FORMAL LANGUAGES AND AUTOMATA THEORY

Module 3

Rijin IK

Assistant Professor
Department of Computer Science and Engineering
Vimal Jyothi Engineering College
Chemperi

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Outline

Course Outcomes

After the completion of the course the student will be able to

1. Classify a given formal language into Regular, Context-Free, Context Sensitive, Recursive or Recursively Enumerable. [Cognitive knowledge level: Understand]
2. Explain a formal representation of a given regular language as a finite state automaton, regular grammar, regular expression and Myhill-Nerode relation. [Cognitive knowledge level: Understand]
3. Design a Pushdown Automaton and a Context-Free Grammar for a given context-free language. [Cognitive knowledge level : Apply]
4. Design Turing machines as language acceptors or transducers. [Cognitive knowledge level: Apply]
5. Explain the notion of decidability. [Cognitive knowledge level: Understand]

Myhill-Nerode Relations (MNR)

Equivalence relation

- Let $R \subseteq \Sigma^*$ be a regular set, and let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for R with no inaccessible states. The automaton M induces an Equivalence relation $\equiv_M$ on Σ^* defined by

$$x \equiv_M y \overset{def}{\iff} \hat{\delta}(q_0, x) = \hat{\delta}(q_0, y).$$

[Equivalence relation$\rightarrow$ it is reflexive, symmetric, and transitive]

- **Reflexive**
  - $\forall_x \in \Sigma^*, \hat{\delta}(q_0, x) = \hat{\delta}(q_0, x)$
- **Symmetric**
  - $x \equiv_M y \implies \hat{\delta}(q_0, x) = \hat{\delta}(q_0, y) \implies \hat{\delta}(q_0, y) = \hat{\delta}(q_0, x) \implies y \equiv_M x$
- **Transitive**
  - suppose $x \equiv_M y \; and \; y \equiv_M z \implies x \equiv_M z$

Myhill-Nerode Relations (MNR)

- The Myhill-Nerode relation for a regular set $R \subseteq \Sigma^*$ is an equivalence relation $\equiv$ on Σ^* that satisfies three key properties:
  1. It is a right congruence: for any $x, y \in \Sigma^*$ and $a \in \Sigma$,
     - $x \equiv_m y \implies xa \equiv_m ya$;
  2. It refines $L(M)$: for any $x, y \in \Sigma*$
     - $x \equiv_m y \implies (x \in L(M) \Leftrightarrow y \in L(M))$;
     - i.e whenever $x \& y$ are related either $x \& y$ are in the language or both $x \& y$ are not in the language.
  3. It is of finite index; that is, it has only finitely many equivalence classes.
     - This is beause there is exactly one equivalence class $x \in \Sigma^* \middle| \hat{\hat{\delta}}(q_0, x) = q$ corresponding to eaeh state q of M.

Myhill-Nerode Relations (MNR)

Formal definition of Myhill-Nerod Relation (MN relation)

Let L be a language over an alphabet set Σ.Then, an MN relation $\equiv$ is an equivalence relation on Σ^*, which is a right congruence of finite index refining L.

Example:

- $L = x \in \{a^* \big| 3 \ divides \ |x|\}$
- $x \equiv y \Leftrightarrow |x| mod3 = |y| mod \ 3$. Equivalence classes are:
  - $[\epsilon] = \{a^i \big| i \mod 3 = 0\}$
  - $[a] = \{a^i \big| i \mod 3 = 1\}$
  - $[a^2] = \{a^i \big| i \mod 3 = 2\}$

Conversion of DFA to MNR

Lemma

If a language L is regular, then there is a Myhill-Nerode relation on Σ^* with respect to L

Proof

L is Regular $\implies$ $\exists$ a DFA M such that L=L(M)

Consider a binary relation $\equiv_M$ on Σ^* induced by$M = (Q, \Sigma, \delta, q_0, F)$, defined as

$$\forall_{x,y} \in \Sigma^*, x \equiv_M y \Leftrightarrow \hat{\delta}(q_0, x) = \hat{\delta}(q_0, y)$$

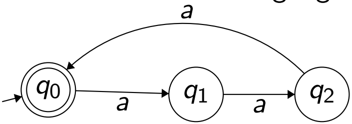
we proved that $\equiv_M$ is an MN relation

Conversion of DFA to MNR

Q:Show the equivalence classes of Myhill Nerod relation for the language $L = \{x \in \{a\}^* \middle| 3 \ divides \ |x|\}$.

