GRAPH THEORY

Module 5

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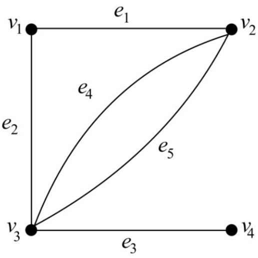
Outline

Incidence Matrix

Incidence Matrix

Let G be a graph with n vertices, e edges, and no self-loops.Define an $n \times e$ matrix $A = [a_{ij}]$, whose n rows correspond to the n vertices and the e columns correspond to the e edges, as follows

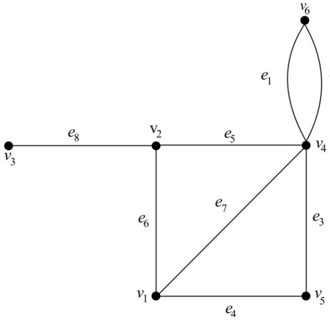
$$a_{ij} = \begin{cases} 1, & \textit{if jth edge } m_j \textit{ is incident on the ith vertex} \\ 0, & \textit{otherwise.} \end{cases}$$



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Problems

1. Find the incidence matrix of the following graph

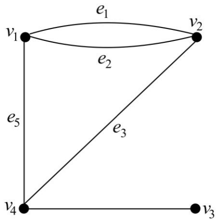


$$
\begin{array}{c c}
 & \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}.
$$

Problems

2 Draw the graph corresponding tho the incidence matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$



Properties of Incidence Matrix

- Each row represents a vertex
- Number of 1s in a row gives the degree of that vertex
- A row with full zeroes indicate an isolated vertex
- Since every edge is incident on exactly two vertices, each column has exactly two 1s
- Parallel edges produce identical columns
- Self loops are not indicated
- If a graph is disconnected and consists of two components G1 and G2, the incidence matrix A(G) of graph G can be written in a block diagonal form as

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix},$$

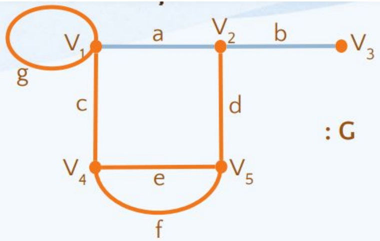
  where A(G1) and A(G2) are the incidence matrices of components G1 and G2.

Properties of Incidence Matrix cont..

- Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.
- Rank of A(G)=n-1 if G is connected; Otherwise A(G)=n-k
- Incidence matrix of a tree is of the order n x n-1 as e=n-1 for a tree

Incidence Matrix

Example



$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right] \end{array}$$

degree
2
3
1
3
3

Theorem

Two graphs G1 and G2 are isomorphic if and only if their incidence matrices A(G1) and A(G2) differ only by permutations of rows and columns.

Proof

- Let the graphs G1 and G2 be isomorphic.Then there is a one-one correspondence between the vertices and edges in G1 and G2 such that the incidence relation is preserved.
- Thus A(G1) and A(G2) are either same or differ only by permutation of rows and columns.
- The converse follows,
- since permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

Incidence Matrix

Theorem

If A(G) is an incidence matrix of a connected graph G with n vertices,then rank of A(G) is n-1.

- Let G be a connected graph with n vertices.
- Now, each row in A(G) is a vector over GF(2) in the vector space of graph G.
- Let the row vectors be denoted by A1, A2,..., An. Then,

$$\text{Then, } A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix}.$$

- Since there are exactly two 1's in every column of A, the sum of all these vectors is 0 (this being a modulo 2 sum of the corresponding entries). Thus vectors A1, A2, ..., An are linearly dependent. Therefore, rank A $<$ n. Hence, rank A$\leq$ n $-$ 1 $eq(1)$.

- Consider the sum of any m of these row vectors, $m \leq n-1$. Since G is connected, A(G) cannot be partitioned in the form

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$$

- such that A(G1) has m rows and A(G2) has n-m rows.
- Thus there exists no m×m submatrix of A(G) for $m \leq n-1$, such that the modulo 2 sum of these m rows is equal to zero.
- As there are only two elements 0 and 1 in this field, the additions of all vectors taken m at a time for m = 1, 2,..., n-1 gives all possible linear combinations of n-1 row vectors.
- Thus no linear combinations of m row vectors of A, for $m \leq n-1$, is zero.Therefore, rank $A(G) \geq n-1$. $eq(2)$
- Therefore,Combining the above two Equation $eq(1)$ & $eq(2)$ it follows that rank A(G) = n-1.

Incidence Matrix

Theorem

The reduced incidence matrix of a tree is non-singular.

