GRAPH THEORY

Module 1

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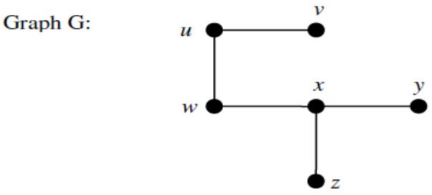
Outline

Graphs

- A graph G = (V, E) consists of a set of objects V={v1, v2, v3, . . . } called vertices (also called points or nodes)
- and other set E = {e1, e2, e3, ...} whose elements are called edges (also called lines or arcs).

- The set V(G) is called the vertex set of G and E(G) is the edge set of G.
- A graph is denoted as G=(V,E)



Graph G with 6 vertices and 5 edges

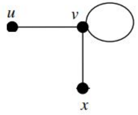
- A graph with p-vertices and q-edges is called a (p, q) graph.

Trivial Graph

- A graph is said to be trivial if a finite graph contains only one vertex and no edge.
- The (1, 0) graph is called trivial graph.

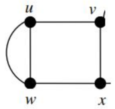
Self-loop.

- An edge having the same vertex as its end vertices is called a self-loop.



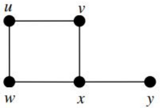
Parallel edges.

- More than one edge associated a given pair of vertices called parallel edges.



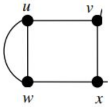
Simple graph.

- A graph that has neither self-loops nor parallel edges is called simple graph.



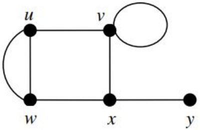
Multigraph.

- Any graph which contain some parallel edges but doesn't contain any self-loop is called multi graph.



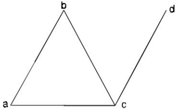
Pseudo Graph.

- A graph that may contain self loop as well as parallel edge is called a pseudo graph.



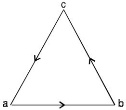
Un-directed graph.

- If an edge consist of unordered pair of elements of V, then the graph is called Un-directed graph.
- In other words, if each edge of the graph G has no direction then the graph is called un-directed graph.



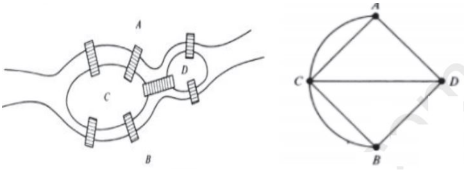
Directed graph.

- If an edge contain ordered pair of elements of V, then the graph is called Directed graph.
- In other words, if each edge of the graph G has a direction then the graph is called directed graph.



Konigsberg bridge problem

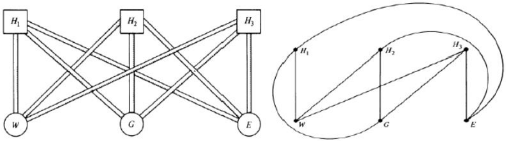
- Two islands C and D were connected to each other and to the banks A and B with seven bridges as shown in figure.
- The problem was to start at any land areas A, B, C or D , walk over each of the seven bridges exactly once, and return to the starting point



- Euler represented this problem by means of a graph. Vertices represent the land areas and the edges represents the bridges.
- Euler proved that a solution for this problem does not exists

Utilities problem

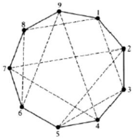
- There are three houses H1, H2 and H3, each to be connected to each of the three utilities water (W), gas (G) and electricity (E) by means of conduits.
- Is it possible to make such connection without any crossover of the conduits?



- A solution for this problem does not exists.
- The graph cannot be drawn in a plane without any edge cross-over

Seating problem

- Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbors at each lunch.
- How many days can this arrangement last?
  - This situation can be represented by a graph with nine vertices such that each vertex represents a member, and an edge joining two vertices represents the relationship of sitting next to each other



- Figure shows two possible seating arrangements
  - 1 2 3 4 5 6 7 8 9 1 (solid lines)
  - 1 3 5 2 7 4 9 6 8 1 (dashed lines).
    - It can be shown by graph-theoretic considerations that there are more arrangements possible.

