UNIT - III

VECTOR DIFFERENTIATION

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1) If $a = x+y+z$, $b = x^2+y^2+z^2$, $c = xy+yz+zx$. Prove that $[\text{grad } a, \text{grad } b, \text{grad } c] = 0$.

Sol:- $a = x+y+z$

$$\text{grad } a = \nabla a = \bar{i}\frac{\partial a}{\partial x} + \bar{j}\frac{\partial a}{\partial y} + \bar{k}\frac{\partial a}{\partial z}$$

$$\frac{\partial a}{\partial x} = 1, \quad \frac{\partial a}{\partial y} = 1, \quad \frac{\partial a}{\partial z} = 1$$

$$\text{grad } a = \nabla a = \bar{i} + \bar{j} + \bar{k}$$

$$\text{grad } b = \nabla b = \bar{i}\frac{\partial b}{\partial x} + \bar{j}\frac{\partial b}{\partial y} + k\frac{\partial b}{\partial z}$$

$$b = x^2 + y^2 + z^2$$

$$\frac{\partial \bar{b}}{\partial x} = 2x \,;\quad \frac{\partial \bar{b}}{\partial y} = 2y \,;\quad \frac{\partial \bar{b}}{\partial z} = 2z$$

$$\text{grad } b = \nabla b = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\text{grad } c = \nabla c = \bar{i}\frac{\partial c}{\partial x} + \bar{j}\frac{\partial c}{\partial y} + \cancel{k}\frac{\partial c}{\partial z}$$

$$C = xy + yz + zx$$

$$\frac{\partial c}{\partial x} = (y+z) \,;\quad \frac{\partial c}{\partial y} = (x+z) \,;\quad \frac{\partial c}{\partial z} = (y+x)$$

$$\text{grad } c = \nabla c = (y+z)\bar{i} + (x+2)\bar{j} + (y+x)\bar{k}$$

[grad a grad b grad c]

$$\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$\Rightarrow 1\big(2y(x+y) - (x+z)(2z)\big) - 1\big(2x(x+y) - (y+z)(2z)\big)$$
$$+ 1\big(2x(x+z) - y+z(2y)\big)$$

$$\Rightarrow 1(2xy+2y^2-2xz-2z^2)-1(2x^2+2xy-2yz-2z^2)$$

$$+1(2x^2+2xz-2y^2-2yz)$$

$$\Rightarrow \cancel{2xy}+\cancel{2y^2}-\cancel{2xz}-\cancel{2z^2}-\cancel{2x^2}-\cancel{2xy}+\cancel{2yz}+\cancel{2z^2}$$

$$+\cancel{2x^2}+\cancel{2xz}-\cancel{2y^2}-\cancel{2yz}$$

$$=0.$$

3) Find the D-D of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x=t, y=t^2, z=t^3$ at the (1,1,1)

Sol:- $f = xy^2+yz^2+zx^2$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + \bar{k}\cdot\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = y^2+2xz \quad , \quad \frac{\partial f}{\partial y} = 2yx+z^2 \quad , \quad \frac{\partial f}{\partial z} = 2yz+x^2$$

$$\nabla f = (y^2+2xz)\,\bar{i} + (2yx+z^2)\,\bar{j} + (2yz+x^2)\,\bar{k}$$

at (1,1,1)

$$\nabla f = 3\bar{i}+3\bar{j}+3\bar{k}$$

$$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$$

$$\bar{r} = t\bar{i}+t^2\bar{j}+t^3\bar{k}$$

$$e = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} \Rightarrow \frac{d\bar{r}}{dt} = \bar{i}+2t\bar{j}+3t^2\bar{k}$$

at (1,1,1)

$$\frac{d\bar{r}}{dt} = \bar{i}+2\bar{j}+3\bar{k}$$

$$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{1^2+2^2+3^2}$$

$$= \sqrt{14}$$

$$D.D = e \cdot \nabla f$$

$$= \frac{(i+2\bar{j}+3\bar{k})}{\sqrt{14}} \cdot (3\bar{i}+3\bar{j}+3\bar{k})$$

$$= \frac{1}{\sqrt{14}}(3+6+9) = \frac{18}{\sqrt{14}}$$ //

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(2.) Find div f and curl f where $f = \text{grad}(x^3+y^3+z^3-3xyz)$

