GRAPH THEORY

Module 4

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Outline

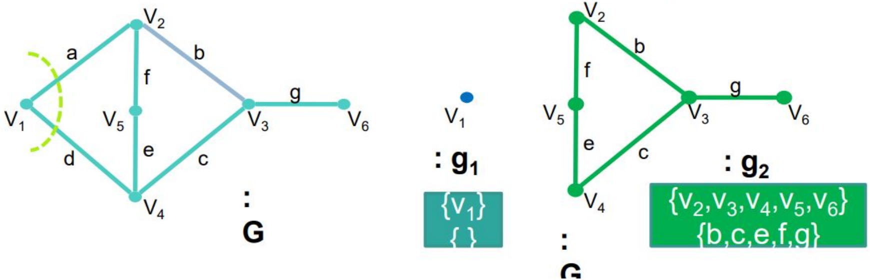
Cut set

CUT-SETS

- In a connected graph G, a cut-set is a set of edges, removal of which leaves the graph disconnected, provided removal of no proper subset of the set disconnects G.
- A cut-set cuts the graph into two components such that no path exists between the two.
- It is the minimal set of edges removal of which reduces the rank of the graph by one.
- A cut-set is also known as
  - Minimalcut-set
  - Propercut-set
  - Co-cycle

Example

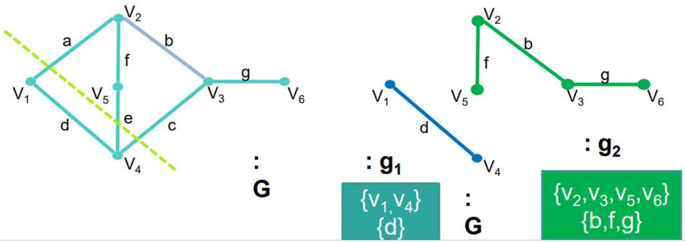
- In graph G, {a,d} is a cut-set; cs_1= {a,d}
- Hence G-cs_1 = 2 sub-graphs of G, g_1 & g_2
- In graph G, n=6, k=1→rank = n – k = 5
- After removing the cut-set, n=6,k=2 → rank = n – k = 4
- Hence removal of a cut-set reduces the rank of the graph by one



Cut set

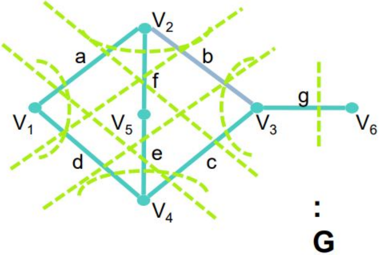
Example

- In graph G, {a,e,c} is another cut-set
- cs_2= {a,e,c}
- Hence G-cs_2 = 2 sub-graphs of G, g_1 & g_2



List all cut -sets

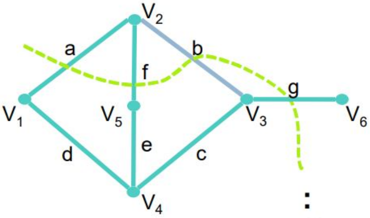
- $cs_1 = \{a,d\}$
- $cs_2 = \{a,b,f\}$
- $cs_3 = \{b,c\}$
- $cs_4 = \{d,e,c\}$
- $cs_5 = \{a,e,c\}$
- $cs_6 = \{a,f,c\}$
- $cs_7 = \{b,f,d\}$
- $cs_8 = \{b,e,d\}$
- $cs_9 = \{g\}$



Cut set

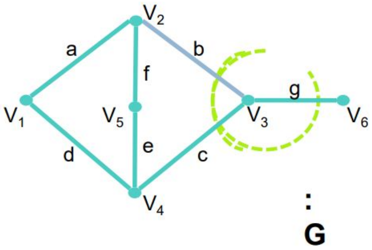
Wrong cut-sets

- Is {a,f,b,g} a cut-set of G ?
- No, coz removal of {a,f,b,g} cuts the graph into three
- Also the proper subset (a,f,b} of {a,f,b,g} is itself a cut-set
- Subset of a cut-set cannot be a cut-set



Wrong cut-sets

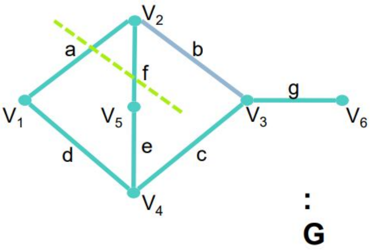
- Is {b,c,g} a cut-set of G ?
- No, coz it cuts the graph into three
- Moreover, subset {b,c} itself is a cut-set



Cut set

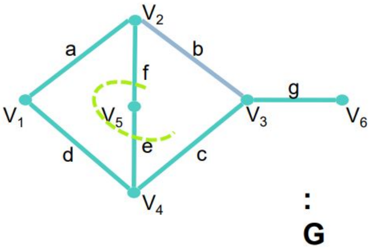
Wrong cut-sets

- Is {a,f} a cut-set of G ?
- No, coz removal of {a,f} does not cut the graph into two



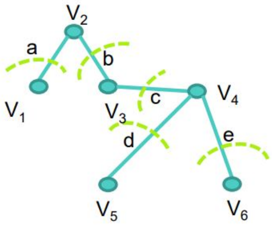
Right cut-sets

- Is {e,f} a cut-set of G?
- Yes, coz {e,f} cuts the graph into two & none of the proper subsets of {e,f} is a cut-set



Cut-set in a tree

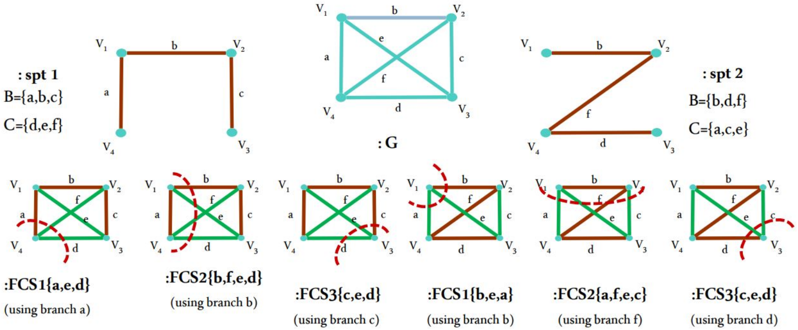
- Since removal of any edge in a tree breaks the tree into two, every edge of a tree is a cut-set
- Cut-sets→ {a} {b} {c} {d} {e}



Cut set

Fundamental cut-set

- W.r.t a given SPT, a cut-set of the graph is said to be fundamental, if it contains exactly one branch of the SPT along with some/all of the chords
- Since each branch can generate 1 cut-set, the no. of fundamental cut-sets possible for a graph is given by the no. of branches in the spanning tree



