for $\beta < \tau$. The family $\mathscr{V} = \{V_\beta: \beta < \tau\}$ is an increasing open cover and by our assumption, for every β there exists an $x_\beta \in \kappa^\lambda \setminus V_\beta$. Let $\delta = \sup\{x_\beta(\gamma): \beta < \tau \text{ and } \gamma < \lambda\}$. By the regularity of κ we have $\delta < \kappa$ and $\mathscr{V}$ is an open cover of the compact space $(\bar{\delta})^\lambda$ which does not have a finite subcover, which is impossible. $\square$

Proof of Theorem 6.7. (i)$\rightarrow$(ii). That λ must be less than κ follows from 6.6. If confinality κ were equal to $\tau < \kappa$, then from κ^2 being normal it would follow that $\kappa \times \bar{\tau}$ is normal, which by 3.7 would in turn imply that κ is τ-paracompact, which is obviously false.

(ii)$\rightarrow$(i). Suppose that κ is regular. We shall prove by induction on $\lambda < \kappa$ that κ^λ is normal. This is obviously true for $\lambda = 0$. Let $0 < n < \omega$ and suppose that κ^{n-1} was shown to be normal. We will show that κ^n is normal. Let K_0 and K_1 be disjoint closed subsets of κ^n. There exists an $i = 0, 1$ such that the n-cardinality (see Example 2.6) of K_i is less than κ, i.e. there exists a $\delta < \kappa$ such that

$$K_i \subset \bigcup_{j=1}^{n} (\kappa^{j-1} \times \bar{\delta} \times \kappa^{n-j}) .$$

Indeed, otherwise we would be able to construct by induction a sequence $\{x_l\}_{l<\omega}$ of points of κ^n such that:

(1) $x_{2l} \in K_0$, $x_{2l+1} \in K_1$, and

(2) $\max\{x_l(j): j < n\} < \min\{x_{l+1}(j): j < n\}$. Then, defining $\hat{x} \in \kappa^n$ by

$$\hat{x}(j) = \sup\{x_l(j): l < \omega\} \quad \text{for } j < n ,$$

we would obtain a point belonging both to K_0 and K_1, which is impossible.

The spaces $\kappa^{j-1} \times \bar{\delta} \times \kappa^{n-j}$ are normal, because κ^{n-1} is normal (by inductive assumption) and δ-paracompact (by Lemma 6.8) (Corollary 3.7). Moreover, those spaces are closed-and-open subsets of κ^n, which easily implies that the sets K_0 and K_1 can be separated by open sets in κ^n.

Suppose now that $\omega \leqslant \lambda < \kappa$ and that spaces κ^α were shown to be normal for $\alpha < \lambda$. Let K_0 and K_1 be disjoint closed subsets of κ^λ and for every $\alpha < \kappa$ let $\pi_\alpha: \kappa^\lambda \rightarrow \kappa^\alpha$ be the projection. Define $L_i^\alpha = \overline{\pi_\alpha(K_i)}$ and $F^\alpha = \pi_\alpha^{-1}(L_0^\alpha \cap L_1^\alpha)$. The family $\{F^\alpha: \alpha < \lambda\}$ is a decreasing family of closed subsets. Lemma 6.8 implies that either $F = \bigcap_{\alpha<\lambda} F_\alpha \neq \emptyset$ or some $F_\alpha = \emptyset$. Obviously, any point belonging to F belongs to $K_0 \cap K_1$, which is impossible. If $F_\alpha = \emptyset$, then $L_0^\alpha \cap L_1^\alpha = \emptyset$ and, by the inductive assumption, the sets L_i^α can be separated by open sets in κ^α, which clearly implies that sets K_i can be separated in κ^λ and thus completes the proof. □

6.9. COROLLARY. *Let κ be an arbitrary cardinal. The space κ^2 is normal if and only if κ is regular.* □

6.10. COROLLARY. *The space $\omega_2^{\omega_1}$ is normal, whereas the space $\omega_2^{\omega_2}$ is not.* □

Theorem 2.10 showed already that no uncountable product $\Pi_{\alpha<\kappa} X_\alpha$ of spaces containing at least two points is *hereditarily* normal. Here, we will only observe that this theorem can be also derived from Theorem 5.1. Indeed, $\Pi_{\alpha<\kappa} X_\alpha$ contains a homeomorphic copy of the Cantor cube $\mathbb{D}^\kappa$, $\kappa > \omega$, and since $\mathbb{D}^\kappa$ is homeomorphic to $\mathbb{D}^\kappa \times \mathbb{D}^\kappa$, hereditary normality of $\mathbb{D}^\kappa$ and Theorem 5.1 would imply that $\mathbb{D}^\kappa$ is perfectly normal and hence first countable which is clearly false.

7. Σ-products

In this section we will discuss the normality of an important and useful class of subspaces of product spaces, called *Σ-products.*

7.1. DEFINITION (CORSON [1959]). Let $X = \Pi_{s\in S} X_s$ be a product space and let p be a fixed point of X. The subspace $\Sigma(p) = \{x \in X: \{s: x_s \neq p_s\}$ is countable$\}$ of X is called a *Σ-product of spaces $\{X_s\}_{s\in S}$ (about p).*

Observe that a Σ-product of spaces $\{X_s\}_{s\in S}$ is a proper subspace of the product space $\Pi_{s\in S} X_s$ if and only if uncountably many spaces X_s contain at least two elements. Such Σ-products we will call *proper.*

7.2. PROPOSITION. *Proper Σ-products contain the space ω_1 as a closed subspace and therefore are never paracompact.*

PROOF. Let us suppose that spaces X_{s_α}, $\alpha < \omega_1$, contain at least two elements. Therefore we can find points $q^\alpha \in X_{s_\alpha}$ so that $p_{s_\alpha} \neq q^\alpha$. For $\beta < \omega_1$ define points $x^\beta \in \Sigma(p)$ by putting:

$$x_s^\beta = \begin{cases} q^\alpha, & \text{if } s = s_\alpha \text{ and } \alpha < \beta \\ p_s, & \text{otherwise}, \end{cases}$$

for $s \in S$. One easily checks that the space $\{x^\beta: \beta < \omega_1\}$ is a closed subspace of $\Sigma(p)$ homeomorphic to ω_1. □

Σ-products of *compact* spaces may be non-normal as the following example shows.

7.3. EXAMPLE. *Σ-product of uncountably many copies of $\bar{\omega}_1$ is not normal.*

PROOF. Let $\Sigma(p)$ denote this Σ-product (p is an arbitrary point of $(\bar{\omega}_1)^{\omega_1}$). Clearly $\Sigma(p)$ is homeomorphic to $\Sigma(p) \times \bar{\omega}_1$ and therefore it is not normal by Proposition 7.2 and Example 2.2. □

Observe, that in view of Theorem 3.19, the same argument shows that if a proper Σ-product of *compact* spaces $\{C_s\}_{s\in S}$ is normal, then all spaces C_s must have *countable tightness.* In fact, also the converse of this statement is true (Theorem 7.5).

On the other hand, Σ-products of metric spaces are always normal (actually, also collectionwise normal; see 7.5).

7.4. THEOREM (GUL'KO [1977], RUDIN [1977]). *Σ-products of metric spaces are normal.*

PROOF. Let Σ denote the Σ-product of metric spaces $\{M_s\}_{s\in S}$ about some point $p \in \Pi_{s\in S} M_s$. Choose some cardinal κ so that the weight of all spaces M_s is not larger than κ. Assume at first that all spaces M_s have σ-locally finite bases consisting of closed-and-open sets, i.e. that the spaces M_s are strongly zero-dimensional (see ENGELKING [1977]).

For $x \in \Sigma$ let $\mathrm{Supp}(x) = \{s \in S: x_s \neq p_s\}$ and for every countable $A \subset S$ denote by r_A the natural retraction $r_A: \Sigma \to \Sigma$ defined for $x \in \Sigma$ by

$$(r_A(x))_s = \begin{cases} x_s, & \text{if } s \in A, \\ p_s, & \text{otherwise}, \end{cases}$$

for $s \in S$. Observe that the spaces $\Sigma_A = r_A(\Sigma)$ are metrizable and choose some metric ρ_A on Σ_A and denote by $\hat{\rho}_A$ the pseudometric on Σ defined by $\hat{\rho}_A(x, y) = \rho_A(r_A(x), r_A(y))$.

Suppose that K_0 and K_1 are disjoint closed subsets of Σ. By Lemma 4.2 it suffices to find a σ-locally finite open covering $\mathcal{U}$ of Σ such that $\bar{U} \cap K_i = \emptyset$ for every $U \in \mathcal{U}$ and some $i = 0, 1$.

Let $\kappa^{<\omega}$ stand for the set of all functions $\phi: n \to \kappa$, $n < \omega$. By induction on n we define for each $\phi \in \kappa^n$ a countable subset A_ϕ of S, a closed-and-open subset V_ϕ of Σ_{A_ϕ}, a pseudo-metric μ_ϕ on Σ and two points x^0_ϕ and x^1_ϕ in Σ, so that the following conditions are satisfied (we denote by U_ϕ the set $r^{-1}_{A_\phi}(V_\phi)$):

(1) the family $\{U_\phi: \phi \in \kappa^n\}$ is a disjoint closed-and-open cover of Σ for every $n < \omega$;

(2) if $\phi' \supset \phi$, then $U_{\phi'} \subset U_\phi$;

(3) if $K_i \cap \bar{U}_\phi \neq \emptyset$, then $x^i_\phi \in K_i \cap \bar{U}_\phi$, for $i = 0, 1$;

(4) $A_\phi = \mathrm{Supp}(x^0_\phi) \cup \mathrm{Supp}(x^1_\phi) \cup A_{\phi|(n-1)}$, for $n > 1$;

(5) $\mu_\phi = \Sigma\{\hat{\rho}_{A\phi|i}: i \leq n\}$ (i.e. algebraic sum of pseudo-metrics $\rho_{A_{\phi|i}}$);

(6) the radius of U_ϕ with respect to $\mu_{\phi|(n-1)}$ is less than $1/n$ for $n > 0$.

Suppose that the construction has been already performed and let

$$\mathcal{U} = \{U_\phi \colon \phi \in \kappa^{<\omega} \text{ and } K_i \cap \bar{U}_\phi = \emptyset, \text{ for some } i = 0, 1\}.$$

Clearly, it suffices to show that $\bigcup\mathcal{U} = \Sigma$. If not, then there exists a $\Phi \in \kappa^\omega$ and an $x_0 \in \Sigma$ such that $x_0 \in \bigcap_{n<\omega} U_{\Phi|n}$ and $U_{\Phi|n} \not\in \mathcal{U}$, for all $n < \omega$. Let $B = \bigcup_{n<\omega} A_{\Phi|n}$ and $Z_0 = r_B(x_0)$. One easily sees that $\lim_{n\to\infty}(r_B(x^i_{\Phi|n})) = r_B(x_0)$, for $i = 0, 1$, and therefore, by (4) $\lim_{n\to\infty} x^i_{\Phi|n} = z_0$, for $i = 0, 1$, and consequently $z_0 \in K_0 \cap K_1$, which is a contradiction.

Now, we show how to perform the construction. Let A_ϕ be an arbitrary countable subset of S, $V_\phi = \Sigma_{A_\phi}$ and $\mu_\phi = \hat{\rho}_{A_\phi}$.

Suppose that the construction has been performed for $n \geq 0$. Let $\phi \in \kappa^n$. The space $\langle V_\phi, \mu_\phi \rangle$ is metrizable and therefore it can be decomposed into a disjoint family $\{W_\alpha \colon \alpha < \kappa\}$ of (possibly empty) open sets of diameter $<1/n$. Define $V_{\phi^\frown\langle\alpha\rangle} = W_\alpha$, for every $\alpha < \kappa$. This establishes sets V_ψ, $\psi \in \kappa^{n+1}$. For every $\psi \in \kappa^{n+1}$ choose points $x^i_\psi \in \bar{V}_\psi \cap K_i$, if there exist such points, otherwise x^i_ϕ are arbitrary. Choose A_ψ to be the countable subset of S satisfying (4) and let μ_ψ be defined using (5). Clearly, all the conditions of the construction are satisfied. This shows that Σ is normal.

Suppose now that Σ is a Σ-product of arbitrary metric spaces $\{M_s\}_{s\in S}$ about some point $p \in \Pi_{s\in S} M_s$. For every $s \in S$ we can find a perfect mapping $f_s \colon M'_s \twoheadrightarrow M_s$ of a strongly zero-dimensional metric space M'_s onto M_s such that $|f_s^{-1}(p_s)| = 1$ (ENGELKING [1977]; Exercise 4.4.T). Clearly, the product $f = \Pi_{s\in S} f_s$ of mappings f_s is perfect, the Σ-product of spaces $\{M'_s\}_{s\in S}$ about $f^{-1}(p)$ is normal by the previous argument, and therefore the Σ-product of spaces $\{M_s\}_{s\in S}$ is normal, as a perfect image of a normal space. □

Theorem 7.4 gave a positive answer to a long outstanding question raised by Corson, who proved the counterpart of 7.4 for complete metric spaces [1959]. The counterpart of 7.4 for separable metric spaces was proved by KOMBAROV and MALYHIN [1973]. Kombarov later generalized Theorem 7.4 obtaining the following beautiful result, whose proof, however, will not be given here.

7.5*. THEOREM (KOMBAROV [1978]). *A proper Σ-product of paracompact p-spaces $\{T_s\}_{s\in S}$ is (collectionwise) normal if and only if all spaces T_s have countable tightness.* □

In particular, a Σ-product of uncountably many copies of a compact space C is normal if and only if C has countable tightness.

8. Some open problems

PROBLEM 1. Suppose that Z is compact (resp. metrizable, paracompact p-space) and $X \times Z$ is normal. Is $X \times Z^2$ normal? (cf. Corollary 4.4).

PROBLEM 2 (Rudin and Starbird). Suppose that T is a paracompact p-space, $X \times T$ is normal and $f: X \twoheadrightarrow Y$ is a closed mapping. Is $Y \times T$ normal? (cf. Corollary 4.6 and Corollary 4.19).

PROBLEM 3 (Rudin and Starbird). Suppose that $X \times Z$ is paracompact and $f: X \twoheadrightarrow Y$ is closed. Is $Y \times Z$ paracompact? (cf. Example 4.15).

PROBLEM 4 (Kunen). Suppose that C is compact, X is normal and $f: X \twoheadrightarrow Y$ is perfect. Is $X \times C$ normal provided that $Y \times C$ is? (cf. Problem 1 and Corollary 4.7).

PROBLEM 5. Characterize the class of all spaces X such that $X \times Y$ is paracompact (Lindelöf) for every paracompact (Lindelöf) space Y (Tamano). Suppose that X belongs to that class. Is X^{ω} paracompact (Lindelöf). Does a closed image of X belong to that class? (cf. Theorem 4.13, Corollary 4.21, Corollary 4.6 and Theorem 3.21).

PROBLEM 6. Suppose that X is Lindelöf and $X \times Y$ is normal for every Lindelöf space Y. Is $X \times Y$ Lindelöf for every Lindelöf space Y?

PROBLEM 7 (Michael). Does there exist a real example of a Lindelöf space X such that the product space $X \times P$ of X with the space of irrationals is not normal? (cf. Example 2.3). This is equivalent to the existence of a separable and complete metric space M such that $X \times M$ is not normal.

PROBLEM 8 (Katětov). Does there exist a real example of a compact non-metrizable space C such that C^2 is hereditarily normal? (cf. Example 5.5).

PROBLEM 9. Does there exist a real example of two compact nonmetrizable spaces C and C' such that $C \times C'$ is perfectly normal? (cf. Corollary 5.2).

PROBLEM 10. Find a reasonably simple proof of Theorem 3.21.

PROBLEM 11. Suppose that C is compact and $X \times C$ is normal. Does there exist a compact zerodimensional space C' such that $X \times C'$ is normal and C is a continuous image of C'?

As is well known, a space X is normal if and only if every continuous function $f: F \to R$ from a closed subspace F of X can be continuously extended onto X. Similarly, a space X is collectionwise normal if and only if every continuous function $f: F \to B$ from a closed subset F of X into a Banach space B can be extended onto X. Thus the problem of preservation of normality or collectionwise normality by product spaces can be considered as part of a broader problem of *extendability of continuous functions from subsets of product spaces.* A considerable number of publications has been devoted to this subject and there are still many open problems in this area. For more information the reader is referred e.g. to my survey article PRZYMUSIŃSKI [1980b] and to more recent papers PRZYMUSIŃSKI [1982], [1983] and WASKÓ [1983].

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CHAPTER 19

Versions of Martin's Axiom

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Contents

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HANDBOOK OF SET-THEORETIC TOPOLOGY
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0. Introducing MA

The native people of Saskatchewan say that in the old days a boy, upon reaching the threshold of manhood, would undertake a special journey into the forest. There he would stay alone, without eating. After several days of fasting the spirit of an animal would appear to him. This animal spirit would guide him and protect him throughout his adult life.

Today, this tradition is no longer observed except by set-theoretic topologists. We search not for the spirits of animals, but of axioms. Set-theoretic principles to guide us and protect us throughout our mathematical lives—to guide us toward answers for questions which might otherwise be intractible and to protect us from inconsistency. One such axiom is Martin's Axiom.

Some historians claim that Martin's Axiom is not named for the little purple bird, but for a mathematician, D.A. Martin. They go on to protest that it was not given to us by the Great Manitou, but in an article by R. Solovay and S. Tennenbaum. Nevertheless, the amount of guidance and protection offered by this axiom is not open to historical controversy.

The many uses of Martin's Axiom and its wealth of applications in topology have been surveyed before. An introduction suitable for a general mathematical audience is SHOENFIELD [1975]. Those wanting more would be interested in RUDIN [1975], [1977], JUHÁSZ [1977] and the original survey MARTIN and SOLOVAY [1970]. One excellent introduction, TALL [1973], was never published. Today, probably the best and most accessible introduction is in chapter II of the textbook KUNEN [1980]. Consequences of Martin's axiom really deserve a whole book of their own, FREMLIN [19 · ·].

Martin's Axiom is relatively consistent with the usual axioms of set theory (SOLOVAY and TENNENBAUM [1971]). This means that if our basic set-theoretic axioms, ZFC, are consistent, then they remain so with the addition of Martin's axiom. It is a comfort to know that if you use Martin's axiom in your mathematical work that you will not be any more inconsistent than you were without it; section I of KUNEN [1980] explains this in detail.

What is Martin's Axiom? We will denote it by MA.

If $\mathbf{P}$ is a partial order which has the ccc and $\mathscr{D}$ is a collection of fewer than $2^{\aleph_0}$ dense subsets of $\mathbf{P}$, then there is a subset $\mathbf{G}$ of $\mathbf{P}$ such that:

(i) if p and q are elements of $\mathbf{G}$, then there is some r in $\mathbf{G}$ such that $r \leqslant p$ and $r \leqslant q$,

(ii) if $p \geqslant q$ and q is in $\mathbf{G}$, then p is in $\mathbf{G}$, and

(iii) $\mathbf{G}$ has non-empty intersection with each dense subset in $\mathscr{D}$.

Two elements p and q of a partial order are said to be *compatible* if there is some r such that $r \leqslant p$ and $r \leqslant q$. A partial order is said to have the countable chain condition, ccc, if there is no uncountable pairwise incompatible subset. A subset $\mathbf{D}$ is said to be *dense* in a partial order $\mathbf{P}$ iff for each p in $\mathbf{P}$ there is some q

in **D** such that $q \leq p$. $2^{\aleph_0}$ is the cardinality of the continuum of real numbers. A subset **G** of **P** satisfying (i), (ii) and (iii) is said to be *generic* for $\mathscr{D}$.

The notation used here is standard. In particular we denote the collection of all subsets of p of size b by $[p]^b$ and all other undefined notation, which is not a misprint, can be found in the textbook KUNEN [1980], sections I and II.

1. The topology of MA

The form in which Martin's Axiom is stated above is certainly the best for the practical purpose of applying the axiom to the solution of problems in set-theoretic topology. There is, however, a topological form which is quite well known. It is not so flexible as the partial order form, but it is simple to state.

No compact Hausdorff space with the ccc *can be the union of less than* $2^{\aleph_0}$ *nowhere dense subsets.*

Recall that a topological space has the ccc means that it has no uncountable collection of pairwise disjoint open sets (see KUNEN [1980] p. 50). One proof of the equivalence is in KUNEN [1980] p. 65, but we need a slightly different form to help with the discussion of the internal structure of Martin's axiom and some of its variations.

It will be instructive to see how this equivalence is accomplished. For each partial order $\langle \mathbf{P}, \leq \rangle$ we associate a topological space $\langle X, \tau \rangle$ in such a way that **P** is in some sense homomorphically embedded as a dense subset of the partial order $\langle \tau, \subseteq \rangle$.

1.1. DEFINITION. A partial order is said to have the κ-cc iff it does not have a collection of κ pairwise incompatible elements. The $\aleph_1$-cc is the ccc. If $\kappa < \lambda$, then the κ-cc is a stronger condition than the λ-cc.

1.2. DEFINITION. If $\langle X, \tau \rangle$ is a topological space, $\mathscr{B}$ is said to be a *π-base* for X iff $\mathscr{B}$ is a dense subset of the partial order $\langle \tau \backslash \{\emptyset\}, \subseteq \rangle$, that is, non-empty open sets ordered by set inclusion. This is one of the most fundamental notions of set-theoretic topology.

1.3. DEFINITION. A subset **Q** of a partial order **P** is said to be *centred* iff for each finite subset $\{p_1, \ldots, p_n\} \subseteq \mathbf{Q}$ there is some $q \in \mathbf{P}$ such that for each $i = 1, \ldots, n$, $q \leq p_i$; q is called a *lower bound* for $\{p_1, \ldots, p_n\}$. Note that q doesn't have to be a member of **Q**. And, **Q** is said to be *maximally centred* if **Q** is not properly contained in any centred subset of **P**.

These definitions will play a large role in what is to follow. In particular, the following lemma is crucial for the construction of the equivalence between the two forms of Martin's Axiom.

1.4. LEMMA. *Suppose $\langle \mathbf{P}, \leqslant\rangle$ is a partial order with the κ^+-cc and $\langle X, \tau\rangle$ is a compact Hausdorff topological space and $\Phi: \mathbf{P} \to (\tau\backslash\{\emptyset\})$ is a map such that*

(i) *$p \leqslant q$ implies $\Phi(p) \subseteq \Phi(q)$,*

(ii) *$\mathbf{Q} \subseteq \mathbf{P}$ is centred iff $\{\Phi(p): p \in \mathbf{Q}\}$ is centred, and*

(iii) *$\{\Phi(p): p \in \mathbf{P}\}$ is a π-base for $\langle X, \tau\rangle$.*

Then the following are equivalent:

(iv) *for each collection of κ dense subsets of $\mathbf{P}$, there is a $\mathbf{G} \subseteq \mathbf{P}$ generic for the collection,*

(v) *the intersection of any collection of κ dense open subsets of X is non-empty.*

The proof is delayed to section 9.1.

We now set out with a partial order $\langle \mathbf{P}, \leqslant\rangle$ and hope to construct a space $\langle X, \tau\rangle$ and a mapping Φ as in the hypothesis of the lemma.

Let $S(\mathbf{P})$ be the set of all maximally centred subsets of $\mathbf{P}$. For each $p \in \mathbf{P}$, let

$$U_p = \{x \in S(\mathbf{P}): p \in x\}$$

and let $\mathscr{B}$ be the collection of non-empty sets of the form

$$\bigcap\{U_p: p \in \mathbf{F}\}$$

where $\mathbf{F} \in [\mathbf{P}]^{<\aleph_0}$. $\mathscr{B}$ forms as base for a topology on $S(\mathbf{P})$; $\{U_p: p \in \mathbf{P}\}$ is a subbase for this topology.

Partially order $\mathscr{B}$ by set inclusion. Let $\phi: \mathbf{P} \to \mathscr{B}$ be the map defined by

$$\phi: p \mapsto U_p.$$

It is straightforward to show that ϕ is a homomorphism, i.e. if $p \leqslant q$ then $U_p \subseteq U_q$. It is not necessarily true that $U_p \subseteq U_q$ implies $p \leqslant q$. However, it is not difficult to show that p and q are compatible iff $U_p \cap U_q \neq \emptyset$; furthermore, a subset $\mathbf{Q} \subseteq \mathbf{P}$ is centred iff $\bigcap\{U_p: p \in \mathbf{Q}\} \neq \emptyset$. It is also straightforward to prove that $\{U_p: p \in \mathbf{P}\}$ is dense in $\langle \mathscr{B}, \subseteq\rangle$, and hence is a π-base for $S(\mathbf{P})$.

Let us look at $S(\mathbf{P})$ as a topological space. Each U_p is clopen; suppose $x \notin U_p$; then $p \notin x$ and by the maximality of x, there are $q_1, \ldots, q_n \in x$ such $\{p, q_1, \ldots, q_n\}$ has no lower bound. That is $\{p, q_1, \ldots, q_n\}$ is not centred and hence $U_p \cap U_{q_1} \cap \cdots \cap U_{q_n} = \emptyset$. So $U_{q_1} \cap \cdots \cap U_{q_n}$ is an open neighbourhood of x which misses U_p. Hence $S(\mathbf{P})$ is zero-dimensional. $S(\mathbf{P})$ is Hausdorff; suppose $x \neq y$; then by the maximality of x there is some $p \in x$ such that $p \notin y$ and therefore $x \in U_p$ and $y \notin U_p$.

We can summarize this as a theorem.

1.5. THEOREM. *For each partial order $\langle \mathbf{P}, \leqslant\rangle$ there is a zero-dimensional Hausdorff space $S(\mathbf{P})$ with a clopen base $\mathscr{B}$, and a map $\phi: \mathbf{P} \to \mathscr{B}$ such that*

(i) *$p \leqslant q$ implies $\phi(p) \subseteq \phi(q)$,*

(ii) *$\mathbf{Q} \subseteq \mathbf{P}$ is centred iff $\{\phi(p): p \in \mathbf{Q}\}$ is centred iff $\bigcap\{\phi(p): p \in \mathbf{Q}\} \neq \emptyset$,*

(iii) *$\{\phi(p): p \in \mathbf{P}\}$ is a π-base for $S(\mathbf{P})$,*

(iv) *$\{\phi(p): p \in \mathbf{P}\}$ is a subbase for $S(\mathbf{P})$.*

Since $S(\mathbf{P})$ is a completely regular space, we can obtain the Čech–Stone compactification $\tilde{S}(\mathbf{P})$. Actually, any compactification of $S(\mathbf{P})$ will suffice for what follows. Let $\mathscr{B}$ be the base for $S(\mathbf{P})$ given in Theorem 1.5. For each $B \in \mathscr{B}$, let, in the space $\tilde{S}(\mathbf{P})$,

$$\tilde{B} = \mathrm{Int}(\bar{B}) .$$

It is straightforward to prove that $\{\tilde{B}: B \in \mathscr{B}\}$ is a base for $S(\mathbf{P})$ as a subspace of $\tilde{S}(\mathbf{P})$. We define a map Φ such that for each $p \in \mathbf{P}$, $\Phi(p) = \tilde{\varphi}(p)$. It is now straightforward to extend Theorem 1.5.

1.6. LEMMA. *Let $\langle \mathbf{P}, \leqslant \rangle$ be any partial order. Let $S(\mathbf{P})$ and ϕ be given as in Theorem 1.5. There is a compactification $\tilde{S}(\mathbf{P})$ of $S(\mathbf{P})$ and a map*

$$\Phi: \mathbf{P} \to \{\textit{non-empty open subsets of } \tilde{S}(\mathbf{P})\}$$

such that

(i) *$p \leqslant q$ implies $\Phi(p) \subseteq \Phi(q)$,*
(ii) *$\mathbf{Q} \subseteq \mathbf{P}$ is centred iff $\{\Phi(p): p \in \mathbf{Q}\}$ is centred, and*
(iii) *$\{\Phi(p): p \in \mathbf{P}\}$ is a π-base for $\tilde{S}(\mathbf{P})$.*

We now put all this together to obtain this theorem.

1.7. THEOREM. *The partial order form and the topological form of Martin's Axiom are equivalent.*

PROOF. First assume the topological form. Let $\mathbf{P}$ be a partial order with the ccc, $\kappa < 2^{\aleph_0}$ and $\{\mathbf{D}_\alpha: \alpha \in \kappa\}$ a family of dense subsets of $\mathbf{P}$. Use Lemma 1.6 to obtain the space $X = \tilde{S}(\mathbf{P})$ and the map Φ to fulfill the hypothesis of Lemma 1.4. It is easy to check that X has the ccc so that we can obtain, via the topological form of MA and Lemma 1.4, a subset $\mathbf{G} \subseteq \mathbf{P}$ generic for $\{\mathbf{D}_\alpha: \alpha \in \kappa\}$.

The following lemma finishes the proof of the theorem.

1.8. LEMMA. *The partial order form of* MA *implies the topological form.*

PROOF. Let X be a compact Hausdorff space with the ccc and $\{U_\alpha: \alpha \in \kappa\}$ be a family of dense open subsets of X; $\kappa < 2^{\aleph_0}$.

For each $\alpha \in \kappa$, let $\mathbf{P}_\alpha$ be a maximal pairwise disjoint collection of non-empty open sets V such that $\bar{V} \subseteq U_\alpha$. Let $\mathbf{P}$ be $\bigcup\{\mathbf{P}_\alpha: \alpha \in \kappa\}$ together with non-empty finite intersections. Under the set inclusion ordering $\mathbf{P}$ is a ccc partial order and for each $\alpha \in \kappa$

$$\{V \in \mathbf{P}: \bar{V} \subseteq U_\alpha\}$$

is dense. If $\mathbf{G} \subseteq \mathbf{P}$ is generic for these sets, then

$$\emptyset \neq \{\bar{V}\colon V \in \mathbf{G}\} \subseteq \{U_\alpha\colon \alpha \in \kappa\},$$

which completes the proof.

The connection between MA and Baire category is now apparent. Just as in the case of the real line, we say that a subset of a topological space is *first category* if it is the union of countably many nowhere dense subsets, and *second category* if it isn't. By remembering the Baire Category theorem and looking at the topological form, we see that Martin's axiom is implied by the continuum hypothesis (i.e. $2^{\aleph_0} = \aleph_1$), and, on the other hand, we cannot replace 'less than $2^{\aleph_0}$' by '$2^{\aleph_0}$' in the statement of the axiom.

1.9. DEFINITION. A topological space is called a *Baire space* if no open subset if first category. That is, a Baire space is one which satisfies the Baire category theorem.

One might conjecture that, as in the case of the Baire category theorem, the hypothesis of compactness in the topological form of MA can be replaced by a weaker assumption. Indeed this is the case—JUHÁSZ [1977] can be consulted for a discussion of this. However, it is interesting that we cannot have complete analogy with the Baire category theorem. The following example was pointed out by R.C. Solomon.

1.10. EXAMPLE. *A regular Baire space with the* ccc *which is the union of* $\aleph_1$ *nowhere dense sets, no matter whether* $2^{\aleph_0} = \aleph_1$ *or* $2^{\aleph_0} > \aleph_1$. The space X is the set of all functions $f\colon \omega_1 \to \{0, 1\}$ such that $\{\alpha \in \omega_1\colon f(\alpha) = 1\}$ is countable. We consider X as a subspace of $2^{\aleph_1}$, the topological (Tychonoff) product of $\aleph_1$ copies of the two point discrete space $\{0, 1\}$. For each $\alpha \in \omega_1$, let $X_\alpha = \{f \in X\colon f(\beta) = 0$ for all $\beta > \alpha\}$. Each X_α is a compact subspace of $2^{\aleph_1}$, and it follows that X is countably compact, and hence Baire since it is regular. X is a dense subset of $2^{\aleph_1}$; this means that each X_α is nowhere dense in X as well as in $2^{\aleph_1}$. Of course, $X = \bigcup\{X_\alpha\colon \alpha \in \omega_1\}$. Furthermore, looking ahead to Theorem 3.6 we note that $2^{\aleph_1}$ has the ccc and therefore so does X.

Martin's axiom was not constructed to be a weak form of the continuum hypothesis (henceforth CH), but to prove the Suslin hypothesis (henceforth SH) and for this we need MA plus not CH (henceforth MA + $\neg$CH.

SH is the hypothesis that every linearly ordered space with the ccc is also separable; for the history of SH see KUNEN [1980] p. 66. The original proof of SH from MA + $\neg$CH used Kurepa's construction of a Suslin tree from a Suslin line—see section 6 of the article by *S.* Todorcević in this volume or KUNEN [1980] p. 72 for the details. However we can easily show that if there is a Suslin line, *L*, then there is a compact Hausdorff space with the ccc which is the union of $\aleph_1$ nowhere dense subsets by following this recipe:

(i) Notice that L has the ccc hereditarily and is first countable, and furthermore so is its Dedekind compactification.

(ii) Using the endpoints of open intervals as a dense set note that every nowhere dense subset is separable.

(iii) Using the ccc property find an interval J of L for which no subinterval is separable.

(iv) Pick $\{x_\alpha : \alpha \in \omega_1\}$ such that for all $\beta \in \omega_1$, $x_\beta \not\in \overline{\{x_\alpha : \alpha \in \beta\}}$.

(v) Notice that $\bigcup\{\overline{\{x_\alpha : \alpha \in \beta\}} : \beta \in \omega_1\}$ is closed and the union of $\aleph_1$ nowhere dense sets.

2. MA + ¬CH

The combination, MA + ¬CH, was proven relatively consistent with the usual axioms of set theory by R. Solovay and S. Tennenbaum (SOLOVAY and TENNENBAUM [1971]). The technique invented to prove this, called iterated forcing, is one of the major accomplishments of the subject of set theory. The proof is long; for complete details consult KUNEN [1980], Chapter VIII or BURGESS [1977]. We will have to be content with a simple sketch. The first step is to prove the following theorem.

2.1. THEOREM. MA *is equivalent to* MA *restricted to partial orders of cardinality less than* $2^{\aleph_0}$.

PROOF. Notice that the partial order used in the proof of Lemma 1.8 had cardinality less than $2^{\aleph_0}$.

Iterated forcing may be thought of as a construction of a chain of models of set theory. First κ is chosen to be a regular cardinal and M_0 a model of set theory such that

$$\kappa = \sup\{2^\lambda : \lambda < \kappa\}$$

is true in M_o; for example a model of the generalized continuum hypothesis and $\kappa = \aleph_2$. A chain of models

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots$$

of length κ is constructed in such a way that for any partial order $\mathbf{P}$ with the ccc which lies in M_0, there is some α and some $\mathbf{G} \in M_{\alpha+1}$ such that $\mathbf{G} \subseteq \mathbf{P}$ and $\mathbf{G}$ is generic for any collection of dense subsets of $\mathbf{P}$; the $\mathbf{G}$ is essentially obtained by forcing with $\mathbf{P}$ over M. However new partial orders may have arisen in M_1 and M_2

etc. Therefore a dovetailing indexing technique (made possible by the cardinal arithmetic hypothesis on κ) is used to enumerate all partial orders with the ccc of cardinality less than κ so that each is confronted at some stage $\alpha < \kappa$.

After this is completed, M_κ is obtained as a kind of limit. It is then possible (in part, because the cofinality of κ is at least $2^{\aleph_0}$) to show that any partial order, with the ccc and cardinality less than $2^{\aleph_0}$ and collection of less than $2^{\aleph_0}$ dense subsets which is in M_κ actually already appears in some M_α, $\alpha < \kappa$. This means that it was given a generic subset at some point in the construction. It is then shown that in M_κ we have $2^{\aleph_0} = \kappa$. But it's easier said than done.

Looking back at the proof of MA + ¬CH implies SH, we notice that we only had to ensure that a compact Hausdorff space with the ccc was not the union of $\aleph_1$ nowhere dense sets—not necessarily $<2^{\aleph_0}$ nowhere dense sets. In fact this property is common to a wide variety of Martin's Axiom proofs. This observation led to the widespread use of a restricted Martin's Axiom, MA($\aleph_1$).

2.2. NOTATION. MA(κ) is the statement "for every ccc partial order **P** with the ccc and every collection $\mathscr{D}$ of less than or equal to κ dense subsets of **P**, there is a $\mathbf{G} \subseteq \mathbf{P}$ generic for $\mathscr{D}$".

That MA($\aleph_0$) is true, and MA($2^{\aleph_0}$) is false comes easily from considering the topological versions. Furthermore we have the following theorem which comes from the proof of Theorem 2.1.

2.3. THEOREM. MA(κ) *iff* MA(κ) *restricted to partial orders of size less than or equal to* κ.

MA + ¬CH is not equivalent to MA($\aleph_1$); in fact MA($\aleph_1$) does not imply MA($\aleph_2$). To see this, we will modify the sketch of the relative consistency of MA + ¬CH given previously.

Begin with M_0, a model of the generalized continuum hypothesis and let $\kappa = \aleph_{\omega_2}$. Undertake the iterated forcing construction, but only consider those partial orders and collections of dense subsets of cardinality $\aleph_1$. Go through the original proof, making any necessary changes, to verify that in the final model M_κ we have MA($\aleph_1$) and $2^{\aleph_0} = \aleph_{\omega_2}$. The exposition in KUNEN [1980], chapter VIII is particularly suitable for this. MA($\aleph_2$) is not true in M_κ since, as we shall see in section 8, MA($\aleph_2$) implies that $2^{\aleph_0} = 2^{\aleph_2}$. However, by Konig's theorem (KUNEN [1980] p. 34), $2^{\aleph_2} \neq \aleph_{\omega_2}$.

In spite of this, results of A. Miller and D. Fremlin show that if θ is the least cardinal such that MA(θ) fails, then θ must have uncountable cofinality (MILLER [1982], FREMLIN [19 · ·]). On the other hand K. Kunen has shown that it is relatively consistent that MA(κ) holds for all $\kappa < \aleph_{\omega_1}$, but MA($\aleph_{\omega_1}$) fails.

2.4. DEFINITION. A partial order is said to be *well met* iff any two compatible elements have an infimum.

Suppose $\mathbf{P}$ is a partial order, $\{\mathbf{D}_\alpha : \alpha \in \kappa\}$ a collection of dense subsets and $\mathbf{C} \subseteq \mathbf{P}$ a centred subset meeting each $\mathbf{D}_\alpha$. If $\mathbf{P}$ is well met, then

$$\{\inf(\mathbf{F}) : \mathbf{F} \in [\mathbf{C}]^{<\aleph_0}\}$$

is a subset of $\mathbf{P}$ generic for $\{\mathbf{D}_\alpha : \alpha \in \kappa\}$. On the other hand, if $\mathbf{P}$ is not well met, no such generic set containing $\mathbf{C}$ may exist. However we still have the following theorem.

2.5. THEOREM. MA *is equivalent to* MA *restricted to well met partial orders.*

PROOF. Just note that in the proof of Lemma 1.8, the partial order used was well met.

Why consider restricted forms of Martin's axiom at all? Since when is half an axiom better than a whole one? Two reasons. Breaking up MA into pieces may help us understand it better, generalize and extend it more appropriately. Also, we may only want some of the consequences of MA and the negation of some others. We isolate that part of MA which gives the desired consequences and try to make it consistent with the negation of the others.

As an example, suppose we desire the following:

(i) $2^{\aleph_0} > \aleph_1$,

(ii) the real line is not the union of fewer than $2^{\aleph_0}$ nowhere dense subsets, and

(iii) there is a Suslin line.

Of course MA $+ \neg$CH implies (i) and (ii) but not (iii). Looking again at the proof of Lemma 1.8 we see the open sets V could have been chosen from any given base (or π-base) for X. Thus, to derive (ii) we only need Martin's axiom restricted to countable partial orders. This has been called MAC (BELL [1980]). The 'C' could stand for 'countable' or 'P.J. Cohen' because MAC is true in any model which is obtained from a model of ZFC plus CH by adding $\aleph_2$ Cohen reals. There is also a Suslin line in such a model (TENNENBAUM [1968]). The article by K. Kunen in this volume has details of such models.

It is valuable, therefore, to find models of set theory in which some parts of MA are true and some parts false. Strangely, until the recent announcement by J. Baumgartner, it wasn't known if Martin's axiom could fail totally. The proof of this theorem will appear in BAUMGARTNER [19··].

2.6. THEOREM. *It is relatively consistent with the usual axioms of set theory that* $2^{\aleph_0} > \aleph_1$ *and every compact Hausdorff space with the* ccc *and no isolated points is the union of* $\aleph_1$ *dense sets.*

Since Martin's axiom is so valuable, let's extend it. A first try at extending MA would be to 'move everything up one cardinal'. We would get the following.

2.7. GUESS. If **P** is a partial order with the $\aleph_2$-cc and $\mathscr{D}$ is a collection of fewer then $2^{\aleph_1}$ dense subsets of **P**, then there is some $\mathbf{G} \subseteq \mathbf{P}$ generic for $\mathscr{D}$.

Look closely; we haven't just changed cardinals—we have weakened the hypothesis of MA and strengthened its conclusion. In fact the guess is false, as this example show.

2.8. EXAMPLE. Let $\langle \mathbf{P}, \leqslant \rangle$ be the partial order defined as follows.

$$\mathbf{P} = \{f : f \text{ is a function with domain a finite subset of } \omega \text{ and with range a subset of } \omega_1\}$$

The ordering is given by

$$f \leqslant g \quad \text{iff} \quad f \supseteq g \text{ as sets of ordered pairs}.$$

Now, $|\mathbf{P}| = \aleph_1$ so **P** has the $\aleph_2$-cc trivially. For each $\alpha \in \omega_1$, let $\mathbf{D}_\alpha$ be the set

$$\{f \in \mathbf{P} : \alpha \text{ is in the range of } f\}.$$

It is easy to see that each $\mathbf{D}_\alpha$ is dense, and so $\{\mathbf{D}_\alpha : \alpha \in \omega_1\}$ is a collection of fewer than $2^{\aleph_1}$ dense sets.

If **G** is a subset of **P** generic for $\{\mathbf{D}_\alpha : \alpha \in \omega_1\}$, then $\bigcup \mathbf{G}$ is a function which maps a subset of ω onto ω_1, giving a contradiction.

This example also shows why we need the 'ccc' in MA + $\neg$CH. In conclusion then, what is needed to extend and to generalize Martin's axiom is an analysis of some properties of partial orders.

3. Partial orders

In this section, we shall introduce some properties of partial orders and show how they are related. A subset **S** of a partial order **P** is said to be *linked* iff every two elements of **S** are compatible. Thus every centred subset of **P** is linked.

3.1. DEFINITION. A partial order **P** is said to have *property K* iff every uncountable subset of **P** contains an uncountable linked subset.

3.2. DEFINITION. A partial order **P** is said to have *precaliber* λ iff every subset of **P** of cardinality λ has a centred subset of cardinality λ.

3.3. DEFINITION. A partial order **P** is said to be λ-*linked* iff it is the union of λ linked subsets; $\aleph_0$-linked is sometimes called σ-linked.

3.4. DEFINITION. A partial order **P** is said to be λ-*centred* iff it is the union of λ centred subsets. $\aleph_0$-centred is sometimes called σ-centred.

This diagram illustrates the relationships among some of these properties intermediate between ccc and countable.

$$
\begin{tikzcd}
& \text{countable} \arrow[d] & \\
& \aleph_0\text{-centred} \arrow[dl] \arrow[dr] & \\
\aleph_0\text{-linked} \arrow[dr] & & \text{precaliber } \aleph_1 \arrow[dl] \\
& \text{property } K \arrow[d] & \\
& \text{ccc} &
\end{tikzcd}
$$

An arrow $A \to B$ means that property A implies property B. No more arrows may be added to the diagram, nor is any arrow reversible. We will demonstrate this with the examples which follow.

Let us first centre on $\aleph_0$-centred. A little topology is helpful here. Note that if a topological space has a dense set of cardinality λ, then the topology is λ-centred; the partial order is set inclusion and the empty set is disregarded. If a compact Hausdorff space has a λ-centred topology then it must also have a dense set of cardinality λ. A compact Hausdorff space which is not separable but has the ccc gives a partial order with the ccc which is not $\aleph_0$-centred. A separable space with no countable π-base gives a partial order which is $\aleph_0$-centred but has no countable dense subset. To get such spaces we can use the following important result of Hewitt, Marczewski and Pondiczery.

3.5. THEOREM. *The (Tychonoff) product of a collection of 2^λ topological spaces, each of density less than or equal to λ, has density less than or equal to λ.*

On the other hand, we know that the product of 2^λ copies of the two point discrete space has density λ. The details are in the article by R. Hodel in this volume. Also in that article is the following theorem.

3.6. THEOREM. *The (Tychonoff) product of any number of separable spaces has the* ccc.

This shows, for example, that if $\lambda = (2^{\aleph_0})^+$ then the product of λ copies of the two point discrete space has the ccc but is not separable.

These theorems give combinatorial results which are often invoked to show that some partial order has ccc or is $\aleph_0$-centred. For example, let $\omega^{\aleph_1}$ be the usual product of $\aleph_1$ copies of the countably infinite discrete space. We can think of this space as the set of all functions with domain ω_1 and range contained in ω. Then a base for the space can be all sets of the form

$$\{f \in {}^{\omega_1}\omega : h \subseteq f\}$$

where h is some finite function with domain a finite subset of ω_1 and range contained in ω, and $h \subseteq f$ as sets of ordered pairs. Since $\omega^{\aleph_1}$ is separable, we can say that there is a countable collection, S, of functions from ω_1 into ω such that any function from a finite subset of ω_1 into ω agrees, on its domain, with some member of S. Because of this we have the following.

3.7. LEMMA. *The partial order of all finite functions from ω_1 into ω ordered by containment is $\aleph_0$-centred.*

We will need a generalization of this later, so let's do it now. We wish to show that the following partial order is $2^{\aleph_0}$-centred.

$$\mathbf{P} = \{f : f \text{ is a function, domain of } f \in [\omega_2]^{\aleph_0}, \text{ range of } f \subseteq \omega_1\}$$

where

$$f \leq g \quad \text{iff} \quad f \supseteq g \text{ as sets of ordered pairs}.$$

This amounts to showing that the following space has a dense subset of cardinality $2^{\aleph_0}$.

$$Y(\mathbf{P}) = \langle (\omega_1)^{\aleph_2}, \tau \rangle$$

where τ is the topology generated by countable intersections of sets of the following form for fixed $\alpha < \aleph_2$ and $\beta \in \omega_1$:

$$\{y \in Y(\mathbf{P}) : \text{the } \alpha\text{th coordinate of } y \text{ is } \beta\}.$$

This is the same as putting the discrete topology on ω_1, taking the $\aleph_2$ power, but instead of restricting on finitely many coordinates as for the Tychonoff product topology, restricting on countably many coordinates. For more about this see the article by S. Williams in this volume.

We could find a dense subset of $Y(\mathbf{P})$ of cardinality $2^{\aleph_0}$ by trying to directly generalize Theorem 3.5. This works. However we can obtain our result directly from Theorem 3.5 by using the following observation of J. Ginsberg (TALL [1979]).

3.8. LEMMA. *If X is a countably compact regular space with density $d(X)$, then there is subset $D \subseteq X$ of cardinality $(d(X))^{\aleph_0}$ such that every non-empty G_δ of X contains an element of D.*

PROOF. Let S be a dense subset of X of cardinality $d(X)$. For each $A \in [S]^{\aleph_0}$ pick x_A as some limit point of A. Let

$$D = S \cup \{x_A : A \in [S]^{\aleph_0}\}.$$

It is straightforward to show that this D satisfies the lemma.

Now let $\langle X, \rho\rangle$ be the compact ordinal space $\omega_1 + 1$ taken to the $\aleph_2$ power with the product topology given by restriction to countably many co-ordinates. Let $\langle X, \tau\rangle$ be the same set with the Tychonoff product topology. By Lemma 3.8 and Theorem 3.5, there is a subset $D \subseteq X$ of cardinality $\aleph_1^{\aleph_0} = 2^{\aleph_0}$ such that every G_δ subset of $\langle X, \tau\rangle$ contains a point of D. Open sets of $\langle X, \rho\rangle$ are G_δ sets of $\langle X, \tau\rangle$ so $\langle X, \rho\rangle$ has density $\leqslant 2^{\aleph_0}$. Therefore $\langle X, \rho\rangle$ has a $2^{\aleph_0}$-centred topology. The space $Y(\mathbf{P})$ mentioned above is homeomorphic to a dense subspace of the (regular) space $\langle X, \rho\rangle$ and hence also has a $2^{\aleph_0}$-centred topology. Now each member of the partial order $\mathbf{P}$ gives an open subset of $Y(\mathbf{P})$ such that $f \in \mathbf{P}$ and $g \in \mathbf{P}$ are compatible iff the associated open sets have non-empty intersection. Thus $\mathbf{P}$ is $2^{\aleph_0}$-centred. Taking the union of each of these $2^{\aleph_0}$ centred subsets of $\mathbf{P}$ gives the following lemma for later use.

3.9. LEMMA. *There exists a collection S of $2^{\aleph_0}$ functions from ω_2 into ω_1 such that any function from a countable subset of ω_2 into ω_1 is contained in some member of S.*

There is, of course, a purely combinatorial proof of this lemma, but after knowing Theorem 3.5 the topological proof seems easier to grasp.

Another combinatorial fact, often used in Martin's Axiom arguments is the Δ-system lemma (see KUNEN [1980] p. 49, or section 2 of the article by J. Roitman in this volume).

3.10. LEMMA. *If $\mathcal{A}$ is any uncountable family of finite sets, there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ and a finite set r (called the root) such that $a \cap b = r$ whenever a and b are distinct members of $\mathcal{B}$. ($\mathcal{B}$ is called a Δ-system).*

We can now describe more examples illustrating the diagram.

3.11. EXAMPLE. *A partial order which has precaliber $\aleph_1$ but is not $\aleph_0$-linked.* Such a partial order was first constructed by F. Galvin and A. Hajnal. S. Todorçević gives an example in Theorem 9.10 of his article in this volume. Note that, in reference to this theorem, if $\mathbf{P}$ doesn't satisfy the $(\aleph_0, \aleph_0)$-chain condition then $\mathbf{P}$ cannot be $\aleph_0$-linked.

3.12. EXAMPLE. *A partial order which is $\aleph_0$-linked but does not have precaliber $\aleph_1$.* Let $\mathcal{M}$ be the collection of all Lebesque measurable subsets of the unit interval. Let $\sim$ be the equivalence relation on $\mathcal{M}$ defined by

$$A \sim B \quad \text{if} \quad (A \backslash B) \cup (B \backslash A) \text{ has measure } 0\,.$$

Let $\mathbf{P}$ be the partial order on all equivalence classes of $\sim$, given by $p \leqslant q$ iff some element of p is contained in some element of q.

We show that $\mathbf{P}$ is $\aleph_0$-linked. First note that for any set A of positive measure, there are rational numbers r and s such that

$$\text{measure of } (A \cap [r, s]) > \tfrac{1}{2}|r - s|\,.$$

This follows immediately from the Lebesque density theorem (see OXTOBY [1971] p. 17). For each rational pair $\langle r, s\rangle$, let $\mathbf{P}(\langle r, s\rangle)$ be

$$\{p \in \mathbf{P}\colon \text{for some } A \in p, \text{ the measure of } (A \cap [r, s]) > \tfrac{1}{2}|r - s|\}\,,$$

which shows that $\mathbf{P}$ is $\aleph_0$-linked.

If the continuum hypothesis is assumed, $\mathbf{P}$ does not have precaliber $\aleph_1$. This argument is due to P. Erdös. Let $\{s_\alpha\colon \alpha \in \omega_1\}$ be an enumeration of the unit interval. Let

$$S_\alpha = \{s_\beta\colon \beta \geqslant \alpha\}\,.$$

Now fix $\alpha \in \omega_1$ and a real number r such that $0 < r < 1$. S_α has measure 1 so there is a compact set $F \subseteq S$ with measure greater than r. Let

$$y = \sup\{x \in [0, 1]\colon F \cap [0, x] \text{ has measure} < r\}$$

and note that $F \cap [0, x]$ has measure exactly r.

What has been shown is that for any $\alpha \in \omega_1$ and real r between 0 and 1 we can find a compact subset of S_α of measure exactly r. This allows us to find a collection $\{F_\alpha\colon \alpha \in \omega_1\}$ such that

(i) for each $\alpha \in \omega_1$, F_α is a compact subset of S_α, and
(ii) if $\alpha \in \beta \in \omega_1$, then F_α and F_β do not have the same measure.

Now let p_α be the equivalence class of F_α. We claim that no uncountable subset of $\{p_\alpha : \alpha \in \omega_1\}$ is centred. To prove this, suppose that $\Gamma \subseteq \omega_1$ indexed an uncountable centred subcollection. For each finite subset Σ of Γ, $\bigcap\{F_\alpha : \alpha \in \Sigma\}$ is non-empty. Hence, by compactness,

$$\bigcap\{F_\alpha : \alpha \in \Gamma\} = \emptyset .$$

However, this gives a contradiction, since

$$\bigcap\{F_\alpha : \alpha \in \Gamma\} \subseteq \bigcap\{S_\alpha : \alpha \in \Gamma\} = \emptyset .$$

One more example completes the discussion of the diagram.

3.13. EXAMPLE. *A partial order with the* ccc *which does not have property K.* The topology of a Suslin line, partially ordered by set inclusion certainly has the ccc. In section 6 of the article by S. Todorčević in this volume or in KUNEN [1980], p. 66 it is shown that the product of such a Suslin partial order with itself does not have the ccc. It can easily be seen that the product of two ccc partial orders has the ccc if one of the partial orders has property K.

3.14. DEFINITION. If $\langle \mathbf{P}, \leqslant \rangle$ and $\langle \mathbf{Q}, \leqslant \rangle$ are two partial orders, the product partial order $\mathbf{P} \times \mathbf{Q}$ is defined by

$$\langle p_1, q_1 \rangle \leqslant \langle p_2, q_2 \rangle \quad \text{iff both } p_1 \leqslant p_2 \text{ and } q_1 \leqslant q_2 .$$

Thus, if $\mathbf{P}$ has property K and $\mathbf{Q}$ has the ccc and $\{\langle p_\alpha, q_\alpha \rangle : \alpha \in \omega_1\}$ is an uncountable subset of the product, find $\Gamma \in [\omega_1]^{\aleph_1}$ such that for all $\alpha, \beta \in \Gamma$, p_α and p_β are compatible. Now find $\alpha, \beta \in \Gamma$ such that q_α and q_β are compatible. Then $\langle p_\alpha, q_\alpha \rangle$ and $\langle p_\beta, q_\beta \rangle$ are compatible.

There are other examples of partial orders which have the ccc without their product having ccc which are constructed using the continuum hypothesis. R. Laver was the first to do this, but the following construction is due to F. GALVIN [1980]. By the way, the reason for the introduction of the continuum hypothesis into this and into Example 3.12 will be evident in section 4.

3.15. THEOREM. *Assume the Continuum Hypothesis. There are two* ccc *partial orders* $\mathbf{P}_0$ *and* $\mathbf{P}_1$ *such that* $\mathbf{P}_0 \times \mathbf{P}_1$ *is not* ccc.

PROOF. Let K_0 and K_1 be two disjoint subsets of $[\omega_1]^2$. For $i = 0$ or 1, let P_i be

$$\{s \in [\omega_1]^{<\aleph_0} : [s]^2 \subseteq K_i\} ;$$

we let $\mathbf{P}_i = \langle P_i, \leqslant \rangle$ where $s \leqslant t$ iff $s \supseteq t$. Note that each P_i contains all singleton subsets of ω_1. This shows that $\mathbf{P}_0 \times \mathbf{P}_1$ is not ccc; the set

$$\{\langle\{\alpha\}, \{\alpha\}\rangle;\ \alpha \in \omega_1\}$$

is an antichain.

Now let us see what properties of K_i are sufficient to ensure that $\mathbf{P}_i$ has ccc. Suppose $\{t_\alpha : \alpha \in \omega_1\}$ is an uncountable subset of P_i. We want to find conditions on K_i which will ensure that two elements of this collection are compatible. Using the Δ-system lemma we can find an uncountable $\Gamma \subseteq \omega_1$ such that $\{t_\alpha : \alpha \in \Gamma\}$ forms a Δ-system with root r. For each $\alpha \in \Gamma$, let $s_\alpha = t_\alpha \backslash r$, so that for α and β in Γ,

$$[t_\alpha \cup t_\beta]^2 = \{t_\alpha]^2 \cup [t_\beta]^2 \cup \{\{x, y\}: x \in s_\alpha \text{ and } y \in s_\beta\}\,.$$

But t_α and t_β are compatible iff $[t_\alpha \cup t_\beta]^2 \subseteq K_i$ and we already have $\{t_\alpha]^2 \subseteq K_i$ and $\{t_\beta]^2 \subseteq K_i$.

Using the notation

$$X \otimes Y = \{\{x, y\}: x \in X \text{ and } y \in Y\}\,,$$

the following lemma completes the proof of this theorem. The proof is in section 9.2.

3.16. LEMMA. *Assume* CH. *There exists two subsets K_0 and K_1 of $[\omega_1]^2$ such that the following holds for both $i = 0$ and $i = 1$. If S is an uncountable collection of pairwise disjoint finite subsets of ω_1 such that for each $s \in S$ we have $[s]^2 \subseteq K_i$, then there are two members s and t of S such that $s \otimes t \subseteq K_i$.*

We can make finer distinctions between $\aleph_0$-linked and $\aleph_0$-centred. We say that a subset $\mathbf{S}$ of a partial order $\mathbf{P}$ is *n-linked* iff every $\mathbf{T} \in [\mathbf{S}]^{\leq n}$ has a lower bound. Thus 'linked' is '2-linked'.

3.17. DEFINITION. A partial order $\mathbf{P}$ is *$\aleph_0$-n-linked* iff it is the union of countably many n-linked subsets.

Example 3.11 is easily seen to be $\aleph_0$-n-linked for each $n \in \omega$, and is known not to be $\aleph_0$-centred (HALMOS [1963], p. 108). M. Bell has constructed, in (BELL [1980]), for each $n \in \omega$, an $\aleph_0$-n-linked partial order which is not $\aleph_0$-$(n+1)$-linked.

It is possible to write a handbook on the theory of partial orders alone. Further work on the topics mentioned here is in GALVIN [1980], and GALVIN and HAJNAL [1976], where there are results such as: if $n \in \omega$ and $\mathbf{P}$ is a partial order such that $\mathbf{P} = \bigcup\{\mathbf{P}_m : m \in \omega\}$ where such $\mathbf{P}_m$ has the n^+-cc, then $\mathbf{P}$ is $\aleph_0$-linked. And there are also open problems such as the following due to A. Horn and A. Tarskii: if $\mathbf{P} = \bigcup\{\mathbf{P}_m : m \in \omega\}$ such that each $\mathbf{P}_m$ has the $\aleph_0$-cc, then is $\mathbf{P} = \bigcup\{\mathbf{P}'_m : m \in \omega\}$ such that each $\mathbf{P}'_m$ has the m^+-cc?

A good reference for the material discussed in this section is the book by COMFORT and NEGREPONTIS [1982].

4. The influence of MA on partial orders

What effect does Martin's axiom have on partial orders with the properties discussed in section 3? Under the continuum hypothesis we could find examples which showed the distinctiveness of each of these properties. Under MA + $\neg$CH, it's a different story.

We could obtain a partial order with the ccc which did not have property K if either $2^{\aleph_0} = \aleph_1$ or there is a Suslin line. However, each partial order with the ccc does have a weak version of precaliber $\aleph_1$.

4.1. LEMMA. *If* $\mathbf{P}$ *is a partial order with the* ccc *and* $\mathbf{S} \in [\mathbf{P}]^{\aleph_1}$, *then there is an* $\mathbf{S}' \in [\mathbf{S}]^{\aleph_1}$ *such that for any finite centred subset* $\mathbf{F} \subseteq \mathbf{S}'$, *there are uncountably many* $p \in \mathbf{S}'$ *such that* $\mathbf{F} \cup \{p\}$ *is centred.*

PROOF. Let $\mathbf{P}$ and $\mathbf{S}$ be as above. Recalling Theorem 1.5, we can assume that $\mathbf{P}$ is well met, without any loss of generosity. Let $\mathbf{S}''$ be the set

$$\{\inf(\mathbf{F}) : \mathbf{F} \in [\mathbf{S}]^{<\aleph_0}\}$$

and let $\mathbf{S}'''$ be the set

$$\{p \in \mathbf{S}'' : p \text{ is compatible with at most } \aleph_0 \text{ elements of } \mathbf{S}''\}.$$

$\mathbf{S}''$ must be countable; otherwise we could choose, by transfinite recursion, an uncountable subset denying that $\mathbf{P}$ has the ccc. We can now let S' be the set $S \backslash S'''$.

With MA we can do more; the following is due to K. Kunen and F. Rowbottom.

4.2. THEOREM. *Assume* MA($\aleph_1$). *Then we have*

(i) *every partial order with the* ccc *also has property* K, *and*

(ii) *every partial order with the* ccc *also has precaliber* $\aleph_1$.

The statement (i) is sometimes denoted by $\mathscr{K}$ and the statement (ii) by $\mathscr{H}$. We need only prove (ii) since it clearly implies (i).

PROOF. Suppose $\mathbf{P}$ is a partial order with the ccc. Let $\mathbf{S} \in [\mathbf{P}]^{\aleph_1}$ and obtain $\mathbf{S}'$ from Lemma 4.1. Enumerate $\mathbf{S}'$ as $\{p_\alpha : \alpha \in \omega_1\}$. Let $\mathbf{Q}$ be the set

$$\{\mathbf{F} \in [S']^{<\aleph_0} : \mathbf{F} \text{ is centred}\}$$

and let set containment partially order $\mathbf{Q}$.

If $\mathbf{C}$ is an uncountable pairwise incompatible subset of $\mathbf{Q}$, for each $\mathbf{F} \in \mathbf{C}$ pick $p_{\mathbf{F}} \in \mathbf{P}$ such that $p_{\mathbf{F}} \leq p$ for all $p \in \mathbf{F}$; we then get an uncountable pairwise incompatible subset $\{p_{\mathbf{F}} : \mathbf{F} \in \mathbf{C}\}$ of $\mathbf{P}$. Hence $\mathbf{Q}$ has the ccc.

Using the property of $\mathbf{S}'$ obtained from Lemma 4.1, we can see that for each $\beta \in \omega_1$. The set

$$\{\mathbf{F} \in \mathbf{Q} : \mathbf{F} \cap \{p_\alpha : \alpha \geq \beta\} \neq \emptyset\}$$

is dense. We use $\mathrm{MA}(\aleph_1)$ to acquire a subset $\mathbf{G} \subseteq \mathbf{Q}$ generic for all these dense sets. Then the set $\bigcup \mathbf{G}$ is an uncountable centred subset of $\mathbf{S}$.

We will show later that $\mathrm{MA}(\aleph_1)$ implies $2^{\aleph_0} = 2^{\aleph_1}$. It is unknown whether $\mathcal{K}$ or $\mathcal{H}$ implies $2^{\aleph_0} = 2^{\aleph_1}$; it is even unknown whether $\mathcal{K}$ or $\mathcal{H}$ implies $\mathrm{MA}(\aleph_1)$; it is also unknown whether $\mathcal{K}$ implies $\mathcal{H}$. But $\mathcal{H}$ *seems* to be weaker than $\mathrm{MA}(\aleph_1)$, as the following characterization shows.

4.3. THEOREM. *The following are equivalent*

(i) $\mathcal{H}$.

(ii) *$\mathcal{H}$ restricted to well met partial orders.*

(iii) *If $\mathbf{P}$ is a well met partial order with the* ccc *and $\{\mathbf{D}_\alpha : \alpha \in \omega_1\}$ is a collection of dense sets, then there is $I \in [\omega_1]^{\aleph_1}$ and $\mathbf{G} \leq \mathbf{P}$ generic for $\{\mathbf{D}_\alpha : \alpha \in I\}$.*

(iv) *If $\mathbf{P}$ is a well met partial order with the* ccc *and $\{\mathbf{D}_\alpha : \alpha \in \omega_1\}$ is a collection of dense sets such that if $\alpha < \beta$, then $\mathbf{D}_\alpha \supseteq \mathbf{D}_\beta$, then there is a $\mathbf{G} \subseteq \mathbf{P}$ generic for $\{\mathbf{D}_\alpha : \alpha \in \omega_1\}$.*

PROOF. To show that (ii) implies (i), just refer to Theorem 1.5. That (ii) implies (iii) implies (iv) is easy. To show that (iv) implies (i), go over the proof of Theorem 4.2 and notice that we can use (iv) instead of $\mathrm{MA}(\aleph_1)$. This is a partial order version of a topological proof in TALL [1974].

Many consequences of $\mathrm{MA}(\aleph_1)$ actually follow from $\mathcal{K}$, for example, all the S-space and L-space results using MA in the article by J. Roitman in this volume. We give three more examples.

4.4. THEOREM. *Assume that every partial order with the ccc also has property K. Then we have*

(i) SH,

(ii) *not* CH, *and*

(iii) *For any $\{x_\alpha : \alpha \in \omega_1\} \subseteq \mathcal{P}(\omega)$ such that if $\alpha \in \beta \in \omega_1$, then $x_\alpha \not\subseteq x_\beta$, there is $I \in [\omega_1]^{\aleph_1}$ such that $x_\alpha \not\subseteq x_\beta$ for all distinct α, β in I.*

PROOF. In section 3 we mentioned that a Suslin line gives rise to a topology which has the ccc but not property K. This gives (i).

Also in section 3 we gave an example, under CH, of a partial order which has the ccc but not property K. This gives (ii).

The third consequence is in an article by E. van Douwen and K. Kunen. They call this combinatorial principle $\neg\downarrow$ (that's not the same as $\uparrow$). Its negation, $\downarrow$, was used to construct an S-space and an L-space and was shown to follow from CH. To prove (iii) we will follow VAN DOUWEN and KUNEN [1982]. The argument is credited to J. Roitman and is a good example of how a topological outlook can simplify combinatorics. It appears in Section 9.3.

None of (i), (ii) or (iii) is equivalent to the hypothesis of Theorem 4.4. In VAN DOUWEN and KUNEN [1982], it is remarked that there is a model of theory in which there is a Suslin line, and CH and $\downarrow$ both fail. Also, there is a model, constructed by R.B. Jensen (in DEVLIN and JOHNSBRATEN [1974]), of SH and CH.

We can extend Theorem 4.2.

4.5. THEOREM. *Assume* MA(κ). *Every partial order of cardinality κ with the* ccc *is also $\aleph_0$-centred.*

PROOF. Suppose $|\mathbf{P}| = \kappa$ and $\mathbf{P}$ has the ccc. Form another partial order $\mathbf{Q}$. Let $\mathbf{Q}$ be the set of all functions f such that

(i) for some $n \in \omega$, domain of $f = n$ $(= \{0, 1, 2, \ldots, n-1\})$, and

(ii) for each $i \in$ domain of f, $f(i)$ is a finite centred subset of $\mathbf{P}$.

The order on $\mathbf{Q}$ is given by $f_1 \leqslant f_2$ iff

(iii) domain $f_1 \supseteq$ domain f_2, and

(iv) for all $i \in$ domain f_2, $f_1(i) \supseteq f_2(i)$.

We first show that $\mathbf{Q}$ has the ccc. Let $\{f_\alpha : \alpha \in \omega_1\} \in [\mathbf{Q}]^{\aleph_1}$. For each $\alpha \in \omega_1$ and $i \in$ domain of f_α, let $p_{\alpha_i} \in \mathbf{P}$ be a lower bound for $f_\alpha(i)$. There is an $n \in \omega$ and an uncountable $\Gamma_0 \subseteq \omega_1$ such that for each $\alpha \in \Gamma_0$, the domain of f_α is n. Applying Theorem 4.2(i) to $\{p_{\alpha_0} : \alpha \in \Gamma_0\}$ we get $\Gamma_1 \in [\Gamma_0]^{\aleph_1}$ such that $\{p_{\alpha_0} : \alpha \in \Gamma_1\}$ is linked. Again applying Theorem 4.2(i), we get $\Gamma_2 \in [\Gamma_1]^{\aleph_1}$ such that $\{p_{\alpha_1} : \alpha \in \Gamma_2\}$ is linked. We continue until we get an uncountable Γ_n such that $\{p_{\alpha_i} : \alpha \in \Gamma_n$ and $i < n\}$ is linked; if $\{\alpha, \beta\} \in [\Gamma_n]^2$, then f_α and f_β are compatible.

Now, for each $p \in \mathbf{P}$, the set

$$\{f \in \mathbf{Q} : p \in f(i) \text{ for some } i \in \text{domain of } f\}$$

is dense in $\mathbf{Q}$. Let $\mathbf{G} \subseteq \mathbf{Q}$ be a subset generic for each of these dense sets. For each $i \in \omega$, define

$$g(i) = \{p \in \mathbf{P} : p \in f(i) \text{ for some } f \in \mathbf{G} \text{ and } i \in \text{domain of } f\}$$

and notice that each $g(i)$ is centred and $\mathbf{P} = \bigcup\{g(i) : i \in \omega\}$.

Theorem 4.5 has a topological translation which is proven in JUHÁSZ [1977]. The translation can be obtained from the proof of Theorem 1.7.

4.6. COROLLARY. *Every compact Hausdorff topological space with the* ccc *and a π-base of cardinality less than $2^{\aleph_0}$ is separable.*

Another example of the use of Theorem 4.4 is the specialization of Aronszajn trees; the details are in section 9 of the article by S. Todorčević in this volume. Other examples are in TALL [1974]. There is, however, much that is unknown.

4.7. PROBLEM. Suppose that every partial order of cardinality less than $2^{\aleph_0}$ with the ccc is also $\aleph_0$-centred. Does $2^{\aleph_0} = 2^{\aleph_1}$? Must $2^{\aleph_0}$ be a regular cardinal?

4.8. PROBLEM. Are the following equivalent? (ATFE?)

(i) MA,

(ii) every partial order of cardinality less than $2^{\aleph_0}$ with the ccc is also $\aleph_0$-centred,

(iii) every partial order of cardinality less than $2^{\aleph_0}$ with the ccc is also $\aleph_0$-linked.

5. MA for restricted kinds of partial orders

Using the notions of section 3 we can form weakened versions of Martin's Axiom as follows.

5.1. DEFINITION. MA(*property K*) is the statement of MA with 'the ccc' replaced by 'property K'. It is usually denoted by just MAK.

5.2. DEFINITION MA(*precaliber* $\aleph_1$) is the statement of MA with 'the ccc' replaced by 'precaliber $\aleph_1$'.

5.3. DEFINITION. MA($\aleph_0$-*linked*) is the statement of MA with 'has the ccc' replaced by 'is $\aleph_0$-linked'.

5.4. DEFINITION. MA($\aleph_0$-*centred*) is the statement of MA with 'has the ccc' replaced by 'is $\aleph_0$-centred'. It is sometimes denoted by MAS.

The 'S' stands for 'separable'; MAS is equivalent to the statement of the topological form of MA with 'ccc' strengthened to 'separable'. An argument similar to Theorem 1.7 directly gives that MA($\aleph_0$-centred) is equivalent to the statement that no compact Hausdorff space with an $\aleph_0$-centred π-base is the union of fewer than $2^{\aleph_0}$ nowhere dense sets. But is is easy to see that a compact Hausdorff space is separable iff it has an $\aleph_0$-centred π-base.

Each of these versions is strictly weaker than full MA, as is evident from the following result of K. Kunen and F. Tall.

5.5. THEOREM. *If there is a model of set theory there is one in which we have*
(i) $2^{\aleph_0} = \aleph_2$,
(ii) MAK, *and*
(iii) *a Suslin line.*

The idea of the proof is to proceed as in the MA + ¬CH proof outlined at the beginning of section 2, with two changes. First, make sure that the original model M_0 contains a Suslin line. Second, only consider partial orders with property K; ignore those partial orders (like the open subsets of a Suslin line) which have the ccc but don't have property K. One then must check that the original Suslin line in M_0 remains a Suslin line in M_κ, as well as other technicalities. The proof is outlined in KUNEN and TALL [1979].

We also have the following in ROITMAN [1979].

5.6. THEOREM. *If M is a model of set theory which has been obtained by adding a single Cohen real to a model of* MA *plus* $2^{\aleph_0} = \aleph_2$, *then in M we have*
(i) $2^{\aleph_0} = \aleph_2$,
(ii) MA($\aleph_0$-*linked*), *and*
(iii) MA *is false.*

The proof is an interesting application of topology to set theory. Refer to the article by J. Roitman in this volume for some of the details.

Are these restricted versions of MA strictly weaker than MA in the presence of SH? The following results are from chapter 4 of the Ph.D. thesis of C. Herink (HERINK [1977]). The proofs are more sophisticated versions of the proof for Theorem 5.5.

5.7. THEOREM. *If there is a model of set theory there is one in which we have*
(i) $2^{\aleph_0} = \aleph_2$,
(ii) SH,
(iii) MA($\aleph_0$-*linked*), *and*
(iv) *there is a partial order with precaliber $\aleph_1$ and $\aleph_1$ dense subsets for which there is no generic subset.*

That is, SH plus MA($\aleph_0$-linked) does not imply MA(precaliber $\aleph_1$).

5.8. THEOREM. *If there is a model of set theory, there is one in which we have*
(i) $2^{\aleph_0} = \aleph_2$,
(ii) SH,
(iii) MA(*precaliber* $\aleph_1$),
(iv) *there is an $\aleph_0$-linked partial order and $\aleph_1$ dense subsets for which there is no generic subset.*

That is, SH plus MA(precaliber $\aleph_1$) does not imply MA($\aleph_0$-linked). The $\aleph_0$-linked partial order used here is the one from Example 3.12. These results still, however, left open the following problem.

5.9. PROBLEM. Does SH plus MAK imply MA?

This problem has recently been solved by S. Todorčević. The answer, as expected, is no.

There is a weak version of MA($\aleph_0$-centred) which we have not yet discussed.

5.10. DEFINITION. MA(finitely centred) is the statement of MA with 'has the ccc' replaced by 'is the union of finitely many centred subsets'.

This is so weak that it is actually true.

5.11. LEMMA. *MA(finitely centred) is true.*

PROOF. We could try a combinatorial argument, but let's look at the topological translation: no compact Hausdorff space with a finite dense set can be the union of fewer than $2^{\aleph_0}$ nowhere dense sets. This does display the value of the topological viewpoint.

We will show that each of the restricted versions, 5.1, 5.2, 5.3 and 5.4 of MA is strictly stronger than MAC, that is, MA restricted to countable partial orders. We will demonstrate a combinatorial principle P_0 which is a consequence of MA($\aleph_0$-centred) but not of MAC.

5.12. DEFINITION. For any ordinal α, let P_α be the combinatorial statement: Given a collection of fewer than $2^{\aleph_\alpha}$ subsets of ω_α such that the intersection of any subcollection of cardinality less than $\aleph_\alpha$ has cardinality $\aleph_\alpha$, there is a subset $B \subseteq \omega_\alpha$ of cardinality $\aleph_\alpha$ such that for any member A of the collection we have $|B \backslash A| < \aleph_\alpha$.

We will mainly be interested in P_0 and P_1. P_0 has been called $P(c)$. P_0 is also equivalent to the statement '$\mathfrak{p} = \mathfrak{c}$' in the article by E. van Douwen in this volume; and the next lemma is a generalization of '$\mathfrak{p} \leqslant \mathfrak{b}$'. The result is due to F. Rothberger (ROTHBERGER [1948]), but this proof can be found in BURKE and VAN DOUWEN [1977].

5.13. LEMMA. *Assume P_α. Suppose $\lambda < 2^{\aleph_\alpha}$, $\aleph_\alpha$ is a regular cardinal and $\{f_\beta : \beta \in \lambda\}$ is a collection of functions from ω_α into ω_α. Then there exists a function $f: \omega_\alpha \to \omega_\alpha$ such that for all $\beta \in \lambda$ there is some $N_\beta \in \omega_\alpha$ such that $f(\xi) > f_\beta(\xi)$ for all $\xi \geqslant N_\beta$.*

PROOF. Without loss of generosity assume that each function f_β is increasing. Consider the collection

$$\mathscr{A} = \{\{\langle\xi, \eta\rangle: \eta \geqslant f_\beta(\xi)\}: \beta \in \lambda\} \cup \{(\omega_\alpha \backslash \eta) \times \omega_\alpha : \eta \in \omega_\alpha\}$$

Using P_α on the set $\omega_\alpha \times \omega_\alpha$, extract $B \subseteq \omega_\alpha \times \omega_\alpha$ for this collection $\mathscr{A}$ such that for each $A \in \mathscr{A}$ we have $|B \backslash A| < \aleph_\alpha$.

Define $\varphi: \omega_\alpha \to \omega_\alpha$ by:

$$\varphi(\eta) = \min\{\xi \geqslant \eta : (\{\xi\} \times \omega_\alpha) \cap B \neq \emptyset\}$$

and define $f: \omega_\alpha \to \omega_\alpha$ by:

$$f(\eta) = \min\{\gamma \in \omega_\alpha : \langle\varphi(\eta), \gamma\rangle \in B\}.$$

For each $\beta \in \omega_\alpha$, let N_β be defined as

$$\sup\{\xi \in \omega_\alpha : \text{for some } \gamma, \langle\xi, \gamma\rangle \in B \text{ and } \gamma \leqslant f_\beta(\xi)\} + 1.$$

It is straightforward to check that this satisfies the conclusion of the lemma.

5.14. LEMMA. MAC *does not imply* P_0, *since*
(i) P_0 *implies that* $2^{\aleph_0}$ *is a regular cardinal, and*
(ii) MAC *does not imply that* $2^{\aleph_0}$ *is a regular cardinal.*

PROOF. We first show (i). Suppose $\kappa < 2^{\aleph_0}$ and $2^{\aleph_0} = \sup\{\lambda_\alpha : \alpha < \kappa\}$, where $\langle\lambda_\alpha\rangle$ is a continuous increasing sequence. Let $\{f_\beta: \beta < 2^{\aleph_0}\}$ enumerate all functions from ω into ω. For each $\alpha < \kappa$, apply Lemma 5.13 to $\{f_\beta: \lambda_\alpha \leqslant \beta < \lambda_{\alpha+1}\}$ obtaining f^α and $\{N_\beta: \lambda_\alpha \leqslant \beta < \lambda_{\alpha+1}\}$. Now apply Lemma 5.13 to $\{f^\alpha: \alpha < \kappa\}$ obtaining f and $\{M_\alpha: \alpha < \kappa\}$. Now, since f is a function from ω into ω, $f = f_\beta$ for some α and some β such that $\lambda_\alpha \leqslant \beta < \lambda_{\alpha+1}$. So for all $n \geqslant N_\beta$ we have $f^\alpha(n) > f(n)$ and for all $n \geqslant M_\alpha$ we have $f(n) > f^\alpha(n)$. So for all $n > \max\{N_\beta, M_\alpha\}$ we have $f(n) > f(n)$.

To show (ii), we can add $\aleph_{\omega_1}$ Cohen reals to a model of CH. In the resulting model we have $2^{\aleph_0} = \aleph_{\omega_1}$ and MAC holding. See the article by K. Kunen in this volume and KUNEN [1980], p. 216.

M. BELL [1980] has taken advantage of 5.14(ii), by using MAC to construct a compact Hausdorff non-separable space of cardinality $2^{\aleph_0}$ with the ccc and the character of each point less than $\mathrm{cf}(2^{\aleph_0})$. If $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$, the space is first countable! By the way, the topology of this space shows that '$<2^{\aleph_0}$' cannot be replaced by '$\leqslant 2^{\aleph_0}$' in Corollary 4.6. Another example of this is Example 3.10.

The next theorem shows that MAC does not imply MA($\aleph_0$-centred). This result was one of the first consequences obtained from MA; it appears in BOOTH [1970].

5.15. THEOREM. MA($\aleph_0$-centred) *implies* P_0.

PROOF. Let $\kappa < 2^{\aleph_0}$ and let $\{A_\alpha : \alpha < \kappa\}$ be a collection of subsets of ω such that

(i) for all $F \in [\kappa]^{<\aleph_0}$, $\bigcap\{A_\alpha : \alpha \in F\}$ is infinite.

Let $\mathbf{P}$ be the set of all pairs $\langle b, F\rangle$ such that

(ii) $b \in [\omega]^{<\aleph_0}$ and $F \in [\kappa]^{<\aleph_0}$.

And let $\langle b_1, F_1\rangle \leqslant \langle b_2, F_2\rangle$ iff

(iii) $b_1 \supseteq b_2$, $F_1 \supseteq F_2$, and

(iv) for all $\alpha \in F_2$, $b_1 \backslash b_2 \subseteq A_\alpha$.

Since there are only $\aleph_0$ possibilities for the first coordinate it is easy to check that $\mathbf{P}$ is $\aleph_0$-centred. For each $\alpha \in \kappa$, consider the set $\{\langle b, F\rangle \in \mathbf{P} : \alpha \in F\}$. It is easy to check that each of these is dense. Furthermore, using (i), we can check that for each $n \in \omega$, the set

$$\{\langle b, F\rangle \in \mathbf{P}\text{: for some } m \geqslant n,\ m \in b\}$$

is dense. Let $\mathbf{G} \subseteq \mathbf{P}$ be generic for all these, and let

$$B = \{b : \langle b, F\rangle \in \mathbf{G} \text{ for some F}\}.$$

It is straightforward to show that B satisfies P_0.

It is a wonderful result of M. Bell that the converse to Theorem 5.15 also holds.

5.16. THEOREM. MA($\aleph_0$-*centred*) *is equivalent to* P_0.

PROOF. Half of this is Theorem 5.15. According to 5.4 we need only show the following Lemma.

5.17. LEMMA. *Assume* P_0. *Suppose* X *is a compact Hausdorff separable space. If* $\kappa < 2^{\aleph_0}$ *and for each* $\alpha \in \kappa$, D_α *is a dense open subset of* X, *then* $\bigcap\{D_\alpha : \alpha \in \kappa\} \neq \emptyset$.

The proof is in section 9.4.

We could have proven Theorem 5.16 in a completely combinatorial manner, omitting topological references entirely. In fact, using section 1, a direct translation of our proof is evident. However it is the topological version that will be more convenient to generalize in the next section. For a different proof of Theorem 5.16, see BELL [1981].

Considering the topological form of MA($\aleph_0$-centred) and Example 1.10, one may ask if there is a separable Baire space which is the union of $\aleph_1$ nowhere dense sets. Indeed there is. Use Theorem 3.5 to obtain a countable dense subset C of $2^{\aleph_1}$. Let Y be the subspace $C \cup X$ of $2^{\aleph_1}$, where X is as in Example 1.10. X is a dense Baire subspace of Y and so Y is also a Baire space; nowhere dense subsets of X are nowhere dense in Y, so Y is the union of $\aleph_1 + \aleph_0$ nowhere dense sets.

The space X from Example 1.10 is, in fact, countably compact. Can we strengthen the above example Y to obtain an example which is also countably compact? Surprisingly, no.

5.18. THEOREM. *The following are equivalent.*

(i) MA($\aleph_0$-*centred*).

(ii) *No compact, Hausdorff separable space is the union of fewer than* $2^{\aleph_0}$ *nowhere dense subsets.*

(iii) *No countably compact, regular separable space is the union of fewer than* $2^{\aleph_0}$ *nowhere dense subsets.*

PROOF. The equivalence of (i) and (ii) was discussed previously; clearly (iii) implies (ii). It remains to show that (i) implies (iii).

We proceed initially as in Lemma 1.8. Let X be a space as in (iii) and let $\{U_\alpha: \alpha \in \kappa\}$ be a family of dense open subsets of X; $\kappa < 2^{\aleph_0}$. Let $\mathbf{P}$ be the partial order of all non-empty open subsets of X ordered by set inclusion. For each

$$\mathbf{D}_\alpha = \{V \in \mathbf{P}: \bar{V} \subseteq U_\alpha\}$$

is dense in $\mathbf{P}$ and hence by MA($\aleph_0$-centred) there is a $\mathbf{G} \subseteq \mathbf{P}$ generic for this collection. For each $\alpha \in \kappa$, pick $V_\alpha \in \mathbf{G} \cap \mathbf{D}_\alpha$.

It remains to show that $\bigcap\{\bar{V}_\alpha: \alpha \in \kappa\} \neq \emptyset$. Consider the collection $\{\omega \cap V_\alpha: \alpha \in \kappa\}$ and apply P_0 to obtain $B \subseteq \omega$ such that $B \setminus V_\alpha$ is finite for each $\alpha \in \kappa$. It is impossible for B to have a limit point in

$$\bigcup\{X \setminus \bar{V}_\alpha: \alpha \in \kappa\}$$

and so, since X is countably compact, this set cannot be all of X. Hence $\bigcap\{\bar{V}_\alpha: \alpha \in \kappa\} \neq \emptyset$ and the proof is complete.

6. Extending a little; extending a lot

As we saw in section 2, MA does not generalize to higher cardinals naively. Some care must be taken when proposing a generalized Martin's axiom (henceforth GMA); nevertheless, since MA is so useful, even a restricted GMA may be worthwhile. What should GMA be like? What consequences should it have? Presumably we would want to have P_1, the cardinal generalization of P_0 (see Definition 5.12) follow from GMA. Since P_0 is equivalent to MA($\aleph_0$-centred), it is with this version of MA that we can begin to generalize.

6.1. DEFINITION. GMA($\aleph_1$-centred) is the statement: Suppose $\mathbf{P}$ is a partial order with the following properties:

(i) $\mathbf{P}$ is $\aleph_1$-centred, and

(ii) every countable centred subset of $\mathbf{P}$ has a lower bound.

Suppose further, that $\kappa < 2^{\aleph_1}$ and for each $\alpha \in \kappa$, $\mathbf{D}_\alpha$ is a dense subset of $\mathbf{P}$. Then there is $\mathbf{G} \subseteq \mathbf{P}$ generic for $\{\mathbf{D}_\alpha: \alpha \in \kappa\}$.

The property of $\mathbf{P}$ described in (ii) is sometimes called *countably compact*. This seems to be the correct way to extend MA($\aleph_0$-centred) because we have the following theorem, which we will prove later in this section.

6.2. THEOREM. *Assume* CH. GMA($\aleph_1$-centred) *is equivalent to* P_1.

This means that GMA($\aleph_1$-centred) is relatively consistent, since it is shown in KUNEN [1980], p. 286 that CH plus P_1 plus $2^{\aleph_1} > \aleph_2$ is relatively consistent. This is not the original proof; we will talk about that later.

Is CH necessary here? It turns out that without CH, GMA($\aleph_1$-centred) can apply only to those partial orders which have an *atom* (i.e. an element p for which $\{q : q \leqslant p\}$ is centred). If p is an atom, $\{q : q \leqslant p\}$ is a generic subset for any collection of dense sets, and so GMA($\aleph_1$-centred) is rendered useless. We verify this with the following lemma.

6.3. LEMMA. *If* **P** *is a countably compact partial order with no atoms, then* **P** *has a pairwise incompatible subset of cardinality* $2^{\aleph_0}$.

PROOF. We shall define, for each finite sequence, $\langle a_1, \ldots, a_n\rangle$, of 0's and 1's (including the empty sequence) an element $p(\langle a_1, \ldots, a_n\rangle)$ of the partial order **P**. Let $p(\langle\ \rangle)$ be any element of **P**. If $p(\langle a_1, \ldots, a_n\rangle)$ has been defined, we know it is not an atom so we can find two incompatible elements below $p(\langle a_1, \ldots, a_n\rangle)$; we let these be $p(\langle a_1, \ldots, a_n, 0\rangle)$ and $p(\langle a_1, \ldots, a_n, 1\rangle)$. Because **P** is countably compact, for each infinite sequence σ of 0's and 1's we can obtain an element $p_\sigma \in \mathbf{P}$ such that

$$p_\sigma \leqslant p(\langle a_1, \ldots, a_n\rangle)$$

for each initial segment $\langle a_1, \ldots, a_n\rangle$ of σ. The collection $\{p_\sigma$: σ is an infinite sequence of 0's and 1's$\}$ is pairwise incompatible and this completes the proof.

We will use a topological translation of GMA($\aleph_1$-centred) for the proof of Theorem 6.2. Unafraid of the paradoxes of self-reference, we shall follow WEISS [1981].

6.4. DEFINITION. The topological form of GMA($\aleph_1$-centred) is the following statement:

No compact Hausdorff space with a dense set of cardinality $\aleph_1$ and a π-base $\mathscr{B}$, such that each non-empty G_δ from $\mathscr{B}$ has non-empty interior, is the union of fewer than $2^{\aleph_1}$ nowhere dense sets.

Here, a G_δ from $\mathscr{B}$ is an intersection of countably many members of $\mathscr{B}$.

6.5. LEMMA. GMA($\aleph_1$-centred) *implies the topological form of* GMA($\aleph_1$-*centred*).

PROOF. Suppose X is a compact Hausdorff space with a dense set of cardinality $\aleph_1$ and $\mathscr{B}$ is a π-base for X such that a non-empty G_δ from $\mathscr{B}$ has non-empty interior. Furthermore suppose that $\kappa < 2^{\aleph_1}$ and for each $\alpha \in \kappa$, U_α is a dense open subset of X. Let $\mathbf{P}$ be the set of all non-empty closed G_δ's from $\mathcal{B}$. Partially order $\mathbf{P}$ with set inclusion. Since X is countably compact we have that $\mathbf{P}$ is a countably compact partial order. Since each element of $\mathbf{P}$ has non-empty interior and X has a dense subset of cardinality $\aleph_1$, we see that $\mathbf{P}$ is $\aleph_1$-centred.

For each $\alpha \in \kappa$, let $\mathbf{D}_\alpha = \{V \in \mathbf{P}: V \subseteq U_\alpha\}$. We claim that each $\mathbf{D}_\alpha$ is dense in $\mathbf{P}$. Let $V \in \mathbf{P}$; since $\mathrm{Int}(V) \neq \emptyset$ and $\mathcal{B}$ is a π-base, we can find $B_0 \in \mathcal{B}$ such that $B_0 \subseteq V \cap U_\alpha$. If B_n has been found we find $B_{n+1} \in \mathcal{B}$ such that $\bar{B}_{n+1} \subseteq B_n$. Then we have

$$\emptyset \neq \bigcap\{\bar{B}_n: n \in \omega\} = \bigcap\{B_n: n \in \omega\}$$

and hence $\bigcap\{B_n: n \in \omega\}$ is an element of $\mathbf{D}_\alpha$ contained in V.

We obtain $\mathbf{G} \subseteq \mathbf{P}$ from GMA($\aleph_1$-centred) which is generic for $\{\mathbf{D}_\alpha: \alpha \in \kappa\}$. We have, by the compactness of X, that

$$\emptyset \neq \bigcap \mathbf{G} \subseteq \bigcap\{U_\alpha: \alpha \in \kappa\},$$

which completes the proof.

The topological form of GMA($\aleph_1$-centred) is not quite so elegant as the topological form of MA($\aleph_0$-centred). Nevertheless, the form we need for Theorem 6.2 is even worse.

6.6. THEOREM. *The following are equivalent.*

(i) GMA($\aleph_1$-*centred*).

(ii) *The topological form of* GMA($\aleph_1$-*centred*).

(iii) *Suppose X is a compact Hausdorff space with a dense subset Y of cardinality $\aleph_1$ and a π-base $\mathcal{B}$ such that*

(a) *$\{B \cap Y: B \in \mathcal{B}\}$ is a base for the subspace Y, and*

(b) *if $\mathcal{B}'$ is a countable centred subcollection of $\mathcal{B}$, then $Y \cap \bigcap \mathcal{B}'$ is a non-empty open set in the subspace Y.*

If $\kappa < 2^{\aleph_0}$, then the intersection of any family of κ dense open subsets of X is non-empty.

This proof appears in section 9.5. The proof of the following lemma appears in section 9.6.

6.7. LEMMA. *Assume* CH *and P_1. Suppose X is a compact Hausdorff space with a dense set Y of cardinality $\aleph_1$ and a π-base $\mathcal{B}$ such that*

(i) *$\{B \cap Y: B \in \mathcal{B}\}$ is a base for the subspace Y, and*

(ii) *if $\mathcal{B}'$ is a countable centred subcollection of $\mathcal{B}$, then $Y \cap \bigcap \mathcal{B}'$ is a non-empty open set in the subspace Y.*

If $\kappa < 2^{\aleph_1}$ and for each $\alpha \in \kappa$, $\mathbf{D}_\alpha$ is a dense open subset of X, then $\bigcap\{\mathbf{D}_\alpha: \alpha \in \kappa\} \neq \emptyset$.

6.8. THEOREM. *Assume* CH. GMA($\aleph_1$-*centred*) *implies* P_1.

PROOF. This is similar to Theorem 5.15. Suppose $\kappa < 2^{\aleph_1}$ and $\{A_\alpha : \alpha \in \kappa\}$ is a collection of subsets of ω_1 such that

(i) for all $F \in [\kappa]^{\leqslant\aleph_0}$, $\bigcap\{A_\alpha : \alpha \in F\}$ is uncountable.

Let **P** be the set of all pairs $\langle b, F\rangle$ such that

(ii) $b \in [\omega_1]^{\leqslant\aleph_0}$ and $F \in [\kappa]^{\leqslant\aleph_0}$.

And let $\langle b_1, F_1\rangle \leqslant \langle b_2, F_2\rangle$ iff

(iii) $b_1 \supseteq b_2$, $F_1 \supseteq F_2$, and

(iv) for all $\alpha \in F_2$, $b_1 \backslash b_2 \subseteq A_\alpha$.

The proof now proceeds analogously to Theorem 5.15.

This theorem completes the logical chain which gives us Theorem 6.2. There is also a corollary.

6.9. COROLLARY. *Assume that* $\kappa^{\aleph_0} < 2^{\aleph_1}$ *for all* $\kappa < 2^{\aleph_1}$. GMA($\aleph_1$-*centred*) *is equivalent to* GMA($\aleph_1$-*centred*) *restricted to partial orders of cardinality less than* $2^{\aleph_1}$.

PROOF. Without CH, both are vacuously true, as seen before in Lemma 6.3. With CH, observe that the cardinality of the partial order **P** in Theorem 6.8 is $\kappa^{\aleph_0}$ for some $\kappa < 2^{\aleph_1}$.

Under CH, the cardinal arithmetic assumption is true if $2^{\aleph_1} < \aleph_\omega$, but false if $2^{\aleph_1} = \aleph_{\omega+1}$. In order to make Corollary 6.9 more like Theorem 1.8, we should remove this cardinal arithmetic hypothesis; however it is not known if this can be done.

With confidence raised from the success at extending MA($\aleph_0$-centred) to GMA($\aleph_1$-centred) in section 6, we can look forward to further generalizations. We could try to extend GMA($\aleph_1$-centred) to higher cardinals—but we won't. We do not yet have all the answers to natural questions about the $\aleph_1$ case, so this is the place to first concentrate our attention. Furthermore there doesn't seem to be, at least from the set-theoretic topology viewpoint, a great need for such a generalization; the known consequences of P_2 do not rival the important consequences of P_0 and P_1 outlined in the next section. Generalization for its own sake may be uninteresting, however, an elegant generalization of the full MA which applies to all cardinals would certainly be satisfying. Unfortunately, it presently seems out of reach.

We will try to strengthen GMA($\aleph_1$-centred) by weakening its hypotheses. The notion of countably compact, in the paragraph after Definition 6.1, is stronger than the simpler notion of countably closed.

6.10. DEFINITION. A partial order **P** is said to be *countably closed* iff every countable decreasing sequence of elements has a lower bound.

How much is countably closed weaker than countably compact? We have the equation

"countably compact = countably closed + well met"

which is explained by the following theorem.

6.11. THEOREM. *In the statement of* GMA($\aleph_1$*-centred*) *the condition* "**P** *is countably compact*" *may be replaced by* "**P** *is countably closed and well met*".

PROOF. Every partial order which is countably closed and well met is surely countably compact. If $\{p_n : n \in \omega\}$ is a centred subset of **P**, we consider the decreasing sequence of elements $\langle \inf\{p_0, \ldots, p_n\} : n \in \omega \rangle$. On the other hand, not every countably compact partial order is well met. However, the partial order **P** used in the proof of Theorem 6.8 is easily seen to be well-met. This, along with Theorem 6.2 completes the proof under CH. However as seen before in Lemma 6.3, without CH there is nothing to prove.

Further scrutiny of the partial order **P** in Theorem 6.8 reveals that is has the property: "every countable centred subset of **P** has an infimum". Hence this property too can replace 'countably compact' in the definition of GMA($\aleph_1$-centred).

The well met condition seems to be very weak—indeed all the partial orders which we have so far constructed in this article are well met. We are therefore tempted to eliminate it.

6.12. DEFINITION. RGMA (Reckless Generalization of Martin's Axiom). If **P** is a countably closed, $\aleph_1$-centred partial order and $\mathcal{D}$ is a collection of less than $2^{\aleph_1}$ dense subsets of **P**, then there is a subset **G** of **P** which is generic for $\mathcal{D}$.

The reason that this generalization is reckless is the following theorem of Shelah (see STANLEY and SHELAH [1982]).

6.13. THEOREM. $2^{\aleph_0} = \aleph_1$ *and* $2^{\aleph_1} > \aleph_2$ *together imply that* RGMA *is false.*

PROOF. We begin by constructing a relation on the set of all sequences of length ω_2 of functions from ω_1 into ω_1. Denote a typical sequence by $\vec{f}$ and its αth element (which is a function) by $\vec{f}(\alpha)$.

We define $\vec{g} \langle^* \vec{f}$ iff for all $\alpha \in \beta \in \omega_2$ there is some $\delta \in \omega_1$ such that $\vec{g}(\alpha)(\delta) \neq \vec{g}(\beta)(\delta)$ and such that either

$$\delta = 0 \quad \text{or} \quad \vec{f}(\alpha) \restriction \delta + 1 = \vec{f}(\beta) \restriction \delta + 1 .$$

That is, either $\vec{g}(\alpha)$ and $\vec{g}(\beta)$ differ at zero or they differ before $\vec{f}(\alpha)$ and $\vec{f}(\beta)$ do.

We can show that there is no infinite $<^*$ descending sequence. To this end, assume that for each $n \in \omega$, $\vec{f}_{n+1} <^* \vec{f}_n$. By a simple cardinality argument, using CH, we get $\alpha \in \beta \in \omega_2$ such that for each $n \in \omega$,

$$\vec{f}_n(\alpha)(0) = \vec{f}_n(\beta)(0)$$

and hence

$$\langle \min\{\delta \in \omega_2 \colon \vec{f}_n(\alpha)(\delta) \neq \vec{f}_n(\beta)(\delta)\} \colon n \in \omega \rangle$$

is an infinite decreasing sequence of ordinals.

Now the following lemma will complete the proof of the theorem. The proof appears in section 9.7.

6.14. LEMMA. *Assume* $2^{\aleph_0} = \aleph_1$ *plus* $2^{\aleph_1} > \aleph_2$ *plus* RGMA. *Then, for all* $\vec{f}$, *there is some* $\vec{g}$ *such that* $\vec{g} <^* \vec{f}$.

Furthermore S. Shelah and L. Stanley display a version of GMA which implies GMA($\aleph_1$-centred) and implies that there is an $\aleph_2$-Suslin tree. K. Kunen has also constructed an axiom with these properties. For the definition of $\aleph_2$-Suslin tree, see the article by S. Todorčević in this volume. However, by terminology alone, we can see that this is not the direction in which we wish to strengthen GMA($\aleph_1$-centred) to something analogous to the full MA.

We therefore leave 'countably compact' alone, and concentrate on weakening the '$\aleph_1$-centred' condition. The first GMA was proposed and proved relatively consistent by R. Laver. Other versions appear in BAUMGARTNER [19··], HERINK [1977], and SHELAH [1978], which is the strongest one to date.

The version of J. Baumgartner can be described as GMA($\aleph_1$-linked), which is just GMA($\aleph_1$-centred) with '$\aleph_1$-centred' replaced by '$\aleph_1$-linked'. The conjuction of GMA($\aleph_1$-linked) and CH has been called BACH (see TALL [1979]).

Until recently we did not know if GMA($\aleph_1$-linked) was strictly stronger than GMA($\aleph_1$-centred). However, J. Steprâns has now shown that this is true and indeed there is an $\aleph_1$-linked partial order which is not $\aleph_1$-centred.

None of the versions of GMA, which have so far been proven to be relatively consistent, is strong enough to imply the $\aleph_2$-SH (i.e. there are no $\aleph_2$-Suslin trees). A partial result, in LAVER and SHELAH [1981], is that if there is a model of set theory with a weakly compact cardinal, then there is one in which both CH and $\aleph_2$-SH hold.

So at least we can hope that $\aleph_2$-SH is consistent with some strengthened form of

GMA($\aleph_1$-centred). Some large cardinal hypothesis is necessary for this, however. STANLEY and SHELAH [1982], have proven that GMA($\aleph_1$-centred), CH, $2^{\aleph_1} > \aleph_2$ and the $\aleph_2$-SH together imply that $\aleph_2$ is a Mahlo cardinal in **L**. The entire situation regarding GMA, then, is still unresolved, and furthermore, may be quite complicated.

7. Some consequences of the combinatorial principles P_0 and P_1

The combinatorial principles P_0 and P_1 have had a lot of employment in set-theoretic topology; P_0 has been applied in most branches of the subject; P_1, somewhat newer, is still having an influence. Although the article by E. van Douwen in this volume shows the value of considering subsets of the integers, most applications of P_0 come via Theorem 5.16 from its other self, MA($\aleph_0$-centred). After the proper acclimatization most people feel that the partial order form is somewhat easier to handle than the integer subset or topological space form. Similarly, we will use Theorem 6.2 to translate P_1 to GMA($\aleph_1$-centred); CH will be assumed in this case.

A complete survey of all applications of P_0 and P_1 in this chapter would be a large undertaking and would also contradict the ignorance of the author in these matters. Therefore, a small sample has been chosen with two objectives in mind. First, we shall contrast P_0 and P_1; second, we shall derive consequences, which have been and still may be useful as lemmas for further work in set-theoretic topology.

Some of the first topological applications of Martin's axiom were to questions of normality of topological spaces (see RUDIN [1975]). A number of diverse results appeared. Later it was realized that all these could be done in a uniform manner (see ALSTER and PRZYMUSIŃSKI [1976], JUHÁSZ and WEISS [1978], PRZYMUSIŃSKI [1981]). In fact, they could all be obtained with just the $\aleph_0$-centred version of MA. The following is not the most general result, but it gives the flavour.

7.1. THEOREM. *Assume* MA($\aleph_0$-*centred*) *plus not* CH. *Suppose ρ and τ are two topologies on a set X such that*

(i) $\rho \subseteq \tau$,

(ii) $\langle X, \rho\rangle$ *is Hausdorff and second countable, and*

(iii) *there is a closed neighbourhood base for τ consisting of sets compact in $\langle X, \rho\rangle$.*

Then, for all H, $K \in [X]^{<2^{\aleph_0}}$ such that $\bar{H} \cap K = H \cap \bar{K} = \emptyset$ in the τ topology, we have open U_H and U_K in τ containing H and K respectively.

PROOF. Let $\mathcal{V}$ be the neighbourhood base for τ obtained from (iii). Let **P** be the set of pairs $\langle V, W\rangle$ such that

(iv) V and W are each unions of finite subcollections of $\mathscr{V}$,

(v) $V \cap (W \cup \bar{K}) = \emptyset$, where the closure is in the τ topology, and

(vi) $(V \cup \bar{H}) \cap W = \emptyset$, where the closure is in the τ topology. Let $\langle V_1, W_1 \rangle \leqslant \langle V_2, W_2 \rangle$ iff

(vii) $V_1 \supseteq V_2$ and $W_1 \supseteq W_2$.

Now let $\mathscr{B}$ be a countable base for $\langle X, \rho \rangle$ which is closed under finite unions. For each $\langle V, W \rangle \in \mathbf{P}$, there is $B \in \mathscr{B}$ such that $V \subseteq B$ and $B \cap W = \emptyset$. $\mathbf{P}$ is $\aleph_0$-centred because for each $B \in \mathscr{B}$,

$$\{\langle V, W \rangle : V \subseteq B \text{ and } B \cap W = \emptyset\}$$

is centred. For each $x \in H \cup K$, the set

$$\{\langle V, W \rangle : x \in \operatorname{Int}(V) \cup \operatorname{Int}(W)\}$$

is dense. Letting $\mathbf{G} \subseteq \mathbf{P}$ be generic for these dense sets gives

$$U_H = \bigcup \{\operatorname{Int}(V) : \text{for some } W, \langle V, W \rangle \in \mathbf{G}\}$$

and

$$U_K = \bigcup \{\operatorname{Int}(W) : \text{for some } V, \langle V, W \rangle \in \mathbf{G}\}$$

which satisfy the theorem.

We can re-phrase a slight weakening of the theorem as follows. Assume MA($\aleph_0$-centred), if X has a countable T_2 point-separating open cover and if each point has a neighbourhood base consisting of sets which are compact in the topology generated by this cover, then disjoint closed sets of cardinality less than $2^{\aleph_0}$ can be separated with disjoint open sets. For an application of this, consult the article by F. Tall in this volume.

We shall derive some other applications.

7.2. COROLLARY. *Assume* MA($\aleph_0$*-centred*). *If X is a separable metrizable space of cardinality less than* $2^{\aleph_0}$, *then every subset of X is a G_δ.*

PROOF. Let X and F be as above. Let $\tilde{X}$ be a metrizable compactification of X, via the Urysohn metrization theorem (see ENGELKING [1977], p. 325), for example. Let this topology on $\tilde{X}$ be denoted by ρ. Let τ be a topology on $\tilde{X}$ formed from ρ by isolating all points of $\tilde{X} \setminus F$. Thus a base for $\langle \tilde{X}, \tau \rangle$ is

$$\rho \cup \{\{x\} : x \in \tilde{X} \setminus F\}$$

and hence local bases in the ρ topology for points in F remain local bases in the τ topology; this will be important later.

Consider the convergent sequence, that is $\omega+1$ with the ordinal topology. Let ρ' denote the topology on the product of $\omega+1$ and $\tilde{X}$ with the ρ topology and similarly τ' denote the topology on the product of $\omega+1$ and $\tilde{X}$ with the τ topology. We have $\langle(\omega+1)\times\tilde{X},\rho'\rangle$ and $\langle(\omega+1)\times\tilde{X},\tau'\rangle$ satisfying the hypotheses of Theorem 7.1 which we now apply. Since $|(\omega+1)\times X|<2^{\aleph_0}$, $(\omega+1)\times X$ is hereditarily normal in the subspace τ' topology. By a theorem of Katětov (ENGELKING [1977], p. 159), X is perfectly normal in the τ topology. Therefore F is a G_δ in the τ topology. However, by the remark at the end of the previous paragraph, F must also be a G_δ in the original ρ topology.

Let MA(κ; $\aleph_0$-centred) denote MA with the restrictions engendered by both MA(κ) and MA($\aleph_0$-centred).

7.3. COROLLARY. *Assume* MA(κ, $\aleph_0$-*centred*); *then* $2^\kappa=2^{\aleph_0}$.

PROOF. It is easy to check that, in the proof of Corollary 7.2, if $|X|=\kappa$, then only MA(κ, $\aleph_0$-centred) is needed. Let X be a separable metrizable space of cardinality κ. X has 2^κ subsets, but since it has a countable base, only $2^{\aleph_0}$ G_δ sets.

We can extend Corollary 7.2 as follows.

7.4. COROLLARY. *Assume* MA($\aleph_0$-*centred*). *If X is a metrizable space of cardinality less than* $2^{\aleph_0}$, *then every subset of X is a G_δ set.*

PROOF. This follows from Corollary 7.2 and the following well-known topological fact (ENGELKING [1977], p. 358). If X is a metrizable space of cardinality at most $2^{\aleph_0}$, then there is a separable metrizable space Y and a continuous one-to-one mapping $f\colon X\to Y$.

7.5. DEFINITION. A *Lusin space* is a regular topological space in which every nowhere dense subset is countable, and yet has no isolated points.

For a discussion of the importance of Lusin spaces in set-theoretic topology see the article by J. Roitman in this volume. It is straightforward to show that every Lusin space is ccc and that every Lusin space contains a Lusin subspace in which every open subset is uncountable.

We now give some further consequences of Theorem 7.1.

7.6. LEMMA. *Assume* MA($\aleph_1$; $\aleph_0$-*centred*). *Suppose X is a regular space which is either*

(i) *a Baire space with a π-base of cardinality* $\leqslant\aleph_1$, *or*

(ii) *a Lusin space.*

Suppose further that $\{S_\alpha\colon \alpha\in\omega_1\}$ is a partition of X into pairwise disjoint sets and $\mathcal{U}$ is a σ-disjoint collection of open subsets of X such that

(iii) *for all $\alpha\in\omega_1$ and for all $U\in\mathcal{U}$ either $S_\alpha\subseteq U$ or $S_\alpha\cap U\neq\emptyset$, and*

(iv) *for all distinct* α *and* β *in* ω_1, *there are disjoint* U *and* V *in* $\mathcal{U}$ *such that* $S_\alpha \subseteq U$ *and* $S_\beta \subseteq V$.

If $A \subseteq \omega_1$, *then* $\bigcup\{S_\alpha : \alpha \in A\}$ *is a* G_δ *subset of* X. *Furthermore, if* S_α *is first category for each* $\alpha \in A$, *then* $\bigcup\{S_\alpha : \alpha \in A\}$ *is first category.*

The proof appears in section 9.8.

7.7. DEFINITION. A *tree*, $\mathbf{T}$, is a partial order such that for each $t \in \mathbf{T}$, $\{s \in \mathbf{T} : s \leq t\}$ is well-ordered. Each tree can therefore be divided into levels (see KUNEN [1980], p. 68). A tree $\mathbf{T}$ is said to be *σ-dense* iff for each collection of subsets $\{\mathbf{D}_n : n \in \omega\}$ such that for each $n \in \omega$ and $t \in \mathbf{T}$ there is some $s \geq t$ such that $\{r \in \mathbf{T} : r \geq s\} \subseteq \mathbf{D}_n$, we have that for each $t \in \mathbf{T}$ there is some $s \geq t$ such that $s \in \bigcap\{\mathbf{D}_n : n \in \omega\}$.

Two elements s and t of $\mathbf{T}$ are said to be *comparable* iff either $s \leq t$ or $t \leq s$. It is straightforward to show that if $\mathbf{T}$ is uncountable but has no uncountable pairwise incomparable subset, then $\mathbf{T}$ is σ-dense.

7.8. LEMMA. *Assume* MA($\aleph_1$; $\aleph_0$*-centred*). *Suppose that* X *is a regular space of cardinality* $\aleph_1$, *without isolated points, which is either*

(i) *a Baire space with a* π*-base of cardinality* $\leq\aleph_1$, *or*

(ii) *a Lusin space.*

Then there is a tree $\mathbf{T}$ *of cardinality* $\aleph_1$ *such that*:

(iii) $\mathbf{T}$ *is* σ*-dense*

(iv) *if* X *has the* ccc, *then every pairwise incomparable subset of* $\mathbf{T}$ *is countable.*

The proof appears in section 9.9.

7.9. DEFINITIONS. A *dense open* subset of a partial order $\mathbf{P}$ is a dense subset $\mathbf{D}$ of $\mathbf{P}$ such that for all p $\mathbf{D}$, $\{q \in \mathbf{P} : q \leq p\} \subseteq \mathbf{D}$. A partial order $\mathbf{P}$ is said to be κ*-dense* iff the intersection of any collection of less than κ dense open sets is dense open; $\aleph_1$-dense is sometimes called σ-dense.

If $\langle \mathbf{T}, \leq \rangle$ is a σ-dense tree, we can define a partial ordering $\leq'$ on $\mathbf{T}$ such that $s \leq' t$ iff $s \geq t$. Under $\leq'$ $\mathbf{T}$ becomes an $\aleph_1$-dense partial order.

P. Davies proved Lemmas 7.6 and 7.8, using them to obtain the following theorem (DAVIES [1979]).

7.10. THEOREM. *Assume* MA($\aleph_1$; $\aleph_0$*-centred*). *The following are equivalent*:

(i) *there is a regular Baire space of cardinality* $\aleph_1$, *without isolated points, but with a* π*-base of cardinality* $\aleph_1$,

(ii) *there is a* σ*-dense tree of cardinality* $\aleph_1$, *and*

(iii) *there is an* $\aleph_1$*-dense partial order of cardinality* $\aleph_1$.

PROOF. That (i) implies (ii) implies (iii) comes from Lemma 7.8 and the discussion above. To see that (iii) implies (i), assume that $\mathbf{P}$ is an $\aleph_1$-dense partial order of cardinality $\aleph_1$. From Theorem 1.5, we obtain $S(\mathbf{P})$. It is straightforward to show that if U is a dense open subset of $S(\mathbf{P})$, then there exists $\mathbf{Q} \subseteq \mathbf{P}$ such that $\bigcup\{\phi(p): p \in \mathbf{Q}\}$ is a dense open subset of U and that $\{q \in \mathbf{P}: q \leqslant p$ for some $p \in \mathbf{Q}\}$ is a dense open subset of $\mathbf{P}$. It now follows readily that $S(\mathbf{P})$ can be used to satisfy (i). This completes the proof.

Note that, under CH, condition (i) of the above theorem is simply satisfied by the real line. We will discuss condition (i) in the absence of CH in the next section. However we can use Lemma 7.8 to obtain the following theorem of K. Kunen, which shows that under MA + $\neg$CH there are no Lusin spaces.

7.11. THEOREM. *Assume* MA($\aleph_1$; $\aleph_0$-*centred*). *There is a Lusin space iff there is a Suslin line.*

PROOF. Using conclusion (iv) of Lemma 7.8 we can obtain a Suslin tree from a Lusin space. A Suslin tree is an uncountable tree with no uncountable linearly ordered subset and no uncountable pairwise incomparable subset. A well-known theorem of D. Kurepa states that there is a Suslin tree iff there is a Suslin line (KUREPA [1935]). The proof is in the article by S. Todorčević in this volume and in KUNEN [1980], p. 72.

On the other hand, it is straightforward to show that the subspace $\{x_\alpha: \alpha \in \omega_1\}$ of the Suslin line constructed at the end of section 1 is indeed a Lusin space.

An argument similar to the above shows that MA + $\neg$CH implies there are no regular ccc Baire spaces of cardinality $\aleph_1$ with no isolated points.

Under CH, there is a variety of Lusin spaces—of course, N. Lusin was the first to construct one (LUSIN [1914]). The addition of GMA($\aleph_1$-centred), in contrast to Theorem 7.11 gives Lusin spaces which have extra properties. It is considered unlikely that CH plus $2^{\aleph_1} > \aleph_2$ plus GMA($\aleph_1$-centred) implies the existence of a Suslin line; however this is, as yet, unknown. Nevertheless, we have the following theorem of F. Tall.

7.12. THEOREM. *Assume* CH, $2^{\aleph_1} > \aleph_2$ *and* GMA($\aleph_1$-*centred*). *There is a Lusin space with weight* $\aleph_2$.

PROOF. Recall that the weight of a space is the smallest cardinality of a base for the topology. We will discover our Lusin space, L, to be a dense subset of $2^{\aleph_2}$, the Tychonoff product of $\aleph_2$ copies of the two point discrete space; since each point of $2^{\aleph_2}$ has character $\aleph_2$ (i.e. $\aleph_2$ is the smallest cardinality of a local base), the same will be true for L.

According to Theorem 3.5, $2^{\aleph_2}$ has a dense subset of cardinality $\aleph_1$. Using CH, we can form a countably compact subset X of $2^{\aleph_2}$ which contains this dense subset but still has cardinality $\aleph_1$. We will use GMA($\aleph_1$-centred) to pick $L \subseteq X$.

Let $\mathbf{P}$ be the set of all ordered pairs $\langle A, B\rangle$ such that

(i) $A \in [X]^{\aleph_0}$,

(ii) $B \in [\{\text{dense open subsets of } 2^{\aleph_2}\}]^{\aleph_0}$.

And let $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff

(i) $A_1 \supseteq A_2$, $B_1 \supseteq B_2$ and

(ii) for all $U \in B_2$, $A_1 \backslash A_2 \subseteq U$.

$\mathbf{P}$ is easily seen to be countably compact; in fact it is countably closed and well met. For each $A_0 \in [X]^{\aleph_0}$ the set

$$\{\langle A, B\rangle \in \mathbf{P}: A = A_0\}$$

is centred and so $\mathbf{P}$ is $\aleph_1$-centred.

Pick a base $\mathcal{V}$ for $2^{\aleph_2}$ of cardinality $\aleph_2$. For each $V \in \mathcal{V}$, we can use the Baire category theorem for countably compact spaces to show that

(iii) $\{\langle A, B\rangle \in \mathbf{P}: A \cap V \neq \emptyset\}$

is dense in $\mathbf{P}$. For each countable $\mathcal{V}' \subseteq \mathcal{V}$ such that $\bigcup \mathcal{V}'$ is dense in $2^{\aleph_2}$, the set

(iv) $\{\langle A, B\rangle \in \mathbf{P}: \bigcup \mathcal{V}' \in B\}$

is dense in $\mathbf{P}$. Since only $\aleph_2$ dense subsets of $\mathbf{P}$ were mentioned above, we can invoke GMA($\aleph_1$-centred) to obtain a $\mathbf{G} \subseteq \mathbf{P}$ generic for the collection of all these dense sets. Let L be the set

$$\{A: \langle A, B\rangle \in \mathbf{G} \text{ for some } B\}.$$

Because $\mathbf{G}$ meets each of the dense sets (iii), L is dense in $2^{\aleph_2}$.

By Theorem 3.6, $2^{\aleph_2}$ has the ccc and so for each dense open $U \subseteq 2^{\aleph_2}$ there is a countable $\mathcal{V} \subseteq \mathcal{V}$ such that $\bigcup \mathcal{V}$ is dense and contained in U. Because $\mathbf{G}$ meets each of the dense sets (iv), (ii) shows that for any dense open U, $|L \backslash U| \leq \aleph_0$. Hence L is a Lusin space.

This proof can be gleaned from TALL [1979], where it is pointed out that L cannot be separable and is hence an L-space. Of course, in Theorem 7.12, $\aleph_2$ can be exchanged for any $\kappa < 2^{\aleph_1}$ such that $\kappa^{\aleph_0} < 2^{\aleph_1}$. A quite similar forcing argument (see JUHÁSZ and WEISS [19··]) can be used to find a model of set theory in which

(i) $2^{\aleph_0} = \aleph_1$; $2^{\aleph_1}$ is anything allowed by Konig's theorem, and

(ii) there is a Lusin subspace of weight $2^{\aleph_1}$.

This is the best possible in this direction.

There is an interesting application of P_1 to trees.

7.13. DEFINITIONS. If $\mathbf{T}$ is a tree, we denote by $\text{Lev}_\alpha(\mathbf{T})$ the αth level of $\mathbf{T}$. A *Kurepa tree* is a tree of height ω_1 with more than $\aleph_1$ cofinal branches, such that $\text{Lev}_\alpha(\mathbf{T})$ is countable for each $\alpha < \omega_1$.

For more on Kurepa trees, including their interaction with topology see JUHÁSZ [1977] or the article by S. Todorčević in this volume.

7.14. THEOREM. *Assume* CH, $2^{\aleph_1} > \aleph_2$ *and* GMA($\aleph_1$-*centred*). *There is a Kurepa tree.*

The proof appears in section 9.10.

We can extend this last theorem with a similar proof to obtain the stronger conclusion $W(\aleph_2)$. See JUHÁSZ [1977] for definition and application. The proof of Theorem 7.14 is taken from lecture notes of I. Juhász at SETOP 1980; for a different proof see TALL [1979].

8. Axiomatic solidarity

Until the late nineteen seventies, it seemed that if a topological statement was independent of the usual axioms of set theory then it was decided one way by MA + ¬CH and the other way by one of the combinatorial principles derived from the axiom of constructibility. At that time some questions were shown to have answers independent of MA; that is, MA + ¬CH did not decide things one way or the other. Now, with the results on S- and L-spaces obtained by Z. Szentmiklóssy, S. Todorčević and U. Avraham and presented in the articles by J. Roitman and by S. Todorčević and U. Avraham in this volume, there can be but one conclusion—MA is simply too weak to rely upon for all important problems in set-theoretic topology.

We are none the less reluctant to dispense with MA entirely. It's success was genuine. It helped clarify many concepts of set-theoretic topology and the theory of partial orders. It brought a unified approach to diverse problems. It still does. For these same reasons we are also reluctant to abandon the axiomatic approach which has been so useful in reducing problems of logic to combinatorial problems.

If MA is too weak, we can strengthen it. One way to do this is to combine MA with other combinatorial and axiomatic principles and find strength in unity. This has been done with noticable success.

8.1. DEFINITIONS. If κ is an ordinal, a subset $C \subseteq \kappa$ is said to be *cub* iff it is closed and unbounded in the order topology on κ; a subset $S \subseteq \kappa$ is said to be *stationary* iff $S \cap C \neq \emptyset$ for each cub $C \subseteq \kappa$. Every cub set is stationary if κ has uncountable cofinality; but not vice-versa—KUNEN [1980], p. 76.

8.2. DEFINITIONS. $E(\kappa)$ is the assertion that κ is a cardinal with a stationary subset of ordinals with countable cofinality, which is not stationary in any limit ordinal less than κ. $\diamondsuit(\kappa, E)$ is the assertion that E is a stationary subset of κ and for each $\beta \in E$ there is a set $S_\beta \subseteq \beta$ such that for any $S \subseteq \kappa$, $\{\beta \in E: S \cap \beta = S_\beta\}$ is stationary in κ.

It turns out that MA plus $2^{\aleph_0} = \aleph_2$ plus $E(\aleph_2)$ plus $\diamondsuit(\aleph_2, E)$, where $E = \{\alpha \in \omega_2: \alpha$ has countable cofinality$\}$ is a relatively consistent set of axioms. This can be established by forcing. Alternatively, a proof can be based on the work of R.B. Jensen, to whom these combinatorial principles are due—some of the details are in DEVLIN [1973].

This combination, or rather some subset of it, has been used to construct a normal collectionwise Hausdorff, not collectionwise normal, first countable space (FLEISSNER [1978]), to construct a Hausdorff space with uncountable net weight, but with each of its subspaces of cardinality $\aleph_1$ having countable net weight (HAJNAL and JUHÁSZ [19··]), and a first countable, separable, locally compact, normal, not countably paracompact space (WEISS [1981]).

We will consider another combination in some detail. The Kurepa hypothesis, KH, is the assertion that there is a Kurepa tree, as defined in 7.13.

8.3. DEFINITION. The weak Kurepa hypothesis is the assertion that there is a tree of height ω_1 and cardinality $\aleph_1$ with more than $\aleph_1$ cofinal branches. It is denoted wKH and its negation is written $\neg$wKH.

Considering the complete binary tree of height ω_1, we see that CH implies wKH. Thus the combination MA + $\neg$wKH can be considered a strengthening of MA + $\neg$CH because $\neg$wKH is a strengthening of $\neg$CH.

8.4. THEOREM. *If there is a model of* ZFC *in which there is an inaccessible cardinal, then there is one in which* MA + $\neg$wKH *holds.*

This was proven by S. Toderčević (TODORČEVIĆ [1981]) and J. Baumgartner (BAUMGARTNER [19··]). We shall not give the proof here; it is quite long and builds upon earlier work of SILVER [1971] who showed the relative consistency of $\neg$KH, MITCHELL [1972], who showed the relative consistency of $\neg$wKH, and DEVLIN [1978], who showed the relative consistency of MA + $\neg$KH. J. Silver showed that the inaccessible cardinal hypothesis is necessary in all these results.

We therefore know that MA + $\neg$wKH is significantly stronger than MA + $\neg$CH, because it has some of the power of an inaccessible cardinal hidden inside it. In order to exploit this strength we first prove the following lemma and theorem of DEVLIN [1975].

8.5. LEMMA. *Suppose* $\mathbf{P}$ *is an* $\aleph_1$*-dense partial order with the* $\aleph_2$*-cc and no atoms and* $\{\mathbf{D}_\alpha : \alpha \in \omega_1\}$ *is a collection of dense open subsets of* $\mathbf{P}$. *Then there is a subset* $\mathbf{T} \subseteq \mathbf{P}$ *such that, in the reverse of the ordering inherited form* $\mathbf{P}$, $\mathbf{T}$ *is a tree of cardinality* $\aleph_1$. *Furthermore*

(i) *For each* $\alpha \in \omega_1$, $\mathrm{Let}_{\alpha+1}(\mathbf{T})$ *is a maximal pairwise incompatible subset of* $\mathbf{D}_\alpha$,

(ii) *If* $\mathbf{B} \subseteq \mathbf{T}$ *and* $\mathbf{B}$ *is linearly ordered and* $p \in \mathbf{P}$ *such that* $p \leq b$ *for each* $b \in \mathbf{B}$, *then there is some* $t \in \mathbf{T}$, *compatible with* p, *such that* $t < b$ *for each* $b \in \mathbf{B}$.

PROOF. In order to define $\mathbf{T}$ we will define $\mathrm{Lev}_\alpha(\mathbf{T})$ for each $\alpha \in \omega_1$. $\mathrm{Lev}_0(\mathbf{T})$ is any maximal pairwise incompatible subset of $\mathbf{P}$. $\mathrm{Lev}_{\alpha+1}(\mathbf{T})$ is a maximal pairwise incompatible subset of $\mathbf{D}_\alpha$ such that each member of $\mathrm{Lev}_{\alpha+1}(\mathbf{T})$ is strictly less than some member of $\mathrm{Lev}_\alpha(\mathbf{T})$. This guarantees (i) and the case of (ii) in which $\mathbf{B}$ has a least element.

If α is a limit ordinal, we let $\mathrm{Lev}_\alpha(\mathbf{T})$ be a maximal pairwise incompatible subset of $\mathbf{P}$ such that for each $p \in \mathrm{Lev}(\mathbf{T})$, we have

(iii) for all $\beta < \alpha$ there is some $q \in \mathrm{Lev}_\beta(\mathbf{T})$ such that $q > p$.

Since each level is a maximal pairwise incompatible subset of $\mathbf{P}$, the $\aleph_1$-dense property of $\mathbf{P}$ ensures that this construction can proceed at limit stages. The point is that if $\mathbf{A}$ is a maximal pairwise incompatible subset of $\mathbf{P}$, then $\{p \in \mathbf{P}: p < a$ for some $a \in \mathbf{A}\}$ is a dense open subset of $\mathbf{P}$. It is straightforward to show that $\mathbf{T}$ has the properties mentioned.

8.6. THEOREM. *Assume* MA($\aleph_1$). *Suppose* $\mathbf{P}$ *is an* $\aleph_1$*-dense partial order with the* $\aleph_2$-cc *and that* $\{\mathbf{D}_\alpha: \alpha \in \omega_1\}$ *is a collection of* $\aleph_1$ *dense open subsets of* $\mathbf{P}$. *Then there is* $\mathbf{G} \subseteq \mathbf{P}$ *generic for this collection.*

PROOF. We can assume that $\mathbf{P}$ has no atoms and obtain $\mathbf{T}$ from Lemma 8.5. A cofinal branch of $\mathbf{T}$ will generate a $\mathbf{G}$ generic for $\{\mathbf{D}_\alpha : \alpha \in \omega_1\}$. Thus, we suppose $\mathbf{T}$ has no cofinal branch and derive a contradiction.

It is a well-known theorem, proved in section 9 of the article by S. Todorčević in this volume, that a tree of cardinality $\aleph_1$ with no uncountable branches is the union of countably many antichains, if MA($\aleph_1$) is assumed. An antichain is a collection of pairwise incomparable elements in the tree ordering of $\mathbf{T}$. Let $\mathbf{T} = \{\mathbf{A}_n: n \in \omega\}$ where each $\mathbf{A}_n$ is an antichain. For each $n \in \omega$, let $\mathbf{E}_n$ be the dense open set

$$\{p \in \mathbf{P}: p \text{ is incompatible with each element of } A_n\}$$
$$\cup\{p \in \mathbf{P}: p \leq a \text{ for some } a \in \mathbf{A}_n\}.$$

Pick any $p \in \mathbf{P}$; let $S_p = \{n \in \omega:\ p \leq a$ for some $a \in \mathbf{A}_n\}$. For each $n \in S_p$, let $a(p, n)$ be the unique element of $\mathbf{A}_n$ such that $p \leq a(p, n)$ and $a(p, n) \in \mathbf{A}_n$. Since $\mathbf{T}$ is a tree the set

$$\mathbf{B}_p = \{a(p, n): n \in S_p\}$$

is linearly ordered. Hence by Lemma 8.5(ii) there is some $t \in \mathbf{T}$ such that t is compatible with p and $t < a(p, n)$ for each $n \in S_p$. Now $t \in \mathbf{A}_m$ for some $m \notin S_p$ and hence $p \notin E_m$. This shows that $\bigcap\{\mathbf{E}_m: m \in \omega\} = \emptyset$ and contradicts that $\mathbf{P}$ is $\aleph_1$-dense, completing the proof.

The conclusion of Theorem 8.6 is interesting in its own right. However, with the added assumption —wKH, S. TODORČEVIĆ [1981] has obtained an interesting strengthening. The notion of an atom of a partial order was defined in section 6.2.

8.7. THEOREM. *Assume* MA + ¬wKH. *Every* $\aleph_1$*-dense partial order with the* $\aleph_2$-cc *and no atoms is, in fact,* $\aleph_2$*-dense.*

PROOF. Suppose that $\mathbf{P}$ is an $\aleph_1$-dense partial order with the $\aleph_2$-cc, but that $\{D_\alpha : \alpha \in \omega_1\}$ is a collection of $\aleph_1$ dense open subsets of $\mathbf{P}$ such that for some $p_0 \in \mathbf{P}$

$$\emptyset = \bigcap\{\mathbf{D}_\alpha : \alpha \in \omega_1\} \cap \{p \in \mathbf{P} : p \leq p_0\}.$$

By restricting our attention to $\{p \in \mathbf{P} : p \leq p_0\}$ we can assume that $\bigcap\{\mathbf{D}_\alpha : \alpha \in \omega_1\} = \emptyset$. Obtain $\mathbf{T}$ from Lemma 8.5.

We will show that $\mathbf{T}$ has more than $\aleph_1$ cofinal branches and obtain a contradiction with ¬wKH. To this end suppose that the set $\mathcal{B}$ of all cofinal branches of $\mathbf{T}$ has cardinality at most $\aleph_1$. For each $\alpha \in \omega_1$, let

$$\mathbf{E}_\alpha = \{p \in \mathbf{P} : p \leq t \text{ for some } t \in \mathrm{Lev}_\alpha(\mathbf{T})\},$$

which is a dense subset $\mathbf{P}$. For each cofinal branch $\mathbf{B} \in \mathcal{B}$, let

$$\mathbf{F}_\mathbf{B} = \{p \in \mathbf{P} : p \leq t \text{ for some } t \in \mathbf{T} \setminus \mathbf{B}\},$$

which is dense since $\bigcap\{\mathbf{D}_\alpha : \alpha \in \omega_1\} = \emptyset$. We now apply Theorem 8.6 to obtain $\mathbf{G} \subseteq \mathbf{P}$ generic for the collection of all $\mathbf{E}_\alpha$ and $\mathbf{F}_\mathbf{B}$. It is straightforward to show that $\mathbf{G}$ must contain a cofinal branch of $\mathbf{T}$ which is not in $\mathcal{B}$. This gives the desired contradiction and completes the proof.

8.8. COROLLARY. *Assume* MA + ¬wKH. *There is no* $\aleph_1$*-dense partial order of cardinality* $\aleph_1$.

From this corollary we get the following (DAVIES [1979]).

8.9. THEOREM. *The existence of a regular Baire space of cardinality* $\aleph_1$ *without isolated points, but with a* π*-base of cardinality* $\aleph_1$ *is independent of* MA + ¬CH.

PROOF. We have MA + ¬wKH deciding that no such space exists, by Corollary 8.8 and Theorem 7.10. On the other hand we can obtain the relative consistency of MA + ¬CH and the existence of an $\aleph_1$-dense partial order of cardinality $\aleph_1$ by choosing a model of CH for M_0 in the sketch of the proof of the relative consistency of MA + ¬CH at the beginning of section 2. In M_0 the complete binary tree of height ω_1 gives a countably closed partial order of cardinality $\aleph_1$. It is a theorem of W.B. Easton (see KUNEN [1980] p. 240 and 265), that a countably closed partial order in a ground model like M_0 remains $\aleph_1$-dense in a ccc forcing extension, like M_κ. Thus the complete binary tree becomes, in M_κ, an $\aleph_1$-dense tree of cardinality $\aleph_1$; of course it is no longer countably closed in M_κ.

Extending Theorem 8.9, K. Kunen has established the relative consistency of the statement that every regular Baire space without isolated points has cardinality $>\aleph_1$. The reader can verify, using Theorems 7.10 and 8.9 and exploiting the duality between trees and linear orders, that MA + $\neg$CH does not decide the existence of a linearly ordered Baire space of cardinality $\aleph_1$ without isolated points. See DAVIES [1979] or TODORČEVIĆ [1981]. This indecision of MA + $\neg$CH further points out its weakness.

There is, however, a strengthening of MA + $\neg$CH which decides those things which we mentioned were independent of MA + $\neg$CH. This axiom, the proper forcing axiom, PFA, developed by R. Laver, S. Shelah and J. Baumgartner is so important that it deserves a chapter all to itself. Indeed, it has one; in that chapter it is shown that PFA implies MA + $\neg$wKH.

As powerful as PFA is, it has drawbacks. The full axiom seems to need the existence of very large cardinals—if, however, these very large cardinals turn out to be not inconsistent, then this is not a drawback, but a virtue. What is perhaps not so virtuous is the surprising fact that PFA implies MA($\aleph_2$) is false; the proof is in the article by J. Baumgartner in this volume. It is true that most major consequences of MA + $\neg$CH use only MA($\aleph_1$), which of course follows from PFA. However, PFA is not a complete strengthening of MA + $\neg$CH.

There is an amusing consequence of MA($\aleph_2$) which does not follow from MA($\aleph_1$) nor from PFA. Consider the following three statements.

I. There is a collection $\{S_\alpha : \alpha \in \omega_2\}$ of infinite subsets of ω_1 such that each uncountable subset of ω_1 includes some S_α.

II. There is a collection $\{S_\alpha : \alpha \in \omega_2\}$ of infinite subsets of ω_1 such that each stationary subset of ω_1 includes some S_α.

III. There is a collection $\{S_\alpha : \alpha \in \omega_2\}$ of infinite subsets of ω_1 such that each cub subset of ω_1 includes some S_α.

Of course I implies II implies III. Since there are only $2^{\aleph_0}$ countable subsets of ω_1, each statement is implied by $2^{\aleph_0} \leq \aleph_2$. If we replace, in each statement, ω_2 by ω_1, then J. Baumgartner has shown that I for ω_1 is consistent with $\neg$CH (BAUMGARTNER [1976]) and in his article in this volume that PFA implies III for ω_1 is false. For topological connections with such statements see BROVERMAN, GINSBERG, KUNEN and TALL [1978].

We have that MA + $2^{\aleph_0} > \aleph_2$ implies that I is false. In fact, we have the following theorem.

8.10. THEOREM. *Assume* MA($\aleph_0$-*centred*). *Let* $\kappa < 2^{\aleph_0}$ *and let* $\{S_\beta : \beta \in \kappa\}$ *be a collection of infinite subsets of* ω_1. *There exists an uncountable* $A \subseteq \omega_1$ *such that* $S \backslash A$ *is infinite for each* $\beta \in \kappa$.

PROOF. Since this is easily seen to be true under CH, we assume $2^{\aleph_0} > \aleph_1$. Let $\mathbf{P}$ be the set of all functions from a finite subset of ω_1 into $\{0, 1\}$. For $\{f, g\} \subseteq \mathbf{P}$ we let $f \leq g$ iff $f \supseteq g$ as sets of ordered pairs. We can apply Lemma 3.7 to show that $\mathbf{P}$ is $\aleph_0$-centred.

For each $\beta \in \kappa$ and $n \in \omega$, the set

$$\mathbf{D}_{\beta_n} = \{f \in \mathbf{P} \colon |\{\alpha \in S_\beta \colon f(\alpha) = 0\}| \geqslant n\}$$

is dense since each S_β is infinite. Furthermore, for each $\gamma \in \omega_1$, the set

$$\mathbf{E}_\gamma = \{f \in \mathbf{P} \colon f(\alpha) = 1 \text{ for some } \alpha \geqslant \gamma\}$$

is also dense. Let $\mathbf{G} \subseteq \mathbf{P}$ be generic for this entire collection of dense sets. We define A to be the set

$$\{\alpha \in \omega_1 \colon f(\alpha) = 1 \text{ for some } f \in \mathbf{G}\}.$$

The sets $\mathbf{E}_\gamma$ ensure that A is uncountable. If $\beta \in \kappa$ and $n \in \omega$, the set $\mathbf{D}_{\beta_n}$ guarantees that $|S \backslash A| \geqslant n$. This completes the (short) proof.

W. Fleissner has proposed a strengthening of MA plus $2^{\aleph_0} > \aleph_2$ which leads to the conclusion that II is also false. We first have a lemma.

8.11. LEMMA. *Assume* MA($\aleph_2$) *and that* III *holds*; *then* II *is false.*

PROOF. Let us go back to Theorem 8.10; we let $\kappa = \aleph_2$. $\mathbf{P}$ and the dense sets $\mathbf{D}_{\beta_n}$ remain the same. If we can produce a stationary $A \subseteq \omega_1$ such that $S \backslash A$ is infinite for each $\beta \in \omega_2$, then we will certainly produce a contradiction to II. The dense sets $\mathbf{E}_\gamma$ from Theorem 8.10 will only guarantee that A is uncountable. We replace these dense sets with others.

Let $\{T_\mu \colon \mu \in \omega_2\}$ witness III. For each $\mu \in \omega_2$ let $\mathbf{F}_\mu$ be the dense subset of $\mathbf{P}$ defined by

$$\mathbf{F}_\mu = \{f \in \mathbf{P} \colon f(\alpha) = 1 \text{ for some } \alpha \in T_\mu\}.$$

A is obtained as in Theorem 8.10; we get $\mathbf{G} \subseteq \mathbf{P}$ generic for the collection of all $\mathbf{F}_\mu$ and $\mathbf{D}_{\beta_n}$. The $\mathbf{F}_\mu$ ensure that A intersects each cub set and is hence stationary.

As a surprise consequence of this we have the following theorem.

8.12. THEOREM. *Assume* MA($\aleph_2$); *then* II *is false.*

PROOF. If III is true, then II is false because of the above lemma. On the other hand, if III is false, then II is false since II implies III!

What is the real reason that MA($\aleph_2$) implies that II is false? Is III true or is III false?

The following is part of set-theoretic folklore. An application is given in FLEISSNER [1980].

8.13. THEOREM. *The following statement is relatively consistent with* MA *plus* $2^{\aleph_0} > \aleph_2$. *There is a collection of* $\aleph_2$ cub *sets such that each* cub *set contains one from this collection.*

PROOF. This is not going to be a real proof, but we will display the ideas involved. Suppose N is a model of set theory which is a generic (forcing) extension of another model M. Suppose that C is a cub subset of ω_1 in M. Then C is in N; but is it a cub in N? Yes it is, because the notion of being cub is absolute—a cub is a cub. If C is a cub in N, is C a cub in M? No, since C may not even be an element of M. However there is a cub C' in M such that $C' \subseteq C$ provided that N was obtained from M by forcing with a partial order which has the ccc. This is an exercise in KUNEN [1980], p. 247.

Now, as outlined in the beginning of section 2, MA + ¬CH can be obtained by forcing with a partial order having the ccc—it is the partial order resulting from the limit process which produced M_κ. If the ground model M_0 was a model of $2^{\aleph_1} = \aleph_2$ and $\kappa = \aleph_3$, then every cub C in M_κ contains a cub C' which is in M_0, and since there can be only $\aleph_2$ cub sets in M_0, we are done.

There is, as yet, no perfect strengthening, no perfect extension or generalization of MA + ¬CH. There is no super axiom which will give a wealth of combinatorial principles in the style of Martin's axiom, like the axiom of constructibility does for $\diamondsuit$ and $\square$ principles.

It seems that Martin's axiom should admit to strengthening and generalization. Few believe that MA represents a great and singular truth about the universe of sets. Instead, it has been a useful combinatorial principle, guiding us to new theorems and protecting us from inconsistency. Nevertheless the need for a stronger axiom of wider scope is now apparent. What is the basic spirit underlying all the partial successes at generalizing and strengthening Martin's axiom? Perhaps we just have not fasted long enough. I'll see you in the forest.

9. Would you like some proof?

In this last section we display some proofs which were omitted in the earlier sections.

9.1. *Proof of Lemma* 1.4. We first prove that (iv) implies (v). Let $\{U_\alpha : \alpha \in \kappa\}$ be a family of dense open subsets of X. For each $\alpha \in \kappa$, we let

$$\mathbf{D}_\alpha = \{p \in \mathbf{P} : \overline{\Phi(p)} \subseteq U_\alpha\}.$$

We claim that each $\mathbf{D}_\alpha$ is dense in $\mathbf{P}$. Let $q \in \mathbf{P}$. Since U_α is dense open, $U_\alpha \cap \Phi(q) \neq \emptyset$; and hence by (iii) and the regularity of X there is $p \in \mathbf{P}$ such that

$\overline{\Phi(p)} \subseteq U_\alpha \cap \Phi(q)$. Thus, from (ii) we get that p and q are compatible, so there is $r \in \mathbf{P}$ such that $r \leqslant p$ and $r \leqslant q$. Since $r \leqslant p$, we have, by (i), that $\overline{\Phi(r)} \subseteq \overline{\Phi(p)} \subseteq U_\alpha$. Hence $r \in \mathbf{D}_\alpha$ and $\mathbf{D}_\alpha$ is dense.

We now get $\mathbf{G} \subseteq \mathbf{P}$ generic for $\{\mathbf{D}_\alpha : \alpha \in \kappa\}$. The collection $\{\Phi(p) : p \in \mathbf{G}\}$ is centred, by (ii). For each $\alpha \in \kappa$, there is $p \in \mathbf{G}$ such that $\overline{\Phi(p)} \subseteq U_\alpha$. Hence

$$\bigcap\{U_\alpha : \alpha \in \kappa\} \supseteq \{\overline{\Phi(p)} : p \in \mathbf{G}\} \neq \emptyset$$

by the compactness of X.

We now prove that (v) implies (iv). Let $\{\mathbf{D}_\alpha : \alpha \in \kappa\}$ be a collection of dense subsets of $\mathbf{P}$. For each $\alpha \in \kappa$, let $\mathbf{A}_\alpha$ be a maximal pairwise incompatible subset of $\mathbf{D}_\alpha$; $|\mathbf{A}_\alpha| \leqslant \kappa$ since $\mathbf{P}$ has the κ^+-cc. Let $U_\alpha = \bigcup\{\Phi(p) : p \in \mathbf{A}_\alpha\}$.

We claim that U_α is dense open in X. Let V be open in X. By (iii) we find $q \in \boldsymbol{P}$ such that $\Phi(q) \subseteq V$. Since $\mathbf{D}_\alpha$ is dense, we find $p \in \mathbf{D}_\alpha$ such that $p \leqslant q$ and hence $\Phi(p) \subseteq V$. Since $\mathbf{A}_\alpha$ is maximal pairwise incompatible, there is $r \in \mathbf{A}_\alpha$ such that r and p are compatible. By (ii) $\Phi(r) \cap \Phi(p) \neq \emptyset$; hence $U_\alpha \cap V \neq \emptyset$.

Let $\mathbf{B}_0 = \bigcup\{A_\alpha : \alpha \in \kappa\}$. Then $|\mathbf{B}_0| \leqslant \kappa$. If $\mathbf{B}_n$ is defined, let $\mathbf{B}'_n$ be a collection of $\leqslant\kappa$ elements of $\mathbf{P}$ such that if p and q are compatible elements of $\mathbf{B}_n$ then $r \leqslant p$ and $r \leqslant q$ for some $r \in \mathbf{B}'_n$. Define $\mathbf{B}_{n+1}$ to be $\mathbf{B}_n \cup \mathbf{B}'_n$ and finally let $\mathbf{B} = \bigcup\{\mathbf{B}_n : n \in \omega\}$. This has the following properties

(vi) $|\mathbf{B}| \leqslant \kappa$, and

(vii) iff $\{p, q\} \subseteq \mathbf{B}$ and p and q are compatible, then $r \leqslant p$ and $r \leqslant q$ for some $r \in \mathbf{B}$.

For each $\{p, q\} \subseteq \mathbf{B}$, let $\mathbf{D}_{pq}$ be the set

$$\{r \in \mathbf{B} : r \leqslant p \text{ and } r \leqslant q\} \cup \{r \in \mathbf{B} : r \text{ and } p \text{ are incompatible}\}$$
$$\cup\{r \in \mathbf{B} : r \text{ and } q \text{ are incompatible}\}.$$

Let $U_{pq} = \bigcup\{\Phi(r) : r \in \mathbf{D}_{pq}\}$. We claim that each U_{pq} is dense. Let V be an open subset of X. Pick $u \in \mathbf{B}$ such that $\Phi(u) \cap V \neq \emptyset$. Is there $t \in \mathbf{P}$ such that $t \leqslant u$ and $t \leqslant p$? If not, then u and p are incompatible and so $u \in D_{pq}$ and hence $U_{pq} \cap V \neq \emptyset$. On the other hand if there is such a $t \in P$, there must be one in $\mathbf{B}$ also, by the construction of $\mathbf{B}$. Now, is there $s \in \mathbf{P}$ such that $s \leqslant t$ and $s \leqslant q$? If not, t and q are incompatible, so $t \in \mathbf{D}_{pq}$ and $U_{pq} \cap V \neq \emptyset$. If there is such an s, then by the construction of $\mathbf{B}$, there must be one in $\mathbf{B}$ as well. Thus $s \in \mathbf{D}_{pq}$ and so $U_{pq} \cap V \neq \emptyset$. Hence U_{pq} is dense open in X.

Now, by (v), let $x \in X$ be in the set

$$\bigcap\{U_\alpha : \alpha \in \kappa\} \cap \bigcap\{U_{pq} : \{p, q\} \subseteq \mathbf{B}\}.$$

Let $\mathbf{G}' = \{p \in \mathbf{B} : x \in \Phi(p)\}$. By (ii), any two elements of $\mathbf{G}'$ are compatible. If $\alpha \in \kappa$, there is some $p \in \mathbf{A}_\alpha$ such that $x \in \Phi(p) \subseteq U_\alpha$; hence $\mathbf{D}_\alpha \cap \mathbf{G}' \neq \emptyset$.

Let $\{p, q\} \subseteq \mathbf{G}'$. Since $\{p, q\} \subseteq \mathbf{B}$, we have $x \in U_{pq}$. That is, $x \in \Phi(r)$ for some

$r \in \mathbf{D}_{pq} \cap \mathbf{B}$. Hence $r \in \mathbf{G}' \cap \mathbf{D}_{pq}$. Since any two elements of $\mathbf{G}'$ are compatible, we must have $r \leq p$ and $r \leq q$ by the definition of $\mathbf{D}_{pq}$.

Now let $\mathbf{G} = \{p \in \mathbf{P}: p \geq q$ for some $q \in \mathbf{G}'\}$. It is now easy to see that $\mathbf{G}$ is generic for $\{\mathbf{D}_\alpha: \alpha \in \kappa\}$. This finishes the proof of the lemma.

9.2. *Proof of Lemma* 3.16. We use CH to enumerate all countably infinite sequences of pairwise disjoint finite subsets of ω_1 by $\{S_\mu: \mu \in \omega_1\}$.

Let each $S_\mu = \langle S_\mu^n: n \in \omega \rangle$. By recursion on $\alpha \in \omega_1$, we shall define disjoint subsets $K_0(\alpha)$ and $K_1(\alpha)$ of α. Then we shall let

$$K_i = \{\{\beta, \alpha\}: \beta \in K_i(\alpha) \text{ and } \alpha \in \omega_1\},$$

or in our notation

$$K_i = \bigcup\{K_i(\alpha) \otimes \{\alpha\}: \alpha \in \omega_1\}.$$

Now, at stage α let $\{\langle i_m, \mu_m, X_m \rangle: m \in \omega\}$ enumerate all triples $\langle i, \mu, X \rangle$ which satisfy

(i) $i \in \{0, 1\}$,

(ii) $\mu < \alpha$ and $\{S_\mu^n: n \in \omega\} \subseteq \alpha$,

(iii) $X \in [\alpha]^{<\aleph_0}$, and

(iv) $\{n \in \omega: S_\mu^n \otimes X \subseteq \bigcup\{K_i(\beta) \otimes \{\beta\}: \beta < \alpha\}\}$ is infinite.

For each $m \in \omega$, let I_m be the infinite set

$$\{n \in \omega: S_{\mu_m}^n \otimes X_m \subseteq \bigcup\{K_{i_m}(\beta) \otimes \{\beta\}: \beta < \alpha\}\}.$$

Let $A = \{S_{\mu_m}^n: n \in I_m \text{ and } m \in \omega\}$. Using a dovetailing argument (easy to do but complicated to write down) we can find two subcollections B_0 and B_1 of A such that $\bigcup B_0$ and $\bigcup B_1$ are disjoint and furthermore, for each $i \in \{0, 1\}$ and $m \in \omega$, $\{n \in I_m: S_{\mu_m}^n \subseteq B_i\}$ is infinite. We let $K_i(\alpha) = B_i$.

By doing the construction this way we have ensured that if i, μ, X and α satisfy (i), (ii) and (iii) and

$$\begin{array}{lll} \text{if} & \{n \in \omega: S_\mu^n \otimes X \subseteq K_i\} & \text{is infinite,} \\ \text{then} & \{n \in \omega: S_\mu^n \otimes (X \cup \{\alpha\}) \subseteq K_i\} & \text{is infinite.} \end{array}$$

Now, if S is any uncountable collection of pairwise disjoint finite subsets of ω_1 and $i \in \{0, 1\}$ such that for each $s \in S$, $[s]^2 \subseteq K_i$, choose $\mu \in \omega_1$ such that $S_\mu \subseteq S$. Choose $\beta \in \omega_1$ such that $\bigcup S_\mu \subseteq \beta$ and choose $t \in S$ such that $t \cap \beta = \emptyset$. Let $t = \{\alpha_1, \ldots, \alpha_n\}$ where

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n.$$

Now with $X = \emptyset$ and $\alpha = \alpha_1$ we can satisfy (i), (ii) and (iii) and furthermore

$$\{n \in \omega : S^n_\mu \otimes X \subseteq K_i\} \quad \text{is infinite}$$

Hence,

$$\{n \in \omega : S^n_\mu \otimes \{\alpha_1\} \subseteq K_i \quad \text{is infinite}\,.$$

We can now repeat this argument with $X = \{\alpha_1\}$ and $\alpha = \alpha_2$ to obtain

$$\{n \in \omega : S^n_\mu \otimes \{\alpha_1, \alpha_2\} \subseteq K_i\} \quad \text{is infinite}\,.$$

We don't stop here; we go, one by one, up to α_n, finally obtaining

$$\{n \in \omega : S^n_\mu \otimes t \subseteq K_i\} \quad \text{is infinite}$$

and thus proving the lemma.

9.3. *Proof of Theorem* 4.4 (iii). Let $\{x_\alpha : \alpha \in \omega_1\}$ be as in (iii). Let $\mathbf{P}$ be the set

$$\{p \in [\omega_1]^{<\aleph_0} : x_\alpha \not\subseteq x_\beta \text{ for distinct } \alpha, \beta \text{ in } p\}$$

partially ordered by set containment. If $\mathbf{P}$ has property K, then $\{\{\alpha\} : \alpha \in \omega_1\}$ has an uncountable linked subset, which gives us the desired conclusion.

Therefore, it remains only to prove that $\mathbf{P}$ has the ccc. To this end, suppose $\{p_\xi : \xi \in \omega_1\}$ is an uncountable set of pairwise incompatible elements. Using Lemma 3.10, form an uncountable Δ-system $\{p_\xi : \xi \in \Gamma_0\}$ with root r. Notice that $\{p_\xi \backslash r : \xi \in \Gamma_0\}$ is an uncountable collection of pairwise incompatible elements of $\mathbf{P}$, and so we can find an uncountable subset $\Gamma_1 \subseteq \Gamma_0$ such that for each $\xi < \eta$ in Γ_1 we have

(iv) $\max(p_\xi \backslash r) < \min(p_\eta \backslash r)$.

Finally, we can find an uncountable subset $\Gamma_2 \subseteq \Gamma_1$ such that for all $\xi \in \Gamma_2$, $|p_\xi \backslash r| = n$.

Now let 2^ω denote the Cantor space (i.e. the countable Tychonoff power of the two point discrete space). We identify each $z \in 2^\omega$ with the subset of ω for which it is the characteristic sequence. Note that the set

$$\{z \in 2^\omega : z \subseteq x\}$$

is closed in the Cantor space, and hence for each $\xi \in \Gamma_2$, so is

$$\{z \in 2^\omega : z \subseteq x_\alpha \text{ for some } \alpha \in p_\xi \backslash r\}\,.$$

If $(2^\omega)^n$ is the nth power of the Cantor space, then for each $\xi \in \Gamma_2$, the set

$$F_\xi = \{\vec{Z} \in (2^\omega)^n\colon \text{ for some coordinate } z \text{ of } \vec{Z} \text{ and for some } \alpha \in p_\xi \backslash r, \text{ we have } z \subseteq x_\alpha\}$$

is closed in $(2^\omega)^n$.

Now note that if ξ, η are in Γ_2 and $\xi < \eta$ then for each $\alpha \in p_\xi \backslash r$ and $\beta \in p_\eta \backslash r$ we have $\alpha < \beta$; hence by the original hypothesis on $\{x_\alpha\colon \alpha \in \omega_1\}$ in (iii) and from (iv) we must have $x_\alpha \nsubseteq x_\beta$. Thus, if $\vec{P}_\xi$ is an element of $(2^\omega)^n$ formed by putting the elements of $p_\xi \backslash r$ in sequence, we must have $\vec{P}_\xi \notin F_\eta$. But on the other hand, since $p_\xi \backslash r$ and $p_\eta \backslash r$ are incompatible, and (iv) is in effect there must be some $\beta \in p_\eta \backslash r$ and some $\alpha \in p_\xi \backslash r$ such that $x_\beta \subseteq x_\alpha$. Thus, if $\vec{P}_\eta$ is an element of $(2^\omega)^n$ formed by putting the elements of $p_\eta \backslash r$ in sequence, we must have $\vec{P}_\eta \in F_\eta$.

This shows that the sequence

$$\langle \bigcap \{F_\xi\colon \xi \in (\Gamma_2 \cap \eta)\}\colon \eta \in \Gamma_2\rangle$$

is an uncountable strictly decreasing sequence of closed subsets of $(2^\omega)^n$, which gives us the desired contradiction and finishes the proof.

9.4. *Proof of Lemma* 5.17. Let X and $\{D_\alpha\colon \alpha \in \kappa\}$ be as in the lemma and identify the countable dense subset of X with ω. Without loss of generosity, we can assume that X has no isolated points. For each $\alpha \in \kappa$ and for each $n \in \omega$ such that $n \in D_\alpha$, we can choose $G(n, \alpha)$ in such a way that

(i) $G(n, \alpha)$ is a compact neighbourhood of n which is contained in D_α, and

(ii) if $m \leqslant n$ and $n \in G(m, \alpha)$, then $G(n, \alpha) \subseteq G(m, \alpha)$. Now fix $n \in \omega$ for awhile. Consider the collection

$$\{\omega \cap G(n, \alpha) \cap D_\beta\colon n \in D_\alpha \text{ and } \alpha, \beta \in \kappa\}$$

of subsets of ω. The intersection of any finite subcollection is an infinite subset of ω since n is not isolated and ω is dense. We can therefore apply P_0 to obtain $B \in [\omega]^{\aleph_0}$ such that

(iii) $|B \backslash (\omega \cap G(n, \alpha) \cap D_\beta)| < \aleph_0$ for all $\beta \in \kappa$ and $\alpha \in \kappa$ such that $n \in D_\alpha$. Enumerate B as $\{x(n, j)\colon j \in \omega\}$.

Now let n vary and fix $\alpha \in \kappa$ for awhile. Define a function $f_\alpha\colon \omega \to \omega$ as follows

$$f_\alpha(n) = \begin{cases} \text{the least } i \in \omega \text{ such that } \{x(n, j)\colon j \geqslant i\} \subseteq G(n, \alpha), & \text{if } n \in D_\alpha, \\ \text{the least } i \in \omega \text{ such that } \{x(n, j)\colon j \geqslant i\} \subseteq D_\alpha, & \text{if } n \notin D_\alpha. \end{cases}$$

Now we can apply Lemma 5.13 to the collection $\{f_\alpha\colon \alpha \in \kappa\}$ and obtain a function $f\colon \omega \to \omega$ such that for all $\alpha \in \kappa$ there is $N_\alpha \in \omega$ such that for all $n \geqslant N_\alpha$ we have $f(n) \geqslant f_\alpha(n)$. That is:

(iv) for all $n \geq N_\alpha$, $\{x(n, j): j \geq f(n)\} \subseteq D_\alpha$, and

(v) for all $n \geq N_\alpha$, $\{x(n, j): j \geq f(n)\} \subseteq G(n, \alpha)$ if $n \in D_\alpha$.

We now define a subset $\{a_k : k \in \omega\}$ of ω. Let $a_0 = 32$ and let

(vi) $a_{k+1} \in \{x(a_k, j): j \geq f(a_k)\}$ such that $a_{k+1} > a_k$.

Now define a function $g: \kappa \to \omega$ by letting $g(\alpha)$ be the second smallest element of the set

$$\{a_k : k \in \omega\} \cap (\omega \backslash N_\alpha) .$$

By (vi) and (iv), for each $\alpha \in \kappa$ we have $g(\alpha) \in D_\alpha$; hence $G(g(\alpha), \alpha)$ is defined and $g(\alpha) \in G(g(\alpha), \alpha) \subseteq D_\alpha$.

We claim that $\{G(g(\alpha), \alpha): \alpha \in \kappa\}$ is centred. Once this is established, by compactness we have

$$\emptyset \neq \bigcap\{G(g(\alpha), \alpha): \alpha \in \kappa\} \subseteq \bigcap\{D_\alpha : \alpha \in \kappa\}$$

which completes the proof.

Let's prove the claim. First, consider $\alpha \in \kappa$ and $g(\alpha) = a_k$ for some $k \in \omega$; we have $a_k > N_\alpha$ and $a_k \in G(a_k, \alpha) \subseteq D_\alpha$. We show by induction that

(vii) $a_i \in G(a_k, \alpha)$ for each $i \geq k$.

Suppose $a_i \in G(a_k, \alpha)$; then $a_i \in D_\alpha$ and $a_i \geq a_k \geq N_\alpha$, so by (v) we have

$$\{x(a_i, j): j \geq f(a_i)\} \subseteq G(a_i, \alpha) .$$

So by (vi) we have that $a_{i+1} \in G(a_i, \alpha)$. But since $a_i \in G(a_k, \alpha)$ we have, by (ii), $G(a_i, \alpha) \subseteq G(a_k, \alpha)$. Hence $a_{i+1} \in G(a_k, \alpha)$.

Now suppose $\{G(g(\alpha_i), \alpha_i): i \leq m\}$ is a finite subcollection of $\{G(g(\alpha), \alpha): \alpha \in \kappa\}$. Let

$$a = \max\{g(\alpha_i): i \leq m\} .$$

Then, by (vii), for each $i \leq m$ we have $a \in G(g(\alpha_i), \alpha_i)$. This completes the proof of the claim and the lemma.

9.5. *Proof of Theorem* 6.6. That (i) implies (ii) is, of course, Lemma 6.5. That (ii) implies (iii) follows immediately from the fact that the only real difference between (ii) and (iii) is that (iii) puts more stringent conditions on the π-base $\mathscr{B}$. It remains to prove that (iii) implies (i).

Let $\mathbf{P}$ be a countably compact partial order such that $\mathbf{P} = \bigcup\{\mathbf{P}_\alpha : \alpha \in \omega_1\}$ and each $\mathbf{P}_\alpha$ is centred. Let $\kappa < 2^{\aleph_1}$ and for each $\beta \in \kappa$, let $\mathbf{D}_\beta$ be a dense subset of $\mathbf{P}$. Apply Theorem 1.5 to obtain the space $S(\mathbf{P})$, the clopen base $\mathscr{B}$ and the map $\phi: \mathbf{P} \to \mathscr{B}$. Let ρ denote the topology on $S(\mathbf{P})$.

Let τ be the topology on $S(\mathbf{P})$ obtained from ρ by making all G_δ sets into open sets. It is easy to see that $\langle S(\mathbf{P}), \tau\rangle$ is Hausdorff and zero-dimensional. We claim that $\{\phi(p): p \in \mathbf{P}\}$ is also a π-base for $\langle S(\mathbf{P}), \tau\rangle$. Let $U \in \tau\backslash\{\emptyset\}$; we will show that $\phi(p) \subseteq U$ for some $p \in \mathbf{P}$. Since $\{\phi(p): p \in \mathbf{P}\}$ is a subbase for ρ we can assume, without loss of generosity, that $U = \bigcap\{\phi(q): q \in \mathbf{Q}\}$ for some $\mathbf{Q} \in [\mathbf{P}]^{<\aleph_0}$. By Theorem 1.4(ii), $\mathbf{Q}$ is centred, and hence the countable compactness of $\mathbf{P}$ gives us a lower bound, p, for $\mathbf{Q}$. Thus, by Theorem 1.4(i),

$$\phi(p) \subseteq \bigcap\{\phi(q): q \in \mathbf{Q}\}\,.$$

Since each $\mathbf{P}_\alpha$ is centred we can find, by Theorem 1.4(ii), $y_\alpha \in \bigcap\{\phi(p): p \in \mathbf{P}_\alpha\}$. Let

$$Y = \{y_\alpha : \alpha \in \omega_1\}:$$

from the claim in the previous paragraph, Y is dense in $\langle S(\mathbf{P}), \tau\rangle$.

Now, let X be any compactification of $\langle S(\mathbf{P}), \tau\rangle$. For each $p \in \mathbf{P}$, let $\Phi(p) = \operatorname{Int}(\overline{\phi(p)})$. Since each $\phi(p)$ is clopen in $\langle S(\mathbf{P}), \tau\rangle$, we have that $\phi(p) \cap S(\mathbf{P})$ is equal to $\Phi(p) \cap S(\mathbf{P})$. It is now straightforward to check that X, Y and the π-base

$$\{\bigcap\{\Phi(p): p \in \mathbf{Q}\}: \mathbf{Q} \in [\mathbf{P}]^{<\aleph_0}\}\backslash\{\emptyset\}$$

satisfy the condition (a) and (b) of (iii).

