UNIT-3

CLASS NO.1

10-1-5

LINEAR TRANSFORMATIONS

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Shorts-4

Definitions :-

$U(F), V(F) \to$ Vector spaces

① **Linear Transformation:** $T : U \to V \to$ function

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

② **Range Space:** $U(F), V(F) \to$ 2 Vector Spaces.

$T : U \to V \to$ L.T.

dim $\to$ Rank $\rho(T)$

$$R(T) = \{T(\alpha) ; \alpha \in U\}$$

$R(T) \subseteq V$.

③ **Null space:** $U(F), V(F) \to$ 2 Vector spaces

$T : U \to V \to$ L.T.

dim $\to$ Nullity $\nu(T)$

denoted by $\to N(T)$.

$$T(\alpha) = \bar{0} \Longleftrightarrow \alpha \in N(T)$$

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$U \longrightarrow$ FDVS

Rank:

$T : U \longrightarrow V$ ~~PA~~ L.T.

Rank $\longrightarrow \rho(T)$

$$\boxed{\rho(T) = \dim R(T)}$$

Revision:

$\longrightarrow$ L.T. $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

$\longrightarrow$ Range space $[R(T)] \longrightarrow T(\alpha);\ \alpha \in U$

↓

డిమెన్షన్ = Value → Dimension

↓

Rank $= \rho(T) = \dim R(T)$

$\longrightarrow$ Null space $N(T)$ < $\alpha \in N(T)$ ↓ $T(\alpha) = \bar{0}$

↓

డిమెన్షన్ Dimension

Nullity $\rightarrow \nu(T) = \dim N(T)$

LONGS. V.V. Imp

① State and Prove Rank-Nullity Theorem.

[or]

Statement:

Let $U(F)$ & $V(F)$ be two Vector Space & $T: U \to V$ is a L.T. Let U is finite dimensional V.S. Then $\delta(T) + \nu(T) = \dim U(F)$

[or] $\rho(T)$

$$\text{rank}(T) + \text{Nullity}(T) = \dim U.$$

PROOF:

The Null space $N(T)$ is a Subspace of a finite dimensional Vector space $U(F)$. [∵ Short Ans]

$\therefore$ $N(T)$ is finite dimensional.

$\therefore$ Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_K\}$ is a Basis of $N(T)$.

$\therefore \dim N(T) = \nu(T) = K.$

$\therefore T(\alpha_1) = \bar{0}, T(\alpha_2) = \bar{0}, \ldots T(\alpha_K) = \bar{0}$

Process
→ ~~RHS~~ LHS
→ RHS →
$S_2\{ \quad \}$
↓
Basis
LI LS.
→ dim
RHS = LHS

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As 's' is L.I., It can be extended to form a Basis of $U(F)$.

Let $S_1 = \{\alpha_1, \alpha_2, \cdots \alpha_K, \theta_1, \theta_2, \cdots \theta_m\}$ is a Basis of $U(F)$.

$$\therefore \dim U(F) = K+m.$$

$$\therefore \rho(T) + \nu(F) = \dim U. \quad [\underline{\text{Notation}}]$$

$$\rho(T) + K = K+m$$

$$\rho(T) = \cancel{K}+m-\cancel{K}$$

$$\rho(T) = m$$

To prove $\rho(T) = \dim R(T) = m$,

Let $S_2 = \{T(\theta_1), T(\theta_2), T(\theta_3), \cdots T(\theta_m)\}$ is a Basis of $R(T)$.

$T(\alpha) = \overline{0} \Rightarrow \alpha \in N(T)$

(i) To P.T. S_2 is L.I.:

$$a_1 T(\theta_1) + a_2 T(\theta_2) + \cdots + a_m T(\theta_m) = \overline{0}$$

$$T[a_1\theta_1 + a_2\theta_2 + \cdots + a_m\theta_m] = \overline{0}$$

$$a_1\theta_1 + a_2\theta_2 + \cdots + a_m\theta_m \in N(T)$$

$$a_1 \omega_1 + a_2 \omega_2 + \cdots + a_m \omega_m = b_1 \alpha_1 + b_2 \alpha_2 + \cdots + b_k \alpha_k$$

$$\therefore a_1 \omega_1 + a_2 \omega_2 + \cdots + a_m \omega_m - b_1 \alpha_1 - b_2 \alpha_2 \cdots - b_k \alpha_k = \bar{0}.$$

= l.c. of the elements of S_1.