Seating problem:A solution exists or not?

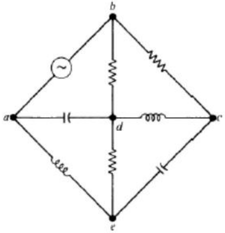
- Yes
- 4 seating arrangements are possible.

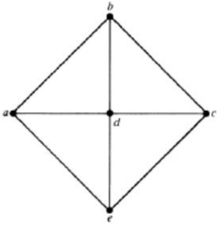
for n people ,the number of possible arrangement is

- $\frac{n-1}{2}$, if n is odd
- $\frac{n-2}{2}$,if n is even

Electrical Network Problem.

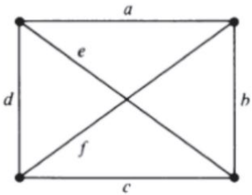
- Topology of a electrical network is studied by means of graphs.
- Vertices represented the electrical network junctions and the edges represented the branches.



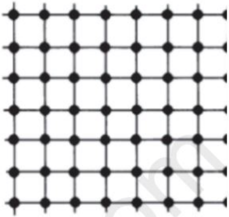


Finite and Infinite Graph

- A graph with finite number of vertices and finite number of edges is called a finite graph, otherwise it is an infinite graph



Finite Graph

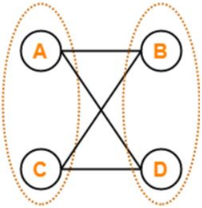


Infinite Graph

Bipartite graphs

Bipartite graphs

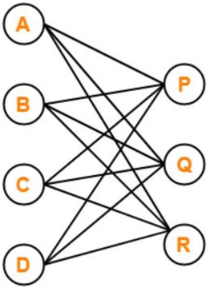
A graph G is bipartite if the node set V can be partioned into two sets V_1 and V_2 in such a way that no nodes from the same set are adjacent



- The vertices of the graph can be decomposed into two sets.
- The two sets are X = {A, C} and Y = {B, D}.
- The vertices of set X join only with the vertices of set Y and vice-versa.
- The vertices within the same set do not join.

Complete Bipartite Graph

- A bipartite graph where every vertex of set X is joined to every vertex of set Y is called as complete bipartite graph.

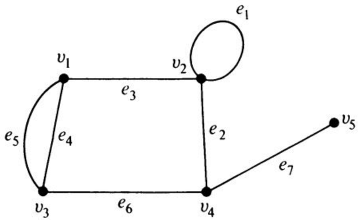


Example of Complete Bipartite Graph

Incidence and Degree

Incidence

- When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to incident with each other.(edge touches that vertices)
- Two non parallel edges said to be **adjacent** if they are incident on a common vertex.

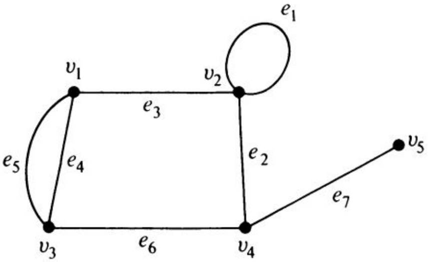


- e_3, e_4, e_5 *incident on the vertex* V_1
- e_3, e_1, e_2 *incident on the vertex* V_2
- e_4, e_5, e_6 *incident on the vertex* V_3
- e_7 *incident on the vertex* V_5

Incidence and Degree

Degree or valency of a vertex

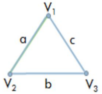
- The number of edges incident on a vertex v_i, with self loop counted twice, is called the degree $d(v_i)$ of vertex v_i.
- For example:
  - $d(v_1) = d(v_3) = d(v_4) = 3$
  - $d(v_2) = 4$
  - $d(v_5) = 1$



A graph in which all vertices are of equal degree is called **regular graph**

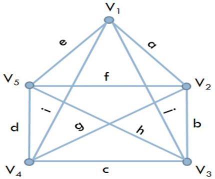
Regular graph

- A graph in which all vertices are of equal degree is called **regular graph** Regular graph with 3 vertices; d(V1)=d(V2)=d(V3)=2
  - A regular graph with degree two is called 2-Regular graph (A regular graph with degree k is called k-Regular graph)



Complete graph

- A graph in which there exists an edge between every pair of vertex is called a complete graph.