Soln:

DFA for the above language



The DFA has 3 states. so MNR partions the Σ^* into 3 classes:

- $C_1 = \{x \in \Sigma^* \middle| \hat{\delta}(q_0, x) = q_0\}$ $[q_0] = \{a^i \middle| i \mod 3 = 0\}$
  - $c_1 = \{\epsilon, a^3, a^6 ....\}$
- $C_2 = \{x \in \Sigma^* \middle| \hat{\delta}(q_0, x) = q_1\}$ $[q_1] = \{a^i \middle| i \mod 3 = 1\}$
  - $c_2 = \{a^1, a^4 ....\}$
- $C_3 = \{x \in \Sigma^* \middle| \hat{\delta}(q_0, x) = q_2\}$ $[q_2] = \{a^i \middle| i \mod 3 = 2\}$
  - $c_3 = \{a^2, a^5 ....\}$

Conversion of MNR to DFA

- Given an MN relation $\equiv$ for a Language L over an alphabet set Σ, one can automatically construct a DFA $M_{\equiv} = (Q, \Sigma, \delta, q_0, F)$, such that $L(M_{\equiv}) = L$.

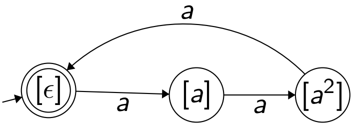
$$\begin{aligned}
Q &= \{[x] \big| x \in \Sigma^*\} \;\; \textit{set of all equivalent class} \\
q_0 &= [\epsilon] \\
\delta([x], a) &= [xa] \\
F &= \{[x] \big| x \in L\}
\end{aligned}$$

Conversion of MNR to DFA

Example: Consider the MN relation represented by the following equivalence classes for the language : $L = \{x \in \{a\}^* \big| 3 \textit{ divides } |x|\}$.
Equivalence classes are:

- $[\epsilon] = \{a^i \big| i \mod 3 = 0\}$
- $[a] = \{a^i \big| i \mod 3 = 1\}$
- $[a^2] = \{a^i \big| i \mod 3 = 2\}$

$$\begin{aligned} Q &= \{[x] \big| x \in \Sigma^*\} \\ q_0 &= [\epsilon] \\ \delta([x], a) &= [xa] \\ F &= \{[x] \big| x \in L\} \end{aligned}$$



The Myhill–Nerode Theorem

- Let $R \subseteq \Sigma^*$. The following statements are equivalent:
  1. R is regular;
  2. There exists a Myhill-Nerode relation for R:
  3. The relation $\equiv_R$ is of finite index.

Applications of MNT

- Proving whether a language is regular or not.
- It can be also used to find the minimal number of states in a Deterministic Finite Automata (DFA).

Example: Is the following language regular

$$L = \{x \in \{a,b\}^* \,\big|\, \text{number of a in x is odd}\}$$

Sln:

- It is enough to prove that $\equiv_L$ is of finite index
- Recall that: $\forall_{x,y} \in \Sigma^*, x \equiv_L y \ iff \ \forall_z \in \Sigma^*, (xz \ \in L \ \Leftrightarrow yz \in L)$
- Equivalence classes of $\equiv_L$:
  - Suppose $x, y, z \in \{a,b\}^*$ such that number of a in x is even and number of a in y is even:
  - **Case 1:** number of a in z is odd

$$\Longrightarrow \quad \text{number of a in (xz) is odd and number of a in (yz) is odd}$$
$$\Longrightarrow \quad xz \in L \ and \ yz \in L$$

  - **Case 2:** number of a in z is even

$$\Longrightarrow \quad \text{number of a in (xz) is even and number of a in (yz) is even}$$
$$\Longrightarrow \quad xz \notin L \ and \ yz \notin L$$

so the equivalent class is $[\epsilon] = \{x \in \{a,b\}^* \,\big|\, \text{number of a in z is even}\}$

Sln cont..:

- Suppose $x, y, z \in \{a, b\}^*$ such that number of a in x is odd and number of a in y is odd:
- **Case 1:** number of a in z is odd

$$\implies \quad \text{number of a in (xz) is even and number of a in (yz) is even}$$
$$\implies \quad xz \notin L \; and \; yz \notin L$$

- **Case 2:** number of a in z is even

$$\implies \quad \text{number of a in (xz) is odd and number of a in (yz) is odd}$$
$$\implies \quad xz \in L \; and \; yz \in L$$

so the equivalent clas is $[a] = \{x \in \{a, b\}^* \big| \text{number of a in z is odd}\}$