Cut set

Properties of a cut-set

- Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G
- In a connected graph G, any minimal set of edges containing at least one branch of every SPT of G is a cut-set
- Every circuit has an even no. of edges in common with any cut-set

Cut set

Theorem

Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G

Proof

- Let G be a connected graph and S be a cut-set of G
- Assume that we have a SPT T that does not have any of its branches in S
- Then removing S from G does not remove any of the branches from G
- Since the spanning tree remains completely in the graph and any SPT shall contain all the vertices of the graph, removal of S still leaves the graph connected
- But it is not possible. Removal of any cut-set must leave the graph disconnected
- Hence our assumption cannot be true
- There can be no SPT without any of its branches in any cut-set of G
- Hence the theorem

Cut set

Theorem

In a connected graph G, any minimal set of edges containing at least one branch of every SPT of G is a cut-set

Proof

- In a connected graph G, let Q be a minimal set of edges containing at least one branch of every SPT of G
- Remove Q from G. The remaining graph will not contain any of the SPTs. That means now the graph is disconnected
- Also, since Q is the minimal set of edges containing branches from all SPTs, returning any one edge to G-Q will create at least one SPT thereby making the graph connected as well
- Then we can say that Q is the minimal set of edges removal of which disconnects G, which is indeed the definition of a cut-set
- Hence Q is a cut-set

Cut set

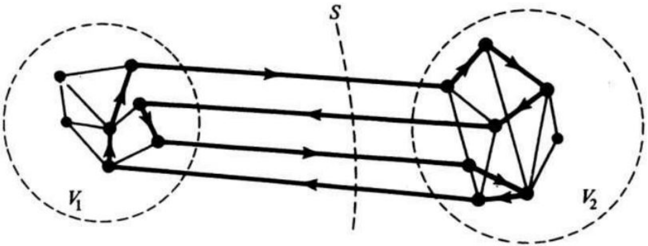
Theorem

Every circuit has an even no. of edges in common with any cut-set

Proof

- Consider a cut-set S in graph G. let the removal of S partition the vertices of G into two disjoint subsets V1 and V2.
- Consider a circuit Γ in G(before the removal of S). If all the vertices of Γ lies entirely within V1 or entirely within V2, then S will have no edge in common with Γ i.e, zero no. of edges in common (even)
- Whereas if some of the vertices of Γ lies in V1 and some in V2, then in order to traverse the circuit we need to go back and forth between V1 and V2 and finally need to reach back at the starting point
- Hence the no. of edges we traverse between V1 and V2 must be even. And these edges could be only from S. Therefore no. of edges common to S and Γ is even

Cut set



Circuit Γ shown in heavy lines, and is traversed along the direction of the arrows

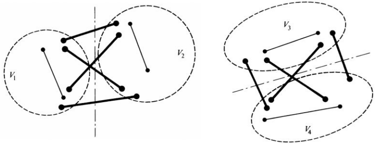
Cut set

Theorem

The ringsum of any two cut-sets in a graph is either a third cut-set or an edge disjoint union of cut-sets

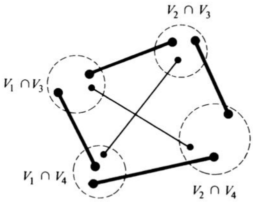
Proof

- Let S1 be a cut-set of the graph that partitions the vertex set V into V1 and V2
- Let S2 be another cut-set of the graph that partitions the V into V3 and V4



- Clearly,
  - $V1 \cup V2 = V$ and $V1 \cap V2 = \phi$
  - $V3 \cup V4 = V$ and $V3 \cap V4 = \phi$

Proof cont..



- Now consider the subset $(V1 \cap V4) \cup (V2 \cap V3)$ as V5 which is in fact $V1 \oplus V3$;
    - similarly consider subset $(V1 \cap V3) \cup (V2 \cap V4)$ as V6 which is same as $V2 \oplus V3$

Proof Cont..

- Now S1 $\oplus$ S2 seem to contain only those edges between V5 and V6. Also there are no other edges between V5 & V6 which implies V5 $\cup$ V6 $=$ V and V5 $\cap$ V6 $=\phi$
- Then S1 $\oplus$ S2 is a cut-set of G if V5 and V6 each remain connected after the removal of S1 $\oplus$ S2 ; otherwise S1 $\oplus$ S2 is the union of cut-sets

Example

- Eg 1: cut-sets $S_1 = \{d,e,f\}$ & $S_2 = \{f,i,h\}$

  $S_1 \oplus S_2 = (S_1 \cup S_2) - (S_1 \cap S_2)$

  $= \{d,e,f,i,h\} - \{ f \}$

  $= \{d,e,i,h\}$

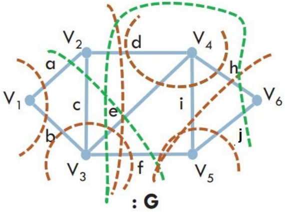
  → again a cut-set
- Eg 2: cut-sets $S_1 = \{a,b\}$ & $S_2 = \{b,c,e,f\}$

  $S_1 \oplus S_2 = \{a,c,e,f\}$

  → again another cut-set
- Eg 3: cut-sets $S_1 = \{d,e,i,h\}$ & $S_2 = \{f,i,j,\}$

  $S_1 \oplus S_2 = \{d,e,f,h,j\}$

  → union of two cut-sets $\{d,e,f\}$ and $\{h,j\}$



Cut set

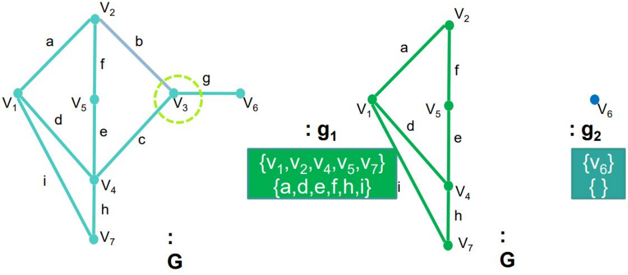
cut-sets

- Cut-set $\rightarrow$ set of edges removal of which disconnects the graph
- Edge connectivity $\rightarrow$ no. of edges in the smallest cut-set
- Cut-set in tree $\rightarrow$ every edge of a tree is a cut-set
- Edge connectivity of any tree $\rightarrow$ is always 1
- Fundamental cut-set $\rightarrow$ a cut-set that contains exactly one branch of the spt
- Number of fundamental cut-sets $\rightarrow$ no. of branches in the spt
- Fundamental circuit $\rightarrow$ a circuit that contains exactly one chord
- Number of fundamental circuits $\rightarrow$ no. of chords in the graph
- Every cut-set will contain at least one branch of every spt