The map Φ also satisfies the conditions (i), (ii) and (iii) of Lemma 1.4. Since $\mathbf{P}$ has the $\aleph_2$-cc, we can conclude that if $\kappa \geqslant \aleph_1$, then there is a $\mathbf{G} \subseteq \mathbf{P}$ which is generic for $\{\mathbf{D}_\beta: \beta \in \kappa\}$. If, however, $\kappa \leqslant \aleph_0$ it is easy to find a subset of the countably compact partial order $\mathbf{P}$ which is generic for $\{\mathbf{D}_\beta: \beta \in \kappa\}$. This completes the proof.

9.6. *Proof of Lemma* 6.7. Let X, Y, $\mathscr{B}$ and $\{\mathbf{D}_\alpha: \alpha \in \kappa\}$ be as in the lemma. Without loss of generosity, assume X has no isolated points. Using CH, obtain $Z \in [X\backslash Y]^{\aleph_1}$ such that Z contains a limit point from each infinite subset of Y. Enumerate Y as

$$\{y_\nu: \nu \text{ is a successor ordinal } <\omega_1\}\,.$$

Enumerate Z, with possible repetitions as

$$\{z_\lambda: \lambda \text{ is a limit ordinal } <\omega_1\}$$

in such a way that

(iii) for any $Y' \in [Y]^{\aleph_0}$, $\{\lambda \in \omega_1: z_\lambda$ is a limit point of $Y'\}$ is uncountable.

Use these enumerations to identify $Y \cup Z$ with ω_1.

Now, for each $\alpha \in \kappa$ and each successor ordinal $\nu \in \omega_1$ such that $\nu \in \mathbf{D}_\alpha$, we can find, by (i), an open $G(\nu, \alpha) \in \mathscr{B}$ such that

(iv) $\nu \in G(\nu, \alpha) \subseteq \overline{G(\nu, \alpha)} \subseteq \mathbf{D}_\alpha$.

Furthermore, by (ii), we can ensure that

(v) if $\mu \leqslant \nu$ and $\nu \in G(\mu, \alpha)$, then $\overline{G(\nu, \alpha)} \subseteq G(\mu, \alpha)$.

Let $\mathscr{G} = \{G(\nu, \alpha) : \nu \in \mathbf{D}_\alpha, \nu$ is a successor ordinal and $\alpha \in \kappa\} \cup \{X\}$.

Fix $\beta \in \omega_1$ for awhile. Consider the collection

$$\{\{\nu \in \omega_1 : \nu \text{ is a successor and } \nu \in G \cap \mathbf{D}_\alpha\} : \beta \in G \in \mathscr{G} \text{ and } \alpha \in \kappa\}$$

of subsets of ω_1. By (ii) and since X has no isolated points, the intersection of any countable subcollection is an uncountable subset of ω_1. We can therefore apply P_1 to obtain $C \in [\omega_1]^{\aleph_1}$ such that

(vi) $|\{\nu \in C : \nu \notin G \cap \mathbf{D}_\alpha\}| \leqslant \aleph_0$ for all $\alpha \in \kappa$ and all $\beta \in G \in \mathscr{G}$. Enumerate C as $\{x(\beta, \xi) : \xi \in \omega_1\}$.

Now let $\beta \in \omega_1$ vary and fix $\alpha \in \kappa$ for awhile. Define a function $f_\alpha : \omega_1 \to \omega_1$ as follows.

If ν is a successor ordinal and $\nu \in \mathbf{D}_\alpha$, then

(vii) $f_\alpha(\nu) =$ the least $\eta \in \omega_1$ such that $\{x(\nu, \xi) : \xi \geqslant \eta\} \subseteq G(\nu, \alpha)$.

If ν is a successor ordinal and $\nu \notin \mathbf{D}_\alpha$, then

(viii) $f_\alpha(\nu) =$ the least $\eta \in \omega_1$ such that $\{x(\nu, \xi) : \xi \geqslant \eta\} \leqslant \mathbf{D}_\alpha$.

If λ is a limit ordinal and $\lambda \in G(\nu, \alpha)$ for some $\nu < \lambda$, then

(ix) $f_\alpha(\lambda) =$ the least $\eta \in \omega_1$ such that

$$\{x(\lambda, \xi) : \xi \geqslant \eta\} \subseteq \bigcap \{G(\nu, \alpha) : \nu < \lambda \text{ and } \lambda \in G(\nu, \alpha)\}.$$

If λ is a limit ordinal and $\lambda \notin G(\nu, \alpha)$ for any $\nu < \lambda$, then

(x) $f_\alpha(\lambda) =$ the least $\eta \in \omega_1$ such that $\{x(\lambda, \xi) : \xi \geqslant \eta\} \subseteq \mathbf{D}_\alpha$.

We can now apply Lemma 5.13 to the collection $\{f_\alpha : \alpha \in \kappa\}$ and obtain a function $f : \omega_1 \to \omega_1$ such that for all $\alpha \in \kappa$ there is $N_\alpha \in \omega_1$ such that:

(xi) for all $\beta \geqslant N_\alpha$ we have $f(\beta) \geqslant f_\alpha(\beta)$.

For each limit ordinal $\delta \in \omega_1 \backslash \{\emptyset\}$, fix an increasing sequence $\langle \delta_n : n \in \omega \rangle$ of successor ordinals such that $\sup\{\delta_n : n \in \omega\} = \delta$. We shall use these sequences to help define a subset $\{a_\gamma : \gamma \in \omega_1\}$ of ω_1. Let $a_0 = 33$. Let

(xii) $a_{\gamma+1} \in \{x(a_\gamma, \xi) : \xi \geqslant f(a_\gamma)\}$ such that $a_{\gamma+1} > a_\gamma$.

If δ is a limit ordinal, use (iii) to let

(xiii) $a_\gamma \in$ closure of $\{a_{\delta_n} : n \in \omega\}$ such that $a_\delta > \sup\{a_\gamma : \gamma < \delta\}$.

Now define a function $g : \kappa \to \omega_1$ by letting $g(\alpha)$ be the second smallest element of the set

$$\{a_\gamma : \gamma \in \omega_1\} \cap (\omega_1 \backslash N_\alpha).$$

Note that if $g(\alpha) = a_\gamma$, then γ is a successor ordinal. By (vii) through (xii), we have $g(\alpha) \in \mathbf{D}_\alpha$. Hence $G(g(\alpha), \alpha)$ is defined, and

$$g(\alpha) \in G(g(\alpha), \alpha) \subseteq \mathbf{D}_\alpha .$$

We claim that $\{G(g(\alpha), \alpha) \colon \alpha \in \kappa\}$ is centred. Once this is established, by compactness we have

$$\emptyset \neq \bigcap\{\overline{G(g(\alpha), \alpha)} \colon \alpha \in \kappa\} \subseteq \bigcap\{\mathbf{D}_\alpha \colon \alpha \in \kappa\}$$

which completes the proof.

Let us embark on the proof of the claim. We first show that for successor ordinals $\gamma \in \omega_1$, we have

(xiv) if $a_\gamma > N_\alpha$ and $\delta \geqslant \gamma$, then $\overline{G(a_{\delta+1}, \alpha)} \subseteq G(a_\gamma, \alpha)$.

We prove this by induction on $\delta \in \omega_1$. There are two cases.

In the first case suppose that δ is a successor ordinal, $\gamma \leqslant \delta$ and, by inductive hypothesis, $\overline{G(a_\delta, \alpha)} \subseteq G(a_\gamma, \alpha)$. Since $a_\delta > N_\alpha$, we have by (vii) and (xi), that

$$\{x(a_\delta, \xi) \colon \xi \geqslant f(a_\delta)\} \subseteq G(a_\delta, \alpha)$$

Hence, by (xii), $a_{\delta+1} \in G(a_\delta, \alpha)$. Therefore, by (v),

$$\overline{G(a_{\delta+1}, \alpha)} \subseteq G(a_\delta, \alpha)$$

and so the inductive hypothesis completes the proof of this case.

In the second case suppose that δ is a limit ordinal, and $\gamma < \delta$. As inductive hypothesis we suppose that for all successor ordinals δ' and δ'' such that $\gamma \leqslant \delta' < \delta'' < \delta$ we have

$$\overline{G(a_{\delta'}, \alpha)} \subseteq G(a_{\delta''}, \alpha) .$$

Recall $\langle \delta_n \colon n \in \omega \rangle$ where $\delta = \sup\{\delta_n \colon n \in \omega\}$. Now, let

$$L = \{x \in X \colon x \text{ is a limit point of } \{a_{\delta_n} \colon n \in \omega\} .$$

We have, by elementary topology,

$$L \subseteq \bigcap\{\overline{G(a_{\delta_n}, \alpha)} \colon \delta_n > \gamma\} = \{G(a_{\delta_n}, \alpha) \colon \delta_n > \gamma\}$$

by our inductive hypothesis. Hence, by (xiii), we have

$$a_\delta \in \bigcap\{G(a_{\delta_n}, \alpha) \colon \delta_n > \gamma\} .$$

Hence by (ix) and (xi) we get

$$\{x(a_\delta, \xi) \colon \xi \geqslant f(a_\delta)\} \subseteq \bigcap\{G(a_{\delta_n}, \alpha) \colon \delta_n > \gamma\} .$$

Thus, by (xii)

$$a_{\delta+1} \in \bigcap\{G(a_{\delta_n}, \alpha): \delta_n > \gamma\}.$$

Therefore, for some δ_n such that $\delta_n > \gamma$, we have from the inductive hypothesis and (v), that

$$a_{\delta+1} \in G(a_{\delta+1}, \alpha) \subseteq \overline{G(a_{\delta+1}, \alpha)} \subseteq G(a_{\delta_n}, \alpha) \subseteq G(a_\gamma, \alpha).$$

This completes the proof of (xiv).

Now suppose $\{G(g(\alpha_i), \alpha_i): i \leqslant m\}$ is a finite subset of $\{G(g(\alpha), \alpha): \alpha \in \kappa\}$. Let $a = \max\{g(\alpha_i): i \leqslant m\}$. Then for each $i \leqslant m$, we have $g(\alpha_i) > N_{\alpha_i}$ and $a \geqslant g(\alpha_i)$, so applying (xiv) gives

$$\overline{G(a, \alpha_i)} \subseteq G(g(\alpha_i), \alpha_i).$$

Therefore $a \in \bigcap\{G(g(\alpha_i), \alpha_i): i \leqslant m\}$ and hence $\{G(g(\alpha), \alpha): \alpha \in \kappa\}$ is centred and the proof is complete.

9.7. *Proof of Lemma* 6.14. Let $\mathbf{P}$ be the set of triples $p = \langle \gamma^p, A^p, g^p \rangle$ such that:

(i) $\gamma^p \in \omega_1 \backslash \{0\}$,

(ii) $A^p \in [\omega_2]^{\aleph_0}$,

(iii) $g^p: A^p \to ({}^{\gamma^p}\omega_1)$, and

(iv) for all α and β in A^p there is $\delta \in \gamma^p$ such that $g^p(\alpha)(\delta) \neq g^p(\beta)(\delta)$ and such that either $\delta = 0$ or $\vec{f}(\alpha) \upharpoonright \delta + 1 = \vec{f}(\beta) \upharpoonright \delta + 1$.

We define $\langle \gamma^q, A^q, g^q \rangle \leqslant \langle \gamma^p, A^p, g^p \rangle$ iff

(v) $A^q \supseteq A^p$, and

(vi) for all $\alpha \in A^q$, $g^q(\alpha) \supseteq g^p(\alpha)$.

Thus $q \leqslant p$ means that q gives more information about a proposed g to prove the lemma. It is easy to see that $\mathbf{P}$ is countably closed.

For each $\beta \in \omega_1$, the set $\{q \in \mathbf{P}: \beta \in A^q\}$ is dense. To see this, choose $p \in \mathbf{P}$ such that $\beta \notin A^p$. We get $q \leqslant p$ by choosing $\gamma^q = \gamma^p$, $A^q = A^p \cup \{\beta\}$ and

$$g^q(\alpha)(\delta) = g^p(\alpha)(\delta) \quad \text{if } \alpha \in A^p \text{, and}$$

$$g^q(\beta)(\delta) = \rho \quad \text{for all } \delta \in \gamma^q$$

where ρ is chosen so that $\rho \notin \{g(\alpha)(0): \alpha \in A^p\}$.

It is also easy to show that for each $\gamma \in \omega_1$, $\{p \in P: \gamma \leqslant \gamma^p\}$ is dense. If $\mathbf{G} \subseteq \mathbf{P}$ is generic with respect to the above mentioned dense sets, then we can define

$$\vec{g} = \bigcup\{g^p: p \in \mathbf{G}\}$$

which satisfies the lemma.

It remains to show that $\mathbf{P}$ is $\aleph_1$-centred. For each $\alpha \in \omega_2$ we fix an enumeration, $\{X^\alpha_\beta: \beta \in \omega_1\}$, of $[\alpha]^{\aleph_0}$. For each $p \in \mathbf{P}$ define a function $F^p: A^p \to \omega_1$, such that for all $\alpha \in A^p$,

$$X^\alpha_{F^p(\alpha)} = A^p \cap \alpha .$$

By Lemma 3.8, there exists a set S of $\aleph_1$ functions from ω_2 into ω_1 such that for all $p \in \mathbf{P}$, there is some $s \in S$ such that $F^p \subseteq s$. For each $s \in S$, let $\mathbf{P}_s = \{p \in \mathbf{P}: F^p \subseteq s\}$. We will show that $\mathbf{P}$ is $\aleph_1$-centred by showing that each $\mathbf{P}_s$ is $\aleph_1$-centred in $\mathbf{P}$.

For p and $q \in \mathbf{P}_s$, let $p \approx q$ iff the following conditions hold.

(vii) $\gamma^p = \gamma^q$, and

(viii) there is an order-preserving bijection $b: A^p \to A^q$ such that for all $\alpha \in A^p$ both $g^p(\alpha) = g^q(b(\alpha))$ and $\vec{f}(\alpha)\restriction(\gamma^p + 1)$ is equal to $\vec{f}(b(\alpha))\restriction(\gamma^q + 1)$.

This is easily seen to be an equivalence relation on $\mathbf{P}_s$. Furthermore, there are only $2^{\aleph_0}$ equivalence classes since the equivalence class of p is determined by only γ^p, the order type of A^p and two sequences of countable functions

$$\langle g^p(\alpha): \alpha \in A^p\rangle \text{ and } \langle \vec{f}(\alpha)\restriction(\gamma^p + 1): \alpha \in A^p\} .$$

Note that if $p, q \in \mathbf{P}_s$ such that $p \approx q$, then g^p and g^q are compatible functions (i.e. they agree on their common domain). To see this, let $b: A^p \to A^q$ be given by (viii). Since F^p and F^q are compatible, for each $\alpha \in A^q \cap A^p$ we have $F^p(\alpha) = F^q(\alpha)$ and $A^p \cap \alpha = A^q \cap \alpha$; since b is order preserving, $b\restriction(A^p \cap \alpha)$ is the identity on $A^p \cap \alpha$, and hence $b\restriction(A^p \cap A^q)$ is the identity function on $A^p \cap A^q$. Thus for each $\alpha \in A^p \cap A^q$, $g^p(\alpha) = g^q(b(\alpha)) = g^q(\alpha)$.

Now let $\{p_0, p_1, \ldots, p_n\} \subseteq \mathbf{P}_s$ such that each $p_i \approx p_0$. We construct $r = \langle \gamma^r, A^r, g^r\rangle \in \mathbf{P}$ such that for each $i = 0, 1, \ldots, n$, $r \leqslant p_i$. Let $\gamma^r = \gamma^{p_0} + 1 = \gamma^{p_1} + 1 = \cdots = \gamma^{p_n} + 1$. Let $A^r = \bigcup\{A^{p_i}: i = 0, 1, \ldots, n\}$. Define g^r by:

$$g^r(\alpha) = \begin{cases} g^{p_0}(\alpha) \cup \{\langle \gamma^{p_0}, 0\rangle\} & \text{if } \alpha \in A^{p_0} , \\ g^{p_i}(\alpha) \cup \{\langle \gamma^{p_0}, i\rangle\} & \text{if } \alpha \in A^{p_i} \setminus \bigcup\{A^{p_j}: j = 0, \ldots, i\} . \end{cases}$$

Since $g^{p_0}, g^{p_1}, \ldots, g^{p_n}$ are compatible functions, $r \leqslant p_i$ for each $i = 0, 1, \ldots, n$.

We check that $r \in \mathbf{P}$, and in particular that (iv) holds for r. Pick two elements α, β in A^r. If $\{\alpha, \beta\} \subseteq A^{p_i}$ for some $i = 0, 1, \ldots, n$ there is no problem. Otherwise, without loss of generality we can assume that i is the least integer such that $\alpha \in A^{p_i}$ and that $\beta \in A^{p_j} \setminus A^{p_i}$ for some $j > i$. Choose $b: A^{p_i} \to A^{p_j}$ from (viii) and let $b(\alpha') = \beta$. If $\alpha = \alpha'$ then $\delta = \gamma^{p_0}$ will satisfy us. If $\alpha \neq \alpha'$, then by (iv) there is some $\delta \in \gamma^{p_0}$ such that $g^{p_i}(\alpha)(\delta) \neq g^{p_i}(\alpha')(\delta)$ and either $\delta = 0$ or $\vec{f}(\alpha)\restriction(\delta + 1) = \vec{f}(\beta)\restriction(\delta + 1)$, and then using (viii) we see that this δ will satisfy us. This completes the proof.

9.8. *Proof of Lemma* 7.6. We first define a map $\mu: X \to \omega_1$ such that

$$\mu(x) = \alpha \text{ iff } x \in S_\alpha .$$

For each $U \in \mathcal{U}$, denote the set $\{\alpha \in \omega_1 \colon S_\alpha \subseteq U\}$ by V_U. It is straightforward to show that $\{V_U \colon U \in \mathcal{U}\}$ is a σ-discrete base for a zero-dimensional Hausdorff topology τ on ω_1. By the metrization theorem of R.H. Bing, J. Nagata and J.M. Smirnov (ENGELKING [1977], p. 352), we have that $\langle \omega_1, \tau \rangle$ is metrizable. If $A \subseteq \omega_1$, we apply Corollary 7.4 to get that A is a G_δ in $\langle \omega_1, \tau \rangle$. But since the map μ is clearly continuous, $\bigcup\{S_\alpha \colon \alpha \in A\}$ is a G_δ in X.

Now suppose each S_α is first category and let

$$Z = \bigcup\{S_\alpha \colon \alpha \in A\},$$

with $W = \operatorname{Int}(\bar{Z})$. Suppose Z is second category; then $|A| = \aleph_1$ and $Z \cap W$ is a dense second category subset of W.

We first suppose that X satisfies (i). Since each S_α is first category we can use transfinite induction to choose disjoint subsets A_1 and A_2 of A such that

(v) each member of the given π-base for X which is contained in W includes some S_α for $\alpha \in A_1$, and

(vi) each member of the given π-base for X which is contained in W includes some S_α for $\alpha \in A_2$.

Therefore

$$(\bigcup\{S_\alpha \colon \alpha \in A_1\} \cap W \quad \text{and} \quad (\bigcup\{S_\alpha \colon \alpha \in A_2\}) \cap W$$

are two disjoint dense G_δ subsets of W, contradicting that X is a Baire space.

We now suppose that X satisfied (ii). We obtain a collection $\{A_\xi \colon \xi \in \omega_1\}$ of pairwise disjoint uncountable subsets of A. Since X is Lusin, for each $\xi \in \omega_1$ there is an open set W_ξ, such that:

(vii) $\{\alpha \in A_\xi \colon S_\alpha \cap V \neq \emptyset\}$ is uncountable for each open $V \subseteq W_\xi$. Since X is ccc there are distinct ξ and η such that $W_\xi \cap W_\eta \neq \emptyset$. Thus

$$(\bigcup\{S_\alpha \colon \alpha \in A_\xi\}) \cap W_\xi \cap W_\eta \quad \text{and} \quad (\bigcup\{S_\alpha \colon \alpha \in A_\eta\}) \cap W_\xi \cap W_\eta$$

are two uncountable disjoint dense G_δ subsets of $W_\xi \cap W_\eta$. This contradicts the fact that X is Lusin and completes the proof.

9.9. *Proof of Lemma* 7.8. The elements of the tree will be open subsets of X. The ordering will be $U \leq V$ iff $V \setminus U$ is first category. We will choose open subsets of X for $\mathbf{T}$ in such a way that the union of any level of $\mathbf{T}$ is a dense open subset. Further, if X has a π-base of cardinality $\aleph_1$, then we will ensure that each member of the π-base includes some member of $\mathbf{T}$. Without loss of generosity we can assume that all open subsets of X are uncountable. We will construct the first $\omega+1$ levels of $\mathbf{T}$ together and hope that you, the reader, will take it from there.

Level zero is just $\{X\}$. Level one is an infinite collection of pairwise disjoint open subsets of X such that one of them is included in the zeroth member of the π-base, and such that the union is dense. Continuing this way, if V is on level one, then the successors of V on level two will be an infinite collection of pairwise disjoint open subsets of V. Again we will ensure that the union of all members of level two is dense and that some member of level two is included in the first member of the π-base.

Now suppose, for each $n \in \omega$ we have picked a pairwise disjoint collection $\mathcal{U}_n$ of open sets for the nth level of the tree $\mathbf{T}$. For each function f,

$$f\colon \omega \to \bigcup\{\mathcal{U}_n : n \in \omega\}$$

such that for each $n \in \omega$, $f(n) \in \mathcal{U}_n$, let

$$G_f = \bigcap\{f(n) : n \in \omega\}$$

and let $\mathcal{G}$ be the collection of all these G_f. Note that $\mathcal{G}$ has cardinality at most $\aleph_1$. Some elements of $\mathcal{G}$ will be used to determine elements of the omegath level of $\mathbf{T}$; some will not. Let

$$\mathcal{G}_1 = \{G \in \mathcal{G} : G \text{ is first category}\}\,.$$

By Lemma 7.6 $\bigcup\mathcal{G}_1$ is first category. Hence, in either case (i) or case (ii) we must have that $\bigcup\mathcal{G}_2$, where $\mathcal{G}_2 = \mathcal{G} - \mathcal{G}_1$, is second category and, in fact, the compliment of a first category set in X.

For each $G \in \mathcal{G}_2$ notice that $\bigcup\{\mathcal{G}_2 \backslash \{G\}\}$ is a G_δ set, by Lemma 7.6, and hence cannot be dense since G is second category. Therefore, there is some open V_G such that

$$(V_G \backslash G) \cup (G \backslash V_G)$$

is first category. The collection $\{V_G : G \in \mathcal{G}_2\}$ is seen to be pairwise disjoint with union dense in X, and becomes the omegath level of $\mathbf{T}$. The remainder of the construction of $\mathbf{T}$ is now straightforward.

The conclusion (iv) is easily verified since an uncountable pairwise incomparable subset of $\mathbf{T}$ is an uncountable pairwise disjoint collection of open subsets of X. So if X has the ccc, then it is easy to show that (iii) holds as well.

It remains to show that (iii) holds if X satisfies (i). In this case every member of the π-base for X contains a member of the tree $\mathbf{T}$, and therefore every maximal pairwise incomparable subset of $\mathbf{T}$ has union dense in X. Let $\{\mathbf{D}_n : n \in \omega\}$ be as in

Definition 7.7 and let $V \in \mathbf{T}$; for each $n \in \omega$ we can find a maximal pairwise incomparable $\mathbf{A}_n \subseteq \mathbf{D}_n$ such that $U \geqslant V$ for each $U \in A_n$. Since X is a Baire space,

$$\bigcap\{\bigcup \mathbf{A}_n : n \in \omega\}$$

is second category, and hence by Lemma 7.6 there is a function $f: \omega \to \bigcup\{\mathbf{A}_n : n \in \omega\}$ with $f(n) \in \mathbf{A}_n$ for each $n \in \omega$ such that $\bigcap\{f(n): n \in \omega\}$ is second category. This gives rise to an element of the tree which will satisfy (iii) and complete the proof.

9.10. *Proof of Theorem* 7.14. For each $\alpha \leqslant \omega_1$, let $\mathcal{P}(\alpha)$ denote the power set of α. Enumerate $\mathcal{P}(\alpha)$ as

(i) $\{i(\alpha, \sigma): \sigma \in \omega_1\} = \mathcal{P}(\alpha)$.

For each $t \in \mathcal{P}(\omega_1)$ we define a function $g_t: \omega_1 \to \omega_1$ by

(ii) $g_t(\alpha) = \sigma$ if $t \cap \alpha = i(\alpha, \sigma)$.

Now, let $S \in [\mathcal{P}(\omega_1)]^{\aleph_2}$. We invoke Theorem 6.8, obtaining P_1 and then invoke Lemma 5.13 on the collection $\{g_t: t \in S\}$ and acquire a function $f: \omega_1 \to \omega_1$ such that for all $t \in S$ there is some $N_t \in \omega_1$ such that

(iii) $f(\alpha) > g_t(\alpha)$ for all $\alpha > N_t$.

We can obtain $S' \in [S]^{\aleph_2}$ and some $\alpha_0 \in \omega_1$ such that

(iv) $f(\alpha) > g_t(\alpha)$ for all $t \in S'$ and $\alpha \geqslant \alpha_0$.

We can now define a tree $\mathbf{T}$ of height $\omega_1 + 1$. For each $\alpha \leqslant \omega_1$, let

(v) $T'_\alpha = \{t \cap (\alpha_0 + \alpha): t \in S'\}$.

and let $\mathrm{Lev}_\alpha(\mathbf{T}) = \{\langle a, \alpha\rangle : a \in T'_\alpha\}$. We order

$$\mathbf{T} = \bigcup\{\mathrm{Lev}_\alpha(\mathbf{T}): \alpha \leqslant \omega_1\}$$

by $\langle a, \alpha\rangle < \langle b, \beta\rangle$ iff $\alpha < \beta$ and for some $t \in S'$ we have $t \cap (\alpha_0 + \alpha) = a$ and $t \cap (\alpha_0 + \alpha) = b$.

For each $\alpha < \omega_1$ we have

$$T'_\alpha = \{i(\alpha_0 + \alpha, g_t(\alpha_0 + \alpha): t \in S'\}$$

by (v) and (ii). Hence, by (iv), we have

$$T'_\alpha \subseteq \{i(\alpha_0 + \alpha, \sigma): \sigma < f(\alpha_0 + \alpha)\}$$

and hence each of the first ω_1 levels of T is countable.

On the other hand $T'_{\omega_1} = S'$. Thus the subtree $\bigcup\{\mathrm{Lev}_\alpha(\mathbf{T}): \alpha < \omega_1\}$ of $\mathbf{T}$ is a Kurepa tree with at least $\aleph_2$ cofinal branches, completing the proof of the theorem.

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CHAPTER 20

Random and Cohen Reals

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
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0. Introduction

Two of the simplest ways of violating CH are by adding Cohen generic reals and by adding random reals. These generic extensions are similar in many respects, but differ greatly in their effects on measure and category. The details of this were first described by Solovoy, but never published in full. This paper is an exposition of these results. We shall give a unified treatment of both extensions simultaneously, since their properties can actually be derived from some abstract properties shared by the ideal of meagre (first category) sets and the ideal of null (measure 0) sets. Thus, the results we prove in this paper would apply also to the extensions arising from any other ideals which share these properties. However, it is an open question whether any such ideals exist.

1. Properties of the meagre and null ideals

For any set I, 2^I denotes the space of all functions from I into 2 with the usual product topology, where $2 = \{0, 1\}$ has the discrete topology. The *Baire* subsets of 2^I are those sets in the σ-algebra generated by the clopen sets, whereas the *Borel* subsets are those in the σ-algebra generated by the open sets. If I is countable, Borel is the same as Baire, whereas if I in uncountable, singletons are examples of Borel sets which are not Baire.

Let $\mathscr{B}(2^I)$ be the family of Baire subsets of 2^I. As usual, $\mathscr{P}(2^I)$ denotes the family of all subsets of 2^I. We shall never need to consider the family of Borel sets.

An *ideal* on 2^I is a subset of $\mathscr{P}(2^I)$ which is closed under finite unions and subsets, contains $\emptyset$ and does not contain 2^I. A σ*-ideal* is closed under countable unions. We are primarily interested in ideals which are *Baire supported* in the following sense.

1.1. DEFINITION. An ideal $\mathscr{S}$ on 2^I is *Baire supported* iff

$$\forall X \in \mathscr{S} \quad \exists Y \in \mathscr{S} \quad (X \subset Y \text{ and } Y \in \mathscr{B}(2^I)) . \qquad \square$$

The two specific ideals we consider in this paper are the ideal of null sets and the ideal of meagre sets.

We consider 2^I to be a measure space by giving it the usual product measure, where in $2 = \{0, 1\}$, $\{0\}$ and $\{1\}$ both have measure $\frac{1}{2}$. Then the *null ideal* is the ideal of sets which have measure 0. This is Baire supported since if X is null, there is a null $Y \supset X$ which is a G_δ of the form

$$\bigcap_{m \in \omega} \bigcup_{n \in \omega} K_{mn} ,$$

with the K_{mn} clopen.

The *meagre* ideal is the ideal of sets of first category. Since 2^I has the *countable chain condition* (ccc), every nowhere dense set is contained in a closed nowhere dense G_δ. Thus, the meagre ideal is Baire supported, and every meagre set is contained in a meagre F_σ of the form

$$\bigcup_{m\in\omega}\bigcap_{n\in\omega} K_{mn},$$

with the K_{mn} clopen.

In this section we discuss some properties possessed by both the null and meagre ideals. We first consider invariance under permuting the index set.

1.2. DEFINITION. If $\Delta: I\to J$, define $\Delta^*: 2^J\to 2^I$ and $\Delta_*: \mathcal{P}(2^I)\to\mathcal{P}(2^J)$ by

$$\Delta^*(f)=f\circ\Delta \quad\text{and}\quad \Delta_*(X)=(\Delta^*)^{-1}(X). \qquad \square$$

In all our applications, Δ will be 1–1, so Δ^* will be onto and Δ_* will be 1–1. Since Δ^* is continuous, $\Delta_*(X)$ will be in $\mathcal{B}(2^J)$ whenever $X\in\mathcal{B}(2^I)$. The meaning of our notation can be made clearer if we consider two special cases.

First, suppose $\Delta: I\to J$ is 1–1 and onto. We may think of Δ as 'identifying' I and J, whence Δ^* and Δ_* are the corresponding identifications of 2^J with 2^I and $\mathcal{P}(2^I)$ with $\mathcal{P}(2^J)$.

Second, suppose $I\subset J$ and $\Delta: I\to J$ is the identity. Then $\Delta^*(f)=f\restriction I$. Let $K=J\backslash I$. We may identify 2^J with $2^I\times 2^K$ by identifying f in 2^J with $\langle f\restriction I, f\restriction K\rangle$. Then Δ^* is just projection onto 2^I.

One approach (which we shall not use) to express the index invariance of an ideal such as the null ideal is in terms of functors. That is, for each I, we have an ideal, $\mathcal{S}(I)$, on 2^I. If $\Delta: I\to J$ is 1–1 and $X\subset 2^I$, then $X\in\mathcal{S}(I)$ iff $\Delta_*(X)\in\mathcal{S}(J)$; one may consider ideals with this sort of invariance as a kind of functor from sets to ideals. However, such an $\mathcal{S}$ is, if it is Baire supported, determined completely by $\mathcal{S}(\omega)$, and it is set-theoretically simpler to define notions directly in terms of the *set*, $\mathcal{S}(\omega)$, rather than in terms of a functor, which is a proper class.

1.3. DEFINITION. If $\mathcal{S}$ is an ideal on 2^ω, $\mathcal{S}$ is *index invariant* iff for all 1–1 $\Delta:\omega\to\omega$ and all $X\subset 2^\omega$, $X\in\mathcal{S}$ iff $\Delta_*(X)\in\mathcal{S}$. $\square$

Such an $\mathcal{S}$ induces an ideal, $\mathcal{S}(I)$, on any 2^I, and the $\mathcal{S}(I)$ will have the same index invariance for all 1–1 $\Delta: I\to J$.

1.4. DEFINITION. If $\mathcal{S}\subset\mathcal{P}(2^\omega)$, define $\mathcal{S}(I)\subset\mathcal{P}(2^I)$ by: $X\in\mathcal{S}(I)$ iff for some 1–1 $\Delta:\omega\to I$ and some $Y\in\mathcal{S}$, $X\subset\Delta_*(Y)$. $\square$

1.5. LEMMA. *Let $\mathscr{S}$ be an index invariant ideal on 2^ω, and let I be infinite*
(a) $\mathscr{S}(\omega)=\mathscr{S}$.
(b) $\mathscr{S}(I)$ *is an ideal on* 2^I.
(c) *If* $\mathscr{S}$ *is an* σ*-ideal, so is* $\mathscr{S}(I)$.
(d) *If* $\Gamma: I\to J$ *is* 1–1 *and* $X\subset 2^I$, *then* $X\in\mathscr{S}(I)$ *iff* $\Gamma_*(X)\in\mathscr{S}(J)$.

PROOF. (a) is immediate from the definitions. For (c), we must show that $\mathscr{S}(I)$ is closed under countable unions if $\mathscr{S}$ is; we shall omit the similar argument for (b) that in any case $\mathscr{S}(I)$ is closed under finite unions. Suppose $\mathscr{S}$ is a σ-ideal and let $X_n\in\mathscr{S}(I)$ for $n\in\omega$. Let $Y_n\in\mathscr{S}$ and $\Delta_n:\omega\to I$ be such that Δ_n is 1–1 and $X_n\subset(\Delta_n)_*(Y_n)$. Let $\Gamma:\omega\to I$ be 1–1 such that $\operatorname{ran}(\Delta_n)\subset\operatorname{ran}(\Gamma)$ for each n. Now, let $\Sigma_n:\omega\to\omega$ be 1–1 and satisfy $\Gamma\circ\Sigma_n=\Delta_n$ for each n. Let $Z_n=(\Sigma_n)_*(X_n)$. Then for each n,

$$X_n\subset(\Delta_n)_*(Y_n)=\Gamma_*(\Sigma_n)_*(Y_n)=\Gamma_*(Z_n)\,.$$

Since $\mathscr{S}$ is index invariant, each $Z_n\in\mathscr{S}$, so, if $\mathscr{S}$ is a σ-ideal,

$$\bigcup_n X_n\subset\bigcup_n\Gamma_*(Z_n)=\Gamma_*(\bigcup_n Z_n)\in\mathscr{S}\,.$$

For (d), let $X\subset 2^I$ and let $\Gamma: I\to J$ be 1–1. If $X\in\mathscr{S}(I)$, let $Y\in\mathscr{S}$ and let $\Delta:\omega\to I$ be 1–1 such that $X\subset\Delta_*(Y)$. Then $\Gamma_*(X)\subset(\Gamma\circ\Delta)_*(Y)$, so $\Gamma_*(X)\in\mathscr{S}(J)$. Conversely, suppose $\Gamma_*(X)\in\mathscr{S}(J)$. Let $\Delta:\omega\to J$ be 1–1 and let $Y\in\mathscr{S}$ be such that $\Gamma_*(X)\subset\Delta_*(Y)$. Let $\Sigma:\omega\to I$ be any 1–1 function such that $\Gamma(\Sigma(k))=\Delta(k)$ whenever $\Delta(k)\in\operatorname{ran}(\Gamma)$. Then $X\subset\Sigma_*(Y)$, so $X\in\mathscr{S}(I)$. □

We remark that if $\mathscr{S}$ is the meagre ideal on 2^ω and $\mathscr{G}(I)$, is defined as per Definition 1.4, then $\mathscr{S}(I)$ is the usual meagre ideal on 2^I. Likewise for the null ideal. When $\mathscr{S}$ is index invariant, we often use $\mathscr{S}$ for $\mathscr{S}(I)$ when I is clear from context.

Another kind of invariance is 0–1 invariance, which says that one may interchange 0's and 1's on any given set of coordinates without affecting the ideal. To formulate this precisely, it is simplest to use group notation. Identify $2=\{0,1\}$ with the additive group $\mathbb{Z}_2$, and on 2^I, let + denote coordinatewise addition. If $f\in 2^I$ and $X\subset 2^I$, let

$$f+X=\{f+g: g\in X\}\,.$$

1.6. DEFINITION. An ideal $\mathscr{S}$ on 2^I is 0–1 *invariant* iff for all $f\in 2^I$ and all $X\subset 2^I$, $X\in\mathscr{S}$ iff $(X+f)\in\mathscr{S}$. □

1.7. LEMMA. *If* $\mathscr{S}$ *is an index invariant and* 0–1 *invariant ideal on* 2^ω, *then* $\mathscr{S}(I)$ *is* 0–1 *invariant for any infinite* I. □

Our intent is to use our ideals to obtain a partial order to use in forcing. In fact, two such orders come to mind.

1.8. DEFINITION. Let $\mathcal{S}$ be an index invariant ideal on 2^ω. For any infinite I, define
(a) $\mathbb{P}(\mathcal{S}, I) = \{B \in \mathcal{B}(2^I): B \notin \mathcal{S}(I)\}$, where $B \leq C$ iff $B \subset C$.
(b) $\mathbb{B}(\mathcal{S}, I) = \mathcal{B}(2^I)/\mathcal{S}(I)$ (i.e., the quotient Boolean algebra with the usual order). □

$\mathbb{B}$ is often preferable to $\mathbb{P}$, since $\mathbb{B}$ is a Boolean algebra while $\mathbb{P}$ is not even separative (see KUNEN [1980], p. 88). However, the elements of $\mathbb{P}$ are Baire sets while the elements of $\mathbb{B}$ are equivalence classes of Baire sets, which sometimes makes $\mathbb{P}$ preferable. In any case, they yield the same forcing. This is proved by exhibiting a dense embedding between them (see Theorem 7.11 of KUNEN [1980]).

1.9. LEMMA. *Let $\mathcal{S}$ be an index invariant ideal on 2^ω and, for $B \in \mathbb{P}(\mathcal{S}, I)$, let $i(B) \in \mathbb{B}(\mathcal{S}, I)$ be its equivalence class. Then*

$$i: \mathbb{P}(\mathcal{S}, I) \to (\mathbb{B}(\mathcal{S}, I) \backslash \{0\})$$

is a dense embedding. □

If $\mathcal{S}$ is the null ideal, then $\mathbb{B}(\mathcal{S}, I)$ is the measure algebra of 2^I and is precisely what Solovay used in defining random real forcing. If $\mathcal{S}$ is the meagre ideal, then $\mathbb{B}(\mathcal{S}, I)$ is isomorphic to the regular open algebra of 2^I (see HALMOS [1950]), but it is sometimes useful to use, as did Cohen, $\mathrm{Fn}(I, 2)$ (finite partial functions from I into 2).

1.10. LEMMA. *Suppose $\mathcal{S}$ is the meagre ideal, and define $i: \mathrm{Fn}(I, 2) \to \mathbb{B}(\mathcal{S}, I)$ by*

$$i(p) = [\{f \in 2^I : p \subset f\}],$$

where $[\cdot \cdot \cdot]$ denotes equivalence class. Then i is a dense embedding.

PROOF. To show i is dense, use the fact that every Baire set has the property of Baire (i.e., equals an open set modulo a meagre set). □

If $I = \omega$, Sacks' perfect set forcing is another example of using an ideal to obtain a partial order, since Prikry showed that Sacks forcing is equivalent to forcing with $\mathbb{P}(\mathcal{C}, \omega)$ where $\mathcal{C}$ is the ideal of countable sets. However, $\mathcal{C}$ is not index-invariant (consider $\Delta: \omega \to \omega$ with $\omega \backslash \mathrm{ran}(\Delta)$ infinite). One can use $\mathcal{C}$ to define an index-invariant ideal; namely, let $\mathcal{G}$ be the ideal countably generated by all sets of the form $\Delta_*(X)$ with $X \in \mathcal{C}$ and Δ a 1–1 map from ω into ω. $\mathcal{C}$ is an example of an index-invariant and 0–1 invariant σ-ideal for which $\mathbb{P}(\mathcal{C}, \omega)$ does not have the countable chain condition (ccc). To see this, for each $f \in 2^\omega$, let

$$K_f = \{g \in 2^\omega : \forall n \in \omega(g(2n+1) = g(2n) + f(n))\}.$$

Then K_f for $f \in 2^\omega$ form a disjoint family of $\mathfrak{c}$ closed sets which are not in $\mathscr{E}$.

The result of this paper about the null and meagre ideals use the ccc in an essential way and will not apply to $\mathscr{E}$. We do not know what happens if one forces with $\mathbb{P}(\mathscr{E}, I)$.

1.11. DEFINITION. A ccc *ideal* is an index-invariant ideal $\mathscr{S}$ on 2^ω such that for every I, $\mathbb{P}(\mathscr{S}, I)$ has the ccc. $\square$

It is easy to see that it is sufficient for ccc that $\mathbb{P}(\mathscr{S}, \omega_1)$ is ccc. We do not know whether it is sufficient that $\mathbb{P}(\mathscr{S}, \omega)$ is ccc.

Both the null and meagre ideals satisfy a form of Fubini's Theorem. This theorem is usually stated about Baire subsets of a product, $2^J \times 2^K$. Since our terminology up to now has just dealt with the spaces 2^I, it will be convenient if we make the identification of $2^J \times 2^K$ and $2^{J \cup K}$ (assuming $J \cap K = \emptyset$).

1.12. DEFINITION. Suppose $I = J \cup K$ and $J \cap K = \emptyset$. If $f \in 2^J$ and $g \in 2^K$, let $\langle f, g \rangle = f \cup g$. If $B \in \mathscr{B}(2^I)$, define the sections, $B_f = \{g : \langle f, g \rangle \in B\}$ (for $f \in 2^J$) and $B^g = \{f : \langle f, g \rangle \in B\}$ (for $f \in 2^K$). If $X \subset 2^J$ and $T \subset 2^K$, $X \times Y$ is $\{f \cup g : f \in X$ and $g \in Y\}$. $\square$

1.13. DEFINITION. A *Fubini ideal* is an index invariant ideal $\mathscr{S}$ such that whenever J and K are disjoint infinite sets and $B \in \mathscr{B}(2^{J \cup K})$, the following are equivalent

(a) $B \in \mathscr{S}(J \cup K)$

(b) $(2^J \setminus \{f \in 2^J : B_f \in \mathscr{S}(K)\}) \in \mathscr{S}(J)$

(c) $(2^K \setminus \{g \in 2^K : B^g \in \mathscr{S}(J)\}) \in \mathscr{S}(K)$. $\square$

Of course, if we had omitted (c), we would have an equivalent notion.

1.14. LEMMA. *If $\mathscr{S}$ is an index invariant ideal, $\mathscr{S}$ is a Fubini ideal iff Definition* 1.13 *holds for countable J, K.* $\square$

There is a certain duality between measure and category which will be made more explicit in the next section. The only abstract property required here is given by the next definition.

1.15. DEFINITION. Ideals $\mathscr{S}$, $\mathscr{T}$ on 2^ω are *dual* iff there is a Baire $Y \subset 2^\omega$ such that $Y \in \mathscr{S}$ and $(2^\omega \setminus Y) \in \mathscr{T}$. $\square$

Finally, for forcing, it will be necessary to relate $\mathscr{S}$ in the ground model with $\mathscr{S}$

in the extension. We need a certain absoluteness for $\mathcal{S}$. Informally, if M is a model of ZFC and B is a Baire set in M, we would like to say, e.g. for the null ideal, that B is null in the universe, V, iff B is null in M. But, this is false. For example, if $B = (2^\omega)^M$ and M is countable, then B is null in V since it is countable, but is not null in M. To express absoluteness correctly, we must first introduce the notion of a *code* for a Baire set, which describes how the set was built up from the clopen sets. Then, if c is a code, the property "the Baire set coded by c is null" will be an absolute property of c.

We may code basic sets in 2^I by elements of $\mathrm{Fn}(I, 2)$ (finite partial functions from I into 2). Then every clopen set in 2^I is a finite disjoint union of basic sets and may thus be coded by a finite antichain in $\mathrm{Fn}(I, 2)$. If X is a countable set of codes, we can let $\langle 1, X\rangle$ code the union of the complements of the sets coded by elements of X. In this way, we get all the Baire sets.

1.16. Definition. For any set I, we define $\mathscr{C}(I)$, the set of codes for Baire subsets of 2^I, as follows:

(a) $\mathscr{C}(I, 0) = \{\langle 0, A\rangle: A$ is a finite antichain in $\mathrm{Fn}(1, 2)\}$.

(b) If $\beta > 0$, $\mathscr{C}(I, \beta) = \mathscr{C}(I, 0) \cup \{\langle 1, A\rangle: |A| \leq \omega$ and $A \subset \bigcup\{\mathscr{C}(I, \alpha): \alpha < \beta\}\}$.

(c) $\mathscr{C}(I) = \bigcup\{\mathscr{C}(I, \alpha): \alpha < \omega_1\}$. □

Observe that if $\alpha \leq \beta$, then $\mathscr{C}(I, \alpha) \subset \mathscr{C}(I, \beta)$; and $\mathscr{C}(I, \alpha) = \mathscr{C}(I)$ whenever $\alpha \geq \omega_1$.

We may define the *evaluation* of a code c, or the Baire set it codes, by recursion on the rank of c, as follows:

1.17. Definition. If $c \in \mathscr{C}(I)$, we define *eval*(c, I) (or eval(c) if I is clear) by

(a) $\mathrm{eval}(\langle 0, A\rangle) = \bigcup_{p\in A}\{f \in 2^I: p \subset f\}$.

(b) $\mathrm{eval}(\langle 1, A\rangle) = \bigcup\{2^I \setminus \mathrm{eval}(c): c \in A\}$. □

1.18. Definition. For $\alpha < \omega_1$,

$$\Sigma^0_\alpha(I) = \{\mathrm{eval}(c): c \in \mathscr{C}(I, \alpha)\}. \quad \square$$

Thus, $\Sigma^0_0(I)$ is the family of clopen sets in 2^I, $\Sigma^0_1(I)$ is the family of open sets, $\Sigma^0_2(I)$ is the family of F_σ sets, etc.

1.19. Lemma. $\mathscr{B}(2^I) = \{\mathrm{eval}(c): c \in \mathscr{C}(I)\}$. □

Note that the property of being a code relativizes upward; that is, if M is a transitive model of ZFC, $c \in M$, and $(c \in \mathscr{C}(I))^M$, then $c \in \mathscr{C}(I)$. The converse is false, even for c in M, because of the restriction on the countability of A in Definition 1.16.

1.20. DEFINITION. If $\mathscr{S}$ is an index-invariant ideal on ω, $\mathscr{S}$ is *absolute* iff for any transitive model M for ZFC and any $c \in M$ with $(c \in \mathscr{C}(I))^M$,

$$[\mathrm{eval}(c) \in \mathscr{S}(I)] \quad \text{iff} \quad [\mathrm{eval}(c) \in \mathscr{S}(I)]^M . \qquad \square$$

To be very formal (which we shall not be in this paper), Definition 1.20 takes place in the metatheory. That is, we introduce $\mathscr{S}$ by a definition:

$$\mathscr{S} = \{x \in \mathscr{P}(2^\omega) \colon \varphi(x)\} ,$$

where $\varphi(x)$ is a formula in the language of set theory. We say that $\mathscr{S}$ (actually φ) is absolute if we have succeeded in proving from ZFC that $\mathscr{S}$ is an index-invariant ideal and that for all transitive M satisfying a suitably large finite fragment of ZFC,

$$\forall c, I \in M[c \in \mathscr{C}(I)^M \to [[\varphi(\mathrm{eval}(c))] \leftrightarrow [\varphi(\mathrm{eval}(c))]^M]] .$$

Actually, if we have this for $I = \omega$, it holds for all $I \in M$, but we do not need this fact here.

Before proving absoluteness for the meagre and null ideals, we discuss absoluteness of unions, intersections, and complements.

It is well-known that each Σ^0_α is closed under finite unions and intersections and that Σ^0_0 is closed under complements as well. We do this now, since our treatment of absoluteness requires that union, intersection, and complement correspond to an explicit map on the codes.

1.21. DEFINITION. Define comp: $\mathscr{C}(I) \to \mathscr{C}(I)$ and maps un and int from $\mathscr{C}(I) \times \mathscr{C}(I)$ into $\mathscr{C}(I)$ as follows:

(a) $\mathrm{comp}(\langle 0, A\rangle) = \langle 0, \{p \in \mathrm{Fn}(I, 2) \colon \mathrm{dom}(p) = \bigcup\{\mathrm{dom}(q) \colon q \in A\}$ and $\forall q \in A(p$ and q are incompatible$)\}\rangle$.

(b) $\mathrm{comp}(c) = \langle 1, \{c\}\rangle$ if $c \notin \mathscr{C}(I, 0)$.

(c) $\mathrm{un}(\langle 0, A\rangle\langle 0, B\rangle) = \langle 0, \{p \in \mathrm{Fn}(I, 2) \colon \mathrm{dom}(p) = \bigcup\{\mathrm{dom}(q) \colon q \in A \cup B\}$ and $\exists q \in A \cup B(q \leq p)\}\rangle$

(d) $\mathrm{int}(\langle 0, A\rangle\langle 0, B\rangle) = \langle 0, \{p \cup q \colon p \in A$ and $q \in B$ and p, q are compatible$\}\rangle$.

(e) $\mathrm{un}(\langle 1, A\rangle, \langle 1, B\rangle) = \langle 1, A \cup B\rangle$

(f) $\mathrm{int}(\langle 1, A\rangle, \langle 1, B\rangle) = \langle 1, \{\mathrm{un}(p, q) \colon p \in A$ and $q \in B\}\rangle$.

(g) $\mathrm{un}(\langle 0, A\rangle, \langle 1, B\rangle) = \mathrm{un}(\langle 1, B\rangle, \langle 0, A\rangle) = \langle 1, B \cup \{\mathrm{comp}(\langle 0, A\rangle)\}\rangle$

(h) $\mathrm{int}(\langle 0, A\rangle, \langle 1, B\rangle) = \mathrm{int}(\langle 1, B\rangle, \langle 0, A\rangle) = \langle 1, \{\mathrm{un}(\langle 0, A\rangle, q) \colon q \in B\}\rangle$. $\square$

1.22. LEMMA. (a) *If* $c \in \mathscr{C}(I, 0)$, *then* $\operatorname{comp}(c) \in \mathscr{C}(I, 0)$ *and* $\operatorname{eval}(\operatorname{comp}(\langle 0, A\rangle)) = 2^I \setminus \operatorname{eval}(\langle 0, A\rangle)$.

(b) *For each* α, *if* c *and* d *are in* $\mathscr{C}(I, \alpha)$, *then* $\operatorname{un}(c, d)$ *and* $\operatorname{int}(c, d)$ *are in* $\mathscr{C}(I, \alpha)$ *and* $\operatorname{eval}(\operatorname{un}(c, d)) = \operatorname{eval}(c) \cup \operatorname{eval}(d)$, $\operatorname{eval}(\operatorname{int}(c, d)) = \operatorname{eval}(c) \cap \operatorname{eval}(d)$.

(c) *If* $c \in \mathscr{C}(I, \alpha)$, *then* $\operatorname{comp}(c) \in \mathscr{C}(I, \alpha + 1)$ *and* $\operatorname{eval}(\operatorname{comp}(c)) = 2^I \setminus \operatorname{eval}(c)$. □

1.23. LEMMA. comp, un, *and* int *are absolute for transitive models of* ZFC. □

We remark that I does not play a very great role here. The functions comp, un, and int do not mention I. Furthermore, if $I \subset J$, then $\mathscr{C}(I, \alpha) \subset \mathscr{C}(J, \alpha)$ for each α; and if $c \in \mathscr{C}(I, \alpha)$, then $\operatorname{eval}(c, J)$ is just the inverse projection of $\operatorname{eval}(c, I)$.

Lemma 1.22(c) gives us a trivial absolute way of taking the complement of a Σ^0_α set and getting a $\Sigma^0_{\alpha+1}$ set. One can prove by a diagonal argument that if $1 \leqslant \alpha < \omega_1$ and $|I| \geqslant \omega$, then Σ^0_α is *not* closed under complements, but we do not need this fact here. Another non-trivial fact not relevant for this paper is that equality is absolute. That is, if $c, d \in \mathscr{C}(I)^M$, then

$$(\operatorname{eval}(c) = \operatorname{eval}(d)) \leftrightarrow (\operatorname{eval}(c) = \operatorname{eval}(d))^M .$$

To prove this, we must note that the statement "$\operatorname{eval}(c) = \operatorname{eval}(d)$" can be effectively transcribed into a Π^1_1 statement of number theory, which, in turn, by a well-known argument of Kleene can be effectively transcribed into the statement that a certain total order is a well-order, which is absolute for transitive models.

A similar argument would show that the meagre and null ideas are absolute, but we shall instead give a more elementary proof. Using recursion theory, one can say why the absoluteness of these ideals is *in principle* more elementary than the absoluteness of equality. The set of all (Gödel numbers of) primitive recursive c such that $\operatorname{eval}(c) = \emptyset$ is complete Π^1_1 set, but the set of such c with $\operatorname{eval}(c)$ null (or meagre) is arithmetical. This forms the basis of the fact that forcing for *arithmetical* sentences is arithmetical; this is due to Feferman for Cohen reals and Sacks for random reals.

1.24. LEMMA. *The null ideal is absolute.*

PROOF. Let M be a transitive model for ZFC. We assume that the rational numbers have been defined so that they and their + and × are the same in every model for ZFC, and that the real numbers, $\mathbb{R}$, have been defined via Dedekind cuts in the rationals, so that $\mathbb{R}^M = \mathbb{R} \cap M$.

Let μ be the usual measure on 2^I. If $\mathscr{I}$ is the null ideal, then $X \in \mathscr{I}(I)$ iff $\mu(X) = 0$. Thus, we may prove absoluteness of $\mathscr{I}$ by proving absoluteness of μ, that is, if $c \in \mathscr{C}(I)^M$, $\mu(\operatorname{eval}(c))^M = \mu(\operatorname{eval}(c))$. This in turn is proved by showing, by induction on β, that $\mu(\operatorname{eval}(c))^M = \mu(\operatorname{eval}(c))$ for all $c \in \mathscr{C}(I, \beta)^M$. For $\beta = 0$, this is true because

$$\mu(\text{eval}(\langle 0, A\rangle)) = \sum_{p\in A} (2^{-|p|}) .$$

(since A is an antichain). Now, assume that $\beta > 0$ and we have proved it for all $\alpha < \beta$. Fix $\langle 1, A\rangle \in \mathscr{C}(I, \beta)^M$. In M, say $A = \{c_n : n \in \omega\}$. Then eval($\langle 1, A\rangle$) is the union of the $2^I\backslash\text{eval}(c_n)$, but its measure is not easily computed from the measures of these sets, since we do not know how they intersect. Instead, let $d_0 = c_0$ and $d_{n+1} = \text{int}(d_n, c_{n+1})$. Then

$$\text{eval}(d_n) = \text{eval}(c_0) \cap \text{eval}(c_1) \cap \cdots \cap \text{eval}(c_n) ,$$

so

$$\text{eval}(\langle 1, A\rangle) = \bigcup_n (2^I\backslash\text{eval}(d_n))$$

and this is an *increasing* union, so

$$\mu(\text{eval}(\langle 1, A\rangle)) = \sup_n (1 - \mu(\text{eval}(d_n))) .$$

Furthermore, each d_n is in $\bigcup_{\alpha<\beta}\mathscr{C}(I, \alpha)$, so $\mu(\text{eval}(d_n)) = \mu(\text{eval}(d_n))^M$. Thus, $\mu(\text{eval}(\langle 1, A\rangle)) = \mu(\text{eval}(\langle 1, A\rangle))^M$. $\square$

1.25. LEMMA. *The meagre ideal is absolute.*

PROOF. This is just like the proof for the null ideal in 1.24, but we need an analog of measure. Let $\mathscr{I}$ be the meagre ideal. If $p \in \text{Fn}(I, 2)$, let $N_p = \{f \in 2^I : p \subset f\}$. If $X \subset 2^I$, let

$$\nu(X) = \{p \in \text{Fn}(I, 2) : (N_p\backslash X) \in \mathscr{I}\} .$$

X has the *property of Baire* iff there is an open U such that $X \Delta U \in \mathscr{I}(I)$; equivalently iff $\nu(X) \cup \nu(2^I\backslash X)$ is dense in $\text{Fn}(I, 2)$, in which case

$$\nu(2^I\backslash X) = \{p : \forall q \in \nu(X)(p \perp q)\} ,$$

where $\perp$ means 'incompatible'.

It is well-known that all Baire sets have the property of Baire. One way to prove this is to prove by induction on c that eval(c) has the property of Baire. At the same time, we may develop a formula for $\nu(\text{eval}(c))$. For $Q \subset \text{Fn}(I, 2)$, let $\text{cp}(Q) = \{p \in \text{Fn}(I, 2) : \forall q \in Q(p \perp q)\}$. Then by induction on β, show that for all $c \in \mathscr{C}(I, \beta)$, eval($c$) has the property of Baire and $\nu(\text{eval}(c))$ is given by

$$\nu(\mathrm{eval}(\langle 0, A\rangle)) = \mathrm{cp}(\mathrm{cp}(A)),$$
$$\nu(\mathrm{eval}(\langle 1, A\rangle)) = \mathrm{cp}(\bigcap_{c\in A} \nu(\mathrm{eval}(c))).$$

Since $\mathrm{eval}(c) \in \mathscr{S}$ iff $\nu(\mathrm{eval}(c)) = \emptyset$, absoluteness of $\mathscr{S}$ is proved exactly as in 1.24; it is somewhat easier here, since we do not have to introduce the d_n. □

To conclude this section, we state one definition which summarizes all the relevant properties of our ideals.

1.26. DEFINITION. $\mathscr{S}$ is a *reasonable ideal* iff

(a) One can prove in ZFC that $\mathscr{S}$ is an index-invariant ccc. Baire supported σ-ideal on 2^ω which contains all singletons and has the Fubini property, and

(b) $\mathscr{S}$ is absolute. □

As remarked after the definition of absoluteness (1.20), reasonableness is really a property defined in the metatheory about the formula used to define $\mathscr{S}$.

In Section 3, we shall discuss using $\mathbb{P}(\mathscr{S}, I)$ to obtain generic extensions and thereby to prove consistency results about $\mathscr{S}$. Index-invariance is used to define $\mathbb{P}(\mathscr{S}, I)$ for arbitrary I and the ccc is used to show cardinals are preserved. The Fubini property will yield an iteration theorem. The assumption that $\mathscr{S}$ (and hence $\mathscr{S}(I)$ for any infinite I) contains all singletons will be used to guarantee that the generic extension is proper. In 1.26(a), we need that all these things are provable in ZFC so that we know they hold in any model M we wish to consider. Absoluteness is used to connect properties of $\mathscr{S}$ in M with those in $M[G]$.

Before entering into consistency proofs, however, we shall, in section 2, discuss some questions one might ask about $\mathscr{S}$ so that we shall, in section 3, have something to prove consistent.

The generality of our approach may be misleading, since we have no concrete examples of reasonable ideals except the null and meagre ideals and trivial variants thereof. What are these variants? One is their intersection. More generally, if $\mathscr{S}$ and $\mathscr{T}$ are dual (Definition 1.15) reasonable ideals, so is $\mathscr{S} \cap \mathscr{T}$. Unfortunately, this is never of much interest, since by duality, $\mathscr{S}$ and $\mathscr{T}$ are disjointly supported, and every property of $\mathscr{S} \cap \mathscr{T}$ is determined in a trivial way from properties of $\mathscr{S}$ and $\mathscr{T}$; also in forcing, every $\mathbb{P}(\mathscr{S} \cap \mathscr{T}, I)$-generic filter is either $\mathbb{P}(\mathscr{S}, I)$-generic or $\mathbb{P}(\mathscr{T}, I)$-generic. For another variant, fix $p \in (0, 1)$ and let $\mathscr{N}_p$ be the null ideal in 2^ω with respect to the product measure in which $\{0\}$ is given measure p and $\{1\}$ is given measure $1-p$; so $\mathscr{N}_{1/2}$ is the usual null ideal. However, this is of no interest in forcing, since $\mathbb{B}(\mathscr{N}_p, I)$ is always isomorphic to $\mathbb{B}(\mathscr{N}_{1/2}, I)$.

Of course, $\mathscr{N}_p$ is not 0–1 invariant unless $p = \frac{1}{2}$. This leads to:

1.27. QUESTION. Let $\mathscr{S}$ be an index-invariant Baire supported ccc 0–1 invariant σ-ideal on 2^ω. Must $\mathscr{S}$ be either the meagre ideal, the null ideal, or the intersection? □

2. Cardinality questions

In the absence of CH, a number of questions are naturally suggested by an attempt to replace 'countable' by '$<\mathfrak{c}$' in well-known properties of the null and meagre sets (where $\mathfrak{c}$ is the cardinal $2^{\aleph_0}$). We consider these questions for general ideals on 2^ω.

2.1. DEFINITION. Let $\mathcal{S}$ be an ideal on 2^ω. Then

(a) ADD($\mathcal{S}$) is the statement

$$\forall \mathcal{X} \subset \mathcal{S} \quad [|\mathcal{X}| < \mathfrak{c} \rightarrow \bigcup \mathcal{X} \in \mathcal{S}] .$$

(b) UNIF($\mathcal{S}$) is the statement

$$\forall X \subset 2^\omega \quad [|X| < \mathfrak{c} \rightarrow X \in \mathcal{S}] .$$

(c) BAIRE($\mathcal{S}$) is the statement

$$\forall \mathcal{X} \subset \mathcal{S} \quad [|\mathcal{X}| < \mathfrak{c} \rightarrow \bigcup \mathcal{X} \neq 2^\omega] . \qquad \square$$

So, ADD($\mathcal{S}$) says that $\mathcal{S}$ is $\mathfrak{c}$-additive. UNIF($\mathcal{S}$) says that $\mathcal{S}$ is uniform, or contains no small sets. If $\mathcal{S}$ is the meagre ideal, BAIRE($\mathcal{S}$) asserts that 2^ω is $\mathfrak{c}$-Baire; i.e., is not the union of $<\mathfrak{c}$ meagre sets.

For any of these three properties about the null or meagre ideals, one might also assert an analogous property about category or measure on the real numbers, $\mathbb{R}$. However, in this regard, $\mathbb{R}$ and 2^ω behave alike. For the proof, first note that the usual dyadic map of 2^ω onto $[0, 1]$ preserves measure and category and is 1–1 except on a countable set; then use the fact that $\mathbb{R}$ is a countable union of intervals.

Now, let $\mathcal{S}$ be any σ-ideal on 2^ω which contains singletons. Under CH, ADD($\mathcal{S}$), UNIF($\mathcal{S}$), and BAIRE($\mathcal{S}$) all hold, so we examine the possibilities of these properties holding or failing under $\neg$CH. Clearly ADD($\mathcal{S}$) implies both UNIF($\mathcal{S}$) and BAIRE($\mathcal{S}$), so only 5 possibilities can occur; namely, ADD($\mathcal{S}$) being true, plus ADD($\mathcal{S}$) being false with the $2 \times 2 = 4$ possibilities for UNIF($\mathcal{S}$) and BAIRE($\mathcal{S}$). If $\mathcal{S}$ is a *reasonable* ideal (Definition 1.26), then $\mathcal{S}$ itself defines a partial order which can be used in forcing to prove the consistency of BAIRE($\mathcal{S}$) and $\neg$UNIF($\mathcal{S}$) (see section 3). In this generality, we do not know about the consistency of the other 4 possibilities. Specifically, we do not know whether MA (Martin's Axiom) implies ADD($\mathcal{S}$), or even UNIF($\mathcal{S}$).

If $\mathcal{S}$ is the null or meagre ideal, then all 5 possibilities are possible. Here, MA does imply ADD($\mathcal{S}$) (see KUNEN [1980] or JECH [1978]), and two of the other 4 possibilities are treated in section 3. But, a new complication arises. We may consider the null and meagre ideals together, yielding $5 \times 5 = 25$ possibilities. However, 11 of these are excluded by the following theorem of Rothberger.

2.2. THEOREM (Rothberger). *Let $\mathcal{S}$ and $\mathcal{J}$ be dual* 0–1 *invariant ideals on* 2^ω. *Then* BAIRE($\mathcal{S}$) *implies* UNIF($\mathcal{J}$).

PROOF. Suppose $\neg$UNIF($\mathcal{J}$). Fix $X \subset 2^\omega$ with $|X| < \mathfrak{c}$ and $X \notin \mathcal{J}$. Also, fix $Y \subset 2^\omega$, as per the definition (1.15) of dual, with $Y \in \mathcal{S}$ and $(2^\omega \setminus Y) \in \mathcal{J}$. Let

$$X + Y = \{x + y : x \in X \text{ and } y \in Y\}.$$

First, note that $X + Y = 2^\omega$, since if $f \notin X + Y$, then $Y \cap (X + f) = \emptyset$ (since addition is mod 2), or $X + f \subset (2^\omega \setminus Y) \in \mathcal{J}$ while $X \notin \mathcal{J}$, contradicting 0–1 invariance of $\mathcal{J}$. Thus,

$$2^\omega = \bigcup\{Y + x : x \in X\},$$

which, by 0–1 invariance of $\mathcal{S}$, implies $\neg$BAIRE($\mathcal{S}$). □

Returning to measure and category, there are still 14 combinations left. Most, but not all, are known to be consistent. However, MILLER [19··] has shown that ADD for the null ideal plus BAIRE for the meagre ideal implies ADD for the meagre ideal, which eliminates one possibility. For more details, see MILLER [1981] or BARTOSZYNSKI [19··].

3. Forcing

We now consider the generic extensions obtained by the partial order $\mathbb{P}(\mathcal{S}, I)$ defined in section 1. It is our intent that $\mathcal{S}$ be a reasonable ideal (Definition 1.26), but some elementary results go through under weaker hypotheses.

As usual in forcing, ctm abbreviates countable transitive model. If M is a ctm for ZFC and G is $\mathbb{P}(\mathcal{S}, I)$-generic over M, we shall see (in 3.2) that G defines a function, $F = \text{ind}(G)$ in 2^I. Our first step (though 3.6) will be to translate results about G into results about F. This will simplify further discussion of the generic extensions, since an element of 2^I is a simpler object than a filter on Baire subsets of 2^I. For this translation, we do not need any absoluteness for $\mathcal{S}$, and we may think of $\mathcal{S}$ as defined entirely within M.

3.1. LEMMA. *Let M be a* ctm *for* ZFC. *Suppose that in M: $\mathcal{S}$ is an index-invariant σ-ideal and $\mathbb{P} = \mathbb{P}(\mathcal{S}, I)$. Let G be $\mathbb{P}$-generic over M. Then*

(1) *If $(B, C \in \mathcal{B}(2^I)$ and $B \subset C)^M$, then $B \in G$ implies $C \in G$.*

(2) *If $(B, C \in \mathcal{B}(2^I)$ and $B = 2^I \setminus C)^M$, then $B \in G$ iff $C \notin G$.*

(3) *If $(\forall n \in \omega (B_n \in \mathcal{B}(2^I))$ and $B = \bigcap_n B_n)^M$, then $\forall n (B_n \in G)$ implies $\bigcap_n B_n \in G$.*

PROOF. (1) is immediate, since the order on $\mathbb{P}$ is inclusion. Likewise in (2), we cannot have $B \in G$ and $C \in G$, since they cannot have a common extension if they are disjoint. Also, since they are complementary,

$$\{X \in \mathscr{B}(2^I): X \notin \mathscr{S} \text{ and } (X \subset B \text{ or } X \subset C)\}^M$$

is dense in $\mathbb{P}$, so one of B, C is in G. Likewise in (3), use the fact (in M) that for σ-ideals,

$$\{X \in \mathbb{P} : X \subset \bigcap_n B_n \text{ or } \exists n (X \cap B_n = \emptyset)\}^M$$

is dense in $\mathbb{P}$. □

3.2. DEFINITION. With the notation of 3.1, define $\operatorname{ind}(G) \in 2^I$ by

$$\operatorname{ind}(G)(i) = y \quad \text{iff} \quad \{f \in 2^I : f(i) = y\}^M \in G. \qquad \square$$

That the function $\operatorname{ind}(G)$ is indeed well-defined follows from Lemma 3.1(2). The next lemma says, informally, that whenever B is a Baire set in M, $\operatorname{ind}(G) \in B$ iff $B \in G$. In particular, if $B \in \mathscr{S}$, then $\operatorname{ind}(G) \notin \mathscr{S}$. To state this precisely, we must use codes.

3.3. LEMMA. *With the notation of* 3.1 *and* 3.2, *if* $c \in \mathscr{C}(I)^M$, *then*
(a) $\operatorname{ind}(G) \in \operatorname{eval}(c)$ *iff* $\operatorname{eval}(c)^M \in G$.
(b) *If* $\operatorname{eval}(c)^M \in \mathscr{S}$, *then* $\operatorname{ind}(G) \notin \operatorname{eval}(c)$.

PROOF. (a) is by induction on c, using 3.1. (b) follows from (a). □

Thus, if $F = \operatorname{ind}(G)$,

$$G = \{\operatorname{eval}(c)^M : c \in \mathscr{C}(I)^M \text{ and } F \in \operatorname{eval}(c)\},$$

so G is determined uniquely from F. It follows that we may express all basic forcing notions in terms of F rather than G.

3.4. DEFINITION. Let M be a ctm for ZFC. Suppose that in M, $\mathscr{S}$ is an index-invariant σ-ideal. Let F be in 2^I. Then F is *$\mathscr{S}$-generic over M* iff there is a (unique by 3.3(a)) $\mathbb{P}(\mathscr{S}, I)^M$-generic (over M) G such that $F = \operatorname{ind}(G)$. In that case, write $M[F]$ for $M[G]$, and, if τ is a name in the forcing language, use τ_F or $\operatorname{val}(\tau, F)$ instead of τ_G for the value of τ in $M[G]$. □

Anthough F is simpler than G, we do not really achieve any simplification until we express the notion of generic (3.5) and the notion of forcing (3.6) directly in terms of F without mentioning G at all. For this we require that $\mathscr{S}$ has the ccc.

3.5. LEMMA. *Let M be a* ctm *for* ZFC. *Suppose that in M, $\mathcal{S}$ is an index-invariant* ccc *σ-ideal. Suppose $I \in M$. Let F be in 2^I. Then F is $\mathcal{S}$-generic over M iff*

$$\forall c \in \mathscr{C}(I)^M[(\mathrm{eval}(c) \in \mathcal{S})^M \to (F \notin \mathrm{eval}(c))]\,. \tag{$*$}$$

PROOF. The implication from left to right is just 3.3(b), so assume ($*$), and we shall try to show that F is $\mathcal{S}$-generic over M. Let

$$G = \{\mathrm{eval}(c)^M : c \in \mathscr{C}(I)^M \text{ and } F \in \mathrm{eval}(c)\}\,.$$

We shall show that G is $\mathbb{P}(\mathcal{S}, I)^M$-generic over M.

We first note that if $c \in \mathscr{C}(I)^M$,

$$\mathrm{eval}(c)^M \in G \quad \text{iff} \quad F \in \mathrm{eval}(c)\,. \tag{$+$}$$

Equivalently, if $d \in \mathscr{C}(I)^M$ and $\mathrm{eval}(c)^M = \mathrm{eval}(d)^M$, then $F \in \mathrm{eval}(c)$ iff $F \in \mathrm{eval}(d)$. Recall that there is an $e \in \mathscr{C}(I)^M$ which codes $\mathrm{eval}(c) \backslash \mathrm{eval}(d)$ both in M and in V (see 1.22; $e = \mathrm{int}(c, \langle 1, \{d\}\rangle)$). Thus, if $\mathrm{eval}(c)^M = \mathrm{eval}(d)^M$, then $\mathrm{eval}(e)^M = \emptyset \in \mathcal{S}$, so $F \notin \mathrm{eval}(e)$; thus $F \in \mathrm{eval}(c)$ implies $F \in \mathrm{eval}(d)$. Interchanging c and d we get $F \in \mathrm{eval}(c)$ iff $F \in \mathrm{eval}(d)$. Actually, as remarked in section 1, $\mathrm{eval}(c) = \mathrm{eval}(d)$; the reader who believes that could have skipped this paragraph.

Next, note that by ($+$) and ($*$), $G \subset \mathbb{P} = \mathbb{P}(\mathcal{S}, I)^M$; that is, if $\mathrm{eval}(c)^M \in \mathcal{S}$, then $\mathrm{eval}(c)^M \notin G$.

Next, G is upward-closed; if $\mathrm{eval}(c)^M \in G$ and $\mathrm{eval}(c)^M \subset \mathrm{eval}(d)^M$, then $\mathrm{eval}(d)^M \in G$. This is proved precisely as was ($+$).

To show G is a filter, suppose $\mathrm{eval}(c)^M \in G$ and $\mathrm{eval}(d)^M \in G$. Recall that there is an $\mathrm{int}(c, d)$ which codes $\mathrm{eval}(c) \cap \mathrm{eval}(d)$ both in M and in V (see 1.22). Then $\mathrm{eval}(\mathrm{int}(c, d))^M \in G$ is a common extension of $\mathrm{eval}(c)^M$ and $\mathrm{eval}(d)^M$.

To show that G is generic, it is sufficient (see A12, p. 239 of KUNEN [1980]) to show that G intersects every maximal antichain of $\mathbb{P}$ in M. By ccc in M, such an antichain is countable. Thus, it is of the form

$$\{\mathrm{eval}(c_n)^M : n \in \omega\}\,,$$

where the sequence, $\langle c_n : n \in \omega \rangle$ is in M. Let $d = \langle 0, \langle c_n : n \in \omega \rangle \rangle$. Then d codes the complement of $\bigcup_n \mathrm{eval}(c_n)$ both in M and in V. Since the antichain is maximal, $\mathrm{eval}(d)^M \in \mathcal{S}$, so $F \notin \mathrm{eval}(d)$, so for some n, $F \in \mathrm{eval}(c_n)$, whence $\mathrm{eval}(c_n)^M \in G$.

Finally, $F = \mathrm{ind}(G)$ is immediate from ($+$). Thus F is $\mathcal{S}$-generic over M. □

3.6. LEMMA. *Let M be a* ctm *for* ZFC. *Suppose that in M: $\mathcal{S}$ is a* ccc *index-invariant σ-ideal and $\mathbb{P} = \mathbb{P}(\mathcal{S}, I)$. Let $\varphi(x_1 \cdots x_n)$ be a formula and $\tau_1, \ldots, \tau_n$ names in the forcing language. Then*

(a) *For any* $c \in \mathscr{C}(I)^M$, $\operatorname{eval}(c)^M \Vdash \varphi(\tau_1 \cdots \tau_n)$ *iff for all* $F \in \operatorname{eval}(c)$ *which are* $\mathscr{S}$*-generic over* M, $\varphi(\tau_{1_F} \cdots \tau_{n_F})^{M[F]}$.

(b) *For any* $F \in 2^I$ *which is* $\mathscr{S}$*-generic over* M, *if* $\varphi(\tau_{1_F} \cdots \tau_{n_F})^{M[F]}$, *then there is a* $c \in \mathscr{C}(I)^M$ *such that* $F \in \operatorname{eval}(c)$ *and* $\operatorname{eval}(c)^M \Vdash \varphi(\tau_1 \cdots \tau_n)$.

(c) *For any* $c \in \mathscr{C}(I)^M$ *with* $(\operatorname{eval}(c) \notin \mathscr{S})^M$, *there is an* $F \in \operatorname{eval}(c)$ *which is* $\mathscr{S}$*-generic over* M. □

We remark that we have not changed the definition of what *names* are; there are still $\mathbb{P}(\mathscr{S}, I)$-names in the usual sense.

From now on, we shall not attempt to keep track of exactly what properties of $\mathscr{S}$ are used where, but shall simply assume that $\mathscr{S}$ is *reasonable* (Definition 1.26). Absoluteness of $\mathscr{G}$ enables us to drop some of the relativizations to M.

3.7. DEFINITION. $B \in \mathscr{B}(2^I)$ is M-coded iff $B = \operatorname{eval}(c)$ for some $c \in \mathscr{C}(I)^M$. □

3.8. LEMMA. *Let* $\mathscr{S}$ *be a reasonable ideal. Let* M *be a* ctm *for* ZFC *with* $I \in M$. *Let* $F \in 2^I$. *Then* F *is* $\mathscr{S}$*-generic over* M *iff* F *is not in any* M*-coded Baire set in* $\mathscr{S}$. □

3.9. DEFINITION. Let M be a ctm for ZFC with $I \in M$. Let $F \in 2^I$. F is *random* over M iff F is null-generic over M. F is *Cohen-generic* over M iff F is meagre-generic over M. □

If $I = \omega$, a random F is sometimes called a 'random real', although sometimes 'random real' means the element of $[0, 1]$ whose dyadic expansion is F. Likewise for Cohen-generic reals.

We continue with our discussion of a general reasonable $\mathscr{S}$, which contains 'random' and 'Cohen-generic' as special cases.

One immediate consequence of absoluteness is that almost every function is generic.

3.10. LEMMA. *Let* $\mathscr{S}$ *be a reasonable ideal, let* M *be a* ctm *for* ZFC, *and let* $I \in M$. *Then*

$$D = \{F \in 2^I : F \text{ is } \mathscr{S}\text{-generic over } M\}$$

is in $\mathscr{B}(2^I)$, *and* $2^I \backslash D \in \mathscr{S}$.

PROOF.

$$D = \bigcap \{\operatorname{eval}(c) : c \in \mathscr{C}(I)^M \text{ and } 2^I \backslash \operatorname{eval}(c) \in \mathscr{S}\},$$

and this is a countable intersection, since M is countable. □

Of course, D is not M-coded.

The fact that $\mathscr{S}$ contains all singletons is used to prove that the generic function is not in the ground model.

3.11. LEMMA. *Let $\mathscr{S}$ be a reasonable ideal, let M be a* ctm *for* ZFC, *and let $I \in M$ with I infinite. If $F \in 2^I$ is $\mathscr{S}$-generic over M, then $F \notin M$.* $\square$

As a preliminary to the iteration theorem, we make the following remark.

3.12. LEMMA. *Let $\mathscr{S}$ be a reasonable ideal. Let M, N be* ctm's *for* ZFC *with $M \subset N$. Let $I \in M$. Suppose $F \in 2^I$ is $\mathscr{S}$-generic over N. Then F is $\mathscr{S}$-generic over M.*

PROOF. By 3.8, since $\mathscr{C}(I)^M \subset \mathscr{C}(I)^N$. $\square$

Regarding Lemma 3.12, observe that if $\mathbb{P}$ is any partial order in M, any filter $\mathbb{P}$-generic over N is $\mathbb{P}$-generic over M also; this is trivial from the definition of 'generic'. However, 3.12 is different and is *not* trivial from the definitions, since $\mathscr{S}$-generic over N is defined (3.4) with respect to $\mathbb{P}(\mathscr{S}, I)^N$, whereas $\mathscr{S}$-generic over M involves $\mathbb{P}(\mathscr{S}, I)^M$. These are not the same partial orders. In the special case of the meagre ideal, they give rise to the same extensions, since they both have the same completion as does $\mathrm{Fn}(I, 2)$ (see Lemma 1.10), but if $\mathscr{S}$ is the null ideal, they give rise to different extensions. That is, in general, adjoining to N a $\mathbb{P}(\mathscr{S}, I)^N$-generic filter need not be the same as adjoining to N a filter which is $\mathbb{P}(\mathscr{S}, I)^M$-generic over N.

The iteration theorem, which will be proved next, will always be used in the following way. We fix an I and force with $\mathbb{P}(\mathscr{S}, I)$ and obtain a generic F. Then, for various J, we wish to consider the extension by $F \upharpoonright J$, followed by the extension by $f \upharpoonright (I \backslash J)$.

3.13. THEOREM. *Let $\mathscr{S}$ be a reasonable ideal, M a* ctm *for* ZFC, *and $F \in 2^I$ $\mathscr{S}$-generic over M. Suppose $J, K \in M$ with $I = J \cup K$ and $J \cap K = \emptyset$. Then*

(1) *$F \upharpoonright J$ is $\mathscr{S}$-generic over M, and*

(2) *if $N = M[F \upharpoonright J]$, then $F \upharpoonright K$ is $\mathscr{S}$-generic over N.*

PROOF. In our notation, we shall identify 2^I with $2^J \times 2^K$ (see 1.12). To prove (1), suppose $c \in \mathscr{C}(J)^M$ and $\mathrm{eval}(c, J) \in \mathscr{S}(J)$. Then also $c \in \mathscr{C}(I)^M$ and

$$\mathrm{eval}(c, I) = \mathrm{eval}(c, J) \times 2^K \in \mathscr{S}(I).$$

Thus, $F \notin \mathrm{eval}(c, I)$ so $F \upharpoonright J \notin \mathrm{eval}(c, J)$.

The same argument will show that $F \upharpoonright K$ is $\mathscr{S}$-generic over M, but we need to prove genericity over N. First, observe that (2) can be expressed in the forcing language as a property of F; that is, if Φ is a $\mathbb{P}(\mathscr{S}, I)$-name for the generic function (i.e. $\Phi_F = F$ for all generic F), then we can write a formula $\psi(\Phi)$ which says that $\Phi \upharpoonright K$ is $\mathscr{S}$-generic over $M[\Phi \upharpoonright J]$ (ψ also mentions names $\check{J}$ and $\check{K}$ for J and K). Then (2) is equivalent to showing that $\mathbb{1} \Vdash \psi(\Phi)$. Suppose this fails; we shall derive a contradiction. Let $c \in \mathscr{C}(I)^M$ with

$$\mathrm{eval}(c)^M \Vdash \neg\psi(\Phi)$$

and $\mathrm{eval}(c) \notin \mathscr{S}(I)$. By Lemma 3.10, let $D_I \in \mathrm{Baire}(2^I)$ with $2^I \backslash D_I \in \mathscr{S}(I)$ and each $F \in D_I$ $\mathbb{P}(\mathscr{S}, I)$-generic over M; also, let $D_J \in \mathrm{Baire}(2^J)$ with $2^J \backslash D_J \in \mathscr{S}(J)$ and each $f \in D_J$ $\mathscr{S}$-generic over M. Since $D_I \cap \mathrm{eval}(c) \neq \mathscr{S}(I)$, the Fubini property (see 1.13) implies that

$$\{f \in 2^J : (D_I \cap \mathrm{eval}(c))_f \notin \mathscr{S}(K)\} \notin \mathscr{S}(J),$$

so fix $f \in D_J$ with $(D_I \cap \mathrm{eval}(c))_f \notin \mathscr{S}(K)$. By Lemma 3.10 applied to $M[f]$, fix a $g \in (D_I \cap \mathrm{eval}(c))_f$ with g $\mathscr{S}$-generic over $M[f]$. Let $H = f \cup g \in 2^I$. Then $H \upharpoonright K = g$ is $\mathscr{S}$-generic over $M[H \upharpoonright J]$, so $\psi(H)$ is true in $M[H]$, but H is $\mathscr{S}$-generic over M and $H \in \mathrm{eval}(c)$, which contradicts (see 3.6(a)) that $\mathrm{eval(c)}^M$ forces $\neg\psi(\Phi)$. □

Although Theorem 3.13 resembles the product theorem for iterated forcing (see KUNEN [1980] VIII, Theorem 1.4 or JECH [1978] Lemma 20.1), it is not the same. The product theorem says that if (the filter defined by) F is $\mathbb{P}(\mathscr{S}, J)^M \times \mathbb{P}(\mathscr{S}, K)^M$-generic over M, then $F \upharpoonright K$ is $\mathbb{P}(\mathscr{S}, K)^M$-generic over N, whereas Theorem 3.13 says that if F is $\mathbb{P}(\mathscr{S}, I)^M$-generic over M, then $F \upharpoonright K$ is $\mathbb{P}(\mathscr{S}, K)^N$-generic over N. These theorems have different hypotheses and different conclusions, since $\mathbb{P}(\mathscr{S}, K)^M$ is not the same as $\mathbb{P}(\mathscr{S}, K)^N$ and $\mathbb{P}(S, J)^M \times \mathbb{P}(\mathscr{S}, J)^M$ is not the same (identifying $2^J \times 2^K$ with 2^I) as $\mathbb{P}(\mathscr{S}, I^M)$. See also the remarks after 3.12.

The iteration theorem can be used to compute the size of the continuum in $M[F]$.

3.14. LEMMA. *Let $\mathscr{S}$ be a reasonable ideal. Let M be a* ctm *for* ZFC, *and assume that in M, κ is a cardinal with $\kappa^\omega = \kappa$. Let $F \in 2^\kappa$ be $\mathscr{S}$-generic over M. Then $2^\omega = \kappa$ in $M[F]$.*

PROOF. In M, $\kappa^\omega = \kappa$ implies that $\mathbb{P}(\mathscr{S}, \kappa)$ has size κ since there are only $\kappa^\omega = \kappa$ Baire sets. This plus the ccc in M implies that $2^\omega \leqslant \kappa$ in $M[F]$ (see KUNEN [1980] or JECH [1978]). To show that $2^\omega \geqslant \kappa$, we assume instead that $F \in 2^{\kappa \times \omega}$; this is permissible since $\mathbb{P}(\mathscr{S}, \kappa)$ and $\mathbb{P}(\mathscr{S}, \kappa \times \omega)^M$ are isomorphic in M. Define $f_\xi \in 2^\omega$ by $f_\xi(n) = F(\xi, n)$. Then $\langle f_\xi : \xi < \kappa \rangle$ in $M[F]$ is a κ-sequence of elements of 2^ω, so we are done if we show that the f_ξ are distinct. But the iteration theorem (3.13) implies that $F \upharpoonright (\{\xi\} \times \omega)$ is $\mathscr{S}$-generic over $N = M[F \upharpoonright ((\kappa \backslash \{\xi\}) \times \omega)]$ so $f_\xi \notin N$ (see 3.11), whereas if $\eta \neq \xi$, then $f_\eta \in N$, so $f_\xi \neq f_\eta$. □

Now that we know what $\mathfrak{c}$ is in $M[F]$, we can decide the properties ADD, UNIF, and BAIRE (see Definition 2.1). Actually, one fact decides all three of them.

3.15. DEFINITION. An $\mathcal{S}$-*Luzin* set is an $X \subset 2^\omega$ such that $|X| = \mathfrak{c}$ and for all $Y \in \mathcal{S}$, $|X \cap Y| \leq \omega$. $\square$

3.16. LEMMA. *If there is an $\mathcal{S}$-Luzin set and $\mathfrak{c} \neq \omega_1$, then* BAIRE($\mathcal{S}$), $\neg$UNIF($\mathcal{S}$), *and* $\neg$ADD($\mathcal{S}$).

PROOF. Let X be an $\mathcal{S}$-Luzin set. If $Y \subset X$ and $|Y| = \omega_1$, then $Y \notin \mathcal{S}$, so $\neg$UNIF($\mathcal{S}$) and hence $\neg$ADD($\mathcal{S}$). If $\mathcal{Y} \subset \mathcal{S}$ and $|\mathcal{Y}| < \mathfrak{c}$, then

$$|X \cap \bigcup \mathcal{Y}| \leq \max(\omega, |\mathcal{Y}|) < \mathfrak{c}$$

so $\bigcup \mathcal{Y} \neq 2^\omega$. $\square$

We now wish to show that $M[F]$ has an $\mathcal{S}$-Luzin set. As a preliminary, we note that every code for a Baire set in $M[F]$ is constructed from a countable piece of F.

3.17. LEMMA. *Let $\mathcal{S}$ be a reasonable ideal. Let M be a* ctm *for* ZFC, *with $I \in M$. Let $F \in 2^I$ be $\mathcal{S}$-generic over M, with $c \in \mathcal{C}(\omega)^{M[F]}$. Then there is a $K \subset I$ with $K \in M$ and K countable in M such that $c \in M[F \restriction K]$.*

PROOF. The key fact is that c may be coded by a subset of ω; that is, there is an $a \subset \omega$ in $M[F]$ such that c lies in every transitive model of ZFC containing a. (*Proof.* Use induction on c, within $M[F]$.) Fix such an a.

Since $a \subset \omega$, $a = \tau_F$, where $\tau \in M^{\mathbb{P}}$ is a term in the forcing language of the form

$$\tau = \bigcup \{\{\check{n}\} \times A_n : n \in \omega\},$$

where each A_n is an antichain in $\mathbb{P} = \mathbb{P}(\mathcal{S}, I)$ (see p. 208 of KUNEN [1980]). In M, $\mathbb{P}$ has ccc, so each A_n is countable. Each $p \in \bigcup_n A_n$ is a Baire subset of 2^I, and we may let K be any countable subset of I such that each such p is a cylinder over the coordinates K (i.e., $p = (\Delta^*)^{-1}(\Delta^*)''p$, where $\Delta : K \to I$ is inclusion). $\square$

3.18. THEOREM. *Let $\mathcal{S}$ be a reasonable ideal. Let M be a* ctm *for* ZFC. *Assume that in M, $\kappa^\omega = \kappa$. Let $F \in 2^\kappa$ be a $\mathcal{S}$-generic over M. Then in $M[F]$ there is an $\mathcal{S}$-Luzin set.*

PROOF. We follow the notation of the proof of 3.14. In particular we assume that $F \in 2^{\kappa \times \omega}$. Let $X = \{f_\xi : \xi < \kappa\}$. We have already seen that in $M[F]$, $|X| = \kappa = \mathfrak{c}$.

Thus to show that X is $\mathcal{S}$-Luzin in $M[F]$, we fix a $c \in \mathscr{C}(\omega)^{M[F]}$ and show that in $M[F]$, if $\text{eval}(c) \in \mathcal{S}$, then $|X \cap \text{eval}(c)| \leq \omega$.

By 3.17, fix $K \subset \kappa$ with $K \in M$, K countable in M, and $c \in M[F \restriction (K \times \omega)]$. Assume $\text{eval}(c) \in \mathcal{S}$. If $\xi \notin K$, then f_ξ is $\mathcal{S}$-generic over $M[F \restriction (K \times \omega)]$. (*Proof.* By 3.13, $F \restriction (\{\xi\} \times \omega)$ is generic over $M[F \restriction (\kappa \backslash \{\xi\} \times \omega)]$ and hence over the smaller $M[F \restriction (K \times \omega)]$ (see 3.12). Using the isomorphism of $\mathbb{P}(\mathcal{S}, \omega)$ with $\mathbb{P}(\mathcal{S}, \{\xi\} \times \omega)$, f_ξ is generic also.) Thus, $f_\xi \notin \text{eval}(c)$.

Thus, $X \cap \text{eval}(c) \subset \{f_\xi \colon \xi \in K\}$, which is countable in M. $\square$

Putting these facts together, we have decided the properties of both measure and category in both the random and Cohen extensions. These all follow from the fact that the null and meagre ideals are dual (Definition 1.15).

3.19. THEOREM. *Let $\mathcal{S}$ and $\mathcal{J}$ be dual,* 0–1 *invariant, reasonable ideals. Let M be a* ctm *for* ZFC, *and assume that in M, κ is a cardinal with $\kappa^\omega = \kappa$ and $\kappa > \omega_1$. Let $F \in 2^\kappa$ be $\mathcal{S}$-generic over M. Then in $M[F]$, we have* BAIRE($\mathcal{S}$), $\neg$UNIF($\mathcal{S}$), $\neg$BAIRE($\mathcal{J}$), *and* UNIF($\mathcal{J}$).

PROOF. BAIRE($\mathcal{S}$) and $\neg$UNIF($\mathcal{S}$) follow from 3.16 and 3.18. Then UNIF($\mathcal{J}$) and $\neg$BAIRE($\mathcal{J}$) follow by Rothberger's Theorem (2.2). $\square$

This settles the cardinality questions for $M[F]$. A related question is what happens to the set $2^\omega \cap M$ in $M[F]$. We shall show that in the random real extension, $2^\omega \cap M$ has outer measure 1 and inner measure 0, and is of first category; dual results apply for the Cohen extension. The general statement of these results is that if we adjoin an $\mathcal{S}$-generic F to M, then in $M[F]$, $2^\omega \cap M$ is not in $\mathcal{S}$, but is in $\mathcal{J}$ for any $\mathcal{J}$ dual to $\mathcal{S}$. The second of these results is easier, so we prove it first.

3.20. THEOREM. *Let $\mathcal{S}$ and $\mathcal{J}$ be dual* 0–1 *invariant reasonable ideals. Let M be a* ctm *for* ZFC, *and let $I \in M$ be infinite. Let $F \in 2^I$ be $\mathcal{S}$-generic over M. Then in $M[F]$, $2^\omega \cap M$ is in $\mathcal{J}$.*

PROOF. Without loss of generality we may assume $\omega \subset I$. Then, since $F \restriction \omega$ is also $\mathcal{S}$-generic over M and $M[F \restriction \omega] \subset M[F]$, we may assume that $\omega = I$. We now produce a Baire set B in $M[F]$ with $B \in \mathcal{J}$ and $2^\omega \cap M \subset B$; this is just like the proof of Rothberger's Theorem (2.2).

In M: By duality, there is a Baire $A \subset 2^\omega$ with $A \in \mathcal{J}$ and $2^\omega \backslash A \in \mathcal{S}$. Say $A = \text{eval}(c)$ with $c \in \mathscr{C}(I)$.

Now, in $M[F]$: Let $B = \text{eval}(c) + F$. By absoluteness and 0–1 invariance of $\mathcal{J}$, $B \in \mathcal{J}$, so we need only show that $2^\omega \cap M \subset B$. Fix $h \in 2^\omega \cap M$. Then $h \in \text{eval}(c) + F$ iff $F \in \text{eval}(c) + h$ iff $F \notin 2^\omega \backslash (\text{eval}(c) + h)$. But $2^\omega \backslash (\text{eval}(c) + h)$ is an

M-coded Baire set and is in $\mathscr{S}$ (by absoluteness and 0–1 invariance of $\mathscr{S}$), and hence does not contain F since F is $\mathscr{S}$-generic. $\square$

We now wish to show that in $M[F]$, $2^\omega \cap M \notin \mathscr{S}$. Of course, $2^\omega \cap M$ is not usually a Baire set in $M[F]$, but recall that by definition, reasonable ideals are Baire supported, so to prove $2^\omega \cap M \notin \mathscr{S}$, it is sufficient to show that there is no Baire $B \in \mathscr{S}$ with $2^\omega \cap M \subset B$; equivalently (taking complements), if B is Baire and $2^\omega \backslash B \in \mathscr{S}$, then $M \cap B \neq \emptyset$. We in fact shall prove a stronger result (the analog to having outer measure 1); namely, if B is Baire and $B \notin \mathscr{S}$, then $M \cap B \neq \emptyset$. To prove this result about all Baire B in $M[F]$, we must first establish a representation in M for such B. Roughly, if $B \subset 2^K$ is an $M[F]$-coded Baire set, where $F \in 2^J$ is $\mathscr{S}$-generic over M, then there is an M-coded Baire set $A \subset 2^J \times 2^K$ and $A_F = B$ ($A_F = \{x : \langle F, x\rangle \in A\}$). The precise statement of this involves codes; also, regarding A, we identify $2^J \times 2^K$ with $2^{J \cup K}$ since we have not discussed codes for subsets of $2^J \times 2^K$; this identification requires $J \cap K = \emptyset$ (see 1.12).

3.21. LEMMA. *Let $\mathscr{S}$ be a reasonable ideal. Let M be a* ctm *for* ZFC, *and suppose in M that $I = J \cup K$ with $J \cap K = \emptyset$. Suppose that τ is a $\mathbb{P}(\mathscr{S}, J)$-name in M and suppose*

$$\mathbb{1} \Vdash \tau \in \mathscr{C}(K)\,,$$

where $\Vdash$ refers to forcing with $\mathbb{P}(\mathscr{S}, J)$. Let Φ be a $\mathbb{P}(\mathscr{S}, J)$-name for the generic function. Then in M there is a $c \in \mathscr{C}(I)$ with

$$\mathbb{1} \Vdash \mathrm{eval}(\tau, K) = \mathrm{eval}(c, I)_\Phi\,.$$

PROOF. Since $\mathbb{P}(\mathscr{S}, J)$ has the ccc and the $\mathscr{C}(K, \alpha)$ are increasing, there is in M and $\alpha < \omega$, such that

$$\mathbb{1} \Vdash \tau \in \mathscr{C}(K, \alpha)\,.$$

We proceed by induction on α. To aid in the induction, we make two preliminary remarks.

Remark 1. Suppose that in M, there are τ_n for $n < \delta$ (where $\delta \leq \omega$) and a maximal antichain $\{\mathrm{eval}(d_n) : n < \delta\}$ in $\mathbb{P}(\mathscr{S}, J)$ such that for each n:

$$\mathrm{eval}(d_n) \Vdash \tau = \tau_n$$

and the lemma holds for τ_n in the sense that there is a c_n such that

$$\mathbb{1} \Vdash \mathrm{eval}(\tau_n, K) = \mathrm{eval}(c_n, I)_\Phi\,.$$

Then the lemma holds for τ as well. To prove this, fix $c \in \mathscr{C}(I)$ such that in M or in any larger model,

$$\operatorname{eval}(c, I) = \bigcup_n (\operatorname{eval}(d_n, I) \cap \operatorname{eval}(c_n, I))$$

(obtain c as in 1.21, 1.22). Then, whenever $F \in 2^J$ is $\mathscr{S}$-generic over M, F is in $\operatorname{eval}(d_n, J)$ for precisely one n, and for this n,

$$\operatorname{eval}(\operatorname{val}(\tau, F), K) = \operatorname{eval}(\operatorname{val}(\tau_n, F), K) = \operatorname{eval}(c_n, I)_F = \operatorname{eval}(c, I)_F$$

(we are already using subscripts for sections, so we are writing out 'val' for the value of a term in a generic extension). F is arbitrary, so

$$\mathbb{1} \Vdash \operatorname{eval}(\tau, K) = \operatorname{eval}(c, I)_\Phi .$$

Remark 2. Suppose that in M, there are τ_n for $n \leqslant \omega$ with

$$\mathbb{1} \Vdash \tau = \langle 1, \{\tau_n : n \in \omega\}\rangle ,$$

and suppose that the lemma holds for each τ_n. Then the lemma holds for τ as well. To prove this, let c_n be as above, and let $c = \langle 1, \{c_n : n \in \omega\}\rangle$.

Now, to proceed with the induction, we consider first $\alpha = 0$. Then, since $\mathscr{C}(K, 0) \subset M$, there is a maximal antichain $\{\operatorname{eval}(d_n) : n < \delta\}$ in $P(\mathscr{S}, J)$ and $e_n \in \mathscr{C}(K, 0)$ such that for each n,

$$\operatorname{eval}(d_n) \Vdash \tau = \check{e}_n .$$

We may then apply Remark 1 with $\tau_n = \check{e}_n$; the lemma holds for τ_n, since we may take $c_n = e_n$.

From now on, suppose $\alpha > 0$ and that we have proved the result for all $\alpha' < \alpha$. Fix τ with

$$\mathbb{1} \Vdash \tau \in \mathscr{C}(K, \alpha) .$$

First, consider the special case where

$$\mathbb{1} \Vdash \tau \in \bigcup_{\beta<\alpha} \mathscr{C}(K, \beta) .$$

Then there is a maximal antichain, $\{\operatorname{eval}(d_n) : n < \delta\}$ in $\mathbb{P}(\mathscr{S}, J)$, and $\beta_n < \alpha$ and τ_n $(n < \delta)$ such that

$$\mathbb{1} \Vdash \tau_n \in \mathscr{C}(K, \beta_n) \quad \text{and} \quad \operatorname{eval}(d_n) \Vdash \tau = \tau_n .$$

Since $\beta_n < \alpha$, the lemma holds for each τ_n, and hence for τ also (by Remark 1).

Next, consider the special case where

$$\mathbb{1} \Vdash \tau \notin \bigcup_{\beta<\alpha} \mathscr{C}(K, \beta).$$

Then there are τ_n for $n \leqslant \omega$ with

$$\mathbb{1} \Vdash \tau = \langle 1, \{\tau_n : n \in \omega\}\rangle$$

and, for each n,

$$\mathbb{1} \Vdash \tau_n \in \bigcup_{\beta<\alpha} \mathscr{C}(K, \beta).$$

By what we have just done, the lemma holds for each τ_n, and hence for τ by Remark 2.

Finally, in the most general case, there is a maximal antichain, $\{\text{eval}(d_n) : n < \delta\}$, and τ_n $(n < \delta)$ such that for each n,

$$\text{eval}(d_n) \Vdash \tau = \tau_n,$$

and either

$$\mathbb{1} \Vdash \tau_n \in \bigcup_{\beta<\alpha} \mathscr{C}(K, \beta)$$

or

$$\mathbb{1} \Vdash \tau_n \in \mathscr{C}(K, \alpha) \setminus \bigcup_{\beta<\alpha} \mathscr{C}(K, \beta).$$

By what we have just done, the lemma holds for each τ_n, and hence, by Remark 1, for τ. □

3.22. THEOREM. *Let $\mathscr{S}$ be a reasonable ideal. Let M be a* ctm *for* ZFC, *and let $J \in M$. Let $F \in 2^J$ be $\mathscr{S}$-generic over M. Then in $M[F]$, if $B \in \mathscr{B}(2^\omega)$ and $B \notin \mathscr{S}$, then $M \cap B \neq \emptyset$.*

PROOF. Translating into forcing, it is sufficient to show the following: Assume in M that τ is a $\mathbb{P}(\mathscr{S}, J)$-name and

$$\mathbb{1} \Vdash \tau \in \mathscr{C}(\omega) \quad \text{and} \quad \text{eval}(\tau) \cap M = \emptyset. \tag{1}$$

Then

$$\mathbb{1} \Vdash \text{eval}(\tau) \in \mathscr{S}.$$

Without loss of generality we may assume $J \cap \omega = \emptyset$. Let $I = J \cup \omega$. Assuming (1), 3.21 now applies (with $K = \omega$), so fix $c \in M \cap \mathscr{C}(I)$ with

$$\mathbb{1} \Vdash \text{eval}(\tau, \omega) = \text{eval}(c, I)_\Phi. \tag{2}$$

Fix $g \in 2^\omega \cap M$. By (1) and (2), there is no $F \in 2^J$ which is $\mathbb{P}(\mathcal{S}, J)$-generic over M such that $\langle g, F\rangle$ (i.e. $g \cup F$) is in eval(c). Since, in V, almost every F is generic, $\text{eval}(c)^g \in \mathcal{S}$ holds in V, and hence in M by absoluteness.

Thus, in M, $\text{eval}(c)^g \in \mathcal{S}$ for every g, so, by the Fubini property, $\text{eval}(c) \in \mathcal{S}$. By the Fubini property again (in V)

$$\{F\colon \text{eval}(c)_F \notin \mathcal{S}\} \in \mathcal{S}. \tag{3}$$

If it were false that $\mathbb{1} \Vdash \text{eval}(\tau) \in \mathcal{S}$, there would be a $d \in M \cap \mathscr{C}(J)$ such that $\text{eval}(d) \notin \mathcal{S}$ and $\text{eval}(d) \Vdash \text{eval}(\tau) \notin \mathcal{S}$. Thus, for any generic $F \in \text{eval}(d)$, $\text{eval}(c)_F \notin \mathcal{S}$, contradicting (3). □

4. Conclusion

This paper has emphasized the facts in common between the random and Cohen extensions, but there are additional facts which can be proved specifically for each one. For example, in the random extension, all functions in ω^ω are dominated by functions in the ground model (see H23 on p. 250 of KUNEN [1980]), whereas this is false for the Cohen model.

One could push this subject somewhat further. One could develop an '$\mathcal{S}$ measure theory', or one might discuss implications between variants of the notion of 'reasonable'. For example, reasonable ideals satisfy a stronger Fubini property: If $I = J \cup K$, $J \cap K = \emptyset$, and $B \in \mathscr{B}(2^I)$, then $\{f \in 2^J : B_f \in \mathcal{S}\}$ is in fact $\mathcal{S}$-measurable (i.e., it equals a Baire set modulo a set in $\mathcal{S}$).

However, we have no examples of reasonable ideals other than the null and meagre ideals and their intersection. The abstract treatment in this paper seemed justified since it gave us a way of developing the basic properties of random and Cohen reals simultaneously, with no extra work. Any further study of reasonable ideals should probably be deferred until other interesting examples of such ideals are found.

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CHAPTER 21

Applications of the Proper Forcing Axiom

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Contents

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HANDBOOK OF SET-THEORETIC TOPOLOGY
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0. Introduction

In recent years, set-theoretic topology has come to look as much like set theory as like topology. Many of the old open problems have been shown to be consistent with and independent of the axioms of set theory, and anyone who plans to work in the field nowadays must be prepared to encounter consistency questions.

As every young set-theoretic topologist knows, the litmus test of consistency is provided by Martin's Axiom (MA) and $\diamond$. Given any proposition about sets of relatively low rank, the reals, say, or ω_1, if the proposition is undecidable in ZFC then the chances are good that MA and $\diamond$ will decide it in different ways.

But what happens when the litmus test fails? In the case of $\diamond$, there are considerably stronger principles available. One can try $\diamond^*$, or $\diamond^+$, or if all else fails, a morass. In the case of MA, however, until recently the only alternative was to strike off into the wilds of forcing on one's own.

The purpose of this article is to present a much more powerful version of MA, the Proper Forcing Axiom (PFA), which seems to be very successful at settling problems left open by MA. PFA is obtained from MA(ω_1) by replacing the countable chain condition by properness, a notion invented and studied extensively by Shelah. In addition to the ccc partial orderings, the proper partial orderings include the countably closed and Axiom A orderings, and many others as well. Furthermore, like the ccc orderings the proper orderings are closed under forcing composition. Thus PFA applies to many more orderings than MA, and this is one source of its power. Perhaps surprisingly, another source of power is the fact that large cardinals are needed in order to obtain the consistency of PFA. This means that PFA can be applied to problems that MA has no hope of solving, simply because of questions of consistency strength.

The reader is presumed to have a solid background in set theory and a good understanding of forcing. An understanding of forcing is more important for using PFA than for using MA, since many applications make use of iterations of length two or three. All the requisite background material may be found in JECH [1978] or KUNEN [1980].

Some specific material on stationary sets and elementary substructures is needed. It may be found in Section 1.

Proper forcing and PFA are presented in Sections 2 and 3, and Sections 4–8 (and part of Section 3) are devoted to applications of PFA. The application sections are more or less independent of one another, and the reader who is on shaky ground with forcing may not want to read them straight through. The easiest sections are probably 4 and 7, and the most difficult is 5. Section 9 is mostly a brief discussion of consistency strength.

Inevitably we have had to make some omissions. While we have tried to present some of the most important applications of PFA, we have concentrated on demonstrating the widest possible variety of techniques. Also, since the primary focus of this article is on the applications of PFA, we have omitted the consistency proof of PFA. This argument would require considerable space, and it seems to have nothing to do with the applications of PFA. In any case, it is available in DEVLIN [1978] and SHELAH [1982]. Nor have we discussed versions of PFA compatible with CH. This part of the subject is still changing rapidly, and an extensive treatment is already available in SHELAH [1982].

1. Stationary sets, elementary substructures, and forcing

This section contains a review of certain background material necessary to approach proper forcing.

The definition of proper forcing makes use of a generalization of the notion of stationary set, due to JECH [1973], which is unlikely to be familiar to everyone. Recall that if α is an ordinal and $C \subseteq \alpha$, then c is *unbounded in* α if $\sup C = \alpha$ and C is *closed in* α if $\sup(C \cap \beta) \in C$ for all $\beta < \alpha$. A set $S \subseteq \alpha$ is *stationary in* α if $S \cap C \neq 0$ for all sets C closed and unbounded in α.

The following theorem, known as *Fodor's Theorem*, may be found in any graduate-level set theory text.

1.1. THEOREM. *Suppose κ is a regular uncountable cardinal.*

(a) *If $\langle C_\alpha : \alpha < \kappa \rangle$ is a sequence of sets closed unbounded in κ, then $C = \{\alpha < \kappa : \forall \beta < \alpha\ \alpha \in C_\beta\}$ is also closed unbounded in κ. C is called the* diagonal intersection *of the C_α.*

(b) *Suppose $S \subseteq \kappa$ is stationary and $f: S \to \kappa$ is such that $f(\alpha) < \alpha$ for all $\alpha > 0$. Such an f is called* regressive *on S. Then there exists stationary $S' \subseteq S$ such that f is constant on S'.*

Now if A is a set and λ is a cardinal, let $[A]^\lambda = \{x \subseteq A : |x| = \lambda\}$. Let $[A]^{\leq\lambda} = \bigcup\{[A]^\mu : \mu \leq \lambda\}$ and let $[A]^{<\lambda} = \bigcup\{[A]^\mu : \mu < \lambda\}$. Suppose λ is regular. A set $C \subset [A]^{<}\lambda$ *is unbounded in* $[A]^{<}\lambda$ if $\forall x \in [A]^{<\lambda} \exists y \in C\ x \subseteq y$; C is *closed* in $[A]^{<\lambda}$ if for any increasing union $\langle x_\alpha : \alpha < \mu \rangle$, $\mu < \lambda$, of elements of C we have $\bigcup\{x_\alpha : \alpha < \mu\} \in C$. We call $S \subseteq [A]^{<\lambda}$ *stationary* if $S \cap C \neq 0$ for all sets C closed and unbounded in $[A]^{<\lambda}$.

1.2. THEOREM. *$C \subseteq [A]^{<\lambda}$ is closed iff C is closed under directed unions of cardinality $<\lambda$, i.e., if $D \subseteq C$, $|D| < \lambda$ and $\forall x, y \in D\ \exists z \in D$, $x, y \subseteq z$, then $\bigcup D \in C$.*

PROOF. Any increasing union is directed, so we need only check the implication from left to right. Suppose C is closed and $D \subseteq C$ is as in the theorem. We verify that $\bigcup D \in C$ by induction on $|D|$. If $|D|$ is finite, then D has a maximal element, so this is trivial. If D is countable then, using the directedness of D, it is easy to find an increasing sequence $\langle x_n : n < \omega \rangle$ of elements of D such that $\forall y \in D$

$\exists n\, y \subseteq x_n$. Then $\bigcup\{x_n : n < \omega\} = \bigcup D \in C$. Suppose $\lambda = |D| > \omega$. Then we may find an increasing sequence $\langle D_\alpha : \alpha < \mathrm{cf}\,\lambda\rangle$ of directed subsets of D such that $\bigcup\{D_\alpha : \alpha < \mathrm{cf}\,\lambda\} = D$ and each $|D_\alpha| < \lambda$. Then by inductive hypothesis each $\bigcup D_\alpha \in C$, and since $\langle \bigcup D_\alpha : \alpha < \mathrm{cf}\,\lambda\rangle$ is an increasing sequence, we have $\bigcup D = \bigcup\{\bigcup D_\alpha : \alpha < \mathrm{cf}\,\lambda\} \in C$.

1.3. THEOREM. (JECH [1973]). *Suppose λ is regular and uncountable, and $|A| \geq \lambda$.*

(a) *If $\langle C_\alpha : \alpha \in A\rangle$ is a sequence of sets closed unbounded in $[A]^{<\lambda}$, then $C = \{x \in [A]^{<\lambda} : \forall \alpha \in x\ x \in C_\alpha\}$ is also closed unbounded. C is called the* diagonal intersection *of the C_α.*

(b) *Suppose $S \subseteq [A]^{<\lambda}$ is stationary and $f: S \to A$ is such that $f(x) \in x$ for all $x \neq 0$. Such an f is called* regressive *on S. Then there exists stationary $S' \subseteq S$ such that f is constant on S'.*

PROOF. (a) It is easy to see that C is closed. Let us verify that C is unbounded. Fix $x \in [A]^{<\lambda}$ and construct increasing $\langle x_n : n < \omega\rangle$ by induction on n. Let $x_0 = x$. Given x_n, for each $\alpha \in x_n$ choose $x_n^\alpha \in C_\alpha$ with $x_n \subseteq x_n^\alpha$; this is possible because each C_α is unbounded. Let $x_{n+1} = \bigcup\{x_n^\alpha : \alpha \in x_n\}$. Now for each $\alpha \in y = \bigcup\{x_n : n < \omega\}$, we have $y = \bigcup\{x_n^\alpha : \alpha \in x_n\}$ since whenever $\alpha \in x_n$ it is true that $x_n \subseteq x_n^\alpha \subseteq x_{n+1}$. Since C_α is closed and $\langle x_n^\alpha : \alpha \in x_n\rangle$ is increasing, we have $y \in C_\alpha$. Hence $y \in C$, $y \supseteq x$, and we are done.

(b) This is really a restated revision of (a). Let $f: S \to A$ be regressive, and suppose f is not constant on any stationary subset of S. For each $\alpha \in A$ let C_α be a closed unbounded set disjoint from $\{x \in S : f(x) = \alpha\}$, and let C be the diagonal intersection of the C_α. Choose $x \in C$, $x \neq 0$, and suppose $f(x) = \alpha$. But $\alpha \in x$ so $x \in C_\alpha$ and $f(x) \neq \alpha$, a contradiction.

Although Theorem 1.3 is due to Jech and not to Fodor, we shall inaccurately refer to it also as Fodor's Theorem, since it generalizes Theorem 1.1.

Note that the intersection of fewer than λ closed unbounded subsets of $[A]^{<\lambda}$ is always closed unbounded. This may be proved directly or derived from Theorem 1.3(a).

Note also that if λ is regular and uncountable, then λ itself is a closed unbounded subset of $[\lambda]^{<\lambda}$, and $S \subseteq \lambda$ is stationary in λ iff S is stationary in $[\lambda]^{<\lambda}$. Thus Theorem 1.3 is a generalization of Theorem 1.1.

For the purposes of proper forcing, we are primarily interested in stationary subsets of $[A]^{<\lambda}$ when $\lambda = \omega_1$. In this case, there is a particularly simple and useful representation theorem for closed unbounded sets.

If $f: [A]^{<\omega} \to A$ and $x \subseteq A$, then we say x is *closed under* f if x is nonempty and $\forall y \in [x]^{<\omega}\, f(y) \in x$. Note that $\{x \in [A]^{\leq\omega} : x$ is closed under $f\}$ is always closed unbounded in $[A]^{\leq\omega}$. If $x \subseteq A$, then the *closure of x under f* is the smallest set $y \supseteq x$ which is closed under f.

1.4. THEOREM. (Kueker [1977]). *Suppose C is closed unbounded in $[A]^{\leqslant\omega}$. Then there exists $f\colon [A]^{<\omega} \to A$ such that $\{x \in [A]^{\leqslant\omega}\colon x$ is closed under $f\} \subseteq C$.*

PROOF. First we obtain an auxiliary function $g\colon \{x \in [A]^{<\omega}\colon x \neq 0\} \to C$. We define $g(x)$ by induction on $|x|$ as follows. If $x = \{\alpha\}$, then let $g(x)$ be any element z of C such that $\alpha \in z$. If $|x| > 1$, then let $g(x)$ be any element z of C such that for any proper subset y of x, $g(y) \subseteq z$. For each x, let $h_x\colon \omega \to g(x)$ be a surjection.

Let $p\colon (\omega - \{0, 1\}) \to (\omega - \{0\}) \times \omega$ be a bijection such that if $p(k) = (m, n)$ then $k \geqslant m$. Now we define $f\colon [A]^{<\omega} \to A$. Let $f(0)$ be arbitrary. Let $<_A$ be a well-ordering of A in order-type $|A|$, and let $f(\{\alpha\})$ be the $<_A$ successor of α under this well-ordering. Thus any set closed under f must be infinite and have order type a limit ordinal under $<_A$. Now suppose $x \in [A]^{<\omega}$, $\|x\| \geqslant 2$. Suppose $p(|x|) = (m, n)$. If $x = \{\alpha_1, \ldots, \alpha_k\}$ in increasing order under $<_A$, then let $f(x) = h_y(n)$, where $y = \{\alpha_1, \ldots, \alpha_m\}$. We claim f works.

1.5. LEMMA. *If z is closed under f, then $\forall x \in [z]^{<\omega}$ if $x \neq 0$, then $g(x) \subseteq z$.*

PROOF. We have already observed that z has limit ordinal type under $<_A$. Suppose $x = \{\alpha_1, \ldots, \alpha_m\}$ in increasing order. Let n be arbitrary and let $k = p(m, n)$. Now choose $\alpha_{m+1}, \ldots, \alpha_k$ in z so that $\alpha_m <_A \alpha_{m+1} <_A \cdots <_A \alpha_k$. Then $f(\{\alpha_1, \ldots, \alpha_k\}) = h_x(n) \in z$ and since n was arbitrary we are done.

But now it is easy to see that if z is closed under f, then $\{g(x)\colon x \in [z]^{<\omega}, x \neq 0\}$ is a directed family of elements of C (since $g(x_1), g(x_2) \subseteq g(x_1 \cup x_2)$); hence if $|z| \leqslant \omega$, then $z = \bigcup\{g(x)\colon x \in [z]^{<\omega}, x \neq 0\} \in C$ by Theorem 1.2. This completes the proof.

A result similar to Theorem 1.4 holds for closed unbounded subsets of $[A]^{<\lambda}$, where λ is an arbitrary regular uncountable cardinal. Since we shall not need this result, we leave it to the reader as an exercise.

1.6. THEOREM. *Suppose C is a closed unbounded subset of $[A]^{\leqslant\omega}$ and $B \subseteq A$. Then $\{x \cap B\colon x \in C\}$ contains a closed unbounded subset of $[B]^{\leqslant\omega}$.*

PROOF. By Theorem 1.4, we may assume that $f\colon [A]^{<\omega} \to A$ is such that $C = \{x \in [A]^{\leqslant\omega}\colon x$ is closed under $f\}$. If $x \in [B]^{\leqslant\omega}$, let $\bar{x}$ denote the smallest set closed under f and containing x as a subset. Then clearly $\{x \in [B]^{\leqslant\omega}\colon x = \bar{x} \cap B\}$ is closed unbounded and is a subset of $\{x \cap B\colon x \in C\}$.

Of course Theorem 1.6 also holds for closed unbounded subsets of $[A]^{<\lambda}$ as well, where λ is any regular uncountable cardinal. One proof uses the generalization of Theorem 1.4 alluded to above.

Next we turn to some elementary model theory. It is not strictly necessary to make use of model theory in connection with proper forcing, but it is extremely convenient, and it will greatly simplify many proofs. The reader is referred to CHANG & KEISLER [1977] for more information on the notions presented here.

We shall only be interested in structures of the form $(A, \in)$, where A is a nonempty set, so the only first-order language we shall use is the language of set theory. If B is a nonempty subset of A, then we say that $(B, \in)$ is an *elementary substructure* of $(A, \in)$, and we write $(B, \in) \prec (A, \in)$, provided that for any formula $\varphi(v_1, \ldots, v_n)$ of the language of set theory and for any $b_1, \ldots, b_n \in B$, the assertion $\varphi(b_1, \ldots, b_n)$ is true in $(B, \in)$ iff it is true in $(A, \in)$. For convenience, we often write $B \prec A$ and drop the $\in$, since it is understood to be included.

The following result is known as the Tarski Criterion for elementary substructure.

1.7. THEOREM. *$B \prec A$ iff B is a nonempty subset of A and for any formula $\varphi(v_0, v_1, \ldots, v_n)$ and any $b_1, \ldots, b_n \in B$, if $\exists x\, \varphi(x, b_1, \ldots, b_n)$ is true in $(A, \in)$, then there exists $b_0 \in B$ such that $\varphi(b_0, b_1, \ldots, b_n)$ is true in $(A, \in)$.*

The proof is an easy induction on the way φ is assembled out of simpler formulas.

The advantage of the Tarski Criterion is that it refers semantically only to the larger structure $(A, \in)$. Using Theorem 1.7, one may now derive the following.

1.8. THEOREM. (a) *Suppose $X \subseteq A$, and A is infinite. Then there exists $B \prec A$ such that $X \subseteq B$ and $|B| = |X| + \aleph_0$.*

(b) *If $\langle B_\alpha : \alpha < \lambda \rangle$ is an increasing sequence of elementary substructures of A, then $\bigcup\{B_\alpha : \alpha < \lambda\} \prec A$.*

1.9. COROLLARY. *Let A be an infinite set (usually uncountable). Then $\{N \in [A]^{\leq\omega} : N \prec A\}$ is a closed unbounded subset of $[A]^{\leq\omega}$.*

If λ is a cardinal, then $H(\lambda)$ denotes the collection of all sets hereditarily of cardinality $<\lambda$. We will usually be interested in elementary substructures of $H(\lambda)$, where λ is regular and uncountable. The reason is that in such $(H(\lambda), \in)$ all the axioms of ZFC hold with the possible exception of the power-set axiom, and it holds also if λ is strongly inaccessible. Thus $(H(\lambda), \in)$ is essentially a model of set theory, and the same is true of any elementary substructure.

Now if $B \prec A$ and $\varphi(x)$ is a formula such that there is a unique $a \in A$ satisfying $\varphi(a)$ (in which case we say a is *defined* by $\varphi(x)$), then that a also belongs to B, for the sentence $\exists x\, \varphi(x)$ is true in $(A, \in)$, hence in $(B, \in)$, so there is $b \in B$ such that $\varphi(b)$ holds in $(b, \in)$. But since $B \prec A$, $\varphi(b)$ also holds in $(A, \in)$, so $b = a$. It follows that if λ is uncountable and $N \prec H(\lambda)$, then $\omega \in N$, for example. A similar remark holds for the case in which a is defined by $\varphi(x, b_1, \ldots, b_n)$, where $b_1, \ldots, b_n \in B$; in this case we say a is definable from $b_1, \ldots, b_n$ by the formula $\varphi(x, b_1, \ldots, b_n)$.

1.10. THEOREM. *Suppose λ is regular and uncountable, and $N \prec H(\lambda)$, N countable. Then for any countable set a, if $a \in N$ then $a \subseteq N$. Hence $\omega_1 \cap N \in \omega_1$.*

PROOF. We have just observed that $\omega \in N$, and a similar argument shows that each $n \in \omega$ is definable in $(H(\lambda), \in)$, hence is an element of N. Now if a is countable and $a \in N$, then the assertion "$\exists f f$ maps ω onto a" holds in $(H(\lambda), \in)$, hence in $(N, \in)$, since ω, $a \in N$. Therefore, there is such a function $f \in N$. But since f, $n \in N$ for each $n \in \omega$ we must have $f(n) \in N$ since $f(n)$ is definable from f and n. Hence $a \subseteq N$.

For the last assertion, note that if $\alpha \in \omega_1 \cap N$ then $\alpha \subseteq N$ by what we have just proved. Hence $\omega_1 \cap N$ is an initial segment of ω_1 and, since N is countable, the initial segment must be proper.

Our forcing terminology is fairly standard, but some points may bear explaining. We regard forcing as taking place over the universe V of set theory. Thus, for a given partial ordering P, we may consider sets $G \subseteq P$ which are P-generic over V (or P-generic, for short) and talk about $V[G]$. Now such a set G is simply a convenient fiction, since for nontrivial partial orderings it is provable that such G do not exist. The reader uncomfortable with this approach may substitute for V a countable transitive model of a sufficiently large fragment of ZFC, or may substitute for $V[G]$ the appropriate Boolean-valued universe of set theory. When it is not convenient to designate a specific generic set G, then we use the notation V^P instead of $V[G]$.

In our partial orderings, $p \leq q$ means that p gives *more* information than q.

Forcing with respect to P is denoted by $\Vdash_P$, or simply by $\Vdash$ if it is clear which partial ordering is intended. If φ is a sentence of the language of forcing, then $\Vdash \varphi$ is used as an abbreviation for $\forall p \in P\, p \Vdash \varphi$.

If $Q \in V^P$ is a partial ordering, then $P * Q$ denotes the forcing composition of P and Q, i.e., forcing first by P, then by Q. Formally, if $\dot{Q}$ is a term denoting Q in V^P, then $P * Q$ consists of all pairs $(p, \dot{q})$ such that $p \in P$ and $\Vdash_P \dot{q} \in \dot{Q}$, and $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ iff $p_1 \leq p_2$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_2$.

See KUNEN [1980] or BAUMGARTNER [1978] for a general treatment of iterated forcing.

We generally use elements of V as names for themselves in the language of forcing, and we use letters with dots over the top, as in the previous paragraph, to denote elements of the generic extension which may not lie in V.

In general, we have made no assumptions about what a term of the language of forcing literally is. There are several different definitions in common use, and this leaves the reader free to choose any of them. Occasionally this presents some difficulties, such as when it is necessary to decide whether a term belongs to $H(\lambda)$ for some regular uncountable λ. In these cases, we take the expedient of 'identifying' the term with some simple set. For example, if $\Vdash_P \dot{A} \subseteq \omega_1$, then we might 'identify' $\dot{A}$ with $\{(p, \alpha)$: $p \in P$, $\alpha \in \omega_1$ and $p \Vdash \alpha \in \dot{A}\}$. Dealing with the latter set is clearly equivalent to dealing with the term itself, and it will be easy to decide whether the set belongs to $H(\lambda)$ for various λ.

2. How to recognize a proper partial ordering

A partial ordering P is said to be *proper* if for any uncountable set X and any stationary set $S \subseteq [X]^{\leq\omega}$, S remains stationary in $[X]^{\leq\omega}$ in V^P. Thus P is proper if it never destroys stationary sets. While this is a simple definition, it is not always easy to verify in practice. There are two other equivalent characterizations which are usually easier to check.

If λ is a regular uncountable cardinal and N is a countable elementary substructure of $H(\lambda)$ with $P \in N$, then we say that $p \in P$ is (P, N)-*generic* if $\forall D \in N$ if D is a maximal incompatible subset of P, then $D \cap N$ is predense below p (i.e., $\forall q \leq p\ \exists r \in D \cap N\ q, r$ are compatible).

Consider the following game $\Gamma(P)$ for two players, I and II. Player I begins (at the 0th move) by choosing $p_0 \in P$ and a maximal antichain A_0 in P. Player II responds by choosing a countable subset B_0^0 of A_0. Next player I chooses another maximal antichain A_1 and II responds by choosing countable $B_0^1 \subseteq A_0$, $B_1^1 \subseteq A_1$. In general, at the nth move ($n > 0$) I chooses a maximal antichain A_n and II chooses countable $B_i^n \subseteq A_i$ for all $i \leq n$. This goes on for ω moves, at which point II wins the game iff $\exists q \leq p_0\ \forall i$: if $B_i = \bigcup\{B_i^n : i \leq n\}$, then B_i is predense below q.

The reader unfamiliar with proper forcing may find the proof of Theorem 2.1 rough going. It is recommended, therefore, that on a first perusal this proof be skipped.

2.1. THEOREM. *The following are equivalent*:

(a) *P is proper.*

(b) *For every regular λ, if $\lambda > |\{D \subseteq P : D$ is a maximal antichain$\}|$, then $\{N < H(\lambda)$: N countable, $P \in N$ and $\forall p \in P \cap N\ \exists q \leq p\ q$ is (P, N)-generic$\}$ contains a closed unbounded subset of $[H(\lambda)]^{\leq\omega}$.*

(c) *Player* II *has a winning strategy in the game $\Gamma(P)$.*

PROOF. (a) *implies* (b). Let $C = \{N < H(\lambda): N$ countable, $P \in N\}$. Then C is clearly closed unbounded. Fix $p \in P$ and let $S_p = \{N \in C: p \in N$ and there is no $q \leq p$ which is (P, N)-generic$\}$. It will suffice to show that each S_p is non-stationary since then by Theorem 1.3 $S = \{N \in C: \exists p \in N\ N \in S_p\}$ is non-stationary and the set in part (b) of the theorem is precisely $C - S$.

Now let G_p be P-generic over V with $p \in G_p$. Working in $V[G_p]$, define a function f as follows. If D is a maximal antichain in P and $D \in V$, then let $f(D)$ be the unique q such that $G_p \cap D = \{q\}$. Now let $C_p = \{N < H(\lambda)^V: N$ countable, $P \in N$, and $\forall D \in N$ if D is a maximal antichain in P then $f(D) \in N\}$. It is clear that C_p is closed unbounded. We claim that $C_p \cap S_p = 0$. Suppose $N \in S_p$ and $q \leq p$ is arbitrary. Since q cannot be (P, N)-generic, there is maximal incompatible

$D \in N$ with $D \cap N$ not predense below q. Hence $\exists q' \leq q$ q' is incompatible with every element of $D \cap N$. But since q was arbitrary we have just shown that $E = \{q' \leq p: \exists$ maximal incompatible $D \in N$ q' is incompatible with every element of $D \cap N\}$ is dense below p. Hence $G_P \cap E \neq 0$. Say $q' \in G_p \cap E$. But if $N \in C_p$, then $f(D) \in N \cap D$ and we must have q', $f(D)$ compatible since both lie in G_p, and this contradicts the fact that q' is incompatible with every element of $D \cap N$.

Of course, since $C_p \in V[G_p]$ this only shows that S_p is non-stationary in $V[G_p]$. But since P is proper, if S_p were stationary in V it would remain stationary in $V[G_p]$. Hence S_p must be non-stationary in V as well.

(b) *implies* (a). Let X be uncountable, and let $S \subseteq [X]^{\leq\omega}$ be stationary. By Theorem 1.4, it will suffice to show that if $p_0 \Vdash \dot{f}: [X]^{<\omega} \to X$ then $\exists q \leq p_0$ $\exists z \in S$ $q \Vdash z$ is closed under $\dot{f}$. For simplicity, we identify $\dot{f}$ with $\{(p, x, a): p \in P$, $x \in [X]^{<\omega}$ and $p \Vdash \dot{f}(x) = a\}$. Let λ be so large that $\dot{f}$, $X \in H(\lambda)$ and (b) applies. Then by (b) $\{N < H(\lambda)$: N countable, $P, \dot{f}, X, p_0 \in N$ and $\forall p \in N \cap P$ $\exists q \leq p$ q is (P, N)-generic$\}$ must contain a closed unbounded set C. It follows from Theorem 1.6 that $\{N \cap X: N \in C\}$ contains a closed unbounded subset of $[X]^{\leq\omega}$. Hence there is $N \in C$ with $N \cap X \in S$. We assert that if $q \leq p_0$ is (P, N)-generic then $q \Vdash N \cap X$ is closed under $\dot{f}$, which will complete this part of the proof.

Let $x \in [N \cap X]^{<\omega}$. Then $x \in N$ so the dense set $D = \{q \in P$: either q is incompatible with p_0 or else $q \leq p_0$ and $\exists a \in X$ $q \Vdash \dot{f}(x) = a\} \in N$ also. Now in $H(\lambda)$ it is true that D contains a maximal incompatible subset E, so such a set E must lie in N as well. Now $E \cap N$ is predense below q. If $q' \leq q$ and $q' \Vdash \dot{f}(x) = a$, say, then q' must be compatible with some $r \in E \cap N$, so $r \Vdash \dot{f}(x) = a$ also. But a is definable from r and x (and $\dot{f}$) so $a \in N$. Hence $q' \Vdash \dot{f}(x) \in N \cap X$. But since q' was arbitrary we have $q \Vdash \dot{f}(x) \in N \cap X$, and since x was arbitrary we have $q \Vdash N \cap X$ is closed under $\dot{f}$, as desired.

(b) *implies* (c). Let λ be any cardinal satisfying (b) and let C be a closed unbounded set inside the set described in (b). We describe a winning strategy for II in $\Gamma(P)$. Suppose I begins by playing p_0 and A_0. Then II finds $N_0 \in C$ such that $p_0, A_0 \in N_0$, and II responds by playing $B_0^0 = N_0 \cap A_0$. If I then plays A_1, then II finds $N_1 \in C$ with $N_0 \subseteq N_1$, $A_1 \in N_1$ and plays $B_0^1 = N_1 \cap A_0$, $B_1^1 = N_1 \cap A_1$. And so forth.

At the end of the game, if $N = \bigcup\{N_n: n \in \omega\}$, then $N \in C$ since C is closed. Since $p_0 \in N$, there is $q \leq p_0$ which is (P, N)-generic. But now for each i we have $\bigcup\{B_i^n: i \leq n\} = \bigcup\{N_n \cap A_i: i \leq n\} = N \cap A_i$, and this set is predense below q. Thus II wins the game.

(c) *implies* (b). Let σ be a winning strategy for II in $\Gamma(P)$. We may assume that σ is defined on the sequences $(p_0, A_0, A_1, A_2, \ldots, A_n)$ of plays of player I and that as long as II plays $\sigma(p_0, A_0, \ldots, A_n)$ at his nth move, he will win the game.

Now let λ be as in (b) and note that $\sigma \in H(\lambda)$. It will suffice to show that if $N < H(\lambda)$, N countable, $P, \sigma \in N$, then $\forall p \in N \cap P$ $\exists q \leq p$ q is (P, N)-generic, since the set of all such N is clearly closed unbounded. Fix such an N. Let $p_0 \in N \cap P$ and let $\langle A_n: n < \omega\rangle$ enumerate the set of all maximal antichains lying in N. Now consider the game in which I plays successively (p_0, A_0), $A_1, A_2, A_3, \ldots$ Since $\sigma \in N$ and $p_0, A_0, \ldots, A_n \in N$ we always have $\sigma(p_0, A_0, \ldots, A_n) = (B_0^n, \ldots, B_n^n) \in N$ and since each B_i^n is countable we have $B_i^n \subseteq N$ as well. Thus at the end of the game $B_i = \bigcup\{B_i^n: n \geqslant i\} \subseteq A_i \cap N$. But II wins the game so there is $q \leqslant p_0$ with each B_i predense below q, and hence each $A_i \cap N$ is predense below q and q is (P, N)-generic. This completes the proof of Theorem 2.1.

It is not difficult to see that in the definition of (P, N)-generic one may replace the maximal antichains by predense sets, or even dense sets. A similar remark holds true for the game $\Gamma(P)$. Instead of playing maximal antichains A_n at each move, player I could play predense sets or dense sets. Of course, II must respond with countable subsets of the A_i at each stage as before, and the winner of the game is determined as before. Details of these equivalences are left to the reader.

The equivalence between (a) and (b) in Theorem 2.1 is due to Shelah, and the equivalence between (b) and (c) is due to C. GRAY [1982], and later but independently to A. TAYLOR [19 · ·]. Some closely related games were introduced and studied by JECH [19 · ·].

Note that the proof of the equivalence between (b) and (c) shows that it is sufficient for (b) to hold for a *single* λ. Although we will not use this observation here, it often simplifies checking properness of a partial ordering.

An an immediate consequence of the definition of proper, we have the following.

2.2. THEOREM. *If P is proper and Q is proper in V^P, then $P*Q$ is proper.*

PROOF. If $S \subseteq [X]^{\leqslant\omega}$ is stationary in V, then it remains stationary in V^P, hence remains stationary in $(V^P)^Q = V^{P*Q}$.

It follows immediately that finite iterations of proper orderings are proper. In fact, considerably more is true. It turns out that a countable-support iteration of proper orderings remains proper regardless of the length of the iteration. This fact, due to Shelah, plays a crucial role in the proof of the relative consistency of the Proper Forcing Axiom, but since it is virtually never required for applications of PFA we shall not attempt a proof of it here. The interested reader is referred to DEVLIN [1978] or SHELAH [1982].

2.3. THEOREM. (a) *Every* ccc *partial ordering is proper.*

(b) *Every countably closed partial ordering is proper.*

PROOF. If P has the ccc, then player II clearly wins the game $\Gamma(P)$ since at each stage he can choose the entire (countable) antichain A_n played by player I.

(b) Again, consider the game $\Gamma(P)$. If I plays p_0, A_0 then II responds by choosing $p_1 \leqslant p_0$ such that $p_1 \leqslant q_1 \in A_0$ and II plays $B_0^0 = \{q_1\}$. At the next stage, II responds by choosing $p_2 \leqslant p_1$ such that for some q_2, $p_2 \leqslant q_2 \in A_1$, and lets $B_0^1 = 0$, $B_1^1 = \{q_2\}$. And so forth. Since P is countably closed, there will be $q \in P$, $q \leqslant p_n$, all $n \in \omega$. But $B_i = \{q_{i+1}\}$ is always predense below q since $q \leqslant p_{i+1} \leqslant q_{i+1}$.

By Theorem 2.2, every finite iteration in which each factor is either ccc or countably closed must be proper. In practice, this situation occurs in the great majority of applications of the Proper Forcing Axiom, so checking properness is usually not particularly difficult (assuming that checking the ccc is not difficult, which is not always the case!).

In BAUMGARTNER [1978], the notion of Axiom A forcing was introduced. A partial ordering $(P, \leqslant)$ satisfies Axiom A iff there exist partial orderings $\leqslant_n$, $n < \omega$, of P such that

(1) $p \leqslant_0 q$ iff $p \leqslant q$,

(2) if $p \leqslant_{n+1} q$, then $p \leqslant_n q$,

(3) if A is a maximal antichain in P (or a predense set, or a dense set) and $p \in P$, then $\forall n\ \exists q \leqslant_n p$ for some countable $A' \subseteq A$, A' is predense below q, and

(4) if $p_{n+1} \leqslant_n p_n$ for all $n \in \omega$, then there is $q \in P$ such that $q \leqslant p_n$ for all n.

2.4. THEOREM. *If P satisfies Axiom* A, *then P is proper.*

PROOF. A strategy for II in $\Gamma(P)$ is easy to find. If I plays p_0, A_0, then II finds countable $B_0^0 \subseteq A_0$ and $p_1 \leqslant_0 p_0$ such that B_0^0 is predense below p_1 by (3). When I plays A_1, II responds by finding $p_2 \leqslant_1 p_1$ and countable $B_1^1 \subseteq A_1$ predense below p_2, etc. In the end, by (4) we will have $q \leqslant p_n$ for all n; and q will demonstrate that II wins.

In fact, ccc and countably closed partial orderings satisfy Axiom A, and finite iterations of Axiom A orderings satisfy Axiom A.

Most ways of adding real numbers, such as Sacks forcing, Prikry–Silver forcing, Laver forcing and Mathias forcing, satisfy Axiom A. Since it is usually easier to check Axiom A than properness, Theorem 2.4 is often quite useful in practice.

An example is given in the next section showing that the converse of Theorem 2.4 is false.

3. The proper forcing axiom. Examples and counterexamples

Here is the object of our study, the Proper Forcing Axiom:

(PFA) If P is a proper partial ordering and $\langle D_\alpha : \alpha < \omega_1 \rangle$ is a sequence of dense subsets of P, then there exists a filter $G \subseteq P$ such that $G \cap D_\alpha \neq 0$ for all $\alpha < \omega_1$.

Since every ccc partial ordering is proper, PFA implies $\mathrm{MA}(\omega_1)$.

One may wonder whether *any* condition on P is needed in PFA. Does the conclusion of PFA hold for arbitrary partial orderings? The answer is no, and any P which collapses ω_1 yields a counterexample. If $\Vdash \dot{f}$ maps ω onto $\check{\omega}_1$, then, letting $D_\alpha = \{p \in P: \exists n\, p \Vdash \dot{f}(n) = \alpha\}$, it is clear that no filter G in P can meet all the D_α, or else

$$f = \{(n, \alpha): \exists p \in G\, p \Vdash \dot{f}(n) = \alpha\}$$

would really map (a subset of) ω onto ω_1!

Well then, what about a version of PFA which would apply to all P which do not collapse ω_1, but may destroy stationary sets? Once again an easy argument shows that this is impossible, but because the counterexample is a trifle more complicated, we defer it until after the first application of PFA later in this section.

Another natural question is whether, as with MA, there are extensions of PFA to handle ω_2 dense sets at a time. Again the answer is no, and the usual ordering P for collapsing ω_2 onto ω_1 is a counterexample. Here $P = \bigcup\{{}^{\alpha}\omega_2: \alpha < \omega_1\}$, ordered by setting $p \leq q$ iff $p \supseteq q$. If $\dot{f}$ is a term denoting $\bigcup H$, where H is P-generic, then $\Vdash \dot{f}$ maps $\check{\omega}_1$ onto $\check{\omega}_2$. Letting

$$D_\alpha = \{p \in P: \exists \beta\, p \Vdash \dot{f}(\beta) = \alpha\} = \{p \in P: \alpha \in \mathrm{range}(p)\}$$

for $\alpha < \omega_2$, we may see as above that no filter G can meet all the D_α. Furthermore P is countably closed, hence is proper.

A more striking answer to the question in the previous paragraph is afforded by the fact, proved in Section 4, that PFA is incompatible even with MA(ω_2).

All this leads ultimately to the question whether PFA is consistent at all. We have the following.

3.1. THEOREM. *If* ZFC + "*there is a supercompact cardinal*" *is consistent, then so is* ZFC + $2^{\aleph_0} = \aleph_2$ + PFA.

Since the proof of Theorem 3.1 requires some sophistication in the theories of large cardinals and iterated forcing, and since it is irrelevant to the applications of PFA, we shall not give it here. In any case, the proof is available elsewhere. See DEVLIN [1978]. It may help, however, to make some brief remarks about why large cardinals are needed for PFA but not for MA.

The usual consistency proof for Martin's Axiom involves arranging, via a finite-support ccc iteration, that MA applies to every partial ordering of cardinality $<2^{\aleph_0}$. The argument is then concluded by the observation that if P is any ccc partial ordering, $\lambda < 2^{\aleph_0}$, and $\langle D_\alpha: \alpha < \lambda\rangle$ are sets dense in P, then there exists ccc $Q \subseteq P$, $|Q| \leq \lambda$ and each $D_\alpha \cap Q$ is dense in Q.

Now the first part of the argument may be repeated for PFA, and no large cardinals are necessary (but countable-support iteration replaces finite support). Thus we arrive at $2^{\aleph_0} = \aleph_2 +$ PFA for orderings of cardinality $\leqslant \aleph_1$. It is the second part of the argument that breaks down. What is needed is a powerful principle that will ensure that the properness of an arbitrary ordering P is 'reflected' to a subordering of cardinality $\leqslant \aleph_1$. Supercompactness is just such a reflection principle.

In fact, the consistency of PFA (and the apparently stronger principle PFA$^+$; see Section 8) requires large cardinals, as we shall see, and this is the source of some of its power. On the other hand, many consequences of PFA do not require any large cardinals at all, so it will be worthwhile to keep track of the consistency assumptions needed for each consequence. We have collected remarks on consistency strength in Section 9.

Several weaker versions of PFA occur in Shelah's handwritten notes from Fall 1978 (SHELAH [1978]), and it is fair to say that PFA itself occurs there implicitly. Theorem 3.1 for PFA and PFA$^+$ were proved by the author early in 1979.

Now let us look at an application of PFA.

It is a well known fact that in any ccc extension of V, every closed unbounded subset of ω_1 contains a closed unbounded set from V. If CH holds in V, then in particular there is a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ of infinite subsets of ω_1 (namely all countable such sets in V) such that every closed unbounded set in the generic extension contains one of the A_α. To put things topologically, if X is the subspace of ${}^{\omega_1}2$ consisting of all functions f such that $\{\alpha : f(\alpha) = 0\}$ contains a closed unbounded set, then in the extension X is not ω_1-Baire. If MA is made true in the usual way, beginning with a model of CH, then we see that in the extension X is not ω_1-Baire.

If we wish to adjoin a closed unbounded set C which does not contain any of a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ of infinite sets, then one natural way to proceed is to force with *finite* approximations of C as conditions. It will turn out to be easier to adjoin the enumerating function of C than to treat C directly. Accordingly, let P consist of all finite functions p such that for some closed unbounded set $C \subseteq \omega_1$, if $f: \omega_1 \to C$ enumerates C in increasing order, then $p \subseteq f$. Let $p \leqslant q$ if $p \supseteq q$. Of course the part of C lying above max range(p) is irrelevant to determining whether or not $p \in P$.

An ordinal $\alpha < \omega_1$ is *indecomposable* if $\alpha = \omega^\beta$ for some β (this is ordinal exponentiation). If α is indecomposable, then $\forall \gamma < \alpha\ \{\beta : \gamma \leqslant \beta < \alpha\}$ has order type α. Thus we have immediately

3.2. LEMMA. *If $p \in P$, $\alpha >$ max range(p) and α is indecomposable, then $p \cup \{(\alpha, \alpha)\} \in P$.*

PROOF. If p is a subset of the enumerating function of C, then $p \cup \{(\alpha, \alpha)\}$ is a subset of the enumerating function of $(C \cap \gamma) \cup (\omega_1 - \gamma)$, where $\gamma =$ max range(p). Hence $p \cup \{(\alpha, \alpha)\} \in P$.

It is also easy to check that the set of indecomposables is closed unbounded in ω_1.

3.3. THEOREM. *P is proper.*

PROOF. Note that for all $\alpha < \omega_1$ there are only countably many $p \in P$ with $p \subset \alpha \times \alpha$. If A is a maximal antichain in P, then for $\alpha < \omega_1$ let $g(\alpha)$ be the least ordinal $\beta < \omega_1$ such that $\forall p \in P$ if $p \subseteq \alpha \times \alpha$, then $\exists q \in A$ q is compatible with p and $q \subseteq \beta \times \beta$. Let $C(A) = \{\beta: \forall \alpha < \beta \; g(\alpha) < \beta\}$. Then $C(A)$ is closed unbounded in ω_1.

We give two proofs that P is proper.

First let us show that player II wins the game $\Gamma(P)$ of Theorem 2.1(c). Suppose player I begins with p_0 and A_0. Player II chooses indecomposable $\alpha_0 \in C(A_0)$ large enough so that $p_0 \subseteq \alpha_0 \times \alpha_0$ and plays $\{p \in A_0: p \subseteq \alpha_0 \times \alpha_0\}$. If player I continues with A_1, then II chooses indecomposable $\alpha_1 \in C(A_0) \cap C(A_1)$, $\alpha_1 > \alpha_0$, and plays

$$B_i^1 = \{p \in A_i: p \subseteq \alpha_1 \times \alpha_1\} \quad \text{for } i = 0, 1 .$$

This continues in the obvious way. We assert that II wins. Let $\alpha = \sup\{\alpha_n: n \in \omega\}$. Then α is indecomposable. Hence by Lemma 3.2 $p' = p_0 \cup \{(\alpha, \alpha)\} \in P$. Also,

$$B_i = \bigcup\{B_i^n: i \leqslant n\} = \{p \in A_i: p \subseteq \alpha \times \alpha\} .$$

Fix i and suppose $q \leqslant p'$. Let $q' = q \cap (\alpha \times \alpha)$. Then $q' \in P$ also, and for some $n \geqslant i$, $q' \subseteq \alpha_n \times \alpha_n$. Since $\alpha_n \in C(A_i)$ there must be $r \in A_i$, $r \subseteq \alpha_n \times \alpha_n$, r compatible with q'. But now r and q are compatible, since if q is a subset of the enumerating function of C_1 and $q' \cup r$ is similarly related to C_2, then $q \cup r$ is a subset of the enumerating function of $(C_2 \cap \gamma) \cup (\alpha - \gamma) \cup (C_1 - \alpha)$, where $\gamma = \max \operatorname{range}(q' \cup r)$.

Another approach, less combinatorial but perhaps more perspicacious in the end, is to use Theorem 2.1(b). Let $\lambda \geqslant \omega_2$ be an arbitrary regular cardinal and let N be a countable elementary substructure of $H(\lambda)$ with $P \in N$ (this will happen automatically since P is definable in $H(\lambda)$). We claim that $\forall p \in P \cap N \; \exists q \leqslant p$, q is (P, N)-generic.

Fix $p \in P \cap N$ and let $q = p \cup \{(\alpha, \alpha)\}$, where $\alpha = \omega_1 \cap N$. Note that if $\beta \in N$ then $\omega^\beta \in N$ also. It follows that $\alpha = \omega^\alpha$, so α is indecomposable and $q \in P$, since $p \subseteq \alpha \times \alpha$. Furthermore, if $A \in N$ is an antichain in P, then $C(A) \in N$ also, since $C(A)$ is definable from A. Now if $\bar{q} \leqslant q$ and $q' = \bar{q} \cap (\alpha \times \alpha)$, then q', being finite, must lie in N. Hence $\exists r \in N \cap A$, r is compatible with q'. But then $r \subseteq \alpha \times \alpha$ and we may see that $\bar{q}$ and r are compatible as in the first proof.

We leave to the reader the easy proof that P does not satisfy Axiom A. This is perhaps the simplest example of a partial ordering which is just barely proper.

3.4. THEOREM. *Assume* PFA. *If* $\langle A_\alpha : \alpha < \omega_1 \rangle$ *is any collection of infinite subsets of* ω_1, *then there is a closed unbounded set* C *such that* $A_\alpha \not\subseteq C$ *for all* $\alpha < \omega_1$.

PROOF. Let P be the ordering for adding a closed unbounded set with finite conditions. It is not difficult to see that for each $\alpha < \omega_1$,

$$D_\alpha = \{p \in P : \exists \gamma \in A_\alpha\, \exists \beta\, p(\beta) < \gamma < p(\beta + 1)\}$$

is dense in P. Note that if $p(\beta) < \gamma < p(\beta + 1)$, then γ cannot be an element of the generic closed unbounded set. The natural thing to do now is to apply PFA, get a filter G meeting all the A_α and let $C = \bigcup\{\text{range}(p) : p \in G\}$. Unfortunately, there is no guarantee that C is closed unbounded!

We must ensure that G meets some more dense sets. Let $E_\alpha = \{p \in P : \alpha \in \text{domain}(p)\}$. Then E_α is dense and if G meets the E_α, then C must at least be unbounded in ω_1 (since $p(\alpha) \geqslant \alpha$ for all $p \in P$). We have to work a little harder to make C closed. For each limit ordinal α, let

$$F_\alpha = \{p \in P : \text{either } \exists \beta\, p(\beta) < \alpha < p(\beta + 1) \text{ or else } \exists \beta\, p(\beta) = \alpha\}\,.$$

Once again, it is not difficult to see that F_α is dense in P, and if G meets all the D_α, E_α and F_α then C will be closed and unbounded as desired.

The reader is invited to use the method above to prove:

3.5. THEOREM. PFA *implies that* X *is* ω_1-*Baire, where* X *is the subspace of* ${}^{\omega_1}2$ *consisting of all functions* f *such that* $\{\alpha : f(\alpha) = 0\}$ *contains a closed unbounded set.*

Theorem 3.5 answers a question of BROVERMAN, GINSBURG, KUNEN and TALL [1978].

A modification of the ordering P will yield the following result, which answers a question of Nyikos (private communication).

3.6. THEOREM. *Assume* PFA. *Suppose that for each* $\alpha < \omega_1$ *a set* $S_\alpha \subseteq \omega_1$ *is given such that, for every limit ordinal* $\beta < \omega_1$, $S_\alpha \cap \beta$ *has order type* $<\beta$. *Then there is a closed unbounded set* C *such that* $\forall \alpha < \omega_1\; C \cap S_\alpha$ *is finite.*

PROOF. Let P' consist of all $p \in P$ for which there is a closed unbounded set C containing only limit ordinals so that p is a subset of the enumerating function of C. Let Q be the set of all pairs (p, x), where $p \in P'$ and $x \in [\omega_1]^{<\omega}$. Let $(p_1, x_1) \leqslant (p_2, x_2)$ iff $p_1 \leqslant p_2$, $x_2 \subseteq x_1$ and $\forall \alpha \in x_2$ range $(p_1 - p_2) \cap S_\alpha = 0$. Now force with Q. Details are left to the reader.

REMARKS. (1) Theorem 3.6 may be used when there is a fixed $\beta < \omega_1$ such that each S_α has order type $<\beta$. Simply replace S_α by $S_\alpha - \beta$, apply Theorem 3.6, and consider the closed unbounded set $C - \beta$.

(2) Theorem 3.6 may also be used when we only know that S_α has order type $<\alpha$, provided that we ask only that $C \cap S_\alpha$ be bounded below α. Once again, replace S_α by S_α – order type (S_α) and apply Theorem 3.6.

Still another modification of the partial ordering P will show why we cannot expect PFA to apply to arbitrary cardinal-preserving orderings (which may destroy stationary sets).

The following combinatorial lemma is fairly well known.

3.7. LEMMA. *If $S \subseteq \omega_1$ is stationary, then for any $\alpha < \omega_1$ there is a closed set $x \subseteq S$ of order type α.*

PROOF. We prove by induction on α that $\forall \gamma < \omega_1 \, \exists x \subseteq S - \gamma \; x$ is closed and has order type $\alpha + 1$. This is clear for $\alpha = 0$ and for successor α since S must be unbounded. Suppose α is limit. Let

$$C = \{\beta \in \omega_1 : \forall \gamma < \beta \; \forall \delta < \alpha \; \exists x \subseteq (S \cap \beta) - \gamma \; x \text{ is closed and has order-type } \delta + 1\}.$$

By inductive hypothesis C is closed unbounded, so there is $\alpha \in C \cap S$. Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence with limit α. By induction on n, determine $x_n \subseteq S \cap \alpha$ so that x_n is closed with order type $\alpha_n + 1$ and $\max(x_n) < \min(x_{n+1})$. The x_n can be found since $\alpha \in C$. Now let $x = \{\alpha\} \cup \bigcup \{x_n : n \in \omega\}$. Then $x \subseteq S$ and x is closed with order type $\alpha + 1$. Moreover we could have arranged $\gamma < \min(x)$ for any fixed γ, so the proof is complete.

We remarked earlier that if p is a subset of the enumerating function of C, then the part of C above max range(p) is irrelevant. We could as well have said $p \in P$ if P is finite and there is a *countable* closed set c such that $p \subseteq f$, where f enumerates c. Now suppose $S \subseteq \omega_1$ is stationary. Let $P_S = \{p$: for some countable closed $c \subseteq S$, if f enumerates c then $p \subseteq f\}$. By Lemma 3.7 we see that forcing with P_S will adjoin a closed unbounded set inside S. If S was chosen so that $\omega_1 - S$ was also stationary, then the stationariness of $\omega_1 - S$ will be destroyed and P_S cannot be proper.

Furthermore PFA cannot apply to both P_S and $P_{\omega_1 - S}$ since otherwise closed unbounded sets would be adjoined inside both S and its complement (recall that this requires only ω_1 dense sets) and these closed unbounded sets would be disjoint, so ω_1 would be collapsed!

The counterexample will be complete if we can show that P_S preserves cardinals. Since $|P_S| = \omega_1$, all cardinals $\geqslant \omega_2$ will be preserved, so we need only check the following.

3.8. THEOREM. *If S is stationary, then P_S does not collapse ω_1.*

PROOF. Suppose $\Vdash \dot{f}: \omega \to \omega_1$ and $p_0 \in P_S$ is arbitrary. For each $n \in \omega$, let $D_n = \{p \in P_S: \exists \alpha\, p \Vdash \dot{f}(n) = \alpha\}$. Now let $\langle N_\xi: \xi < \omega_1\rangle$ be a sequence of countable elementary substructures of $H(\omega_2)$ such that p_0, S, P_S, $\langle D_n: n \in \omega\rangle \in N_0$ and $\forall \xi$ $\omega_1 \cap N_\xi < \omega_1 \cap N_{\xi+1}$ and if ξ is limit, then $N_\xi = \bigcup\{N_\eta: \eta < \xi\}$. It follows that $C = \{N_\xi \cap \omega_1: \xi < \omega_1\}$ is closed unbounded, so there is ξ such that $\alpha = N_\xi \cap \omega_1 \in S$.

Let $q = p_0 \cap \{(\alpha, \alpha)\}$. We claim $q \in P_S$. First choose $x_0 \in N_\xi$ so that p is a subset of the enumerating function of x_0, x_0 is closed, and $x_0 \subseteq S$. Since Lemma 3.7 holds in $H(\omega_2)$, hence in N_ξ, we may find sets $x_n \in N_\xi (n \geqslant 0)$, as in the conclusion of the proof of Lemma 3.7, which are such that q is a subset of the enumerating function of $\{\alpha\} \cup \bigcup\{x_n: n \in \omega\}$. (Note that the order type of $\bigcup\{x_n: n \in \omega\}$ is $\Sigma\{\alpha_n + 1: n \in \omega\} = \sup\{\alpha_n: n \in \omega\} = \alpha$ since α is indecomposable and the α_n are chosen with $\sup \alpha$.) Hence $q \in P_S$. Furthermore we may now argue that q is (P_S, N_ξ)-generic, as in the second proof of Theorem 3.3. But now it follows that each $D_n \cap N_\xi$ is predense below q, so

$$\forall n \quad q \Vdash \dot{f}(n) \in \{\beta: \exists p \in D_n \cap N_\xi\; p \Vdash \dot{f}(n) = \beta\}.$$

Hence if $\bar{\beta} = \sup\{\beta: \exists n\; \exists p \in D_n \cap N_\xi\, p \Vdash \dot{f}(n) = \beta\}$, then we have $q \Vdash \dot{f}: \omega \to \bar{\beta} < \omega_1$ and we are done.

Finally, let us make some brief remarks about an example, due to Shelah, which shows why in the definition of properness more than the preservation of stationary subsets of ω_1 is required. Theorem 3.1, the relative consistency result for PFA, is proved by iterating proper partial orderings with countable support. As we remarked in Section 2, such an iteration always remains proper, but the crucial thing is that it does not collapse ω_1. Rather surprisingly, it turns out that iterations of orderings preserving the stationary subsets of ω_1 may in fact collapse ω_1.

If $f, g: \omega_1 \to \omega_1$, let us say $f \lessdot g$ iff $\exists \alpha\, \forall \beta > \alpha\; f(\beta) < g(\beta)$. We say g *eventually dominates* f. It is clear that $\lessdot$ is a well-founded relation, and that for any set A of functions $f: \omega_1 \to \omega_1$, if $|A| \leqslant \omega_1$, then $\exists g\, \forall f \in A\; f \lessdot g$. Hence there is a sequence $\langle f_\alpha: \alpha < \omega_2\rangle$ such that $\alpha < \beta$ implies $f_\alpha \lessdot f_\beta$. Now there is an obvious countably closed (hence proper) partial ordering which will adjoin f_{ω_2} such that $\forall \alpha < \omega_2\, f_\alpha \lessdot f_{\omega_2}$.

Given $\langle f_\alpha: \alpha \leqslant \omega_2\rangle$ as above, there is also an obvious partial ordering which will adjoin a function $g \lessdot f_{\omega_2}$ such that $\forall \alpha < \omega_2\, f_\alpha \lessdot g$. Conditions are pairs (s, x) such that $s \in \bigcup\{{}^\alpha\omega_1: \alpha < \omega_1\}$, $x \in [\omega_2]^\omega$, $\forall \beta \in \text{domain}(s)$ $s(\beta) < f_{\omega_2}(\beta)$, and $\forall \beta \geqslant \text{domain}(s)$ $\sup\{f_\alpha(\beta): \alpha \in x\} < f_{\omega_2}(\beta)$. Let $(s, x) \leqslant (t, y)$ iff $t \subseteq s$ and $y \subseteq x$. We leave to the reader the non-trivial proof that this ordering preserves stationary subsets of ω_1. But iterating this kind of ordering countably many times clearly produces a descending sequence under the relation $\lessdot$, so ω_1 must be collapsed!

4. Gaps and the cardinal-collapsing trick

The applications of PFA that we have seen so far have been, in a sense, much like the standard applications of Martin's Axiom. In each case there is a natural partial ordering for the application which is, if not ccc, at least proper, and therefore PFA can be applied. In many applications of PFA, however, the 'natural' partial ordering lies only in some generic extension of the universe and therefore some preliminary forcing is required. This involves a couple of tricks which do not seem to have analogues in Martin's Axiom arguments.

The first of these tricks will be introduced here, where it will be applied to the study of Hausdorff-style gaps in $P(\omega)$. As a by-product we obtain the perhaps surprising fact that PFA is incompatible with $\mathrm{MA}(\omega_2)$. The section concludes with a proof from $\mathrm{PFA}+2^{\aleph_0}=\aleph_2$ that not every compact space of weight $2^{\aleph_0}$ is a continuous image of $\beta\omega-\omega$.

With the usual set-theoretic operations, $P(\omega)$ is a Boolean algebra. Let Fin denote the ideal of finite sets. Then the quotient algebra $P(\omega)/\mathrm{Fin}$ consists of equivalence classes $[a]$ for $a\subseteq\omega$, and we have $[a]<[b]$ iff $a-b$ is finite and $b-a$ is infinite. Let us extend this notation to write $a<b$ if $[a]<[b]$.

If κ, λ are cardinals then a *(κ,λ^*)-gap* is a pair $(\bar a,\bar b)$ of sequences such that $\bar a=\langle a_\alpha:\alpha<\kappa\rangle$, $\bar b=\langle b_\beta:\beta<\lambda\rangle$, $\forall\alpha,\ \beta<\kappa\ \ a_\alpha<a_\beta$, $\forall\alpha,\ \beta<\lambda\ \ b_\beta<b_\alpha$ and $\forall\alpha<\kappa\ \forall\beta<\lambda\ a_\alpha<b_\beta$. We say a gap $(\bar a,\bar b)$ is *filled* by $a\subseteq\omega$ iff $\forall\alpha<\kappa\ \forall\beta<\lambda$ $a_\alpha<a<b_\beta$.

We will generally be interested only in (κ,λ^*)-gaps where $\kappa\leq\lambda$ and κ, λ are regular. By passing to the complements of the sets involved, anything that can be proved about (κ,λ^*)-gaps will be true of (λ,κ^*)-gaps in dual form, so we assume $\kappa\leq\lambda$. If κ is singular, then there is clearly a $(\mathrm{cf}\,\kappa,\lambda^*)$-gap cofinal in the given (κ,λ^*)-gap in the obvious sense.

Perhaps the best known result about gaps is the following.

4.1. THEOREM. (HAUSDORFF [1936]). *There exists an unfilled (ω_1,ω_1^*)-gap.*

There is a natural candidate for a partial ordering to adjoin a subset of ω which fills a given (κ,λ^*)-gap $(\bar a,\bar b)$. Let $P(\bar a,\bar b)$ consist of all triples $\langle s,x,y\rangle$, where $s\in{}^n2$ for some $n<\omega$, $x\in[\kappa]^{<\omega}$ and $y\in[\lambda]^{<\omega}$. Let $\langle s_1,x_1,y_1\rangle\leq\langle s_2,x_2,y_2\rangle$ iff $s_1\supseteq s_2$, $x_1\supseteq x_2$, $y_1\supseteq y_2$, and $\forall i\in\mathrm{domain}(s_1)-\mathrm{domain}(s_2)$ if $i\in\bigcup\{a_\alpha:\alpha\in x_2\}$ then $s(i)=1$ and if $i\notin\bigcap\{b_\beta:\beta\in y_2\}$ then $s(i)=0$. It is not difficult to see that if G is $P(\bar a,\bar b)$-generic, then $a=\{i:\exists\langle s,x,y\rangle\in G\ s(i)=1\}$ will fill the gap.

The following facts about $P(\bar a,\bar b)$ are due to Kunen in unpublished work.

4.2. THEOREM. *Let $(\bar a,\bar b)$ be a (κ,λ^*)-gap.*

(a) *If the gap is filled, then $P(\bar a,\bar b)$ has the* ccc.

(b) *If* $\mathrm{cf}\,\kappa\neq\omega_1$ *or* $\mathrm{cf}\,\lambda\neq\omega_1$, *then $P(\bar a,\bar b)$ has the* ccc.