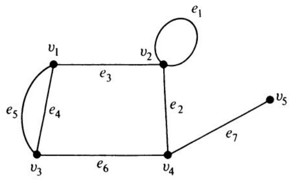


Incidence and Degree

Handshaking Lemma

Sum of degrees of all vertices in G is twice the number of edges in G

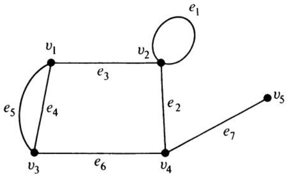
$$\sum_{i=1}^{n} d(v_i) = 2e$$



$$\begin{aligned} sum\ of\ degrees\ of\ vertices &= d(v1) + d(v2) + d(v3) + d(v4) + d(v5) \\ &= 3 + 4 + 3 + 3 + 1 \\ &= 14 \\ &= twice\ the\ number\ of\ edges \end{aligned}$$

Incidence and Degree

Theorem1.1:
The number of vertices of odd degree in a graph is always even



$$\begin{aligned}
\textit{the number of vertices of odd degree} &= d(v1), d(v3), d(v4), d(v5) \\
&= 3, 3, 3, 1 \\
&= 4(\textit{even})
\end{aligned}$$

The number of vertices of odd degree in a graph is always even

(Proof)Sum of the degrees of all vertices in G

$$\sum_{i=1}^{n} d(v_i) = 2e \tag{1}$$

If we consider the vertices with odd and even degree separately.

$$\sum_{i=1}^{n} d(v_i) = \sum_{even} d(v_j) + \sum_{odd} d(v_k) \tag{2}$$

$$\sum_{even} d(v_j) + \sum_{odd} d(v_k) = 2e \tag{3}$$

$$even + \sum_{odd} d(v_k) = even \tag{4}$$

$$\sum_{odd} d(v_k) = even - even \tag{5}$$

$$\sum_{odd} d(v_k) = even \tag{6}$$

Find if a degree sequence can form a simple graph

i Sort the degrees in descending order

ii Delete the first element(say V). Subtract 1 from the next V elements.

iii Repeat 1 and 2 until one of the stopping conditions is met.

Stopping conditions:

a All the elements remaining are equal to 0 (Simple graph exists).

b Negative number encounter after subtraction (No simple graph exists).

c Not enough elements remaining for the subtraction step (No simple graph exists).

Q1: Can there be a graph with degree sequence 3 2 2 1

Constructing a graph from a degree sequence

i Sort the degrees in descending order

ii Connect the highest degree d to the next d vertices

iii Take away(remove) the first degree (value of d) and reduce the following d degrees by one

iv Repeat the steps until all degrees are zero

Incidence and Degree

Q1: How many edges are there in a graph with 10 vertices and each of degree 6 ?

Q2: Draw graphs representing problems of

- a two houses and three utilities;
- b four houses and four utilities, say, water, gas, electricity, and telephon

Q3 A graph has 5 vertices with degree as: a. 2,3,1,1,2 check the given degree sequence form a graph

Q4 Can there be a graph with degree sequence (5, 5, 4, 3, 2, 1)

Q5 Can there be a graph with degree sequence (1, 1, 1)? Explain.

Q6 Could there exist a graph with the following degrees of vertices: (a) 4, 3, 3, 1 (b) 4, 3,3,2,2 (c) 5,4,4,2,2,1? (If yes, can we provide an example and If no, can we explain why?

Q7 There are 25 telephones in CSE Dept. Is it possible to connect them with wires so that each telephone is connected with exactly 7 others.