Proof:

- A tree is a connected graph with n vertices & (n-1) edges
- Its reduced incidence matric is a square matrix of the order $(n-1)x(n-1)$
- And its rank is (n-1)
- Therefore the reduced incidence matrix is non-singular

Note:A matrix is non-singular if its inverse exists or its determinant is non zero

Incidence Matrix

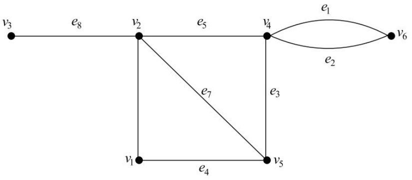
Reduced Incidence Matrix

- When we remove any 1 row from the incidence matrix, the resultant matrix is known as reduced incidence matrix - Af
- It is of the order (n-1) x e
- The vertex corresponding to the deleted row in called reference vertex
- Any vertex can be made the reference vertex
- (n-1) rows contain the same information as n rows
- We can easily construct the deleted row from the remaining rows, as each column can contain exactly two 1s

Circuit Matrix

Circuit Matrix

- Let G be a graph with n vertices, e edges and b circuits
- Circuit matrix denoted by B(G) is a $b \ x \ e$ matrix where rows corresponds to b circuits and columns, e edges
- $b_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ edge is included in } i^{th} \text{ circuit} \\ 0 & \text{otherwise} \end{cases}$



G_1

$$B(G_1) = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The graph G1 has four different cycles Z1 = {e1, e2}, Z2 = {e3, e5, e7}, Z3 = {e4, e6, e7} and Z4 = {e3, e4, e6, e5}.

Properties of circuit matrix

- Each row represents a circuit
- Number of 1s in a row indicates no. of edges included in the circuit
- There can be no row with full zeroes
- Number of 1s in a column indicates no. of circuits in which that particular edge is included
- A column with full zeroes indicate a non-circuit edge or an edge that is not a part of any circuit (like pendant edges or cut edge)
- A row with two 1s indicates a pair of parallel edges
- A row with a single 1 indicates a self loop
- If the graph G is separable (or disconnected) and consists of two blocks (or components) H1 and H2, then the cycle matrix B(G) can be written in a block-diagonal form as

$$B(G) = \begin{bmatrix} B(H_1) & 0 \\ 0 & B(H_2) \end{bmatrix},$$

where B(H1) and B(H2) are the cycle matrices of H1 and H2.

Circuit Matrix

Properties of circuit matrix cont..

- Permutation of any two rows or columns in a cycle matrix corresponds to relabeling the cycles and the edges.
- Rank of B(G) = e-n+1 if G is connected
- Rank of B(G) = e-n+k if G is disconnected
- Two graphs G1 & G2 will have same circuit matrices if and only if they are 2 – isomorphic
  - Since circuit matrix do not define the graph completely, we cannot specify complete isomorphism, only 2 – isomorphism can be specified

Circuit Matrix

Theorem

Let B and A be, respectively, the circuit matrix and the incidence matrix (of a self-loop-free graph) whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row A; that is,

$$A.B^T = B.A^T = 0(mod2)$$

Proof:

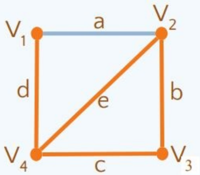
- Consider a vertex V and a circuit ρ in G
- There can be 2 cases
  - **Case1:** V is in ρ, then the no.of edges in ρ, that is incident on V is exactly 2
  - **Case2:** V is not in ρ, then no edge of ρ is incident on v
- consider ith row of A (non zero entries $\rightarrow$ incident on ith vertex) and jth row of B(which is the j^{th} column of B^T)

Proof cont..

- If ith vertex is not in jth circuit then there is no 1 in their common positions, so their dot product is zero
- If ith vertex is in jth circuit, there would be exactly two 1s in common position. Sum of their dot product turns to be 2, which is 0 (since mod 2)
- Since dot product of any two arbitrary rows is zero (one from A & one from B), $A.B^T = B.A^T = 0(mod2)$

Circuit Matrix

Example



: G

$$A(G) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$B(G) = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Circuit Matrix

$$
A.\,B^T = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array}
\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}
\cdot
\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}
=
$$

$$
\begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \pmod 2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0
$$

Circuit Matrix

Theorem

If B is a cycle(circuit) matrix of a connected graph G with n vertices and e edges, then rank B = e-n+1.

Proof

- Let A be the incidence matrix of the connected graph G.
- Then $AB^T \cong 0(mod2)$.
- Using Sylvester's theorem , we have rank A+ rank $B^T \leq e$ so that rank A+ rank $B \leq e$.
- Therefore, rank $B \leq e - rankA$.
- As rank A = n-1, we get rank $B \leq e - (n-1) = e - n + 1$. $eq(1)$
- We have the result, rank of $B_f = e - n + 1$
- Sience B_f(Fundamental Circuit Matrix) is a sub matrix of B
  - Rank of $B \geq e - n + 1$ $eq(2)$
- Combining $eq(1)$ & $eq(2)$, we get rank B = e-n+1.

Circuit Matrix

Theorem

If B is a cycle matrix of a disconnected graph G with n vertices, m edges and k components, then rank B = m-n+k.