CUT-VERTEX

CUT-VERTEX

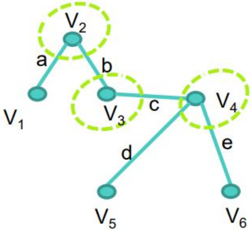
- In a connected graph G, a cut-vertex is a set of vertices removal of which leaves the graph disconnected, provided removal of no proper subset of the set disconnects G.
- A cut-vertex cuts the graph into two or more components, such that no path exists between the components



In graph G, $\{V_3\}$ is a cut-vertex; $cv_1 = \{V_3\}$
Hence G-cv_1 = 2 sub-graphs of G, g_1 & g_2

Cut-vertex in a tree

- Since removal of any vertex other than the pendant vertices breaks the tree, every vertex of a tree is a cut-vertex
- Cut-vertices→ $\{V_2\}$ $\{V_3\}$ $\{V_4\}$



Cut Vertex

- Cut-vertex $\rightarrow$ set of vertices removal of which disconnects the graph
- Vertex connectivity $\rightarrow$ no. of vertices in the smallest cut-vertex
- Cut-vertex in tree $\rightarrow$ Every vertex (other than pendant vertex) in a tree is a cut-vertex
- Vertex connectivity of any tree $\rightarrow$ always 1
- Separable graph$\rightarrow$ graph whose vertex connectivity is 1
- Edge connectivity $\rightarrow$ cannot exceed the smallest degree
- The vertex connectivity $\rightarrow$ cannot exceed edge connectivity
- A graph is K-connected $\rightarrow$ vertex connectivity is K

Note:-

- Recall the concept of Fundamental circuits which was formed corresponding to a spanning tree by adding a chord to it.
- Recall the concept of Fundamental cutsets containing exactly one branch of a spanning tree.
- Also remember that every circuit has an even number of edges in common with any cut-set.
- Also every cut set in a connected graph G must contain at least one branch of any spanning tree.
- Number of fundamental cutset $=$ number of branches in the SPT
- Number of fundamental circuit $=$ number of chords in the SPT

Fundamental circuits

Theorem

W.r.t a given SPT T, a chord ci that determines a fundamental circuit ρ occurs in every fundamental cut-set associated with the branches in ρ and in no other

Proof

- T is the given SPT
  - let ρ be the fundamental circuit determined by the chord ci
  - $\rho \ in = \{ci, b1, b2, \ldots, bk\}$
  - Let S1 be the fundamental cut-set associated with branch b1
  - $S1 = \{b1, c1, c2, \ldots, cq\}$
  - Since the number of edges common toρ and S1 must be even, ci must be in S1
  - The same is true for fundamental cut-sets made by branches b2,b3,. . . bk

Fundamental circuits

Proof Cont...

- On the other hand suppose that ci occurs in some fundamental cut-set Sk+1 made by a branch other than b1,b2,. . . .bk.
  - Since none of the branches is in Sk+1, there is only 1 edge ci - common to Sk+1 and the fundamental circuit ρ which is not possible
- Hence the theorem

Fundamental circuits

Theorem

With respect to a given spanning tree T, a branch bi that determined a fundamental cut-set S is contained in every fundamental circuit associated with the chord in S, and in no others

Proof

- T is the given SPT
  - Let S be the fundamental cut-set determined by the branch bi
  - $S = \{bi, c1, c2, \ldots, cq\}$
  - Let ρ be the fundamental circuit determined by the chord C1
  - $\rho_1 = \{c1, b1, b2, \ldots, bk\}$
  - Since the no. of edges common to S and ρ_1 must be even, bi must be in ρ_1.
  - The same is true for the fundamental circuits made by chords C2, C3 . . . Cq

Proof Cont..

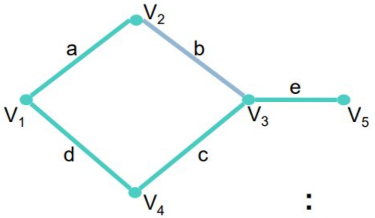
- On the other hand, suppose that bi occurs in some fundamental circuit ρ_{q+1} made by a chord other than C1,C2, ... Cq. Since none of the chords C1, C2, ... Cq is in ρ_{q+1}, there is only ρ_1 edge bi common to a circuit ρ_{q+1} & cut-set S, which is not possible
- Hence the theorem

Edge connectivity (Ec)

Edge connectivity (Ec)

- Minimum no. of edges removal of which disconnects the graph or reduces the rank of the graph by one
- It is given by the size of the smallest cut-set

- Cut-sets are
  - {a,d}
  - {a,b}
  - {a,c}
  - {b,c}
  - {b,d}
  - {c,d}
  - {e}

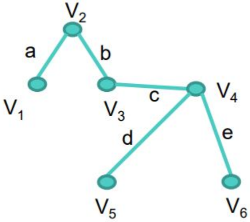


: G

- Smallest cut-set is {e} → contains one element
- Hence edge connectivity E_c of graph G is 1

Edge connectivity of a tree

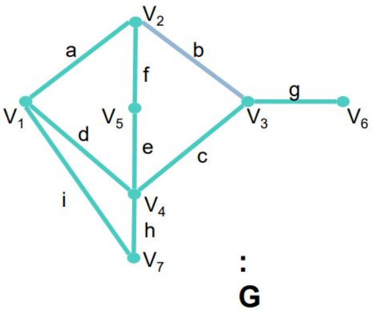
- Since a tree can be broken by the removal of a single edge, edge connectivity of a tree is always 1
- Cut-sets of the tree are
- {a} , {b} , {c} , {d} , {e}
- Hence E_c is 1



Vertex connectivity (Vc)

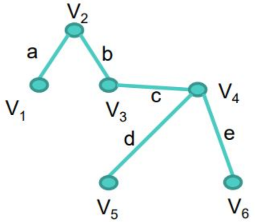
- Minimum no. of vertices removal of which disconnects the graph
- It is given by the size of the smallest cut-vertex

- Cut-vertices are
  - $\{V_3\}$
  - $\{V_1, V_4\}$
  - $\{V_2, V_4\}$
- Smallest cut-vertex is $\{V_3\}$ → contains one element
- Hence vertex connectivity V_c of graph G is 1



Vertex connectivity of a tree

- Since a tree can be broken by the removal of a single non-pendant vertex, vertex connectivity of a tree is always 1
- Cut-vertices of the tree are
- $\{V_2\}$, $\{V_3\}$, $\{V_4\}$
- Hence V_c is 1



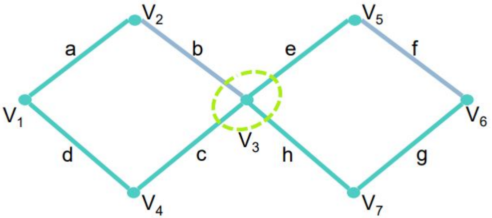
Separable graph

- A connected graph is said to be separable if its vertex connectivity is one
- If removal of a single vertex disconnects the graph, then it is separable
- The vertex, removal of which disconnects the graph is called an articulation point or cut-vertex or cut-node
- In such a graph, there would be a subgraph g such that g & $\overline{g}$ have only 1 vertex in common