Equivalence classes of $\equiv_L$:

1. $[\epsilon] = \{x \in \{a, b\}^* \big| \text{number of a in z is even}\}$
2. $[a] = \{x \in \{a, b\}^* \big| \text{number of a in z is odd}\}$

Thus $\equiv_L$ is of finite index and hence the language is regular

Context Free Grammars

- Context-free grammars (CFGs) are used to describe context-free languages.
- A context-free grammar is a set of recursive rules used to generate patterns of strings.
- A context-free grammar can describe all regular languages and more, but they cannot describe all possible languages.
  - i.e It is **more powerfull than Regular grammar**
- Widely used in Programming Languages-syntax,parsing.

Applications of Context Free Grammar (CFG)

- For defining programming languages
- For parsing the program by constructing syntax tree
- For translation of programming languages
- For describing arithmetic expressions
- For construction of compilers
- Document Type Definition in XML is a Context Free Grammars which describes the HTML tags and the rules to use the tags in a nested fashion.

Formal definition of Context Free Grammars

- A context-free grammar (CFG) is denoted $G = (V, T, P, S)$, where
  - V and T are finite sets of variables and terminals, respectively.
  - P is a finite set of productions; each production is of the form $A \rightarrow \alpha$,
    - where A is a variable and α is a string of symbols from $(V \cup T)^*$.
  - S is a special variable called the start symbol.

Language generated by CFG

- The notation $S \overset{*}{\underset{G}{\Longrightarrow}} w$ represents the fact that we can derive the string w from the start symbol of the grammar G in zero or more steps
- The language generated by G [denoted $L(G)$] is $\{w|\ w\ is\ inT^*\ and\ S \overset{*}{\underset{G}{\Longrightarrow}} w\}$. That is, a string is in L(G) if:
  - The string consists solely of terminals.
  - The string can be derived from S.

*A language A is said to be a context free language if there exsits a context free grammar G such that $A = L(G)$
*The production rules are context-independent, meaning the replacement of non-terminal symbols can occur without regard to the surrounding context.

C language is an example for Context Free Language.

Example:

- Write CFG for the language $L = \{a^n b^n \middle| n \geq 1\}$

$$\begin{aligned} L &= \{ab, aabb, aaabbb, aaaabbbb, aaaaabbbbb, \ldots\ldots\ldots\ldots\} \\ G &= (\{S\}, \{a, b\}, P, S) \end{aligned}$$

P:

$$\begin{aligned} S &\rightarrow aSb \middle| ab \\ &(Or) \\ S &\rightarrow aSB \\ S &\rightarrow aB \\ B &\rightarrow b \end{aligned}$$

Proving correctness of CFGs

- $L \subseteq L(G)$: Every string in L can be generated by G.
- $L(G) \subseteq L$: G only generate strings of L.

Example:

$$\begin{array}{rcl} G & = & (V, \Sigma, S, P) \\ V & = & \{S\} \\ \Sigma & = & \{0, 1\} \\ S & = & S \\ P & = & \{S \rightarrow 0S1|S1|\epsilon\} \\ & or & \\ G & : & S \rightarrow 0S1|S1|\epsilon \end{array}$$

$L(G) = L$ where $L = \{0^i 1^j || i \leq j\}$ prove $L(G) = L$?

Proving correctness of CFGs

Proof: Need to show that $L \subseteq L(G)$ and $L(G) \subseteq L$.

$\Rightarrow$ Suppose $x \in L$. Must show x can be generated by G.

Proof by induction on the length of x, $\left|x\right|$.

Basis:

- $|x| = 0$. Then $x = \epsilon$ and $S \rightarrow \epsilon$ is a rule in G.

Induction Hypothesis:

- Suppose all words in L shorter than x can be generated in the grammar and that $x \notin \epsilon$.
- Need to show x can also be generated.

Induction Step: since $x \in L, x = 0^i 1^j$

- **Case 1:** $i = 0$ *so* $x = 1^j = 1^{j-1}1$
  - Since $1^{j-1} \in L$ and $|1^{j-1}| < |x|, S \overset{*}{\Rightarrow} 1^{j-1}$ by the I.H.
  - Then use rule $S \rightarrow S1$ followed by the derivation $S \overset{x}{\Rightarrow} 1^{j-1}$ to construct x.
  - Therefore, $x \in L(G)$

Proof cont..