Example

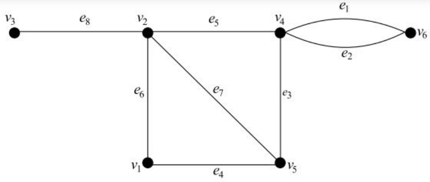


Fig. 10.5

Clearly,

$$AB^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

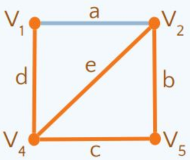
$$= \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix} \equiv 0(\text{mod}2).$$

Fundamental Circuit Matrix

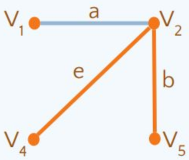
- A fundamental circuit matrix, denoted by $B_f(G)$ is a circuit matrix, in which each row corresponds to a fundamental circuit of G wrt some spt
- If we rearrange the columns of $B_f(G)$ such that the chord corresponding to the first row is placed at the first column, and the one corresponding to the second row is placed at the second column and so on. Then we can write,
- $B_f(G) = [I\mu : B_t]$
- where $I\mu$ is the identity matrix of order $\mu = e - n + 1$ and Bt is the remaining submatrix of order $\mu x(n-1)$ corresponding to the branches of the spt
- Then we can say that rank of $B_f(G) = \mu = e - n + 1$

Fundamental Circuit Matrix

Example



: G



: SPT

Branches: a, b, e
Chord: c, d

- Since 2 chords, 2 fundamental circuits
- $\mu = 2, e = 5$

$$B_f(G) = \begin{array}{c} \\ b_1 \\ b_2 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}_{\mu \times e} \end{array}$$

Fundamental Circuit Matrix

- Now, rearrange the columns, in the order of chords

$$B_f(G) = \begin{array}{c} \\ b_1 \\ b_2 \end{array} \begin{array}{c} \begin{array}{ccccc} c & d & a & b & e \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array} \right] \end{array}$$

$\mu \times \mu$ $\quad$ $\mu \times (n-1)$ $\quad$ $\mu \times e$

$$= [I_2 : B_{2x3}]$$

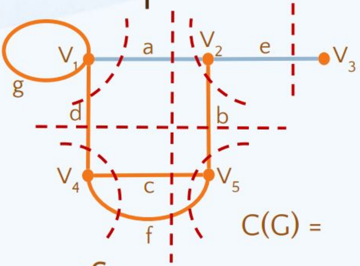
$$= [I\mu : B_t]$$

Cut-set matrix

- Let G be a graph with n vertices, e edges and b cut-sets
- Cut-set matrix denoted by C(G) is a c x e matrix where rows corresponds to c cut-sets and columns, e edges
- The matrix element
- $c_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ edge is included in } i^{th} \text{ cutset} \\ 0 & \text{otherwise} \end{cases}$
- Cut-sets that contain just 1 edge, implies a pendant edge
- Cut-sets that contain 2 edges (rows with two 1s) may or may not be a parallel edge pair

Cut-set matrix

Example



: G

$$C(G) = \begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline C_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C_2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ C_3 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ C_4 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ C_5 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ C_6 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ C_7 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}$$

Rank of A(G) = n -1 = 5-1 = 4

Properties of cut-set matrix

- Each row represents a cut-set
- Number of 1s in a row indicates no. of edges included in the cut-set
- There can be no row with full zeroes
- Number of 1s in a column indicates no. of cut-sets in which that particular edge is included
- A column with full zeroes indicate an edge that is not a part of any cut-set. Self loops do not appear in any cut-set. Hence a self loop will have full zeroes in its column
- Parallel edges will have identical columns
- A column with full zeroes indicate a self loop

Properties of cut-set matrix cont...

- If G is a non-separable graph, every set of edges incident on a vertex is a cut set. Hence every row of A(G) will appear in C(G). so we can say C(G) contains A(G)
- If G is a separable graph, incidence matrix of each block will be in C(G) i.e, C(G) contains A(g1), A(g2) etc
- Permutations of any two rows or columns of C(G) simply correspond to rebelling of circuits & edges of the same graph

Properties of cut-set matrix cont...

- Rank of C(G) = n-1 if G is connected
- Rank of C(G) = n-k if G is disconnected
- If B(G) is the circuit matrix of G & C(G) the cut-set matrix of G, both of whose columns are arranged in the same order of edges, then

$$B.C^T = C.B^T = 0(mod 2)$$

Fundamental cut-set matrix

- Cut-sets of a graph that corresponds to each branch of a spanning tree of G are known as fundamental cut-sets
- Since no. of branches = n − 1,
    - no. of fundamental cut-sets = n − 1
    - therefore, order of the matrix $C_f(G) = (n-1)xe$
    - $Cf_(G)$ is a Submatrix of C(G)
- We can obtain $C_f(G)$ from C(G) by retaining those rows that correspond to fundamental cut-sets and removing the remaining rows. The removed rows can be reconstituted from the fundamental rows
- Thus a fundamental cut-set matrix denoted by Cf (G) is a cut-set matrix, in which each row corresponds to a fundamental cut-set of G wrt to some spt

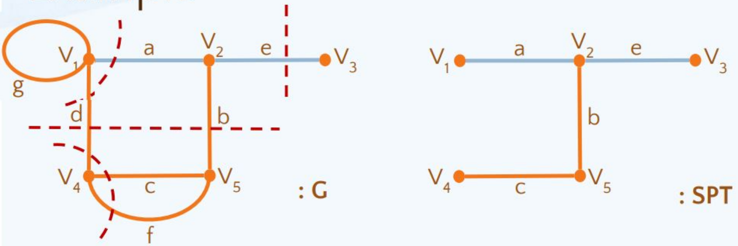
Fundamental cut-set matrix

- If we arrange the columns of Cf (G)such that the branch that is corresponding to the first cut-set is at last column & the branch related to the second cut-set is at second-last column and so on. Then we can write
- $C_f(G) = [C_c : I_{n-1}] = [C_c : I_r]$
- where $I_r$is the identify matrix of order (n-1)x(n-1) and C_c is the remaining submatrix of order $(n-1)x(e-n+1)$