Since, S_1 is Basis $\Rightarrow$ S_1 is L.I.

$$\therefore \quad a_1 = 0,\ a_2 = 0,\ \cdots\ a_m = 0$$
$$b_1 = 0,\ b_2 = 0,\ \cdots\ b_k = 0.$$

$\therefore$ S_2 is L.I.

ii. To Prove $L(S_2) = R(T)$. :-

Let $\beta \in$ range of $R(T)$. $\exists\ \alpha \in U \Rightarrow T(\alpha) = \beta$.

$\alpha \in U$

$$\alpha = c_1 \alpha_1 + c_2 \alpha_2 + \cdots + c_k \alpha_k + d_1 \omega_1 + d_2 \omega_2 + \cdots d_m \omega_m$$

Taking Transformation on b.s.

$$T(\alpha) = c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots + c_k T(\alpha_k) + d_1 T(\omega_1) + d_2 T(\omega_2) + \cdots + d_m T(\omega_m)$$

$$T(\alpha) = c_1 \cdot 0 + c_2 \cdot 0 + \cdots c_k \cdot 0 + d_1 T(\omega_1) + d_2 T(\omega_2) + \cdots d_m T(\omega_m)$$

$$T(\alpha) = d_1 T(\omega_1) + d_2 T(\omega_2) + \cdots d_m T(\omega_m)$$

$$T(\alpha) \in L(S_2)$$

$$\beta \in L(S_2).$$

$\therefore \beta \in L(S_2)$.

$\therefore$ S_2 is a Basis of $R(T)$.

$\dim R(T) = m$.

$$\boxed{\therefore \rho(T) + \nu(T) = \dim U}$$

* Singular Transformation: $T: U \to V \to$ L.T

Singular ആയാൽ $\to$ $N(T) \to$ [atleast of Non-Zero vectors]

* Non-Singular Transformation: $T: U \to V \to$ L.T.

"Invertible"

Non-Singular ആയാൽ

$N(T) \xrightarrow{(\,)}$ [atleast one zero vecter.]

$$\boxed{\text{Singler} \downarrow \neq 0}$$

② If $T: R^3 \to R^3$ is invertible operator defined by $T(x,y,z) = (2x, 4x-y, 2x+3y-z)$. find T^{-1}.

SOL:

$$T(x,y,z) = (a,b,c).$$

$$T^{-1}(a,b,c) = (x,y,z) \longrightarrow ?$$

$$\therefore T(x,y,z) = (a,b,c)$$

$$(2x, 4x-y, 2x+3y-z) = (a,b,c)$$

$$\therefore \quad 2x = a \Rightarrow \boxed{x = \frac{a}{2}}$$

$$4x - y = b$$

$$\cancel{4}\left(\frac{a}{\cancel{2}}\right) - b = y$$

$$\boxed{y = 2a-b}$$

$$\therefore \quad 2x+3y-z = c$$

$$\cancel{2}\left(\frac{a}{\cancel{2}}\right) + 3(2a-b) - z = c.$$

$$a + 6a - 3b - z = c$$

$$\boxed{z = 7a-3b-z}$$

$$\therefore T^{-1}(a,b,c) = \left[\frac{a}{2}, 2a-b, 7a-3b-z\right] \;//$$

③ S.T. the Linear Operator T defined by $T(x,y,z) = (x+z, x-z, y)$ is Invertible & Hence find T^{-1}.