HW1: Convince yourself that the maximum degree of any vertex in a simple graph with n vertices is n − 1.

HW2: Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-l)}{2}$.

HW3: Draw the graph of the Wheatstone bridge circuit.

HW4: Draw graphs of the following chemical compounds:$(a)CH_4, (b)C_2H_6, (c)C_6H_6, (d)N_2O_3$.(Hint: Represent atoms by vertices and chemical bonds between them by edges.)

HW5: Draw a graph with 64 vertices representing the squares of a chessboard. Join these vertices appropriately by edges, each representing a move of the knight. You will see that in this graph every vertex is of degree two, three, four, six, or eight. How many vertices are of each type?

Q1: Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-l)}{2}$.

- We have that is a simple graph, no parallel or loop exist. Therefore the degree of each vertex will be one less than the total number of vertices (at most). ie, degree=n-1.
- We know that the sum of the degree in a simple graph always even ie,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

here d(v)=n-1 : we have n vertices the total degree is n(n-1)

$$n(n-1) = 2E$$

$$E = \frac{n(n-1)}{2}$$

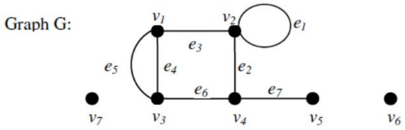
Isolated vertex & Pendant vertex

Isolated vertex

- A vertex having no incident edge is called isolated vertex.
- Isolated vertices are vertices with zero degree.

Pendant vertex

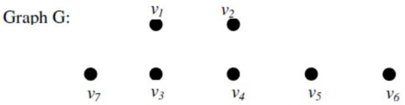
- A vertex of degree one is called a pendant vertex or an end vertex.



- The vertices v_6 and v_7 are isolated vertices.
- The vertex v_5 is a pendant vertex.

Null graph

- A graph without any edges is called null graph
- Every vertex in null graph is an isolated vertex



Isomorphism

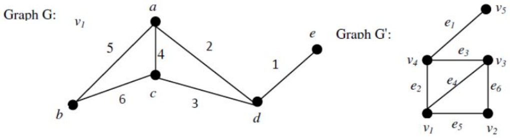
Isomorphism

Two graphs G and G' are said to be isomorphic to each other if there is a one-to one correspondence (bijection) between their vertices and between their edges such that the incidence relationship is preserved.

The two isomorphic graph must have

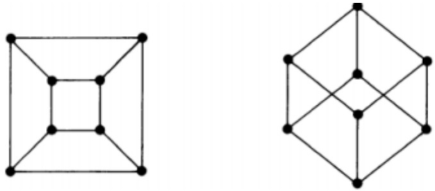
- same number of vertices
- same number of edges
- equal number of vertices with a given degree

QN: Check whether the following pair of graphs are isomorphic or not

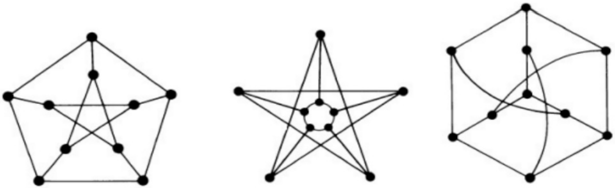


	G1	G2
Vertices	{a,b,c,d,e}	{v1,v2,v3,v4,v5}
Total no. of vertices	5	5
Vertices with degree (arranged in decending order)	a, c, d, b, e	v1, v3, v4, v2, v5
Total number of edges= sum of degrees of all vertices/2	$(3+3+3+2+1)/2 = 6$	$(3+3+3+2+1)/2 = 6$

Since G1 and G2 have same number of vertices and same number of edges, G1 and G2 are isomorphic Graphs