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Sol:- $f = \text{grad}\,(x^3+y^3+z^3-3xyz)$

$$f = \bar{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + \bar{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$
$$+ \bar{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$f = \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$$

(i) ~~grad~~ div $f = \nabla \cdot f$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$= 6x + 6y + 6z$$

(ii) curl $f = \nabla \times f$.

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right] - \bar{j}\left[\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right] + \bar{k}\left[\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right]$$

$$= i[-3x + 3x] - j[-3y + 3y] + k[-3z + 3z]$$

$$= 0$$

* 2) Prove that grad $(A \cdot B) = (B \cdot \nabla) A + (A \cdot \nabla) B +$
$B \times \text{Curl } A + A \times \text{Curl } B.$

Sol:- $A \times \text{curl } B = A \times (\nabla \times B)$

$$= A \times \Sigma i \times \frac{\partial B}{\partial x}$$

$$= \Sigma A \times \left(i \times \frac{\partial B}{\partial x} \right)$$

$$\boxed{a \times (b \times c) = (a \cdot c) b - (a \cdot b) c}$$

$$= \Sigma \left\{ \left(A \cdot \frac{\partial B}{\partial x} \right) i - (A \cdot i) \frac{\partial B}{\partial x} \right\}$$

$$= \Sigma i \left(A \cdot \frac{\partial B}{\partial x} \right) - \Sigma (A \cdot i) \frac{\partial B}{\partial x}$$

$$= \Sigma i \left(A \cdot \frac{\partial B}{\partial x} \right) - \left(A \cdot \Sigma i \frac{\partial}{\partial x} \right) B$$

$$= \Sigma i \left(A \cdot \frac{\partial B}{\partial x} \right) - (A \cdot \nabla) B$$

$$A \times \text{curl } B = \Sigma i \left(A \cdot \frac{\partial B}{\partial x} \right) - (A \cdot \nabla) B$$

$$A \times \text{curl } B + (A \cdot \nabla) B = \Sigma i \left(A \cdot \frac{\partial B}{\partial x} \right) \rightarrow (1)$$

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$$B \times \text{curl } A = \Sigma i \left(B \cdot \frac{\partial A}{\partial x} \right) - (B \cdot \nabla) A$$

$$B \times \text{curl } A + (B \cdot \nabla) A = \Sigma i \left(B \cdot \frac{\partial A}{\partial x} \right) \rightarrow (2)$$

Adding eq^n ① & ②,

$$A \times \text{curl } B + (A \cdot \nabla) B + B \times \text{curl } A + (B \cdot \nabla) A = \Sigma i \left(A \cdot \frac{\partial B}{\partial x} \right) + \Sigma i \left(B \cdot \frac{\partial A}{\partial x} \right)$$

$$= \Sigma i \left(A \cdot \frac{\partial B}{\partial x} + B \cdot \frac{\partial A}{\partial x} \right)$$

$$= \Sigma i \frac{\partial}{\partial x} (A \cdot B)$$

$$= \nabla (A \cdot B)$$

$$= \text{grad} (A \cdot B)$$

$$\therefore \text{grad} (A \cdot B) = (B \cdot \nabla) A + (A \cdot \nabla) B + B \times \text{curl } A + A \times \text{curl } B.$$

9) Prove that $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$.