(c) *If $\kappa=\lambda=\omega_1$ and the gap is not filled, then there is a* ccc *partial ordering Q which adjoins an uncountable antichain in $P(\bar a,\bar b)$.*

PROOF. (a) Let $\langle p_\alpha:\alpha<\omega_1\rangle$ be a sequence of elements of $P(\bar a,\bar b)$, where $p_\alpha=\langle s_\alpha, x_\alpha,y_\alpha\rangle$. Suppose a fills the gap. For each α fix $k_\alpha\in\omega$ such that

$$\forall\xi\in x_\alpha\quad\forall\eta\in y_\alpha\quad a_\xi-k_\alpha\subset a-k_\alpha\subseteq b_\eta-k_\alpha\,.$$

Without loss of generality, assume that if $s_\alpha \in {}^n 2$ then $n \geq k_\alpha$. Now choose $\alpha, \beta < \omega_1$, $\alpha < \beta$, such that $s_\alpha = s_\beta$ and $k_\alpha = k_\beta$. Then it is easy to check that p_α and p_β are compatible. Note that this argument works even if κ and λ are ordinals and not cardinals.

(b) Suppose, on the contrary, that $A = \langle\langle s_\alpha, x_\alpha, y_\alpha\rangle: \alpha < \omega_1\rangle$ is an antichain in $P(\bar{a}, \bar{B})$. First suppose cf $\kappa > \omega_1$. Then there exists $\xi < \kappa$ such that $x_\alpha \subseteq \xi$ for all $\alpha < \omega_1$. But now a_ξ fills the 'gap' $(\langle \alpha_\eta: \eta < \xi\rangle, \bar{b})$ so by (a) $P(\langle a_\eta: \eta < \xi\rangle, \bar{b})$ has the ccc, which is impossible since A is in fact an antichain in the latter ordering. If cf $\kappa < \omega_1$, then there must exist $\xi < \kappa$ such that $x_\alpha \subseteq \xi$ for uncountably many α, and this is sufficient to obtain a contradiction as above. The case cf $\lambda \neq \omega_1$ is exactly similar.

(c) For each $\alpha < \omega_1$, let $p_\alpha = \langle 0, \{\alpha\}, \{\alpha\}\rangle \in P(\bar{a}, \bar{b})$. Let Q consist of all finite $z \in [\omega_1]^{<\omega}$ such that $\{p_\alpha: \alpha \in z\}$ is an antichain. Of course, $z_1 \leq z_2$ iff $z_2 \subseteq z_1$.

We assert that Q has the ccc. If not, then there is an antichain $\langle z_\alpha: \alpha < \omega_1\rangle$ which, without loss of generality, we may assume forms a Δ-system with $\max(z_\alpha) < \min(z_\beta)$ whenever $\alpha < \beta$. Now for each α there is $k_\alpha \in \omega$ such that $\forall \xi$, $\eta \in z_\alpha$ if $\xi \leq \eta$ then $a_\xi - k_\alpha \subseteq a_\eta - k_\alpha \subseteq b_\eta - k_\alpha \subseteq b_\xi - k_\alpha$. Again without loss of generality, we may assume that $k_\alpha = k$ for all α. Let $c_\alpha = a_\xi - k$ and $d_\alpha = b_\eta - k$, where $\xi = \min(z_\alpha)$ and $\eta = \max(z_\alpha)$. If there exist α, β, $\alpha \neq \beta$, such that $c_\alpha \not\subseteq d_\beta$, then it is straightforward to check that $\{p_\gamma: \gamma \in z_\alpha \cup z_\beta\}$ is an antichain in $P(\bar{a}, \bar{b})$; hence z_α and z_β are compatible in Q, a contradiction. Thus $c_\alpha \subseteq d_\beta$ for all α and β. But now we assert that $a = \bigcup\{c_\alpha: \alpha < \omega_1\}$ fills the gap, a contradiction that will complete the proof. Suppose $\gamma < \omega_1$. Then for some α, $\gamma < \min(z_\alpha)$ so $a_\gamma < c_\alpha \subseteq a$. Also $a = \bigcup\{c_\alpha: \alpha < \omega_1\} \subseteq d_\alpha < b_\gamma$ so we are done.

Finally, we must show that Q can add an uncountable antichain in P. Let A_G be a name for the set $\{p_\alpha: \alpha \in \bigcup G\}$, where G is Q-generic. If $\Vdash_Q A_G$ is countable, then since Q has the ccc, there is $\alpha < \omega_1$ such that $\Vdash_Q A_G \subseteq \{p_\beta: \beta < \alpha\}$, which is impossible, since if $\gamma \geq \alpha$, then $\{\gamma\} \Vdash p_\gamma \in A_G$. Hence there is $z \in Q$ such that $z \Vdash A_G$ is uncountable.

Now we are ready for an application of PFA.

4.3. THEOREM. *Assume* PFA. *If κ and λ are regular, $\kappa \leq \lambda$, and $(\bar{a}, \bar{b})$ is an unfilled (κ, λ^*)-gap, then either $\kappa = \omega$ and $\lambda \geq \omega_2$ or else $\kappa = \lambda = \omega_1$.*

PROOF. First suppose $\kappa = \omega$. by Theorem 4.2(b), $P(\bar{a}, \bar{b})$ has the ccc, and we may fill the gap using only λ dense subsets of $P(\bar{a}, \bar{b})$. If $\lambda \leq \omega_1$ this could be done with MA (ω_1), hence with PFA. Therefore $\lambda \geq \omega_2$.

Now suppose $\omega_1 \leq \kappa \leq \lambda$ and $\lambda > \omega_1$. Let R be the partial ordering for collapsing λ onto ω_1; elements of R are functions in $\bigcup\{{}^\alpha\lambda: \alpha < \omega_1\}$, ordered by extension. Now in V^R, cf κ = cf $\lambda = \omega_1$ (we cannot have cf κ, cf $\lambda = \omega$ since R is countably complete) so in V^R $(\bar{a}, \bar{b})$ has a cofinal (ω_1, ω_1^*)-gap $(\bar{a}_1, \bar{b}_1)$. Also, since R adds no new subsets of ω, $(\bar{a}_1, \bar{b}_1)$ remains unfilled. Hence by Theorem 4.2(c) there is a ccc ordering Q which adds an uncountable antichain in $P(\bar{a}_1, \bar{b}_1)$, hence in $P(\bar{a}, \bar{b})$.

But look: R is proper since it is countably closed, and Q is proper in V^R since it is ccc there. Hence $R*Q$ is proper and adds an uncountable antichain in $P(\bar{a}, \bar{b})$. Moreover, to decide the elements of the antichain requires only ω_1 dense sets. Hence by PFA there is an uncountable antichain lying in V, so $P(\bar{a}, \bar{b})$ does not have the ccc, and this contradicts Theorem 4.2(b) since cf $\lambda > \omega_1$.

The trick mentioned at the beginning of this section is the cardinal-collapsing trick involving the countably closed partial ordering R. As we remarked earlier, in ZFC alone it is true that PFA applies to countably closed orderings. The extra strength comes from the coupling of the countably closed ordering R with the ccc ordering Q in V^R. This kind of argument occurs frequently in applications of PFA.

In unpublished work, Kunen showed that the existence of unfilled $(\omega_1, (2^{\aleph_0})^*)$- and $(2^{\aleph_0}, (2^{\aleph_0})^*)$-gaps was consistent with and independent of MA. Thus in Theorem 4.3 PFA cannot be replaced by MA.

It is conceivable that PFA implies $2^{\aleph_0} = \aleph_2$, although this has not yet been settled. If it should happen that PFA + $2^{\aleph_0} > \aleph_2$ is consistent, then it is natural to hope for as great a generalization of MA as possible, namely that PFA + $2^{\aleph_0} > \aleph_2$ + MA is consistent. Rather surprisingly, this turns out to be impossible.

4.4. THEOREM. PFA *implies* $\neg$MA(ω_2).

PROOF. MA(ω_1) is already enough to show that if $\langle a_\alpha : \alpha < \omega_1 \rangle$ and b are such that $a_\alpha < a_\beta < b$ whenever $\alpha < \beta$, then $\exists c\, \forall \alpha\, a_\alpha < c < b$. This fact may be used repeatedly to construct (under MA(ω_1)) an increasing sequence $\langle a_\alpha : \alpha < \omega_2 \rangle$. Now let $\langle b_\beta : \beta < \lambda \rangle$ be a sequence such that $\langle \bar{a}, \bar{b} \rangle$ is an unfilled gap. Such a construction is possible since one can simply keep adding elements b_β until the gap is no longer filled. We may suppose λ is regular.

If $\lambda \leqslant \omega$, then MA(ω_2) and Theorem 4.2(b) combine to show that the gap is filled. If $\lambda > \omega$, then Theorem 4.3 implies that such an unfilled gap cannot exist. The only conclusion is that PFA and MA(ω_2) are incompatible.

We conclude this section with another application of Theorem 4.3. Let us say that a linearly ordered set $(X, <_X)$ is *embeddable* in $P(\omega)/\mathrm{Fin}$ iff there is a mapping $f: X \to P(\omega)/\mathrm{Fin}$ such that $\forall x, y \in X\; x <_X y$ iff $f(x) < f(y)$.

4.5. THEOREM. *Assume that* PFA *holds or, more generally, only that* $2^{\aleph_1} = 2^{\aleph_0}$ *and the conclusion of Theorem* 4.3 *holds. Then there is a linear ordering of cardinality* $2^{\aleph_0}$ *which is not embeddable in* $P(\omega)/\mathrm{Fin}$.

PROOF. Let κ be the least cardinal such that $2^\kappa > 2^{\aleph_0}$. Then κ is regular, and since $2^{\aleph_1} = 2^{\aleph_0}$, we have $\kappa > \omega_1$. Let $X = \bigcup\{{}^\alpha 2: \alpha < \kappa\}$, ordered lexicographically. Suppose $f \in {}^\kappa 2$ and both $\{\alpha: f(\alpha) = 0\}$ and $\{\alpha: f(\alpha) = 1\}$ are cofinal in κ. Let

$$l(f) = \{s \in X: \exists\alpha \text{ domain}(s) = \alpha + 1, s|\alpha = f|\alpha, s(\alpha) = 0, f(\alpha) = 1\},$$

and let

$$r(f) = \{s \in X: \exists\alpha \text{ domain}(s) = \alpha + 1, s|\alpha = f|\alpha, s(\alpha) = 1, f(\alpha) = 0\}.$$

Then $l(f)$ is well-ordered in type κ and $r(f)$ is conversely well-ordered in type κ^*.

Now suppose by way of contradiction that $h: X \to P(\omega)/\text{Fin}$ is an embedding. Note that if $f, g \in {}^\kappa 2$ are as above, then there cannot be an element a of $P(\omega)/\text{Fin}$ such that

$$\forall s \in l(f) \quad \forall t \in r(f) \quad h(s) < a < h(t),$$

and

$$\forall s \in l(g) \quad \forall t \in r(g) \quad h(s) < a < h(t).$$

Since there are $2^\kappa > 2^{\aleph_0}$ such functions f, it follows that there is one with the property that for no $a \in P(\omega)/\text{Fin}$ is it true that $\forall s \in l(f)\, \forall t \in r(f)\; h(s) < a < h(t)$. It follows that $h''l(f)$ and $h''r(f)$ determine an unfilled (κ, κ^*)-gap, contradicting Theorem 4.3.

REMARK. It is easy to prove that MA implies that all linear orderings of cardinality $<2^{\aleph_0}$ are embeddable in $P(\omega)/\text{Fin}$. LAVER [1979] showed it consistent that all linear orderings of cardinality $2^{\aleph_0}$ are embeddable in $P(\omega)/\text{Fin}$, and WOODIN [19 · ·] extended Laver's result to obtain the consistency with MA of the latter assertion.

It is also not difficult to prove that under MA all Boolean algebras of cardinality $<2^{\aleph_0}$ may be embedded in $P(\omega)/\text{Fin}$ in the obvious sense, and hence that every compact Hausdorff space of weight $<2^{\aleph_0}$ is a continuous image of $\beta\omega - \omega$. On the other hand, if X is a linear ordering not embeddable in $P(\omega)/\text{Fin}$, then the *interval algebra* over X, i.e., the subalgebra of $P(X)$ generated by all half open intervals of the form $\{y \in X: y \leq_X x\}$ for $x \in X$, cannot be embedded in $P(\omega)/\text{Fin}$ either, for it contains a copy of X. Thus we have the following result.

4.6. COROLLARY. *Assume* PFA.

(a) *There is a Boolean algebra of cardinality* $2^{\aleph_0}$ *not embeddable in* $P(\omega)/\text{Fin}$.

(b) *There is a compact Hausdorff space of weight* $2^{\aleph_0}$ *which is not the continuous image of* $\beta\omega - \omega$.

An elaboration of Laver's method shows that the negations of (a) and (b) are relatively consistent with $\neg$CH, but we do not know whether they are relatively consistent with MA + $\neg$CH.

We might also remark that under CH both (a) and (b) are false. See COMFORT & NEGREPONTIS [1974].

5. TOP and the closed-unbounded-set trick

Here we present another trick useful in PFA arguments, and we use it to derive a powerful combinatorial assertion, which we refer to as the Thinning-out Principle:

(TOP) Suppose $A, B \subseteq \omega_1$ are uncountable and $\langle S_\alpha : \alpha \in B\rangle$ is such that $S_\alpha \subseteq \alpha$ for all α. Suppose also that for any uncountable $X \subseteq A$ there exists $\beta < \omega_1$ such that $\{X\} \cup \{S_\alpha : \alpha \in B, \alpha > \beta\}$ has the finite intersection property. Then there exists uncountable $X \subseteq A$ such that $\forall \alpha \in B (X \cap \alpha) - S_\alpha$ is finite. Hence in particular there are uncountable $X \subseteq A$ and $Y \subseteq B$ such that $\forall \alpha \in Y\, X \cap \alpha \subseteq S_\alpha$.

This is a slight strengthening of a proposition introduced in BAUMGARTNER [1981] and called there G_2. Since the proposition has turned out to have numerous applications, it seems desirable to give it a more informative name. For example, $\mathrm{MA}(\omega_1)$ + TOP, which follows from PFA, is already enough to imply the nonexistence of S-spaces and the partition relation $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$, both consistency results originally due to Todorčević. See the article of Roitman in this volume for the S-space proof. After we show that PFA implies TOP, we will present an application for directed sets due to K. DEVLIN and J. STEPRANS [19··].

First, however, let us verify that the final sentence of TOP is a consequence of the rest of it. Suppose $X \subseteq A$ is uncountable and $\forall \alpha \in B$ $(X \cap \alpha) - S_\alpha$ is finite. Let $z_\alpha = (X \cap \alpha) - S_\alpha$. We may find uncountable $Y \subseteq B$ such that $\{z_\alpha : \alpha \in Y\}$ forms a Δ-system with kernel Δ. Now let $Y' \subseteq Y$ be uncountable such that $Y - Y'$ is also uncountable, and let $X' = X - \bigcup\{z_\alpha : \alpha \in Y'\}$. Then X' is uncountable, and $\forall \alpha \in Y'\, X' \cap \alpha \subseteq S_\alpha$, since $(X' \cap \alpha) - S_\alpha \subseteq z_\alpha$, which is disjoint from Y.

5.1. THEOREM. PFA *implies* TOP.

PROOF. Let A, B and $\langle S_\alpha : \alpha \in B\rangle$ be as in the hypothesis of TOP.

By PFA, it will suffice to find a proper partial ordering P such that in V^P the conclusion of TOP is true, for it will require only ω_1 dense sets to determine the identity of each element of X as well as each $(X \cap \alpha) - S_\alpha$ for $\alpha \in B$.

We will obtain P as a three-stage iteration as follows. Let P_1 be the usual ordering for adjoining ω_1 Cohen reals, and let $V_1 = V^{P_1}$. In V_1, let P_2 be $\{(c, C): c$ is a countable closed subset of ω_1 and C is a closed unbounded subset of $\omega_1\}$. Let $(c_1, C_1) \leq (c_2, C_2)$ iff c_1 is an end-extension of c_2, $C_1 \subseteq C_2$, and $c_1 - c_2 \subseteq C_2$. If G_2 is a generic set for P_2 then $\bar{C} = \bigcup\{c : \exists C (c, C) \in G_2\}$ is clearly a closed unbounded set in ω_1, and by an easy denseness argument it is clear that for any closed unbounded set $C \in V_1$, $\bar{C} - C$ is countable. P_2 is usually referred to as the ordering for "shooting a club through all existing clubs." Note that if Z is any countable pairwise compatible subset of P_2, then Z has a greatest lower bound in P_2. Hence, in particular P_2 is countably closed and proper. Let $V_2 = V_1^{P_2}$.

Now we define $P_3 \in V_2$ as follows. Let $\bar{C}$ be the generic closed unbounded set described above, and let P_3 consist of all pairs (x, y) such that $y \in [B]^{<\omega}$, $x \in [A]^{<\omega}$ and $\forall \alpha, \beta \in x$ if $\alpha < \beta$ then $\exists \gamma \in \bar{C}\ \alpha < \gamma < \beta$. Let $(x_1, y_1) \leq (x_2, y_2)$ iff $x_2 \subseteq x_1$, $y_2 \subseteq y_1$ and $\forall \alpha \in y_2\ (x_1 - x_2) \cap \alpha \subseteq S_\alpha$. If G_3 is P_3-generic over V_2 and $X = \bigcup\{x : \exists y (x, y) \in G_3\}$, then X is clearly a cofinal subset of A which satisfies the conclusion of TOP. We will spend the rest of the proof showing that P_3 has the ccc in V_2 and hence $P = P_1 * P_2 * P_3$ is proper, being the composition of proper orderings.

The fact that P_3 has the ccc is the essence of the closed-unbounded-set trick. If the obvious analogue of P_3 were defined back in V, then there is no reason to believe that it would have the ccc, and it might not even be proper. However, the closed unbounded set $\bar{C}$ grows very, very rapidly and therefore the elements of X are spread very, very far apart. As we shall see, this accounts for the fact that P_3 has the ccc. This trick is due to Shelah.

Note, incidentally, that all the elements of P_3 lie in V and that the order relation between any two of them can be determined in V as well. Of course $P_3 \notin V$ since P_3 is defined from $\bar{C}$.

Now we work in V_1. It will suffice to show that if $p_0 \in P_2$ and

$$p_0 \Vdash \dot{A} \text{ is an antichain in } P_3,$$

then there exists $p^* \leq p_0$ such that

$$p^* \Vdash \dot{A} \text{ is countable.}$$

For convenience we will identify $\dot{A}$ with

$$\{(p, x, y) : p \in P_2,\ x, y \in [\omega_1]^{<\omega} \text{ and } p \Vdash (x, y) \in \dot{A}\}.$$

Let λ be a regular cardinal so large that $P_2, \dot{A} \in H(\lambda)$, the collection of all sets hereditarily of cardinality $<\lambda$ ($\lambda = (2^{\omega_1})^+$ will do). Now let $\langle N_\xi : \xi < \omega_1 \rangle$ be a sequence of countable elementary substructures such that

(1) $p_0, P_2, \dot{A}, \langle S_\alpha : \alpha \in B \rangle \in N_0$,
(2) $\forall \xi\ N_\xi \cap \{N_\xi\} \subseteq N_{\xi+1}$,
(3) if ξ is limit, then $N_\xi = \bigcup\{N_\eta : \eta < \xi\}$.

In particular, by (2) and (3), $C^* = \{N_\xi \cap \omega_1 : \xi < \omega_1\}$ is closed unbounded in ω_1.

At this point it would be appropriate to confess that the proof underway is not the shortest possible proof of TOP from PFA. It may be the most informative, however, for it will allow the reader to check that no large cardinals are required for the consistency of TOP. See the discussion in Section 9. The shorter proof is obtained by modifying the present one as follows. All that is required of the filter $\bar{G} \subseteq P_2 \cap N_0$, defined after Lemma 5.3 below, is that $\bar{G} \cap \bar{D} \neq 0$ for all dense sets $\bar{D}$ as in Lemma 5.8 below. Since there are only $\aleph_1$ such sets $\bar{D}$ we may obtain $\bar{G}$ by an application of PFA rather than as a genuinely generic set. This argument renders the Cohen ordering P_1 unnecessary.

Let $\alpha_0 = N_0 \cap \omega_1$. Since N_0 is well-founded under the $\in$-relation and satisfies the axiom of extensionality (since $H(\lambda)$ does), it follows that N_0 is canonically isomorphic to a unique countable transitive set a. Let $j: (N_0, \in) \to (a, \in)$ be the isomorphism.

5.2. LEMMA. *In V_1 there are reals Cohen-generic over $V[a]$.*

PROOF. Let $V_1 = V[\langle r_\beta : \beta < \omega_1 \rangle]$, where the r_β are the Cohen reals adjoined by P_1. Now since a is countable and transitive there must be $\alpha < \omega_1$ such that $a \in V[\langle r_\beta : \beta < \alpha \rangle]$. Of course, if $\gamma \geqslant \alpha$, then r_γ is Cohen-generic over $V[\langle r_\beta : \beta < \alpha \rangle]$, hence over $V[a]$.

5.3. LEMMA. *There is $G \in V_1$ which is $j(P_2)$-generic over $V[a]$.*

PROOF. $P_2 \cap N_0$ is countable, and $\forall p \in P_2 \cap N_0\ \exists q, r \in P_2 \cap N_0\ q, r \leqslant p$ and q and r are incompatible. Hence $P_2 \cap N_0$ and the Cohen ordering have isomorphic dense subsets, so they are equivalent as regards forcing (i.e., a generic subset of one canonically determines a generic subset of the other).

Let G be as in Lemma 5.3, and without loss of generality assume $p_0 \in G$.

Let $\bar{G} = j^{-1}(G)$. Since G is pairwise compatible in $j(P_2)$, $\bar{G}$ is pairwise compatible in $P_2 \cap N_0$, hence in P_2. Therefore $\bar{G}$ has a greatest lower bound (c', C'), say. Let $p^* = (c', C' \cap C^*)$, where $C^* = \{N_\xi \cap \omega_1 : \xi < \omega_1\}$ as above. Note that since $\bar{G} \subseteq N_0$ we must have $\sup(c') \leqslant N_0 \cap \omega_1 = \alpha_0$, since for each $(c, C) \in \bar{G}$ we have $\sup c < \alpha_0$.

Now we will finish the proof by showing $p^* \Vdash \dot{A} \subseteq N_0$.

Before we get started, let us state the following lemma, which will prove useful to the argument.

5.4. LEMMA. *Suppose $\varphi(x_0, \ldots, x_n)$ is a formula of first-order logic, $a_1, \ldots, a_n \in N_\xi$ and $\alpha \in \omega_1$, $\alpha \notin N_\xi$. If*

$$(H(\lambda), \in) \models \varphi(\alpha, a_1, \ldots, a_n),$$

then $\{\beta < \omega_1 : (H(\lambda), \varepsilon) \models \varphi(\beta, a_1, \ldots, a_n)\}$ is uncountable.

PROOF. Suppose not. Let γ be the least upper bound of $\{\beta < \omega_1: (H(\lambda), \in) \models \varphi(\beta, a_1, \dots, a_n)\}$. Then γ is definable in $H(\lambda)$ from $a_1, \dots, a_n$. Since $(N_\xi, \in) < (H(\lambda), \in)$ and $a_1, \dots, a_n \in N_\xi$, γ is also definable in N_ξ from $a_1, \dots, a_n$; hence $\gamma \in N_\xi$. But $\alpha < \gamma$ so this is impossible.

If $p^* \nVdash \dot{A} \subseteq N_0$, then there exist $p^{**} \leqslant p^*$, $\bar{x}$, $\bar{y} \in [\omega_1]^{<\omega}$ such that $p^{**} \Vdash (\bar{x}, \bar{y}) \in \dot{A}$, but $(\bar{x}, \bar{y}) \notin N_0$. Let

$$\bar{x}_0 = \bar{x} \cap N_0\,, \quad \bar{y}_0 = \bar{y} \cap N_0\,, \quad \bar{x}_1 = \bar{x} - \bar{x}_0\,, \quad \bar{y}_1 = \bar{y} - \bar{y}_0\,.$$

Then $(\bar{x}_0, \bar{y}_0) \in N_0$ and $p^{**} \Vdash (\bar{x}_0, \bar{y}_0) \in P_3$. We will find $(x, y) \in N_0$ such that $(x, y) \leqslant (\bar{x}_0, \bar{y}_0)$, (x, y) is compatible with $(\bar{x}, \bar{y})$, and $p^* \Vdash (x, y) \in \dot{A}$. This contradiction will then complete the proof.

For $p \in P_2$, define $D_i(p)$ by induction on $i \in \omega$ as follows. Let $D_0(p) = \{x: \exists q \leqslant p \quad \exists y (\bar{x}_0 \cup x, y) \leqslant (\bar{x}_0, \bar{y}_0) \quad \text{and} \quad q \Vdash (\bar{x}_0 \cup x, y) \in \dot{A}\}$. Let $D_{i+1}(p) = \{x \in [\omega_1]^{<\omega}: \{\alpha: x \cup \{\alpha\} \in D_i(p)\}$ is uncountable$\}$. Note that if $p^{**} \leqslant p$, then $\bar{x}_1 \in D_0(p)$.

5.5. LEMMA. *Let $n = |\bar{x}_1|$. Then $\forall p \in \bar{G}\ 0 \in D_n(p)$.*

PROOF. Say $\bar{x}_1 = \{\beta_1, \dots, \beta_n\}$, where $\beta_1 < \cdots < \beta_n$. We prove by induction on $i \leqslant n$ that $\{\beta_1, \dots, \beta_{n-i}\} \in D_i(p)$. If $p \in \bar{G}$, then $p^{**} \leqslant p^* \leqslant p$, so $\bar{x}_1 \in D_0(p)$ and the case $i = 0$ is finished. Assume the lemma is true for $i < n$. First suppose $i < n - 1$ (the other case is easier). Since $p^{**} \leqslant p^* = (c', C' \cap C^*)$, $p^* \Vdash C - C^* \subseteq c'$, and $\beta_{n-i-1}, \beta_{n-i} \geqslant \alpha_0 \geqslant \sup c'$, there must be some $\gamma \in C^*$ with $\beta_{n-i-1} < \gamma < \beta_{n-i}$ (recall that the elements of $\bar{x}$ must be "spread out" with respect to the generic closed unbounded set adjoined by P_2).

Say $\gamma = N_\xi \cap \omega_1$. Then $\beta_1, \dots, \beta_{n-i-1} \in N_\xi$ but $\beta_{n-i} \notin N_\xi$. Moreover $(H(\lambda), \in) \models \{\beta_1, \dots, \beta_{n-i}\} \in D_i(p)$. Hence, since also $p \in N_0 \subseteq N_\xi$, we have by Lemma 5.4 that $\{\alpha: \{\beta_1, \dots, \beta_{n-i-1}, \alpha\} \in D_i(p)\}$ is uncountable, so $\{\beta_1, \dots, \beta_{n-i-1}\} \in D_{i+1}(p)$, as desired.

If now $i = n - 1$, then a similar argument will work using N_0 in place of N_ξ.

5.6. LEMMA. *$\forall p \in P_2 \cap N_0$, if $0 \in D_n(p)$, then*

$$\exists x \in D_0(p) \cap N_0 \quad x \subseteq \bigcap\{S_\alpha: \alpha \in \bar{y}_1\}\,.$$

PROOF. We prove by reverse induction on $i \leqslant n$ that $\exists x \in D_i(p)$ $x \subseteq \bigcap\{S_\alpha: \alpha \in \bar{y}_1\}$. For $i = n$ this is trivial. Just take $x = 0$. Now suppose the lemma holds for $i + 1$. Choose $x \in D_{i+1}(p) \cap N_0$, $x \subseteq \{S_\alpha: \alpha \in \bar{y}_1\}$, and let $Z = \{\alpha: x \cup \{\alpha\} \in D_i(p)\}$. Then Z is uncountable, and since Z is definable from p, $x \in N_0$ we must have $Z \in N_0$. By the hypothesis of TOP there is an ordinal $\beta < \omega_1$ such that $\{Z\} \cup \{S_\alpha: \alpha \in B, \alpha > \beta\}$ has the finite intersection property. The least such β must be definable from Z and $\langle S_\alpha : \alpha \in B\rangle$, hence must lie in N_0, so $\beta < \alpha_0$. Choose $\gamma \in B$, $\beta < \gamma < \alpha_0$ (note B is cofinal in ω_1 so $B \cap \alpha_0$ is cofinal in α_0 since $N_0 \prec H(\lambda)$), and let $\delta \in Z \cap S_\gamma \cap \bigcap\{S_\alpha : \alpha \in \bar{y}_1\}$. But then $x \cup \{\delta\} \in D_i(p)$, $x \cup \{\delta\} \subseteq \bigcap\{S_\alpha : \alpha \in \bar{y}_1\}$, and since $\delta < \gamma < \alpha_0$ we have $x \cup \{\delta\} \in N_0$. This completes the proof of Lemma 5.6.

Let $\bar{E} = \{p \in P_2 : 0 \in D_n(p)\}$. Then $\bar{E}$ is definable in N_0, hence lies in N_0. Let $E = j(\bar{E})$.

5.7. LEMMA. *$\exists p \in G\, E$ is dense below p in $j(P_2)$.*

PROOF. Let $E' = \{p \in j(P_2)$: either E or $j(P_2) - E$ is dense below $p\}$. Then E' is dense, for if $p \in j(P_2)$ and there is no $q \leqslant p$ with E dense below q, then $j(P_2) - E$ is already dense below p. Hence $\exists p \in E' \cap G$. If $j(P_2) - E$ is dense below p, then $G - E \neq 0$ by genericity of G, contradicting Lemma 5.5. Hence E is dense below p.

Let p be as in Lemma 5.7, and let $\bar{p} = j^{-1}(p)$.

Let $\bar{D} = \{q \in N_0 \cap P_2 : q \leqslant \bar{p}$ and $\exists x \in N_0\, x \subseteq \bigcap\{S_\alpha : \alpha \in \bar{y}_1\}$ and $\exists y \in N_0\, q \Vdash (\bar{x}_0 \cup x, y) \in \dot{A}$ and $(x_0 \cup x, y) \leqslant (\bar{x}_0, \bar{y}_0)\}$.

5.8. LEMMA. *$\bar{D}$ is dense below p in $P_2 \cap N_0$.*

PROOF. Let $p_1 \leqslant \bar{p}$ be arbitrary, $p_1 \in N_0$. By Lemma 5.7 $\exists q_1 \in E\, q_1 \leqslant j(p_1)$. Let $p_2 = j^{-1}(q_1)$. By Lemma 5.6 $\exists x \in D_0(p) \cap N_0\; x \subseteq \bigcap\{S_\alpha : \alpha \in \bar{y}_1\}$. By the definition of $D_0(p)$ there exist $q \leqslant p_2$ and y such that $q \Vdash (\bar{x}_0 \cup x, y) \in \dot{A}$ and $(\bar{x}_0 \cup x, y) \leqslant (\bar{x}_0, \bar{y}_0)$. Moreover the latter existential assertion may be made in $H(\lambda)$ and since all parameters occurring in it belong to N_0 it must be true in N_0 also, i.e., q and y may be found in N_0.

Now we are nearly done. Let $D = j''\bar{D}$. Then D is dense below p in $j(P_2)$, but since $\bar{D}$ need not belong to N_0 (since $\bar{y}_1$ occurs in its definition) we need not have $D \in a$. Nevertheless we do have

5.9. LEMMA. *$D \in V[a]$.*

PROOF. The map $j : N_0 \to a$ is the identity on every transitive subset of N_0, so in particular j is the identity on α_0, and hence on all finite subsets of α_0. It follows that D may be defined as $\{q \in a : q \leqslant p$ and $\exists x \in [\alpha_0]^{<\omega}\, x \subseteq \bigcap\{S_\alpha : \alpha \in \bar{y}_1\}$ and $\exists y \in [\alpha_0]^{<\omega}\; q \Vdash^* (\bar{x}_0 \cup x, y) \in \dot{A}$ and $(\bar{x}_0 \cup x, y) \leqslant (x_0, y_0)\}$, where $q \Vdash^* (\bar{x}_0 \cup x, y) \in \dot{A}$ stands for the expression $(q, \bar{x}_0 \cup x, y) \in j(\dot{A})$: recall that we identify $\dot{A}$ with $\{(p, x, y) : p \Vdash (x, y) \in \dot{A}\}$. Since all parameters in the definition of D lie in $V[a]$ and all quantifiers are bounded (so they mean the same in $V[a]$ as they do in V_1), we have $D \in V[a]$.

But now since G is $j(P_2)$-generic over $V[a]$ we have $G \cap D \neq 0$. Hence $\bar{G} \cap \bar{D}$ is nonempty. But if $q \in \bar{G} \cap \bar{D}$ and x, y are as in the definition of D, then clearly $(\bar{x}_0 \cup x, y)$ and $(\bar{x}, \bar{y})$ are compatible, whereas $p^{**} \Vdash (\bar{x}, \bar{y}) \in \dot{A}$ and $q \Vdash (\bar{x}_0 \cup x, y) \in \dot{A}$, a contradiction since $p^{**} \leqslant q$. This completes the proof that PFA implies TOP.

We conclude now with an application of TOP to the theory of directed sets. A partial ordering $(D, \leqslant_D)$ is *directed* iff for any $d_1, d_2 \in D$ there is $d_3 \in D$ with $d_1, d_2 \leqslant_D d_3$.

The conclusion of Theorem 5.10 was originally derived directly from PFA by K. DEVLIN and J. STEPRANS [19··].

5.10. THEOREM. *Assume* $\mathrm{MA}(\omega_1) + \mathrm{TOP}$, *and let* $(D, \leqslant_D)$ *be a directed set of cardinality* ω_1. *If every uncountable subset of D contains a countable unbounded set, then there exists an uncountable subset of D every infinite subset of which is unbounded.*

PROOF. Without loss of generality, assume $D = \omega_1$. Consider the partial ordering $P = \{(x, d) \colon x \in [D]^{<\omega},\ d \in D\}$, where $(x, d) \leqslant (y, e)$ iff $y \subseteq x$, $e \leqslant_D d$ and $\forall \alpha \in x - y\ \alpha \not\leqslant_D e$. If G is a generic filter over P, then it is easy to see that $X = \bigcup\{x \colon \exists d (x, d) \in G\}$ has the property that every infinite subset is unbounded. Moreover, it is only necessary for G to meet ω_1 dense sets in order for this to be true. Hence we will be done by $\mathrm{MA}(\omega_1)$ if it is the case that P has the ccc. We use TOP to prove that P has the ccc.

Suppose that P has an uncountable antichain $\langle (x_\alpha, d_\alpha) \colon \alpha < \omega_1 \rangle$. Without loss of generality we may assume that $|x_\alpha| = n$ for all α and that n is minimal among all such uncountable antichains. We may also assume that $\{x_\alpha \colon \alpha < \omega_1\}$ forms a Δ-system. Since subtracting off the kernel of the Δ-system from each x_α would give rise to an antichain with smaller x_α's, it must be the case that the kernel is empty and the x_α are disjoint.

Since we are assuming that every uncountable subset of D is in particular unbounded, it follows that for all $d \in D$, $\{e \in D \colon e \leqslant_D d\}$ is countable or finite. Hence, since D is identified with ω_1, for every $\alpha < \omega_1$ there is some $\xi_\alpha < \omega_1$ such that $\{e \in D \colon e \leqslant_D d_\alpha\} \subseteq \xi_\alpha$. Thus we may find $B \subseteq \omega_1$ such that if our antichain is reindexed as $\langle (x_\alpha, d_\alpha) \colon \alpha \in B \rangle$ then $\forall \alpha\ S_\alpha = \{e \in D \colon e \leqslant_D d_\alpha\} \subseteq \alpha$. Finally, by thinning out B if necessary, we may assume that whenever $\alpha, \beta \in B$ and $\alpha < \beta$, then $x_\beta \cap \alpha = 0$; hence in particular $x_\beta \cap S_\alpha = 0$. Therefore, since (x_α, d_α) and (x_β, d_β) are incompatible, we must have $x_\alpha \cap S_\beta \neq 0$.

Choose $a_\alpha \in x_\alpha$ for each α and let $A = \{a_\alpha \colon \alpha \in B\}$.

5.11. LEMMA. *If* $X \subseteq A$ *is uncountable, then there exists* $\beta < \omega_1$ *such that* $\{X\} \cup \{S_\alpha \colon \alpha \in B,\ \alpha > \beta\}$ *has the finite intersection property.*

PROOF. Suppose not, and let X be a counterexample. Let $Y=\{\alpha \in B\colon a_\alpha \in X\}$. We construct inductively $Z=\{\beta_\xi\colon \xi<\omega_1\} \subseteq Y$ and an antichain $\langle (z_\alpha, e_\alpha)\colon \alpha \in Z\rangle$ as follows. Given β_η for all $\eta<\xi$, choose $\beta=\beta_\xi \in Y$ so large that $\forall \eta<\xi$ $x_\beta \cap \{e \in D\colon e \leqslant_D e_{\beta_\eta}\}=0$ and $\beta_\eta<\beta_\xi$. Since the lemma is presumed false, there exist $\alpha_1, \ldots, \alpha_m \in B$, $\alpha_1, \ldots, \alpha_m>\beta$ such that $X \cap S_{\alpha_1} \cap \cdots \cap S_{\alpha_m}=0$. Let $z_{\beta_\xi}=x_{\beta_\xi}-\{a_{\beta_\xi}\}$, and use the fact that D is directed to choose e_{β_ξ} such that $d_{\alpha_1}, \ldots, d_{\alpha_m} \leqslant_D e_{\beta_\xi}$. Let us check that $(z_{\beta_\eta}, e_{\beta_\eta})$, $(z_{\beta_\xi}, e_{\beta_\xi})$ are incompatible for $\eta<\xi$. Now $\beta_\eta<\alpha_i$ for all i so by the choice of B we have $x_{\beta_\eta} \cap S_{\alpha_i} \neq 0$. We cannot have $a_{\beta_\eta} \in x_{\beta_\eta} \cap S_{\alpha_i}$ for all i, or else $a_{\beta_\eta} \in X \cap S_{\alpha_1} \cap \cdots \cap S_{\alpha_m}$ contrary to assumption. Hence $\exists i\, \exists b \in x_{\beta_\eta} \cap S_{\alpha_i}\ b \neq a_{\beta_\eta}$. But then $b \in z_{\beta_\eta}$ and $b \leqslant_D d_{\alpha_i} \leqslant_D e_{\beta_\xi}$, so $(z_{\beta_\eta}, e_{\beta_\eta})$, $(z_{\beta_\xi}, e_{\beta_\xi})$ are incompatible.

But this is now a contradiction since for $\alpha \in Z$, $|z_\alpha|=|x_\alpha|-1=n-1$, contrary to the choice of n.

We may now apply TOP to find uncountable $X \subseteq A$, $Y \subseteq B$ such that $\forall \alpha \in Y$ $X \cap \alpha \subseteq S_\alpha$. But this means that every element of $X \cap \alpha$ is bounded by d_α, contrary to the assumption that every uncountable subset of D has a countable unbounded subset.

6. The real numbers and the Continuum-Hypothesis trick

Here we shall use a variation on the cardinal-collapsing trick to show that PFA implies all $\aleph_1$-dense sets of reals are order-isomorphic, every uncountable collection of sets of integers contains an uncountable chain or antichain under inclusion, every uncountable Boolean algebra has an uncountable pairwise incomparable set, and every function from an uncountable set of reals into the reals is monotonic on an uncountable set.

A set A of real numbers is $\aleph_1$-*dense* provided that A intersects every interval in exactly $\aleph_1$ points. Of course, if CH holds, then it is easy to construct many pairwise non-order-isomorphic $\aleph_1$-dense sets. See, e.g., BAUMGARTNER [1982], which contains a general discussion of order-types of real numbers.

It is more interesting to ask whether all $\aleph_1$-dense sets can be order-isomorphic (hence in particular homeomorphic) the consistency of this proposition with MA was established in BAUMGARTNER [1973], and Shelah ([ABRAHAM et al. [19··]; see also BAUMGARTNER [1982]) subsequently found a different proof which used a version of the closed-unbounded set trick. Shelah's argument also yielded the consistency of $2^{\aleph_0}>\aleph_2$ with the assertion that all $\aleph_1$-dense sets are order-isomorphic. The proof presented here is an amalgam of the two arguments which seems a little simpler than both.

Suppose A and B are $\aleph_1$-dense. A natural partial ordering for making A and B order-isomorphic is the set Q of all finite order-preserving functions from A into B. Unfortunately, Q does not have the ccc, since for each $a \in A$ $\{\{(a, b)\}\colon b \in B\}$ is an uncountable antichain. In fact, a moment's thought will show that Q

collapses ω_1, so it isn't even proper. It may be possible, however, to find a subordering of Q which does have the ccc and which is still strong enough to make A and B isomorphic. If this were true in general, then MA(ω_1) would be sufficient to imply that all $\aleph_1$-dense sets are isomorphic. As it turns out, this is not the case. Nevertheless, we can show that if CH holds then such a subordering exists, and this is sufficient for the application of PFA. This is the Continuum-Hypothesis trick.

6.1. THEOREM. *Assume* CH. *Suppose A and B are $\aleph_1$-dense and Q is the partial ordering of finite functions mapping A into B. Then there is a subordering of Q which has the* ccc *and which forces A and B to be order-isomorphic.*

PROOF. The idea of the proof is roughly as follows. If X is an uncountable antichain in Q and each $q \in X$ has cardinality n, then X may be identified with a subspace of $2n$-dimensional Euclidean space. Therefore, X has a countable dense subset $\bar{X}$. Since CH holds, it is possible to enumerate all such $\bar{X}$ in order-type ω_1. If we can construct a subordering $\bar{Q}$ of Q which avoids each $\bar{X}$ from some point on, then $\bar{Q}$ will have ccc.

If $q \in Q$, then we may write q canonically as $\{(a_0, b_0)(a_1, b_1), \ldots, (a_{n-1}, b_{n-1})\}$ where $a_0 < a_1 < \cdots < a_{n-1}$, and we may identify q with the point $(a_0, b_0, a_1, b_1, \ldots, a_{n-1}, b_{n-1})$ in $2n$-dimensional Euclidean space. Thus we may speak of an open set U being a neighborhood of q, etc. We refer to the pair (a_i, b_i) as the ith *element* of q and we write $q_i = (a_i, b_i)$. If $U = I_0 \times I_1 \times \cdots \times I_{2n-1}$ is a product of open intervals with rational endpoints and U is a neighborhood of q, then we say q is *separated by* U iff $\forall i < 2n-2\ I_i \cap I_{i+2} = 0$.

6.2. LEMMA. *Suppose p and q are both separated by U, p and q have no coordinates in common, and p and q are incompatible in Q. Then there exists $i < n$ such that p_i and q_i are incompatible, i.e., $\{p_i, q_i\} \notin Q$.*

PROOF. Trivial.

Let us say that p and q are *totally distinct* if they have no coordinates in common.

Now suppose $X \subseteq \{q \in Q: |q| = n\}$. We say that $q \in Q$ is *consistent with* X iff

(1) q is totally distinct from all the elements of X and $|q| = n$,

(2) q is incompatible with every element of X,

(3) $\forall i < n\ \forall U$ if U is a neighborhood of q, then $\exists p \in U \cap X\ p - \{p_i\}$ is compatible with $q - \{q_i\}$.

In practice X is usually an antichain and to say that q is consistent with X is in part to say that it can be added to X to obtain a larger antichain.

If X is as above, $q \in Q$, $|q| = n-1$ and $a \in A$, then let $B_i(X, q, a) =$

$\{b \in B: q \cup \{(a, b)\}$ is consistent with X and (a, b) is the ith element of $q \cup \{(a, b)\}\}$. If $b \in B$, then let $A_i(X, q, b) = \{a \in A: q \cup \{(a, b)\}$ is consistent with X and (a, b) is the ith element of $q \cup \{(a, b)\}\}$.

6.3. LEMMA. *Each $A_i(X, q, b)$ and $B_i(X, q, a)$ has cardinality at most 2.*

PROOF. Suppose $a \in A_i(X, q, b)$. Let $\langle U_m : m \in \omega\rangle$ be a descending neighborhood base for $q \cup \{(a, b)\}$. By (3), for each m there is $p^m \in U_m \cap X$ such that $p^m - \{p^m_i\}$ is compatible with q. Since we could have chosen U_0 so that $q \cup \{(a, b)\}$ is separated by U_0, we have by Lemma 6.2 that p^m_i and (a, b) are incompatible, i.e., if $p^m_i = (a^m_i, b^m_i)$, then either

(4) $a^m_i < a$ and $b < b^m_i$, or

(5) $a < a^m_i$ and $b^m_i < b$.

If (4) occurs for infinitely many m then we say a is of *type* 1, while if (5) occurs for infinitely many m we say a is of *type* 2.

If $|A_i(X, q, b)| \geq 3$, then $\exists a_1, a_2 \in A_i(X, q, b)$ $a_1 < a_2$ and a_1 and a_2 have the same type, say type 1. But now if $a_2 = a$ as above, then by choosing m large enough we must have $a_1 < a^m_i < a_2$ and $b < b^m_i$, and this means that $q \cup \{(a_1, b)\}$ is compatible with p^m, contradicting (2). All the other cases are similar to this one.

Now let $\langle X_\alpha : \alpha < \omega_1\rangle$ enumerate all countable subsets of $\{q \in Q: |q| = n\}$ for each n. Suppose $A = \{a_\alpha : \alpha \in \omega_1\}$ and $B = \{b_\alpha : \alpha \in \omega_1\}$. Let C be the set of all $\alpha < \omega_1$ such that for all $\beta, \gamma < \alpha$ and for all $q \subseteq \{(a_\xi, b_\eta): \xi, \eta < \alpha\}$, we have $A_i(X_\beta, q, b_\gamma) \subseteq \{a_\xi : \xi < \alpha\}$ and $B_i(X_\beta, q, a_\gamma) \subseteq \{b_\xi : \xi < \alpha\}$. Note $0 \in C$. By Lemma 6.3, C is closed unbounded. Since A and B are $\aleph_1$-dense, it is easy to find a closed unbounded set $C' \subseteq C$ such that $0 \in C'$ and if $\{c_\alpha : \alpha < \omega_1\}$ enumerates C' in increasing order, $A_\alpha = \{a_\xi : c_\alpha \leq \xi < c_{\alpha+1}\}$ and $B_\alpha = \{b_\xi : c_\alpha \leq \xi < c_{\alpha+1}\}$, then A_α and B_α are always dense in the reals. (Just choose the elements of C' sufficiently far apart.)

Now we are ready to define the ccc subordering $\bar{Q}$ of Q. If $\alpha, \beta < \omega_1$, then α and β are *neighbors* if $\exists n \in \omega$ $n \neq 0$ and either $\alpha + n = \beta$ or $\beta + n = \alpha$. Let $\bar{Q}$ consist of all $q \in Q$ such that

(6) $\forall (a, b) \in q$, if $a \in A_\alpha$ and $b \in B_\beta$, then α and β are neighbors,

(7) $\forall \alpha$ there is at most one pair $(a, b) \in q$ such that *either* $a \in A_\alpha$ and $\exists \beta < \alpha$ $b \in B_\beta$, or else $b \in B_\alpha$ and $\exists \beta < \alpha$ $a \in A_\beta$.

6.4. LEMMA. *$\forall a \in A$ $\{q \in \bar{Q}: a \in$ domain $q\}$ is dense in $\bar{Q}$ and $\forall b \in B$ $\{q \in \bar{Q}: b \in$ range $q\}$ is dense in $\bar{Q}$.*

PROOF. This is easy since the A_α and B_α are all dense in the reals. If $a \in A_\alpha$ and $q \in \bar{Q}$, $a \notin$ domain q, then choose n so large that $B_{\alpha+n} \cap$ range $q = 0$ and $A_{\alpha+n} \cap$ domain$(q) = 0$, choose $b \in B_{\alpha+n}$ so that $q \cup \{(a, b)\} \in Q$ and now note that $q \cup \{(a, b)\} \in \bar{Q}$. This proves the first half of the lemma. The second half is similar.

It follows that in $V^{\bar{Q}}$, A and B are order-isomorphic. It remains only to show that $\bar{Q}$ has the ccc.

Suppose X is an uncountable antichain in $\bar{Q}$. Without loss of generality we may assume that every element of X has cardinality n, and n is minimal among all uncountable antichains in $\bar{Q}$.

6.5. LEMMA. *Without loss of generality we may assume that the elements of X are pairwise disjoint.*

PROOF. Since the elements of X are finite, there is an uncountable Δ-system $Y \subseteq X$ with kernel Δ. If $\Delta \neq 0$, then $\{q - \Delta : q \in Y\}$ is an uncountable antichain with $|q - \Delta| < |q| = n$, contradicting minimality of n. Thus $\Delta = 0$ and the elements of Y are pairwise disjoint. Just replace X by Y.

6.6. LEMMA. *Without loss of generality we may assume that the elements of X are pairwise totally distinct.*

PROOF. This follows immediately from Lemma 6.5 and condition (6) in the definition of $\bar{Q}$.

Now let $\langle U_m : m \in \omega \rangle$ enumerate a basis for $2n$-dimensional Euclidean space. For each $m \in \omega$ and each $i \leqslant n$, let $X(i, m) \subseteq X$ be a maximal set such that $X(i, m) \subseteq U_m$ and $\{q - \{q_i\} : q \in X(i, m)\}$ is an antichain. By the minimality of n, each $X(i, m)$ is countable. Hence there is some α such that $X_\alpha = \bigcup\{X(i, m) : i < n, m \in \omega\}$.

By Lemma 6.6 there must be $q \in X$ such that $\forall (a, b) \in q$, if $(a, b) \in A_\beta \times B_\gamma$, then $\beta, \gamma > \alpha$.

6.7. LEMMA. *q is consistent with X_α.*

PROOF. (1) and (2) are obvious. For (3), fix i and U. Choose m so that $U_m \subseteq U$ and $q \in U_m$. By the choice of $X(i, m)$ there must be $p \in X(i, m)$ such that $p - \{p_i\}$ is compatible with $q - \{q_i\}$, and p satisfies (3).

Now let δ be the largest ordinal such that $(A_\delta \times B) \cap q = 0$ or $(A \times B_\delta) \cap q = 0$. For concreteness, assume $(A_\delta \times B) \cap q = 0$. By (7), $(A_\delta \times B) \cap q$ is a singleton $\{(a, b)\}$ where $b \in B_\gamma$ for some $\gamma < \delta$. Say $(a, b) = q_i$.

6.8. LEMMA. $a \in A_i(X_\alpha, q - \{q_i\}, b)$.

PROOF. Trivial from Lemma 6.7.

But now we clearly have $\alpha < c_\delta$, $q - \{q_i\} \subseteq \{(a_\beta, b_\gamma) : \beta, \gamma < c_\delta\}$ and $b \in \{b_\gamma : \gamma < c_\delta\}$. Hence by the definition of C we would have to have $A_i(X_\alpha, q - \{q_i\}, b) \subseteq \{\alpha_\gamma : \gamma < c_\delta\}$, contradicting the fact that $a \in A_\delta$. This contradiction completes the proof of Theorem 6.1.

6.9. THEOREM. PFA *implies that all* $\aleph_1$*-dense sets of reals are order-isomorphic.*

PROOF. Let A and B be $\aleph_1$-dense. Let P be the partial ordering for collapsing $2^{\aleph_0}$ onto $\aleph_1$ with countable conditions. Then P is countably complete, hence proper, and since P adds no new reals, CH holds in V^P. Hence by Theorem 6.1 there is a ccc partial ordering $\bar{Q} \in V^P$ such that in $(V^P)^{\bar{Q}}$, A and B are order-isomorphic. But $P*\bar{Q}$ is proper and we require only $\aleph_1$ dense sets to decide the value of the isomorphism for every $a \in A$. Therefore if PFA holds there must be such an isomorphism.

Now let us turn to the problem of showing it consistent that every uncountable family X of subsets of ω has an uncountable chain or antichain under inclusion. The idea is quite simple: if X has no uncountable chain, then introduce an uncountable antichain generically. The obvious ordering for introducing such an antichain is $Q(X)$, the family of all finite antichains (pairwise incomparable sets) in X. Unfortunately, rather as in the case of $\aleph_1$-dense sets, it need not be true that $Q(X)$ has the ccc. There is, however, a way around the difficulty which uses the Continuum-Hypothesis trick.

6.10. THEOREM. *Assume* CH. *Suppose X is an uncountable family of subsets of ω with no uncountable chain under inclusion. Then there exists uncountable $Y \subseteq X$ such that $Q(Y)$ has the* ccc.

PROOF. The proof is quite similar to Theorem 6.1 so we shall treat it rather briefly.

6.11. LEMMA. *There is a one-to-one mapping φ from $P(\omega)$ into the reals such that if $x \subseteq y$, then $\varphi(x) \leq \varphi(y)$.*

PROOF. Just let $\varphi(x) = \Sigma\{3^{-i}: i \in x\}$.

Thus any $q \in Q(X)$ may be canonically ordered by letting $q = \{x_0, \ldots, x_{n-1}\}$, where $\varphi(x_0) < \cdots < \varphi(x_{n-1})$. We associate with q the point $\varphi(q) = (\varphi(x_0), \ldots, \varphi(x_{n-1}))$ in n-dimensional Euclidean space. We refer to x_i as the ith element of q and sometimes we write $q_i = x_i$. If k is the smallest integer such that $\forall i, j < n$, if $i \neq j$, then $x_i \cap k \not\subseteq x_j \cap k$, then we refer to $(x_0 \cap k, \ldots, x_{n-1} \cap k)$ as the *type* of q. Note that there are only countably many types and that if q, $r \in Q(X)$ have the same type and are incompatible, then $\exists i < n\, q_i$ and r_i are comparable, i.e., either $q_i \subseteq r_i$ or $r_i \subseteq q_i$.

If $A \subseteq \{q \in Q(X): |q| = n\}$ and $q \in Q(X)$, $|q| = n$, then we say q is *consistent with A* if

(8) q is incompatible with and disjoint from every element of A, and

(9) $\forall i < n \forall U$ if U is a neighborhood of $\varphi(q)$, then there exists $p \in A$ such that $\varphi(p) \in U$, p has the same type as q and $p - \{p_i\}$ is compatible with $q - \{q_i\}$.

With A as above, $q \in Q(X)$, $|q| = n - 1$, and $i < n$, let $X_i(A, q) = \{x \in X: q \cup \{x\}$ is consistent with A and x is the ith element of $q\}$.

6.12. Lemma. *Every $X_i(A, q)$ is countable or finite.*

Proof. Suppose not. Fix $x \in X_i(A, q)$ and let $\langle U_m: m \in \omega\rangle$ be a descending neighborhood base for $\varphi(q \cup \{x\})$. By (9) choose $p^m \in A$ so that $\varphi(p^m) \in U_m$, $p^m - \{p_i^m\}$ is compatible with $q - \{q_i\}$ and p^m has the same type as q. Then either

(10) $\varphi(p_i^m) < \varphi(q_i) = x$, or

(11) $x = \varphi(q_i) < \varphi(p_i^m)$.

If (10) occurs infinitely often, then x is *left-approachable*; if (11) occurs infinitely often, then x is *right-approachable*. Note that if (10) holds, then $\varphi(p_i^m) \subseteq \varphi(q_i)$ since p^m and q have the same type, while $p^m - \{p_i^m\}$ and $q - \{q_i\}$ are compatible. Similarly for (11).

Now suppose $Z \subseteq X_i(A, q)$ is an uncountable set of left-approachable elements, such that $\{q \cup \{x\}: x \in Z\}$ all have the same type. The case for right-approachability is exactly similar. Suppose $x_1, x_2 \in Z$ and $\varphi(x_1) < \varphi(x_2)$. If $x_2 = x$ as above, then we can find m so that $\varphi(x_1) < \varphi(p_i^m) < \varphi(x_2)$. But then we have $\varphi(x_1) \subseteq \varphi(p_i^m) \subseteq \varphi(x_2)$ and Z is an uncountable chain in X, contrary to the hypothesis of Theorem 6.10.

Now let $\langle A_\alpha: \alpha < \omega_1\rangle$ enumerate all countable $A \subseteq \{q \in Q(X): |q| = n\}$ for each $n \in \omega$. We define $Y = \{y_\alpha: \alpha < \omega_1\}$ by induction on α. Given $\{y_\beta: \beta < \alpha\}$, let $Z_\alpha = \bigcup\{X_i(A_\beta, q): \beta < \alpha, \exists n A_\beta \subseteq \{p \in Q(\{y_\beta: \beta < \alpha\}): |p| = n\}, q \in Q(\{y_\beta: \beta < \alpha\}), |q| = n - 1, i < n\}$. By Lemma 6.12, Z_α is countable so choose $y_\alpha \in X - (Z_\alpha \cup \{y_\beta: \beta < \alpha\})$.

Checking that $Q(Y)$ has the ccc is now just like finishing the proof of Theorem 6.1. Details are left to the reader.

6.13. Theorem. PFA *implies that every uncountable family of subsets of ω contains an uncountable chain or antichain.*

Proof. The proof is virtually the same as Theorem 6.9, with only one minor complication. It is conceivable that in the $Q(Y)$ obtained from Theorem 6.10 there are elements which cannot be extended, i.e., they may be *finite* maximal antichains in Y. In order to be certain that an uncountable antichain may be adjoined by $Q(Y)$, we must find $p \in Q(Y)$ such that $\forall q \in Q(Y)$, if $q \supseteq p$, then there are uncountably many elements $y \in Y$ such that $q \cup \{y\} \in Q(Y)$. There is a standard way of finding such p. Let $A \subseteq Q(Y)$ be a maximal antichain in $Q(Y)$ consisting of elements of $Q(Y)$ which cannot be extended. Then A is countable since $Q(Y)$ has the ccc. Choose $y \in Y$, $y \notin \bigcup A$. Then $p = \{y\}$ clearly works. Now repeat the proof of Theorem 6.9, replacing $\bar{Q}$ by the set of elements of $Q(Y)$ which extend p.

As S. Todorčević has pointed out, Theorem 6.13 immediately yields some further results.

6.14. THEOREM. (Todorčević). *Assume that every uncountable family of subsets of* ω *contains an uncountable chain or antichain. Then*

(a) *every function from an uncountable set of reals into the reals is monotonic on an uncountable set, and*

(b) *every uncountable Boolean algebra has an uncountable pairwise incomparable subset.*

PROOF. (a) Suppose $f: A \to R$, where R denotes the reals and $A \subseteq R$ is uncountable. If f is constant on an uncountable set we are done, so assume otherwise. Without loss of generality, we may assume f is one-to-one. Now let the set Q of rational numbers be partitioned into disjoint sets Q_1 and Q_2, each dense in Q. For each $r \in A$, let

$$x_r = \{q \in Q_1 : r \leqslant q\} \cup \{q \in Q_2 : f(r) \leqslant q\}\,.$$

Suppose r, $s \in A$ and $r < s$. If x_r and x_s are comparable, then $x_r \subseteq x_s$ and $f(r) < f(s)$. If x_r and x_s are incomparable, then $f(r) > f(s)$. Hence an uncountable chain or antichain in $\{x_r : r \in A\}$ gives rise to an uncountable monotonic mapping.

(b) Suppose B is an uncountable Boolean algebra with no uncountable pairwise incomparable subset. Then by Theorem 3 of BAUMGARTNER & KOMJATH [1981], B is representable as a field of subsets of ω. We have two cases.

Case 1. For some $b \in B$, both $\{c \in B : c \leqslant b\}$ and $\{c \in B : c \leqslant \omega - b\}$ are uncountable. Without loss of generality, B has cardinality ω_1, so there is a bijection h mapping $X = \{c \in B : c \leqslant b\}$ onto $Y = \{c \in B : c \leqslant \omega - b\}$. Now, if φ is the mapping of Lemma 6.11, then h induces a map h^* from $\varphi(X)$ onto $\varphi(Y)$. By (a) of the present theorem there is uncountable $X' \subset X$ such that h^* is monotonic on $\varphi(X')$. If h^* is increasing on $\varphi(X')$, then $\{(b - c) \cup h(c) : c \in X'\}$ is an uncountable pairwise incomparable set, while if h^* is decreasing on $\varphi(X')$ then $\{c \cup h(c) : c \in X'\}$ is uncountable and pairwise incomparable.

Case 2. Otherwise. Call $b \in B$ *small* if $\{c \in B : c \leqslant b\}$ is countable. Then $\forall b \in B$ either b or $\omega - b$ is small. But now it is easy to find by induction a sequence $\langle b_\alpha : \alpha < \omega_1\rangle$ of small elements such that if $\alpha < \beta$ then $b_\beta \nleqslant b_\alpha$. It follows that $\{b_\alpha : \alpha < \omega_1\}$ is well-founded under $\subseteq$. Any uncountable chain in $\{b_\alpha : \alpha < \omega_1\}$ would be an uncountable family of subsets of ω well-ordered by $\subseteq$, which is impossible. Hence there is an uncountable antichain in $\{b_\alpha : \alpha < \omega_1\}$ and we are done.

6.15. COROLLARY. PFA *implies* (a) *and* (b) *of Theorem* 6.14.

The consistency of the propositions that every uncountable family of subsets of ω has an uncountable chain or antichain and that every uncountable Boolean algebra has an uncountable pairwise incomparable subset was established by BAUMGARTNER [1980]. The consistency of (a) of Theorem 6.14 is due to U. ABRAHAM [1981]. ABRAHAM and SHELAH [1981] have shown that none of the propositions in this section, with the possible exception of (b) of Theorem 6.14, follows from Martin's Axiom alone. See also ABRAHAM et al. [19··].

7. Trees, $\diamondsuit$, and $\square$

It is well-known that Martin's Axiom has many implications for the theory of trees, so it should not be surprising that the same is true of PFA.

Let us recall some of the terminology on trees from Todorčević's article in this volume. A *tree* is a partially ordered set $(T, \leqslant_T)$ with the property that $\forall t \in T$ $\{s \in T: s <_T t\}$ is well-ordered by $\leqslant_T$. The *height of t in T*, $\mathrm{ht}_T(f)$, is the order type of $\{s \in T: s <_T t\}$. The αth *level of T* is $R_\alpha T = \{t \in T: \mathrm{ht}_T(t) = \alpha\}$. The *height* of T is the smallest α such that $R_\alpha(T) = 0$. A *branch* is a maximal linearly ordered subset of T; the length of a branch is its order type. A branch is *cofinal* if its length is the height of T. A κ*-tree* is a tree of height κ such that $|R_\alpha(T)| < \kappa$ for all $\alpha < \kappa$, where κ is regular. A κ*-Aronszajn tree* is a κ-tree with no branches of length κ. Two $\aleph_1$-Aronszajn trees T and U are *club-isomorphic* if there is a closed unbounded set $C \subseteq \omega_1$ such that $\{t \in T: \mathrm{ht}_T(t) \in C\}$ and $\{u \in U: \mathrm{ht}_U(u) \in C\}$ are isomorphic (with respect to the induced tree orderings). Then we already have

7.1. THEOREM. PFA *implies that all* $\aleph_1$*-Aronszajn trees are club-isomorphic.*

7.2. THEOREM. PFA *implies that there are no* $\aleph_2$*-Aronszajn trees.*

These are Theorems 5.10 and 7.7 of Todorčević's article in this volume. The reader is referred to that paper for more complete discussions than can be given here. Let us remark, however, that ABRAHAM and SHELAH [19··] have shown that the conclusion of Theorem 7.1 does not follow from MA. The same is true for Theorem 7.2 as well, but for an entirely different reason. Namely, by a result of Silver if it is consistent that there are no $\aleph_2$-Aronszajn trees, then it is consistent that there is a weakly compact cardinal (in fact $\aleph_2$ must be weakly compact in L), and MA has no such large-cardinal strength! This is the first evidence we have had that the consistency strength of PFA is greater than MA. Henceforth we shall be seeing many more examples of this phenomenon.

Let us restrict attention for the moment to trees of height $\leqslant\omega_1$, which for convenience we shall call *short* trees. Such a tree T is called *special* if it is the union of countably many antichains (pairwise incomparable sets), or equivalently if there is a map $f: T \to \omega$ such that $\forall s, t \in T$, if $s <_T t$, then $f(s) \neq f(t)$.

In Theorem 9.4 of Todorčević's article in this volume, it is shown that MA(κ) implies that every short tree of cardinality κ with no uncountable branches is special. We shall begin by extending this result to apply to trees with uncountable branches.

Let T be a short tree, possibly with uncountable branches. We say that T is *special* (in the new sense) if there is $f\colon T \to \omega$ such that $\forall s, t, u \in T$ if $f(s) = f(t) = f(u)$ and $s <_T t, u$ then t and u are comparable in T.

7.3. THEOREM. *Both definitions of "special" coincide for short trees with no uncountable branches.*

PROOF. It is clear that the old definition implies the new one. Suppose T is short with no uncountable branches, and $f\colon T \to \omega$ is as in the definition of special in the new sense. Define $g\colon T \to \omega \times \omega$ as follows. Suppose $t \in T$ and $f(t) = n$. Let $s \leqslant_T t$ be minimal such that $f(s) = n$. Now $A_s = \{u \in T\colon s \leqslant_T u, f(u) = n\}$ is linearly ordered, and since T has no uncountable branches, A_s must be countable. Let $g_s\colon A_s \to \omega$ be one-to-one and let $g(t) = (n, g_s(t))$. It is easy to check that $g^{-1}(m, n)$ is an antichain for each m and n.

Now any tree which is special in the old sense cannot have uncountable branches, so we shall obtain a wider class of trees by considering trees special in the new sense. Henceforth 'special' will mean special in the new sense.

Suppose T is a short tree and B is a collection of branches through T. We say B is *non-stationary* if there is a one-to-one map $f\colon B \to T$ such that $f(b) \in b$ for all $b \in B$; otherwise B is *stationary*. Note that if $|T| = \aleph_1$ and B is a collection of uncountable branches through T, then B is non-stationary iff $|B| \leqslant \aleph_1$.

7.4. THEOREM. *If T is special, then the set B of uncountable branches through T is non-stationary.*

PROOF. Suppose $f\colon T \to \omega$ is as in the definition of special. Let b be an element of B. Then there is some n such that $\{t \in b\colon f(t) = n\}$ is cofinal in b. Choose $t \in b$ with $f(t) = n$ and let $g(b) = t$. It is clear from the definition of special that g is one-to-one.

7.5. THEOREM. *Suppose that every short tree of cardinality $\leqslant\kappa$ is special. Then for every short tree T of cardinality $\leqslant\kappa$, if the set of uncountable branches through T is non-stationary, then T is special.*

PROOF. Let B be the set of uncountable branches through a short tree T with $|T| \leqslant \kappa$, and suppose B is non-stationary.

7.6. LEMMA. *There is a one-to-one map $g\colon B \to T$ such that $g(b) \in b$ for all $b \in B$ and whenever $g(b_1) <_T g(b_2)$, then $g(b_2) \notin b_1$.*

PROOF. Let $f\colon B \to T$ be as in the definition of non-stationary. Fix $b \in B$. Then $\{b' \in B\colon f(b') <_T f(b)\}$ is countable since f is one-to-one, so there must be $t \in b$ such that $\forall b'$ if $f(b') <_T f(b)$ then $t \notin b'$, and $f(b) \leqslant_T t$. Let $g(b) = t$. Now suppose $g(b_1) <_T g(b_2)$. Then $f(b_1)$ and $f(b_2)$ must have been comparable. If $f(b_1) <_T f(b_2)$, then $g(b_2)$ was chosen outside b_1 and we are done. If $f(b_2) <_T f(b_1)$, then $g(b_1)$ was chosen outside b_2, contradicting our assumption that $g(b_1) <_T g(b_2) \in b_2$.

Let $g\colon B \to T$ be as in Lemma 7.6, and let $S = \{t \in T\colon \forall b \in B$, if $t \in b$, then $t \leqslant_T g(b)\}$. Then S is a tree under the ordering inherited from T and the range of g is included in S.

7.7. LEMMA. *S has no uncountable branches.*

PROOF. If b were an uncountable branch through S, then clearly $\bar{b} = \{t \in T\colon \exists s \in b\ t <_T s\} \in B$. But then since $g(\bar{b}) \in \bar{b}$ we would have $s \in b$ with $g(\bar{b}) \leqslant_T s$, contradicting the fact that $s \in S$.

By the hypothesis of the theorem, S is special. Let $f\colon S \to \omega$ be as in the definition of special in the old sense. If $t \in T - S$, then there is a branch b with $t \in b$ and $g(b) \leqslant_T t$. Moreover by Lemma 7.6 this branch b is unique. Let $f(t) = f(g(b))$. This definition extends the domain of f to all of T. We claim f shows that T is special (in the new sense).

Suppose s, t, $u \in T$, $f(s) = f(t) = f(u) = n$, say, and $s \leqslant_T t, u$. Fix unique b with $t \in b$ and $g(b) \leqslant_T t$. Then also $s \in b$. If $s <_T g(b)$, then there must be a branch $c \neq b$ with $s \in c$ and $g(c) \leqslant_T s$. Note that $g(b)$, $g(c) \in S$ by Lemma 7.6. Since $g(c) <_T g(b)$ we have by the choice of f that $f(g(c)) \neq f(g(b))$. But $f(g(c)) = f(s) = n = f(t) = f(g(b))$, contradiction. Hence $b = c$. A similar argument applies to s and u. Thus s, t, $u \in b$ and t and u must be comparable.

Since MA(κ) implies that all trees of cardinality $\leqslant\kappa$ with no uncountable branches are special (see Theorem 9.4 of Todorčević's article in this volume), we have the following corollary.

7.8. COROLLARY. MA(κ) *implies that for any short tree T, if $|T| \leqslant \kappa$ and the set of uncountable branches through T is non-stationary, then T is special.*

For our next application of PFA, we need to know a little more.

7.9. THEOREM. *Suppose T is a short tree, $|T| \leqslant \kappa$, and the set of uncountable branches through T is non-stationary. Then there is a* ccc *partial ordering $Q(T)$ such that in $V^{Q(T)}$, T is special.*

PROOF. It suffices to find $Q(T)$ so that the set S of the proof of Theorem 7.5 is special in $V^{Q(T)}$. But it is well known that for trees with no uncountable branches, the set of finite approximations to a specializing map $f\colon S \to \omega$ has the ccc. See Todorčević's article in this volume, especially Theorem 5.9 and Remark 9.6(i).

A *Kurepa tree* is an $\aleph_1$-tree with more than $\aleph_1$ branches. A *weak Kurepa tree* is a tree of height ω_1 and cardinality $\aleph_1$ with more than $\aleph_1$ branches. Kurepa's Hypothesis (KH) asserts that there is a Kurepa tree; the weak Kurepa Hypothesis (wKH) asserts that there is a weak Kurepa tree.

7.10. THEOREM. PFA *implies that every tree of height* ω_1 *and cardinality* $\aleph_1$ *is special, and therefore* wKH *is false.*

PROOF. Let T be a tree of height ω_1 and cardinality $\aleph_1$. Let B be the set of uncountable branches through T. If $|B| \leqslant \aleph_1$, then B is non-stationary and we are done immediately by Theorem 7.9 (and the observation that only $\aleph_1$ dense sets are needed to decide the values of the specializing function on each element of T).

Suppose, then, that $|B| = \lambda > \aleph_1$. Let P be the partial ordering for collapsing λ onto $\aleph_1$ with countable conditions. Then P is countably closed, hence proper, and CH holds in V^P since CH holds in V^P whenever P is countably closed and adjoins a subset of ω_1. See, e.g., KUNEN [1980], Ex. G3, p. 246, or JECH [1978], Ex. 19.7, p. 183.

7.11. LEMMA. *In* V^P *there are no new uncountable branches through T.*

PROOF. Suppose on the contrary that $\dot{b}$ is a term, $p_0 \in P$, and $p_0 \Vdash \dot{b}$ is a new uncountable branch through T (i.e., $\dot{b} \notin B$). Since $\dot{b}$ must be a new branch, we have immediately that

(*) $\forall q \leqslant p_0$, if $q \Vdash t \in \dot{b}$, then $\exists q_1, q_2 \leqslant q\ \exists t_1, t_2 \geqslant_T t\ t_1$ and t_2 are incomparable, $q_1 \Vdash t_1 \in \dot{b}$ and $q_2 \Vdash t_2 \in \dot{b}$.

To see that (*) holds, fix $q \leqslant p_0$ and suppose $q \Vdash t \in \dot{b}$. Let $a = \{t' \in T\colon \exists q' \leqslant q\ q' \Vdash t' \in \dot{b}\}$. If a is a branch through T, then clearly $q \Vdash \dot{b} = a$, contrary to the assumption that $\dot{b}$ was a new branch. Since a must contain elements from every level of T, the only alternative is that there exist incomparable $t_1, t_2 \in a$. But if $q_1, q_2 \leqslant q$ are such that $q_1 \Vdash t_1 \in \dot{b}$, $q_2 \Vdash t_2 \in \dot{b}$, then we are in the situation of (*). Note that $t \leqslant_T t_1, t_2$ since $q \Vdash t \in \dot{b}$.

Using the observation (*) repeatedly, we may now find sequences $\langle q_\sigma\colon \sigma \in \bigcup\{{}^n2\colon n \in \omega\}\rangle$ and $\langle t_\sigma\colon \sigma \in \bigcup\{{}^n2\colon n \in \omega\}\rangle$ by induction on the length of σ such that $p_0 \geqslant q_\sigma$, $q_\sigma \Vdash t_\sigma \in \dot{b}$, $t_{\sigma 0}$ and $t_{\sigma 1}$ are incomparable, and if $\sigma \subseteq \tau$, then $q_\tau \leqslant q_\sigma$ and $t_\sigma \leqslant_T t_\tau$.

Now for each $f \in {}^\omega 2$, choose q_f and t_f so that $q_f \leqslant q_{f|n}$ and $t_{f|n} \leqslant_T t_f$ for all n (this is possible since P is countably closed and $\dot{b}$ is forced to be uncountable). But now it is clear that whenever $f \neq g$ then t_f and t_g are incomparable, since if n is minimal such that $f|n \neq g|n$ then $t_{f|n}$ and $t_{g|n}$ are already incomparable. Thus the t_f form an antichain of size $2^{\aleph_0}$, which is impossible since $|T| = \aleph_1$ and CH is false, by PFA.

It follows from Lemma 7.11 that in V^P the set B of uncountable branches through T has cardinality $\aleph_1$, hence is non-stationary. By Theorem 7.9 applied in

V^P, there is ccc $Q(T)$ such that in $(V^P)^{Q(T)}$, T is special. But $P*Q(T)$ is proper, and specializing T requires only $\aleph_1$ dense sets. This completes the proof.

For a more complete discussion of KH and wKH, the reader is referred to Section 8 of Todorčević's article in this volume.

In general, Martin's Axiom serves as a kind of antithesis to the Axiom of Constructibility. For example, MA + $\neg$CH denies almost every kind of $\diamondsuit$ principle on ω_1, and PFA goes even further in this direction, as is shown by Theorem 3.4. On the other hand, MA does not deny $\square_{\omega_1}$, since if MA is made true by ccc forcing over a model of $\square_{\omega_1}$, then it is easy to see that $\square_{\omega_1}$ remains true in the extension. In this case also, PFA goes farther.

7.12. THEOREM. PFA *implies that* $\square_{\omega_1}$ *is false.*

PROOF. It follows from theorem 7.4 of Todorčević's article in this volume that if $\square_{\omega_1}$ holds, then there is an $\aleph_2$-Aronszajn tree. Now we are done by Theorem 7.2.

In view of these results, it is quite surprising that $\diamondsuit_{\omega_2}$ is not only consistent with PFA, it is even implied by PFA + $2^{\aleph_0} = \aleph_2$! After this observation was made by the author, Todorčević noticed that in fact $\diamondsuit_{\omega_2}$ follows from $2^{\aleph_0} = \aleph_2$ and the failure of wKH together with the nonexistence of $\aleph_2$-Aronszajn trees. See DEVLIN [1978]. Subsequent work of Todorčević and the author has led to Theorem 7.13 below.

If $S \subseteq \omega_2$, then the principle $\diamondsuit^+(S)$ asserts that there exists a sequence $\langle S_\alpha : \alpha \in S\rangle$ such that $\forall x \in S_\alpha\ x \subseteq \alpha$, $|S_\alpha| \leq \omega_1$ and $\forall A \subseteq \omega_2\ \exists$ closed unbounded $C \subseteq \omega_2$ $\forall \alpha \in C \cap S\ C \cap \alpha,\ A \cap \alpha \in S_\alpha$. We refer to $\langle S_\alpha : \alpha \in S\rangle$ as a $\diamondsuit^+$-sequence. It is well-known that if $\diamondsuit^+(S)$ holds, then $\diamondsuit(E)$ holds for every stationary $E \subseteq S$. See KUNEN [1980], for example.

7.13. THEOREM. *Assume that* wKH *fails and* $2^{\aleph_0} = \aleph_2$. *Then* $\diamondsuit^+(\{\alpha < \omega_2 \colon \operatorname{cf}\alpha = \omega_1\})$ *holds. Hence in particular* PFA *implies* $\diamondsuit(E)$ *for every stationary* $E \subseteq \{\alpha < \omega_2 \colon \operatorname{cf}\alpha = \omega_1\}$.

PROOF. First let us observe that $2^{\aleph_1} = \aleph_2$. This is a consequence of a rather general fact (Theorem 3.4 of BAUMGARTNER [1976]), but for convenience we shall give a direct proof. Let $\langle s_\alpha : \alpha < \omega_2\rangle$ enumerate $\bigcup\{{}^\alpha 2 \colon \alpha < \omega_1\}$. Now for $f \in {}^{\omega_1}2$ let β_f be such that $\{f|\alpha : \alpha < \omega_1\} \subseteq \{s_\gamma \colon \gamma < \beta_f\}$. If $2^{\aleph_1} > \aleph_2$, then there would be $X \subseteq {}^{\omega_1}2$ and $\beta < \omega_2$ such that $|X| \geq \aleph_3$ and $\forall f \in X\ \beta_f = \beta$. But that means that $\{s_\gamma \colon \gamma < \beta\}$ would form a weak Kurepa tree, since for each $f \in X \{f|\alpha : \alpha < \omega_1\}$ would be a branch through $\{s_\gamma \colon \gamma < \beta\}$. Thus $2^{\aleph_1} = \aleph_2$.

For each $\alpha < \omega_2$, let $\langle s^\alpha_\xi : \xi < \omega_2\rangle$ enumerate ${}^\alpha 2$. For any $f \in {}^\alpha 2$, where $\alpha \leq \omega_2$, define $g_f \colon \alpha \to \omega_2$ so that $g_f(\beta) = \gamma$ iff $f|\beta = s^\beta_\gamma$. Now fix $\alpha < \omega_2$ with $\operatorname{cf}\alpha = \omega_1$. We wish to define S_α.

Let $W_\alpha = \{f \in {}^\alpha 2 \colon \forall \beta < \alpha\ \ g_f(\beta) < \alpha\}$, and let $T_\alpha = \{f|\beta : \beta < \alpha,\ f \in W_\alpha\}$. If $W_\alpha \neq 0$, then T_α is a tree of cardinality ω_1 and height α. Moreover, T_α has a cofinal set of levels which forms a tree U_α of height ω_1. Since any cofinal branch through T_α would determine an uncountable branch through U_α, it follows from the failure of wKH that T_α has at most $\aleph_1$ cofinal branches. Since each $f \in W_\alpha$ determines a branch $\{f|\beta: \beta < \alpha\}$, it follows that $|W_\alpha| \leq \aleph_1$. For each $f \in W_\alpha$, let $A_f = \{\beta < \alpha: f(\beta) = 1\}$ and let $C_f = \{\beta < \alpha: \forall \gamma < \beta \ g_f(\gamma) < \beta\}$. Note that C_f is closed in α. Let $S_\alpha = \{A_f: f \in W_\alpha\} \cup \{C_f: f \in W_\alpha\}$. We claim $\langle S_\alpha: \alpha < \omega_2, \operatorname{cf} \alpha = \omega_1\rangle$ is a $\diamondsuit^+$-sequence.

Suppose $A \subseteq \omega_2$. Let $f: \omega_2 \to 2$ be the characteristic function of A (i.e., $f(\alpha) = 1$ iff $\alpha \in A$). Let $C = \{\alpha < \omega_2: \forall \beta < \alpha \ g_f(\beta) < \alpha\}$. Then C is closed unbounded in ω_2. Suppose $\alpha \in C$, $\operatorname{cf}\alpha = \omega_1$. We must show $C \cap \alpha$, $A \cap \alpha \in S_\alpha$. Clearly $f|\alpha \in W_\alpha$. But then $A \cap \alpha = A_{f|\alpha} \in S_\alpha$ and $C \cap \alpha = C_{f|\alpha} \in S_\alpha$ so we are done.

8. Reflection principles and PFA$^+$

The following assertion is the strongest version of PFA that we know of. We shall denote it by PFA$^+$.

(PFA$^+$) Suppose P is a proper partial ordering, $\langle D_\alpha: \alpha < \omega_1\rangle$ are dense sets and $\langle \dot{S}_\alpha: \alpha < \omega_1\rangle$ is a sequence of terms such that $\forall \alpha < \omega_1 \Vdash \dot{S}_\alpha$ is stationary in ω_1. Then there exists a filter $G \subseteq P$ such that

(1) $\forall \alpha < \omega_1 \ G \cap D_\alpha \neq 0$, and
(2) $\forall \alpha < \omega_1 \ S_\alpha(G) = \{\xi < \omega_1: \exists p \in G \, p \Vdash \xi \in \dot{S}_\alpha\}$ is stationary in ω_1.

We do not know whether PFA$^+$ is a consequence of PFA, although we conjecture that it is not. Nevertheless, it is not difficult to see that the usual consistency proof of PFA (Theorem 3.1; see DEVLIN [1978]) actually yields the consistency of PFA$^+$. A similar remark holds for MA$^+(\omega_1)$, the enhancement of MA(ω_1) obtained by adding the analogue of (2). All the usual consistency proofs of MA(ω_1) seem to yield the consistency of MA$^+(\omega_1)$, but we have not been able to deduce MA$^+(\omega_1)$ from MA(ω_1).

The situations are not completely parallel, however. It *is* possible to deduce from MA(ω_1) the weaker version of MA$^+(\omega_1)$ in which only one stationary set $\dot{S}_\alpha$ is considered. As we shall see below, the analogous weakening of PFA$^+$ is still very strong indeed, even for countably closed orderings, while PFA for countably closed orderings is simply a theorem of ZFC!

The following two easy facts about stationary sets will be needed for our first application of PFA$^+$.

8.1. LEMMA. *Suppose α is an ordinal and* $\operatorname{cf} \alpha = \kappa > \omega$. *Let $f: \kappa \to \alpha$ enumerate a closed unbounded subset of α in increasing order. Then for all $A \subseteq \alpha$, A is stationary in α iff $f^{-1}(A) = \{\xi < \kappa: f(\xi) \in A\}$ is stationary in κ.*

PROOF. Left to the reader.

8.2. LEMMA. *Suppose α is an ordinal and $\operatorname{cf}\alpha = \kappa > \omega$. Then for all $A \subseteq \{\xi < \kappa : \operatorname{cf}\xi = \omega\}$, A is stationary in α iff $\{x \in [\alpha]^{\leq\omega} : \sup x \in A\}$ is stationary in $[\alpha]^{\leq\omega}$.*

PROOF. First suppose A is non-stationary. If $C \subseteq \alpha$ is a closed unbounded set disjoint from A, then $\{x \in [\alpha]^{\leq\omega} : \sup x \in C\}$ is a closed unbounded set in $[\alpha]^{\leq\omega}$ disjoint from $\{x \in [\alpha]^{\leq\omega} : \sup x \in A\}$. Now suppose C is a closed unbounded set in $[\alpha]^{\leq\omega}$ disjoint from $\{x \in [\alpha]^{\leq\omega} : \sup x \in A\}$. By Theorem 1.4, we may assume that for some $g : [\alpha]^{<\omega} \to \alpha$, $C = \{x \in [\alpha]^{\leq\omega} : x$ is closed under $g\}$. Let $X \subseteq \alpha$ be a cofinal subset of order type κ. Then for each $\beta < \alpha$ there is a minimal $h(\beta) < \alpha$ such that the closure of $X \cap \beta$ under g is a subset of $h(\beta)$. Let $D = \{\xi < \alpha : \forall \beta < \xi\; h(\beta) < \xi\}$. Then D is closed unbounded in α, so if A is stationary in α, then $\exists \xi \in D \cap A$ ξ is a limit point of X. Clearly, if Y is the closure under g of $X \cap \xi$, then $Y \subseteq \xi$ also. Now choose a countable subset of $X \cap \xi$ cofinal in ξ and let x be its closure under g. Since $x \subseteq Y$ we must have $\sup x = \xi \in A$, contradiction, since $x \in C$.

8.3. THEOREM. *Assume* PFA$^+$. *If α is any ordinal with $\operatorname{cf}\alpha > \omega_1$ and $A \subseteq \{\xi < \alpha : \operatorname{cf}\xi = \omega\}$ is stationary in α, then $\exists \beta < \alpha$ $\operatorname{cf}\beta = \omega_1$ and $A \cap \beta$ is stationary in β.*

PROOF. Let P consist of all countable closed subsets of α, with $p \leqslant q$ iff p is an end-extension of q. Then P is countably closed, and it is clear that if G is P-generic, then in $V[G]$, $\bigcup G$ is a closed unbounded subset of α with order type ω_1. If $A \subseteq \{\xi < \alpha : \operatorname{cf}\xi = \omega\}$ is stationary in α in V, then by Lemma 8.2 and the fact that P is proper, A remains stationary in α in $V[G]$. Thus if $f \in V[G]$ enumerates $\bigcup G$ in increasing order, we have by Lemma 8.1 that $S = f^{-1}(A)$ is stationary in ω_1. Let $\dot{S}$ be a term denoting S.

Now apply PFA$^+$ to $\dot{S}$. By choosing $\aleph_1$ dense sets judiciously, we can ensure that if H is a filter meeting all of them and $S(H) = \{\xi : \exists p \in H\, p \Vdash \xi \in \dot{S}\}$ is stationary, then $\bigcup H$ is closed unbounded in some ordinal $\beta < \alpha$ with $\operatorname{cf}\beta = \omega_1$, and if $g : \omega_1 \to \bigcup H$ enumerates $\bigcup H$ in increasing order, then $S(H) = \{\xi : g(\xi) \in A\}$. But then by Lemma 8.1 we see that $A \cap \beta$ is stationary in β.

A completely similar argument using all of PFA$^+$ yields the following.

8.4. THEOREM. *Under the hypotheses of Theorem 8.3, if $\forall \gamma < \omega_1 A_\gamma \subseteq \{\xi < \alpha : \operatorname{cf}\xi = \omega\}$ is stationary in α, then $\exists \beta < \alpha$ $\operatorname{cf}\beta = \omega_1$ and $\forall \gamma < \omega_1$ $A_\gamma \cap \beta$ is stationary in β.*

This kind of stationary-set reflection principle for α the successor of a singular cardinal already implies the consistency of the existence of many measurable cardinals. See KANAMORI & MAGIDOR [19 · ·], p. 222. Thus PFA$^+$ is quite strong indeed. Notice also that Theorems 8.3 and 8.4 apply to arbitrarily large ordinals, a situation without precedent in the case of MA alone.

There are some similar results for stationary sets of branches through trees. For terminology on trees, see Section 7 and Todorčević's article in this volume. For the rest of this section, we shall be interested only in trees of height ω.

Suppose T is a tree of height ω. Recall that a set B of branches through T is called *non-stationary* if there exists one-one $f: B \to T$ such that $f(b) \in b$ for all $b \in B$. Otherwise B is *stationary*. This terminology is justified by the following result.

8.5. THEOREM. *Let T be a tree of height ω and let B be a set of infinite branches through T. Then B is stationary iff $S(B) = \{x \in [B \cup T]^{\leq\omega} : \exists b \in B \;\; b \subseteq x$ but $b \notin x\}$ is stationary in $[B \cup T]^{\leq\omega}$.*

PROOF. First suppose $S(B)$ is stationary, $f: B \to T$ and $f(b) \in b$ for all $b \in B$. For each $x \in S(B)$, fix $b_x \subseteq x$, $b_x \in B - x$, and let $g(x) = f(b_x)$. Then g is regressive on $S(B)$ so there is stationary $S' \subseteq S(B)$ on which g is constant. If $x \in S'$, then since S' is stationary there is $y \in S'$ with $b_x \in y$. But then $b_x \neq b_y$ but $f(b_x) = g(x) = g(y) = f(b_y)$ so f is not one–one.

Now suppose $S(B)$ is not stationary. By Theorem 1.4, there is g: $[B \cup T]^{<\omega} \to B \cup T$ such that $\{x \in [B \cup T]^{\omega} : x$ is closed under $g\}$ is disjoint from $S(B)$. Without loss of generality we may assume that g is chosen so that if x is closed under g and $b \in B \cap x$, then $b \subseteq x$ and if $t \in x$ and $s <_T t$, then $s \in x$.

To complete the proof it will suffice to establish the following lemma.

8.6. LEMMA. *If $X \subseteq B \cup T$ is closed under g then $B \cap X$ is non-stationary.* (*Note*: *X may be uncountable*).

PROOF. The proof is by induction on $|X|$. If $|X| = \aleph_0$, then the lemma is clear. Suppose $|X| > \aleph_0$, and let $\langle X_\alpha : \alpha < \mathrm{cf}(|X|)\rangle$ be an increasing sequence of sets, each closed under g, such that $\bigcup\{X_\alpha : \alpha < \mathrm{cf}(|X|)\} = X$, each $|X_\alpha| < |X|$, and if α is a limit ordinal then $X_\alpha = \bigcup\{X_\beta : \beta < \alpha\}$. We construct, by induction on $\alpha < \mathrm{cf}(|X|)$, a function $f_\alpha : B \cap X_\alpha \to T \cap X_\alpha$ showing that $B \cap X_\alpha$ is non-stationary. Moreover, the f_α will be constructed so that, if $\beta < \alpha$, then $f_\beta \subseteq f_\alpha$. The cases $\alpha = 0$ and α limit present no problem. The only difficulty lies in producing $f_{\alpha+1}$.

We know by inductive hypothesis that there is some function $\bar{f}: B \cap X_{\alpha+1} \to T \cap X_{\alpha+1}$ showing that $B \cap X_{\alpha+1}$ is non-stationary. Suppose $b \in B \cap X_{\alpha+1}$, $b \notin X_\alpha$. Now $b \not\subseteq X_\alpha$ since otherwise the closure of b under g would contain b (since this set cannot lie in $S(B)$) and we would have $b \in X_\alpha$. By our assumptions about g, it follows that there is $n \in \omega$ such that all elements of b of height $<n$ lie in X_α, while all elements of b of height $\geqslant_n$ lie in $X_{\alpha+1} - X_\alpha$. Suppose $\bar{f}(b)$ has height m. Now let

$f_{\alpha+1}(b)$ be the element of b of height $m+n$. It is routine to see that $f_{\alpha+1}$ works (of course $f_{\alpha+1}$ is defined to agree with f_α on $B \cap X_\alpha$). This completes the proof.

8.7. THEOREM. *Assume* PFA$^+$. *Let T be a tree of height ω and let B be a stationary set of branches through T. Then there is $B' \subseteq B$, $|B'| = \aleph_1$, such that B' is also stationary.*

PROOF. Clearly $|B| \geq \aleph_1$. Let P be the usual countably closed ordering for collapsing $|B|$ onto $\aleph_1$. Since P is proper, we have by Theorem 8.5 that B remains stationary in V^P. Let $\dot{f}$ be a term such that $\Vdash \dot{f}: B \cup T \to \omega_1$ is a bijection. Let $\dot{S}$ be a name for $\{\alpha < \omega_1: \exists b \in B\ \forall t \in b\ f(t) < \alpha$ but $f(b) \geq \alpha\}$. Then by Theorem 8.5, $\Vdash \dot{S}$ is stationary.

For each $\alpha < \omega_1$ and $n < \omega$, let $D_{\alpha n} = \{p \in P$: either $\exists t \in T\ p \Vdash \dot{f}(t) = \alpha$ or else $\exists b \in B\ p \Vdash \dot{f}(b) = \alpha$ and if s is the element of b of level n, then $\exists \beta\ p \Vdash \dot{f}(s) = \beta\}$. For each $\alpha < \omega_1$, let $E_\alpha = \{p \in P$: either $p \Vdash \alpha \notin \dot{S}$ or else $p \Vdash \alpha \in \dot{S}$ and $\exists b \in B$ $\exists \beta \geq \alpha\ p \Vdash \dot{f}(b) = \beta$ and $p \Vdash \{\dot{f}(t): t \in b\} \subseteq \alpha\}$. Clearly the $D_{\alpha n}$ and E_α are dense. By PFA$^+$, let $G \subseteq P$ be a filter meeting the $D_{\alpha n}$ and E_α, and such that

$$S(G) = \{\alpha < \omega_1: \exists p \in G\ p \Vdash \alpha \in \dot{S}\}$$

is stationary. Let $f = \{(x, \alpha): \exists p \in G\ p \Vdash \dot{f}(x) = \alpha\}$. Then f is a bijection between some $X \subseteq B \cup T$ and ω_1, and since $S(G)$ is stationary it follows by Theorem 8.5 and the observation following Theorem 1.3 that $B' = X \cap B$ is stationary. (Just consult the $D_{\alpha n}$ and E_α; details are left to the reader.)

It turns out that the conclusion of Theorem 8.7 implies that the so-called gap-one two-cardinal transfer property $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$ fails whenever κ is a singular strong limit cardinal of cofinality ω. We shall not attempt the proof here, but we refer the reader to BEN-DAVID [1978], where similar results are obtained.

9. Concluding remarks

One of the greatest advantages of PFA over MA, namely its large-cardinal consistency strength, is also one of its disadvantages. For example, none of the consequences of PFA deduced in Sections 3–6 require any large cardinals at all.

How is it possible to tell whether large cardinals are needed, and if they are, how does one tell exactly which cardinals are needed? In a sense, this is a problem that set theorists face all the time, and there is no easy answer. In the case of PFA, one might envision a sequence of versions of PFA graded according to consistency strength. Unfortunately, no such sequence is available at present.

At the lowest level, there is the following result of SHELAH [1982].

9.1. THEOREM. (Shelah). *Let* PFA$^-$ *denote* PFA *for partial orderings of cardinality* $\leq \aleph_1$. *If* ZF *is consistent, then so is* ZFC $+ 2^{\aleph_0} = \aleph_2 +$ PFA$^-$.

Whereas Theorem 9.1 may seem to be the answer, a quick check will show that PFA$^-$ applies only in Section 3 and in Theorem 7.1 (Theorem 5.10 of Todorčević's article in this volume). What about the results in Sections 4–6?

There are several *ad hoc* approaches which may be applied. The consistency of the consequences of PFA in Section 6 may be established by ordinary finite-support (or countable-support) iterated ccc forcing, making use of Theorems 6.1 and 6.10 and the fact that iterating fewer than ω_2 steps leaves CH true.

A similar approach will work for Theorem 4.3 as well, provided that one makes use of a $\diamond$ sequence on $\{\alpha < 2^\omega : \mathrm{cf}\,\alpha = \omega_1\}$ to guess initial segments of potential gaps as they occur, and then uses Theorem 4.2(c) to destroy them in ccc fashion.

The case of TOP in Section 5 is more difficult. In SHELAH [1982], Shelah defines the κ-properness isomorphism condition (κ-pic) and shows that iterated κ-pic forcing retains the κ-pic. One may check that the consistency of TOP may be established by countable-support iterated $\aleph_2$-pic forcing, but all these details are left to the reader. Shelah states a version of PFA which applies to $\aleph_2$-pic orderings as Axiom I_b in Lemma 3.1 of SHELAH [1982], but the axiom is really a disguised iteration condition and so is not completely satisfactory. Nevertheless it is the best version of PFA not requiring large cardinals which is currently available.

The failure of wKH (see Theorem 7.10) requires at least an inaccessible cardinal. See Section 8 of Todorčević's article in this volume for a discussion. A consistency proof for $\neg$wKH + MA which uses only an inaccessible cardinal may be found in Section 8 of BAUMGARTNER [1978]. Essentially each tree of height ω_1 and cardinality $\aleph_1$ is made special in three steps: first adjoin $\aleph_2$ Cohen reals, then collapse $2^{\aleph_1}$ onto $\aleph_1$ and note that the tree has only $\aleph_1$ uncountable branches at this point, and finally use Theorem 7.9 to force the tree to be special. This is repeated for κ steps, where κ is inaccessible.

The consistency of $\neg$wKH + MA was independently obtained by TODORČEVIĆ [1981]. The consistency of $\neg$wKH itself is due to MITCHELL [1972].

In theorem 7.2 a weakly compact cardinal is required. See Section 7 of Todorčević's article in this volume. Both this proof and the proof of Theorem 7.10 require only compositions of countably closed and ccc partial orderings. All such orderings satisfy Axiom A, and since finite compositions of Axiom A orderings still satisfy Axiom A, these results could have been deduced from PFA(Axiom A) in the following unpublished result of the author (a proof is implicitly contained in BAUMGARTNER [1978].

9.2. THEOREM. *Let* PFA(Axiom A) *denote* PFA *for Axiom* A *orderings* P *such that* $|P| \leq 2^{\aleph_0}$. *If* ZFC + "*there is a weakly compact cardinal*" *is consistent, then so is* ZFC $+ 2^{\aleph_0} = \aleph_2 +$ PFA(Axiom A).

As we remarked in Section 8, PFA^+ is strong enough to imply the consistency of the existence of measurable cardinals, but nothing stronger than the consistency of a weakly compact cardinal is known to follow from PFA itself.

Shelah has also found versions of PFA which are relatively consistent with CH and apply to certain orderings which do not add reals. In fact, it was the search for such axioms which originally led Shelah to the definition of proper. These axioms imply Souslin's Hypothesis, for example. However, since the statement of these axioms does not seem to have reached final form, and since the applications are frequently more elaborate than the applications of PFA presented here, we have chosen not to include them. See Section 5 of SHELAH [1982]. The reader who has made it this far should have no difficulty continuing with SHELAH [1982].

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CHAPTER 22

Borel Measures

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

1. Introduction

The aim of this chapter is a systematic introduction to certain recent developments in the study of measures in topological spaces.

We hope to present a self-contained exposition, to guide the beginner from some simple and well known results to the frontiers of present knowledge. For this reason we assume very little, and give proofs in detail except for a few peripheral results. The reader is expected to know only the rudiments of measure theory, and no set theory beyond the ability to manipulate ordinals. Our expectation of the reader's knowledge of topology is somewhat higher, because he can easily consult other chapters of the Handbook.

The first systematic study of the relationship between measure and topology appeared in the long paper of Varadarajan [1961]. Although many important papers preceded it, Varadarajan's work may be taken as a beginning of topological measure theory. The subject now has an extensive literature, and its fair share of differing treatments and terminology; the survey article of GARDNER [1982] will give some idea of what has followed. Therefore, we had to be rather selective with the choice of material.

Contrary to the preference of Varadarajan and many others, we have chosen the theory of Borel measures as a framework. The emphasis is mainly on those topics, all quite recent, which involve the foundation of mathematics. The stimulants of these were Martin's axiom, publicized by MARTIN and SOLOVAY [1970], and Jensen's axiom $\diamond$ of JENSEN [1972]. Typically, Martin's axiom gives positive answers to certain questions we consider, while the continuum hypothesis or $\diamond$ provide counterexamples. Many of the counterexamples stem from the ingenious construction of OSTASZEWSKI [1976]. As far as we know, almost all results of this type are included here, with a couple of notable exceptions: the paper of CHOKSI and FREMLIN [1979] on completion regular measures, and some recent unpublished work of Fremlin in which Martin's axiom is used.

It is our intention that the chapter will be of use in several ways. For those already familiar with the theory of Borel measures, Section 11 will clarify the relation between the approach of SCHWARTZ [1973] and that of the authors (GARDNER [1975] and PFEFFER [1977]) to the study of Radon spaces. To the reader who knows the theory of Baire measures, Section 14 may help explain the link to Borel measures. And for the topologist to whom all this is new, we hope to prove that measure theory provides a powerful tool in working with various covering properties.

Here and there are scattered some very incomplete historical remarks. As we paid only minimal attention to these, many deserving authors may not have received due credit.

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2. Conventions and definitions

If A and B are sets, we denote by $\mathscr{P}(A)$ and A^B the families of all subsets of A and all maps from B into A, respectively. If $f\colon B \to A$ is a map and $C \subset B$, then $f \restriction C$ is the restriction of f to C.

Throughout this chapter, we shall identify an ordinal with the set of all smaller ordinals. Thus if α and β are ordinals, then $\alpha \in \beta$ and $\alpha \subset \beta$ have the same meaning as $\alpha < \beta$ and $\alpha \leqslant \beta$, respectively. The cardinals are identified with the initial ordinals, and denoted by $0, 1, \ldots, \omega_0, \omega_1, \ldots$. If κ is a cardinal, then 2^κ denotes both the set of all maps from κ into $2 = \{0, 1\}$ and the cardinality of this set. As usual, we shall write ω and $\mathfrak{c}$ instead of ω_0 and 2^{ω_0}, respectively. The cardinality of a set A is denoted by $|A|$.

By $\mathbb{R}$ we shall denote the set of all real numbers. Unless specified otherwise, $\mathbb{R}$ will always be equipped with its usual order topology. Let T be a set, and let $X \subset \mathbb{R}^T$. The topology in X induced by the product topology in $\mathbb{R}^T$ is called the *Euclidean topology* in X and it is denoted by $\mathscr{E}$.

Let X be a set, and let $\mathscr{M}$ be a σ-algebra in X. A *measure* on $\mathscr{M}$ is a σ-additive function $\mu\colon \mathscr{M} \to [0, +\infty]$ such that $\mu(\emptyset) = 0$. A measure μ on $\mathscr{M}$ is called *trivial* if $\mu(X) = 0$, *finite* if $\mu(X) < +\infty$, and *σ-finite* if $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathscr{M}$ and $\mu(X_n) < +\infty$, $n = 1, 2, \ldots$. A *probability* on $\mathscr{M}$ is a measure μ on $\mathscr{M}$ for which $\mu(X) = 1$. A measure μ on $\mathscr{M}$ which takes only the values 0 and 1 is called *2-valued*. If μ is a measure on $\mathscr{M}$, we call the triple $(X, \mathscr{M}, \mu)$ a *measure space*.

By λ we shall denote the *Lebesgue measure* on the σ-algebra Λ of all *Lebesgue measurable* subsets of $\mathbb{R}$.

Throughout, a space will always mean a *Hausdorff* topological space. If X is a space and $A \subset X$, we denote by A^- and A° the *closure* and the *interior* of A, respectively. A *neighborhood* of a point $x \in X$ or a set $A \subset X$ is a set $U \subset X$ (not necessarily open) such that $x \in U^\circ$ or $A \subset U^\circ$, respectively.

Let X be a space. By $\mathscr{G}(X)$, $\mathscr{F}(X)$, and $\mathscr{K}(X)$ we shall denote the families of all open, closed and compact subsets of X, respectively. The *Borel σ-algebra* $\mathscr{B}(X)$ in X is the smallest σ-algebra in X containing $\mathscr{G}(X)$. The elements of $\mathscr{B}(X)$ are called *Borel sets*. When no confusion can arise, we shall write $\mathscr{G}$, $\mathscr{F}$, $\mathscr{K}$, and $\mathscr{B}$ instead of $\mathscr{G}(X)$, $\mathscr{F}(X)$, $\mathscr{K}(X)$, and $\mathscr{B}(X)$, respectively.

2.1. DEFINITION. A *Borel measure* in a space X is a measure μ on $\mathscr{B}$ which is *locally finite*, i.e., each $x \in X$ has a neighborhood $U \in \mathscr{B}$ with $\mu(U) < +\infty$.

We remark that a Borel measure is not usually taken to be locally finite by those working outside of *topological* measure theory. For example, the famous Hausdorff measures (see ROGERS [1970]) are generally not locally finite.

We shall agree throughout that the statement "μ is a Borel measure in X" tacitly implies that X is a space.

If μ is a Borel measure in X and $A \subset X$, we let

$$\mu^*(A) = \inf\{\mu(G): A \subset G, G \in \mathcal{G}\},$$

$$\mu_*(A) = \sup\{\mu(F): F \subset A, F \in \mathcal{F}\}.$$

2.2. DEFINITION. Let μ be a Borel measure in X. A set $B \in \mathcal{B}$ is called
(i) μ*-outer regular* if $\mu(B) = \mu^*(B)$;
(ii) μ*-inner regular* if $\mu(B) = \mu_*(B)$;
(iii) μ*-Radon* if $\mu(B) = \sup\{\mu(K): K \subset A, K \in \mathcal{K}\}$.
Accordingly, μ is called *outer regular, inner regular*, and *Radon* if each $B \in \mathcal{B}$ is μ-outer regular, μ-inner regular, and μ-Radon, respectively. If μ is both outer and inner regular, it is called *regular.*

Clearly, a *finite* Borel measure in X is regular whenever it is either inner regular or outer regular. For infinite Borel measures, this simple relationship is generally false (see 6.5 and 12.7).

As we are dealing with Hausdorff spaces only, each Radon measure is also inner regular. In general, the converse is false (see 5.2, 5.4, or 5.5).

A nonempty family $\mathcal{A}$ of sets is called directed *upwards* or *downwards* if for each A, B in $\mathcal{A}$ there is a C in $\mathcal{A}$ such that $A \cup B \subset C$ or $C \subset A \cap B$, respectively. If $\mathcal{A}$ is directed upwards and $A = \bigcup\mathcal{A}$, we write $\mathcal{A} \nearrow A$. Similarly, if $\mathcal{A}$ is directed downwards and $A = \bigcap\mathcal{A}$, we write $\mathcal{A} \searrow A$.

2.3. DEFINITION. A Borel measure μ in X is called
(i) *weakly τ-additive* if

$$\sup\{\mu(G): G \in \mathcal{G}_0\} = \mu(X) \quad \text{for each } \mathcal{G}_0 \subset \mathcal{G} \text{ with } \mathcal{G}_0 \nearrow X;$$

(ii) *τ-additive* if

$$\sup\{\mu(G): G \in \mathcal{G}_0\} = \mu(G_0) \quad \text{for each } \mathcal{G}_0 \subset \mathcal{G} \text{ with } \mathcal{G}_0 \nearrow G_0.$$

2.4. REMARK. It is easy to see that a *finite* Borel measure in X is
(i) weakly τ-additive if and only if

$$\inf\{\mu(F): F \in \mathcal{F}_0\} = 0 \quad \text{for each } \mathcal{F}_0 \subset \mathcal{F} \text{ with } \mathcal{F}_0 \searrow \emptyset;$$

(ii) τ-additive if and only if

$$\inf\{\mu(F): F \in \mathcal{F}_0\} = \mu(F_0) \quad \text{for each } \mathcal{F}_0 \subset \mathcal{F} \text{ with } \mathcal{F}_0 \searrow F_0.$$

For an arbitrary Borel measure μ in X the previous statement is false (see 5.1).

2.5. DEFINITION. The *support* of a Borel measure μ in X is the set of all $x \in X$ such that $\mu(U) > 0$ for each open neighborhood U of x.

Let μ be a Borel measure in X with support S. It is clear from the definition that S is a *closed* subset of X. If $\mu(X - S) = 0$, we say that μ is *fully supported.* If $S = \emptyset$, the measure μ is called *locally trivial*; for each $x \in X$ has a neighborhood $U \in \mathcal{G}$ with $\mu(U) = 0$. We shall see in 5.5 that a locally trivial measure need not be trivial.

Important examples illustrating these definitions are given in Section 5.

3. Restrictions and extensions of Borel measures

The purpose of this section is to summarize briefly some measure-theoretic facts which will be needed in the sequel. However, this summary is *not* a prerequisite for all that follows. The reader may choose to skip it at this point, and return only when the need arises.

3.1. DEFINITION. An *outer measure* in a set X is a function $\varphi: \mathcal{P}(X) \to [0, +\infty]$ such that

(i) $\varphi(\emptyset) = 0$;
(ii) $\varphi(A) \leqslant \varphi(B)$ for each $A \subset B \subset X$;
(iii) $\varphi(\bigcup_{n=1}^{\infty} A_n) \leqslant \sum_{n=1}^{\infty} \varphi(A_n)$ for each sequence of $A_n \subset X$, $n = 1, 2, \ldots$.

Given an outer measure φ in a set X, we say that $E \subset X$ is *φ-measurable* whenever

$$\varphi(M) = \varphi(M \cap E) + \varphi(M - E)$$

for each $M \subset X$. This unintuitive definition, due to CARATHÉODORY [1918], yields surprising but standard results.

3.2. THEOREM. *If φ is an outer measure in a set X, then the following hold.*

(i) *The family $\mathcal{M}$ of all φ-measurable subsets of X is a σ-algebra in X.*
(ii) *If $E_n \in \mathcal{M}$, $n = 1, 2, \ldots$, are disjoint and $A \subset X$, then*

$$\varphi\Big[\bigcup_{n=1}^{\infty} (A \cap E_n)\Big] = \sum_{n=1}^{\infty} \varphi(A \cap E_n)\,.$$

In particular, $\varphi \restriction \mathcal{M}$ is a measure on $\mathcal{M}$.

(iii) *If $A \subset X$ and $\varphi(A) = 0$, then $A \in \mathcal{M}$.*

3.3. THEOREM. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let*

$$\mu^{\#}(A) = \inf\{\mu(B)\colon A \subset B,\ B \in \mathcal{M}\}$$

for each $A \subset X$. Then $\mu^{\#}$ is an outer measure in X, and the following hold.

(i) *The* σ-*algebra* $\mathcal{M}^{\#}$ *of all* $\mu^{\#}$-*measurable subsets of* X *contains* $\mathcal{M}$, *and* $\mu^{\#}\restriction\mathcal{M} = \mu$.

(ii) *A set* $E \subset X$ *belongs to* $\mathcal{M}^{\#}$ *if and only if* $E \cap B \in \mathcal{M}^{\#}$ *for each* $B \in \mathcal{M}$ *with* $\mu(B) < +\infty$.

(iii) *A set* $E \subset X$ *with* $\mu^{\#}(E) < +\infty$ *belongs to* $\mathcal{M}^{\#}$ *if and only if there is a set* $B \in \mathcal{M}$ *such that* $\mu^{\#}[(E-B)\cup(B-E)] = 0$.

The proofs of Theorems 3.2 and 3.3 are not difficult, and they can be found in many textbooks on real analysis. For a brief self-contained treatment, we refer to PFEFFER [1977], (12-3)–(12-11), pp. 156–159.

The next proposition is the main vehicle for applying these results to Borel measures.

3.4. PROPOSITION. *If* X *is a space and* $Y \subset X$, *then* $\mathcal{B}(Y) = \{B \cap Y : B \in \mathcal{B}(X)\}$. *In particular,* $\mathcal{B}(Y) = \{B \in \mathcal{B}(X) : B \subset Y\}$ *whenever* $Y \in \mathcal{B}(X)$.

PROOF. The family $\mathcal{A} = \{B \cap Y : B \in \mathcal{B}(X)\}$ is a σ-algebra in Y containing $\mathcal{G}(Y)$. Hence $\mathcal{B}(Y) \subset \mathcal{A}$. On the other hand, the family $\mathcal{H} = \{B \subset X : B \cap Y \in \mathcal{B}(Y)\}$ is a σ-algebra in X containing $\mathcal{G}(X)$. Thus $\mathcal{B}(X) \subset \mathcal{H}$, and so $\mathcal{A} \subset \mathcal{B}(Y)$. □

3.5. CONSTRUCTION. Let μ be a Borel measure in X. If $Y \subset X$, then by 3.3, 3.4 and 3.2(ii), $\mu_Y = \mu^{\#}\restriction\mathcal{B}(Y)$ is a Borel measure in Y. We say that μ_Y is the *restriction* of the Borel measure μ.

If $Y \in \mathcal{B}(X)$, it follows from 3.4 and 3.3(i) that $\mu_Y = \mu\restriction\mathcal{B}(Y)$. However, the reader should compare this with 5.3.

3.6. PROPOSITION. *Let* μ *be an outer regular Borel measure in* X, *and let* $Y \subset X$. *Then* $\mu^{\#} = \mu^{*}$, *and* μ_Y *is outer regular.*

PROOF. It is easy to see that $\mu^{\#} = \mu^{*}$. If $B \in \mathcal{B}(Y)$, we have

$$\begin{aligned}\mu_Y(B) &= \mu^{*}(B) = \inf\{\mu(G) : B \subset G,\ G \in \mathcal{G}(X)\}\\ &\geqslant \inf\{\mu^{*}(G \cap Y) : B \subset G,\ G \in \mathcal{G}(X)\}\\ &= \inf\{\mu_Y(H) : B \subset H,\ H \in \mathcal{G}(Y)\} \geqslant \mu_Y(B)\,. \qquad \square\end{aligned}$$

Note. It is easy to verify that for any Borel measure μ in X, μ^{*} is an outer measure in X. However, if μ is not outer regular, the relationship between μ and μ^{*} is not particularly useful.

3.7. CONSTRUCTION. Let X be a space, $Y \subset X$, and let μ be a *finite* Borel measure in Y. In view of 3.4, we can define a Borel measure ${}_X\mu$ in X by letting ${}_X\mu(B) = \mu(B \cap Y)$ for each $B \in \mathcal{B}(X)$. We say that ${}_X\mu$ is the *extension* of the Borel measure μ.

It is easy to check that $({}_X\mu)_Y = \mu$.

Note that if μ is not finite, ${}_X\mu$ may not be locally finite. Thus unlike restriction, extension cannot be defined for arbitrary Borel measures.

3.8. PROPOSITION. *Let X be a space, $Y \subset X$, and let μ be a finite Borel measure in Y. Then μ is τ-additive if and only if ${}_X\mu$ is τ-additive.*

PROOF. Let μ be τ-additive, $\mathscr{G}_0 \subset \mathscr{G}(X)$, and $\mathscr{G}_0 \nearrow G_0$. If $\mathscr{H} = \{G \cap Y: G \in \mathscr{G}_0\}$, then $\mathscr{H} \subset \mathscr{G}(Y)$ and $\mathscr{H} \nearrow G_0 \cap Y$. Thus

$$ {}_X\mu(G_0) = \mu(G_0 \cap Y) = \sup\{\mu(H): H \in \mathscr{H}\} = \sup\{{}_X\mu(G): G \in \mathscr{G}_0\}. $$

Conversely, let ${}_X\mu$ be τ-additive, $\mathscr{A}_0 \subset \mathscr{G}(Y)$, and $\mathscr{A}_0 \nearrow U_0$. For each $U \in \mathscr{A}_0$ there is a $U\hat{} \in \mathscr{G}(X)$ with $U\hat{} \cap Y = U$. Let $\mathscr{V}$ be the family of all unions $\bigcup\{U\hat{}: U \in \mathscr{A}_1\}$ where $\mathscr{A}_1$ is a finite subcollection of $\mathscr{A}_0$. Then $\mathscr{V} \subset \mathscr{G}(X)$, $\mathscr{V} \nearrow V_0$, and $V_0 \cap Y = U_0$. Hence

$$ \mu(U_0) = {}_X\mu(V_0) = \sup\{{}_X\mu(V): V \in \mathscr{V}\} = \sup\{\mu(U): U \in \mathscr{A}_0\}. \qquad \square $$

3.9. REMARK. If Y is a *closed* subspace of X and μ is a finite inner regular Borel measure in Y, then it is easy to see that ${}_X\mu$ is also inner regular. Example 5.5 shows that this is false if $Y \subset X$ is not closed.

3.10. REMARK. It is immediately verifiable that the restrictions and extensions of 2-valued Borel measures are also 2-valued.

Let X be a space, and let $Z \subset Y \subset X$. If μ is a finite Borel measure in Z, then it is obvious that ${}_X\mu = {}_X({}_Y\mu)$. On the other hand, if μ is a Borel measure in X, then some proof is needed to obtain that $\mu_Z = (\mu_Y)_Z$.

3.11. LEMMA. *Let μ be a Borel measure in X, and let $Y \subset X$. Then $(\mu_Y)^\# = \mu^\# \restriction \mathscr{P}(Y)$.*

PROOF. Let $A \subset Y$. Using 3.4, we obtain

$$ \begin{aligned} (\mu_Y)^\#(A) &= \inf\{\mu^\#(C): A \subset C, C \in \mathscr{B}(Y)\} \geqslant \mu^\#(A) \\ &= \inf\{\mu(B): A \subset B, B \in \mathscr{B}(X)\} \\ &\geqslant \inf\{\mu^\#(B \cap Y): A \subset B, B \in \mathscr{B}(X)\} \geqslant (\mu_Y)^\#(A). \qquad \square \end{aligned} $$

3.12. COROLLARY. *Let μ be a Borel measure in X, and let $Z \subset Y \subset X$. Then $\mu_Z = (\mu_Y)_Z$.*

3.13. COROLLARY. *Let μ be a Borel measure in X, and let $Y \subset X$ be $\mu^\#$-measurable. Then $E \subset Y$ is $\mu^\#$-measurable if and only if it is $(\mu_Y)^\#$-measurable.*

PROOF. Let $E \subset Y$ be $(\mu_Y)^{\#}$-measurable, and let $M \subset X$. By 3.11,

$$\begin{aligned}\mu^{\#}(M) &= \mu^{\#}(M \cap Y) + \mu^{\#}(M - Y)\\ &= \mu^{\#}(M \cap Y \cap E) + \mu^{\#}(M \cap Y - E) + \mu^{\#}(M - Y)\\ &\geq \mu^{\#}(M \cap E) + \mu^{\#}[(M \cap Y - E) \cup (M - Y)]\\ &= \mu^{\#}(M \cap E) + \mu^{\#}(M - E) \geq \mu^{\#}(M),\end{aligned}$$

and so E is $\mu^{\#}$-measurable. In view of 3.11, the converse is obvious. □

The following proposition is useful for estimating the cardinality of a Borel σ-algebra.

3.14. PROPOSITION. *If X is a space, then*
(i) $|\mathscr{G}| \geq \omega \Rightarrow |\mathscr{B}| \geq \mathfrak{c}$; (ii) $|\mathscr{G}| \leq \mathfrak{c} \Rightarrow |\mathscr{B}| \leq \mathfrak{c}$.

We omit the simple proof (see PFEFFER [1977], (7-10), p. 94).

4. Set-theoretic preliminaries

Unless specified otherwise, our universe throughout will be the Zermelo–Fraenkel set theory including the axiom of choice (abbreviated as ZFC). However, at some points we shall use additional set-theoretic axioms which we introduce in this section. The reader may prefer to postpone its reading until the time when these axioms are actually used.

4.1. *Continuum Hypothesis* (abbreviated as CH). This familiar axiom given by

(CH) $\qquad \mathfrak{c} = \omega_1$

is well known and it does not require any comments.

4.2. *Jensen's Axiom* (abbreviated as $\diamondsuit$). Giving ω_1 the order topology, we call a set $A \subset \omega_1$ *stationary* if $A \cap F \neq \emptyset$ for each uncountable closed set $F \subset \omega_1$. Intuitively, we may think of stationary sets as 'substantial' subsets of ω_1 (see 5.5). In particular, stationary sets are uncountable.

($\diamondsuit$) For each $\alpha \in \omega_1$ there is an $S_\alpha \subset \alpha$ such that the set $\{\alpha \in \omega_1 \colon A \cap \alpha = S_\alpha\}$ is stationary for each $A \subset \omega_1$.

One might think of S_α as an a priori 'guess' for $A \cap \alpha$. Then $\diamondsuit$ says that no matter which $A \subset \omega_1$ we choose, we 'guessed' correctly for 'substantially' many $\alpha \in \omega_1$.

4.3. PROPOSITION. $\diamondsuit \Rightarrow$ CH.

PROOF. Choose an $A \subset \omega$. Since the set $\{\alpha \in \omega_1 : A \cap \alpha = S_\alpha\}$ is stationary, it contains a $\beta > \omega$. Thus $A = A \cap \beta = S_\beta$, and we have $\mathcal{P}(\omega) \subset \{S_\alpha : \alpha \in \omega_1\}$. □

4.4. *Martin's Axiom* (abbreviated as MA). This axiom is usually formulated in terms of partially ordered sets or Boolean algebras (see JECH [1978], §23). However, an equivalent topological formulation will be more suitable for our purposes (see RUDIN [1977], Theorem 2, p. 492).

We say that a space X satisfies the *countable chain condition* (abbreviated as ccc) if each disjoint family of open subsets of X is countable. A space X satisfies the ccc whenever either of the following conditions holds:

(i) X is separable;

(ii) X is the support of a σ-finite Borel measure in X.

The sufficiency of (i) is obvious; the sufficiency of (ii) will be proved in 12.2.

(MA) In every nonempty compact space satisfying the ccc, the intersection of fewer than $\mathfrak{c}$ open dense sets is nonempty.

The next proposition clarifies the role of the ccc in MA.

4.5. PROPOSITION. CH *holds if and only if in every nonempty compact space the intersection of fewer than $\mathfrak{c}$ open dense sets is nonempty.*

PROOF. If CH holds, then fewer than $\mathfrak{c}$ means countably many. Thus it suffices to apply the Baire category theorem (see ENGELKING [1977], 3.9.4, p. 254).

Conversely, let $\omega_1 < \mathfrak{c}$. Give $\omega_1 + 1$ the order topology, and $X = (\omega_1 + 1)^\omega$ the product topology. Then X is a compact space. Let A be the set of all isolated ordinals from ω_1, and for each $\alpha \in A$ set $G_\alpha = \{x \in X : \alpha \in x(\omega)\}$. The sets G_α are open dense subsets of X, and $\bigcap_{\alpha \in A} G_\alpha = \emptyset$. Indeed, if $x \in \bigcap_{\alpha \in A} G_\alpha$, then $x : \omega \to A$ is surjective contrary to $\omega < |A|$. □

We note that the space X from the previous proof does not satisfy the ccc; for $H_\alpha = \{x \in X : x(0) = \alpha\}$, $\alpha \in A$, are open and disjoint.

4.6. COROLLARY. CH $\Rightarrow$ MA.

The assumptions that CH or $\diamondsuit$ or MA + $\neg$CH holds are consistent with ZFC, i.e., if ZFC is consistent, then so are ZFC + CH, ZFC + $\diamondsuit$, and ZFC + MA + $\neg$ CH (see JECH [1978]).

Whenever the axioms CH, $\diamondsuit$, MA, or MA + $\neg$CH are used in proving a theorem or constructing an example, we shall always indicate this by placing the appropriate axiom in parentheses immediately after the heading Theorem or Example (e.g., see 4.8 and 5.8).

Using MA + $\neg$ CH, we shall prove some results of TALL [1973], which will be needed in Sections 11 and 12.

4.7. LEMMA. *Let X be a space which satisfies the* ccc, *and let $\{H_\alpha: \alpha < \omega_1\} \subset \mathcal{G}$ be such that $H_\beta \subset H_\alpha$ for each $\alpha < \beta < \omega_1$. Then there is a $\gamma < \omega_1$ such that $H_\alpha^- = H_\gamma^-$ whenever $\gamma \leq \alpha < \omega_1$.*

PROOF. If the lemma does not hold, then for each $\alpha < \omega_1$ there is a $\beta(\alpha)$ such that $\alpha < \beta(\alpha) < \omega_1$ and $H_{\beta(\alpha)}^- \subsetneqq H_\alpha^-$. Let $\alpha_0 = \beta(0)$, and if $\alpha_\gamma < \omega_1$ is defined for all $\gamma < \kappa < \omega_1$, let $\alpha_\kappa = \beta(\tau)$ where $\tau = \sup\{\alpha_\gamma: \gamma < \kappa\}$. Then $\{H_{\alpha_\gamma} - H_{\beta(\alpha_\gamma)}^-: \gamma < \omega_1\}$ is an uncountable disjoint family of open sets; a contradiction. $\square$

If X is a set, $B \subset X$, and $\mathcal{A} \subset \mathcal{P}(X)$, we let

$$\mathrm{st}(B, \mathcal{A}) = \{A \in \mathcal{A}: A \cap B \neq \emptyset\}.$$

If $x \in X$, we write $\mathrm{st}(x, \mathcal{A})$ instead of $\mathrm{st}(\{x\}, \mathcal{A})$; thus

$$\mathrm{st}(x, \mathcal{A}) = \{A \in \mathcal{A}: x \in A\}.$$

A family $\mathcal{A} \subset \mathcal{P}(X)$ is called

(i) *point-finite* or *point-countable* if $\mathrm{st}(x, \mathcal{A})$ is, respectively, finite or countable for each $x \in X$;

(ii) *star-finite* or *star-countable* if $\mathrm{st}(A, \mathcal{A})$ is, respectively, finite or countable for each $A \in \mathcal{A}$.

4.8. THEOREM (MA + $\neg$ CH). *Let X be a locally compact space satisfying the* ccc. *Then each point-countable family $\mathcal{G}_0 \subset \mathcal{G}$ is countable.*

PROOF. Let $\mathcal{G}_0 \subset \mathcal{G}$ be an uncountable family, and let $\alpha \mapsto G_\alpha$ be an injection of ω_1 into $\mathcal{G}_0$. If $H_\alpha = \bigcup_{\alpha \leq \beta < \omega_1} G_\beta$, then by 4.7 there is a $\gamma < \omega_1$ such that $H_\alpha^- = H_\gamma^-$ whenever $\gamma \leq \alpha < \omega_1$. Choose an open set $E \subset H_\gamma$ with E^- compact, and let $E_\alpha = E \cap H_\alpha$. Since E is open, E^- satisfies the ccc, and $E^- = E_\alpha^-$ whenever $\gamma \leq \alpha < \omega_1$. Thus by MA + $\neg$ CH applied to E^-, the intersection $D = \bigcap_{\gamma \leq \alpha < \omega_1} E_\alpha$ is not empty. However, $|\mathrm{st}(x, \mathcal{G}_0)| \geq \omega_1$ for each $x \in D$. $\square$

A space X is called *metacompact* or *metalindelöf* if each open cover of X has an open point-finite or point-countable refinement, respectively. Some basic properties of metacompact and metalindelöf spaces can be found in Burke's Chapter 9.

4.9. COROLLARY (MA + $\neg$ CH). *Each locally compact metalindelöf space which satisfies the* ccc *is Lindelöf.*

We say that a space X satisfies the ccc *locally* if each $x \in X$ has a neighborhood U which satisfies the ccc.

4.10. COROLLARY (MA + $\neg$ CH). *Each locally compact metalindelöf space which satisfies the* ccc *locally is paracompact.*

PROOF. Let X be a metalindelöf space which satisfies the ccc locally. Given an open cover of X, find an open point-countable refinement $\mathscr{G}_0$ such that each $G \in \mathscr{G}_0$ satisfies the ccc. By 4.8, $\mathscr{G}_0$ is star-countable and the corollary follows from Burke's Chapter 9, Section 3.12. □

The last set-theoretic device we need is the notion of a measurable cardinal. We introduce it now.

4.11. DEFINITION. A Borel measure μ in X is called *diffused* if $\mu(\{x\}) = 0$ for each $x \in X$.

4.12. DEFINITION. A cardinal κ is called *real-valued measurable* if there is a discrete space X with $|X| = \kappa$ and a diffused Borel probability ν in X. If the probability ν is 2-valued, then κ is called 2*-valued measurable.*

Thus a cardinal κ is real-valued (2-valued) measurable whenever there is a nontrivial (2-valued) probability defined on *all* subsets of κ.

We note that the previous definition, which is most convenient for our considerations, is different from the usual definition of measurable cardinals (see JECH [1978], §27). In particular, according to our definition, each cardinal larger than a measurable cardinal is also measurable.

In the following two theorems, we shall summarize some facts about measurable cardinals. For proofs we refer to JECH [1978].

4.13. THEOREM. (i) *The first real-valued or* 2*-valued measurable cardinal is, respectively, weakly or strongly inaccessible.*

(ii) *The cardinal* $\mathfrak{c}$ *is not* 2*-valued measurable.*

(iii) *If* $\mathfrak{c}$ *is not real-valued measurable, then the real-valued and* 2*-valued measurable cardinals coincide.*

(iv) MA *implies that* $\mathfrak{c}$ *is not real-valued measurable.*

4.14. THEOREM. *It is consistent with* ZFC *to assume that measurable cardinals do not exist. Moreover, in the* ZFC *set theory the following statements are equiconsistent (i.e., if one is consistent with* ZFC*, then so are the others):*

(i) *a* 2*-valued measurable cardinal exists;*

(ii) *a real-valued measurable cardinal exists;*

(iii) *the cardinal* $\mathfrak{c}$ *is real-valued measurable;*

(iv) *Lebesgue measure* λ *can be extended to a measure defined on* $\mathscr{P}(\mathbb{R})$.

Note. The relative consistency of $\diamondsuit$ was proved in JENSEN [1972], and that of

MA + ¬ CH in SOLOVAY and TENNENBAUM [1971]. Theorem 4.13(i)–(iii) is due to ULAM [1931]; (iv) is due to KUNEN [1968]. SOLOVAY [1971] obtained the equiconsistency of (i) and (iv) in Theorem 4.14.

5. Some examples

We shall present several important examples which will be used throughout for illustration and counterexamples.

5.1. *Weighted counting measure.* Let X be a space, $f: X \to [0, +\infty)$, and let

$$\mu(B) = \sum_{x \in B} f(x)$$

for each $B \in \mathscr{B}$. Then μ is a measure on $\mathscr{B}$ which, however, need not be locally finite. We shall call it the *weighted counting measure* in X with the *weight* f. If $f(x) = 1$ for each $x \in X$, we say that μ is the *counting measure* in X. If $f(x) = 1$ for an $x \in X$ and $f(y) = 0$ for each $y \in X - \{x\}$, then μ is called the *Dirac measure* in X concentrated at x; it is denoted by δ_x.

If ν is a finite Borel measure in X, there is a *unique decomposition* $\nu = \nu_1 + \nu_2$, where ν_1 is a diffused Borel measure in X (see 4.11), and ν_2 is a weighted counting measure in X. Indeed it suffices to let

$$\nu_2(B) = \sum_{x \in B} \nu(\{x\})$$

for each $B \in \mathscr{B}$, and $\nu_1 = \nu - \nu_2$. Clearly, ν_1 and ν_2 are both finite.

Let X be a discrete space, and let μ be a weighted counting measure in X with the weight f. Then it is clear that μ is a fully supported, τ-additive, outer regular, and Radon measure whose support is $S = \{x \in X: f(x) > 0\}$. Letting

$$\mathscr{F}_0 = \{F \subset S: |S - F| < \omega\},$$

we have $\mathscr{F}_0 \subset \mathscr{F}$ and $\mathscr{F}_0 \searrow \emptyset$. However, if $\mu(X) = +\infty$ then $\mu(F) = +\infty$ for each $F \in \mathscr{F}_0$.

5.2. *Sorgenfrey interval.* Let $X = [0, 1)$ have the topology generated by all intervals $[a, b) \subset X$. Then X is a hereditarily Lindelöf space (see KELLEY [1955], prbl. K, p. 59), and the Borel subsets of X coincide with the Euclidean Borel subsets of $[0, 1)$. Thus $\mu = \lambda \restriction \mathscr{B}$ is a fully supported Borel measure in X. It follows from 6.8 and 10.2 that μ is regular and τ-additive. Since μ is diffused and since compact subsets of X are countable, μ is not Radon.

5.3. *Split interval.* Let $X = [-1, 1)$ have the topology generated by all sets $[-b, -a) \cup [a, b) \subset X$. It is easy to see that X is compact. Since X is the union of two Sorgenfrey intervals $[-1, 0)$ and $[0, 1)$, it is also hereditarily Lindelöf. Moreover, each $B \in \mathscr{B}(X)$ is a Euclidean Borel subset of $[-1, 1)$. Thus $\mu = \lambda \restriction \mathscr{B}(X)$ is a fully supported Borel measure in X. By 6.8, μ is outer regular and Radon. In particular, $\mu^* = \mu^\#$ (see 3.6). By 6.9, μ is also τ-additive.

If $G \in \mathscr{G}(X)$ contains $[-1, 0)$ or $[0, 1)$, then $X - G$ is countable. Thus $\mu^*([-1, 0)) = \mu^*([0, 1)) = 2$, and the set $Y = [0, 1)$ is not μ^*-measurable. Moreover, $\mu_Y = 2\lambda \restriction \mathscr{B}(Y)$ (see 3.5).

Note. The space X from this example is due to P.S. Alexandroff (see ALEXANDROFF and URYSOHN [1929], Ch. V, §1.3). It is sometimes called 'two arrows', particularly in the Russian literature.

5.4. *Bernstein sets.* A set $S \subset [0, 1]$ is called a *Bernstein set* if both S and $[0, 1] - S$ contain only countable compact subsets. The existence of such sets follows from the axiom of choice (see HEWITT and STROMBERG [1965], (10.54), p. 146), and from the definition it is clear that $\lambda_*(S) = 0 < \lambda^*(S)$. In particular, $S \notin \Lambda$.

Let X be a Bernstein set with the Euclidean topology, and let $\mu = \lambda_X$ (see 3.5). Then μ is a fully supported, regular and τ-additive Borel measure in X whose support is X (see 6.8 and 10.2). Since μ is diffused, the definition of a Bernstein set implies that μ is not Radon.

5.5. *Dieudonné measure.* Let κ be a limit ordinal which is not cofinal with ω. Give κ the order topology and denote by $\mathscr{H}$ the family of all closed unbonded (abbreviated as cub) subsets of κ. The following claim is a special case of a more general result about the cub subsets of κ (see KUNEN [1980], Ch. II, Lemma 6.8, p. 78).

CLAIM. $\mathscr{H}$ is closed under countable intersections.

PROOF. Let F and H be in $\mathscr{H}$, and let $\alpha \in \kappa$. Find $x_n \in F$ and $y_n \in H$ so that $\alpha < x_n < y_n < x_{n+1}$ for each $n \in \omega$. Since $\sup x_n = \sup y_n$ belongs to $F \cap H$, we see that $F \cap H \in \mathscr{H}$. Let $F_n \in \mathscr{H}$, $n \in \omega$, and let $\beta \in \kappa$. Replacing F_n by $\bigcap_{k \in n} F_k$, we may assume that $F_0 \supset F_1 \supset \cdots$. If $z_n \in F_n$ are such that $\beta < z_n < z_{n+1}$ for each $n \in \omega$, then $\sup z_n$ belongs to $\bigcap_{n \in \omega} F_n$. Consequently, $\bigcap_{n \in \omega} F_n \in \mathscr{H}$. $\square$

Let $\mathscr{M}$ be the family of all sets $M \subset \kappa$ for which there is an $F \in \mathscr{H}$ with $F \subset M$ or $F \subset \kappa - M$. Using the claim, it is easy to see that $\mathscr{M}$ is a σ-algebra in κ containing $\mathscr{F}(\kappa)$, and hence also $\mathscr{B}(\kappa)$. Moreover, we can define a Borel measure μ in κ as follows: for each $B \in \mathscr{B}(\kappa)$, let $\mu(B) = 1$ if there is an $F \in \mathscr{H}$ with $F \subset B$, and $\mu(B) = 0$ otherwise.

From the definition it is clear that μ is inner regular and locally trivial. Being finite, μ is also regular. Since $\{\alpha: \alpha \in \kappa\} \nearrow \kappa$, μ is neither weakly τ-additive nor Radon.

Let $\nu = {}_{\kappa+1}\mu$ be the extension of μ (see 3.7). Then ν is a Borel measure in $\kappa + 1$ which is neither regular, nor Radon or τ-additive; for $\nu(K) = 0$ for each $K \in \mathscr{K}(\kappa)$. Since $\kappa + 1$ is compact, ν is weakly τ-additive (see 7.2). Finally, $\{\kappa + 1\}$ is the support of ν.

The measures μ and ν are called the *Dieudonné measures* in κ and $\kappa + 1$, respectively. They were essentially introduced in Dieudonné [1939].

5.6. *Measurable cardinal.* Let X be a discrete space of measurable cardinality (real-valued or 2-valued), and let μ be a diffused Borel probability in X. Then μ is locally trivial and regular, but neither weakly τ-additive nor Radon; for $\mathscr{K}(X) \nearrow X$.

Let $X^+ = X \cup \{\infty\}$ be a one point compactification of X, and let $\nu = {}_{X^+}\mu$ be the extension of μ (see 3.7). Then ν is a weakly τ-additive, non-τ-additive, non-regular, and non-Radon, Borel measure in X whose support is $\{\infty\}$.

5.7. *Haydon's space.* Among all sets $A \subset [0, 1]$ for which $\lambda^*(A) > 0$ choose one, Y, with the least cardinality; suppose $|Y| = \kappa$. Select a bijection $t \mapsto \alpha_t$ from Y onto κ, and set

$$X = \{(\alpha, t) \in \kappa \times Y: \alpha \leq \alpha_t\}.$$

Give κ the order topology, Y the Euclidean topology, and topologize X as a subspace of $\kappa \times Y$.

From our choice of Y it follows easily that κ is a limit ordinal which is not cofinal with ω. Thus we can define the Dieudonné measure μ in κ (see 5.5). In Y we shall consider the Borel measure λ_Y (see 3.5).

As Y is second countable, each $G \in \mathscr{G}(\kappa \times Y)$ is a countable union of open rectangles. Indeed, if $\{V_n: n \in \omega\}$ is an open base of Y, let

$$U_n = \bigcup\{H \in \mathscr{G}(\kappa): H \times V_n \subset G\},$$

$n \in \omega$, and observe that $G = \bigcup_{n\in\omega}(U_n \times V_n)$. It follows that the product measure $\mu \times \lambda_Y$ is a Borel measure in $\kappa \times Y$. Since μ and λ_Y are regular, it follows from 6.1 that each $G \in \mathscr{G}(\kappa \times Y)$ is $(\mu \times \lambda_Y)$-inner regular. By 6.2, $\mu \times \lambda_Y$ is regular, and so is the restriction $\nu = (\mu \times \lambda_Y)_X$ (see 3.5 and 3.6).

Let $G \in \mathscr{G}(\kappa \times Y)$, $X \subset G$, and let $G^\alpha = \{t \in Y: (\alpha, t) \in G\}$ for each $\alpha \in \kappa$. Then

$$|Y - G^\alpha| \leq |\{t \in Y: \alpha_t < \alpha\}| < \kappa,$$

and $\lambda^*(Y - G^\alpha) = 0$ by our choice of κ. Thus $\lambda_Y(G^\alpha) = \lambda^*(Y)$ for each $\alpha \in \kappa$. By the Fubini theorem

$$(\mu \times \lambda_Y)(G) = \int_\kappa \lambda_Y(G^\alpha)\, d\mu(\alpha) = \lambda^*(Y),$$

and we have

$$\nu(X) = (\mu \times \lambda_Y)^*(X) = \lambda^*(Y) > 0 .$$

As ν is locally trivial, it is neither Radon nor weakly τ-additive.

Haydon's space was introduced in HAYDON [1974]. Other examples in the chapter are based on it (see 7.12 and 10.5).

Note. For $\alpha \in \kappa$ and $t \in Y$, let

$$X^\alpha = \{t \in Y : (\alpha, t) \in X\} = \{t \in Y : \alpha \leqslant \alpha_t\},$$

$$X_t = \{\alpha \in \kappa : (\alpha, t) \in X\} = \{\alpha \in \kappa : \alpha \leqslant \alpha_t\} .$$

Since

$$\int_Y \mu(X_t)\, d\lambda_Y(t) = 0 < \lambda^*(Y) = \int_\kappa \lambda_Y(X^\alpha)\, d\mu(\alpha),$$

it follows from the Fubini theorem that X is not $(\mu \times \lambda_Y)^*$-measurable.

5.8. *JKR-space* (CH). JUHÁSZ, KUNEN, and RUDIN [1976] showed that if CH holds, then there is a topology $\mathcal{T}$ in $\mathbb{R}$ which is finer than the Euclidean topology (i.e., $\mathscr{E} \subset \mathcal{T}$), and such that the space $X = (\mathbb{R}, \mathcal{T})$ has the following properties.

(i) X is first countable, locally countable, and locally compact.

(ii) X is zero-dimensional, perfectly normal, and hereditarily separable.

(iii) For each $G \in \mathcal{T}$ there is an $E \in \mathscr{E}$ such that $E \subset G$ and $|G - E| \leqslant \omega$. In particular, the Borel subsets of X coincide with the Euclidean Borel subsets of $\mathbb{R}$.

The construction of the topology $\mathcal{T}$ is given in Roitman's Chapter 7, Section 4.1.

It follows from (iii) that $\mu = \lambda \restriction \mathscr{B}$ is a locally trivial, σ-finite, Borel measure in X which is neither Radon nor weakly τ-additive. By (ii) and 6.8, μ is regular.

5.9. *Ostaszewski's space* ($\diamondsuit$). Assuming $\diamondsuit$, OSTASZEWSKI [1976] showed that there is a topology $\mathcal{T}$ in ω_1 such that the space $X = (\omega_1, \mathcal{T})$ has the following properties.

(i) X is first countable, locally countable, and locally compact.

(ii) X is zero-dimensional, perfectly normal, and hereditarily separable.

(iii) $\alpha \in \mathcal{T}$ for each $\alpha \in \omega_1$, and if $G \in \mathcal{T}$ then G or $X - G$ is countable.

The construction of the topology $\mathcal{T}$ is given in Roitman's Chapter 7, Section 5.3.

It follows from (iii) that B or $X - B$ is countable for each $B \in \mathcal{B}$. Thus we can define a Borel measure μ in X by letting $\mu(B) = 0$ or $\mu(B) = 1$ according to whether $B \in \mathcal{B}$ is countable or uncountable, respectively. It is easy to see that μ is locally trivial and regular, but neither Radon nor weakly τ-additive.

Note. The full power of $\diamond$ is not needed for the construction of Ostaszewski's space: a weaker axiom denoted by $\clubsuit$ is sufficient (see Roitman's Chapter 7, Section 5.3). However, $\clubsuit$ + CH is equivalent to $\diamond$. Since we shall always use Ostaszewski's space in connection with CH, the introduction of $\clubsuit$ is not warranted.

5.10. *Kunen's space* (CH). Assuming CH, KUNEN [1981] constructed a zero-dimensional compact space Y with the following properties:

(i) Y is hereditarily Lindelöf;

(ii) there is a diffused Borel probability ν in Y such that $A \subset Y$ is nowhere dense if and only if $\nu^*(A) = 0$.

By (i) and 6.8, ν is Radon. It follows from (ii) that Y is not separable, and that $\nu(G) > 0$ for each nonempty open set $G \subset Y$. Thus Y is the support of ν, and by 12.2, Y satisfies the ccc. By (i) and ENGELKING [1977], Theorem 3.3.4, p. 197, Y is first countable. It follows from CH and JUHÁSZ [1971], 2.16, p. 23 that $|Y| = \omega_1$. Finally, as ν is diffused, Y has no isolated points.

The construction of the space Y and measure ν is accomplished by repeated applications of *projective limits*. The process is rather involved.

If $\alpha \leqslant \beta \leqslant \omega_1$, let $p_{\beta\alpha}: 2^\beta \to 2^\alpha$ be the natural projection of the *dyadic spaces*, i.e., $p_{\beta\alpha}(f) = f \upharpoonright \alpha$ for each $f \in 2^\beta$. Inductively, we shall define closed subspaces Y_α of 2^α and Radon probabilities ν_α in Y_α, $\alpha \leqslant \omega_1$, so that

(a) Y_α is the support of ν_α;

(b) if $\alpha \leqslant \beta \leqslant \omega_1$ and $\pi_{\beta\alpha} = p_{\beta\alpha} \upharpoonright Y_\beta$, then $\pi_{\beta\alpha}(Y_\beta) = Y_\alpha$ and

$$\nu_\beta[\pi_{\beta\alpha}^{-1}(B)] = \nu_\alpha(B) \quad \text{for each } B \in \mathcal{B}(Y_\alpha).$$

We note that each Y_α, $\alpha \leqslant \omega_1$, has a compact open base $\mathcal{U}_\alpha$ consisting of all sets

$$U(f, T) = \{g \in Y_\alpha : g \upharpoonright T = f \upharpoonright T\}$$

where $f \in Y_\alpha$ and $T \subset \alpha$ is finite. When the induction is completed, we let $Y = Y_{\omega_1}$ and $\nu = \nu_{\omega_1}$.

For $n < \omega$, let $Y_n = 2^n$ and let ν_n be the weighted counting measure in Y_n with the constant weight 2^{-n} (see 5.1).

Let $\omega \leqslant \beta \leqslant \omega_1$, and suppose that Y_α and ν_α have been defined for each $\alpha < \beta$.

First we show by induction that given $f_\alpha \in Y_\alpha$, $\alpha < \beta$, there are $f_\gamma \in Y_\gamma$, $\alpha < \gamma < \beta$, such that $f_\gamma \upharpoonright \kappa = f_\kappa$ whenever $\alpha \leqslant \kappa \leqslant \gamma < \beta$. Thus assume that such

f_γ's exist for $\alpha < \gamma < \tau < \beta$. If $\tau = \gamma + 1$, choose $f_\tau \in Y_\tau$ so that $f_\tau \restriction \gamma = f_\gamma$. If τ is a limit ordinal, find $f_\tau \in 2^\tau$ and $g_\gamma \in Y_\tau$ such that $f_\gamma = f_\tau \restriction \gamma = g_\gamma \restriction \gamma$ whenever $\alpha \leq \gamma < \tau$. Since Y_τ is closed, $f_\tau \in Y_\tau$.

Let β be a limit ordinal. We set $Y_\beta = \bigcap_{\alpha<\beta} p_{\beta\alpha}^{-1}(Y_\alpha)$ and topologize it as a subspace of 2^β. Thus Y_β is compact, and it follows from the previous paragraph that $\pi_{\beta\alpha}(Y_\beta) = Y_\alpha$ for each $\alpha \leq \beta$.

Let $\mathscr{H}$ be the family of all sets $\pi_{\beta\alpha}^{-1}(B)$ where $B \in \mathscr{B}(Y_\alpha)$ and $\alpha < \beta$. Then $\mathscr{H}$ is an algebra in Y_β, and $\mathscr{U}_\beta \subset \mathscr{H} \subset \mathscr{B}(Y_\beta)$. If $H \in \mathscr{H}$, then for some $\alpha < \beta$, $\pi_{\beta\alpha}(H) \in \mathscr{B}(Y_\alpha)$ and $H = \pi_{\beta\alpha}^{-1}[\pi_{\beta\alpha}(H)]$. Setting

$$v(H) = \nu_\alpha[\pi_{\beta\alpha}(H)]\,,$$

it follows from (b) that $v(H)$ is independent of α, and that v is an additive function on $\mathscr{H}$. If $\mathscr{H}_0 = \mathscr{H} \cap \mathscr{K}(Y_\beta)$, then $(Y_\beta, \mathscr{H}_0, v)$ is a *volume space* in the sense of PFEFFER [1977], p. 169. By ib. (13-7)–(13-10), pp. 169, 170, there is a Radon probability ν_β in Y_β such that $\nu_\beta(K) = v(K)$ for each $K \in \mathscr{H}_0$. As $\mathscr{U}_\beta \subset \mathscr{H}_0$, the support of ν_β is Y_β. Let $\alpha < \beta$ and $B \in \mathscr{B}(Y_\alpha)$. If C is a compact subset of $\pi_{\beta\alpha}^{-1}(B)$, then $\pi_{\beta\alpha}(C)$ is a compact subset of B, and

$$C \subset \pi_{\beta\alpha}^{-1}[\pi_{\beta\alpha}(C)] \subset \pi_{\beta\alpha}^{-1}(B)\,.$$

It follows that

$$\begin{aligned}\nu_\alpha(B) &= \sup\{\nu_\alpha(K) \colon K \subset B,\, K \in \mathscr{K}(Y_\alpha)\}\\ &= \sup\{\nu_\beta[\pi_{\beta\alpha}^{-1}(K)] \colon K \subset B,\, K \in \mathscr{K}(Y_\alpha)\} = \nu_\beta[\pi_{\beta\alpha}^{-1}(B)]\,.\end{aligned}$$

Consequently, $\nu_\beta(H) = v(H)$ for each $H \in \mathscr{H}$, and conditions (a) and (b) are verified.

The informed reader naturally observes that (Y_β, ν_β) is isomorphic to the projective limit of $\{(Y_\alpha, \nu_\alpha) \colon \alpha < \beta\}$, and that we have actually proved a highly specialized version of the Prokhorov theorem (see SCHWARTZ [1973], Ch. 1, Theorem 22, p. 81). Also ν_ω is the *Haar probability* in 2^ω (see HALMOS [1950], §58). Assuming its existence, we could have started the induction at ω.

Let $\beta = \alpha + 1$. For $\gamma \leq \alpha$, the space Y_γ is second countable; for 2^γ is metrizable and separable. Thus using CH, we have $\mathscr{F}(Y_\gamma) = \{F_\gamma^\xi \colon \xi < \omega_1\}$. Choose a map $\gamma \mapsto (\gamma', \gamma'')$ from ω_1 onto $\omega_1 \times \omega_1$ so that $\gamma' \leq \gamma$ for each $\gamma < \omega_1$. Let $F_\gamma = \pi_{\gamma\gamma'}^{-1}(F_{\gamma'}^{\gamma''})$, $\gamma \leq \alpha$, and set

$$N = \bigcup\{\pi_{\alpha\gamma}^{-1}(F_\gamma) \colon \nu_\gamma(F_\gamma) = 0,\ \gamma \leq \alpha\}\,.$$

Clearly, $\nu_\alpha(N) = 0$. If $\nu_\alpha(F_\alpha) = 0$, set $C_\alpha = \emptyset$. If $\nu_\alpha(F_\alpha) > 0$, let C_α be a compact subset of $F_\alpha - N$ such that $\nu_\alpha(C_\alpha \cap G) > 0$ for each $G \in \mathscr{G}(Y_\alpha)$ with $C_\alpha \cap G \neq \emptyset$.

Since ν_α is a Radon measure, the existence of such C_α follows from 6.9 and 6.14(ii). Now we let $Y_\beta = (Y_\alpha \times \{0\}) \cup (C_\alpha \times \{1\})$ and topologize it as a subspace of 2^β. If $B \in \mathscr{B}(Y_\beta)$, we let

$$B_0 = \pi_{\beta\alpha}[B \cap (Y_\alpha \times \{0\})]\,, \qquad B_1 = \pi_{\beta\alpha}[B \cap (C_\alpha \times \{1\})]\,,$$

and set

$$\nu_\beta(B) = \nu_\alpha(B_0 - C_\alpha) + \tfrac{1}{2}[\nu_\alpha(B_0 \cap C_\alpha) + \nu_\alpha(B_1)]\,.$$

A straightforward verification shows that Y_β and ν_β have the desired properties. As ν_ω is already diffused, it is clear that so is ν.

Claim 1. Let $A \subset Y$ be nowhere dense. Then A is second countable and $\nu^*(A) = 0$.

PROOF. Let $\mathscr{U}$ be a maximal disjoint subfamily of $\{U \in \mathscr{U}_{\omega_1}: U \cap A^- = \emptyset\}$, and let $E = Y - \bigcup\mathscr{U}$. Since A is nowhere dense, $\mathscr{U}$ is also a maximal disjoint subfamily of $\mathscr{G}(Y)$. By the ccc, $\mathscr{U}$ is countable. Hence there is an $\alpha < \omega_1$ and a family $\mathscr{V} \subset \mathscr{U}_\alpha$ such that $\mathscr{U} = \{\pi^{-1}_{\omega_1\alpha}(V): V \in \mathscr{V}\}$. Clearly, $Y_\alpha - \bigcup\mathscr{V} = F^\xi_\alpha$ for some $\xi < \omega_1$. Find $\beta < \omega_1$ with $(\beta', \beta'') = (\alpha, \xi)$. Then $\beta' \leq \beta$, and

$$E = \pi^{-1}_{\omega_1\alpha}(F^\xi_\alpha) = \pi^{-1}_{\omega_1\beta}[\pi^{-1}_{\beta\beta'}(F^{\beta''}_{\beta'})] = \pi^{-1}_{\omega_1\beta}(F_\beta) \supset \pi^{-1}_{\omega_1\beta}(C_\beta) = \pi^{-1}_{\omega_1\beta+1}(C_\beta \times \{1\})\,.$$

As $C_\beta \times \{1\}$ is an open subset of $Y_{\beta+1}$, it follows from the maximality of $\mathscr{U}$ in $\mathscr{G}(Y)$ that $C_\beta = \emptyset$. By our construction

$$\nu(E) = \nu_{\omega_1}[\pi^{-1}_{\omega_1\beta}(F_\beta)] = \nu_\beta(F_\beta) = 0\,,$$

and since $A \subset E$, we also have $\nu^*(A) = 0$. Moreover, $F_\beta \cap C_\gamma = \emptyset$ whenever $\beta \leq \gamma < \omega_1$, and it follows that $\pi_{\omega_1\beta} \restriction E$ is injective. Consequently, A is second countable; for it is homeomorphic to a subspace of a second countable space Y_β. $\square$

Claim 2. Let $A \subset Y$ and $\nu^*(A) = 0$. Then A is nowhere dense.

PROOF. Choose a nonempty $G \in \mathscr{G}(Y)$ and find an $H \in \mathscr{G}(Y)$ such that $A \subset H$ and $\nu(H) < \nu(G)$. Then $\nu(G - H) > 0$, and so there is a compact set $C \subset G - H$ with $\nu(C) > 0$. By Claim 1, $C^\circ \neq \emptyset$. Now $C^\circ \cap A = \emptyset$, so $C^\circ \cap A^- = \emptyset$, and it follows that $G \not\subset A^-$. $\square$

Claim 3. The space Y is hereditarily Lindelöf.

PROOF. Let $\mathscr{U} \subset \mathscr{G}(Y)$, let $\mathscr{V} = \bigcup\{\mathscr{G}(U): U \in \mathscr{U}\}$, and let $\mathscr{V}_0$ be a maximal disjoint subfamily of $\mathscr{V}$. Then $A = \bigcup\mathscr{U} - \bigcup\mathscr{V}_0$ is nowhere dense. By Claim 1, there is a countable family $\mathcal{U}_1 \subset \mathcal{U}$ such that $A \subset \bigcup \mathcal{U}_1$. By the ccc, $\mathcal{V}_0$ is countable, and hence there is a countable family $\mathcal{U}_0 \subset \mathcal{U}$ with $\bigcup \mathcal{U}_0 \supset \bigcup \mathcal{V}_0$. Consequently, $\bigcup \mathcal{U} = (\bigcup \mathcal{U}_0) \cup (\bigcup \mathcal{U}_1)$. $\square$

5.11. *Density topology.* For $A \in \Lambda$, let $\phi(A)$ denote the set of all $t \in \mathbb{R}$ such that

$$\lim_{h \to 0+} \frac{\lambda(A \cap [t-h, t+h])}{2h} = 1 .$$

The map $\phi : \Lambda \to \mathcal{P}(\mathbb{R})$ given by $A \mapsto \phi(A)$ is called the *Lebesgue lower density.* From the definition it is clear that $\phi(\emptyset) = \emptyset$, $\phi(\mathbb{R}) = \mathbb{R}$, and $G \subset \phi(G)$ for each $G \in \mathscr{E}$.

If A, B are subsets of $\mathbb{R}$, then $A \sim B$ signifies that

$$\lambda^*[(A - B) \cup (B - A)] = 0 .$$

Claim 1. If A, B are in Λ, then the following properties hold:
(i) $\phi(A) \sim A$;
(ii) $A \sim B \Rightarrow \phi(A) \sim \phi(B)$;
(iii) $A \subset B \Rightarrow \phi(A) \subset \phi(B)$;
(iv) $\phi(A \cap B) = \phi(A) \cap \phi(B)$.

Claim 2. The family $\mathcal{T} = \{A \in \Lambda : A \subset \phi(A)\}$ is a regular T_1 topology in $\mathbb{R}$ which is finer than the Euclidean topology, i.e., $\mathscr{E} \subset \mathcal{T}$. In the space $X = (\mathbb{R}, \mathcal{T})$, the following conditions are equivalent for each $A \subset X$:
(i) $\lambda^*(A) = 0$;
(ii) A is nowhere dense;
(iii) A is closed and discrete.

To prove the implication (iii)$\Rightarrow$(i) in Claim 2, it suffices to observe that each set $A \subset \mathbb{R}$ with $\lambda^*(A) > 0$ contains a Lebesgue nonmeasurable subset, and that all subsets of a closed discrete set are closed. Otherwise, the proofs of Claims 1 and 2 can be found in OXTOBY [1971], Theorem 3.21, p. 18, and Theorems 22.5, 22.6, 22.9, pp. 88–90.

Claim 3. Each subset of X is a union of a closed discrete set and a set which satisfies the ccc.

PROOF. Since $\mathscr{E} \subset \mathcal{T} \subset \Lambda$, $\lambda \restriction \mathscr{B}(X)$ is a σ-finite Borel measure in X whose support is X. By 12.2, X satisfies the ccc. If $A \subset X$, then

$$A = [A \cap (A^-)^\circ] \cup (A \cap [A^- - (A^-)^\circ]) .$$

The set $A \cap (A^-)^\circ$ is dense in an open set $(A^-)^\circ$, and hence satisfies the ccc. As $A \cap [A^- - (A^-)^\circ]$ is nowhere dense, it is closed and discrete by Claim 2. □

A *Sierpinski set* is an *uncountable* set $S \subset \mathbb{R}$ such that $|S \cap A| \leq \omega$ for each $A \subset \mathbb{R}$ with $\lambda^*(A) = 0$.

Claim 4. A set $S \subset \mathbb{R}$ is a Sierpinski set if and only if S is a nonseparable hereditarily Lindelöf subspace of X.

PROOF. Let $S \subset X$ be nonseparable and Lindelöf. Then S is uncountable and contains only countable closed discrete subsets. It follows from Claim 2 that S is a Sierpinski set.

Conversely, let S be a Sierpinski set. By Claim 2, countable subsets of X are closed, and hence S is not separable. By the same Claim, nowhere dense subsets of S are countable. From this and Claim 3, it follows that S satisfies the ccc. Now the proof that S is hereditarily Lindelöf is the same as that of 5.10, Claim 3. □

Claims 3 and 4 are due to TALL [1976]. The next claim was first proved by SIERPINSKI [1934].

Claim 5 (CH). There is a Sierpinski set.

PROOF. By 3.14 and CH, the family of all Euclidean Borel sets $B \subset \mathbb{R}$ with $\lambda(B) = 0$ can be enumerated as $\{B_\alpha : \alpha \in \omega_1\}$. For $\alpha \in \omega_1$, choose a $t_\alpha \in B_\alpha - \bigcup_{\beta \in \alpha} B_\beta$ whenever $B_\alpha - \bigcup_{\beta \in \alpha} B_\beta \neq \emptyset$. As each $A \subset \mathbb{R}$ with $\lambda^*(A) = 0$ is contained in some B_α, it is clear that the set S of all t_α's is a Sierpinski set. □

If there is a Sierpinski set, then using the translation invariance of λ, it is easy to see that there is one in $[0, 1]$. Let S be a Sierpinski set with the density topology. It follows from Claims 2 and 4, 10.2, and 10.3 that $\lambda^* \restriction \mathscr{B}(S)$ is a nontrivial, regular, τ-additive, Borel measure in S. By 6.14(ii), the support Y of $\lambda^* \restriction \mathscr{B}(S)$ is also a Sierpinski set. We summarize these facts.

Claim 6 (CH). There is a Sierpinski set $Y \subset [0, 1]$ such that giving Y the density topology, $\nu = \lambda^* \restriction \mathscr{B}(Y)$ is a regular, τ-additive, Borel measure in Y whose support is Y.

Since $\mathscr{G}(X) \subset \Lambda$, each $G \in \mathscr{G}(Y)$ is a union of a Euclidean Borel subset of Y and a countable set. It follows that $\mathscr{B}(Y)$ consists of all Euclidean Borel subsets of Y. In particular, by 3.14, $|\mathscr{B}(Y)| = \mathfrak{c}$.

6. Regularity and τ-additivity of Borel measures

We shall prove the classical results for the regularity of finite Borel measures, and establish a few basic connections between regularity, τ-additivity, and support properties.

6.1. LEMMA. *Let μ be a Borel measure in X, $A_n \in \mathscr{B}$, $n = 1, 2, \ldots$, and let $A = \bigcup_{n=1}^{\infty} A_n$. If each A_n is, respectively, μ-outer regular, μ-inner regular or μ-Radon, then so is A.*

PROOF. We may assume that $\mu(A_n) < +\infty$, $n = 1, 2, \ldots$; for if $\mu(A_n) = +\infty$ for some integer $n \geq 1$, then $\mu(A) = +\infty$, and the lemma clearly holds.

Let each A_n be μ-outer regular. Choose $\varepsilon > 0$ and $G_n \in \mathscr{G}$ such that $A_n \subset G_n$ and $\mu(G_n - A_n) < \varepsilon \cdot 2^{-n}$, $n = 1, 2, \ldots$. Then $G = \bigcup_{n=1}^{\infty} G_n$ is open, $A \subset G$, and

$$\mu(G - A) \leq \mu\Big[\bigcup_{n=1}^{\infty}(G_n - A_n)\Big] \leq \sum_{n=1}^{\infty} \mu(G_n - A_n) < \varepsilon .$$

It follows that A is μ-outer regular.

The proofs of the remaining two cases are identical. We shall carry it out only when each A_n is μ-Radon. Choose an $a < \mu(A)$, and select an integer $N \geq 1$ such that $\varepsilon = \mu(\bigcup_{n=1}^{N} A_n) - a$ is positive. Find $K_n \in \mathscr{K}$ with $K_n \subset A_n$ and $\mu(A_n - K_n) < \varepsilon/N$, $n = 1, \ldots, N$. Then $K = \bigcup_{n=1}^{N} K_n$ is compact, $K \subset \bigcup_{n=1}^{N} A_n \subset A$, and

$$\mu\Big(\bigcup_{n=1}^{N} A_n - K\Big) \leq \mu\Big[\bigcup_{n=1}^{N}(A_n - K_n)\Big] \leq \sum_{n=1}^{N} \mu(A_n - K_n) < \varepsilon .$$

Thus

$$\mu(K) > \mu\Big(\bigcup_{n=1}^{N} A_n\Big) - \varepsilon = a ,$$

and A is μ-Radon. $\square$

6.2. PROPOSITION. *Let μ be a finite Borel measure in X. If each open set is μ-inner regular, then μ is regular.*

PROOF. Let $\mathscr{A}$ be the family of all Borel sets which are both μ-inner and μ-outer regular. By our assumption $\mathscr{G} \subset \mathscr{A}$. Since μ is finite, it follows from 6.1 that $\mathscr{A}$ is a σ-algebra in X. Consequently, $\mathscr{A} = \mathscr{B}$. $\square$

6.3. LEMMA. *Let μ be an outer regular Borel measure in X such that each open set of finite measure is, respectively, μ-inner regular or μ-Radon. Then each Borel set of finite measure is μ-inner regular or μ-Radon.*

PROOF. We shall consider only the case when each open set of finite measure is μ-inner regular. The other case is similar. Let $A \in \mathscr{B}$ and $\mu(A) < +\infty$. Choose $\varepsilon > 0$ and $G \in \mathscr{G}$ such that $A \subset G$ and $\mu(G - A) < \varepsilon$. Next find $H \in \mathscr{G}$ for which $G - A \subset H$ and $\mu(H) < \varepsilon$. By our assumption there is a closed set $F \subset G$ with $\mu(F) > \mu(G) - \varepsilon$. Thus $C = F - H$ is a closed subset of A, and

$$\mu(C) \geqslant \mu(F) - \mu(H) > \mu(G) - 2\varepsilon \geqslant \mu(A) - 2\varepsilon . \quad \square$$

6.4. COROLLARY. *Let μ be a finite Borel measure in X. If each open set is μ-Radon, then μ is outer regular and Radon.*

6.5. EXAMPLE. Let Y be an uncountable discrete space. Then $X = Y \times [0, 1]$ is locally compact and metrizable. For each $B \in \mathscr{B}(X)$, let

$$\mu(B) = \sum_{y \in Y} \lambda(\{t \in [0, 1] : (y, t) \in B\}).$$

It is easy to see that μ is a Radon measure in X, and that the closed set $Y \times \{0\}$ is not μ-outer regular.