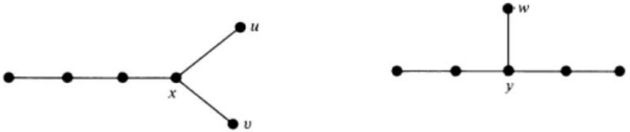
	G1	G2
Vertices and	A1, A2, A3, A4, A5, A6, A7, A8	B1, B2, B3, B4, B5, B6, B7, B8
Total number of vertices	8	8
Vertices with degree (arranged in decending order)	A1, A2, A3, A4, A5, A6, A7, A8	B1, B2, B3, B4, B5, B6, B7, B8
Total number of edges= sum of degrees of all vertices/2	(3+3+3+3+3+3+3+3)/2=12	(3+3+3+3+3+3+3+3)/2=12
Since G1 and G2 have same number of vertices and same number of edges, G1 and G2 are isomorphic Graphs		



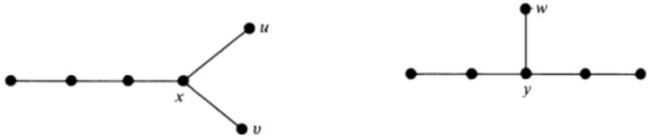
	G1	G2	G3
Vertices and	A1, A2, A3, A4, A5 , A6, A7, A8, A9, A10	B1, B2, B3, B4, B5, B6, B7, B8, B9, B10	C1, C2, C3, C4, C5, C6, C7, C8, C9, C10
Total number of vertices	10	10	10
Vertices with degree (arranged in decending order)	A1, A2, A3, A4, A5, A6, A7, A8, A9, A10	B1, B2, B3, B4, B5, B6, B7, B8, B9,B10	C1, C2, C3, C4, C5, C6, C7, C8, C9,C10
Total number of edges= sum of degrees of all vertices/2	(3+3+3+3+3+3+3+3 +3+3)/2=15	(3+3+3+3+3+3+3+3+ 3+3)/2=15	(3+3+3+3+3+3+3+3+ 3+3)/2=15
Since G1,G2 and G3 have same number of vertices and same number of edges, G1, G2 and G3 are isomorphic Graphs			

Isomorphism

The following two graphs are not isomorphic, because x is adjacent to two pendant vertex is not preserved.



Isomorphism

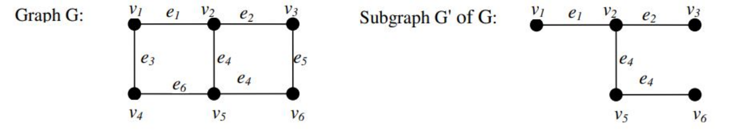


	G1	G2
Vertices and	$A1, A2, A3, A4, A5, A6$	$B1, B2, B3, B4, B5, B6$
Total number of vertices	6	6
Vertices with degree (arranged in decending order)	$A4, A2, A3, A1, A5, A6$	$B3, B2, B5, B1, B4, B6$
Total number of edges= sum of degrees of all vertices/2	(3+2+2+1+1+1)/2=5	(3+2+2+1+1+1)/2=5
G1 and G2 have same number of vertices and same number of edges, but does not follow 1-to-1 correspondance G1 and G2 are not isomorphic Graphs		

Sub graphs

Sub graphs

A graph G' is said to be a subgraph of a graph G, if all the vertices and all the edges of G' are in G, and each edge of G' has the same end vertices in G' as in G.

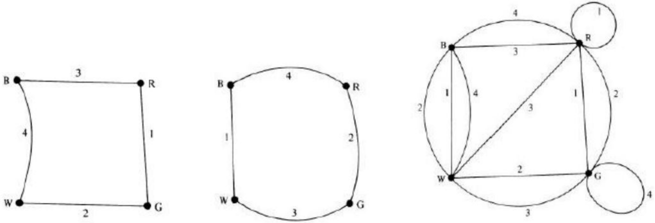


Properties of sub graph

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph C is a subgraph of G.
4. A single edge in G, together with its end vertices, is also a subgraph of G.