Circuit Matrix

Fundamental cut-set matrix

Example



$$C_f(G) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Fundamental cut-set matrix

- Now, rearrange the columns, in the order of branches

$$
C_f(G) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} g & f & d & c & b & a & e \end{array} \\
\left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

b x e

r x r

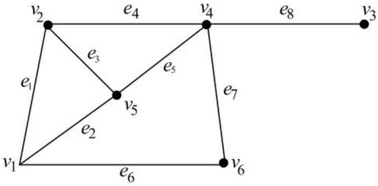
$= [C_{4\times3} : I_4]$

$= [C_c : I_r]$

Adjacency Matrix

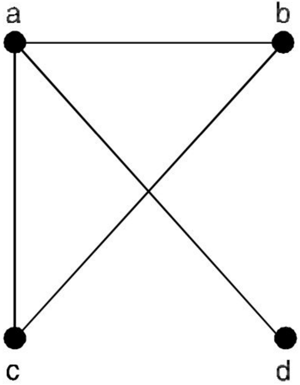
The adjacency matrix of a graph G with n vertices and no parallel edges is an n $\times$ n symmetric binary matrix $X = [x_{ij}]$ x defined over the ring of integers such that

$$x_{ij} = \begin{cases} 1, & if\ v_i v_j \in E, \\ 0, & otherwise. \end{cases}$$



$$X = \begin{array}{c c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Problems: Find the adjacency matrix of the following graph

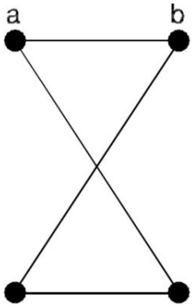


$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrix

Problems: Draw the graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



Adjacency Matrix

Properties of adjacency matrix

- Each row corresponds to a vertex
- The no. of 1s in a row indicates the degree of the vertex (as diagonal element indicates self loop, they are counted as '2')
- A row with full zeroes indicates isolated vertex
- The no. of 1s in a column indicates the degree of the vertex
- A column with full zeroes indicates isolated vertex
- Parallel edges cannot be indicated
- A self loop is indicated when a diagonal element is 1
- A graph G is disconnected having components G1 and G2 if and only if the adjacency matrix X(G) is partitioned as

$$X(G) = \begin{bmatrix} X(G_1) & : & O \\ .. & : & .. \\ O & : & X(G_2) \end{bmatrix}$$

  where X(G1) and X(G2) are respectively the adjacency matrices of the components G1 and G2

Adjacency Matrix

Properties of adjacency matrix cont..

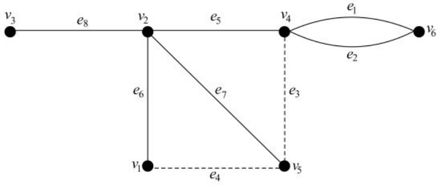
- Permutation of rows and the corresponding columns imply reordering the vertices.
- Rank of X(G)=n

Path Matrix

A path matrix is defined for a specific pair of vertices in a graph, say (x, y), and is written as P(x,y). The rows in P(x, y) correspond to different paths between vertices x and y, and the columns correspond to the edges in G. That is, the path matrix for (x, y) vertices is P(x, y) = $[p_{ij}]$, where

$$p_{ij} = \begin{cases} 1, & if\ \ jth\ edge\ lies\ in\ the\ ith\ path, \\ 0, & otherwise. \end{cases}$$

Find P(v3, v4)



The different paths between the vertices v_3 and v_4 are

$p_1 = \{e_8, e_5\}$, $p_2 = \{e_8, e_7, e_3\}$ and $p_3 = \{e_8, e_6, e_4, e_3\}$.

The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \left[\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \end{array}.$$

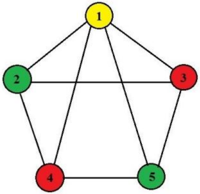
Properties of Path Matrix

- Each row represents a path b/w Vi & Vj
- Number of 1s in a row indicates no. of edges in the path Pij
- A row with full zeroes not possible
- The no. of 1s in a column indicates the no. of paths in which that particular edge is involved
- A column with full zeroes indicates that the edge is not present in any of the paths b/w Vi & Vj
- Parallel edges produce 2 identical rows except for a single edge
- Self loops can be included in any path
- The ring sum of any 2 rows in P_{ij} corresponds to a circuit or an edge disjoint union of circuits

Graph Colouring Problem(PROPER COLORING)

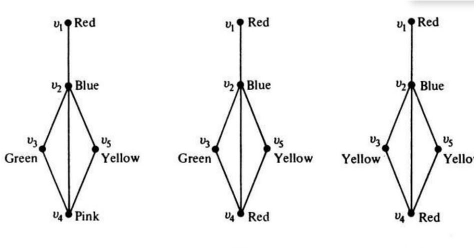
- A colouring of simple graph is the assignment of a colour to each vertex of the graph so that no two adjacent vertices are assigned the same colour
- A graph in which every vertex has been assigned a color according to a proper coloring is called a properly colored graph.