The measure μ from the previous example is not σ-finite. Extending Corollary 6.4 to σ-finite Borel measures is a nontrivial task which we shall consider in depth in Section 12. However, there is a class of Borel measures to which Proposition 6.2 and Corollary 6.4 extend quite easily.

6.6. DEFINITION. A Borel measure μ in X is called *moderated* if there are *open* sets $G_n \subset X$ such that $\mu(G_n) < +\infty$, $n = 1, 2, \ldots$, and $\bigcup_{n=1}^{\infty} G_n = X$.

Clearly, each moderated Borel measure is σ-finite, but not vice versa (see 12.6 or 12.7). On the other hand, an outer regular σ-finite Borel measure is moderated.

6.7. THEOREM. *Let μ be a moderated Borel measure in X. If each open set of finite measure is, respectively, μ-inner regular or μ-Radon, then μ is regular, or outer regular and Radon.*

PROOF. As the proofs are identical, it suffices to consider the case when each open set of finite measure is μ-Radon. Let $A \in \mathscr{B}(X)$, and find $G_n \in \mathscr{G}(X)$ such that $\mu(G_n) < +\infty$, $n = 1, 2, \ldots$, and $\bigcup_{n=1}^{\infty} G_n = X$. By 6.4, the restrictions $\mu_{G_n} = \mu \restriction \mathscr{B}(G_n)$ (see 3.5) are outer regular. Thus $A \cap G_n$, $n = 1, 2, \ldots$, is μ_{G_n}-outer regular, and since G_n is open, also μ-outer regular. By 6.1, $A = \bigcup_{n=1}^{\infty}(A \cap G_n)$ is μ-outer regular. Therefore, μ is outer regular. It follows from 6.3 that each $A \cap G_n$ is μ-Radon, and hence by 6.1, so is A. $\square$

6.8. COROLLARY. *Let each open subset of a space X be, respectively, F_σ or σ-compact. Then each moderated Borel measure in X is regular, or outer regular and Radon.*

We shall see in 12.5 that if open sets are σ-compact, Corollary 6.8 remains valid when 'moderated' is replaced by 'σ-finite'. Examples 12.6 or 12.7 show that this is not so if open sets are merely F_σ.

An interesting generalization of Corollary 6.8 is given in 10.9.

6.9. PROPOSITION. *Let μ be a Borel measure in X. If each open set is μ-Radon, then μ is τ-additive.*

PROOF. Let $\mathcal{G}_0 \subset \mathcal{G}$ and $\mathcal{G}_0 \nearrow G_0$. If $K \subset G_0$ is compact, then $K \subset G$ for some $G \in \mathcal{G}_0$. Hence

$$\mu(G_0) = \sup\{\mu(K): K \subset G_0, K \in \mathcal{K}\} \leq \sup\{\mu(G): G \in \mathcal{G}_0\}.$$

The reverse inequality is obvious. □

6.10. PROPOSITION. *Let μ be a Borel measure in X, and let $G \in \mathcal{G}(X)$ be such that $\mu_G = \mu \restriction \mathcal{B}(G)$ is weakly τ-additive. If X is regular or locally compact, then G is μ-inner regular or μ-Radon, respectively.*

PROOF. Suppose that X is locally compact. Then each $x \in G$ has an open neighborhood U for which U^- is a compact subset of G. If $\mathcal{H}$ is the family of all finite unions of such neighborhoods, then $\mathcal{H} \nearrow G$. By our assumption,

$$\begin{aligned}\mu(G) &= \sup\{\mu(U): U \in \mathcal{H}\} \leq \sup\{\mu(U^-): U \in \mathcal{H}\} \\ &\leq \sup\{\mu(K): K \subset G, K \in \mathcal{K}\} \leq \mu(G).\end{aligned}$$

The proof when X is regular is completely analogous. □

A direct consequence of 6.10 and 6.7 is the following corollary.

6.11. COROLLARY. *Let μ be a moderated τ-additive Borel measure in X. If X is regular or locally compact, then μ is regular, or outer regular and Radon, respectively.*

For Radon measures, Corollaries 6.8 and 6.11 will be substantially generalized in Sections 11 and 12.

Examples 5.2 or 5.4 show that the assumption of local compactness cannot be omitted in 6.10 and 6.11. The following example, due to ADAMSKI [1980], shows that the assumption of regularity of X is also essential.

6.12. EXAMPLE. Let S be a Bernstein set (see 5.4), and let $X = [0, 1]$ have the topology generated by the Euclidean topology together with the set S. Then X is a second countable space which is not regular; for closed subsets of S are closed in the Euclidean topology of S, and hence countable. Using 3.5 and 3.7, we let $\mu = {}_X(\lambda_S)$. It follows from 7.4 and 10.2 that μ is τ-additive; in particular, $\mu_S = \lambda_S$ is τ-additive (see 3.8). But since $\mu(S) > 0$, the open set S is not μ-inner regular.

6.13. PROPOSITION. *Let μ be a finite, weakly τ-additive Borel measure in X. If μ is also regular, then it is τ-additive.*

PROOF. Let $\mathscr{G}_0 \subset \mathscr{G}$ and $\mathscr{G}_0 \nearrow G_0$. Choose $a < \mu(G_0)$ and $F \in \mathscr{F}$ such that $F \subset G_0$ and $a < \mu(F)$. If $\mathscr{H} = \{G \cup (X - F)\colon G \in \mathscr{G}_0\}$, then $\mathscr{H} \subset \mathscr{G}$ and $\mathscr{H} \nearrow X$. Thus

$$\begin{aligned}\mu(X) &= \sup\{\mu(H)\colon H \in \mathscr{H}\}\\ &\leqslant \sup\{\mu(G) + \mu(X - F)\colon G \in \mathscr{G}_0\}\\ &= \sup\{\mu(G)\colon G \in \mathscr{G}_0\} + \mu(X - F),\end{aligned}$$

and we have

$$a < \mu(F) \leqslant \sup\{\mu(G)\colon G \in \mathscr{G}_0\}. \qquad \square$$

6.14. PROPOSITION. *Let μ be a Borel measure in X.*

(i) *If μ is weakly τ-additive and locally trivial, it is trivial.*

(ii) *If μ is τ-additive, it is fully supported.*

PROOF. If $\mathscr{G}_0 = \{G \in \mathscr{G}\colon \mu(G) = 0\}$, then $\mathscr{G}_0 \nearrow X - S$ where S is the support of μ. The proposition follows. $\square$

7. Completeness properties for finite Borel measures

The terminology in the following definition is motivated by analogy with 2-valued Baire measures (see Section 14) and the associated notion of real-compactness (see also GILLMAN and JERISON [1960]).

7.1. DEFINITION. A space X is called

(i) *Borel measure-complete* if each finite Borel measure in X is τ-additive;

(ii) *weakly Borel measure-complete* if each finite Borel measure in X is weakly τ-additive;

(iii) *Borel measure-compact* if each finite regular Borel measure in X is τ-additive.

Using 6.13, we immediately obtain:

$$\begin{array}{ccccc} \text{Borel} & \Rightarrow & \text{weakly Borel} & \Rightarrow & \text{Borel} \\ \text{measure-complete} & & \text{measure-complete} & & \text{measure-compact} \end{array}.$$

Examples showing that these implications cannot be reversed are given in 7.3 and 7.12.

7.2. PROPOSITION. *Each compact space is weakly Borel measure-complete.*

PROOF. If $\mathscr{F}_0 \subset \mathscr{F}$ and $\mathscr{F}_0 \searrow \emptyset$, then $F = \emptyset$ for some $F \in \mathscr{F}_0$. The proposition follows from 2.4(i). □

7.3. EXAMPLE. By 7.2 and 5.5, the set $\omega_1 + 1$ equipped with the order topology is a weakly Borel measure-complete space which is not Borel measure-complete.

The relationship between Borel measure-complete and weakly Borel measure-complete spaces is quite simple.

7.4. PROPOSITION. *A space X is Borel measure-complete if and only if it is hereditarily weakly Borel measure-complete. In particular, if X is a Borel measure-complete space, then so is each $Y \subset X$.*

PROOF. Let X be hereditarily weakly Borel measure-complete, and let μ be a finite Borel measure in X. Let $\mathscr{G}_0 \subset \mathscr{G}(X)$ and $\mathscr{G}_0 \nearrow G_0$. Then $\mathscr{G}_0 \subset \mathscr{G}(G_0)$, and by our assumption the restriction $\mu_{G_0} = \mu \upharpoonright \mathscr{B}(G_0)$ (see 3.5) is weakly τ-additive. Hence

$$\mu(G_0) = \mu_{G_0}(G_0) = \sup\{\mu_{G_0}(G) \colon G \in \mathscr{G}_0\} = \sup\{\mu(G) \colon G \in \mathscr{G}_0\}.$$

Conversely, let X be Borel measure-complete, let $Y \subset X$, and let μ be a finite Borel measure in Y. The extension ${}_X\mu$ of μ (see 3.7) is τ-additive, and by 3.8, so is μ. □

7.5. LEMMA. *Let μ be a finite Borel measure in X. If μ is not weakly τ-additive, then there is a locally trivial Borel probability ν in X. Moreover, ν is, respectively, regular or 2-valued whenever μ is.*

PROOF. Find $\mathscr{G}_0 \subset \mathscr{G}$ with $\mathscr{G}_0 \nearrow X$ and

$$\sup\{\mu(G) \colon G \in \mathscr{G}_0\} = a < \mu(X).$$

For $n = 1, 2, \ldots$, there are $G_n \in \mathscr{G}_0$ such that $G_n \subset G_{n+1}$ and $\mu(G_n) > a - 1/n$. If $G_0 = \bigcup_{n=1}^{\infty} G_n$, then

$$\nu = \frac{1}{\mu(X) - a} \cdot {}_X(\mu_{X-G_0})$$

(see 3.5 and 3.7) is a Borel probability in X; for $\mu(G_0) = a$.

Let $G \in \mathcal{G}_0$. If $\nu(G) > 0$, then $\mu(G - G_0) > 0$, and so

$$\lim \mu(G \cup G_n) = \mu(G \cup G_0) = \mu(G_0) + \mu(G - G_0) > a\,.$$

Thus $\mu(G \cup G_N) > a$ for some integer $N \geq 1$, and we obtain a contradiction by finding an $H \in \mathcal{G}_0$ with $G \cup G_N \subset H$. As $\mathcal{G}_0$ is directed upwards such an H indeed exists. Consequently, $\nu(G) = 0$ for each $G \in \mathcal{G}_0$, and ν is locally trivial; for $\bigcup \mathcal{G}_0 = X$.

If μ is 2-valued, then clearly so is ν (see 3.10). If μ is regular, then ν is regular by 3.6 and 3.9. □

7.6. THEOREM. *A space X is*

(i) *Borel measure-complete if and only if each Borel probability in X is fully supported*;

(ii) *weakly Borel measure-complete if and only if each Borel probability in X has a nonempty support*;

(iii) *Borel measure-compact if and only if each regular Borel probability in X has a nonempty support.*

PROOF. Properties (ii) and (iii) follow from 6.14(i) and 7.5.

To prove (i), let μ be a Borel probability in X, and let S be the support of μ. If $\mu(X - S) = a > 0$, then $\nu = (1/a)\mu_{X-S}$ (see 3.5) is a locally trivial Borel probability in $X - S$. By (ii) and 7.4, X is not Borel measure-complete. The converse follows from 6.14(ii). □

7.7. PROPOSITION. *Let Y be an F_σ subset of a space X. If X is weakly Borel measure-complete or Borel measure-compact, then so is Y, respectively.*

PROOF. Let μ be a Borel probability in Y. Since Y is F_σ, there is an $F \in \mathcal{F}(X)$ with $\mu(F) > 0$. Then $\nu = (1/\mu(F)) \cdot {}_X(\mu_F)$ is a Borel probability in X (see 3.5 and 3.7). Moreover, by 3.6, 3.9, and 6.2, ν is regular whenever μ is. As $\nu(X - F) = 0$, the support of ν is contained in F, and consequently in the support of μ. The proposition follows from 7.6(ii) and (iii). □

Example 5.5 shows that the assumption of Y being F_σ is essential in Proposition 7.7.

The next theorem is a direct consequence of 6.8.

7.8. THEOREM. *Let each open subset of a space X be F_σ. Then conditions* (i)–(iii) *of Definition* 7.1 *are equivalent.*

7.9. THEOREM. *Let X be a Borel measure-complete space. If X is regular or locally compact, then each moderated Borel measure in X is, respectively, regular, or outer regular and Radon.*

PROOF. Let X be regular, and let μ be a moderated Borel measure in X. It follows from 7.4 and 6.10 that each open set $G \subset X$ with $\mu(G) < +\infty$ is μ-inner regular. Thus μ is regular by 6.7.

If X is locally compact the proof is similar. □

It follows from 7.4 and 10.2 that the spaces from Examples 6.12 and 5.2 are Borel measure-complete. Thus without regularity or local compactness of X, the conclusion of Theorem 7.9 may be false.

As respectable spaces are usually regular, Theorem 7.9 is a satisfactory answer to the question of regularity of Borel measures in Borel measure-complete spaces. In contrast to this, local compactness is a very restrictive condition. Consequently, for Radon measures, a better result than that given by Theorem 7.9 is needed. We address this problem in Section 11.

Before constructing an example of a space which is Borel measure-compact but not weakly Borel measure-complete, we need a general result due to JOHNSON [1982].

7.10. PROPOSITION. *Let μ and ν be finite Borel measures in X and Y, respectively, and let X be first countable. Then there is a Borel measure χ in $X \times Y$ such that*

$$\chi(B) = \int_X \nu(B^x)\,\mathrm{d}\mu(x)$$

for each $B \in \mathscr{B}(X \times Y)$; here $B^x = \{y \in Y : (x, y) \in B\}$ for each $x \in X$.

PROOF. It suffices to show that the function $x \mapsto \nu(B^x)$ is $\mathscr{B}(X)$-measurable for each $B \in \mathscr{B}(X \times Y)$. Let $G \in \mathscr{G}(X \times Y)$, $x \in X$, and $\nu(G^x) > r \geqslant 0$. Choose a nested open neighborhood base $\{U_n : n \in \omega\}$ at x. If

$$V_n = \bigcup\{V \in \mathscr{G}(Y) : U_n \times V \subset G\},$$

then $V_n \subset V_{n+1}$, $n \in \omega$, and $\bigcup_{n\in\omega} V_n = G^x$. Thus $\nu(V_N) > r$ for some $N \in \omega$, and we have $\nu(G^t) \geqslant \nu(V_N) > r$ for each $t \in U_N$. This implies that $\{x \in X : \nu(G^x) > r\}$ is open, and the $\mathscr{B}(X)$-measurability of the function $x \mapsto \nu(G^x)$ follows. Extending this result to Borel subsets of $X \times Y$ is routine; however, it requires a structure theorem for Borel σ-algebras. The reader who wishes to fill in all the details is referred to PFEFFER [1977], (7-5), (7-6), p. 92, and (8-7), p. 104. □

7.11. REMARK. To indicate the asymmetric role of μ and ν, the Borel measure χ of Proposition 7.10 is denoted by $\nu \cdot \mathrm{d}\mu$. Using Fubini's theorem, it is easy to see that $\nu \cdot \mathrm{d}\mu$ is an extension of the product measure $\mu \times \nu$.

7.12. EXAMPLE (CH). Let $Y \subset [0, 1]$ be the Sierpinski set of 5.11, Claim 6, and let $t \mapsto \alpha_t$ be a bijection from Y onto ω_1. Give ω_1 the order topology, Y the density topology, and topologize

$$X = \{(\alpha, t) \in \omega_1 \times Y : \alpha \leq \alpha_t\}$$

as a subspace of $\omega_1 \times Y$. If $A \subset \omega_1 \times Y$ and $\alpha \in \omega_1$, let $A^\alpha = \{t \in Y : (\alpha, t) \in A\}$.

Claim 1. X is not weakly Borel measure-complete.

PROOF. Let μ be the Dieudonné measure in ω_1 (see 5.5), let $\nu = \lambda^* \restriction \mathscr{B}(Y)$, and let $\chi = (\nu \cdot \mathrm{d}\mu)_X$ (see 7.11 and 3.5). Since $\nu \cdot \mathrm{d}\mu(\alpha \times Y) = 0$ for each $\alpha \in \omega_1$, χ is locally trivial. Choose a $B \in \mathscr{B}(\omega_1 \times Y)$ with $X \subset B$. Given $\alpha \in \omega_1$,

$$Y - B^\alpha \subset \{t \in Y : \alpha_t < \alpha\}$$

is countable, and so $\nu(B^\alpha) = \nu(Y)$. Thus $\nu \cdot \mathrm{d}\mu(B) = \nu(Y)$, and it follows that χ is not weakly τ-additive; for $\chi(X) = (\nu \cdot \mathrm{d}\mu)^\#(X) = \nu(Y) > 0$. $\square$

To establish that X is Borel measure-compact is much harder. Proceeding towards a contradiction (see 7.6(iii)), we shall assume that there is a locally trivial, regular Borel probability η in X, and let

$$\psi(B) = \eta[(\omega_1 \times B) \cap X]$$

for each $B \in \mathscr{B}(Y)$. As Y is hereditarily Lindelöf (see 5.11, Claim 4), it follows from 10.2 that ψ is a τ-additive Borel probability in Y. Using CH, enumerate the families $\{B \in \mathscr{B}(Y) : \psi(B) > 0\}$ and $\{B \in \mathscr{B}(Y) : \psi(B) = 0\}$ as $\{P_\alpha : \alpha \in \omega_1\}$ and $\{N_\alpha : \alpha \in \omega_1\}$, respectively. This can be done, as $|\mathscr{B}(Y)| = \mathfrak{c}$ (see 5.11), and since by the local triviality of η, both η and ψ are diffused. Since

$$\psi(P_\alpha - \bigcup_{\beta \in \alpha} N_\beta) = \psi(P_\alpha) > 0,$$

we can select points $t_\alpha \in Y$ such that $t_\alpha \in P_\alpha - \bigcup_{\beta \in \alpha}(N_\beta \cup \{t_\beta\})$ and $\alpha \leq \alpha_{t_\alpha}$ for each $\alpha \in \omega_1$. Letting

$$E = \{t_\alpha : \alpha \in \omega_1\} \quad \text{and} \quad D = \{(\alpha, t_\alpha) : \alpha \in \omega_1\},$$

we see that $D \subset X$. We obtain a contradiction by showing that D is a Borel set which is not η-outer regular.

If $N \in \mathscr{B}(Y)$ and $\psi(N) = 0$, then $N = N_\alpha$ for some $\alpha \in \omega_1$, and so $E \cap N \subset \{t_\beta : \beta \leq \alpha\}$ is countable. As the map $\alpha \mapsto t_\alpha$ is injective, E is itself a 'Sierpinski set' with respect to the measure ψ.

Claim 2. Let $B \subset X$ be such that $|B^\alpha| \leq 1$ for each $\alpha \in \omega_1$. Then $B \in \mathscr{B}(X)$ and $\eta(B) = 0$.

PROOF. Clearly, $B = \{(\alpha, b_\alpha) : \alpha \in T\}$ for some $T \subset \omega_1$. By 5.11, Claim 2, $F_\alpha = \{b_\beta : \beta \in \alpha\}$ is a closed subset of Y for each $\alpha \in \omega_1$. Thus

$$G_\alpha = [(\alpha + 1) \times (Y - F_\alpha)] \cap X$$

is an open subset of X, and $B \cap G_\alpha = \{(\alpha, b_\alpha)\}$ for each $\alpha \in T$. It follows that B is discrete and hence Borel; for every discrete set is relatively closed in an open set. As ω_1 is not a real-valued measurable cardinal (see 4.13(i)), it is immediate that $\eta(B) = 0$. □

In particular, $D \in \mathscr{B}(X)$ and $\eta(D) = 0$.

Claim 3. Let $B \in \mathscr{B}(X)$ be such that $\psi(B^\alpha) = 0$ for all but countably many $\alpha \in \omega_1$. Then $\eta(B) = 0$.

PROOF. Since η is locally trivial and Y is Lindelöf,

$$\eta[(\alpha \times Y) \cap X] = 0 \quad \text{for each } \alpha \in \omega_1 .$$

Thus replacing B by $B \cap [(\omega_1 - \alpha) \times Y]$ with $\alpha \in \omega_1$ sufficiently large, we may assume that $\psi(B^\alpha) = 0$ for each $\alpha \in \omega_1$. Suppose that $\eta(B) > 0$, and choose an $F \in \mathscr{F}(X)$ with $F \subset B$ and $\eta(F) > 0$. Let $G = \bigcup_{\alpha \in \omega_1} (F^\alpha)^\circ$. As ψ is τ-additive, we have

$$\eta[(\omega_1 \times G) \cap X] = \psi(G) = 0 .$$

The sets $F^\alpha - (F^\alpha)^\circ$ are nowhere dense, and hence countable by 5.11, Claim 2; for Y is a Sierpinski set. If $H = F - [(\omega_1 \times G) \cap X]$, then $\eta(H) = \eta(F) > 0$, and

$$H^\alpha = F^\alpha - G \cap X^\alpha \subset [F^\alpha - (F^\alpha)^\circ] \cup (Y - X^\alpha)$$

for each $\alpha \in \omega_1$. Since $Y - X^\alpha = \{t \in Y : \alpha_t < \alpha\}$ is countable, Claim 2 implies that $\eta(H) = 0$; a contradiction. □

Recall that a stationary set was defined in 4.2.

Claim 4. There is an $\varepsilon > 0$ such that the set $\{\alpha \in \omega_1 : t_\alpha \notin B\}$ is stationary for each $B \in \mathscr{B}(Y)$ with $\psi(B) < \varepsilon$.

PROOF. Suppose that for each $n = 1, 2, \ldots$ there is a $B_n \in \mathscr{B}(Y)$ such that $\psi(B_n) < 1/n$ and $\{\alpha \in \omega_1 : t_\alpha \notin B_n\}$ is not stationary. By the Claim in 5.5,

$$M = \bigcup_{n=1}^{\infty}\{\alpha \in \omega_1 : t_\alpha \notin B_n\} = \{\alpha \in \omega_1 : t_\alpha \notin \bigcap_{n=1}^{\infty} B_n\}$$

is also a nonstationary set. However, this is impossible, as $\psi(\bigcap_{n=1}^{\infty} B_n) = 0$ implies that $\omega_1 - M$ is countable. $\square$

Claim 5. If $G \in \mathcal{G}(X)$ and $D \subset G$, then $\eta(G) \geqslant \varepsilon$, where $\varepsilon > 0$ is as in Claim 4.

PROOF. Suppose that $\eta(G) < \varepsilon$ for some $G \in \mathcal{G}(X)$ with $D \subset G$. Inductively, we shall construct sets $V_\alpha \in \mathcal{G}(Y)$, $\alpha \in \omega_1$, satisfying the following conditions:

(i) $\psi(V_\alpha - \bigcup_{\beta \in \alpha} V_\beta) > 0$ for each $\alpha \in \omega_1$;

(ii) if $W_\alpha = (\omega_1 \times V_\alpha) \cap X$, then $\eta(W_\alpha - G) = 0$.

It is clear that (i) contradicts the finiteness of ψ (see 12.1).

For each α with $0 < \alpha < \omega_1$ find a $\beta_\alpha \in \alpha$ and an open neighborhood U_α of t_α such that

$$X \cap \bigcup_{\alpha \in \omega_1} (\{\beta : \beta_\alpha < \beta \leqslant \alpha\} \times U_\alpha) \subset G .$$

Since the map $\alpha \mapsto \beta_\alpha$ is regressive, we can find a $\gamma \in \omega_1$ for which the set $T = \{\alpha \in \omega_1 : \beta_\alpha = \gamma\}$ is uncountable (see JECH [1978], Theorem 22, p. 59). If $H_\beta = \bigcup\{U_\alpha : \alpha \in T - \beta\}$, $\beta \in \omega_1$, then $\psi(H_\beta) > 0$ because $E \cap H_\beta$ is uncountable. As $\{\psi(H_\beta) : \beta \in \omega_1\}$ is a decreasing transfinite sequence of positive numbers, there are $r > 0$ and $\delta > \gamma$ such that $\psi(H_\beta) = r$ for each $\beta \geqslant \delta$. If $V_0 = H_\delta$ and $\beta \geqslant \delta$, then

$$(W_0 - G)^\beta \subset X^\beta \cap (H_\delta - H_\beta) ,$$

and consequently $\psi[(W_0 - G)^\beta] = 0$. By Claim 3, $\eta(W_0 - G) = 0$.

Next assume that V_α satisfying (i) and (ii) have been defined for each $\alpha < \kappa < \omega_1$, and let $W = \bigcup_{\alpha \in \kappa} W_\alpha$. Then

$$\eta(W - G) = \eta[\bigcup_{\alpha \in \kappa} (W_\alpha - G)] \leqslant \sum_{\alpha \in \kappa} \eta(W_\alpha - G) = 0 ,$$

and hence

$$\psi(\bigcup_{\alpha \in \kappa} V_\alpha) = \eta(W) \leqslant \eta(G) < \varepsilon .$$

By our choice of ε, the set $S = \{\alpha \in \omega_1 : t_\alpha \notin \bigcup_{\beta \in \kappa} V_\beta\}$ is stationary. Using again that $\alpha \mapsto \beta_\alpha$ is a regressive map, we can find a $\gamma' \in \omega_1$ for which the set $T' = \{\alpha \in S : \beta_\alpha = \gamma'\}$ is uncountable. Letting $H'_\beta = \bigcup\{U_\alpha : \alpha \in T' - \beta\}$, $\beta \in \omega_1$, we see as before that there are $r' > 0$ and $\delta' > \gamma'$ such that $\psi(H'_\beta) = r'$ for each $\beta \geqslant \delta'$. If $V_\kappa = H'_{\delta'}$ and $\beta \geqslant \delta'$, then

$$(W_\kappa - G)^\beta \subset X^\beta \cap (H'_{\delta'} - H_\beta),$$

and consequently, $\psi[(W_\kappa - G)^\beta] = 0$. By Claim 3, $\eta(W_\kappa - G) = 0$. Since $E \cap (V_\kappa - \bigcup_{\alpha\in\kappa} V_\alpha)$ is uncountable, $\psi(V_\kappa - \bigcup_{\alpha\in\kappa} V_\alpha) > 0$. □

We conclude that $\eta^*(D) > 0$, which contradicts the regularity of η. The Borel measure-compactness of X follows.

Note. The previous example is taken from GRUENHAGE and GARDNER [1978]. Its main feature is the *regularity* of the space X. With no set-theoretic assumptions, a *nonregular* space which is Borel measure-compact but not weakly Borel measure-complete is constructed in JOHNSON [1980].

8. Completeness properties for 2-valued Borel measures

In this section, we shall generalize the completeness properties defined in 7.1 by replacing the finite Borel measures with 2-valued Borel measures.

8.1. DEFINITION. A space X is called

(i) *Borel-complete* if each 2-valued Borel measure in X is τ-additive;

(ii) *weakly Borel-complete* if each 2-valued Borel measure in X is weakly τ-additive;

(iii) *closed-complete* if each 2-valued regular Borel measure in X is τ-additive.

We note that in the literature closed-complete spaces are sometimes referred to as *α-realcompact* (see DYKES [1970] and GARDNER [1975]).

From 6.13 we obtain

$$\begin{array}{ccc}
\text{Borel measure-complete} & \Rightarrow & \text{Borel-complete} \\
\Downarrow & & \Downarrow \\
\text{weakly Borel measure-complete} & \Rightarrow & \text{weakly Borel-complete} \\
\Downarrow & & \Downarrow \\
\text{Borel measure-compact} & \Rightarrow & \text{closed-complete}
\end{array}$$

Since the Dieudonné measure is 2-valued (see 5.5), the space from 7.3 is weakly Borel-complete but not Borel-complete. It follows from 5.7 that Haydon's space is not Borel measure-compact. We show in 8.9 that it is always weakly Borel-complete, and Borel-complete whenever CH holds. Finally, SIMON [1971] proved that the Dowker space of RUDIN [1971] is closed-complete but not weakly Borel-complete.

In view of 3.10 and 7.5, the proofs of the following five statements are identical to those of the corresponding statements in Section 7 (i.e., 7.4, 7.6–7.9).

8.2. PROPOSITION. *A space X is Borel-complete if and only if it is hereditarily weakly Borel-complete. In particular, if X is a Borel-complete space, then so is each $Y \subset X$.*

8.3. THEOREM. *A space X is*

(i) *Borel-complete if and only if each* 2*-valued Borel probability in X is fully supported*;

(ii) *weakly Borel-complete if and only if each* 2*-valued Borel probability in X has a nonempty support*;

(iii) *closed-complete if and only if each* 2*-valued regular Borel probability in X has a nonempty support.*

8.4. PROPOSITION. *Let Y be an F_σ subset of a space X. If X is weakly Borel-complete or closed-complete, then so is Y, respectively.*

8.5. THEOREM. *Let each open subset of a space X be F_σ. Then conditions* (i)–(iii) *of Definition* 8.1 *are equivalent.*

8.6. THEOREM. *Let X be a Borel-complete space. If X is regular or locally compact, then each* 2*-valued Borel measure in X is, respectively, regular, or outer regular and Radon.*

Since we consider only Hausdorff spaces, the support of a 2-valued Borel measure in X is either empty or a singleton. Using this, 8.3(i), (iii), and 6.14(ii), we can characterize Borel-completeness and closed-completeness in terms of Dirac measures (see 5.1).

8.7. PROPOSITION. *A space X is*

(i) *Borel-complete if and only if each* 2*-valued Borel probability in X is a Dirac measure*;

(ii) *closed-complete if and only if each* 2*-valued regular Borel probability in X is a Dirac measure.*

By 7.2, the set $\omega_1 + 1$ with the order topology is a weakly Borel-complete space. Yet, the Dieudonné measure in $\omega_1 + 1$ (see 5.5) is a 2-valued probability which is not a Dirac measure.

8.8. THEOREM. *Let X be a space in which each singleton is G_δ. Then X is Borel-complete if and only if it is weakly Borel-complete.*

PROOF. Let X be weakly Borel-complete, and choose a 2-valued Borel probability μ in X. By 8.3(ii), the support of μ is a singleton $\{x\} \subset X$. As $\{x\}$ is G_δ and μ is 2-valued, $\mu(\{x\}) = 1$. It follows from 8.7(i) that X is Borel-complete; for $\mu = \delta_x$. The converse is obvious. □

8.9. EXAMPLE. Let X be Haydon's space (see 5.7), and let ν be a 2-valued Borel probability in X. Letting

$$\chi(B) = \nu[(\kappa \times B) \cap X]$$

for each $B \in \mathcal{B}(Y)$, it is easy to see that χ is a 2-valued Borel probability in Y. Since Y is Lindelöf and first countable, $\chi = \delta_s$ for some $s \in Y$. Thus $\nu(X_s) = 1$, and as X_s is a compact subset of X, the support of ν is not empty. By 8.3(ii), X is weakly Borel-complete.

If $\kappa = \omega_1$, then X is first countable, and hence Borel-complete by 8.8. In particular, X is Borel-complete whenever CH holds.

However, if $\kappa > \omega_1$, then $\alpha_{t_0} \geqslant \omega_1$ for some $t_0 \in Y$. Consequently, X contains a subspace $\{(\alpha, t_0) \in \kappa \times Y: \alpha < \omega_1\}$ which is homeomorphic to ω_1. It follows from 5.5 and 8.2 that X is not Borel-complete in this case.

A property of a space is called *topological* if it is invariant under homeomorphisms, i.e., if X and Y are homeomorphic spaces and one has the property, then so does the other. In this sense, all properties defined in 7.1 and 8.1 are topological. Therefore, it is natural to ask whether they can be characterized in purely topological terms without referring to measures. Using ultrafilters, we shall give such a characterization of the properties defined in 8.1. We do not know how to characterize the properties defined in 7.1; in fact, we shall see some indications that this may not be possible within the ZFC set theory.

Let $\mathscr{A}$ be a family of sets. An *$\mathscr{A}$-filter* is a collection $\Phi \subset \mathscr{A}$ such that

(i) $\emptyset \notin \Phi$;

(ii) if $A, B \in \Phi$, then $A \cap B \in \Phi$;

(iii) if $A \in \Phi$, then $B \in \Phi$ for each $B \in \mathscr{A}$ which contains A.

8.10. DEFINITION. Let X be a space, $\mathscr{A} \subset \mathscr{P}(X)$, and let Φ be an $\mathscr{A}$-filter. We say that

(i) Φ is *fixed* if $\bigcap \Phi \neq \emptyset$;

(ii) Φ *converges* to an $x \in X$ if every neighborhood of x contains an $F \in \Phi$;

(iii) Φ has the *countable intersection property* (abbreviated as cip) if $\bigcap \Phi_0 \neq \emptyset$ for each countable family $\Phi_0 \subset \Phi$;

(iv) Φ is an *$\mathscr{A}$-ultrafilter* if there is no $\mathscr{A}$-filter Ψ with $\Phi \subsetneq \Psi$.

8.11. CONSTRUCTION. Let X be a space, let $\mathscr{A} \subset \mathscr{P}(X)$ be closed under countable intersections, and let Φ be an $\mathscr{A}$-ultrafilter with cip. Denote by $\mathscr{M}$ the family of all sets $A \subset X$ for which either A or $X - A$ contains a set $B \in \Phi$. For each $A \in \mathscr{M}$ let $\mu(A) = 1$ if A contains a set $B \in \Phi$, and $\mu(A) = 0$ otherwise.

It is easy to check that $\mathscr{M}$ is a σ-algebra in X containing $\mathscr{A}$, and that μ is a 2-valued measure on $\mathscr{M}$. The measure space $(X, \mathscr{M}, \mu)$ is said to be *associated* with Φ.

On the other hand, if $(X, \mathscr{M}, \mu)$ is a measure space with a 2-valued measure μ, and $\mathscr{A} \subset \mathscr{M}$, then $\Phi = \{A \in \mathscr{A}: \mu(A) = 1\}$ is an $\mathscr{A}$-filter with cip. Moreover, Φ is an $\mathscr{A}$-ultrafilter whenever $\mathscr{A}$ is closed also with respect to complementation.

8.12. THEOREM. *A space X is*
(i) *Borel-complete if and only if each $\mathscr{B}$-ultrafilter with* cip *is fixed*;
(ii) *weakly Borel-complete if and only if each $\mathscr{B}$-ultrafilter with* cip *converges*;
(iii) *closed-complete if and only if each $\mathscr{F}$-ultrafilter with* cip *is fixed.*

PROOF. To avoid trivialities, we shall assume that X is a nonempty space. Let Φ be a $\mathscr{B}$-ultrafilter with cip or an $\mathscr{F}$-ultrafilter with cip. If $(X, \mathscr{M}, \mu)$ is the measure space associated with Φ (see 8.11), then $\mathscr{B} \subset \mathscr{M}$. Hence μ can be restricted to a 2-valued Borel probability in X also denoted by μ.

Let Φ be a $\mathscr{B}$-ultrafilter. If X is weakly Borel-complete, then the support of μ is a singleton $\{x\} \subset X$ (see 8.3(ii)), and it is clear that Φ converges to x. Moreover, if X is Borel-complete, then by 8.7(i), $\bigcap \Phi = \{x\}$.

Let Φ be an $\mathscr{F}$-ultrafilter. Then μ is regular, and so if X is closed-complete, then $\mu = \delta_x$ for some $x \in X$ (see 8.7(iii)). It follows that $\bigcap \Phi = \{x\}$.

Conversely, let ν be a 2-valued Borel probability in X, and let

$$\Gamma = \{B \in \mathscr{B}: \nu(B) = 1\} \quad \text{and} \quad \Delta = \{F \in \mathscr{F}: \nu(F) = 1\}.$$

Since $\mathscr{B}$ is a σ-algebra, Γ is a $\mathscr{B}$-ultrafilter with cip. If Γ is fixed, it is easy to show that $\bigcap \Gamma = \{x\}$ for some $x \in X$. Thus $\nu = \delta_x$, and X is Borel-complete by 8.7(i). If Γ converges to an $x \in X$, then $\{x\}$ is the support of ν, and X is weakly Borel-complete by 8.3(ii). Let ν be regular. Then it is not difficult to verify that Δ is an $\mathscr{F}$-ultrafilter with cip. If Δ is fixed, then as above we see that ν is a Dirac measure. Consequently, X is closed-complete by 8.7(ii). $\square$

Note. Clearly, each fixed $\mathscr{B}$-ultrafilter is convergent, but not vice versa. For instance, the $\mathscr{B}$-ultrafilter of all Borel subsets of $\omega_1 + 1$ whose Dieudonné measure (see 5.5) is 1 converges to ω_1 but is not fixed. On the other hand, it is easy to see that an $\mathscr{F}$-ultrafilter converges if and only if it is fixed.

We remark that Borel-complete and weakly Borel-complete spaces were studied in HAGER, REYNOLDS, and RICE [1972] and RICE and REYNOLDS [1980], respectively.

9. Completeness properties of small spaces

In general, there are no criteria for deciding whether a closed-complete space X is also Borel measure-compact. However, we show that if $|X| < c$, then a necessary and sufficient condition can be obtained by looking at the Lebesgue measure of the subsets of reals with cardinality $|X|$.

Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $A \in \mathcal{M}$ is called an *atom* of μ if $\mu(A) > 0$, and for each $B \in \mathcal{M}$ with $B \subset A$ either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. We need a lemma which is a very special case of the Liapunoff theorem (see LINDENSTRAUSS [1966]).

9.1. LEMMA. *Let $(X, \mathcal{M}, \mu)$ be a measure space. If μ has no atoms, then*

$$\{\mu(A): A \in \mathcal{M}\} = [0, \mu(X)] .$$

PROOF. Let $0 < a < \mu(X)$, and let $\mathcal{A}_1 = \{A \in \mathcal{M}: \mu(A) \geq a\}$. We show that there is a set $C \in \mathcal{A}_1$ such that $\mu(A) = \mu(C)$ for each $A \in \mathcal{A}_1$ with $A \subset C$. If $\mu(A) = +\infty$ for each $A \in \mathcal{A}_1$, it suffices to let $C = X$. If $\mu(A) < +\infty$ for some $A \in \mathcal{A}_1$, we construct inductively families $\mathcal{A}_n$ and sets $A_n \in \mathcal{A}_n$ such that

$$\mu(A_n) < \inf\{\mu(A): A \in \mathcal{A}_n\} + \frac{1}{n},$$

and $\mathcal{A}_{n+1} = \{A \in \mathcal{A}_n: A \subset A_n\}$, $n = 1, 2, \ldots$. Let $A \in \mathcal{A}_1$ and $A \subset \bigcap_{n=1}^{\infty} A_n$. Then $A \in \mathcal{A}_n$ and

$$\mu(A) \leq \mu\Big(\bigcap_{n=1}^{\infty} A_n\Big) \leq \mu(A_n) < \mu(A) + \frac{1}{n},$$

$n = 1, 2, \ldots$. It follows that $C = \bigcap_{n=1}^{\infty} A_n$ is the desired set.

Similarly, we can find a set $D \in \mathcal{M}$ such that $D \subset C$, $\mu(D) \leq a$, and $\mu(B) = \mu(D)$ for each $B \in \mathcal{M}$ with $D \subset B \subset C$ and $\mu(B) \leq a$. It is clear that $C - D$ is an atom of μ unless $\mu(C) = \mu(D) = a$. $\square$

We note that the previous proof does not use transfinite induction (cf. HALMOS [1950], §41, Exercise (2), p. 174).

9.2. THEOREM. *If $\kappa < c$ is a cardinal, then the following conditions are equivalent.*

(i) *If $C \subset \mathbb{R}$ and $|C| \leq \kappa$, then $\lambda^*(C) = 0$.*

(ii) *If $(X, \mathcal{M}, \mu)$ is a measure space, $|X| \leq \kappa$, and $\mu(X) > 0$, then μ has an atom.*

PROOF. If $C \subset \mathbb{R}$ is such that $|C| \leq \kappa$ and $\lambda^*(C) > 0$, then λ_C (see 3.5) is a nontrivial Borel measure in C which has no atoms.

Conversely, let $(X, \mathcal{M}, \mu)$ be a measure space with $|X| \leq \kappa$, where μ is a nontrivial measure which has no atoms. We may assume that $\mu(X) = 1$; for μ can either be restricted to a suitable subset of X (see 9.1) or multiplied by a suitable constant.

Let ν be the Haar probability in the dyadic space 2^{ω} (see 5.10). For $f \in 2^{\omega}$ and

$p \in \omega$, let

$$U(f, p) = \{g \in 2^\omega : g \upharpoonright p = f \upharpoonright p\}.$$

By a repeated application of 9.1, we can define sets $A(f, p) \in \mathcal{M}$, where $f \in 2^\omega$ and $p \in \omega$, so that the following conditions are satisfied.

(i) $A(f, 0) = X$.

(ii) If $f \upharpoonright p = g \upharpoonright p$, then $A(f, p) = A(g, p)$; if in addition $f(p) \neq g(p)$, then $A(f, p) = A(f, p+1) \cup A(g, p+1)$ and $A(f, p+1) \cap A(g, p+1) = \emptyset$.

(iii) $\mu[A(f, p)] = \nu[U(f, p)]$ (see 9.1).

Letting $D = \{f \in 2^\omega : \bigcap_{p\in\omega} A(f, p) \neq \emptyset\}$, it is easy to see that $|D| \leqslant |X| \leqslant \kappa$. Suppose that $\nu^*(D) < 1$. As the sets $U(f, p)$ form a countable base for the topology of 2^ω, there are $U(f_n, p_n)$, $n \in \omega$, such that

$$D \subset \bigcup_{n\in\omega} U(f_n, p_n) \quad \text{and} \quad \sum_{n\in\omega} \nu[U(f_n, p_n)] < 1.$$

Using (iii), we obtain a contradiction by showing that $X = \bigcup_{n\in\omega} A(f_n, p_n)$.

If $x \in X$, then $x \in \bigcap_{p\in\omega} A(f, p)$ for some $f \in D$. There is an $n \in \omega$ such that $f \in U(f_n, p_n)$. Thus $f \upharpoonright p_n = f_n \upharpoonright p_n$, and it follows from (ii) that $x \in A(f_n, p_n)$.

We conclude that $\nu^*(D) = 1$. Letting

$$C = \left\{\sum_{n\in\omega} f(n) 2^{-n} : f \in D\right\},$$

it is well known and easy to prove that $\lambda^*(C) = \nu^*(D) = 1$, and we also have $|C| = |D| \leqslant \kappa$. $\square$

We note that Theorem 9.2 may be regarded as a special case of the famous classification theorem of MAHARAM [1942].

In the next theorem we shall use the weighted counting measure which was defined in 5.1.

9.3. THEOREM. *Let $\kappa < c$ be a cardinal such that $\lambda^*(C) = 0$ for each $C \subset \mathbb{R}$ with $|C| \leqslant \kappa$, and let X be a space with $|X| \leqslant \kappa$. Then*

(i) *X is Borel-complete if and only if each finite Borel measure in X is a weighted counting measure;*

(ii) *X is closed-complete if and only if each finite regular Borel measure in X is a weighted counting measure.*

PROOF. We shall prove only (ii). The proof of (i) is completely analogous and somewhat easier.

Let X be closed-complete, and let μ be a finite, diffused, regular, Borel measure in X. In view of 5.1, it suffices to show that μ is trivial. Hence suppose that $\mu(X)>0$. By 9.2, μ has an atom $F \in \mathscr{F}$; for μ is regular. Clearly,

$$\nu = \frac{1}{\mu(F)} x(\mu_F)$$

(see 3.5 and 3.7) is a 2-valued, diffused, Borel probability in X, which is regular by 3.6 and 3.9. This contradicts 8.7(ii).

Since each 2-valued weighted counting probability in X is a Dirac measure, the converse follows from 8.7(ii). □

9.4. THEOREM. *If $\kappa < \mathfrak{c}$ is a cardinal, the following conditions are equivalent.*

(i) $\lambda^*(C)=0$ *for each* $C \subset \mathbb{R}$ *with* $|C| \leqslant \kappa$.

(ii) *Each weakly Borel-complete space X with $|X| \leqslant \kappa$ is weakly Borel measure-complete.*

(iii) *Each closed-complete space X with $|X| \leqslant \kappa$ is Borel measure-compact.*

PROOF. (i)⇒(ii). Let X be a space with $|X| \leqslant \kappa$, and let μ be a locally trivial Borel probability in X. By (i) and 9.2, μ has an atom $A \in \mathscr{B}(X)$, and so

$$\nu = \frac{1}{\mu(A)} x(\mu_A)$$

(see 3.5 and 3.7) is a 2-valued, locally trivial, Borel probability in X. Now (ii) follows from 7.6(ii) and 8.3(ii).

(i)⇒(iii). This follows immediately from 9.3(ii) and 7.6(iii).

$\neg$(i)⇒$\neg$(ii) & $\neg$(iii). Let X be Haydon's space (see 5.7). Then $|X| \leqslant \kappa$ and by 8.9, X is weakly Borel-complete. However, it follows from 5.7 that X is not Borel measure-compact. □

Note. Under MA, MARTIN and SOLOVAY [1970] proved that $\lambda^*(C)=0$ for each $C \subset \mathbb{R}$ with $|C| < \mathfrak{c}$.

The results of this section were obtained in GARDNER and GRUENHAGE [1981].

10. Completeness and covering properties

We show that under a mild cardinality restriction, weak θ-refinability implies the completeness properties from 7.1. In view of Proposition 7.4, we consider only weak Borel measure-completeness and Borel measure-compactness.

Recall that the measurable cardinals were defined in 4.12.

10.1. LEMMA. *In any space X the following conditions are equivalent.*

(i) *There is a (inner regular) Borel measure μ in X, a set $Y \in \mathscr{B}(X)$ with $0 < \mu(Y) < +\infty$, and a disjoint cover $\mathscr{G}_0 \subset \mathscr{G}(Y)$ of Y such that $\mu(G) = 0$ for each $G \in \mathscr{G}_0$.*

(ii) *There is a (closed) discrete set $T \subset X$ of real-valued measurable cardinality.*

PROOF. (i)$\Rightarrow$(ii). Letting

$$\nu(\mathscr{H}) = \frac{1}{\mu(Y)}\mu(\bigcup\mathscr{H})$$

for each $\mathscr{H} \subset \mathscr{G}_0$, we see that the cardinality of $\mathscr{G}_0$ is real-valued measurable. For each nonempty $G \in \mathscr{G}_0$ select an $x_G \in G$, and set $T = \{x_G: G \in \mathscr{G}_0, G \neq \emptyset\}$. Then T is discrete, $T \in \mathscr{F}(Y)$, and $|T| = |\mathscr{G}_0|$. If μ is inner regular, there is an $F \in \mathscr{F}(X)$ with $F \subset Y$ and $\mu(F) > 0$. Repeating the above argument for the set F and its cover $\{G \cap F: G \in \mathscr{G}_0\} \subset \mathscr{G}(F)$, we see that $T \in \mathscr{F}(X)$.

(ii)$\Rightarrow$(i). Let ν be a diffused Borel probability in T. Then $\mu = {}_X\nu$ (see 3.7) is a Borel probability in X which is inner regular whenever T is closed (see 3.9 and 5.6). Each discrete set is relatively closed in an open set, and hence Borel. Thus it suffices to let $Y = T$. □

A space X is called, respectively, *weakly θ-refinable* or *weakly $\delta\theta$-refinable* if each open cover of X has an open refinement $\bigcup_{n=1}^{\infty}\mathscr{U}_n$ such that for every $x \in X$ there is an integer $n_x \geq 1$ with $1 \leq |\mathrm{st}(x, \mathscr{U}_{n_x})| < \omega$ or $1 \leq |\mathrm{st}(x, \mathscr{U}_{n_x})| \leq \omega$. For some basic properties of weakly θ-refinable and weakly $\delta\theta$-refinable spaces we refer to Burke's Chapter 9.

10.2. THEOREM. *Let X be a weakly θ-refinable space. If X contains no discrete, or no closed discrete, subsets of real-valued measurable cardinality, then X is weakly Borel measure-complete, or Borel measure-compact, respectively.*

PROOF. Let μ be a locally trivial Borel probability in X. For each $x \in X$ let U_x be an open neighborhood of x for which $\mu(U_x) = 0$. By Burke's Chapter 9, Section 3.6, we can find families $\mathscr{U}_n \subset \mathscr{G}(X)$ such that $\bigcup_{n=1}^{\infty}\mathscr{U}_n$ is a refinement of $\{U_x: x \in X\}$, and for each $x \in X$ there is an integer $n_x \geq 1$ with $|\mathrm{st}(x, \mathscr{U}_{n_x})| = 1$. If

$$X_n = \{x \in X: |\mathrm{st}(x, \mathscr{U}_n)| = 1\},$$

$n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty}X_n = X$ and $X_n \in \mathscr{B}(X)$; for

$$X_n = \{x \in X: |\mathrm{st}(x, \mathscr{U}_n)| \geq 1\} - \{x \in X: |\mathrm{st}(x, \mathscr{U}_n)| \geq 2\}$$

is a difference of two open sets. Find an integer $N \geq 1$ with $\mu(X_N) > 0$, and set $Y = X_N$. Then $\mathscr{G}_0 = \{G \cap Y : G \in \mathscr{U}_N\}$ is a disjoint subfamily of $\mathscr{G}(Y)$ and $\bigcup \mathscr{G}_0 = Y$. The theorem follows from 10.1 and 7.6(ii), (iii). $\square$

Using 7.4 and 7.9, we obtain the following corollary.

10.3. COROLLARY. *Let X be a hereditarily weakly θ-refinable space which contains no discrete subsets of real-valued measurable cardinality. If X is regular or locally compact, then each moderated Borel measure in X is regular, or outer regular and Radon, respectively.*

It follows from 5.6 that the cardinality restrictions in Theorem 10.2 and Corollary 10.3 are essential. The reader can easily verify that if we restrict our attention to 2-valued Borel measures only, we can replace real-valued measurable cardinals by 2-valued measurable cardinals.

We show that the sufficient condition from Theorem 10.2 is by no means necessary.

10.4. LEMMA. *Let Y be a space, let Z be a second countable space, and let $\pi: Y \times Z \to Y$ be the natural projection. If $X \subset Y \times Z$ is weakly $\delta\theta$-refinable, then so is $\pi(X)$.*

PROOF. We may assume that $\pi(X) = Y$. Given an open cover $\mathscr{Y}$ of Y, $\mathscr{U} = \{U \times Z : U \in \mathscr{Y}\}$ is an open cover of $Y \times Z$. If X is weakly $\delta\theta$-refinable, we can find an open refinement $\mathscr{V} = \bigcup_{p=1}^{\infty} \mathscr{V}_p$ of $\mathscr{U}$ such that for each $x \in X$ there is an integer $p_x \geq 1$ with $1 \leq |\mathrm{st}(x, \mathscr{V}_{p_x})| \leq \omega$. Let $H_1, H_2, \ldots$ be a countable open base for the topology of Z. For $V \in \mathscr{V}$ let

$$V_q = \bigcup \{G \in \mathscr{G}(Y) : G \times H_q \subset V\},$$

$q = 1, 2, \ldots$, and set

$$\mathscr{W}_p = \{V_q \times H_q : V \in \mathscr{V}_p, q = 1, 2, \ldots\},$$

$$\mathscr{Y}_p = \{V_q : V \in \mathscr{V}_p, q = 1, 2, \ldots\},$$

$p = 1, 2, \ldots$. Since $V = \bigcup_{q=1}^{\infty} (V_q \times H_q)$, we have $1 \leq |\mathrm{st}(x, \mathscr{W}_{p_x})| \leq \omega$ for every $x \in X$. Consequently $1 \leq |\mathrm{st}(\pi(x), \mathscr{Y}_{p_x})| \leq \omega$, and the lemma follows; for $\bigcup_{p=1}^{\infty} \mathscr{Y}_p$ is an open refinement of $\mathscr{Y}$. $\square$

10.5. EXAMPLE. By LAVER [1976], Theorem 1.2, there is a set $Z \subset [0, 1]$ such that $|Z| = \omega_1$, and with respect to the Euclidean topology in Z, each finite diffused Borel measure in Z is trivial. Select a bijection $t \mapsto \alpha_t$ from Z onto ω_1, and let

$$X = \{(\alpha, t) \in \omega_1 \times Z : \alpha \leqslant \alpha_t\}.$$

Give ω_1 the order topology, Z the Euclidean topology, and topologize X as a subspace of $\omega_1 \times Z$. By 10.4 and 13.5, X is not weakly $\delta\theta$-refinable. Yet we show that X is Borel measure-complete.

Let μ be a finite diffused Borel measure in X. If

$$\nu(B) = \mu[(\omega_1 \times B) \cap X] \quad \text{for each } B \in \mathscr{B}(Z),$$

then ν is a finite Borel measure in Z. Since $(\omega_1 \times \{t\}) \cap X$ is countable for each $t \in Z$, ν is diffused. Thus by our choice of Z, $\mu(X) = \nu(Z) = 0$. It follows from 5.1 that each finite Borel measure in X is a weighted counting measure. By 7.6(i), X is Borel measure-complete.

Note. It is worthwhile to compare the previous example with Examples 5.7 and 7.12.

If κ is a limit ordinal which is not cofinal with ω, then in its order topology, κ is countably compact but not compact. According to Burke's Chapter 9, Section 9.4, such a κ is not weakly $\delta\theta$-refinable. Thus it follows from 10.4 that Haydon's space (see 5.7) is not weakly $\delta\theta$-refinable.

For a locally compact space X, Corollary 10.3 generalizes Corollary 6.8; however, this is not so if X is merely regular. Employing ideas of CHOQUET [1959], we obtain a completely different generalization of Corollary 6.8.

10.6. LEMMA. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let*

$$\mu^{\#}(A) = \inf\{\mu(B) : A \subset B, B \in \mathcal{M}\}$$

for each $A \subset X$. If $A_1 \subset A_2 \subset \cdots \subset X$ and $A = \bigcup_{n=1}^{\infty} A_n$, then $\mu^{\#}(A) = \lim \mu^{\#}(A_n)$.

PROOF. By 3.3, $\mu^{\#}$ is an outer measure in X. Thus

$$\lim \mu^{\#}(A_n) \leqslant \mu^{\#}(A)$$

(see 3.1(ii)), and we may assume that $\lim \mu^{\#}(A_n) < +\infty$. There are $B_n \in \mathcal{M}$ such that $A_n \subset B_n$ and $\mu(B_n) = \mu^{\#}(A_n)$, $n = 1, 2, \ldots$. If $C_n = \bigcup_{k=1}^{n} B_k$, then

$$\mu^{\#}(A) \leqslant \mu\Big(\bigcup_{n=1}^{\infty} C_n\Big) = \lim \mu(C_n),$$

and the lemma will be established by showing that $\mu(C_n) = \mu(B_n)$ for $n = 1, 2, \ldots$.

Trivially, $\mu(C_1) = \mu(B_1)$. Proceeding by induction, suppose that $\mu(C_n) = \mu(B_n)$ for an integer $n \geqslant 1$. Now $A_n \subset C_n \cap B_{n+1}$ and

$$\begin{aligned}\mu(C_{n+1}) + \mu(C_n \cap B_{n+1}) &= \mu(C_n \cup B_{n+1}) + \mu(C_n \cap B_{n+1}) \\ &= \mu(C_n) + \mu(B_{n+1}),\end{aligned}$$

so by the inductive hypothesis,

$$\begin{aligned}\mu(B_{n+1}) \leqslant \mu(C_{n+1}) &= \mu(C_n) + \mu(B_{n+1}) - \mu(C_n \cap B_{n+1}) \\ &\leqslant \mu(B_n) + \mu(B_{n+1}) - \mu^{\#}(A_n) = \mu(B_{n+1}). \qquad \square\end{aligned}$$

Let $\mathbb{N}$ be the set of all positive integers, and let $\mathscr{H}$ be a family of sets. We say that a set A is *Souslin-$\mathscr{H}$* if for each $f \in \mathbb{N}^\omega$ and each $p \in \mathbb{N}$ there are sets $H(f \restriction p) \in \mathscr{H}$ such that

$$A = \bigcup_{f \in \mathbb{N}^\omega} \bigcap_{p \in \mathbb{N}} H(f \restriction p).$$

We stress that the set $H(f \restriction p)$ depends only on $f \restriction p$, i.e., $H(f \restriction p) = H(g \restriction p)$ whenever $f \restriction p = g \restriction p$.

Souslin-$\mathscr{H}$ sets have been studied for many years. The following standard result is proved in ROGERS and JAYNE [1980], §2.3.

10.7. PROPOSITION. *If $\mathscr{H}$ is a family of sets, then the collection of all Souslin-$\mathscr{H}$ sets is closed under countable unions and intersections.*

Next we prove a special version of Choquet's capacitability theorem (see CHOQUET [1959], Theorem 1, p. 84).

10.8. THEOREM. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\mathscr{H} \subset \mathcal{M}$ be a family which is closed under finite unions and countable intersections. Then*

$$\mu^{\#}(A) = \sup\{\mu(H) \colon H \subset A, H \in \mathscr{H}\}$$

for each Souslin-$\mathscr{H}$ set $A \subset X$ with $\mu^{\#}(A) < +\infty$.

PROOF. Let $A = \bigcup_{f \in \mathbb{N}^\omega} \bigcap_{p \in \mathbb{N}} H(f \restriction p)$, where $H(f \restriction p) \in \mathscr{H}$ for each $f \in \mathbb{N}^\omega$ and $p \in \mathbb{N}$. Replacing $H(f \restriction p)$ by $\bigcap_{n=1}^{p} H(f \restriction n)$, we may assume that every $\{H(f \restriction p)\}_{p \in \mathbb{N}}$ is a decreasing sequence. If $p, n \in \mathbb{N}$ and $s \in \mathbb{N}^p$, let $\langle s, n\rangle \in \mathbb{N}^{p+1}$ be such that $\langle s, n\rangle \restriction p = s$ and $\langle s, n\rangle(p) = n$. In accordance with this notation, $\langle n\rangle \colon 1 \to \mathbb{N}$

is a map defined by $\langle n\rangle(0) = n$. Given $\alpha \leq \omega$ and f, g in $\mathbb{N}^\alpha$, we write $f \leq g$ to denote that $f(n) \leq g(n)$ for each $n \in \alpha$. For $s \in \mathbb{N}^p$, we set

$$A(s) = \bigcup\{\bigcap_{q\in\mathbb{N}} H(f\restriction q): f \in \mathbb{N}^\omega, f\restriction p = s\},$$

$$B(s) = \bigcup\{A(r): r \in \mathbb{N}^p, r \leq s\}.$$

Then $A(s) = \bigcup_{n\in\mathbb{N}} A(\langle s, n\rangle)$, and it follows that $B(s) = \bigcup_{n\in\mathbb{N}} B(\langle s, n\rangle)$. Moreover,

$$A = \bigcup_{n\in\mathbb{N}} A(\langle n\rangle) = \bigcup_{n\in\mathbb{N}} B(\langle n\rangle),$$

and $B(r) \subset B(s)$ for each r, s in $\mathbb{N}^p$ with $r \leq s$. Choose an $a < \mu^\#(A)$, and use 10.6 to find an $n \in \mathbb{N}$ for which $\mu^\#[B(\langle n\rangle)] > a$. Inductively, we construct an $h \in \mathbb{N}^\omega$ such that $\mu^\#[B(h\restriction p)] > a$ for each $p \in \mathbb{N}$. The set

$$H_p = \bigcup\{H(f\restriction p): f \in \mathbb{N}^\omega, f \leq h\},$$

$p \in \mathbb{N}$, belongs to $\mathcal{H}$. As $A(f\restriction p) \subset A \cap H(f\restriction p)$, we have $A(f\restriction p) \subset A \cap H_p$ for each $f \in \mathbb{N}^\omega$ with $f \leq h$. Consequently $B(h\restriction p) \subset A \cap H_p$, and hence

$$a < \mu^\#[B(h\restriction p)] \leq \mu^\#(A \cap H_p).$$

Now $H_{p+1} \subset H_p$, so if $H = \bigcap_{p\in\mathbb{N}} H_p$ and $\mu^\#(A) < +\infty$, it follows that

$$a \leq \lim \mu^\#(A \cap H_p) = \mu^\#(A \cap H);$$

for by 3.2(ii), $\mu^\#$ is a finite measure on $\{A \cap M: M \in \mathcal{M}\}$.

We complete the proof by showing that $H \subset A$.

If $x \in H$, there are $f_p \in \mathbb{N}^\omega$ such that $f_p \leq h$ and $x \in H(f_p\restriction p)$ for each $p \in \mathbb{N}$. Inductively, we can construct an $f \in \mathbb{N}^\omega$ such that given $q \in \mathbb{N}$, $f\restriction q = f_p\restriction q$ for infinitely many $p \in \mathbb{N}$. Choosing such a p with $p \geq q$ we obtain

$$H(f_p\restriction p) \subset H(f_p\restriction q) = H(f\restriction q).$$

Since q is arbitrary, $x \in \bigcap_{q\in\mathbb{N}} H(f\restriction q)$. □

10.9. COROLLARY. *Let each open subset of a space X be, respectively, Souslin-$\mathcal{F}$ or Souslin-$\mathcal{K}$. Then each moderated Borel measure in X is regular, or outer regular and Radon.*

Letting $\mathcal{H} = \mathcal{F}$ or $\mathcal{H} = \mathcal{K}$ in Theorem 10.8, the corollary follows from 6.7.

It is possible to show that if all open subsets of X are Souslin-$\mathcal{K}$, then they are, in fact, σ-compact. Thus for Radon measures, Corollaries 10.9 and 6.8 are identical.

They will be properly generalized in 11.15. On the other hand, we do not know any stronger result for regular measures.

The following important theorem will be needed in Section 11 (see 11.14).

10.10. THEOREM. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then each Souslin-$\mathcal{M}$ set is $\mu^{\#}$-measurable.*

PROOF. Let A be a Souslin-$\mathcal{M}$ set. By 3.3(ii), it suffices to show that $A \cap B$ is $\mu^{\#}$-measurable for each $B \in \mathcal{M}$ with $\mu(B) < +\infty$. Thus in view of 10.7, we may assume that $\mu^{\#}(A) < +\infty$. Using 10.8, we can find an $H \in \mathcal{M}$ with $H \subset A$ and $\mu(H) = \mu^{\#}(A)$. By the definition of $\mu^{\#}$ (see 3.3), there is an $E \in \mathcal{M}$ such that $A \subset E$ and $\mu(E) = \mu^{\#}(A)$. Since $A - H \subset E - H$ and $\mu(E - H) = 0$, the theorem follows from 3.3(iii). □

Note. Theorem 10.2 was essentially proved in GARDNER [1975], but parts of it were made explicit in BLAIR [1977] and DALGAS [1978]. Corollary 10.3 was independently given in PFEFFER [1977]. Example 10.5 is from GRUENHAGE and GARDNER [1978]. Finally, a direct proof of Theorem 10.10 (i.e., without referring to 10.8) can be found in SAKS [1937], Ch. II, §5.

11. Radon spaces

11.1. DEFINITION. A space X is called *Radon* if each finite Borel measure in X is Radon.

By 6.9 every Radon measure is τ-additive, and according to 6.11, each finite τ-additive Borel measure in a locally compact space is Radon. Thus each Radon space is Borel measure-complete, and every locally compact Borel measure-complete space is Radon (see 7.9). We shall improve these results by obtaining a necessary and sufficient condition under which a completely regular, Borel measure-complete space is Radon. In carrying out this program the following definition will be useful.

11.2. DEFINITION. A space X is called *pre-Radon* if each finite τ-additive Borel measure in X is Radon.

Using this definition, the next proposition summarizes what we have already said.

11.3. PROPOSITION. *Every compact space is pre-Radon. A space is Radon if and only if it is pre-Radon and Borel measure-complete.*

Following the ideas of KNOWLES [1967] and SUNYACH [1969], we shall investigate when a subspace of a pre-Radon space is pre-Radon.

11.4. LEMMA. *Let μ be a finite regular Borel measure in Y, and let $X \subset Y$. If μ is τ-additive, then so is μ_X.*

PROOF. Let $\mathscr{G}_0 \subset \mathscr{G}(X)$ and $\mathscr{G}_0 \nearrow G_0$. For each $G \in \mathscr{G}_0$ find a $G^\wedge \in \mathscr{G}(Y)$ with $G^\wedge \cap X = G$. Let $\mathscr{H}$ be the family of all unions $\bigcup\{G^\wedge : G \in \mathscr{G}_1\}$ where $\mathscr{G}_1$ is a finite subcollection of $\mathscr{G}_0$. Then $\mathscr{H} \subset \mathscr{G}(Y)$, $\mathscr{H} \nearrow H_0$, and $H_0 \cap X = G_0$. By our assumption, for each $\varepsilon > 0$ there is an $H \in \mathscr{H}$ such that $\mu(H_0 - H) < \varepsilon$ and $H \cap X \in \mathscr{G}_0$. Thus

$$\mu_X(G_0 - H \cap X) = \mu^\#[(H_0 - H) \cap X] \leqslant \mu(H_0 - H) < \varepsilon,$$

and the τ-additivity of μ_X follows. $\square$

Recall that the outer measure $\mu^\#$ and $\mu^\#$-measurability were defined in Section 3, particularly in Theorem 3.3.

11.5. DEFINITION. (i) A subspace X of a space Y is called *Radon measurable* in Y if it is $\mu^\#$-measurable for every finite Radon measure μ in Y.

(ii) A space X is called *universally Radon measurable* if X is Radon measurable in Y for each space Y containing X as a subspace.

11.6. THEOREM. *Let X be a subspace of a pre-Radon space Y. Then the following conditions are equivalent.*

(i) *X is universally Radon measurable.*

(ii) *X is Radon measurable in Y.*

(iii) *X is a pre-Radon space.*

PROOF. (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iii). Let μ be a finite τ-additive Borel measure in X. According to 3.8, ${}_Y\mu$ is a finite τ-additive Borel measure in Y, and hence it is Radon. By (ii), X is $({}_Y\mu)^\#$-measurable, and by 3.3(i) and 3.4, so is each $B \in \mathscr{B}(X)$. Given $B \in \mathscr{B}(X)$, it follows from 3.3(iii) that there are sets E and H in $\mathscr{B}(Y)$ such that $E \subset B \subset H$ and ${}_Y\mu(E) = {}_Y\mu(H)$. As

$${}_Y\mu(E) = \mu(E \cap X) \leqslant \mu(B) \leqslant \mu(H \cap X) = {}_Y\mu(H),$$

we have

$$\mu(B) = {}_Y\mu(E) = \sup\{{}_Y\mu(K) \colon K \subset E, K \in \mathcal{K}(Y)\}$$

$$\leqslant \sup\{\mu(K) \colon K \subset B, K \in \mathcal{K}(X)\} \leqslant \mu(B)\,;$$

for $E \subset B \subset X$. Consequently, μ is Radon.

(iii)$\Rightarrow$(i). Let Z be a space containing X as a subspace, and let μ be a finite Radon measure in Z. It follows from 6.9 and 11.4 that μ_X is a finite τ-additive Borel measure in X. By (iii), μ_X is Radon, and we can find a σ-compact set $F \subset X$ such that

$$\mu^{\#}(X - F) = \mu_X(X - F) = 0\,.$$

Since $F \in \mathcal{B}(X)$, X is $\mu^{\#}$-measurable by 3.3(iii). □

11.7. COROLLARY. *Let X be a Radon space. A subspace $Y \subset X$ is Radon if and only if it is Radon measurable in X. In particular, each $Y \in \mathcal{B}(X)$ is a Radon space.*

This corollary follows immediately from 11.6, 7.4, and 11.3. Its meaning is illustrated by Examples 5.3 and 5.4.

For a general space X, there may be no pre-Radon space Y containing X as a subspace. For instance, the space X of Example 6.12 is not pre-Radon, yet it is universally Radon measurable (observe that $S^- = X$, and remembering that all spaces are *Hausdorff*, use ENGELKING [1977], 3.12.5(a)). However, if X is completely regular, then by 11.3, Y can be any compact space containing X; in particular, Y can be the Stone–Čech compactification βX of X.

11.8. COROLLARY. *Let X be a completely regular space. Then X is pre-Radon if and only if it is Radon measurable in its Stone–Čech compactification βX.*

Combining 7.4, 10.2, 11.3, and 11.8, we obtain the most general known criterion for deciding whether a completely regular space is Radon.

11.9. THEOREM. *Let X be a completely regular, hereditarily weakly θ-refinable space which contains no discrete subset of real-valued measurable cardinality. Then X is Radon if and only if it is Radon measurable in its Stone–Čech compactification βX.*

Corollary 11.8 is very important. It implies that a completely regular space X which is a Borel subset of βX is pre-Radon. In particular, all *Čech-complete spaces* (see ENGELKING [1977], §3.9, p. 252) are pre-Radon. Thus each locally compact space is pre-Radon (a result we have already established in 6.11), and also by ENGELKING [1977], Theorem 4.3.26, p. 343, each *completely metrizable space* is pre-Radon. Applying 11.9, we obtain the theorem of MARCZEWSKI and SIKORSKI [1948].

11.10. THEOREM. *Let X be a completely metrizable space which contains no discrete subset of real-valued measurable cardinality. Then X is a Radon space.*

Theorem 11.9 will not be found in certain texts concerning Radon spaces; e.g., the influential book of SCHWARTZ [1973]. To explain an alternate approach used by L. Schwartz and others, we need to introduce $\mathscr{K}$-analytic spaces.

As before, we use $\mathbb{N}$ to denote the set of all positive integers. We topologize $\mathbb{N}^\omega$ as a product of discrete spaces.

Let X be a space. A map $K: \mathbb{N}^\omega \to \mathscr{K}(X)$ is called *upper semi-continuous* (abbreviated as usc) if for each $f \in \mathbb{N}^\omega$ and each $G \in \mathscr{G}(X)$ with $K(f) \subset G$ there is a neighborhood U of f such that $K(g) \subset G$ for each $g \in U$. If there is an usc map $K: \mathbb{N}^\omega \to \mathscr{K}(X)$ such that $\bigcup_{f \in \mathbb{N}^\omega} K(f) = X$, we say that X is a *$\mathscr{K}$-analytic space.* For a detailed study of $\mathscr{K}$-analytic spaces, we refer to ROGERS and JAYNE [1980].

11.11. REMARK. Perhaps the best known $\mathscr{K}$-analytic spaces are *Souslin spaces*, i.e., continuous images of $\mathbb{N}^\omega$. To see that a Souslin space X is $\mathscr{K}$-analytic, it suffices to choose a continuous surjection $\pi: \mathbb{N}^\omega \to X$, and let $K(f) = \{\pi(f)\}$ for each $f \in \mathbb{N}^\omega$. It follows from KURATOWSKI [1966], §36, II that each open subspace of a Souslin space is again Souslin, and hence $\mathscr{K}$-analytic.

We note that Souslin spaces are called *analytic spaces* by some authors.

11.12. PROPOSITION. *Each $\mathscr{K}$-analytic space is Lindelöf.*

PROOF. Let $\mathscr{U}$ be an open cover of a $\mathscr{K}$-analytic space X. Choose an usc map $K: \mathbb{N}^\omega \to \mathscr{K}(X)$ with $\bigcup_{f \in \mathbb{N}^\omega} K(f) = X$. For each $f \in \mathbb{N}^\omega$ there is a finite family $\mathscr{U}_f \subset \mathscr{U}$ which covers $K(f)$. As K is usc, we can find an open neighborhood V_f of f with

$$\bigcup_{g \in V_f} K(g) \subset \bigcup \mathscr{U}_f .$$

Now $\mathbb{N}^\omega$ is Lindelöf, so $\mathbb{N}^\omega = \bigcup_{f \in C} V_f$ where C is a countable subset of $\mathbb{N}^\omega$. It follows that $\bigcup_{f \in C} \mathscr{U}_f$ is a countable subcover of $\mathscr{U}$. $\square$

The next theorem relates $\mathscr{K}$-analytic spaces to the Souslin-$\mathscr{H}$ sets defined in Section 10.

11.13. THEOREM. *Let X be a space.*

(i) *If X is Souslin-$\mathscr{K}(X)$, then it is $\mathscr{K}$-analytic.*

(ii) *If X is $\mathscr{K}$-analytic, then it is Souslin-$\mathscr{F}(Y)$ for any space Y containing X.*

PROOF. (i) If X is Souslin-$\mathscr{K}(X)$, then $X = \bigcup_{f\in\mathbb{N}^\omega}\bigcap_{p\in\mathbb{N}}K(f\restriction p)$ where $K(f\restriction p)\in\mathscr{K}(X)$ for each $f\in\mathbb{N}^\omega$ and $p\in\mathbb{N}$. Replacing $K(f\restriction p)$ by $\bigcap_{n=1}^{p}K(f\restriction n)$, we may assume that every $\{K(f\restriction p)\}_{p\in\mathbb{N}}$ is a decreasing sequence. Letting $K(f)=\bigcap_{p\in\mathbb{N}}K(f\restriction p)$ for each $f\in\mathbb{N}^\omega$, it is easy to see that $f\mapsto K(f)$ is an usc map. This implies that X is $\mathscr{K}$-analytic.

(ii) Let X be a $\mathscr{K}$-analytic space contained in a space Y, and let $K:\mathbb{N}^\omega\to\mathscr{K}(X)$ be an usc map with $\bigcup_{f\in\mathbb{N}^\omega}K(f)=X$. A base for the topology of $\mathbb{N}^\omega$ is given by the sets

$$U(f\restriction p)=\{g\in\mathbb{N}^\omega: g\restriction p=f\restriction p\}$$

where $f\in\mathbb{N}^\omega$ and $p\in\mathbb{N}$. Let $F(f\restriction p)$ be the closure in Y of the set $\bigcup_{g\in U(f\restriction p)}K(g)$. Clearly, $K(f)\subset\bigcap_{p\in\mathbb{N}}F(f\restriction p)$. If $H\in\mathscr{F}(Y)$ is a neighborhood of $K(f)$, then $\bigcup_{g\in U(f\restriction q)}K(g)\subset H$ for some $q\in\mathbb{N}$; for K is usc. As H is closed in Y,

$$\bigcap_{p\in\mathbb{N}}F(f\restriction p)\subset F(f\restriction q)\subset H.$$

Being compact, $K(f)$ is the intersection of its closed neighborhoods in Y. Consequently $K(f)=\bigcap_{p\in\mathbb{N}}F(f\restriction p)$ and it follows that $X=\bigcup_{f\in\mathbb{N}^\omega}\bigcap_{p\in\mathbb{N}}F(f\restriction p)$ is a Souslin-$\mathscr{F}$(Y) set. □

11.14. COROLLARY. *Each regular $\mathscr{K}$-analytic space is normal and pre-Radon.*

PROOF. By 11.12 and ENGELKING [1977], Theorem 3.8.2, p. 247, each regular $\mathscr{K}$-analytic space is normal, and hence completely regular. Now the corollary follows from 11.8, 11.13(ii), and 10.10. □

The following theorem is a consequence of 11.14, 11.12, and 11.9.

11.15. THEOREM. *If each open subset of a regular space X is $\mathscr{K}$-analytic, then X is Radon.*

11.16. COROLLARY. *Each regular Souslin space is Radon.*

This corollary, which follows from 11.11, is pivotal in the exposition of SCHWARTZ [1973]. It is quite useful as Souslin spaces are encountered frequently in functional analysis. We also note that by a different proof, Theorem 11.15 and Corollary 11.16 can be established without assuming the *regularity* of X (see SCHWARTZ [1973], pp. 122–126, and ROGERS and JAYNE [1980], Theorem 2.8.3).

In Theorem 11.6 we saw that a pre-Radon space is universally Radon measurable. For a Radon space, we can prove a stronger result.

11.17. THEOREM. *Let μ be a Borel measure in Y, and let $X \subset Y$ be a Radon space. Then X is $\mu^{\#}$-measurable.*

PROOF. By 3.3(ii), it suffices to show that $X \cap B$ is $\mu^{\#}$-measurable for each $B \in \mathscr{B}(Y)$ with $\mu(B) < +\infty$. According to 3.3(i) and 3.13, $X \cap B$ is $\mu^{\#}$-measurable whenever it is $(\mu_B)^{\#}$-measurable (see 3.5). Thus, in view of 11.7, we may assume that μ is finite, in which case μ_X is a Radon measure in X. There is a σ-compact set $F \subset X$ such that

$$\mu^{\#}(X - F) = \mu_X(X - F) = 0\,.$$

As $F \in \mathscr{B}(Y)$, the theorem follows from 3.3(iii). □

Theorem 11.17 gives a precise meaning to the frequently quoted statement that Radon spaces are *universally Borel measurable.*

The best known sufficient condition for a locally compact space to be Radon is given in terms of hereditary weak θ-refinability (see 10.3). Weakly θ-refinable spaces came into existence as a clever generalization of metacompact spaces. Generalizing metacompactness in another, more obvious, direction produces metalindelöf spaces. It is natural, then, to ask whether locally compact, hereditarily metalindelöf spaces are Radon. For a certain class of these spaces, we shall see (Theorem 11.18 and Examples 11.20 and 11.21) that, irrespective of measurable cardinals, this question cannot be answered within the ZFC set theory.

11.18. THEOREM (MA + ¬CH). *Let X be a locally compact and hereditarily metalindelöf space. Then X is Radon whenever either of the following conditions holds.*

(i) *X satisfies the* ccc.

(ii) *X satisfies the* ccc *locally, and contains no discrete subset of real-valued measurable cardinality.*

The theorem follows immediately from 4.9, 4.10, and 10.3.

Before presenting some examples, we need a simple observation.

11.19. LEMMA. *Let X be a nonseparable space with $|X| = \omega_1$. Then there is an uncountable point-countable family of open subsets of X.*

PROOF. Let $X = \{x_\alpha : \alpha \in \omega_1\}$, and let $G_\beta = X - \{x_\alpha : \alpha \in \beta\}^-$ for each $\beta \in \omega_1$. Then $G_\beta \subset G_\alpha$ whenever $\alpha < \beta < \omega_1$, and $\bigcap_{\alpha \in \omega_1} G_\alpha = \emptyset$. Since X is not separable, $G_\alpha \neq \emptyset$ for each $\alpha \in \omega_1$. It follows that $\{G_\alpha : \alpha \in \omega_1\}$ is the desired family. □

11.20. EXAMPLE (CH). Using CH, we construct a locally compact, hereditarily metalindelöf space X, which satisfies the ccc locally but is not Radon.

Let Y and ν be as in 5.10. Applying 11.19, find a point-countable family $\{G_\alpha : \alpha \in \omega_1\} \subset \mathscr{G}(Y)$ of distinct nonempty sets. Give ω_1 the order topology, and topologize $X = \bigcup_{\alpha \in \omega_1} (\alpha \times G_\alpha)$ as a subspace of $\omega_1 \times Y$. Clearly, X is locally compact. Since Y satisfies the ccc, and α is countable, each $\alpha \times G_\alpha$ satisfies the ccc.

To show that X is hereditarily metalindelöf, choose an $A \subset X$. If $\mathscr{U}$ is an open cover of A, let $\mathscr{V}$ be a common open refinement of $\mathscr{U}$ and $\{\alpha \times G_\alpha : \alpha \in \omega_1\}$. As α is countable and G_α is Lindelöf, there is a subcover $\mathscr{W}$ of $\mathscr{V}$ such that $\{W \in \mathscr{W} : W \subset \alpha \times G_\alpha\}$ is countable for each $\alpha \in \omega_1$. The point countability of $\mathscr{W}$ follows from that of $\{G_\alpha : \alpha \in \omega_1\}$.

Let μ be the Dieudonné measure in ω_1 (see 5.5), and let $\chi = \nu \cdot \mathrm{d}\mu$ (see 7.11). It is immediate that χ is a locally trivial Borel measure in X. Since $\nu(G_\alpha) > 0$ for each $\alpha \in \omega_1$, the sets $U_n = \bigcup\{\alpha : \nu(G_\alpha) \geq 1/n\}$ form an open cover of ω_1. Thus $U_N = \omega_1$ for some integer $N \geq 1$, and it follows from 7.10 that $\chi(X) \geq 1/N$. Consequently, χ is not a Radon measure, and as it is finite, X is not a Radon space.

11.21. EXAMPLE (CH). Using the JKR-space (5.8), we can improve the previous example, and construct a locally compact, hereditarily metalindelöf space X which satisfies the ccc (globally) but is not Radon.

The construction is identical to that of Example 11.20, except that we employ the JKR-space $(\mathbb{R}, \mathscr{T})$ instead of ω_1. If $A \subset X$ and $t \in \mathbb{R}$, set $A^t = \{y \in Y : (t, y) \in A\}$.

Let $\mathscr{G}_0 \subset \mathscr{G}(X)$ be an uncountable disjoint family, and let D be a countable dense subset of $\mathbb{R}$ (see 5.8(ii)). For some $t \in D$, the family $\{G^t : G \in \mathscr{G}_0\}$ is uncountable. As Y satisfies the ccc, this is a contradiction. Consequently, X satisfies the ccc.

The remaining properties of X are established by the same arguments as in Example 11.20. Note that if $\diamondsuit$ were assumed instead of CH, we could have modified Example 11.20 by merely introducing Ostaszewski's topology in ω_1 (see 5.9).

The next two examples are based on a general construction which we now describe. Its special case (Example 11.23) is sometimes called the "Alexandroff duplicate" (see ALEXANDROFF and URYSOHN [1929], Ch. 1, §2.7).

11.22. CONSTRUCTION. Let X be a compact space, and let $\mathscr{T}$ be a locally compact topology in X which is finer than $\mathscr{G} = \mathscr{G}(X)$, i.e., $\mathscr{G} \subset \mathscr{T}$. We give $Y = X \times 2$ the topology generated by all sets $H \times \{1\}$ where $H \in \mathscr{T}$, and $G \times 2 - K \times \{1\}$ where $G \in \mathscr{G}$ and K is $\mathscr{T}$-compact.

It is easy to see that with this topology, Y is a compact space such that the subspaces $X \times \{0\}$ and $X \times \{1\}$ are homeomorphic to $(X, \mathscr{G})$ and $(X, \mathscr{T})$, respectively. Moreover, by ENGELKING [1977], Theorem 3.3.4, p. 197, the characters of X and Y are the same.

11.23. EXAMPLE. Let $X = [0, 1]$ have the Euclidean topology, and let $\mathcal{T}$ be the discrete topology in X. Topologizing $Y = X \times 2$ as in 11.22, we see that Y is a first countable compact space. It follows immediately from 11.9 and 11.7 that Y is a Radon space if and only if $\mathfrak{c}$ is not a real-valued measurable cardinal.

11.24. EXAMPLE (CH). Let $X = [0, 1]$ have the Euclidean topology, and let $\mathcal{T}$ be the JKR-topology in X defined in 5.8. Topologizing $Y = X \times 2$ as in 11.22, we obtain a first countable compact space Y which is not Radon; for by 5.8, the open subspace $X \times \{1\}$ is not Radon (cf. 11.7).

Examples 11.23 and 11.24 represent an interesting dichotomy: a space with certain properties exists whenever $\mathfrak{c}$ is either very large by being real-valued measurable (see 4.13(i)), or as small as possible by CH. We shall see in Examples 11.25 and 11.26 that a similar situation occurs when we consider the product of two Radon spaces.

Note. After this writing was finished, D.H. Fremlin kindly informed us that *without* any set-theoretic axioms (i.e., in ZFC) he found a locally compact refinement of the Euclidean topology in $[0, 1]$ which can be used in the Alexandroff duplicate (see 11.22) to produce a non-Radon, first countable, compact space.

11.25. EXAMPLE. According to 6.8, the split interval X (see 5.3) is a compact Radon space. If $Y = X \times X$, then $D = \{(x, -x): x \in X\}$ is a discrete subset of Y. Since

$$Y = \bigcup_{i,j=0}^{1} [i-1, i) \times [j-1, j)$$

it follows from LUTZER [1972] that Y is hereditarily weakly θ-refinable. Thus by 11.9 and 11.7, Y is a Radon space if and only if $\mathfrak{c}$ is not a real-valued measurable cardinal.

11.26. EXAMPLE (CH). Let $X = [-1, 1]$ and $Y = \{(x, x) \in X \times X: x \in [0, 1]\}$. With the help of CH, we construct Hausdorff topologies σ and τ in X such that the following conditions are satisfied.

(i) The spaces (X, σ) and (X, τ) are zero-dimensional, compact, and perfectly normal.

(ii) Y is a Borel subset of $Z = (X, \sigma) \times (X, \tau)$ and in the relative topology, each $K \in \mathcal{K}(Y)$ is countable.

(iii) Y as a subspace of Z has the same Borel sets as Y with the Euclidean topology.

According to (i) and 6.8, the spaces (X, σ) and (X, τ) are Radon. Using (iii), we let

$$\mu(B) = \lambda(\{x \in [0, 1]: (x, x) \in B\})$$

for each $B \in \mathscr{B}(Y)$. Then μ is a Borel probability in Y, which is not Radon by (ii). It follows from (ii) and 11.7 that Z is not a Radon space.

The topologies σ and τ are constructed inductively.

For $A \subset X$, denote by A^{-} the Euclidean closure of A, and let $\dot{-}A = \{-x: x \in A\}$. By CH, there is an enumeration $\{a_\alpha: \alpha \in \omega_1\}$ of $[0, 1]$. Letting $A_\alpha = \{a_\beta: \beta \in \alpha\}$ for $\alpha \leqslant \omega_1$, we may assume that $A_\omega^{-} = [0, 1]$. Applying CH again, find an enumeration $\{S_\alpha: \alpha \in \omega_1\}$ of all countable subsets of $[0, 1]$. We may assume that $S_0 = A_\omega$. For each $\beta \in \omega_1 - \omega$, the family

$$\mathscr{S}_\beta = \{S_\alpha: \alpha \in \beta, S_\alpha \subset A_\beta, a_\beta \in S_\alpha^{-}\}$$

is countable and nonempty; for $S_0 \in \mathscr{S}_\beta$. Let $n \mapsto S_{\beta,n}$ be a map from ω onto $\mathscr{S}_\beta$ such that the set $N_{S,k} = \{n \in \omega: S_{\beta,3n+k} = S\}$ is infinite for each $S \in \mathscr{S}_\beta$ and $k = 0, 1, 2$. Recall from Section 2 that neighborhoods need not be open.

Claim 1. For each ordinal α with $\omega \leqslant \alpha \leqslant \omega_1$ there are zero-dimensional topologies σ_α and τ_α in X having the following properties.

(a) For each $x \in A_\alpha$, the points x and $-x$ have disjoint neighborhoods in σ_α, and also in τ_α. However, if $x \in [0, 1] - A_\alpha$, then σ_α and τ_α neighborhoods of x and $-x$ coincide with those of $\{x, -x\}$.

(b) The neighborhoods of each set $\{x, -x\} \subset X$ in the topologies σ_α and τ_α coincide with those in the Euclidean topology.

(c) If $\omega \leqslant \beta < \alpha \leqslant \omega_1$, then $\sigma_\beta \subset \sigma_\alpha$ and $\tau_\beta \subset \tau_\alpha$.

(d) Given $x \in A_\alpha$ and a Euclidean neighborhood G of the set $\{x, -x\}$, there are $U \in \sigma_\alpha$ and $V \in \tau_\alpha$ such that $x \in U \cap V$, $U \cup V \subset G$, $U \cap (\dot{-} V) = \emptyset$, $U \cap V \subset [0, 1]$, and $|U \cap V| \leqslant \omega$.

(e) Let $U \in \sigma_\alpha$, $V \in \tau_\alpha$, and $S \in \mathscr{S}_\kappa$ where $\kappa \in \omega_1 - \omega$. If $\{a_\kappa, -a_\kappa\} \cap U \cap V \neq \emptyset$, then the sets $U \cap S$, $V \cap S$, $U \cap (\dot{-} S)$, and $V \cap (\dot{-} S)$ are infinite. Moreover, if $a_\kappa \in U \cap V$, then the set $S \cap U \cap V$ is also infinite.

PROOF. Let σ_ω and τ_ω be the topologies in X generated, respectively, by the bases

$$[-y, -x) \cup [x, y) \quad \text{and} \quad (-y, -x] \cup (x, y]$$

where $x, y \in A_\omega$. Keeping in mind that $A_\omega^{-} = [0, 1]$, it is straightforward to verify that σ_ω and τ_ω are zero-dimensional topologies in X satisfying (a)–(e). Note that in view of (a), the topologies σ_ω and τ_ω are not Hausdorff (in fact, not T_1).

Proceeding inductively, we assume that zero-dimensional topologies σ_β and τ_β satisfying (a)–(e) have been defined for each β with $\omega \leqslant \beta < \alpha \leqslant \omega_1$. If α is a limit ordinal, it suffices to let σ_α and τ_α be the topologies in X generated by $\bigcup_{\beta\in\alpha}\sigma_\beta$ and $\bigcup_{\beta\in\alpha}\tau_\beta$, respectively. Hence suppose that $\alpha=\beta+1$.

Now the trick is to add new neighborhoods to a_β and $-a_\beta$ so that (a) is satisfied, and (b)–(e) preserved. For $n\in\omega$, let

$$G_n=\left\{x\in[0,1]:|x-a_\beta|<\frac{1}{n+1}\right\},$$

and choose $x_n\in G_n\cap S_{\beta,n}$ so that the map $n\to x_n$ is injective. There are disjoint sets

$$H_n\subset G_n\cup(\dot{-}G_n)-\{a_\beta,-a_\beta\}$$

open in the Euclidean topology, and such that $\{x_n,-x_n\}\subset H_n$ for each $n\in\omega$.

If $n\in\omega$ is not a multiple of 3, choose $U_n\in\sigma_\beta$ and $V_n\in\tau_\beta$ so that they are Euclidean neighborhoods of the set $\{x_n,-x_n\}$, and $U_n\cup V_n\subset H_n$. Note that such U_n and V_n exist already in σ_ω and τ_ω, respectively.

If $n\in\omega$ is a multiple of 3, use (d) to find $U_n\in\sigma_\beta$ and $V_n\in\tau_\beta$ such that $x_n\in U_n\cap V_n$, $U_n\cup V_n\subset H_n$, $U_n\cap(\dot{-}V_n)=\emptyset$, $U_n\cap V_n\subset[0,1]$, and $|U_n\cap V_n|\leqslant\omega$. This is possible, because $x_n\in S_{\beta,n}\subset A_\beta$.

As σ_β and τ_β are zero-dimensional, we may assume that all U_n and V_n are closed in σ_β and τ_β, respectively. Let

$$U=\{a_\beta\}\cup\bigcup\{U_{3n+k}:n\in\omega;\,k=0,1\},$$

$$V=\{a_\beta\}\cup\bigcup\{V_{3n+k}:n\in\omega;\,k=0,2\},$$

and define σ_α and τ_α as the topologies in X generated by the subbases $\sigma_\beta\cup\{U,X-U\}$ and $\tau_\beta\cup\{V,X-V\}$, respectively.

Noting that in σ_α, U and $X-U$ are closed disjoint neighborhoods of a_β and $-a_\beta$, respectively, we see that σ_α is a zero-dimensional topology in X satisfying (a). Clearly, the same holds for τ_α.

Since each U_n is closed and open in σ_β, and $U_n\subset G_n$, σ_α and σ_β neighborhoods of every $x\in X-\{a_\beta,-a_\beta\}$ are the same. Similarly, τ_α and τ_β neighborhoods of every $x\in X-\{a_\beta,-a_\beta\}$ coincide. From this, the inductive hypothesis, and the construction of U and V, it is easy to deduce that σ_α and τ_α satisfy (b)–(e). □

Letting $\sigma=\sigma_{\omega_1}$ and $\tau=\tau_{\omega_1}$, it follows from 1(a), (b) that σ and τ are zero-dimensional Hausdorff topologies in X. By 1(b), the spaces (X,σ) and (X,τ) are compact (cf. 5.3). From 1(b), we also conclude that the double diagonal $D=\{(x,y)\in Z: x=\pm y\}$ is closed in Z. Finally, by 1(d), Y is a locally countable open subset of D, which proves (ii).

Claim 2. Let $F\subset X$ be closed in the topology σ or τ. Then there is an $H\subset[0,1]$, closed in the Euclidean topology, such that $F\subset H\cup(\dot{-}H)$ and $|H\cup(\dot{-}H)-F|\leqslant\omega$.

PROOF. There are $\alpha, \beta \in \omega_1$ such that

$$S_\alpha \subset F \cap [0,1] \subset S_\alpha^- \quad \text{and} \quad \dot{-} S_\beta \subset F \cap [-1,0] \subset \dot{-} S_\beta^- .$$

Letting $H = S_\alpha^- \cup S_\beta^-$, we have $F \subset H \cup (\dot{-}H)$. Choose a $\gamma \in \omega_1$ with $\alpha \cup \beta \cup \omega \subset \gamma$ and $S_\alpha \cup S_\beta \subset A_\gamma$. If $\gamma < \kappa < \omega_1$ and $a_\kappa \in H$, then either S_α or S_β belongs to $\mathcal{S}_\kappa$. It follows from Claim 1(e) that $\{a_\kappa, -a_\kappa\} \subset F$ whenever $a_\kappa \in H$ and $\gamma < \kappa < \omega_1$. □

Claim 2 together with Claim 1(b) imply that closed subsets of (X, σ) and (X, τ) are G_δ. This completes the proof of (i).

Claim 3. If $F \subset Y$ is closed in the topology of Y inherited from Z, then there is an $H \subset Y$, closed in the Euclidean topology, such that $F \subset H$ and $|H - F| \leqslant \omega$.

PROOF. There is an $\alpha \in \omega_1$ such that

$$S_\alpha \subset \{x \in [0,1] \colon (x, x) \in F\} \subset S_\alpha^- .$$

Choose a $\beta \in \omega_1$ with $\alpha \cup \omega \subset \beta$ and $S_\alpha \subset A_\beta$. If $\beta < \kappa < \omega_1$ and $a_\kappa \in S_\alpha^-$, then $S_\alpha \in \mathcal{S}_\kappa$. By Claim 1(e), $(a_\kappa, a_\kappa) \in F$ whenever $a_\kappa \in S_\alpha^-$ and $\beta < \kappa < \omega_1$. Thus it suffices to let $H = \{(x, x) \colon x \in S_\alpha^-\}$. □

Now (iii) follows easily from Claim 3.

Note. Corollary 11.8 was proved by KNOWLES [1967], but our proof bears more resemblance to that of SUNYACH [1969]. The paper of GARDNER and PFEFFER [1980] contains Theorem 11.18 and Example 11.21. Examples 11.24 and 11.26 are due to JUHÁSZ, KUNEN, and RUDIN [1976] and WAGE [1980], respectively. D.H. Fremlin and R.G. Haydon constructed Example 11.25.

12. Regularity of σ-finite Borel measures

Theorem 6.7 gives a useful sufficient condition for the regularity of moderated Borel measures. Our goal is to extend this result to σ-finite measures. Example 6.5 shows that we cannot hope to do this for non-σ-finite measures.

12.1. LEMMA. *Let $(X, \mathcal{M}, \mu)$ be a measure space with a σ-finite measure μ, and let $\mathcal{A} \subset \mathcal{M}$ be a point-finite family. Then $\mu(A) = 0$ for all but countably many $A \in \mathcal{A}$.*

PROOF. There are $X_p \in \mathcal{M}$ such that $\mu(X_p) < +\infty$, $p = 1, 2, \ldots$, and $X = \bigcup_{p=1}^{\infty} X_p$. Suppose that $\mathcal{A}_+ = \{A \in \mathcal{A} \colon \mu(A) > 0\}$ is uncountable. Since

$$\mathscr{A}_+ = \bigcup_{p,q=1}^{\infty} \left\{ A \in \mathscr{A} : \mu(A \cap X_p) > \frac{1}{q} \right\},$$

there are positive integers p_0, q_0, and distinct sets $A_k \in \mathscr{A}$ such that $\mu(A_k \cap X_{p_0}) > 1/q_0$ for $k = 1, 2, \ldots$. If $B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (A_k \cap X_{p0})$, then $\mu(B) \geqslant 1/q_0$. In particular $B \neq \emptyset$, contrary to the point-finiteness of $\mathscr{A}$. □

Since each disjoint family is point-finite, we obtain the following corollary.

12.2. COROLLARY. *Let μ be a σ-finite Borel measure in X. If X is the support of μ, then X satisfies the* ccc.

Note. Lemma 12.1 cannot be extended to point-countable families. To see this, consider the Dieudonné measure in ω_1 (see 5.5) and the point-countable family $\mathscr{A} = \{\omega_1 - \alpha : \alpha \in \omega_1\}$.

12.3. THEOREM. *Let μ be a σ-finite Borel measure in X. If X is metacompact, then μ is moderated whenever either of the following conditions holds.*

(i) *X contains no discrete subsets of real-valued measurable cardinality.*

(ii) *Each open subset of X is μ-Radon.*

PROOF. For each $x \in X$ let V_x be an open neighborhood of x with $\mu(V_x) < +\infty$, and let $\mathscr{U}$ be a point-finite open refinement of $\{V_x : x \in X\}$. By 12.1, the family $\mathscr{U}_+ = \{U \in \mathscr{U} : \mu(U) > 0\}$ is countable. Let $\mathscr{U}_0 = \mathscr{U} - \mathscr{U}_+$ and $G_0 = \bigcup \mathscr{U}_0$. Clearly, the theorem will be proved by showing that $\mu(G_0) = 0$.

(i) Suppose that $\mu(G_0) > 0$, and let

$$B_n = \{x \in G_0 : |\mathrm{st}(x, \mathscr{U}_0)| = n\}, \quad n = 1, 2, \ldots.$$

Then $B_n \in \mathscr{B}(X)$ and $G_0 = \bigcup_{n=1}^{\infty} B_n$; for

$$B_n = \{x \in G_0 : |\mathrm{st}(x, \mathscr{U}_0)| \geqslant n\} - \{x \in G_0 : |\mathrm{st}(x, \mathscr{U}_0)| \geqslant n + 1\}$$

is a difference of two open sets, and $\mathscr{U}_0$ is a point-finite cover of G_0. Since μ is σ-finite, there are $X_m \in \mathscr{B}(X)$ such that $\mu(X_m) < +\infty$, $m = 1, 2, \ldots$, and $X = \bigcup_{m=1}^{\infty} X_m$. Thus we can find positive integers M, N for which $\mu(B_N \cap X_M) > 0$. Let $Y = B_N \cap X_M$, and let $\mathscr{G}_0$ be the family of all sets $Y \cap U_1 \cap \cdots \cap U_N$ where $U_1, \ldots, U_N$ are distinct sets from $\mathscr{U}_0$. It follows from the definition of B_N that $\mathscr{G}_0 \subset \mathscr{G}(Y)$ is a disjoint family and $\bigcup \mathscr{G}_0 = Y$. By 10.1, this is a contradiction.

(ii) Since $\mu(K) = 0$ for each $K \in \mathscr{K}(G_0)$, $\mu(G_0) = 0$ whenever G_0 is μ-Radon. □

From Theorems 12.3 and 6.7 we obtain the following important corollaries.

12.4. COROLLARY. *Let μ be a σ-finite Borel measure in a metacompact space X which contains no discrete subset of real-valued measurable cardinality. If each open set of finite measure is, respectively, μ-inner regular or μ-Radon, then μ is regular, or outer regular and Radon.*

12.5. COROLLARY. *Let μ be a σ-finite Borel measure in a metacompact space X. If each open set is μ-Radon, then μ is outer regular and Radon.*

12.6. EXAMPLE. We show that the cardinality restriction in Corollary 12.4 is essential. Let Y be the set of all rational numbers with the Euclidean topology, let Z be a discrete space of real-valued measurable cardinality, and let ν be a diffused Borel probability in Z. In the metrizable space $X = Y \times Z$, we define a σ-finite inner regular Borel measure μ by setting

$$\mu(B) = \sum_{y \in Y} \nu(\{z \in Z: (y, z) \in B\})$$

for each $B \in \mathscr{B}(X)$. Let $G \in \mathscr{G}(X)$ and $\mu(G) > 0$. Find a $y \in Y$ such that $\nu(\{z \in Z: (y, z) \in G\}) > 0$, and denote by Z_n the set of all $z \in Z$ for which $(t, z) \in G$ whenever $|t - y| < 1/n$, $n = 1, 2, \ldots$. Since G is open,

$$\bigcup_{n=1}^{\infty} Z_n = \{z \in Z: (y, z) \in G\}$$

and we obtain that $\nu(Z_N) > 0$ for some integer $N \geqslant 1$. Thus

$$\mu(G) \geqslant \mu\left(\left\{t \in Y: |t - y| < \frac{1}{N}\right\} \times Z_N\right)$$

$$= \sum \left\{\nu(Z_N): t \in Y, |t - y| < \frac{1}{N}\right\} = +\infty .$$

Consequently, $\mu(G) = 0$ or $\mu(G) = +\infty$ for each $G \in \mathscr{G}(X)$. It follows that μ is not outer regular, and that each open set of finite measure is μ-Radon.

A space X is called *θ-refinable* if each open cover of X has open refinements $\mathscr{U}_n$, $n = 1, 2, \ldots$, such that for every $x \in X$ there is an $n_x \geqslant 1$ with $|\mathrm{st}(x, \mathscr{U}_{n_x})| < \omega$. The θ-refinable spaces are discussed in Burke's Chapter 9.

The next example shows that metacompact cannot be replaced by hereditarily θ-refinable in Corollary 12.5.

12.7. EXAMPLE. For nonnegative integers k and n, let $q_{k,n} = (k2^{-n}, 2^{-n})$, $Q_n = \{q_{k,n}: k = 0, \ldots, 2^n\}$, and $Q = \bigcup_{n=1}^{\infty} Q_n$. In $X = [0, 1] \cup Q$, we define a topology as follows: the points of Q are isolated, and a neighborhood base at $t \in [0, 1]$ is given by the sets

$$U(t, \varepsilon) = \{t\} \cup \{q_{k,n} \in Q : 2|k2^{-n} - t| < 2^{-n} < \varepsilon\}$$

where $\varepsilon > 0$. Thus $U(t, \varepsilon)$ consists of t and of all points from Q which lie inside the open wedge in $[0, 1] \times [0, 1]$ with the vertex $(t, 0)$, height ε, and the slopes of the sides equal to ± 2.

It is easy to check that with this topology X becomes a zero-dimensional space, which is separable and locally compact. Each open subset of X is F_σ; for all subsets of $[0, 1]$ are closed. Since the set $[0, 1]$ is uncountable, closed and discrete, X is not Lindelöf. As X is separable it is not metacompact (the argument for this is similar to that of DUGUNDJI [1966], Ch. VIII, 7.4, p. 176).

Let $E \subset X$. Each open cover of E has an open refinement

$$\mathcal{U} = \{\{x\} : x \in E \cap Q\} \cup \{U(t, \varepsilon_t) : t \in E \cap [0, 1]\},$$

where $0 < \varepsilon_t < 1$ for each $t \in E \cap [0, 1]$. Thus letting

$$\mathcal{U}_n = \{\{x\} : x \in E \cap Q\} \cup \{U(t, \varepsilon_t/n) : t \in E \cap [0, 1]\},$$

$n = 1, 2, \ldots$, it is easy to see that E is θ-refinable.

Given $x \in X$, let $f(x) = 2^{-n}$ if $x \in Q_n$, and $f(x) = 0$ otherwise. Let μ be the weighted counting measure in X with the weight f (see 5.1). Since $|U(t, \varepsilon) \cap Q_n| \leqslant 1$, $n = 0, 1, \ldots$, we have $\mu[U(t, \varepsilon)] \leqslant 2$. It follows that μ is a σ-finite Radon measure in X. We shall prove that μ is not outer regular.

Let $G \in \mathcal{G}(X)$ contain $[0, 1]$. For each $t \in [0, 1]$ find an $\varepsilon_t > 0$ with $U(t, \varepsilon_t) \subset G$, and consider the Euclidean topology of $[0, 1]$. By the Baire category theorem, there is a nonempty open interval $J \subset [0, 1]$ and an $\varepsilon > 0$ such that the set $\{x \in J : \varepsilon_t > \varepsilon\}$ is dense in J. Thus $Q \cap (J \times [0, \varepsilon]) \subset G$. Choose an integer $N \geqslant 1$ for which $|Q_N \cap (J \times [0, \varepsilon])| \geqslant 2$. Then

$$|Q_{N+n} \cap (J \times [0, \varepsilon])| \geqslant 2^n, \quad n = 0, 1, \ldots,$$

and it follows that

$$\mu(G) \geqslant \sum_{n=0}^{\infty} 2^{-(N+n)} \cdot 2^n = +\infty .$$

As $\mu([0, 1]) = 0$, the set $[0, 1]$ is not μ-outer regular.

The same argument which showed that the space X of Example 12.7 is not metacompact will also show that X is not metalindelöf. Therefore it is natural to ask whether Corollary 12.5 remains correct if X is a metalindelöf space. We shall see that this question cannot be answered within the ZFC set theory.

12.8. DEFINITION. Let μ be a Borel measure in X. A *concassage* of μ is a disjoint family $\mathcal{D}$ of nonempty compact subsets of X such that

(i) $\mu(D \cap G) > 0$ for each $D \in \mathscr{D}$ and each $G \in \mathscr{G}$ with $D \cap G \neq \emptyset$;
(ii) $\mu(B) = \Sigma\{\mu(B \cap D) \colon D \in \mathscr{D}\}$ for each $B \in \mathscr{B}$ which is μ-Radon.

12.9. REMARK. Taking $G = X$ in 12.8(i), we see that $\mu(D) > 0$ for each $D \in \mathscr{D}$. Thus by 12.1, every concassage of a σ-finite Borel measure is countable.

12.10. PROPOSITION. *Let μ be a Borel measure in X. If each open set of finite measure is μ-Radon, then μ has a concassage.*

PROOF. By Zorn's lemma there is a maximal disjoint family $\mathscr{D}$ of nonempty compact subsets of X which satisfies condition (i) of Definition 12.8. We prove that $\mathscr{D}$ also satisfies condition (ii).

Let $B \in \mathscr{B}(X)$ be μ-Radon, and let $B \cap \bigcup\mathscr{D} = \emptyset$. If $\mu(B) > 0$, we can find a compact set $K \subset B$ with $\mu(K) > 0$. Since K is contained in an open set G of finite measure, it follows from 6.4 that $\mu_K = \mu \restriction \mathscr{B}(K)$ is a Radon measure in K. Thus μ_K has a nonempty compact support S. Adding S to $\mathscr{D}$, we obtain a contradiction to the maximality of $\mathscr{D}$.

If $K \in \mathscr{K}(X)$, find an open set G of finite measure which contains K, and deduce as before that μ_K is a Radon measure in K. By 12.1, the family $\mathscr{D}_0 = \{D \in \mathscr{D} \colon D \cap G \neq \emptyset\}$ is countable. Since the set $K - \bigcup\mathscr{D}_0$ is μ-Radon, and $(K - \bigcup\mathscr{D}_0) \cap \bigcup\mathscr{D} = \emptyset$, we have

$$\mu(K) = \mu(K \cap \bigcup\mathscr{D}_0) = \Sigma\{\mu(K \cap D) \colon D \in \mathscr{D}\}.$$

If $B \in \mathscr{B}(X)$, then

$$\mu(K) = \sum\{\mu(K \cap D) \colon D \in \mathscr{D}\} \leqslant \sum\{\mu(B \cap D) \colon D \in \mathscr{D}\}$$

for each $K \in \mathscr{K}(B)$. Hence

$$\mu(B) \leqslant \sum\{\mu(B \cap D) \colon D \in \mathscr{D}\}$$

whenever the set B is μ-Radon. As $\mathscr{D}$ is disjoint, the reverse inequality is clear. $\square$

Note. The mere existence of a concassage does not mean much. For instance, it is easy to check that the only concassage of the measure μ from Example 12.6 is the empty family. However, under the assumptions of Proposition 12.10 this cannot happen if there is an open set of finite positive measure.

12.11. THEOREM (MA + $\neg$CH). *Let μ be a σ-finite Borel measure in a metalindelöf space X. If each open set is μ-Radon, then μ is outer regular and Radon.*

PROOF. For each $x \in X$ let V_x be an open neighborhood of x with $\mu(V_x) < +\infty$, and let $\mathscr{U}$ be a point-countable open refinement of $\{V_x : x \in X\}$. By 12.10 and 12.9, there is a countable concassage $\mathscr{D}$ of μ. Let

$$\mathscr{U}_+ = \{U \in \mathscr{U} : U \cap (\bigcup \mathscr{D}) \neq \emptyset\}$$

and $G_0 = \bigcup(\mathscr{U} - \mathscr{U}_+)$. From 6.9 we find that $\mu(G_0) = 0$. As $\mathscr{U}_+$ is a point-countable family of open sets, 12.2 and 4.8 imply that the collection $\{U \in \mathscr{U}_+ : U \cap D \neq \emptyset\}$ is countable for each $D \in \mathscr{D}$. Thus

$$\mathscr{U}_+ = \bigcup_{D \in \mathscr{D}} \{U \in \mathscr{U}_+ : U \cap D \neq \emptyset\}$$

is countable, and so μ is moderated. The theorem follows from 6.7. $\square$

12.12. EXAMPLE (CH). Assuming CH, we construct a zero-dimensional space Z such that

(i) Z is locally compact, and $|Z| = \omega_1$;

(ii) each open subset of Z is F_σ;

(iii) each subset of Z is metalindelöf and θ-refinable;

(iv) there is a σ-finite diffused Radon measure χ in Z which is not outer regular (and so by 12.5, Z is not metacompact).

The main idea is to destroy the separability of the space X defined in 12.7 by replacing the points of Q with copies of Kunen's space (see 5.10). We shall adhere to the notation of Example 12.7.

Let Y and ν be as in 5.10. By 11.19, there is a point-countable family $\mathscr{H} \subset \mathscr{G}(Y)$ with $|\mathscr{H}| = \omega_1$. We may assume that $\mathscr{H}$ consists of compact sets. Indeed, choosing a compact open subset $H^\frown \subset H$ for each $H \in \mathscr{H}$, the family $\mathscr{H}^\frown = \{H^\frown : H \in \mathscr{H}\}$ is point-countable, and $|\mathscr{H}^\frown| = \omega_1$; for the map $H \mapsto H^\frown$ is countable-to-one. We may also assume that $\nu(H) \geq r$ for some $r > 0$ and each $H \in \mathscr{H}$. By CH, there is a bijection $t \mapsto {}^tH$ between $[0, 1]$ and $\mathscr{H}$.

We let $Z = [0, 1] \cup (Q \times Y)$, and $B_q = \{q\} \times B$ for each $q \in Q$ and $B \subset Y$. We define a topology in Z by prescribing the neighborhood bases. If $x \in Q \times Y$, then x belongs to some Y_q, $q \in Q$, and we let the neighborhood base at x in Z be the same as that in Y_q. The neighborhood base at $t \in [0, 1]$ is given by the sets

$$V(t, \varepsilon) = \{t\} \cup ([U(t, \varepsilon) - \{t\}] \times {}^tH)$$

where $\varepsilon > 0$. It follows immediately from 5.10 and 12.7 that Z is a zero-dimensional space which satisfies conditions (i) and (ii).

Let $E \subset Z$. Since $Q \times Y$ is hereditarily Lindelöf, it is hereditarily metacompact. Thus each open cover of E has an open refinement

$$\mathscr{V} = \mathscr{G}_0 \cup \{V(t, \varepsilon_t) : t \in E \cap [0, 1]\}$$

where $0<\varepsilon_t<1$ for each $t\in E\cap[0,1]$, and $\mathscr{G}_0\subset\mathscr{G}(Q\times Y)$ is a point-finite family. If $x\in E$ is contained in uncountably many $V(t,\varepsilon_t)$, then x belongs to uncountably many ${}^tH_q$ for a fixed $q\in Q$. As this contradicts the point-countability of $\mathscr{H}$, we conclude that E is metalindelöf. Considering the covers

$$\mathscr{V}_n=\mathscr{G}_0\cup\{V(t,\varepsilon_t/n): t\in E\cap[0,1]\},$$

$n=1,2,\ldots$, of E, we see that E is also θ-refinable.

Let f be as in 12.7. For $B\in\mathscr{B}(Z)$, set

$$\chi(B)=\sum_{q\in Q} f(q)\nu(B\cap Y_q).$$

As ν is a diffused Radon probability in Y, it is easy to verify that χ is a σ-finite diffused Radon measure in Z (the proof of the local finiteness of χ parallels that of the local finiteness of μ in 12.7). Since $\nu({}^tH)\geqslant r>0$ for each $t\in[0,1]$, the same argument we used in 12.7 reveals that the set $[0,1]$ is not χ-outer regular.

Combining Theorem 12.11 and Example 12.12, we conclude that within the ZFC set theory we cannot decide whether σ-finite Radon measures in metalindelöf spaces are always outer regular.

Note. The paper of GRUENHAGE and PFEFFER [1978] contains Theorem 12.3 and Example 12.7. An example similar to that in 12.7 is also in BOURBAKI [1952], Ch. IV, §1, exercise 5. Theorem 12.11 and Example 12.12 were given in GARDNER and PFEFFER [1980].

13. σ-finiteness of diffused, outer regular, Radon measures

The counting measure in an uncountable discrete space (see 5.1) is outer regular and Radon but not σ-finite. However, this measure is not diffused (see 4.11). In this section we investigate whether a diffused, outer regular, Radon measure is σ-finite. Our main result is the surprising Theorem 13.15.

A subset E of a space X is called *locally countable* if each $x\in X$ has a neighborhood U for which $E\cap U$ is countable.

13.1. LEMMA. *Let μ be a diffused, outer regular, Radon measure in X with a concassage $\mathscr{D}$, and let $E\subset\bigcup\mathscr{D}$ be such that $|E\cap D|\leqslant\omega$ for each $D\in\mathscr{D}$. Then E is locally countable, and contains no uncountable set $B\in\mathscr{B}(X)$.*

PROOF. Since μ is locally finite, it follows from the definition of a concassage (see 12.8) that E is locally countable. Consequently, each compact subset of E is countable. As μ is diffused and Radon, $\mu(B)=0$ for each $B\in\mathscr{B}(X)$ with $B\subset E$.

Let $A \subset E$ be uncountable, and let $G \in \mathscr{G}(X)$ contain A. Then $\mu(G) = +\infty$; for G meets uncountably many members of $\mathscr{D}$. From the outer regularity of μ, we deduce that $A \notin \mathscr{B}(X)$. □

13.2. THEOREM. *Let μ be a diffused, outer regular, Radon measure in X with a concassage $\mathscr{D}$. Then μ is σ-finite if and only if each uncountable, locally countable set $E \subset \bigcup\mathscr{D}$ contains an uncountable set $B \in \mathscr{B}(X)$.*

PROOF. Suppose that each uncountable, locally countable set $E \subset \bigcup\mathscr{D}$ contains an uncountable set $B \in \mathscr{B}(X)$. For each $D \in \mathscr{D}$ choose an $x_D \in D$. Applying 13.1 to the set $E = \{x_D : D \in \mathscr{D}\}$, we see that $\mathscr{D}$ is countable. Since $\mu(X - \bigcup\mathscr{D}) = 0$, the measure μ is σ-finite.

Conversely, let μ be σ-finite. By 12.9, $\mathscr{D}$ is countable, and so $\bigcup\mathscr{D}$ is Lindelöf. It follows that each locally countable set $E \subset \bigcup\mathscr{D}$ is countable. □

Theorem 13.2 is very useful, because it can be applied to a large class of spaces.

13.3. THEOREM. *Let X be a weakly δθ-refinable space, and let E be a locally countable subset of X. Then E is a countable union of discrete sets; in particular $E \in \mathscr{B}(X)$.*

PROOF. For each $x \in X$ choose an open neighborhood V_x of x with $|E \cap V_x| \leqslant \omega$. There is an open refinement $\bigcup_{n=1}^{\infty}\mathscr{U}_n$ of $\{V_x : x \in X\}$ such that for every $x \in X$ there is an integer $n_x \geqslant 1$ with $1 \leqslant |\mathrm{st}(x, \mathscr{U}_{n_x})| \leqslant \omega$. Letting

$$E_n = \{x \in E : 1 \leqslant |\mathrm{st}(x, \mathscr{U}_n)| \leqslant \omega\},$$

$n = 1, 2, \ldots$, we have $\bigcup_{n=1}^{\infty} E_n = E$. Fix an integer $n \geqslant 1$. Since $\mathscr{H} = \{E_n \cap U : U \in \mathscr{U}_n\}$ is a point-countable family of countable sets, $\mathscr{H}$ is star-countable. It follows from Section 3.11 in Burke's Chapter 9 that $E_n = \bigcup\mathscr{A}$ where $\mathscr{A} \subset \mathscr{G}(E_n)$ is a disjoint collection of countable sets $A = \{x_{A,k} : k \in \omega\}$. The sets $B_k = \{x_{A,k} : A \in \mathscr{A}\}$, $k \in \omega$, are discrete, and $\bigcup_{k\in\omega} B_k = E_n$. The theorem follows. □

13.4. COROLLARY. *Each diffused, outer regular, Radon measure in a weakly δθ-refinable space is σ-finite.*

13.5. COROLLARY. *The set ω_1 with the order topology is not a weakly δθ-refinable space.*

PROOF. Clearly, each subset of ω_1 is locally countable. However, it follows from 4.13(i) and 5.5 that $\mathscr{B}(\omega_1) \subsetneqq \mathscr{P}(\omega_1)$. □

Note. Although Corollary 13.5 is well known to topologists (see Burke's Chapter 9, Section 9.4), the above proof is nonstandard. It demonstrates the idea that some topological properties of a space X can be deduced by looking at Borel measures in X.

Next we show that those spaces which do not satisfy the condition of Theorem 13.2 exhibit a definite pathology.

13.6. LEMMA. *Let X be a space, and let E be a locally countable subset of X which contains no uncountable set $B \in \mathscr{B}(X)$. Then E is hereditarily separable.*

PROOF. It suffices to show that E is separable. By Zorn's lemma, there is a maximal disjoint family $\mathscr{U} \subset \mathscr{G}(E)$ such that $1 \leqslant |U| \leqslant \omega$ for each $U \in \mathscr{U}$. The maximality of $\mathscr{U}$ implies that $\bigcup\mathscr{U}$ is dense in E. For each $U \in \mathscr{U}$ choose an $x_U \in U$, and let $B = \{x_U : U \in \mathscr{U}\}$. As B is discrete, $B \in \mathscr{B}(X)$. By our assumption, B is countable, and hence so is $\mathscr{U}$ and $\bigcup\mathscr{U}$. $\square$

An *S-space* is a hereditarily separable space which is not Lindelöf. S-spaces are studied extensively in Roitman's Chapter 7 and Abraham and Todorčević's Chapter 8.

Note. Our definition of an S-space, which is most suitable for our purposes, is slightly different from that given in Roitman's Chapter 7, Section 1.1. In particular, we do *not* require an S-space to be regular. However, there is no danger of confusion, as the reader will always be able to reconcile the obvious differences.

13.7. THEOREM. *Let μ be a diffused, outer regular, Radon measure in X with a concassage $\mathscr{D}$. If $\bigcup\mathscr{D}$ contains no S-space, then μ is σ-finite.*

PROOF. Let $E \subset \bigcup\mathscr{D}$ be a locally countable set containing no uncountable set $B \in \mathscr{B}(X)$. By 13.6 and our assumption, E is Lindelöf and hence countable. The theorem follows from 13.2. $\square$

Using the theorem of Todorčević (see Roitman's Chapter 7, Section 7.2), we obtain the following corollary.

13.8. COROLLARY. *It is consistent with the* ZFC *set theory to assume that each diffused, outer regular, Radon measure in a regular space is σ-finite.*

In general, the sufficient condition of Theorem 13.7 is not necessary (see 13.10). However, we can prove a partial converse using MA + $\neg$CH.

Recall that a space X has *countable tightness* if for each $E \subset X$ and each $x \in E^-$ there is a countable $C \subset E$ such that $x \in C^-$. In particular, X has countable tightness whenever it is first countable or hereditarily separable.

13.9. PROPOSITION (MA + ¬CH). *Let μ be a diffused, outer regular, Radon measure in X with a concassage $\mathcal{D}$, and let each $D \in \mathcal{D}$ have countable tightness. Then μ is σ-finite if and only if $\bigcup\mathcal{D}$ contains no S-space.*

PROOF. By Section 6.4 in Roitman's Chapter comment 5, no $D \in \mathcal{D}$ contains an S-space. If μ is σ-finite, $\mathcal{D}$ is countable (see 12.9), and so $\bigcup\mathcal{D}$ contains no S-space. The converse follows from 13.7. $\square$

The next example shows that under CH the "only if" part of Proposition 13.9 is false.

13.10. EXAMPLE (CH). Let Y be the compact space from 11.24, and let $Z = Y \times [0, 1]$. Since Y is hereditarily separable, so is Z. Choose a countable set $\{y_n: n = 1, 2, \ldots\}$ dense in Y, and for $B \in \mathcal{B}(Z)$ let

$$\mu(B) = \sum_{n=1}^{\infty} 2^{-n}\lambda(\{t \in [0, 1]: (y_n, t) \in B\}) .$$

It is easy to see that μ is a diffused, outer regular, Radon probability in Z whose support is Z. As Z is compact, $\{Z\}$ is a concassage of μ. It follows from 5.8 that Z contains an S-space (e.g., $X \times \{1\} \times \{0\}$).

On the other hand, the family $\mathcal{D} = \{\{y_n\} \times [0, 1]: n = 1, 2, \ldots\}$ is also a concassage of μ, and clearly, $\bigcup\mathcal{D}$ contains no S-space.

13.11. PROPOSITION (MA + ¬CH). *Let X be a locally compact and hereditarily separable space. Then every Borel measure in X is moderated.*

PROOF. As X is hereditarily separable, so is the one point compactification of X. By Section 6.4, comment 5, in Roitman's Chapter 7, X is Lindelöf, and the proposition follows. $\square$

For diffused, outer regular, Radon measures, we can relax the separability condition.

13.12. LEMMA. *Let μ be a diffused, outer regular, Radon measure in X with a concassage $\mathcal{D}$. If each $D \in \mathcal{D}$ is hereditarily separable, then so is $\bigcup\mathcal{D}$.*

PROOF. Let $E \subset \bigcup\mathcal{D}$. For each $D \in \mathcal{D}$ find a countable set $E_D \subset E \cap D$ which is dense in $E \cap D$. Then $H = \bigcup\{E_D: D \in \mathcal{D}\}$ is a dense subset of E. By 13.1 and 13.6, H is separable, and the separability of E follows. $\square$

13.13. THEOREM (MA + ¬CH). *Let μ be a diffused, outer regular, Radon measure in X with a concassage $\mathcal{D}$, and let each $D \in \mathcal{D}$ be hereditarily separable. If $\bigcup\mathcal{D}$ is locally compact, then μ is σ-finite.*

PROOF. By 13.12, $Y = \bigcup \mathscr{D}$ is hereditarily separable, and so is the one point compactification of X. In view of Roitman's Chapter 7, Section 6.4, comment 5, the theorem follows from 13.7. □

Note. As each member of a concassage satisfies the ccc, the assumptions of Theorem 13.13 are not outrageous.

The following example shows that Proposition 13.11 and Theorem 13.13 fail if CH is assumed.

13.14. EXAMPLE (CH). Assuming CH, we construct a zero-dimensional S-space X with the following properties:

(i) X is first countable and locally compact;

(ii) X is perfectly normal;

(iii) there is a diffused, outer regular, Radon measure in X which is not σ-finite.

The construction of X is involved. In principle, it is an elaboration of the JKR-space (see 5.8) with many technical details added.

Let $I = 2^\omega$ be the dyadic space. If $B \in \mathscr{B}(I)$, we denote by $\|B\|$ the Haar probability of B (see 5.10). For $f \in I$ and $n \in \omega$, let

$$U(f, n) = \{g \in I : g \restriction n = f \restriction n\}.$$

The sets $U(f, n)$ form a compact open base for the topology of I, and it is easy to see that $\|U(f, n)\| = 2^{-n}$.

With the help of CH, we choose a fixed bijection $\alpha \mapsto f_\alpha$ from ω_1 onto I such that $\{f_\alpha : \alpha \in \omega\}$ is dense in I. We shall give ω_1 the unique topology for which the bijection $\alpha \mapsto f_\alpha$ is a homeomorphism. Thus throughout this example, I and ω_1 are homeomorphic spaces.

Let $X = \omega_1 \times I$, and let $\pi : X \to \omega_1$ be the natural projection. If $E \subset X$ and $\alpha \in \omega_1$, we let $E^\alpha = \{f \in I : (\alpha, f) \in E\}$. By CH, there is an enumeration $\{S_\alpha : \alpha \in \omega_1\}$ of all sets $S \subset X$ such that $|\pi(S)| = \omega$ and $|S^\gamma| = 1$ for each $\gamma \in \pi(S)$. We may assume that $\pi(S_0) = \omega$. For each $\beta \in \omega_1 - \omega$, the family

$$\mathscr{S}_\beta = \{S_\alpha : \alpha \in \beta,\ S_\alpha \subset \beta \times I,\ \beta \in \pi(S_\alpha)^-\}$$

is countable and nonempty; for $S_0 \in \mathscr{S}_\beta$. Let $n \mapsto S_{\beta,n}$ be a map from ω onto $\mathscr{S}_\beta$ such that the set $N_S = \{n \in \omega : S_{\beta,n} = S\}$ is infinite for each $S \in \mathscr{S}_\beta$. Thus ω is a disjoint union of $\{N_S : S \in \mathscr{S}_\beta\}$ whenever $\mathscr{S}_\beta \neq \emptyset$.

We topologize X by inductively defining topologies on $\alpha \times I$ for each $\alpha \leqslant \omega_1$.

Claim 1. Given $(\kappa, f) \in X$ and p, q in ω, there are sets $U(\kappa, f; p, q) \subset X$ satisfying the following conditions.

(a) If $f \restriction p = g \restriction p$, then

$$U(\kappa, f; p, q) = U(\kappa, g; p, q);$$

if $f(p) \neq g(p)$, then

$$U(\kappa, f; p+1, q) \cap U(\kappa, g; p+1, q) = \emptyset.$$

(b) If $p' \geq p$ and $q' \geq q$, then

$$U(\kappa, f; p', q') \subset U(\kappa, f; p, q) \subset (\kappa + 1) \times I.$$

(c) $U^\kappa(\kappa, f; p, q) = U(f, p)$.

(d) $\Sigma_{\beta \leq \kappa} \|U^\beta(\kappa, f; p, q)\| \leq 2^{-p} + 2^{-q}$.

Moreover, given $\gamma \leq \omega_1$, there is a topology $\mathcal{T}_\gamma$ in $\gamma \times I$ such that the following holds.

(e) For each $(\kappa, f) \in \gamma \times I$ the family $\{U(\kappa, f; p, q): p, q \in \omega\}$ is a compact, open, neighborhood base at (κ, f) in the topology $\mathcal{T}_\gamma$.

(f) If $G \in \mathcal{G}(\omega_1)$, then $(G \cap \gamma) \times I \in \mathcal{T}_\gamma$; in particular, $\mathcal{T}_\gamma$ is a Hausdorff topology (see (a)).

(g) If $\omega \leq \beta < \gamma$ and $S \in \mathcal{S}_\beta$, then $\{\beta\} \times I \subset S^-$, where the closure is taken in $\mathcal{T}_\gamma$.

PROOF. We proceed by induction on γ. If $\gamma < \omega$, it is clear that the conditions of the claim will be satisfied if we set

$$U(\kappa, f; p, q) = \{\kappa\} \times U(f, p)$$

for each $(\kappa, f) \in \gamma \times I$ and p, q in ω.

Assume that the claim holds for each $\gamma < \delta \leq \omega_1$. If δ is a limit ordinal, let $\mathcal{T}_\delta$ be the topology in $\delta \times I$ generated by $\bigcup_{\gamma<\delta} \mathcal{T}_\gamma$. Since by (b) and (e), $\mathcal{T}_\beta \subset \mathcal{T}_\gamma$ whenever $\beta < \gamma < \delta$, it is easy to see that the claim holds also for δ. Thus suppose that $\omega \leq \gamma$ and $\delta = \gamma + 1$.

Let $\{G_n: n \in \omega\} \subset \mathcal{G}(\omega_1)$ be a nested neighborhood base at γ. For each $n \in \omega$ choose a $\kappa_n \in G_n \cap \pi(S_{\gamma,n})$ so that the map $n \mapsto \kappa_n$ is injective, and find $f_n \in I$ with $(\kappa_n, f_n) \in S_{\gamma,n}$. Since $\{\kappa_n\}$ is a sequence of distinct points converging to γ, it follows from (e) and (f) that there are disjoint sets

$$U_n = U(\kappa_n, f_n; p_n, p_n)$$

such that $\pi(U_n) \subset G_n$ and $p_n \geq n + 2$ for each $n \in \omega$.

If $A \subset \omega$ is infinite, there is a unique order-preserving map $n \mapsto a_n$ from ω onto A. For $i = 0, 1$ let ${}^iA = \{a_{2n+i}: n \in \omega\}$. Given $f \in I$ and $S \in \mathcal{S}_\gamma$, we define inductively sets $N_S(f, p) \subset N_S$ so that $N_S(f, 0) = N_S$, and

$$N_S(f, p+1) = {}^{f(p)}N_S(f, p)$$

for each $p \in \omega$. We set

$$N(f, p) = \bigcup\{N_S(f, p): S \in \mathscr{S}_\gamma\}.$$

Now let

$$U(\gamma, f; p, q) = [\{\gamma\} \times U(f, p)] \cup \bigcup\{U_n: n \in N(f, p), n \geq q\}$$

for each $f \in I$ and p, q in ω. These sets clearly satisfy conditions (a)–(c). Verification of the remaining conditions is laborious but presents no real difficulties.

(d) We have

$$\sum_{\beta \leq \gamma} \|U^\beta(\gamma, f; p, q)\| \leq \|U(f, p)\| + \sum_{n \geq q} \sum_{\beta \leq \kappa_n} \|U_n^\beta\|$$

$$\leq 2^{-p} + \sum_{n \geq q} (2^{-p_n} + 2^{-p_n}) \leq 2^{-p} + \sum_{n \geq q} 2^{-n-1} = 2^{-p} + 2^{-q}.$$

(e) Let $\mathscr{T}_\delta$ be the topology in $\delta \times I$ generated by the family

$$\{U(\kappa, f; p, q): (\kappa, f) \in \delta \times I; p, q \in \omega\}.$$

By the inductive hypothesis, $U(\gamma, f; p, q) \cap (\gamma \times I)$ belongs to $\mathscr{T}_\gamma$. Using this, it is straightforward to verify that for each $(\kappa, f) \in \delta \times I$, the family

$$\{U(\kappa, f; p, q): p, q \in \omega\}$$

is an open neighborhood base at (κ, f).

The set $\{\gamma\} \times U(f, p)$ is compact, and by the inductive hypothesis, so are the sets U_n, $n \in \omega$. Thus the compactness of $U = U(\gamma, f; p, q)$ will be established by showing that any infinite set E of points $(\kappa_n, g_n) \in U_n$ with $n \in N(f, p)$ and $n \geq q$ has a cluster point (γ, g) in U. To this end, we define inductively a $g \in U(f, p)$. First observe that in order to define $N(g, l)$, $l \in \omega$, we only need to know $g \restriction l$. Let $g(i) = f(i)$ for each $i \in p$. Then the set

$$\{n \in N(g, p): E \cap U_n \neq \emptyset\} = \{n \in N(f, p): n \geq q\}$$

is infinite. Suppose that $g(i)$ has been defined for $i \in l$, with $l \geq p$, so that the set

$$\{n \in N(g, l): E \cap U_n \neq \emptyset\}$$

is infinite. If the set

$$\{n \in \bigcup_{S \in \mathcal{S}_\gamma} {}^0N_S(g, l): E \cap U_n \neq \emptyset\}$$

is infinite, let $g(l) = 0$; otherwise, let $g(l) = 1$. From this definition of g it is clear that each neighborhood $U(\gamma, g; l, k)$ contains infinitely many points of E.

(f) Let $G \in \mathcal{G}(\omega_1)$ be such that $\gamma \in G$; for if $\gamma \notin G$, then $(G \cap \delta) \times I = (G \cap \gamma) \times I$ belongs to $\mathcal{T}_\gamma \subset \mathcal{T}_\delta$ by the inductive hypothesis. As $\{G_n : n \in \omega\}$ is nested, there is a $q \in \omega$ such that $G_n \subset G$ whenever $n \in \omega$ and $n \geq q$. It follows that

$$\pi[U(\gamma, f; 0, q)] \subset G$$

for each $f \in I$. This, (b), and the inductive hypothesis imply that $(G \cap \delta) \times I \in \mathcal{T}_\delta$.

(g) By the inductive hypothesis, it suffices to show that $\{\gamma\} \times I \subset S^-$ for each $S \in \mathcal{S}_\gamma$. Let $S \in \mathcal{S}_\gamma$, $f \in I$, and let p, q be in ω. Since the set $S \cap U(\gamma, f; p, q)$ contains the infinite set

$$\{(\kappa_n, f_n): n \in N_S(f, p), n \geq q\},$$

we see that $(\gamma, f) \in S^-$. $\square$

Using Claim 1, we give X the topology $\mathcal{T} = \mathcal{T}_{\omega_1}$. It follows from 1(e) that the space X is zero-dimensional, first countable, and locally compact. Moreover, by 1(f), the map $\pi: X \to \omega_1$ is continuous.

Claim 2. If $A \subset X$, then $|\{\alpha \in \pi(A)^-: (A^-)^\alpha \neq I\}| \leq \omega$.

PROOF. Since $\pi(A)$ is separable, there is an $\alpha \in \omega_1$ such that $S_\alpha \subset A$ and $\pi(S_\alpha)^- = \pi(A)^-$. Find a $\kappa \in \omega_1$ with $\alpha \cup \omega \subset \kappa$ and $S_\alpha \subset \kappa \times I$. If $\beta \in \pi(S_\alpha)^-$ and $\beta > \kappa$, then $S_\alpha \in \mathcal{S}_\beta$. Thus by 1(g), $\{\beta\} \times I \subset S_\alpha^- \subset A^-$ for each $\beta \in \pi(A)^-$ with $\beta > \kappa$. $\square$

Claim 3. The space X is a perfectly normal S-space.

PROOF. Let E and F be closed disjoint subsets of X. We call a set $G \in \mathcal{T}$ *suitable* if $G^- \cap E = \emptyset$ or $G^- \cap F = \emptyset$. We show that X can be covered by a countable collection of suitable sets. From this the normality of X follows by a standard argument (see KELLEY [1955], Ch. 4, Lemma 1, p. 113). Claim 2 implies that $|\pi(E)^- \cap \pi(F)^-| \leq \omega$. Thus $[\pi(E)^- \cap \pi(F)^-] \times I$, being Lindelöf, can be covered by a countable family of suitable sets. Since ω_1 is second countable (it is homeomorphic to I), we can find a countable open cover $U_1, U_2, \ldots$ of $\omega_1 - \pi(E)^- \cap \pi(F)^-$ such that $U_n^- \cap \pi(E)^- = \emptyset$ or $U_n^- \cap \pi(F)^- = \emptyset$ for each $n = 1, 2, \ldots$. As $\pi: X \to \omega_1$ is a continuous map, the sets $\pi^{-1}(U_n)$, $n = 1, 2, \ldots$, are suitable. Moreover.

$$\bigcup_{n=1}^{\infty} \pi^{-1}(U_n) = X - [\pi(E)^- \cap \pi(F)^-] \times I.$$

Given a closed set $F \subset X$, find $G_n \in \mathcal{G}(\omega_1)$ $n = 1, 2, \ldots$, such that $\bigcap_{n=1}^{\infty} G_n =$

$\pi(F)^-$. By the continuity of π and Claim 2, the set

$$E = \pi(F)^- \times I - F = \pi^{-1}[\pi(F)^-] - F$$

is locally compact and Lindelöf. Hence E is a countable union of compact sets $K_1, K_2, \ldots$. It follows that the sets

$$H_n = G_n \times I - K_n = \pi^{-1}(G_n) - K_n\,,$$

$n = 1, 2, \ldots$, are open, and $\bigcap_{n=1}^{\infty} H_n = F$.

Let $A \subset X$. Since $\pi(A)$ is separable, there is a countable set $S \subset A$ with $\pi(S)^- = \pi(A)^-$. By Claim 2, $A^- - S^-$ is hereditarily separable. The separability of A follows.

Finally, X is not Lindelöf, for the open cover $\{\pi^{-1}(\alpha)\colon \alpha \in \omega_1\}$ of X (see Claim 1, (b)) has no countable subcover. □

Claim 4. If $B \in \mathscr{B}(X)$, then $|\{\alpha \in \pi(B)\colon B^\alpha \neq I\}| \leq \omega$.

PROOF. It is easy to see that the family $\mathscr{A}$ of all sets $A \subset X$ for which $|\{\alpha \in \pi(A)\colon A^\alpha \neq I\}| \leq \omega$ is a σ-algebra in X. By claim 2, $\mathscr{F}(X) \subset \mathscr{A}$, and consequently $\mathscr{B}(X) \subset \mathscr{A}$. □

Claim 5. There is a diffused, outer regular, Radon measure μ in X which is not σ-finite. The family $\{\{\alpha\} \times I\colon \alpha \in \omega_1\}$ is a concassage of μ.

PROOF. If $B \in \mathscr{B}(X)$, it follows from Claim 1(c) that $B^\kappa \in \mathscr{B}(I)$ for each $\kappa \in \omega_1$. Thus we let

$$\mu(B) = \sum_{\kappa \in \omega_1} \|B^\kappa\| \quad \text{for every } B \in \mathscr{B}(X)\,.$$

In view of Claim 1, (b)–(d), it is immediate that μ is a diffused Borel measure in X with a concassage $\{\{\alpha\} \times I\colon \alpha \in \omega_1\}$. It is also easy to see that μ is Radon and not σ-finite. To show that μ is also outer regular, we choose a $B \in \mathscr{B}(X)$. If $|\pi(B)| > \omega$, then $\mu(B) = +\infty = \mu^*(B)$ by Claim 4. If $|\pi(B)| \leq \omega$, let $\pi(B) = \{\beta_k\colon k \in \omega\}$. Given $\varepsilon > 0$, find $G_k \in \mathscr{G}(I)$ such that $B^{\beta_k} \subset G_k$ and

$$\|G_k\| < \|B^{\beta_k}\| + \varepsilon \cdot 2^{-k-2}\,.$$

It is well known (and, in fact, not hard to prove directly from the definition of I) that for each $k \in \omega$ there is a disjoint family $\{U(f_{k,n}, p_{k,n})\colon n \in T_k\}$, $T_k \subset \omega$, whose union is G_k. Thus

$$\sum_{n \in T_k} 2^{-p_{k,n}} = \sum_{n \in T_k} \|U(f_{k,n}, p_{k,n})\| = \|G_k\| < \|B^{\beta_k}\| + \varepsilon \cdot 2^{-k-2}$$

for each $k \in \omega$. Choose $q_{k,n} \in \omega$ so that $\Sigma_{n\in\omega} 2^{-q_{k,n}} < \varepsilon \cdot 2^{-k-2}$ for each $k \in \omega$, and let

$$U_{k,n} = U(\beta_k, f_{k,n}; p_{k,n}, q_{k,n}).$$

It follows from Claim 1, (e) and (c), that the set $G = \bigcup_{k\in\omega}\bigcup_{n\in T_k} U_{k,n}$ is open in X and contains B. By the same Claim, (b) and (d),

$$\begin{aligned}\mu(G) &\leqslant \sum_{k\in\omega}\sum_{n\in T_k} \mu(U_{k,n}) \\ &= \sum_{k\in\omega}\sum_{n\in T_k}\sum_{\beta\leqslant\beta_k} \|U^{\beta}_{k,n}\| \leqslant \sum_{k\in\omega}\sum_{n\in T_k} (2^{-p_{k,n}} + 2^{-q_{k,n}}) \\ &\leqslant \sum_{k\in\omega} (\|B^{\beta_k}\| + \varepsilon \cdot 2^{-k-2} + \varepsilon \cdot 2^{-k-2}) = \mu(B) + \varepsilon. \qquad \square\end{aligned}$$

Example 13.14 in conjunction with Corollary 13.8 yields the following theorem.

13.15. THEOREM. *In the* ZFC *set theory, it is impossible to decide whether all diffused, outer regular, Radon measures in regular spaces are σ-finite.*

In view of Proposition 13.11, the question of Theorem 13.15 cannot be decided even for measures defined in locally compact, hereditarily separable spaces.

Without proof, it was announced in SZENTMIKLÓSSY [1978] that under MA + $\neg$CH, a compact, hereditarily normal space contains no S-space. While no proof of this has been published, recently I. Juhász kindly informed us that Szentmiklóssy's claim is indeed correct. Since the one point compactification of a locally compact, hereditarily normal space is again hereditarily normal, Szentmiklóssy's result implies that Proposition 13.11 remains valid when 'hereditarily separable' is replaced by 'hereditarily normal'. Thus it follows from 13.14(ii) and ENGELKING [1977], Theorem 2.1.6, p. 96, that the question of Theorem 13.15 is undecidable also in locally compact, hereditarily normal spaces.

Note. The results of this section are taken from GARDNER and PFEFFER [1979]. However, in that paper, Example 13.14 was constructed by means of $\diamond$.

14. Baire measures

Throughout this section, a space will always mean a *completely regular* space (and, of course, all spaces are still assumed to be Hausdorff).

Let X be a space. By $\mathscr{C}(X)$ we denote the collection of all continuous real-valued functions defined on X. A set $A \subset X$ is called a *zero set* or a *cozero set* if there is an $f \in \mathscr{C}(X)$ such that

$$A = f^{-1}(0) \quad \text{or} \quad A = f^{-1}(\mathbb{R} - \{0\}),$$

respectively. We denote by $\mathscr{F}^*(X)$ and $\mathscr{G}^*(X)$ the families of all zero sets and all cozero sets in X, respectively. The *Baire σ-algebra* $\mathscr{B}^*(X)$ in X is the smallest σ-algebra in X containing $\mathscr{G}^*(X)$. The elements of $\mathscr{B}^*(X)$ are called the *Baire sets.* When no confusion can arise, we shall write $\mathscr{C}$, $\mathscr{G}^*$, $\mathscr{F}^*$, and $\mathscr{B}^*$ instead of $\mathscr{C}(X)$, $\mathscr{G}^*(X)$, $\mathscr{F}^*(X)$, and $\mathscr{B}^*(X)$, respectively. It is clear that $\mathscr{G}^* \subset \mathscr{G}$, $\mathscr{F}^* \subset \mathscr{F}$, and $\mathscr{B}^* \subset \mathscr{B}$.

While the Baire sets can be defined in any topological space X, if X is not completely regular, there is generally no meaningful relationship between $\mathscr{B}^*$ and the topology of X. We note that $\mathscr{B}^*$ is the smallest σ-algebra in X such that all functions from $\mathscr{C}$ are $\mathscr{B}^*$-measurable (see PFEFFER [1977], (7-12), p. 94).

14.1. DEFINITION. A *Baire measure* in X is a *finite* measure μ on $\mathscr{B}^*$.

We note that any measure μ on $\mathscr{B}^*$ is called a Baire measure by many authors. However, as we shall be interested only in finite Baire measures, it seems expedient to include this condition into the definition.

There are strong similarities between Baire and Borel measures, and many arguments about Borel measures can be translated verbatim into arguments about Baire measures, and vice versa. Consequently, in places we shall simply replace the proofs concerning Baire measures by references to the corresponding statements about Borel measures. We mention that a rather successful attempt to unify the theories of Borel and Baire measures has been made by working with measures defined on lattices. However, the unified theory would have only a few applications in this section. The interested reader may consult BACHMAN and SULTAN [1978], [1980].

14.2. THEOREM. *Let μ be a Baire measure in X. Then*

$$\mu(B) = \inf\{\mu(U) \colon B \subset U,\ U \in \mathscr{G}^*\} = \sup\{\mu(Z) \colon Z \subset B,\ Z \in \mathscr{F}^*\}$$

for each $B \in \mathscr{B}^*$.

PROOF. If $f \in \mathscr{C}$, then

$$f^{-1}(\mathbb{R} - \{0\}) = \bigcup_{n=1}^{\infty} f_n^{-1}(0)$$

where $f_n = \min(|f| - 1/n, 0)$, $n = 1, 2, \ldots$. Consequently,

$$\mu(G) = \sup\{\mu(Z) \colon Z \subset G,\ Z \in \mathscr{F}^*\}$$

for each $G \in \mathscr{G}^*$. Now the proof is analogous to that of Proposition 6.2. □

Although we have not defined regularity for Baire measures, the previous theorem actually says that each Baire measure is 'regular'. This fact makes the theory of Baire measures simpler than that of Borel measures. Historically, Baire measures were developed first (see the major paper of VARADARAJAN [1961]), and much of the Borel theory was spawned by this earlier work. The main link between the two theories (Theorem 14.14) was established by MAŘÍK [1957].

Let μ be a Baire measure in X. By analogy with Definition 2.3, we say that μ is *τ-additive* whenever

$$\sup\{\mu(U): U \in \mathscr{U}_0\} = \mu(U_0)$$

for each $\mathscr{U}_0 \subset \mathscr{G}^*$ and $U_0 \in \mathscr{G}^*$ with $\mathscr{U}_0 \nearrow U_0$. In view of 14.2, the argument employed in 6.13 shows that μ is τ-additive if and only if

$$\sup\{\mu(U): U \in \mathscr{U}_0\} = \mu(X)$$

for each $\mathscr{U}_0 \subset \mathscr{G}^*$ with $\mathscr{U}_0 \nearrow X$, or equivalently,

$$\inf\{\mu(Z): Z \in \mathscr{Z}_0\} = 0$$

for each $\mathscr{Z}_0 \subset \mathscr{F}^*$ with $\mathscr{Z}_0 \searrow \emptyset$ (see 2.4(i)). Thus for Baire measures τ-additivity and weak τ-additivity are the same properties.

Following Definition 2.5, the *support* of μ is the set S (necessarily closed) of all $x \in X$ such that $\mu(U) > 0$ for each cozero neighborhood U of x. If the support of μ is empty, μ is called *locally trivial*. A *Baire Dirac measure* in X is a Dirac measure in X (see 5.1) restricted to the Baire sets.

14.3. DEFINITION. A space X is called
(i) *measure-compact* if each Baire measure in X is τ-additive;
(ii) *realcompact* if each 2-valued Baire measure in X is τ-additive.

As nothing new is gained by defining weakly τ-additive Baire measures, measure-compactness and realcompactness are the only completeness properties for Baire measures.

Realcompact spaces were introduced by HEWITT [1948], and they are sometimes called real-complete spaces or Q-spaces. Measure-compact spaces were originally called Φ-spaces, B-compact spaces, and almost Lindelöf spaces. The term measure-compact, which is now widely accepted, is due to MORAN [1968].

The proofs of the following two theorems are similar to those of 7.6(iii), 8.3(iii), 8.7(ii), and 8.12(iii).

14.4. THEOREM. *A space X is measure-compact if and only if each Baire probability in X has a nonempty support.*

14.5. THEOREM. *For a space X the following conditions are equivalent:*

(i) *X is realcompact;*

(ii) *each 2-valued Baire probability in X has a nonempty support;*

(iii) *each 2-valued Baire probability in X is a Baire Dirac measure in X;*

(iv) *each $\mathscr{F}^*$-ultrafilter with* cip *is fixed.*

14.6. *Construction.* Let X be a space, $Y \subset X$, and let μ be a Baire measure in Y. If $f \in \mathscr{C}(X)$ and $G = f^{-1}(\mathbb{R} - \{0\})$, then $g = f \restriction Y$ belongs to $\mathscr{C}(Y)$ and $G \cap Y = g^{-1}(\mathbb{R} - \{0\})$. It follows that $B \cap Y \in \mathscr{B}^*(Y)$ for each $B \in \mathscr{B}^*(X)$; for the collection of all $B \subset X$ with $B \cap Y \in \mathscr{B}^*(Y)$ is a σ-algebra in X containing $\mathscr{G}^*(X)$. Thus we can define a Baire measure ${}_X\mu$ in X by letting ${}_X\mu(B) = \mu(B \cap Y)$ for each $B \in \mathscr{B}^*(X)$. We say that ${}_X\mu$ is the *extension* of the Baire measure μ.

Note. This extension of Baire measures is completely analogous to the extension of Borel measures defined in 3.7. Hence one might expect that by analogy with 3.5, the restriction of Baire measures could also be defined. To see that this is generally not possible, observe that in Example 14.10 the Baire measure μ in Y has no meaningful restriction to the set $D \in \mathscr{F}^*(Y)$.

14.7. PROPOSITION. *Let X be a space, and let $Y \in \mathscr{F}(X) \cup \mathscr{B}^*(X)$. If X is measure-compact or realcompact, then so is Y, respectively.*

PROOF. Given a Baire probability μ in Y, we define a Baire probability ν in X as follows.

(i) If $Y \in \mathscr{F}(X)$, let $\nu = {}_X\mu$.

(ii) If $Y \in \mathscr{B}^*(X)$, use 14.2 to find a $Z \in \mathscr{F}^*(X)$ with $Z \subset Y$ and ${}_X\mu(Z) > 0$. Then let

$$\nu(B) = \frac{1}{{}_X\mu(Z)}\, {}_X\mu(B \cap Z) = \frac{1}{\mu(Z)}\, \mu(B \cap Z)$$

for each $B \in \mathscr{B}^*(X)$. Clearly, ν is 2-valued whenever μ is. Since either Y or Z is closed in X, it is easy to see that the support of ν is contained in that of μ (cf. the proof of Proposition 7.7). The proposition follows from 14.4 and 14.5. □

14.8. PROPOSITION. *Let X be a realcompact space in which each singleton is G_δ. Then Y is realcompact for each $Y \subset X$.*

PROOF. Let $Y \subset X$, and let μ be a 2-valued Baire probability in Y. Then ${}_X\mu$ (see 14.6) is a 2-valued Baire probability in X, which is a Baire Dirac measure δ_x for some $x \in X$ (see 14.5(iii)). By our assumption, $\{x\} \in \mathscr{B}^*(X)$. Since

$$\mu(\{x\} \cap Y) = {}_X\mu(\{x\}) = \delta_x(\{x\}) = 1\,,$$

we have $x \in Y$, and the proposition follows from 14.5(iii). □

14.9. EXAMPLE. Let X be Haydon's space, and let ν be the Borel measure in X defined in 5.7. Considering the Baire measure $\chi = \nu \restriction \mathscr{B}^*(X)$ in X, we see immediately from 14.4 that X is not measure-compact. On the other hand, the same argument we employed in 8.9 implies that X is realcompact. If CH holds, then X is first countable, and so hereditarily realcompact by 14.12.

The following example is due to MORAN [1968].

14.10. EXAMPLE. Let $Y = X \times X$ where X is the Sorgenfrey interval (see 5.2). Since X is hereditarily Lindelöf, it is realcompact by 14.5(ii). Now with the help of 14.5(iii), it is not difficult to establish the realcompactness of Y. Hence by 14.8, Y is hereditarily realcompact. LUTZER [1972] proved that Y is hereditarily weakly θ-refinable. From this, 7.4, and 10.2 it follows that Y is Borel measure-complete whenever $\mathfrak{c}$ is not a real-valued measurable cardinal. Despite this, we prove next that Y is not measure-compact.

Claim. If $D = \{(t, u) \in Y : t + u = 1\}$, then for each $B \in \mathscr{B}^*(Y)$, $B \cap D$ is a Borel subset of D with respect to the Euclidean topology of D.

PROOF. As $D \in \mathscr{F}^*(Y)$, it suffices to prove the claim for each $Z \in \mathscr{F}^*(Y)$ with $Z \subset D$. Thus let $f \in \mathscr{C}(Y)$ be such that $Z = f^{-1}(0)$ is contained in D. If $x = (t, u)$ is in Y, then

$$N_k(x) = Y \cap \left(\left[t, t + \frac{1}{k}\right) \times \left[u, u + \frac{1}{k}\right)\right),$$

$k = 1, 2, \ldots$, is a neighborhood base at x in Y. For $n, k = 1, 2, \ldots$, set

$$C_n = \left\{x \in D : |f(x)| \geq \frac{1}{n}\right\}, \qquad U_n = \left\{x \in Y : |f(x)| < \frac{1}{2n}\right\},$$

$$U_{n,k} = \{x \in Z : N_k(x) \subset U_n\}\,,$$

and let $B = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} U_{n,k}^{-}$ where $U_{n,k}^{-}$ is the Euclidean closure of $U_{n,k}$. As $Z = \bigcup_{k=1}^{\infty} U_{n,k}$ for each $n = 1, 2, \ldots$, we have $Z \subset B$. Given $x \in C_n$, there is an integer $p \geq 1$ such that $|f(y)| > 1/2n$ for each $y \in N_p(x)$; it follows that $x \notin U_{n,k}^{-}$ for $k = 1, 2, \ldots$. Thus $C_n \cap U_{n,k}^{-} = \emptyset$ for each $n, k = 1, 2, \ldots$, and we conclude that $B \subset Z$; for

$$(D - Z) \cap B = \left(\bigcup_{n=1}^{\infty} C_n\right) \cap B = \emptyset\,. \qquad \square$$

Using the claim, we can define a Baire probability μ in Y by letting

$$\mu(B) = \lambda(\{t \in [0, 1): (t, 1-t) \in B\})$$

for each $B \in \mathscr{B}^*(Y)$. As $D \cap N_1(x) = \{x\}$ for each $x \in D$, the probability μ is locally trivial.

14.11. PROPOSITION. *Each measure-compact space is weakly Borel measure-complete, and each realcompact space is weakly Borel-complete.*

PROOF. Let X be a measure-compact space, and let μ be a Borel probability in X. By 14.4, the support of the Baire probability $\nu = \mu \restriction \mathscr{B}^*$ is not empty. Since X is completely regular, cozero sets form a base for the topology of X. It follows that the supports of μ and ν are identical, and X is weakly Borel measure-complete by 7.6(ii). The rest of the proof is similar. □

Corollary 14.15 gives conditions on a space X under which the converse of Proposition 14.15 is also true.

14.12. LEMMA. *Let X be a normal space. If $F \in \mathscr{F}$, $G \in \mathscr{G}$, and $F \subset G$, then there is a $Z \in \mathscr{F}^*$ such that $F \subset Z \subset G$.*

PROOF. By Urysohn's lemma, there is an $f \in \mathscr{C}$ so that $f \restriction F = 1$ and $f \restriction (X-G) = 0$. Then let $g = \min(f - \frac{1}{2}, 0)$ and $Z = g^{-1}(0)$. □

14.13. COROLLARY. *Let X be a normal space, and let μ be a Baire measure in X. If ν is a regular Borel measure in X with $\nu \restriction \mathscr{B}^* = \mu$, then*

$$\nu(G) = \sup\{\mu(Z): Z \subset G, Z \in \mathscr{F}^*\}$$

for each $G \in \mathscr{G}$. In particular, if μ is 2-valued, then so is ν.

14.14. THEOREM. *Let X be a normal, countably paracompact space, and let μ be a Baire measure in X. Then there is a unique regular Borel measure ν in X such that $\nu \restriction \mathscr{B}^* = \mu$.*

PROOF. For $G \in \mathscr{G}$, let

$$v(G) = \sup\{\mu(Z): Z \subset G, Z \in \mathscr{F}^*\}.$$

Let $G_1, G_2, \ldots$ be in $\mathscr{G}$, and let $Z \in \mathscr{F}^*$ be contained in $\bigcup_{n=1}^{\infty} G_n$. By 14.12 and ENGELKING [1977], Theorem 5.2.3, p. 393, there are $Z_n \in \mathscr{F}^*$ such that $Z_n \subset G_n$,

$n = 1, 2, \ldots$, and $Z \subset \bigcup_{n=1}^{\infty} Z_n$. Thus

$$\mu(Z) \leqslant \sum_{n=1}^{\infty} \mu(Z_n) \leqslant \sum_{n=1}^{\infty} v(G_n),$$

and also, as Z is arbitrary,

$$v\left(\bigcup_{n=1}^{\infty} G_n\right) \leqslant \sum_{n=1}^{\infty} v(G_n).$$

Letting

$$\varphi(A) = \inf\{v(G): A \subset G, G \in \mathscr{G}\}$$

for each $A \subset X$, it is easy to see that φ is an outer measure in X (see 3.1) and that a set $E \subset X$ is φ-measurable whenever

$$\varphi(G) \geqslant \varphi(G \cap E) + \varphi(G - E) \quad \text{for each } G \in \mathscr{G}.$$

Let E and G be open sets. Given $\varepsilon > 0$, choose Z and Y in $\mathscr{F}^*$ so that $Z \subset G \cap E$, $Y \subset G - Z$, and

$$\varphi(G \cap E) = v(G \cap E) < \mu(Z) + \varepsilon, \qquad \varphi(G - Z) = v(G - Z) < \mu(Y) + \varepsilon.$$

Then

$$\begin{aligned}\varphi(G \cap E) + \varphi(G - E) &\leqslant \mu(Z) + \mu(Y) + 2\varepsilon \\ &= \mu(Z \cup Y) + 2\varepsilon \leqslant v(G) + 2\varepsilon = \varphi(G) + 2\varepsilon,\end{aligned}$$

and the φ-measurability of E follows. Thus by 3.2, $\nu = \varphi \restriction \mathscr{B}$ is a finite Borel measure, which is outer regular and hence regular. As $\nu(G) = v(G) = \mu(G)$ for each $G \in \mathscr{G}^*$ (see 14.2), we have $\nu \restriction \mathscr{B}^* = \mu$. The uniqueness of ν follows from 14.13. □

14.15. COROLLARY. *If X is a normal, countably paracompact space, then the following conditions are equivalent.*

(i) *X is measure-compact (realcompact).*

(ii) *X is weakly Borel measure-complete (weakly Borel-complete).*

(iii) *X is Borel measure-compact (closed-complete).*

PROOF. As (ii)$\Rightarrow$(iii) is obvious, and (i)$\Rightarrow$(ii) follows from 14.11, it remains to prove (iii)$\Rightarrow$(i).

Given a Baire probability μ in X, let ν be the unique regular Borel probability in X with $\nu \restriction \mathscr{B}^* = \mu$ (see 14.14). Since it is easy to see that the supports of μ and ν coincide, it suffices to apply 7.6(iii), 8.3(iii), 14.4, and 14.5(ii). □

Countable paracompactness is essential in the previous corollary; for SIMON [1971] proved that the Dowker space of RUDIN [1971] is closed-complete but not realcompact.

By ENGELKING [1977], 5.3D, p. 407, each normal metacompact space is countably paracompact. From this, 14.15, and 10.2, we obtain a result of HAYDON [1974], Proposition 3.2.

14.16. COROLLARY. *Let X be a normal metacompact space which contains no closed discrete subset of real-valued measurable cardinality. Then X is measure-compact.*

For an example showing that the normality assumption in Corollary 14.16 cannot be omitted, we refer to MORAN [1970], example 7.1.

15. Products of real lines and Radon measures

Being metrizable and separable, the countable product $\mathbb{R}^\omega$ is measure-compact by 14.16. For uncountable products, the strikingly complex situation will be described by Theorem 15.3. Although this beautiful result is due to FREMLIN [1977], we shall present a more transparent proof obtained recently by KOUMOULLIS [1982].

For $i = 1, 2$, let $\mathscr{A}_i$ be a σ-algebra in a set X_i. The smallest σ-algebra in $X_1 \times X_2$ containing all sets $A_1 \times A_2$ with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$ will be denoted by $\mathscr{A}_1 \otimes \mathscr{A}_2$. A map $f: X_1 \to X_2$ is called *$(\mathscr{A}_1, \mathscr{A}_2)$-measurable* if $f^{-1}(A) \in \mathscr{A}_1$ for each $A \in \mathscr{A}_2$.

Again, throughout this section, all spaces will be completely regular and Hausdorff.

15.1. LEMMA. *Let $\{X_\alpha : \alpha \in \kappa\}$ be a collection of second countable spaces, $X = \Pi_{\alpha \in \kappa} X_\alpha$, and let $\pi_\alpha : X \to X_\alpha$, $\alpha \in \kappa$, be the natural projections. If $\mathscr{A}$ is a σ-algebra in a set Y, then a map $f: Y \to X$ is $[\mathscr{A}, \mathscr{B}^*(X)]$-measurable if and only if the maps $\pi_\alpha \circ f: Y \to X_\alpha$ are $[\mathscr{A}, \mathscr{B}^*(X_\alpha)]$-measurable for all $\alpha \in \kappa$.*

PROOF. Note that $\mathscr{G}^*(X_\alpha) = \mathscr{G}(X_\alpha)$ for each $\alpha \in \kappa$. It follows from ROSS and STONE [1964], Theorem 4, that the cozero sets in X are of the form $G \times \Pi_{\alpha \in \kappa - T} X_\alpha$ where $T \subset \kappa$ is countable, and $G \in \mathscr{G}^*(\Pi_{\alpha \in T} X_\alpha)$. As $\Pi_{\alpha \in T} X_\alpha$ is second countable, each cozero subset of X is a countable union of basic open sets in X. If $\pi_\alpha \circ f$ is $[\mathscr{A}, \mathscr{B}^*(X_\alpha)]$-measurable for each $\alpha \in \kappa$, then $f^{-1}(U) \in \mathscr{A}$ for each basic open set U in X, and the $[\mathscr{A}, \mathscr{B}^*(X)]$-measurability of f follows. The converse is obvious. □

Throughout, we shall assume that the set $\mathbb{N}$ of all positive integers is endowed with the Euclidean, i.e. discrete, topology.