S. **Proof**:- $\nabla\times(\nabla\times A) = \sum i\times\frac{\partial}{\partial x}(\nabla\times A)$

$$i\times\frac{\partial}{\partial x}(\nabla\times A) = i\times\frac{\partial}{\partial x}\left(\bar{i}\times\frac{\partial A}{\partial x}+\bar{j}\times\frac{\partial A}{\partial y}+\bar{k}\times\frac{\partial A}{\partial z}\right)$$

$$= i\times\left(i\times\frac{\partial^2 A}{\partial x^2}+\bar{j}\times\frac{\partial^2 A}{\partial x\partial y}+\bar{k}\times\frac{\partial^2 A}{\partial x\partial z}\right)$$

$$= i\times\left(i\times\frac{\partial^2 A}{\partial x^2}\right)+i\times\left(J\times\frac{\partial^2 A}{\partial x\partial y}\right)+i\times\left(k\times\frac{\partial^2 A}{\partial x\partial z}\right)$$

$\because$ $\boxed{a\times(b\times c) = (a\cdot c)b - (a\cdot b)c}$

$$= \left(i\cdot\frac{\partial^2 A}{\partial x^2}\right)\bar{i} - (i\cdot i)\frac{\partial^2 A}{\partial x^2} + \left(i\cdot\frac{\partial^2 A}{\partial x\partial y}\right)\bar{j} - (i\cdot j)\frac{\partial^2 A}{\partial x\partial y} + \left(i\cdot\frac{\partial^2 A}{\partial x\partial z}\right)\bar{k} - (i\cdot k)\frac{\partial^2 A}{\partial x\partial z}$$

$$= \left(i\cdot\frac{\partial^2 A}{\partial x^2}\right)\bar{i} - \frac{\partial^2 A}{\partial x^2} + \left(i\cdot\frac{\partial^2 A}{\partial x\partial y}\right)\bar{j} + \left(i\cdot\frac{\partial^2 A}{\partial x\partial z}\right)k$$

$$= i\times\frac{\partial}{\partial x}(\nabla\times A) = i\cdot\frac{\partial}{\partial x}\left(i\cdot\frac{\partial A}{\partial x}\right)+\bar{j}\frac{\partial}{\partial y}\left(i\cdot\frac{\partial A}{\partial x}\right)+\bar{k}\frac{\partial}{\partial z}\left(i\cdot\frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$= \left(i \cdot \frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left(i \cdot \frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$\cancel{\Sigma\left(i \times \frac{\partial}{\partial x}(\nabla \times A)\right) = \nabla\left(\Sigma i \cdot \partial A\right.}$$

$$= i \cdot \frac{\partial}{\partial x}(\nabla \times A) = \nabla\left(i \cdot \frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$= \Sigma\left(i \times \frac{\partial}{\partial x}(\nabla \times A)\right) = \nabla\left(\Sigma i \cdot \frac{\partial A}{\partial x}\right) - \Sigma \frac{\partial^2 A}{\partial x^2}$$

$$\therefore \nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A \;//.$$

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SHORTS

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3) $A = 5t^2\bar{i} + t\bar{j} - t^3\bar{k}$ and $B = \sin t\,\bar{i} - \cos t\,\bar{j}$.

(i) $\frac{d}{dt}(A \cdot B)$ (ii) $\frac{d}{dt}(A \times B)$ (iii) $\frac{d}{dt}(A \cdot A)$

Sol:- $A \cdot B = 5t^2 \sin t\,\bar{i} - t\cos t\,\bar{j}$ $\quad uv' - vu'$

$$\frac{d}{dt}(A \cdot B) = [10t \sin t + 5t^2 \cos t) - (\cos t + (-\sin t)t)$$

$$= 10t \sin t + 5t^2 \cos t - \cos t + t \sin t$$

$$= (10t + t)\sin t + (5t^2 - 1)\cos t.$$

$$A \times B = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$= \bar{i}[-t^3\cos t] - \bar{j}[t^3\sin t] + \bar{k}[-5t^2\cos t - t\sin t]$