- A graph G that requires "k" different colors for its proper coloring and no less is called a k-chromatic graph.
- The number k is called chromatic number of G.



Chromatic Number(K-chromatic graph)

The chromatic number of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color



(a) (b) (c)

the above graph is initially coloured with 5 different colour then 4 and finally three so the chromatic number is 3 i.e the graph is 3-chromatic

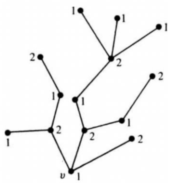
Properties of Chromatic Number

- A graph consisting of only isolated vertices is 1-chromatic.
- A graph with one or more edges (not a self-loop, of course) is at least 2- chromatic (also called bichromatic).
- Every tree with two or more vertices is 2-chromatic
- A complete graph of n vertices is n-chromatic, as all its vertices are adjacent.
- A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.

Chromatic Number

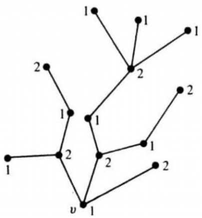
Theorem

Every tree with two or more vertices is 2-chromatic.



- Select any vertex v in the given tree T.
- Consider T as a rooted tree at vertex v. Paint v with color 1. Paint all vertices adjacent to v with color 2.
- Next, paint the vertices adjacent to these (those that just have been colored with 2) using color 1.
- Continue this process till every vertex in T has been painted.

Chromatic Number



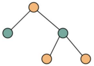
- Now in T we find that all vertices at odd distances from v have color 2, while v and vertices at even distances from v have color 1.
- Now along any path in T the vertices are of alternating colors.
- Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same color.
- Thus T has been properly colored with two colors. Moreover one color would not have been enough.

Theorem

A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.

Proof.

- Let G be a connected graph with cycles of only even length and let T be a spanning tree in G.
- Then, T can be coloured with two colours. Now add the chords to T one by one.



- As G contains cycles of even length only, the end vertices of every chord get different colours of T .
- Thus G is coloured with two colours and hence is 2-chromatic.
- Conversely, if G has a circuit of odd length, we would need at least three colors just for that circuit. Thus the theorem

Theorem (An upper bound for k)

For any graph $G, k(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of a vertex in G.

Proof.

- We prove the result by induction on n, the number of vertices of G.
- If n = 1, we have $\Delta(G) = 0$ and k = 1. Thus $k(G) \leq \Delta(G) + 1$,
- Assume the result is true for a simple graph with n vertices.
- Let G be a simple graph with n + 1 vertices.
- Let v be a vertex of G and consider G1 = G - v.
- Now G1 is a simple graph with n vertices and $\Delta(G1) \leq \Delta(G)$ and also by induction hypothesis, $k(G1) \leq \Delta(G1) + 1 \leq \Delta(G) + 1$
- This means that G1 can be properly colored by at most $\Delta(G) + 1$ colors.

Proof Cont...

- Now considering the graph G, the vertex v has at most $\Delta(G)$ adjacent vertices.
- Since $\Delta(G1) \leq \Delta(G) + 1$, it follows that not all the $\Delta(G) + 1$ colors are required to color the possible $\Delta(G)$ adjacent vertices of v and hence an unused color among the $\Delta(G) + 1$ colors could be used to color v.
- Hence.$k(G) \leq \Delta(G) + 1$

Chromatic Polynomial

Chromatic Polynomial

- A given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors.
- The computation of the total number of ways of proper coloring of a graph G of n vertices using λ or fewer colors is expressed by means of a polynomial called chromatic polynomial $P_n(\lambda)$ of G

Chromatic Polynomial

Let c_i be the different ways of properly coloring G using exactly i different colors. Then

$$P_n(\lambda) = \sum_{i=1}^{n} c_i \binom{\lambda}{i}$$

$$P_n(\lambda) = c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + \dots + c_n \frac{\lambda(\lambda-1)(\lambda-2).....(\lambda-n+1)}{n!}$$

Chromatic Polynomial of a Complete Graph

Theorem

A graph of n vertices is a complete graph if and only if its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2)...(\lambda - n + 1)$

Proof.

- With λ colors, there are λ different ways of coloring any selected vertex of a graph.
- A second vertex can be colored properly in exactly $\lambda - 1$ ways, the third in $\lambda - 2$ ways, the fourth in $\lambda - 3$ ways, . . . , and the n_{th} in $\lambda - n + 1$ ways if and only if every vertex is adjacent to every other.
- That is, if and only if the graph is complete

Chromatic Polynomial of a tree

Theorem

An n-vertex graph is a tree if and only if its chromatic polynomial

$$P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$$

Proof:

- We will prove the result using induction on n, the number of vertices of the tree.
- When n $=$ 1, we have an isolated vertex and hence $P_n(\lambda) = \lambda$
- When n $=$ 2, we have k_2 vertex and hence $P_n(\lambda) = \lambda(\lambda - 1)$
- Assume the result is true for all trees upto k vertices.
  ie $P_k(\lambda) = \lambda(\lambda - 1)^{k-1}$

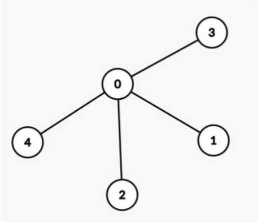
Chromatic Polynomial of a tree

Proof cont...