15.2. PROPOSITION. *Let κ be a cardinal, and let χ be a regular Borel probability in Y such that $Y = \bigcup_{\alpha\in\kappa} A_\alpha$ where $A_\alpha \in \mathscr{B}(Y)$ and $\chi(A_\alpha) = 0$ for each $\alpha \in \kappa$. Then there is a probability μ on $\mathscr{B}^*(\mathbb{N}^\kappa) \otimes \mathscr{B}(Y)$ such that $\mathbb{N}^\kappa \times Y = \bigcup_{\alpha\in\kappa}(U_\alpha \times G_\alpha)$, where $U_\alpha \in \mathscr{G}^*(\mathbb{N}^\kappa)$, $G_\alpha \in \mathscr{G}(Y)$, and $\mu(U_\alpha \times G_\alpha) = 0$ for each $\alpha \in \kappa$.*

PROOF. As χ is regular, there are $G_{\alpha,n} \in \mathscr{G}(Y)$ such that

$$A_\alpha \subset G_{\alpha,n+1} \subset G_{\alpha,n} \subset G_{\alpha,1} = Y$$

and $\chi(G_{\alpha,n}) < 1/n$ for each $\alpha \in \kappa$ and $n = 1, 2, \ldots$. Let $B_\alpha = \bigcap_{n=1}^\infty G_{\alpha,n}$, and define a map $y \mapsto f_y$ from Y to $\mathbb{N}^\kappa$ by setting

$$f_y(\alpha) = \begin{cases} 1 & \text{if } y \in B_\alpha\,, \\ \min\{n \in \mathbb{N} : y \notin G_{\alpha,n}\} & \text{if } y \notin B_\alpha\,, \end{cases}$$

for each $y \in Y$ and $\alpha \in \kappa$. Since

$$\{y \in Y : f_y(\alpha) = n\} = \begin{cases} B_\alpha & \text{if } n = 1\,, \\ G_{\alpha,n-1} - G_{\alpha,n} & \text{if } n \geq 2\,, \end{cases}$$

it follows from 15.1 that the map $y \mapsto f_y$ is $[\mathscr{B}(Y), \mathscr{B}^*(\mathbb{N}^\kappa)]$-measurable. Now letting $\varphi(y) = (f_y, y)$ for each $y \in Y$, it is easy to see that the map $\varphi : Y \to \mathbb{N}^\kappa \times Y$ is $[\mathscr{B}(Y), \mathscr{B}^*(\mathbb{N}^\kappa) \otimes \mathscr{B}(Y)]$-measurable. If

$$\mu(A) = \chi[\varphi^{-1}(A)]$$

for each $A \in \mathscr{B}^*(\mathbb{N}^\kappa) \otimes \mathscr{B}(Y)$, then μ is a well-defined probability on $\mathscr{B}^*(\mathbb{N}^\kappa) \otimes \mathscr{B}(Y)$. For $\alpha \in \kappa$ and $n = 1, 2, \ldots$, let

$$H_{\alpha,n} = \{f \in \mathbb{N}^\kappa : f(\alpha) = n\} \times G_{\alpha,n}\,.$$

Then $\bigcup_{\alpha\in\kappa}\bigcup_{n=1}^\infty H_{\alpha,n} = \mathbb{N}^\kappa \times Y$, and $\mu(H_{\alpha,n}) = 0$ for each $\alpha \in \kappa$ and $n = 1, 2, \ldots$; for

$$\varphi^{-1}(H_{\alpha,n}) = \begin{cases} A_\alpha & \text{if } n = 1 \\ \emptyset & \text{if } n \geq 2\,. \end{cases}$$

By the σ-additivity of χ, the cardinal κ is infinite (in fact, uncountable). Hence $|\kappa \times \mathbb{N}| = \kappa$, and the proof is completed. $\square$

15.3. THEOREM. *If κ is a cardinal, then the following conditions are equivalent.*

(i) *$\mathbb{R}^\kappa$ is measure-compact.*

(ii) *$\mathbb{N}^\kappa$ is measure-compact.*

(iii) *If ν is a Radon measure in X and $\{B_\alpha: \alpha \in \kappa\} \subset \mathscr{B}(X)$ is such that $\nu(B_\alpha) = 0$ for each $\alpha \in \kappa$, then $\nu_*(\bigcup_{\alpha\in\kappa} B_\alpha) = 0$.*

(iv) *If ν is a Radon measure in X and $\{F_\alpha: \alpha \in \kappa\} \subset \mathscr{F}(X)$ is such that $\nu(F_\alpha) = 0$ for each $\alpha \in \kappa$, then $\nu_*(\bigcup_{\alpha\in\kappa} F_\alpha) = 0$.*

PROOF. (i)$\Rightarrow$(ii) follows from 14.7, and (iii)$\Rightarrow$(iv) is trivial.

$\neg$(iii)$\Rightarrow\neg$(ii). If (iii) does not hold, there is a Radon measure ν in a space X and a family $\{B_\alpha: \alpha \in \kappa\} \subset \mathscr{B}(X)$ such that $\nu(B_\alpha) = 0$ for each $\alpha \in \kappa$ and $\nu_*(\bigcup_{\alpha\in\kappa} B_\alpha) > 0$. As ν is Radon we can find a compact set $Y \subset \bigcup_{\alpha\in\kappa} B_\alpha$ with $\nu(Y) > 0$. Let $\chi = (1/\nu(Y))\nu_Y$ (see 3.5) and $A_\alpha = B_\alpha \cap Y$ for $\alpha \in \kappa$. By 3.6 and 15.2, there is a probability μ on $\mathscr{B}^*(\mathbb{N}^\kappa) \otimes \mathscr{B}(Y)$, and sets $U_\alpha \in \mathscr{G}^*(\mathbb{N}^\kappa)$, $G_\alpha \in \mathscr{G}(Y)$ such that $\mu(U_\alpha \times G_\alpha) = 0$ for each $\alpha \in \kappa$, and $\bigcup_{\alpha\in\kappa}(U_\alpha \times G_\alpha) = \mathbb{N}^\kappa \times Y$. Given $f \in \mathbb{N}^\kappa$, there is a finite set $T(f) \subset \kappa$ with

$$\{f\} \times Y \subset \bigcup_{\alpha\in T(f)} (U_\alpha \times G_\alpha);$$

for $\{f\} \times Y$ is compact. Let $U_f = \bigcap_{\alpha\in T(f)} U_\alpha$, and define a Baire probability ψ in $\mathbb{N}^\kappa$ by setting $\psi(B) = \mu(B \times Y)$ for each $B \in \mathscr{B}^*(\mathbb{N}^\kappa)$. If $f \in \mathbb{N}^\kappa$, then

$$\psi(U_f) = \mu(U_f \times Y) \leq \sum_{\alpha\in T(f)} \mu(U_\alpha \times G_\alpha) = 0 .$$

Consequently, ψ is locally trivial and $\neg$(ii) follows from 14.4.

(iv)$\Rightarrow$(i). Suppose that μ is a locally trivial Baire probability in $\mathbb{R}^\kappa$. Let $\mathbb{R}^- = \mathbb{R} \cup \{-\infty, +\infty\}$ be the extended reals viewed as a compactification of $\mathbb{R}$, and let $X = (\mathbb{R}^-)^\kappa$. By 14.6 and 14.14, there is a regular Borel measure ν in X such that $\nu(B) = \mu(B \cap \mathbb{R}^\kappa)$ for each $B \in \mathscr{B}^*(X)$. As X is compact, ν is Radon. The sets

$$Z_{\alpha\pm} = \{f \in X: f(\alpha) = \pm\infty\} \subset X - \mathbb{R}^\kappa$$

belong to $\mathscr{F}^*(X)$; for $\mathbb{R}^-$ is homeomorphic to $[0, 1]$. Thus $\nu(Z_{\alpha\pm}) = 0$ for each $\alpha \in \kappa$, and by (iv),

$$\nu_*(X - \mathbb{R}^\kappa) \leq \nu_*[\bigcup_{\alpha\in\kappa} (Z_{\alpha+} \cup Z_{\alpha-})] = 0 .$$

Using the local triviality of μ, find a family $\mathscr{U} \subset \mathscr{G}^*(X)$ such that $\mathbb{R}^\kappa \subset \bigcup\mathscr{U}$ and $\nu(U) = \mu(U \cap \mathbb{R}^\kappa) = 0$ for each $U \in \mathscr{U}$. Since ν is Radon, $\nu(\bigcup\mathscr{U}) = 0$. Also

$$\nu(X - \bigcup \mathcal{U}) \leq \nu_*(X - \mathbb{R}^\kappa) = 0\,,$$

which contradicts $\nu(X) = \mu(\mathbb{R}^\kappa) = 1$. □

An application of 15.3(iv) to Lebesgue measure λ in $\mathbb{R}$ yields the next corollary.

15.4. COROLLARY. *$\mathbb{R}^c$ and $\mathbb{N}^c$ are not measure-compact spaces.*

15.5. COROLLARY (CH). *$\mathbb{R}^{\omega_1}$ and $\mathbb{N}^{\omega_1}$ are not measure-compact spaces.*

It is interesting to note that Corollary 15.5 is consistent with $\neg$CH. More precisely, by HECHLER [1973], Theorem 1, it is consistent with ZFC + $\neg$CH to assume that $\mathbb{R}^\kappa$ and $\mathbb{N}^\kappa$ are not measure-compact spaces for any regular cardinal $\kappa \geq \omega_1$. However, we shall see that things are quite different when MA is assumed.

15.6. PROPOSITION (MA). *Let $\kappa < c$ be a cardinal, and let ν be a Radon measure in X. If $\{F_\alpha : \alpha \in \kappa\} \subset \mathcal{F}(X)$ and $\nu(F_\alpha) = 0$ for each $\alpha \in \kappa$, then $\nu_*(\bigcup_{\alpha \in \kappa} F_\alpha) = 0$.*

PROOF. If $\nu_*(\bigcup_{\alpha\in\kappa} F_\alpha) > 0$, we can find a compact set $K \subset \bigcup_{\alpha\in\kappa} F_\alpha$ with $\nu(K) > 0$. Let S be the support of the finite Radon measure $\mu = \nu_K$ in K (see 3.5 and 3.6). Then S satisfies the ccc (see 12.2), and $\mu(S - F_\alpha) = \mu(S)$ for each $\alpha \in \kappa$. It follows from the definition of S (see 2.5) that $S - F_\alpha$ is a dense subset of S. However

$$\bigcap_{\alpha\in\kappa} (S - F_\alpha) = S - \bigcup_{\alpha\in\kappa} F_\alpha = \emptyset\,,$$

which contradicts MA. □

15.7. COROLLARY (MA). *$\mathbb{R}^\kappa$ and $\mathbb{N}^\kappa$ are measure-compact spaces for each cardinal $\kappa < c$.*

15.8. COROLLARY (MA + $\neg$CH). *$\mathbb{R}^{\omega_1}$ and $\mathbb{N}^{\omega_1}$ are measure-compact spaces.*

Thus we see from Corollaries 15.5 and 15.8 that the measure-compactness of $\mathbb{R}^{\omega_1}$ and $\mathbb{N}^{\omega_1}$ cannot be decided within the ZFC set theory.

If $X = \mathbb{R}$ and $\nu = \lambda$ is the Lebesgue measure in $\mathbb{R}$, then $\nu_*(\bigcup_{\alpha\in\kappa} F_\alpha) = 0$ can be replaced by $\nu^*(\bigcup_{\alpha\in\kappa} F_\alpha) = 0$ in Proposition 15.6 (see MARTIN and SOLOVAY [1970]). The next example shows that this is not true for a general Radon measure.

15.9. EXAMPLE. Let $X = [0,1]^{\omega_1}$, and let $\nu = \lambda^{\omega_1}$ be the product of the Lebesgue measures in $[0,1]$. Then ν is a Radon measure in X. Given $\alpha \in \omega_1$, let $F_\alpha = \{f \in X : f(\alpha) = f(\alpha+1)\}$, and define a map $\pi_\alpha : X \to [0,1]^2$ by $\pi_\alpha(f) = (f(\alpha), f(\alpha+1))$ for each $f \in X$. Then $F_\alpha \in \mathcal{F}(X)$ and $\nu(F_\alpha) = 0$; for $\lambda^2[\pi_\alpha(F_\alpha)] = 0$.

If $F \in \mathscr{F}(X)$ and $F \subset X - \bigcup_{\alpha\in\omega_1} F_\alpha$, then $\lambda^2[\pi_\alpha(F)] < 1$ for each $\alpha \in \omega_1$. Thus $\lambda^2[\pi_\alpha(F)] \leqslant r$ for some $r < 1$ and infinitely many $\alpha \in \omega_1$. It follows that $\nu(F) = 0$, and so $\nu^*(\bigcup_{\alpha\in\omega_1} F_\alpha) = 1$. This result is clearly independent of any set-theoretic axioms.

Note. The observation that $\mathbb{R}^c$ is not measure-compact was first made in MORAN [1968], by another method. The stronger result that $\mathbb{N}^c$ is not measure-compact was proved initially by KEMPERMAN and MAHARAM [1970]; indeed, Proposition 15.2 is based on their ideas, as well as those of HECHLER [1973]. Example 15.9 is due to R.G. Haydon, and it is also given in FREMLIN [1977].

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CHAPTER 23

Banach Spaces and Topology

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Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

Introduction

The theory of Banach spaces has of course its foundation on set-theoretic topology; thus the classical work of BANACH [25] rests heavily on the deeper (diagonal) properties of topology: completeness of the reals gives the Hahn–Banach theorem, Baire's category theorem proves the open mapping theorem and the uniform boundedness principle, while Tychonoff's compactness theorem proves the Alaoglou theorem; EBERLEIN's theorem [44] on weakly compact subsets of Banach spaces is established with a delicate compactness (diagonal) argument as well.

Relatively recently—in the last fifteen years or so—the deeper parts of set-theoretic topology have been transformed essentially to a branch of set theory, mainly with the successful introduction of infinitary combinatorial methods: witness the monographs of JUHÁSZ [67], COMFORT–NEGREPONTIS [34], JUHÁSZ [68], COMFORT–NEGREPONTIS [35], RUDIN [107] and ROGERS [100]. A similar transformation is now well under way with a significant part of the theory of Banach spaces. This exciting development of the last eight years (beginning symbolically with the appearance in 1974 of Rosenthal's beautiful result, described in Section 1 (Theorem 1.9)) forms the total content of this survey.

My aim is to convince set-theorists and set-theoretic topologists to apply their considerable talents and research efforts to the fertile domain of Banach space theory, and to contribute with their formidable set-theoretic techniques (philosophically of a dialectical nature) towards the resolution of some of the celebrated open questions in the area, not on an *ad hoc* and incidental basis, but with a view for an essential reunion of Set Theory with Analysis. A prototypical example of such a question would be the construction of an injective Banach space, not of the form $C(K)$ for a compact extremally disconnected space K.

The interaction between set-theoretic topology and Banach spaces is not only internal, i.e., in the sense of using analytic methods in the study of (mainly compact) topological spaces, and conversely of using topological arguments in the study of (mainly non-separable) Banach spaces. The interaction is to a large extent also due to the fact that both branches use the same set-theoretic methods: mainly infinitary combinatorics, compactness/diagonal arguments, ultraproducts.

Set-theoretic, topological, and Banach spaces problems and methods coalesce and interact in problems concerning calibers (and more generally chain conditions of compact spaces), existence of independent families, and isomorphic embeddings of l^1_α into Banach spaces. These considerations occupy the first four Sections of this survey. Some of the work of Argyros, Bourgain, Rosenthal, Shelah, Talagrand, Tsarpalias and Zachariadies is being developed here. Argyros' positive solution of Pelczynski's conjecture (in Section 4) is certainly an impressive success of infinitary combinatorics in Banach space theory.

The example of Kunen–Haydon–Talagrand, assuming CH, is described in detail in Section 5. It is remarkable that the same construction occurred independently for topological reasons (Kunen), and for Banach space reasons (Haydon), while measure-theoretic properties of this (Talagrand), or of a similar space (Losert) are very interesting too.

In Section 6 we witness a complete topologization (with substantial simplification as well) of some of the deeper parts of Banach space theory: the study of weak compactness. The Amir–Lindenstrauss theorem is thus significantly generalized and totally topologized by Gul'ko's deep theorem: Every Gul'ko-compact space is Corson-compact. We describe most of the significant consequences, and some examples separating the concepts of uniform Eberlein-compact, Eberlein-compact, Talagrand-compact, Gul'ko-compact, and Corson-compact, with one notable (and still open) exception.

In Section 7 Kunen's example of an S-space (assuming CH) gives an example in Banach space theory (related to Shelah's answer of a conjecture on Banach spaces), while in Section 8 there is an introduction to Maurey's work on fixed points of contractions on weakly compact (but generally non norm compact) convex sets.

I feel that the topics included in this survey are important and deep mathematics, that will meet with confidence the test of time. But, of course, a selection has been made, and thus many interesting topics that could fall in the subject matter of this survey were omitted for simple lack of space (and energy on my part). Among them the following should be mentioned: Milutin isomorphism theorem and generalizations, Szlenk index theorem, Bourgain–Rosenthal index theorems, $\mathscr{P}_1$-Banach spaces and injective Banach spaces, Haydon's and Talagrand's Grothendieck spaces of the form $C(K)$.

I am pleased to thank by colleagues, collaborators, and present or former students Spiros Argyros, Nikos Kalamidas, Sophocles Mercourakis, and Theodosios Zachariades for valuable help in the preparation of this paper.

0. Preliminaries

Infinitary combinatorics

The *ordinals* are defined in such a way that an ordinal is the set of smaller ordinals. A *cardinal* is an ordinal not in one-to-one correspondence with any smaller ordinal. The *cofinality* of a cardinal α, denoted by $\mathrm{cf}(\alpha)$, is the least ordinal β such that α is the cardinal sum of β many cardinals, each smaller than α. A cardinal α is *regular* if $\alpha = \mathrm{cf}(\alpha)$, and *singular* if $\mathrm{cf}(\alpha) < \alpha$. The least cardinal strictly greater than β is denoted by β^+. A cardinal α is a *successor cardinal* if it is of the form $\alpha = \beta^+$ for some β; every successor cardinal is regular. The (cardinality of the) natural numbers is denoted by ω. The cardinality of a set A is denoted by $|A|$. The cardinality of the family $\mathscr{P}(\alpha)$ of all subsets of (a set of cardinality) α is denoted by 2^α. We set, for a set S, $\mathscr{P}_\kappa(S) = \{A\colon A \subset S, |A| < \kappa\}$, and $S^{[\omega]} = \{A\colon A \subset S, |A| = \omega\}$. If α, κ are cardinals, we say that α is *strongly κ-inaccessible* (in symbols, $\kappa \ll \alpha$) if $\kappa < \alpha$ and $\beta^\lambda < \alpha$ for all $\lambda < \kappa$ and $\beta < \alpha$.

We will make use of the following infinitary combinatorics.

0.1. ARGYROS' LEMMA ([5], [35] (Lemma 1.1)). *Let α be an infinite regular cardinal and $\kappa \ll \alpha$. Assume that for every $A \subset \alpha$, with $|A| = \alpha$, there is a partition $\mathcal{P}_A$ of A such that $|\mathcal{P}_A| < \alpha$. Then, there is a family $\{A_\eta : \eta < \kappa\}$ of subsets of α such that*

$$\begin{aligned} &|A_\eta| = \alpha && \text{for } \eta < \kappa\,, \\ &A_{\eta+1} \in \mathcal{P}_{A_\eta} && \text{for } \eta < \kappa\,, \quad \text{and} \\ &A_{\eta'} \subset A_\eta && \text{for } \eta < \eta' < \kappa\,. \end{aligned}$$

We recall that a subset S of a cardinal α is called *stationary* if S intersects every closed and unbounded subset of α (in the order toplogy on α).

0.2. FODOR'S THEOREM. *Let S be a stationary subset of an uncountable regular cardinal α, and let $f: S \to \alpha$ be a function such that $f(\xi) < \xi$ for $\xi \in S$, $\xi \neq 0$. Then, there is a stationary subset T of S, and $\zeta < \alpha$, such that $f(\xi) = \zeta$ for all $\xi \in T$.*

0.3. ERDOS–RADO THEOREM ON QUASI-DISJOINT SETS. *Let $\omega \leq \kappa \ll \alpha$, with α regular and let $\{S_\xi : \xi < \alpha\}$ be a family of sets such that $|S_\xi| < \kappa$ for $\xi < \alpha$. Then there are $A \subset \alpha$ with $|A| = \alpha$ and a set S such that*

$$S_\xi \cap S_\eta = S \quad \text{for } \xi, \eta \in A,\ \xi \neq \eta\,.$$

0.4. HAJNAL'S THEOREM ON FREE SETS. *Let α be an infinite cardinal, $\kappa < \alpha$, and $f: \alpha \to \mathcal{P}_\kappa(\alpha)$ a function such that $\xi \notin f(\xi)$ for $\xi < \alpha$. Then there is $A \subset \alpha$, with $|A| = \alpha$, such that $\xi \notin f(\zeta)$ for $\xi, \zeta \in A$ (i.e., A is free for f).*

0.5. LEMMA ([5], [35] (A.7)). *If $\kappa \ll \alpha$, α singular, and $\kappa \ll \mathrm{cf}(\alpha)$, then there is a family $\{\alpha_\sigma : \sigma < \mathrm{cf}(\alpha)\}$ of cardinals such that*

$$\begin{aligned} &\alpha_\sigma \text{ is regular} && \text{for } \sigma < \mathrm{cf}(\alpha)\,, \\ &\kappa \ll \alpha_\sigma && \text{for } \sigma < \mathrm{cf}(\alpha)\,, \\ &\mathrm{cf}(\alpha) < \alpha_\tau < \alpha_\sigma < \alpha && \text{for } \tau < \sigma < \mathrm{cf}(\alpha)\,, \quad \text{and} \\ &\alpha = \sum_{\sigma < \mathrm{cf}(\alpha)} \alpha_\sigma\,. \end{aligned}$$

0.6. THE NASH–WILLIAMS [90], GALVIN–PRIKRY [51] THEOREM. *Let X be a closed subset of $\omega^{[\omega]}$ (where $\omega^{[\omega]}$ is considered as a subset of the Cantor set $\{0, 1\}^\omega$). Then there is $A \in \omega^{[\omega]}$, such that either $A^{[\omega]} \subset X$ or $A^{[\omega]} \subset \omega^{[\omega]} \setminus X$.*

General references for set theory and infinitary combinatorics: COMFORT and NEGREPONTIS [34], [35], JUHÁSZ [67], [68], WILLIAMS [125]; for Martin's axiom RUDIN [107], and the forthcoming monograph of FREMLIN [50].

Topology

A space K has *caliber* α is for every family $\{U_\xi\ \xi < \alpha\}$ of non-empty open subsets of K, there is $A \subset \alpha$, with $|A| = \alpha$, such that $\bigcup_{\xi \in A} U_\xi \neq \emptyset$.

A space K satisfies *condition* (K_n) (where $1 \leq n < \omega$) if for every family $\{U_\xi\colon \xi < \omega^+\}$ of non-empty open subsets of K, there is $A \subset \omega^+$, with $|A| = \omega^+$, such that every n elements of the family $\{U_\xi\colon \xi \in A\}$ has non-empty intersection.

The *Souslin number* $S(K)$ of a space K is the smallest cardinal number α, such that there is no family of α many pairwise disjoint non-empty open subsets of K. By the *Erdos–Tarski theorem*, $S(K)$ is an uncountable regular cardinal for an infinite space K. A space K has the *countable chain condition* (ccc) if $S(K) \leq \omega^+$.

Of course, K has caliber $\omega^+ \Rightarrow K$ has (K_n) for all $1 \leq n < \omega \Rightarrow K$ has ccc.

A partially ordered set $(\mathbb{P}, \leq)$ *satisfies the* ccc if every family of pairwise incompatible elements of $\mathbb{P}$ is countable.

We denote by $\beta\tau$ the *Stone–Čech compactification of* (a discrete set of cardinality) τ. The space $\beta\tau$ is identified with the space of all untrafilters on τ. Every bounded function $f\colon \tau \to \mathbb{R}$ has a unique continuous extension $\bar{f}\colon \beta\tau \to \mathbb{R}$. More generally, for any completely regular Hausdorff space X, we denote by βX its Stone–Čech compactification.

If K, L are compact Hausdorff spaces, then a continuous function f from K onto L is *irreducible* if for every closed subset F of K, with $F \neq K$, we have $f(F) \neq L$. A straightforward application of Zorn's lemma yields the following statement: if $f\colon K \to L$ is continuous, onto, with K and L compact Hausdorff spaces, then there is a closed subset F of K, such that $f(F) = L$, and $f|F$ is irreducible.

A compact Hausdorff space K is *extremally disconnected* if the closure of every open subset of K is open-and-closed. The Stone–Čech compactification of a discrete set is extremally disconnected.

0.7. GLEASON'S THEOREM. *If K is a compact Hausdorff space, then there is a pair $(G(K), \pi)$, such that $G(K)$ is an extremally disconnected, compact space, and $\pi\colon G(K) \to K$ is an irreducible function. The space $G(K)$ is called the Gleason space of K.*

The *weight* (resp. *density character*) of K is denoted by $w(K)$ (resp., $d(K)$).

The *pseudoweight* $\pi(x, K)$ *of K at $x \in K$* is the least cardinal a such that there is a family $\mathscr{B}$ of non-empty open subsets of K, with $|\mathscr{B}| = \alpha$ and such that if V is open in K and $x \in V$ then there is $B \in \mathscr{B}$ with $B \subset V$.

Corson-compact spaces and their subclasses will be defined and studied in Section 6.

A compact Hausdorff space K is *angelic* if it is a *Frechét–Urysohn* space, i.e., if for every $A \subset K$ and $x \in \bar{A}$ there is a sequence $(x_n) \subset A$, with $x_n \to x$. General references for topological concepts: COMFORT and NEGREPONTIS [34], [35], JUHÁSZ [67], [68], KELLEY [70], KURATOWSKI [77].

Banach spaces and measure theory

If X and Y are Banach spaces and $T: X \to Y$ is a linear map, then T is an *isomorphism* (resp. an *isometry*) if T is one-to-one bicontinuous (resp. norm preserving) from X onto Y.

Two Banach spaces are *isomorphic* (resp. *isometric*) if there is an isomorphism (resp. an isometry) from one onto the other.

If $T: X \to Y$ is a map, such that $T: X \to T(X)$ is an isomorphism (resp. an isometry) then T is called an *isomorphic* (resp. *isometric*) *embedding of X into Y*. If there is an isomorphic embedding of X into Y, we write $X \hookrightarrow Y$.

0.8. THE OPEN MAPPING THEOREM. *If X and Y are Banach spaces, and $T: X \to Y$ is linear, bounded, and onto, then T is an open mapping.*

A consequence of the open mapping theorem is the following. A linear mapping $T: X \to Y$ is an isomorphic embedding if and only if there are real numbers $m, M > 0$, such that

$$m\|x\| \leq \|T(x)\| \leq M\|x\| \quad \text{for all } x \in X .$$

The unit sphere (or ball) of a Banach space is denoted by S_X; thus

$$S_X = \{x \in X: \|x\| \leq 1\} .$$

If X is a Banach space, X^* denotes its *dual*; $X^* = \{f: X \to \mathbb{R}: f \text{ is continuous}$ and linear$\}$. The *weak* topology* on X^* is the topology induced on X^* by X (i.e., $x_i^* \to x^*$ is weak* convergent in X^* if $x_i^*(x) \to x^*(x)$ for $x \in X$).

0.9. ALAOGLOU'S THEOREM. *For every Banach space X, the unit sphere S_{X^*} of X^* in the weak* topology is a compact Hausdorff space.*

We identify a Banach space X with a closed subspace of its second dual X^{**}, via the *natural* isometric *embedding*

$$\chi: X \to X^{**}$$

given by $(\chi(x)(x^*) = x^*(x)$ for $x^* \in X^*$.

The *weak topology* on X is the topology induced on X by X^* (i.e., $x_i \to x$ weakly in X if $x^*(x_i) \to x^*(x)$ for all $x^* \in X$).

0.10. The norm-closed convex subsets of X coincide with the weakly-closed convex subsets of X (a consequence of the *Hahn–Banach theorem*); on such sets the weak density character coincides with the norm density character.

The restriction of the weak* topology of X^{**} on X coincides with the weak topology on X.

0.11. Every weakly compact subset of a Banach space is norm-bounded (a consequence of the *uniform boundedness principle*).

0.12. GOLDSTINE'S THEOREM. *For every Banach space X, S_X is dense in $S_{X^{**}}$, where $S_{X^{**}}$ has the weak* topology.*

If $\chi(X) = X^{**}$, then X is called *reflexive* Banach space. The unit sphere of a reflexive Banach space is weakly compact (from Alaoglou's theorem). The reader is referred to Diestel's monograph [42] for a survey of the deep work of James on weak compactness (not used in this paper).

If $T: X \to Y$ is a bounded linear operator between Banach spaces, then $T^*: Y^* \to X$ is the *dual operator*, given by $T^*(y^*) = y^* \circ T$.

0.13. We note that:

(a) T is (weakly, weakly)-continuous.

(b) $\|T^*\| = \|T\|$ and T^* is (weak*, weak*)-continuous.

(c) If T is 1–1, then the range of T is dense in X.

(d) If T is an isomorphic embedding, then T^* is onto.

(e) If $X = Z^*$ for some Banach space Z, and $T: Z^* \to Y$ is (weak*, weakly)-continuous, then $T^*(Y^*) \subset X$.

Given a compact Hausdorff space K, $C(K)$ denotes the space of real-valued continuous functions on K, with supremum norm.

If $A \subset C(K)$ we say that A *separates points of* K if for all $x, y \in K$ with $x \neq y$, there is $f \in A$ with $f(x) \neq f(y)$.

0.14. STONE–WEIERSTRASS THEOREM. *If A is a sub-algebra of $C(K)$ that contains* 1, *and separates points of K, then A is dense in $C(K)$.*

The canonical operator $T: X \to C(S_{X^*})$ given by $(T(x))(x^*) = x^*(x)$ is a linear isometry (where S_{X^*} has the weak* topology).

We denote by $M(K)$ the space of all finite real-valued regular Borel (or, Baire) measures on K (with $\|\mu\| = |\mu|(K)$, where $|\mu|$ is the total variation of μ).

0.15. RIESZ REPRESENTATION THEOREM. *The dual $C(K)^*$ is isometrically isomorphic to $M(K)$ (via the isomorphism $T\colon M(K)\to C(K)^*$ given by $(T(\mu))(f)=\int f\,\mathrm{d}\mu$).*

Thus $S_{M(K)}$ is weak* compact (by Alaoglou's theorem); also the space $M_1^+(K)$ of all *probability measures* $\mu\in M(K)$ (i.e., $\mu\in M(K)$, $\mu\geqslant 0$, and $\mu(K)=1$) is weak* compact.

By $L^p(\mu)$, for $1\leqslant p<\infty$, we denote the Banach space $L^p(\Omega,\Sigma,\mu)$ of equivalence classes of real μ-measurable functions f on (Ω,Σ) such that $\|f\|_p=(\int|f|^p\,\mathrm{d}\mu)^{1/p}<\infty$ (where Ω is a set, Σ is a σ-algebra of Ω, and μ is a positive measure on Σ). By $L^\infty(\mu)$ we denote the Banach space $L^\infty(\Omega,\Sigma,\mu)$ of equivalence classes of real μ-measurable μ-essentially bounded functions f on Ω, with

$$\|f\|_\infty=\mu\text{-}\operatorname*{ess\,sup}_{\omega\in\Omega}|f(\omega)|\,.$$

If $1\leqslant p\leqslant\infty$ $\Omega=\{0,1\}^I$, Σ is the σ-algebra of Borel sets of Ω, and μ is the (unique) Haar probability (product) measure on the (compact abelian) topological group $\{0,1\}^I$, we write $L^p\{0,1\}^I$ for $L^p(\Omega,\Sigma,\mu)$.

0.16. We identify $L^\infty(\mu)$ with $(L^1(\mu))^*$ for any measure μ.

0.17. If μ is a non-negative measure on K, we identify $L^1(\mu)$ with the subspace of $M(K)$ consisting of all measures λ with λ *absolutely continuous* (in symbols, $\lambda\ll\mu$) with respect to μ, by the *Radon–Nikodym Theorem*. The element $h\in L^1(\mu)$ that corresponds to λ, i.e., for which

$$\lambda(A)=\int_A h\,\mathrm{d}\mu\quad\text{for every Borel set } A \text{ in } K\,.$$

is called the *Radon–Nikodym derivative* of λ with respect to μ, in symbols $h=\mathrm{d}\lambda/\mathrm{d}\mu$). By the regularity of the measures λ, μ h can be chosen to be a Baire-measurable function.

Given a set Γ, $l^\infty(\Gamma)$ denotes the Banach space of all bounded functions $f\colon\Gamma\to\mathbb{R}$, with $\|f\|=\sup_{\gamma\in\Gamma}|f(\gamma)|$, and

$$c_0(\Gamma)=\{f\in l^\infty(\Gamma)\colon \text{for all } \varepsilon>0\,\{\gamma\in\Gamma\colon|f(\gamma)|>\varepsilon\}\text{ is finite}\}\,.$$

Also, $l^1(\Gamma)$ denotes the Banach space of all functions $f\colon\Gamma\to\mathbb{R}$ such that $\Sigma_{\gamma\in\Gamma}|f(\gamma)|<\infty$, with the obvious norm. The *usual basis of* $l^1(\Gamma)$ is the set $\{e_\gamma\colon\gamma\in\Gamma\}$, where $e_\gamma(\delta)=1$ if $\gamma=\delta$, $=0$ if $\gamma\neq\delta$, for $\delta,\gamma\in\Gamma$. A subset $\{x_\gamma\colon\gamma\in\Gamma\}$ of a Banach space X is said to be equivalent to the usual basis of $l^1(\Gamma)$ if there is an isomorphic embedding $T\colon l^1(\Gamma)\to X$, such that $T(e_\gamma)=x_\gamma$ for $\gamma\in\Gamma$. If $|\Gamma|=\alpha$, we write sometimes l^1_α, l^∞_α for $l^1(\alpha)$, $l^\infty(\alpha)$ respectively; and, we write c_0, l^1, l^∞ for $c_0(\omega)$, $l^1(\omega)$, $l^\infty(\omega)$, respectively. We identify in a natural way $l^1(\Gamma)$ with the dual $(c_0(\Gamma))^*$; and $l^\infty(\Gamma)$ with the dual $(l^1(\Gamma))^*$, and also with $C(\beta\Gamma)$, where $\beta\Gamma$ is the Stone-Čech compactification of the discrete set Γ.

The *support of a measure* $\lambda\in M(K)$, denoted $\text{supp}(\lambda)$, is the set of all $x\in K$, for which $\lambda(U)>0$ for every open set U containing x. The support of a measure is a closed subset of K.

A non-negative probability measure $\mu\in M(K)$ is called a *strictly positive measure on* K if $\text{supp}(\mu)=K$; and a strictly positive measure μ is called a *normal measure on* K if $\mu(F)=0$ for every closed nowhere dense subset F of K.

If $G(K)$ is the Gleason space of K, $\pi\colon G(K)\to K$ the canonical mapping, and $\mu\in M(K)$, then: (a) if μ is strictly positive, then there is a strictly positive measure ν on $G(K)$, such that $\pi_*(\nu)=\mu$; and, (b) if μ is a normal measure, then there is a normal measure ν on $G(K)$, such that $\pi_*(\nu)=\mu$. (Both statements follow from an application of the Hahn–Banach theorem, and the fact that π is irreducible). (The image measures $\pi_*(\nu)$ are defined below, just above 0.34).

0.18. If K has a strictly positive measure, then K has property (K_n) for all $1\leq n<\omega$ (cf. [35] (Theorem 6.15) or the paper of ARGYROS and KALAMIDAS [11] for a more general result).

0.19. If μ is a σ-finite measure, then, by *Dixmier's Theorem*, there is a compact Hausdorff extremally disconnected space Ω, with a normal measure ν on Ω, such that $C(\Omega)$ is isometric to $L^\infty(\mu)$. The converse is true as well: The Banach $C(\Omega)$, where is Ω is a compact space with a strictly positive normal measure μ, is isometric to $L^\infty(\mu)$.

A subspace Y of a Banach space X is *complemented in* X if there is a bounded linear map P from X onto Y such that $P(y)=y$ for $y\in Y$. Such a map P is called a *projection* from X onto Y.

0.20. For any measure $(\Omega,\Sigma.\lambda)$ $L^\infty(\lambda)$ is an *injective* Banach space, and in fact a *$\mathcal{P}_1$-Banach space* (i.e., $L^\infty(\lambda)$ is complemented, with a projection operator of norm 1, in every Banach space in which it is isometrically contained).

0.21. The unit sphere of $L^1(\mu)$ is weak* dense in the unit sphere of $M(K)$, if μ is a strictly positive measure on a compact space K.

Given a Banach space X, the (*functional*) *dimension of* X, denoted $\dim X$, is the smallest cardinal number α for which there is a subset of cardinality α with linear span norm-dense in X. If K is a compact space, then $w(K)=\dim C(K)$. (By an application of the Stone–Weierstrass Theorem).

0.22. If X be a Banach space with $\dim X = \alpha$, then there is a family $\{x_\xi : \xi < \alpha\}$ of elements of X, with $\|x_\xi\| \leqslant 1$ for $\xi < \alpha$, and a real number $\theta > 0$, such that $\|x_\xi - x_\zeta\| \geqslant \theta$ for $\xi < \zeta < \alpha$.

Given a family $\{X_i : i \in I\}$ of Banach spaces, we denote by

$$\Big(\sum_{i\in I} \oplus X_i\Big)_1$$

The Banach space consisting of all $x = (x_i)_{i\in I}$, with $x_i \in X_i$ for $i \in I$, and $\Sigma_{i\in I} \|x_i\| < \infty$, under the obvious norm. If $X_i = X$ for all $i \in I$, and I is the cardinal number α, we denote

$$\Big(\sum_{i\in I} \oplus X_i\Big)_1 \quad \text{by} \quad \Big(\sum_{\alpha} \oplus X\Big)_1 .$$

A measure space (Ω, Σ, μ) is *purely atomless* if there is no $A \in \Sigma$ for which $\{B \in \Sigma : B \subset A\} = \{A, \emptyset\}$.

0.23. CARATHEODORY'S THEOREM. *Let (Ω, Σ, μ) be a finite measure space, such that μ is purely atomless, and $L^1(\mu)$ is separable. Then $L^1(\mu)$ is isometric to $L^1\{0, 1\}^\omega$.*

0.24. MAHARAM'S THEOREM. *Let (Ω, Σ, μ) be a purely atomless finite measure space. Then there is a countable (finite or infinite) set of distinct infinite cardinals (α_n), such that $L^1(\mu)$ is isometric to $(\Sigma_n \oplus L^1\{0, 1\}^{\alpha_n})_1$.*

Another form of Maharam's theorem is used in Theorem 4.11(iii)$\Rightarrow$(i), below. The reader is referred to LACEY [78] (§14), or SEMANDENI [110] (§26) for details.

The Banach space l^1_α has the following properties:

0.25. If X is a Banach space, with $\dim X = \alpha$, then there is a bounded linear operator T of l^1_α onto X.

0.26. (Lifting property of l^1_α). If $T : X \to Y$ is a bounded linear operator of X onto Y, and $T_1 : l^1_\alpha \to Y$ is a bounded linear operator, then there is a bounded linear operator $T_2 : l^1_\alpha \to X$ such that $T \circ T_2 = T_1$.

0.27. For any finite measure space (Ω, Σ, μ), $l^1_{\omega^+}$ does not embed isomorphically into $L^1(\mu)$.

If $f : X = \Pi_{i\in I} X_i \to \mathbb{R}$, we say that f *depends on the set* J of coordinates (with $J \subset I$) if there is $f : \Pi_{i\in J} X_i \to \mathbb{R}$ such that $f = g \circ \pi_J$ (where π_J is the natural projection from X onto $\Pi_{i\in J} X_i$). If J can be chosen countable (finite, resp.) then f is said to *depend on a countable* (*finite*, resp.) *set of coordinates.*

0.28. We note: (a) If X_i is compact for all $i \in I$, then every $f \in C(X)$ depends on a countable set of coordinates; and, the set of $f \in C(X)$ that depend on a finite set of coordinates is (by the Stone–Weierstrass Theorem) uniformly dense in $C(X)$;
(b) Every $f \in L^\infty\{-1, 1\}^I$ depends on a countable set of coordinates.

We say that $A \subset X$ is *depends on* J if the characteristic function χ_A of A depends on J.

0.29. DUNFORD–PETTIS COMPACTNESS THEOREM. *A subset $K \subset L^1(\mu)$ is weakly compact if and only if K is closed, bounded and equi-integrable (i.e., for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\mu(A) < \delta$, then $\int_A |f|\, d\mu < \varepsilon$ for all $f \in K$).*

A linear bounded operator $T: X \to Y$ is *weakly compact* if $T(S_X)$ is a relatively weakly compact subset of Y.

0.30. If T is weakly compact, then T^* is also weakly compact.
A Banach space X has the *Dunford–Pettis property* if for every Banach space Y, every weakly compact operator $T: X \to Y$, and every weakly compact subset K of X, the set $T(K)$ is norm-compact in Y.

0.31. DUNFORD–PETTIS THEOREM. *$L_1(\mu)$ and $C(K)$ have the Dunford–Pettis property for any measure space (Ω, Σ, μ) and every compact space K.*

A measure $\lambda \geq 0$ is *singular with respect to* $\mu \geq 0$ (in symbols $\lambda \perp \mu$) if there is a Borel set A in K with $\lambda(A) = \mu(K \setminus A) = 0$. By the regularity of the measures λ, μ there is then a Baire set B in K with $\lambda(B) = \mu(K \setminus B) = 0$.

0.32. Give two non-negative measures $\lambda, \mu \in M(K)$ *Lebesgue's decomposition theorem* states that there are non-negative measures $\lambda_1, \lambda_2 \in M(K)$ such that

$$\lambda_1 \ll \mu \quad \text{and} \quad \lambda_2 \perp \mu\,.$$

If $\mu \in M(K)$, and F is a closed subset of K with $\mu(F) > 0$, then $\mu \mid F \in M(F)$ is defined by

$$(\mu \mid F)(B) = \frac{\mu(B \cap F)}{\mu(F)} \quad \text{for } B \text{ Borel, } B \subset F\,.$$

If $K \cdot L$ are compact spaces, $f: K \to L$ is continuous, and $\mu \in M(K)$, then $f_*(\mu) \in M(L)$ is defined by

$$(f_*(\mu))(B) = \mu(f^{-1}(B)) \quad \text{for } B \text{ Borel, } B \subset L\,.$$

0.33. KOLMOGOROV EXTENSION THEOREM. *Let $(K_i, \pi_{ij}, \mu_i)_{i,j\in I}$ be an inverse system of compact spaces K_i, with connecting mappings $\pi_{ij}: K_i \to K_j$ continuous onto for $i > j$, $i, j \in I$, and μ_i probability regular Borel measure on K_i, with $\pi_{ij}(\mu_i) = \mu_j$ for $i > j$, $i, j \in I$. Let K be the inverse limit of this system. Then there is a probability regular Borel measure μ on K, with $\pi_{i*}(\mu) = \mu_i$ for $i \in I$ (where $\pi_i: K \to K_i$ denotes the natural mappings for $i \in I$).*

General references for measure-theory: HALMOS [59]; for Banach spaces: DISTEL [42], DUNFORD and SCHWARTZ [43], LACEY [78], LINDENSTRAUSS and TZAFRIRI [80], [81], SEMADENI [110], YOSHIDA [126]. Some recent general surveys on Banach spaces (and in some cases their relation to topology) are the following: ARCHANGEL'SKIĬ [19], [20] (especially for topics treated in Section 6), HAYDON [63] (especially for topics treated in Sections 4 and 5), NEGREPONTIS [91] (especially for topics treated in Sections 2, 4, 5), PELCZYNSKI [97], ROSENTHAL [104], [106] (especially for topics treated in Section 1), WAGE [124] (especially for topics treated in Section 6).

1. Rosenthal's Theorem for isomorphic embedding of l^1 into Banach spaces

The principal result of this Section is Rosenthal's Theorem (1.9), based on the set-theoretic result 1.5 (whose proof uses the infinitary topological version of Ramsey's Theorem). Useful for the whole survey is also Rosenthal's criterion proved in 1.2. Work by Rosenthal, Odell, Bourgain, Fremlin, Talagrand, and Godefroy on Rosenthal-compact spaces is described mostly without proof.

1.1. DEFINITION. Let S be a set, $Y \subset S$, and $\{(A_i, B_i): i \in I\}$ a family of ordered pairs of subsets of S. The family $\{(A_i, B_i): i \in I\}$ is *independent on* Y if for every two (possibly empty) disjoint finite subsets F, G of I,

$$\bigcap_{i\in F} A_i \cap \bigcap_{i\in G} B_i \cap Y \neq \emptyset .$$

In case $Y = S$, we simply say that $\{(A_i, B_i): i \in I\}$ is *independent.*

NOTATION. Note that we do not require that $A_i \cap B_i = \emptyset$. We set $(+1)A_i = A_i$, $(-1)A_i = B_i$ for $i \in I$.

1.2. PROPOSITION (ROSENTHAL'S criterion [102]). *Let S be a set, α a cardinal number, $f_\xi: S \to \mathbb{R}$ for $\xi < \alpha$ a function, such that*

$$\|f_\xi\|_\infty \ (= \sup_{s\in S} |f_\xi(s)|) \leq M \qquad \textit{for } \xi < \alpha ,$$

and there is $r \in \mathbb{R}$ and $\delta > 0$, such that setting $A_\xi = f_\xi^{-1}(r + \delta, \infty)$ and $B_\xi =$

$f_\xi^{-1}(-\infty, r)$ for $\xi < \alpha$, the family $\{(A_\xi, B_\xi): \xi < \alpha\}$ is independent. Then,

$$\frac{\delta}{2}\sum_{k=1}^{n}|\alpha_k| \leq \left\|\sum_{k=1}^{n}\alpha_k f_{\xi_k}\right\|_\infty \leq M\sum_{k=1}^{n}|\alpha_k|$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $\xi_1 < \cdots < \xi_n < \alpha$. Hence, $\{f_\xi: \xi < \alpha\}$ is equivalent (in $l^\infty(S)$) to the usual l^1_α-basis.

PROOF. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $\xi_1 < \cdots < \xi_n < \alpha$. We set $P = \{k: 1 \leq k \leq n, \alpha_k > 0\}$ and $N = \{k: 1 \leq k \leq n, \alpha_k \leq 0\}$. By independence there are

$$x \in \bigcap_{k\in P} A_{\xi_k} \cap \bigcap_{k\in N} B_{\xi_k} \quad \text{and} \quad y \in \bigcap_{k\in N} A_{\xi_k} \cap \bigcap_{k\in P} B_{\xi_k}.$$

Then,

$$\sum_{k\in P}|\alpha_k|(r+\delta) \leq \sum_{k\in P} a_k f_{\xi_k}(x),$$

$$\sum_{k\in P}|\alpha_k|(-r) \leq \sum_{k\in P}\alpha_k(-f_{\xi_k}(y)) = -\sum_{k\in P}\alpha_k f_{\xi_k}(y),$$

and

$$\sum_{k\in N}\alpha_k(r+\delta) \leq -\sum_{k\in N}\alpha_k f_{\xi_k}(y),$$

$$\sum_{k\in N}\alpha_k(-r) \leq \sum_{k\in N} -\alpha_k(-f_{\xi_k}(x)) = \sum_{k\in N}\alpha_k f_{\xi_k}(x).$$

Hence,

$$\begin{aligned}\Big(\sum_{k=1}^{n}|\alpha_k|\Big)\delta &\leq \sum_{k=1}^{n}\alpha_k f_{\xi_k}(x) - \sum_{k=1}^{n}\alpha_k f_{\xi_k}(y)\\ &\leq \left|\sum_{k=1}^{n}\alpha_k f_{\xi_k}(x)\right| + \left|\sum_{k=1}^{n}\alpha_k f_{\xi_k}(y)\right|\\ &\leq 2\left\|\sum_{k=1}^{n}\alpha_k f_{\xi_k}\right\|_\infty \leq 2M\sum_{k=1}^{n}|\alpha_k|.\end{aligned}$$

1.3. EXAMPLE (ARGYROS–KALAMIDAS [69]; Haydon (unpublished)). Assume the continuum hypothesis. There is a totally disconnected, compact Hausdorff ccc space K, such that

$l^1_{\omega^+}$ embeds isomorphically into $C(K)$, and
there is no continuous function from K onto $\{0, 1\}^{\omega^+}$.

Thus, if $\{f_\xi: \xi < \omega^+\}$ is equivalent to the $l^1_{\omega^+}$-basis in $C(K)$, there is no $r \in \mathbb{R}$, $\delta > 0$, such that, setting $A_\xi = \{x \in K: f_\xi(x) > r + \delta\}$, $B_\xi = \{x \in K: f_\xi(x) < r\}$ for $\xi < \omega^+$, the family $\{(A_\xi, B_\xi)\colon \xi < \omega^+\}$ is independent. Thus, families of functions that turn out to be equivalent to l^1_α do not always arise from Rosenthal's criterion (1.2). In Sections 3 and 4 (cf. Theorems 3.4, 4.11) we shall look into this matter more closely, and obtain positive results as well.

1.4. DEFINITION. Let S be a set, and $\{(A_n, B_n)\colon n < \omega\}$ a family of ordered pairs of subsets of S, with $A_n \cap B_n = \emptyset$ for $n < \omega$. The family $\{(A_n, B_n)\colon n < \omega\}$ *converges* if for every $x \in S$,

$$\begin{aligned} &\text{either the set } \{n < \omega\colon x \in A_n\} \text{ is finite},\\ &\text{or the set } \{n < \omega\colon x \in B_n\} \text{ is finite}. \end{aligned}$$

1.5. THEOREM (ROSENTHAL [102]). *Let S be a set, and $\{(A_n, B_n)\colon n < \omega\}$ a family of ordered pairs of subsets of S, with $A_n \cap B_n = \emptyset$ for $n < \omega$. Then, there is an infinite subset M of ω such that*

$$\begin{aligned} &\text{either } \{(A_n, B_n)\colon n \in M\} \text{ converges},\\ &\text{or } \{(A_n, B_n)\colon n \in M\} \text{ is independent}. \end{aligned}$$

PROOF (FARAHAT [48]). For $n < \omega$, the set

$$\begin{aligned} F_n = \{M &= (m_0 < m_1 < \cdots < m_n < \cdots)\\ &\in \omega^{[\omega]}\colon (+1)A_{m_0} \cap (-1)A_{m_1} \cap \cdots \cap (-1)^n A_{m_n} \neq \emptyset\}, \end{aligned}$$

and $F = \bigcap_{n<\omega} F_n$. It is easy to see that F_n is closed in $\omega^{[\omega]}$, and hence F is closed in $\omega^{[\omega]}$. By Theorem 0.6, there is $M_0 \in \omega^{[\omega]}$ such that either $M_0^{[\omega]} \subset F$, or $M_0^{[\omega]} \subset \omega^{[\omega]} \backslash F$.

Claim. $M_0^{[\omega]} \subset F$.

(It is enough to prove that for every infinite subset $M \subset \omega$, $M^{[\omega]} \cap F \neq \emptyset$. In fact, if $M \in \omega^{[\omega]}$, then by assumption $\{(A_n, B_n)\colon n \cap\}$ does not converge, hence, there is $x_0 \in S$, such that $M_0 = \{n \in M\colon x_0 \in A_n\}$ is infinite, and $M_1 = \{n \in M\colon x_0 \in B_n\}$ is infinite. We then can find inductively $M_2 = \{m_n\colon n < \omega\}$, such that $m_0 < m_1 < \cdots < m_n < \cdots$, and such that $m_{2k} \in M_0$, $m_{2k+1} \in M_1$ for $k < \omega$, so that $x_0 \in A_{m_{2k}}$, $x_0 \in B_{m_{2k+1}}$ for $k < \omega$. Then $M_2 \in M^{[\omega]} \cap F$).

Say $M_0 = \{k_0 < k_1 < \cdots < k_n < \cdots\}$. We set $M = \{k_{2n}\colon 1 \leq n < \omega\}$, and we claim that the family $\{(A_n, B_n)\colon n \in M\}$ is independent.

Indeed, let $p \geq 1$, $\varepsilon_1, \ldots, \varepsilon_p \in \{-1, 1\}$, and we must prove that $\bigcap_{i=1}^p \varepsilon_1 A_{k_{2i}} \neq \emptyset$. We proceed as follows:

if $\varepsilon_1 = 1$, set $m_1^{(0)} = m_1^{(1)} = k_2$,
if $\varepsilon_1 = -1$, set $m_1^{(0)} = k_1$, $m_1^{(1)} = k_2$;
for $1 < i \leq p$,
if $\varepsilon_{i-1} = 1$, $\varepsilon_i = 1$, set $m_i^{(0)} = k_{2i-1}$, $m_i^{(1)} = k_{2i}$,
if $\varepsilon_{i-1} = 1$, $\varepsilon_i = -1$, set $m_i^{(0)} = m_i^{(1)} = k_{2i}$,
if $\varepsilon_{i-1} = -1$, $\varepsilon_i = 1$, set $m_i^{(0)} = m_i^{(1)} = k_{2i}$, and
if $\varepsilon_{i-1} = -1$, $\varepsilon_i = -1$, set $m_i^{(0)} = k_{2i-1}$, $m_i^{(1)} = k_{2i}$.

We finally set

$$\tilde{M} = \{m_1^{(0)} \leqslant m_1^{(1)} < m_2^{(0)} \leqslant m_2^{(1)} < \cdots < m_p^{(0)} \leqslant m_p^{(1)} < k_{2p+1} < k_{2p+2} < \cdots\}.$$

Then $\tilde{M}$ is infinite, and $\tilde{M} \subset M_0$, hence $\tilde{M} \in F$. It follows that

$$\bigcap_{i=1}^{p} \varepsilon_i A_{k_{2i}} \neq \emptyset .$$

1.6. LEMMA. *Let $\{f_n : n < \omega\}$ be a sequence of real-valued functions on a set S, such that*

$$|f_n(x)| \leqslant M_x \quad \text{for all } n < \omega,\ x \in S$$

(*where M_x is a positive real number for $x \in S$*), *and*

$$\{f_n : n < \omega\} \quad \textit{has no point-wise convergent subsequence.}$$

Then, there are an infinite set $N \subset \omega$, $r \in \mathbb{R}$ and $\delta > 0$, such that for all infinite subsets $M \subset N$, there is $x_M \in S$ with

$$(*) \qquad \begin{array}{ll} \{m \in M : f_m(x_M) > r + \delta\} & \textit{an infinite set, and} \\ \{m \in M : f_m(x_M) < r\} & \textit{and infinite set.} \end{array}$$

PROOF. For any infinite set $M \subset \omega$, we denote by $\underline{\lim}_{m \in M} f_m(x)$ the number $\underline{\lim}_k f_{m_k}(x)$, and by $\overline{\lim}_{n \in M} f_m(x)$ the number $\overline{\lim}_k f_{m_k}(x)$, where $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$. Let $\{(q_k, \delta_k) : k < \omega\}$ be an enumeration of all pairs of rational numbers (q, δ), with $\delta > 0$. We say that the triple (N, r, δ) works if it satisfies the conclusion of the lemma. Suppose no such triple works. In particular (ω, q_0, δ_0) does not work, hence there is infinite $N_1 \subset \omega$, so that $(*)$ fails for all $x \in S$. Assume inductively that infinite sets $\omega \supset M_1 \supset \cdots \supset M_n$ have been chosen, so that (N_k, q_k, δ_k) does not work. We let N_{n+1} be an infinite subset of N_n that witnesses the failure of $(*)$. By the Cantor diagonalization process, we find an infinite set $N \subset \omega$, such that $N_n \backslash N$ is finite for all n.

We claim that $(f_n)_{n \in N}$ converges pointwise. (Indeed, if not, by the pointwise boundedness of $(f_n)_{n<\omega}$, there are $x \in S$, and a rational pair (q, δ), with $\delta > 0$, so that

$$\underline{\lim}_{m \in N} f_m(x) < r < r + \delta \overline{\lim}_{m \in N} f_m(x) .$$

There is $k < \omega$, with $(q_k, \delta_k) = (q, \delta)$. Since $N_k \backslash N$ is finite, we have

$$\underline{\lim}_{m\in N_k} f_m(x) < q_k < q_k + \delta_k < \overline{\lim}_{m\in N_k} f_m(x) .$$

But by the definition of N_k, it follows that

$$\text{either} \quad \overline{\lim}_{m\in N_k} f_m(x) \leq q_k + \delta_k ,$$

$$\text{or} \quad \underline{\lim}_{m\in N_k} f_m(x) \geq q_k .$$

This contradiction proves the Lemma.

1.7. THEOREM. *Let S be a set, and $\{f_n : n < \omega\}$ a sequence of real-valued functions on S, such that*

$$\|f_n\| = \sup_{x\in S} |f_n(x)| \leq M \quad \textit{for all } n < \omega$$

(where M is some positive real number), and

$$\{f_n : n < \omega\} \quad \textit{has no pointwise convergent subsequence} .$$

Then, there are an infinite set $M \subset \omega$, and $\delta > 0$, such that

$$\frac{\delta}{2} \sum_{k=1}^{n} |\alpha_k| \leq \left\| \sum_{k=1}^{n} \alpha_k f_{m_k} \right\| \quad \textit{for any } n < \omega \ \alpha_1, \ldots, \alpha_n \in \mathbb{R},\ m_1, \ldots, m_n \in N .$$

PROOF. We choose an infinite subset $N_1 \subset \omega$, $r \in \mathbb{R}$, and $\delta > 0$ satisfying the conclusion of Lemma 1.6. It follows that setting

$$A_n = \{x \in S : f_n(x) > r + \delta\} , \qquad B_n = \{x \in S : f_n(x) < r\} \quad \text{for } n \in N_1$$

the sequence $\{(A_n, B_n) : n \in N_1\}$ has no convergent subsequence. From Theorem 1.5, there is an infinite $N \subset N_1$, such that

$$\{(A_n, B_n) : n \in N\} \quad \text{is independent} .$$

The conclusion now follows from Rosenthal's criterion (1.2).

1.8. DEFINITION. A sequence (x_n) in a Banach space X is a *weak Cauchy sequence* if the real sequence $(x^*(x_n))$ converges for every $x^* \in X^*$.

1.9. THEOREM (ROSENTHAL [102]). *Let $(x_n)_{n<\omega}$ be a bounded sequence in a Banach space X.*

Either $(x_n)_{n<\omega}$ has a weak Cauchy subsequence, or $(x_n)_{n<\omega}$ has a subsequence, equivalent to the usual l^1-basis (and hence $l^1 \hookrightarrow X$).

PROOF. Let $\|x_n\| \leqslant M$ for $n = 1, 2, \ldots$, and (x_n) has no weak Cauchy subsequence. Let S_{X^*} be unit ball of X^*, in the weak* topology. Define $f_n(x^*) = x^*(x_n)$ for $x^* \in S_{X^*}$. Then $f_n : S_{X^*} \to \mathbb{R}$ is a sequence of functions such that

$$\|f_n\|_\infty = \sup_{\|x^*\|=1} |f_n(x^*)| = \sup_{\|x^*\|=1} |x^*(x_n)| = \|x_n\| \leqslant M \quad \text{for } n < \omega,$$

and (f_n) has no convergent subsequence.

It follows from Theorem 1.7 that there are an infinite set $N \subset \omega$ and $\delta > 0$, such that

$$\frac{\delta}{2} \sum_{k=1}^{n} |\alpha_k| \leqslant \left\| \sum_{k=1}^{n} \alpha_k f_{m_k} \right\|_\infty \quad \left(= \sup_{x \in S} \left| \sum_{k=1}^{n} \alpha_k f_{m_k}(x) \right| \right)$$

$$\text{for } n < \omega,\ \alpha_1, \ldots, \alpha_n \in \mathbb{R},\ m_1, \ldots, m_n \in N.$$

Hence,

$$\frac{\delta}{2} \sum_{k=1}^{n} |\alpha_k| \leqslant \left\| \sum_{k=1}^{n} \alpha_k x_{m_k} \right\| \leqslant M \sum_{k=1}^{n} |\alpha_k|$$

for $n < \omega$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $m_1, \ldots, m_n \in N$. This shows that the subsequence $(x_n)_{n \in N}$ is equivalent to the usual l^1-basis, and hence that $l^1 \hookrightarrow X$.

1.10. DEFINITIONS. Let X be a Polish space (i.e., a complete separable metric space). A function $f : X \to \mathbb{R}$ is a *Baire-1* function is there is a sequence $(f_n)_{n<\omega}$ of continuous real-valued functions on X, such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for } x \in X.$$

The family of all Baire-1 functions on X with the pointwise convergence is denoted by $\mathscr{B}_1(X)$.

The classical, and very beautiful characterization of Baire-1 functions on Polish spaces is the following theorem of Baire (for a proof see HAUSDORFF [60] or KURATOWSKI [77]).

1.11. BAIRE'S CHARACTERIZATION THEOREM [22]. *Let X be a Polish space, and $f : X \to \mathbb{R}$. The following are equivalent*:

(a) $f \in \mathscr{B}_1(X)$

(b) $f^{-1}(U)$ *is an F_σ-set for every open set U in $\mathbb{R}$; and*

(c) $f \mid K$ *has a point of continuity for every non-empty closed subset K of X.*

1.12. DEFINITION. A compact Hausdorff space K is called *Rosenthal-compact* if K is homeomorphic to a subspace of $\mathscr{B}_1(X)$, for some Polish space X.

The principal information concerning Rosenthal-compact spaces is the following

1.13. THEOREM (ROSENTHAL [105], BOURGAIN–FREMLIN–TALAGRAND [30]). *If K is a Rosenthal-compact space, then*

(a) *K is sequentially compact, and*

(b) *for all $A \subset K$, and $f \in \bar{A}$, there is a sequence (f_n) in A such that $f_n \to f$ (pointwise) (i.e., K is angelic).*

A very readable self-contained purely topological proof of this deep theorem is given in ROSENTHAL [106] (Section 3).

1.14. EXAMPLE. (a) Every compact metric space is a (separable) Rosenthal-compact.

(b) The space $[0, 1] \times \{0, 1\}$ given the order topology in its lexicographic order is a separable, non-metrizable Rosenthal-compact.

(c) The Helly space of all non-decreasing functions from $[0, 1]$ to $[0, 1]$ is a separable, non-metrizable Rosenthal-compact (since every non-decreasing function on $[0, 1]$ is certainly Baire-1).

(d) The space of all $f: [0, 1] \to \mathbb{R}$, with $f(0) = 0$, and of total variation at most 1 is a separable, non-metrizable Rosenthal-compact.

We state, also without proof GODEFROY's characterization in [52] of the separable Rosenthal-compact spaces:

1.15. THEOREM. *A compact separable Hausdorff space K is Rosenthal-compact if and only if for every countable dense subset D of K the space $C(K)$ with the pointwise topology on D (i.e., $f_i \to f$ if and only if $f_i(d) \to f(d)$ for all $d \in D$) is analytic (i.e., is the continuous image of the space of irrationals).*

(The proof makes use of results of ROSENTHAL [105], and BOURGAIN–FREMLIM–TALAGRAND [30]).

The main interest in Banach spaces for Rosenthal-compact spaces arises from the following theorem 1.17 (due to Odell and Rosenthal).

1.16. LEMMA. *Let K be a compact Hausdorff space, $f: K \to \mathbb{R}$ a bounded function, without any point of continuity and $\mathcal{F}$ a uniformly bounded subset of $C(K)$, such that f is in the pointwise closure of $\mathcal{F}$. Then*

(a) *there are a closed, non-empty subset L of K, $r \in \mathbb{R}$, $\delta > 0$ such that for every open, non-empty subset U of L, there are y_U, $z_U \in U$ with $f(z_U) < r < r + \delta < f(y_U)$; and*

(b) *ther is a sequence $(f_n) \subset \mathcal{F}$ such that, setting*

$$A_n = \{x \in K: f_n(x) < r + \delta\}, \qquad B_n = \{x \in K: r < f_n(x)\} \quad \text{for } n < \omega,$$

the sequence $\{(A_n, B_n): n < \omega\}$ is independent (and hence (f_n) is equivalent to the usual l^1-basis).

PROOF. (a) We set $C_n = \{x \in K$: the oscillation of f at $x \geq 1/n\}$ for $n < \omega$. Then C_n is closed in K, and since f does not have a point of continuity, $K = \bigcup_{n<\omega} C_n$. Hence, by Baire's category theorem, there is n_0 such that

$$U_0 = (C_{n_0})^0 \neq \emptyset .$$

We set $\delta = 1/n_0 > 0$, and $K_0 = \bar{U}_0$. Let $\{q_n : n < \omega\}$ be an enumeration of the set of rational numbers. We set

$$D_n = \{x \in K_0\text{: if } U \text{ is open in } K_0 \text{ and } x \in U\text{, then } f(U) \cap (-\infty, q_n) \neq \emptyset \text{ and } f(U) \cap (q_n + \delta, \infty) \neq \emptyset\} .$$

Then D_n is closed in K_0, and $K_0 = \bigcup_{n<\omega} B_n$, by the choice of δ and k_0. By Baire's category theorem, there is n_1 such that

$$V_0 = (D_{n_1})^0 \neq \emptyset$$

We set $L = \bar{V}_0$, $r = q_{n_1}$. It is clear that the triple (L, r, δ) satisfies the conclusion of (a).

(b) We choose $y_1^{\emptyset}$, $y_2^{\emptyset} \in L$ with $f(y_2^{\emptyset}) < r < r + \delta < f(y_1^{\emptyset})$. Since f is in the pointwise closure of $\mathscr{F}$, there is $f_1 \in \mathscr{F}$, with

$$f_1(y_2^{\emptyset}) < r < r + \delta < f_1(y_1^{\emptyset}) .$$

Suppose that $f_1, \ldots, f_n \in \mathscr{F}$ are chosen so that $\{(A_k, B_k): k \leq n\}$ is independent. For every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, with $\varepsilon_k \in \{-1, 1\}$ for $1 \leq k < n$, we have, by inductive assumption that

$$\bigcap_{k=1}^{n} \varepsilon_k A_k \quad \text{is open, and non-empty} ,$$

hence there are y_1^{ε}, $y_2^{\varepsilon} \in \bigcap_{k=1}^{n} \varepsilon_k A_k$, such that $f(y_2^{\varepsilon}) < r < r + \delta < f(y_1^{\varepsilon})$. Since f is in the pointwise closure of $\mathscr{F}$, there is $f_{n+1} \in \mathscr{F}$, with

$$f_{n+1}(y_2^{\varepsilon}) < r < r + \delta < f_{n+1}(y_1^{\varepsilon})$$

for every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, with $\varepsilon_k = \{-1, 1\}$, $1 \leq k \leq n$.

It is then clear that the sequence (f_n) satisfies condition (b), i.e. $\{(A_n, B_n): n < \omega\}$ is independent. From Rosenthal's criterion (1.2), (f_n) is equivalent to the usual l^1-basis.

1.17. THEOREM (ODELL–ROSENTHAL [93]). *Let X be a separable Banach space, such that l^1 does not embed isomorphically in X. Then the unit ball $S_{X^{**}}$, with the weak* topology, is a separable Rosenthal-compact space. In addition, if X^* is not (norm-) separable, then $S_{X^{**}}$ is a non-metrizable, separable, Rosenthal-compact space.*

PROOF. We let K be the unit ball S_{X^*} of X^* with the weak* topology. Since X is separable, K is a compact metrizable space (0.9), and hence a Polish space. The correspondence

$$x^{**} \to x^{**} \mid K \quad \text{for } x^{**} \in S_{X^{**}}$$

is clearly a homeomorphism in the weak* topology of $S_{X^{**}}$; thus we identify $S_{X^{**}}$ with a family of functions on K. Suppose there is $x^{**} \in S_{X^{**}}$, such that $x^{**} \mid K$ is not a Baire-1 function on K. By Baire's characterization theorem (1.11), there is a non-empty closed subset F of K, such that $x^{**} \mid F$ has no point of continuity. The natural embedding of X into X^{**} embeds S_X into $S_{X^{**}}$ in an isometric way, and with S_X weak* dense in $S_{X^{**}}$ by Goldstine's theorem (0.12). Furthermore, via the identification $x^{**} \to x^{**} \mid K$, S_X is identified, in an isometric way, with a subset of $C(K)$. Thus x^{**} is in the pointwise closure of S_X. By Lemma 1.16, there is a sequence $(x_n) \subset S_X$, such that $(x_n)_{n<\omega}$ is equivalent to the usual l^1-basis, contradicting the assumptions of the theorem on X.

It follows that $S_{X^{**}}$ is Rosenthal-compact.

Since X is separable, it follows that S_X is norm-separable, and hence (0.10) weakly separable, and since S_X is weak* dense in $S_{X^{**}}$, it follows that $S_{X^{**}}$ is separable.

Finally, if X^* is not norm-separable, then S_{X^*} is not norm-separable, and hence, $S_{X^{**}}$ is not metrizable in weak* topology.

The proof of the theorem is complete.

1.18. REMARK. An example of a separable Banach space X, such that l^1 does not embed isomorphically in X, and X^* is not separable has been given by JAMES in [66].

We give, without proof a long list of characterizations of isomorphic embeddings of l^1 into (mainly) separable Banach spaces. The list is taken from ROSENTHAL [106] (Section 1), where an outline of the proof, and detailed references are given.

1.19. THEOREM. *Let X be a separable Banach space. The following are equivalent:*

(1) *l^1 does not embed isomorphically into X.*

(2) *X is weak*-sequentially dense in X^{**}.*

(3) $|X^{**}| = |X|$.

(4) *Every bounded sequence of X has a weak Cauchy subsequence.*

(5) *Every bounded sequence of X^{**} has a weak*-convergent subsequence.*

(6) *Every bounded subset of X is weakly sequentially dense in its weak closure.*

(7) *Every bounded subset of X^{**} is weak*-sequentially dense in its weak* closure.*

(8) *Every bounded weak*-closed convex subset of X^* is the norm-closed convex hull of the set of its extreme points.*

(9) *If $Y \subset X^*$ and Y has the Dunford–Pettis property, then every weakly compact subset of Y is norm-compact.*

(10) $L^1\{0, 1\}^\omega$ *does not embed isomorphically into* X^*.

(11) $l^1_{\omega^+}$ *does not embed isomorphically into* X^*.

(12) *There is no bounded linear operator from X onto $C(\{0, 1\}^\omega)$.*

(13) $M(\{0, 1\}^\omega)$ *does not embed isomorphically into* X^*.

(*The equivalences* $(1)\Leftrightarrow(4)\Leftrightarrow(8)\Leftrightarrow(9)\Leftrightarrow(10)\Leftrightarrow(13)$ *do not require separability of* X).

The following examples indicate that Rosenthal's theorem (1.9) cannot be generalized to a characterization of isomorphic embeddings of l^1_α for uncountable cardinals α.

1.20. EXAMPLE (HAYDON [61] [§3]). There is a compact totally disconnected Hausdorff space K, and an uncountable subset F of the Banach space $C(K)$, such that

F contains no weak Cauchy sequence, and
$l^1_{\omega^+}$ does not embed isomorphically into $C(K)$.

An example of a Banach space X with similar properties has also been given by Hagler.

We end this section with a

1.21. PROBLEM. Assume CH. Is there a non-separable ccc Rosenthal-compact space? (By a simple result of GODEFROY [52] (Proposition 8), such a space, if it exists, cannot have a strictly positive measure). We note that if we assume Martin's axiom plus the denial of the continuum hypothesis then every Rosenthal compact ccc space is separable.

2. Calibers, independent families of comapct spaces K and universal isomorphic embeddings of l^1_α into $C(K)$

The principal results of this Section are Theorems 2.7 and 2.8 by Argyros and Tsarpalias (and independently by Shelah). Further work by Hagler, Argyros, Negrepontis, Sapirovskii, Balcar and Franek is described without proof.

2.1. DEFINITION. Let S be a set, α a cardinal, and $\{(A_\xi, B_\xi): \xi < \alpha\}$ a family of ordered pairs of subsets of S. We say that the family has *property* $(*)$ if for every $A \subset \alpha$, with $|A| = \alpha$, there are $\xi, \zeta \in A$ with $A_\xi \cap B_\zeta \neq \emptyset$.

2.2. LEMMA. *Let K be a compact space, $f_\xi \in C(K)$, with $\|f_\xi\| \leq 1$, for $\xi < \alpha$, and $\|f_\xi - f_\zeta\| \geq \theta > 0$ for $\xi < \zeta < \alpha$. Then there are $A \subset \alpha$, with $|A| = \alpha$, and a family $\{(A_\xi, B_\xi): \xi \in A\}$ of ordered pairs of disjoint open subsets of K, such that*

(i) *the family $\{(A_\xi, B_\xi): \xi \in A\}$ has property $(*)$, and*

(ii) $\inf(f_\xi(B_\xi)) - \sup(f_\xi(A_\xi)) > \theta/2$.

PROOF. Let $I_1, \ldots, I_n$ be a finite open cover of the closed interval $[-1, 1]$, such that $\operatorname{diam}(I_k) < \theta/8$, $k = 1, \ldots, n$. For $\xi < \alpha$, and $1 \leq k \leq n$, set $C_{\xi,k} = f_\xi^{-1}(I_k)$.

Claim. There are k, l, $1 \leq k$, $l \leq n$, and $A \subset \alpha$, with $|A| = \alpha$, such that $|A| = \alpha$, and if $B \subset A$, $|B| = \alpha$, there are ξ, $\zeta \in B$ and $x \in C_{\xi,k} \cap C_{\zeta,l}$ such that $|f_\xi(x) - f_\zeta(x)| > \theta$.

[In fact, assume the contrary. We note that if $\xi < \zeta < \alpha$, then (since $\|f_\xi - f_\zeta\| \geq \theta$), there are $k(\xi, \zeta)$, $l(\xi, \zeta)$ and

$$x \in C_{\xi,k(\xi,\zeta)} \cap C_{\zeta,l(\xi,\zeta)} \quad \text{such that} \quad |f_\xi(x) - f_\zeta(x)| > \theta\,.$$

Let $\{(k_m, l_m): m = 1, \ldots, s\}$ be an enumeration of those (k, l), with $1 \leq k$, $l \leq n$, such that $\operatorname{diam}(I_k \cup I_l) > \theta$. Since the claim fails, we find inductively sets $\alpha \supset A_1 \supset \cdots \supset A_s$, with $|A_s| = \alpha$,

$$m < m' < s\,, \quad \xi, \zeta \in A_{m'} \;\Rightarrow\; C_{\xi,k_m} \cap C_{\zeta,l_m} = \emptyset\,.$$

But then if ξ, $\zeta \in A_s$, $\xi \neq \zeta$, then $\|f_\xi - f_\zeta\| \leq \theta$ a contradiction].

Let $A \subset \alpha$, and k, l satisfy the claim. Then

$$\operatorname{diam}(I_k \cup I_l) \geq \theta\,, \qquad \operatorname{diam}(I_k), \operatorname{diam}(I_l) < \theta/8$$

hence $I_k \cap I_l = \emptyset$, and

$$\begin{array}{ll} \text{either} & \inf I_l - \sup I_k > \theta/2\,, \\ \text{or} & \inf I_k - \sup I_l > \theta/2\,. \end{array}$$

Without loss of generality assume that the first case holds. We set

$$A_\xi = C_{\xi,k}\,, \qquad B_\xi = C_{\xi,l} \quad \text{for } \xi \in A\,,$$

and the family $\{(A_\xi, B_\xi): \xi \in A\}$ satisfies (i) and (ii).

2.3. DEFINITION. Let K be a topological space, α a cardinal and $\{(A_\xi, B_\xi): \xi < \alpha\}$ a family of ordered pairs of open subsets of K. A non-empty open set V of K is a *representative for the family* $\{(A_\xi, B_\xi): \xi < \alpha\}$ if for every cardinal $\beta < \alpha$ and every family $\{U_i: i < \beta\}$ of non-empty open subsets of V, there is $A \subset \alpha$, with $|A| = \alpha$, such that

$$A_\xi \cap U_i \neq \emptyset\,, \quad \text{and} \quad B_\xi \cap U_i \neq \emptyset \quad \text{for all } i < \beta,\ \xi \in A\,.$$

2.4. LEMMA. *Let K be a compact space, α regular cardinal, $\kappa = S(K) \leqslant \alpha$, and $\{(A_\xi, B_\xi): \xi < \alpha\}$ a family of ordered pairs of open subsets of K, with property $(*)$. Then there are $\xi_1, \xi_2 < \alpha$, such that $A_{\xi_1} \cap B_{\xi_2}$ is a representative for the family $\{(A_\xi, B_\xi): \xi < \alpha\}$.*

PROOF. Suppose that the conclusion of the lemma does not hold. Let $I \subset \alpha$, with $|I| = \alpha$. Since the family $\{(A_\xi, B_\xi): \xi < \alpha\}$ has property $(*)$, there are $\xi_I, \zeta_I \in I$, with $A_{\xi_I} \cap B_{\zeta_I} \neq \emptyset$. Then $A_{\xi_I} \cap B_{\zeta_I}$ is not a representative for $\{(A_\xi, B_\xi): \xi \in I\}$. Hence, there are a cardinal $\beta_I < \alpha$, and a family $\{V_j: j < \beta_I\}$ of non-empty open subsets of $A_{\xi_I} \cap B_{\zeta_I}$, and $R_I \subset \alpha$, $|R_I| < \alpha$, such that for every $\xi \in I \backslash R_I$ there is $j_\xi < \beta_I$ and $\varepsilon_\xi \in \{-1, 1\}$, such that $V_{j_\xi} \cap \varepsilon_\xi A_\xi = \emptyset$.

We define $\phi_I: I \backslash R_I \to \beta_I \times \{-1, 1\}$ by $\phi_I(\xi) = (j_\xi, \varepsilon_\xi)$, and

$$\mathcal{P}_I = \{\phi_I^{-1}(\{(j, \varepsilon)\}): (j, \varepsilon) \in \beta_I \times \{-1, 1\}\} \cup \{I \cap R_I\}.$$

It is clear that $\mathcal{P}_I$ is a partition of I, with $|\mathcal{P}_I| \leqslant \beta_I < \alpha$.

By Lemma 0.1, there is a family $\{I_\eta: \eta < \kappa\}$ of subsets of α, such that

$$\begin{array}{ll} |I_\eta| = \alpha & \text{for } \eta < \kappa, \\ I_{\eta+1} \in \mathcal{P}_{I_\eta} & \text{for } \eta < \kappa, \quad \text{and} \\ I_{\eta'} \subset I_\eta & \text{for } \eta < \eta' < \kappa \end{array}$$

For $\eta < \kappa$ there are $j_\eta < \beta_{I_\eta}$, $\varepsilon_\eta \in \{-1, 1\}$, such that

$$I_{\eta+1} = \phi_I^{-1}(\{j_\eta, \varepsilon_\eta\}).$$

Without loss of generality assume that $\varepsilon_\eta = \varepsilon$ for $\eta < \kappa$. We note that if $\eta < \eta' < \kappa$, then

$$\xi_{I_{\eta'}}, \zeta_{I_\eta} \in I_{\eta'} \subset I_{\eta+1},$$

hence $V_{j_\eta} \cap \varepsilon A_{\xi_{I_{\eta'}}} = V_{j_\eta} \cap \varepsilon B_{\zeta_{I_{\eta'}}} = \emptyset$, hence $V_{j_\eta} \cap A_{\xi_{I_{\eta'}}} \cap B_{\zeta_{I_{\eta'}}} = \emptyset$, and since $V_{j_{\eta'}} \subset A_{\xi_{I_{\eta'}}} \cap B_{\zeta_{I_{\eta'}}}$, we finally have that $V_{j_\eta} \cap V_{j_{\eta'}} = \emptyset$. Hence the family $\{V_{j_\eta}: \eta < \kappa\}$ is a family of pairwise disjoint non-empty open subsets of K, contradicting the fact that $S(K) = \kappa$.

2.5. LEMMA. *Let K be a compact space with $S(K) = \kappa$ and α a regular cardinal such that $\kappa \leqslant \alpha$. Then α is a caliber of K.*

PROOF. Let $\{U_\xi: \xi < \alpha\}$ be a family of non-empty open subsets of K. From Lemma 2.4, setting $A_\xi = B_\xi = U_\xi$, there are $\xi_1, \xi_2 < \alpha$ such that $U_{\xi_1} \cap U_{\xi_2}$ is a representative for the family $\{(U_\xi, U_\xi)\colon \xi < \alpha\}$. We set

$$\mathcal{P} = \{A \subset \alpha\{U_\xi \cap U_{\xi_1} \cap U_{\xi_2}\colon \xi \in A\} \text{ has the finite intersection property}\},$$

and choose $A \in \mathcal{P}$, A maximal (by Zorn's lemma). Suppose that $|A| < \alpha$. We set

$$U_F = U_{\xi_1} \cap U_{\xi_2} \cap \bigcap_{\xi \in F} U_\xi \neq \emptyset \quad \text{for } \emptyset \neq F \in \mathcal{P}_\omega(A).$$

Thus the family $\{U_F\colon \emptyset \neq F \in \mathcal{P}_\omega(A)\}$ is a family of non-empty open subsets of the representative, and $|\mathcal{P}_\omega(A)| < \alpha$. Hence, by the defining property for a representative (2.3) there is $\xi \in \alpha \backslash A$, such that

$$U_F \cap U_\xi \neq \emptyset \quad \text{for all } \emptyset \neq F \in \mathcal{P}_\omega(A).$$

Then $A \cup \{\xi\}$ properly contains A, and $A \cup \{\xi\} \in \mathcal{P}$, contradicting the maximality of A.

2.6. LEMMA. *Let K be a compact space, α, β, α_i for $i < \beta$ cardinals, such that $\alpha = \Sigma_{i<\beta}\alpha_i$ and $\beta < \alpha_i$ for all $i < \beta$, $\{(A_\xi, B_\xi)\colon \xi < \alpha\}$ a family of ordered pairs of open subsets of K, $\{V_i\colon i < \beta\}$ a family of open subsets of K with the finite intersection property, and V_i a representative for the family $\{(A_\xi, B_\xi)\colon \xi < \alpha_i\}$ for $i < \beta$. Then there is $A \subset \alpha$ with $|A| = \alpha$, such that $\{(A_\xi, B_\xi)\colon \xi \in A\}$ is an independent family.*

PROOF. We set

$$\mathcal{A} = \{A \subset \alpha\colon \{(A_\xi, B_\xi)\colon \xi \in A\} \text{ is independent on every finite intersection of elements of } \{V_i\colon i < \beta\}\},$$

and order $\mathcal{A}$ by set-inclusion. The family $\mathcal{A}$ is inductive, and by Zorn's lemma, there is a maximal element $A \in \mathcal{A}$. We claim that $|A| = \alpha$. In fact, suppose that $|A| < \alpha$; then there is $i < \beta$, such that $|A| < \alpha_i$. For $H \in \mathcal{P}_\omega(\beta)$, $G \in \mathcal{P}_\omega(A)$, and $\varepsilon\colon G \to \{-1, 1\}$, we set

$$U(H, G, \varepsilon) = V_i \cap \bigcap_{j \in H} V_j \cap \bigcap_{\xi \in G} \varepsilon_\xi A_\xi;$$

by the assumptions and the definition of $\mathcal{A}$, the set $U(H, G, \varepsilon)$ is a non-empty open subset of V_i. Since V_i is a representative for the family $\{(A_\xi, B_\xi)\colon \xi < \alpha_i\}$, and

$$|\{(H, G, \varepsilon)\colon H \in \mathcal{P}_\omega(\beta), G\mathcal{P}_\omega(A), \varepsilon\colon G \to \{-1, 1\}\}| < \alpha_i,$$

it follows that there is $\xi < \alpha_i$, such that

$$A_\xi \cap U(H, G, \varepsilon) \neq \emptyset \quad \text{and} \quad B_\xi \cap U(H, G, \varepsilon) \neq \emptyset$$

for all triples (H, G, ε), as above. Hence, $A \cup \{\xi\} \in \mathscr{A}$, and since clearly $\xi \notin A$, we have a contradiction to the maximality of A in $\mathscr{A}$.

Hence $|A| = \alpha$, and the proof of the lemma is complete.

2.7. THEOREM. *If K is a compact space, with $S(K) = \kappa$, and α a cardinal such that $\kappa \ll \alpha$, and $\kappa \ll \mathrm{cf}(\alpha)$, then α is a caliber of K.*

PROOF. From 0.5, 2.5 and 2.6 (with $U_\xi = A_\xi = B_\xi$).

2.8. THEOREM. *Let K be a compact space, with $S(K) = \kappa$, and α a cardinal such that $\kappa \ll \alpha$, $\alpha \ll \mathrm{cf}(\alpha)$, and $\alpha \leq w(K)$. Then*

(i) *there is a continuous onto function $\phi: K \to [0, 1]^\alpha$; and,*

(ii) *if X is a closed linear subspace of $C(K)$ with* $\operatorname{diam} X \geq \alpha$, *then there is an isomorphic embedding of l^1_α into X.*

PROOF. There is (by 0.5) a family $\{\alpha_\sigma : \sigma < \mathrm{cf}(\alpha)\}$ of cardinals, such that $\kappa \ll \alpha_\sigma$ for $\sigma < \mathrm{cf}(\alpha)$, α_σ regular for $\sigma < \mathrm{cf}(\alpha)$, and

$$\alpha = \sum_{\sigma < \mathrm{cf}(\alpha)} \alpha_\sigma .$$

Ad (i). Since $\alpha \leq w(K)$, it follows, from the Stone–Weierstrass Theorem, that $\alpha \leq \dim(C(K))$. So, by 0.22, there is a family $\{f_\xi : \xi < \alpha\}$ of elements of $C(K)$, with $\|f_\xi\| \leq 1$, and $\theta > 0$ such that $\|f_\xi - f_\zeta\| \geq \theta$ for all $\xi < \zeta < \alpha$. By Lemma 2.2, there is $A \subset \alpha$, with $|A| = \alpha$, and a family $\{(A_\xi, B_\xi) : \xi \in A\}$ of ordered pairs of non-empty open sets of K, such that it has property (*) and $\bar{A}_\xi \cap \bar{B}_\xi = \emptyset$ for $\xi \in A$. From Lemmas 2.4, and 2.6, it follows that there is $B \subset A$, with $|B| = \alpha$, such that the family $\{(A_\xi, B_\xi) : \xi \in B\}$ is independent. Let $\phi_\xi : K \to [0, 1]$ be a continuous function, such that $\phi_\xi \mid \bar{A}_\xi = 0$, $\phi_\xi \mid \bar{B}_\xi = 1$ for $\xi \in B$, and let $M_\xi = \phi_\xi(K)$ for $\xi \in B$. It is clear that there is a continuous onto function $\psi = \Pi_{\xi \in B} M_\xi \to [0, 1]^\alpha$. Hence, $\phi = \psi \circ (\phi_\xi)_{\xi \in B}$ is a continuous function of K onto $[0, 1]^\alpha$.

Ad (ii). Let X be a closed linear subspace of $C(K)$ with $\dim X = \alpha$. By Lemma 0.22, there is a family $\{f_\xi : \xi < \alpha\}$ of elements of X such that $\|f_\xi\| \leq 1$, and $\theta > 0$ such that $\|f_\xi - f_\zeta\| \geq \theta$ for all $\xi < \zeta < \alpha$. It follows from Lemmas 2.2, 2.4, and 2.6, and from the fact that, since $\kappa \ll \mathrm{cf}(\alpha)$, K has caliber $\mathrm{cf}(\alpha)$, that there is $A \subset \alpha$, with $|A| = \alpha$ and an independent family $\{(A_\xi, B_\xi) : \xi \in A\}$ of ordered pairs of non-empty open subsets of K, with

$$\inf(f_\xi(B_\xi)) - \sup(f_\xi(A_\xi)) > \theta/2 \quad \text{for } \xi \in A .$$

It follows from Rosenthal's lemma (1.2), that the family $\{f_\xi\colon \xi \in A\}$ is equivalent to the usual l_α^1-basis.

2.9. REMARK. The results of this section (together with some additional results) were proved in the theses of ARGYROS [5] and TSARPALIAS [121], and they appear in ARGYROS–TSARPALIAS [17], [18] and in the monograph [35] (Chapter 5). Similar results, with different techniques, were also obtained by SHELAH [112].

2.10. EXAMPLES. If we assume GCH, then the above results (Theorems 2.7, 2.8) are essentially the best possible, as follows from the following examples.

(a) If β is a cardinal, κ a regular cardinal with $\mathrm{cf}(\beta) < \kappa < \beta$. Then there is an extremally disconnected compact Hausdorff space K such that $S(K) = \kappa$ and K does not have caliber β^+ (ARGYROS [7]; see also [35] (Theorem 5.28)).

(b) In fact under the same set theoretic assumptions on κ and α (as in (a)), there is an extremally disconnected compact Hausdorff space K such that $S(K) = \kappa$, and for which there is a closed subspace X of $C(K)$ of $\dim X = \alpha$ such that l_α^1 does not embed isomorphically into X. (ARGYROS [8]; see also [35] (Theorem 6.29); the above property of the space is described in ARGYROS–TSARPALIAS [18]).

We mention without proof the following two significant results on independent families.

2.11. THEOREM (SAPIROVSKII [108]; see also JUHÁSZ [68]). *Let α be an infinite and K a totally disconnected compact Hausdorff space. There is a continuous function from K onto $\{0, 1\}^\alpha$ if and only if there is a non-empty closed subset F of K such that the pseudo-weight of F at x is at least α for all $x \in F$.*

2.12. THEOREM (BALCAR–FRANEK [23]). *If K is a compact Hausdorff extremally disconnected space of weight equal to $\alpha \geqslant \omega$, then there is a continuous function ϕ from K onto $[0, 1]^\alpha$ (and hence there is an isometric embedding of l_α^1 into $C(K)$).*

2.13. REMARK. HAGLER [57], ARGYROS [5], ARGYROS–NEGREPONTIS [14], KALAMIDAS [69], ZACHARIADES [127] have various results on the embeddability of l_α^1 into α-dimensional subspaces of $C(K)$ for dyadic, or more general type of spaces K.

3. Isomorphic embeddings of l_α^1 in Banach spaces X and independent families on the dual unit ball S_{X^*}

The principal result of this very short Section is Talagrand's combinatorial theorem 3.4, which for most cardinals α, reduces the functional analytic question of whether l_α^1 can be isomorphically embedded in a Banach space X to the purely set-theoretic question of whether $[0, 1]^\alpha$ is the continuous image of the unit ball S_{X^*} of the dual space X^* in the weak* topology. (Theorem 4.11 in the next section establishes, independently of Theorem 3.4, an even stronger statement for a somewhat more restricted class of cardinals).

3.1. THEOREM (Talagrand). *Let α be a cardinal number with $\mathrm{cf}(\alpha) > \omega$, S a set, $\{(X_i, Y_i): i < \alpha\}$ an independent family on S, n a natural number $n \geq 1$, and $X_{i,m} \subset X_i$, $Y_{i,m} \subset Y_i$ with $i < \alpha$, $1 \leq m \leq n$, such that*

$$X_i \times Y_i = \bigcup_{1 \leq m \leq n} (X_{i,m} \times Y_{i,m}) \quad \textit{for } i < \alpha .$$

Then, there is m, $1 \leq m \leq n$, and $I \subset \alpha$, with $|I| = \alpha$, such that the family $\{(X_{i,m}, Y_{i,m}): i \in I\}$ is independent on S.

PROOF (Argyros). We proceed by induction on n. The case $n = 1$ is trivial. Suppose now that $n \geq 1$, that the theorem holds for n, and that

$$X_i \times Y_i = \bigcup_{1 \leq m \leq n+1} X_{i,m} \times Y_{i,m} \quad \text{for } i < \alpha .$$

We define $T_i: S \to \{1, 0, -1\}$ by $T_i | X_i = 1$, $T_i | Y_i = -1$, and $T_i | S \setminus (X_i \cup Y_i) = 0$ for $i < \alpha$. We define

$$T: \beta(S) \to \{-1, 0, -1\}^{\alpha}$$

by $T = \Pi_{i<\alpha} \tilde{T}_i$ where $\tilde{T}_i: \beta(S) \to \{1, 0, -1\}$ is the unique continuous extension of T_i, and $\beta(S)$ denotes the Stone–Čech compactification of the (discrete) set S. It is clear, since the family $\{(X_i, Y_i): i < \alpha\}$ is independent, that $T(\beta S) \supset \{1, -1\}^{\alpha}$. We set

$$Z = T^{-1}(\{1, -1\}^{\alpha}) ,$$

a closed subset of $\beta(S)$. By remark above Gleason's Theorem 0.7, there is a closed subset Z of Y, such that $T(Y) = \{1, -1\}^{\alpha}$, and $T | Y$ is irreducible. We set

$$\tilde{X}_i = \mathrm{cl}_{\beta(S)} X_i \cap Y , \qquad \tilde{X}_{i,m} = \mathrm{cl}_{\beta(S)} X_{i,m} \cap Y ,$$
$$\tilde{Y}_i = \mathrm{cl}_{\beta(S)} Y_i \cap Y , \qquad \tilde{Y}_{i,m} = \mathrm{cl}_{\beta(S)} Y_{i,m} \cap Y$$

for $i < \alpha$, and $1 \leq m \leq n$, and we note that

$$\tilde{X}_i \times \tilde{Y}_i = \bigcup_{i \leq m \leq n+1} \tilde{X}_{i,m} \times \tilde{Y}_{i,m} \quad \text{for } i < \alpha .$$

Claim 1. For every $i<\alpha$, either $\tilde{Y}_{i,m}=\tilde{Y}_i$ for all $1\leq m\leq n$, or there is an open-and-closed subset C_i of $\tilde{Y}_i$, and m_0, $1\leq m_0\leq n$, such that

$$\tilde{X}_i \times C_i = \bigcup_{\substack{m=1\\ m\neq m_0}}^{n} \tilde{X}_{i,m} \times \tilde{Y}_{i,m}\,.$$

(Indeed, if the first possibility fails, then there is m_0 with $\tilde{Y}_{i,m_0}\neq \tilde{Y}_i$. Set $C_i = \tilde{Y}_i \setminus \tilde{Y}_{i,m_0}$.)

Claim 2. Let $\{(D_i, C_i): i<\alpha\}$ be a family of ordered pairs of open-and-closed subsets of Y, such that $D_i\cap C_i=\emptyset$ for $i<\alpha$, and either $D_i=\tilde{X}_i$ or $C_i=\tilde{Y}_i$ for $i<\alpha$. Then there is $I\subset\alpha$, with $|I|=\alpha$, such that $\{(D_i, C_i): i\in I\}$ is independent.

(Without loss of generality assume that $D_i=\tilde{X}_i$ for $i<\alpha$. Since $T\,|\,Y$ is irreducible, there is an open-and-closed subset W_i of $\{1,-1\}^\alpha$, with $(T\,|\,Y)^{-1}(W_i)\subset C_i$. Let F_i be the finite subset of α, on which W_i depends. By Theorem 0.4 on free sets, and the fact that the space $\{1,-1\}^\alpha$ has caliber α (for all cardinals α, with $\mathrm{cf}(\alpha)>\omega$, cf. Theorem 3.18(a), [35]) it follows that there is $I\subset\alpha$, with $|I|=\alpha$, such that $\{W_i: i\in I\}$ has the finite intersection property, and $i\notin F_j$ for $i,j\in I$, $i\neq j$.

It is now easy to prove that if $i_1,\ldots,i_p, j_1,\ldots,j_q$ are distinct elements of I, then

$$\bigcap_{k=1}^{p} \pi_{i_k}^{-1}(\{1\}) \cap \bigcap_{l=1}^{q} W_{j_l} \neq \emptyset\,.$$

It follows that

$$\emptyset \neq (T\,|\,Y)^{-1}\Big(\bigcap_{k=1}^{p} \pi_{i_k}^{-1}(\{1\}) \cap \bigcap_{l=1}^{q} W_{j_l}\Big) \subset \Big(\bigcap_{k=1}^{p} D_{i_k} \cap \bigcap_{l=1}^{q} C_{j_l}\Big)\,.$$

The theorem follows now from the inductive assumption and Claims 1 and 2.

3.2. COROLLARY. *Let α be a cardinal number with $\mathrm{cf}(\alpha)>\omega$, K a compact Hausdorff space, $\phi: K\to[0,1]^\alpha$ a continuous onto function, and E as linear subspace of $C(K)$, such that E distinguishes points of K. Then, there are real numbers s, t, with $s<t$, and a family $\{f_i: i\in I\}\subset E$, with $|I|=\alpha$, such that the family $\{(f_i^{-1}(-\infty, s], f_i^{-1}[t,+\infty)): i\in I\}$ is independent.*

PROOF. For $i<\alpha$, we denote by $\pi_i: [0,1]^\alpha\to[0,1]$ the natural ith projection. We set, for $i<\alpha$,

$$X_i = (\pi_i\circ\phi)^{-1}(\{0\}) \quad\text{and}\quad Y_i = (\pi_i\circ\phi)^{-1}(\{1\})\,.$$

Of course, the family $\{(X_i, Y_i): i<\alpha\}$ is independent. For $x, y\in K$, with $x\neq y$, there is $f_{x,y}\in E$, with $f_{x,y(x)}\neq f_{x,y}(y)$. Since E is a linear subspace, we assume without loss of generality, that there are rational numbers $s_{x,y} < t_{x,y}$ such that

$$f_{x,y}(x) < s_{x,y} < t_{x,y} < f_{x,y}(y).$$

Thus

$$X_i \times Y_i \subset \bigcup_{(x,y)\in X_i\times Y_i} (f_{x,y}^{-1}(-\infty, s_{x,y}) \times f_{x,y}^{-1}(t_{x,y}, +\infty))$$

for $i < \alpha$. Since $X_i = Y_i$ is a compact space for $i < \alpha$, and $\mathrm{cf}(\alpha) > \omega$, the result follows easily from Theorem 3.1.

3.3. REMARK. The following slightly more general result is proved as in Corollary 3.2.

Let α be cardinal number with $\mathrm{cf}(\alpha) > \omega$, K a compact Hausdorff space, $\phi: K \to \{0, 1\}^\alpha$ a continuous onto function, $\lambda < \mathrm{cf}(\alpha)$, and $\{E_j: j < \lambda\}$ a family of linear subspaces of $C(K)$, such that $\bigcup_{j<\lambda} E_j$ distinguishes points of K. Then, there are real numbers s, t, with $s < t$, $j_0 < \lambda$, and a family $\{f_i: i \in I\} \subset E_{j_0}$, with $|I| = \alpha$, such that the family $\{(f_i^{-1}(-\infty, s], f_i^{-1}[t, +\infty): i \in I\}$ is independent.

3.4. THEOREM. *Let α be a cardinal number, with $\mathrm{cf}(\alpha) > \omega$, and X a Banach space. Then, there is an isomorphic embedding of l_α^1 into X if and only if there is a continuous onto function*

$$\phi: S_{X^*} \to [0, 1]^\alpha,$$

where S_{X^*} denotes the unit sphere of the dual space X^*, in the weak topology.*

PROOF. ($\Rightarrow$) Let T be an isomorphic embedding l_α^1 into X. Then the dual operator

$$T^*: X^* \to l_\alpha^\infty,$$

is linear, continuous, and onto. Furthermore the unit ball $S_{l_\alpha^\infty}$ in its weak* topology is homeomorphic to the cube $[-1, 1]^\alpha$. By the open mapping theorem for Banach spaces, there is a real number $k > 0$, such that

$$kS_{l_\alpha^\infty} \subset T^*(S_{X^*}).$$

We set $F = ((T^*)^{-1}(kS_{l_\alpha^\infty})) \cap S_{X^*}$. Then F is a closed subset of S_{X^*} (in its weak* topology) and $T^* | F: F \to [-k, k]^\alpha$ is a continuous onto function (where we have identified $kS_{l_\alpha^\infty}$ with its homeomorph $[-k, k]^\alpha$, and we have used that T^* is weak* continuous, as well). By Tietze's extension theorem, there is a continuous function $\phi: S_{X^*} \to [-k, k]^\alpha$, such that ϕ extends $T^* | F$. Of course, $[-k, k]^\alpha$ is homeomorphic to $[0, 1]^\alpha$.

($\Leftarrow$) Let $X \hookrightarrow C(S_{X^*})$ be the canonical embedding (where S_{X^*} has the weak* topology). The result follows from the previous theorem and Rosenthal's criterion (1.2).

3.5. COROLLARY. *Let α be a cardinal number, with* $\mathrm{cf}(\alpha) > \omega$, $\lambda < \mathrm{cf}(\alpha)$, *and* $\{K_i : i < \lambda\}$ *a family of compact spaces, with* $K: \Pi_{i<\lambda} K_i$. *Then,* $[0,1]^\alpha$ *is the continuous image of* K *if and only if* $[0,1]^\alpha$ *is the continuous image of* K_i *for some* $i < \lambda$.

PROOF. The ($\Leftarrow$) direction is trivial, while the ($\Rightarrow$) direction follows, from Remark 3.3, upon setting $E_i = C(K_i)$, $i < \lambda$.

3.6. COROLLARY. *Let τ be an infinite cardinal,* $\alpha = 2^\tau$, $\lambda < \mathrm{cf}(\alpha)$, *and* $\{K_i : i < \lambda\}$ *a family of compact spaces, with* $K = \Pi_{i<\lambda} K_i$. *Then, the Stone–Čech compactification* $\beta\tau$ *of (the discrete space of cardinality)* τ *embeds homeomorphically into* K *if and only if* $\beta\tau$ *embeds into* K_i *for some* $i < \lambda$.

PROOF. The ($\Leftarrow$) direction is trivial, while for the converse we note (since $[0,1]^{2^\tau}$ contains a dense subset of cardinality τ) that, in general $\beta\tau$ embeds into K if and only if $[0,1]^{2^\tau}$ is the continuous image of K, and we apply the previous Corollary (3.5).

3.7. REMARK. The contents of this Section are due to TALAGRAND [119]. The present topological proof of the combinatorial Theorem 3.1, by Argyros, is a great simplification over the original proof.

4. Large subsets of $L^\infty(\mu)$ far apart in L_1-norm and Pelczynski's Conjecture

The principal results of this Section are Theorem 4.5 (by Argyros–Bourgain–Zachariades), Theorem 4.6 (by Argyros–Zachariades), and Theorem 4.9 (by Argyros). The combinatorial content of this method arose from Argyros' solution of Pelczynski's conjecture (given in 4.9).

4.1. DEFINITIONS. We denote by $\pi_A : \{-1,1\}^I \to \{-1,1\}^A$, the natural projection, with $A \supset I$. Let

$$P_A : L^1\{-1,1\}^A \to L^1\{-1,1\}^I$$

be the induced on $L^1\{-1,1\}^A$ linear operator, i.e. $P_A(f) = f \circ \pi_A$.

Finally, by

$$\mathscr{E}_A^\infty : L^\infty\{-1,1\}^I \to L^\infty\{-1,1\}^A$$

we denote the operator, conjugate to P_A. Also, let

$$Q_A: L^\infty\{-1, 1\}^A \to L^\infty\{-1, 1\}^I$$

be the induced on $L^\infty\{-1, 1\}^A$ linear operator, i.e., $Q_A(f) = f \circ \pi_A$, and

$$\mathscr{E}_A^1: L^1\{-1, 1\}^I \to L^1\{-1, 1\}^A$$

be the operator, conjugate to Q_A, restricted to $L^1\{-1, 1\}^I$.

The range of $\mathscr{E}_A^1$ on $L^1\{-1, 1\}^I$ is in fact contained in $L^1\{-1, 1\}^A$; and if $f \in L^\infty\{-1, 1\}^I$, then $\mathscr{E}_A^\infty(f) = \mathscr{E}_A^1(f)$.

4.2. *The Walsh Functions.* For $S \subset I$, S finite non-empty set

$$P_S: \{-1, 1\}^I \to \{-1, 1\}$$

by $P_S(x) = \Pi_{i \in S}\pi_i(x)$. We also set $p_\emptyset = 1$.

We observe the following properties:

(1) The linear combinations of $\{p_S: S \subset I, S \text{ finite}\}$ is dense in $L\{-1, 1\}^I$.

(2) If I is a finite set, and $f \in L^\infty\{-1, 1\}^I$, then

$$f = \sum_{S \subset I} \alpha_S p_S ,$$

where $\alpha_S = \int f \cdot p_S$ (integrate with respect to Haar measure on $\{-1, 1\}^I$).

(3) If $f \in L^\infty\{-1, 1\}^I$ depends on the set N_f of coordinates, and S is a finite subset of I, with $S \not\subset N_f$, then $\int f \cdot p_S = 0$. (Indeed, let $i \in S \backslash N_f$. Then, by Fubini's theorem,

$$\int f \cdot p_S = \int f p_{S \backslash \{i\}} \cdot p_{\{i\}} = \int f \cdot p_{S \backslash \{i\}} \cdot \int p_{\{i\}} = 0 ,$$

since $\int p_{\{1\}} = 0$).

(4) The family $\{p_S: S \subset I\}$ is an orthogonal basis for $L^2\{-1, 1\}^I$ (follows from (3)).

(5) If $f \in L^\infty\{-1, 1\}^I$, N_f a set of coordinates on which f depends, and $K \subset I$, then $\mathscr{E}_K^\infty(f) = \mathscr{E}_{K \cap N_f}^\infty(f)$. (This follows easily from properties (1) and (3).)

(6) If $\{M_n: n < \omega\}$ is a sequence of finite, pairwise different subsets of I, then $p_{M_n} \to 0$ weakly* in $L^\infty\{-1, 1\}^I$. (This follows easily from properties (1) and (3).)

The uncountable case

4.3. LEMMA. *Let $f \in L^\infty\{-1, 1\}^I$ and $\varepsilon > 0$. Then, there is $A \subset I$, with A finite, such that $\|\mathscr{E}_A^\infty(f) - f\|_1 < \varepsilon$.*

PROOF. There is $g \in L^1\{-1, 1\}^I$, depending on a finite set N_g of coordinates, such that $\|f - g\|_1 < \varepsilon/2$. Then

$$\|\mathscr{E}^1_{N_g}(f) - g\|_1 = \|\mathscr{E}^1_{N_g}(f) - \mathscr{E}^1_{N_g}(g)\|_1 \leq \|f - g\|_1 < \varepsilon/2\,,$$

hence,

$$\|\mathscr{E}^\infty_{N_g}(f) - f\|_1 = \|\mathscr{E}^1_{N_g}(f) - f\|_1 \leq \|\mathscr{E}^1_{N_g}(f) - g\|_1 + \|g - f\|_1 < \varepsilon$$

The main combinatorial tool for the proof of Theorem 4.5 is the following version of Hajnal's Theorem (0.4) on free sets.

4.4. LEMMA. *Let α be a cardinal number, $\alpha > \omega^+$ and $\{N_\xi, \xi < \alpha\}$ a family of countable subsets of α, $\{A_\xi, \xi < \alpha\}$ a family of finite pairwise disjoint subsets of α.*
Then, there is $A \subset \alpha$, with $|A| = \alpha$, such that

$$N_\xi \cap A_\zeta = \emptyset \quad \textit{for } \xi, \zeta \in A,\ \xi \neq \zeta\,.$$

PROOF. We define $\phi\colon \alpha \to \alpha$ by

$$\begin{aligned} \phi(\zeta) &= \xi \quad \text{if } \zeta \in A_\xi \\ &= \zeta \quad \text{if } \zeta \in \alpha \backslash \Big(\bigcup_{\eta < \alpha} A_\eta \Big)\,. \end{aligned}$$

We apply Hajnal's Theorem on free sets (0.4) on the family $\{\phi(N_\xi)\colon \xi < \alpha\}$. Thus, there is $A \subset \alpha$, with $|A| = \alpha$, such that $\zeta \notin \phi(N_\xi)$ for ζ, $\xi \in A$, $\zeta \neq \xi$. The conclusion follows.

4.5. THEOREM (Argyros–Bourgain–Zachariades). *Let α be a cardinal number, $\alpha < \omega^+$, $\{f_\xi\colon \xi < \alpha\} \subset L^\infty\{-1, 1\}^\alpha$, such that*

$\|f_\xi\|_\infty \leq M$ *for all* $\xi < \alpha$, *and*
there is $\theta > 0$, *with* $\|f_\xi - f_\zeta\| > \theta$ *for* $\xi < \zeta < \alpha$.

Then there is $A \subset \alpha$, with $|A| = \alpha$, such that the family $\{f_\xi\colon \xi \in A\}$ is equivalent to the l^1_α-basis in $L^\infty\{-1, 1\}^\alpha$.

PROOF. *Claim* 1. There are $A_1 \subset \alpha$, with $|A_1| = \alpha$, and a family $\{I_\xi\colon \xi \in A_1\}$ of finite subsets of α, such that, setting

$$J_\xi = \bigcup_{\substack{\zeta < \xi \\ \zeta \in A_1}} I_\zeta\,,$$

we have

$$\|\mathscr{E}^\infty_{I_\xi}(f_\xi) - \mathscr{E}^\infty_{I_\xi \cap J_\xi}(f_\xi)\|_1 > \theta/5\,.$$

Proof of Claim 1. We proceed by transfinite induction. Let $\xi < \alpha$, and suppose that we have defined $\{i_\zeta, \zeta < \xi\}$, $\{I_\zeta, \zeta < \xi\}$ finite subsets of α, such that

$$\|\mathscr{E}^\infty_{I_\zeta}(f_{i_\zeta}) - \mathscr{E}^\infty_{I_\zeta \cap J_\zeta}(f_{i_\zeta})\|_1 > \theta/5\,,$$

where we have set $J_\zeta = \bigcup_{\eta<\zeta} I_\eta$. We define i_ξ, I_ξ.

We set $J_\xi = \bigcup_{\zeta<\xi} I_\zeta$ and $\bar{i}_\xi = \sup\{i_\zeta\colon \zeta < \xi\}$. Assume that for every $\zeta > \bar{i}_\xi$, and every finite subset A of I, we have

$$\|\mathscr{E}^\infty_{A}(f_\zeta) - \mathscr{E}^\infty_{A \cap J_\xi}(f_\zeta)\|_1 \leq \theta/5\,.$$

From Lemma 4.2, for every $\zeta > \bar{i}_\zeta$ there is a finite set $A_\zeta \subset I$, such that

$$\|\mathscr{E}^\infty_{A_\zeta}(f_\zeta) - f_\zeta\|_1 < \theta/5\,.$$

Hence, for $\zeta_1, \zeta_2 > \bar{i}_\xi$, $\zeta_1 \neq \zeta_2$ we have

$$\|\mathscr{E}^\infty_{A_{\zeta_1}}(f_{\zeta_1}) - \mathscr{E}^\infty_{A_{\zeta_2}}(f_{\zeta_2})\|_1 > \theta/2\,,$$

and thus

$$\|\mathscr{E}^\infty_{A_{\zeta_1} \cap J_\xi}(f_{\zeta_1}) - \mathscr{E}^\infty_{A_{\zeta_2} \cap J_\xi}(f_{\zeta_2})\|_1 > \theta/8\,,$$

a contradiction to to facts that $|J_\xi| < \alpha$, and that $L^1\{-1, 1\}^{J_\xi}$ has (functional) dimension less that α. The proof of the claim is complete.

Claim 2. For every $\xi \in A_1$, there is $d_\xi \in L^\infty\{-1, 1\}^{I_\xi}$, with

$$\|d_\xi\|_\infty \leq 1\,, \qquad \int f_\xi \cdot d_\xi > \theta/10\,,$$

$$\int d_\xi p_S = 0 \quad \text{for all finite subsets } S \subset J_\xi\,.$$

Proof of Claim 2. From Claim 1, there is $g_\xi \in L^\infty\{-1, 1\}^{I_\xi}$, such that $\|g_\xi\|_\infty \leq 1$, and

$$|(\mathscr{E}^\infty_{I_\xi}(f_\xi))(g_\xi) - (\mathscr{E}^\infty_{i_\xi \cap J_\xi}(f_\xi)(g_\xi)| > \theta/5\,.$$

We have that

$$g_\xi = \sum_{S \subset I_\xi} \alpha_S p_S\,, \quad \text{with } \alpha_S = \int g \cdot p_S\,,$$

and thus, setting

$$d_1^\xi = \sum_{S \subset I_\xi \cap J_\xi} \alpha_S p_S\,, \qquad d_2^\xi = \sum_{\substack{S \subset I_\xi \\ S \not\subset J_\xi}} \alpha_S p_S\,,$$

we have $g_\xi = d_1^\xi + d_2^\xi$, and

$$\theta/5 < |\mathscr{E}^\infty_{I_\xi}(f_\xi)(d_1^\xi) + \mathscr{E}^\infty_{I_\xi}(f_\xi)(d_2^\xi) - \mathscr{E}^\infty_{I_\xi \cap J_\xi}(f_\xi)(d_1^\xi) - \mathscr{E}^\infty_{I_\xi \cap J_\xi}(f_\xi)(d_2^\xi)| .$$

Observing that, inside the absolute value, the first and third terms are equal, while the fourth is zero, by property 4 of the Walsh functions, we conclude that

$$\left| \int f_\xi d_2^\xi \right| = |(\mathscr{E}^\infty_{I_\xi}(f_\xi))(d_2^\xi)| > \theta/5 , \qquad \text{with } \|d_2^\xi\| \leq 2 .$$

We finally set $d_\xi = \pm d_2^\xi/2$, where the sign is chosen so that

$$\int f_\xi d_\xi > \theta/10 .$$

It is clear that $\int d_\xi p_S = 0$ if S is finite $\subset J_\xi$. The proof of the claim is complete.

We set $A_\xi = I_\xi \backslash J_\xi$ for $\xi \in A_1$, and we have that $\{A_\xi : \xi \in A_1\}$ is a pairwise disjoint family of finite subsets of α. We also choose a countable set N_ξ of coordinates, so that f_ξ depends on N_ξ and $I_\xi \subset N_\xi$ for $\xi \in A_1$.

It follows from Lemma 4.4 that there is $A \subset A_1$, with $|A| = \alpha$, such that

$$N_\xi \cap A_\zeta = \emptyset \quad \text{for } \xi, \zeta \in A, \xi \neq \zeta . \tag{$*$}$$

Claim 3. For all $\xi_1, \ldots, \xi_r \in A$, $\xi_1 < \cdots < \xi_r < \alpha$,

(i) $\int d_{\xi_1} \cdots d_{\xi_r} = 0$,

(ii) if $\xi \in A$ and $\int f_\xi d_{\xi_1} \cdots d_{\xi_r} \neq 0$, then $r = 1$ and $\xi_1 = \xi$.

Proof of Claim 3. (i) The function $d_{\xi_1} \cdots d_{\xi_{r-1}}$ depends on the set J_{ξ_r} and $d_{\xi_r} = \Sigma\{\alpha_S p_S : S \subset I_{\xi_r}, S \not\subset J_{\xi_r}\}$. From Claim 2, we have that $\int d_{\xi_1} \cdots d_{\xi_r} = 0$.

(ii) *Case* 1: $\xi_r \neq \xi$. Let N_ξ be a countable set, such that f_ξ depends on N_ξ. Then $f_\xi \cdot d_\xi \cdots d_{\xi_{r-1}}$ depends on $N_\xi \cup J_{\xi_r}$. If $S \subset I_{\xi_r}$ and $S \not\subset J_{\xi_r}$, then $S \not\subset N \cap J_{\xi_r}$ (since $S \subset J_{\xi_r}$, and property $(*)$). We now use Claim 2.

Case 2: $\xi_r = \xi$, $r > 1$. Since $I_\xi \subset N_\xi$, $f_\xi \cdot d_{\xi_1} \cdots d_{\xi_{r-2}} \cdot d_{\xi_r}$ depends on $N_\xi \cup J_{\xi_{r-1}}$, and $d_{\xi_{r-1}}$ plays the role that d_{ξ_r} plays in Case 1). The proof of the claim is complete.

Claim 4. The family $\{f_\xi : \xi \in A\}$ is equivalent to the usual l^1_α-basis.

Proof of Claim 4. Let $\xi_1 < \cdots < \xi_r < \alpha$, $\xi_1, \ldots, \xi_r \in A$ and $c_1, \ldots, c_r \in \mathbb{R}$ for some natural number $r \geq 1$. We set $\varepsilon_i = 1$ if $c_i \geq 0$ and $\varepsilon_i = -1$ if $c_i < 0$ for $1 \leq i \leq r$, and set

$$g = \prod_{i=1}^r (1 - \varepsilon_i d_{\xi_i}) - 1 .$$

Thus $g \in L^1\{-1, 1\}^\alpha$, and g depends on a finite set of coordinates. We note that $\|g\|_1 \leq 2$, since $1 - \varepsilon_i d_{\xi_i} \geq 0$ for $1 \leq i \leq r$, and $\int \Pi_{i=1}^r (1 - \varepsilon_i d_{\xi_i}) = 1$, using Claim 3(i).

Hence,

$$\|c_1 f_{\xi_1} + \cdots + c_r f_{\xi_r}\|_\infty \geq \frac{1}{2}|c_1 f_{\xi_1}(g) + \cdots + c_r f_{\xi_r}(g)| = \qquad \text{(using Claim 3)}$$

$$= \frac{1}{2}(\varepsilon_1 c_1 f_{\xi_1}(d_{\xi_1}) + \cdots + \varepsilon_r c_r f_{\xi_r}(d_{\xi_r})| =$$

$$= \frac{1}{2}(\varepsilon_1 c_1 f_{\xi_1}(d_{\xi_1}) + \cdots + \varepsilon_r c_r f_{\xi_r}(d_{\xi_r}) \geq \qquad \text{(by Claim 2)}$$

$$\geq \frac{\theta}{20}(\varepsilon_1 c_1 + \cdots + \varepsilon_r c_r) = \frac{\theta}{20}(|c_1| + \cdots + |c_r|).$$

Since, in addition, $\|f_\xi\|_\infty \leq M$ for all $\xi \in A$, the claim follows.

The case $\alpha = \omega^+$ with MA + $\neg$CH

4.6. THEOREM (Argyros–Zachariades). *Assume Martin's axiom and the denial of the continuum hypothesis. Let $\{f_\xi : \xi < \omega^+\} \subset L^\infty\{-1, 1\}^{\omega+}$ be such that*

$\|f_\xi\|_\infty \leq M$ for all $\xi < \omega^+$ and
there is $\theta > 0$, with $\|f_\xi - f_\zeta\|_1 \geq \theta$ for $\xi < \zeta < \omega^+$.

Then, there is $A \subset \omega^+$, with $|A| = \omega^+$, such that the family $\{f_\xi : \xi \in A\}$ is equivalent to the $l^1_{\omega^+}$-basis in $L^\infty\{-1, 1\}^{\omega+}$.

PROOF. *Claim* 1. There are $A_1 \subset \omega^+$, with $|A_1| = \omega^+$, and a family $\{I_\xi : \xi \in A_1\}$ of finite subsets of ω^+, such that

(i) $\int f_\xi p_{I_\xi} \neq 0$ for all $\xi \in A_1$, and choosing N_ξ a countable set on which f_ξ depends, and $M_\xi = \bigcup_{\zeta<\xi, \zeta\in A_1} N_\xi$, we have

(ii) $I_\xi \not\subset M_\xi$ for $\xi \in A_1$.

Proof of Claim 1. We proceed by transfinite induction. Let $\xi < \omega^+$, and suppose that we have defined $\{i_\zeta : \zeta < \xi\}$, $\{I_\zeta : \zeta < \xi\}$ finite subsets of ω^+, such that the analogues of (i) and (ii) are satisfied. Since $M_\xi = \bigcup_{\zeta<\xi} N_{i_\zeta}$ is countable, $L^1\{-1, 1\}^{M_\xi}$ is separable, our hypothesis $\|f_\eta - f_\zeta\|_1 \geq \theta$ for $\eta < \zeta < \omega^+$ and property (1) of Walsh functions imply that there are $i_\xi > \sup\{i_\zeta : \zeta < \xi\}$, and a finite subset $I_\zeta \subset N_{i_\xi}$ of ω^+, such that $f_{i_\xi} p_{I_\xi} \neq 0$, and $I_\xi \not\subset M_\xi$. We set $A_1 = \{i_\xi : \xi < \omega^+\}$. The proof of the claim is complete.

By the Erdös–Rado Theorem (0.3) on quasi-disjoint sets, there are $A_2 \subset A_1$, with $|A_2| = \omega^+$, and a finite set I, such that $I_\xi \cap I_\zeta = I$ for $\xi, \zeta \in A_2$, $\xi \neq \zeta$. It is clear that $I_\xi \backslash I \neq \emptyset$ for $\xi \in A_2$.

We consider the case where $I \neq \emptyset$ (the case $I = \emptyset$ being similar and simpler). From the definition of $\mathscr{E}^\infty_{I_\xi}$ it follows that

$$\int \mathscr{E}^\infty_{I_\xi}(f_\xi) \cdot p_{I_\xi} = \int f_\xi \cdot p_{I_\xi} \neq 0\,.$$

It follows from property (3) of the Walsh functions that the function $\mathscr{E}^\infty_{I_\xi}(f_\xi)$ depends *on all* of I_ξ, and on no proper subset of I_ξ. Hence for all $\xi \in A_2$ there are $e_\xi \in \{-1, 1\}^I$, r_ξ, δ_ξ rational numbers, with $\delta_\xi < 0$, such that

$$\{x \in \{e_\xi\} \times \{-1, 1\}^{I_\xi \setminus I} \colon \mathscr{E}^\infty_{I_\xi}(f_\xi) < r_\xi\} \neq \emptyset\,,$$

and

$$\{x \in \{e_\xi\} \times \{-1, 1\}^{I_\xi \setminus I} \colon \mathscr{E}^\infty_{I_\xi}(f_\xi) > r_\xi + \delta_\xi\} \neq \emptyset$$

It follows, that there are $A_3 \subset A_2$, with $|A_3| = \omega^+$, and $e \in \{-1, 1\}^I$, r, δ rational numbers, with $\delta > 0$, such that

$$e_\xi = e\,, \quad r_\xi = r\,, \quad \delta_\xi = \delta \qquad \text{for } \xi \in A_3\,.$$

For a finite subset F of A_3 we set $A(F) = \int_{\xi \in F} I_\xi$. We say that F has *property* (*) if for every $\xi \in F$, there are non-empty sets Δ_ξ, E_ξ such that

$$\begin{aligned}
&\Delta_\xi,\ E_\xi \quad \text{depends on} \quad I_\xi\,, \\
&\pi_I(\Delta_\xi) = \pi_I(E_\xi) = \{e\}\,, \\
&\mathscr{E}^\infty_{A(F)}(f_\xi)(x) < r \qquad \text{for } x \in \Delta_\xi\,, \quad \textit{and} \\
&\mathscr{E}^\infty_{A(F)}(f_\xi)(x) > r + \delta \quad \text{for } x \in E_\xi\,.
\end{aligned}$$

(Of course, in general the sets Δ_ξ, E_ξ depend on F, as well; if we want to indicate this dependence we write $\Delta_{\xi,F}$, $E_{\xi,F}$ for Δ_ξ, E_ξ, respectively.) We set $\mathbb{P} = \{F\colon F$ finite subset of A_3, F has property (*)$\}$ and we set $F_1 < F_2$ if $F_1 \supset F_2$, for F_1, $F_2 \in \mathbb{P}$. It is clear that every singleton of A_3, is in $\mathbb{P}$, and thus in particular $\mathbb{P} \neq \emptyset$.

Claim 2. The partially ordered set $(\mathbb{P}, <)$ has the ccc.

Proof of Claim 2. Let $\{F_\sigma\colon \sigma < \omega^+\} \subset \mathbb{P}$. From the Erdös–Rado Theorem on quasi-disjoint sets (0.3), there are $\Delta \subset \omega^+$, with $|\Delta| = \omega^+$, and a set F, such that $F_\sigma \cap F_\tau = F$ for $\sigma, \tau \in \Delta$, $\sigma \neq \tau$. We assume that $F \neq \emptyset$ (the case $F = \emptyset$ being analogous and simpler). We set $A(F) = \bigcup_{\xi \in F} I_\xi$ (as above), $N_F = \bigcup_{\xi \in F} N_\xi$, and $A_\sigma = A(F_\sigma)$ $(= \bigcup_{\xi \in F_\sigma} I_\xi)$, $N_\sigma = N_{F_\sigma}$ $(= \bigcup_{\xi \in F_\sigma} N_\xi)$. By our initial choice of I_ξ, we have that $A_\sigma \subset N_\sigma$ for $\sigma \in \Delta$.

By a simple transfinite induction, there are $\Delta_1 \subset \Delta$, with $|\Delta_1| = \omega^+$, and $\nu \in \mathbb{N}$, such that

$$\begin{aligned}
&|A_\sigma| = \nu && \text{for } \sigma \in \Delta_1\,, \\
&N_F \cap A_\sigma = A(F) && \text{for } \sigma \in \Delta_1\,, \quad \text{and} \\
&N_\sigma \cap A_\tau \subset A(F) && \text{for } \sigma < \tau,\ \sigma, \tau \in \Delta_1\,.
\end{aligned}$$

We choose $\sigma_0 \in \Delta_1$, such that $\Lambda = \{\sigma \in \Delta_1\colon \sigma < \sigma_0\}$ is infinite.

We note (from property (5) of Walsh functions and above conditions) that if

$\sigma \in \Lambda$ and $\xi \in F_\sigma$, then

$$\mathscr{E}^\infty_{A(F_\sigma \cup F_{\sigma_0})}(f_\xi) = \mathscr{E}^\infty_{A(F_\sigma \cup F_{\sigma 0}) \cap N_\xi}(f_\xi) = \mathscr{E}^\infty_{A(F_\sigma) \cap N_\xi}(f_\xi) = \mathscr{E}^\infty_{A(F_\sigma)}(f_\xi) \,. \tag{1}$$

Subclaim. For every $\varepsilon > 0$, there is a finite set $I(\varepsilon) \subset \Lambda$, such that

$$\|\mathscr{E}^\infty_{A(F_\sigma \cap F_{\sigma_0})}(f_\xi) - \mathscr{E}^\infty_{A(F_{\sigma 0})}(f_\xi)\|_\infty < \varepsilon \quad \text{for } \sigma \in \Lambda \backslash I(\varepsilon), \xi \in A(F_{\sigma 0}) \,.$$

Proof of subclaim. Let $\varepsilon > 0$. By property (2) of Walsh functions,

$$\mathscr{E}^\infty_{A(F_\sigma \cup F_{\sigma 0})}(f_\xi) = \sum \{\alpha_M \cdot p_M : M \subset A(F_\sigma \cup F_{\sigma 0})\}$$

where $\alpha_M = \int f_\xi p_M$, and

$$\mathscr{E}^\infty_{A(F_{\sigma 0})}(f_\xi) = \sum \{\alpha_M \cdot p_M : M \subset A(F_{\sigma 0})\} \,.$$

We set $M_0 = 2^{2\nu} - 2^\nu$, and we note (from property (6) of Walsh functions) that there is a finite subset $I(\varepsilon) \subset \Lambda$, such that

$$\left| \int f_\xi p_M \right| = |\alpha_M| < \frac{\varepsilon}{M_0} \quad \text{for } \sigma \in \Lambda \backslash I(\varepsilon), \xi \in A_F, M \subset A(F_\sigma \cup F_{\sigma 0}) \,.$$

Hence,

$$\|\mathscr{E}^\infty_{A(F_\sigma \cup F_{\sigma_0})}(f_\xi) - \mathscr{E}^\infty_{A(F_{\sigma_0})}(f_\xi)\|_\infty = \left\| \sum \{\alpha_M p_M : M \subset A(F_\sigma \cup F_{\sigma 0}), M \not\subset A(F_{\sigma 0})\} \right\|_\infty$$

$$\leqslant \sum \{|\alpha_M| : M \subset A(F_\sigma \cup F_{\sigma 0}), M \subset A(F_{\sigma 0})\} \leqslant \varepsilon \,,$$

proving the subclaim.

We set $\varepsilon_{\sigma 0} = \min\{\varepsilon_\xi : \xi \in A(F_{\sigma 0})\}$, where

$$\varepsilon_\xi = \min\{\min(\mathscr{E}^\infty_{F_{\sigma_0}}(f_\xi)[E_\xi]) - (r + \delta), r - \max(\mathscr{E}^\infty_{F_{\sigma_0}}(f_\xi)[\Delta_\xi]\}) \,,$$

and we choose $\varepsilon > 0$, and $\varepsilon > 0$, and $\varepsilon < \varepsilon_{\sigma 0}$. We finally prove that if $\sigma \in \Lambda \; I(\varepsilon)$, then $F_\sigma \cup F_{\sigma 0} \in \mathbb{P}$.

It follows from equality (1) and the subclaim that

$$\|\mathscr{E}^\infty_{A(F_\sigma \cup F_{\sigma 0})}(f_\xi) - \mathscr{E}^\infty_{A(F_{\sigma 0})}(f_\xi)\|_\infty < \varepsilon \quad \text{for } \xi \in A(F_{\sigma 0}) \,,$$

and

$$\mathscr{E}^\infty_{A(F_\sigma \cup F_\sigma)}(f_\xi) = \mathscr{E}^\infty_{A(F_\sigma)}(f_\xi) \qquad \text{for } \xi \in A(F_\sigma) \,.$$

From the second of these too equalities it follows (with the notation indicated immediately following the definitions of property (*) above) that

$$\Delta_{\xi,F_\sigma} = \Delta_{\xi,F_\sigma \cup F_{\sigma 0}}, \qquad E_{\xi,F_\sigma} = E_{\xi,F_\sigma \cup F_{\sigma 0}} \quad \text{for } \xi \in A(F_\sigma);$$

and, from the first of these equalities and the mannex in which ε is defined, it follows that

$$\Delta_{\xi,F_{\sigma 0}} = \Delta_{\xi,F_\sigma \cup F_{\sigma 0}}, \qquad E_{\xi,F_{\sigma 0}} = E_{\xi,F_\sigma \cup F_{\sigma 0}} \quad \text{for } \xi \in A(F_{\sigma 0});$$

It follows immediately that $F_\sigma \cup F_{\sigma_0} \in \mathbb{P}$, and that $(\mathbb{P}, <)$ has ccc, proving Claim 2.

From MA + $\neg$CH, it follows that there is $A \subset A_\xi$, with $|A| = \omega^+$, such that every finite subset of A is in $\mathbb{P}$. It follows now from Rosenthal's Lemma (1.2) applied to every finite subfamily of $\{f_\xi : \xi \in A\}$, that $\{f_\xi : \xi \in A\}$ is equivalent to the usual $l^1_{\omega^+}$-basis (with isomorphism constant $\delta/2$, which incidentally is unrelated to the constant θ).

The proof of the theorem is complete.

Pelczynski's Conjecture

The power of Theorems 4.5 and 4.6 may be seen by the fact that a positive solution to Pelczynski's Conjecture (4.9) is an easy consequence of these theorems. We first consider Pelczynski's result that led to the conjecture; Theorem 4.9 contains the answer to Pelczynski's Conjecture. In Section 5 below, we consider the Kunen–Haydon–Talagrand example, which assuming CH, gives a negative answer to Pelczynski's Conjecture for $\alpha = \omega^+$.

4.7. THEOREM (Pelczynski). *Let α be an infinite cardinal, X a Banach space, and $l^1_\alpha \hookrightarrow X$ an isomorphic embedding. Then the space $M(\{0,1\}^\alpha)$ is isomorphically embedded in the dual space X^*. In particular, then $L^1\{0,1\}^\alpha \hookrightarrow X^*$ and $l^1_{2^\alpha} \hookrightarrow X^*$.*

PROOF. For every Banach space of dimension at most α, there is a linear bounded operator $T: l^1_\alpha \to C\{0,1\}^\alpha$, T onto (cf. 0.25). So, there is such an operator $T: l^1_\alpha \to C\{0,1\}^\alpha$. Furthermore, $(C\{0,1\}^\alpha)^{**} = L^\infty(\lambda)$ for some measure λ, from Riesz's representation theorem (0.15), the fact that $M(\{0,1\}^\alpha) = L^1(\lambda))$ for some measure λ, and the fact that $(L^1(\lambda))^* = L^\infty(\lambda)$ (0.16). Also, $L^\infty(\lambda)$ is an injective Banach space (0.20). Hence the operator $T: l^1_\alpha \to C\{0,1\}^\alpha \quad (C\{0,1\}^\alpha)^{**}$, has a bounded linear extension $\tilde{T}: X \to C(\{0,1\}^\alpha)^{**}$. The dual operator

$$\tilde{T}^*: (C\{0,1\}^\alpha)^{***} = (M\{0,1\}^\alpha)^{**} \to X^*$$

has the property that $\tilde{T}^* \mid M\{0,1\}^\alpha$ is an isomorphic embedding.

Of course, $L^1\{0,1\}^\alpha \hookrightarrow M\{0,1\}^\alpha$, and $l^1_{2^\alpha} \hookrightarrow M\{0,1\}^\alpha$ (since, in general $l^1_{|K|} \hookrightarrow M(K)$).

PELCZYNSKI, in [95], conjectured that the converse of this theorem holds, as well; and, he proved this conjecture for $\alpha = \omega$. For $\alpha = \omega^+$, and assuming the continuum hypothesis, HAYDON [62] produced a counterexample. This example is described in Section 5. For $\alpha > \omega^+$, and for $\alpha = \omega^+$ with Martin's axiom and the negation of the continuum hypothesis, Pelczynski's conjecture was proved correct by ARGYROS [6]. We now deduce Argyros' results (Theorem 4.9) from the main theorems of this section.

The following lemma, following an idea of FAKHOURY [47], is used for the positive answer to Pelczynski's conjecture for $\alpha = \omega$.

4.8. LEMMA. *Let $\{f_n : n < \omega\} \subset L^\infty\{-1, 1\}^\omega$ be such that*

$\|f_n\|_\infty \leq M$ for all $n < \omega$, and
there is $\theta > 0$, such that $\|f_n - f_m\|_1 > \theta$ for $n < m < \omega$.

Then there is $A \subset \omega$, with $|A| = \omega$, such that the family $\{f_n : n \in A\}$ is equivalent to the l^1-basis in $L^\infty\{-1, 1\}^\omega$.

PROOF. If the conclusion fails, then from Rosenthal's Theorem (1.9), there is $A \subset \omega$, with $|A| = \omega$, such that $\{f_n : n \in A\}$ is weakly Cauchy. Let $T: L^\infty\{-1, 1\}^\omega \to L^1\{-1, 1\}^\omega$ denote the identity operator. This operator is weakly compact (by Dunford–Pettis Theorem (0.29)). We now use the fact that the Banach space $L^\infty\{-1, 1\}^\omega$ (in fact, every Banach space of the form $C(\Omega)$, with Ω a compact space) has the Dunford–Pettis property (0.31). Thus, $\{f_n : n \in A\}$ must be norm-convergent in $L^1\{-1, 1\}^\omega$, clearly impossible from the condition of the theorem.

4.9. THEOREM. *Let α be an infinite cardinal such that either $\alpha \neq \omega^+$, or $\alpha = \omega^+$ and Martin's Axiom with the negation of the continuum hypothesis hold, and let X be a Banach space such that $L^1\{0, 1\}^\alpha \hookrightarrow X^*$. Then $l^1_\alpha \hookrightarrow X$.*

PROOF. Let $T: L^1\{0, 1\}^\alpha \hookrightarrow X^*$ be an isomorphic embedding. Then the dual operator $T^*: X^{**} \to L^\infty\{0, 1\}^\alpha$ is onto. By the open mapping theorem (0.8), there is $k > 0$ such that

$$S_{L^\infty\{0,1\}^\alpha} \subset T^*(kS_{X^{**}}) .$$

By Goldstine's Theorem (0.12), S_X is weak*-dense in $S_{X^{**}}$, and hence $T^*(kS_X)$ is weak*-dense in $S_{L^\infty\{0,1\}^\alpha}$.

Assume first that $\alpha > \omega^+$.

Claim. There are $\{x_\xi : \xi < \alpha\} \subset kS_X$, $\{\eta_\xi : \xi < \alpha\} \subset \alpha$

$$\int T^*(x_\xi)\pi_{\eta_\xi}\, \mathrm{d}\mu > \frac{1}{2} \quad \text{for } \xi < \alpha , \quad \text{and}$$

$$\text{if } \xi < \zeta < \alpha, \text{ then } \int T^*(x_\xi)\pi_{\eta_\zeta}\, \mathrm{d}\mu = 0$$

(where μ is the Haar measure on $\{0, 1\}^\alpha$, and $\pi_\eta: \{0, 1\}^\alpha \to \{0, 1\}$ is the usual ηth projection, $\eta < \alpha$).

(We proceed inductively, and let $\zeta < \alpha$. We note (since $\alpha > \omega^+$) that

$$|\{\eta < \alpha: \text{there is } \xi < \zeta \text{ with } \textstyle\int T^*(x_\xi)\pi_\eta \, \mathrm{d}\mu \neq 0\}| < \alpha ,$$

and hence there is $\eta_\zeta < \alpha$ such that $\int T^*(x_\xi)\pi_{\eta_\zeta} \, \mathrm{d}\mu = 0$ for all $\xi < \zeta$. From the weak* density of $T^*(kS_X)$ in $S_{L^\infty\{0,1\}^\alpha}$ there is $x_\zeta \in kS_X$, such that $\int T^*(x_\zeta)\pi_{\eta_\zeta} \, \mathrm{d}\mu > \frac{1}{2}$.) Then

$$\|T^*(x_\xi) - T^*(x_\zeta)\|_1 \geq |T^*(x_\xi)(\pi_{\eta_\zeta}) - T^*(x_\zeta)(\pi_{\eta_\zeta})| = \left|0 - \frac{1}{2}\right| = \frac{1}{2}$$

for $\xi < \zeta < \alpha$. From Theorems 4.5 and 4.6, there is $A \subset \alpha$, with $|A| = \alpha$, such that $\{T^*(x_\xi): \xi \in A\}$ is equivalent to the usual l^1_α-basis. The lifting property of the space l^1_α (0.26) implies that $\{x_\xi: \xi \in A\}$ is equivalent to the usual l^1_α-basis.

The case $\alpha = \omega$ follows from Lemma 4.8, the lifting property of l^1 (0.26), the weak compactness of the natural inclusion operator $L^\infty \hookrightarrow L^1$; the details are left to the reader.

4.10. COROLLARY. *Let α be a cardinal, such that*

either $\alpha > \omega^+$ and $\mathrm{cf}(\alpha) > \omega$,
or MA + ¬CH holds and $\alpha = \omega^+$.

Let $\{f_\xi: \xi < \alpha\} \subset L^\infty\{-1, 1\}^\alpha$ be such that

$\|f_\xi\| \leq M$ for all $\xi < \alpha$, and
$|\{\zeta < \alpha$: there is $\xi < \alpha$ such that f_ζ depends on the coordinate $\zeta\}| = \alpha$.

Then there is $A \subset \alpha$, with $|A| = \alpha$, such that the family $\{f_\xi: \xi \in A\}$ is equivalent to the l^1_α-basis in $L^\infty\{-1, 1\}^\alpha$.

The proof of this corollary follows if we prove that there is $\theta > 0$ and $A \subset \alpha$, with $|A| = \alpha$, such that $\|f_\xi - f_\zeta\|_1 > \theta$ for $\xi, \zeta \in A$, $\xi \neq \zeta$. This is proved, using transfinite induction as in the proof of Claim 1 of Theorem 4.6. The details are omitted.

The final result of this section is an improvement of Theorem 3.4 (for a somewhat more restricted class of cardinals than the class for which Theorem 3.4 is valid).

4.11. THEOREM. *Let α be a cardinal, such that either $\alpha > \omega^+$ and $\mathrm{cf}(\alpha) > \omega$, or MA + ¬CH holds and $\alpha = \omega^+$, and let X be a Banach space. The following are equivalent:*

(i) $l^1_\alpha \hookrightarrow X$;
(ii) *there is a continuous onto function* $\phi: S_{X^*} \to [0, 1]^\alpha$;
(iii) $l^1_\alpha \hookrightarrow C(S_{X^*})$
(*where* S_{X^*} *denotes the unit sphere of the dual space* X^*, *in the weak* topology*).

PROOF. (i)$\Rightarrow$(ii) follows from (the simple implication of) Theorem 3.5.

(ii)$\Rightarrow$(iii) follows from the fact that $C[0, 1]^\alpha$ contains isometrically l^1_α.

(iii)$\Rightarrow$(i). From Pelczynski's Theorem (4.7), $L^1\{0, 1\}^\alpha$ is isomorphic to a subspace of $M(S_{X^*})$. It follows that there is an α-homogeneous measure ν on S_{X^*}, and thus the measure algebra of ν is isomorphic to the (standard) measure algebra of ($\{0, 1\}^\alpha$, Borel sets, Haar probability measure). Hence, there is a continuous Banach algebra homomorphism $T: C(S_{X^*}) \to L^\infty\{0, 1\}^\alpha$, such that the image of T is a weak* dense subalgebra of $L^\infty\{0, 1\}^\alpha$. Thus the set $\{\zeta < \alpha$: there is $x \in X$ such that $T(x)$ depends on $\zeta\}$ has cardinality α (where x is identified with an elements of $C(S_{X^*})$ via the canonical embedding), since X distinguishes elements of S_{X^*}, and hence, by the Stone–Weierstrass Theorem (0.14) $X \cup \{1\}$ generates a dense subalgebra of $C(S_{X^*})$. The result now follows from Corollary 4.10.

5. The Kunen–Haydon–Talagrand example

The remarkable example that will be described below (in Theorem 5.9) serves as a counterexample to numerous conjectures mainly concerning Banach spaces of dimension ω^+, and assuming the continuum hypothesis. Thus, there is a compact space K, such that:

(a) $l^1_{\omega^+}$ cannot be isomorphically embedded in $C(K)$, but $L^1\{0, 1\}^{\omega^+}$ is isomorphically embedded in its dual space $M(K)$ (this settles in the negative Pelczynski's conjecture, for $\alpha = \omega^+$, assuming CH [95], [62]; cf. results on Section 4).

(b) The Banach space $(\Sigma_{\omega^+} \oplus (\Sigma_{n=1} \oplus l^\infty_n)_1)_1$ cannot be isomorphically embedded in $C(K)$, but $L^1\{0, 1\}^{\omega^+}$ is isomorphically embedded as a complemented subspace in its dual space $M(K)$ (this settles in the negative Hagler–Stegall's conjecture, for $\alpha = \omega^+$, assuming CH [58], [62]; cf. positive results by ARGYROS in [6], for all other cardinals, and for ω^+ under Martin's Axiom, and the denial of CH).

(c) $l^1_{\omega^+}$ cannot be isomorphically embedded in $C(K)$, but there is a family $\{f_\xi: \xi < \omega^+\}$ of elements in $C(K)$, with $\|f_\xi\| \leq 1$, and with $\|f_\xi - f_\zeta\|_1 > \theta$ for $\xi < \zeta < \omega^+$ (for some real number $\theta > 0$) (this negative result should be compared with the positive results by Argyros, Bourgain, Zachariades for any infinite cardinal $\alpha = \omega^+$, and for $\alpha = \omega^+$ under Martin's Axiom, and the denial of CH, proved in Section 4).

(d) The dual $M(K)$ (of $C(K)$) in its weak* topology is separable, while the dual $M(GK)$ (of $C(GK)$), where GK is the Gleason space of K, in its weak* topology is not separable (this settles a question by MÄGERL and NAMIOKA [83], and shows that contrary to the weak* separability of $M^+_1(K)$ as witnessed by Corollary 5.6, the weak* separability of $M(K)$ is *not* a chain condition).

(e) K is a Corson-compact space that has a strictly positive measure (since $M(K)$ is weak* separable), and thus K has property (K_n) for all $n < \omega$, but K does not have caliber ω^+ (cf. Section 6 in the monograph [35] for some (non-Corson-compact) spaces, and the papers of ARGYROS–MERCOURAKIS–NEGREPONTIS [12, 13] for some other Corson-compact such spaces).

(f) K is a (Corson-) compact space with a strictly positive measure, and the space $M_1^+(K)$ in its weak* topology does not have caliber ω^+ (for all other regular uncountable cardinals, the analogue of this example cannot exist, as proved by ARGYROS–NEGREPONTIS [16]).

(g) $M(K)$ is weak* separable, but $M_1^+(K)$ is not weak* separable (equivalently, $C(K)$ cannot be isomorphically embedded in l^∞) (this example was constructed by TALAGRAND [118]).

(h) K is an L-space, i.e., K is hereditarily Lindelöf, but not separable (KUNEN [76]).

5.1. LEMMA. *Let K be a compact Hausdorff space, such that K is not separable, and $|K| = \omega^+$. Then K does not have caliber ω^+.*

PROOF. Let $\{x_\alpha: \alpha < \omega^+\}$ be a well-ordering of K, and set $V_\alpha = X\backslash\{x_\beta: \beta < \alpha\}^-$ for $\alpha < \omega^+$. Then $\{V_\alpha: \alpha < \omega^+\}$ is a family of non-empty (since K is not separable) open sets in K; but, if $A \subset \omega^+$, and $|A| = \omega^+$, then $\bigcap_{\alpha\in A} V_\alpha = \emptyset$.

5.2. DEFINITIONS. Let K be a compact Hausdorff space.

We denote by $\mathcal{T}^*(K)$ the family of all non-empty open subsets of K, For $\mathcal{T} \subset \mathcal{T}^*(K)$, we set

$$\kappa(\mathcal{T}) = \inf\{(\text{cal } \mathcal{F})/N: 1 \leq N < \omega, \mathcal{F} = \{U_1, \ldots, U_N\} \subset \mathcal{T}\}$$

where cal $\mathcal{T}$ = cal$\{U_1, \ldots, U_N\}$ = largest integer k such that there is $S \subset \{1, \ldots, N\}$, with $|S| = k$ and $\bigcup_{i\in S} U_i \neq \emptyset$.

We say that K has *property* (**) if there is a sequence $\{\mathcal{T}_n: n < \omega\}$, with $\mathcal{T}^*(K) = \bigcup_{n<\omega} \mathcal{T}_n$ and $\kappa(\mathcal{T}_n) > 0$ for $n < \omega$; and, K has *property* C_α for some $\alpha > 0$ if there is a sequence $\{\mathcal{T}_n: n < \omega\}$, with $\mathcal{T}^*(K) = \bigcup_{n<\omega} \mathcal{T}_n$ and $\kappa(\mathcal{T}_n) \geq \alpha$ for $n < \omega$.

5.3. LEMMA (Kelley). *Let K be a compact Hausdorff space. Then*

(a) *If μ is a regular Borel probability measure on K, $\theta > 0$, and $\mathcal{T} = \{U \in \mathcal{T}^*(K): \mu(\bar{U}) \geq \theta\}$ then $\kappa(\mathcal{T}) \geq \theta$.*

(b) *If $\mathcal{T} \subset \mathcal{T}^*(K)$ with $\kappa(\mathcal{T}) \geq \theta > 0$, then there is a regular Borel probability measure μ on K, with $\mu(\bar{U}) \geq \theta$ for $U \in \mathcal{T}$.*

5.4. COROLLARY. *A compact Hausdorff space K has a strictly positive measure if and only if K has property* (**).

(We will not prove here these (not very difficult) statements, but refer to the monograph [35] (Section 6), or KELLEY's paper [71]).

5.5. THEOREM. *Let K be a compact Hausdorff space. The following are equivalent.*

(1) *$C(K)$ can be isomorphically embedded in l^∞.*
(2) *K satisfies condition C_α for some $0<\alpha<1$.*
(3) *K satisfies condition C_α for all $0<\alpha<1$.*
(4) *$M_1^+(K)$ is weak* separable.*
(5) *$S_{M(K)}$ is weak* separable.*
(6) *$C(K)$ can be isometrically embedded in l^∞.*

PROOF. (1)$\Rightarrow$(2). Let $T\colon C(K)\to l^\infty$ be a linear isomorphism; we may assume that there is a constant $\lambda>0$ such that

$$\lambda\|f\|\leqslant\|T(f)\|\leqslant\|f\| \quad \text{for } f\in C(K).$$

Since $l^\infty=C(\beta\omega)$, where $\beta\omega$ is the Stone–Čech compactification of (the discrete set) ω, we have

$$\mathcal{T}^*(\beta\omega)=\bigcup_{n<\omega}\mathcal{T}_n,$$

with $\bigcap\mathcal{T}_n\neq\emptyset$ for $n<\omega$. For every $U\in\mathcal{T}^*(X)$, we find $f_U\in C(K)$ with $\|f_U\|=1$, $\operatorname{supp}(f_U)=\{x\in K\colon f_U(x)\neq 0\}^-\subset U$, and $0\leqslant f_U(x)\leqslant 1$ for $x\in K$, and we set

$$V_U=\{y\in\beta\omega\colon |T(f_U)|>\lambda/2\}\neq\emptyset.$$

We set

$$\mathcal{S}_n=\{U\in\mathcal{T}^*(X)\colon V_U\in\mathcal{T}_n\} \quad \text{for } n<\omega.$$

Then, $\mathcal{T}^*(X)=\bigcup_{n<\omega}\mathcal{S}_n$. We claim that

$$\kappa(\mathcal{S}_n)\geqslant\lambda/8 \quad \text{for } n<\omega.$$

In fact, let $U_1,\dots,U_m\in\mathcal{S}_n$. Then there is $y_0\in V_{U_1}\cap\cdots\cap V_{U_m}$, and we have

$$m\frac{\lambda}{2}<|T(f_{U_1})(y_0)|+\cdots+|T(f_{U_m})(y_0)|\leqslant 2|T(f_{U_{i_1}})(y_0)+\cdots+T(f_{U_{i_p}})(y_0)|$$

for some $\{i_1,\dots,i_p\}\subset\{1,\dots,m\}$. Hence,

$$m\frac{\lambda}{4}\leqslant|T(f_{U_{i_1}}+\cdots+f_{U_{i_p}})(y_0)|\leqslant\|f_{i_1}+\cdots+f_{i_p}\|,$$

and thus there is $x_0\in K$, such that

$$m\,\frac{\lambda}{8} \leqslant f_{i_1}(x_0) + \cdots + f_{i_p}(x_0) \leqslant \chi_{U_{i_1}}(x_0) + \cdots + \chi_{U_{i_p}}(x_0)$$
$$\leqslant \chi_{U_1}(x_0) + \cdots + \chi_{U_m}(x_0)\,,$$

hence,

$$\frac{\lambda}{8} \leqslant \frac{\sum_{i=1}^{m} \chi_{U_i}(x_0)}{m} \leqslant \frac{\left\| \sum_{i=1}^{m} \chi_{U_i} \right\|}{m} = \frac{\mathrm{cal}\{U_1, \ldots, U_m\}}{m}\,.$$

It follows that $\kappa(\mathcal{S}_n) \geqslant \lambda/8$. Thus K has C_α with $\alpha = \lambda/8$.

(2)$\Rightarrow$(3). Suppose that K satisfies C_α for some $0 < \alpha < 1$; thus $\mathcal{T}^*(K) = \bigcup_{n<\omega} \mathcal{T}_n$, with $\kappa(\mathcal{T}_n) \geqslant \alpha$ for $n < \omega$. From Kelley's Lemma (5.3(b)), there are regular Borel probability measures μ_n on K, with $\mu_n(\bar{U}) \geqslant \alpha$ for $U \in \mathcal{T}_n$, $n < \omega$. Let $0 < \beta < 1$. We choose $\varepsilon < 0$, such that $(\alpha - \varepsilon)/(\alpha + \varepsilon) \geqslant \beta$. For $U \in \mathcal{T}^*(K)$, we set

$$\sigma(U) = \mathrm{supp}\{\mu_n(U) \colon n < \omega\}\,, \qquad \tau(U) = \inf\{\sigma(V) \colon \emptyset \neq V \subset U\}\,.$$

We note that, for $U \subset V$, we have

$$\sigma(U) \leqslant \sigma(V)\,, \qquad \alpha \leqslant \tau(V) \leqslant \tau(U)\,.$$

We set

$$\mathcal{B} = \{U \in \mathcal{T}^*(K) \colon \sigma(U) \leqslant \tau(U) + \varepsilon\}\,,$$

and we note that if $V \in \mathcal{T}^*(K)$, then there is $U \in \mathcal{B}$, with $U \subset V$. (In fact, for $V \in \mathcal{T}^*(K)$, there is $U \in \mathcal{T}^*(K)$, with $\sigma(U) \leqslant \tau(V) + \varepsilon \leqslant \tau(U) + \varepsilon$; hence $U \in \mathcal{B}$.)

Let now $\{U_m \colon m < \omega\}$ be a maximal family of pairwise disjoint elements of $\mathcal{B}$; such a family in countable, since K satisfies C_α, hence it certainly has ccc. We set, for $m, n < \omega$,

$$\mathcal{T}_{n,m} = \{V \in \mathcal{T}^*(X) \colon \mu_n(V \cap U_m)/\mu_n(U_m) \geqslant \beta\}\,,$$

and we claim that

$$\mathcal{T}^*(K) = \bigcup_{m,n<\omega} \mathcal{T}_{n,m}\,.$$

In fact let $V \in \mathcal{T}^*(K)$. There is $U \in \mathcal{B}$, with $U \subset V$, and there is $m < \omega$, with $U \cap U_m \neq \emptyset$. Then $\tau(U_m) \leqslant \tau(U \cap U_m) \leqslant \sigma(U \cap U_m)$, hence there is $n < \omega$, with $\tau(U_m) - \varepsilon \leqslant \mu_n(U \cap U_m)$. Also, $\mu_n(U_m) \leqslant \sigma(U_m) \leqslant \tau(U_m) + \varepsilon$, and thus

$$\frac{\mu_n(V \cap U_m)}{\mu_n(U_m)} \geqslant \frac{\mu_n(U \cap U_m)}{\mu_n(U_m)} \geqslant \frac{\tau(U_m)-\varepsilon}{\tau(U_m)+\varepsilon} \geqslant \frac{\alpha-\varepsilon}{\alpha+\varepsilon} \geqslant \beta .$$

Since $\mu_{n,m}(B) = \mu_n(B \cap U_m)/\mu_n(U_m)$ is a Borel regular probability measure on K, it follows from Lemma 5.3(a), that K has C_β.

(3)$\Rightarrow$(4). Since K satisfies $C_{1-1/n}$ for $n = 2, 3, \ldots$, it follows, using Kelley's Lemma (5.3(b)), that there is a sequence $\{\mu_k : k < \omega\} \subset M_1^+(K)$, such that for any $U \in \mathcal{T}^*(K)$ and $0 < \theta < 1$, there is $k < \omega$, with $\mu_k(U) \geqslant \theta$. We claim that the weak* closed convex hull of $\{\mu_k : k < \omega\}$ is equal to $M_1^+(K)$. (If this is not the case, then there is $x_0 \in K$, with δ_{x_0} not in this hull where, with δ_{x_0} we denote Dirac's measure at x_0. By the Hahn–Banach theorem, there is $f \in C(K)$, with $f(x_0) > \sup\{\mu_k(f) : k < \omega\}$. Without loss of generality, we may assume, adding a constant to f if necessary, that $f \geqslant 0$. Then, we have $f(x_0) > \|f\|$, a contradiction).

(4)$\Rightarrow$(5). If $\{\mu_n : n < \omega\}$ is a countable weak* dense subset of $M_1^+(K)$, then it is easy to see that $\{q\mu_n : n < \omega,\ q \text{ rational } -1 \leqslant q \leqslant 1\}$ is countable, and weak* dense in $S_{M(K)}$.

(5)$\Rightarrow$(6). Let $\{\mu_n : n < \omega\}$ be a countable weak* dense subset of $S_{M(K)}$. We let $T: C(K) \to l^1$ by $T(f)(n) = \mu_n(f)$ for $n < \omega$. It is easy to see that T is a linear isometry.

(6)$\Rightarrow$(1) is trivial.

5.6. COROLLARY. *Let K be a compact Hausdorff space, and we denote by $G(K)$ the Gleason space of K. Then, $M_1^+(K)$ is weak* separable if and only if $M_1^+(G(K))$ is weak* separable.*

(This is clear from the equivalence (4)$\Leftrightarrow$(2), and the fact that the validity of (2) for K easily implies the validity of (2) for $G(K)$).

5.7. REMARK. In Theorem 5.6, the implications (2)$\Rightarrow$(3)$\Rightarrow$(4) appear in [83] (Theorem 2.3), the implication (1)$\Rightarrow$(6) in [118] (Th. II.1), and the equivalences (4)$\Leftrightarrow$(5)$\Leftrightarrow$(6) in [46] (Corollary 12).

5.8. LEMMA. *Let K be a compact metric space, μ a regular Borel probability measure on K, L a compact space, and $f: K \to L$ a continuous onto function, such that $f_*(\mu)\{y\} = 0$ for all $y \in L$. Then, for every $\varepsilon > 0$, there is a closed subset M of K, such that*

$\mu(M) \geqslant 1 - \varepsilon$,

$\mu \mid M$ is a strictly positive measure, and

$f \mid M$ is an irreducible function.

PROOF. Let $\varepsilon > 0$ be given. We choose a countable base $\{B_n : n < \omega\}$ of open subsets of K. We define two sequences M_n, U_n, $1 \leqslant n < \omega$, such that

M_n is a closed subset of K,
U_n is an open subset of L,
$M_1 = \operatorname{supp}(\mu)$,
if $M_n \cap B_n \neq \emptyset$, then $\mu(M_n \cap B_n \cap f^{-1}(U_n)) > 0$,

$$\mu(f^{-1}(U_n)) \leq \frac{\varepsilon}{2^n}, \min\left\{\frac{1}{2^k}\mu(M_k \cap B_k \cap f^{-1}(U_k)\colon k < n, M_k \cap B_k \neq \emptyset\right\},$$

and

$$M_{n+1} = \operatorname{supp}(\mu \mid (M_n \cap \bar{B}_n) \cup (M_n \setminus f^{-1}(U_n))).$$

We proceed inductively: If M_n is defined, we define U_n and M_{n+1}.

Case 1: $M_n \cap B_n = \emptyset$. We set $U_n = \emptyset$, and $M_{n+1} = M_n$.

Case 2: $M_n \cap B_n \neq \emptyset$. We choose $x_n \in M_n \cap B_n$; by the assumptions, $f_*(\mu)\{(fx_n)\} = 0$, hence, by the regularity of the measure $f_*(\mu)$, there is an open set U_n in L, $f(x_n) \in U_n$, and such that

$$\mu(f^{-1}(U_n)) \leq \frac{\varepsilon}{2^n}, \min\left\{\frac{1}{2^k}\mu(M_k \cap B_k \cap f^{-1}(U_k)\colon k < n, M_k \cap B_k \neq \emptyset\right\},$$

and

$$\mu(M_n \cap B_n \cap f^{-1}(U_n)) > 0.$$

and, we set $M_{n+1} = \sup(\mu \mid (M_n \cap \bar{B}_n) \cup (M_n \setminus f^{-1}(U_n))$,

It is clear that the conditions are satisfied. We set

$$M = \bigcap_{1 \leq n < \omega} M_n.$$

It is clear that M is a closed subset of K. Furthermore,

$$\mu(M_n) \geq 1 - \varepsilon\left(\frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) \quad \text{for all } n,$$

hence $\mu(M) \geq 1 - \varepsilon$.

We remark, that if $1 \leq k < n < \omega$, then

$$\mu(M_n \cap B_n \cap f^{-1}(U_n)) \geq (1 - \left(\frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{n-1}}\right))\mu(M_k \cap B_k \cap f^{-1}(U_k) \geq$$

$$(*) \qquad \geq \left(\frac{1}{2} + \frac{1}{2^n}\right)\mu(M_k \cap B_k \cap f^{-1}(U_k)).$$

Let V be an open subset of K, with $M \cap V \neq \emptyset$. There is $k < \omega$, with $\bar{B}_k \subset V$ and $M \cap B_k \neq \emptyset$. Thus $M_k \cap B_k \neq \emptyset$, and hence, by the properties of the sequences

M_k, U_k, we have

$$(M_k \cap B_k \cap f^{-1}(U_k)) > 0\,,$$

while, from property (*), we have, for $n > k$,

$$\mu(M_n \cap B_k \cap f^{-1}(U_k)) \geqslant \left(\frac{1}{2} + \frac{1}{2^n}\right)\mu(M_k \cap B_k \cap f^{-1}(U_k))\,.$$

Hence,

$$\mu(M \cap B_k \cap f^{-1}(U_k)) \geqslant \frac{1}{2}\,\mu(M_k \cap B_k \cap f^{-1}(U_k)) > 0$$

It follows that

$$\mu(M \cap V) > 0$$

(i.e., $\mu \mid M$ is strictly positive), and

$$f(M \cap B_k) \cap U_k \neq \emptyset\,,$$

hence $f(M \backslash V) \subsetneqq f(M)$ (i.e., $f \mid M$ is irreducible).

We are now ready to construct the Kunen–Haydon–Talagrand space.

5.9. THEOREM. *Assume the continuum hypothesis. There is a compact Hausdorff space K such that*:

(1) *K is a totally disconnected Corson-compact space of cardinality and weight ω^+ (and thus without caliber ω^+).*

(2) *$M_1^+(K)$, $S_{M(K)}$ are not weak* separable (hence, $C(K)$ cannot be isomorphically embedded in l^∞).*

(3) *K has a strictly positive normal Borel regular probability measure μ, with $L^1(\mu)$ non-separable.*

(4) *K is a L-space (i.e., hereditarily Lindelöf and non-separable).*

(5) *$M(K)$ is linearly isomorphic to*

$$\left(L^1\{0,1\}^{\omega^+} \oplus \left(\sum_{\omega^+} \oplus L^1\{0,1\}^{\omega}\right)_1 \oplus l^1_{\omega^+}\right)_1$$

(and hence $L^1\{0,1\}^{\omega^+}$ is isomorphic to a complemented subspace of $M(K)$; $|M(K)| = \omega^+$, and thus $M_1^+(K)$ in weak topology does not have caliber ω^+).*

(6) *$l^1_{\omega^+}$ cannot be embedded isomorphically in $C(K)$.*

(7) *$M(K)$ is weak* separable.*

(8) *$M(GK)$ is not weak* separable (where GK denotes the Gleason space of X).*

(9) *There is a family $\{f_\xi : \xi < \omega^+\} \subset C(K)$, with $\|f_\xi\| \leqslant 1$ for $\xi < \omega^+$, and $\|f_\xi - f_\zeta\|_1 > \theta$ for $\xi < \zeta < \omega^+$ and for some positive real number θ.*

PROOF. The space K will be a closed subset of $\{0, 1, -1\}^{\omega^+}$. We set

$$A_\alpha = \{0, 1, -1\}^\alpha \times \{0\}^{\omega^+ \setminus \alpha} \subset \{0, 1, -1\}^{\omega^+}$$

and we denote by $P_\alpha\colon \{0, 1, -1\}^\omega \to A_\alpha$ the natural projection, for $\alpha < \omega^+$.