Separable graph

Example

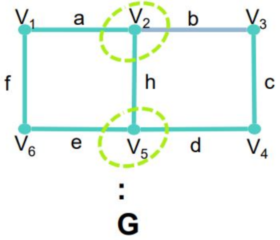
- Smallest cut-vertex is $\{V_3\}$; contains 1 element
- Hence G is a separable graph
- V_3 is the articulation point
- Removal of V_3 disconnects the graph



: G

Example

- Smallest cut-vertex is $\{V_2, V_5\}$; contains 2 elements
- Hence G is a non-separable graph



CUT-VERTEX

Theorem

A vertex V in a connected graph G is a cut–vertex iff these exists two vertices x & y in G such that every path between x & y passes through V

Proof

- Let V be a cut-vertex of graph G
  - Then removal of V from G must disconnect the graph into two components, such that the components are not empty. Each component must contain at least an isolated vertex
  - Let x be a vertex from first component & y from the other component
  - If there exists a path between x & y, other than through vertex V, then removal of V will not disconnect the graph. But since V is a cut-vertex, removal of V must disconnect the graph
  - So there can be no path between x & y other than through V

Proof Cont..

Conversely

- If x & y are two vertices of G such that all paths between x & y are through vertex V
  - Then removal of V from G makes x & y not reachable from each other as all paths between x & y have been broken
  - Since no path exists between x & y, then x & y must be lying in different components, which implies that the graph has been disconnected by the removal of V
  - Any vertex V, removal of which disconnects a graph is a cut vertex. Hence here, V is a cut-vertex
  - Hence the theorem

Edge connectivity (Ec) & Vertex connectivity (Vc)

Theorem

The edge connectivity of a graph cannot exceed the degree of the vertex with the smallest degree in G

Proof

- Let Vi be the vertex with the smallest degree.
  - Let d(Vi) represent the degree of Vi
  - Vertex Vi can be separated from the graph by removing all the d(Vi) edges incident on it
  - Hence d(Vi) is the edge connectivity of the graph
  - Hence the theorem

Theorem

The vertex connectivity of any graph G can never exceed the edge connectivity of G

Proof

- Let α denote the edge connectivity of G
    - Then, there must exist a cut–set with α edges. Let it be S.
    - S partitions the vertex set of the graph into two. Let they be V1 & V2
    - By removing at most α vertices from V1 (or V2) on which the α edges were incident, we can bring the same effect on the graph i.e, we can disconnect the graph in the same way how S disconnected the graph. However if any other edges were incident on these vertices, they too would get deleted
    - However the vertex connectivity would be α itself

Theorem

The maximum vertex connectivity one can achieve with a graph G of n vertices and e edges ($e \geq n - 1$) is the integral part of the number $\frac{2e}{n}$ that is, $\lfloor \frac{2e}{n} \rfloor$

Proof

- Tht total degree 2e of the graph is to be distributed among the n vertices.
- Hence there exist at least one vertex with degree $\leq \frac{2e}{n}$
  - As the edge connectivity and hence vertex connectivity cannot exceed this number, this is an upper bound for the vertex connectivity.

Edge connectivity (Ec) & Vertex connectivity (Vc)

Proof cont..

- Now to show this value is actually attainable, first construct a regular graph on n vertices with degree $\leq \frac{2e}{n}$ and then add the remaining edges arbitrarily.ie $(e - \frac{n}{2}\lfloor \frac{2e}{n} \rfloor)$
- Now, each vertex is connected to at least $\lfloor \frac{2e}{n} \rfloor$ other vertices and hence to disconnect even a single vertex $\lfloor \frac{2e}{n} \rfloor$ needs to be removed
- hence the vertex connectivity of this graph $= \lfloor \frac{2e}{n} \rfloor$

K-connected graph

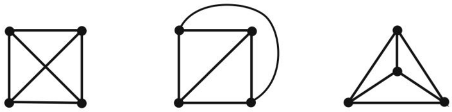
- A graph whose vertex connectivity is K
- Every pair of vertices in a k-connected graph is joined by at least k non-intersecting paths

Planar Graphs

Planar Graphs

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect

- A graph that cannot be drawn on a plane without a crossover between its edges is called **nonplanar**
- A drawing of a geometric representation of a graph on any surface such that no edges intersect is called an **embedding** of G.



Planar graph K_4 Two plane embeddings of K_4

Properties of Planar Graphs

- For a given graph G, there can be many embeddings.
- To declare that a graph G is nonplanar, we have to show that of all possible geometric representations of G none can be embedded in a plane.
- A geometric graph G is planar if there exists a graph isomorphic to G that is embedded in a plane.
- An embedding of a planar graph G on a plane is called a plane representation of G.

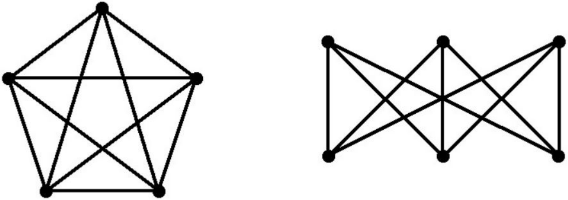
Kuratowski's graphs

Two specific non planar graphs are called Kuratowski's Graphs, after the polish mathematician kasimir Kuratowski, who discovered their unique property

Kuratowski's graphs

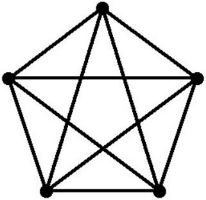
Kuratwoski's two graphs are non planar

1. Kuratowski's first graph is K_5, the complete graph on 5 vertices.
2. Kuratowski's second graph is the complete bipartite graph $K_{3,3}$

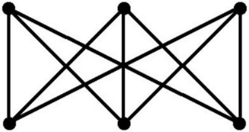


(a) $\mathbf{K_5}$ (b) $\mathbf{K_{3,3}}$

Kuratowski's graph



(a) K_5



(b) $K_{3,3}$

- Both are regular graphs
- Removal of one vertex or one edge makes the graph planar
- Kuratowski's first graph is the non-planar graph with smallest number of vertices
- Kuratowski's second graph is the non-planar graph with smallest number of edges.
- Thus both are simplest non-planar graphs.

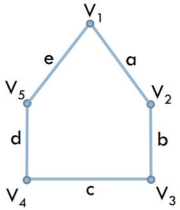
Theorem

The complete graph with 5 vertices (K5) is non planar

Proof:

- Let the 5 vertices of the graph be V1, V2,V3, V4 and V5
- Since it is a complete graph, every vertex needs to be connected to every other vertex by an edge
- There must be a circuit going from V1 to V2 to V3 to V4 to V5 and back to V1 ; that is a pentagon that divides the region into 2 - inside & outside of the pentago

:G



Proof cont..