- **Case 2:** $i \neq 0$ *so* $x = 0^i 1^j$
  - Then $x' = 0^{i-1}1^{j-1} \in L$ since $i - 1 \leq j - 1$.
  - $|x'| < |x|$ so by the I.H., $x' \in L(G)$ and $S \stackrel{*}{\Longrightarrow} 0^{i-1}1^{j-1}$
  - Then use rule $S \rightarrow 0S1$ followed by the derivation $S \stackrel{*}{\Longrightarrow} 0^{i-1}1^{j-1}$ to construct x.
  - Thus $S \stackrel{*}{\Longrightarrow} x$ *so* $x \in L(G)$.

$\Leftarrow$Must show if $x \in L(G)$ *the* $x \ \in L$.
Proof by induction on k, the number of steps in x's derivation.
Basis:

- 1 step $(S \Rightarrow \epsilon)$: Then $x = \epsilon$ and $x \in L$.

Induction Hypothesis:

- Assume every word in L(G) with a shorter derivation than x is in L.
- Either $x = 0w1$ *or* $x = w1$ where $w \in L(G)$ and w can be derived from S in less than k steps;
  - i.e. $S \stackrel{<k}{\Longrightarrow} w$

Proving correctness of CFGs

Proof cont..

- By I.H. we know $w \in L$ so it has at least as many 1's as 0's
- Then by applying either $S \to 0S1$ *or* $S \to S1$ followed by the derivation $S \stackrel{*}{\Rightarrow}$ w we obtain x in the form 0^i1^j where we are only increasing the difference of 1's to 0's so $i \leq j$ and $x \in L$.

Therefore, $L(G) \subseteq L$ and $L \subseteq L(G)$ so $L = L(G)$.

Derivation Trees and ambiguity

ϵ –Productions

- A production of the form $A \to \epsilon$, where A is a variable, is called a null production or ϵ –Productions

Unit Productions

- A production of the form $A \to B$ whose right-hand side consists of a single variable is called a unit production.
- All other productions, including those of the form $A \to a$ and ϵ -productions, are nonunit productions.

Recursive Productions

- If a production's left side occurs in right side
- $S \to aS$ (Directly recursive)
- $S \to aA,\ A \to b|bS$ (indirectly recursive)

Derivation Trees and ambiguity

Derivation

- Derivation is the process of applying productions repeatedly to expand non-terminals in terms of terminals or non-terminals, until there are no more non-terminals.
- A derivation can be either Leftmost derivation or Right most derivation.

Leftmost derivation

- If at each step in a derivation a production is applied to the leftmost variable, then the derivation is said to be leftmost.
- **Example:**
  - Consider the grammar $G = (\{S, A\}, \{a, b\}, P, S)$, where P consists of

$$\begin{aligned} S &\rightarrow aAS|a \\ A &\rightarrow SbA|SS|ba \end{aligned}$$

  - The corresponding rightmost derivation is

$$S \implies aAS \implies aSbAS \implies aabAS \implies aabbaS \implies aabbaa.$$

Rightmost derivation:

- A derivation in which the rightmost variable is replaced at each step is said to be rightmost.
- **Example:**
  - Consider the grammar $G = (\{S, A\}, \{a, b\}, P, S)$, where P consists of

$$\begin{array}{rcl} S & \rightarrow & aAS|a \\ A & \rightarrow & SbA|SS|ba \end{array}$$

  - The corresponding rightmost derivation is

$$S \implies aAS \implies aAa \implies aSbAa \implies aSbbaa \implies aabbaa.$$

*****Note:** "If w is in L(G) for CFG G, then w has at least one parse tree, and corresponding to a particular parse tree, w has a unique leftmost and a unique rightmost derivation."

Sentential Form:

- A sentential form is any string consisting of non-terminals and/or terminals that is derived from a start symbol.
- A string of terminals and variables α is called a sentential form if $S \stackrel{*}{\Rightarrow} \alpha$

Example:

- Consider the grammar $G = (\{S, A\}, \{a, b\}, P, S)$, where P consists of

$$\begin{aligned} S &\rightarrow aAS|a \\ A &\rightarrow SbA|SS|ba \end{aligned}$$

- The corresponding rightmost derivation is
  $S \implies aAS \implies aAa \implies aSbAa \implies aSbbaa \implies aabbaa.$