We fix a mapping $\tau\colon \omega^+ \to \omega^+ \times \omega^+$, 1–1 and onto, and such that if $\tau(\alpha) = (\beta, \gamma)$ then $\beta \leqslant \alpha$. We define K_α, μ_α, $G_{\alpha\beta}$, $D_{\alpha,\beta}$, f_α, $H_{\alpha,m}$, B_α, C_α, D_α, M_α, N_α for $\omega \leqslant \alpha < \omega^+$, $\beta < \omega^+$, $m < \omega$, such that the following conditions are satisfied:

(A) 1. K_α is closed subset of A_α,
2. μ_α is a Borel regular probability measure on K_α,
3. $K_\beta \subset K_\alpha$, $K_\beta = P_\beta(K_\alpha)$, $\mu_\beta = P_{\beta_*}(\mu_\alpha)$ for $\beta < \alpha$,
4. $\mu_\alpha\{x\} = 0$ for $x \in K_\alpha$;

(B) 1. $\{G_{\alpha\beta}\colon \beta < \omega^+\}$ is a well-ordering of the family of all Borel μ_α-null subsets of K_α, such that
2. G_{α_0} is a countable dense subset of K_α,
3. $\{D_{\alpha\beta}\colon \beta < \omega^+\}$ is a well-ordering of the family of all closed subsets of K_α, with $\mu_\alpha(D_{\alpha\beta}) > 0$;

(C) 1. B_α is a closed subset of K_α,
2. $P_\gamma(B_\alpha) \cap G_{\gamma,\delta} = \emptyset$,
3. $\mu_\alpha(B_\alpha) \geqslant 1 - 1/n$, where n is the unique natural number, determined by the equality $\alpha + 1 = \lambda + n$, with λ limit ordinal and $n < \omega$,
4. $\mu_\alpha \mid B_\alpha$ is a strictly positive measure,
5. P_ω is irreducible on B_α;

(D) 1. $f_\alpha = \mathrm{d}P_{\omega^*}(\mu_\alpha \mid B_\alpha)/\mathrm{d}\mu_\omega \in L^1(\mu_\omega)$,
2. $H_{\alpha,m}$ is closed in $P_\omega(B_\alpha)$ for $m < \omega$,
3. $\mu_\omega \mid H_{\alpha,m}$ is strictly positive measure for $m < \omega$,
4. $f_\alpha \mid H_{\alpha,m} \geqslant 1/m$ for $m < \omega$,
5. $\bigcup_{m<\omega} H_{\alpha,m}$ is dense in $P_\omega(B_\alpha)$,
6. $C_{\alpha,m}$ is countable dense subset of $H_{\alpha,m}$ for $m < \omega$,
7. $P_\omega(B_\alpha) \cap C_{\gamma,m} = \emptyset$ for $\omega \leqslant \gamma < \alpha$, $m < \omega$;

(E) 1. B_α is the union of two pairwise disjoint closed sets C_α, D_α,
2. $\mu_\alpha(D_\alpha) > 0$, $D_\alpha \subset (P_\gamma^{-1}(D_{\tau(\alpha)})) \cap K_\alpha$, where $\tau(\alpha) = (\gamma, \delta)$,
3. $M_\alpha = \{(x \mid \alpha, 1, 0, 0, \ldots)\colon x \in C_\alpha\}$, $N_\alpha = \{(x \mid \alpha, -1, 0, 0, \ldots)\colon x \in D_\alpha\}$;

(F) 1. $K_\omega = A_\omega$,
2. $\mu_\omega =$ the canonical Haar measure on K_ω,
3. if $\alpha = \beta + 1$, we set $K_\alpha = K_\beta \cup M_\beta \cup N_\beta$,
4. if $\alpha = \beta + 1$, we set

$$\mu_\alpha = \mu_\beta \mid K_\beta \setminus B_\beta + \frac{1}{2}\mu_\beta \mid B_\beta + \phi_{\beta^*}\Big(\frac{1}{2}\mu_\beta \mid C_\beta\Big) + \psi_{\beta^*}\Big(\frac{1}{2}\mu_\beta \mid D_\beta\Big),$$

where $\phi_\beta\colon C_\beta \to M_\beta$, $\psi_\beta\colon D_\beta \to N_\beta$ are the homeomorphisms given by $\phi_\beta(x) = (x \mid \beta, 1, 0, 0, \ldots)$, $\psi_\beta(x) = (x \mid \beta, -1, 0, 0, \ldots)$,

5. K_α is the closure in A_α of $\bigcup_{\beta<\alpha} K_\beta$ for limit ordinals α, $\omega<\alpha<\omega^+$, and
6. μ_α is the weak* limit in $M(K_\alpha)$ of $\{\mu_\beta: \beta<\alpha\}$ for limit ordinals α, $\omega<\alpha<\omega^+$.

The definition of these objects is by transfinite induction. For $\alpha=\omega$ K_ω, μ_ω is defined by F.1, 2. Let $\omega\leqslant\alpha<\omega^+$. We define K_α, μ_α, using (F) (and assuming that all objects have been defined for all ordinals smaller than α). It is clear that A.1, A.2, A.3, A.4 are satisfied. Using the continuum hypothesis, we define $G_{\alpha\beta}$, $D_{\alpha\beta}$ for $\beta<\omega^+$, by (B). We choose B^1_α, a closed subset of K_α, such that some preliminary forms of C.1, C.2, C.3, C.7, and E.2 are satisfied, namely:

$$P_\gamma(B^1_\alpha)\cap G_{\gamma,\delta}=\emptyset \qquad \text{for } \omega\leqslant\gamma\leqslant\alpha,\ \delta\leqslant\alpha\,,$$

$$P_\omega(B^1_\alpha)\cap C_{\gamma,m}=\emptyset \qquad \text{for } \omega\leqslant\gamma<\alpha,\ m<\omega\,,$$

$$\mu_\alpha(B^1_\alpha)\geqslant 1-1/n^2 \qquad \text{for the natural } n \text{ indicated in C.3}\,,$$

$$\mu_\alpha(B^1_\alpha\cap P^{-1}_\gamma(D_{\tau(\alpha)}))>0 \qquad \text{for } \tau(\alpha)=(\gamma,\delta)\,.$$

Such a choice is possible since the countable union

$$\bigcup_{\substack{\omega\leqslant\gamma\leqslant\alpha\\ \delta\leqslant\alpha}} P^{-1}_\gamma(G_{\gamma,\delta})\cup\bigcup_{\substack{\omega\leqslant\gamma<\alpha\\ m<\omega}} P^{-1}_\gamma(C_{\gamma,m})\cap K_\alpha$$

is a Borel μ_α-null set in K_α, μ_α is a Borel regular probability measure, and $\mu_\alpha(P^{-1}_\gamma(D_{\tau(\alpha)}\cap K_\alpha)>0$.

Next, using Lemma 5.8, we find two disjoint closed subsets C_α, D_α of B^1_α, such that setting $B_\alpha=C_\alpha\cup D_\alpha$, conditions C.3, C.4, C.5, and E.1, E.2 are satisfied.

We note that the measure $P_{\omega^*}(\mu_\alpha\mid B_\alpha)$ is absolutely continuous with respect to μ_ω (since in fact $P_{\omega^*}(\mu_\alpha\mid B_\alpha)\leqslant\mu_\omega$), and we let f_α be the Radon–Nikodym derivative, as in D.1 (0.17). The definitions of $H_{\alpha,m}$, $C_{\alpha,m}$ are then made, without difficulty, so that they satisfy conditions D.2, D.3, D.4, D.5, and D.6. Finally M_α, N_α are defined by E.3.

The inductive definitions are now complete.

Finally, we set

K the closure in $\{0,1,-1\}^{\omega^+}$ of $\bigcup_{\alpha<\omega} K_\alpha$, and

μ the weak*-limit in $M(K)$ of $\{\mu_\alpha: \alpha<\omega^+\}$ (using 0.33).

We verify that K, μ have the indicated properties.

Ad(1). If $x\in K$, then $P_\omega(x)\in K_\omega$, and $\mu_\omega\{P_\omega(x)\}=0$. Hence there is $\delta<\omega^+$, such that $\{P_\omega(x)\}=G_{\omega,\delta}$. It follows that $x\in P^{-1}_\omega(P_\omega(x))\subset K_\delta$. Thus $K=\bigcup_{\alpha<\omega^+} K_\alpha$, and hence K is Corson-compact.

It follows (since each K_α is compact metric) that $|K|=\omega^+$. Since every countable subset of K is contained in some K_α, it follows that K is not separable. That K does not have caliber ω^+ follows from 5.2.

Ad(2). (a) We claim first that if B is a Baire subset of K, with $\mu(B)=0$, then there is $\alpha<\omega$ such that $B\subset K_\alpha$.

(In fact, B depends on a countable set of coordinates. Hence, there is $\beta<\omega^+$, such that $P_\beta(B)$ is a Borel subset of K_β, with $\mu_\beta(P_\beta(B))=0$. Hence, there is $\delta<\omega^+$, such that $P_\beta(B)=G_{\beta,\delta}$. Then, $B\subset K_{\delta+1}$).

We prove now that $M_1^+(K)$ is not weak separable. Let $(\lambda_n)\subset M_1^+(K)$. By Lebesgue's decomposition theorem (0.32), there are non-negative measures $\lambda_{n,1}$, $\lambda_{n,2}\in M(K)$ such that

$$\lambda_n=\lambda_{n,1}+\lambda_{n,2}\,,\qquad \lambda_{n,1}\perp\mu,\ \lambda_{n,2}\ll\mu\quad\text{for } n<\omega\,.$$

Since $\lambda_{n,1}\perp\mu$, there is a Baire set B_n in K such that $\mu(B_n)=0$, $\lambda_{n,1}(B_n)=\lambda_{n,1}(K)$ for $n<\omega$. By Radon–Nikodym Theorem (0.17), there is a Baire-measurable function $h_n\in L^1(\mu)$, such that $h_n=\mathrm{d}\lambda_{n,2}/\mathrm{d}\mu$ for $n<\omega$. Hence, by claim (a), there is $\alpha<\omega^+$ such that $B_n\subset K_\alpha$ for $n<\omega$ and $h_n=g_n\circ P_\alpha$, for g_n Borel function on K_α. We set

$$f=\chi_{P_{\alpha+1}^{-1}(L_\alpha)}\,,\quad\text{where } L_\alpha=K_{\alpha+1}\backslash K_\alpha\,.$$

Then,

$$\int f d\lambda_n=\int fg_n\circ P_\alpha\,\mathrm{d}\mu=\int_{L_\alpha}g_n\circ P_\alpha\,\mathrm{d}\mu=\frac{1}{2}\int_{P_\alpha(L_\alpha)}g_n\,\mathrm{d}\mu_\alpha\,,$$

and since $\|\lambda_n\|\leq 1$, we have that

$$\int|g_n|\,\mathrm{d}\mu_\alpha\leq 1\quad\text{for } n<\omega\,.$$

Thus, $|\int f\,\mathrm{d}\lambda_\mu|\leq\frac{1}{2}$ for $n<\omega$. It follows that (λ_n) is not weak* dense in $M_1^+(K)$ (since for some function $f\in C(K)$, with $\|f\|=1$, we have $|f\,\mathrm{d}\lambda_n|\leq\frac{1}{2}$ for $n<\omega$).

That $S_{M(K)}$ is not weak* separable, and that $C(K)$ does not embed isomorphically in l^∞ follows now from Theorem 5.8.

Ad(3). It is easy to prove inductively, using E2, E4, and E6, that μ_α is a strictly positive measure in K_α for $\alpha<\omega^+$. Let U be a non-empty open subset of K. Then there is $\alpha<\omega^+$ and a non-empty open subset V of K_α, such that $P_\alpha^{-1}(V)\subset U$. Then

$$\mu(U)\geq\mu(P_\alpha^{-1}(V))=\mu_\alpha(V)>0\,,$$

i.e., μ is a strictly positive measure.

To prove that μ is a normal measure, let F be a closed subset of K, with $\mu(F)>0$. Since $F=\bigcap_{\alpha<\omega^+}P_\alpha^{-1}(P_\alpha(F))$, and μ is a regular measure, it follows that there is $\alpha < \omega^+$, such that $\mu(F) = \mu(P_\alpha^{-1}(P_\alpha(F))) = \mu_\alpha(P_\alpha(F)) > 0$. There is $\gamma < \omega^+$, such that $P_\alpha(f) = D_{\alpha,\gamma}$ (condition B.3). There is $\beta > \alpha$, such that $\tau(\beta) = (\alpha, \gamma)$ (by the property of τ). Condition E.2, E.3, and F.3 imply that $P_\alpha^{-1}(P_\alpha(F) \cap K_{\beta+1}$ has non-empty interior in $K_{\beta+1}$. Hence $P_\alpha^{-1}(P_\alpha(F))$ has non-empty interior in K. If F has empty interior, then there is a non-empty open set W such that $W \subset P_\alpha^{-1}(P_\alpha(F)) \backslash F$. Since μ is a strictly positive measure on K, and $\mu(F) = \mu(P_\alpha^{-1}(P_\alpha(F)))$, this is a contradiction.

It is clear that $\dim L^1(\mu) \leqslant \omega^+$. If $L^1(\mu)$ were separable, then the fact that the unit sphere of $L^1(\mu)$ is dense in the unit sphere of $M(K)$ in its weak*-topology (since μ is a strictly positive measure on K cf. (0.21)) would imply that the unit sphere of $M(K)$ is weak*-separable, contrary to above property. Thus, $\dim L^1(\mu) = \omega^+$.

Ad(4). Let $X \subset K$, and $\{U_i : i \in I\}$ a family of open subsets of K, with $U = \bigcup_{i \in I} U_i \supset X$. Since K has a strictly positive measure, it is a ccc space, and hence, as it is easy to verify, there is a countable set $J_1 \subset I$, such that $V = \bigcup_{i \in J_1} U_i$ is dense in U. Thus the set $U \backslash V$ is a nowhere dense subset of K, and since μ is a normal measure, it follows that $\mu(U \backslash V) = 0$. By the regularity of μ, there a (closed) Baire set B, such that $U \backslash V \subset B$, and $\mu(B) = 0$.

By claim (a) in Ad(2), there is $\alpha < \omega^+$ with $U \backslash V \subset B \subset K_\alpha$, and thus $U \backslash V$ is a separable metric space; hence there is a countable set $J_2 \subset I$, with $\bigcup_{i \in J_2} U_i \supset U \backslash V$. Thus $\{U_i : i \in J_1 \cup J_2\}$ is a countable subcover of X of the original cover $\{U_i : i \in I\}$. Since K is not separable (by (1)), it follows that K is an L-space.

Ad(5). Let $\{\nu_i : i \in I\}$ be a maximal family of probability measures in $M(K)$, such that

$$\nu_i \perp \nu_j \quad \text{for } i, j \in I,\ i \neq j,$$

$$\nu_i \perp \mu \quad \text{for } i \in I, \quad \text{and}$$

$$\nu_i \text{ is purely non-atomic}.$$

Since $\nu_i \perp \mu$, it follows that there is a Baire set B_i in K, such that

$$\nu_i(K \backslash B_i) = 0 \quad \text{and} \quad \mu(B_i) = 0.$$

By (a) in Ad(2) there is $\alpha_i < \omega^+$, such that $B_i \subset K_{\alpha_i}$ for $i \in I$.

It follows that $|I| \leqslant |\bigcup_{\alpha < \omega^+} M(K_\alpha)| = \omega^+$, since each K_α is a compact metric space. On the other hand, it is clear that $|I| \geqslant \omega^+$. Thus, we assume that $I = \omega^+$.

Since every ν_i is a purely non-atomic measure, supported in a compact metric space, it follows from Caratheodory's Theorem (0.23), that $L^1(\nu_i)$ is isomorphically isometric to $L^1\{0, 1\}^\omega$.

Thus $\{\mu\} \cup \{\nu_i : i < \omega^+\} \cup \{\delta_x : x \in K\}$ is a maximal family in $M(K)$ of pairwise orthogonal probability measures in $M(K)$.

From Maharam's Theorem (0.24), the fact that $\dim L^1(\mu) = \omega^+$, and that μ is purely non-atomic, it follows that $L^1(\mu)$ is equal to $L^1\{0,1\}^{\omega^+}$, or to $(L^1\{0,1\}^{\omega^+}\oplus L^1\{0,1\}^{\omega})_1$ (in fact the former case is true since μ is a homogeneous measure, but we do not step to prove this here). Hence, it follows easily from Radon–Nikodym Theorem, and Lebesgue decomposition theorem in a standard way that $M(K)$ is isomorphic to

$$\left(L^1\{0,1\}^{\omega^+}\oplus\left(\sum_{\omega^+}\oplus L^1\{0,1\}^{\omega}\right)_1\oplus l^1_{\omega^+}\right)_1.$$

Then $|M(K)|=\omega^+$ follows from the form that $M(K)$ has. That $M_1^+(K)$ does not have caliber ω^+ follows from (2), the fact that $|M(K)|=\omega^+$, and Lemma 5.1.

Ad(6). Suppose that $l^1_{\omega_+}\hookrightarrow C(K)$. By the Theorem of Pelczynski (4.7), $l^1_{2^{(\omega)^+}}\hookrightarrow M(K)$. But since for any finite measure ν, and any uncountable cardinal β, l^1_β cannot be embedded in $L^1(\nu)$ (0.27), the result follows immediately from (5).

Ad(7). Let $\{g_n: n<\omega\}$ be a norm-dense subset in $C(K_\omega)$, and let $\mu_n=g_n\circ P_\omega\,\mathrm{d}\mu\in M(K)$. We now prove that $\{\mu_n: n<\omega\}$ is weak*-dense in $M(K)$. In fact, it is enough to prove that if $f\in C(K)$, and $f\neq 0$, then there is $n<\omega$, such that $\int f(t)\,\mathrm{d}\mu_n(t)\neq 0$. It is enough to prove, that there is a bounded Borel function $g: K_\omega\to\mathbb{R}$, such that $\int f(t)g\circ P_\omega(t)\,\mathrm{d}\mu\neq 0$.

Let $f\in C(K)$, $f\neq 0$. Let α be the least ordinal such that $f\,|\,K_\alpha\neq 0$. By the continuity of f, and the fact that for limit ordinals λ, $\omega<\lambda<\omega^+$, $\overline{\bigcup_{\beta<\lambda}K_\beta}=K_\lambda$, it follows that $\alpha=\beta+1$. Thus, $f\,|\,K_{\beta+1}\neq 0$, and $f\,|\,K_\beta=0$. There is $\varepsilon>0$ and V non-empty basic open subset in $\{0,1,-1\}^{\omega^+}$, such that $f\,|\,V\cap K\geqslant\varepsilon$, and $V\cap K_{\beta+1}\neq 0$. Thus, $V=V_1\times V_2$, where V_1 depends on a finite subset of $[\omega,\beta]$, and V_2 depends on a finite subset J of $[\beta+1,\omega^+)$. It is clear that $K_\beta\cap V=\emptyset$, and hence $x_\beta=1$ for $x\in V_1$; also if $\gamma\in J$, and $x\in V_2$, then $x_\gamma=0$, since $V\cap K_{\beta+1}\neq\emptyset$. We set $L_\gamma=K_{\gamma+1}\backslash K_\gamma$ for any $\gamma<\omega^+$. The set $\bigcup_{\gamma\in J}P_\gamma(L_\gamma)$ is nowhere dense in $K_{\beta+1}$ (on account of the fact $G_{\beta+1,0}$ is dense in $K_{\beta+1}$, and $P_\gamma(L_\gamma)\cap G_{\beta+1,0}$ for all $\gamma\geqslant\beta+1$), and hence there is a non-empty open set $W_1\subset V_1\cap K_{\beta+1}$, such that $P^{-1}_{\beta+1}(W)\cap(\bigcup_{\gamma\in J}L_\gamma)=\emptyset$; hence, $P^{-1}_{\beta+1}(W)\subset V\cap K$. It follows that $W\subset K_{\beta+1}\backslash K_\beta$ (since $f\,|\,K_\beta=0$, and $f\,|\,V\cap K\geqslant\varepsilon$). Since $P_\omega\,|\,P_\beta(L_\beta)$ is irreducible, it follows that $P_\omega(L_\beta\backslash W)\neq P_\omega(L_\beta)$, hence there is a non-empty open subset T of $P_\omega(L_\beta)$, such that $P_\omega(L_\beta\backslash W)\cap T=\emptyset$, i.e.,

$$P^{-1}_\omega(T)\cap L_\beta\subset W.$$

Furthermore, there is $n<\omega$, such that $T\cap H_{\beta,n}\neq\emptyset$ (and of course, $f_\beta\,|\,T\cap H_{\beta,n}\geqslant 1/n$).

Since $f\,|\,K_\beta=0$, there is a non-empty basic open set U in $\{0,1,-1\}^{\omega^+}$, such that $|f|\,|\,U\cap K<\varepsilon/4n$. Let I be the finite subset of ω^+ on which U depends. It is clear that $I\subset[\beta,\omega^+)$, and that if $\gamma\in I$, and $x\in U$ then $x_\gamma=0$. The set $\bigcup_{\gamma\in I\backslash\{\beta\}}P_\omega(L_\gamma)$ is nowhere dense in $H_{\beta,n}$ (on account of the fact $C_{\beta,n}$ is dense in $H_{\beta,n}$, and $\bigcup_{\gamma\in I\backslash\{\beta\}}P_\omega(L_\gamma)\cap C_{\beta,m}=\emptyset$), and hence there is a non-empty set A, open in $T\cap H_{\beta,n}$,

$P_\omega^{-1}(A) \cap L_\gamma = \emptyset$ for all $\gamma \in I \setminus \{\beta\}$; hence $E_0 = \{x \in K: P_\omega(x) \in A$ and $x_\beta = 0\} \subset U \cap K$. Of course, $E_1 = \{x \in K: P_\omega(x) \in A$ and $x_\beta = 1\} \subset V \cap K$.

Hence we have

$$\int_K f(t)\chi_A \circ P_\omega(t)\,\mathrm{d}\mu(t) = \int_{E_0} f(t)\chi_A \circ P_\omega(t)\,\mathrm{d}\mu(t) + \int_{E_1} f(t)\chi_A \circ P_\omega(t)\,\mathrm{d}\mu(t) \geqslant$$
$$\geqslant \varepsilon\mu(E_1) - \frac{\varepsilon}{4n}\mu(E_0),$$

and

$$\mu(E_1) = \mu_{\beta+1}(L_\beta \cap P_\omega^{-1}(A)) = \frac{1}{2}\mu_\beta(P_\beta(L_\beta) \cap P_\omega^{-1}(A))$$
$$= \frac{1}{2} P_\omega(\mu_\beta \mid P_\beta(L_\beta)(A))$$
$$\geqslant \frac{1}{2}\cdot\frac{1}{n}\mu_\omega(A),$$
$$\mu(E_0) \leqslant \mu(P_\omega^{-1}(A)) = \mu_\omega(A).$$

Hence,

$$\int_K f(t)\chi_A \circ P_\omega(t)\,\mathrm{d}\mu(t) \geqslant \frac{\varepsilon}{2n}\mu_\omega(A) - \frac{\varepsilon}{4n}\mu_\omega(A) > 0,$$

Since $\mu_\omega \mid H_{\beta,n}$ is a strictly positive measure. This completes the proof that $M(K)$ is weak*-separable.

Ad(8). The Gleason space GK has a strictly positive normal Borel regular probability measure ν (on account of (3)). Then, by a Theorem of Dixmier (0.19), $C(GK)$ is isometrically isomorphic to $L^\infty(\nu)$. Hence, by a result by W. ARVESON and H. ROSENTHAL [101], (p. 218), $M(GK)$ is not weak*-separable.

Ad(9). There is a family $\{f_\xi: \xi < \omega^+\} \subset C(K)$ such that

$\|f_\xi\| \leqslant 1$ for $\xi < \omega^+$ and
there is $\theta > 0$, such that $\|f_\xi - f_\zeta\|_1 > \theta$ for $\xi < \zeta < \omega^+$.

This follows immediately from the facts, that $C(K)$ is L^1-norm dense in $L^1(\mu)$, and $\dim L^1(\mu) = \omega^+$.

The proof of the theorem is now complete.

5.10. REMARKS. (a) A space with properties (1)–(4) has been found by KUNEN in 1975, and published in [76].

(b) A space with properties (1), (2), (3), (5), and (6) of Theorem 5.9 has been essentially described by HAYDON [62] (except that his space was not Corson-compact, but first countable and 2) was not explicitly mentioned.

(c) A space with properties (1), (2), (3), (5), (6), (7) of Theorem 5.9 has been given by TALAGRAND [118] (except that the measure μ was not normal). Talagrand also proved there Lemma 5.8).

(d) A construction similar to the Kunen–Haydon–Talagrand one, and in a sense quite closely related to Talagrand's space was given by LOSERT in [82]: Assuming CH, there is a compact separable space with a probability measure μ such that μ has a uniformly distributed sequence, but μ does not have a well-distributed sequence.

(e) A variant of Haydon's example, consistent with the denial of the continuum hypothesis, has been given by FLEISSNER and the author in [49].

(f) TALAGRAND has also proved in [118], assuming CH, the existence of a compact space K, with $M_1^+(K)$ weak*-separable, and such that K has no strictly positive measure μ with $L^1(\mu)$ separable.

6. Corson-compact spaces and subclasses—Applications to Banach spaces

The main results in this Section are: Gul'ko Theorem (6.23) that every Gul'ko-compact space is Corson-compact, the classical Amir–Linenstrauss Theorem on W.C.G. Banach space (6.31), and various applications of there (closely related) theorems; the Argyros–Negrepontis Theorem (6.40) on ccc Gul'ko compact spaces; and the three striking examples of an Eberlein-compact space that is not a uniform Eberlein-compact space (6.53), of a Talagrand-compact space that is not an Eberlein-compact space (6.55), and of a Corson-compact space that is not a Gul'ko-compact space (6.58).

6.1. DEFINITION. If K is a compact Hausdorff space, then the *pointwise topology* on $C(K)$ is determined by the requirement: a net $(f_i)_{i\in I}$ converges in the pointwise topology to f (for f_i, $f \in C(K)$) if $\lim_{i\in I} f_i(x) = f(x)$ for all $x \in K$.

The pointwise topology on $C(K)$ is clearly weaker than or equal to the weak topology on $C(K)$.

6.2. DEFINITIONS. Let K be a compact Hausdorff space. Then

(a) K is called *Eberlein-compact* if K is homeomorphic to a subset of a Banach space in its weak topology.

(b) K is called *uniform Eberlein-compact* if K is homeomorphic to a subset of a Hilbert space in its weak topology.

(c) K is called *Talagrand-compact* if $C(K)$ in its weak topology is a $\mathcal{K}$-analytic set. (A topological space is said to be *$\mathcal{K}$-analytic* if it is the continuous image of a $K_{\sigma\delta}$-space Y, i.e. a space Y of the form

$$Y = \bigcap_{n<\omega} \bigcup_{m<\omega} K_{m,n}\,, \quad \text{with } K_{m,n} \text{ compact}\,.$$

(d) K is called *Gul'ko-compact* if $C(K)$ in its weak topology is a $\mathcal{K}$-countably determined set. (A topological space Y is *$\mathcal{K}$-countably determined* if there is a compact space Z containing Y and a sequence $\{K_n: n < \omega\}$ of closed subsets of Z, such that for every $y \in Y$, there is $\emptyset \neq N_y \subset \omega$, such that $y \in \bigcap_{n \in N_y} K_n \subset Y$).

(e) K is called *Corson-compact* if K is homeomorphic to a subset of $\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma: \text{supp}(x) \text{ is countable}\}$, where $\text{supp}(x) = \{\gamma \in \Gamma: x_\gamma \neq 0\}$ for $x \in \mathbb{R}^\Gamma$, for some set Γ. (Of course $\mathbb{R}^\Gamma$ has the Cartesian topology.)

6.3. REMARKS. (a) It is clear that a Hausdorff continuous image of a Talagrand-compact (resp., Gul'ko-compact) space is Talagrand-compact (resp., Gul'ko-compact).

(b) Of course every uniform Eberlein-compact is an Eberlein-compact.

It is not difficult to see that every Talagrand-compact space is Gul'ko-compact (cf. TALAGRAND [116] (Proposition 1.1)).

(c) If K is a Corson-compact space, that $w(K) = d(K)$ and K is an angelic space.

(d) If K is a Gul'ko-compact space, then $C(K)$ is Lindelöf in the weak-topology, and hence in the pointwise topology (since every $\mathcal{K}$-countably determined space is Lindelöf).

REMARK. $\mathcal{K}$-analytic spaces were introduced by CHOQUET [33]; Corson-compact spaces, and generally the class of Corson spaces, consisting of all subsets of $\Sigma(\mathbb{R}^\Gamma)$ were introduced and studied by CORSON [36]; the term Eberlein-compact was formulated in the survey paper of LINDENSTRAUSS [79], where several results and open questions are given (e.g. the question, answered in the negative by POL [99] (cf. 7.8(c) below) and TALAGRAND [116] (cf. 6.55 below), whether the Lindelöf property for $C(K)$ in the weak topology implies that K is an Eberlein-compact).

The following three results are classical.

6.4. THEOREM (Krein). *Let K be a weakly compact subset of a Banach space X. Then the closed convex hull of K is also weakly compact.*

6.5. THEOREM (EBERLEIN [44]). *A weakly compact subset K of a Banach space is sequentially compact, and if $A \subset K$, then A is sequentially dense in its closure* (*i.e. K is an angelic space*).

6.6. THEOREM (GROTHENDIECK [53]). *A bounded subset L of $C(K)$, with K a compact Hausdorff space, is weakly compact if and only if L is compact in the point-wise topology.*

6.7. DEFINITION. A Banach space X is *weakly compactly generated* (W.C.G.) if there is a weakly compact subset K of X, such that X is equal to the closed linear space generated by K.

6.8. EXAMPLES. The following are the principal examples of W.C.G. Banach spaces:

(a) $c_0(\Gamma)$ for every set Γ (in fact the set $K = \{e_\gamma : \gamma \in \Gamma\} \cup \{0\}$, where $e_\gamma(\delta) = 0$ if $\gamma \neq \delta$, $= 1$ if $\gamma = \delta$ is a weakly compact subset of $c_0(\Gamma)$, in fact homeomorphic to the one-point compactification of the discrete set Γ, generating $c_0(\Gamma)$).

(b) Every reflexive Banach space X (because the unit ball of X is weakly compact). In particular every Hilbert space is W.C.G.

(c) Every separable Banach space X (if (x_n) is a sequence that is dense in the unit ball of X, then the set $K = \{x_n/n : n = 1, 2, \ldots\} \cup \{0\}$ is weakly compact).

(d) $L^1(\mu)$ for every finite measure μ (the image of the unit ball of the Hilbert space $L^2(\mu)$ under the natural inclusion $L^2(\mu) \subset L^1(\mu)$ generates $L^1(\mu)$).

(e) $C(K)$ for K an Eberlein-compact space. (This is a consequence of Grothendieck's characterization (6.6) of weakly compact subsets of $C(K)$, for any compact space K, as bounded and compact in the pointwise topology, and of Lebesgue's dominated convergence theorem.)

6.9. THEOREMS. (a) *If X is a* W.C.G. *Banach space, then there is an Eberlein-compact space K, and* 1–1 *bounded linear and (weak*, weak)-continuous operator $T: X^* \to C(K)$.*

(b) *A compact Hausdorff space K is Eberlein-compact if and only if $C(K)$ is* W.C.G.

(c) *If X is a* W.C.G. *Banach space, then the unit sphere S_{X^*} of the dual space X^*, in its weak* topology, is Eberlein-compact.*

Simple proofs of these theorems can be found in ROSENTHAL [103] (§3).

6.10. THEOREM (TALAGRAND [114]). *If X is a* W.C.G. *Banach space, then X is $\mathcal{K}$-analytic in its weak topology.*

PROOF (ROGERS–JAYNE [100]). Let K be a weakly compact subset of X, whose linear span is dense in X. For any finite sequence $q_1, \ldots, q_k$, set

$$K(q_1, \ldots, q_k) = q_1 K + \cdots + q_k K,$$

and note that $K(q_1, \ldots, q_k)$ is weakly compact in X. Let $\{K^{(n)} : n < \omega\}$ be an enumeration of the sequence

$$\{K(q_1, \ldots, q_k) : q_1, \ldots, q_k \text{ rationals}\}.$$

Since the linear span of K is (norm-)dense in X, we have

$$X \subset \bigcup_{n<\omega} K^{(n)} + \frac{1}{p+1} S_X \subset \bigcup_{n<\omega} K^{(n)} + \frac{1}{p+1} S_{X^{**}},$$

hence

$$X \subset \bigcap_{p<\omega} \bigcup_{n<\omega} \left(K^{(n)} + \frac{1}{p+1} S_{X^{**}} \right).$$

Let

$$x^{**} \in \bigcap_{p<\omega} \bigcup_{n<\omega} K^{(n)} + \frac{1}{p+1} S_{X^{**}}.$$

For every $p<\omega$ there is $x_p \in \bigcup_{n<\omega} K^{(n)} \subset X$, with $\|x_p - x^{**}\| \leq 1/(p+1)$. So $(x_p)_{p<\omega}$ is a Cauchy sequence in X^{**}, hence in X, and so it converges to a point of X, which can only be x^{**}. Thus

$$X = \bigcap_{p<\omega} \bigcup_{n<\omega} \left(K^{(n)} + \frac{1}{p} S_{X^{**}} \right).$$

This shows that X is a $\mathcal{K}_{\sigma\delta}$-set in the weak* topology of X^{**}. Since the weak topology on X is equal to the topology on X induced by the weak* topology on X^{**}, the result follows.

REMARK. Theorem 6.10 was proved by TALAGRAND [114], in response to a problem of CORSON [37]. The proof of ROGERS–JAYNE [100] is similar to Talagrand's original proof, but more elementary in the sense that it avoids using Krein's Theorem (6.4).

Gul'ko's Theorem

In §§6.11–6.23 below, K denotes a compact Hausdorff space and the topological concepts involved (closure, density, continuity, etc.) in $C(K)$ are always considered in the pointwise topology of $C(K)$.

Also if M is closed in K, and $p: K \to M$ is continuous we set $p^*: C(K) \to C(K)$ by $p^*(f) = (f \mid M) \circ p$.

6.11. DEFINITION. Let $F \subset C(K)$, F distinguishes points of K, $M \subset K$, $L \subset F$. We say that the pair (M, L) is *F-conjugate* if

(a) for all $x \in K$, there is $p(x) \in M$, such that $f(x) = f(p(x))$ for all $f \in L$, and

(b) for all $f \in F$, there is $q(f) \in L$, such that $f(x) = (q(f))(x)$ for all $x \in M$.

6.12. LEMMA. *Let $F \subset C(K)$, F distinguish points of K, $M \subset K$, $L \subset F$, with (M, L) F-conjugate. Then,*

(a) *$p(x)$, $q(f)$ are unique for all $x \in K$, $f \in F$;*

(b) *the mappings $p: K \to M$, $q: F \to L$ defined by* (a) *are continuous retractions* (*and hence M, L are closed subsets of K, F, respectively*); *and*

(c) $p^* \mid F = q$.

PROOF. It follows from the definition of conjugate pair that L distinguishes points of M and M distinguishes points of L. Hence, $p(x)$, $q(f)$ are unique.

Let now $x_i \to x$ be a convergent net in K. If $f \in F$, then

$$f(p(x_i)) = (q(f))(p(x_i)) = q(f)(x_i) \to q(f)(x) = (q(f))(p(x)) = f(p(x));$$

thus, $p(x_i) \to p(x)$, i.e., p is continuous. In a similar way q is continuous. The fact that the mappings p, q are retractions follows immediately from the uniqueness of $p(x)$, $q(f)$, (c) follows easily.

6.13. LEMMA. *Let $F \subset C(K)$, F distinguish points of K, $M \subset M_1 \subset K$, $L \subset L_1 \subset F$, with the pairs (M, L) and (M_1, L_1) F-conjugate. Then for the corresponding retractions of Lemma* 6.12(b), *we have*

$$p_1 \circ p = p \circ p_1 = p \quad \textit{and} \quad q_1 \circ q = q \circ q_1 = q .$$

We omit the simple proof of this lemma.

6.14. DEFINITION. Let $F \subset C(K)$, $M \subset K$, $L \subset F$. We say that the pair (M, L) is *F-preconjugate* if

(a) $\{x \mid L: x \in M\}$ is dense in $\{x \mid L: x \in K\}$ (in the pointwise topology of $L \subset C(K)$), and

(b) $\{f \mid M: f \in L\}$ is dense in $\{f \mid M: f \in F\}$ (in the pointwise topology of $L \subset F \subset C(K)$).

Here and below we use the *notation*: $x \mid L: L \to \mathbb{R}$ to mean $(x \mid L)(f) = f(x)$ for $f \in L$.

6.15. LEMMA. *Let $F \subset C(K)$, $A \subset K$, $B \subset F$, τ an infinite cardinal, and $|A| \leqslant \tau$, $|B| \leqslant \tau$. Then there is an F-preconjugate pair (M, L) such that $A \subset M$, $B \subset L$, and $|M| \leqslant \tau$, $|L| \leqslant \tau$.*

PROOF. We define inductively

$$A = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots \subset K, \quad B \subset L_0 \subset L_1 \subset \cdots \subset L_n \subset \cdots \subset F,$$

with $|M_n| \leqslant \tau$, $|L_n| \leqslant \tau$ for $n < \omega$, such that for $n < \omega$

$$\begin{aligned} &\{f \mid M_n : f \in L_n\} && \text{is dense in } \{f \mid M_n : f \in F\}, \quad \text{and} \\ &\{x \mid L_n : x \in M_{n+1}\} && \text{is dense in } \{x \mid L_n : x \in K\}. \end{aligned}$$

We set $M_0 = A$. To choose L_0, we consider $G = \{f \mid M_0 : f \in F\}$. In the pointwise topology, we have $d(G) \leqslant w(G) \leqslant |M_0| \leqslant \tau$, hence there is a dense subset E of G, with $|E| \leqslant \tau$. We now choose, for every $e \in E$, $f_e \in E$ with $f_e \mid M_0 = e$, and set

$$L_0 = B \cup \{f_e : e \in E\}.$$

The remaining steps of the induction proceed analogously. We finally set $M = \bigcup_{n<\omega} M_n$, $L = \bigcup_{n<\omega} L_n$, and prove that the pair (M, L) is F-preconjugate. In fact, let $x \in K$, $\varepsilon > 0$, $f_1, \ldots, f_k \in L$; then there is $n < \omega$ such that $f_1, \ldots, f_k \in L_n$, and since $\{x \mid L_n : x \in M_{n+1}\}$ is dense in $\{x \mid L_n : x \in K\}$, there is $y \in M_{n+1} \subset M$ with $|f_i(x) - f_i(y)| < \varepsilon$ for $i = 1, \ldots, k$. This verifies condition (a) of an F-preconjugate pair; condition (b) is verified analogously.

6.16. LEMMA. *Let $F \subset C(K)$, ξ a limit ordinal, and suppose that (M_η, L_η) are F-preconjugate pairs for $\eta < \xi$, with $M_\eta \subset M_\zeta$, $L_\eta \subset L_\zeta$ for $\eta < \zeta < \xi$. Then the pair $(\bigcup_{\eta<\xi} M_\eta, \bigcup_{\eta<\xi} L_\eta)$ is F-preconjugate.*

PROOF. We set $M = \bigcup_{\eta<\xi} M_\eta$, $L = \bigcup_{\eta<\xi} L_\eta$. To prove that (M, L) is F-preconjugate, let $x \in K$, $f_1, \ldots, f_k \in L$, and $\varepsilon > 0$. Since ξ is a limit ordinal, there is $\eta < \xi$, such that $f_1, \ldots, f_k \in L_\eta$; Since (M_η, L_η) is F-preconjugate, $\{x \mid L_\eta : x \in M_\eta\}$ is dense in $\{x \mid L_\eta : x \in K\}$. Hence, there is $y \in M_\eta \subset M$, with $|f_i(x) - f_i(y)| < \varepsilon$ for $i = 1, \ldots, k$. This verifies condition (a) of an F-preconjugate pair; condition (b) is verified analogously.

6.17. LEMMA. *Let $F \subset C(K)$, and suppose that (M, L) is an F-preconjugate pair. We set $P = \mathrm{cl}_K M$, $Q = \mathrm{cl}_F L$. Then*

$$\{x \mid Q : x \in P\} = \{x \mid Q : x \in K\}$$

PROOF. Let $x \in K$. For $A \in \mathcal{P}_\omega(L)$, there is $x_A \in M$ such that

$$|f(x) - f(x_A)| < 1/|A| \quad \text{for } f \in A.$$

(since (M, L) is an F-preconjugate pair). The net $\{X_A : A \in \mathcal{P}_\omega(L)\} \subset M$ has a limit point y in P (by the compactness of K). It is then easy to verify that $x \mid L = y \mid L$, and by continuity of x, y (considered as functions on F), we have $x \mid Q = y \mid Q$.

6.18. LEMMA. *Let $F_n \subset C(K)$ for $n < \omega$, $A \subset K$, $B \subset \bigcup_{n<\omega} F_n = F$, τ an infinite cardinal, and $|A| \leqslant \tau$, $|B| \leqslant \tau$. Then there are sets $A \subset M \subset K$, $B \subset L \subset F$, with $|M| \leqslant \tau$, $|L| \leqslant \tau$, such that $(M, L \cap F_n)$ is F_n-preconjugate pair for $n < \omega$.*

PROOF. Let $\omega = \bigcup_{n<\omega} D_n$ be a partition of ω, with D_n infinite for $n < \omega$. We define pairs (M_k, L_k) for $k < \omega$, such that

$$M_0 \subset \cdots \subset M_k \subset M_{k+1} \subset \cdots, \quad |M_k| \leqslant \tau$$

$$L_0 \subset \cdots \subset L_k \subset L_{k+1} \subset \cdots, \quad |L_k| \leqslant \tau,$$

and if $k \in D_n$, then $(M_k, L_k \cap F_n)$ is an F_n-preconjugate pair for $k, n < \omega$. We proceed inductively using Lemma 6.15. For example, let n_0 be the unique natural number such that $0 \in D_{n_0}$. By Lemma 6.15, there is M_0, L_0 such that

$$A \subset M_0 \subset K\,, \qquad |M_0| \leqslant \tau\,,$$
$$B \cap F_{n_0} \subset L_0 \subset F_{n_0}\,, \quad |L_0| \leqslant \tau\,,$$

and $(M_0, L_0 \cap F_{n_0})$ an F_{n_0}-preconjugate pair.

The rest of the induction proceeds analogously. We set

$$M = \bigcup_{k<\omega} M_k\,, \qquad L = \bigcup_{k<\omega} L_k$$

Let $n < \omega$. Since $M = \bigcup_{k \in D_n} M_k$, $L = \bigcup_{k \in D_n} L_k$, and D_n is infinite, it follows from Lemma 6.16 that $(M, L \cap F_n)$ is an F_n-preconjugate pair.

6.19. LEMMA. *Let K be a Gul'ko compact space, S the unit ball of $C(K)$ in pointwise topology, S embedded homeomorphically in $[-1, 1]^K$ by the natural embedding $f \to (f(x), x \in K)$, and let $\{K_n: n < \omega\}$ be a sequence of compact subsets of $[-1, 1]^K$, closed under finite intersections, and such that*

(*) *for every $f \in S$, there is $\emptyset \neq N_f \subset \omega$, with $f \in \bigcap_{nN_f} K_n \subset S$.*

Let $M \subset K$, $L \subset \bigcup_{n<\omega}(K_n \cap S)$ be such that

(**) *$(M, L \cap K_n \cap S)$ is a $(K_n \cap S)$-preconjugate pair for $n < \omega$,*

and set $P = \mathrm{cl}_K M$, $Q = \mathrm{cl}_S L$. Then (P, Q) is an S-conjugate pair.

PROOF. Let $f \in S$. If $n \in N_f$, we set

$$V(f, n) = \{g \in \mathrm{cl}_{[-1,1]^K}(K_n \cap Q): g\,|\,M = fM\} \subset K_n \subset [-1, 1]^K\,;$$

Since $(M, L \cap K_n \cap S)$ is a $(K_n \cap S)$-preconjugate pair, and $f \in K_n \cap S$, there is a net $(g_i)_{i \in I} \subset L \cap K_n \cap S \subset K_n \cap Q$, such that $\lim_{i \in I} g_i\,|\,M = f\,|\,M$; if g is a limit point of this net, then it is easy to see that $g \in V(f, n)$, so that $V(f, n)$ is a non-empty compact subset of $[-1, 1]^K$.

The family $\{V(f, n): n \in N_f\}$ has the finite intersection property, since if $A \subset D_f$ A finite, and $\bigcap_{n \in A} K_n = K_m$ (since the K_n's are closed under finite intersections), then

$$V(f, m) \subset \bigcap_{n \in A} V(f, n)\,.$$

We set

$$V_f = \bigcap_{n\in D_f} V(f,n) \subset \bigcap_{n\in D_f} K_n \subset S$$

and we conclude by these remarks that V_f is non-empty, compact, and by (*) a subset of S. Hence clearly from the definition of V_f, $V_f \subset Q$; this means precisely that there is $g \in V_f \subset Q$, with $g\,|\,M = f\,|\,M$, and by the continuity of f and g, with $g\,|\,P = f\,|\,P$. Thus we have verified condition (b) for an S-conjugate pair.

By (**) and Lemma 6.17 it follows that

$$\{x\,|\,Q\cap K_n\colon x\in P\} = \{x\,|\,Q\cap K_n\colon x\in K\} \quad \text{for } n<\omega\,.$$

From (*), which implies that $\bigcup_{n<\omega} K_n \supset S$, it follows that

$$\{x\,|\,Q\colon x\in P\} = \{x\,|\,Q\colon x\in K\}\,,$$

condition (a) for an S-conjugate pair. The proof of the lemma is now complete.

6.20. LEMMA. *If K is a Gul'ko compact space, and S in the unit ball of $C(K)$ in the pointwise topology, then $w(K) = d(K) = d(S)$.*

PROOF. We omit the elementary, but somewhat delicate proof of this lemma, and refer instead to TALAGRAND [116] (Th. 6.2, Prop. 6.3).

6.21. REMARKS. According to a result of CORSON [37], if for a compact space K, $C(K)$ in its pointwise topology is Lindelöf, then K has *countable tightness* (i.e., if $A \subset K$, and $x \in \bar{A}$, then there is a countable subset C of A, with $x \in \bar{C}$). It follows easily that every Gul'ko-compact space is angelic.

6.22. LEMMA. *Let K be a Gul'ko compact space, of weight τ, and let $\{x_\xi\colon \omega\leqslant\xi<\tau\}$ be a dense subset of K, and $\{f_\xi\colon \omega\leqslant\xi<\tau\}$ a dense subset of the unit ball S of $C(K)$ in the pointwise topology. Then there is a family $\{(P_\xi, Q_\xi)\colon \omega\leqslant\xi<\tau\}$ such that*

(1) *(P_ξ, Q_ξ) is S-conjugate pair for $\omega\leqslant\xi<\tau$.*

(2) *$P_\xi \subset P_\zeta$, $Q_\xi \subset Q_\zeta$ for $\omega\leqslant\xi<\tau$.*

(3) *If ξ is a limit ordinal, $\omega<\xi<\tau$, then $P_\xi = \mathrm{cl}_K(\bigcup_{n<\xi} P_n)$ and $Q_\xi = \mathrm{cl}_S(\bigcup_{n<\xi} Q_\eta)$.*

(4) *$x_\xi \in P_{\xi+1}$, $f_\xi \in Q_{\xi+1}$ for $\omega\leqslant\xi<\tau$.*

(5) *$w(P_\xi) = d(P_\xi) = d(Q_\xi) \leqslant |\xi|$ for $\omega\leqslant\xi<\tau$.*

(6) *$\lim_{\eta<\xi} p_\eta(x) = p_\xi(x)$ for $x\in K$ and ξ a limit ordinal, $\omega<\xi<\tau$, where p_ξ is the retraction from K onto P_ξ given by Lemma 6.12.*

(7) *For every $x\in K$ the set $\{\xi\colon \omega\leqslant\xi<\tau, \text{ and } p_\xi(x)\neq p_{\xi+1}(x)\}$ is countable.*

PROOF. A family $\{(P_\xi, Q_\xi)\colon \omega \leqslant \xi < \tau\}$ satisfying properties 1 to 5 is easily constructed, using Lemmas 6.12, 6.13, 6.16, 6.18, 6.19, 6.20. We verify property (6). Let $x \in K$, and ξ a limit ordinal $\omega < \xi < \tau$. Let y be a limit point of the net $(p_\eta(x))_{\eta<\xi}$, $\varepsilon > 0$, and $f \in Q_\xi = \mathrm{cl}_S(\bigcup_{\eta<\xi} Q_\eta)$. There is $\eta < \xi$ and $f_\eta \in Q_\eta$ such that

$$|f(x) - f_\eta(y)| \leqslant \varepsilon/3 \quad \text{and} \quad |f(y) - f_\eta(y)| \leqslant \varepsilon/3\,.$$

We next choose ζ, with $\eta < \zeta < \xi$, such that

$$|f_\eta(p_\zeta(x)) - f_\eta(y)| \leqslant \varepsilon/3$$

(using the fact that y is a limit point of the net $(p_{\xi'}(x)_{\xi'<\xi})$.

We note that $f_\eta(p_\zeta(x)) = f_\eta(p_\eta p_\zeta(x)) \times f_\eta(p_\eta(x)) = f_\eta(x)$ (using Lemma 6.13). Hence, we conclude that

$$|f(y) - f(x)| \leqslant \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, we have $f(y) = f(x)$. The uniqueness of p_ξ given from Lemma 6.12, and the compactness of K imply that $p_\xi(x) = \lim_{\eta<\xi} p_\eta(x)$.

To prove (7) suppose there is $x \in K$, such that the set $A = \{\xi\colon p_\xi(x) \neq p_{\xi+1}(x)\}$ is uncountable. There is then $B \subset A$, B uncountable, $p_\xi(x) \neq p_\zeta(x)$ for all $\xi, \zeta \in B$, $\xi \neq \zeta$. Let $\eta = \sup B \leqslant \tau$. It is easy to see that we can choose B and, so that $\mathrm{cf}(\eta) \geqslant \omega^+$. (If $\mathrm{cf}(\eta) = \omega$, and $n > \omega^+$, then there is η_0 with $\omega^+ \leqslant \eta_0 < \eta$, so that $\mathrm{cf}(\eta_0) \geqslant \omega^+$, and $\eta_0 \cap B$ is uncountable; we then choose η_0 instead of η.) It follows from property (6) that $p_\eta(x) = \lim_{\xi\in B} p_\xi(x)$. From Remark 6.21, there is $\{\xi_n\colon n < \omega\} \subset B$, with $\lim_{n<\omega} p_{\xi_n}(x) = p_\eta(x)$. Then, for $\zeta = \sup_{n<\omega} \xi_n$, we have that $p_\zeta(x) = p_\xi(x) = p_\eta(x)$ for $\xi \in B$ and $\zeta \leqslant \xi < \eta$, a contradiction.

6.23. THEOREM. *If K is a Gul'ko compact space, then K is a Corson-compact space.*

PROOF. The proof proceeds by induction on the weight of K. A (Gul'ko) compact space of countable weight is metrizable, and hence Corson-compact.

Let τ be an uncountable cardinal, and suppose that the theorem holds for all cardinals $\alpha < \tau$. Let K be a Gul'ko compact space of weight τ, and let $\{(P_\xi, Q_\xi)\colon \omega \leqslant \xi < \tau\}$ be a family with the properties of Lemma 6.22 for K. By the inductive assumption, there is a homeomorphism $\phi_\xi\colon p_\xi(K) \to (\mathbb{R}^{\Gamma_\xi})$ for some sets Γ_ξ, $\omega \leqslant \xi < \tau$, with $\Gamma_\omega = \omega$, and $\{\Gamma_\xi\colon \omega \leqslant \xi < \tau\}$ pairwise disjoint. We set $\Gamma = \omega \cup \bigcup\{\Gamma_{\xi+1}\colon \omega \leqslant \xi < \tau\}$, and define $\phi\colon K \to \Sigma(\mathbb{R}^\Gamma)$ by

$$\phi(x)(n) = \phi_\omega(P_\omega(x))(n) \quad \text{for } n < \omega\,,$$

$$\phi(x)(\gamma) = \frac{1}{2}(\phi_{\xi+1}(P_{\xi+1}(x))(\gamma) - \phi_{\xi+1}(P_\xi(x))(\gamma)) \quad \text{for } \gamma \in \Gamma_{\xi+1},\ \omega \leqslant \xi < \tau\,.$$

We verify without difficulty using Lemma 6.22 that ϕ is a well-defined homeomorphic embedding.

REMARK. In the stated form, Theorem 6.23 has been proved by GUL'KO [56]. Most of the concepts and arguments needed for its proof were given somewhat earlier in VASAK [122], in functional analytic form.

For $J \subset \Gamma$, we set $r_J : \mathbb{R}^\Gamma \to \mathbb{R}^J$ by

$$\begin{aligned} r_J(x)_\gamma &= x_\gamma \quad \text{if } \gamma \in J \\ &= 0 \quad \text{if } \gamma \in \Gamma \setminus J . \end{aligned}$$

Then r_J is a continuous retraction of $\mathbb{R}^\Gamma$ onto $\mathbb{R}^J$ (or more precisely onto $\mathbb{R}^J \times \{0\}^{\Gamma \setminus J}$). If $K \subset \mathbb{R}^\Gamma$ and $J \subset \Gamma$, then J is called *K-good* if $r_J(K) \subset K$.

6.24. LEMMA. (BENYAMINI [27]). *Let K be a compact subset of $\Sigma(\mathbb{R}^\Gamma)$ and I an infinite subset of Γ. Then there is J with $I \subset J \subset \Gamma$, $|I| = |J|$, and J K-good.*

PROOF. We define inductively suitable sets $I = J_0 \subset J_1 \subset \cdots \subset J_n \subset \cdots$ for $n < \omega$, with $|J_n| = |I|$, and set $J = \bigcup_{n<\omega} J_n$. Suppose J_n has been defined. We note that $r_{J_n}(K)$ has density character at most $|I|$, so let $x_\alpha^{(n)}$, $\alpha < |I|$, be elements of K, with $\{r_{J_n}(x^{(n)}) : \alpha < |I|\}$ dense in $r_{J_n}(K)$. Set $J_{n+1} = J_n \cup \bigcup\{\operatorname{supp}(x_\alpha^{(n)}) : \alpha < |I|\}$. The compactness of K now easily implies the lemma.

6.25. LEMMA. *Let K be a compact subset of $\Sigma(\mathbb{R}^\Gamma)$, with $\Gamma = \bigcup\{\operatorname{supp}(x) : x \in K\}$. Then there is a family $\{K_\xi : \omega \leqslant \xi \leqslant |\Gamma|\}$ of closed subsets of K, and $r_\xi : K \to K_\xi$ continuous retractions for $\omega \leqslant \xi \leqslant |\Gamma|$ such that*

(1) *$r_\xi \circ r_\zeta = r_\zeta \circ r_\xi = r_\zeta$ for $\omega \leqslant \zeta < \zeta \leqslant |\Gamma|$.*

(2) *$w(K_\xi) = d(K_\xi) \leqslant |\xi|$ for $\omega \leqslant \xi \leqslant |\Gamma|$.*

(3) *If ξ is a limit ordinal, $\omega < \xi \leqslant |\Gamma|$, then $K_\xi = \overline{\bigcup_{\eta<\xi} K_\eta}$ and $\lim_{n<\xi} r_\eta(x) = r_\xi(x)$ for all $x \in K$.*

(4) *$K_{|\Gamma|} = K$, and $r_{|\Gamma|}$ is the identity on K.*

(5) *$\{\xi : \omega \leqslant \xi \leqslant |\Gamma|, \|r_\xi^*(f) - r_{\xi+1}^*(f)\| \geqslant \varepsilon\}$ is finite for every $f \in C(K)$, $\varepsilon > 0$ (where $r_\xi^* : C(K) \to C(K)$ is given by $r_\xi^*(f) = f \circ r_\xi$).*

The existence of K_ξ, r_ξ, $\omega \leqslant \xi \leqslant |\Gamma|$, satisfying properties (1) to (4) of this lemma is easily proved by transfinite induction, using Lemma 6.24.

The simple verification of property (5) (using the fact that every Corson-compact space is angelic) is left to the reader.

REMARK. If K is a compact subset of $\Sigma(\mathbb{R}^\Gamma)$ and $J \subset \Gamma$ with $r_J(K) \subset K$, we set $F = \{r_{\{\gamma\}} = \pi_\gamma : \gamma \in \Gamma\}$, $M = r_J(K)$, $L = \{r_{\{\gamma\}} : \gamma \in \Gamma\}$, and we note that the pair (M, L) is F-conjugate.

6.26. THEOREM (GUL'KO [54]). *The Hausdorff continuous image of a Corson-compact space is Corson-compact.*

The proof is based on lemmas similar to Lemmas 6.24, 6.25 above, where the sets J are also 'good' with respect to the given continuous function $\phi: K \to L$.

6.27. THEOREM [12, 13]. *Let K be a Corson-compact space. Then there are a set Γ and a one-to-one bounded linear operator $T: C(K) \to c_0(\Gamma)$.*

PROOF. The proof proceeds by induction on the weight of K. A (Corson-) compact space K of countable weight is metrizable, and in this case certainly such an operator exists (with $\Gamma = \omega$).

Let τ be an uncountable cardinal, and suppose that the theorem holds for all cardinals $\alpha < \tau$. Let K be a Corson-compact space of weight τ, so we may assume $K \subset \Sigma(\mathbb{R}^\tau)$, with $\tau = \bigcup\{\mathrm{supp}(x): x \in K\}$. Let K_ξ, r_ξ, $\omega \leqslant \xi \leqslant \tau$, be given as in Lemma 6.25. By the inductive assumption there is a 1–1, bounded, linear operator $T_\xi: C(K_\xi) \to c_0(\Gamma_\xi)$ for some set Γ_ξ, with $\|T_\xi\| \leqslant 1$. We may assume that the sets Γ_ξ are pairwise disjoint (and disjoint with ω) and we set $\Gamma = \omega \cup \bigcup\{\Gamma_{\xi+1}: \omega \leqslant \xi < \tau\}$. We set $T: C(K) \to c_0(\Gamma)$ by

$$T(f)(n) = T_\omega(r_\omega^* f)(n) \qquad \text{for } n < \omega\,,$$

$$T(f)(\gamma) = \frac{1}{2} T_{\xi+1}(r_{\xi+1}^* f - r_\xi^* f)(\gamma) \quad \text{for } \gamma \in \Gamma_{\xi+1},\ \omega \leqslant \xi < \tau\,.$$

It follows without difficulty, from properties (1) to (5) of Lemmas 6.25 that T has the required properties.

REMARK. The operator T given in Theorem 6.27 is also continuous if $C(K)$ and $c_0(\Gamma)$ have the pointwise topology.

The Amir–Lindenstrauss Theorem and other consequences

6.28. DEFINITION. A norm $\|\cdot\|$ of a Banach space X is *strictly convex* if for every $x, y \in X$, with $\|x\| = \|y\| = 1$, we have $\|(x+y)/2\| < 1$.

A Banach space X is *strictly convexifiable* if there is a strictly convex norm $|||\cdot|||$ on X equivalent to the original norm of X (i.e., such that the identity operator $(X, \|\cdot\|) \to (X, |||\cdot|||)$ is an isomorphism).

6.29. COROLLARY [12, 13]. *If K is a Corson-compact space, then $C(K)$ is strictly convexifiable.*

PROOF. According to a classical result by DAY [41], $c_0(\Gamma)$ is strictly convexifiable with a norm, say $|\cdot|$. Let $T: C(K) \to c_0(\Gamma)$ be a 1–1, linear bounded operator, given by Theorem 6.27. Then $|||x||| = \|x\| + |T(x)|$ for $x \in C(K)$ defines an equivalent strictly convex norm on $C(K)$.

6.30. REMARK. Following a technique of TROYANSKI [120], there is on $C(K)$ for K Corson-compact, an equivalent locally uniformly convex norm. For various renorming theorems we refer the reader to DIESTEL [42] (Chapters 4 and 5).

6.31. COROLLARY (AMIR–LINDENSTRAUSS [4]). *If X is a* W.C.G. *Banach space, then there is a set Γ and a one-to-one, linear bounded operator $T: X \to c_0(\Gamma)$ (and hence X is strictly convexifiable).*

PROOF. By 6.9(c), S_{X^*} is Eberlein-compact in the weak* topology, hence by Theorems 6.10, 6.3(b), 6.23 a Corson compact space. Since X is isometric to a subspace of $C(S_{X^*})$ the result follows from Theorem 6.37 and Corollary 6.29.

6.32. REMARK. The proof of the Amir–Lindenstrauss theorem does not require all the sequence of Lemmas leading to the proof of Theorem 6.23. The reader will be able to construct a direct proof using only Lemmas 6.12, 6.13, 6.15, 6.16, 6.17 above. The original proof of the fundamental result by AMIR–LINDENSTRAUSS [4] was functional analytic.

6.33. COROLLARY [4]. *If K is an Eberlein-compact space, then there is a set Γ such that K is homeomorphic to a subset of $c_0(\Gamma)$ in the weak topology (which coincides with the pointwise topology).*

PROOF. If K is an Eberlein-compact space, then K is homeomorphism to a subset of a Banach space X. We assume without loss of generality that X is W.C.G. (replacing X if necessary with the subspace of X generated by K). By Corollary 6.31 there is a bounded linear 1–1 operator $T: X \to c_0(\Gamma)$. Then T is (weak, weak)-continuous as well, and hence $T|K$ is a homeomorphic embedding of K into $c_0(\Gamma)$ with the weak topology. (The fact that the weak and the pointwise topology coincide on $c_0(\Gamma)$ follows form the fact, mentioned below in 7.4, that the one-point compactification of Γ is a scattered space).

6.34. THEOREM (BENYAMINI–RUDIN–WAGE [28]). *The Hausdorff continuous image of an Eberlein-compact is Eberlein-compact.*

We omit a proof of this theorem. The technique of a proof due to GUL'KO [54], or BENYAMINI [27], follows the same ideas as the proof of the theorem (6.26) that the continuous image of a Corson-compact is Corson-compact, with $\Sigma(\mathbb{R}^\Gamma)$ replaced by its subset of $c_0(\Gamma)$, in pointwise topology. For still another proof, see NEGREPONTIS–TSARPALIAS [92], MICHAEL–RUDIN [86].

6.35. COROLLARY [4]. *If X is a* W.C.G. *Banach space, then there is a* 1–1 *bounded linear operator $T: X^* \to c_0(\Gamma)$, such that T is also (weak*, weak)-continuous. In particular X^* is strictly convexifiable.*

PROOF. There is (by Theorem 6.9(a)) an Eberlein-compact K, and a 1–1 linear bounded (weak*, weak)-continuous operator $F: X^* \to C(K)$. Since every Eberlein-compact is Corson-compact, it follows from Theorem 6.27, that there is a 1–1 linear bounded operator $G: C(K) \to c_0(\Gamma)$. The composition $T = G \circ F$ satisfies the requirement.

6.36. COROLLARY [4]. *Let X be a* W.C.G. *Banach space. Then there is a subset K of X, such that*

K generates X, and

K in its weak topology is homeomorphic to the one-point compactification of a discrete set.

PROOF. Let $T: X^* \to c_0(\Gamma)$ be a 1–1, bounded linear, (weak*, weak)-continuous operator. The dual operator T^* maps the dual $l^1(\Gamma)$ of $c_0(\Gamma)$ into X, since T is (weak*, weak)-continuous, and since T is 1–1, T has dense range in X. Let $\{e_\gamma: \gamma \in \Gamma\}$ be the elements of the usual basis of $l^1(\Gamma)$. Then $K = \{T^*(e_\gamma): \gamma \in \Gamma\} \cup \{0\} \subset X$, has the required property.

6.37. COROLLARY (ROSENTHAL'S characterization of Eberlein-compact spaces [103]). *A compact Hausdorff space K is Eberlein-compact if and only if there is a sequence $\{\mathscr{B}_n: n < \omega\}$ such that*

(a) *If $n < \omega$ and $B \in \mathscr{B}_n$, then B is an open F_σ-set of K.*

(*b*) *If $x, y \in K$ and $x \neq y$, then there is $n < \omega$ and $B \in \mathscr{B}_n$ such that either $x \in B$ and $y \notin B$, or $x \notin B$; and $y \in B$.*

(*c*) *If $x \in K$ and $n < \omega$, then x belongs to a finite number of sets in $\mathscr{B}_n$ (i.e., each $\mathscr{B}_n$ is point-finite).*

We omit the proof of this theorem. A similar characterization, with (c) replaced by condition

(c)′ If $x \in K$ and $n < \omega$, then there is $N_n < \omega$ such that x belongs to at most N_n elements of $\mathscr{B}_n$

holds for uniform Eberlein-compact spaces (cf. BENYAMINI–STARBIRD [29])).

6.38. *Dense subsets.* (a) Every Eberlein-compact space has a dense metrizable G_δ-set (BENYAMINI–RUDIN–WAGE [28], NAMIOKA [89]).

(b) Every Talagrand-compact space has a dense G_δ-set consisting of G_δ-points (TALAGRAND [117]).

It is an open question whether every Talagrand-compact space has a dense metrizable subset.

(c) Every Corson-compact space has a dense subspace with point-countable basis (SAPIROVKSKII [109]).

If we assume the continuum hypothesis, then the Kunen–Haydon–Talagrand space of Section 5 is a Corson-compact space that does not have a dense metrizable subset.

Gul'ko-compact spaces and ccc

NOTATIONS. We denote by Σ the Baire space ω^ω. S denotes the set of finite sequences of natural numbers. For $s \in S$ we denote by $|s|$ the length (i.e., the domain) of s. For $s, t \in S$, we write $s < t$ if s is equal to the first $|s|$ terms of the sequence t. If $\sigma \in \Sigma$, and $n < \omega$, $\sigma | n$ denotes the finite sequence of the first n terms of σ. Is $s \in S$, and $n < \omega$, $s \frown n$ denotes the finite sequence of length $|s| + 1$, whose first $|s|$ terms is s, and whose last term is n.

6.39. LEMMA. (a) *A topological space X is $\mathcal{K}$-countably determined if and only if, there are a subset Σ' of the space Σ of irrationals, a compact Hausdorff space K, and a closed subset F of $K \times \Sigma'$ such that (denoting by π_1: $K \times \Sigma' \to S$ the projection $\pi_1(x, y) = x$) $X = \pi_1(F)$.*

(b) *Let X be a $\mathcal{K}$-countably determined space. There is a family $\{A_s : s \in S\}$ of subsets of X such that*

$$A_\emptyset = X, \qquad \bigcup_{k<\omega} A_{s\frown k} = A_s \quad \text{for } s \in S,$$

and for every $x \in X$ there is $\sigma \in \Sigma$ such that (i) $x \in \bigcap_{k<\omega} A_{\sigma|k}$ *and* (ii) *if $x_k \in A_{\sigma|k}$ for $k < \omega$, then the sequence (x_k) has a limit point in X.*

PROOF. For the proof of (a) the reader is referred to TALAGRAND [116]

(b) There are $\Sigma' \subset \Sigma$, a compact space K, and a closed subset F of $K \times \Sigma'$ such that $\pi_1(F) = X$. There is a family $\{B_s : s \in S\}$ of subsets of Σ' such that

$$B_\emptyset = \Sigma', \qquad \bigcup_{k<\omega} B_{s\frown k} = B_s \quad \text{for } s \in S,$$

$\{B_s : |s| = n\}$ is a family of open-and-closed pairwise disjoint subsets of Σ', $\operatorname{diam}(B_s) < 1/|s|$ for $s \in S$, and

$$\bigcap_{k<\omega} B_{\sigma|k} = \{\sigma\} \quad \sigma \in \Sigma',$$
$$= \emptyset \qquad \sigma \in \Sigma \setminus \Sigma'.$$

We set $A_s = \pi_1(\pi_2^{-1}(B_s) \cap F)$ for $s \in S$. If $x \in X$, than there is $\sigma \in \Sigma'$ such that $(x, \sigma) \in F$. It is clear that

$$x \in \bigcap_{k<\omega} A_{\sigma|k} .$$

Let $x_k \in A_{\sigma|k}$ for $k < \omega$. Then there is $\sigma_k \in \Sigma'$ such that $(f_k, \sigma_k) \in \pi_2^{-1}(B_{\sigma|k}) \cap F$, hence $\sigma_k \in B_{\sigma|k}$ for $k < \omega$. Hence $\sigma_k \to \sigma$. Since K is compact and $X \subset K$ there is a limit point $y \in K$ of the sequence (x_k). Since F is closed in $K \times \Sigma'$, it is clear that $(y, \sigma) \in F$, and so $y \in F$.

6.40. THEOREM (ARGYROS–NEGREPONTIS [15]). *If K is a Gul'ko compact space, then $S(K) = w(K)^+$ (where $S(K)$ is the Souslin number of K). In particular, a* ccc *Gul'ko-compact space is metrizable.*

PROOF. For simplicity we assume that K is a ccc Gul'ko compact space. By Theorem 6.23, every Gul'ko-compact space is Corson-compact. Suppose that K is non-metrizable. Since the continuous image of a Gul'ko-compact space is (obviously) Gul'ko-compact, we may assume that the weight of K is exactly ω^+. Thus, $K \subset \Sigma[0, 1]^{\omega^+}$. Furthermore, we assume without loss of generality that $\pi_\xi \mid K \neq 0$ for all $\xi < \omega^+$, and in fact that there is θ, $0 < \theta < 1$, such that

$$V_\xi = (\pi_\xi \mid K)^{-1}(\theta, 1] \neq \emptyset \quad \text{for } \xi < \omega^+$$

(where $\pi_\xi \colon [0, 1]^{\omega^+} \to [0, 1]$ denotes the ξth projection).

Let X be the unit sphere of $C(K)$. i.e., $X = \{f \in C(K) \colon \|f\| \leqslant 1\}$. Since $C(K)$ is weakly $\mathscr{K}$-countably determined, X is too, and so there is a family $\{A_s \colon s \cup S\}$ of subsets of X satisfying the properties of Lemma 6.39(b). We set

$$\Pi = \{\pi_\xi \colon \xi < \omega^+\} \subset X = A_\emptyset , \quad \text{and} \quad B_s = \{\xi < \omega^+ \colon \pi_\xi \in A_s\} \quad \text{for } s \in S .$$

For every $s, t \in S$, with $|s| = |t|$ we define x_s^t, ξ_ξ^t, V_s^t, C_s^t such that

(i) For every $m < \omega$,

$$\bigcup_{p<\omega} V_{\widehat{s,m}}^{\widehat{t,p}} \quad \text{is dense } n \quad \bigcup\{V_\xi \cap V_s^t \colon \xi \in C_s^t \quad B_{\widehat{s,m}}\} .$$

(ii) V_s^t is open in K,

$$V_{s'}^{t'} \subset V_s^t \quad \text{for } t' > t,\ s' > s , \quad \text{and} \quad x_s^t \in V_s^t \quad \text{if } V_s^t \neq \emptyset .$$

(iii) $C_s^t \subset B_s$, $C_s^t \cap \operatorname{supp}(x_s^t) = \emptyset$,

$$C_{s'}^{t'} \subset C_s^t \quad \text{for } t' > t,\ s' > s , \qquad C_\emptyset^\emptyset = \omega^+ ,$$

$$\bigcup_{p,m<\omega} C_{\widehat{s,m}}^{\widehat{t,p}} \supset C_s^t \setminus \bigcup\{\operatorname{supp}(x_{s'}^{t'}) \colon |t'| = |s'| \leqslant |t| + 1\} ,$$

and if $\xi \in C_s^t$, then $V_\xi \cap V_s^t \neq \emptyset$.

(iv) If $x \in V_s^t$, then $\pi_{\xi_s^t}(x) > \theta$.

(v) $\xi_{\widehat{s,m}}^{\widehat{t,p}} \in C_s^t \cap B_{\widehat{s,m}}$ if $C_s^t \cap B_{s,m} \neq \emptyset$.

We proceed inductively; assume that $n < \omega$, and that for every $s, t \in S$, with $|s| = |t| \leq n$, x_s^t, ξ_s^t, V_s^t, C_s^t have been defined and they satisfy (i) through (v). Let $t, s \in S$, with $|t| = |s| = n$;

Since K satisfies ccc there are $\xi_{\widehat{s,m}}^{\widehat{t,p}} \in C_s^t \cap B_{\widehat{s,m}}$ for $p < \omega$ and $m < \omega$, such that

$$\bigcup_{p<\omega} (V_{\xi_{\widehat{s,m}}^{\widehat{t,p}}} \cap V_s^t) \quad \text{is dense in} \quad \bigcup\{V_\xi \cap V_s^t : \xi \in C_s^t \cap B_{\widehat{s,m}}\}.$$

We set $V_{\widehat{s,m}}^{\widehat{t,p}} = V_s^t \cap V_{\xi_{\widehat{s,m}}^{\widehat{t,p}}}$ for p, $m < \omega$. If $V_{\widehat{s,m}}^{\widehat{t,p}} \neq \emptyset$, we choose $x_{\widehat{s,m}}^{\widehat{t,p}} \in V_{\widehat{s,m}}^{\widehat{t,p}}$ (and otherwise, we choose $x_{s,\,m}^{t,\,p} \in K$, arbitrarily). We set (for p, $m < \omega$)

$$C_{\widehat{s,m}}^{\widehat{t,p}} = \{\xi \in C_s^t \cap B_{\widehat{s,m}} \setminus \operatorname{supp}(x_{\widehat{s,m}}^{\widehat{t,p}}) : V_{\widehat{s,m}}^{\widehat{t,p}} \cap V_\xi \neq \emptyset\}.$$

We note that if $x \in V_{\widehat{s,m}}^{\widehat{t,p}}$, then $\pi_{\xi_{\widehat{s,m}}^{\widehat{t,p}}}(x) > \theta$, since $V_{\widehat{s,m}}^{\widehat{t,p}} \subset V_{\xi_{\widehat{s,m}}^{\widehat{t,p}}}$.

The recursive definitions are complete; it is clear that they satisfy properties (i) through (v).

We choose $\xi \in \omega^+ \setminus \bigcup\{\operatorname{supp}(x_s^t) : t, s \in S |t| = |s|\}$. By the conditions of Lemma 6.39(b) for the family $\{A_s : s \in S\}$, there is $\sigma \in \Sigma$, such that

$$\pi_\xi \in \bigcap_{k<\omega} A_{\sigma|k},$$

and if $f_k \in A_{\sigma|k}$ for $k < \omega$, then the sequence (f_k) has a weak limit point in X. From property (iii), we choose inductively natural numbers $t_0, \ldots, t_k, \ldots$ for $k < \omega$, such that

$$\xi \in C_{s(0),\ldots,\sigma(k)}^{t_0,\ldots,t_k} \quad \text{for } k < \omega.$$

Setting $\tau = (t_0, t_1, \ldots, t_k, \ldots) \in \Sigma$, we have that

$$\xi \in \bigcap_{k<\omega} C_{\sigma|k}^{\tau|k}.$$

By property (iii), $V_{\sigma|k}^{\tau|k} \neq \emptyset$ for $k < \omega$; and hence by property (ii), $x_{\sigma|k}^{\tau|k} \in V_{\sigma|k}^{\tau|k}$ for $n < \omega$. Since K is Corson-compact, it is clear that there is $x \in K$, and a subsequence of $(x_{\sigma|k}^{\tau|k})$ (which for convenience we denote also by $(x_{\sigma|k}^{\tau|k})$), such that

$$x_{\sigma|k}^{\tau|k} \to x.$$

By property (v), we have that $\pi_{\xi_{\sigma|k}^{\tau|k}} \in A_{\sigma|k}$ for $k < \omega$. By the conditions of Lemma 6.39(b) for the family $\{A_s : s \in S\}$ and Eberlein's Theorem for weak convergence (6.5), there is $g \in X$, and a subsequence of $(\pi_{\xi_{\sigma|k}^{\tau|k}})$ which for convenience we denote also by $(\pi_{\xi_{\sigma|k}^{\tau|k}})$ such that

$$\pi_{\xi_{\sigma|k}^{\tau|k}} \to g \quad \text{pointwise}\,.$$

We note that $g \in X \subset C(K)$, i.e., that g is a continuous function. From property (iv), we have that

$$\pi_{\xi_{\sigma|l}^{\tau|l}}(x_{\sigma|k}^{\tau|k}) > \theta \quad \text{for all } l > k\,,$$

hence, by the pointwise convergence, $g(x_{\sigma|k}^{\tau|k}) \geqslant \theta$ for all $k < \omega$, and hence by continuity of g, $g(x) \geqslant \theta > 0$. From properties (iii) and (v), we have that

$$\pi_{\xi_{\sigma|l}^{\tau|l}}(x_{\sigma|k}^{\tau|k}) = 0 \quad \text{for all } l < k\,,$$

hence by continuity of the projections, and convergence of $x_{\sigma|k}^{\tau|k} \to x$, we have that $\pi_{\xi_{\sigma|l}^{\tau|l}}(x) = 0$ for all $l < \omega$; hence by the pointwise convergence, we have that $g(x) = 0$. This contradiction completes the proof.

REMARK. For Eberlein-compact spaces, this is a result of BENYAMINI–RUDIN–WAGE [28]. Theorem 6.40 answers in the positive a question of TALAGRAND [115], [116].

6.41. *'Pathological' Corson-compact* ccc *Spaces* (Argyros–Mercourakis–Negrepontis). Assume the continuum hypothesis.

(a) There is a Corson-compact space K with ccc, such that $K \times K$ does not have ccc.

(b) For every $n \geqslant 2$ there is a Corson-compact space K with property (K_n) and not (K_{n+1}).

(c) There is a Corson-compact space K with property $(*)$, but without any strictly positive measure.

Any of the spaces K in (a), (b), (c) proves that, assuming CH,

(d) There is a (Corson-)compact space K, such that $C(K)$ is strictly convexifiable, and K does not have a strictly positive measure.

It is an open problem to characterize those compact space K such that $C(K)$ is strictly convexifiable (DASHIELL–LINDENSTRAUSS [40]).

For detailed definitions, and constructions of (a), (b), (c) and other 'pathological' Corson-compact ccc spaces the reader is referred to [35], [12], and [13].

6.42. THEOREM (ALSTER [2]). *A Corson-compact scattered space is Eberlein-compact.*

6.43. THEOREM (ALSTER–POL [3], GUL'KO [55]). *If K is a Corson-compact space, then $C(K)$ is Lindelöf in the pointwise topology.*

We note that if K is Corson-compact, then $C(K)$ need not be Lindelöf in the weak topology: In fact, assuming CH, the Kunen–Haydon–Talagrand space K of Section 5 is a Corson-compact, such that $C(K)$ is not Lindelöf in the weak topology.

6.44. THEOREM (MERCOURAKIS [13]). *For a compact Hausdorff space, K is Corson-compact and every measure $\mu \in M(K)$ has metrizable support if and only if $S_{M(K)}$ is Corson-compact in the weak* topology.*

The Kunen–Haydon–Talagrand example of Section 5, is an example of a Corson-compact K with a measure on K supported on a non-metrizable set.

6.45. PROBLEM (Mercourakis). If K is a Corson-compact space, L a compact space and $C(K)$, $C(L)$ are isomorphic Banach spaces, does it follow that L is Corson-compact? (The answer is yes if on K every measure has metrizable support, by 6.44. In particular, the answer is yes if MA + $\neg$CH is assumed, because then every Corson-compact ccc space is metrizable).

6.46. THEOREM (MERCOURAKIS [13]). *Every Eberlein-compact of weight at most 2^{ω} is Rosenthal-compact.*

6.47. THEOREM (VAN MILL [87]). *There is a non-metrizable homogeneous uniform Eberlein-compact space.*

Examples

6.48. DEFINITIONS. If T be a non-empty set, a family $\mathscr{A}$ of subsets of T is called *adequate* if

(a) $A \in \mathscr{A}$, $B \subset A \Rightarrow B \in \mathscr{A}$;
(b) $\{t\} \in \mathscr{A}$ for all $t \in T$; and
(c) if $A \subset T$, and every finite subset of A belongs to $\mathscr{A}$, then $A \in \mathscr{A}$.

If $\mathscr{A}$ is an adequate family of T, we set

$$K = K_{\mathscr{A}} = \{\chi_A : A \in \mathscr{A}\} \subset \{0, 1\}^T .$$

It is immediate to see that K is a closed subset of $\{0, 1\}^T$ and hence compact. We often identify A with χ_A, and thus consider K as a space of certain subsets of T.

If $\mathscr{A}$ is an adequate family of T, then we consider $T^* = T \cup \{\infty\}$, and define on T^* a topology as follows: every element of T is isolated; a sub-basis for the neighborhoods of ∞ is the family

$$\{\{\infty\} \cup (T \setminus A) : A \in \mathscr{A}\}$$

We state without proof the following fact, due to TALAGRAND [116] (Theórème 4.2).

6.49. PROPOSITION. *Let $\mathscr{A}$ be an adequate family of T. Then*
(a) *$K_{\mathscr{A}}$ is Eberlein-compact if and only if T^* is σ-compact.*
(b) *$K_{\mathscr{A}}$ is Talagrand-compact if and only if T^* is a $\mathscr{K}$-analytic space.*

6.50. LEMMA [29]. *Let K be a weakly compact subset of a Hilbert space and D a discrete subset of K, with exactly one limit point (in K). Then there is a family $\{V_d: d \in D\}$ of open subsets of K, with $d \in V_d$ for $d \in D$, and a partition $D = \bigcup_{1 \leqslant n < \omega} D_n$ of d such that: if $d, \ldots, d_{n+1}$ are $n+1$ distinct elements of D_n, then $\bigcap_{k=1}^{n+1} V_{d_k} = \emptyset$ for all $n < \omega$.*

PROOF. Without loss of generality, we assume that K is contained in the unit ball of the Hilbert space, and that the unique limit point of K is 0. We denote by $\langle x, y\rangle$ the inner product of elements x, y in the Hilbert space.

For $d, e \in D$, we set $d \sim e$ if there are $1 \leqslant m < \omega$ and $d = d_1, d_2, \ldots, d_m = e$ in D, so that $\langle d_i, d_{i+1}\rangle \neq 0$ for $i = 1, 2, \ldots, m-1$. It is clear (from the fact that for every $d \in D$, the set $\{e \in D: \langle d, e\rangle \neq 0\}$ is countable (since 0 is the only limit point of D)) that the $\sim$-equivalence classes are countable sets. If $d, e \in D$, and $d \not\sim e$, then $\langle d, e\rangle = 0$. Thus, there is a countable partition $D = \bigcup_{m<\omega} E_m$ of D, such that if $d, e \in E_m$, $m < \omega$ then $\langle d, e\rangle = 0$. Partitioning the sets $\{E_m: m < \omega\}$ further, we can find a countable partition $D = \bigcup_{1 \leqslant n < \omega} D_n$, such that if $d, e \in D_n$, $1 < n < \omega$, then $\langle d, e\rangle = 0$, and

$$d \in D_n \Rightarrow \|d\|^4 > 1/n\,.$$

We now set $V_d = \{x \in K: \langle x, d\rangle^2 > 1/n\}$ for $d \in D_n$. We verify that the family $\{V_d: d \in D\}$ satisfies the requirements of the Lemma. Let $1 \leqslant n < \omega$, and $d_1, \ldots, d_{n+1}$ distinct elements of D_n. If there were $x \in \bigcap_{k=1}^{n+1} V_{d_k}$, then (from Bessel's inequality for Hilbert spaces)

$$1 < \frac{n+1}{n} \leqslant \sum_{k=1}^{n} \langle x, d_k\rangle^2 \leqslant \|x\|^2 \leqslant 1\,, \quad \text{a contradiction}\,.$$

Since this lemma has some relevance to them, we now open a brief parenthesis on

6.51. *Simultaneous Extension Operators.* Let K be a compact Hausdorff space, and F a compact Hausdorff space containing K. A bounded linear operator $T: C(K) \to C(F)$ such that $T(f)\,|\,K = f$ for all $f \in C(K)$ is called a *simultaneous extension operator.* We set

$$\eta(K, F) = \inf\{\|T\|: T: C(K) \to C(F) \text{ is a simultaneous extension operator}\}$$

(and $\eta(K, F) = +\infty$, if no such operator exists). The classical result of Borsuk, Dugundji, and Kakutani states that if K is a compact metric space, and F any compact space then $\eta(K, F) = 1$. (See BADE [21] for a classical proof, and LINDENSTRAUSS–TZAFRIRI [80] (II.4.9) for a very short proof based on Michael's selection theorem).

(A) THEOREM (CORSON–LINDENSTRAUSS [38]). *If K is the one-point compactification of an uncountable discrete set, then*

(a) *for any compact space F $\eta(K, F)$ is either an odd natural number $\geqslant 1$, or $+\infty$; and*

(b) *denoting by $\lambda S_{M(K)}$ the space of all measures $\mu \in M(K)$ with $\|\mu\| \leqslant \lambda$, in the weak* topology, then*

$$\eta(K, (2n+1)S_{M(K)} = 2n+1 \quad \text{for } n < \omega, \qquad \text{and} \qquad \eta(K, \beta(M(K)) = \infty .$$

(B) THEOREM (BENYAMINI [26]). *If K is the unit ball of a non-separable Hilbert space in its weak topology, and $\lambda S_{M(K)}$ is the space of all measures $\mu \in M(K)$ with $\|\mu\| \leqslant \lambda$ in the weak* topology, then*

$$\eta(K, \lambda S_{M(K)}) = \lambda \quad \text{for all } \lambda \in \mathbb{R},\ \lambda \geqslant 1, \quad \text{and} \quad \eta(K, \beta(M(K)) = \infty$$

We note that these results concern uniform Eberlein-compact spaces K; the proof of Benyamini's theorem uses a variation of Lemma 6.50.

We also remark that if K is to have the Borsuk–Dugundji–Kakutani property, then K must certainly have a strictly positive measure (and if $\omega(K) \leqslant 2^{\omega}$ then $M_1^+(K)$ must be weak* separable).

Incidentally, other results on simultaneous extension operators can be found in PELCZYNSKI [96], and SEMADENI [110].

6.52. *The BENYAMINI–STARBIRD Space* [29]. Set $\Gamma = [0, 1] \times \Pi_{n=2}^{\infty}\{1, 2, \ldots, n\}$. We denote by $P_m : \Gamma \to \{1, 2\} \times \{1, 2, 3\} \times \cdots \times \{1, 2, \ldots, m\}$ the projection function given by $P_m(\tau, n_2, n_3, \ldots) = (n_2, \ldots, n_m)$ for $2 \leqslant m < \omega$. We set

$$\mathscr{A} = \{A \subset \Gamma\colon \text{there is } 2 \leqslant m < \omega \text{ such that if } \sigma, \tau \in A,\ \sigma \neq \tau, \text{ then } \\ P_m(\sigma) = P_m(\tau) \text{ and } P_{m+1}(\sigma) \neq P_{m+1}(\tau)\} .$$

It is clear that $\mathscr{A}$ is an adequate family of subsets of Γ, consisting of finite sets. The Benyamini–Starbird space is the space $K = K_{\mathscr{A}}$ determined by the adequate family $\mathscr{A}$. Since K is a compact subset of $\{0, 1\}^{\Gamma}$, consisting of elements whose coordinate differ from 0 on finite sets, it follows that K is a scattered Eberlein compact subset of $c_0(\Gamma)$.

6.53. THEOREM [29]. *The Benyamini–Starbird space K is a scattered Eberlein-compact space, but not a uniform Eberlein-compact space.*

PROOF. We have just noted above that K is scattered and Eberlein-compact. Suppose that K is a uniform Eberlein-compact space. We set

$$D = \{\chi_{\{\gamma\}}: \gamma \in \Gamma\},$$

a discrete subset of K, with 0 its unique limit point. From Lemma 6.50 there is a family $\{V_d: d \in D\}$ of open subsets of K, with $d \in V_d$ for $d \in D$, and a partition $D = \bigcup_{1 \leq n < \omega} D_n$ of D such that for any $n+1$ $d_1, \ldots, d_{n+1}$ elements f D_n, $\bigcap_{k=1}^{n+1} V_{d_k} = \emptyset$.

We assume that V_d is basic open in K, and so it has the form

$$V_d = \Big(\{1\}_d \times \{0\}_{d_1} \times \cdots \times \{0\}_{d_k} \times \prod_{\gamma \in \Gamma \setminus F_d} \{0, 1\}\Big) \cap K,$$

where $F_d = \{d, d_1, \ldots, d_k\} \subset \Gamma$. We remark that if $A \subset \Gamma$, then

$$\chi_A \in V_d \Leftrightarrow \chi_A \in K \text{ and there is } B \text{ with } A \cap F_d = B \cap F_d, \text{ and } \chi_B \in V_d .$$

Claim. Let $m \geq 2$ and $n_2, \ldots, n_{m-1}$ be given. There is $1 \leq j \leq m$ such that $D_{m-1} \cap P_m^{-1}(n_2, \ldots, n_{m-1}, j)$ is countable.

Proof of Claim. Assume that $\Gamma_j = D_{m-1} \cap P_m^{-1}(n_2, \ldots, n_{m-1}, j)$ is uncountable for $j = 1, \ldots, m$. From the Erdös–Rado Theorem (0.3), we assume without loss of generality that there are finite sets $A_1, \ldots, A_m$ such that for $j = 1, \ldots, m$, if $d \in \Gamma_j$ then $F_d = A_j \cup B_d$, and $\{B_d: d \in \Gamma_j\}$ are pairwise disjoint sets.

Since $A_1 \cup \cdots \cup A_m$ is finite, we also assume without loss of generality that $(A_1 \cup \cdots \cup A_m) \cap (\Gamma_1 \cup \cdots \cup \Gamma_m) = \emptyset$.

We now choose $d_j \in \Gamma_j$, with $d_j \notin F_{d_{j'}}$, if $j \neq j'$. Indeed, let $d_1 \in \Gamma_1$. Since B_{d_1} is finite, and $\{B_d: d \in \Gamma_2\}$ consists of disjoint sets, there is $d_2 \in \Gamma_2$, with $d_1 \notin B_{d_2}$, $d_2 \notin B_{d_1}$, and so on inductively.

However, $\{d_1, \ldots, d_m\} \subset \bigcap_{j=1}^m V_{d_j}$; in fact, we have $\{d_1, \ldots, d_m\} \subset K$, $d_j \in V_{d_j}$, V_{d_j} depends only on the set F_{d_j}, and $\{d_1, \ldots, d_m\} \cap F_{d_j} = \{d_j\} \cap F_{d_j}$. Thus $d_1, \ldots, d_m \in V_{d_j}$ for $j = 1, \ldots, m$. This is a contradiction to the fact that $\bigcap_{j=1}^m V_{d_j} = \emptyset$, proving the claim.

We define inductively a sequence $\{n_j, j \geq 2\}$, $1 \leq n_j \leq j$ as follows. By the claim, there is n_2 such that $D_1 \cap P_2^{-1}(n_2)$ is countable. Suppose $n_2, \ldots, n_{m-1}$ have been defined, and we choose n_m, using the claim, so that $D_{m-1} \cap P_m^{-1}(n_2, \ldots, n_m)$ is countable. Then the set

$$E = D \cap \bigcap_{m \geq 2} P_m^{-1}(n_2, \ldots, n_m) \subset \bigcap_{m \geq 2} (D_{m-1} \cap P_m^{-1}(n_2, \ldots, n_m))$$

is countable, a contradiction to the fact that $\bigcap_{m \geq 2} P_m^{-1}(n_2, \ldots, n_m) = \{(\tau, n_2, \ldots, n_m, \ldots): \tau \in [0, 1]\}$ is countable.

REMARK. Recently the Benyamini–Starbird example has been considerably simplified by Argyros and Mercourakis.

6.54. REMARK. If K is a closed subspace of $\{0, 1\}^\alpha$, and order type $\operatorname{supp}(x) < \omega$ for all $x \in K$, then K is Eberlein-compact (since clearly $K \subset c_0(\alpha)$). We have asked in ARGYROS–MERCOURAKIS–NEGREPONTIS [12] what happens if we have order type $\operatorname{supp}(x) \leqslant \omega$ for all $x \in K$. The two examples below (6.55 and 6.58) show that we can have very pathological Corson-compact spaces, that satisfy this 'rareness' condition.

6.55. THEOREM. *There is a compact Hausdorff totally disconnected space K such that*

(a) *K is Talagrand-compact,*
(b) *K is not Eberlein-compact,*
(c) *$K \subset \{0, 1\}^{2^\omega}$, and $x \in K \Rightarrow$ ordertype* $\operatorname{supp}(x) \leqslant \omega$, *and*
(d) *K is Rosenthal-compact.*

PROOF. We first construct the space K.

Let $\Delta = [0, 1] \times \{0, 1\}$ with the 'porc-épic' topology, i.e. the topology in which every element of $[0, 1] \times \{1\}$ is isolated, and a basis for the neighborhoods of $(x, 0) \in [0, 1] \times \{0\}$ is of the form $\{V \times \{0, 1\} \setminus \{(x, 1)\}: V \in \mathscr{B}\}$ where $\mathscr{B}$ is a basis of neighborhoods of x in the usual topology of $[0, 1]$. The space Δ is also called the "Alexandroff duplicate' of $[0, 1]$.

We note that Δ is an Eberlein-compact space, while the subspace $X = \mathbb{J} \times \{0, 1\} \subset \Delta$, where $\mathbb{J}$ denotes the set of irrational numbers in $[0, 1]$ is a $K_{\sigma\delta}$-set in Δ, and hence a $\mathscr{K}$-analytic space. (Note also that K is not σ-compact.)

Let $\Sigma = \omega^\omega$ be the Baire space, and fix $\phi: \Sigma \to \mathbb{J}$ a homeomorphism. Also, let $\{\sigma_\xi: \xi < 2^\omega\}$ be a 1–1 well-ordering of Σ. We denote by $P_m: \omega^\omega \to \omega^m$ the natural projection $P_m(n_1, \ldots, n_m, \ldots) = (n_1, \ldots, n_m)$. A finite subset $\{\xi_1 < \cdots < \xi_n\} \subset 2^\omega$ is *ϕ-admissible* if

(a) $|\phi(\sigma_{\xi_i}) - \phi(\sigma_{\xi_j})| \leqslant 1/i$ for $1 \leqslant i < j \leqslant n$, and
(b) there is $k < \omega$, with $P_k(\sigma_{\xi_i}) = P_k(\sigma_{\xi_j})$ but $P_{k+1}(\sigma_{\xi_i}) \neq P_{k+1}(\sigma_{\xi_j})$ for all $1 \leqslant i < j \leqslant n$.

We set

$$\mathscr{A} = \{A \subset \Sigma\text{: every finite subset of } \{\xi < 2^\omega: \sigma_\xi \in A\} \text{ is } \phi\text{-admissible}\}.$$

It is clear that $\mathscr{A}$ is an adequate family; we set

$$K = K_{\mathscr{A}} \subset \{0, 1\}^\Sigma .$$

We next verify that K has the required properties. If follows from (b), that if $A \in \mathscr{A}$, then A is a closed and discrete subset of the Polish space Σ, hence A is countable (hence K is Corson-compact), and also $\chi_A : \Sigma \to \{0, 1\}$ is a Baire-1 function (hence K is Rosenthal-compact).

Claim 1. If $A \in \mathscr{A}$, then order type $A \leqslant \omega$.

(Indeed suppose that there is $A = \{\sigma_{\xi_n} : n \leqslant \omega\} \in \mathscr{A}$ of order type $\omega + 1$. It follows from condition (a), then $|\phi(\sigma_{\xi_n}) - \phi(\sigma_{\xi_\omega})| \leqslant 1/n$ for $n < \omega$, and hence $\phi(\sigma_{\xi_n}) \to \phi(\sigma_{\xi_\omega})$, hence $\sigma_{\xi_n} \to \sigma_{\xi_\omega}$ as $n \to \infty$, impossible since A is necessarily a closed discrete subset of Σ).

Claim 2. K is not Eberlein-compact.

(By Proposition 6.49(a) it is enough to prove that $\Sigma^* = \Sigma \cup \{\infty\}$, supplied with the topology determined by $\mathscr{A}$, as in 6.48, is not σ-compact. In fact, suppose that $\Sigma = \bigcup_{n<\omega} K_n$, with $K_n \cup \{\infty\}$ compact in Σ^*. By the Baire category theorem, for Σ with the usual topology, there is $n_0 < \omega$ such that $\mathrm{Int}_{\mathbb{R}}\mathrm{cl}_{\mathbb{R}}(K_{n_0}) \neq \emptyset$; thus, there are $n_1, \ldots, n_k < \omega$, such that the set

$$V = P_k^{-1}(n_1, \ldots, n_k) \subset \mathrm{Int}_{\mathbb{R}}\mathrm{cl}_{\mathbb{R}}(K_{n_0}) .$$

For $m < \omega$, we choose $\sigma_m \in V$, such that $P_{k+1}(\sigma_m) = (n_1, \ldots, n_k, m)$. Thus $\{\sigma_m : m < \omega\} \subset V \subset \mathrm{cl}_{\mathbb{R}}(K_{n_0})$. We set for $m < \omega$

$$V_m = P_{k+1}^{-1}(n_1, \ldots, n_k, m) .$$

Then V_m is open neighborhood of σ_m, and $V_m \cap \Sigma_{n_0} \neq \emptyset$ for $m < \omega$. We choose $\tau_m \in V_m \cap \Sigma_{m_0}$ for $m < \omega$. Then

$$P_k(\tau_m) = (n_1, \ldots, n_k) \quad \text{for } m < \omega , \quad \text{and}$$

$$P_{k+1}(\tau_m) \neq P_{k+1}(\tau_{m'}) \quad \text{for } m < m' < \omega .$$

Let $A = \{\xi < 2^\omega :$ there is $m < \omega$ with $\sigma_\xi = \tau_m\}$, so A is an infinite subset of 2^ω. We choose $B \subset A$, with order type $(B) = \omega$, say $B = \{\xi_0 < \cdots < \xi_n < \cdots : n < \omega\}$. The sequence $\{\phi(\sigma_{\xi_n}) : n < \omega\}$ has a necessarily rational limit point x (in $[0, 1]$). Hence, there is a subsequence of $\{\xi_n : n < \omega\}$ (which we assume without loss of generality that it coincides with $\{\xi_n : n < \omega\}$ satisfying condition (a) and converging to x. Then

$$C = \{\sigma_{\xi_n} : n < \omega\} \subset K_{n_0} , \quad \text{and} \quad C \in \mathscr{A} .$$

Hence $K_{n_0} \cup \{\infty\}$ has an infinite closed discrete subset (as a subset of Σ^*), and hence Σ^* is not σ-compact).

Claim 3. K is Talagrand-compact.

(We define $F : X \to \Sigma^*$ by

$$F(t, 0) = \infty \qquad \text{for } t \in \mathbb{J} ,$$

$$F(t, 1) = \phi^{-1}(t) \quad \text{for } t \in \mathbb{J} .$$

It is easy to see that F is continuous and onto, and since X is $\mathscr{K}$-analytic, it follows that Σ^* is $\mathscr{K}$-analytic. Hence, Proposition 6.49(b), $C(K)$ is $\mathscr{K}$-analytic in its weak topology, i.e. K is Talagrand compact).

The proof of the theorem is complete.

6.56. REMARK. (a) ROSENTHAL in [103] (§1, 2) first constructed a Banach space X such that X is not W.C.G., but X is weakly $\mathscr{K}$-analytic. In fact Rosenthal proved

(1) There is a closed subspace X of $L^1\{0,1\}^{2^\omega}$. such that X is not W.C.G.

(2) Assuming CH, there is a non-separable closed subspace X of $L^1\{0,1\}^{\omega_+}$, such that every weakly compact subset of X is metrizable. (A Lusin-type construction!)

(3) Assuming MA + —CH every subspace of $L^1\{0,1\}^{\omega_+}$ is W.C.G.

(b) TALAGRAND in [115], [116] constructed the example given in Theorem 6.55(a), (b). The improvements stated in 6.55(c), (d) are from [12, 13] (and are due to Mercourakis).

6.57. PROBLEM. Is there a Gul'ko-compact space K, such that K is not Talagrand-compact?

The answer to this question should be positive.

6.58. THEOREM. *There is a compact subset K of $\{0,1\}^{\omega_+}$ such that $x \in K \Rightarrow$ order type$(x) \leq \omega$ (hence K is Corson-compact), and K is not Gul'ko-compact.*

PROOF. We define a family $\{N_\xi: \xi < \omega\}$ of subsets of ω as follows: We choose $\{N_k: k<\omega\}$, such that $|N_k| = \omega$, $N_k \cap N_l = \emptyset$ for $k<l<\omega$. Let $\omega \leq \xi < \omega^+$, and suppose that $\{N_\zeta: \zeta < \xi\}$ have been defined, and are infinite and almost disjoint. Let $\{\zeta_n: n<\omega\}$ be a 1–1 well-ordering of ξ. We choose $F_n^\xi \subset N_{\zeta_n^\xi} \setminus \bigcup_{k<n} N_{\zeta_k^\xi}$, such that $|F_n^\xi| = n$ for $n<\omega$. We set $N_\zeta = \bigcup_{n<\omega} F_n^\xi$. It is clear $\{N_\xi: \xi<\omega^+\}$ is a family of infinite, almost disjoint subsets of ω.

We define $\phi: (\omega^+)^{[2]} \to \omega$ by $\phi(\xi,\zeta) = |N_\xi \cap N_\zeta|$.

A subset $J \subset \omega^+$ is *ϕ-admissible* if for all $\xi, \zeta \in J$, with $\xi < \zeta$, we have

$$|\{\eta \in J: \xi < \eta < \zeta\} \leq \phi(\xi,\zeta).$$

We set $\mathscr{A} = \{J \subset \omega^+: J$ is a ϕ-admissible set$\}$. It is clear that $\mathscr{A}$ is an adequate family of subsets of ω^+. We set $K = K_{\mathscr{A}}$. It is clear that if $x \in K$, then $x = \chi_J$ for $J \in \mathscr{A}$, and order type$(J) \leq \omega$. (It follows that K is Corson-compact.)

We finally prove that K is not Gul'ko-compact. It it is, then according to Lemma 6.39(b) there is a family $\{K_s: s \in S\}$ of subsets of ω^+ such that

$$K_\emptyset = \omega^+, \qquad \bigcup_{k<\omega} K_{\widehat{s,k}} = K_s \quad \text{for } s \in S,$$

and for every $\xi < \omega^+$, there is $\sigma \in \Sigma$ such that $\xi \in \bigcap_{k<\omega} K_{\sigma|k}$, and if $\xi_k \in K_{\sigma|k}$ for $k<\omega$, then $\{\xi_k: k<\omega\}$ is not ϕ-admissible.

We set $L = \bigcup\{K_s: K_s$ not a stationary set in $\omega^+\}$. Then $\omega^+ \setminus L$ contains a closed, unbounded subset of ω^+, and thus if G is a stationary subset of ω^+ then $|G\setminus L| = \omega^+$.

For $s \in S$, $|s| = k$, with $K_s \not\subset L$, we set $f^s: K_s \to \omega^+$ by $f_s(\xi) = \max\{\zeta_1^\xi, \ldots, \zeta_k^\xi\} < \xi$. From Fodor's Theorem (0.2), there is $L_s \subset K_s$, L_s stationary, and $\xi_s < \omega^+$, such that $f^s(\xi) = \xi_s$ for $\xi \in L_s$.

We choose $\xi < \omega^+$, with $\xi \in L$ and

$\xi > \operatorname{supp}\{\xi_s: s \in S, K_s \subset L\}$.

By above property of the family $\{K_s: s \in S\}$, there is $\sigma \in \Sigma$, such that $\xi \in \bigcap_{k<\omega} K_{\sigma|k}$, and if $\xi_k \in K_{\sigma|k}$ for $k < \omega$, then $\{\xi_k: k < \omega\}$ is not ϕ-admissible. We choose $\zeta_0 > \xi$, $\zeta_1 \in L_{s_0}$, and inductively, $\zeta_k > \zeta_{k-1}$, $\zeta_k \in L_{s_k}$.

We claim that $\{\zeta_k: k < \omega\}$ is ϕ-admissible. In fact, if $k < n$, then $\phi(\zeta_k, \zeta_n) \geq n$, since in the enumeration of $\zeta_n = \{\zeta_k^{\zeta_n}: k < \omega\}$ ζ_k appears after the n-th position; hence

$$|N_{\zeta_k} \cap N_{\zeta_n}| = \phi(\zeta_k \zeta_n) \geq n = |\{\zeta: k < l < n\}| = n - k\,.$$

This contradiction proves that K is not Gul'ko-compact.

6.59. Remarks. Alster and Pol [3] were the first to construct a non-Talagrand-compact (that turned out to be also a non-Gul'ko-compact [12, 13]) Corson-compact space.

Assuming CH, an example of a non-Gul'ko Corson-compact space, each element of which has order-type for its support at most $\omega + 1$ is given in [12, 13] using a family constructed for a different purpose by Hajnal. By a remark of Mercourakis an the author, a similar example can be had, without recourse to CH, using instead a family of subsets of [2] constructed for a different purpose by Galvin–Hajnal (cf. [35] (Theorem 6.32)).

The present example 6.58 is due to Argyros [9].

7. Kunen's example of an S-space and Banach spaces

This Section is devoted to the construction, assuming CH, of an S-space by Kunen (Theorem 7.1), and the application (in Theorem 7.7) that this space has in Banach spaces, strengthening earlier results by Shelah.

7.1. Theorem (Kunen [74]). *Assume the continuum hypothesis. There is a Hausdorff topological space X, such that X is locally compact locally countable, first countable, and*

X is not Lindelöf, and

X^n is hereditarily separable for $1 \leq n < \omega$.

PROOF. We choose $X = \{x_\alpha : \alpha < \omega^+\} \subset [0, 1]$, with $x_\alpha \neq x_\beta$ for $\alpha < \beta < \omega^+$. We set $X_\alpha = \{x_\beta : \beta < \alpha\}$ for $\alpha < \omega^+$. We will define a topology $\mathcal{T}$ on X.

By the continuum hypothesis, there is a well-ordering $\{A_\alpha : \alpha < \omega^+\}$ of the set $\bigcup_{1 \leq n < \omega} (X^n)^{[\omega]}$.

If $A \subset X^n$, and σ is a permutation of $\{1, \ldots, n\}$, we set

$$A^\sigma = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in X^n : (x_1, \ldots, x_n) \in A\}.$$

We define $\mathcal{T}_\alpha$, $\mathcal{B}_\alpha$, $\mathcal{F}_\alpha$, $(y_k^\alpha)_{k<\omega}$, $(B_k^\alpha)_{k<\omega}$ for $\alpha < \omega^+$, such that:

(1) $\mathcal{T}_\alpha$ is a topology on X_α.

(2) $\mathcal{T}_\beta \subset \mathcal{T}_\alpha$ for $\beta < \alpha$.

(3) $\mathcal{T}_\alpha \mid X_\beta = \mathcal{T}_\beta$ for $\beta < \alpha$ (where $\mathcal{T}_\alpha \mid X_\beta$ denotes the topology on X_β induced by $\mathcal{T}_\alpha$).

(4) $\mathcal{B}_\alpha$ is the family of all $\mathcal{T}_\alpha$-open-and-compact subsets of X_α.

(5) $\mathcal{B}_\alpha$ is countable.

(6) $\mathcal{F}_\alpha = \{A$; there are $1 \leq n < \omega$, $\beta \leq \alpha$, with $A = A_\beta \subset X_\alpha^n$, and $x_\alpha \in \mathrm{cl}_{\mathbb{R}}(A)\} \cup \{A$: there are $1 \leq m$, $n < \omega$, $\beta \leq \alpha$, $B_1, \ldots, B_m \in \mathcal{B}_\alpha$, such that $A_\beta \subset X_\alpha^{m+n}$, $(x_\alpha, \ldots, x_\alpha) \in \mathrm{cl}_{\mathbb{R}}{}^n(A)$, and $A = \pi_{\{m+1, \ldots, m+n\}}((\pi_{\{1, \ldots, m\}}^{-1}(B_1 \times \cdots \times B_m)) \cap A_\beta) \subset X_\alpha^n\}$ (where $\pi_{\{1, \ldots, m\}}: X^{m+n} \to X^m$ denotes the usual projection function, etc.).

(7) (a) If $x_\alpha \notin \mathrm{cl}_{\mathbb{R}}\{x_\beta : \beta < \alpha\}$, then $y_k^\alpha = 0$ for $k < \omega$.

(b) If $x_\alpha \in \mathrm{cl}_{\mathbb{R}}\{x_\beta : \beta < \alpha\}$, then $\lim_{k \to \infty} y_k^\alpha = x_\alpha$ (in $\mathbb{R}$); $y_k^\alpha \in X_\alpha$ for $k < \omega$; and, if $A \in \mathcal{F}_\alpha$, and $A \subset X_\alpha^n$, then $\{k < \omega : y_{k+1}^\alpha, \ldots, y_{k+n}^\alpha \in A\}$ is infinite.

(8) (a) If $x_\alpha \notin \mathrm{cl}_{\mathbb{R}}\{x_\beta : \beta < \alpha\}$, then $B_k^\alpha = \{0\}$ for $k < \omega$.

(b) If $y_\alpha \in \mathrm{cl}_{\mathbb{R}}\{x_\beta : \beta < \alpha\}$, then

$$\begin{aligned} &y_k^\alpha \in B_k^\alpha && \text{for } k < \omega, \\ &B_k^\alpha \in \mathcal{B}_\alpha && \text{for } k < \omega, \\ &B_k^\alpha \cap B_l^\alpha = \emptyset && \text{for } y_k^\alpha \neq y_l^\alpha. \end{aligned}$$

(9) If α is a limit ordinal, then

$$\mathcal{T}_\alpha = \{U \subset X_\alpha : U \cap X_\beta \in \mathcal{T}_\beta \text{ for } \beta < \alpha\}$$

(10) If $x_\alpha \notin \mathrm{cl}_{\mathbb{R}}\{x_\beta : \beta < \alpha\}$, then $\mathcal{T}_{\alpha+1}$ is the topology on $X_{\alpha+1}$ generated by $\mathcal{T}_\alpha \cup \{x_\alpha\}$.

(11) If $x_\alpha \in \mathrm{cl}_{\mathbb{R}}\{x_\beta : \beta < \alpha\}$, then $\mathcal{T}_{\alpha+1}$ is the topology on $X_{\alpha+1}$ generated by

$$\mathcal{T}_\alpha \cup \left\{\{x_\alpha\} \cup \left\{\bigcup_{k \geq m} B_k^\alpha : m < \omega\right\} : m < \omega\right\}.$$

We proceed inductively. Let $\alpha < \omega^+$, and suppose that $\mathcal{T}_\beta$, $\mathcal{B}_\beta$, $(y_k^\beta)_{k<\omega}$, $(B_k^\alpha)_{k<\omega}$ have been defined for $\beta < \alpha$.

If α is a limit ordinal, we define $\mathcal{T}_\alpha$ by (9), and if α is a nonlimit ordinal, then we define $\mathcal{T}_\alpha$ by (10), or (11).

It is clear that (1), (2), and (3), are satisfied. It is then clear (by (2)), if $\mathcal{B}_\alpha$ is defined by (4), that $\mathcal{B}_\alpha = \bigcup_{\beta<\alpha}\mathcal{B}_\beta$ and hence (5) is satisfied. If $x_\alpha \notin \mathrm{cl}_{\mathbb{R}}\{x_\beta\colon \beta<\alpha\}$, then $\mathcal{F}_\alpha = \emptyset$ (consistent with (6)), $y_k^\alpha = 0$, $B_k^\alpha = \{0\}$ for $k<\omega$ (consistent with (7)(a), (8)(a)). If $x_\alpha \in \mathrm{cl}_{\mathbb{R}}\{x_\beta\colon \beta<\alpha\}$, we define $\mathcal{F}_\alpha$ by (6). We note that $\mathcal{F}_\alpha$ is a countable family, and hence we can enumerate $\mathcal{F}_\alpha = \{D_p\colon p<\omega\}$, such that each element of $\mathcal{F}_\alpha$ appears infinitely often in the enumeration. We define the sequence $\{y_k^\alpha\colon k<\omega\}$ in blocks, inductively. Let $p<\omega$, and suppose that $\{y_k^\alpha\colon k\leqslant N_p\}$ has been defined; if $D_p \subset X_\alpha^n$ (for a unique n), then we set $N_{p+1} = N_{p+n}$, and we choose y_k^α, $N_p<k\leqslant N_p+n$, so that

$$(y_k^\alpha\colon N_p<k\leqslant N_p+n)\in D_p\,,$$

and

$$|y_k^\alpha - x_\alpha| < 1/p \quad \text{for } N_p<k\leqslant N_p+n\,.$$

It is clear that (7)(b) is satisfied. Finally, we choose $B_k^\alpha \in \mathcal{B}_\alpha$, such that (8)(b) is satisfied. (This is possible on account of (2), (10), and (11).)

The inductive definitions are complete.

We set $\mathcal{T} = \{U\subset X\colon U\cap X_\alpha \in \mathcal{T}_\alpha \text{ for } \alpha<\omega^+\}$ a topology on X. It is clear that $(X,\mathcal{T})$ is locally countable, locally compact, and first countable, and that it is not a Lindelöf space (since $X_\alpha\in\mathcal{T}$ for all $\alpha<\omega^+$). It remains to prove that X^n is hereditarily separable for $n = 1, 2, \ldots$. We need the following lemma.

LEMMA. *For every $p\geqslant 1$ and $E\subset X^p$, there is $\beta_0<\omega^+$ such that for all $\xi_1,\ldots,\xi_p > \beta_0$, with $(x_{\xi_1},\ldots,x_{\xi_p})\in \mathrm{cl}_{\mathbb{R}^p}(E)$, then $(x_{\xi_1},\ldots,x_{\xi_p})\in\mathrm{cl}_{\mathcal{T}^p}(E)$.*

PROOF. We proceed inductively on $p\geqslant 1$.

Let $p=1$, and $E\subset X$. There is a countable set $D\subset E$, such that D is $\mathbb{R}$-dense in E. There is $\beta_1<\omega^+$, such that $D = A_{\beta_1}$, and $\beta_1\leqslant\beta_0<\omega^+$, such that $D\subset X_{\beta_0}$. Let now $\xi>\beta_0$ and $x_\xi\in\mathrm{cl}_{\mathbb{R}}(E)$. Then $x_\xi\in\mathrm{cl}_{\mathbb{R}}(D)$, and from (6) we have $D\in\mathcal{F}_\xi$. It follows from (7)(b) and (11) that $x_\xi\in\mathrm{cl}_{\mathcal{T}}(D)\subset\mathrm{cl}_{\mathcal{T}}(E)$.

Assume that the statement holds for all $p'<p$, and we will prove it for p. If $m<p$, then the topology $\mathcal{T}^m\times\mathbb{R}^{p-m}$ on X^p (which is $\mathcal{T}$ for the first m factors) is easily seen to be hereditarily separable from the inductive assumption. Hence, there is $\beta_1<\omega^+$, such that E and $E\cap(X_{\beta_1})^p$ have equal closures in $\mathcal{T}^m\times\mathbb{R}^{p-m}$ and there is $\alpha<\omega^+$, such that $E\cap(X_{\beta_1})^p = A_\alpha$. It follows from the fact that $\mathbb{R}^{p-m}$ is a second-countable space and from the inductive assumptions, that there is $\beta_2>\beta_1$, α such that for all basic open sets U in $\mathbb{R}^{p-m}$ and for all $\eta_1,\ldots,\eta_m>\beta_2$ such that $(x_{\eta_1},\ldots,x_{\eta_m})\in\mathrm{cl}_{\mathbb{R}^m}(\pi_{\{1,\ldots,m\}}(E\cap(X^m\times U)))$, it follows that

$$(x_{\eta_1},\ldots,x_{\eta_m})\in\mathrm{cl}_{\mathcal{T}^m}(\pi_{\{1,\ldots,m\}}(E\cap(X^m\times U)))\,.$$

We note that for a fixed $E \subset X^p$, β_2 depends on m; writing temporarily $\beta_2 = \beta_2^m$, we choose $\beta_0 < \omega^+$, with $\beta_2^m < \beta_0$ for all $m < p$.

Claim 1. For all $m < p$, all $\eta_1, \ldots, \eta_m$, β with $\beta_0 < \eta_1, \ldots, \eta_m < \beta$, with $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathcal{T}^m \times \mathbb{R}^{p-m}}(E)$ we have that $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathcal{T}_p}(E)$.

(In fact, a basic open neighborhood U of $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta)$ in $\mathcal{T}^p$ has the form $U = K_1 \times \cdots \times K_m \times B_{m+1} \times \cdots \times B_p$, where $K_1, \ldots, K_m \in \mathcal{B}_\beta$ (since $\eta_1, \ldots, \eta_m < \beta$). It follows from the assumptions, that

$$(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathcal{T}^m \times \mathbb{R}^{p-m}}(E \cap (X_{\beta_1})^p) = \mathrm{cl}_{\mathcal{T}^m \times \mathbb{R}^{p-m}} A_\alpha .$$

We set

$$A = \pi_{\{m+1, \ldots, p\}}(A_\alpha \cap \pi^{-1}_{\{1, \ldots, m\}}(K_1 \times \cdots \times K_m)) .$$

It is clear that $(x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathbb{R}^{p-m}}(A)$. Thus $A \in \mathcal{F}_\beta$. From (7)(b), we have that $\{k < \omega : y^\beta_{k+1}, \ldots, y^\beta_{k+p-m} \in A\}$ is infinite. Hence, from (11), there is k such that $(y^\beta_{k+1}, \ldots, y^\beta_{k+p-m}) \in B_{m+1} \times \cdots \times B_p$. From the definition of A, there is $(x_1, \ldots, x_m) \in K_1 \times \cdots \times K_m$, such that $(x_1, \ldots, x_m, y^\beta_{k+1}, \ldots, y^\beta_{k+p-m}) \in A_\alpha \subset E$. Thus $(x_1, \ldots, x_m, y^\beta_{k+1}, \ldots, y^\beta_{k+p-m})$ is an element of $U \cup E$. Hence $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathcal{T}^p}(E)$).

Claim 2. For all $\beta_1, \ldots, \beta_{p-m} < \omega^+$, and $\eta_1, \ldots, \eta_m > \beta_0$, if $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_{\beta_{p-m}}) \in \mathrm{cl}_{\mathbb{R}^m \times \mathbb{R}^{p-m}}(E)$, then $(x_{\eta_1}, \ldots, x_{\eta_m}, x_{\beta_1}, \ldots, x_{\beta_{p-m}}) \in \mathrm{cl}_{\mathcal{T}^m \times \mathbb{R}^{p-m}}(E)$.

(In fact, let $K \times U$ be a basic open set in $\mathcal{T}^m \times \mathbb{R}^{p-m}$, containing $(x_{\eta_1}, \ldots, x_{\eta_m}, x_{\beta_1}, \ldots, x_{\beta_{p-m}})$. Since $(x_{\eta_1}, \ldots, x_{\eta_m}, x_{\beta_1}, \ldots, x_{\beta_{p-m}})$ is in $\mathrm{cl}_{\mathbb{R}^m \times \mathbb{R}^{p-m}}(E)$, it is easy to see that

$$(x_{\eta_1}, \ldots, x_{\eta_m}) \in \mathrm{cl}_{\mathbb{R}^m}(\pi_{\{1, \ldots, m\}} E \cap (X^m \times U)))$$

for every basic open set U in $\mathbb{R}^{p-m}$, containing $(x_{\beta_1}, \ldots, x_{\beta_{p-m}})$. Since $\eta_1, \ldots, \eta_m > \beta_0$, we have from (*), that

$$\begin{aligned}(x_{\eta_1}, \ldots, x_{\eta_m}) \in \mathrm{cl}_{\mathbb{R}^m}(\pi_{\{1, \ldots, m\}}(E \cap (X^m \times U))) \Rightarrow (x_{\eta_1}, \ldots, x_{\eta_m}) \\ \in \mathrm{cl}_{\mathcal{T}_m}(\pi_{\{1, \ldots, m\}}(E \cap (X^m \times U)) .\end{aligned}$$

Hence, there is $(x_1, \ldots, x_m) \in \pi_{\{1, \ldots, m\}}(E \cap (X^m \times U)) \cap K$, and hence there is $(y_{m+1}, \ldots, y_p) \in U$, such that $(x_1, \ldots, x_m, y_{m+1}, \ldots, y_p) \in (K \times U) \cap E)$.

Let now $m < p$, $\beta_0 < \eta_1, \ldots, \eta_m < \beta < \omega^+$, and $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathbb{R}^p}(E)$. For Claim 2, it follows that $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots x_\beta) \in \mathrm{cl}_{\mathcal{T}^m \times \mathbb{R}^{p-m}}(E)$, and from Claim 1, it follows that $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathcal{T}_p}(E)$.

We have thus proved that:

(**) If $E \subset X^p$, then there is $\beta_0 < \omega^+$, such that if $m < p$, $\beta_0 < \eta_1, \ldots, \eta_m < \beta$, and $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathbb{R}^p}(E)$, then $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathcal{T}^p}(E)$.

We now prove the general case of the lemma. Let $E \subset X^p$; for every permutation σ of $\{1, \ldots, p\}$, we let β_0^σ be the ordinal whose existence is guaranteed by (**) for the set E^σ, and we find $\beta_0 < \omega^+$ such that $\beta_0 > \beta_0^\sigma$ for every permutation σ of $\{1, \ldots, p\}$. Let now $\xi_1, \ldots, \xi_p > \beta_0$, so that $(x_{\xi_1}, \ldots, x_{\xi_p}) \in \mathrm{cl}_{\mathbb{R}^p}(E)$. Let σ be a permutation of $\{1, \ldots, p\}$ such that $(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)})$ has the form $(\eta_1, \ldots, \eta_m, \beta, \ldots, \beta)$ where $m < p$ and $\eta_1, \ldots, \eta_m < \beta$. It is clear that

$$(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathcal{T}^p}(E^\sigma)\,.$$

From property (**), it follows that $(x_{\eta_1}, \ldots, x_{\eta_m}, x_\beta, \ldots, x_\beta) \in \mathrm{cl}_{\mathcal{T}^p}(E^\sigma)$, and hence, clearly, $(x_{\xi_1}, \ldots, x_{\xi_p}) \in \mathrm{cl}_{\mathcal{T}^p}(E)$.

The proof of the lemma is now complete.

With the aid of the lemma it is now easy to prove by induction that X^p is hereditarily separable for all $p < \omega$.

The proof of the theorem is now complete.

7.2. LEMMA. *Let $X = \Pi_{n<\omega} X_n$ be a topological product, such that every finite subproduct is hereditarily separable. Then X is hereditarily separable.*

PROOF. Let $Y \subset X$. For every $F \subset \omega$, F finite, we choose a countable set $D_F \subset Y$, such that $\pi_F(D_F)$ is dense in $\pi_F(Y)$. The set $D = \bigcup\{D_F : F \subset \omega,\ F \text{ finite}\}$ is countable and dense in Y.

7.3. THEOREM (ZENOR [128]). *If the power X^ω is hereditarily separable, then $C(X)$ is hereditarily Lindelöf in the pointwise topology.*

PROOF. We set $\Psi : X^\omega \times C(X) \to \mathbb{R}^\omega$ by $\Psi(x, f) = (f(x_n))$ where $x = (x_n) \in X^\omega$, $f \in C(X)$. For a basic open set U in $\mathbb{R}^\omega$, and $x \in X^\omega$, we set

$$N(x, U) = \{f \in C(X) : \Psi(x, f) \in U\}$$

It is easy to see that the family $\{N(x, U) : x \in X^\omega,\ U \text{ open basic in } \mathbb{R}^\omega\}$ is a basis for the pointwise topology of $C(X)$. We choose a countable basis $\mathcal{B}$ or $\mathbb{R}^\omega$, consisting of basic open sets in the Cartesian topology. Then, the family

$$\mathcal{P} = \{N(x, U) : x \in X^\omega,\ U \in \mathcal{B}\}$$

is a basis for the pointwise topology of $C(X)$.

It is enough to prove that for any $\mathcal{C} \subset \mathcal{P}$, there is a countable family $\mathcal{D} \subset \mathcal{C}$, with $\bigcup \mathcal{D} = \bigcup \mathcal{C}$.

For $U \in \mathcal{B}$, we set

$$Y_U = \{x \in X^\omega : N(x, U) \in \mathcal{C}\} \subset X^\omega\,.$$

By assumption, Y_U is separable, hence there is a countable set D_U dense in Y_U. We set

$$\mathcal{D} = \{N(x, U): x \in D_U, U \in \mathcal{B}\}$$

and it is not difficult to verify that $\mathcal{D}$ is countable, $\mathcal{D} \subset \mathcal{C}$, and $\bigcup \mathcal{D} = \bigcup \mathcal{C}$.

7.4. THEOREM. *If K is a compact, Hausdorff, scattering space, then the weak topology on $(C(K)$ coincides with the pointwise topology on $C(K)$.*

We do not give the proof of this simple result. We refer the reader to SEMADENI's text [110] (Theorem 19.7.6).

7.5. DEFINITION. A family $\{x_\alpha: \alpha < \omega^+\}$ in a topological space X is *right-separated* if $x_\alpha \notin \overline{\{x_\beta: \beta > \alpha\}}$ for all $\alpha < \omega^+$.

7.6. PROPOSITION. *A topological space X is hereditarily Lindelöf if and only if there does not exist any right-separated family in X.*

PROOF. ($\Rightarrow$) If $Y = \{x_\alpha: \alpha < \omega^+\}$ is right-separated, then, setting $U_\alpha = Y \backslash \{x_\beta: \beta < \alpha\}^-$ for $\alpha < \omega^+$, we note that $\{U_\alpha: \alpha < \omega^+\}$ is an open cover of Y, without any countable subcover.

($\Leftarrow$) Suppose there is $Y \subset X$, Y not Lindelöf. Thus there is an open cover $\mathcal{U}$ of Y without any countable subcover. For any $\alpha < \omega^+$, we assume inductively that we have chosen x_β, and $U_\beta \in \mathcal{U}$ such that $x_\beta \in U_\beta \backslash \bigcup_{\gamma<\beta} U_\gamma$ for $\beta < \alpha$. Since $\mathcal{U}$ has no countable subcover, there is $x_\alpha \in Y \backslash \bigcup_{\beta<\alpha} U_\beta$, and since $\mathcal{U}$ is a cover, there is $U_\alpha \in \mathcal{U}$, with $x_\alpha \in U_\alpha$. The family $\{x_\alpha: \alpha < \omega^+\}$ is clearly right-separated.

7.7. THEOREM (KUNEN [75]). *Assume the continuum hypothesis. There is a scattered compact Hausdorff space K, such that K is not metrizable* (*and hence $C(K)$ is not separable*), *and for every $F \subset C(K)$ with $|F| = \omega^+$ there is $f \in F$ with $f \in \overline{\mathrm{conv}}(F \backslash \{f\})$* (*where $\overline{\mathrm{conv}}(H)$ denotes the closed convex hull of H in $C(K)$*).

PROOF. Let X be the topological space, given in Theorem 7.1 and let K be the one-point compactification of X. It is clear that K is compact Hausdorff scattered space. Since K is not metrizable (by the local countability of X), $C(K)$ is not separable. By Lemma 7.2, K^ω is hereditarily separable, and by Theorem 7.3 $C(K)$ is hereditarily Lindelöf in the pointwise topology, and hence, since K is scattered, by 7.4, in the weak topology. By Proposition 7.6, in $C(K)$ there is no weakly right-separated family. Let $F = \{f_\alpha: \alpha < \omega^+\} \subset C(K)$, with $f_\alpha \neq f_\beta$ for $\alpha < \beta < \omega^+$. There is $\alpha < \omega^+$, such that $f_\alpha \in \overline{\{f_\beta: \beta > \alpha\}}$ (the closure taken in the weak topology). Hence $f_\alpha \in \overline{\mathrm{conv}}(F \backslash \{f_\alpha\})$ (where the closed convex hull of $F \backslash \{f_\alpha\}$ in the weak and in the norm topology coincide, by 0.10).

The proof of the theorem is complete.

7.8. REMARKS: (a) Kunen's example (assuming CH) should be contrasted with the following classical theorem of MARKUSHEVICH [84] (cf. [81] (1.f.3)): Let X be a separable Banach space; then there is $(x_n) \subset X$ and $(x_n^*) \subset X^*$, such that

$$x^* \in X^*, x^*(x_n) = 0 \text{ for } n < \omega \Rightarrow x^* = 0\,,$$

$$x \in X, x_n^*(x) = 0 \quad \text{ for } n < \omega \Rightarrow x = 0\,,$$

and

$$x_n^*(x_m) = 0 \quad \text{if } n \neq m, x_n^*(x_n) = 1 \quad \text{for } n, m < \omega$$

Such a sequence (x_n) is called a *Markushevich basis for X*. Of course, $x_n \notin \overline{\text{conv}}(\{x_m : m < \omega, m \neq n\})$ for all $n < \omega$ for such a sequence (x_n). Thus a non-separable Banach space need not have the analogue of a Markushevich basis.

(b) SHELAH [113] was the first to construct a non-separable Banach space X, such that among any ω^+ elements of X there is one that belongs to the closure of the convex hull of the rest (solving a problem by Davis and Johnson). However, Shelah proved the existence of a Banach space, assuming the diamond principle for ω^+, and furthermore his space X was not of the form $X = C(K)$ for some compact space K. On the other hand, Shelah's example X had an additional property: every bounded linear operator $T: X \to X$ has the form $T = \lambda I + T_1$, where T_1 has separable range, $\lambda \in \mathbb{R}$, and I is the identity operator (cf. also Shelah's earlier paper [111]). It is not known if Kunen's example, or some modification, can have this additional property.

(c) If K is a Corson-compact space, then $C(K)$ is Lindelöf in its pointwise topology (see [3], and [55] for a stronger result). The converse is not true. Assuming CH, Kunen's example K is not a Corson-compact space (since K is separable, but not metrizable), but $C(K)$ is (hereditarily) Lindelöf in the pointwise topology. The continuum hypothesis is not necessary for such an example. An example, considered by WAGE [123], was proved by POL in [99], to be a non-Corson-compact scattered compact space, with pointwise Lindelöf space of continuous functions.

8. Fixed points of contractions in weakly compact convex subsets of Banach spaces

The main result of this Section is Maurey's Theorem (8.24) on fixed points for contractions on weakly compact convex subsets of c_0.

8.1. DEFINITION. Let C be a subset of a Banach space. A mapping $T: C \to C$ is called a *contraction* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$.

8.2. DEFINITION. Let C be a non-empty closed, convex subset of a Banach space, and $T: C \to C$ a contraction. A non-empty, closed, convex subset D of C such that $T(D) \subset D$ is called *T-minimal* if for every non-empty, closed convex subset E of D, with $T(E) \subset E$, we have $E = D$.

8.3. PROPOSITION. *If C is non-empty, convex and weakly compact in a Banach space X, and $T\colon C \to C$ a contraction, then there is a T-minimal subset of C.*

The proof follows from a straightforward application of Zorn's lemma.

8.4. REMARK. If T does not have a fixed point, then every T-minimal set is not a single point.

8.5. PROPOSITION. *If D is T-minimal, then* $D = \overline{\text{conv}}(T(D))$ (*where* $\overline{\text{conv}}\, T(D)$ *denotes the closed convex hull of* $T(D)$).

PROOF. Set $E = \text{conv}(T(D))$. Then E is convex, $E \subset D$ since D is convex, and thus $T(E) \subset T(D) \subset E$. By continuity $T(\bar{E}) \subset \overline{T(E)} \subset \bar{E}$. Since D is T-minimal, it follows that $E = \overline{\text{conv}}\, T(D) = D$.

8.6. DEFINITION. If C is a topological space, and $\phi\colon C \to \mathbb{R}$ then ϕ is a *lower-semicontinuous* (l.s.c.) function if $\{x \in C\colon \phi(x) \leq \theta\}$ is closed for all $\theta \in \mathbb{R}$.

If C is a compact space, and ϕ l.s.c. on C, then ϕ attains a minimum.

8.7. LEMMA. *Let C be a weakly compact, convex subset of a Banach space, $T\colon C \to C$ a contraction, with C T-minimal, and $\phi\colon C \to \mathbb{R}$ a* (*weakly*) l.s.c. *convex function, such that $\phi(T(x)) \leq \phi(x)$ for all $x \in C$. Then ϕ is a constant function.*

PROOF. By the assumptions, there is $x_0 \in C$ such that $\phi(x_0)$ is the minimum value of ϕ. The set $D = \{x \in C\colon \phi(x) \leq \phi(x_0)\}$ is non-empty, convex, closed, and $T(D) \subset D$ (since if $x \in D$, then $\phi(Tx) \leq \phi(x) \leq \phi(x_0)$). By the T-minimality of C, we conclude $D = C$, i.e., ϕ is constant.

8.8. COROLLARY. *Let C be a weakly compact, convex subset of a Banach space, $T\colon C \to C$ a contraction, with C T-minimal. Then* $\sup\{\|y - x\|\colon y \in C\} = \text{diam}(C)$ *for all $x \in C$* (*where* diam(C) *denotes the diameter of* C).

PROOF. The function $\phi\colon C \to \mathbb{R}$ given by

$$\phi(x) = \sup\{\|y - x\|\colon y \in C\}$$

satisfies the conditions of the lemma. In fact, ϕ is clearly l.s.c. and convex. We note that

$$\phi(x) = \sup\{\|x - T(y)\|\colon y \in C\}. \tag{1}$$

In fact, $\sup\{\|x - T(y)\|: y \in C\} \leq \phi(x)$. Suppose that for some $x \in C$, $\alpha = \sup\{\|x - T(y)\|: y \in C\} < \phi(x)$. Then, there is $y \in C$, with $\alpha < \|x - y\|$. Since C is T-minimal, $C = \overline{\mathrm{conv}}(T(C))$ (Proposition 8.5). Thus there is a convex combination $\lambda_1 y_1 + \cdots + \lambda_n y_n \in C$ of elements $y_1, \ldots, y_n \in C$, such that

$$\|\lambda_1 T(y_1) + \cdots + \lambda_n T(y_n) - y\| < (\|x - y\| - \alpha)/2\,,$$

whence a contradiction follows easily.

We next verify that $\phi(Tx) \leq \phi(x)$ for all $x \in C$. In fact, by the above equality (1), and the fact that T is a contraction,

$$\phi(Tx) = \sup\{\|Tx - Ty\|: y \in C\} \leq \sup\{\|x - y\|: y \in C\} = \phi(x)\,.$$

By Lemma 8.7, ϕ is a constant function. It follows from the definition of ϕ, that its constant value can only be equal to $\mathrm{diam}(C)$.

We need the classical

8.9. THEOREM (Fixed-point theorem of BANACH [24]). *If (C, ρ) is a complete metric space, $T: C \to C$, and there is $0 < \theta < 1$ such that $\rho(Tx, Ty) \leq \theta\rho(x, y)$ for all $x, y \in C$, then T has a (unique) fixed point.*

8.10. PROPOSITION. *Let C be a convex closed and bounded subset of a Banach space, $T: C \to C$ a contraction, and $\varepsilon > 0$. Then, there is $x \in C$, such that $\|T(x) - x\| \leq \varepsilon$.*

PROOF. We may assume, without loss of generality, that $\varepsilon < 1$. We choose $x_0 \in C$, and we define $S: C \to C$ by

$$S(x) = \varepsilon x_0 + (1 - \varepsilon)T(x)\,.$$

Then $S(C) \subset C$, since C is a convex set. Also, for $x, y \in C$,

$$\begin{aligned}\|S(x) - S(y)\| &= \|\varepsilon x_0 + (1 - \varepsilon)Tx - \varepsilon x_0 - (1 - \varepsilon)T_y\| \\ &= \|(1 - \varepsilon)Tx - Ty\| \leq (1 - \varepsilon)\|x - t\|\,.\end{aligned}$$

By Banach's fixed-point theorem (8.9) (since C is certainly a complete metric space), there is a (unique) $x \in C$ such that $S(x) = x$, i.e. $\varepsilon x_0 + (1 - \varepsilon)Tx = x$. Then $Tx - x = \varepsilon T(x) - \varepsilon x_0$, and hence, since C is bounded,

$$\|Tx - x\| \leq \varepsilon\|Tx - x_0\| \leq \varepsilon \cdot \mathrm{diam}(C)\,.$$

8.11. COROLLARY. *Let C be a closed, bounded, convex subset of a Banach space, and $T: C \to C$ a contraction. Then there is a sequence (x_n) in C such that*

$$\lim_{n\to\infty} \|T(x_n) - x_n\| = 0\,.$$

8.12. DEFINITION. Let C be a closed, bounded and convex subset of a Banach space, $T: C \to C$ a contraction, and p a non-trivial ultrafilter on ω. A sequence (x_n) in C, is called *T-quasi-fixed* (*relative to* p) if $\lim_p \|T(x_n) - x_n\| = 0$ (where for a sequence (x_n) in C, $\lim x_n = x$ means that for every open neighborhood U of x there is $A \in p$ such that $x_n \in U$ for $n \in A$).

8.13. PROPOSITION. *Let C be a weakly compact, convex set in a Banach space, $T: C \to C$ a contraction, with C T-minimal, p is a non-trivial ultrafilter on ω, and (x_n) a T-quasi-fixed sequence (relative to p) in C. Then*

$$\lim_p \|x - x_n\| = \operatorname{diam}(C) \quad \textit{for all } x \in C .$$

PROOF. We define $\Psi: C \to \mathbb{R}$ by $\Psi(x) = \lim_p \|x - x_n\|$. It is easy to see that Ψ is (weakly) l.s.c., convex, and (using that T is a contraction and that $\lim_p \|Tx_n - x_n\| = 0$) that $\Psi(Tx) \leq \Psi(x)$ for all $x \in C$. It follows from Lemma 8.7 that Ψ is a constant function. In order to find the constant value, we set $y = \text{weak-}\lim_p x_n \in C$. From the fact that the $\|\cdot\|$ function is weakly l.s.c. it follows that

$$\|x - y\| \leq \lim_p \|x - x_n\| = \Psi(x) \quad \text{for } x \in C .$$

Hence, from Corollary 8.8,

$$\sup_{y \in C} \|x - y\| = \operatorname{diam}(C) \leq \Psi(x) \leq \operatorname{diam}(C) \quad \text{for } x \in C .$$

8.14. DEFINITIONS. Let X be a Banach space, and p a (non-trivial) ultrafilter on ω. We set

$$l^\infty(X) = \{x = (x_n) \subset X : \sup_n \|x_n\| < \infty\} .$$

It is immediate that $l^\infty(X)$ is a Banach space with $\|x| = \sup_n \|x_n\|$.

We set

$$N_p = \{x = (x_n) \in l^\infty(X) : \lim_p \|x_n\| = 0\} .$$

It is easy to verify that N_p is a closed linear subspace $l^\infty(X)$.

The *ultrapower* $\tilde{X} = X^\omega/p$ *of* X (*modulo* p) is the quotient Banach space $l^\infty(X)/N_p$. So $\tilde{X}$ is a Banach space. An element $\tilde{x}$ of $\tilde{X}$ is of the form

$$(x_n)/p = \{(y_n) \in l^\infty(X) : \lim_p \|x_n - y_n\| = 0\} .$$

It is easy to verify that $\|\tilde{x}\| = \lim_p \|x_n\|$.

The *canonical embedding* $X \hookrightarrow \tilde{X}$ is given by the diagonal mapping $x \to (x, x, \ldots)/p$. The canonical embedding is a linear isometry.

If $C \subset X$, we set

$$\tilde{C} = \{\tilde{x} \in \tilde{X};\ \text{there are } x_n \in C \text{ for } n < \omega, \text{ with } \tilde{x} = (x_n)/p\}.$$

Of course, $C \subset \tilde{C}$, via the canonical embedding.

Let now C be a closed, bounded, convex set in the Banach space X, and $T: C \to C$ a contraction. We set $\tilde{T}: \tilde{C} \to \tilde{C}$ by

$$\tilde{T}(\tilde{x}) = \tilde{T}((x_n)/p) = (T(x_n)/p),$$

where $\tilde{x} = (x_n)/p$, and $x_n \in C$ for $n < \omega$.

It is easy to verify that $\tilde{T}$ is well-defined (i.e., independent of the choice of the representative in the equivalence class $\tilde{x}$), $\tilde{T}$ is an extension of T, $\tilde{T}(\tilde{C}) \subset \tilde{C}$, and $\tilde{T}$ is a contraction.

8.15. REMARK. Ultraproducts of Banach spaces have developed into an important and powerful tool of the theory of Banach spaces, following Krivine's pioneering ideas. Some basic papers are [39], [72], [73], and the survey paper by HEINRICH [64]. My favorite reference for ultrafilters and general ultraproduct structures is [34].

8.16. PROPOSITION. *Let C be a closed, bounded, convex subset of a Banach space, and $T: C \to C$ a contraction. Then*

(a) *$\tilde{T}$ has fixed points.*

(b) *If C is weakly compact and T-minimal, and $\tilde{x}$ is a fixed point of $\tilde{T}$, then*

$$\|\tilde{x} - x\| = \operatorname{diam}(C) = \operatorname{diam}(\tilde{C}) \quad \textit{for all } x \in C.$$

PROOF. (a) According to Corollary 8.11, there is a T-quasi-fixed sequence (x_n) modulo p (since p is a non-trivial ultrafilter). We set $\tilde{x} = (x_n)/p \in \tilde{C}$. Since $\|\tilde{T}(\tilde{x}) - \tilde{x}\| = \lim_p \|Tx_n - x_n\| = 0$, we have $\tilde{T}(\tilde{x}) = \tilde{x}$.

(b) Let $\tilde{T}(\tilde{x}) = \tilde{x} \in \tilde{C}$. Choose $x_n \in C$, such that $\tilde{x} = (x_n)/p$. Then (x_n) is a T-quasi-fixed sequence modulo p. By Proposition 8.13

$$\|\tilde{x} - x\| = \lim_p \|x_n - x\| = \operatorname{diam}(C) \quad \text{for } x \in C.$$

Furthermore, if $\tilde{x} = (x_n)/p$, $\tilde{y} = (y_n)/p$, with $x_n, y_n \in C$ for $n < \omega$, then

$$\|\tilde{x} - \tilde{y}\| = \lim_p \|x_n - y_n\| \leq \operatorname{diam}(C).$$

Thus $\operatorname{diam}(\tilde{C}) = \operatorname{diam}(C)$.

8.17. DEFINITION. A Banach space X is called *uniformly convex* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, it follows that $\|x + y\|/2 \leq 1 - \delta$.

8.18. REMARK. Every uniformly convex Banach space is reflexive, by the theorem of MILMAN [88], PETTIS [98]. A Banach space X is uniformly convex if and only if X is *super-reflexive* (i.e., all ultrapowers of X reflexive) (JAMES [65], ENFLO [45]).

8.19. THEOREM (BROWDER [32]). *Let X be a uniformly convex Banach space, C a closed, convex, bounded subset of X, and T: $C \to C$ a contradiction. Then T has a fixed point.*

PROOF. Every uniformly convex Banach space is reflexive (8.18) and hence C is weakly compact (0.9). Suppose that T does not have a fixed point. By Proposition 8.3, we assume without loss of generality that C is T-minimal, and $\operatorname{diam}(C) = 1$. Let p be a non-trivial ultrafilter on ω. The ultrapower $\tilde{X}$ of X modulo p is trivially seen to be uniformly convex. By Proposition 8.16(a), there is a fixed point $\tilde{x}$ of $\tilde{T}$. We choose $x, y \in C$, with $x \neq y$ (since $\operatorname{diam}(C) > 0$). Then, from Proposition 8.16(b),

$$\|x - \tilde{x}\| = \|y - \tilde{x}\| = \|(x + y)/2 - \tilde{x}\| = 1\,,$$

contradicting the uniform convexity of $\tilde{X}$.

8.20. COROLLARY. *If $1 < p < \infty$, and T: $C \to C$ is a contraction of a closed, bounded, convex subset C of $L^p(\mu)$, then T has a fixed point.*

PROOF. According to CLARKSON's theorem, (cf. [42] (Ch. 3, §1)) $L^p(\mu)$ is a uniformly convex Banach space so that Theorem 8.19 applies.

8.21. REMARK. The earlier fixed point theorems of the type considered in this section were due to BROWDER [31] (for Hilbert spaces), and [32] (where Theorem 8.19 was proved).

The compactness property

8.22. DEFINITION. If x, y, z are elements of a Banach space X, and $\|x - z\| = \|y - z\| = \frac{1}{2}\|x - y\|$, then the point z is called a *quasi-middle* of x and y.

8.23. PROPOSITION. *Let C be a closed, bounded, convex subset of a Banach space, T: $C \to C$ a contraction, and $\tilde{x}$, $\tilde{y}$ fixed points of $\tilde{T}$: $\tilde{C} \to \tilde{C}$. Then, there is $\tilde{z} \in \tilde{C}$, such that*

$\tilde{z}$ is a fixed point of $\tilde{C}$, and

$\tilde{z}$ is a quasi-middle of $\tilde{x}$ and $\tilde{y}$.

PROOF. We set

$$K = \{w \in \tilde{C}\colon w \text{ is a quasi-middle of } \tilde{x} \text{ and } \tilde{y}\}.$$

We claim that K is a non-empty closed, convex subset of $\tilde{C}$, and $\tilde{T}(K) \subset K$. (In fact, $K \neq \emptyset$ since $(\tilde{x}+\tilde{y})/2 \in K$. It is immediate that K is closed, and bounded. Let $w_1, w_2 \in K$, $0 \leq \lambda \leq 1$. Then

$$\begin{aligned}\|\tilde{x} - (\lambda w_1 + (1-\lambda)w_2)\| &= \|\lambda(\tilde{x} - w_1) + (1-\lambda)(\tilde{x} - w_2)\| \\ &\leq \|\tilde{x} - w_1\| + (1-\lambda)\|\tilde{x} - w_2\| \leq \tfrac{1}{2}\|\tilde{x} - \tilde{y}\|,\end{aligned}$$

and similarly $\|\tilde{y} - (\lambda w_1 + (1-\lambda)w_2)\| \leq \frac{1}{2}\|\tilde{x} - \tilde{y}\|$. It follows from the triangle inequality that

$$\|\tilde{x} - (\lambda w_1 + (1-\lambda)w_2)\| = \|\tilde{y} - (\lambda w_1 + (1-\lambda)w_2)\| = \tfrac{1}{2}\|\tilde{x} - \tilde{y}\|,$$

i.e., that $\lambda w_1 + (1-\lambda)w_2 \in K$.

Let $w \in K$. So $\|\tilde{x} - w\| = \|\tilde{y} - w\| = \frac{1}{2}\|\tilde{x} - \tilde{y}\|$. Then

$$\|\tilde{x} - \tilde{T}(w)\| = \|\tilde{T}(\tilde{x}) - \tilde{T}(w)\| \leq \|\tilde{x} - w\| = \tfrac{1}{2}\|\tilde{x} - \tilde{y}\|,$$

and similarly $\|\tilde{y} - \tilde{T}(w)\| \leq \frac{1}{2}\|\tilde{x} - \tilde{y}\|$. So,

$$\|\tilde{x} - \tilde{y}\| \leq \|\tilde{x} - \tilde{T}(w)\| + \|\tilde{T}(w) - \tilde{y}\| \leq \|\tilde{x} - \tilde{y}\|,$$

hence $\|\tilde{x} - \tilde{T}(w)\| = \|\tilde{y} - \tilde{T}(w)\| = \frac{1}{2}\|\tilde{x} - \tilde{y}\|$, i.e., $\tilde{T}(w) \in K$. Thus $\tilde{T}(K) \subset K$.

Thus, K is a closed, bounded, convex subset of $\tilde{X}$, and $\tilde{T}\colon \tilde{K} \to \tilde{K}$ is a contraction).

By Proposition 8.11, there is a sequence $(t_m) \subset K$, with

$$\|\tilde{T}(t_m) - t_m\| \leq 1/m \quad \text{for } m = 1, 2, \ldots .$$

We choose $x_n, y_n, t_{m,n} \in C$, with

$$\tilde{x} = (x_n)/p, \qquad \tilde{y} = (y_n)/p, \qquad t_m(t_{m,n})/p \quad \text{for } m = 1, 2, \ldots .$$

(where p is the ultrafilter on ω, such that $\tilde{X}$ is the ultrapower of X modulo p). We have

$$\|\tilde{x} - t_m\| = \lim_p \|x_n - t_{m,n}\| = \tfrac{1}{2}\|\tilde{x} - \tilde{y}\|,$$

$$\|\tilde{y} - t_m\| = \lim_p \|y_n - t_{m,n}\| = \tfrac{1}{2}\|\tilde{x} - \tilde{y}\|,$$

and

$$\|\tilde{T}(t_m) - t_m\| = \lim_p \|T(t_{m,n}) - t_{m,n}\| \leq 1/m \quad \text{for } m = 1, 2, \ldots .$$

We choose $A_m \in p$, $m = 1, 2, \ldots$, such that

$$\omega \supset A_1 \supset A_2 \supset \cdots \supset A_m \supset \cdots, \qquad \bigcap_{m=1}^{\infty} A_m = \emptyset,$$

$$\|T(t_{k,n}) - t_{k,n}\| \leq \frac{1}{k},$$

$$\left| \|y_n - t_{k,n}\| - \frac{1}{2}\|\tilde{x} - \tilde{y}\| \right| < \frac{1}{k},$$

$$\left| \|x_n - t_{k,n}\| - \frac{1}{2}\|\tilde{x} - \tilde{y}\| \right| < \frac{1}{k}$$

for $n \in A_m$, $k = 1, 2, \ldots, m$. We set

$$t_n = t_{m,n} \quad \text{for } n \in A_m \backslash A_{m+1},\ m = 1, 2, \ldots, \qquad \text{and} \qquad t = (t_n)/p.$$

Then $t \in K$, and $\tilde{T}(t) = t$

8.24. THEOREM (MAUREY [85]). *Let C be a non-empty weakly compact, convex subset of c_0, and $T: C \to C$ a contraction. Then T has a fixed point.*

PROOF. Suppose that T does not have a fixed point. By Proposition 8.3, we assume that C is T-minimal. So, $\operatorname{diam}(C) > 0$, and without loss of generality assume $\operatorname{diam}(C) = 1$. By Proposition 8.11, there is $(x_n) \subset C$, such that

$$\lim_n \|T(x_n) - x_n\| = 0.$$

We may assume, by passing to a subsequence of (x_n) if necessary, since C is weakly compact (and using Eberlein's theorem), that there is $x \in C$, such that $x_n \to x$ weakly. Furthermore, by a translation of C, we assume that $x = 0$, i.e., that $x_n \to 0$ weakly. By proposition 8.13,

$$\lim_p \|x_n\| = \operatorname{diam}(C) = 1,$$

for any non-trivial ultrafilter p on ω. We claim that there is a subsequence (k_n) such that

$$\|\min\{|x_n|, |x_{k_n}|\}\| \leq 1/n \quad \text{for } n = 1, 2, \ldots .$$

(In fact, suppose $k_1 < \cdots < k_{n-1}$ have been defined. Since $x_n \in C \subset c_0$, there is a natural number N_n such that

$$m \leqslant n_n \Rightarrow |x_n(m)| \leqslant 1/n .$$

Since $x_n \to 0$ weakly, there is $k_n > k_{n-1}$, such that

$$k \geqslant k_n \Rightarrow |x_k(m)| \leqslant 1/n \quad \text{for } m < N_n .$$

Then, of course, $\|\min\{|x_n|, |x_{k_n}|\}\| \leqslant 1/n$).

It is then easy to verify that

$$\lim_n \|x_n - x_{k_n}\| = 1 .$$

We choose a non-trivial ultrafilter p on ω, and we set $\tilde{x} = (x_n)/p$, $\tilde{y} = (y_{k_n})/p$ in the ultrapower c_0^ω/p. Then $\tilde{x}, \tilde{y} \in \tilde{C}$, $\tilde{x}, \tilde{y}$ are fixed point for $\tilde{T}$, and $\|\tilde{x} - \tilde{y}\| = 1$.

By the compactness property (Proposition 8.23), there is $\tilde{z} = (z_n)/p \in \tilde{C}$, such that $\tilde{z}$ is a fixed point for $\tilde{T}$, and $\|\tilde{x} - \tilde{z}\| = \|\tilde{y} - \tilde{z}\| = \frac{1}{2}$.

We have $\lim_p \|z_n\| = 1$, and $\lim_p \|x_n - z_n\| = \lim_p \|x_{k_n} - z_n\| = \frac{1}{2}$. Furthermore, using the fact that for any three real numbers α, β, γ the inequality $|\alpha| \leqslant \max\{|\alpha - \beta|, |\alpha - \gamma|\} + \min\{|\beta|, |\gamma|\}$ holds, we have

$$|z_n| \leqslant \max\{|z_n - x_n|, |z_n - x_{k_n}|\} + \min\{|x_n|, |x_{k_n}|\} ,$$

it follows that $\|\tilde{z}\| = \lim_p \||\, z_n \,|\| \leqslant \frac{1}{2}$, a contradiction proving the theorem.

8.25. REMARK. ODELL and STERNFELD in [94] had proved, with concrete methods, a special case of Theorem 8.24, for the case that C is the closed convex hull of a weakly convergent sequence in c_0. Some early references on fixed point theorems of the type considered here are given in the Odell–Sternfeld paper, and in DIESTEL's monograph [42] (Chapter 2, §5).

8.26. EXAMPLE. ALSPACH in [1] has constructed a weakly compact convex subset K of $L^1[0, 1]$, and a mapping $T: K \to K$ with $\|T(x) - T(y)\| \leqslant \|x - y\|$ for all x, $y \in K$, such that T has no fixed point. We describe briefly this example. We set

$$K = \left\{ f \in L^1[0, 1]: \int f \,\mathrm{d}\mu = 1 \text{ and } 0 \leqslant f \leqslant 2 \ \mu\text{-a.e} \right\} .$$

The weak compactness of K follows directly from the Dunford–Pettis Theorem (0.29). We set $T: K \to K$ by

$$\begin{aligned} Tf(t) &= \min\{2f(2t), 2\} && \text{for } C \leqslant t \leqslant \tfrac{1}{2}, \\ &= \max\{2f(2t-1)-2, 0\} && \text{for } \tfrac{1}{2} \leqslant t \leqslant 1. \end{aligned}$$

This example is related to Baker's transformation in ergodic theory.

8.27. REMARK. Alspach's counter-example should be contrasted with the following positive result by MAUREY [85]: If X is a reflexive Banach subspace of $L^1[0, 1]$, K a closed bounded, and convex subset of X, and $T: K \to K$ a contraction, then T has a fixed point. In the proof Maurey uses ultrapowers of Banach spaces, and probabilistic techniques.

8.28. PROBLEM. If K is a closed and bounded convex subset of a reflexive Banach space, and $T: K \to K$ a contraction, does T necessarily have a fixed point?

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CHAPTER 24

Topological Groups

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HANDBOOK OF SET-THEORETIC TOPOLOGY
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1. Introduction: Seven major results

1.1. DEFINITION. A *topological group* is a pair $\langle G, \mathcal{T}\rangle$ for which

(i) $\langle G, \mathcal{T}\rangle$ is a topological space,

(ii) G is a group (that is, a non-empty set together with a binary operation satisfying the usual group axioms), and

(iii) the function from $G \times G$ to G defined by $\langle a, b\rangle \to ab^{-1}$ is continuous.

1.2. In this chapter we investigate some aspects of the theory of topological groups which may please or intrigue the set-theoretic topologist; if euphony did not dictate against it, our limited goal might be described more accurately by titling the chapter "On the topological theory of topological groups". In any event we fall prey unblushingly to the topologist's conventional infatuation with cardinal invariants, pathological counter-examples to attractive conjectures, and the like; and regretfully, being governed by certain natural limitations of space, energy and mathematical competence, we ignore almost completely the literature concerning harmonic analysis, Lie groups, topological vector spaces, transformation groups, topological semi-groups and related topics. In Sections 2 through 10, we give proofs when they are available in ZFC within reasonably brief compass. We tend to record without proof or verification those theorems or examples which are not definitive in the sense that (i) the only known argument uses additional axioms—typically the continuum hypothesis (CH) or Martin's Axiom (MA), and yet (ii) the independence of the statement in question has not been established.

1.3. We use without proof, often without drawing explicit attention to their use, many facts which follow without difficulty directly from Definition 1.1. Here are some examples: (a) In the presence of (i) and (ii), condition (iii) of 1.1 may be replaced by the condition that the functions $\langle a, b\rangle \to ab$ and $a \to a^{-1}$ are both continuous; left-translation by a fixed element, right-translation by a fixed element, and inversion, are homeomorphisms of G onto G; a subgroup with non-empty interior is open; an open subgroup is closed.

1.4. In this section we state seven major results in the topological theory of topological groups, together with closely related material. Their proofs are sufficiently lengthy and non-trivial that it is impossible to include them within the number of pages here allowed; with regret, we settle for references to the original sources, and to expository texts when these are available.

Here are the fundamental constructions we cite: Separation and metrizability (1.6–1.10); uniform continuity and completion (1.11–1.13); Haar measure (1.14–1.19); the 'sufficiently many characters' theorem and Pontrjagin duality (1.20–1.31); some structure theorems for locally compact groups (1.32–1.35); dyadicity of compact groups (1.36–1.43); and the existence of continuous functions onto cubes of the form $[0, 1]^{w(G)}$ (1.44–1.46).

1.5. We introduce notation locally when it is called for, but certain standing conventions concerning terminology, symbols and notations are used throughout. We record these conventions now.

The symbols α, β, and γ denote cardinal numbers, while ξ, ζ, η are ordinals. The first infinite cardinal is denoted ω. The cardinal successor of an (infinite) cardinal α is denoted α^+; the symbols ω^+, ω_1 and $\aleph_1$ are identical in meaning and are interchangeable.

The symbol λ, or λ_G, is reserved to denote left Haar measure on a locally compact group G; similarly μ and ν are used (only) to denote measures.

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{T}$ and $\mathbb{C}$ denote respectively the sets of integers, rational numbers, real numbers, complex numbers of modulus 1, and complex numbers; in each case and context the usual algebraic and topological properties are assumed and used to the degree appropriate and necessary.

As was suggested in 1.1, we write our groups multiplicatively. That is, for a, $b \in G$ the 'product' is denoted ab or $a \cdot b$, and the inverse of a is denoted a^{-1}. (We adopt the exceptions to this rule dictated by common sense. When G is, say, the additive group $\mathbb{Z}$ or $\mathbb{R}$, we use $a+b$ and $-a$ as expected.) For A, $B \subset G$ and $p \in G$ we write

$$AB = \{ab : a \in A, b \in B\}, \qquad A^2 = AA, \qquad A^{n+1} = AA^n \quad (2 \leq n < \omega),$$
$$Ap = A\{p\} \quad \text{and} \quad pA = \{p\}A.$$

We note that in general the relation $A^n = \{a^n : a \in A\}$ is false.

The identity or neutral element of a group G is denoted e_G or, quite frequently when confusion is impossible, simply by e. (In certain specific contexts as above, the symbol 0 replaces e.)

For G a topological group and $p \in G$, we denote by $\mathcal{N}_G(p)$ or $\mathcal{N}(p)$ the set of open neighborhoods of p in G. In particular $\mathcal{N}(e)$, also written $\mathcal{N}_G(e)$ or $\mathcal{N}(e_G)$, is the open neighborhood system at e and we have

$$\mathcal{N}(p) = \{pU : U \in \mathcal{N}(e)\} = \{Up : U \in \mathcal{N}(e)\} \quad \text{for all } p \in G.$$

For $\{G_i : i \in I\}$ a set of groups, we write

$$\bigoplus_{i \in I} G_i = \{x \in \Pi_{i \in I} G_i : |\{i \in I : x_i \neq e_i\}| < \omega\}.$$

If $G_i = G$ for all $i \in I$, then $\bigoplus_{i \in I} G_i$ may be written $\bigoplus_I G$. We use this notation only for Abelian groups G_i. The group $\bigoplus_{i \in I} G_i$ is given only the discrete topology.

For a group G, the symbol G_d denotes G with the discrete topology.

We note in 1.6 that every topological group satisfying the T_0 separation property is a completely regular, Hausdorff space (i.e., a Tychonoff space). Throughout this chapter after 1.6, every topological group is understood (i.e., is assumed) to be a completely regular, Hausdorff topological group. An analogous convention applies throughout this chapter to topological spaces: the four expressions 'space', 'topological space', 'Tychonoff space' and 'completely regular, Hausdorff space' have identical meanings and are used interchangeably.

For spaces X and Y we write

$$C(X, Y) = \{f \in Y^X : f \text{ is continuous}\}.$$

The symbol $C(X)$ abbreviates $C(X, \mathbb{R})$, and

$$C^*(X) = \{f \in C(X) : f \text{ is a bounded function}\}.$$

For groups G and H, the symbol $\mathrm{Hom}(G, H)$ denotes as usual the set of homomorphisms from G into H.

For topological groups G and H, the notation $G \simeq H$ indicates that there is a topological isomorphism from G onto H.

The symbol $[\ ]$ does double duty: (i) For G a group and $A \subset G$, the symbol $[A]$ denotes the subgroup of G generated by A; (ii) for sets S and T and $f \in T^S$ with $A \subset S$, we write $f[A] = \{f(x) : s \in A\}$.

The overbar $\bar{\ }$ does triple duty: (i) for G a locally bounded group, $\bar{G}$ is the Weil completion of G (see 1.13); (ii) for $a \in \mathbb{C}$, and for $A = (a_{ij})$ a matrix with complex entries, the symbols $\bar{a}$ and $\bar{A} = (\overline{a_{ij}})$ denote the usual complex conjugation; (iii) for spaces X, Y and K with K compact and $Y \subset K$ and $f \in C(X, Y)$, the symbol $\bar{f}$ denotes the Stone extension from $\beta(X)$ into K of f—that is, $\bar{f}$ is the unique function $g \in C(\beta(X), K)$ such that $g|X = f$. (The extension $\bar{f}$ is well-defined only when K has been specified. In most applications, as in 7.5 below, one has $K = \beta(Y)$.)

The closure and the interior in a space X of a subset A of X are denoted $\mathrm{cl}_X A$ and $\mathrm{int}_X A$, respectively.

1.6. The following theorem, proved by an onion-skin argument reminiscent of the proof by URYSOHN [1925] that disjoint closed subsets of a normal space are contained in disjoint zero-sets, is cited by WEIL [1973] (page 13) as due in its essentials independently to Leon Pontrjagin (unpublished letter) and KAKUTANI [1936]. As with Urysohn's theorem, it is remarkable here that continuous functions are shown to exist in profusion in a context to which they appear quite foreign. A careful proof of this result and of closely related statements is given by HEWITT and ROSS [1963] (§8) and by MONTGOMERY and ZIPPIN [1955] (1.17 ff.).

THEOREM. *Every topological group which satisfies the T_0 separation property is a completely regular, Hausdorff space.*

1.7. Henceforth in this chapter, every (hypothesized) topological group is assumed to be a completely regular, Hausdorff space, i.e., a Tychonoff space.

1.8. The following theorem, closely related to 1.6, is due independently to KAKUTANI [1936] and (part (i)) to BIRKHOFF [1936].