$= \bar{i}[-t^3\cos t] - \bar{j}[t^3\sin t] + \bar{k}[-t(5t\cos t + \sin t)]$

$$\frac{d}{dt}(A\times B) = (-3t^2\cos t + t^3\sin t)\bar{i} - (3t^2\sin t + t^3\cos t)\bar{j} + \bar{k}[-10t\cos t + 5t^2\sin t - \sin t - t\cos t]$$

$$= (-3t^2\cos t + t^3\sin t)\bar{i} - (3t^2\sin t + t^3\cos t)\bar{j} + (-10t\cos t + 5t^2\sin t - t\cos t)\bar{k}$$

(iii) $A\cdot A) = (5t^2\cdot 5t^2) + (t\cdot t) + (t^3\cdot t^3)$

$= 25t^4 + t^2 + t^6$

$\frac{d}{dt}(A\cdot A) = 100t^3 + 2t + 6t^5$ //.

3) If $\phi = 2xz^4 - x^2y$ find the value of $\left|\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}\right|$ at $(2,-2,1)$.

Sol:- $\phi = 2xz^4 - x^2y$

$$\frac{\partial\phi}{\partial x} = 2z^4 - 2xy \text{ at } (2,-2,1) = 10$$

$$\frac{\partial\phi}{\partial y} = -x^2 \text{ at } (2,-2,1) = -4$$

$$\frac{\partial\phi}{\partial z} = 8xz^3 \text{ at } (2,-2,1) = -16$$

$$\left|\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}\right|$$

$$= |10\bar{i} - 4\bar{j} - 16\bar{k}|$$

$$= \sqrt{(10)^2 + (-4)^2 + (-16)^2} = \sqrt{100 + 16 + 256} = \sqrt{372}$$

$$= \sqrt{4 \times 93} = 2\sqrt{93}$$

3) Show that $3y^4z^2\bar{i} + 4x^3z^2\bar{j} - 3x^2y^2\bar{k}$ is solenoidal.

Sol:- Solenoidial $\Rightarrow$ div $f = 0$ $\Rightarrow \nabla \cdot \bar{f} = 0$

Given,

$$f = 3y^4z^2\,\bar{i} + 4x^3z^2\bar{j} - 3x^2y^2\bar{k}$$

$$\nabla \cdot \bar{f} = \left(i \cdot \frac{\partial}{\partial x} + j \cdot \frac{\partial}{\partial y} + k \cdot \frac{\partial}{\partial z}\right) \cdot \left(3y^4z^2\bar{i} + 4x^3z^2\bar{j} - 3x^2y^2\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0 + 0 + 0$$

$$= 0$$

$\therefore$ It is solenoidial

8.) S.T $\nabla^2\left(\frac{1}{r}\right) = 0$

Sol:- $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

partial derivative w.r.t 'x',

$$\cancel{2}r \cdot \frac{\partial r}{\partial x} = \cancel{2}x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla^2\left(\frac{1}{r}\right) = \Sigma \frac{\partial}{\partial x^2}\left(\frac{1}{r}\right)$$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right)$$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{-1}{r^2} \cdot \frac{\partial r}{\partial x}\right)$$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{-1}{r^2} \cdot \frac{x}{r}\right)$$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right)$$

$$= \Sigma \frac{r^3(-1) + x \quad 3r^2 \cdot \frac{\partial r}{\partial x}}{(r^3)^2}$$

$$= \Sigma \frac{-r^3 + 3x \cdot r^{\cancel{2}}}{r^6} \cdot \frac{x}{r}$$

$$= \Sigma \frac{-r^3}{r^6} + \frac{3x^2 \cdot \cancel{r}}{r^6}$$

$$= \Sigma \frac{-1}{r^3} + \frac{3x^2}{r^5}$$

$$= \frac{-3}{r^3} + \frac{3}{r^5}\left(x^2+y^2+z^2\right)$$

$$= \frac{-3}{r^3} + \frac{3}{r^5} \cdot \left(r^2\right)$$

$$= \frac{-3}{r^3} + \frac{3}{r^3} = 0 = R.H.S.$$