Edge-Disjoint Subgraphs

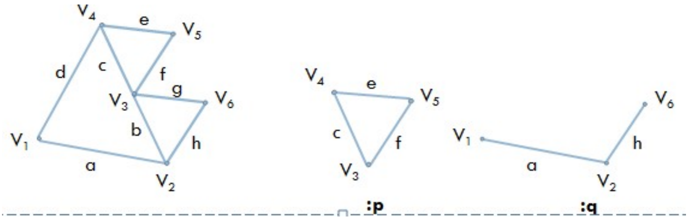
- Two (or more) subgraphs g1, and g2 of a graph G are said to be edge disjoint if g1, and g2 do not have any edges in common.
- The following two graphs are edge-disjoint sub-graphs of the graph G



Note that although edge-disjoint graphs do not have any edge in common, they may have vertices in common.

Vertex disjoint

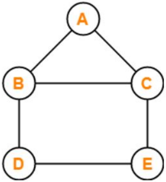
- Sub-graphs that do not even have vertices in common are said to be vertex disjoint.
- Obviously they cannot have any edges in common.
- p and q are vertex-disjoint subgraphs of G.



A subgraph that contains all the vertices of the original graph is called A **spanning subgraph** .

Walks/Trail

- A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices.
- No edge appears more than once.
- It is also called as an edge train or a chain.



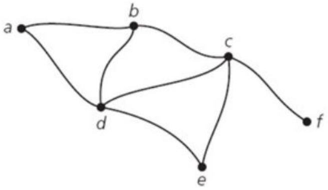
- In this graph, few examples of walk are-
  - a , b , c , e , d (Length = 4)
  - d , b , a , c , e , d (Length = 5)
  - e , c , b , a , c , e , d (Length = 6)

All about a Walk/Trail

- No edge can appear more than once
- Vertex may appear more than once
- Walk is a sub graph of G
- Vertices with which a walk Starts and ends are called terminal vertices of the walk.

Walks/Trail

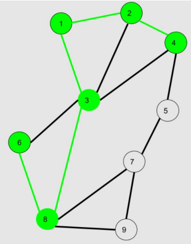
- **Closed Walk:** If a walk begins and ends with same vertex, then it is called closed walk
- **Open Walk:** If a walk is not closed is called Open Walk



- (b,c),(c,d) and (d,b) provide a b-b (closed walk)
- (a,b),(b,c),(c,d), and (d,e) provide a a-e (open walk)

Paths

- An open walk in which no vertex appears more than once is called path.
- The number of edges in the path is called length of a path.



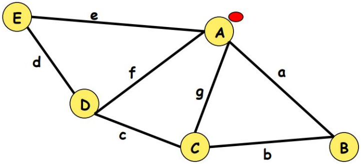
Here $6->8->3->1->2->4$ is a Path
Length=5

All about a Path

- No edge can appear more than once
- No vertex can appear more than once
- A path of G is a subgraph of G
- Starts and ends at different vertices, called the terminal vertices of the path
- Terminal vertices are of degree 1
- All other vertices are of degree 2
- The no. of edges in a path is called its length
- An edge itself could be a path
- A path is a walk with no vertex repetition

Circuits/Cycle

- A closed walk in which no vertex (except initial and final vertex) appears more than once is called a circuit.
- That is, a circuit is a closed, nonintersecting walk.
- Also called as:Cycle, Elementary cycle, Circular path, Polygon



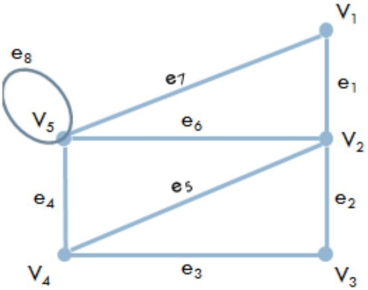
✓ A - a - B - b - C - c - D - d - E - e - A

Walks, Paths and Circuits

Item	**Vertex**	**edge**	**Initial vertex = final vertex ?**
walk	Can repeat	Cannot repeat	no
Closed walk	Can repeat	Cannot repeat	yes
Open walk	Can repeat	Cannot repeat	no
Path	Cannot repeat	Cannot repeat	no
Circuit	Cannot repeat	Cannot repeat	yes