$S \stackrel{*}{\Rightarrow} aAa$, $S \stackrel{*}{\Rightarrow} aSbAa$, $S \stackrel{*}{\Rightarrow} aSbbaa$,$S \stackrel{*}{\Rightarrow} aabbaa$. are Sentential Form

Derivation Trees (or) Parse tree:

- The derivations in a CFG can be represented using trees. Such trees representing derivations are called derivation trees.
- Let G = (V, T, P, S) be a CFG. A tree is a derivation (or parse) tree for G if:
  1. Every vertex has a label, which is a symbol of $V \cup T \cup \{\epsilon\}$.
  2. The label of the root is S(start symbol).
  3. If a vertex is interior and has label A, then A must be in V.
  4. If n has label A and vertices $n_1, n_2, n_3, ..., n_k$ are the sons of vertex n, in order from the left, with labels $X_1, X_2, ..., X_k$, respectively, then $A \rightarrow X_1X_2.......X_k$ must be a production in P.
  5. If vertex n has label ϵ, then n is a leaf and is the only son of its father.

Derivation Trees (or) Parse tree:

Consider the grammar $G = (\{S, A\}, \{a, b\}, P, S)$, where P consists of

$$\begin{aligned} S &\rightarrow aAS|a \\ A &\rightarrow SbA|SS|ba \end{aligned}$$

Construct a derivation tree for the string "aabbaa"

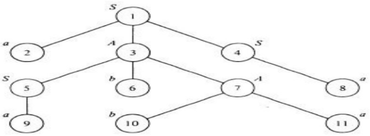


Figure: Derivation tree

$S \implies aAS \implies aSbAS \implies aabAS \implies aabbaS \implies aabbaa.$

Yield of a Tree: The final string obtained by concatenating the labels of the leaves of the tree from left to right, ignoring the Nulls.(eg:-*aabbaa*.)

Ambiguity in context free grammars:

- A context-free grammar G is said to be ambiguous if it has two parse trees for some word.

(or)

- A word which has more than one leftmost derivation or more than one rightmost derivation is said to be ambiguous.

*Note: A CFL for which every CFG is ambiguous is said to be an inherently ambiguous CFL

Example:

- $G = (\{S\}, \{a, b, +, *\}, P.S)$, where P consists of $S \rightarrow S + S|S * S|a|b$ We have two derivation trees for $a + a * b$

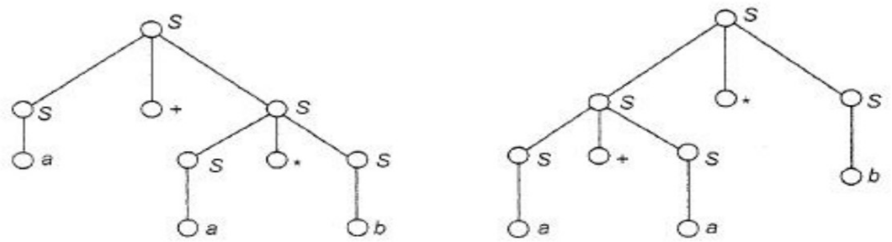


Figure: Two derivation trees for $a + a * b$

Simplified Context free grammar

Reduction of context free grammar

1. We must eliminate useless symbol
   - Eliminate non generating symbols
   - Eliminate non reachable symbols
2. We must eliminate ϵ-production
3. We must eliminate Unit production

Eliminating useless symbol

- Those variables or terminals that do not appears in any derivation of a terminal string from the start symbol
- A symbol γ in CFG is usefull if and ony if
  1. $\gamma \stackrel{*}{\Rightarrow} w$ where $w \in L(G)$ and $w \in T^*$ that is γ leads to a string of terminals. Here γ is said to be generating.
  2. If there is a derivation $S \stackrel{*}{\Rightarrow} \alpha\gamma\beta \stackrel{*}{\Rightarrow} w$ and $w \in L(G)$, for some α and β then γ is said to be reachable.
- So first eliminate symbols that are not generating and then eliminate those symbols are not reachable.

Useful symbol

- A symbol X in a CFG $G = \{V, T, P, S\}$ is called useful if there exist a derviation of a terminal string from S where X appears somewhere, else it is called useless.
    - A symbol X is called generating if some terminal string can be dervied from X.
    - A symbol X is called reachable if it can be reached from S.

