7B

UNIT-2

Laplace transforms - 2

CLASS - 1

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Topics :-

→ Multiplication with 't' → Derivative

$t^n . f(t)$ $\qquad \boxed{t^2 \sin 5t}$

→ Division by 't' → Integration

$\boxed{\frac{f(t)}{t^n}}$ Ex: $\frac{\sin 3t}{t^2}$

→ Initial Value theorem & final value theorem

→ Some Special functions

Dirac Delta, error, Bessel function.

LIVE → TM ✓
video → Tm [Rec]
Pdf → TM → [SM]
ST4U → DOWN

Formula's :-

$$\rightarrow L\left(t^n \times f(t)\right) = (-1)^n \frac{d^n}{dp^n} f(p)$$

$$\rightarrow L\left(\frac{f(t)}{t}\right) = \int_p^\infty f(p)\, dp$$

① Find the laplace transforms of

(i) $t\cos at$ (ii) $t^2 \sin at$

(iii) $t^3 e^{-3t}$

$L(t^n \cdot f(t))$

$\downarrow$

$(-1)^n \cdot \frac{d}{dp}[f(p)]$

$(-1)^n \frac{d}{dp}(Lap)$

$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

SOL:

(i) $L(t\cos at) = ?$

$$f(p) = L(\cos at) = \frac{p}{p^2+a^2}.$$

$$\therefore L(t^1 \cos at) = (-1)^n \frac{d^1}{dp^1}[f(p)]$$

$$= (-1)^1 \frac{d}{dp}\left(\frac{p}{p^2+a^2}\right)$$

$$= -\left[\frac{(p^2+a^2)(1) - p(2p+0)}{(p^2+a^2)^2}\right]$$

$\boxed{x^n = nx^{n-1}}$

$p^2 = 2p^{2-1}$

$= 2p$

$$= -\left[\frac{p^2+a^2-2p^2}{(p^2+a^2)^2}\right]$$

$$= -\left[\frac{a^2-p^2}{p^4+a^4+2p^2a^2}\right] //$$

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iii) $L\left(t^2 \sin at\right) = ?$

$$f(p) = L(\sin at) = \frac{a}{p^2+a^2}.$$

$$L\left(t^2 \sin at\right) = (-1)^2 \frac{d^2}{dp^2}\left(\frac{a}{p^2+a^2}\right)$$

$$= \frac{d}{dp}\left[\frac{d}{dp}\left(\frac{a}{p^2+a^2}\right)\right]$$

$$\boxed{\left(\frac{u}{v}\right)^1 = \frac{vu^1 - uv^1}{v^2}}$$

$$= \frac{d}{dp}\left[\frac{(p^2+a^2)(0) - a(2p+0)}{(p^2+a^2)^2}\right]$$

$$= \frac{d}{dp}\left[\frac{-2ap}{(p^2+a^2)^2}\right]$$

$$= -2a\frac{d}{dp}\left[\frac{p}{(p^2+a^2)^2}\right]$$

$$* = -2a\left[\frac{(p^2+a^2)^2(1) - p \cdot 2(p^2+a^2)^{2-1}(2p+0)}{(p^2+a^2)^4}\right]$$

$$= -2a\left[\frac{(p^2+a^2)^2 - 4p^2(p^2+a^2)}{(p^2+a^2)\ 4}\right] //..$$

iii) $L\left(t^3 e^{-3t}\right) = ?$

$$f(p) = L\left(e^{-3t}\right) = \frac{1}{p-(-3)} = \frac{1}{p+3}.$$

$$\therefore L\left(t^3 e^{-3t}\right) = (-1)^3 \frac{d^3}{dp^3}\left(\frac{1}{p+3}\right)$$

$$= -\frac{d^2}{dp^2}\left[\frac{d}{dp}\left(\frac{1}{p+3}\right)\right]$$

$$= -\frac{d^2}{dp^2}\left[\frac{-1}{(p+3)^2}\right] \qquad \boxed{\frac{d}{dp}\left(\frac{1}{p}\right) = \frac{-1}{p^2}}$$

$$= \frac{d}{dp}\left[\frac{d}{dp}\left(\frac{1}{(p+3)^2}\right)\right] \qquad \boxed{\frac{1}{x^2} \rightarrow \frac{-2}{x^3}}$$

$$x^{-2} = -2x^{-2-1}$$

$$= \frac{d}{dp}\left[\frac{-2}{(p+3)^3}\right] \qquad = -2x^{-3}$$

$$= \boxed{\frac{-2}{x^3}}$$

$$= -2\frac{d}{dp}\left[\frac{1}{(p+3)^3}\right] \qquad \frac{1}{x^3} \rightarrow \frac{-3}{x^4}$$

$$= -2.\left(\frac{-3}{(p+3)^4}\right) = \frac{6}{(p+3)^4}$$

② P.T. $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\frac{1}{p}$

Hence find $L\left[\frac{\sin at}{t}\right]$.

SOL:

Since, $L(\sin t) = \frac{1}{p^2+1^2} = \frac{1}{p^2+1}$.

and $L\left[\frac{\sin t}{t}\right] = \int_p^\infty f(p)\,dp$

$= \int_p^\infty \frac{1}{p^2+1}\,dp$

Study material

$\left[\int \frac{1}{1+x^2}dx = \tan^{-1}x\right]$

$= \left[\tan^{-1}p\right]_p^\infty$

$= \tan^{-1}\infty - \tan^{-1}p$

$= \cancel{\tan^{-1}}\left(\cancel{\tan}\frac{\pi}{2}\right) - \tan^{-1}p$

$= \frac{\pi}{2} - \tan^{-1}p$

$= \cot^{-1}p.$

$\left[90 - \tan^{-1}x = \cot^{-1}x\right]$

$= \tan^{-1}\frac{1}{p}$ //..