SOL:

$$T(x,y,z) = \bar{0}$$

$$(x+z, x-z, y) = (0,0,0)$$

$$x+z=0 \;;\; x-z=0 \;;\; \boxed{y=0}$$

$$\begin{array}{r} x+\cancel{z}=0 \\ x-\cancel{z}=0 \\ \hline 2x=0 \end{array} \Rightarrow \boxed{x=0}$$

$\therefore$ $x+z=0 \Rightarrow 0+z=0$

$$\boxed{z=0}$$

$\therefore$ $x=0, y=0, z=0$

$\therefore$ T is Invertible.

$\therefore$ $T(x,y,z) = (a,b,c)$

$T^{-1}(a,b,c) = (x,y,z) \longrightarrow ?$

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$\therefore T(x, y, z) = (a, b, c)$

$(x+z, x-z, y) = (a, b, c)$

$$\therefore \begin{aligned} x + \cancel{z} &= a \\ x - \cancel{z} &= b \end{aligned} \qquad \boxed{y = c}$$

$$2x = a+b$$

$$\boxed{x = \frac{a+b}{2}}$$

$$x + z = a \Rightarrow \frac{a+b}{2} + z = a$$

$$a+b+2z = 2a$$

$$2z = 2a - a - b$$

$$2z = a - b$$

$$\boxed{z = \frac{a-b}{2}}$$

$$\therefore T^{-1}(a, b, c) = (x, y, z)$$

$$= \left(\frac{a+b}{2}, c, \frac{a-b}{2}\right)$$

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UNIT-3

CLASS NO. 2

(4) The mapping $T: V_3(R) \to V_2(R)$ is defined by $T(x,y,z) = (x-y, x-z)$. Show that T is Linear Transformation.

SOL:

T.L.T
$T(a\alpha+b\beta)$
$= aT(\alpha)+bT(\beta)$

Let $\alpha = (x_1, y_1, z_1)$
$\beta = (x_2, y_2, z_2)$.

LHS

$$T(a\alpha+b\beta) = T\left[a(x_1,y_1,z_1)+b(x_2,y_2,z_2)\right]$$

$$= T\left[ax_1+bx_2, ay_1+by_2, az_1+bz_2\right]$$

$$= \left[(ax_1+bx_2)-(ay_1+by_2), (ax_1+bx_2)-(az_1+bz_2)\right]$$

$$= \left[a(x_1-y_1)+b(x_2-y_2), a(x_1-z_1)+b(x_2-z_2)\right]$$

$$= a\left[x_1-y_1, x_1-z_1\right]+b\left[x_2-y_2, x_2-z_2\right]$$

$$= a \cdot T(x_1,y_1,z_1)+bT(x_2,y_2,z_2)$$

$$= aT(\alpha)+b\cdot T(\beta)$$

$$= \text{RHS}$$

$\therefore T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)$. $\therefore T$ is L.T.

⑤ If $T: V_3(R) \longrightarrow V_2(R)$ is defined as $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$. Prove that T is L.T.

Sol: Let $\alpha = (p_1, q_1, r_1)$

$\beta = (p_2, q_2, r_2)$

$$T(a\alpha + b\beta) = T[a(p_1, q_1, r_1) + b(p_2, q_2, r_2)]$$

$$= T[ap_1 + bp_2, aq_1 + bq_2, ar_1 + br_2]$$

$$= [(ap_1 + bp_2) - (aq_1 + bq_2), (ap_1 + bp_2) + (ar_1 + br_2)]$$

$$= [a(p_1 - q_1) + b(p_2 - q_2), a(p_1 + r_1) + b(p_2 + r_2)]$$

$$= a[p_1 - q_1, p_1 + r_1] + b[p_2 - q_2, p_2 + r_2]$$

$$= a.T(p_1, q_1, r_1) + bT(p_2, q_2, r_2)$$

$$= aT(\alpha) + b.T(\beta)$$

$$\therefore T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

$\therefore T$ is L.T.

⑥ If $T: V_4(R) \to V_3(R)$ is a L.T. defined by $T(a,b,c,d) = (a-b+c+d,\ a+2c-d,\ a+b+3c-3d)\ \forall\ a,b,c,d \in R$. Then Verify $\rho(T) + \nu(T) = \dim V_4(R)$.