- Now consider a tree T with k + 1 vertices.
- Now every tree with $n \geq 2$ has atleast 2 pendant vertices.
- Now removing one of the pendant vertices results in a tree T1 with k vertices.
- This tree T1 can be colored in $\lambda(\lambda-1)^{k-1}$ ways by induction hypothesis.
- After coloring all the vertices inT1 attach back the removed pendant vertex to T1 to obtain the tree T.
- Now this pendant vertex can be colored in $\lambda - 1$ ways. without affecting the proper coloring.
- Hence by product rule, the total number of ways to color the vertices in T is $\lambda(\lambda-1)^{k-1}(\lambda-1) = \lambda(\lambda-1)^k$

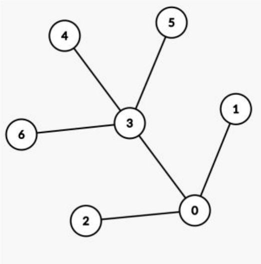
Chromatic Polynomial

Problem:

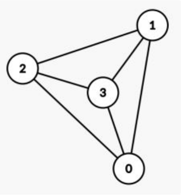


Chromatic Polynomial

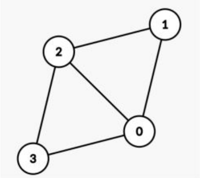
Problem:



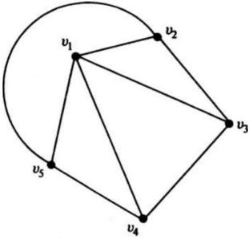
Problem:



Problem:

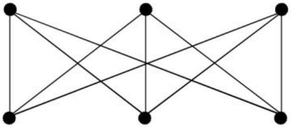


Problem:

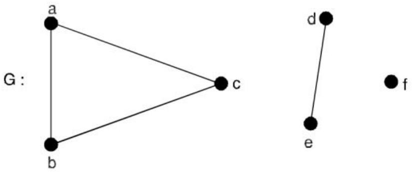


Chromatic Polynomial

Problem:



Problem:



Problem:

THEOREM 8-6

Let a and b be two nonadjacent vertices in a graph G. Let G' be a graph obtained by adding an edge between a and b. Let G'' be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edges. Then

$$P_n(\lambda) \text{ of } G = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''.$$

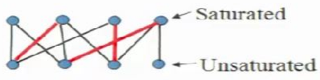
Matching

Matching

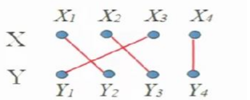
A matching in a graph is a subset of edges in which no two edges are adjacent.

- A single edge in a graph is a matching.

- The vertices incident to the edges of a matching M are **staturated**by M; the others are **unsaturated**
  - we say M-saturated and M-unsaturated



- A **perfect matching or Complete matching** in a graph is a matching that saturates every vertex



Matching

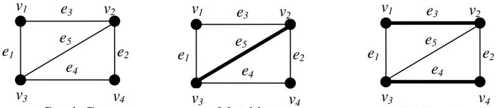
Matching

A matching in a graph is a subset of edges in which no two edges are adjacent.

- A single edge in a graph is a matching.

Maximal matching

- A maximal matching is a matching to which no edge in the graph can be added.
- A graph may have many different maximal matchings, and of different sizes. Among these, the maximal matchings with the largest number of edges are called the **largest maximal matchings.**
- The number of edges in a largest maximal matching is called the **matching number** of the graph.



Maximal matching

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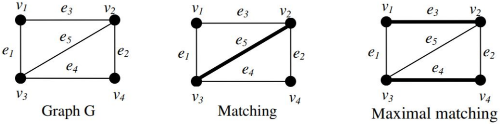
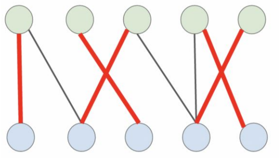


Matching

Complete Matching In a bipartite graph

- In a bipartite graph having a vertex partition V1 and V2, a complete matching of vertices in set V1 into those in V2 is a matching in which there is one edge incident with every vertex in V1.
- In other words, every vertex in V1 is matched against some vertex in V2.

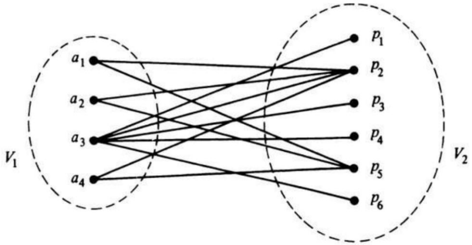


- A complete matching (if it exists) is a largest maximal matching, whereas the converse is not necessarily true.

Matching

Complete Matching In a bipartite graph cont...

- For the existence of a complete matching of set V1 into set V2, first we must have at least as many vertices in V2 as there are in V1.
  - Note that as per our definition, V2 can have more vertices than V1 in a complete matching from V1 into V2.
  - But even the existence of as many vertices in V2 as there are in V1 will not guarantee a complete matching from V1 into V2.