- Now we need V1 to be connected to V3 & V4.
  - V1 can be connected to V3 along an edge inside the pentagon. Similarly V1 can be connected to V4 also, inside the pentagon

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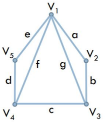

- Next we need V2 to be connected to V4 & V5
  - Drawing an edge inside the pentagon is not possible as it will intersect the previously drawn edges.
    - So let as draw these 2 edges along the outside region

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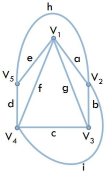

Proof cont..

- Now V1, V2, & V4 have degrees 4 each. V3 & V5 have degrees 3 each. So the remaining edge to be drawn is between V3 & V5 . We cannot draw this edge inside or outside, without intersecting previous edges
- Hence this graph cannot be embedded in a plane
- So it is non-planar

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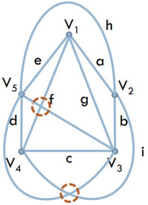

Bi-partite graph with 6 vertices (K3,3)

Theorem

Kuratowski's 2nd graph is non-planar

Proof:

- The graph is bi-partite graph with 6 vertices
- Here the vertex set V is divided into two V' & V''
- Every vertex in V' is connected to every vertex in V'' by an edge
- V'={V1,V2,V3} & V''={V4,V5,V6}
- E = {a,b,c,d,e,f,g,h,i}

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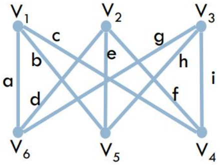
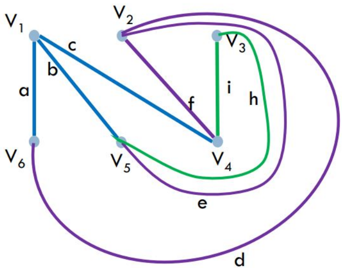


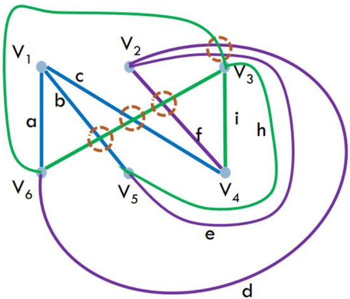
Proof cont..

- From vertex V1, draw edges towards V4,V5 & V6. Now V1 has degree 3
- From vertex V2, draw an edge toward V4
- From V3 draw an edge towards V4
- Again at V2, draw edge towards V5 & V6 using curved lines so as to avoid intersecting
- Also from V3 to V5



Proof cont..

- Now 1 more edge is required between V3 & V6. We cannot draw the edge without intersecting previous edges
- Hence the proof



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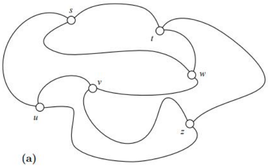
Different representations of planar graphs

- Straight line Representation
- Plane Representation
- Embedding on a Sphere

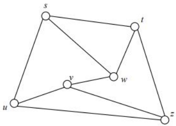
Different representations of planar graphs

Straight line Representation

- Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.
- It is necessary for the graph to be simple because a self-loop or one of two parallel edges cannot be drawn by a straight line segment.



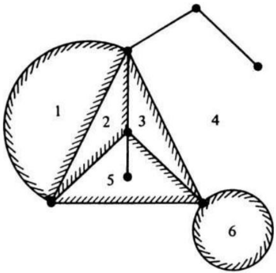
(a)



(b)

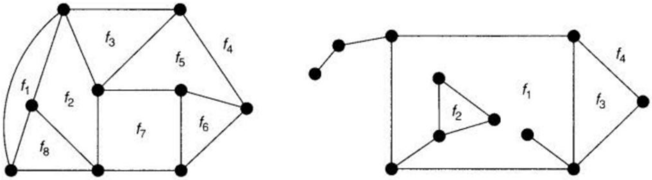
Plane Representation

- A plane representation of a graph divides the plane into regions (also called windows, faces, or meshes)



Regions in a Planar Graph

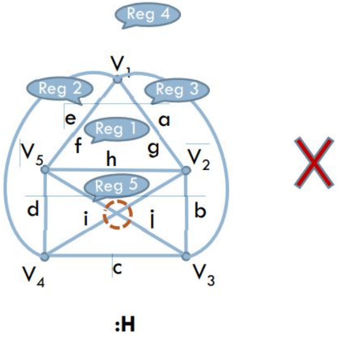
- If G is a planar graph, then any plane drawing of G divides the set of points of the plane not lying on G into **regions (faces or windows or meshes)**



- A region is characterized by the set of edges (or the set of vertices) forming its **boundary**
- Note that a region is not defined in a nonplanar graph or even in a planar graph not embedded in a plane.

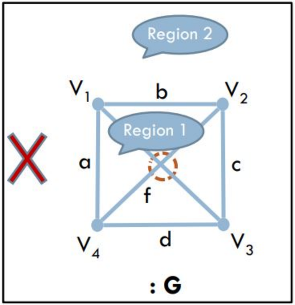
non-planar

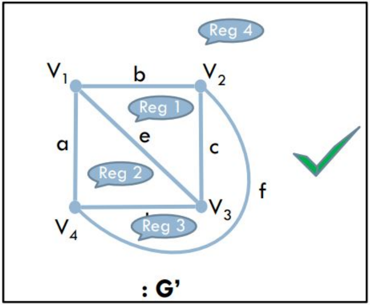
- Since the faces do not have proper boundaries, regions cannot be defined for non-planar graphs



planar but not an embedding

- Though planar, G is not an embedding; hence cannot have regions defined
- G' is an embedding of the same planar graph G; hence 4 regions can be identified



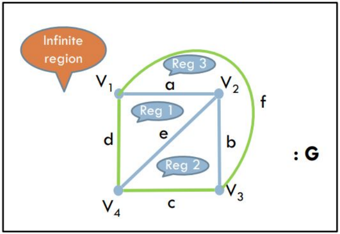


Different representations of planar graphs

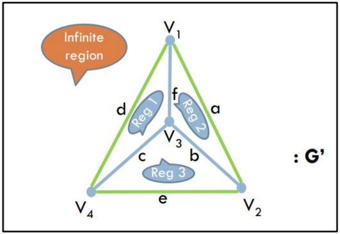
Infinite Region

- The portion of the plane lying outside a graph embedded in a plane is called the infinite region
- Also known as unbounded, outer or exterior region
- Also there is nothing special about the infinite region, In fact any region could be made an infinte one by redrawing the planar graph.