THEOREM. *Let G be a topological group.*

(i) *G is metrizable if and only if G is first countable* (*in the sense that there is* $\{U_n : n < \omega\} \subset \mathcal{N}(e)$ *such that every* $U \in \mathcal{N}(e)$ *contains one of the sets* U_n).

(ii) *If the conditions of* (i) *are satisfied, then the compatible metric ρ for G may be chosen left-invariant* (*in the sense that* $\rho(pa, pb) = \rho(a, b)$ *for all* $a, b, p \in G$).

1.9. It is not difficult to see that if ρ is a compatible metric for the topological group G and ρ is both left- and right-invariant, then for sequences a_n and b_n in G one has $a_n b_n \to e$ if and only if $b_n a_n \to e$. From this we see that not every metrizable topological group admits a compatible two-sided invariant metric. For example let G be the matrix group $G = \mathrm{SL}(2, \mathbb{R})$ defined and topologized as in 9.8 below, and for $0 < n < \omega$ define $A(n) = a(n)_{i,j}$ and $B(n) = b(n)_{i,j}$ by

$$a(n)_{11} = a(n)_{12} = b(n)_{12} = b(n)_{22} = 1/n\,,$$

$$a(n)_{22} = b(n)_{11} = n,\ a(n)_{21} = b(n)_{21} = 0\,.$$

Then $A(n) \cdot B(n) \to e_G$, $B(n) \cdot A(n) \not\to e_G$.

It is noted by HEWITT and ROSS [1963] (8.6), however, that every compact metrizable topological group does admit a two-sided invariant metric.

1.10. HEWITT and ROSS [1963] (4.22(a), (d)) have recorded some examples showing that even for countable, Abelian topological groups, first countability (that is, metrizability) is not a consequence of the condition that $\{e\}$ be a G_δ. For another example let $\omega^+ \leq \alpha \leq 2^\omega$ and, using the Hewitt–Marczewski–Pondiczery theorem (see 11.2 in the article in this Handbook by Hodel), let G be a countable, dense subgroup of the group $K = \{-1, +1\}^\alpha$. Of course each point of G is a G_δ in G, and K itself is not metrizable; it then follows easily, as in 2.7 below, that the dense subgroup G of K is not metrizable.

1.11. Given a topological group $G = \langle G, \mathcal{T}\rangle$ and a subgroup H of G, it is tempting to strive for a stronger topological group topology $\mathcal{U}$ on G by defining H and each of its (say) left-translates to be $\mathcal{U}$-open; specifically, one would define

$$\mathcal{U} = \{aH \cap V : a \in G,\ V \in \mathcal{T}\}$$

in the hope that $\langle G, \mathcal{U}\rangle$ is a topological group. This proposed topology $\mathcal{U}$ for G may suffer from two fatal flaws: the functions $\langle a, b\rangle \to ab$ and $a \to a^{-1}$ may fail to be $\mathcal{U}$-continuous. (In case the subgroup H is normal—for example, in case G is Abelian as in 6.17 below—this potential difficulty does not arise and

$\langle G, \mathcal{U}\rangle$ is indeed a topological group.) Such considerations serve adequately to introduce the following result, an elegant and serviceable axiomatization of the topology of a topological group due to WEIL [1927]. The proof is straightforward. For help and additional references concerning early definitions and characterizations of topological groups, the reader may consult MONTGOMERY and ZIPPIN [1955] (1.15 and page 18) or HEWITT and ROSS [1963] (4.5 and Notes to §4).

THEOREM. *For every topological group $G = \langle G, \mathcal{T}\rangle$ the open neighborhood system $\mathcal{N} = \mathcal{N}(e)$ at e satisfies*

(i) $\bigcap \mathcal{N} = \{e\}$,

(ii) *for $U, V \in \mathcal{N}$ there is $W \in \mathcal{N}$ such that $W \subset U \cap V$,*

(iii) *for $U \in \mathcal{N}$ there is $V \in \mathcal{N}$ such that $VV^{-1} \subset U$,*

(iv) *for $a \in U \in \mathcal{N}$ there is $V \in \mathcal{N}$ such that $aV \subset U$, and*

(v) *for $U \in \mathcal{N}$ and $x \in G$ there is $V \in \mathcal{N}$ such that $xVx^{-1} \subset U$.*

Conversely: Given a group G and a family $\mathcal{N}$ of subsets of G satisfying (i), (ii), (iii), (iv) *and* (v), *there is a topological group topology $\mathcal{T}$ for G for which $\mathcal{N}$ is an open basis at e; the family*

$$\{aU: a \in G, U \in \mathcal{N}\} = \{Ua: a \in G, U \in \mathcal{N}\}$$

is an open basis for $\mathcal{T}$.

1.12. A topological group G admits two natural uniform structures. When these coincide the functions $\langle a, b\rangle \to ab^{-1}$ and $a \to a^{-1}$ are uniformly continuous, hence continuously extendable over the uniform completion of G. In this case, then, the uniform completion of G is itself a topological group (with G a topological subgroup). In order that $\langle a, b\rangle \to ab$ be (say) left-uniformly continuous and hence extendable over the left-completion, it is enough that $a \to a^{-1}$ be left-uniformly continuous. (*Proof.* Given $V \in \mathcal{N}(e)$ there is then $W = W^{-1} \in \mathcal{N}(e)$ such that $W^{-1}x^{-1} \subset x^{-1}V$, and hence $Wb \subset bV$, for all $x, b \in G$. Now given $U \in \mathcal{N}(e)$ choose $V \in \mathcal{N}(e)$ so that $VV \subset U$, and then choose W as above with $W \subset V$. Then for $a, b \in G$ we have

$$(aW)(bW) \subset abVW \subset abVV \subset abU,$$

as required.) More generally, as is proved in the seminal work of WEIL [1937], it is enough that $a \to a^{-1}$ be uniformly continuous on some $V \in \mathcal{N}(e)$, and for this in turn it suffices that some $V \in \mathcal{N}(e)$ be bounded in the sense of the following definition.

DEFINITION. Let $G = \langle G, \mathcal{T}\rangle$ be a topological group and let $A \subset G$. Then

(i) A is *bounded* if for all $U \in \mathcal{N}(e)$ there is finite $F \subset G$ such that $A \subset FU$;

(ii) G is *locally bounded* if some $V \in \mathcal{N}(e)$ is bounded; and

(iii) G is *totally bounded* if G itself is bounded.

The apparent preference conferred in (ii) and (iii) upon the left uniform structure by the inclusions $V \subset FU$ and $G \subset FU$ (rather than $V \subset UF$ and $G \subset UF$) is illusory: If (for example) some $V \in \mathcal{N}(e)$ is bounded in the sense of (i) and if $U \in \mathcal{N}(e)$, then there is finite $F \subset G$ such that $V \subset FU^{-1}$; we then have $V^{-1} \in \mathcal{N}(e)$, and $V^{-1} \subset UF^{-1}$ with F^{-1} finite.

1.13. The completions of a locally bounded topological group G with respect to its two natural uniform structures coincide. We refer to this completion as the Weil completion of G; throughout this chapter the symbol $\bar{G}$ is used exclusively to denote the Weil completion of a locally bounded topological group G. Considerations evolving from the observations of 1.12 allow WEIL [1937] to prove the following result.

THEOREM. *Let G be a locally bounded topological group and $\bar{G}$ its completion.*

(i) *$\bar{G}$ is a locally compact topological group containing G as a (dense) topological subgroup; $\bar{G}$ is unique in the sense that for any locally compact topological group K containing G as a dense topological subgroup there is a topological isomorphism i from $\bar{G}$ onto K such that $i(x) = x$ for all $x \in G$.*

(ii) *If in addition G is totally bounded, then $\bar{G}$ is compact.*

When conditions akin to those described in 1.12 fail for a topological group G, its completions will be distinct and they will not admit the structure of a topological group. For examples of this phenomenon, see 5.10–5.12 below.

1.14. DEFINITION. *A measure space* is a triple $\langle X, \mathcal{S}(\mu), \mu\rangle$ with X a set, $\mathcal{S}(\mu)$ a σ-algebra of subsets of X, and μ a function from $\mathcal{S}(\mu)$ to $\mathbb{R} \cup \{\infty\}$ such that

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(A) \geqslant 0$ for $A \in \mathcal{S}(\mu)$ and

(iii) $\mu(\bigcup_{n<\omega} A_n) = \Sigma_{n<\omega}\mu(A_n)$ whenever $\{A_n : n < \omega\}$ is a pairwise disjoint sequence of elements of $\mathcal{S}(\mu)$.

If $X = \langle X, \mathcal{T}\rangle$ is a topological space and if $\mathcal{S}(\mu)$ contains every Borel set of X (that is, every set in the σ-algebra generated by $\mathcal{T}$), then the measure μ is said to be a *Borel measure* on X. If this occurs then for every compact subset F of X and for every open subset U of X we have $F \in \mathcal{S}(\mu)$ and $U \in \mathcal{S}(\mu)$ and for (arbitrary) $A \subset X$ one may define

$$\mu^*(A) = \inf\{\mu(U): A \subset U,\ U \text{ is open}\}, \quad \text{and}$$
$$\mu_*(A) = \sup\{\mu(F): F \subset A,\ F \text{ is compact}\}.$$

A Borel measure μ is said to be *regular* provided

(i) if A is open then $\mu(A) = \mu_*(A)$, and

(ii) if $A \in \mathcal{S}(\mu)$ then $\mu(A) = \mu^*(A)$.

With this terminology the Haar measure theorem takes the following form.

1.15. THEOREM. *For every locally compact topological group G there is a regular Borel measure $\lambda = \lambda_G$ on G such that*

(i) $\lambda(U) > 0$ *for* $\emptyset \neq U \in \mathcal{T}$;

(ii) $\lambda(F) < \infty$ *for compact* $F \subset G$; *and*

(iii) $\lambda(aA) = \lambda(A)$ *for* $a \in G$, $A \in \mathcal{S}(\lambda)$.

Further, λ is unique in the sense that if μ is a regular Borel measure on G satisfying (the analogues of) (i), (ii) *and* (iii), *then there is a positive constant c such that* $\mu(A) = c\lambda(A)$ *for all* $A \in \mathcal{S}(\lambda) \cap \mathcal{S}(\mu)$—*in particular, for all Borel sets A of G.*

1.16. We make several comments concerning Theorem 1.15 and its use in this Chapter.

(i) A measure λ as in 1.15 is called a (left) Haar measure for G. We use the symbols λ and λ_G only in this context.

(ii) Condition 1.15(iii) is referred to as left translation-invariance.

(iii) When G itself is compact we have from 1.15(iii) that $\lambda(G) < \infty$; in this case we follow standard practice and assume $\lambda(G) = 1$.

(iv) Let $\mathcal{S}(\bar{\mu})$ denote the family of subsets E of G for which there exist A, $B \in \mathcal{S}(\mu)$ with $A \subset E \subset B$ and $\mu(B \backslash A) = 0$, and define $\bar{\mu}(E) = \mu(A)$. Then $\langle X, \mathcal{S}(\bar{\mu}), \bar{\mu}\rangle$ is a measure space in the sense of Definition 1.14, complete in the sense that if $B \subset A \in \mathcal{S}(\bar{\mu})$ and $\bar{\mu}(A) = 0$, then $B \in \mathcal{S}(\bar{\mu})$ and $\bar{\mu}(B) = 0$. Since properties (i), (ii) and (iii) of Theorem 1.15 are preserved under passage from $\langle G, \mathcal{S}(\lambda), \lambda\rangle$ to $\langle G, \mathcal{S}(\bar{\lambda}), \bar{\lambda}\rangle$, we may (and we do) assume without loss of generality that our Haar measures are complete, i.e., that $\lambda = \bar{\lambda}$ and $\mathcal{S}(\lambda) = \mathcal{S}(\bar{\lambda})$.

For theorems relating to the definition of Haar measures and their Hopf or Carathéodory extensions, the reader may consult HEWITT and ROSS [1963] (§11 and 13), HEWITT and STROMBERG [1965] (§10 and 11), HALMOS [1950] (§54) or BERBERIAN [1965].

1.17. Theorem 1.15 is the culmination of work by many mathematicians, including (in an order approximately chronological) S. Lie, H. Weyl, J. von Neumann, A. Haar himself, S. Kakutani, H. Cartan, and A. Weil. For a description of the specific contributions of these workers and for references to the literature, the reader may consult HEWITT and ROSS [1963] (Notes to §15 and §16) and NACHBIN [1965].

1.18. Let G be a compact topological group and U an open subset of G. It is not difficult to see, using the regularity of the measure $\lambda = \lambda_G$, that there is a sequence $\{F_n : n < \omega\}$ of compact sets such that $\bigcup_n F_n \subset U$ and $\lambda(\bigcup_{n<\omega} F_n) = \lambda(U)$. Since the topological space G is normal, for $n < \omega$ there is a cozero set W_n of G such that $F_n \subset W_n \subset U$, and then the set $W = \bigcup_{n<\omega} W_n$ is a cozero set of G such that $W \subset U$ and $\lambda(W) = \lambda(U)$. The following theorem contains an equally useful, less trivial assertion.

THEOREM. *Let G be a compact topological group and U an open subset of G. Then there are cozero sets V and W of G such that $W \subset U \subset V$ and $\lambda(W) = \lambda(U) = \lambda(V)$.*

1.19. Theorem 1.18, which is adequate to our purposes in 6.14 below, is a special case of a result proved by HEWITT and ROSS [1963] (19.30); their proof is based on an argument of HALMOS [1950] (§64) using the Kakutani-Kodaira theorem (3.7 below). (There is a compact, normal G_δ subgroup N of G such that $\lambda(U) = \lambda(UN)$. The quotient group G/N, topologized as usual, is metrizable by 1.8 above. The canonical homomorphism $\varphi: G \to G/N$ takes U to an open set $\varphi[U]$, and the set $V = \varphi^{-1}(\varphi[U]) = UN$ is as required.)

1.20. For a proof of the following result, which is the theorem of GEL'FAND and RAĬKOV [1943], [1944], as well as for definitions and references to related literature, the reader should consult HEWITT and ROSS [1963] (§22).

THEOREM. *Let G be a locally compact topological group and let $e \neq a \in G$. Then there is a continuous, irreducible unitary representation h of G such that $h(a) \neq h(e)$.*

1.21. When G is compact, the Gel'fand–Raĭkov theorem takes the following form; here for $0 < n < \omega$ the symbol $U(n, \mathbb{C})$ denotes the group of complex $n \times n$ unitary matrices, defined and topologized as usual as in 9.8 below.

THEOREM. *Let G be a compact topological group and let $e \neq a \in G$. Then there are a positive integer n and a continuous homomorphism $h: G \to U(n, \mathbb{C})$ such that $h(a) \neq h(e)$.*

1.22. When G is Abelian the continuous, irreducible unitary representations of G are in effect continuous homomorphisms into $\mathbb{T}$, and 1.20 reads as follows.

THEOREM. *Let G be a locally compact Abelian topological group and let $e \neq a \in G$. Then there is continuous $h \in \mathrm{Hom}(G, \mathbb{T})$ such that $h(a) \neq 1$.*

1.23. For an Abelian topological group G, we denote by $\hat{G}$ the set of continuous homomorphisms from G to $\mathbb{T}$. Then $\hat{G}$, itself a group under pointwise multiplication, is called the dual group of G, and its elements are called characters on G. With this convention, 1.22 becomes the statement that for a locally compact, Abelian topological group G, the group $\hat{G}$ distinguishes (or separates) points of G; in the terminology of HEWITT and ROSS [1963], G has sufficiently many characters.

1.24. Let G be a locally compact, Abelian topological group and for compact $F \subset G$ and $\varepsilon > 0$ define

$$P(F, \varepsilon) = \{\chi \in \hat{G}: |\chi(x) - 1| < \varepsilon \text{ for all } x \in F\}.$$

A brief inspection indicates that the family

$$\mathcal{N} = \{P(F, \varepsilon): \text{compact } F \subset G,\ \varepsilon > 0\}$$

satisfies the conditions of Theorem 1.11; thus the family

$$\{\chi \cdot U: \chi \in \hat{G},\ U \in \mathcal{N}\}$$

is a basis for a topological group topology $\mathcal{T}$ for $\hat{G}$. In fact, $\mathcal{T}$ is the topology of uniform convergence on compact subsets of G, i.e., the compact-open topology. The topology $\mathcal{T}$ was defined and studied in important special cases by PONTRJAGIN [1934] and VAN KAMPEN [1935], and in full generality in 1941 by WEIL [1951].

Whenever a locally compact, Abelian topological group G is given, references to 'the topological group $\hat{G}$' should be interpreted as references to $\langle \hat{G}, \mathcal{T} \rangle$ with $\mathcal{T}$ as above.

1.25. It is proved in the references cited above that the dual group $\hat{G}$ is itself locally compact. Iterating, $\hat{\hat{G}}$ is locally compact; the way is paved for the following beautiful result, normally referred to as the Pontrjagin duality theorem.

THEOREM. *Let G be a locally compact Abelian topological group and define $\upsilon: G \to \hat{\hat{G}}$ by*

$$\upsilon(x)\chi = \chi(x) \quad \textit{for } x \in G,\ \chi \in \hat{G}.$$

Then υ is a topological isomorphism of G onto $\hat{\hat{G}}$.

1.26. Theorem 1.25 is rich with consequences and sequels. Here we list several taken from HEWITT and ROSS [1963]; the reader is referred there for proofs. (Strictly speaking, some of what follows is logically available prior to 1.25 and is in fact necessary to the proof of 1.25.) Throughout 1.27–1.31, we assume that G, G_i ($i \in I$) and so forth are locally compact topological Abelian groups and that dual groups are topologized as in 1.24. As usual, the symbol $\simeq$ may be read 'is topologically isomorphic with'. In each case there is a specific and easily discerned function which establishes the indicated topological isomorphism; since the functions themselves, rather than simply their existence, are required for most applications, any full and careful statement of these results (as in HEWITT and ROSS [1963]) will involve their explicit definition.

1.27. (i) *G is compact if and only if $\hat{G}$ is discrete;*
(ii) *G is discrete if and only if $\hat{G}$ is compact.*

1.28. (i) *If each G_i is compact, then $(\Pi_{i\in I}G_i)^{\wedge} \simeq \bigoplus_{i\in I}\hat{G}_i$,*
(ii) *If each G_i is discrete, then $(\bigoplus_{i\in I}G_i)^{\wedge} \simeq \Pi_{i\in I}\hat{G}_i$.*

1.29. *If H is a closed subgroup of G and $a \in G\backslash H$, there is $\chi \in \hat{G}$ such that $\chi(x) = 1$ for all $x \in H$, and $\chi(a) \neq 1$.*

1.30. *If H is a closed subgroup of G, then $(G/H)^{\wedge} \simeq \{\chi \in \hat{G}: \chi[H] = \{1\}\}$.*

1.31. *If H is a closed subgroup of G and $A = \{\chi \in \hat{G}: \chi[H] = \{1\}\}$, then A is a closed subgroup of $\hat{G}$ and $\hat{G}/A \simeq \hat{H}$.*

In 1.32–1.35 we cite without proof four results on the structure of locally compact topological groups. Theorems 1.32 and 1.33 are due to WEIL [1951] and VAN KAMPEN [1935] respectively; proofs are available also in HEWITT and ROSS [1963] (9.8, 24.30). As to Theorem 1.34, the existence of the indicated decomposition is due to ROBERTSON [1968] (an unpublished manuscript), its uniqueness to ARMACOST and ARMACOST [1978]; the latter paper contains complete proofs. Theorem 1.35, due to FREUDENTHAL [1936] for metrizable groups and to WEIL [1951] (§32) in the generality here quoted, is proved also by DIXMIER [1969] (16.4.6); for additional results on the structure of (not necessarily Abelian) locally compact groups, see GROSSER and MOSKOWITZ [1967] and T. WILCOX [1968].

1.32. THEOREM. *Every locally compact, compactly generated, Abelian topological group G satisfies $G \simeq \mathbb{R}^m \times \mathbb{Z}^k \times F$ with m and k non-negative integers and with F a compact group.*

1.33. THEOREM. *Every locally compact Abelian topological group G satisfies $G \simeq \mathbb{R}^m \times H$ with m a non-negative integer and with H a locally compact group containing a compact-open subgroup. If $G \simeq \mathbb{R}^n \times K$ similarly, then $n = m$ and $K \simeq H$.*

1.34. In the following theorem the rational group $\mathbb{Q}$ is given the discrete topology. The group $\hat{\mathbb{Q}}$ is then compact, and $\bigoplus_\alpha \mathbb{Q}$ denotes as in 1.4 the weak sum of α copies of $\mathbb{Q}$ with the discrete topology.

THEOREM. *Every locally compact, Abelian topological group G satisfies $G \simeq \mathbb{R}^m \times (\bigoplus_\alpha \mathbb{Q}) \times (\hat{\mathbb{Q}})^\beta \times E$ with m a non-negative integer, with α and β cardinal numbers, and with E a topological group such that if $S = E$ or $S = \hat{E}$, then every element of the maximal divisible subgroup of S lies in some compact subgroup of S. If similarly $G \simeq \mathbb{R}^n \times (\bigoplus_\gamma \mathbb{Q}) \times (\hat{\mathbb{Q}})^\delta \times F$, then $n = m$, $\gamma = \alpha$, $\delta = \beta$, and $F \simeq E$.*

1.35. The reader unfamiliar with the concept of a maximally almost periodic group may wish to consult 9.7 below for the definition and a few characterizations.

THEOREM. *Let G be a locally compact, connected topological group. Then G satisfies $G \simeq \mathbb{R}^m \times F$, with m a non-negative integer and with F a compact group, if and only if G is maximally almost periodic.*

1.36. DEFINITION. A topological space Y is *dyadic* if for some cardinal number α there is a continuous function from $\{0, 1\}^\alpha$ onto Y.

The following powerful and useful result is due to KUZ'MINOV [1959].

THEOREM. *Every compact topological group is dyadic.*

We note in the statement of Kuz'minov's theorem that the algebraic structure of the space $\{0, 1\}^\alpha$ is suppressed or ignored; in particular it is not asserted that the continuous surjection in question is a homomorphism.

Unfortunately there appears to be no valid proof of Kuz'minov's theorem available in the English language. When the given compact group is Abelian, Kuz'minov's theorem reduces to a result of VILENKIN [1958]. Here Pontrjagin duality and standard theorems concerning the structure and behavior of (discrete) Abelian groups can be brought into play; Vilenkin's proof has been adapted and fleshed out in full detail by HEWITT and ROSS [1963] (25.35). The same authors prove (9.15) another result closely related to Kuz'minov's theorem, due independently to KUZ'MINOV [1959] and IVANOVSKIĬ [1958]: Every (not necessarily Abelian) infinite, zero-dimensional, compact topological group G is homeomorphic to the space $\{0, 1\}^\alpha$ with α equal to the weight of G.

1.37. For many applications, the unadorned statement that a given space Y is dyadic in the sense of the definition above—that is, the statement that for some $\alpha \geq \omega$ there is a continuous function from $\{0, 1\}^\alpha$ onto Y—is inadequate: It is necessary to know that $w(Y)$ is a possible value for α. (Here as elsewhere, $w(Y)$ denotes the weight of Y.) We prove this now, using an intermediate result (Theorem 1.39) of independent interest.

NOTATION. For $\{X_i: i \in I\}$ a set of sets and $J \subset I$, we write $X_J = \Pi_{i\in J} X_i$; in particular, $X_I = \Pi_{i\in I} X_i$. For $p \in X_I$ we write

$$p_J = \pi_J(p) = \langle p_i: i \in J\rangle \in X_J.$$

DEFINITION. Let $\alpha \geq \omega$, let $\{X_i: i \in I\}$ be a set of sets, let Y be a set, let $A \subset X_I$ and let $f: X_I \to Y$.

(i) If there is $J \subset I$ such that $A = \pi_J^{-1}(\pi_J[A])$ and such that $|J| \leq \alpha$, then we say that A *depends on* J, and that A *depends on* $\leqslant\alpha$ *coordinates.*

(ii) If there is $J \subset I$ such that $f(p) = f(q)$ whenever $p, q \in X_I$ and $p_J = q_J$, and if $|J| \leqslant \alpha$, then we say that f *depends on* J, and that f *depends on* $\leqslant\alpha$ *coordinates.*

1.38. LEMMA. *Let β be a cardinal number, $\{X_i: i \in I\}$ a set of topological spaces, and $A = \bigcup_{\eta<\beta} A_\eta \subset X_I$.*

(i) *If A depends on $J \subset I$, then* cl A *depends on J; and*

(ii) *if A_η depends on $J_\eta \subset I$, then A depends on $\bigcup_{\eta<\beta} J_\eta$.*

PROOF. (i) Let $x \in \operatorname{cl} A$, $y \in X_I$ and $x_J = y_J$. There is a net $\langle x(s): s \in S\rangle$ of elements of A such that $\lim_s x(s) = x$. For $s \in S$ we define $y(s) \in X_I$ by

$$\begin{aligned} y(s)_i &= x(s)_i \quad \text{if } i \in J, \\ &= y_i \qquad\quad \text{if } i \in I\backslash J. \end{aligned}$$

Since $y(s)_J = x(s)_J$ and $x(s) \in A$, we have $y(s) \in A$. From $\lim_s y(s) = y$ it then follows that $y \in \operatorname{cl} A$.

(ii) We leave the (routine) verification to the reader.

1.39. The density character of a topological space X is denoted $d(X)$. As usual, a space X is said to be separable if $d(X) \leqslant \omega$.

LEMMA. *Let $\alpha \geqslant \omega$ and $\{X_i: i \in I\}$ a set of spaces with each $d(X_i) \leqslant \alpha$, and let U be an open subset of X_i. Then* cl U *depends on* $\leqslant\alpha$ *coordinates.*

PROOF. Let $\{V_\eta: \eta < \beta\}$ be a maximal family of pairwise disjoint, non-empty basic open subsets of X_I with each $V_\eta \subset U$, and set $V = \bigcup_{\eta<\beta} V_\eta$. It is well known that $\beta \leqslant \alpha$. (This relation is derived in this Handbook by Hodel (Corollary 11.3) from the Hewitt–Marczewski–Pondiczery theorem.) It follows from maximality that V is dense in U, so that $\operatorname{cl} U = \operatorname{cl} V$. For $\eta < \beta$ there is finite $J_\eta \subset I$ on which V_η depends, so the required conclusion follows from Lemma 1.38.

1.40. THEOREM. *Let $\alpha \geqslant \omega$ and $\{X_i: i \in I\}$ a set of spaces with each $d(X_i) \leqslant \alpha$, and let $f \in C(X_I, Y)$ with Y a space such that $w(Y) \leqslant \alpha$. Then f depends on $\leqslant\alpha$ coordinates.*

PROOF. Let $\mathscr{B}$ be a basis for Y with $|\mathscr{B}| \leqslant \alpha$, and using 1.39 for $B \in \mathscr{B}$ choose $J(B) \subset I$ such that $|J(B)| \leqslant \alpha$ and $\operatorname{cl}(f^{-1}(B))$ depends on $J(B)$. Now set $J = \bigcup_{B\in\mathscr{B}} J(B)$. Then $|J| \leqslant \alpha$, and it is not difficult to see that f depends on J. Indeed if x, $y \in X_I$ with $x_J = y_J$ and $f(x) \neq f(y)$, then since Y satisfies the Hausdorff separation axiom there is $B \in \mathscr{B}$ such that $f(x) \in \mathscr{B}$ and $f(y) \notin \operatorname{cl} B$ and we have

$$x \in f^{-1}(B) \subset \operatorname{cl}(f^{-1}(B))$$

and

$$f(y) \notin \mathrm{cl}(B) \supset \mathrm{cl}(f[f^{-1}(B)]) \supset f[\mathrm{cl}(f^{-1}(B))]$$

and hence $y \notin \mathrm{cl}(f^{-1}(B))$. This contradiction completes the proof.

1.41. THEOREM. *Let Y be a dyadic space with $w(Y) = \alpha \geqslant \omega$. Then α is the least cardinal number for which there is a continuous function from $\{0, 1\}^\alpha$ onto Y.*

PROOF. As is well known (see for example ENGELKING [1977] (Theorem 3.7.19)), continuous surjections between compact spaces cannot raise weight. Thus for every continuous surjection $f: \{0, 1\}^\gamma \to Y$ we have $\gamma \geqslant \alpha$ and it is enough to show that the value $\gamma = \alpha$ can be achieved.

Given such $f: \{0, 1\}^\gamma \to Y$ there is by 1.40 a set $J \subset \gamma$ such that $|J| \leqslant \alpha$ and f depends on J. The function $g: \{0, 1\}^J \to Y$ given by

$$g(x_J) = g(\pi_J(x)) = f(x) \quad \text{for } x \in \{0, 1\}^\gamma$$

is then well-defined (since $f(x) = f(y)$ whenever $x_J = y_J$) and we have $|J| = \alpha$, as required.

1.42. That under appropriate conditions every continuous function from a product space X_I to a space Y depends (for suitable small α) on $\leqslant\alpha$ coordinates is a phenomenon noted and studied by many authors. Extensive references to the literature are given by COMFORT and NEGREPONTIS [1982] (Notes for Chapter 10); their 10.12–10.14, from which 1.37–1.41 above is drawn, is an argument introduced by BOCKSTEIN [1948] to show that disjoint, open subsets in a product of separable metric spaces remain disjoint when projected onto a suitable countable subproduct. In addition to the new contributions they contain, the works of ENGELKING [1966] and HUŠEK [1976] are valuable surveys of the literature of continuous functions defined on product spaces and their subspaces; the latter classifies into six groupings the patterns of proof used by the authors cited.

1.43. The proof that a particular space Y is dyadic, or is homeomorphic to a space of the form $\{0, 1\}^\alpha$, is likely to carry already the information $\alpha = w(Y)$; this is indeed the case with the theorems of Kuz'minov, Vilenkin and Ivanovskiĭ cited in 1.36. For purposes of reference throughout this chapter, then, it seems best to adopt the following statement as our authoritative version of the theorem of KUZ'MINOV [1959].

THEOREM. *For every compact topological group G with $w(G) = \alpha \geqslant \omega$ there is a continuous surjection from $\{0, 1\}^\alpha$ onto G.*

1.44. We turn now to the seventh and last of the fundamental results cited in this section, a theorem of ŠAPIROVSKIĬ [1975]. For our purposes a special case of his very general result is sufficient. We state this in 1.45.

For a space X and $x \in X$ we denote by $\pi\chi(x, X)$ the π-character of X at x, that is, the least cardinal of a family $\mathscr{B}$ of open subsets of X with the property that every neighborhood of x contains an element B of $\mathscr{B}$ (it is not required that $x \in B$). It is shown by ŠAPIROVSKIĬ [1975] that if $\alpha \geqslant \omega$ and X is a space containing a compact subspace F such that $\pi\chi(x, F) \geqslant \alpha$ for all $x \in F$, then F (and hence X) maps continuously onto $[0, 1]^\alpha$.

It is easy to show (see 3.5(iii) and 3.6(i) below) that for a compact topological group G we have $\pi\chi(x, G) = w(G)$ for all $x \in G$. Taking $F = G$ in the paragraph above we have then the following statement; for ease of exposition we refer in this chapter to this specialized statement as Šapirovskiĭ's theorem.

1.45. THEOREM. *For every compact topological group G with $w(G) = \alpha \geqslant \omega$ there is a continuous surjection from G onto* $[0, 1]^\alpha$.

A complete proof is given by JUHÁSZ [1980] (3.18). Related results have been proved by GERLITS [1978/81] and by BALCAR and FRANĚK [1982].

1.46. What we have called Šapirovskiĭ's theorem can be deduced from Kuz'minov's theorem using other work of Gerlits as follows.

(i) GERLITS [1976] (Theorem 4) shows that every homogeneous dyadic space X with $\chi(X) = \alpha \geqslant \omega$ maps continuously onto $[0, 1]^\alpha$.

(ii) Let $\alpha \geqslant \omega$. A space $X = \langle X, \mathscr{T} \rangle$ is said to have calibre α if for every $\{U_\xi \colon \xi < \alpha\} \subset \mathscr{T}$ there is $A \subset \alpha$ such that $|A| = \alpha$ and $\bigcap_{\xi \in A} U_\xi \neq \emptyset$.

It has been shown by SHELAH [1977] that if X has calibre α with $\mathrm{cf}(\alpha) > \omega$, then every power of X also has calibre α. (For a proof of this and related results, see COMFORT and NEGREPONTIS [1982] (§3).)

It is shown by GERLITS [1980] (Corollary 3) that if a product space X_I has calibre $\alpha > \omega$ and if $f \in C(X_I, Y)$ is a surjection, then either f depends on $<\alpha$ coordinates or Y contains a homeomorph of $\{0, 1\}^\alpha$.

Suppose now that G is a compact topological group (or indeed, any dyadic space) such that $w(G) = \alpha$ with $\mathrm{cf}(\alpha) > \omega$. Since $\{0, 1\}$ has calibre α, it follows from the two results just cited that G contains a homeomorph of $\{0, 1\}^\alpha$. This subspace, and hence G itself, then maps continuously onto $[0, 1]^\alpha$.

The argument cited in (i) suffices to deduce Šapirovskiĭ's theorem in full generality from Kuz'minov's theorem. The argument of (ii) is available only when $w(G) = \alpha$ satisfies $\mathrm{cf}(\alpha) > \omega$, since for $\mathrm{cf}(\alpha) = \omega$ no infinite (Hausdorff) space has calibre α.

2. Topological pathology (upper bounds)

As is clear from several articles in this volume, topologists enjoy finding counterexamples—that is, spaces designed to destroy naive or attractive conjectures. In Section 5 below, conforming to this custom, we describe the construction of a few topological groups which are, in some topological sense, bizarre or pathological. The present section may be viewed as establishing some upper bounds to the topological pathology available in groups. One cannot achieve in the class of topological groups all that can be achieved in the class of Tychonoff spaces.

2.1. Let us note first, however, that a topological group can contain a space as bad as any (Tychonoff) space can be. It is a fundamental theorem of TYCHONOFF [1929] that the Tychonoff space X embeds homeomorphically into $[0, 1]^{w(X)}$; the topological inclusion $[0, 1] \subset \mathbb{T}$ affords an embedding of X into the group $G = \mathbb{T}^{w(X)}$. We note that G enjoys a number of pleasant properties: It is compact, connected and Abelian; and, if X is infinite, no topological group containing a copy of X can have smaller weight than does G.

2.2. If X has a base of open-and-closed sets, a similar device yields an embedding of X into the zero-dimensional group $\{-1, +1\}^{w(X)}$.

2.3. Although 2.1 does not show it, a Tychonoff space X can in fact be embedded in a suitably defined Abelian topological group $A(X)$ as a closed subspace, indeed in such a way that every continuous function from X to an Abelian topological group extends to a continuous homomorphism over $A(X)$. Early definitions and investigations of $A(X)$ and its properties are given by MARKOV [1941], [1945]; the following remarks are based largely on the exposition of SMITH–THOMAS [1974].

(In 9.20–9.21 below we consider the non-Abelian analogue $F(X)$ of $A(X)$ and the question of defining a non-discrete topological group topology on it.)

As a set we write

$$A(X) = \{\Sigma k_x \cdot x \colon x \in X, k_x \in \mathbb{Z}, |\{x \in X \colon k_x \neq 0\}| < \omega\}.$$

The 'empty word' (with each $k_x = 0$) serves as the identity e for $A(X)$, which is an Abelian group under the operation $(\Sigma k_x \cdot x) + (\Sigma l_x \cdot x) = \Sigma(k_x + l_x) \cdot x$. For $p \in X$ we denote by $\eta(p)$ that element $\Sigma k_x \cdot x$ of $A(X)$ for which $k_x = 1$ if $x = p$, $k_x = 0$ if $x \neq p$; we write $\eta(p) = 1 \cdot p \in A(G)$ and, similarly abusing notation, we designate elements of $A(X)$ by symbols of the form $\Sigma_{i=1}^{n} k_i \cdot x_i$ (with $n < \omega$, $k_i \in Z$, $x_i \in X$).

With every continuous function f from X into an Abelian topological group G we associate that homomorphism $\bar{f}\colon A(X) \to G$ which is (well-) defined by the rule $\bar{f}(\Sigma k_x \cdot x) = \Sigma k_x \cdot f(x)$, and we write $\mathcal{S} = \{f^{-1}(U)\colon f \in C(X, G),\ G$ an Abelian topological group, U open in $G\}$ and $\bar{\mathcal{S}} = \{\bar{f}^{-1}(U)\colon f \in C(X, G),\ G$ an Abelian topological group, U open in $G\}$. The topology for $A(X)$ is defined by the requirement that $\bar{\mathcal{S}}$ is a sub-base. It is easy to see that, so topologized, $A(X)$ is a (possibly non-Hausdorff) topological group. Since $\mathcal{S}$ is a sub-base for the topology of X and

$$\{\eta[S]\colon S \in \mathcal{S}\} = \{\eta[X] \cap \tilde{S}\colon \tilde{S} \in \tilde{\mathcal{S}}\},$$

the function η is a homeomorphism from X onto $\eta[X] \subset A(X)$.

To see that $A(X)$ is a Hausdorff space it is enough to show that if $a = \Sigma_{i=1}^{n} k_i \cdot x_i \in A(X)$ with $a \neq e$, then a and e have disjoint neighborhoods U and V. To do this, assume without loss of generality that $k_1 \neq 0$ and choose $f \in C(X)$ such that $f(x_1) = 1/k_1$, $f(x_i) = 0$ for $1 < i \leq n$. Then $\bar{f}(a) = 1$ and $\bar{f}(e) = 0$ and we may take $U = \bar{f}^{-1}(\frac{1}{2}, \frac{3}{2})$, $V = \bar{f}^{-1}(-\frac{1}{2}, \frac{1}{2})$.

It remains finally to show that $\eta[X]$ is closed in $A(X)$. To this end let $a = \Sigma_{i=1}^{n} k_i \cdot x_i \in A(X)$ with $a \notin \eta[X]$ and consider separately three cases as follows.

Case 1. $k_i = 0$ for $1 \leq i \leq n$ (that is, $a = e$). Define $f(x) = 1$ for all $x \in X$. We have $\bar{f}(a) = 0$, and $\bar{f}(\eta(x)) = f(x) = 1$ for all $x \in X$.

Case 2. $k_i \neq 0$ for exactly one value of i (say $k_1 \neq 0$). Then $k_1 \neq 1$. Choose $f \in C(X, [0, 1])$ with $f(x_1) = 1$, $f(x_i) = 0$ for $1 < i \leq n$. We have $\bar{f}(a) = k_1 \notin [0, 1]$, and $\bar{f}(\eta(x)) = f(x) \in [0, 1]$ for all $x \in X$.

Case 3. $k_i \neq 0$ for at least two values of i (say $k_1 \neq 0$, $k_2 \neq 0$). Choose $f \in C(X, [-1, +1])$ with $f(x_i) = k_i/|k_i|$ for $i = 1, 2$ and $f(x_i) = 0$ for $2 < i \leq n$. We have $\bar{f}(a) = |k_1| + |k_2| \geq 2$, and $\bar{f}(\eta(x)) = f(x) \in [-1, 1]$ for all $x \in X$.

In each of the three cases the set $\bar{f}^{-1}(\bar{f}(a) - \frac{1}{2}, \bar{f}(a) + \frac{1}{2})$ is a neighborhood in $A(X)$ of a, disjoint from $\eta[X]$. The proof is complete.

2.4. Markov's original proof of the existence of $A(X)$ and its non-Abelian analogue, based on a complicated theory of multi-norms, was simplified by NAKAYAMA [1943] and by GRAEV [1948], [1950]. The latter worked with 'pointed spaces' $\langle X, p\rangle$ and took for $A(X)$ the group freely generated by $X\backslash\{p\}$; here the embedding $X \to A(X)$ satisfies $p \to e$.

Let $\{\langle G_i, f_i\rangle\colon i \in I\}$ be a listing of all pairs with G_i an Abelian topological group and f_i a continuous function from X onto a dense subspace of G_i, and with the additional technical condition that (as a set) G_i is a subset of the cardinal number $\exp\exp|X|$. (This condition assures us that I is a set rather than a proper class, so the product $G = \Pi_{i\in I} G_i$ is a topological group.) The evaluation function $h\colon X \to G$ given by $h_i(x) = f_i(x)$ is a homeomorphism into G, and the subgroup S of G generated by $h[X]$ can serve as a model for $A(X)$ in the sense that S is freely generated by $h[X]$ and for every continuous function f_i from X onto a dense subset of a topological Abelian group G_i there is a continuous homomorphism $\bar{f}\colon S \to G_i$ such that $\bar{f}_i \circ h = f_i$ (namely, $\bar{f}_i = \pi_i|S$). This method of defining $A(X)$, as well as what in modern terminology would be called epi-reflections of the class of Tychonoff spaces into several other classes of topological groups, was described independently by KAKUTANI [1944] and SAMUEL [1948]. A careful exposition of the construction of the non-Abelian analogue of $A(X)$, following KAKUTANI [1944], is given by HEWITT and ROSS [1963] (Theorem 8.8).

2.5. Not only for $n=1$, but in fact for every integer $n>0$, is there a closed subspace X_n of $A(X)$ homeomorphic to X^n. Indeed, as SMITH-THOMAS [1974] shows, one may take $X_n = \{\Sigma_{i=1}^n 2^{i-1} \cdot x_i \colon x_i \in X\}$.

2.6. Let us note in passing that 2.3 yields immediately the existence of topological groups G which are not normal (in the topologist's usual sense of the word). Since X is closed in $A(X)$ and every closed subspace of a normal space is normal, it is enough to take $G = A(X)$ with X a non-normal Tychonoff space. The same sort of reasoning is used in conjunction with 2.5 by SMITH-THOMAS [1974] to show that a number of familiar topological properties (e.g., Lindelöf, paracompact, countably compact) need not be preserved under passage from X to $A(X)$.

We return to the theme of this section: not all the pathology available in the class of Tychonoff spaces is available in the class of topological groups. We give several illustrative examples.

2.7. THEOREM. *Let G be a topological group.*

(i) *If $\alpha \geqslant \omega$ and there is $S \subset G$ such that* $\operatorname{int}\operatorname{cl} S \neq \emptyset$ *and $\chi(S) = \alpha$, then $\chi(G) = \alpha$.*

(ii) *If G has a dense, first countable subspace, then G is first countable.*

PROOF. It is enough to prove (i), since (ii) is a consequence.

Recall first for any Tychonoff space X that if D is dense in X and $x \in D$, then $\chi(x, D) = \chi(x, X)$. Then in the present case, taking $X = \operatorname{cl} S$ and $D = S$ and choosing $x \in S \cap (\operatorname{int}\operatorname{cl} S)$, we have

$$\chi(G) = \chi(x, G) = \chi(x, \operatorname{int}\operatorname{cl} S) \leqslant \chi(x, \operatorname{cl} S) = \chi(x, S).$$

2.8. THEOREM. *Every locally compact topological group G with $|G| = \alpha < 2^\omega$ is discrete.*

PROOF. For $\alpha = \omega$ this is immediate from the Baire category theorem for locally compact spaces. For the general case ($\alpha < 2^\omega$), suppose that G is locally compact and non-discrete and apply the ČECH–POSPÍŠIL [1938] theorem (see 7.15 of the article by R. Hodel in this Handbook) to a compact set K of the form $K = \operatorname{cl} U$ with U an open neighborhood of e: one has $\chi(x, K) \geqslant \omega$ for all $x \in K$ and hence $|G| \geqslant |K| \geqslant 2^\omega$.

2.9. THEOREM. *Every locally compact topological group G is paracompact.*

PROOF. Following MICHAEL [1956], again let K be a compact set of the form $K = \operatorname{cl} U$ with U an open, symmetric neighborhood of e, and set $K(n) = \{x_1 \cdot x_2 \cdot \dots \cdot x_n \colon x_i \in K\}$ and $H = \bigcup_{n<\omega} K(n)$. Since multiplication in G is continuous, each set $K(n)$ is the continuous image of the compact power K^n and hence compact; thus H is σ-compact and hence paracompact. Like every subgroup with non-empty interior, H is open in G and hence also closed. Thus G, the union of pairwise disjoint open-and-closed paracompact subspaces, is itself paracompact.

2.10. THEOREM. *In a homogeneous Tychonoff space X in which each countable, discrete subspace is C^*-embedded, every compact subspace is finite.*

PROOF. We use a result of KUNEN [1972] to the effect that there are non-principal ultrafilters p, q over the discrete space ω (that is, $p, q \in \beta(\omega)\backslash\omega$) such that for every $f \in \omega^\omega$ the Stone extension $\bar{f}: \beta(\omega) \to \beta(\omega)$ satisfies $\bar{f}(p) \neq q$, $\bar{f}(q) \neq p$. (This and related results of Kunen and others are proved in detail in the text of COMFORT and NEGREPONTIS [1974] (Chapter 10).)

Suppose then that the statement fails for the (homogeneous) space X and let ω be a countably infinite discrete subspace of X with compact closure. Since ω is C^*-embedded, the space $\mathrm{cl}(\omega)$ is homeomorphic to $\beta(\omega)$ and there are p, $q \in \beta(\omega)\backslash\omega$ as given by Kunen's theorem. Suppose that there is a homeomorphism h of X onto X such that $h(p) = q$. From $p \in \mathrm{cl}(\omega) \cap \mathrm{cl}(h^{-1}[\omega])$ it follows, as noted in a related context by FROLÍK [1967c], [1971], that

$$p \in \mathrm{cl}(\omega \cap \mathrm{cl}(h^{-1}[\omega])) \cup \mathrm{cl}(h^{-1}[\omega] \cap \mathrm{cl}(\omega))$$

for otherwise $p \in \mathrm{cl}(\omega\backslash\mathrm{cl}(h^{-1}[\omega])) \cap \mathrm{cl}(h^{-1}[\omega]\backslash\mathrm{cl}(\omega))$ and the discrete set $(\omega\backslash\mathrm{cl}(h^{-1}[\omega])) \cup (h^{-1}[\omega]\backslash\mathrm{cl}(\omega))$ is not C^*-embedded. We consider two cases separately.

Case 1. $p \in \mathrm{cl}(\omega \cap \mathrm{cl}(h^{-1}[\omega]))$. If $p \in \mathrm{cl}(\omega \cap h^{-1}[\omega])$ we define $f(n) = h(n)$ for $n \in \omega \cap h^{-1}[\omega]$, $f(n) = 0$ for $n \in \omega\backslash h^{-1}[\omega]$; then $f \in \omega^\omega$ and we have $\bar{f}(p) = h(p) = q$, a contradiction. Thus

$$p \in \mathrm{cl}(\omega \cap (\mathrm{cl}(h^{-1}[\omega])\backslash h^{-1}[\omega])) .$$

We choose $g: \omega \to \mathrm{cl}(\omega)\backslash\omega$ so that $g[\omega]$ is discrete and $g(k) = h(k)$ for all k such that $h(k) \in \mathrm{cl}(\omega)\backslash\omega$. Since $g[\omega]$ is discrete we can write $\omega = \bigcup_{k<\omega} A_k$ with $\{A_k: k < \omega\}$ a pairwise disjoint family and with $g(k) \in \mathrm{cl}(A_k)$ for all $k < \omega$. Now define, again following Frolík, $f(n) = k$ for $n \in A_k$. Then $f \in \omega^\omega$, and from $f(g(k)) = k$ for $k < \omega$ we have $(\bar{f} \circ g)^-(p) = p$ and hence $p = (\bar{f} \circ g)^-(p) = \bar{f}(\bar{g}(p)) = \bar{f}(h(p)) = \bar{f}(q)$, a contradiction.

Case 2. $p \in \mathrm{cl}(h^{-1}[\omega] \cap \mathrm{cl}(\omega))$. Here $q \in \mathrm{cl}(\omega \cap \mathrm{cl}(h[\omega]))$ and one may repeat the argument of Case 1 with p, q and h replaced by q, p and h^{-1}, respectively.

2.11. A space X is a k-space if its topology is determined by its compact subspaces in the following sense: if $A \subset X$ and $A \cap K$ is closed for every compact $K \subset X$, then A is closed in X. Of course locally compact spaces, and metrizable spaces, are k-spaces. It is clear that every non-discrete k-space has an infinite compact subspace, so we have the following corollary to Theorem 2.10: Every homogeneous k-space in which every countable discrete subspace is C^*-embedded is discrete.

2.12. We cite some consequences, and predecessors in the literature, of 2.10 and 2.11. This is achieved most conveniently in the context of three classes of (Tychonoff) spaces, as follows. A space is extremally disconnected if each open set has open closure; a space is basically disconnected if each cozero-set has open closure; and, a space is an F-space if disjoint cozero-sets are contained in disjoint zero-sets (equivalently, if each cozero-set is C^*-embedded). It is clear that every extremally disconnected space is basically disconnected, and that every basically disconnected space is an F-space.

Let us note further that every countable discrete subspace $D = \{x_n : n < \omega\}$ in an F-space X is C^*-embedded (we do not need the fact, remarked by GILLMAN and JERISON [1960] (Exercise 14N.5), that the word 'discrete' may be omitted). Indeed let $\{U_n : n < \omega\}$ be a family of pairwise disjoint cozero-sets in X with $x_n \in U_n$ and, given $f: D \to [0, 1]$, define $g(x) = f(x_n)$ for all $x \in U_n$; since $U = \bigcup_{n<\omega} U_n$ is a cozero-set and $g \in C(U, [0, 1])$, there is $\bar{g} \in C(X, [0, 1])$ such that $f \subset g \subset \bar{g}$, as required.

It is a consequence of 2.11 that a topological group which is an F-space and a k-space is discrete. RAJAGOPALAN [1964a], using an argument based on the existence of Haar measure, achieved this conclusion for locally compact, extremally disconnected topological groups; RAJAGOPALAN and SOUNDARARAJAN [1971] showed that every extremally disconnected, k-space topological group is discrete. An intermediate result, to the effect that there is no infinite compact subspace in a topological group in which disjoint open F_σ subsets have disjoint closures (e.g., a basically disconnected group) is given by ARHANGEL'SKIĬ [1968] (Theorem 16 and footnote). The fact that no infinite extremally disconnected compact Hausdorff space is homogeneous is an Editor's Note appended by Z. Frolík to related results of EFIMOV [1972] (footnote, p. 105).

3. Cardinal invariants

For a space X we denote by $d(X)$, $w(X)$, $\pi w(X)$, $\chi(X)$, $\pi\chi(X)$, $\psi(X)$, $o(X)$, $c(X)$, $t(X)$ and $wc(X)$ the density character, weight, π-weight, character, π-character, pseudocharacter, open-set number, cellularity, tightness and weak covering number, respectively; each is used here as by Hodel earlier in this Handbook. In addition we denote by $\kappa(X)$ the compact-covering number of X, the least cardinal of a family of compact subsets of X whose union is X.

This section is organized as follows. 3.1 records certain relations given by the results of Kuz'minov and Šapirovskiĭ for compact groups; 3.9 gives the corresponding relations for locally compact, σ-compact groups, using the Kakutani–Kodaira theorem (3.7) where this is helpful, and 3.12 concerns (arbitrary) locally compact groups; 3.16 improves the statement that $o(G) = (o(G))^\omega$ for all topological groups G.

3.1. According to the theorems of KUZ'MINOV [1959] and ŠAPIROVSKIĬ [1975], 1.43 and 1.45 above, for every compact topological group G with $w(G) = \alpha \geqslant \omega$ there are continuous surjections $f: X = \{0, 1\}^\alpha \to G$ and $g: G \to Y = [0, 1]^\alpha$. We know that $|X| = |Y| = 2^\alpha$; that $d(X) = d(Y) = \log(\alpha)$ (see the so-called Hewitt–Marczewski–Pondiczery theorem, 11.2 in Hodel's article); that $o(X) = o(Y) = 2^\alpha$; and that $c(X) = c(Y) = \omega$. From these equalities and elementary properties of continuous functions we have the following result.

THEOREM. *Every compact topological group G with $w(G) = \alpha \geqslant \omega$ satisfies* (i) $|G| = 2^\alpha$; (ii) $d(G) = \log(\alpha)$; (iii) $o(G) = 2^\alpha$; *and* (iv) $c(G) = \omega$.

Apropos of (i) we may say, paraphrasing the comment of KURATOWSKI [1958] (p. 387) concerning analytic subsets of the real line, that "les groupes compacts 'réalisent' donc l'hypothèse generalizée du continu".

The relation $\chi(G) = \alpha$, as well as the relations (i)–(iv), hold not only for compact G but also for locally compact, σ-compact G; see Theorem 3.9 below.

It is well to notice that the relation $w(G) = \log(|G|)$, in view of 3.1 a tempting conjecture for compact topological groups G, is not a theorem of ZFC. Suppose for example that there exist cardinals α, γ with $\omega \leqslant \gamma < \alpha$ and $2^\gamma = 2^\alpha$. The (compact, Abelian) topological groups $\{-1, +1\}^\gamma$ and $\{-1, +1\}^\alpha$ are not homeomorphic since they have weight γ and α respectively, and with $G = \{-1, +1\}^\alpha$ we have

$$w(G) = \alpha > \gamma \geqslant \log(2^\gamma) = \log(|G|)\,.$$

3.2. THEOREM 3.1(i), (ii) were proved for Abelian groups using delicate computations involving the character group by KAKUTANI [1943] and, more directly, by HARTMAN and HULANICKI [1958]; both papers used the generalized continuum hypothesis. It was ITZKOWITZ [1972] who first used arguments related to Kuz'minov's theorem to prove 3.1(ii) in its present general form.

I do not know who first noted 3.1(iv). Let us remark in any event that the existence of Haar measure (1.5 above) furnishes a quick proof quite independent of Kuz'minov's theorem; see in this connection 3.8(iv) and 3.11(c) below.

3.3. LEMMA. *Let G be a topological group.*

(i) *If D is dense in G and $U \in \mathcal{N}(e)$, then $G = DU$;*

(ii) *if G is locally compact, then $\kappa(G) \leqslant d(G)$.*

PROOF. (i) Given $p \in G$ there is $x \in D \cap pU^{-1}$, and then $p \in xU \subset DU$.

(ii) There is $U \in \mathcal{N}(e)$ with $\mathrm{cl}_G U$ compact, and with D dense in G we have from (i) that $G = \bigcup_{x \in D} x \cdot (\mathrm{cl}_G U)$.

3.4. LEMMA. *Let G be a topological group and $\mathcal{B}$ a local base at e, and for $B \in \mathcal{B}$ let $D_B \subset G$ satisfy $G = D_B \cdot B$. Then $\{xB\colon B \in \mathcal{B},\ x \in D_B\}$ is a base for G.*

PROOF. It is enough to show that if $p \in G$ and $U \in \mathcal{N}(e)$, then there are $B \in \mathcal{B}$ and $x \in D_B$ such that $p \in xB \subset pU$. Given such p and U choose $V \in \mathcal{N}(e)$, $B \in \mathcal{B}$ and $x \in D_B$ so that $V^{-1}V \subset U$, $B \subset V$ and $p \in xB$; then $p \in xB \subset pB^{-1}B \subset pV^{-1}V \subset pU$, as required.

The two preceding simple lemmas yield some cardinal relations, as follows.

3.5. THEOREM. *Let G be a topological group. Then*

(i) $w(G) = d(G) \cdot \chi(G)$;

(ii) $w(G) \leqslant \kappa(G) \cdot \chi(G)$; *and*

(iii) *if G is locally compact, then* $w(G) = \kappa(G) \cdot \chi(G)$.

PROOF. We retain the notation of 3.4 and we assume, the statements being otherwise obvious, that G is non-discrete.

(i) The inequality $\geqslant$ follows from the inequalities $w(G) \geqslant d(G)$, $w(G) \geqslant \chi(G)$, $w(G) \geqslant \omega$. For $\leqslant$, let D be dense in G and use 3.3(i) and $D_B = D$ for all $B \in \mathcal{B}$.

(ii) For compact $K \subset G$ and $B \in \mathcal{B}$ there is finite $F \subset K$ such that $K \subset F \cdot B$. Thus one may choose $|D_B| \leqslant \omega \cdot \kappa(G)$ and (ii) is immediate.

(iii) follows from (i), (ii), and 3.3(ii).

3.6. THEOREM. *Let G be a topological group. Then*

(i) $\pi\chi(G) = \chi(G)$; *and* (ii) $\pi w(G) = w(G)$.

PROOF. Again we assume, the statements being otherwise obvious, that G is non-discrete.

(i) Since $\pi\chi(G) \leqslant \chi(G)$ is clear, it is enough to show that if $\mathcal{B}$ is a π-base at e, then $\{BB^{-1}\colon B \in \mathcal{B}\}$ is a base at e. Given $U \in \mathcal{N}(e)$ find $V \in \mathcal{N}(e)$ and then $B \in \mathcal{B}$ such that $B \subset V \subset VV^{-1} \subset U$. Then $e \in BB^{-1} \subset U$, as required.

(ii) From (i) and 3.5(i) and the relation $d(G) \leqslant \pi w(G)$ it follows that

$$w(G) = d(G) \cdot \chi(G) \leqslant \pi w(G) \cdot \pi\chi(G) \leqslant \pi w(G) \cdot \pi w(G) = \pi w(G).$$

The cardinal relations given in 3.12(ii)–(v) for (arbitrary) locally compact groups G are based on the special case where G is σ-compact. For 3.8(v), in turn, the following structural analysis is crucial.

3.7. THEOREM. (Kakutani and Kodaira [1944]). *If G is a locally compact, σ-compact topological group, then there is a compact, normal subgroup N of G such that G/N is metrizable.*

PROOF. (We outline the proof given in detail by HEWITT and Ross [1963] (§8.7).) Let $G = \bigcup_{n<\omega} K_n$ with each K_n compact, with $K_n \subset K_{n+1}$ for $n < \omega$, and with int $K_0 \neq \emptyset$. Recursively define $V_n \in \mathcal{N}(e)$ so that

$$V_0 \subset K_0\,, \qquad V_{n+1}^2 \subset V_n\,, \qquad V_n = V_n^{-1}\,,$$

and

$$xV_{n+1}x^{-1} \subset V_n \quad \text{for all } x \in K_n\,,$$

and set $N = \bigcap_{n<\omega} V_n$. Then $xNx^{-1} \subset N$ for all $x \in G$, so N is a normal subgroup of G. The set N is closed, and hence G/N satisfies the Hausdorff separation axiom, because cl $V_{n+1} \subset V_n$ for $n < \omega$; the canonical homomorphism from G onto G/N takes $\{V_n: n < \omega\}$ to a local base at the identity of G/N, so G/N is metrizable (cf. 1.8(i) above).

The following natural extension of Theorem 3.7 is often useful: If G is a locally compact, σ-compact topological group and $\{U_n: n < \omega\} \subset \mathcal{N}(e)$, then there is a compact, normal subgroup N of G such that G/N is metrizable and $N \subset \bigcap_{n<\omega} U_n$. For the proof it is enough to adjoin to the list of conditions concerning V_n the additional requirement that $V_n \subset U_n$.

3.8. Let us note in passing that not every locally compact, σ-compact topological group is compactly generated. For an example it is enough to consider any infinite torsion group with the discrete topology, e.g., the group $\bigoplus_\omega \{-1, +1\}$.

3.9. THEOREM. *Let G be a non-discrete, locally compact, σ-compact topological group. Then*

(i) $w(G) = \chi(G)$; (ii) $|G| = 2^{w(G)}$; (iii) $o(G) = 2^{w(G)}$; (iv) $c(G) = \omega$; *and* (v) $d(G) = \log(\chi(G))$.

PROOF. (i) is immediate from 3.5. From (i) and Theorems 3.1 and 7.5 of Hodel's article in this Handbook we have

$$|G| \leq o(G) \leq 2^{w(G)} = 2^{\chi(G)} \leq |G|\,,$$

from which (ii) and (iii) follow.

We prove (iv). Let $U \in \mathcal{N}(e)$ with cl U compact (hence $\lambda(U) < \infty$) and let $A \subset G$ with $|A| \leq \omega$ and $G = AU$. If $\{U_\xi: \xi < \omega^+\}$ is a faithfully indexed cellular family in G there is a $a \in A$ such that $|\{\xi < \omega^+: U_\xi \cap aU \neq \emptyset\}| = \omega^+$. But then from the countable additivity of λ and the fact that λ assigns positive measure to every non-empty open subset of G it will follow that $\lambda(U) = +\infty$, a contradiction.

To prove (v) we note first that if G is any topological group and H a closed subgroup, and if $\{aH\colon a \in A\}$ is dense in G/H and B is dense in H, then AB is dense in G. (The straightforward proof, noted by VILENKIN [1948] in the case that H is normal in G and by HEWITT and ROSS [1963] (5.38(f)) in general, uses only the fact that the function $x \to xH$ is open from G onto G/H.) Thus

$$d(G) \leq d(H) \cdot d(G/H)$$

whenever G is a topological group and H a closed subgroup.

In the present case with G σ-compact let N be as given by the Kakutani–Kodaira theorem (3.7). We claim that $d(N) \leq \log(\chi(G))$. If $|N| < \omega$ this is clear; if $|N| \geq \omega$, then from 3.1(ii) and (i) of the present theorem, both applied to the compact group N, we have

$$d(N) = \log(w(N)) = \log(\chi(N)) \leq \log(\chi(G))$$

as required. Since G/N, like every σ-compact metrizable space, is separable, we then have

$$d(G) \leq d(N) \cdot d(G/N) \leq \log(\chi(G)) \cdot \omega = \log(\chi(G)) .$$

The relation $d(G) \geq \log(w(G)) \geq \log(\chi(G))$ is noted by Hodel (Theorem 3.3(b)) in this Handbook.

3.10. Clearly if in Theorem 3.9 the hypothesis that G is σ-compact is omitted, some of the resulting statements are false. Indeed for all cardinals α, β, γ with $\beta \geq \alpha \geq \omega$ and $\gamma \geq 2^{\omega}$ there are non-discrete locally compact topological groups G_0 and G_1 with $\chi(G_0) = \alpha$, $w(G_0) = c(G_0) = \beta$ and $|G_1| = w(G_1) = c(G_1) = \gamma$: Let D and E be discrete groups with $|D| = \beta$ and $|E| = \gamma$ and set $G_0 = D \times \mathbb{T}^{\alpha}$ and $G_1 = E \times \mathbb{T}$. Simple examples like these suggest the role to be played by the compact-covering number $\kappa(G)$ in modifying the computations of 3.9.

3.11. LEMMA. *Let G be a locally compact topological group and H an open, compactly generated subgroup of G. Then $\omega \cdot \kappa(G) = \omega \cdot |G/H|$.*

PROOF. It is clear that $\kappa(G) \leq \kappa(H) \cdot |G/H| \leq \omega \cdot |G/H|$. If $|G/H| \leq \omega$, then G is σ-compact and hence $\omega \cdot \kappa(G) = \omega \cdot |G/H|$; and if $|G/H| > \omega$ then, since each compact subset of G meets at most finitely many cosets of H, no compact-cover $\mathcal{A}$ of G can satisfy $|\mathcal{A}| < |G/H|$.

3.12. Let us now collect the information given by the foregoing results concerning cardinal functions for locally compact groups.

THEOREM. *Let G be a non-discrete, locally compact topological group. Then* (i) $w(G) = \kappa(G) \cdot \chi(G)$; (ii) $d(G) = \kappa(G) \cdot \log(\chi(G))$; (iii) $|G| = \kappa(G) \cdot 2^{\chi(G)}$; (iv) $o(G) = 2^{w(G)}$; *and* (v) $c(G) = \kappa(G) \cdot \omega$.

PROOF. (i) is 3.5(iii), restated for convenience. Choosing an open, compactly generated subgroup H of G and applying 3.9 to H, we have

$$d(G) = |G/H| \cdot d(H) = \kappa(G) \cdot d(H) = \kappa(G) \cdot \log(\chi(H))$$
$$= \kappa(G) \cdot \log(\chi(G)),$$
$$|G| = |G/H| \cdot |H| = \kappa(G) \cdot |H| = \kappa(G) \cdot 2^{\chi(H)} = \kappa(G) \cdot 2^{\chi(G)},$$

and

$$o(G) = o(H)^{|G/H|} = 2^{w(H) \cdot |G/H|} = 2^{w(G)},$$

which are (ii), (iii) and (iv), respectively.

(v) It is clear that $c(G) \geq c(H) = \omega$ and $c(G) \geq |G/H|$. For every cellular family $\mathcal{U}$ of G we have

$$|\mathcal{U}| \leq |\{U \cap xH : U \in \mathcal{U},\ x \in G,\ U \cap xH \neq \emptyset\}| \leq |G/H| \cdot c(H).$$

Combining these we have

$$\kappa(G) \cdot \omega = |G/H| \cdot \omega \leq c(G) \leq |G/H| \cdot \omega = \kappa(G) \cdot \omega,$$

as required.

3.13. Theorem 3.12(ii) has been noted independently (a) by SOUNDARARAJAN [1971b] (Theorem 1.7) in conversation with M. Rajagopalan, with an additional set-theoretic hypothesis shown redundant by STEPHENSON [1972] and (b) by Kenneth A. Ross, who communicated it to COMFORT and ITZKOWITZ [1977] (Lemma 2.3). The inequality $\geq$ of 3.9(ii) and 3.12(iii), a consequence of the theorem of ČECH and POSPÍŠIL [1938], was reproved by MROWKA [1958] and HULANICKI [1960]. The full equality of 3.12(iii) is given by HULANICKI [1958] in the case $\kappa(G) \leq 2^{\chi(G)}$.

3.14. We cite some miscellaneous consequences of some of the cardinality relationships recorded above.

THEOREM. *Let G be a locally compact topological group.*

(i) *If H is a closed subgroup G, then $d(H) \leq d(G)$;*

(ii) *if an addition G is Abelian, then $\chi(\hat{G}) = \kappa(G)$ and $w(G) = w(\hat{G})$.*

PROOF. (i) If H is discrete there is $U \in \mathcal{N}_G(e)$ such that $U \cap H = \{e\}$ and then with $V = V^{-1} \in \mathcal{N}_G(e)$ such that $V^2 \subset U$ the family $\{xV : x \in H\}$ is faithfully indexed and pairwise disjoint; then for each dense $D \subset G$ and $x \in H$ we have $D \cap xV \neq \emptyset$ and hence

$$|D| \geq |H| = d(H).$$

If H is non-discrete then, using $\kappa(H) \leq \kappa(G)$ and $\chi(H) \leq \chi(G)$ we have from 3.12(ii) that

$$d(H) = \kappa(H) \cdot \log(\chi(H)) \leq \kappa(G) \cdot \log(\chi(G)) = d(G),$$

as required.

(ii) We first prove $\chi(\hat{G}) = \kappa(G)$. If G is compact, then $\hat{G}$ is discrete and we have $\chi(\hat{G}) = 1 = \kappa(G)$. We assume in what follows that G is not compact.

There is a family $\mathcal{K}$ of compact subsets of G such that $|\mathcal{K}| = \kappa(G)$ and every compact subset of G is a subset of an element of $\mathcal{K}$. It is then clear, defining $P(F, \varepsilon)$ as in 1.24 above, that the family

$$\{P(F, 1/n) \colon F \in \mathcal{K}, 0 < n < \omega\}$$

is a base for the topology of $\hat{G}$. Since G is not compact we then have

$$\chi(G) \leq |\mathcal{K}| \cdot \omega = \kappa(G) \cdot \omega = \kappa(G).$$

To show $\kappa(G) \leq \chi(\hat{G})$ for G non-compact it is by Pontrjagin duality enough to show $\kappa(\hat{G}) \leq \chi(G)$ for G non-discrete. Let $\{U_\xi \colon \xi < \alpha\}$ be a local base at $e \in G$ of compact neighborhoods and for $\xi < \alpha = \chi(G)$ define

$$F_\xi = \{\Psi \in \hat{G} \colon |\Psi(x) - 1| \leq \tfrac{1}{2} \text{ for all } x \in U_\xi\}.$$

A delicate (but not difficult) argument shows that the sets F_ξ are compact subsets of $\hat{G}$. It is clear that $\hat{G} = \bigcup_{\xi<\alpha} F_\xi$, so that $\kappa(\hat{G}) \leq \alpha = \chi(G)$.

The proof that $\chi(\hat{G}) = \kappa(G)$ for every locally compact Abelian group is complete. Again by duality we have $\chi(G) = \kappa(\hat{G})$, and from Theorem 3.5(iii) now follows

$$w(G) = \kappa(G) \cdot \chi(G) = \chi(\hat{G}) \cdot \kappa(\hat{G}) = w(\hat{G}).$$

From statement (ii) above and 1.8 it follows that a locally compact, Abelian topological group G is metrizable if and only if $\hat{G}$ is σ-compact; a compact, Abelian topological group G is metrizable if and only if the (discrete) group $\hat{G}$ is countable. Concerning (ii), see HEWITT and STROMBERG [1962] and HEWITT and ROSS [1963] (24.14, 24.48).

That the inequality of (i) can fail when G is not assumed to be locally compact is a theorem of GINSBURG, RAJAGOPALAN, and SAKS [1976]. COMFORT and ITZKOWITZ [1977] give a minor technical improvement on their result, as well as a proof of 3.14(i).

We note in 7.10 below that the inequality $d(G) < d(H)$ can occur when H is a dense subgroup of a compact Abelian topological group.

3.15. It cannot be proved in ZFC that for every topological group G the number $o(G)$ has the form 2^α. I. Juhász and J. Roitman have remarked independently in conversation that the construction of HAJNAL and JUHÁSZ [1974] can be modified to yield a model of $\mathrm{CH} + (2^{(\omega^+)} > \omega^{++})$ containing not only a space but in fact a topological subgroup G of $\{-1, +1\}^{(\omega^+)}$ such that $|G| = \omega^{++}$ and G is hereditarily separable. As indicated in 5.1 of Roitman's article in this Handbook, from CH follows $|G|^\omega = |G|$ and then from hereditary separability of G follows $|G| \leq o(G) \leq |G|^\omega = |G|$ and hence $o(G) = \omega^{++} \neq 2^\alpha$.

For related phenomena deriving from the HFD sets of HAJNAL and JUHÁSZ [1974], see also 5.22 and 5.23 below.

SHELAH [1978] has announced that it is consistent with ZFC that there exist infinite Hausdorff spaces X such that $o(X) \neq (o(X))^\omega$; indeed, $\mathrm{cf}(o(X)) = \omega$ is possible. Let us show in 3.16, arguing essentially as in JUHÁSZ [1980] (4.9), that such pathology is not available in the class of infinite topological groups.

For cardinal numbers α and β (not necessarily infinite), we write $\alpha^{\underset{\smile}{\beta}} = \Sigma\{\alpha^\kappa : \kappa$ is a cardinal, $\kappa < \beta\}$. For X a space we denote by $S(X)$ the Souslin number of X, i.e., the least cardinal number that is not the cardinal of a cellular family in X; the number $S(X)$ is denoted $\hat{c}(X)$ by JUHÁSZ [1980].

It is clear that $S(X) \leq o(X)$ for every space X. Indeed if $\{U_i : i \in I\}$ is a faithfully indexed cellular family in X with $|I| = o(X)$ and for $J \subset I$ we set $U(J) = \bigcup_{i \in J} U_i$, then $\{U(J) : J \subset I\}$ is a faithfully indexed family of open subsets of X; hence $o(X) \geq 2^{|I|} = 2^{o(X)}$, a contradiction.

3.16. THEOREM. *The identity $o(G) = (o(G))^{\underset{\smile}{S(G)}}$ holds for every infinite topological group G.*

PROOF. Since $S(G) \leq o(G)$ it is enough to show $o(G) = (o(G))^\beta$ for all $\beta < S(G)$, for then

$$(o(G))^{\underset{\smile}{S(G)}} = \Sigma_{\beta < S(G)} (o(G))^\beta \leq S(G) \cdot o(G) = o(G)\,.$$

If G is discrete the statement is obvious. We assume that G is not discrete, we choose $U \in \mathcal{N}(e)$ so that $o(U)$ is minimal (hence $o(V) = o(U)$ whenever $V \in \mathcal{N}(e)$, $V \subset U$) and, given $\beta < S(G)$, we choose $\{x_\xi : \xi < \beta\} \subset G$ and $\{U_\xi : \xi < \beta\} \subset \mathcal{N}(e)$ so that $\{x_\xi U_\xi : \xi < \beta\}$ is a faithfully indexed cellular family in G. We choose $V \in \mathcal{N}(e)$ so that $VV^{-1} \subset U$, for $\xi < \beta$ we choose $V_\xi \in \mathcal{N}(e)$ so that $V_\xi V_\xi^{-1} \subset U_\xi$ and $V_\xi \subset V$, and we choose $A \subset G$ maximal with respect to the property that

$$\{x_\xi V_\xi : \xi < \beta\} \cup \{aV : a \in A\}$$

is a cellular family in G. It is clear that

$$(1) \quad o(G) \geq \prod_{\xi<\beta} o(x_\xi V_\xi) \times \prod_{a\in A} o(aV) = (o(U))^{\beta+|A|}.$$

We note that

$$\{x_\xi U : \xi < \beta\} \cup \{aU : a \in A\}$$

is a cover for G: If $p \in G$ then either there is $\xi < \beta$ such that $x_\xi V_\xi \cap pV \neq \emptyset$ or there is $a \in A$ such that $aV \cap pV \neq \emptyset$, and hence $p \in x_\xi V_\xi V^{-1} \subset x_\xi U$ or $p \in aVV^{-1} \subset aU$. From this follows

$$(2) \quad o(G) \leq \prod_{\xi<\beta} o(x_\xi U) \times \prod_{a\in A} o(aU) = (o(U))^{\beta+|A|};$$

combining (1) and (2) we find $o(G) = (o(U))^{\beta+|A|}$ and hence $o(G) = (o(G))^{\beta}$, as required.

3.17. Since every infinite (Hausdorff) space X has an infinite cellular family (that is, $S(X) \geq \omega^+$), we have in particular from Theorem 3.16 that every infinite topological group G satisfies $o(G) = (o(G))^{\omega}$. This result is due to JUHÁSZ [1980] (4.9).

3.18. Among the many authors who have achieved results related to those considered in this section, one may cite in particular ISMAIL [1981] and ARHANGEL'SKIĬ [1979], [1980]. The former shows that the topological restrictions imposed here are in some cases unnecessarily severe—and further, many of the cardinal identities given here are valid for homogeneous topological spaces (not assumed to be topological groups). The following two results are among those given by Arhangel'skiĭ.

(i) If G is a topological group with a subset A such that $\mathrm{wc}(A) \leq \alpha$ and the subgroup generated by A is dense in G, then G is a subgroup of a product group $\Pi_{i\in I} G_i$ with each $\psi(G_i) \leq \alpha$.

(ii) If G is a compact topological group and F a compact subspace which generates G algebraically, then $w(G) = t(F) = w(F)$; thus, not every compact topological space can generate algebraically a compact topological group.

Let us note, finally, that the bibliographies of the books of HEWITT and ROSS [1963], [1970] contain many references to articles devoted in part to topological groups and their cardinal invariants, while ARHANGEL'SKIĬ [1980] contains not only new contributions of the author but also a survey of some recent literature.

4. Measurability conditions and the group Hom(G, $\mathbb{T}$)

Here we explore three aspects of a hypothesized locally compact topological group G: the Haar measure of G, the class of homomorphisms from G to the circle group $\mathbb{T}$, and the relation between these two. Here are some typical questions to which answers are available: Is every subset of G measurable (4.20)? Is every subgroup measurable (4.10, 4.15)? Is every homomorphism measurable (4.18, 4.24)? An important tool is the theorem of Steinhaus and Weil (4.6) which asserts, in the case $G = \mathbb{R}$, that for every set $A \subset \mathbb{R}$ with positive Lebesgue measure the difference set $\{x - y: x, y \in A\}$ contains an interval about zero.

Our study of subgroups $\mathscr{H}$ of Hom(G, $\mathbb{T}$) and the topologies they define on G(4.2) suggests the structure Hom($\hat{G}$, $\mathbb{T}$) and the natural embedding $\zeta: G \to$ Hom($\hat{G}$, $\mathbb{T}$). This defines the 'Bohr compactification' $\tilde{G}$ (4.26), a vehicle leading to the proof that every homomorphism from G to $\mathbb{T}$ is uniformly approximable on finite sets by continuous homomorphisms, i.e., by elements of $\hat{G}$ (4.30). A consequence is a version of Kronecker's theorem (4.32), and an elementary result concerning monothetic tori.

Although the point will be made locally and frequently it is worth noting globally once that essentially everything of value in this section can be found in the work of HEWITT and ROSS [1963], [1965], [1970]. It is futile to hope to improve upon their high standards of taste and exposition; one may only hope, by choosing some of the most striking and attractive results, to retain them.

4.1. We begin with a simple but useful observation.

LEMMA. *Let G and H be topological groups and $h: G \to H$ a homomorphism continuous at some point $x \in G$. Then h is uniformly continuous on G.*

PROOF. Given $V \in \mathscr{N}_H(e)$ we must find $U \in \mathscr{N}_G(e)$ such that if $p, q \in G$ with $p \in qU$, then $h(p) \in h(q) \cdot V$. It is enough to choose $U \in \mathscr{N}_G(e)$ so that $y \in xU$ implies $h(y) \in h(x) \cdot V$, for then $p \in qU$ implies $xq^{-1} p \in xU$, hence

$$h(x) \cdot h(q^{-1}) \cdot h(p) \in h(x) \cdot V$$

and then $h(p) \in h(q) \cdot V$.

4.2. We recall from 1.23 that for $G = \langle G, \mathscr{T} \rangle$ an Abelian topological group which is perhaps not locally compact, the symbol $\hat{G} = \langle G, \mathscr{T} \rangle^{\wedge}$ is defined to be the set of continuous homomorphisms from G to $\mathbb{T}$.

For G a group and $\mathscr{H}$ a point-separating subgroup of Hom(G, $\mathbb{T}$), we denote by $\mathscr{T}_{\mathscr{H}}$ the topology induced on G by $\mathscr{H}$. It is easy to see that $\langle G, \mathscr{T}_{\mathscr{H}} \rangle$ is an Abelian topological group; the Hausdorff separation axiom is assured by the hypothesis that $\mathscr{H}$ separates points of G. The family of all sets of the form

$$U(\mathscr{F}, \varepsilon) = \{x \in G : |\chi(x) - 1| < \varepsilon \text{ for all } \chi \in \mathscr{F}\},$$

with $\varepsilon > 0$ and finite $\mathscr{F} \subset \mathscr{H}$, is a basis at e for the topology $\mathscr{T}_{\mathscr{H}}$.

It is an important tool in a rigorous development of Pontrjagin duality that

(*) for an Abelian group G the family $\mathrm{Hom}(G, \mathbb{T})$ distinguishes points of G;

in turn, (*) is a consequence of that duality. While we have agreed not to attempt a full development of that theory in this chapter, and to use that theory when it is convenient to do so, we nevertheless include a proof of (*) in Section 9 below, in the interest of making that section self-contained. The following statement, taken from COMFORT and ROSS [1964], is given here as a lemma to Theorem 4.15(ii); statement (*) serves to show that its hypotheses are easily and frequently satisfied.

THEOREM. *Let G be an Abelian group and $\mathscr{H}$ a point-separating subgroup of* $\mathrm{Hom}(G, \mathbb{T})$. *Then $\langle G, \mathscr{T}_{\mathscr{H}}\rangle$ is a totally bounded topological group, and* $\langle G, \mathscr{T}_{\mathscr{H}}\rangle\hat{} = \mathscr{H}$.

PROOF. Since $\mathscr{H}$ is point-separating, the embedding function $i: G \to \mathbb{T}^{\mathscr{H}}$ defined by $i(x)_\chi = \chi(x)$ for $x \in G$ and $\chi \in \mathscr{H}$ is one-to-one. The topology $\mathscr{T}_{\mathscr{H}}$ is the topology which makes i a homeomorphism from G onto the subgroup $i[G]$ of $\mathbb{T}^{\mathscr{H}}$. Since $\mathbb{T}^{\mathscr{H}}$ is compact, this subgroup is totally bounded. Suppressing now the function i and viewing $\langle G, \mathscr{T}_{\mathscr{H}}\rangle$ as a subspace of $\mathbb{T}^{\mathscr{H}}$, we denote by $\bar{G}$ the closure in $\mathbb{T}^{\mathscr{H}}$ of G: this is the Weil completion (1.13) of $\langle G, \mathscr{T}_{\mathscr{H}}\rangle$.

It is clear that $\mathscr{H} \subset \langle G, \mathscr{T}_{\mathscr{H}}\rangle\hat{}$. To prove the reverse inclusion let $\psi \in \langle G, \mathscr{T}_{\mathscr{H}}\rangle\hat{}$ and note that ψ, since it is uniformly continuous, extends to an element of $\hat{\bar{G}}$ (also denoted ψ). Finally since $\bar{G}$ is closed in $\mathbb{T}^{\mathscr{H}}$ there is $\Omega \in (\mathbb{T}^{\mathscr{H}})\hat{}$ such that $\Omega|\bar{G} = \psi$; this follows from 1.30 and is proved in detail in HEWITT and ROSS [1963] (24.12). Since $\hat{\mathbb{T}} = \mathbb{Z}$ and $(\mathbb{T}^{\mathscr{H}})\hat{} = \bigoplus_{\mathscr{H}}\mathbb{T}$, there are $\{\chi_k : 1 \leqslant k \leqslant n\} \subset \mathscr{H}$ and $\{m_k : 1 \leqslant k \leqslant n\} \subset \mathbb{Z}$ such that $\Omega: \mathbb{T}^{\mathscr{H}} \to \mathbb{T}$ is given by the rule

$$t = \langle t_\chi : \chi \in \mathscr{H}\rangle \to \prod_{k=1}^{n} (t_{\chi_k})^{m_k}.$$

In particular for $x = \langle x_\chi : \chi \in \mathscr{H}\rangle \in G$ we have

$$\psi(x) = \Omega(x) = \prod_{k=1}^{n} \chi_k(x)^{m_k}$$

and hence $\psi = \prod_{k=1}^{n} \chi_k^{m_k} \in \mathscr{H}$.

4.3. COROLLARY. *Let $G = \langle G, \mathscr{T}\rangle$ be a totally bounded Abelian topological group and set $\mathscr{H} = \langle G, \mathscr{T}\rangle\hat{}$. Then*

(i) $\mathscr{T} = \mathscr{T}_{\mathscr{H}}$ *and*

(ii) *if* $\mathcal{J}$ *is a point-separating subgroup of* $\operatorname{Hom}(G, \mathbb{T})$ *such that* $\mathcal{T} = \mathcal{T}_{\mathcal{J}}$, *then* $\mathcal{J} = \mathcal{H}$.

PROOF. (i) Each element of $\langle G, \mathcal{T}\rangle\hat{}$, being uniformly continuous on G, extends continuously over the Weil completion $\bar{G}$. From Pontrjagin duality it follows that the topology of $\bar{G}$ is the topology induced by $\hat{\hat{G}}$. Thus the topology $\mathcal{T}$ is that induced by $\{\chi|G: \chi \in \hat{\bar{G}}\}$, i.e., by $\mathcal{H}$.

(ii) From $\mathcal{T} = \mathcal{T}_{\mathcal{H}}$ and $\mathcal{T} = \mathcal{T}_{\mathcal{J}}$ we have $\mathcal{H} = \langle G, \mathcal{T}_{\mathcal{H}}\rangle \hat{=} \langle G, \mathcal{T}\rangle \hat{=} \langle G, \mathcal{T}_{\mathcal{J}}\rangle \hat{=} \mathcal{J}$, as required.

We note in passing a consequence of 4.2 and 4.3. There is a family of cardinality $\mathfrak{c}$ of totally bounded, metrizable topological group topologies for $\mathbb{Z}$. (The number $\mathfrak{c}$ here is clearly maximal, since the number of metrics on $\mathbb{Z}$ does not exceed $|\mathbb{R}^{\mathbb{Z}\times\mathbb{Z}}| = \mathfrak{c}$.) Indeed for $\zeta \in \mathbb{T}$ such that ζ has infinite order define $h \in \operatorname{Hom}(\mathbb{Z}, \mathbb{T})$ by $h_\zeta(n) = \zeta^n$ and set

$$\mathcal{H}(\zeta) = \{h_\zeta^k : k \in \mathbb{Z}\} \subset \operatorname{Hom}(\mathbb{Z}, \mathbb{T}).$$

The topology $\mathcal{T}_{\mathcal{H}(\zeta)}$ induced on $\mathbb{Z}$ by $\mathcal{H}(\zeta)$ is given by the requirement that the isomorphism h_ζ be a homeomorphism. For elements ζ, η of infinite order in $\mathbb{T}$ we have $\mathcal{H}(\zeta) = \mathcal{H}(\eta)$ if and only if $\zeta = \eta$ or $\zeta = \bar{\eta}$; thus the family

$$\{\mathcal{T}_{\mathcal{H}(\zeta)}: \zeta \in \mathbb{T},\ \zeta \text{ has infinite order, } \operatorname{Re}(\zeta) > 0\}$$

is as required.

It is noted by COMFORT and ROSS [1964] (Theorem 1.9) that for G an Abelian group and $\mathcal{H}$ a subgroup of $\operatorname{Hom}(G, \mathbb{T})$, the group $\mathcal{H}$ is point-separating if and only if $\mathcal{H}$ is dense in $(G_d)\hat{}$. Since $(\mathbb{Z}_d)\hat{} = \mathbb{T}$ admits $2^{\mathfrak{c}}$ distinct dense subgroups, it follows from 4.3(ii) that $\mathbb{Z}$ admits $2^{\mathfrak{c}}$ distinct totally bounded topological group topologies.

4.4. For every locally compact topological group G and $A \subset G$, as in 1.14 we write

$$\lambda_*(A) = \sup\{\lambda(F): F \subset A,\ F \text{ is compact}\}.$$

According to 1.15, the relation $\lambda_*(A) = \lambda(A)$ holds whenever A is λ-measurable with $0 \leq \lambda(A) < +\infty$. It can fail, however, for $\lambda(A) = +\infty$. The following two examples, of which the first is a special case of the second, are from HEWITT and ROSS [1963] (11.22 and 16.14); as usual, $\mathbb{R}_d$ denotes the group $\mathbb{R}$ with the discrete topology.

(i) Let $G = \mathbb{R}_d \times \mathbb{R}$ and $A = \mathbb{R}_d \times \{0\}$. Since every subset S of A is closed in G, every such S is λ-measurable; it is not difficult to see for $S \subset A$ that $\lambda(S) = 0$ if $|S| \leq \omega$, and $\lambda(S) = +\infty$ if $|S| > \omega$. Thus $\lambda(A) = +\infty$. But every compact subset of A is finite, and hence $\lambda_*(A) = 0$.

(ii) Let G be any non-discrete, locally compact group that is not σ-compact, let H be an open, σ-compact subgroup of G, and let A be a subset of G containing exactly one point from each (left) coset of H. Again A and each of its subsets is closed and hence λ-measurable, and every compact $F \subset A$ satisfies $|F| < \omega$ and hence $\lambda(F) = 0$. Thus $\lambda_*(A) = 0$. Every open set U of G containing A satisfies $\lambda(xH \cap U) > 0$ for all cosets xH, and from $|G/H| > \omega$ follows $\lambda(U) = +\infty$. From the regularity of λ (see 1.15) we have $\lambda(A) = +\infty$.

4.5. The unpleasant pathology just described cannot occur when G is σ-compact.

THEOREM. *Let G be a locally compact topological group and A a λ-measurable subset of G. If $\lambda(A) < +\infty$ or A is contained in a σ-compact subset of G, then $\lambda(A) = \lambda_*(A)$.*

PROOF. Since λ is a regular Borel measure we have $\lambda(A) = \lambda_*(A)$ for all λ-measurable $A \subset G$ with $\lambda(A) < +\infty$. Suppose now that ($\lambda(A) = +\infty$ and) that $A \subset \bigcup_{n<\omega} K_n$ with K_n compact and with $K_n \subset K_{n+1}$ for $n < \omega$. Then

$$\lambda(A \cap K_n) \leq \lambda(K_n) < \infty$$

and, as is well known (see for example HEWITT and STROMBERG [1965] (10.13)), $\lim_n \lambda(A \cap K_n) = +\infty$. That $\lambda(A) = \lambda_*(A)$ now follows from the corresponding relations for the sets $A \cap K_n$.

Most results in the theory of topological groups concerning the existence of Haar non-measurable sets are based on some version of the following well known theorem, which is due to STEINHAUS [1920] in the case $G = \mathbb{R}$ and to WEIL [1951] (p. 50) in general. The present elegant, elementary proof is taken from STROMBERG [1972]; see also HALMOS [1950] (§54.B).

4.6. THEOREM. *Let G be a locally compact topological group and A a λ-measurable subset of G such that $\lambda(A) > 0$. If either $\lambda(A) < +\infty$ or A is contained in a σ-compact subset of G, then there is $U \in \mathcal{N}(e)$ such that $U \subset AA^{-1}$.*

PROOF. By 4.5 there is compact $F \subset A$ such that $0 < \lambda(F) + \infty$, and since λ is regular there is open $V \subset G$ such that $F \subset V$ and $\lambda(V) < 2\lambda(F)$. For every $x \in F$ there is $U_x \in \mathcal{N}(e)$ such that $U_x \cdot x \subset V$; hence there is $U \in \mathcal{N}(e)$ such that $UF \subset V$. It follows that $U \subset FF^{-1} \subset AA^{-1}$, for if $x \in U$ and $xF \cap F = \emptyset$, then from $V \supset xF$ and $V \supset F$ we have

$$2\lambda(F) > \lambda(V) \geq \lambda(xF) + \lambda(F) = 2\lambda(F),$$

a contradiction.

4.7. Two comments will indicate potential difficulties encountered in attempting to weaken the hypotheses of Theorem 4.6.

(i) The example of 4.4(i) shows that the requirement that either $\lambda(A) < +\infty$ or A is contained in a σ-compact subset of G cannot be omitted. Here with

$$A = \mathbb{R}_d \times \{0\} \subset \mathbb{R}_d \times \mathbb{R} = G$$

we have $\lambda(A) = +\infty$ and (retaining multiplicative notation) the set AA^{-1}, which is A, contains no neighborhood in G of e.

(ii) Even when G is compact, the condition in 4.6 that A be measurable with $\lambda(A) > 0$ cannot be weakened to the condition that A has positive outer measure (in the sense that $\lambda(S) > 0$ whenever $S \supset A$ and S is λ-measurable). Let G be a compact topological group with a proper, dense, subgroup A such that $|G/A| < \omega$. (Such pairs $\langle G, A\rangle$ are easy to find. In 6.15 below we show that every compact, totally disconnected topological Abelian group G with $w(G) > \omega$ contains a proper subgroup H with $|G/H| < \omega$ such that H is even G_δ-dense in G.) Like every proper, dense subgroup of a topological group, A has empty interior in G; that is, the conclusion of 4.6 fails for the pair $\langle G, A\rangle$. But it is clear with $|G/A| = n < \omega$ that $\lambda(S) \geq 1/n$ for every λ-measurable set S such that $S \supset A$: Otherwise, writing $G/A = \{x_kA\colon 1 \leq k \leq n\}$ faithfully indexed, we have

$$G = \bigcup_{k=1}^{n} x_kA = \bigcup_{k=1}^{n} x_kS$$

and hence $1 = \lambda(G) \leq \Sigma_{k=1}^{n} \lambda(x_kS) = \Sigma_{k=1}^{n} \lambda(S) < 1$, a contradiction. (The more sophisticated argument of 4.11 below shows that in fact $\lambda(S) = 1$.)

4.8. BECK, CORSON and SIMON [1958] offer an extension of Theorem 4.6: If G is a locally compact topological group and if $A, B \subset G$ with A λ-measurable of (finite) positive measure and with B of (finite) positive outer measure, then there is $U \in \mathcal{N}(e)$ such that $U \subset (AB \cap BA)$.

4.9. The converse to Theorem 4.6 is false. Let λ be Haar measure (Lebesgue measure) on $\mathbb{R}$ and A the Cantor 'middle third' set, a subset of $[0, 1]$. Then $\lambda(A) = 0$ but, as RANDOLPH [1940], [1941] has shown, every $p \in [-1, 1]$ has the form $p = x - y$ for appropriately chosen $x, y \in A$.

4.10. We repeat for emphasis the fact that our framework of discourse in this chapter is ZFC, i.e., Zermelo–Fraenkel set theory with the axiom of choice. As we approach the project of finding or constructing nonmeasurable sets, the point to be made is not simply that the axiom of choice seems convenient or useful. (Though surely it is. The early construction of VITALI [1905] is achieved by selecting one point from each coset of $\mathbb{Q}$ in $\mathbb{R}$ and showing that the resulting set is not Lebesgue-measurable; and the argument of F. BERNSTEIN [1908] (§3) involves well-ordering the set $\mathbb{R}$ in order to define a set A such that A and $\mathbb{R}\backslash A$ meet every uncountable closed subset of $\mathbb{R}$, so that $\lambda_*(A) = \lambda_*(\mathbb{R}\backslash A) = 0$.) Rather, the axiom of choice or one of its relatives is essential to our endeavor. Using the axioms of ZF and the axiom of determinacy (a statement easily seen incompatible with the axiom of choice), MYCIELSKI and ŚWIERCZKOWSKI [1964] have proved that every subset of $\mathbb{R}$ is Lebesgue-measurable.)

The following definition is suggested by terminology of HEWITT and ROSS [1965].

DEFINITION. Let G be a locally compact topological group and let $A \subset G$. If there is a measure μ on G such that

(i) μ is (left-) translation-invariant,
(ii) every Borel set of G is μ-measurable,
(iii) A is μ-measurable, and
(iv) $\mu(S) = \lambda(S)$ for every Borel set S of G,

then A is an *ultimately measurable* subset of G.

If there is no such measure μ, then A is *ultimately non-measurable.*

It has been proved by SOLOVAY [1971] that if the system [ZFC + "there is a measurable cardinal"] is consistent, then the system [ZFC + "Lebesgue measure extends to a measure defined on every set of reals"] is consistent (and conversely). As has been noted by HALMOS [1950] and others, however, there is (in ZFC) no extension of Lebesgue measure making every set of reals measurable which is also translation-invariant.

Evidently the definition just given concerns not the question of simultaneously rendering all sets measurable, but the more modest question whether a given non-measurable set can be 'made measurable'. Let us turn to this now in a general context, following HEWITT and ROSS [1963] (16.13(d)).

THEOREM. *Let G be a locally compact topological group and H a dense subgroup such that $|G/H| = \omega$. Then*

(i) *H is not measurable*; *and*
(ii) *if G is compact, then H is ultimately non-measurable.*

PROOF. (i) Suppose that H is measurable and let N be an open, σ-compact subgroup of G. We note that

$$|N/(H \cap N)| \leq |G/H| = \omega .$$

The set $H \cap N$ is measurable. If $\lambda(H \cap N) = 0$, then from $|N/(H \cap N)| \leq \omega$ we have $\lambda(N) = 0$, contradicting the fact that $\lambda(U) > 0$ for every non-empty open subset U of G. And if $\lambda(H \cap N) > 0$, then since N is σ-compact we have from 4.6 that there is $U \in \mathcal{N}(e)$ such that $U \subset (H \cap N) \cdot (H \cap N)^{-1} = H \cap N$. From

$\mathrm{int}_G H \neq \emptyset$ it then follows that H is open-and-closed; hence $H = G$, a contradiction.

(ii) Suppose that μ is a translation-invariant extension of λ such that $H \in \mathcal{M}(\mu)$. We have $\mu(G) = \lambda(G) = 1$; but if $\mu(H) = 0$, then $\mu(G) = 0$, and if $\mu(H) > 0$, then $\mu(G) = +\infty$.

Let us use the theorem just proved to show that every locally compact, σ-compact, non-discrete Abelian topological group G contains a dense non-measurable subgroup H; if in addition G is compact, then H may be taken ultimately non-measurable. According to Theorem 3.9 such a group G contains a dense subset, hence a dense subgroup N, such that $|N| < |G|$; we have $|G/N| = |G| > \omega$. It is known (see for example HEWITT and ROSS [1963] (16.13) or FUCHS [1970]) that every infinite Abelian group contains a subgroup of countably infinite index. In particular, G/N has such a subgroup. This has the form H/N for a suitably chosen subgroup H of G, and from a standard theorem of group theory we have the (algebraic) isomorphism $G/H \cong (G/N)/(H/N)$ and hence $|G/H| = \omega$. From $N \subset H \subset G$ it follows that H is dense in G, and the required statements follow from the theorem proved.

Let G be a locally compact, non-discrete, Abelian topological group and N an open, σ-compact subgroup. According to what has just been shown, some dense subgroup of N is not λ_N-measurable (and hence is not λ_G-measurable). It is tempting to conjecture that G itself must contain a dense subgroup that is not λ_G-measurable. This conjecture is false. We show in 7.2 below that there are locally compact, non-discrete Abelian topological groups which contain no proper dense subgroups whatever.

4.11. Our next principal result is 4.15(ii), which is in striking contrast with Theorem 4.10(ii). To prove it we use a special case of a theorem of VAROPOULOS [1965]; his more general result, not needed here, involves a more delicate statement and proof which are applicable when G is not assumed σ-compact.

We recall from 1.16 our convention that for every locally compact topological group G, the Haar measure $\lambda = \lambda_G$ is complete; that is, if $A \subset G$ and there is λ-measurable $B \subset G$ such that $A \subset B$ and $\lambda(B) = 0$, then A is λ-measurable (and $\lambda(A) = 0$).

THEOREM. *Let G be a locally compact, σ-compact topological group and H a dense subgroup. Then either H is measurable with $\lambda(H) = 0$, or $\lambda_*(G \backslash H) = 0$.*

PROOF. Since G is σ-compact there is a Borel set B of G such that $H \subset B$ and $\lambda_*(B \backslash H) = 0$. We assume in what follows that $\lambda(B) > 0$, since otherwise H is measurable with $\lambda(H) = 0$.

We note for $x \in H$ that $H \subset x^{-1}B$ and hence $\lambda(x^{-1}B \,\Delta\, B) = 0$.

Now for every Borel set A of G we set $\mu(A) = \lambda(A \cap B)$. For $x \in H$ we have

$$\mu(xA) = \lambda(xA \cap B) = \lambda(A \cap x^{-1}B),$$

and from $(A \cap x^{-1}B)\,\Delta\,(A \cap B) \subset x^{-1}B\,\Delta\,B$ and $\lambda(x^{-1}B\,\Delta\,B) = 0$ it follows that $\lambda(A \cap x^{-1}B) = \lambda(A \cap B)$ and hence $\mu(xA) = \mu(A)$. From the density of H in G we have $\mu(xA) = \mu(A)$ for all $x \in G$ and all Borel sets A of G.

Since $\mu(G) = \lambda(B) > 0$, the measure μ is not the trivial (zero) measure. From the uniqueness of Haar measure there is a real constant c (namely, $c = \lambda(B)$) such that $\mu = c\lambda$.

Now suppose there is a Borel set S of G such that $S \subset G\backslash H$ and $\lambda(S) > 0$. Then either $\lambda(S \cap B) > 0$ or $\lambda(S\backslash B) > 0$, and hence from 4.5 there is compact $F \subset G$ with $\lambda(F) > 0$ such that either $F \subset S \cap B$ or $F \subset S\backslash B$. In the former case from $S \cap B \subset B\backslash H$ we have

$$\lambda_*(B\backslash H) \geqslant \lambda(F) > 0$$

and in the latter case we have

$$\lambda(F) = (1/c)\mu(F) = (1/c)\lambda(F \cap B) = 0\,.$$

The proof is complete.

Let us note in passing two statements related to the result just proved.

4.12. COROLLARY. *Let G be a locally compact, σ-compact topological group and H a proper, dense, measurable subgroup of G. Then $\lambda(H) = 0$.*

PROOF. If the statement fails then, choosing $x \in G\backslash H$, there is (by 4.5) compact $F \subset xH$ such that $\lambda(F) > 0$; we then have

$$\lambda_*(G\backslash H) \geqslant \lambda_*(xH) \geqslant \lambda(F) > 0\,,$$

contradicting 4.11.

4.13. If G is a locally compact topological group and H a dense subgroup, then from 4.11 it follows that for every open, σ-compact subgroup N of G either $\lambda_G(H \cap N) = \lambda_N(H \cap N) = 0$ or $\lambda_{G^*}(N\backslash H) = \lambda_{N^*}(N\backslash H) = 0$. It may occur, however, that H itself is measurable with $\lambda_G(H) = +\infty$. For an example to this effect consider $G = \mathbb{R}_d \times \mathbb{R}$ as in 4.7(i) and set $H = \mathbb{R}_d \times \mathbb{Q}$. Then H, an F_σ in G, is measurable, and $\lambda_G(H) = +\infty$ (see 4.7(i)). For every open, σ-compact subgroup N of G there is countable $S \subset \mathbb{R}_d$ such that $N = S \times \mathbb{R}$, and since $\lambda_N(\{\langle s, r\rangle\}) = 0$ for all $\langle s, r\rangle \in N$ and $|H \cap N| \leqslant \omega$ we have $\lambda_N(H \cap N) = 0$.

4.14. The names of Jankowska-Wiatr and O. Nikodym are cited by MARCZEWSKI [1935] (Szpilrajn) as contributing helpful ideas to his extension of n-dimensional Lebesgue measure to a large translation-invariant σ-algebra. Subsequent work in $\mathbb{R}^n$ by KODAIRA and KAKUTANI [1950] and by KAKUTANI and OXTOBY [1950] was continued into the more general context by HULANICKI [1959], [1964], ITZKOWITZ [1965a], [1965b], and HULANICKI and RYLL–NARDZEWSKI [1979]. Theorem 4.15(ii) below is a very special case of a powerful result due to HEWITT and ROSS [1965]. (Strictly speaking it is 4.16(ii) which follows from their work, but the equivalence between subgroups of finite index and homomorphisms with finite range, which we indicate, poses no difficulty.)

For the most part these authors are concerned with the question of finding extensions μ of Haar measure $\lambda = \lambda_G$ so that the function space $L_2(G, \mu)$ has dimension which is large relative to the dimension of $L_2(G, \lambda)$; this is equivalent to arranging (simultaneously) that many λ-nonmeasurable homomorphisms from G to $\mathbb{T}$ become μ-measurable. Our goal here is more modest and hence more easily achieved: We handle just a single homomorphism, or a single subgroup.

While all that follows is from HEWITT and ROSS [1965], we note in particular the elegant and crucial proof that $|K/G| < \omega$; this appears within their Theorem 3.12.

4.15. THEOREM. *Let G be a locally compact topological group and H a proper, dense subgroup such that $|G/H| < \omega$. Then*

(i) *H is not λ-measurable*; *and*

(ii) *if G is compact and Abelian, then H is ultimately measurable.*

PROOF. (i) The proof of 4.10(i) is valid without change.

(ii) [The idea of this proof is to embed G, with a slightly finer topology, into a new compact group K in such a way that G has full outer λ_K-measure and H becomes closed in G.]

To avoid confusion in what follows we write $G = \langle G, \mathcal{T}\rangle$. Let $\{x_kH: 1 \leq k \leq n\}$ be a faithful indexing of G/H.

Let φ be the canonical homomorphism of G onto G/H, define

$$\mathcal{F} = \{f \circ \varphi : f \in \operatorname{Hom}(G/H, \mathbb{T})\},$$

and let $\mathcal{H}$ be the subgroup of $\operatorname{Hom}(G, \mathbb{T})$ generated by $\mathcal{F} \cup \langle G, \mathcal{T}\rangle^{\wedge}$. Since the topology $\mathcal{T}$ is induced on G by $\langle G, \mathcal{T}\rangle^{\wedge}$ (see 4.3), the topology $\mathcal{T}_{\mathcal{H}}$ induced on G by $\mathcal{H}$ satisfies $\mathcal{T} \subset \mathcal{T}_{\mathcal{H}}$. (This is our only use in this proof that the locally compact group $\langle G, \mathcal{T}\rangle$ is totally bounded, i.e., is compact.) The group $\langle G, \mathcal{T}_{\mathcal{H}}\rangle$ is totally bounded (4.2); we denote by K its Weil completion.

Since $H = \bigcap\{\ker \psi: \psi \in \langle G, \mathcal{T}_{\mathcal{H}}\rangle^{\wedge}, H \subset \ker \psi\}$, the set H (and hence each set x_kH) is closed in $\langle G, \mathcal{T}_{\mathcal{H}}\rangle$; hence each x_kH is open in $\langle G, \mathcal{T}_{\mathcal{H}}\rangle$, and from $G = \bigcup_{k=1}^{n} x_kH$ it follows that $K = \bigcup_{k=1}^{n} \operatorname{cl}_K(x_kH)$ and each set $\operatorname{cl}_K x_kH$ is open-and-closed in K.

We wish now for every Borel set A of K to define $\mu(A \cap G)$ by the rule $\mu(A \cap G) = \lambda_K(A)$. The proof below that μ is well-defined depends on the assertion that $|K/G| < \omega$.

To see that $|K/G| < \omega$, note first (see 4.1) that the elements of $\mathscr{H}$ are uniformly continuous on $\langle G, \mathscr{T}_{\mathscr{H}}\rangle$ and hence extend continuously over K. That is, the groups $\mathscr{H}$ and $\hat{K}$ are naturally isomorphic; we write $\hat{G} = \langle G, \mathscr{T}_{\mathscr{H}}\rangle\hat{} \subset \mathscr{H} = \hat{K}$ and we write

$$\mathscr{A} = \{f \in \hat{\hat{K}} : f(\chi) = 1 \text{ for all } \chi \in \langle G, \mathscr{T}\rangle\hat{}\}.$$

Since $|\mathscr{H}/\hat{G}| < \omega$ we have $|\mathrm{Hom}(\mathscr{H}/\hat{G}, \mathbb{T})| < \omega$ and from the isomorphism $\mathscr{A} \simeq \mathrm{Hom}(\mathscr{H}/\hat{G}, \mathbb{T})$ of 1.30 then follows $|\mathscr{A}| < \omega$.

Identifying K with $\hat{\hat{K}}$, for $p \in K$ we write $p|\langle G, \mathscr{T}\rangle\hat{} \in \langle G, \mathscr{T}\rangle\hat{\hat{}}$ and we note that (in the notation of 1.25) there is $x \in G$ such that $p|\langle G, \mathscr{T}\rangle\hat{} = \upsilon_G(x)$. That is, for $p \in K$ there is $x \in G$ such that $p(\chi) \cdot \upsilon_G(x^{-1})(\chi) = 1$ for all $\chi \in \langle G, \mathscr{T}\rangle\hat{}$ and hence $p \cdot \upsilon_G(x^{-1}) \in \mathscr{A}$. We have $\hat{\hat{K}} = \mathscr{A} \cdot \upsilon_G[G]$ with $|\mathscr{A}| < \omega$ and hence

$$|K/G| = |\hat{\hat{K}}/\upsilon_G[G]| \leq |\mathscr{A}| < \omega.$$

Now set $\mathscr{S} = \{A \cap G : A \text{ is Borel in } K\}$ and for $A \cap G \in \mathscr{S}$ define $\mu(A \cap G) = \lambda_K(A)$. To see that μ is well-defined we note that if A_0 and A_1 are Borel sets of K with $A_0 \cap G = A_1 \cap G$, then $A_0 \Delta A_1 \subset K \backslash G$; since $|K/G| < \omega$ it is impossible that $\lambda_K(G) = 0$, so from 4.11 we have $\lambda_{K^*}(K \backslash G) = 0$ and hence $\lambda_K(A_0 \Delta A_1) = 0$ and $\lambda_K(A_0) = \lambda_K(A_1)$.

For $x \in G$ and A a Borel set of K we have

$$\mu(x(A \cap G)) = \mu(xA \cap G) = \lambda_K(xA) = \lambda_K(A) = \mu(A \cap G).$$

It is clear that $\mathscr{S}$ is a σ-algebra of subsets of G. Further, since $\mathscr{T} \subset \mathscr{T}_{\mathscr{H}}$, every open set U of $\langle G, \mathscr{T}\rangle$ arises in the form $U = V \cap G$ for some open $V \subset K$. Thus $\mathscr{S}$ contains every Borel set of $\langle G, \mathscr{T}\rangle$.

That μ agrees with λ_G on all Borel sets of $\langle G, \mathscr{T}\rangle$, and is therefore an extension of λ_G, follows from the uniqueness of the measure λ_G and the relation

$$\mu(G) = \mu(K \cap G) = \lambda_K(K) = 1 = \lambda_G(G).$$

We note, finally, that $\mathrm{cl}_K H$ is an (open-and-closed) Borel set of K with $H = (\mathrm{cl}_K H) \cap G$, and that $\mu(H) = \lambda_K(\mathrm{cl}_K H) = 1/n$. The proof is complete.

For another argument which gives the existence of non-measurable, dense subgroups of many (not necessarily Abelian) compact topological groups, see Theorem 6.14 below.

4.16. Let us turn now briefly from subgroups and the question of their (ultimate) measurability to homomorphisms into $\mathbb{T}$. Here the natural setting is Abelian topological groups and the natural question is similar to those considered above:

Given a (non-measurable) homomorphism $\chi: G \to \mathbb{T}$, is there a translation-invariant extension μ of λ_G relative to which χ is measurable? If so we say, with HEWITT and ROSS [1965], that χ is ultimately measurable; if not, we say that χ is ultimately non-measurable.

THEOREM. *Let G be a compact Abelian topological group and let $\chi \in \mathrm{Hom}(G, \mathbb{T})$.*
(i) *If $|\chi[G]| = \omega$, then χ is ultimately non-measurable.*
(ii) *If $|\chi[G]| < \omega$, then χ is ultimately measurable.*

PROOF. We set $H = \ker \chi$ and $N = \mathrm{cl}_G H$, and we note that $|G/N| < \omega$ since, according to 3.1(i), the compact topological group G/N cannot satisfy $|G/N| = \omega$. It follows in particular that N is open-and-closed in G; it is clear then that in both (i) and (ii) we may assume without loss of generality that $N = G$, i.e., that H is dense in G.

Statement (i) now follows from 4.15(i), since any translation-invariant extension of λ_G relative to which χ is measurable makes $H = \chi^{-1}(\{1\})$ a measurable subset of G. To prove (ii) we may note that in the notation of the proof of 4.15(ii) we have $\chi \in \mathscr{F} \subset \mathscr{H}$ and that proof was effected precisely by associating with $\mathscr{H}$ a compact group K on which the elements of $\mathscr{H}$ (including those in $\mathscr{F}$) become continuous and hence λ_K-measurable. Referring in any event to the statement of 4.15(ii) rather than to its proof, we note μ is a translation-invariant extension of λ_G relative to which H is measurable. From $|\chi[G]| < \omega$ it is clear that χ is μ-measurable.

4.17. It is not by chance or coincidence that in the proof of 4.15(ii), while seeking to make a non-measurable set $H = \ker \chi$ into a measurable set by defining an extended σ-algebra, we went so far as to make H closed. This was inevitable since, as we now show, Haar-measurability and continuity are equivalent for homomorphisms between locally compact, σ-compact topological groups. Again, Theorem 4.6 of Steinhaus and Weil is essential to the proof.

THEOREM. *Let G and H be locally compact topological groups with H σ-compact, and let $h: G \to H$ be a homomorphism that is measurable* (*with respect to the Haar measures of G and H*). *Then h is continuous.*

PROOF. By 4.1 it is enough to show that for all $V \in \mathscr{N}_H(e)$ there is $U \in \mathscr{N}_G(e)$ such that $h[U] \subset V$. We choose $W \in \mathscr{N}_H(e)$ such that $WW^{-1} \subset V$ and, using the fact that H is σ-compact, we choose countable $D \subset H$ such that $H = WD$. We have $G = \bigcup_{x \in D} h^{-1}(Wx)$, and each set $h^{-1}(Wx)$ is λ_G-measurable.

Let K be any compact neighborhood of e in G. Since $0 < \lambda(K) < \infty$ and $K = \bigcup_{x \in D}(K \cap h^{-1}(Wx))$, there is $\bar{x} \in D$ such that $0 < \lambda(K \cap h^{-1}(W\bar{x})) < \infty$. We have from 4.6, writing $A = K \cap h^{-1}(W\bar{x})$, that there is $U \in \mathscr{N}_G(e)$ such that

$U \subset AA^{-1}$. The set U is as required: For $p \in U$ there are a, $b \in A$ such that $p = ab^{-1}$, and then

$$h(p) = h(a) \cdot h(b)^{-1} \in (W\bar{x})(W\bar{x})^{-1} = WW^{-1} \subset V.$$

4.18. COROLLARY. *Let G be a locally compact topological group and $\chi: G \to \mathbb{T}$ a measurable homomorphism. Then χ is continuous.*

4.19. Let us digress to consider briefly homomorphisms from the additive group of real numbers to itself. As might be expected, such functions can be quite badly discontinuous.

THEOREM. *There is a homomorphism $g: \mathbb{R} \to \mathbb{R}$ which assumes every real value on every open interval.*

PROOF. Let $H = \{x(r, n): r \in \mathbb{R},\ n < \omega\}$ be a Hamel basis for $\mathbb{R}$ over the rationals, faithfully indexed by the set $\mathbb{R} \times \omega$, and let $\{B_n: n < \omega\}$ be a basis for the topology of $\mathbb{R}$. For $\langle r, n\rangle \in \mathbb{R} \times \omega$ let $\tilde{x}(r, n)$ be a non-zero rational multiple of $x(r, n)$ such that $\tilde{x}(r, n) \in B_n$. The set $\tilde{H} = \{\tilde{x}(r, n): r \in \mathbb{R},\ n < \omega\}$ is again a faithfully indexed Hamel basis, and the function $f: \tilde{H} \to \mathbb{R}$ defined by $f(\tilde{x}(r, n)) = r$ extends to a homomorphism $g: \mathbb{R} \to \mathbb{R}$ as required.

Every homomorphism $h: \mathbb{R} \to \mathbb{R}$ satisfies $h(r) = h(1) \cdot r$ for all rational numbers r; thus h, if continuous, has the form $h(x) = cx$ (with $c = h(1)$) for all $x \in \mathbb{R}$. But if h is not continuous then its graph, since it contains a non-linear rational subspace of $\mathbb{R} \times \mathbb{R}$ whose dimension over $\mathbb{Q}$ is at least 2, is dense in $\mathbb{R} \times \mathbb{R}$. For a complete and elementary proof that graph h is dense in $\mathbb{R} \times \mathbb{R}$, see HEWITT and ZUCKERMAN [1969].

If A is the graph of a discontinuous homomorphism $h: \mathbb{R} \to \mathbb{R}$, then A contains no non-trivial, bounded, connected set. (*Proof.* Suppose that C is a bounded, connected subset of A with $\langle a, h(a)\rangle$, $\langle b, h(b)\rangle \in C$ and $a < b$, and suppose $h(x) < M$ whenever $\langle x, h(x)\rangle \in C$. Since A is dense there is $\bar{x} \in \mathbb{R}$ such that $a < \bar{x} < b$ and $h(\bar{x}) > M$, and then the sets

$$\{\langle x, h(x)\rangle \in C: x < \bar{x}\} \quad \text{and} \quad \{\langle x, h(x)\rangle \in C: x > \bar{x}\}$$

are complementary, non-empty open subsets of C.) Nevertheless A itself may be connected. For this and related topological pathology available in this context, see JONES [1942a], [1942b].

4.20. Our final result concerning non-measurability, also due to HEWITT and ROSS [1963] (16.13(a)) is related to 4.15 as follows: The assumption that the underlying group be compact is abandoned, and simultaneously the conclusion that the non-measurable subset be algebraically closed is sacrificed.

THEOREM. *For every locally compact, non-discrete topological group G there is an ultimately non-measurable set $A \subset G$.*

PROOF. Choose $U \in \mathcal{N}(e)$ with $\lambda(U) < \infty$ and $V = V^{-1} \in \mathcal{N}(e)$ with $V^3 \subset U$, and let D be a countably infinite subset of V. Let H be the subgroup of G generated by D and let A be a set containing exactly one point from each non-empty set of the form $(Hx) \cap V$. We verify three statements concerning the set A.

(i) $V \subset (H \cap V^2)A \subset V^2 A \subset V^3 \subset U$;

(ii) $|H \cap V^2| = \omega$; and

(iii) if h_1, h_2 are distinct elements of $H \cap V^2$, then $h_1 A \cap h_2 A = \emptyset$.

(i) Only the first inclusion is not obvious. For $x \in V$ we have $Hx \cap V \neq \emptyset$ and hence there is $a = hx \in Hx \cap V \cap A$. From $x = h^{-1}a \in HA$ follows $h^{-1} = xa^{-1} \in VV^{-1} = V^2$, so that $x \in (H \cap V^2)A$, as required.

(ii) We have $D = D \cap V \subset H \cap V^2 \subset H$.

(iii) If $h_1 a_1 = h_2 a_2$ with $a_i \in A \cap Hx_i$, then from $x_1 x_2^{-1} \in H$ follows $a_1 = a_2$ and hence $h_1 = h_2$.

Suppose now that A is μ-measurable. Then either $\mu(A) = 0$ or $\mu(A) > 0$. In the former case we have

$$0 < \lambda(V) = \mu(V) \leq \mu((H \cap V^2)A) = 0$$

and in the latter case we have

$$\infty > \lambda(U) = \mu(U) \geq \mu((H \cap V^2)A) = \infty .$$

4.21. We refer the reader to 9.4 below for brief comments concerning the existence of a compact homogeneous topological space X, closely associated with the circle group $\mathbb{T}$ and bearing a Borel measure μ which is essentially $\lambda_{\mathbb{T}}$, with the remarkable property that $\mu(A) = \mu(B)$ for every two homeomorphic Borel sets A and B of X.

4.22. We turn now to a counting argument, four decades old and due to KAKUTANI [1943], according to which the number of (continuous) characters on an infinite discrete Abelian group G is exactly $2^{|G|}$. Strictly speaking, with Theorem 3.1(i) and the elements of Pontrjagin duality available to us, this material is redundant: the compact group $\hat{G}$, which is $\mathrm{Hom}(G, \mathbb{T})$, is easily seen to have weight $|G|$, and hence $|\hat{G}| = 2^{w(\hat{G})} = 2^{|G|}$. The elegant simplicity of the argument, and the fact that it affords a proof of the Abelian case of 3.1(i) quite independent of Kuz'minov's theorem, amply justify its inclusion here.

THEOREM. *Let G be an Abelian group with $|G| = \alpha \geq \omega$. Then $|\mathrm{Hom}(G, \mathbb{T})| = 2^{\alpha}$.*

PROOF. Surely $|\mathrm{Hom}(G, \mathbb{T})| \leq |\mathbb{T}^G| = 2^{\omega \cdot |G|} = 2^{\alpha}$. We prove the reverse inequality.

In the Pontrjagin topology the space $\mathrm{Hom}(G, \mathbb{T})$ is a compact (Hausdorff) space without isolated points. From the theorem of ČECH and POSPÍŠIL [1938], recorded in this Handbook in the article of Hodel, it follows that $|\mathrm{Hom}(G, \mathbb{T})| \geqslant 2^\omega$. The proof is complete in case $\alpha = \omega$ and we assume in what follows that $\alpha > \omega$.

Let $\{x_\xi : \xi < \alpha\}$ be a faithful indexing of G, for $\xi < \alpha$ set $H_\xi = [\{x_\eta : \eta < \xi\}]$, and set $A = \{x_\xi \in G : x_\xi \notin H_\xi\}$. We claim that $[A] = G$. If the claim fails we choose $\xi < \alpha$ minimal with respect to the property $x_\xi \in G \backslash [A]$ and we note that since $x_\xi \in H_\xi$ there are $\{\eta_k : k < m\} \subset \xi$ and $\{n_k : k < m\} \subset \mathbb{Z}$ with $m < \omega$ such that $x_\xi = \Pi_{k<m} x_{\eta k}^{n_k}$; from $x_{\eta_k} \in [A]$ then follows $x_\xi \in [A]$. The claim is proved. We note that from $[A] = G$ and $|G| = \alpha > \omega$ follows $|A| = \alpha$.

We construct elements $\chi = \bigcup_{\xi<\alpha} \chi_\xi$ of $\mathrm{Hom}(G, \mathbb{T})$ with each $\chi_\xi \in \mathrm{Hom}(H_{\xi+1}, \mathbb{T})$. First, define $\chi_0(x) = 1$ for all $x \in H_1$. Suppose now that $\xi < \alpha$ and that $\chi_\eta \in \mathrm{Hom}(H_{\eta+1}, \mathbb{T})$ has been defined for all $\eta < \xi$. In order to extend the function $\bigcup_{\eta<\xi} \chi_\eta \in \mathrm{Hom}(\bigcup_{\eta<\xi} H_{\eta+1}, \mathbb{T})$ to an element $\chi_\xi \in \mathrm{Hom}(H_{\xi+1}, \mathbb{T})$, we consider three cases.

Case 1. $x_\xi \notin A$. Then $x_\xi \in H_\xi = H_{\xi+1}$ and we set $\chi_\xi = \bigcup_{\eta<\xi} \chi_\eta$.

Case 2. $x_\xi \in A$ and there is a positive integer r such that $x_\xi^r \in H_\xi$. We choose r minimal (noting $r \geqslant 2$), we choose $\{\eta_k : k < m\} \subset \xi$ and $\{n_k : k < m\} \subset \mathbb{Z}$ with $m < \omega$ so that $x_\xi^r = \Pi_{k<m} x_{\eta k}^{n_k}$, we choose $t \in \mathbb{T}$ so that $t^r = \Pi_{k<m} \chi_{\eta_k}(x_{\eta k}^{n_k})$, and we define χ_ξ to be that unique element of $\mathrm{Hom}(H_\xi, \mathbb{T})$ for which $\bigcup_{\eta<\xi} \chi_\eta \subset \chi_\xi$ and $\chi_\xi(x_\xi) = t$.

Case 3. For every integer $r \geqslant 1$ we have $x_\xi^r \notin H_\xi$. Here we choose any $t \in \mathbb{T}$ and we define χ_ξ to be that unique element of $\mathrm{Hom}(H_\xi, \mathbb{T})$ for which $\bigcup_{\eta<\xi} \chi_\eta \subset \chi_\xi$ and $\chi_\xi(x_\xi) = t$.

It is now lcear, the family $\{\chi_\xi : \xi < \alpha\}$ having been defined as above, that the function $\chi = \bigcup_{\xi<\alpha} \chi_\xi$ satisfies $\chi \in \mathrm{Hom}(G, \mathbb{T})$. For every $\xi < \alpha$ with $\chi_\xi \in A$ we had at least two choices for $\chi_\xi(x_\xi)$—namely $r \geqslant 2$ choices for x_ξ as in case 2 and $|\mathbb{T}| = 2^\omega$ choices for x_ξ as in case 3. Thus the number of functions χ so constructed is at least $2^{|A|} = 2^\alpha$, as required.

4.23. Here, as promised, for Abelian groups we recapture by elementary means the statement of Theorem 3.1(i).

COROLLARY. *For every infinite, compact Abelian group G the cardinal number $|G|$ has the form $|G| = 2^\alpha$; one may choose $\alpha = |\hat{G}|$.*

PROOF. Define $\alpha = |\hat{G}|$. Then from Pontrjagin duality and 4.22 applied to the discrete group $\hat{G}$ we have $|G| = |\hat{\hat{G}}| = |\mathrm{Hom}(\hat{G}, \mathbb{T})| = 2^\alpha$.

4.24. COROLLARY. *Let G be a compact, Abelian topological group with $|G| = \alpha \geqslant \omega$. Then G has 2^α 'discontinuous characters', i.e., $|\mathrm{Hom}(G, \mathbb{T}) \backslash \hat{G}| = 2^\alpha$.*

PROOF. Since $|\mathrm{Hom}(G, \mathbb{T})| = 2^\alpha$ and $\hat{G}$ is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$, it is enough to show $\mathrm{Hom}(G, \mathbb{T}) \neq \hat{G}$.

Set $\beta = |\hat{G}|$ and let D denote the group G with the discrete topology. If $\mathrm{Hom}(G, \mathbb{T}) = \hat{G}$ then as sets we have

$$\hat{D} = \mathrm{Hom}(D, \mathbb{T}) = \mathrm{Hom}(G, \mathbb{T}) = \hat{G}$$

and since both G and $\hat{D}$ are compact we have

$$\alpha = |G| = 2^{|\hat{G}|} = 2^\beta \text{ and } \beta = |\hat{D}| = 2^{|D|} = 2^\alpha.$$

For additional references concerning "discontinuous characters" and related phenomena in the context of locally compact (not necessarily compact) Abelian topological groups, see 11.1 below.

4.25. The natural statement in parallel to 4.24 for infinite compact (not necessarily Abelian) topological groups is that they admit many discontinuous homomorphisms into compact topological groups; this statement is a natural conjecture. In fact, however, we indicate in 9.16–9.18 below the existence of infinite non-Abelian compact topological groups G for which every homomorphism $G \to K$ with K a compact topological group is continuous.

4.26. We have seen ample evidence in this section of the fact that a (locally) compact Abelian topological group may admit many discontinuous homomorphisms into $\mathbb{T}$. It is remarkable that, in a sense made precise in 4.30, all such homomorphisms are approximable by continuous homomorphisms into $\mathbb{T}$, i.e., by characters. The appropriate context for this result is the so-called Bohr compactification, to which we now turn; a consequence is one formulation of the Kronecker approximation theorem.

DEFINITION. Let G be a locally compact Abelian topological group and D the group $\hat{G}$ with the discrete topology. The *Bohr compactification* of G, denoted $\tilde{G}$, is the compact group $\tilde{G} = \hat{D}$ together with the function $\zeta\colon G \to \tilde{G}$ defined by the rule $\zeta(x)(\chi) = \chi(x)$ for $x \in G$, $\chi \in D$.

The space $\tilde{G}$ coincides in important special cases with a compact topological group associated with a (not necessarily Abelian) topological group G by means of the almost periodic functions introduced in the case $G = \mathbb{R}^n$ by H. BOHR [1932]. (A great deal of information concerning properties and uses of these functions is available in volumes II and III of the collected works of BOHR [1952]. For a discussion of the so-called almost periodic compactification and for references to the substantial literature concerning representations into compact groups, see WEIL [1951] (Chapter 7) and HEWITT and ROSS [1970] (Chapter 7 and 33.26)). This explains the term 'Bohr compactification', which is by now so widely used and accepted that any argument against its propriety becomes useless and itself improper. The reader should be aware, however, that the Bohr compactification of G is not a compactification of G in the topologist's traditional sense: the properties of the function ζ are set down in the following theorem and it is only in the extreme and uninteresting case—namely when G itself is compact, so that $\tilde{G} = G$—that ζ is a homeomorphism of G into $\tilde{G}$.

In what follows we continue, given a locally compact Abelian topological group G, to denote by D the group $\hat{G}$ with the discrete topology. The notation $P(F, \varepsilon) \in \mathcal{N}_{\tilde{G}}(e)$ with $\varepsilon > 0$ and compact $F \subset G$ is used as in 1.24.

4.27. THEOREM. *Let G be a locally compact Abelian topological group and $\tilde{G}$ the Bohr compactification of G. Then*

(i) *the function $\zeta: G \to \tilde{G}$ is an isomorphism*;

(ii) *the function $\zeta: G \to \tilde{G}$ is continuous*; *and*

(iii) *$\zeta[G]$ is dense in $\tilde{G}$.*

PROOF. (i) It is clear that ζ is a homomorphism. And ζ is a one-to-one function because $\hat{G}$ separates points of G.

(ii) We show that for $P(F, \varepsilon) \in \mathcal{N}_{\tilde{G}}(e)$ there is $U \in \mathcal{N}(e)$ such that $\zeta[U] \subset P(F, \varepsilon)$. Since D is discrete the compact set $F \subset D$ is finite and it is enough to choose $U \in \mathcal{N}_G(e)$ so that $|\chi(x) - 1| < \varepsilon$ for all $\chi \in F$ and $x \in U$.

(iii) Let H denote the closure in $\tilde{G}$ of $\zeta[G]$ and suppose that $H \neq \tilde{G}$. From 1.29 there is $\Omega \in \hat{\tilde{G}}$ such that $\Omega(\psi) = 1$ for all $\psi \in H$ but Ω is not the function identically equal to 1 on $\tilde{G}$. From 1.25 there is $\chi \in D$ such that $\Omega = \upsilon_{\tilde{G}}(\chi)$ and hence $\Omega(\psi) = \psi(\chi)$ for all $\psi \in \tilde{G}$. Then for $x \in G$ we have

$$1 = \Omega(\zeta(x)) = \zeta(x)(\chi) = \chi(x)$$

and hence χ is the function identically equal to 1 on G. It follows that $\Omega(\psi) = \psi(\chi) = 1$ for all $\psi \in \tilde{G}$, a contradiction.

4.28. THEOREM. *Let G be an infinite locally compact Abelian topological group and let $\varepsilon > 0$ and $\{\chi_k : k < n\} \subset \hat{G}$ with $n < \omega$. Then there is $x \in G$ with $x \neq e$ such that*

$$|\chi_k(x) - 1| < \varepsilon \quad \textit{for all } k < n.$$

PROOF. We write $U = P(F, \varepsilon) \in \mathcal{N}_{\tilde{G}}(e)$ with $F = \{\chi_k : k < n\}$ and we claim $|U| \geq \omega$. Indeed if $|U| < \omega$, then the compact group $\tilde{G}$ is discrete and hence finite; from the fact that $\zeta: G \to \tilde{G}$ is a one-to-one function it would follow that $|G| < \omega$, a contradiction. Since $|U| \geq \omega$ and $\zeta[G]$ is dense in $\tilde{G}$ there is $x \in G$ such that $\zeta(x) \in U$ and $\zeta(x) \neq e \in \tilde{G}$. Clearly $x \neq e \in G$, and x is as required.

4.29. THEOREM. *Let G be an infinite locally compact Abelian topological group and let $\varepsilon > 0$ and $\{x_k : k < n\} \subset G$ with $n < \omega$. Then there is $\chi \in \hat{G}$ with $\chi \neq e \in \hat{G}$ such that*

$$|\chi(x_k) - 1| < \varepsilon \quad \textit{for all } k < n .$$

PROOF. We have as usual $\upsilon_G(x) \in \hat{\hat{G}}$ for $x \in G$. Now according to the previous theorem, applied to $\hat{G}$ in place of G, there is $\chi \in \hat{G}$ with $\chi \neq e \in \hat{G}$ such that

$$|\chi(x_k) - 1| = |\upsilon_G(x_k)(\chi) - 1| < \varepsilon \quad \text{for all } k < n .$$

4.30. Here is the statement showing $\hat{G}$ dense in $\mathrm{Hom}(G, \mathbb{T})$.

THEOREM. *Let G be a locally compact Abelian topological group, let $\psi \in \mathrm{Hom}(G, \mathbb{T})$, and let $\varepsilon > 0$ and $\{x_k : k < n\} \subset G$ with $n < \omega$. Then there is $\chi \in \hat{G}$ such that $|\chi(x_k) - \psi(x_k)| < \varepsilon$ for $k < n$.*

PROOF. Let E denote the group $\hat{\hat{G}}$ with the discrete topology; the Bohr compactification of $\hat{G}$, denoted $\tilde{\hat{G}}$, is the group $\tilde{\hat{G}} = \hat{E}$ (together with the natural function $\zeta: \hat{G} \to \hat{E}$). As sets we have $E = \hat{\hat{G}}$, so we may view the canonial homeomorphism $\upsilon = \upsilon_G: G \to \hat{\hat{G}}$ as a (discontinuous) function from G onto E. We define $U \in \mathcal{N}_{\hat{E}}(e)$ by $U = P(F, \varepsilon)$ with F the compact set $F = \{\upsilon(x_k) : k < n\} \subset E$ and we note that $\psi \circ \upsilon^{-1} \in \hat{E}$. Since $(\psi \circ \upsilon^{-1}) \cdot U$ is a neighborhood in $\hat{E}$ of $\psi \circ \upsilon^{-1}$ and $\zeta[\hat{G}]$ is dense in $\hat{E}$, there is $\chi \in \hat{G}$ such that $\zeta(\chi) \in (\psi \circ \upsilon^{-1}) \cdot U$. For $k < n$ we have

$$|(\zeta\chi)(\upsilon x_k) - (\psi \circ \upsilon^{-1})(\upsilon x_k)| < \varepsilon$$

and hence $|\chi(x_k) - \psi(x_k)| < \varepsilon$, as required.

4.31. Theorems 4.28 and 4.29, together with our treatment of the elementary properties of the function $\zeta: G \to \tilde{G}$, are from HEWITT and ROSS [1963] (26.13, 26.14).

The classical theorem of KRONECKER [1884] is the statement that for every two real numbers a and r with r irrational and for $\varepsilon > 0$, there are $m, s \in \mathbb{Z}$ such that $|mr - (s + a)| < \varepsilon$. As is clear from the proof of 4.32, Theorem 4.30 is a very general form of Kronecker's theorem; in fact we derive the '$(n-1)$-dimensional' Kronecker theorem from 4.30.

Concerning Kronecker's theorem we refer the reader as follows: to HARDY and WRIGHT [1960] for equivalent formulations, some whimsical or entertaining, and for alternative proofs; to KOKSMA [1936] for an exhaustive bibliography of later work inspired by the theorem; and to HEWITT and ZUCKERMAN [1950], the original source of Theorem 4.30, for sharpened statements and several applications.

4.32. For real numbers r and a and $\varepsilon > 0$, we write $|r - a| < \varepsilon \pmod 1$ if there is $s \in \mathbb{Z}$ such that $|r - (s + a)| < \varepsilon$.

We say that a (finite) set $\{r_k : k < n\}$ of real numbers is rationally independent if the equation $\Sigma_{k<n} q_k r_k = 0$ with $q_k \in \mathbb{Q}$ has only the solution $q_k = 0$ $(k < n)$.

COROLLARY. *Let $\{r_k : k < n\}$ (with $n < \omega$) be a rationally independent set of real numbers with $r_0 = 1$ and let $\{a_k : 0 < k < n\} \subset [0, 1)$. For all $\varepsilon > 0$ there is $m \in \mathbb{Z}$ such that*

$$|mr_k - a_k| < \varepsilon \pmod 1 \quad \textit{for } 0 < k < n .$$

PROOF. The character $\Omega : \mathbb{R} \to \mathbb{T}$ given by $\Omega(r) = \exp(2\pi i r)$ satisfies $\ker \Omega = \mathbb{Z}$ and $\mathbb{R}/(\ker \Omega) = \mathbb{T}$. Hence there is $\delta > 0$ such that if r, $s \in \mathbb{R}$ and $|\exp(2\pi i r) - \exp(2\pi i s)| < \delta$, then $|r - s| < \varepsilon \pmod 1$.

Let H be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ such that $\{r_k : k < n\} \subset H$ and define $\psi : H \to \mathbb{T}$ so that

$$\psi(r_0) = \psi(1) = 1 , \qquad \psi(r_k) = \exp(2\pi i a_k) \quad \text{for } 0 < k < n ,$$

and $\psi(h)$ is arbitrary in $\mathbb{T}$ for $h \in H \setminus \{r_k : k < n\}$; extend ψ to a homomorphism (also denoted ψ) from $\mathbb{R}$ to $\mathbb{T}$. Since $\psi(r + m) = \psi(r)$ for all $r \in \mathbb{R}$, $m \in \mathbb{Z}$, the function φ from $\mathbb{T}$ to $\mathbb{T}$ given by $\varphi(\exp(2\pi i r)) = \psi(r)$ is a well-defined homomorphism of $\mathbb{T}$ to $\mathbb{T}$. It then follows from Theorem 4.30 that there is $\chi \in \hat{\mathbb{T}}$ such that

$$|\chi(\exp(2\pi i r_k)) - \varphi(\exp(2\pi i r_k))| < \delta \quad \text{for } 0 < k < n .$$

There is $m \in \mathbb{Z}$ such that $\chi(w) = w^m$ for all $w \in \mathbb{T}$. From $\varphi(\exp(2\pi i r_k)) = \exp(2\pi i a_k)$ now follows

$$|\exp(2\pi i m r_k) - \exp(2\pi i a_k)| < \delta$$

and hence $|mr_k - a_k| < \varepsilon \pmod 1$ for $0 < k < n$, as required.

It should be clear why it is in the result just proved that we do not define a_0 and attempt to arrange $|mr_k - a_k| < \varepsilon \pmod 1$ also for $k = 0$: We have $mr_0 = m \in \mathbb{Z}$, and hence $mr_0 = 0 \pmod 1$, for all $m \in \mathbb{Z}$.

4.33. A topological group is said to be monothetic if it contains a dense cyclic subgroup. We bring this section to an end with a result which is probably the first or basic non-trivial result in the theory of compact monothetic groups. For generalizations and references to the literature, see HEWITT and ROSS [1963] (§25).

THEOREM. *Let α be a cardinal number and $G = \mathbb{T}^\alpha$. Then G is monothetic if and only if $\alpha \leqslant \mathfrak{c}$.*

PROOF. Suppose first that $\alpha > \mathfrak{c}$. From $w(G) = \alpha$ and Theorem 3.1(ii) we have $d(G) = \log(\alpha) > \omega$; thus no countable subset of G is dense, so G is not monothetic.

The argument that G is monothetic in case $\alpha \leq \mathfrak{c}$ runs much as in 4.32. Let H be a subset of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ with $|H| = \alpha$ and with $1 \notin H$, and define $x = \langle x_r : r \in H \rangle \in \mathbb{T}^H$ by $x_r = \exp(2\pi i r)$. Now let finite $F = \{r_k : 0 < k < n\} \subset H$, $\{t_k : 0 < k < n\} \subset \mathbb{T}$ and $\delta > 0$, and let U be the basic open set

$$U = \{p \in \mathbb{T}^H : |pr_k - t_k| < \delta \text{ for } 0 < k < n\}.$$

As in the proof of 4.32 there is $m \in \mathbb{Z}$ such that

$$|\exp(2\pi i m r_k) - t_k| < \delta \quad \text{for } 0 < k < n,$$

i.e., $x^m \in U$.

5. Topological groups with special topological properties

The five problems treated here arise quite independently; accordingly, this section has five parts which may be read or skipped independently and in any order.

This section is organized as follows. (i) 5.1 cites the existence of an infinite topological group with remarkably few homeomorphisms; (ii) 5.2–5.4 shows the existence of Abelian topological groups G for which $\hat{G}$ does not separate points; (iii) 5.5–5.9 describes an approach toward the construction (not yet achieved in ZFC) of non-discrete, extremally disconnected topological groups; (iv) 5.10–5.17 shows by example that the (uniform) completion of a topological group need not be a topological group; and (v) 5.18–5.23 discusses the S and L problem within the context of topological groups.

5.1. Translation by a (fixed) element of a topological group is of course a surjective homeomorphism. VAN MILL [1982b] has announced the existence of a connected, locally connected, separable metrizable topological group G with no surjective homeomorphisms other than translations. (Let μ denote Lebesgue measure restricted to $[0, 1]$ and define

$$\mathcal{A} = \{A \subset [0, 1] : A \text{ is } \mu\text{-measurable}\}$$

and $\mathcal{N} = \{A \in \mathcal{A} : \mu(A) = 0\}$. Van Mill's group G is a subgroup of $\mathcal{A}/\mathcal{N}$, in which the group structure is given by

$$[A] + [B] = [A \,\Delta\, B]$$

and the metric d is defined by

$$d([A],[B]) = \lambda(A \,\Delta\, B)\,.$$

Assuming CH, G can be chosen so that every continuous function from G into G is either a constant function or a translation.

5.2. It is a crucial and non-trivial step in the development of Pontrjagin duality theory, and a simple logical consequence of that theory, that if G is a locally compact Abelian topological group then the group $\hat{G}$ of continuous homomorphisms from G to $\mathbb{T}$ distinguishes points of G. This powerful result makes natural the question, raised in a variety of contexts by many authors, whether the same conclusion holds for all Abelian topological groups. In view of Theorem 4.2 above, this question may be (equivalently) rephrased as follows: Does every Abelian topological group $G = \langle G, \mathcal{T}\rangle$ admit a totally bounded topological group topology $\mathcal{T}'$ such that $\mathcal{T}' \subset \mathcal{T}$? The answer is "No". The following argument, chosen for its simplicity, is taken from HOOPER [1968].

5.3. First we set our notation. Let c_0 denote as usual the set of real sequences $a = \langle a_k : k < \omega\rangle$ with $\lim_k a_k = 0$, topologized by the metric generated by the norm $\|\ \|$ defined by the rule

$$\|a\| = \sup\{|a_k| : k < \omega\}\,.$$

Let $H = \{a \in c_0\colon \text{each } a_k \in \mathbb{Z}\}$ and set

$$K = \{a \in H\colon \sum_{k<\omega} a_k \text{ is even}\}\,.$$

(The indicated summation is finite since for $a \in H$ there is $N < \omega$ such that $a_k = 0$ for all $k \geqslant N$.)

The (usual) group operation for c_0 is defined coordinatewise and is denoted by $+$; the identity e satisfies $e_k = 0$ for all $k < \omega$. For $\varepsilon > 0$ we define $S_\varepsilon \in \mathcal{N}(e)$ by $S_\varepsilon = \{a : \|a\| < \varepsilon\}$.

In what follows we will wish to consider c_0/H, c_0/K and H/K as topological groups. To ensure the Hausdorff separation axiom, we show now that H is closed in c_0 and K is closed in H (hence also in c_0).

If $a \in c_0 \backslash H$ there are $k < \omega$ and $m \in \mathbb{Z}$ such that $m < a_k < m+1$, and then with $\varepsilon = \min\{a_k - m,\ m+1-a_k\}$ we have $(a + S_\varepsilon) \in \mathcal{N}(a)$ and $(a + S_\varepsilon) \cap H = \emptyset$. Thus H is closed in c_0. That K is closed in H follows from the fact that H is a discrete space: for $a \in H$ we have $(a + S_{1/2}) \cap H = \{a\}$.

Let φ denote the canonical homomorphism from c_0 onto c_0/K.

THEOREM. *Define* c_0, H, K *and* φ *as above and set* $G = c_0/K$. *Then* $\hat{G}$ *does not distinguish points of* G.

PROOF. For $n < \omega$ define $b(n) \in H \subset c_0$ by $b(n)_k = 1$ if $k = n$ and $b(n)_k = 0$ if $k \neq n$. We have $b(n) \in H \backslash K$ (in fact $|H/K| = 2$ and $H \backslash K = b(n) + K$). To complete the proof it is enough to show for all $\chi \in \hat{G}$ that $\chi(b(n) + K) = 1$.

Suppose there is $\chi \in \hat{G}$ with $\chi(b(n) + K) \neq 1$. Since $b(n) + b(n) \in K$ and χ is a homomorphism, we have $\chi(b(n) + K) = -1$.

In what follows we denote scalar multiplication in c_0 by the symbol $\cdot$.

For $n < \omega$ and $0 < k < \omega$ the number $t = (\chi\varphi)((1/k) \cdot b(n))$ satisfies $t^k = -1$ and hence there is $m \in \mathbb{Z}$ such that $t = \exp((2m + 1)\pi i/k)$. That is: the continuous function from the interval $[0, 1/k]$ to $\mathbb{T}$ given by $r \to (\chi\varphi)(r \cdot b(n))$ satisfies $0 \to 1$, $1/k \to \exp((2m + 1)\pi i/k$. From connectivity and the fact that $\chi(-a + H)$ is the conjugate of $\chi(a + H)$ it follows that there is real $r_{n,k}$ with $|r_{n,k}| \leq 1/k$ such that $(\chi\varphi)(r_{n,k} \cdot b(n)) = \exp(\pi i/k)$.

Now for $0 < k < \omega$ define $c(k) \in c_0$ by $c(k) = \Sigma_{n<k} r_{n,k} \cdot b(n)$. Then

$$\chi(c(k) + H)) = (\chi\varphi)(c(k)) = \prod_{n<k} (\chi\varphi(r_{n,k} \cdot b(n)) = (\exp(\pi i/k))^k = -1 ,$$

but from $\|c(k)\| = \sup\{|c(k)_n|: n < \omega\} \leq 1/k$ we have $c(k) \to 0$ in c_0 and hence the contradiction

$$\chi(c(k) + H)) \to \chi(0 + H) = \chi(H) = +1 .$$

5.4. The example just given provokes a few miscellaneous remarks, as follows.

(i) Up to homeomorphism there is just one topological group of cardinality two, and H/K is it. It has two characters; the non-constant one distinguishes points. Here H/K is a closed subgroup of $G = c_0/K$, but the non-constant character on H/K does not extend to a continuous character of G. This phenomenon constrasts with the situation prevailing among locally compact Abelian topological groups (1.31).

(ii) COTLAR and RICABARRA [1954] have developed the following unexpected condition concerning the separation of points by (continuous) characters: Given an arbitrary Abelian topological group G and $e \neq x \in G$, there is $\chi \in \hat{G}$ such that $\chi(x) \neq 1$ if and only if there is $U \in \mathcal{N}(e)$ such that (a) there is finite $F \subset G$ such that $G = FU$ and (b) $x \notin U^6$.

(iii) The example of 5.3 can be strengthened: there are Abelian topological groups for which the only continuous homomorphism into $\mathbb{T}$ is the function identically equal to 1. See HEWITT and ROSS [1963] for examples drawn from functional analysis, and PRODANOV [1980] for an elementary example, algebraically a subgroup of $\bigoplus_\omega \mathbb{Z}$, whose properties are established using arguments drawn strictly from the vocabulary of topological groups.

(iv) It is noted by HEWITT and ROSS [1963] (§23) that, although little attention has been paid to computing character groups of Abelian topological groups that are not locally compact, the characterization is in principle complete for every real topological linear space E (viewed as a topological group under addition) whose dual space E^* is known. The following theorem, due to HEWITT and ZUCKERMAN [1950] and SMITH [1952], does the trick: for every $\chi \in \hat{E}$ there is $f \in E^*$ such that $\chi(x) = \exp(2\pi i f(x))$ for all $x \in E$.

(v) The example of 5.3 addresses the question whether, given an (Abelian) group which is already topologized, there is a coarser totally bounded topological group topology. That question is related to, but should be distinguished from, the following: Given a group G, does it admit a totally bounded topological group topology? It follows from 4.2 that the answer is "Yes" in case G is Abelian; we treat the general case in 9.9–9.12 below.

5.5. Students of the theory of rings of continuous functions have considered several classes of topological spaces which, while their definitions impose strong separation properties, nevertheless contain some non-discrete spaces. Among the classes studied intensively and exhaustively by GILLMAN and JERISON [1960], and classified both topologically and in terms of algebraic properties of rings of real-valued continuous functions, are the three we defined in 2.12 above: the extremally disconnected spaces, the basically disconnected spaces, and the F-spaces. Let us cite here a fourth and related class, the P-spaces: A space is a P-space if every cozero set is closed (equivalently, if every G_δ-set is open). It is clear that every P-space is basically disconnected, hence is an F-space. It is a theorem of ISBELL [1955] (cf. also GILLMAN and JERISON [1960] (12H.6) or BLAIR [1961]) that every extremally disconnected P-space of (Ulam) non-measurable cardinality is a discrete space.

The American Mathematical Society [1979] (54G05, 54G10) has classified these spaces as 'Peculiar'. Here we investigate briefly the extent to which the peculiarity available within the class of topological spaces can be achieved among topological groups. We give the details of the required arguments just to the extent that they are available in ZFC. The question whether there is in ZFC a non-discrete, extremally disconnected topological group, raised by ARHANGEL'SKIĬ [1967], has been solved (in the affirmative) only by appealing to one or another additional axiom (5.8).

5.6. The question of the existence of non-discrete topological groups which are P-spaces poses no difficulty: Let $\langle G, \mathscr{T}\rangle$ be any topological group with $\psi(G) > \omega$, and give G the topology $\mathscr{T}_\delta$ generated by the G_δ-sets of $\langle G, \mathscr{T}\rangle$. It is easy to see that $\langle G, \mathscr{T}\rangle$ is as required. (Some uses and consequences of this simple device are described in sections 6 and 7 of this chapter. For example, we show in Theorem 6.4 that if $\langle G, \mathscr{T}\rangle$ is compact then the dense subgroups of $\langle G, \mathscr{T}\rangle$ which are pseudocompact are exactly those subgroups which remain dense in $\langle G, \mathscr{T}_\delta\rangle$.)

5.7. Let α and β be cardinals with $\alpha \geqslant \beta \geqslant \omega$, let $\mathscr{F}$ be a β-uniform filter on α (i.e., $|F| \geqslant \beta$ for all $F \in \mathscr{F}$) such that $\alpha \backslash \{\xi\} \in \mathscr{F}$ for all $\xi < \alpha$, and set

$$G = \{A \subset \alpha : |A| < \beta\}.$$

As in 5.1 let Δ denote the symmetric difference function from $G \times G$ to G defined by $A \Delta B = (A \backslash B) \cup (B \backslash A)$, for $A \in G$ and $F \in \mathscr{F}$ set

$$U_A(F) = \{A \Delta S : S \subset F, |S| < \beta\} \quad (= U_F(A))$$

and let $\mathscr{T}(\mathscr{F})$ be the topology for G for which the family $\mathscr{S} = \{U_A(F) : A \in G, F \in \mathscr{F}\}$ is a subbase. With these conventions we have the following result.

THEOREM. (i) *$\langle G, \mathscr{T}(\mathscr{F})\rangle$ is a non-discrete topological group; the family $\mathscr{S}$ is a basis of $\mathscr{T}(\mathscr{F})$.*

(ii) *The family $\{U_F(\emptyset) : F \in \mathscr{F}\}$ is a basis at $\emptyset = e_G$ consisting of open-and-closed subgroups of $\langle G, \mathscr{T}(\mathscr{F})\rangle$.*

PROOF. (i) It is clear that $\langle G, \Delta\rangle$ is a group with identity $e_G = \emptyset$. The inversion function is the function $A \to A$ so inversion is continuous. To check that $\Delta : G \times G \to G$ is continuous at $\langle A, B\rangle$ we must show that if $F \in \mathscr{F}$ then there are D, $E \in \mathscr{F}$ such that $X \Delta Y \in U_F(A \Delta B)$ whenever $X \in U_D(A)$, $Y \in U_E(B)$. It is enough to take $D = E = F$, for if $X = A \Delta S$, $Y = B \Delta T$ with $S \subset D$, $T \subset E$, $|S| < \beta$, $|T| < \beta$, then $|S \Delta T| < \beta$ and hence

$$X \Delta Y = A \Delta S \Delta B \Delta T = (A \Delta B) \Delta (S \Delta T) \in U_F(A \Delta B).$$

A similar routine computation shows that if $C \in U_E(A) \cap U_F(B)$ with E, $F \in \mathscr{F}$, then

$$C \in U_{E \cap F}(C) \subset U_E(A) \cap U_F(B);$$

hence $\mathscr{S}$ is a basis for $\mathscr{T}(\mathscr{F})$.

For every basic neighborhood $U_F(A)$ of $A \in G$ there is $\xi \in F \backslash A$. We have

$$A \neq A \cup \{\xi\} = A \Delta \{\xi\} \in U_F(A),$$

so the topology $\mathscr{T}(\mathscr{F})$ is not the discrete topology.

If $A, B \in G$ with $A \neq B$, then there is $\xi \in A \Delta B$ and with $F = \alpha \backslash \{\xi\} \in \mathscr{F}$ we have $U_F(A) \cap U_F(B) = \emptyset$; thus the topology $\mathscr{T}(\mathscr{F})$ satisfies the Hausdorff separation axiom required in our definition of a topological group.

(ii) It is clear that if $A, B \in U_F(\emptyset)$, then $A \Delta B^{-1} = A \Delta B \in U_F(\emptyset)$.

5.8. The theorem just given (with $\alpha = \beta = \omega$ and with $\mathscr{F}$ an ultrafilter on α) is the context, in ZFC appropriately augmented, for the construction of non-discrete, extremally disconnected topological groups. We comment briefly.

(i) It was SIROTA [1969] who gave the first such example. His construction requires the use of a so-called k-ultrafilter, whose existence is shown assuming CH.

(ii) LOUVEAU [1972] proved that if p is taken to be an ultrafilter which is absolute in the sense of CHOQUET [1968], then the group $\langle G, \mathcal{T}(p)\rangle$ is extremally disconnected. Indeed in this case a subset U of G is $\mathcal{T}(p)$-open if and only if for all $A \in U$ and $F \in p$ there is $n < \omega$ such that $\{n\} \in U_F(\emptyset) \cap (U \Delta A)$. (Here as usual $U \Delta A$ denotes $\{B \Delta A : B \in U\}$.) The property in question transfers from (open) U to its closure, so that cl U is $\mathcal{T}(p)$-open whenever U is, i.e., $\langle G, \mathcal{T}(p)\rangle$ is extremally disconnected.

An ultrafilter p on ω is (by definition) absolute if for every partition $\{A_n : n < \omega\}$ of ω either there is $n < \omega$ such that $A_n \in p$ or there is $A \in p$ such that $|A \cap A_n| = 1$ for all $n < \omega$. In his original work, CHOQUET [1968] deduced the existence of absolute ultrafilters from CH. It is not difficult to modify his proof, or (alternatively) the proof of W. RUDIN [1956] that $\beta(\omega)\backslash\omega$ contains P-points if CH is assumed, to show that MA implies the existence of absolute ultrafilters. Thus we may say that in effect LOUVEAU [1972] has solved Arhangel'skiĭ's problem, assuming ZFC + MA.

Choquet's absolute ultrafilters are often called selective ultrafilters. Several alternative definitions and applications have been collected from the literature and recorded by COMFORT and NEGREPONTIS [1974] (§9).

(iii) For $\omega \leqslant \kappa \leqslant \mathfrak{c}$, let $P(\kappa)$ be the statement that if $\{A_\xi : \xi < \alpha\}$ is a family of subsets of ω with $\alpha < \kappa$ such that $|\bigcap_{\xi \in F} A_\xi| = \omega$ for all finite $F \subset \alpha$, then there is infinite $A \subset \omega$ such that $|A \backslash A_\xi| < \omega$ for all $\xi < \alpha$. Statement $P(\omega)$ is a theorem of ZFC; $P(\omega^+)$ has been considered by ROTHBERGER [1948] (Lemma III.3) in connection with his study of Hausdorff's gaps and Ω-limits; BOOTH [1969] derives $P(\mathfrak{c})$ from ZFC + MA, he attributes independent derivations to J. Silver and K. Kunen, and he notes that $P(\mathfrak{c})$ is equivalent to the existence in $\beta(\omega)\backslash\omega$ of (a dense set of) $P(\mathfrak{c})$-points. (An ultrafilter p on ω is a $P(\kappa)$-point if for every $\{A_\xi : \xi < \alpha\} \subset p$ with $\alpha < \kappa$ there is $A \in p$ such that $|A \backslash A_\xi| < \omega$ for all $\xi < \alpha$. The P-points of $\beta(\omega)\backslash\omega$ are the $P(\omega^+)$-points.)

With the preceding paragraph as preamble we can summarize the contribution to Arhangel'skiĭ's problem of MALYHIN [1975] as follows: Assuming $P(\mathfrak{c})$ (which he denotes R_1) as an axiom, there is a $P(\mathfrak{c})$-point $p \in \beta(\omega)\backslash\omega$ such that the topological group $\langle G, \mathcal{T}(p)\rangle$ is extremally disconnected. Malyhin shows also that it is no coincidence that his example, like those of Sirota and Louveau, is based on a group for which (in multiplicative notation) each element x satisfies $x^2 = e$: Every extremally disconnected topological group contains such a (necessarily Abelian) subgroup as an open-and-closed subgroup.

5.9. A locally compact topological group, if it is a P-space or extremally disconnected, must be discrete: It is easy to see that every compact P-space is finite, and in any event we have noted in 2.12 that every topological group which is an F-space and a k-space is discrete.

5.10. It is proved in the fundamental memoir of WEIL [1937] that every topological group G, viewed as a uniform space in (say) its left uniform structure, may, like any uniform space, be embedded uniformly into a complete uniform space. (When G is totally bounded we have agreed to denote this completion by the symbol $\bar{G}$. To preserve the integrity of that notation let us here, in the general case, denote this uniform completion of G by cG.) In order that cG itself admit the structure of a topological group with G a topological subgroup it is sufficient that the function $x \to x^{-1}$ be uniformly continuous on some $W \in \mathcal{N}_G(e)$; for in this case it is simple to see, as WEIL [1937] (§5) has shown, that the functions $x \to x^{-1}$ and $\langle x, y \rangle \to xy$ extend continuously over cG and $cG \times cG$, respectively.

Are there topological groups G which arise naturally for which the uniform completion cG in not a topological group? DIEUDONNÉ [1944] addresses this question, which he attributes to N. BOURBAKI, and he gives the following two examples of topological groups "qui ne peuvent être complétés".

5.11. THEOREM. *Let G be the group of homeomorphisms of* $[0, 1]$ *onto* $[0, 1]$, *topologized by the metric*

$$\rho(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\} .$$

Then the metric completion $\langle cG, \sigma \rangle$ of $\langle G, \rho \rangle$ is not a topological group.

PROOF. We leave it to the reader to verify the fact, from which the present statement derives its significance or interest, that the group G, with the topology induced by the metric ρ, is indeed a topological group.

Now for $0 < n < \omega$ define $f_n \in G$ by the requirements that $f_n(0) = 0$, $f_n(\frac{1}{2}) = 1/n$, $f_n(1) = 1$, and f_n is linear on the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. It is clear that the sequence $\{f_n : 0 < n < \omega\}$ is ρ-Cauchy; hence there is $f \in cG$ such that $f_n \to f(\sigma)$. If $\langle cG, \sigma \rangle$ is a topological group then $f_n^{-1} \to f^{-1}(\sigma)$, and hence $\{f_n^{-1} : 0 < n < \omega\}$ is ρ-Cauchy. That this is a contradiction may be seen using elementary properties of the function space $C([0, 1], [0, 1])$, or directly as follows: from $f_n^{-1}(1/n) = \frac{1}{2}$ and $f_n^{-1}(1/(2n)) = \frac{1}{4}$ we have

$$\rho(f_n^{-1}, f_{2n}^{-1}) \geq |f_n^{-1}(1/(2n)) - f_{2n}^{-1}(1/(2n))| = |\tfrac{1}{4} - \tfrac{1}{2}| = \tfrac{1}{4} .$$

It may be noted that in the theorem just proved the metric completion $\langle cG, \sigma \rangle$ of $\langle G, \rho \rangle$ coincides with the completion of G in its right uniform structure: denoting by e the identity element of G and defining $U_\varepsilon \in \mathcal{N}(e)$ for $\varepsilon > 0$ by the rule

$$U_\varepsilon = \{f \in G : \rho(f, e) < \varepsilon\},$$

we have for $f, g \in G$ that the three conditions $fg^{-1} \in U_\varepsilon$, $f \in U_\varepsilon \circ g$, $\rho(f, g) < \varepsilon$ are equivalent. (The additional conditions $g^{-1}f \in U_\varepsilon$, $f \in g \circ U_\varepsilon$, $\rho(f^{-1}, g^{-1}) < \varepsilon$ are themselves mutually equivalent, but for suitable f, g and ε they are not equivalent to the first three conditions. Indeed we have seen that for all $\varepsilon > 0$ there is $N < \omega$ such that $f_n f_m^{-1} \in U_\varepsilon$ for m, $n > N$, but for $\varepsilon = \frac{1}{4}$ there is no $N < \omega$ such that $f_m^{-1} f_n \in U_\varepsilon$ for m, $n > N$.)

5.12. In the next example we consider, for contrast and simplicity, the completion of the given group G with respect to its left uniform structure.

THEOREM. *Let G be the set of permutations of ω with the topology $\mathcal{T}$ of pointwise convergence. (Here for finite $F \subset \omega$ one sets*

$$U_F = \{f \in G : f(k) = k \textit{ for all } k \in F\},$$

and $\mathcal{T}$ is defined so that $\{U_F$: finite $F \subset \omega\}$ is a local base at e.) Then G is a topological group whose completion cG is not a topological group.

PROOF. Again we leave to the reader the proof that $G = \langle G, \mathcal{T} \rangle$ is a topological group. We note from 1.8(i) above that G, being first countable, is in fact metrizable.

For $n < \omega$ define $f_n \in G$ by the rule

$$\begin{aligned} f_n(k) &= k + 1 && \text{for } 0 \leq k \leq n, \\ &= 0 && \text{for } k = n + 1, \\ &= k && \text{for } k > n + 1. \end{aligned}$$

For every finite $F \subset \omega$ there is $N < \omega$ such that if m, $n > N$, then $f_n \in f_m \circ U_F$ (that is, $f_n(k) = f_m(k)$ for $k \in F$). Thus the sequence $\{f_n : n < \omega\}$ is (left-) Cauchy and there is $f \in cG$ such that $f_n \to f$. If cG is a topological group then $f_n^{-1} \to f^{-1}$ and hence f_n^{-1} is (left-) Cauchy. For the required contradiction it is enough to note that with $F = \{0\}$ there is no $N < \omega$ such that $f_n^{-1} \in f_m^{-1} \circ U_F$ for all m, $n > N$: For $m \neq n$ we have

$$f_n^{-1}(0) = n + 1 \neq m + 1 = f_m^{-1}(0)$$

and hence $f_n^{-1} \notin f_m^{-1} \circ U_F$.

It is noted by DIEUDONNÉ [1944] that when the set ω is topologized by the metric d defined by the rule

$$d(k, k') = 0 \quad \text{if } k = k', \\ = 1 \quad \text{if } k \neq k',$$

then ω is a complete metric space and G is a group of isometries of ω.

5.13. The two groups just discussed, although their completions do not admit the structure of a topological group, do nevertheless admit compatible (i.e., equivalent) metrics with respect to which they are already complete. We show this for 5.11 and 5.12 in 5.14 and 5.15, respectively.

5.14. The following statement and its proof are given by BESSAGA and PEŁCZYŃSKI [1975] (page 121).

THEOREM. *Let $\langle X, d\rangle$ be a compact metric space and let G be the group of homeomorphisms of X onto X with the metric ρ defined by*

$$(*) \qquad \rho(f, g) = \sup\{d(f(x), g(x)) \colon x \in X\}.$$

Let $\sigma(f, g) = \rho(f, g) + \rho(f^{-1}, g^{-1})$ for $f, g \in G$. Then σ is a compatible metric for G, and $\langle G, \sigma\rangle$ is a complete metric space.

PROOF. For $f_n, f \in G$ we have $f_n \to f(\rho)$ if and only if $f_n^{-1} \to f^{-1}(\rho)$. It follows that $f_n \to f(\rho)$ if and only if $f_n \to f(\sigma)$, i.e., the metrics ρ and σ are equivalent.

In what follows we view the relation (*) as defining a metric not only on G but in fact on the larger space $C(X, X)$. It is well-known that $\langle C(X, X), \rho\rangle$ is a complete metric space.

Now let f_n be a σ-Cauchy sequence in G. Since the sequences f_n and f_n^{-1} are ρ-Cauchy there are $f, g \in C(X, X)$ such that $f_n \to f(\rho)$ and $f_n^{-1} \to g(\rho)$. We show $f \in G$.