Complete Matching In a bipartite graph cont...

- A necessary condition for existence of a complete matching from V1 into V2 is that every subset of r vertices in V1 must collectively be adjacent to at least r vertices in V2, for all values of r = 1, 2, ...,$|V1|$

Theorem

A complete matching of V1 into V2 in a bipartite graph exists if and only if every subset of r vertices in V1 is collectively adjacent to r or more vertices in V2 for all values of r.

Theorem

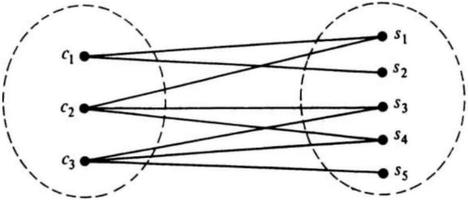
In a bipartite graph a complete matching of V1 into V2 exists if there is a positive integer m for which the following condition is satisfied degree of every vertex in V1 $\geq$ m $\geq$ degree of every vertex in V2

Proof:

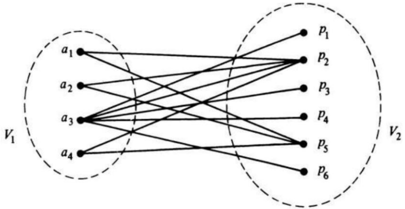
- Consider a subset of r vertices in V1.
- These r vertices have at least $m \ x \ r$ edges incident on them.
- Each $m \ x \ r$ edge is incident to some vertex in V2.
- Since the degree of every vertex in set V2 is no greater than m
  - these $m \ x \ r$ edges are incident on at least $\frac{m \ x \ r}{m} = r$ vertices in V2.
- Thus any subset of r vertices in V1 is collectively adjacent to r or more vertices in V2.
- Therefore, according to earlier result, there exists a complete matching of V1 into V2.

Matching

Proof Cont..



- Here degree of every vertex in $V1 \geq 2 \geq$ degree of every vertex in V2.
- Therefore there exists a complete matching.

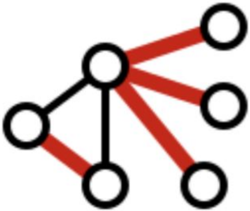


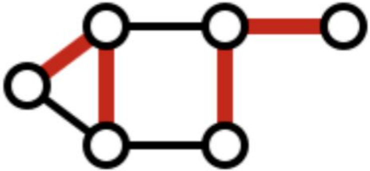
Coverings

Coverings

In a graph G, a set g of edges is said to cover G if every vertex in G is incident on at least one edge in g. A set of edges that covers a graph G is said to be an **edge covering**, a **covering subgraph**, or a **covering of G**.

- A graph is its own cover.
- A spanning tree, a Hamiltonian circuit are also covers.

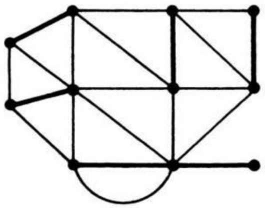


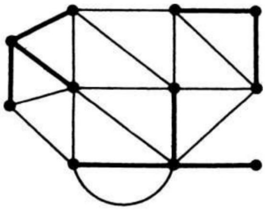


Minimal Covers

A covering from which no edge can be removed without destroying its ability to cover the graph.

- A graph and two of its minimal coverings are shown in heavy lines.





Coverings

- A covering exists for a graph if and only if the graph has no isolated vertex.
- A covering of an n-vertex graph will have at least $\lceil \frac{n}{2} \rceil$ edges.($\lceil n \rceil$ denotes the smallest integer not less than x.)
- Every pendant edge in a graph is included in every covering of the graph.
- Every covering contains a minimal covering.
- If we denote the remaining edges of a graph by (G – g), the set of edges g is a covering if and only if, for every vertex V, the degree of vertex in (G – g) $\leq$ (degree of vertex v in G) – 1.

Coverings

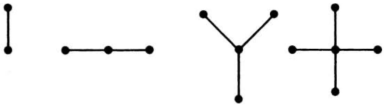
- No minimal covering can contain a circuit, for we can always remove an edge from a circuit without leaving any of the vertices in the circuit uncovered. Therefore, a minimal covering of an n-vertex graph can contain no more than $n - 1$ edges.
- A graph, in general, has many minimal coverings, and they may be of different sizes (i.e., consisting of different numbers of edges). The number of edges in a minimal covering of the smallest size is called the **covering number** of the graph.

Coverings

THEOREM

A covering g of a graph is minimal if and only if g contains no paths of length three or more.

- Suppose that a covering g contains a path of length three, and it is v1e1v2e2v3e3v4.
- Edge e2 can be removed without leaving its end vertices v2 and v3 uncovered.
  - Therefore, g is not a minimal covering.
- Conversely, if a covering g contains no path of length three or more, all its components must be star graphs as shown.



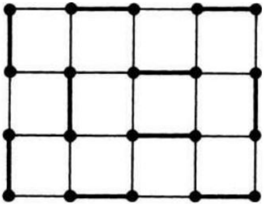
- From a star graph no edge can be removed without leaving a vertex uncovered. That is, g must be a minimal covering.