Regions of graph G are
Region 1 = {a,d,e}
Region 2 = {e,b,c}
Region 3 = {a,b,f}
Infinite region = {c,d,f}

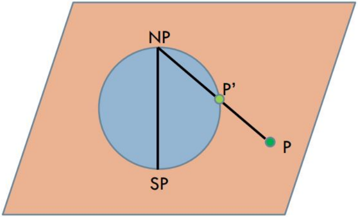


Regions of graph G' are
Region 1 = {c,d,f}
Region 2 = {a,b,f}
Region 3 = {e,c,b}
Infinite region = {a,e,d}



Embedding on a Sphere

- To eliminate the distribution between finite & infinite regions, a planar graph can be embedded on the surface of a sphere
- It is done by stereographic projection of a sphere on a plane



Different representations of planar graphs

Theorem

A graph can be embedded on the surface of a sphere if and only if it can be embedded on a plane

Proof:

- Place the sphere on the plane and note the point of contact as SP (south pole)
  - From the point SP, draw a straight line perpendicular to the plane. The point where this line meets the circumference of the sphere is noted as NP
  - For any point P on the plane, there is a corresponding point p' on the sphere and vice versa
  - To obtain P', draw a straight line from P to meet NP. Point where this line intersects the circumference of the sphere is the point p'
  - Thus we can say that there is a one-to-one correspondence between the points on the sphere and the finite points on the plane
  - Points at infinity corresponds to NP
  - Hence the theorem

Different representations of planar graphs

Theorem

A planar graph may be embedded in a plane such that any specific region can be made the infinite region

Proof:

- A planar graph embedded in the surface of a sphere divides the surface into different regions.
    - Each region on the sphere is finite, the infinite region has been mapped on to the point NP
    - Now, it is clear that by suitably rotating the sphere, we can make any specific region to be the infinite region on the plane
    - Hence the theorem

Euler's formula

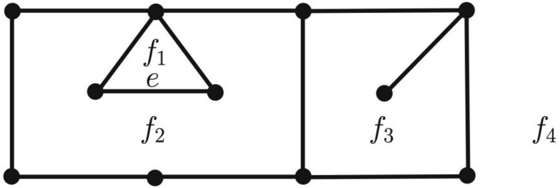
Theorem

A connected planar graph with n vertices and e edges has e-n+2 regions

Proof.

- Let G be a connected planar graph with n vertices, e edges and f regions.
    - To show that $f = e - n + 2$ *or* $n - e + f = 2$
    - We will use induction on f to prove the result.
- When $f = 1$, G is a tree and hence $e = n - 1$.
    - Hence $n - e + f = n - (n - 1) + 1 = 2$.
- Now assume that the result is true for all connected planar graphs with $f - 1$ reqions, $f \geq 2$. Suppose G has f regions. Since $f \geq 2$, G is not a tree and hence contains a cycle say C.

Proof cont..

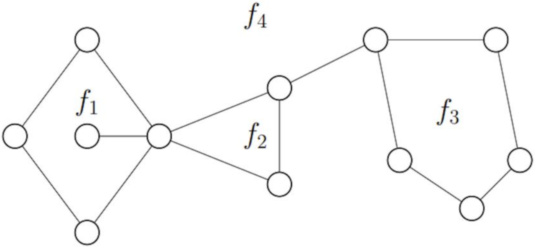


- Let e be an edge of C.
    - Then e belongs to exactly two regions, say f1 and f2 of G and the deletion of e from G results in the formation of a single region from f1 and f2.
    - Also, since e is not a cut edge of G; G-e is connected.
    - Further, the number of regions of G-e is f-1.
- So applying induction to G-e
    - we get $n-(e-1)+(f-1)=2$and this
    - implies that $n-e+f=2$.
- Hence the proof.

Euler's formula

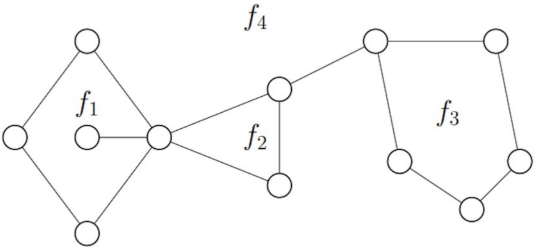
Note:

- All plane embeddings of a given planar graph have the same number of faces.
- Let G be a connected plane graph. Each edge of G belongs to one or two faces of G:
- A cut edge of G belongs to exactly one face, and conversely, if an edge belongs to exactly one face of G; it must be a cut edge of G
- An edge of G that is not a cut edge belongs to exactly two faces and conversely.



Note:

- The union of the vertices and edges of G incident with a face f of G is called the **boundary** of f
- The vertices and edges of a planar graph G belonging to the boundary of a face of G are said to be incident with that face.
- The number of edges incident with a face f is called its **degree.**
- In counting the degree of a face, a **cut edge is counted twice.**



Euler's formula

Results from Euler's formula

Theorem

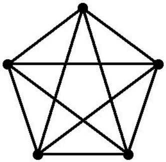
If G is a simple connected planar graph with at least three vertices, then $e \leq 3n - 6$ *and* $e \geq \frac{3}{2}f$

Proof.

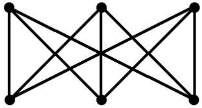
- The sum of the degrees of the regions is equal to twice the number of edges($2e$). But each region must have degree ≥ 3
  - (since the graphs discussed here are simple graphs)
- Hence $2e \geq 3f$ or e$\geq \frac{3}{2}f$
- Combining with Euler's formula
- $e \geq \frac{3}{2}(e - n + 2)$
- or $e \leq 3n - 6$
- This result is useful for checking planarity of a given graph.

Euler's formula

Problem Non planarity of K5



(a) K_5



(b) $K_{3,3}$

Solution

- $In\ k5,\ \ n = 5\ and\ e = 10. Now\ e = 10 \nleq 3x5 - 6 = 9$ Hence k5 is non planar

This inequality is only a necessary condition for planarity of a graph and not a sufficient condition. This is because K3,3 satisfies the inequality but is still non planar.

Results from Euler's formula

Theorem

The number of edges in a planar bipartite graph of order n is at most $2n - 4$, *or* $e \leq 2n - 4$

Proof

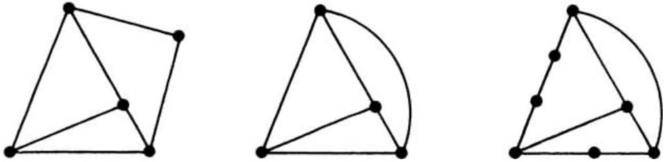
- Let G be a planar bipartite graph with n vertices and e edges.
- Consider a planar embedding of G.
  - Since, G is bipartite, G has no cycle of length three. So, each face in the planar embedding contains at least four edges.
- Hence $2e \geq 4f$
- and applying Euler's formula
  - $2e \geq 4(e - n + 2) = 4e - 4n + 8$ and this implies $e \leq 2n - 4$

Using above result, show that K3,3 is non planar.