Sol:

Standard Basis Set $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$.

$$T(a,b,c,d) = (a-b+c+d,\ a+2c-d,\ a+b+3c-3d)$$

$$\therefore T(1,0,0,0) = (1-0+0+0,\ 1+2(0)-0,\ 1+0+3(0)-3(0))$$

$$T(1,0,0,0) = (1,1,1)$$

$$T(0,1,0,0) = (-1,0,1)$$

$$T(0,0,1,0) = (1,2,3)$$

$$T(0,0,0,1) = (1,-1,-3)$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \quad \frac{R_4}{-2}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis of $R(T) = \{(1,1,1), (0,1,2)\}$

$\therefore$ $\dim R(T) = \rho(T) = 2.$

Also, $\therefore$ $\alpha \in N(T) \Rightarrow T(\alpha) = \bar{0}.$

$T(a,b,c,d) = \bar{0}$

$[a-b+c+d,\ a+2c-d,\ a+b+3c-3d] = (0,0,0)$

$a-b+c+d = 0,\ a+2c-d = 0$

$a+b+3c-3d = 0.$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix} \quad \frac{R_3}{2}$$

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$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix} \quad \frac{R_3}{2}$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis of $N(T) = \{(1,-1,1,1), (0,1,1,-2)\}$

dim of $N(T) = \nu(T) = 2$.

$\downarrow$ Nullity

By Rank-Nullity theorem,

$\therefore$ LHS $= \rho(T) + \nu(T)$

$= 2 + 2$

$= 4$

$= \dim V_4(\mathbb{R})$

$\therefore$ LHS = RHS

$\therefore$ Rank-Nullity theorem is Verified.

UNIT - 3

SHORTS

① Let $U(F)$ and $V(F)$ be two Vector Spaces and $T: U \rightarrow V$ is a L.T. then P.T. Nullspace $N(T)$ is a Subspace of $U(F)$.

SOL:

Let $N(T) = \{T(\alpha) = \bar{0} / \alpha \in U\}$

PROCESS

→ Unit 1 Longs - ①

↓

$a\alpha + b\beta \in W$ ✓

✓ $N(T)$ is N.E

→ $a\alpha + b\beta \in N(T)$

$\therefore T(\bar{0}) = 0 \Rightarrow \bar{0} \in N(T)$

$\therefore N(T)$ is Non-empty Set.

Now, $\alpha, \beta \in N(T)$

$\Rightarrow T(\alpha) = \bar{0}$; $T(\beta) = \bar{0}$.

$\therefore T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

$= a(\bar{0}) + b(\bar{0})$

$T(a\alpha + b\beta) = \bar{0}$

$\boxed{T(\alpha) = \bar{0} \Rightarrow \alpha \in N(T)}$

$\therefore a\alpha + b\beta \in N(T)$.

$\therefore N(T)$ is Subspace of $U(F)$.

② Let $U(F)$ and $V(F)$ be two Vector Spaces. Let $T: U \to V$ be a L.T. Then Range Set $R(T)$ is a Subspace of $V(F)$.

PROOF :

Let $\bar{0} \in U \Rightarrow T(\bar{0}) = 0 \in R(T)$

$\therefore$ $R(T)$ is Non-empty.

$\alpha_1, \alpha_2 \in U$; $\beta_1, \beta_2 \in R(T)$ s.t.

$T(\alpha_1) = \beta_1$, $T(\alpha_2) = \beta_2$

$T(\alpha_1) = \beta_1$

$T(\alpha_2) = \beta_2$

$\to$ Non-empty

$\to a\alpha + b\beta \in R(T)$

$\alpha \in U$

$\downarrow$

$T(\alpha) \in R(T)$

$a, b \in F$, $\alpha_1, \alpha_2 \in U$

$\therefore$ $a\alpha_1 + b\alpha_2 \in U$. [$\because$ U is V.S.]