Coverings

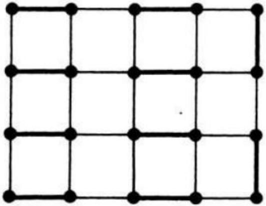
Dimer covering or a 1-factor

A covering in which every vertex is of degree one is called a dimer covering or a 1-factor.

- A dimer covering is obviously a matching because no two edges in it are adjacent.
- Moreover, a dimer covering is a maximal matching.



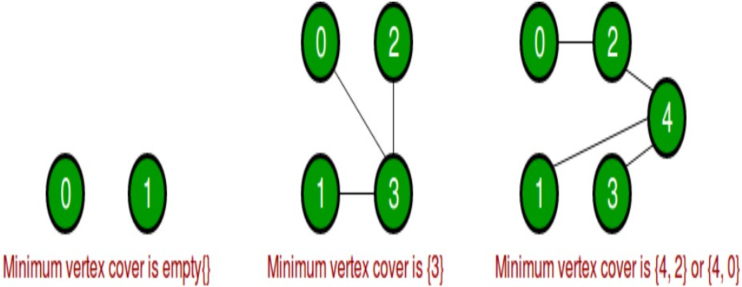
(a)



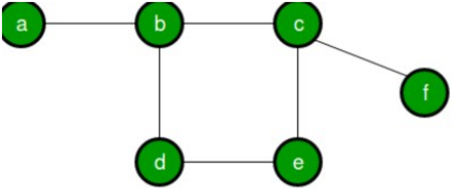
(b)

Vertex Covering

A subset of V is called a vertex covering of G if every edges of G is incident with or covered by a vertex in subset of V.



Vertex Covering



Minimum vertex cover is {b,c,d } or {b,c,e }

FOUR COLOURING PROBLEM

- The proper coloring of regions in a planar graph, is done by coloring no to adjacent regions with same color.
- Two regions are said to be adjacent if they have a common edge between them.
- The proper coloring of regions is also called map coloring.

The Four Coloring Theorem

Every map (i.e., a planar graph) can be properly colored with four colors. The four-color conjecture.

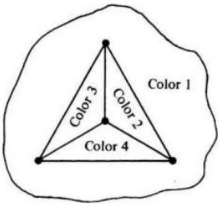


Fig. 8-14 Necessity of four colors.

Vertex Coloring Versus Region Coloring

- we know that a graph has a dual if and only if it is planar.
- Therefore, coloring the regions of a planar graph G is equivalent to coloring the vertices of its dual G*, and vice versa.
- Thus the four-color conjecture can be restated as follows:
- **Every planar graph has a chromatic number of four or less.**

Five-Color Theorem

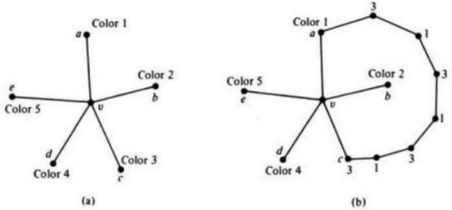
The vertices of every planar graph can be properly colored with five colors.

Proof:

- The theorem will be proved by induction
- Since the vertices of all graphs (self-loop free, of course) with 1, 2,3,4, or 5 vertices can be properly coloured with five colour
- let us assume that vertices of every planar graph with $n - 1$ vertices can be properly colored with five colors.
- Then, if we prove that any planar graph G with n vertices will require no more than five colors, we shall have proved the theorem.
- Consider the planar graph G with n vertices.
- Since G is planar, it must have at least one vertex with degree five or less
- Let this vertex be v. Let G' be a graph (of n -1 vertices) obtained from G by deleting vertex v (i.e., v and all edges incident on v).

Proof cont..

- Graph G' requires no more than five colors, according to the induction hypothesis.
- Suppose that the vertices in G' have been properly colored, and now we add to it v and all edges incident on v.
- If the degree of v is 1, 2, 3, or 4, we have no difficulty in assigning a proper color to v.
- This leaves only the case in which the degree of v is five, and all the five colors have been used in coloring the vertices adjacent to v,



Proof cont..

- Suppose that there is a path in G' between vertices a and c colored alternately with colors 1 and 3,
- Then a similar path between b and d, colored alternately with colors 2 and 4, cannot exist; otherwise, these two paths will intersect and cause G to be nonplanar.
- If there is no path between b and d colored alternately with colors 2 and 4, starting from vertex b we can interchange colors 2 and 4 of all vertices connected to b through vertices of alternating colors 2 and 4.
- This interchange will paint vertex b with color 4 and yet keep G' properly colored.
- Since vertex d is still with color 4, we have color 2 left over with which to paint vertex v.
- Had we assumed that there was no path between a and c of vertices painted alternately with colors 1 and 3, we would have released color 1 or 3 instead of color 2.
- And thus the theorem.

GREEDY COLOURING ALGORITHM

1. Color first vertex with first colour.
2. Do following for remaining V-1 vertices
   - Consider the currently picked vertex
   - Colour it with the lowest numbered colour that has not been used on any previously colored vertices adjacent to it
   - If all previously used colors appear on vertices adjacent to v, assign a new color to it

GREEDY COLOURING ALGORITHM