Kuratowski's Theorem

Kuratowski's Theorem

A necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them.



Detection of Planarity(Elementary reduction)

In order to check whether a given graph is planar or not the following steps of Elementary reduction can be used

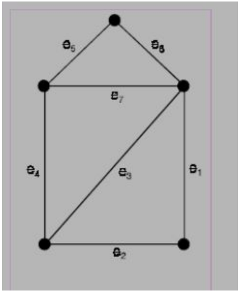
- If the graph is disconnected, we need to check whether each component is planar. If all components are planar, then the disconnected graph is said to be planar
  - If the graph is a separable graph, we need to check whether each block is planar. It all blocks are planar, then the separable graph is planar
  - Now, let our graph G = {G1, G2, ... Gk}
  - Where each Gi is a non-separable block of G
  - Test each Gi for planarity
- Remove all self loops as self loops does not affect planarity
- Similarly remove all parallel edges as they too do not have anything to do with planarity
- Merge edges in series, by ignoring their common vertex

Detection of Planarity(Elementary reduction) cont..

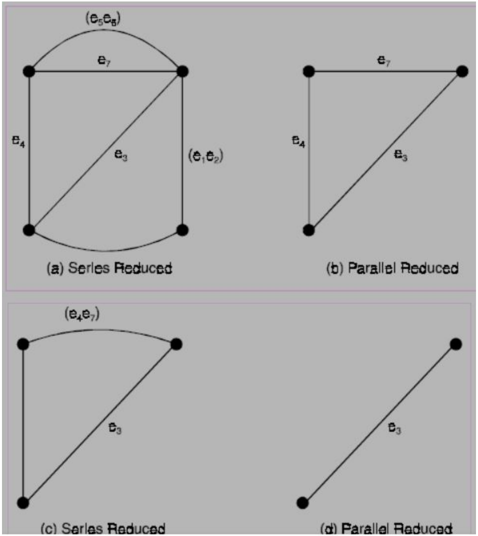
- Repeat step 3 and 4 repeatedly until no more edges can be deleted.
- Now the resulting graph may contain
  - A single edge
  - A complete graph with 4 vertices
  - A non separate graph with $n \geq 5$ & $e \geq 7$
- If it is (i) or (ii) then our graph is planar; no need of further clarification
- But if it is (iii) continue to next step
- Check whether $e \leq 3n - 6$ for the resultant graph
  - If the condition is satisfied, then our graph is planar. But if not, may or may not be planar. So we need to check further
- Check whether the resultant graph contain either of Kuratowski's graphs or their homeomorphic graphs
  - If our graph contains K5 , K3,3 or graphs homeomorphic to K5 and K3,3 , then our graph is certainly non-planar

Detection of Planarity

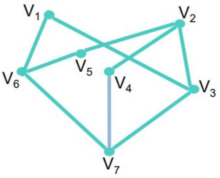
Q: Check whether the given graph is planar by the method of elementary reduction



Ans:


(e5e6)
e7
e4
e3
(e1e2)
(a) Series Reduced
e7
e4
e3
(b) Parallel Reduced
(e4e7)
e3
(c) Series Reduced
e3
(d) Parallel Reduced

Q: Check whether the given graph is planar by the method of elementary reduction



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Q: Check whether the given graph is planar by the method of elementary reduction

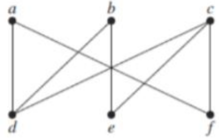


Figure 1: *

(a)

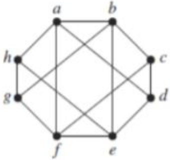


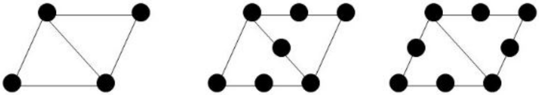
Figure 2: *

(b)

Homeomorphic graphs

Homeomorphic graphs

Two graphs are said to be homeomorphic if one can be obtained from the other by creating edges in series or by merging edges in series

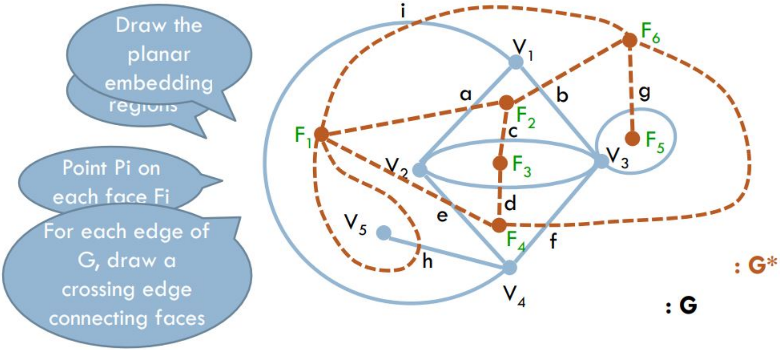


Geometric dual of a planar graph

In order to obtain the geometric dual of a planar graph

- Start with a plane representation of the planar graph (planar embedding)
- Name the regions or faces are $F_1, F_2, F_3 \ldots \ldots F_{e-n+2}$
- Place a point P_i in each face F_i
- For each edge of G, draw a line crossing the edge connecting the two faces on either sides; For an edge lying entirely in a region, draw a self loop at the point that passes through the edge
- Name the new graph as G* which forms the dual of G

Example
Let G be the plane representation of a graph



Properties of duals

- An edge forming a self-loop in G yields a pendant edge in G*
- A pendant edge in G yields a self-loop in G*
- Parallel edges in G produce edges in series in G*
- Graph G* is also embedded in the plane and is therefore planar.
- Dual of G* is G
- If n, e, f ,r, μ denote as usual the numbers of vertices, edges, regions, rank, and nullity of a connected planar graph G, and n*,e*,f*,r*,$\mu*$ are the corresponding numbers in dual graph G*, then

$$n* = f, e* = e, f* = n$$

and hence

$$r* = \mu, \mu* = r$$

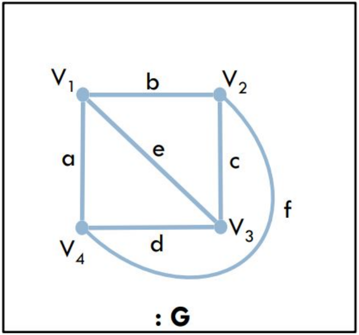
Geometric dual of a planar graph

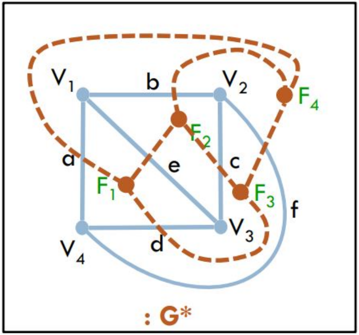
All duals of G

- A planar graph G may have different planar embeddings. For each planar embedding, we can obtain a corresponding geometric dual
- A planar graph G will have a unique dual if & only if it has a unique planar embedding
- If G & G' are isomorphic, then their corresponding duals G* & G'* may not be isomorphic