Note first that if $x_n, x \in X$ and $x_n \to x(d)$, then $f_n(x_n) \to f(x)(d)$. This follows from the relations

$$d(f_n(x_n), f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x))$$

$$\leq \rho(f_n, f) + d(f(x_n), f(x))$$

and the facts that $f_n \to f(\rho)$ and $f(x_n) \to f(x)$ (d). Similarly from $x_n \to x(d)$ follows $f_n^{-1}(x_n) \to g(x)$ (d).

(i) *f maps onto X.* Given $p \in X$ there is $x_n \in X$ such that $f_n(x_n) = p$. Without loss of generality we assume, using the fact that X is compact, that there is $x \in X$ such that $x_n \to x(d)$. Then we have $p = f_n(x_n) \to f(x)(d)$, as required.

(ii) *f is one-to-one.* It is enough to show that $g(f(x)) = x$ for all $x \in X$. Defining $x_n = x$ we have $x_n \to x(d)$ and hence

$$x = f_n^{-1}(f_n(x_n)) \to g(f(x))\ (d)\,.$$

5.15. THEOREM. *The group G of permutations of ω, with the topology of pointwise convergence on ω, admits a compatible complete metric.*

PROOF. The discrete space ω has a compatible complete metric, hence the product (power) space ω^ω does also. Now we set

$$A = \{f \in \omega^\omega : f \text{ is one-to-one}\} \quad \text{and} \quad B = \{f \in \omega^\omega : f[\omega] = \omega\}\,.$$

Then as sets we have $G = A \cap B \subset \omega^\omega$, and the topology of G is the topology inherited from the complete metric space ω^ω. It is clear that A is closed in ω^ω, and hence a G_δ. Further for $k < \omega$ the set

$$B(k) = \{f \in \omega^\omega : k \in f[\omega]\}$$

is open, and $B = \bigcap_{k<\omega} B(k)$. Thus G is a G_δ in a complete metric space. It is well known (see for example HAUSDORFF [1924], ČECH [1937] (§III), or ENGELKING [1977] (4.3.23)) that every such space admits a compatible complete metric.

5.16. It is well known that there exist metrizable topological groups with no compatible metric. (The group $\mathbb{Q}$ comes most quickly to mind. Of course its completion relative to the usual metric is the complete topological group $\mathbb{R}$.)

5.17. The results of 5.11–5.16 suggest the following question. I first heard this from van Douwen in conversation in 1979 and I believe it remains unsolved.

Question. Does every metrizable topological group admit an equivalent metric with respect to which its completion is a topological group?

Let us note that if in the metrizable group G there is $W \in \mathcal{N}(e)$ such that W is Abelian or W is bounded (cf. 1.12 above), then any metric for G which is (say) left translation-invariant (as in 1.8(ii)) is as required. The proof is given by the argument of Weil in 5.10 above and the fact that in this case the function $x \to x^{-1}$ is uniformly continuous on W.

5.18. The article by Roitman in this Handbook considers carefully and at length questions concerning axiom systems which determine the existence or non-existence of (strong) S-spaces and (strong) L-spaces. We will not repeat here any of those consistency statements; our present aim is simply to show that the strong

S-space and L-space phenomena available in any given model of ZFC are available within the class of topological groups.

We adopt the (standard) definitions and terminology given in Roitman's article, and this as well.

DEFINITION. Let G be a topological group. Then

(i) G is a *strong S-group* if, for $0<n<\omega$, G^n is hereditarily separable but not hereditarily Lindelöf;

(ii) G is a *strong L-group* if, for $0<n<\omega$, G^n is hereditarily Lindelöf but not hereditarily separable.

Before passing to the proof of Theorem 5.20, which is (together with 5.21) due to ROITMAN [1980], let us discuss briefly an alternative approach given earlier by ZENOR [1980] and based on the following lemma.

LEMMA. *Let Y, X and S be topological spaces with $w(X)\leqslant\omega$ and let Φ: $Y\times X\to S$ be a function such that* (a) *$x\to\Phi(y, x)$ is continuous for each $y\in Y$ and* (b) *Y has the weakest topology making each function $y\to\Phi(y, x)$ with $x\in X$ continuous. Then* (i) *if every $X^n(n<\omega)$ is hereditarily separable, then Y is hereditarily Lindelöf, and* (ii) *if every X^n $(n<\omega)$ is hereditarily Lindelöf, then Y is hereditarily separable.*

Now let H denote sequential (separable) Hilbert space, and use the topology of pointwise convergence on X to make the vector space $C(X, H)$ a topological group. Through repeated shrewd use of the lemma just stated, making suitable choices for Y, S and Φ, one can prove the following statement.

THEOREM. *Let X be a space. Then*

(i) *X^n is hereditarily separable (all $n<\omega$) if and only if $C(X, H)$ is hereditarily Lindelöf; and*

(ii) *X^n is hereditarily Lindelöf (all $n<\omega$) if and only if $C(X, H)$ is hereditarily separable.*

Upon consideration of the spaces X, $C(X, H)$, and $C(C(X, H), H)$, it becomes clear that there is a strong S-space if and only if there is a strong S-group; and, there is a strong L-space if and only if there is a strong L-group.

Details required to complete the preceding arguments are given in the papers of ZENOR [1980] and ROITMAN [1980]. The former paper is more general than our present superficial exposition indicates for it considers, in contrast to current S-and-L fashion, for an arbitrary infinite cardinal number α the properties of being hereditarily α-separable and hereditarily α-Lindelöf.

5.19. We turn now to material from ROITMAN [1980]. The proof of the following lemma is left to the reader.

LEMMA. *Let P be one of these two properties: hereditarily separable, hereditarily Lindelöf.*

(i) *The continuous image of a space with P has P;*

(ii) *if X is a space of the form $X = \bigcup_{n<\omega} X_n$ and each X_n has P, then X has P.*

5.20. We recall that for G a group and $X \subset G$, the symbol $[X]$ denotes the smallest subgroup of G that contains X.

THEOREM. *Let G be a topological group and let $X \subset G$.*

(i) *If X is a strong S-space, so is $[X]$;*

(ii) *if X is a strong L-space, so is $[X]$.*

PROOF. We claim first that $[X]$ is an S-space in (i), an L-space in (ii). Indeed for $n < \omega$ and $f: n = \{k: 0 \leq k \leq n-1\} \to \{-1, +1\}$ set

$$X(f, n) = \left\{\prod_{k<n} x_k^{f(k)}: x_k \in X\right\}.$$

The function $\langle x_k: k < n\rangle \to \Pi_{k<n} x_k^{f(k)}$ is a continuous surjection from X^n onto $X(f, n)$, and

$$[X] = \bigcup\{X(f, n): n < \omega, f \in \{-1, +1\}^n\}.$$

The claim then follows from Lemma 5.19.

It remains to show for all $n < \omega$ that $[X]^n$ is an S-space in (i), an L-space in (ii). The subgroup $[X^n]$ of G^n generated by X^n is, according to what has been proved already, hereditarily separable in (i) and hereditarily Lindelöf in (ii); thus it is enough to show $[X]^n \subset [X^n]$. (The reverse containment also holds, but we do not need it.)

Let $p = \langle p_k: k < n\rangle \in [X]^n$ and for $k < n$ define $p(k) \in G^n$ by the rule

$$\begin{aligned} p(k)_i &= p_k \quad \text{if } i = k\,, \\ &= e \quad \text{if } i \neq k\,. \end{aligned}$$

Clearly $p(0) \in [X] \times \Pi_{0<i<n}\{e\} \subset [X^n]$, and more generally $p(k) \in [X^n]$ for all $k < n$. Since $[X^n]$ is a subgroup of G^n and

$$p = p(0) \cdot p(1) \cdot \cdots \cdot p(n-1)$$

we have $p \in [X^n]$, as required.

We note finally for $0 < n < \omega$ that $[X]^n$ is not hereditarily Lindelöf in (i) because X^n is not, and is not hereditarily separable in (ii) because X^n is not. The proof is complete.

5.21. COROLLARY. (i) *There is a strong S-group if and only if there is a strong S-space*;

(ii) *there is a strong L-group if and only if there is a strong L-space.*

PROOF. the 'only if' statements are clear, since a topological group is a topological space. For the 'if' statements it is enough, according to Theorem 5.20, to show that every topological space X embeds homeomorphically into a topological group; for this use the embedding $i: X \to \mathbb{R}^{C(X)}$ given by $i(x)_f = f(x)$, or the variant discussed in 2.1 above, or the free Abelian topological group $A(X)$ described in 2.3.

5.22. The HFD construction of Hajnal and Juhász is, as indicated in some detail by Roitman in this Handbook, a handy tool for the construction of S-spaces. Every HFD is hereditarily normal (cf. Hajnal and Juhász [1974]). This yields the following theorem, whose proof we omit.

THEOREM (HAJNAL and JUHÁSZ [1976]). *Assume* CH. *The group* $\{-1, +1\}^{(\omega^+)}$ *contains a dense subgroup which is hereditarily separable, hereditarily normal, and non-Lindelöf.*

5.23. In the proof of Theorem 5.20, even the partial and preliminary conclusion that the group $[X]$ is an S-space used the full hypothesis that X is a strong S-space; were X assumed simply to be an S-space, the argument furnishes no information whatever about $[X]$. (A similar remark concerns $[X]$ as an L-group.) This observation generates two questions, as follows.

(i) If an S-space exists, must an S-group exist?

(ii) If an L-space exists, must an L-group exist?

In any event it cannot be proved in ZFC that every S-group is a strong S-group, nor that every L-group is a strong L-group: Working in ZFC + CH, ROITMAN [1980] has constructed an S-group whose square is not an S-group, and an L-group whose square is not an L-group.

6. Pseudocompact topological groups

It was HEWITT [1948] who introduced the class of pseudocompact spaces: A (Tychonoff) space X is pseudocompact if and only if every real-valued continuous function on X is bounded, that is, $C^*(X) = C(X)$. He showed *inter alia* that X is pseudocompact if and only if X is G_δ-dense in βX (see 6.1 below).

A pseudocompact topological group G is totally bounded (6.3 below), hence by Theorem 1.2 of Weil is dense in a (unique) compact group $\bar{G}$. In this section we show that every pseudocompact topological group G is G_δ-dense in $\bar{G}$; further, $\bar{G} = \beta(G)$. This latter relation makes possible the construction or definition of many topological spaces (indeed, topological groups) whose Stone–Čech compactification can be identified in concrete form.

DEFINITION. A subspace X of a space Y is *G_δ-dense in Y* if every non-empty G_δ of Y meets X.

6.1. LEMMA. *For a topological space X, the following statements are equivalent.*
(i) *X is pseudocompact*;
(ii) *every locally finite family of non-empty open subsets of X is finite*;
(iii) *X is G_δ-dense in each of its compactifications*;
(iv) *X is G_δ-dense in $\beta(X)$.*

PROOF. (i)⇒(ii). Let $\{U_n: n<\omega\}$ be a faithfully indexed locally finite family of non-empty open subsets of X, for $n<\omega$ choose $x_n \in U_n$ and $f_n \in C(X)$ such that $f_n(x_n)=n$, $f_n(x)=0$ for all $x\in X\backslash U_n$, and $f\geqslant 0$, and set $f=\Sigma_{n<\omega} f_n$. Since each point of X has a neighborhood throughout which f agrees with the sum of finitely many continuous functions, the unbounded function f is itself continuous, contradicting (i).

(ii)⇒(i). Let $f\in C(X)\backslash C^*(X)$, say with $f\geqslant 0$, choose $x_0 \in X$ and recursively for $n<\omega$ choose $x_{n+1}\in X$ so that $f(x_{n+1}) > f(x_n)+1$. Then

$$\{f^{-1}(f(x_n)-1, f(x_n)+1): n<\omega\}$$

is a faithfully indexed family of non-empty open subsets of X, contradicting (ii).

(i)⇒(iii). Let $B(X)$ be a compactification of X and suppose there is $p\in \bigcap_{n<\omega} U_n \subset B(X)\backslash X$ with U_n open in $B(X)$. Choose $f_n \in C(B(X), [0,1])$ with $f_n(p)=0$, $f_n(x)=1$ for $x\in B(X)\backslash U_n$, and set $f=\Sigma_{n<\omega} f_n/2^n$ and $g=f\,|\,X$. Since $f\in C(B(X))$ and $f(p)=0$ we have $1/g \in C(X)\backslash C^*(X)$, a contradiction.

(iii)⇒(iv). This is clear.

(iv)⇒(i). If (i) fails there is $f\in C(X)\backslash C^*(X)$ with $f\geqslant 1$. Let $\bar{g}$ denote the Stone extension of the (continuous) function $g=1/f \in C(X, (0,1])$. Since $\inf g = 0$, the set $\bar{g}^{-1}(\{0\})$, a G_δ in $\beta(X)$, is non-empty; but $\bar{g}^{-1}(\{0\})\cap X=\emptyset$, contradicting (iv).

The equivalence (i)⇔(ii) of 6.1 was given by GLICKSBERG [1952]. See also MARDEŠIĆ and PAPIĆ [1955], and BAGLEY, CONNELL and MCKNIGHT [1958].

6.2. Let us note in passing, apropos of 6.1, that a space X which is G_δ-dense in some one of its compactifications $B(X)$ need not be pseudocompact. One may take for X an uncountable discrete space and for $B(X)$ its one-point compactification. Alternatively, recall that there exist pseudocompact spaces X_0, X_1 such that $X=X_0\times X_1$ is not pseudocompact, and set $B(X)=B(X_0)\times B(X_1)$. (The usual example in this connection, based on work of NOVÁK [1953] and TERASAKA [1952], is described in this Handbook by Vaughan (Theorem 3.1); that the product space in question is not pseudocompact is clear from the fact that it contains the countably infinite discrete space ω as an open-and-closed subspace. See also GILLMAN and JERISON [1960] (9.15).) Since X_i is G_δ-dense in $\beta(X_i)$ $(i = 0, 1)$, the space X is G_δ-dense in $B(X)$ (cf. 8.6 below).

6.3. LEMMA. *Every pseudocompact topological group is totally bounded.*

PROOF. If G is a counterexample there is $U \in \mathcal{N}(e)$ such that every finite $F \subset G$ satisfies $FU \neq G$. Let $x_0 \in G$ and recursively, if $n < \omega$ and x_k has been chosen for all $k < n$, let $x_n \in G \backslash \bigcup_{k<n} x_k U$. We claim that for $V \in \mathcal{N}(e)$ with $V = V^{-1}$ and $V^4 \subset U$, the family $\{x_n V: n < \omega\}$ is locally finite. Indeed let $p \in G$ and suppose that pV meets both $x_k V$ and $x_n V$, say with $k < n$; then $x_n \in x_k V V^{-1} V V^{-1} \subset x_k U$, a contradiction.

6.4 It follows from 6.3 that every pseudocompact topological group has a Weil completion. Let us show that a totally bounded topological group G is pseudocompact if and only if it is G_δ-dense in $\bar{G}$.

THEOREM. *For K a compact topological group and G a dense subgroup of K, the following are equivalent.*

(i) *G is pseudocompact*;

(ii) *G is G_δ-dense in K*;

(iii) *$G \times G$ is pseudocompact.*

PROOF. (i)$\Rightarrow$(ii). This follows from Lemma 6.1 ((i)$\Rightarrow$(iii)).

(ii)$\Rightarrow$(i). Let $\{x_n U_n: n < \omega\}$ be a locally finite open family in G with $U_n \in \mathcal{N}(e)$, for $n < \omega$ choose $\tilde{U}_n$ open in K with $\tilde{U}_n \cap G = U_n$, and define

$$V_n = \bigcup_{k>n} x_k U_k\,, \qquad \tilde{V}_n = \bigcup_{k>n} x_k \tilde{U}_k\,, \quad \text{and} \quad \tilde{U} = \bigcap_{n<\omega} \tilde{U}_n\,.$$

We have $\mathrm{cl}_G V_n = \bigcup_{k>n} \mathrm{cl}_G x_k U_k$ and hence $\bigcap_{n<\omega} \mathrm{cl}_G V_n = \emptyset$.

Since K is compact there is $x \in K$ such that every neighborhood in K of x contains x_k for infinitely many $k < \omega$, and since $x\tilde{U}$ is a non-empty G_δ of K there is $y \in \tilde{U}$ such that $xy \in G$. Since every neighborhood in K of xy contains $x_k y$ for infinitely many $k < \omega$ we have for each $n < \omega$ that

$$xy \in (\mathrm{cl}_K(\bigcup_{k>n} x_k \tilde{U}_k)) \cap G = (\mathrm{cl}_K(\tilde{V}_n \cap G)) \cap G = (\mathrm{cl}_K V_n) \cap G = \mathrm{cl}_G V_n\,,$$

a contradiction.

(ii)$\Rightarrow$(iii). $G \times G$ is G_δ-dense in $K \times K$, and hence $G \times G$ is pseudocompact by the implication (ii)$\Rightarrow$(i) applied to $G \times G$.

(iii)$\Rightarrow$(i). This is immediate from the fact that the continuous image of a pseudocompact space is pseudocompact.

We defer to 8.6 below the proof, which follows easily from 6.4, that the product of any set of pseudocompact topological groups is pseudocompact.

6.5. THEOREM. *For K a compact topological group and G a dense subgroup of K, the following are equivalent.*

(i) *G is pseudocompact;*

(ii) *every $f \in C(G)$ extends to $\bar{f} \in C(K)$;*

(iii) *K is the Stone–Čech compactification of G (in symbols: $K = \beta(G)$).*

PROOF. (i)$\Rightarrow$(ii). It is enough to show that f is uniformly continuous on G. For then, for every $\varepsilon > 0$ there is $U \in \mathcal{N}_K(e)$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in G$ and $x \in yU$ and it follows since f is bounded that for $p \in K$ the real numbers

$$\inf\{\sup\{f(x): x \in pU\}: U \in \mathcal{N}_K(e)\}, \qquad \sup\{\inf\{f(x): x \in pU\}: U \in \mathcal{N}_K(e)\}$$

are equal; denoting their common value by $\bar{f}(p)$ yields a function $\bar{f}$ as required.

We proceed by contradiction. Replacing f by a suitable real multiple if necessary, we assume that for all $U \in \mathcal{N}(e)$ there are $x, y \in G$ with $x \in yU$ and $|f(x) - f(y)| > 1$.

We set $U_0 = G$, we choose $x_1, y_1 \in G$ so that $x_1 \in y_1 U_0$ and $|f(x_1) - f(y_1)| > 1$, and we choose $U_1 \in \mathcal{N}(e)$ such that if $x \in x_1 U_1$ and $y \in y_1 U_1$ then

$$|f(x) - f(y)| > 1, \qquad |f(x) - f(x_1)| < 1, \quad \text{and} \quad |f(y) - f(y_1)| < 1.$$

If $k < \omega$ and x_k, y_k, U_k have been defined we choose $x_{k+1}, y_{k+1} \in G$ so that $x_{k+1} \in y_{k+1} U_k$ and $|f(x_{k+1}) - f(y_{k+1})| > 1$, and we choose $U_{k+1} \in \mathcal{N}(e)$ so that $U_{k+1} = U_{k+1}^{-1}$, $U_{k+1}^3 \subset U_k$, and if $x \in x_{k+1} U_{k+1}$ and $y \in y_{k+1} U_{k+1}$, then

$$|f(x) - f(y)| > 1, \qquad |f(x) - f(x_{k+1})| < \frac{1}{(k+1)},$$

$$\text{and} \quad |f(y) - f(y_{k+1})| < \frac{1}{(k+1)}.$$

By Theorem 6.4 the space $G \times G$ is pseudocompact. Hence the family $\{x_k U_{k+1} \times y_k U_{k+1}: k < \omega\}$ is not locally finite and there is $\langle \bar{x}, \bar{y} \rangle \in G \times G$ such that every neighborhood of $\langle \bar{x}, \bar{y} \rangle$ meets $x_k U_{k+1} \times y_k U_{k+1}$ for infinitely many $k < \omega$. Since $U_{k+1} \subset U_k$, it follows in particular that for each neighborhood W of $\langle \bar{x}, \bar{y} \rangle$ there are $k < \omega$ and

$$\langle x, y \rangle \in W \cap (x_k U_k \times y_k U_k).$$

For such $\langle x, y \rangle$ we have $|f(x) - f(y)| > 1$; it follows from the continuity of f that $|f(\bar{x}) - f(\bar{y})| \geq 1$.

Now let V be a neighborhood in G of $\bar{x}^{-1}\bar{y}$, and let $N < \omega$. By continuity of the function $\langle x, y\rangle \to x^{-1}y$ at $\langle \bar{x}, \bar{y}\rangle$ there is $n > N$ such that V has non-empty intersection with the set $(x_n U_{n+1})^{-1}(y_n U_{n+1})$; since

$$(x_n U_{n+1})^{-1}(y_n U_{n+1}) = U_{n+1}^{-1} x_n^{-1} y_n U_{n+1} \subset U_{n+1}^{-1} U_{n-1}^{-1} y_n^{-1} y_n U_{n+1} \subset U_{n-2}$$

and $U_{k+1} \subset U_k$ for all $k < \omega$, it follows that V meets U_k for all $k < \omega$, i.e.,

$$\bar{x}^{-1}\bar{y} \in \mathrm{cl}_G U_k \subset U_k^2 \subset U_{k-1}\,.$$

Finally let U be a neighborhood of $\bar{x}$ and $\varepsilon > 0$, and choose $N < \omega$, $n < \omega$ and x so that $2/N < \varepsilon$, $n > N$ and $x \in U \cap x_n U_{n+1}$. From

$$x\bar{x}^{-1}\bar{y} \in x_n U_{n+1} U_{n+1} \subset x_n U_n$$

we have $|f(x\bar{x}^{-1}\bar{y}) - f(x_n)| < 1/n$, and from

$$x \in x_n U_{n+1} \subset x_n U_n$$

we have $|f(x) - f(x_n)| < 1/n$. It follows that for every neighborhood U of $\bar{x}$ and $\varepsilon > 0$ there is $x \in U$ such that

$$|f(x\bar{x}^{-1}\bar{y}) - f(x)| < 1/n + 1/n < 2/N < \varepsilon\,.$$

By continuity we have

$$|f(\bar{y}) - f(\bar{x})| = |f(\bar{x}\bar{x}^{-1}\bar{y}) - f(\bar{x})| < \varepsilon$$

and hence $f(\bar{x}) = f(\bar{y})$. This contradicts the relation $|f(\bar{x}) - f(\bar{y})| \geq 1$. The proof is complete.

(ii)$\Rightarrow$(iii). This is obvious.

(iii)$\Rightarrow$(i). It follows from (iii) and Theorem 1.43 above of Kuz'minov that $\beta(G)$ is a dyadic space. Thus to establish (i) it is enough to prove the following quite general result, due to ENGELKING and PEŁCZYŃSKI [1963]: X is pseudocompact whenever $\beta(X)$ is dyadic. If the statement fails let $\{r_n: n < m\}$ be a dense subset of $\mathbb{R}$, let $\{U_n: n < \omega\}$ be a locally finite family of pairwise disjoint non-empty open subsets of X, choose $x_n \in U_n$, let $f_n \in C(X)$ satisfy $f_n(x_n) = r_n$ and $f_n(x) = 0$ for all $x \in X\backslash U_n$, and set $f = \Sigma_{n<\omega} f_n$. Then $f \in C(X)$ and since $f(x_n) = r_n$ the Stone extension $\bar{f}: \beta(X) \to \beta(\mathbb{R})$ of f satisfies $\bar{f}[\beta(X)] = \beta(\mathbb{R})$; hence $\beta(\mathbb{R})$, the continuous image of the dyadic space $\beta(X)$, is itself dyadic.

Now let h be a homeomorphism of $\mathbb{R}$ onto $(-1, 1)$ and denote by $\bar{h}: \beta\mathbb{R} \to [-1, 1]$ the Stone extension of h. It follows from Theorem 1.40, denoting by g a continuous function from $\{0, 1\}^\alpha$ onto $\beta(\mathbb{R})$, that $\bar{h} \circ g$ depends on countably many coordinates; that is, there are countable $S \subset \alpha$ and $F \in C(\{0,1\}^S, [-1,1])$ such that $\bar{h} \circ g = F \circ \pi_S$. It then follows, fixing $p \in \{0,1\}^{\alpha \setminus S}$ and defining $M = \{0,1\}^S \times \{p\}$, that the compact metric space M satisfies

$$\begin{aligned}[-1,1] = \bar{h}[\beta(\mathbb{R})] &= (\bar{h} \circ g)[\{0,1\}^\alpha] = (F \circ \pi_S)[\{0,1\}^\alpha] \\ &= (F \circ \pi_S)[M] = \bar{h}[g[M]] .\end{aligned}$$

No proper closed subspace of $\beta(\mathbb{R})$ is carried by $\bar{h}$ onto $[-1,1]$, that is, $g(M) = \beta(\mathbb{R})$. Since continuous surjections between compact spaces do not raise weight (cf. ENGELKING [1977] (Theorem 3.7.19)) the space $\beta(\mathbb{R})$ is then metrizable. This contradiction completes the proof.

6.6. Theorems 6.4 and 6.5 are rooted in the work of GLICKSBERG [1959], who showed that a product $X = \Pi_i X_i$ of non-trivial spaces is pseudocompact if and only if $\beta(X) = \Pi_i \beta(X_i)$. In the context of topological groups, he recognized the utility of the condition that $G \times G$ is pseudocompact. The various equivalences of 6.4 and 6.5 are achieved by COMFORT and ROSS [1966] by using *inter alia* an argument to the effect that if G is G_δ-dense in K, then every $f \in C(G)$ is constant on the cosets of an appropriately chosen compact, normal, G_δ subgroup of K. This argument, based on Theorem 3.7 above of KAKUTANI and KODAIRA [1944] and an argument of HALMOS [1950] (§64), is derived from two results of ROSS and STROMBERG [1965]: (a) In a locally compact topological group, the closure of each open subset is a Baire set; and (b) for every Baire set E in a compact topological group K, there is a compact, normal, G_δ subgroup N of K such that $E = EN$. It was DE VRIES [1975] who recognized the availability of the present simplified approach to the equivalences of 6.4 and 6.5, avoiding completely the concepts of Baire sets and functions constant on cosets. Specifically, the proof given here that (ii)$\Rightarrow$(i) in 6.4 is from de Vries' Proposition 4.2, while our proof that (i)$\Rightarrow$(ii) in 6.5, reminiscent of topological arguments given by GLICKSBERG [1959], FROLÍK [1960] and KISTER [1962], is essentially as in de Vries' Lemma 3.2.

The use of Kuz'minov's theorem and the Engelking–Pełczynski result to prove the implication (iii)$\Rightarrow$(i) of 6.5 is also a suggestion of DE VRIES [1975]. At least two alternative proofs are available, as follows. (a) COMFORT and ROSS [1966] show directly that if G is a totally bounded topological group and every $f \in C^*(G)$ is uniformly continuous (a condition guaranteed by (iii) of 6.5), then every $f \in C(G)$ is uniformly continuous; condition (i) of 6.5 is then immediate from 6.7 below. (b) VAN DOUWEN [1979a] shows that every topological space X such that $\beta(X)$ is homogeneous is pseudocompact. (The same paper shows further that several additional conditions are equivalent to those of 6.4 and 6.5, including: $\beta(G)$ is homogeneous; $\beta(G)$ is dyadic; $\beta(G)$ is the continuous image of a supercompact topological space. VAN DOUWEN [1979a] gives also an example of a pseudocompact, homogeneous topological space with no homogeneous compactification.)

6.7. Of course a uniformly continuous function $f \in C(G)$ with G a topological group need not be bounded: the case $G = \mathbb{R}$, $f(x) = x$ is a simple example. Responding to a question of KISTER [1962], COMFORT and ROSS [1966] described a non-discrete, non-pseudocompact topological group G for which every $f \in C(G)$ is uniformly continuous. Let us note in passing however that if G is a totally bounded topological group and $f \in C(G)$ is uniformly continuous, then f is bounded. Indeed there is $U \in \mathcal{N}(e)$ such that if $x, y \in G$ and $x \in yU$, then $|f(x) - f(y)| < 1$, and there is finite $F \subset G$ such that $G = FU$; it is then clear that $|f(x)| < \max\{|f(y)|: y \in F\} + 1$ for all $x \in G$.

6.8. Given a compact topological group, one may construct a dense, pseudocompact subgroup by selecting a point from each nonempty G_δ and taking the subgroup generated by the resulting set. We consider this method of defining pseudocompact groups in section 7 below. A less tiring method of finding pseudocompact groups, indeed the richest source of examples, is available through the Σ-product construction defined and studied in detail by CORSON [1959].

DEFINITION. Let $\mathcal{S} = \{X_i: i \in I\}$ be a set of topological spaces and for $i \in I$ let $p_i \in X_i$. The *Σ-product of $\mathcal{S}$* at $p = \langle p_i: i \in I\rangle$, denoted $\Sigma(\mathcal{S}, p)$ or simply $\Sigma(p)$, is the space

$$\Sigma(p) = \left\{x \in \prod_{i \in I} X_i: |\{i \in I: x_i \neq p_i\}| \leqslant \omega\right\}$$

in the topology inherited from $\Pi_{i \in I} X_i$.

When each X_i is a topological group and $p_i = e_i$ it is clear that the Σ-product $\Sigma(p)$, now denoted $\Sigma(e)$, is itself a topological group.

6.9. As usual, we say that a space X is countably compact if every infinite subset of X has an accumulation point; and X is ω-bounded if every countably infinite subset of X has compact closure. The reader is referred to the article by Vaughan in this Handbook for a comprehensive treatment of these spaces and their properties. For our purposes here it is enough to note that (a) every ω-bounded space is countably compact, and (b) every countably compact space is pseudocompact. Indeed, (a) is obvious and (b) is immediate from Lemma 6.1: If $\{U_n: n < \omega\}$ is a locally finite family of non-empty open subsets of X and we choose $x_n \in U_n$, then $\{x_n: n < \omega\}$ has no accumulation point in X. (Alternatively for (b): If $f \in C(X)\backslash C^*(X)$ there is $x_n \in X$ such that $|f(x_n)| > n$, and again $\{x_n: n < \omega\}$ has no accumulation point in X.)

6.10. THEOREM. *Let $\{X_i: i \in I\}$ be a set of compact spaces and $p_i \in X_i$ for $i \in I$, and let X be the Σ-product $X = \Sigma(p)$. Then X is ω-bounded (hence countably compact, hence pseudocompact).*

PROOF. Given a countable subset A of X let

$$J = \{i \in I: \text{some } x \in A \text{ has } x_i \neq p_i\}$$

and set $C = (\Pi_{i\in J} X_i) \times \{p_{I\setminus J}\}$ where $p_{I\setminus J} = \pi_{I\setminus J}(p)$. Then C is compact and $A \subset C$, and since $|J| \leq \omega$ we have $C \subset \Sigma(p)$.

6.11. COROLLARY. *Let $\{G_i: i \in I\}$ be a set of compact topological groups and let $G = \Sigma(e)$ and $K = \Pi_{i\in I} G_i$. Then the topological group G is ω-bounded (hence countably compact, hence pseudocompact), and $K = \beta(G)$.*

6.12. The present formulation of Corollary 6.11 is undoubtedly appropriate for this article, but even from the time of its original expression, achieved independently by GLICKSBERG [1959] and KISTER [1962], it has been known to be true with 'group' replaced throughout by 'space'. In any event the result as given here allows us to identify in concrete form, and to recognize as commonplace and manageable spaces, the Stone–Čech compactification of various naturally defined spaces. This is an unusual luxury since for many spaces the Stone–Čech compactification, normally defined existentially in terms of ultrafilters or some other derivative of the axiom of choice, is a strange space, topologically bizarre and unfamiliar (see for example $\beta(\omega)$ and its subspaces described in this Handbook by van Mill).

The availability of Corollary 6.11 for compact spaces (rather than for the more restrictive class of compact topological groups) is deduced from the fact that the Σ-product $G = \Sigma(e)$ projects onto every subproduct of the form $\Pi_{i\in J} G_i$ with $J \subset I$, $|J| \leq \omega$. For a summary of results concerning the extendability of continuous functions under these circumstances, and for extensive relevant bibliographical citations, see ENGELKING [1966], HUŠEK [1976] and COMFORT and NEGREPONTIS [1982] (10.4–10.7).

We return to pseudocompact topological groups in Section 7 below and briefly in Theorem 8.6, but the remarks in this Chapter by no means exhaust what is known about them. We ignore completely, for example, the fundamental role they play in the continuing development of topological Galois theory, citing from the works of SOUNDARARAJAN [1970], [1971a], [1983] only the following elegant result: For a normal, separable, algebraic extension E of a field K, the dense, pseudocompact subgroups of the (compact) Galois group $G = G(E/K)$ are exactly those subgroups H of G with the property that every K-isomorphism: $F \to E$ with F countably generated over K extends to an element of H.

We conclude this section with a few miscellaneous results about pseudocompact topological groups.

For a space X we denote by PX the set X with the topology generated by the G_δ subsets of X. That is, a subset U of X is open in PX if and only if U is a union of G_δ subsets of X. Every space of the form PX is a P-space, that is, every G_δ in PX is open. For extended remarks on P-spaces and their properties, see GILLMAN and JERISON [1960], especially 4JKL and 14.29.

6.13. THEOREM. *Let G be a non-discrete pseudocompact topological group. Then*
(i) $|G| \geq \mathfrak{c}$;
(ii) $d(PG) \geq \mathfrak{c}$; *and*
(iii) $\mathrm{cf}(d(PG)) > \omega$.

PROOF. (i) Let λ denote (left) Haar measure on the Weil completion $\bar{G}$ of G. For $n < \omega$ there is $U_n \in \mathcal{N}_{\bar{G}}(e)$ such that $\lambda(U_n) < 1/n$, and by the natural extension of Theorem 3.7 cited earlier there is a compact, normal subgroup N of $\bar{G}$ such that $N \subset \bigcap_{n<\omega} U_n$ and $\bar{G}/N$ is metrizable. Since $\lambda(N) = 0$ we have $|\bar{G}/N| > \omega$ and hence $|\bar{G}/N| = \mathfrak{c}$. Since G meets each coset of N in $\bar{G}$ (by 6.4), we have $|G| \geq \mathfrak{c}$.

(ii) This follows from (i): Any dense subgroup of PG, since it is G_δ-dense in G and hence in $\bar{G}$, is pseudocompact.

(iii) It is a familiar consequence of the theorem of J. KÖNIG [1904] and the relation $\mathfrak{c}^\omega = (2^\omega)^\omega = 2^\omega = \mathfrak{c}$ that $\mathrm{cf}(\mathfrak{c}) > \omega$. We assume therefore in what follows that $d(PG) > \mathfrak{c}$.

It is not difficult to verify that for any compact space X the space PX is a Baire space in the sense that the countable intersection of dense, open subsets is dense. In particular $P\bar{G}$, and hence its G_δ-dense subspace PG, is a Baire space.

Suppose now that $\mathrm{cf}(d(PG)) = \omega$, let $D = \bigcup_{n<\omega} D_n$ be dense in PG with each $|D_n| < d(PG)$, and let E_n be the closed subgroup of PG generated by D_n. Since PG is a P-space, the (dense) set $\bigcup_{n<\omega} E_n$ is closed in PG; that is, $PG = \bigcup_{n<\omega} E_n$. Since PG is a Baire space there is $n < \omega$ such that E_n has non-empty interior in PG (and hence E_n is open in PG). There are $\{U_k: k < \omega\} \subset \mathcal{N}_G(e)$ and $\{\tilde{U}_k: k < \omega\} \subset \mathcal{N}_{\bar{G}}(e)$ such that $\bigcap_{k<\omega} U_k \subset E_n$ and $\tilde{U}_k \cap G = U_k$, and then by the natural extension of Theorem 3.7 there is a compact, normal subgroup N of $\bar{G}$ such that $N \subset \bigcap_{k<\omega} \tilde{U}_k$ and $\bar{G}/N$ is metrizable. We have

$$|G/E_n| \leq |G/(N \cap G)| \leq |\bar{G}/N| \leq \mathfrak{c} < d(PG),$$

and from $d(E_n) < d(PG)$ we have the contradiction

$$d(PG) \leq d(E_n) \cdot |G/E_n| < d(PG).$$

The arguments in 6.13 are from COMFORT and ROBERTSON [1984], but other proofs are available and perhaps preferable. For example, VAN DOUWEN [1980b] has noted that 6.13(i) is valid not only for non-discrete pseudocompact topological groups G but also for pseudocompact spaces X with no isolated point: This follows from the fact that X is G_δ-dense in $\beta(X)$ and 'an obvious Cantor tree argument'. Similarly, 6.13(iii) for compact topological groups follows from a very general topological result of CATER, ERDŐS and GALVIN [1978]; see in this connection 7.9 below.

6.14. THEOREM. *Let K be a compact topological group with Haar measure λ and let G be a proper dense, pseudocompact subgroup of K. Then G is not λ-measurable.*

PROOF. We show first that G has full outer measure in the sense that if U is open in K and $U \supset G$, then $\lambda(U) = 1$. Suppose instead that $\lambda(U) < 1$ and use 1.18 to choose a cozero set V of K such that $V \supset U$ and $\lambda(V) = \lambda(U) < 1$. Like every cozero set, the set V is a countable union of closed sets; hence $K \backslash V$, which is non-empty because $\lambda(K \backslash V) = 1 - \lambda(V) > 0$, is a G_δ. This contradicts the fact that G meets every non-empty G_δ of K (Theorem 6.4).

It follows that if G is λ-measurable then $\lambda(G) = 1$. From $\lambda(G) > 0$ and 4.6 it then follows that GG^{-1}, which is G, has non-empty interior in K. Then G is open, hence also closed, and we have the contradiction $G = K$.

The statement that a subset of a compact (Abelian) group has full outer measure if and only if it meets each non-empty Baire set, from which Theorem 6.14 follows, is offered by ITZKOWITZ [1965b] without elaboration. In a variety of contexts associated with the extension of Haar measure to a translation-invariant measure defined on a larger σ-algebra, several authors have shown in effect that proper, dense pseudocompact subgroups of compact groups have full outer measure; see in this connection KAKUTANI and OXTOBY [1950], HARTMAN and HULANICKI [1958], and HULANICKI [1959], [1964].

6.15. A pseudocompact topological group G can be of finite index in its Weil completion $\bar{G}$. Let $\mathcal{S} = \{G_i : i \in I\}$ with $|I| > \omega$ and each $G_i = \{-1, +1\}$, let H be the Σ-product $\Sigma(\mathcal{S}, e)$, set $K = \Pi_{i \in I}\, G_i$, define $f \in K$ by $f_i = -1$, and then, as suggested by GLICKSBERG [1959] (§4), let G be a subgroup of K containing H and maximal with respect to the property $f \notin G$. Since $H \subset G \subset K$ with H pseudocompact and dense in K, G is pseudocompact and $K = \bar{G}$. It is clear that $|K/G| = 2$.

This example suggests how to show, with COMFORT and SOUNDARARAJAN [1982], that every compact, totally disconnected Abelian topological group K with $w(K) > \omega$ has a proper, dense, pseudocompact subgroup G such that $|K/G| < \omega$. Like every Abelian torsion group, the dual group $\hat{K}$ of K has the form $\hat{K} = \bigoplus_{p \in P} \hat{K}_p$ where $\hat{K}_p$ is a p-group in the sense that the order of each element of $\hat{K}_p$ is an integer of the form p^k (cf. for example FUCHS [1970] (8.4)); since $|\hat{K}| = w(K) > \omega$ there is p such that $|\hat{K}_p| > \omega$. The socle S of $\hat{K}_p$, consisting of all elements of $\hat{K}_p$ of order $\leq p$, has the form $S = \bigoplus_\alpha \mathbb{Z}(p)$ for some $\alpha > \omega$, and the annihilator A of S in K and the canonical homomorphism φ satisfy

$$\varphi : K \to K/A = \hat{S} = (\mathbb{Z}(p))^\alpha .$$

With H the Σ-product $H = \Sigma(e) \subset \hat{S}$ and $\tilde{H}$ a maximal proper subgroup of $\hat{S}$ containing H, we have: $\tilde{H}$ is pseudocompact and $|\hat{S}/\tilde{H}| = p < \omega$. From the Baire category theorem it follows that φ is an open function; thus $\varphi[E]$ contains a non-empty G_δ in $\hat{S}$ whenever E is a non-empty G_δ in K. It follows, then, writing $G = \varphi^{-1}(\tilde{H})$, that ($|K/G| = p < \omega$ and) G is G_δ-dense in K, hence pseudocompact.

6.16. Not every pseudocompact topological group is countably compact. The following example is due to KISTER [1962]. Let D be a proper dense subgroup of the circle group $\mathbb{T}$, set $K = \mathbb{T}^\alpha$ with $\alpha > \omega$, and set

$$G = \{x \in K: |\{\xi < \alpha: x_\xi \notin D\}| \leq \omega\};$$

that is, writing $H = \Sigma(e) \subset K$, we have $G = H \cdot D^\alpha$. Again from $H \subset G \subset K$ it follows that G is pseudocompact. The group G is not countably compact since, choosing $x(n) \in D (n < \omega)$ with $\lim_n x(n) = x \in \mathbb{T} \backslash D$ and defining $\Delta(p) = \langle \Delta(p)_\xi: \xi < \alpha \rangle \in K$ for each $p \in \mathbb{T}$ by the rule $\Delta(p)_\xi = p$, we have $\Delta(x(n)) \in G$, $\Delta(x) \in K \backslash G$, and $\lim_n \Delta(x(n)) = \Delta(x)$.

6.17. On an Abelian group, a compact topological group topology of uncountable weight is not maximal among pseudocompact topological group topologies. Let us restate this result more carefully and prove it.

THEOREM. *If $K = \langle K, \mathcal{T} \rangle$ is a compact, Abelian topological group with $w(K) > \omega$, then there is a pseudocompact, topological group topology $\mathcal{T}'$ for K such that $\mathcal{T} \subset \mathcal{T}'$, $\mathcal{T} \neq \mathcal{T}'$.*

PROOF. We note first that if C is a closed subgroup of K, then $\psi(K) \leq \psi(C) \cdot \psi(K/C)$. Indeed if $\{U_i: i \in I\}$ and $\{V_j: j \in J\}$ are families of open subsets of C and K/C respectively with $\{e\} = \bigcap_i U_i$ and $\{H\} = \bigcap_j V_j$, and if sets $\tilde{U}_i$, $\tilde{V}_j$ are chosen open in K so that $U_i = \tilde{U}_i \cap C$, $\tilde{V}_j = \varphi^{-1}(V_j)$ with φ the canonical homomorphism from K onto K/C, then it is easy to verify that

$$\{e\} = (\bigcap_i \tilde{U}_i) \cap (\bigcap_j \tilde{V}_j).$$

In the present case we take for C the component of the identity, and we note that since the cardinal functions w, χ and ψ coincide for compact topological groups (§3) we have either $w(C) > \omega$ or $w(K/C) > \omega$.

In the latter case there is, as in 6.15 above applied to K/C, a dense pseudocompact subgroup $\tilde{H}$ of K/C such that $|(K/C)/\tilde{H}| < \omega$. As before with $\varphi: K \to K/C$ the canonical homomorphism, the group $H = \varphi^{-1}(\tilde{H})$ is pseudocompact, dense and of finite index in K. The required pseudocompact topological group topology $\mathcal{T}'$ for K is the topology generated by $\mathcal{T} \cup \{xH: x \in K\}$.

In case $w(C) = \alpha > \omega$ we choose by 1.31 a continuous homomorphism φ from $K = \langle K, \mathcal{T} \rangle$ onto $\mathbb{T}^\alpha$, and using the divisibility of $\mathbb{T}$ we choose a (necessarily discontinuous) homomorphism $\Omega: \mathbb{T}^\alpha \to \mathbb{T}$ such that $\Omega(t) = \Pi_{\xi<\alpha} t_\xi$ whenever $t = \langle t_\xi: \xi < \alpha \rangle \in \bigoplus_\alpha \mathbb{T}$. Let us show, defining $H = \mathrm{graph}(\Omega \circ \varphi) \subset K \times \mathbb{T}$, that H is G_δ-dense in $K \times \mathbb{T}$.

Given $r \in \mathbb{T}$ and a non-empty G_δ F of K, the set $\varphi[F]$ contains a non-empty G_δ

of $\mathbb{T}^\alpha$ and this in turn contains a non-empty G_δ S which depends on countably many coordinates, that is, for which there is countable $A \subset \alpha$ with $S = \pi_A[S] \times \mathbb{T}^{\alpha \setminus A}$. Choose $\bar{s} \in S \subset \mathbb{T}^\alpha$ and $\eta \in \alpha \setminus A$, and define $\bar{t} \in \mathbb{T}^\alpha$ so that $\bar{t}_\xi = 1$ for all $\xi < \alpha$ with $\xi \neq \eta$, and $\bar{t}_\eta = r \cdot (\Omega(\bar{s}))^{-1}$. Since $\bar{s}\bar{t} \in S \subset \varphi[F]$ there is $p \in F$ such that $\varphi(p) = \bar{s}\bar{t}$, and then

$$(\Omega \circ \varphi)(p) = \Omega(\bar{s}\bar{t}) = \Omega(\bar{s}) \cdot \Omega(\bar{t}) = \Omega(\bar{s}) \cdot \bar{t}_\eta = r,$$

i.e., $\langle p, r \rangle \in F \times \{r\}$.

The projection from H onto K is a one-to-one function, continuous when H has the (pseudocompact) topology inherited from $K \times \mathbb{T}$ with $K = \langle K, \mathcal{T} \rangle$. The required pseudocompact topology $\mathcal{T}' \supset \mathcal{T}$ for K is now defined by the requirement that this projection be a homeomorphism onto $\langle K, \mathcal{T}' \rangle$.

The theorem just proved for compact, Abelian groups is given by COMFORT and SOUNDARARAJAN [1982] for the totally disconnected case, by COMFORT and ROBERTSON [1982] in general.

7. Dense subgroups with special properties

Given properties P and Q and a group G with P, one may ask whether G contains a dense subgroup with Q. In this section we give several results responding to such questions. Typically, P is compactness; and Q is countable compactness or a related topological property, or a 'smallness' property defined in terms of cardinality, or a combination of such properties.

7.1. Of course, a discrete topological group has no proper dense subgroup whatever. On the other hand it is known (cf. HEWITT [1943] (Theorem 47)) that every locally compact (Hausdorff) space without isolated points has complementary dense subsets, and it becomes natural to ask, as did DIETRICH [1972], whether every non-discrete locally compact topological group has a proper dense subgroup. (It is clear that if H is a dense subgroup of a topological group G then each coset xH $(x \in G)$ is dense in G and hence $G \setminus H$ is dense in G.) The following example, due to RAJAGOPALAN and SUBRAHMANIAN [1976], shows that the answer to Dietrich's question is "No".

7.2. Set $K = \{-1, +1\}^\omega$ and $G = (\mathbb{Z}(2^\infty))^\omega$, where $\mathbb{Z}(2^\infty)$ denotes the group of complex numbers of the form $\exp(2\pi i n/2^m)$ with $m, n \in \mathbb{Z}$. The group G is given the smallest topological group topology which makes K open and induces on K (and each of its cosets) the usual compact topology of K. Explicitly: each coset xK $(x \in G)$ is homeomorphic to K, and a subset U of G is open in G if and only if $U \cap xK$ is open in xK for all $x \in G$. (It is easy to check directly in this case that G, so topologized, is indeed a topological group. Theorem 1.11, a more general result, is also applicable here.)

THEOREM. *The group G is a locally compact, non-discrete, divisible Abelian group with no proper dense subgroup.*

PROOF. It is enough to show, the other statements being obvious, that if H is a dense subgroup of G and $p \in G$, then $p \in H$. Since $pK \in \mathcal{N}(p)$ there is $x \in K$ such that $px \in H$, and since G is divisible there is $y \in G$ such that $y^2 = x$. From $yK \in \mathcal{N}(y)$ there is $z \in K$ such that $yz \in H$, and since elements of K have order 2 we have

$$x = y^2 = y^2e = y^2z^2 = (yz)^2 \in H^2 = H$$

and hence $p \in Hx^{-1} \subset HH^{-1} = H$, as required.

Rajagopalan and Subrahmanian have described several classes of groups each of which admits a proper dense subgroup, and have characterized the locally compact Abelian topological groups with this property. More recently, KABENYUK [1980] and KHAN [1980] have shown (independently) that a non-discrete locally compact Abelian topological group G has no proper dense subgroup if and only if (a) G has an open, compact subgroup of bounded order and (b) $\{x^p : x \in G\}$ is open in G for all primes p.

7.3. An infinite subset A of a topological group generates a subgroup H such that $|H| = |A|$. It follows that every topological group G such that $d(G) < |G|$ contains a dense subgroup H such that $|H| < |G|$ (in particular, every non-discrete locally compact separable topological group G contains a dense subgroup H with $|H| < |G|$, since by Theorem 2.8 we have $|G| \geqslant 2^{\omega}$); hence we have the following result.

THEOREM. *Let G be a non-discrete, locally compact topological group such that $\kappa(G) < |G|$. Then G contains a dense subgroup H such that $|H| < |G|$.*

PROOF. From 3.12 we have $\log(\chi(G)) < 2^{\chi(G)} \leqslant |G|$ and $d(G) < |G|$.

It follows that every infinite, compact topological group G contains a dense subgroup H such that $|H| < |G|$. Let us improve this statement.

7.4. THEOREM. *Let G be an infinite, compact group with $w(G) = \alpha$. Then G contains a dense, σ-compact subgroup H such that $|H| \leqslant \alpha < 2^{\alpha} = |G|$.*

PROOF. There is by Theorem 1.43 a continuous function f from $\{0, 1\}^{\alpha}$ onto G. We set

$$A_n = \{x \in \{0, 1\}^{\alpha} : |\{\xi < \alpha : x_{\xi} \neq 0\}| \leqslant n\} \quad \text{for } n < \omega$$

and we note, with CORSON [1959], that each set A_n is compact and that the set $A = \bigcup_n A_n$ is dense in $\{0, 1\}^{\alpha}$; further, we have $|A_n| = \alpha$ for $n < \omega$. The set $f[A]$ is then σ-compact, dense in G, and of cardinality not exceeding α. From the continuity of multiplication and inversion it then follows that the subgroup H of G generated by $f[A]$ has the required properties.

7.5. If K is a compact topological group with $w(K) = \alpha \geq \omega$, one may define without difficulty a family $\mathcal{G}$ of non-empty G_δ subsets of K such that $|\mathcal{G}| \leq \alpha^\omega$ and every non-empty G_δ subset of K contains an element of $\mathcal{G}$. If one chooses a point from each element of $\mathcal{G}$, the subgroup H of K generated by the set so formed satisfies $|H| \leq \alpha^\omega$ and is G_δ-dense in K (hence pseudocompact). This simple argument shows that a compact topological group K with $w(K) = \alpha \geq \omega$ contains a dense, pseudocompact subgroup H with $|H| \leq \alpha^\omega$. The reasoning appears so direct and free of inessentials that one is tempted to believe that the result is optimal, but in fact it can be improved in at least two ways: the hypothesis $w(K) = \alpha$ may be weakened to $w(K) \leq 2^\alpha$, and the subgroup H may be chosen 'more than pseudocompact'—countably compact, for example, or even p-compact for an element p of ω^* chosen in advance. Let us define our terminology and notation.

DEFINITION. Let p be a non-principal ultrafilter over ω (that is, $p \in \omega^*$ in the notation adopted by van Mill in this Handbook) and let X be a topological space. Then X is *p-compact* if for every function $f: \omega \to X$ the Stone extension $\bar{f}: \beta(\omega) \to \beta(X)$ satisfies $\bar{f}(p) \in X$.

As is noted by Vaughan in this Handbook, the covenient definition adopted here is equivalent (in our context, Tychonoff spaces) to the formulation in terms of p-limits introduced by A. BERNSTEIN [1970] in connection with problems in the theory of nonstandard analysis. It was Bernstein who initiated and completed the systematic study of the behavior of p-compact spaces under the usual topological operations, though it may be noted that his p-limit concept coincides in important special cases with the 'producing' relation introduced by FROLÍK [1967a], [1967b], with one of the orderings considered by KATĚTOV [1961/62], [1968], and with a definition given independently by SAKS [1972].

It is not difficult to see that, in parallel with the definition of p-compact, a (Tychonoff) space X is countably compact if and only if for every $f: \omega \to X$ the Stone extension $\bar{f}: \beta(\omega) \to \beta(X)$ satisfies $\bar{f}(p) \in X$ for some $p \in \omega^*$. From this observation or otherwise it follows that a space which is p-compact for some $p \in \omega^*$ is countably compact. (As noted in the article by Vaughan (Example 4.8), there are countably compact spaces which are p-compact for no $p \in \omega^*$.)

7.6. THEOREM. *Let $\alpha \geq \omega$, let K be a compact topological group with $w(K) \leq 2^\alpha$, and let $\emptyset \neq A \subset \omega^*$. Then*

(i) *K has a dense subgroup G such that $|G| \leq \alpha^\omega \cdot |A|^\omega$ and G is p-compact for all $p \in A$;*

(ii) *every subgroup H of K extends to a subgroup G such that $|G| \leq |H|^\omega \cdot |A|^\omega$ and G is p-compact for all $p \in A$.*

PROOF. It is enough to prove (ii). For from 3.1(ii) we have $d(K) \leqslant \log(2^\alpha) \leqslant \alpha$ and hence K contains a dense subgroup of cardinality $\leqslant \alpha$; applying (ii) to such a subgroup yields G as in (i).

We prove (ii). We assume without loss of generality that $|H| > 1$. Let us agree for $S \subset K$ and $f\colon \omega \to S$ that the symbols $\bar{f}$ and f' denote the Stone extensions of f viewed as functions into $\beta(S)$ and K respectively; that is, $\bar{f}[\beta(\omega)] \subset \beta(S)$ and $f'[\beta(\omega)] \subset K$. With this convention we set $H_0 = H$ and recursively, if $\xi < \omega^+$ and H_η has been defined for all $\eta < \xi$, we let H_ξ be the subgroup of K generated by the set

$$\{f'(p)\colon p \in A, f\colon \omega \to \bigcup_{\eta<\xi} H_\eta\}\,.$$

Now define $G = \bigcup_{\xi<\omega^+} H_\xi$. Then $|H_0| \leqslant |H|^\omega \cdot |A|^\omega$ and, assuming $|H_\eta| \leqslant |H|^\omega \cdot |A|^\omega$ for all $\eta < \xi$, we have

$$|H_\xi| \leqslant |\bigcup_{\eta<\xi} H_\eta|^\omega \cdot |A| \leqslant |\xi| \cdot |H|^\omega \cdot |A|^\omega \cdot |A| = |H|^\omega \cdot |A|^\omega$$

and hence

$$|G| \leqslant \Sigma_{\xi<\omega^+} |H_\xi| \leqslant \omega^+ \cdot |H|^\omega \cdot |A|^\omega = |H|^\omega \cdot |A|^\omega\,.$$

It remains to prove that G is p-compact for all $p \in A$. We recall, perhaps from GILLMAN and JERISON [1960] (6.12) or from the theory of perfect functions as developed by HENRIKSEN and ISBELL [1958], that for any compactification $B(X)$ of a (Tychonoff) space X, the usual continuous function $\pi\colon \beta(X) \to B(X)$ satisfies $\pi[\beta(X) \backslash X] = B(X) \backslash X$. Now suppose in the present case that there are $f\colon \omega \to G$ and $p \in A$ such that $\bar{f}(p) \in \beta(G) \backslash G$, and set $B(G) = \mathrm{cl}_K G$ and $\pi\colon \beta(G) \twoheadrightarrow B(G)$. Since the functions $\pi \circ \bar{f}$ and f' agree on ω they are identical, and hence $f'(p) \in B(G) \backslash G$. But there is $\xi < \omega^+$ such that $f[\omega] \subset H_\xi$, and then $f'(p) \in H_{\xi+1} \subset G$. This contradiction completes the proof.

7.7. COROLLARY. *Let $\alpha \geqslant \omega$. Every compact topological group K with $w(K) \leqslant 2^\alpha$ contains a dense, countably compact subgroup G such that $|G| \leqslant \alpha^\omega$.*

PROOF. It is enough to choose $p \in \omega^*$ and to set $A = \{p\}$ in Theorem 7.6.

It is sometimes convenient to replace α in Corollary 7.7 by $\log(\alpha)$, thus obtaining the following (equivalent) statement: For $\alpha \geqslant \omega$, every compact topological group K with $w(K) \leqslant \alpha$ contains a dense, countably compact subgroup G such that $|G| \leqslant (\log \alpha)^\omega$.

Let us remark, continuing the notation of 7.7, that if $w(K) = 2^\alpha$ then the subgroup G constructed is a proper subgroup since

$$|G| \leqslant \alpha^\omega \leqslant \alpha^\alpha = 2^\alpha < 2^{2^\alpha} = |K|\,.$$

On the other hand if K is metrizable (i.e., $w(K) \leqslant \omega$), then $G = K$; indeed it is well known (see for example GILLMAN and JERISON [1960] (3D.2)) that every pseudocompact metrizable space is compact. In any event one cannot prove in ZFC that every compact topological group K with $w(K) > \omega$ contains a dense pseudocompact subgroup G such that $|G| < |K|$. Indeed if $2^\omega = 2^{(\omega^+)}$, then every dense pseudocompact subgroup G of every compact topological group K with $w(K) = \omega^+$ satisfies $|G| = |K| = 2^{(\omega^+)}$; see also 7.9 below.

7.8. Participating in the argument used to prove Theorem 7.6 are the inductive construction and an attention to bounds on cardinality taken from the original article of A. BERNSTEIN [1970] and the observation by GINSBURG and SAKS [1975] and SAKS [1978] that the reasoning put forth in the context of general topology carries over without obstruction to topological groups. (Indeed in the proof of 7.6 the expression 'the subgroup of K generated by' is redundant: Given $f, g: \omega \to S$ with S a subgroup of K there are functions $m, i \to S$ such that $f'(p) \cdot g'(p) = m'(p)$ and $(f'(p))^{-1} = i'(p)$. Thus $\{f'(p): f: \omega \to S\}$ is a subgroup.) When $A = \{p\}$ with $p \in \omega^*$, the space G of 7.6 is constructed independently both by WOODS [1975] and by GINSBURG and SAKS [1975] who refer to it as 'the maximal p-compact extension' of H and 'the minimal p-compact extension' of H, respectively. In any event G enjoys and is characterized by the property that every $f \in C(H, X)$ with X p-compact for all $p \in A$ extends to $\bar{f} \in C(G, X)$.

SAKS [1978] (6.4) offers a generalization of 7.6 involving ultrafilters uniform over various (possibly distinct) discrete spaces. Corollary 7.7 appears in COMFORT and SAKS [1973]. The same statement, with 'pseudocompact' in place of 'countably compact', is given by H. WILCOX [1966], [1971], and by ITZKOWITZ [1965a], [1965b] for the case that K is Abelian.

7.9. We noted in 7.7 that if $2^\omega = 2^{(\omega^+)}$, then no dense, pseudocompact subgroup G of the compact topological group $K = \{-1, +1\}^{(\omega^+)}$ satisfies $|G| < |K|$. Let us show now, with COMFORT and ROBERTSON [1981], [1984], that compact groups K with no 'small' dense, pseudocompact subgroups arise whenever one assumes the generalized continuum hypothesis (GCH). Indeed, it is enough to assume the so-called singular cardinals hypothesis (SCH): $\kappa^\lambda \leqslant 2^\lambda \cdot \kappa^+$ for all infinite cardinal numbers κ and λ. The relevance of SCH to these topics came to our attention from CATER, ERDŐS and GALVIN [1978]; 7.9(v) for $K = \{-1, +1\}^\alpha$ is a special case of their Theorem 2.5. Concerning this axiom they write (here we paraphrase slightly): "Clearly, SCH follows from the generalized continuum hypothesis, but is much weaker. In fact, models of set theory violating SCH are not easy to come by; Prikry and Silver (see JECH [1978] (Section 37) and MAGIDOR [1977a], [1977b] have constructed such models assuming the consistency of very large (e.g., supercompact) cardinals, and DEVLIN and JENSEN [1975] have shown that some large cardinal assumption is necessary."

For G a pseudocompact topological group, let $m(G)$ denote the least cardinality of a dense, pseudocompact subgroup of G. We have from Theorem 6.4 that $m(K) = d(PK)$ for every compact topological group K.

THEOREM. *Let K and K' be compact topological groups with $w(K) = w(K') = \alpha \geq \omega$. Then*

(i) $m(K) = m(K')$, *that is, the cardinal $m(K)$ depends only on the cardinal $\alpha = w(K)$, not on the algebraic structure of K.*

(ii) $m(K) \geq \mathfrak{c}$ *and* $\mathrm{cf}(m(K)) > \omega$.

(iii) $\log \alpha \leq m(K) \leq (\log \alpha)^\omega$.

(iv) *If* $(\log \alpha)^\omega < 2^\alpha$, *then* $m(K) < |K|$.

(v) *Assume* SCH. *If* $(\log \alpha)^\omega = 2^\alpha$, *then* $m(K) = |K|$.

PROOF. By 1.43 and 1.45 there are continuous, onto functions $\{0, 1\}^\alpha \to K$ and $K \to [0, 1]^\alpha$. These remain continuous when $\{0, 1\}^\alpha$, K and $[0, 1]^\alpha$ are replaced by $P(\{0, 1\}^\alpha)$, PK, and $P([0, 1]^\alpha)$ respectively; since the first and last of these three spaces are homeomorphic, it follows from elementary properties of continuous functions that

$$d(P(\{0, 1\}^\alpha)) = d(PK) = d(P([0, 1]^\alpha)) .$$

The relation $m(K) = d(PK)$ now gives (i); and with 6.13 it gives (ii), too. We have $\log \alpha = d(K) \leq d(PK)$ from 3.1(ii), and $m(K) \leq (\log \alpha)^\omega$ is immediate from (the remark following) 7.7. Thus (iii) is proved. Since $|K| = 2^\alpha$, statement (iv) follows from (iii) and it remains only to prove (v).

If $2^\alpha = 2^\omega$, then from (ii) we have

$$2^\alpha = 2^\omega \leq m(K) \leq |K| = 2^\alpha$$

and hence $m(K) = 2^\alpha$. We assume in what follows that $2^\omega < 2^\alpha$. Then from

$$2^\omega < 2^\alpha = (\log \alpha)^\omega \leq 2^\omega \cdot (\log \alpha)^+$$

follows $\alpha^+ \leq 2^\alpha = (\log \alpha)^\omega \leq (\log \alpha)^+ \leq \alpha^+$ and hence $\alpha^+ = 2^\alpha = (\log \alpha)^\omega = (\log \alpha)^+$ and $\alpha = \log \alpha$. Since

$$\alpha = \log \alpha \leq m(K) \leq (\log \alpha)^\omega = (\log \alpha)^+ = \alpha^+ \quad \text{and} \quad \mathrm{cf}(m(K)) > \omega$$

by (ii) and (iii), it is enough to show $\mathrm{cf}(\alpha) = \omega$. If $\mathrm{cf}(\alpha) > \omega$, then for every countable $A \subset \alpha$ there is $\xi < \alpha$ such that $A \subset \xi$ and from SCH we have

$$\alpha^\omega \leq \Sigma_{\xi<\alpha} |\xi|^\omega \leq \Sigma_{\xi<\alpha} 2^\omega \cdot |\xi|^+ \leq \alpha \cdot 2^\omega \cdot \alpha < 2^\alpha = \alpha^\omega .$$

This contradiction completes the proof.

Since it is set-theoretic in flavor rather than topological, we omit here and leave to the interested reader the proof, available in ZFC alone without additional hypotheses, that the condition $(\log \alpha)^\omega = 2^\alpha$ of (v) is equivalent to the condition that either $\omega \leqslant \alpha < \log(\beth_1^+)$ or there is ν with $\mathrm{cf}(\nu) = \omega$ such that $\beth_\nu \leqslant \alpha < \log(\beth_{\nu+1}^+)$; here as usual for ordinal numbers ξ the beth cardinals $\beth_\xi$ are defined by the relations $\beth_0 = \aleph_0 = \omega$, $\beth_{\xi+1} = 2^{\beth_\xi}$, and $\beth_\xi = \Sigma_{\zeta<\xi} \beth_\zeta$ for non-zero limit ordinals ξ.

7.10. We have considered in this section several topological properties which can arise in the context $H \subset G$ with G a topological group and H a dense subgroup. Let us note in addition a curiosity available in this context: the relation $d(G) < d(H)$ is possible, even with H totally bounded and with $G = \bar{H} = \beta(H)$. Let $G = \{-1, +1\}^\alpha$ with $\alpha > \omega$ and, with notation and terminology as in 6.8ff., let H be the Σ-product $H = \Sigma(e) \subset G$. Then $d(G) = \log \alpha$, while $d(H) \geqslant \alpha$ is easily established: If $D \subset H$ with $|D| < \alpha$ then with

$$A = \{\xi < \alpha \colon \text{some } x \in D \text{ has } x_\xi = -1\}$$

we have $|A| \leqslant \omega \cdot |D| < \alpha$ and hence there is $\eta \in \alpha \backslash A$; the set $\pi_\eta^{-1}(\{-1\}) \cap H$ is a non-empty, open subset of H disjoint from D, so D is not dense in H. Thus to ensure $d(G) < d(H)$ it is enough to choose α so that $\omega \leqslant \log \alpha < \alpha$. For a specific example with $\beta(H)$ separable and H not separable (as in COMFORT [1963]), take $\alpha = 2^\omega$.

7.11. Let us note, finally, the existence of a substantial body of literature concerning the so-called totally dense subgroups of a given topological group G, i.e., those subgroups H of G such that $H \cap K$ is dense in K for all closed subgroups K of G. For example, SOUNDARARAJAN [1971a] has shown (i) every compact topological group G with a closed normal subgroup H such that G/H has a proper, totally dense subgroup must itself have a proper, totally dense subgroup, but the corresponding statement when G is assumed to be (only) locally compact fails; (ii) a locally compact group G has a totally dense cyclic subgroup if and only if $G = \mathbb{Z}$ or, for some prime p, G is the compact group of p-adic integers; using a result of H. WILCOX [1966] to the effect that the set of metric elements in a compact, Abelian group is a dense, pseudocompact subgroup, KHAN [1984] has shown that a locally compact, Abelian group G contains a proper totally dense subgroup if and only if every element of G is compact but not every element of G has finite order; and GRANT [1979] has shown, denoting by $t\mathbb{T}$ the torsion subgroup of the circle group $\mathbb{T}$, that $(t\mathbb{T})^\alpha$ is totally dense in $\mathbb{T}^\alpha$ for all $\alpha \geqslant \omega$. Using Grant's result and other ideas COMFORT and SOUNDARARAJAN [1982] have shown (with H a totally dense subgroup of a compact topological group G) that (i) $|G| \leqslant 2^{|H|}$; (ii) if G is Abelian then $|G| \leqslant |H|^\omega$; (iii) if G is connected Abelian then $|G| = |H|$; and (iv) if G is totally disconnected and H countably compact, then $G = H$, COMFORT and

ROBERTSON [1981], [1984] have shown that it is undecidable in ZFC whether there exists a compact, Abelian topological group G with a totally dense, pseudocompact subgroup H such that $|H| < |G|$.

8. Products of topological groups

Of course, the product of topological groups is a topological group.

It is common for topologists to inquire, given topological properties P and Q (generally with $P \subset Q$), whether every product of (finitely many, or fewer than α, or arbitrarily many) spaces with P has Q. In this section we discuss a few instances of the corresponding question in the context of topological groups. Here the literature appears to be not coherently developed or fully seasoned, and we are able to consider, using *ad hoc* methods, only a handful of properties.

8.1. THEOREM. *Let $\{G_n: n < \omega\}$ be a countable set of topological groups and let $G = \Pi_{n<\omega} G_n$.*

(i) *If each G_n is locally compact, then G is paracompact*; *and*

(ii) *if each G_n is σ-compact, then G is a Lindelöf space.*

PROOF. (i) Each G_n is paracompact by Theorem 2.9, and a G_δ (indeed, an open set) in $\beta(G_n)$. It is a theorem of FROLÍK [1960b] that a space X is paracompact and a G_δ in $\beta(X)$ if and only if there is a perfect function from X onto some complete metric space. In the present case then it is enough to choose perfect surjections $f_n: G_n \to M_n$ with M_n complete metric and to define $f: G \to \Pi_{n<\omega} M_n$ by $f(x)_n = f_n(x_n)$.

(ii) Here we ignore the algebraic properties of the spaces G_n and we show directly that every product $X = \Pi_{n<\omega} X_n$ of countably many σ-compact spaces X_n is a Lindelöf space. This theorem and its proof are due to HAGER [1969].

We show first, with HENRIKSEN, ISBELL and JOHNSON [1961] (Lemma 2.2), that if S and Y are topological spaces with $S \subset Y$ and if there is a countable family $\mathscr{A}$ of compact subsets of Y such that $S \subset \bigcup \mathscr{A}$ and

$$\bigcap\{A \in \mathscr{A}: x \in A\} \subset S \quad \text{for all } x \in S,$$

then S is a Lindelöf space. We suppose without loss of generality that $\bigcap \mathscr{F} \in \mathscr{A}$ whenever $\mathscr{F} \subset \mathscr{A}$, $|\mathscr{F}| < \omega$. Let $\mathscr{U}$ be a cover of S by (relatively) open subsets of S and let $\mathscr{V}$ be a family of open subsets of Y such that $\mathscr{U} = \{V \cap S: V \in \mathscr{V}\}$. For $x \in S$ the family

$$\{A \in \mathscr{A}: x \in A\} \cup \{Y \backslash \bigcup \mathscr{V}\}$$

does not have the finite intersection property and hence for every $x \in S$ there is

$A_x \in \mathscr{A}$ such that $x \in A_x \subset \bigcup \mathscr{V}$. Since $\{A_x : x \in S\}$ is a countable family and every A_x is compact, there is $\{V_n : n < \omega\} \subset \mathscr{V}$ such that

$$\bigcup_{x \in S} A_x \subset \bigcup_{n<\omega} V_n ;$$

it is clear that $\{V_n \cap S : n < \omega\}$ is a countable cover of S (by elements of $\mathscr{U}$), as required.

In the present case we take $X = S$ and $Y = \Pi_{n<\omega}\beta(X_n)$ and, writing $X_n = \bigcup_{m<\omega} X_{n,m}$ with each $X_{n,m}$ compact, we take for $\mathscr{A}$ the family of all sets which are finite intersections of sets of the form $\pi_n^{-1}(X_{n,m})$ with $\pi_n : Y \to \beta(X_n)$. The hypotheses being satisfied, it follows that X is a Lindelöf space.

A subset A of a space X is said to be a *Baire set* of X if A is an element of the σ-algebra generated by the zero-sets of X. It has been shown by a number of authors, including Frolík, M. Sion, Negrepontis and E.R. Lorch, that a Baire set in a compact space is a Lindelöf space. (This and related results, together with bibliographical citations, are treated in some detail by COMFORT and NEGREPONTIS [1975] (§9).) Thus in the same vein as the content of Theorem 8.1 we have the following result: If $\{G_n : n < \omega\}$ is a countable set of topological groups (indeed, topological spaces) with each G_n a Baire set in $\beta(G_n)$, then $\Pi_{n<\omega} G_n$ is a Lindelöf space.

There are other classes $\mathscr{C}$ of spaces well-established in the literature for which simultaneously (a) every σ-compact space belongs to $\mathscr{C}$ and (b) every countable product of elements of $\mathscr{C}$ is a Lindelöf space. The interested reader may look into the Lindelöf Σ-spaces (that is, the $\Sigma(\omega)$-spaces) of NAGAMI [1969] and the so-called totally Lindelöf spaces of VAUGHAN [1978] (§6); the former of these classes is countably productive and the latter is not.

8.2. We have just seen that under suitable conditions the product of countably many topological groups is a Lindelöf space. Let us note now that, other hypotheses remaining unchanged, the countability requirement cannot be relaxed. The following well-known result is due to STONE [1948], the proof to ROSS and STONE [1964].

THEOREM. *For $\alpha > \omega$, the space $\mathbb{Z}^\alpha$ is not a normal space.*

PROOF. For $i = 0, 1$ we set

$$F_i = \{x \in \mathbb{Z}^\alpha : n \in \mathbb{Z},\, n \neq i \text{ implies } |\{\xi < \alpha : x_\xi = n\}| \leq 1\} .$$

If $\mathbb{Z}^\alpha$ is normal, then since the sets F_i are closed and disjoint there is $f \in C(\mathbb{Z}^\alpha)$ such that $f(x) = i$ for all $x \in F_i$. From Theorem 1.40 there is a (faithfully indexed) countable subset $A = \{\xi_n : n < \omega\}$ of α such that if $p, q \in \mathbb{Z}^\alpha$ and $p_A = q_A$, then $f(p) = f(q)$. We define p, q by the rules

$$p_\xi = q_\xi = n \quad \text{if } \xi = \xi_n \in A\,,$$

$$p_\xi = 0 \quad \text{if } \xi \in \alpha \backslash A\,, \qquad \text{and} \qquad q_\xi = 1 \quad \text{if } \xi \in \alpha \backslash A\,.$$

Since $p \in F_0$ and $q \in F_1$ and $p_A = q_A$, we have $0 = f(p) = f(q) = 1$, a contradiction.

(Parenthetical remark: The statement that the closed sets F_0, F_1 are not contained in disjoint open subsets of $\mathbb{Z}^\alpha$ has been strengthened by POL and PUZIO–POL [1976] to the statement that they are not contained in disjoint G_δ subsets of $\mathbb{Z}^\alpha$.)

From the result just proved and the fact that a closed subspace of a normal space is normal, it follows that if a product $\Pi_{i \in I} X_i$ of non-empty spaces is normal, then all but at most countably many of the spaces X_i are countably compact. We note in passing that NOBLE [1967] has strengthened the conclusion of this statement: there is countable $A \subset I$ such that $\Pi_{i \in I \backslash A} X_i$ is countably compact.

8.3. We gave in 2.6 above an alternative proof of the following immediate consequence of 8.2.

COROLLARY. *Not every topological group is a normal space.*

8.4. We see from 8.2 that the product of uncountably many Lindelöf topological groups need not be a Lindelöf space. It is a more delicate enterprise to show that the class of Lindelöf topological groups is not closed under passage to products of finitely many of its elements. The following example is due to VAN DOUWEN [1983].

THEOREM. *There exist Lindelöf topological groups G and H such that $G \times H$ is not a Lindelöf space.*

PROOF. We need first the following statement, due to PRZYMUSIŃSKI [1980] (especially 4.6).

There is $A \subset \mathbb{R}$ with $|A| = |\mathbb{R} \backslash A| = \mathfrak{c}$ such that if $A^n \subset U \subset \mathbb{R}^n$ with $n < \omega$ and U open in $\mathbb{R}^n$, then there is countable $C \subset \mathbb{R}$ such that

$$\mathbb{R}^n \backslash U \subset \{x \in \mathbb{R}^n : x_i \in C \text{ for some } i\}\,;$$

furthermore if the space X is (defined to be) the set $\mathbb{R}$ with the topology defined by the requirement that a subset S of $\mathbb{R}$ is open in X if and only if S has the form $S = T \cup B$ with T open in $\mathbb{R}$ and $B \subset \mathbb{R} \backslash A$, then X^n is a Lindelöf space for all $n < \omega$.

Let X be so defined, and set $Y = \mathbb{R}\backslash A$ in the topology inherited from $\mathbb{R}$. Then Y^n is a Lindelöf space for all $n < \omega$; but $X \times Y$, since it contains $\{\langle y, y\rangle : y \in Y\}$ as a closed, discrete subspace of cardinality $\mathfrak{c}$, is not a Lindelöf space.

Now let $A(X)$ be the free Abelian topological group on X, defined and topologized as in 2.3 above and with the group operation denoted by the symbol $+$, and define $G = A(X)$. For $1 \leqslant n < \omega$ and $f: \{k : 0 \leqslant k < n\} \to \{+, -\}$ set

$$G(n, f) = \{\Sigma f(k)x_k : 0 \leqslant k < n, x_k \in X\}.$$

Each set $G(n, f)$ is a continuous image of the Lindelöf space X^n and is therefore itself a Lindelöf space; hence G, the union of the (countably many) sets $G(n, f)$, is itself a Lindelöf space.

Similarly embed $Y \subset A(Y)$, define $H = A(Y)$, and note that H is a Lindelöf topological group.

The spaces X and Y are closed in G and H respectively; hence $X \times Y$ is closed in $G \times H$. Since $X \times Y$ is not a Lindelöf space, $G \times H$ is not a Lindelöf space. The proof is complete.

Van Douwen has noted that the construction above can be given without recourse to the free group construction of Markov *et al.* Indeed it is known that for a Lindelöf space X the canonical embedding of TYCHONOFF [1929] and ČECH [1937]: $X \to \mathbb{R}^{C(X)}$ given by $x \to \langle f(x) : f \in C(X)\rangle$ is a homeomorphism of X onto a closed subspace of $\mathbb{R}^{C(X)}$; this statement, due in its essentials to HEWITT [1948], is explored at some length by GILLMAN and JERISON [1960] (Chapter 11). Thus in the foregoing construction the groups $A(X)$ and $A(Y)$ may be replaced by $\mathbb{R}^{C(X)}$ and $\mathbb{R}^{C(Y)}$, respectively.

8.5. Van Douwen has suggested the following question, which he attributes to Arhangel'skiĭ: Is there a Lindelöf topological group G such that $G \times G$ is not a Lindelöf space? He suggests or relays an even more striking question: Can the Sorgenfrey line be embedded into some Lindelöf topological group as a closed subspace?

8.6. We have referred above in 6.2 to the fact that (independently) NOVÁK [1953] and TERASKA [1952] have constructed a pseudocompact space X such that $X \times X$ is not pseudocompact; indeed, X may be taken countably compact. We show now that no such examples exist in the class of topological groups.

THEOREM. *Let $\{G_i : i \in I\}$ be a set of pseudocompact topological groups and let $G = \Pi_{i \in I} G_i$. Then G is pseudocompact.*

PROOF. Set $K = \Pi_{i \in I} \bar{G}_i$, where $\bar{G}_i$ denotes as usual the Weil completion of G_i. Since K is a compact topological group in which G is dense, we have $K = \bar{\bar{G}}$; that is, K is the Weil completion of G. It is not difficult to see that G is G_δ-dense in K: every non-empty G_δ of K contains a non-empty set A of the form $\bigcap_{n<\omega}(\bigcap_{i\in I}\pi_i^{-1}(U_{i,n}))$ with $U_{i,n}$ open in X_i, and choosing $x_i \in \bigcap_{n<\omega} U_{i,n}$ defines a point $x = \langle x_i : i \in I\rangle \in A$. That G is pseudocompact now follows from Theorem 6.4.

The theorem just given is due to COMFORT and ROSS [1966]. The present rather efficient treatment has profited from the simplifications in the proof of Theorem 6.4 by DE VRIES [1975] (see 6.6 above).

8.7. It is natural to ask, in view of Theorem 8.6, whether the product of every set of countably compact topological groups is countably compact. It was not until 1979 that VAN DOUWEN [1980a] announced a counterexample. His construction is pleasing in that only two groups are required, but annoying in that it depends irretrievably upon Martin's Axiom (MA). At the present writing it remains unknown whether in ZFC one can define two (or more) countably compact topological groups whose product is not countably compact.

THEOREM (VAN DOUWEN [1980a]). *Assume* MA. *There are countably compact topological groups G and H, both subgroups of the compact group $\{-1, +1\}^c$, such that $|G \cap H| = \omega$. Hence $G \times H$, since it contains the 'diagonal'*

$$\Delta = \{\langle x, y\rangle \in G \times H : x = y\} = \{\langle x, x\rangle : x \in G \cap H\}$$

as a countably infinite, closed subgroup, is not countably compact.

I shall not duplicate here the proof given by VAN DOUWEN [1980a]. It proceeds in two steps: first, using MA, the construction of a countably compact subgroup S of $\{-1, +1\}^c$ such that every convergent sequence of S is eventually constant; second, given S, the construction in ZFC of subgroups G and H of S with the required properties. It is shown that S may be chosen dense or non-dense in $\{-1, +1\}^c$, and (in each case) separable or non-separable. (An addendum to the paper asserts the existence in ZFC + MA of a countably compact group whose square is not countably compact.)

Furthermore, assuming MA and $2^\omega = 2^{(\omega^+)}$ (a consequence of (MA) + (— CH)), the group S and subsequently G and H may be chosen to be not only initially ω-compact, i.e., countably compact, but even initially ω^+-compact. (As usual, for $\alpha \geq \omega$ a space is said to be initially α-compact if every open cover of cardinality $\leq\alpha$ has a finite subcover.) Let us explore briefly with van Douwen the implications of this concerning the necessity of MA in his construction.

First, let KA (for Kunen's Axiom) be the statement that some free ultrafilter on ω is generated by ω^+ sets, that is, there is $p \in \omega^*$ such that $\chi(p, \omega^*) = \omega^+$. (That $\chi(p, \omega^*) \leq \alpha$ is equivalent to the set-theoretic statement that there is $\mathscr{B} \subset p$

with $|\mathscr{B}| \leq \alpha$ such that for every $A \in p$ some $B \in \mathscr{B}$ satisfies $|B \backslash A| < \omega$.) Although POSPÍŠIL [1939] and JUHÁSZ [1967], [1969] have shown in ZFC that

$$|\{p \in \omega^*: \chi(p, \omega^*) = \mathfrak{c}\}| = 2^{\mathfrak{c}},$$

it is known (cf. KUNEN [1972] (p. 303)) that if ZFC is consistent, then so is the system (ZFC) + (KA) + ($2^\omega = 2^{(\omega+)} = \omega^{++}$).

Next we observe that if $p \in \omega^*$ and $\chi(p, \omega^*) \leq \alpha$, then every initially α-compact topological space X is p-compact. Indeed $\chi(p, \beta(\omega)) \leq \alpha$, so there is a local base $\mathscr{B}$ for p in $\beta(\omega)$ such that $|\mathscr{B}| \leq \alpha$, and for every $f: \omega \to X$ the Stone extension $\bar{f}: \beta(\omega) \to \beta(X)$ satisfies

$$\{\bar{f}(p)\} = \bigcap_{B \in \mathscr{B}} \mathrm{cl}_{\beta(X)} \bar{f}[B].$$

But $\bigcap_{B \in \mathscr{B}} \mathrm{cl}_X \bar{f}[B] \neq \emptyset$ (since $|\mathscr{B}| \leq \alpha$ and X is initially α-compact), and hence $\bar{f}(p) \in X$ as required.

From the preceding paragraph it follows that in ZFC + KA the product of every set of initially ω^+-compact spaces is countably compact. For it is easy to prove (see the original paper of A. BERNSTEIN [1970], or the article by Vaughan in this Handbook for a non-non-standard proof) that for every $p \in \omega^*$ the product of p-compact spaces is p-compact and hence countably compact.

Now we argue as follows. Suppose that the construction of two initially ω^+-compact topological groups G and H whose product is not countably compact can be carried out without MA—in ZFC alone, say, or in (ZFC) + ($2^\omega = 2^{(\omega^+)}$). The argument then remains valid in the system (ZFC) + (KA) + ($2^\omega = 2^{(\omega^+)}$). Here, however, the groups G and H are both p-compact (with p as given by KA), and hence $G \times H$ is countably compact.

Thus we see that MA, though perhaps not essential for van Douwen's theorem, is essential for his proof.

8.8. There is an example in ZFC showing that not every countably compact topological group is initially ω^+-compact: the Σ-product $G = \Sigma(e) \subset \{-1, +1\}^{(\omega^+)}$ of CORSON [1959], defined and discussed above in 6.8ff., is countably compact, but the family $\{\pi_\xi^{-1}(\{+1\}) \cap G: \xi < \omega^+\}$ is an open cover of G of cardinality ω^+ which does not even have a countable subcover.

8.9. We repeat for emphasis that it is not known whether it can be shown in ZFC that there is a set of countably compact topological groups whose product is not countably compact. The following result, which has been known for some years, reduces the construction of such a family (albeit, a large one) to the question whether for all $p \in \omega^*$ there is a countably compact topological group that is not p-compact.

THEOREM. *The following statements are equivalent.*

(i) *There is $p \in \omega^*$ such that every countably compact topological group is p-compact;*

(ii) *every product of countably compact topological groups is countably compact.*

PROOF. This is immediate from results recorded in this Handbook by Vaughan (Theorems 4.7 and 4.11). Every product of p-compact spaces is p-compact and hence countably compact; this shows (i)⇒(ii). Suppose for the coverse that (i) fails, for $p \in \omega^*$ let $G(p)$ be a countably compact topological group that is not p-compact, and set $G = \Pi_{p\in\omega^*} G(p)$. It can be seen directly that G is not countably compact: If for $p \in \omega^*$ the function $f_p\colon \omega \to G(p)$ has a Stone extension $\bar{f}_p\colon \beta(\omega) \to \beta(G(p))$ such that $\bar{f}_p(p) \notin G(p)$, then the function $f\colon \omega \to G$ defined by $f(n)_p = f_p(n)$ satisfies $\bar{f}[\omega^*] \cap G = \emptyset$ (see the last paragraph of 7.5 above). In any event there is no $p \in \omega^*$ such that G is p-compact and therefore $G^{2^{\mathfrak{c}}}$, which is $\Pi_{p\in\omega^*}(G(p))^{2^{\mathfrak{c}}}$, is not countably compact.

9. Topologizing a group

This section may be viewed as a selective survey of the literature responding to a question apparently first posed by MARKOV [1945]; Does every infinite group admit a non-discrete (Hausdorff) topological group topology? Abelian groups do (9.2), but for $\alpha \geqslant \omega$ there is a non-Abelian group G with $|G| = \alpha$ and with no such topology (9.5, 9.6). The availability of such counterexamples lends interest to the project of finding large classes of groups whose members do admit non-discrete topological group topologies. We show here that free groups (9.21), and large groups of permutations (9.24(ii)) (including the full group of all permutations on an infinite set), do admit such topologies.

Various authors have refined Markov's question: Can a given infinite group G be given a non-discrete topological group topology which is (i) metrizable? (ii) totally bounded? Again for G Abelian the answer in each case is "Yes" (9.3 and 9.2), but there are in ZFC non-Abelian groups with no totally bounded topological group topology (9.12).

Every group G with a totally bounded topological group topology has a largest such topology (9.13(ii)). There are infinite groups G for which this largest topology is compact (9.16–9.18), but no such group can be Abelian (9.13(iv)).

9.1. THEOREM. *Let G be an Abelian group and let $e \neq a \in G$. Then there is $\chi \in \operatorname{Hom}(G, \mathbb{T})$ with $\chi(a) \neq 1$.*

PROOF. Let A be the subgroup of G generated by a. If $|A| = n < \omega$ define

$$\chi(a^k) = \exp(2\pi i k/n) \quad \text{for } 1 \leqslant k \leqslant n$$

and if $|A| = \omega$ choose $t \in \mathbb{T}$ with $t \neq 1$ and set

$$\chi(a^k) = t^k \quad \text{for } k \in \mathbb{Z}.$$

We have $\chi \in \text{Hom}(A, \mathbb{T})$. A routine argument using Zorn's lemma shows that there is a pair $\langle H, \Psi \rangle$, with H a subgroup of G and $\Psi \in \text{Hom}(H, \mathbb{T})$, maximal with respect to the property $A \subset H$ and $\Psi | A = \chi$. Thus to complete the present proof it is enough to show $H = G$. Suppose, that there is $x \in G \backslash H$. If there is an integer $n > 1$ with $h = x^n \in H$ we choose the least such n and then $t \in \mathbb{T}$ such that $t^n = \Psi(h)$, and if there is no such n we choose arbitrary $t \in \mathbb{T}$; and we define $\Psi'(hx^k) = \Psi(h) \cdot t^k$ for $h \in H$, $k \in \mathbb{Z}$. It is easy to see that Ψ' is a (well-defined) element of $\text{Hom}([H \cup \{x\}], \mathbb{T})$ with $\Psi' | A = \Psi | A = \chi$, so the pair $([H \cup \{x\}], \Psi')$ contradicts the maximality of (H, Ψ).

9.2. THEOREM. *Every infinite Abelian group G admits a non-discrete totally bounded topological group topology.*

PROOF. According to 9.1, the group $\text{Hom}(G, \mathbb{T})$ separates points of G. It then follows from Theorem 4.2 that the topology $\mathcal{T}$ induced on G by $\text{Hom}(G, \mathbb{T})$ is a totally bounded topological group topology. Since a discrete topological group is totally bounded if and only if it is finite, the group $\langle G, \mathcal{T} \rangle$ is non-discrete.

9.3. THEOREM. *For every group G with infinite center there is a faithfully indexed family $\{\mathcal{T}_\xi : \xi < 2^\omega\}$ such that each $\langle G, \mathcal{T}_\xi \rangle$ is a non-discrete dense subgroup of a locally compact metrizable topological group.*

PROOF. Let H be a countably infinite subgroup of the center of G and $\mathcal{H}$ a countable, point-separating subgroup of $\text{Hom}(H, \mathbb{T})$. Since $|\text{Hom}(H, \mathbb{T})| = 2^\omega$ (4.22), there is a faithfully indexed family $\{\mathcal{H}_\xi : \xi < 2^\omega\}$ of (sub-) groups such that $\mathcal{H} \subset \mathcal{H}_\xi \subset \text{Hom}(H, \mathbb{T})$ and $|\mathcal{H}_\xi| = \omega$. (*Proof.* For $A \subset \text{Hom}(H, \mathbb{T})$ let $[A]$ denote as usual the subgroup generated by A, and for $\chi, \psi \in \text{Hom}(H, \mathbb{T})$ write $\chi \sim \psi$ if $[H \cup \{\chi\}] = [H \cup \{\psi\}]$. It is clear for each χ that $|\{\psi : \chi \sim \psi\}| = \omega$; so there is $\{\chi_\xi : \xi < 2^\omega\} \subset \text{Hom}(H, \mathbb{T})$ such that $\chi_\xi \not\sim \chi_\eta$ for $\xi < \eta < 2^\omega$. Set $\mathcal{H}_\xi = [\mathcal{H} \cup \{\chi_\xi\}]$.)

Let $\mathcal{U}_\xi$ be the topology induced by $\mathcal{H}_\xi$ on H. Then $\langle H, \mathcal{U}_\xi \rangle$ is totally bounded topological group (4.2), and $\mathcal{U}_\xi \neq \mathcal{U}_\eta$ when $\xi < \eta < 2^\omega$ (4.3). Further $\langle H, \mathcal{U}_\xi \rangle$ is first countable, since

$$\{\{x \in H : |\chi(x) - 1| < 1/n\} : \chi \in \mathcal{H}_\xi, 0 < n < \omega\}$$

is a local base for $\mathcal{U}_\xi$ at e. Now define

$$\mathcal{T}_\xi = \{xU : x \in G, U \in \mathcal{U}_\xi\}.$$

It is easy to see (and it follows from 1.11 above) that $\langle G, \mathcal{T}_\xi \rangle$ is a topological group. Since H with the topology $\mathcal{U}_\xi$ is an open-and-closed subgroup of $\langle G, \mathcal{T}_\xi \rangle$ and is non-discrete and first countable, $\langle G, \mathcal{T}_\xi \rangle$ itself is non-discrete and first countable. Since $\langle H, \mathcal{U}_\xi \rangle$ is totally bounded and is open in $\langle G, \mathcal{T}_\xi \rangle$, the latter is locally bounded. Thus the Weil completion (1.13) of $\langle G, \mathcal{T}_\xi \rangle$ is locally compact. Like any group with a dense, first countable subgroup, the Weil completion of $\langle G, \mathcal{T}_\xi \rangle$ is first countable (2.7(ii)), hence metrizable (1.8(i)).

9.4. Theorem 9.1, an easy algebraic exercise, has been noted and exploited by KAKUTANI [1943] in connection with computations of cardinal numbers associated with compact Abelian topological groups (see also 4.22 above). Its use in 4.2 and 9.2 follows COMFORT and ROSS [1964]. It was KERTÉSZ and SZELE [1953] who first showed that every infinite Abelian group admits a non-discrete (and metrizable) topological group topology. Theorem 9.3 reflects generalizations by COMFORT and ROSS [1964] and SHARMA [1981]. For a proof that every infinite Abelian group G admits $2^{2^{|G|}}$ totally bounded topological group topologies, and for related results, see REMUS [1983] and BERHANU, COMFORT and REID [1984].

9.5. In 1976 Shelah achieved in ZFC without additional axioms a solution (in the negative, by an example) to the following difficult problem, posed years ago by A.G. KUROSH and B. JÓNSSON and others: Does every group G such that $|G| = \omega^+$ contain a proper subgroup H such that $|H| = \omega^+$? In connection with his construction Shelah was able, assuming the continuum hypothesis, to prove the following result.

THEOREM. *Assume* CH. *There is a group* $\tilde{G}$ *with* $|\tilde{G}| = \omega^+$ *such that the only topological group topology for* $\tilde{G}$ *is the discrete topology.*

Proofs of these two results are given in SHELAH [1980]. Markov's question (cited in the first sentence of this section) was left open for countable groups; indeed it is remarked in SHELAH [1980] (page 375) that every countable subgroup H of $\tilde{G}$ does admit a non-discrete topological group topology.

9.6. The definitive solution to Markov's problem—that is, the proof in ZFC of the existence of a countable group with no non-discrete topological group topology—is located by A. Ju. Ol'šanskiĭ in groups $A(m, n)$ of S.I. ADIAN [1980] (§13.4) related to the Burnside groups $B(m, n)$. (The groups $B(m, n)$ are freely generated by m generators, subject to the relation $x^n = e$. That $B(m, n)$ is infinite for $m > 1$ and for n sufficiently large and odd was proved by NOVIKOV and ADIAN [1968]. The groups $A(m, n)$ which solve Markov's problem have the property that there is $s \in A(m, n)$ such that the subgroup $\langle s \rangle$ of $A(m, n)$ generated by s is the center of $A(m, n)$, and $A(m, n)/\langle s \rangle = B(m, n)$.)

We show in 9.12 that there are groups which admit no totally bounded topological group topology.

If $G = \langle G, \mathcal{T} \rangle$ is a totally bounded topological group, then the inclusion function from G to its Weil completion is a point-separating continuous homomorphism (indeed, an isomorphism) onto a compact topological group. Conversely it is easy to check, just as in the Abelian case treated in 4.2, that if G is a group and for $e \neq a \in G$ there are a compact group K_a and $h_a \in \text{Hom}(G, K_a)$ with $h_a(a) \neq e$, then the function $i: G \to \Pi_{a \in G} K_a$ defined by $i(x)_a = h_a(x)$ is an isomorphism from G into a compact group; the topology $\mathcal{T}$ for G defined by the requirement that i be a homeomorphism onto $i[G] \subset \Pi_{a \in G} K_a$ makes $\langle G, \mathcal{T} \rangle$ a totally bounded topological group. Thus we see that a group admits a totally bounded topological group topology if and only if its points are distinguished by homomorphisms into compact groups. Similarly, a topological group $G = \langle G, \mathcal{T} \rangle$ admits a totally bounded topological group topology $\mathcal{T}'$ such that $\mathcal{T}' \subset \mathcal{T}$ if and only if the points of G are distinguished by $\mathcal{T}$-continuous homomorphisms into compact groups. A group $G = \langle G, \mathcal{T} \rangle$ with this property is maximally almost periodic in the sense of VON NEUMANN [1934]; at the other extreme are von Neumann's minimally almost periodic groups, that is, those topological groups G for which every continuous $h \in \text{Hom}(G, K)$ with K a compact topological group is the trivial homomorphism. (As the terminology suggests, von Neumann's maximally and minimally almost periodic groups are defined in terms of almost periodic functions rather than in terms of homomorphisms into compact groups. For proofs of the equivalences indicated but not proved above, the interested reader might consult VON NEUMANN [1934].)

According to the preceding paragraph, finding a group with no totally bounded topological group topology is equivalent to finding a group that is not maximally almost periodic. In 9.11 we do more: we find an infinite minimally almost periodic group. The existence of such a group is noted by VON NEUMANN [1934] (§18). In 9.9–9.12 we follow VON NEUMANN and WIGNER [1940].

9.8. For an integer $n > 0$ and $A = (a_{ij})$ a real or complex $n \times n$ matrix, we denote by A', $\bar{A}$ and A^* the transpose, conjugate, and adjoint respectively of A. These are defined as usual: A' is that matrix $B = (b_{ij})$ such that $b_{ij} = a_{ji}$; $\bar{A}$ is the matrix $C = (c_{ij})$ such that $c_{ij} = \overline{a_{ij}}$ (the complex conjugate); and $A^* = \overline{A'} = \bar{A}'$.

We use the usual inner product $\langle\, , \rangle$ on $\mathbb{C}^n$ given by $\langle u, v \rangle = \Sigma_{i=1}^n u_i \bar{v}_i$. We use also the fact that $\langle A^*(v), w \rangle = \langle v, A(w) \rangle$ for all $n \times n$ matrices A and $v, w \in \mathbb{C}^n$; in particular $\langle A'A(v), v \rangle = \langle A(v), A(v) \rangle$ when $v \in \mathbb{R}^n$ and A is a real matrix.

We denote by $E(n)$, or simply by E when confusion is impossible, that $n \times n$ matrix $E(n) = (e_{ij})$ such that

$$\begin{aligned} e_{ij} &= 1 \quad \text{if } i = j, \\ &= 0 \quad \text{if } i \neq j. \end{aligned}$$

Now let us recall certain standard terminology and notation.

DEFINITION. Let $0 < n < \omega$.

(i) The complex general linear group $\mathrm{GL}(n, \mathbb{C})$ is the set of $n \times n$ complex matrices A such that $\det A \neq 0$; the group operation is the usual multiplication of matrices.

(ii) The complex special linear group $\mathrm{SL}(n, \mathbb{C})$ is the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of those matrices A in $\mathrm{GL}(n, \mathbb{C})$ such that $\det A = 1$.

(iii) The complex unitary group $\mathrm{U}(n, \mathbb{C})$ is the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of those matrices A in $\mathrm{GL}(n, \mathbb{C})$ such that $A^{-1} = A^*$.

(iv) The complex orthogonal group $\mathrm{O}(n, \mathbb{C})$ is the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of those matrices A in $\mathrm{GL}(n, \mathbb{C})$ such that $A' = A^{-1}$.

(v) The complex special orthogonal group $\mathrm{SO}(n, \mathbb{C})$ is the subgroup $\mathrm{SL}(n, \mathbb{C}) \cap \mathrm{O}(n, \mathbb{C})$.

The real general linear, special linear, unitary, orthogonal and special orthogonal groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{U}(n, \mathbb{R})$, $\mathrm{O}(n, \mathbb{R})$ and $\mathrm{SO}(n, \mathbb{R})$ are defined just as in (i), (ii), (iii), (iv) and (v) but with a_{ij} required to be real.

We note that $\mathrm{U}(n, \mathbb{R}) = \mathrm{O}(n, \mathbb{R})$ and $\mathrm{SU}(n, \mathbb{R}) = \mathrm{SO}(n, \mathbb{R})$.

The group $\mathrm{GL}(n, \mathbb{C})$ is naturally embedded (topologically, not algebraically) into the n^2-dimensional Euclidean space $\mathbb{C}^{n^2}$. Since the functions $A \to \det A$, $A \to A'$ and $A \to \bar{A}$ are continuous on $\mathrm{GL}(n, \mathbb{C})$, it is clear that $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{U}(n, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ are respectively open, closed, closed, closed and closed in $\mathbb{C}^{n^2}$; hence each group is locally compact. Further $\mathrm{U}(n, \mathbb{C})$ is bounded in $\mathbb{C}^{n^2}$: for $A = (a_{ij}) \in \mathrm{U}(n, \mathbb{C})$ we have $\Sigma_{k=1}^{n} a_{ik}\overline{a_{ik}} = 1$ for $1 \leqslant i \leqslant n$ and hence each a_{ik} satisfies $|a_{ik}| \leqslant 1$. Thus $\mathrm{U}(n, \mathbb{C})$ is a compact topological group.

Similar considerations show that the groups $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{U}(n, \mathbb{R}) = \mathrm{O}(n, \mathbb{R})$ and $\mathrm{SU}(n, \mathbb{R}) = \mathrm{SO}(n, \mathbb{R})$ are locally compact; here $\mathrm{O}(n, \mathbb{R})$, and hence its closed subgroup $\mathrm{SO}(n, \mathbb{R})$, is compact.

(We note parenthetically that according to the conventions adopted above the groups $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ are not compact for $1 < n < \omega$. The group $\mathrm{SO}(2, \mathbb{C})$ contains a closed subgroup topologically isomorphic to the multiplicative group of non-zero real numbers. Indeed $\mathrm{SO}(2, \mathbb{C})$, an Abelian group, is topologically isomorphic to $\mathrm{GL}(1, \mathbb{C})$, the multiplicative group of non-zero complex numbers.)

9.9. LEMMA. *Let n be a positive integer and $A \in \mathrm{GL}(n, \mathbb{C})$ and suppose for all $k < \omega$ there are $B_k \in \mathrm{GL}(n, \mathbb{C})$ and an integer $m(k) > 0$ such that k divides $m(k)$ and $A^{m(k)} = B_k A B_k^{-1}$. Then all eigenvalues of A are equal to* 1.

PROOF. Let $\{\lambda_i : 1 \leqslant i \leqslant n\}$ be the eigenvalues of A, each listed with its correct multiplicity. It is easy to see that the eigenvalues of $A^{m(k)}$ are $\{\lambda_i^{m(k)} : 1 \leqslant i \leqslant n\}$ and the eigenvalues of $B_k A B_k^{-1}$ are $\{\lambda_i : 1 \leqslant i \leqslant n\}$. That is, as sets we have

$$(*) \qquad \{\lambda_i : 1 \leqslant i \leqslant n\} = \{\lambda_i^{m(k)} : 1 \leqslant i \leqslant n\} .$$

For fixed j with $1 \leqslant j \leqslant n$ we have $\lambda_j^{m(k)} \in \{\lambda_i : 1 \leqslant i \leqslant n\}$; since $m(k) \geqslant k$ there are distinct integers m_1, m_2 such that $\lambda_j^{m_1} = \lambda_j^{m_2}$. It follows for each j that λ_j is a root of unity; hence there is a positive integer k such that $\lambda_i^k = 1$ for all i $(1 \leqslant i \leqslant n)$. For this k we have $\lambda_i^{m(k)} = 1$ for all i, and relation $(*)$ gives the desired conclusion.

9.10. COROLLARY. *Let n be a positive integer and G a group, and let $h \in \mathrm{Hom}(G, \mathrm{SL}(n, \mathbb{C}))$ and $x \in G$. Suppose for all $k < \omega$ there are $y(k) \in G$ and an integer $m(k) > 0$ such that k divides $m(k)$ and $x^{m(k)} = y_k x y_k^{-1}$. Then all eigenvalues of $h(x)$ are equal to one.*

PROOF. Set $A = h(x)$, $B_k = h(y_k)$ in Lemma 9.9.

9.11. Here are the promised statement and proof that the (infinite, non-Abelian) group $\mathrm{SL}(2, \mathbb{C})$ is minimally almost periodic; indeed $\mathrm{SL}(2, \mathbb{C})_{\mathrm{d}}$—that is, the group $\mathrm{SL}(2, \mathbb{C})$ with the discrete topology—is minimally almost periodic.

THEOREM. *Let K be a compact topological group and $h \in \mathrm{Hom}(\mathrm{SL}(2, \mathbb{C}), K)$. Then $h(x) = e_K$ for all $x \in \mathrm{SL}(2, \mathbb{C})$.*

PROOF. We consider first the case $K = U(n, \mathbb{C})$ with $0 < n < \omega$. We note that if $A \in U(n, \mathbb{C})$ and every eigenvalue of A is equal to 1, then A is the identity matrix $A = E(n)$. For a unitary matrix A is normal in the sense that $AA^* = A^*A$ (indeed, $A^* = A^{-1}$), and it is well known and easy to prove (see for example GANTMACHER [1959] (IX, §10)) or MACLANE and BIRKHOFF [1979] (X, §10)) that the normal matrices in $\mathrm{GL}(n, \mathbb{C})$ are precisely those matrices A for which $\mathbb{C}^n$ has an orthonormal basis of eigenvectors. In the present case, choosing such a basis $\{b_k : 1 \leqslant k \leqslant n\} \subset \mathbb{C}^n$ and writing arbitrary $x \in \mathbb{C}^n$ in the form $x = \Sigma_{k=1}^n c_k b_k$ with $c_k \in \mathbb{C}$, we have from $A(b_k) = 1 \cdot b_k = b_k$ that $A(x) = x$ for all x; it follows that $A = E(n)$.

We set $H = \ker h = h^{-1}(\{E(n)\})$. In order to show that every $x \in \mathrm{SL}(2, \mathbb{C})$ satisfies $x \in H$ it is enough, according to the preceding paragraph and Corollary 9.10, to show that for all $x \in \mathrm{SL}(2, \mathbb{C})$ and $k < \omega$ there are $y_k \in \mathrm{SL}(2, \mathbb{C})$ and an integer $m(k) > 0$ such that k divides $m(k)$ and $x^{m(k)} = y_k x y_k^{-1}$.

For notational simplicity, for $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ we write $x = (a, b, c, d)$.

If $x = (1, b, 0, 1)$ then with $y_k = (k, 0, 0, 1/k) \in \mathrm{SL}(2, \mathbb{C})$ we have $y_k x y_k^{-1} = x^{k^2}$ and hence $x \in H$; and if $x = (1, 0, c, 1)$, then with y_k as above we have $y_k^{-1} x y_k = x^{k^2}$ and hence $x \in H$.

Finally let $x = (a, b, c, d) \in \mathrm{SL}(2, \mathbb{C})$. If $c \neq 0$, then from what has been shown we have

$$(1, (a-1)/c, 0, 1),\ (1, 0, c, 1),\ (1, (d-1)/c, 0, 1) \in H$$

and hence

$$x = (1, (a-1)/c, 0, 1) \cdot (1, 0, c, 1) \cdot (1, (d-1)/c, 0, 1) \in H\,;$$

while if $c = 0$, then $a \neq 0$ and from what has been shown we have

$$(-b, a, -d, 0),\ (0, -1, 1, 0) \in H$$

and hence

$$x = (a, b, 0, d) = (-b, a, -d, 0) \cdot (0, -1, 1, 0) \in H\,.$$

The proof is complete in the case $K = \mathrm{U}(n, \mathbb{C})$. For the general case it is enough to recall from 1.21, the Gel'fand–Raĭkov theorem, that if $h \in \mathrm{Hom}(\mathrm{SL}(2, \mathbb{C}), K)$ with K compact satisfies $h(x) \neq e_K$ for some $x \in \mathrm{SL}(2, \mathbb{C})$, then there is $n > 0$ and a (continuous) homomorphism $j\colon K \to \mathrm{U}(n, \mathbb{C})$ such that $j(h(x)) \neq E(n)$. The composition $j \circ h$ is then a non-trivial homomorphism from $\mathrm{SL}(2, \mathbb{C})$ into $\mathrm{U}(n, \mathbb{C})$, contrary to what has been proved.

9.12. COROLLARY. *The group* $\mathrm{SL}(2, \mathbb{C})$ *admits no totally bounded topological group topology.*

PROOF. According to the discussion of 9.7, this statement is equivalent to the statement of Theorem 9.11.

9.13. Here we digress briefly to prove certain statements concerning the set of totally bounded topological group topologies on a (given) group. Parts (i), (ii) and (iii) for Abelian groups are from COMFORT and ROSS [1964], while (iv), which is one motivation for 9.18, is credited by COMFORT and SAKS [1973] to Lewis C. Robertson.

THEOREM. *Let G be a group and Λ the set of topologies $\mathcal{T}$ such that $\langle G, \mathcal{T}\rangle$ is a totally bounded topological group.*

(i) *If G is Abelian, then $\Lambda \neq \emptyset$;*

(ii) *if $\Lambda \neq \emptyset$ there is $\mathcal{L} \in \Lambda$ such that every $\mathcal{T} \in \Lambda$ satisfies $\mathcal{T} \subset \mathcal{L}$;*

(iii) *if $\Lambda \neq \emptyset$, then $\mathcal{L}$ is the unique element of Λ such that every $h \in \mathrm{Hom}(\langle G, \mathcal{L}\rangle, \langle H, \mathcal{S}\rangle)$ with $\langle H, \mathcal{S}\rangle$ a totally bounded topological group is continuous*; *and*

(iv) *if G is Abelian and $|G| \geq \omega$, then $\langle G, \mathcal{L}\rangle$ is not pseudocompact.*

PROOF. (i) is immediate from 9.2.

(ii) and (iii). Let $\langle\langle H_j, \mathcal{S}_j, h_j\rangle\colon j \in I\rangle$ be an enumeration of all triples such that

(a) $\langle H_j, \mathcal{S}_j\rangle$ is a totally bounded topological group;

(b) $h_j \in \mathrm{Hom}(G, H_j)$; and

(c) the underlying set of the group H_j is a subset of G, that is, $H_j \subset G$ as sets.

(Condition (c) is a technical device designed to ensure that the indicated family of triples is a set and not a proper class. The number of subsets H of G is $\exp|G|$; for each H, the number of topologies $\mathcal{S}$ on H is at most $\exp(\exp|H|)$ and the number of homomorphisms from G to H is at most $\exp|G|$. Thus in fact for $|G| \geq \omega$ we have

$$|I| \leq \exp|G| \cdot \exp(\exp|G|) \cdot \exp|G| = \exp(\exp|G|).)$$

Now define $i: G \to \Pi_{j \in J}\langle H_j, \mathcal{S}_j\rangle$ by $i(x)_j = h_j(x)$. Like every product of totally bounded topological groups, the group $\Pi_{j \in J}\langle H_j, \mathcal{S}_j\rangle$ is totally bounded. Since $\Lambda \neq \emptyset$, the homomorphism i is a one-to-one function. Let $\mathcal{L}$ be the topology for G defined by the requirement that the isomorphism $i: G \to i[G] \subset \Pi_{j \in J}\langle H_j, \mathcal{S}_j\rangle$ is a homeomorphism.

For $\mathcal{T} \in \Lambda$ there is $j \in J$ such that

$$\langle H_j, \mathcal{S}_j, h_j\rangle = \langle G, \mathcal{T}, \pi_j \circ i\rangle$$

and $h_j(x) = x$ for all $x \in G$. For $U \in \mathcal{T}$ we have $U = (\pi_j \circ i)^{-1}(U) \in \mathcal{L}$ and hence $\mathcal{T} \subset \mathcal{L}$. The proof of (ii) is complete.

One may assume without loss of generality that for h, H and $\mathcal{S}$ as in (iii) there is $j \in J$ such that ($H_j \subset G$ and) $\langle H, \mathcal{S}, h\rangle = \langle H_j, \mathcal{S}_j, h_j\rangle$. Then for $U \in \mathcal{S}$ we have

$$h^{-1}(U) = i^{-1}(\pi_j^{-1}(U)) = (\pi_j \circ i)^{-1}(U) \in \mathcal{L},$$

so that h is continuous. That $\mathcal{L}$ has the indicated uniqueness property is clear: If $\mathcal{T} \in \Lambda$ with $\mathcal{T} \neq \mathcal{L}$ then the identity homomorphism $\langle G, \mathcal{T}\rangle \to \langle G, \mathcal{L}\rangle$ is not continuous.

(iv) We note first that every subgroup A of G is $\mathcal{L}$-closed. Indeed for $x \in G \backslash A$ there is $\chi \in \mathrm{Hom}(G, \mathbb{T}) = \langle G, \mathcal{L}\rangle\hat{}$ such that $A \subset \ker \chi$ and $\chi(x) \neq 1$; that is,

$$A = \bigcap\{\ker \chi : \chi \in \langle G, \mathcal{L}\rangle\hat{}, A \subset \ker \chi\}.$$

Suppose now that $\langle G, \mathcal{L}\rangle$ is pseudocompact and, as in 4.10, let A be a subgroup of G such that $|G/A| = \omega$. Since A is $\mathcal{L}$-closed, G/A is a (Hausdorff) topological group. Like every continuous image of a pseudocompact space, G/A is pseudocompact. In summary: G/A is a countably infinite pseudocompact topological group. This contradicts Theorem 6.13(i). (The required contradiction may be derived from Theorem 3.1(i), a familiar result, rather than from the less readily accessible 6.13(i), through the observation that G/A, being countable and pseudocompact, is Lindelöf and pseudocompact and hence compact.)

9.14. We have noted already that an algebraic property, to be Abelian, can have significant topological consequences. (One may contrast 9.2 with 9.5, and with

9.12.) Here in 9.18 we give another result indicating a powerful topological difference between the class of Abelian groups and the class of groups: The maximal totally bounded topological group topology on an infinite Abelian group is not compact (nor even pseudocompact (9.13(iv))), but on a non-Abelian group it may be compact.

The only examples of this kind I have found discussed in the literature arise in the context of Lie groups. Since that theory lies beyond the scope of this article as defined in section 1 and is in any case the subject of several reliable treatises (see below for some references), the present exposition is less detailed and complete than that which is usually offered elsewhere in this chapter.

The material of 9.15–9.16 is taken from VAN DER WAERDEN [1933]. I am grateful to Lewis C. Robertson for several informative conversations concerning van der Waerden's paper and the specialized arguments of 9.17 which serve to illuminate the case of the rotation group $SO(3, \mathbb{R})$.

With ρ a conveniently chosen compatible metric for a metrizable topological group G as in 9.15–9.17, and $\varepsilon > 0$, we set

$$B_\varepsilon(a) = \{x \in G : \rho(x, a) < \varepsilon\} \quad \text{and} \quad S_\varepsilon(a) = \{x \in G : \rho(x, a) = \varepsilon\}.$$

The notations are chosen to suggest the (open) ball, and the (closed) sphere or shell, of radius ε centered at $a \in G$.

9.15. Given a topological group H and $a \in H$, in the following lemma and theorem we denote by $M_H(a)$, or simply by $M(a)$ when confusion is impossible, the set

$$M(a) = \{cbab^{-1}a^{-1}c^{-1} : c \in H\}.$$

LEMMA. *Let K be a compact metrizable topological group and $\{a_n : n < \omega\} \subset K$ such that $a_n \to e_K$. Then for every (open) $W \in \mathcal{N}(e)$ there is n such that $M(a_n) \subset W$.*

PROOF. If the statement fails then for all $n < \omega$ there are b_n, c_n, $x_n \in K$ such that $x_n = c_n b_n a_n b_n^{-1} a_n^{-1} c_n^{-1} \in K \backslash W$. Passing to subsequences if necessary, we assume without loss of generality that there are b, $c \in K$ such that $b_n \to b$ and $c_n \to c$. From continuity we have $x_n \to cbeb^{-1}e^{-1}c^{-1} = e$, and since $K \backslash W$ is compact and each $x_n \in K \backslash W$ we have $e = \lim_n x_n \in K \backslash W$, a contradiction.

9.16. THEOREM. *Let G be a compact, connected, simple Lie group with trivial center, and let h be a homomorphism from G into any compact topological group K. Then h is continuous.*

PROOF. We may assume without loss of generality that K is metrizable. For according to the Gel'fand–Raĭkov theorem (1.21 above) there are sufficiently many continuous homomorphisms from the (arbitrary compact) group K into compact metrizable groups (indeed, into the unitary groups $U(n, \mathbb{C})$) to distinguish points; that is, there is a topological isomorphism φ from K onto a sub-group $\varphi[K]$ of a product of the form $\Pi_{i\in I}M_i$ with each M_i metrizable. The condition that $h\colon G \to K$ is continuous is equivalent to the condition that each composition $\pi_i \circ \varphi \circ h\colon G \to M_i$ is continuous.

We assume also without loss of generality, replacing if necessary the function $h\colon G \to K$ by $h'\colon G \to G \times K$ defined by $h'(x) = \langle x, h(x)\rangle$, that h is a one-to-one function. (Here we note that $G \times K$ is compact metrizable since G and K are, and that h' is continuous if and only if h is continuous.)

We note first that $e_K = h(e_G)$ is not isolated in $h[G]$. Indeed if e_K is isolated in $h[G]$ then $h[G]$ is a discrete subgroup of the compact group K and hence $h[G]$ is finite; this contradicts the fact that $h\colon G \to K$ is a one-to-one function.

Let $\dim G = m < \omega$. To show that h is continuous we choose $V \in \mathcal{N}_K(e)$ and we will find $U \in \mathcal{N}_G(e)$ such that $h[U] \subset V$. We choose $W \in \mathcal{N}_K(e)$ such that $W^m \subset V$. Since e_K is not isolated in $h[G]$ there is $\{a_n\colon n < \omega\} \subset G$ such that $h(a_n) \neq e_K$ and $h(a_n) \to e_K$. From lemma 9.15 there is $n < \omega$ such that $M_K(h(a_n)) \subset W$. We choose such n and for notational simplicity in what follows we set $a = a_n$. We claim there is $U \in \mathcal{N}_G(e)$ such that $M_G(a) \supset U$.

Since $h(a) \neq e_K$ we have $a \neq e_G$ and hence a is not in the center of G. It follows that for all $N \in \mathcal{N}_G(e)$ there is $b \in N$ such that $ab \neq ba$; indeed otherwise, since G is connected and hence $[N] = G$, we have $ab = ba$ for all $b \in G$.

There is $\varepsilon > 0$ such that every element of $B_\varepsilon(e_G)$ lies on a one-paramenter subgroup—that is, for $b \in B_\varepsilon(e_G)$ there is an analytic homomorphism $\beta\colon \mathbb{R} \to G$ such that $\beta(1) = b$. We choose $b \in B_\varepsilon(e_G)$ and β such that $ab \neq ba$ and we set $b_t = \beta(-t)$ and $a_t = b_t a b_t^{-1} a^{-1}$. The two (continuous) homomorphisms $t \to b_t$ and $t \to a b_t a^{-1}$ from $\mathbb{R}$ to G differ at $t = 1$. Since any two continuous homomorphisms from $\mathbb{R}$ agree on a closed subgroup, there is $\delta > 0$ such that $b_t \neq a b_t a^{-1}$ (hence, $a_t \neq e_G$) for $0 < |t| \leq \delta$.

The hypothesis that G is simple means that conjugation of a_δ by elements of G leaves invariant no proper subspace of the m-dimensional tangent space at e_G; that is, there is $\{c_i\colon 1 \leq i \leq m\} \subset G$ such that $\{c_i a_\delta c_i^{-1}\colon 1 \leq i \leq m\}$ spans the Lie algebra. It then follows from the regularity of exponentiation at the origin, using canonical coordinates and non-trivial properties of the adjoint representation, that the set

$$\left\{\prod_{i=1}^{m} c_i a_{t_i} c_i^{-1}\colon |t_i| \leq \delta\right\},$$

which is $\{\Pi_{i=1}^m c_i b_{t_i} a b_{t_i}^{-1} a^{-1} c_i^{-1}\colon |t_i| \leq \delta\}$, contains a neighborhood U of e_G, as required. (The reader acquainted with the subtleties of transition from a Lie group to the associated Lie algebra and back will have no difficulty supplying the missing details. Others may wish to consult, for different parts of the argument, the texts of PONTRJAGIN [1939], CHEVALLEY [1946], MONTGOMERY and ZIPPIN [1955], COHN [1957], HELGASON [1962], or HOCHSCHILD [1965].)

Now let $p = \prod_{i=1}^{m} p_i \in U$ with $p_i = c_i b_{t_i} a b_{t_i}^{-1} a^{-1} c_i^{-1} \in M_G(a)$. Then

$$h(p_i) \in M_K(h(a)) \subset W$$

and hence $h(p) \in W^m \subset V$. Thus $h[U] \subset V$. The proof is complete.

9.17. It is instructive to consider the simplifications in the argument of VAN DER WAERDEN [1933] (9.16 above) which result when one takes for G the group $G = \mathrm{SO}(3, \mathbb{R})$. The following exposition, which avoids mention of Lie algebras and the exponential map, profits from conversations with Lewis C. Robertson and follows in substance an argument suggested in conversation by Paul R. Halmos (August, 1982).

We denote by $E = E(3)$ the identity (matrix) in $\mathrm{GL}(3, \mathbb{C})$. For $v = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$ we set

$$\|v\| = |\langle v, v\rangle|^{1/2} = (v_1^2 + v_2^2 + v_3^2)^{1/2} ;$$

we set $S^2 = \{v \in \mathbb{R}^3 : \|v\| = 1\}$, and for every 3×3 real matrix A we define

$$\|A\| = \sup\{\|A(v)\| : v \in S^2\} .$$

We write $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Finally for $\theta \in [-\pi, \pi]$ we define $E_\theta \in G = \mathrm{SO}(3, \mathbb{R})$ by

$$E_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The verification that every homomorphism $h: G \to K$ with K compact is continuous proceeds in seven steps, as follows.

(i) The group G acts transitively on S^2. (*Proof.* It is enough to note that for every $v = \langle v_1, v_2, v_3 \rangle \in S^2$ there are θ and φ such that

$$v = \langle (\sin\varphi)(\cos\theta), (\sin\varphi)(\sin\theta), \cos\varphi \rangle ,$$

and that

$$\begin{pmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.)$$

(ii) If $C \in G$ and $C(e_3) = e_3$, then there is $\theta \in [-\pi, \pi]$ such that $C = E_\theta$. (*Proof.* The rows of C are an orthonormal basis for $\mathbb{R}^3$ and the columns of C are an orthonormal basis for $\mathbb{R}^3$; since $C(e_3) = e_3$ it is then clear that $c_{31} = c_{32} = c_{13} = c_{23} = 0$ and $c_{33} = 1$, while the 2×2 'upper left corner' of C is an element of $\mathrm{SO}(2, \mathbb{R})$.)

(iii) If $D = (d_{ij})$ is a real 3×3 diagonal matrix, then $\|D\| = \max\{|d_{ii}| : i \leqslant i \leqslant 3\}$. (*Proof.* For notational simplicity we assume $|d_{11}| \geqslant |d_{ii}|$ $(i = 2, 3)$. Then for $v = \langle v_i, v_2, v_3\rangle \in S^2 \subset \mathbb{R}^3$ we have

$$\|D(v)\|^2 = \sum_{i=1}^{3} (d_{ii}v_i)^2 \leqslant |d_{11}|^2 \cdot \sum_{i=1}^{3} v_i^2 = |d_{11}|^2 \cdot \|v\|^2 = d_{11}^2$$

and hence $\|D\| \leqslant |d_{11}|$. For the reverse inequality we note from $e_1 \in S^2$ that

$$\|D\| \geqslant \|D(e_1)\| = \|\langle d_{11}, 0, 0\rangle\| = |d_{11}| \,.)$$

(iv) For $A \in G$ there is unique $\varphi \in [0, \pi]$ such that $\|A - E\| = (2 - 2\cos(\varphi))^{1/2}$. (*Proof.* This is immediate from the inequalities $0 \leqslant \|A - E\| \leqslant \|A\| + \|E\| = 1 + 1 = 2$.)

(v) Let $A \in G$ and let $\varphi \in [0, \pi]$ satisfy $\|A - E\| = (2 - 2\cos(\varphi))^{1/2}$. Then there is $B \in G$ such that $BAB^{-1} = E_\varphi$. (*Proof.* It can be shown algebraically without difficulty, and it is obvious geometrically since A is a rotation of $\mathbb{R}^3$ about some line through the origin, that there is $w \in S^2$ such that $A(w) = w$; this is essentially the statement that 1 is an eigenvalue of A. From (i) there is $B \in G$ such that $B^{-1}(e_3) = w$, and with $C = BAB^{-1}$ we have $C(e_3) = e_3$; it then follows from (ii) that there is $\theta \in [-\pi, \pi]$ such that $C = E_\theta$. Denoting by F the diagonal matrix with diagonal entries $(1, -1, -1)$, we note that $FBA(FB)^{-1}(e_3) = e_3$ and $FCF^{-1} = E_{-\theta}$. Thus we may assume without loss of generality, replacing if necessary B by FB (and hence θ by $-\theta$), that $\theta \in [0, \pi]$. It remains only to show $\theta = \varphi$, i.e., that $\|A - E\| = (2 - 2\cos(\theta))^{1/2}$.

A routine computation shows, writing $T_\theta = E_\theta - E$ and denoting by D the diagonal matrix with diagonal entries $((2 - 2\cos(\theta))^{1/2}, (2 - 2\cos(\theta))^{1/2}, 0)$, that $T'_\theta T_\theta = D^2 = D'D$. Now for $v \in \mathbb{R}^3$ we have

$$\begin{aligned}\|T_\theta(v)\|^2 &= \langle T_\theta(v), T_\theta(v)\rangle = \langle T'_\theta T_\theta(v), v\rangle \\ &= \langle D'D(v), v\rangle = \langle D(v), D(v)\rangle = \|D(v)\|^2 ,\end{aligned}$$

and hence $\|T_\theta\| = \|D\|$. It then follows from (iii) that

$$\|A - E\| = \|BAB^{-1} - BEB^{-1}\| = \|E_\theta - E\| = \|T_\theta\| = \|D\| = (2 - 2\cos(\theta))^{1/2} ,$$

as required.)

(vi) Let $A, X \in G$ with $\|A - E\| = \|X - E\|$. Then there is $C \in G$ such that $X = CAC^{-1}$. (*Proof.* This is immediate from (iv) and (v). We note in passing that from equalities as in (v) it follows conversely that if $A, X, C \in G$ with $X = CAC^{-1}$, then $\|A - E\| = \|X - E\|$.)

(vii) Every $h \in \mathrm{Hom}(G, K)$ with K compact is continuous. (*Proof.* According to 9.15 and the early parts of the proof of 9.16, it is enough to show for $E \neq A \in G$ that there is $\varepsilon > 0$ such that $M(A) \supset B_\varepsilon(E)$. Choose $B \in G$ such that $AB \neq BA$ and set $\varepsilon = \|BAB^{-1}A^{-1} - E\|$. Like every connected, locally Euclidean space, the group G is arc-wise connected; hence there is a continuous function $[0, 1] \to G$ so that $t \to B_t$ with $B_0 = E$, $B_1 = B$. For $t \in [0, 1]$ we set $A_t = B_tAB_t^{-1}A^{-1}$. Since $\|A_0 - E\| = 0$ and $\|A_1 - E\| = \varepsilon$, it follows from continuity that for $0 \leqslant r \leqslant \varepsilon$ there is $t(r) \in [0, 1]$ such that $\|A_{t(r)} - E\| = r$. From (vi) we have

$$\{CA_{t(r)}C^{-1}: C \in G\} \supset S_r(E) = \{X \in G: \|X - E\| = r\},$$

and hence

$$M(A) \supset \bigcup\{S_r(E): 0 \leqslant r < \varepsilon\} = B_\varepsilon(E),$$

as required.) The proof is complete.

9.18. COROLLARY. *Let $G = \langle G, \mathcal{T}\rangle$ be a compact, connected, simple Lie group with trivial center (for example, let G be the rotation group* $\mathrm{SO}(3, \mathbb{R})$*). If $\mathcal{T}'$ is a totally bounded topological group topology for G such that $\mathcal{T} \subset \mathcal{T}'$, then $\mathcal{T} = \mathcal{T}'$.*

PROOF. Suppose that the conclusion fails and let K be the Weil completion of $\langle G, \mathcal{T}'\rangle$. Then the inclusion function $f: \langle G, \mathcal{T}\rangle \to K$ (defined by $h(x) = x$ for $x \in G$) is a discontinuous homomorphism, contradicting 9.16 and 9.17.

9.19. The preceding portions of this section have attempted to deal with questions that may be called quite broad or general, that is, the questions of topologizing non-trivially (and with a totally bounded topology, if possible) an arbitrary group or an arbitrary Abelian group. We turn now to consider two much more restrictive problems which concern, however, certain groups which arise frequently and naturally: the free groups (9.20–9.22) and certain groups of permutations (9.23–9.24).

9.20. DEFINITION. Let S be a non-empty set. A *word on S* is a symbol w such that either $w = e$ or

$$w = s_1^{\varepsilon(1)} \cdot s_2^{\varepsilon(2)} \cdot \cdots \cdot s_n^{\varepsilon(n)} = \prod_{i=1}^{n} s_i^{\varepsilon(i)}$$

with each $s_i \in S$ and each $\varepsilon(i) \in \{-1, +1\}$. (It is not required that the function $i \to s_i$ be one-to-one.) A word w is reduced if $w = e$ or if $\varepsilon(i) = \varepsilon(i + 1)$ whenever $s_i = s_{i+1}$.

The *free group on S*, denoted $F(S)$, is the set of all reduced words on S with the operation defined as follows.

First, $e \cdot w = w \cdot e = w$ for all $w \in F(S)$. Suppose now that $k < \omega$ and that $v \cdot w$ has been defined for all $v, w \in F(S)$ with (length v) + (length w) $< k$, and let $v = \Pi_{i=1}^{n} s_i^{\varepsilon(i)} \in F(S)$, $w = \Pi_{j=1}^{m} t_j^{\delta(j)} \in F(S)$ with $n + m = k$. Let u denote the juxtaposition $(\Pi_{i=1}^{n} s_i^{\varepsilon(i)}) \cdot (\Pi_{j=1}^{m} t_j^{\delta(j)})$. If the word u is reduced, we define $vw = u$. If u is not reduced, then $s_n = t_1$ and $\varepsilon(n) = -\delta(1)$; in this case we set $v' = \Pi_{i=1}^{n-1} s_i^{\varepsilon(i)}$ (or $v' = e$ if $n = 1$) and $w' = \Pi_{j=2}^{m} t_j^{\delta(j)}$ (or $w' = e$ if $m = 1$) and we define vw to be the product $v'w'$ (which has been defined by our inductive hypothesis).

In the terminology of 9.7 above, part (i) of the following theorem is the statement that every (discrete) free group $F(S)$ is maximally almost periodic. In fact, the proof shows more: every such group is *residually finite* in the sense that its points are distinguished by homomorphisms into topological groups which are not only compact but indeed finite. This result, known to VON NEUMANN and WIGNER [1940] (§3) and strengthened subsequently into the context of free topological groups by NAKAYAMA [1943], was discovered independently (together with part (ii)) by HALL [1950]; our treatment parallels that of HALL [1950].

THEOREM. *Let S be a non-empty set and let $G = F(S)$. Then*
(i) *G admits a non-discrete, totally bounded topological group topology $\mathcal{T}$;*
(ii) *if in addition $|S| < \omega$, then $\mathcal{T}$ may be taken metrizable.*

PROOF. Let $\mathcal{U}$ be the set of normal subgroups of G with finite index.
(i) According to 1.11 above it is enough to verify that
(1) $\bigcap \mathcal{U} = \{e\}$,
(2) for $U, V \in \mathcal{U}$ there is $W \subset U \cap V$ such that $W \in \mathcal{U}$,
(3) for $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $VV^{-1} \subset U$,
(4) for $x \in U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $xV \subset U$, and
(5) for $U \in \mathcal{U}$ and $x \in G$ there is $V \in \mathcal{U}$ such that $xVx^{-1} \subset U$.
For then the family $\{xU : x \in G, U \in \mathcal{U}\}$ is a base for a (Hausdorff) topological group topology $\mathcal{T}$ for G, and in the present case from $|G/U| < \omega$ for $U \in \mathcal{U}$ it is clear that $\langle G, \mathcal{T} \rangle$ is totally bounded.

Conditions (5), (4) and (3) are obvious: choose $V = U$.

To prove (2), set $W = U \cap V$. It is clear that W is a normal subgroup of G. The required condition $|G/W| < \omega$ follows from $|G/U| < \omega$, $|G/V| < \omega$ and the relations

$$|G/W| = |G/(U \cap V)| = |G/U| \cdot |U/(U \cap V)| \leq |G/U| \cdot |G/V| .$$

To complete the proof of (i) it is enough to verify (1). Here we follow the argument of KUROSH [1956] (§36).

Let $e \neq w \in F(S)$ and let $A = \{s_k : 1 \leq k \leq m\}$ be those elements of S (listed without repetition) which appear in the representation of w. We write

$$w = s_{f(n)}^{\varepsilon(n)} \cdot \cdots \cdot s_{f(2)}^{\varepsilon(2)} \cdot s_{f(1)}^{\varepsilon(1)}$$

with $f: \{i: 1 \leqslant i \leqslant n\} \to \{k: 1 \leqslant k \leqslant m\}$ and $\varepsilon: \{i: 1 \leqslant i \leqslant n\} \to \{-1, +1\}$.

Now let P denote the group of permutations of the set $\{i: 1 \leqslant i \leqslant n+1\}$. We will define a homomorphism $h: G \to P$ such that $h(w) \neq e_P$; in fact we will arrange $h(w)(1) = n+1$. To define h it is enough to define $h(s)$ for all $s \in S$; for it is clear that any function from S to P (or to any group) extends uniquely to a homomorphism from $F(S)$.

For $s \in F(S) \backslash A$ we choose for $h(s)$ any element whatever of P. For $s = s_k \in A$ with $1 \leqslant k \leqslant m$ we define $h(s) = p_k \in P$ with p_k chosen so that, for all i with $1 \leqslant i \leqslant n$ and with $f(i) = k$, $p_k = p_{f(i)}$ satisfies

$$(*) \qquad \begin{cases} p_{f(i)}(i) = i+1 & \text{if } \varepsilon(i) = +1, \\ p_{f(i)}(i+1) = i & \text{if } \varepsilon(i) = -1. \end{cases}$$

(For fixed k with $1 \leqslant k \leqslant m$ it may occur that there exist i and i' with $1 \leqslant i < i' \leqslant n$ and $f(i) = f(i') = k$. The fact that the word w is reduced guarantees however that the relations $(*)$ well-define a function p_k which is one-to-one from a certain subset of $\{i: 1 \leqslant i \leqslant n+1\}$ into $\{i: 1 \leqslant i \leqslant n+1\}$. Thus for all k there is $p_k \in P$ such that p_k satisfies $(*)$ whenever $f(i) = k$.)

It is now clear, with

$$h(w) = h(s_{f(n)})^{\varepsilon(n)} \circ \cdots \circ h(s_{f(2)})^{\varepsilon(2)} \circ h(s_{f(1)})^{\varepsilon(1)} \in P,$$

that $h(w)$ satisfies

$$h(w)(1) = p_{f(n)}^{\varepsilon(n)} \circ \cdots \circ p_{f(2)}^{\varepsilon(2)} \circ p_{f(1)}^{\varepsilon(1)}(1) = n+1$$

as asserted; hence $h(w) \neq e_P$.

The normal subgroup $U = \ker h = h^{-1}(\{e_P\})$ satisfies $w \notin U$ and $|G/U| = |h[G]| \leqslant |P| = (n+1)! < \omega$, as required.

(ii) The family $\mathscr{U}$ is a local base for $\langle G, \mathscr{T} \rangle$ at e. Thus to prove (ii) it is, according to 1.8(i), enough to show $|\mathscr{U}| \leqslant \omega$. We set $\mathscr{U}(n) = \{U \in \mathscr{U}: |G/U| = n\}$ and we show $|\mathscr{U}(n)| < \omega$ for $n < \omega$.

Let P be the group of permutations of $\{i: 1 \leqslant i \leqslant n\}$ and for $U \in \mathscr{U}(n)$ let $\{x_{i,U}: 1 \leqslant i \leqslant n\}$ be a (complete) set of coset representatives for G/U with $x_{1,U} = e_G$. For $U \in \mathscr{U}(n)$ and $x \in G$ we define $p_U(x) \in P$ by the rule

$$p_U(x)(i) = j \quad \text{if } x x_{i,U} U = x_{j,U} U.$$

It is not difficult to see that $p_U \in \mathrm{Hom}(G, P)$. Indeed let $x, y \in G$ and, given i with $1 \leq i \leq n$, find j and k so that $yx_{i,U}U = x_{j,U}U$ and $xx_{j,U}U = x_{k,U}U$. Then $xyx_{i,U}U = x_{k,U}U$ and hence $p_U(xy)(i) = k$; and from $p_U(y)(i) = j$ and $p_U(x)(j) = k$ follows

$$(p_U(x) \circ p_U(y))(i) = k .$$

Then $p_U(xy) = p_U(x) \circ p_U(y)$, i.e., $p_U \in \mathrm{Hom}(G, P)$.

For $U \in \mathcal{U}(n)$ and $x \in G$ we have $p_U(x)(1) = 1$ if and only if $x \in U$. It follows that if $U, V \in \mathcal{U}(n)$ with $p_U = p_V$, then $U = V$, that is, the function $U \to p_U$ from $\mathcal{U}(n)$ into $\mathrm{Hom}(G, P)$ is one-to-one. Since two homomorphisms on G which agree on S are identical, we then have

$$|\mathcal{U}(n)| \leq |\mathrm{Hom}(G, P)| \leq |P|^{|S|} = n!^{|S|} < \omega ,$$

as required.

9.22. When $0 < |S| < \omega$, as in 9.20(ii) above, we have of course $|G| = |F(S)| = \omega$. It is perhaps tempting to conjecture that every countable group G contains only countably many (normal) subgroups of finite index, but HALL [1950] offers the following nice counterexample.

Let $|S| = \omega$ and $G = F(S)$, and for every $T \subset S$ define $h_T \in \mathrm{Hom}(G, \{-1, +1\})$ by specifying

$$\begin{aligned} h_T(s) &= +1 \quad \text{if } s \in T, \\ &= -1 \quad \text{if } s \in S \backslash T, \end{aligned}$$

and extending to a homomorphism h_T defined throughout G. Then $\ker(h_T)$ is a normal subgroup of G with $|G/\ker(h_T)| = 2$, and for $T, U \subset S$ with $T \neq U$ we have

$$\ker(h_T) \cap S = T \neq U = \ker(h_U) \cap S$$

and hence $\ker(h_T) \neq \ker(h_U)$. Thus G is a countable group with 2^ω (distinct) subgroups of index 2.

9.23. It is a natural to raise the question whether, given a topological space X, the group of homeomorphisms of X onto X admits non-discrete topological group topologies. The work of FLETCHER and LIU [1975] contains several contributions in this direction, as well as relevant bibliographical references. Further, according to an oft-cited result of DE GROOT [1959], every group is algebraically isomorphic to the group of all homeomorphisms of some space X onto itself; in fact X may be chosen to be a connected and locally connected complete metric space, or compact and connected. Thus at least in principle the question of topologizing homeomorphism groups is no less general than the question of MARKOV [1945] cited in the introduction to this section.

It is a more modest enterprise to attempt to topologize (as a topological group) a homeomorphism group in the special case that the underlying space is discrete; this is nothing other than a permutation group. In what follows we are guided by SHARMA [1981], whose "method of ideals" allows him to topologize various groups of homeomorphisms and permutations. Theorem 9.24(ii) was first proved by TAIMANOV [1977].

9.24. For a set A we denote by $A!$ the set of permutations of A, and for $f \in A!$ we write

$$S(f) = \{x \in A: f(x) \neq x\}.$$

(The set $S(f)$ is called the support of f.)

THEOREM. *Let α and β be cardinals with $\alpha \geqslant \beta \geqslant \omega$, let A be a set such that $|A| = \alpha$, and let G be a subgroup of $A!$. Suppose further that either*

(i) *there is $B \subset A$ such that $G = \{f \in A!: S(f) \subset B$ and $|S(f)| < \beta\}$, or*

(ii) $|G| > \alpha$.

Then the topology of pointwise convergence of $A!$ induces on G a non-discrete topological group topology.

PROOF. As in 5.12 we leave it to the reader to verify that with the topology of pointwise convergence the group $A!$ (and hence its subgroup G) is a topological group. To prove that G so topologized is non-discrete, it is enough to show, defining

$$U_F = \{f \in G: f(x) = x \text{ for all } x \in F\} \quad \text{for } F \subset A,$$

that for finite $F \subset A$ there is $f \in U_F$ such that $f \neq e_G$.

(i) Choose distinct $p, q \in B \backslash F$ and define $f(p) = q$, $f(q) = p$, and $f(x) = x$ for $x \in A \backslash \{p, q\}$.

(ii) For $E \subset A$ with $|E| = |F|$ set

$$G(E) = \{g \in G: g[F] = E\}.$$

Since $G = \bigcup_E G(E)$ and $|\{E \subset G: |E| = |F|\}| = \alpha < |G|$, there is $E \subset G$ with $|E| = |F|$ such that $|G(E)| > \alpha$. Since $|E^F| < \omega$, there are distinct $g, h \in G(E)$ such that $g|F = h|F$. The function $f = h^{-1} \circ g$ is then as required.

10. Not all homogeneous spaces are topological groups

A natural companion to the question "Can a (given) group be topologized?" is the question "Can a (given) space be given a group structure?"—it being understood in each case that a topological group is desired.

In order that it support the structure of a topological group, a space must of course be homogeneous.

10.1 Let us note first that several homogeneous spaces which are familiar to practicing point-set topologists do not admit a topological group structure. We give three examples.

(i) The sequential space S_ω of ARHANGEL'SKIĬ and FRANKLIN [1968]. As the authors show, this space is homogeneous and not first countable. But S_ω is weakly first countable in the sense of ARHANGEL'SKIĬ [1966] (that is, for $x \in S_\omega$ one can assign a sequence $\{B_n(x): n < \omega\}$ of sets with $x \in B_n(x) \subset S_\omega$ in such a way that a subset U of S_ω is open if and only if for every $x \in U$ there is $n < \omega$ such that $B_n(x) \subset U$); and, as indicated by NYIKOS [1981] in a paper with several applications to pathological topological groups, every weakly first countable topological group is metrizable.

(ii) The long half-line, L; to form this space we interpolate between each countable ordinal ξ and its ordinal successor $\xi + 1$ a copy of the real interval $(0, 1)$ and, removing the least point, we give the resulting space L the order topology. It is then easy to see, using the facts that every interval of the form $(0, \xi]$ is homeomorphic to $(0, 1]$ and every two intervals of the form $[\xi, \omega^+)$ are homeomorphic, that L is homogeneous. But L, since it is first countable but not metrizable, admits no compatible topological group structure (cf. 1.8(i) above).

For a variation on this last argument we note that a topological group G is involutorily homogeneous in the sense that for every pair $\{x, y\}$ of elements there is a homeomorphism of G exchanging x and y—indeed, the function $a \to xa^{-1}y$ is as required. The space L lacks this property since, as a moment's reflection indicates, any homeomorphism exchanging x and y (with $x < y$) would take the metrizable space $(0, x]$ onto the non-metrizable space $[y, \omega^+)$.

(iii) The SORGENFREY line [1947], here denoted S. We are claiming more than the fact that the real line with its usual additive structure and with the Sorgenfrey topology is not a topological group (that much is obvious because inversion is not continuous): we are claiming that no algebraic structure makes S into a topological group. For this it is enough to appeal again to 1.8(i) above: S is first countable but it is not metrizable because, as shown by SORGENFREY [1947], the space $S \times S$ is not normal.

10.2. It is a more challenging exercise to find homogeneous spaces which do not admit the structure of a topological group and which are compact. Here I acknowledge with thanks several helpful suggestions and comments received from ERIC VAN DOUWEN (letter of September, 1981).

(i) Every Euclidean n-sphere $S^n = \{x \in \mathbb{R}^{n+1}: \Sigma_{k=0}^n x_k^2 = 1\}$ is homogeneous, but S^n admits a compatible group structure if and only if n is 0, 1 or 3. An explicit statement and proof are given by SAMELSON [1940/41], while already five years earlier É. CARTAN [1936] (page 237) had written: "...l'espace sphérique a $2n+1 \geqq 5$ dimensions est orientable, simplement connexe et de caractéristique nulle; si c'était l'espace d'un groupe, ce groupe serait clos, done serait un groupe de Lie; d'après les théorèmes que nous verrons un peu plus loin, ce groupe serait semi-simple et son troisième nombre de Betti serait positif, alors que celui de l'espace donné est nul."

An H-space is a topological space X and a continuous 'multiplication': $X \times X \to X$ such that for some $e \in X$ one has $xe = ex = x$ for all $x \in X$; that is, from the axioms of a topological group have been deleted associativity and the existence of inverses. ADAMS [1960] showed that S^n is an H-space if and only if n is 0, 1, 3 or 7. An intermediate result, to the effect (for $n > 0$) that S^n is not parallelizable (hence, is not a topological group) unless n is 1, 3 or 7, is due to KERVAIRE [1958] and to BOTT and MILNOR [1958].

(ii) The Hilbert cube $H = \Pi_{k<\omega} [-1/k, +1/k]$, a subspace of sequential Hilbert space ℓ^2, is compact in the inherited metric defined by $\|x - y\| = (\Sigma_{k<\omega} (x_k - y_k)^2)^{1/2}$. KELLER [1931] showed that the infinite-dimensional compact convex subsets of ℓ^2 are homogeneous and mutually homeomorphic, and TYCHONOFF [1935], in work substantially extended by KLEE [1955] and others, proved that every compact convex subset of a locally convex topological linear space has the fixed-point property. (Let us note that the fixed-point property for H is immediate from the classical Brouwer fixed-point theorem. Let π_n denote the natural projection from H onto $B_n = (\Pi_{k\leq n} [-1/k, 1/k]) \times \Pi_{n<k<\omega} \{0\}$, and for $f \in C(H, H)$ let x_n be a fixed point of $\pi_n \circ (f \mid B_n)$. Since

$$\|f(x_n) - x_n\| \leq \Big(\sum_{n<k<\omega} \frac{1}{k^2} \Big)^{1/2} \to 0 ,$$

any convergent subsequence of x_n must converge to a fixed point of f.) Since translation in a group by an element other than e moves every element, it is immediate that H does not admit the structure of a topological group.

(iii) The double arrow space of ALEXANDROFF and URYSOHN [1929] is the subspace $A = (0, 1] \times \{0\} \cup [0, 1) \times \{1\}$ of the lexicographically ordered square $[0, 1] \times [0, 1]$. That A is homogeneous is clear. To see that A does not admit a topological group structure we recall from 1.8(i) that a first countable topological group is metrizable; but A is first countable, separable and not second countable (hence, not metrizable). Alternatively we may note that $w(A) = 2^\omega$ and, like every separable, first countable space, A satisfies $\pi w(A) = \omega$; now use Theorem 3.6(ii).

(iv) Let L be a lexicographically ordered space of the form $L = \{0, 1\}^\gamma$ with

$\eta = \omega^\xi$ (ordinal exponentiation), ξ an infinite, countable limit ordinal. (The reader unfamiliar with ordinal exponentiation should consult SIERPIŃSKI [1958], especially pp. 306–323. We remark in passing that $|\omega^\xi| < \omega^+$ whenever $|\xi| < \omega^+$.) It is not difficult to see that L is complete in the sense that every subset of L has a supremum (hence also, an infinum); hence L is compact (cf. HAAR and KÖNIG [1911] or ENGELKING [1977], problem 3.12.3(a)). To see that L is homogeneous, we follow MAURICE [1964] (Theorem II.2.5): Given $p, q \in L$, let $\{I_n: n < \omega\}$ and $\{J_n: n < \omega\}$ be decreasing sequences of open-and-closed sets such that $\bigcap_n I_n = \{p\}$ and $\bigcap_n J_n = \{q\}$, say with $I_0 = J_0 = L$ and with $I_1 \cap J_1 = \emptyset$, and let f_n be an order-preserving function from $I_n \backslash I_{n+1}$ onto $J_n \backslash J_{n+1}$; the function

$$f = \Big(\bigcup_{n<\omega} f_n\Big) \cup \{\langle p, q\rangle\},$$

a homeomorphism of L onto L, satisfies $f(p) = q$.

To see that L does not support the structure of a topological group it is enough, according to Theorem 3.1(iv), to show that the compact space L does not satisfy the countable chain condition. Since $\xi \geq \omega$ the limit ordinal η satisfies $\eta > \omega$. For $f \in \{0, 1\}^\omega$ we define $l(f)$, $r(f) \in L$ by the rules

$$l(f)(n) = r(f)(n) = f(n) \quad \text{for } n < \omega,$$
$$l(f)(\zeta) = 0 \qquad \text{and} \qquad r(f)(\zeta) = 1 \quad \text{for } \omega \leq \zeta < \eta.$$

For $f \in \{0, 1\}^\omega$ the open interval $U(f) = (l(f), r(f))$ is non-empty; it contains for example the function $\bar{f}$ defined by

$$\bar{f}(n) = f(n) \quad \text{for } n < \omega, \qquad \bar{f}(\omega) = 0, \qquad \bar{f}(\zeta) = 1 \quad \text{for } \omega < \zeta < \eta.$$

Thus there is in L a cellular family of cardinality 2^ω, and L does not satisfy the countable chain condition.

(v) A space is an Eberlein compact if it is homeomorphic to a compact subspace of some Banach space E in its weak topology, that is, in the topology induced by the dual space E^*. Responding to a question of ARHANGEL'SKIĬ [1978], VAN MILL [1982c] has constructed a homogeneous, non-metrizable Eberlein compact space M. It has been known for some time that an Eberlein compact space with the countable chain condition must be metrizable. In any event van Mill proves that his compact space M is non-metrizable by exhibiting explicitly a continuous function from M onto a space which lacks the countable chain condition, so again from 3.1(iv) above it is clear that the compact, homogenous space X cannot support a topological group structure.

10.3. It is an interesting question, brought to my attention by van Douwen, whether there are a homogeneous, compact (Hausdorff) space X and a cellular family $\mathcal{U}$ in X

such that $|\mathcal{U}| > 2^\omega$. The following argument, taken from MAURICE [1964], shows that for η an ordinal the lexicographically ordered space $\{0, 1\}^\eta$ cannot serve as an example. For $\eta < \omega$ this is clear. Suppose then that $\eta \geqslant \omega$ and that $X = \{0, 1\}^\eta$ is homogeneous. Let x_n be a strictly increasing sequence in X and set $x = \sup_{n<\omega} x_n$. Since $x_n \to x$ in X, there is by homogeneity a faithfully indexed sequence with limit 0 (the least element of X). Then $\{0\}$, and hence $\{p\}$ for each $p \in X$, is a G_δ in X. It is then clear that $\eta < \omega^+$, so that $|X| \leqslant 2^\omega$ and X has no cellular family $\mathcal{U}$ with $|\mathcal{U}| > 2^\omega$.

10.4. Let G be a compact topological group with Haar measure λ and as usual for $A \subset G$ set

$$\lambda_*(A) = \sup\{\lambda(F)\colon F \subset A,\ F \text{ is compact}\}\,.$$

When H is a subgroup of G with $\lambda_*(G\backslash H) = 0$, then as in the proof of 4.15(ii) one can use λ to define a measure μ on H so that every subset A of H of the form $A = B \cap H$ with B Borel in G is μ-measurable: the rule $\mu(B \cap H) = \lambda(B)$ well-defines μ on a σ-algebra of subsets of H which contains the Borel sets of H.

Motivated by a question in Boolean algebras, VAN DOUWEN [1979b] with $G = \mathbb{T}$ has managed to construct a subgroup H of $\mathbb{T}$ with $\lambda_*(\mathbb{T}\backslash H) = 0$ so that the indicated measure μ has this remarkable property: $\mu(A_0) = \mu(A_1)$ whenever A_0 and A_1 are homeomorphic Borel sets of H. The measure μ in turn induces a measure ν on a compactification $B(H)$ of H by the rule $\nu(B) = \mu(B \cap H)$ for B Borel in $B(H)$, and the measure space $\langle B(H), \nu\rangle$ has these properties:

(i) $B(H)$ is a compact, homogeneous, zero-dimensional topological space;

(ii) every open set of $B(H)$ is an F_σ:

(iii) homeomorphic Borel sets B_0, B_1 of $B(H)$ satisfy $\nu(B_0) = \nu(B_1)$; and

(iv) if B_1 and B_2 are subsets of $B(H)$, with both compact or neither compact, and if $\nu(B_0) = \nu(B_1)$, then B_0 and B_1 are homeomorphic.

10.5. It is a well known theorem of SIERPIŃSKI [1956] (§59) that every countably infinite metric space without isolated points is homeomorphic to the space $\mathbb{Q}$ of rational numbers in its usual topology; every such space, then, is homogeneous and in fact susceptible to a compatible topological group structure. It is tempting to conjecture that every homogeneous subset of $\mathbb{R}$ admits a topological group structure, but VAN MILL [1982a] has constructed a counterexample which is homeomorphic (by a translation) to its complement in $\mathbb{R}$. Van Mill's space A is a union of additive cosets of $\mathbb{Q}$, hence is homogeneous, but it has nevertheless a certain rigidity: there is a countable dense subset D of A such that $h[D] \cap D \neq \emptyset$ for every homeomorphism h of A. That A admits no compatible group structure then follows from the fact that for every infinite group G and any two subsets E and F of G with $|E| + |F| < |G|$ there is $p \in G$ such that $(pE) \cap F = \emptyset$ (one may choose for p any element of $G\backslash FE^{-1}$).

11. Concluding remarks and acknowledgements

If we had been allowed an article twice as long as this one, and twice the time to prepare it, we might have touched upon twice as many topics. In this section, with no attempt at horizontal or vertical completeness, we identify a few of the attractive omitted topics and we offer a brief guide to some of the relevant literature.

11.1. *Discontinuous characters and closed subgroups.* It is a theorem of HEWITT [1963] that if G is an Abelian group with locally compact topological group topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\mathcal{T}_1 \neq \mathcal{T}_2$, then some $\chi \in \mathrm{Hom}(G, \mathbb{T})$ is $\mathcal{T}_2$-continuous and not $\mathcal{T}_1$-continuous; RAJAGOPALAN [1964b], ROSS [1964] and VAROPOULOS [1964] showed that in fact there are at least $2^{\mathfrak{c}}$ such homomorphisms. While offering a simplified proof of Hewitt's theorem, ROSS [1965] showed that some subgroups of G must be $\mathcal{T}_2$-closed but not $\mathcal{T}_1$-closed, and he raised the following question: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are locally compact topological group topologies on the Abelian group G with the same closed subgroups, must $\langle G, \mathcal{T}_1\rangle$ and $\langle G, \mathcal{T}_2\rangle$ be topologically isomorphic? RICKERT [1967] and RAJAGOPALAN [1968] showed that under various mild and natural additional hypotheses the answer is "Yes"; recently MOSKALENKO [1981] has constructed an example showing that the answer to Ross' question is "No".

Given a locally compact Abelian topological group $\langle G, \mathcal{T}_1\rangle$, the search for strictly finer topological group topologies $\mathcal{T}_2$ with the same closed subgroups simplifies considerably if one abandons the requirement that $\langle G, \mathcal{T}_2\rangle$ be locally compact. JANAKIRAMAN and SOUNDARARAJAN [1982] have given several characterizations of those compact, totally disconnected Abelian topological groups which admit strictly finer totally bounded topological group topologies with the same closed subgroups, showing that such groups exist in profusion; and COMFORT and SOUNDARARAJAN [1982] have shown that with $\mathcal{T}_1$ the usual topology for $\mathbb{T}$ there is a family $\{\mathcal{T}_\xi : 1 < \xi < \mathfrak{c}\}$ of totally bounded topological group topologies for $\mathbb{T}$, each with the same closed subgroups as has $\langle \mathbb{T}, \mathcal{T}_1\rangle$, such that for $1 < \xi < \xi' < \mathfrak{c}$ there is a $\mathcal{T}_1$-dense, proper, $(\mathcal{T}_\xi \vee \mathcal{T}_{\xi'})$-closed subgroup of $\mathbb{T}$.

11.2. *Locally compact topologies.* Given locally compact topological group topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for an Abelian group G, let $[\mathcal{T}_1, \mathcal{T}_2]$ denote the set of locally compact topological group topologies $\mathcal{T}$ for G such that $\mathcal{T}_1 \subset \mathcal{T} \subset \mathcal{T}_2$; and let $\mathcal{T}_d$ denote the discrete topology for G. Closely related to the questions touched upon in 11.1 are questions concerning the size and structure of the 'intervals' $[\mathcal{T}_1, \mathcal{T}_2]$. Here again it was HEWITT [1963] who initiated this fruitful line of inquiry; he proved *inter alia* that if $\langle G, \mathcal{T}_1\rangle$ denotes either $\mathbb{T}$ or $\mathbb{R}$ with its usual topology, then $|[\mathcal{T}_1, \mathcal{T}_d]| = 2$. (In more direct language, this is the statement that the only locally compact topological group topology for $\mathbb{T}$ or $\mathbb{R}$ which contains the usual topology is the discrete topology.) We cite just a few of the papers deriving from this work.

(i) RICKERT [1967]. Either $|[\mathcal{T}_1, \mathcal{T}_d]| < \omega$ or $|[\mathcal{T}_1, \mathcal{T}_d]| \geq \mathfrak{c}$.

(ii) JANAKIRAMAN and RAJAGOPALAN [1973]. Either $|[\mathcal{T}_1, \mathcal{T}_2]| < \omega$ or $|[\mathcal{T}_1, \mathcal{T}_2]| \geq \mathfrak{c}$.

(iii) MILLER and RAJAGOPALAN [1975]. If $|[\mathcal{T}_1, \mathcal{T}_2]| < \omega$, then $|[\mathcal{T}_1, \mathcal{T}_2]|$ has the form 2^n and $[\mathcal{T}_1, \mathcal{T}_2]$ is isomorphic as a partially ordered set to the power set of a set of cardinality n.

11.3. *Unique compact topologies.* STEWART [1960] has shown that if $\langle G, \mathcal{T}\rangle$ is a compact, connected topological group with totally disconnected center, then $\mathcal{T}$ is the only topology making G a compact topological group. Making full use of the work of HULANICKI [1964], SOUNDARARAJAN [1969] has shown that a compact, Abelian topological group $\langle G, \mathcal{T}\rangle$ admits no locally compact topological group topology other than $\mathcal{T}$ and the discrete topology if and only if $G \simeq \Delta_p \times F$ with Δ_p the p-adic integers and with F a finite group. Subsequently and independently of Soundararajan's work, KALLMAN [1976] and CORWIN [1976] achieved the 'if' portion of Soundararajan's theorem for classes of groups wider than the class of compact Abelian topological groups.

11.4. *Minimal topological groups.* A topological group is said to be minimal if no strictly coarser (Hausdorff) topology makes it a topological group; and G is said to be totally minimal if every quotient group G/H (with H a closed, normal subgroup of G) is minimal.

We cite here a few theorems from the extensive literature on the subject. Many of these may be viewed as approximations or preliminary steps toward the resolution of the following question, which is apparently the principal remaining unsolved problem in this area: Is every minimal Abelian topological group totally bounded? [*Note added August 1983.* An affirmative answer to this question was announced this month by Luchesar Stojanov at the Eger (Hungary) Colloquium on Topology. A publication is anticipated.]

(i) GRANT [1979], DIKRANJAN and STOJANOV [1982]. Let G be the torsion subgroup of $\mathbb{T}$. Then every power of G is totally minimal.

(ii) Every minimal Abelian topological group G in which $\hat{G}$ separates points is totally bounded. Hence (STEPHENSON [1971], COMFORT and GRANT [1981]) if a product of locally compact Abelian topological groups is minimal, then it is compact.

(iii) DIEROLF and SCHWANENGEL [1979]. There is a (non-Abelian) locally compact, non-compact, minimal topological group which is not totally bounded.

(iv) PRODANOV [1971/72]. For p a prime and $n < \omega$ let $U_n(p) = \{p^n \cdot a : a \in \mathbb{Z}\}$ and let $Z(p)$ denote $\mathbb{Z}$ with the (totally bounded) topology for which $\{U_n(p) : n < \omega\}$ is a base at 0; let $\overline{Z(p)}$ denote as usual the Weil completion of $Z(p)$. An infinite, compact Abelian topological group G has the property that every subgroup is minimal if and only if there is p such that $G \simeq \overline{Z(p)}$.

(v) DOÏTCHINOV [1972]. The groups $Z(p) \times Z(p)$ are not minimal.

(vi) PRODANOV [1977]. Every totally minimal Abelian topological group is totally bounded.

(vii) DIEROLF and SCHWANENGEL [1977]. There is a (non-Abelian) totally minimal topological group which is not totally bounded.

11.5. *Groups of homeomorphisms.* We have cited without proof in 9.23 above the theorem of DE GROOT [1959] to the effect that every group is isomorphic to the group of homeomorphisms of some space X. KANNAN and RAJAGOPALAN [1978] have given a surprising generalization: For every group G and subgroup H, and for every metric space X, there is a metric space Y, containing X as a closed subspace, for which the group of homeomorphisms of Y onto Y is isomorphic with G and the group of isometries of Y onto Y is isomorphic with H.

The appropriate analogue of de Groot's theorem for semi-groups is formulated and proved by PAALMAN–DE MIRANDA [1966]; MAGILL [1975/76] gives a readable account of her work and related results.

Let us note also that WHITTAKER [1963] has shown that under suitable supplementary hypotheses the space X defined by de Groot is unique. If (for example) X and Y are compact, locally Euclidean manifolds with isomorphic homeomorphism groups $H(X)$ and $H(Y)$, then X and Y are homeomorphic; indeed for any isomorphism φ of $H(X)$ onto $H(Y)$ there is a homeomorphism h of X onto Y such that $\varphi(f) = hfh^{-1}$ for all $f \in H(X)$.

11.6. *Weaker structures.* We cite just two of many theorems indicating that a set with algebraic and topological structure may be a topological group under conditions which apparently guarantee much less.

(i) ELLIS [1957a], [1957b]. Let G be a group with a locally compact (Hausdorff) topology in which the functions $a \to ab$ and $a \to ba$ are continuous for all $b \in G$. Then $\langle a, b\rangle \to ab^{-1}$ is continuous from $G \times G$ to G; hence G is a topological group.

(ii) ZELAZKO [1960]. Let G be an Abelian group with a complete metric ρ for which the function $a \to ab$ is continuous (relative to the topology on G defined by ρ). Then $\langle a, b\rangle \to ab^{-1}$ is continuous from $G \times G$ to G; hence G is a topological group.

It is a problem of WALLACE [1965] whether a countably compact topological semi-group with two-sided cancellation must be a topological group. The argument of GELBAUM, KALISH and OLMSTED [1951], like the argument they quote due to Iwasawa, responds affirmatively when G is compact.

11.7. *Other surveys.* Several articles in this Handbook deal directly or tangentially with topological groups. For further reading on the general theory, liberally sprinkled with counterexamples chosen carefully to delineate the bounds of applicability, and for an exhaustive treatment of integration theory and harmonic analysis over locally compact groups, HEWITT and ROSS [1963], [1970] are indispensable. PALMER [1978] considers 35 properties possessed by some compact and by some Abelian groups, and describes fully the diagram of implications among them. ARHANGEL'SKIĬ [1980] studies the topological properties of topological groups, with emphasis on cardinal invariants. ROELCKE and DIEROLF [1981] discuss at length various uniform structures on topological groups, and their uniform completions. COMFORT and GRANT [1982] has much in common with the present article, and contains a detailed list of references concerning minimal topological groups. The reader interested in topological semi-groups and their generalizations may consult WALLACE [1955], HOFMANN and MOSTERT [1966], and MAGILL [1975/76].

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Index

This Index is in two parts. The first lists symbols, ordered by the first page on which they occur. The second lists terms and proper names in alphabetical order. Greek letters are alphabetized in the second index under the English spelling of their name; e.g., "Δ-system" is alphabetized as if it were "delta-system".

Special Symbols

General Index