Example:

$$\begin{aligned} S &\rightarrow AB/a \\ A &\rightarrow b \end{aligned}$$

Solution:

- Identify nongenerating Symbol
- $S \rightarrow AB/a$ here B is nongenerating.
- The CFG become

$$\begin{aligned} S &\rightarrow a \\ A &\rightarrow b \end{aligned}$$

- A is non reachable
- The CFG become

$$S \rightarrow a$$

Simplified Context free grammar

Eliminate ϵ-production

- ϵ-production are those productions that are of the form $X \rightarrow \epsilon$ these are also called null production
- A variable that derive ϵ ($A \stackrel{*}{\Rightarrow} \epsilon$) then we say A is nullable
- IN CFG, if there is ϵ- production we can remove it without changing the meaning of the grammar.
  - If $A \rightarrow \epsilon$ is a production to be eliminated then we look all production whose right side contain A
  - And replace each occurence of A in each of these productions to obtain the non ϵ-Productions.
  - Now these resultant non ϵ -Production must be added to the grammar to keep the language generated the same.

Simplified Context free grammar

- **Example:**

$$\begin{aligned} S &\rightarrow aA \\ A &\rightarrow b|\epsilon \end{aligned}$$

- $A \rightarrow \epsilon$ is the ϵ-production
- Only one production $S \rightarrow aA$ whose right side contain A , so replace A by ϵ
- we get $S \rightarrow a$
- add this new production to keep the language generated by this grammar same

$$\begin{aligned} S &\rightarrow aA \\ S &\rightarrow a \\ A &\rightarrow b \end{aligned}$$

Eliminate Unit production

- A production i of the form $A \rightarrow B$ is called unit production
- It increase the cost of derivation in a grammar
- If there exist unit production in the grammar
  - Select a unit production $A \rightarrow B$ such that there exist a production $B \rightarrow \alpha$ where α is a terminal
    - For every nonunitproduction $B \rightarrow \alpha$
    - Add production $A \rightarrow \alpha$ to the grammar
    - Eliminate $A \rightarrow B$ from the grammar

Example:

$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow a \\ B &\rightarrow C|b \\ C &\rightarrow D \\ D &\rightarrow E \\ E &\rightarrow a \end{aligned}$$

Solution

Unit Productions are

$$\begin{aligned} B &\rightarrow C \\ C &\rightarrow D \\ D &\rightarrow E \end{aligned}$$

Simplified Context free grammar

- Remove unit production $B \rightarrow C$ if there exists a production whose left side has C and right side contain a terminal but there is no such production in G,
- Similar things hold for production $C \rightarrow D$.
- Now we try to remove Unit production $D \rightarrow E$, because there is a production $E \rightarrow a$
- Eliminate $D \rightarrow E$ and introduce $D \rightarrow a$
- Grammar becomes

$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow a \\ B &\rightarrow C|b \\ C &\rightarrow D \\ D &\rightarrow a \\ E &\rightarrow a \end{aligned}$$

Simplified Context free grammar

- Now we can remove $C \to D$ by using $D \to a$

$$\begin{aligned} S &\to AB \\ A &\to a \\ B &\to C|b \\ C &\to a \\ D &\to a \\ E &\to a \end{aligned}$$

Similarly remove $B \to C$ by using $C \to a$

$$\begin{aligned} S &\to AB \\ A &\to a \\ B &\to a|b \\ C &\to a \\ D &\to a \\ E &\to a \end{aligned}$$

Simplified Context free grammar

- $C \rightarrow a, D \rightarrow a, E \rightarrow a$ are useless symbols so the CFG become

$$
\begin{aligned}
S &\rightarrow AB \\
A &\rightarrow a \\
B &\rightarrow a|b
\end{aligned}
$$

Chomsky Normal Form(CNF)

- Any context-free language without ϵ is generated by a grammar in which all productions are of the form $A \rightarrow BC$ or$A \rightarrow a$.
- Here, A, B, and C, are variables and a is a terminal.

Converting CFG into CNF

1. Arrange that all bodies of length 2 or more consist only of variable
2. Break bodies of length ≥ 3 into cascade of productions each with a body consisting of two variables.

Chomsky Normal Form(CNF)

Steps for converting CFG into CNF

Step 2: Simplify the grammar.

a. Eliminate ϵ –productions
b. Eliminate unit productions
c. Eliminate Useless symbols.

Chomsky Normal Form(CNF)

Step 3: Eliminate terminals from RHS if they exist with other terminals or non-terminals.