Self dual Graphs

- If a planar graph G is isomorphic to its own dual, it is called a self dual graph
- Example

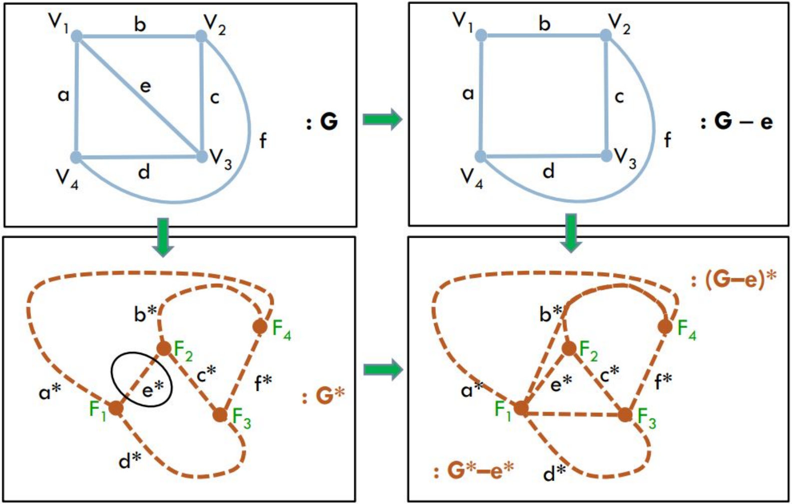




Dual of a Subgraph

- Let G* be the dual of G
- Let 'a' be an edge in G & a*, the corresponding edge in G*
- To find the dual of G-a; that is the dual of the graph G after deleting the edge 'a' i.e, (G-a)*
- This can be directly obtained from G*
- If 'a' was a boundary of 2 regions in G, then by deleting a* from G*, we can obtain (G-a)* ; deleting the edge will require to fuse the end vertices
- Else if 'a' was a not any boundary in G, then a* would be a self loop in G*. Then deleting the self loop yields (G-a)*
- G – q

Dual of a Subgraph



Geometric dual of a planar graph

Dual of a homeomorphic graph

- Let G* be the dual of G
- Let ‘a’ be an edge in G & a*, the corresponding edge in G*
- Suppose we create a new vertex in G by introducing a vertex of degree 2 on edge a. This will create a new edge as well. Let it be b. Now the dual of G+b will contain a new edge b* which appears as an edge parallel to a*
- Similarly merging 2 edges in series will simply eliminate one of the corresponding parallel edges in G*
- Thus dual of a homeomorphic graph of G can be obtained from G*

Combinatorial Dual

- G* is said to be combinatorial dual of G if there is a one to one correspondence between the edges of G & G* such that if g is any subgraph of G & h is the corresponding subgraph of G* then
  - Rank (G*-h) = rank (G*) – nullity (g)

Geometric dual of a planar graph

Theorem

A necessary and sufficient condition for two planar graphs G1 & G2 to be duals of each other is that, there is a one-to-one correspondence between the edges of G1 & G2 such that a set of edges in G1 forms a circuit if & only if the corresponding set in G2 forms a cut-set

Proof:

- Since every edge of G will be intersected by exactly one edge of G*, there must ne a one to one correspondence between the edges of G1 & G2
- Now, consider a planar representation of G & its dual G*. Let ρ be an arbitrary circuit in G. ρ will form will form some simple closed curve in G, dividing the plane into 2 areas, one inside ρ & the other outside
- Now the vertices of G* can be viewed as two non empty disjoint subsets, those vertices that represent regions inside ρ & those that represents regions outside ρ and this partition is brought by the set of edges in $\rho*$. Hence $\rho*$ is a cut-set in G*

Geometric dual of a planar graph

Proof cont...

- Similarly every cut-set S* in G* will have a unique circuit S in G
- **Conversely:**Suppose there are two planar graphs G & G' such that there is one to one correspondence between their edges and also one to one correspondence between the cut-sets of G & the circuits of G' and vice versa
- Let G* be a dual of G
- Then there is a one to one correspondence between the cut-sets of G & the circuits of G' & also between the cut-sets of G & the circuits of G*
- Therefore there is a one to one correspondence between the circuits of G' & G* implying that G' & G* are 2-isomorphic. Then G' must be a dual of G
- (Based on the theorem: Two graphs are 2- isomorphic if & only if they have circuit correspondence)

Geometric dual of a planar graph

Theorem

A graph has a dual if and only if it is planar

Proof:

- **Let us prove that a non planar graph does not have a dual**
- Let G be a non-planar graph. Then according to Kuratowski's theorem, G contains either K5 or K3,3 or a graph homeomorphic to them
- Any graph can have a dual only if every subgraph of that graph & every graph homeomorphic to that graph has a dual
- From the above 2 statements, we can say that if K5 and K3,3 cannot have a dual then none of the non-planar graphs can have a dual

Proof cont..

- To prove that K3,3 do not have a dual, assume the contradiction that K3,3 has a dual D
- Since K3,3 has 9 edges, so must be D
- All cut-sets in K3,3 must have corresponding circuits in D & vice versa
- Since K3,3 do not have any cut-set containing 2 edges, D cannot have any circuit containing 2 edges. That means D cannot contain any parallel edges
- Since every circuit in K3,3 is of length 4 or 6, D cannot have any cut set with less than 4 edges, which implies every vertex in D has degree of at least 4
- Since D has no parallel edges & every vertex has degree of minimum 4, D must contain at least 5 vertices, each of degree 4 or D may contain more than 5 vertices with larger degrees

Proof cont..

- D must then at least contain $\frac{5*4}{2} = 10$ edges, contradicting to the fact that D has only 9 edges
- So there can be no such D. Hence K3,3 cannot have a dual
- Similarly we can prove that K5 do not have a dual
- Assume the contradiction that K5 has a dual, H
- Since K5 has 10 edges, H must also have 10 edges
- All cut-sets in K5 must have corresponding circuits in H vice versa
- Since K5 do not have any cut-set with 2 edges, it cannot have a circuit with 2 edges. That means H has no parallel edges
- Since every cut-set in K5 contains 4 or 6 edges, H can have circuits of length 4 or 6 only

Geometric dual of a planar graph

Proof cont..

- Consider a circuit of length 6 (hexagon) in H. Now, we cannot add the remaining 4 edges, without creating parallel edges or circuits of length three
- So in order to add the remaining 4 edges without violating the rules (parallel edge & circuits of length 3) we assume H to have 7 vertices, with degree at least 3
- Then H must have $\frac{7*3}{2} = 11$ edges, contradicting that H has 10 edges
- So there can be no such dual for K5