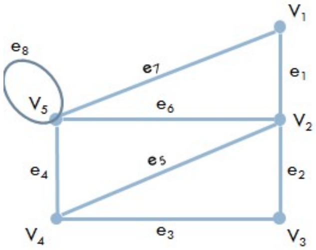
Q: In the given graph, trace a

- Walk
- Closed walk
- Path
- Circuit



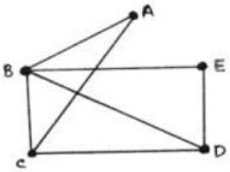
Q: For the given graph H, trace

- 2 edge disjoint subgraphs
- 2 vertex disjoint subgraphs
- A walk
- A path
- A closed walk
- A circuit

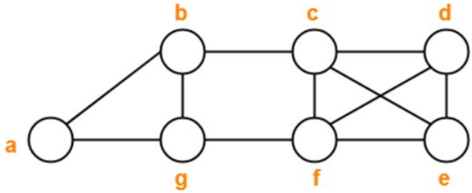


Q:Using the graph classify each sequence as a walk, path or a circuit.

- E-B-D-E
- A-C-D-E-B-A
- B-D-E-B-C
- A-B-C-D-B-A



Consider the following graph-



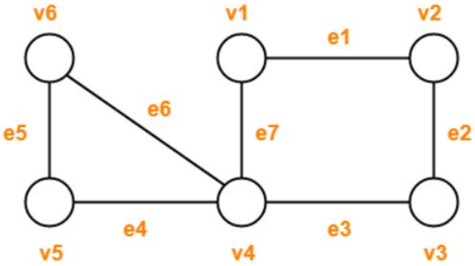
Decide which of the following sequences of vertices determine walks.

For those that are walks, decide whether it is a circuit, a path, a cycle or a trail.

1. a , b , g , f , c , b
2. b , g , f , c , b , g , a
3. c , e , f , c
4. c , e , f , c , e
5. a , b , f , a
6. f , d , e , c , b

Walks, Paths and Circuits

Consider the following graph-



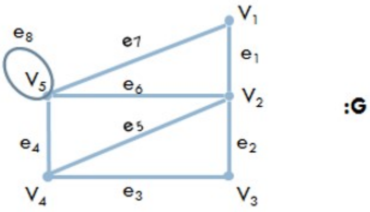
1. v1e1v2e2v3e2v2
2. v4e7v1e1v2e2v3e3v4e4v5
3. v1e1v2e2v3e3v4e4v5
4. v1e1v2e2v3e3v4e7v1

Connected Graph

- A graph G is connected if there is at least one path between every pair of vertices in G; else G is disconnected.

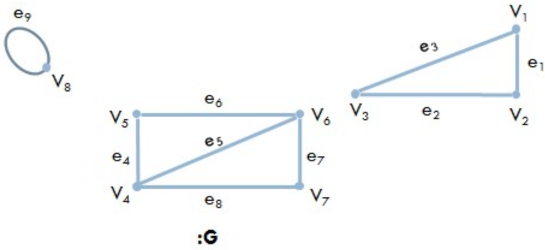
Or

- A graph G is connected if we can reach any vertex from any other vertex by travelling along the edges.



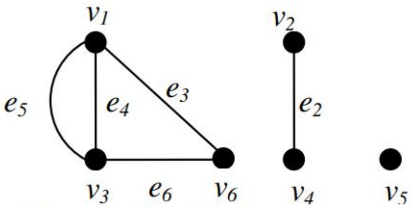
Disconnected Graph

- A disconnected graph consists of two or more connected graphs
- each of these connected subgraphs is called a **component**.
- Hence each component itself is a graph.



COMPONENTS

- A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.



Disconnected Graph H with 3 components

Connected Graph,Disconnected Graph & Components

THEOREM 1-2

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in subset V_1 and the other in subset V_2.