- Consider a production in P,of the form $A \rightarrow X_1X_2X_3.....X_m$ where $m \geq 2$.If X_i is a terminal,
- Introduce a new variable Ca and a production $C_a \rightarrow a$.
- Then replace X_i by C_a.

Step 4:

- Consider a production $A \rightarrow B_1B_2B_3.....B_m$ where $m \geq 3$,
- Create new variables $D_1, D_2, ...D_{m-2}$ and replace $A \rightarrow B_1B_2B_3...B_m$ by the set of productions $\{A \rightarrow B_1D_1, D_1 \rightarrow B_2D_2, ...D_{m-3} \rightarrow B_m - 2D_{m-2}, D_{m-2} \rightarrow B_{m-1}B_m\}$

Chomsky Normal Form(CNF)

Example: Consider the grammar $(\{S, A, B\}, \{a, b\}, P, S)$ that has the productions:

$$\begin{aligned} S &\rightarrow bA|aB \\ A &\rightarrow bAA|aS|a \\ B &\rightarrow aBB|bS|b \end{aligned}$$

Sln:

Step 1: Simplify the grammar.

- The given grammar does not contain ϵ –productions, unit productions and useless symbols.
- It is in optimized form.

Chomsky Normal Form(CNF)

Step 2:

- The only productions already in proper form are $A \to a$ and $B \to b$.
- So we may begin by replacing terminals on the right by variables, except in the case of the productions $A \to a$ and $B \to b$.
- $S \to bA$ is replaced by $S \to C_bA$ and $C_b \to b$.
- Similarly, $A \to aS$ is replaced by $A \to C_aS$ and $C_a \to a$; $A \to bAA$ is replaced by $A \to C_bAA$; $S \to aB$ is replaced by $S \to C_aB$; $B \to bS$ is replaced by $B \to C_bS$, and $B \to aBB$ is replaced by $B \to C_aBB$.

Step 3:

- In the next stage, the production $A \to C_bAA$ is replaced by $A \to C_bD1$ and $D1 \to AA$, and the production $B \to C_aBB$ is replaced by $B \to C_aD2$ and $D2 \to BB$.

Chomsky Normal Form(CNF)

The productions for the grammar in CNF are :

$$
\begin{aligned}
S &\rightarrow C_bA|C_aB \\
A &\rightarrow C_aS|C_bD_1|a \\
B &\rightarrow C_bS|C_aD_2|b \\
C_b &\rightarrow b \\
D1 &\rightarrow AA \\
D2 &\rightarrow BB \\
Ca &\rightarrow a \\
Cb &\rightarrow b
\end{aligned}
$$

Normal Form

Left Recursion

- A grammar which is of the form $A \rightarrow A\alpha|\beta$
- To avoid left recursion

$$\begin{aligned} A &\rightarrow \beta A' \\ A' &\rightarrow \alpha A' \\ A' &\rightarrow \epsilon \end{aligned}$$

- After removing ϵ production the grammar become

$$\begin{aligned} A &\rightarrow \beta|\beta A' \\ A' &\rightarrow \alpha|\alpha A' \end{aligned}$$

Greibach Normal Form(GNF)

- Every context free language L without ϵ can be generated by a grammar for which every production is of the form $A \rightarrow a\alpha$ or $A \rightarrow a$
- Where A is a variable , a is a terminal and α is a string of variables.

To convert given grammar into GNF following Two lemma's are considered

Lemma1:

- Define an A-production to be a production with variable A on the left.
- Let $G = (V, T, P, S)$ be a CFG. Let $A \rightarrow \alpha_1 B \alpha_2$ be a production in P and $B \rightarrow \beta_1|\beta_2|....|\beta_r$ be the set of all B-productions.
- Let $G1 = (V, T, P1, S)$ be obtained from G by deleting the production $A \rightarrow \alpha_1 B \alpha_2$ from P and adding the productions $A \rightarrow \alpha_1\beta_1\alpha_2|\alpha_1\beta_2\alpha_2|...|\alpha_1\beta_r\alpha_2$.
- Then $L(G) = L(G1)$.