$\therefore$ $T(a\alpha_1 + b\alpha_2) \in R(T)$

$\therefore$ $T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2)$

$= a\beta_1 + b\beta_2$

$\in R(T)$.

$\therefore$ $a\beta_1 + b\beta_2 \in R(T)$

$\therefore$ $R(T)$ is Subspace.

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③ The mapping $T: V_3(R) \to V_1(R)$ is defined by $T(a,b,c) = a^2+b^2+c^2$. Can T be a L.T.?

PROOF :

Let $\alpha = (a,b,c)$

$\beta = (x,y,z)$

$$LHS = T(p\alpha + q\beta) = T\left[p(a,b,c) + q(x,y,z)\right]$$

$$= T\left[pa+qx,\ pb+qy,\ pc+qz\right]$$

$$= (pa+qx)^2 + (pb+qy)^2 + (pc+qz)^2$$

$$RHS = pT(\alpha) + qT(\beta)$$

$$= p.T(a,b,c) + qT(x,y,z)$$

$$= p.(a^2+b^2+c^2) + q(x^2+y^2+z^2)$$

$$\therefore T(p\alpha + q\beta) \neq pT(\alpha) + qT(\beta).$$

$\therefore$ T is NOt a L.T.

④ If the mapping $T: R^3 \to R^2$ defined by $T(x, y, z) = (|x|, 0)$ is a L.T.?

PROOF:

Let $\alpha = (x_1, y_1, z_1)$

$\beta = (x_2, y_2, z_2)$

$$LHS = T(a\alpha + b\beta)$$

$$= T(a(x_1, y_1, z_1) + b(x_2, y_2, z_2))$$

$$= T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

$$= (|ax_1 + bx_2|, 0)$$

$$RHS = aT(\alpha) + bT(\beta)$$

$$= aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$$

$$= a(|x_1|, 0) + b(|x_2|, 0)$$

$$\therefore T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$$

$\therefore$ T is not a L.T.

⑤ Describe Explicity of the L.T. $T: R^2 \to R^2$ such that $T(2,3) = (4,5)$, $T(1,0) = (0,0)$.

SOL:

First of all, we have to S.T. $(2,3)$ & $(1,0)$ is L.I.

→ L.I.
→ L.S.
→ $T(x,y) = ?$
$aT(2,3) + bT(1,0)$

$\therefore \; a(2,3) + b(1,0) = 0$

$(2a+b, 3a+0) = (0,0)$

$2a + b = 0$

$3a = 0 \Rightarrow \boxed{a=0}$

$2(0) + b = 0 \Rightarrow \boxed{b=0}$

$\therefore$ $(2,3)$, $(1,0)$ is Linearly Independent.

$a(2,3) + b(1,0) = (x,y)$

$(2a+b, 3a+0) = (x,y)$

$2a + b = x \quad ; \quad 3a = y$

$b = x - 2a \qquad \boxed{a = \frac{y}{3}}$

$b = x - 2\left(\frac{y}{3}\right)$

$$\boxed{b = \frac{3x - 2y}{3}}$$

$$\therefore T(x,y) = aT(2,3) + bT(1,0)$$

$$= \frac{y}{3}(4,5) + \frac{3x-2y}{3}(0,0)$$

$$T(x,y) = \left(\frac{4y}{3}, \frac{5y}{3}\right)$$

REVISION

LONGS - UNIT-1

1. $a\alpha + b\beta \in W$
2. $a\alpha + \beta \in W$
3. $L[L(S)]$

model paper

SHORTS

→ 3 Units 1, 4, 5

UNIT-2

1. $\dim(W_1 + W_2)$
2. $\dim\left(\frac{V}{W}\right)$

$\frac{75}{75}$

UNIT-3

1. Rank-Nullity
2. Invertible

UNIT-4

1. Cayley-Hamilton theorem
2. ch. vector, ch. values.

ALL THE BEST

UNIT-5

1. Cauchy-Schwarz
2. Bessels
3. GSO.

Yt

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