Proof:

$\Rightarrow$

- Let G be a disconnected graph
- Consider an arbitary vertex U in G
  - Let V_1 be the set of all vertices reachable from U
  - Since G is disconnected, V_1 doesnot contain all the vertices of G.
  - Let V_2 be the set of remaining vertices $V_2 = V - V_1$
  - ie $V_1 \cap V_2 = \phi$
  - No vertex in V_1 is joined to any other vertex in V_2 by edge
  - So there is no edge in G whose one end vertex is in V_1 and other in V_2

Proof cont...

$\Leftarrow$

- Let G be a graph Whose vertex can be partition in to two nonempty disjoint subsets V_1 & V_2
    - No edge of G has one end point in V_1 and other end point in V_2
- ie $V_1 \cap V_2 = \phi$
- Let u and w be any two vertices in G such that u$\in V_1$ and w$\in V_2$
- There is no path between vertices u & w, since there is no edge joining
- There for the graph is disconnected

THEOREM 1-3

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof:

- Let G be a graph with all even vertices except vertices v_1, and v_2, which are odd
- Since every component of a graph can be considered as a graph itself, no graph can have an odd number of odd vertices.
- Therefore, in graph G, v_1 and v_2 must belong to the same component
- If they lie in the same component there must be a path between them.

THEOREM 1-4

A simple graph with n vertices and k components can have at most

$$\frac{(n-k)(n-k+1)}{2}$$

edges.

Proof:

- Let G be a simple graph with n vertices and k components
- let the components be $G_1, G_2, G_3, G_4......$
- Let the number of vertices in G_i be n_i
- then $n = n_1 + n_2 + n_3 + ... + n_k$

Proof cont...

- The maximum possible number of edges in G_i is
$$\frac{n_i(n_i - 1)}{2}$$

- Thus maximum number of edges in G is
$$\sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$

Proof cont...

- from algebraic inequality for any set of positive integers $n_1, n_2, n_3, n_4 \ldots\ldots n_k$

$$\sum_{i=1}^{k} (n_i - 1) = n - k \tag{1}$$

squaring both sides(1)

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n - k)^2 \tag{2}$$

rewrite(2)

$$\sum_{i=1}^{k} (n_i - 1)^2 \leq (n - k)^2 \tag{3}$$

Proof cont...

$$\sum_{i=1}^{k}(n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2 \quad (4)$$

$$\sum_{i=1}^{k} n_i^2 - 2n + K \leq n^2 - 2nk + k^2 \quad (5)$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - K \quad (6)$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \quad (7)$$

Proof cont...

- therefor , the maximum number of edges in G is

$$\begin{aligned}
\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} &= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{1}{2}\sum_{i=1}^{k} n_i \\
&= \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{n}{2} \\
&\quad \textit{Apply eq(7)in the above equation we get} \\
&\leq \frac{1}{2}[n^2-(k-1)(2n-k)] - \frac{n}{2} \\
&\leq \frac{(n-k)(n-k+1)}{2}
\end{aligned}$$

Problems

At a party of N people, some pair of people are friends with the same number of people at the party.

- In a group of N people(vertex), a fellow may have 0, 1, 2, ..., N-1 friends(degrees). Assume to the contrary that all N people have different number of friends. Then for each number in the sequence 0, 1, 2, ..., N-1 there must be a fellow with exactly this number of friends. In particular, there is at least one with N-1 friends. But, if so, all others have this fellow as a friend, implying that there is no one with no friends at all. Therefore, the only possible numbers of friends come from the shortened sequence: 1, 2, 3, ..., N-1. By the Pigeonhole Principle, there are at least two with the same number of friends

Problems

If 10 people each shake hands with each other, how many handshakes took place? What does this question have to do with graph theory?

- This is asking for the number of edges in. Each vertex (person) has degree (shook hands with) 9 (people). So the sum of the degrees is 90. However, the degrees count each edge (handshake) twice, so there are 45 edges in the graph. That is how many handshakes took place.