Greibach Normal Form(GNF)

Lemma2:

- Let $G = (V, T, P, S)$ be a CFG. Let $A \rightarrow A\alpha_1|A\alpha_2|...|A\alpha_r$ be the set of A-productions for which A is the leftmost symbol of the right-hand side.
- Let $A \rightarrow \beta_1|\beta_2|...|\beta_s$ be the remaining A-productions.
- Let $G1 = (V \cup \{B\}, T, P1, S)$ be the CFG formed by adding the variable B to V and replacing all the A-productions by the productions:

$$\left.\begin{aligned} A \rightarrow &\quad \beta_i \\ A \rightarrow &\quad \beta_i B \end{aligned}\right\} 1 \leq i \leq s \qquad\qquad \left.\begin{aligned} B \rightarrow &\quad \alpha_i \\ B \rightarrow &\quad \alpha_i B \end{aligned}\right\} 1 \leq i \leq r$$

- Then $L(G1) = L(G)$.

Algorithm for converting a CFG into GNF

1. Remove Unit productions, useless symbols and ϵ-production, if any.
2. For every terminal symbol introduce a new non terminal.
   - $A \rightarrow Ba$ become $A \rightarrow BC_a$, $C_a \rightarrow a$
3. Introduce an order among nonterminals by renaming them.
4. Using substitutions, rewrite productions (except Left Recursive) if required, to ensure that all productions of the form $X_i \rightarrow X_j\alpha$ satisfy the condition $i < j$
5. Remove Left Recursions, if any.
6. Remove unit productions,useless symbol if any added in step 4.
7. Obtain the CFG in GNF by applying substitutions.

Greibach Normal Form(GNF)

Example: Convert the following grammar into GNF

$$\begin{aligned} S &\rightarrow AB|CC \\ C &\rightarrow b|SC \\ A &\rightarrow a \\ B &\rightarrow b \end{aligned}$$

Sln:

- The given CFG is in CNF
- Renaming the nonterminals in G with index values
  - Replace, S with X_1,A with X_2 ,B with X_3,C with X_4
- Equivalent CFG G'

$$\begin{aligned} X_1 &\rightarrow X_2X_3|X_4X_4 \\ X_4 &\rightarrow b|X_1X_4 \\ X_2 &\rightarrow a \\ X_3 &\rightarrow b \end{aligned}$$

Example cont..

- The production $X_4 \rightarrow X_1X_4$ does not satisfy our requirement we need $i < j$
- Now, substitute the value of X_1 in X_4 we will get $X_4 \rightarrow X_2X_3X_4 | X_4X_4X_4$
- Substitute $X_2 \rightarrow a$ into $X_4 \rightarrow X_2X_3X_4$ then $X_4 \rightarrow aX_3X_4$
- The production $X_4 \rightarrow X_4X_4X_4$ is left recursive remove the leftrecursive
- the grammar become

$$
\begin{aligned}
X_1 &\rightarrow X_2X_3 | X_4X_4 \\
X_4 &\rightarrow b | aX_3X_4 | bX_5 | aX_3X_4X_5 \\
X_2 &\rightarrow a \\
X_3 &\rightarrow b \\
X_5 &\rightarrow X_4X_4 | X_4X_4X_5
\end{aligned}
$$

Example cont..

- Againg perform substitution then the grammar become

$$\begin{aligned}
X_1 &\rightarrow aX_3|X_4X_4 \\
X_4 &\rightarrow b|aX_3X_4|bX_5|aX_3X_4X_5 \\
X_2 &\rightarrow a \\
X_3 &\rightarrow b \\
X_5 &\rightarrow X_4X_4|X_4X_4X_5
\end{aligned}$$

- Againg substitute the value of X_4 then the grammar become

$$\begin{aligned}
X_1 &\rightarrow aX_3|bX_4|aX_3X_4X_4|bX_5X_4|aX_3X_4X_5X_4 \\
X_4 &\rightarrow b|aX_3X_4|bX_5|aX_3X_4X_5 \\
X_2 &\rightarrow a \\
X_3 &\rightarrow b \\
X_5 &\rightarrow X_4X_4|X_4X_4X_5
\end{aligned}$$

Example cont..

- Againg substitute the value of X_4 then the grammar become

$$
\begin{aligned}
X_1 &\rightarrow aX_3|bX_4|aX_3X_4X_4|bX_5X_4|aX_3X_4X_5X_4 \\
X_4 &\rightarrow b|aX_3X_4|bX_5|aX_3X_4X_5 \\
X_2 &\rightarrow a \\
X_3 &\rightarrow b \\
X_5 &\rightarrow bX_4|aX_3X_4X_4|bX_5X_4|aX_3X_4X_5X_4| \\
&\quad\ bX_4X_3|aX_3X_4X_4X_5|bX_3X_4X_5|aX_3X_4X_5X_4X_5
\end{aligned}
$$