$\left[\cot^{-1}x = \tan^{-1}\frac{1}{x}\right.$

$\left.\cot^{-1}\frac{1}{x} = \tan^{-1}x\right]$

iii) $L\left[\frac{\sin at}{t}\right] = ?$

$$L(\sin at) = \frac{a}{p^2+a^2}.$$

$$L\left[\frac{\sin at}{t}\right] = \int_p^\infty \frac{a}{p^2+a^2}\,dp$$

$$= a\int_p^\infty \frac{1}{p^2+a^2}\,dp$$

$$= \left[\cancel{a}\cdot\frac{1}{\cancel{a}}\tan^{-1}\left(\frac{p}{a}\right)\right]_p^\infty \qquad \left[\because \int\frac{1}{x^2+a^2}dx \quad \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \left[\tan^{-1}\left(\frac{p}{a}\right)\right]_p^\infty$$

$$= \tan^{-1}\left(\frac{\infty}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{p}{a}\right)$$

$$= \cot^{-1}\left(\frac{p}{a}\right) = \tan^{-1}\left(\frac{a}{p}\right)$$

UNIT-2

7B

CLASS-2

① Evaluate i) $\int_0^\infty t e^{-2t} \sin t \, dt$

ii) $\int_0^\infty t e^{-2t} \cos t$.

Sol:

$$\int_0^\infty t e^{-2t} \sin t \, dt = \int_0^\infty e^{-pt} (t \sin t) \, dt$$

where, $\boxed{p=2}$

T ↓ A ↓ B ↓ L ↓ E

$\therefore \quad L(t \sin t) = ?$

$$\therefore \quad L[f(t)] = L(\sin t) = \frac{1}{p^2+1}.$$

$$L(t^1 \sin t) = (-1)^1 \frac{d}{dp}\left(\frac{1}{p^2+1}\right)$$

$\dfrac{vu' - uv'}{v^2}$

$$= -\frac{d}{dp}\left(\frac{1}{p^2+1}\right)$$

$$= -\left[\frac{(p^2+1)(0) - 1(2p+0)}{(p^2+1)^2}\right]$$

$$= +\left[\frac{2p}{(p^2+1)^2}\right].$$

Putting $p=2 \Rightarrow$

$$\therefore \int_0^\infty t e^{-2t} \sin t \, dt = \frac{2 \times 2}{(4+1)^2} = \frac{4}{25}$$

ii) $\int_0^\infty t e^{-2t} \cos t \, dt = ?$

$$L(\cos t) = \frac{p}{p^2+1}$$

$$L(t \cos t) = (-1)^1 \frac{d}{dp}\left(\frac{p}{p^2+1}\right)$$

$$= -\left[\frac{(p^2+1)1 - p(2p+0)}{(p^2+1)^2}\right]$$

$$= -\left[\frac{p^2+1-2p^2}{(p^2+1)^2}\right]$$

$$= -\left[\frac{1-p^2}{(p^2+1)^2}\right]$$

$$= \frac{p^2-1}{(p^2+1)^2}$$

Taking $p=2$,

$\therefore$ we have, $\int_0^\infty t e^{-2t} \cos t \, dt$

$$= \frac{4-1}{(4+1)^2} = \frac{3}{25}$$

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② Evaluate (i) $L\left[e^{-4t}\,\frac{\sin 3t}{t}\right]$.

(ii) $\int_{p}^{\infty} t \sin^{-3t} \sin t$

Sol:

(i) $L(\sin 3t) = \frac{3}{p^2+9}$.

$$L\left(\frac{\sin 3t}{t}\right) = \int_{p}^{\infty} \frac{3}{p^2+9}\,dp$$

$$= 3\int_{p}^{\infty} \frac{1}{p^2+3^2}\,dp$$

$$= \left[\cancel{3}\times\frac{1}{\cancel{3}}\tan^{-1}\left(\frac{p}{3}\right)\right]_{p}^{\infty} \quad \left[\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \left[\tan^{-1}\left(\frac{p}{3}\right)\right]_{p}^{\infty}$$

$$= \tan^{-1}\left(\frac{\infty}{3}\right) - \tan^{-1}\left(\frac{p}{3}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{p}{3}\right)$$

$$= \cot^{-1}\left(\frac{p}{3}\right)$$

(1st shifting theorem)

$$\therefore L\left[e^{-4t}\,\frac{\sin 3t}{t}\right] = \cot^{-1}\left[\frac{p-(-4)}{3}\right] = \cot^{-1}\left(\frac{p+4}{3}\right)$$

$$= \tan^{-1}\left(\frac{3}{p+4}\right) //.$$

(ii) $\displaystyle\int_0^\infty t e^{-3t} \sin t \, dt = ?$

$$\therefore \; L(\sin t) = \frac{1}{p^2+1}.$$

$$L(t^1 \sin t) = (-1)^1 \frac{d}{dp}\left(\frac{1}{p^2+1}\right)$$

$$= -\left[\frac{(p^2+1)\,0 - 1(2p+0)}{(p^2+1)^2}\right]$$

$$= -\left[\frac{-2p}{(p^2+1)^2}\right]$$

$$= \frac{2p}{(p^2+1)^2}.$$

Putting $p = 3$;

$$\int_0^\infty t e^{-3t} \sin t \, dt = \frac{2\times 3}{(9+1)^2}$$

$$= \frac{\cancel{6}\,3}{\cancel{100}\,50}$$

$$= \frac{3}{50}$$ //..

③ Evaluate $L\left[e^{2t}\left(3t^5-2t^4+4e^{-5t}-3\sin 6t+4\cos 4t\right)\right]$.

SOL: Given that,

$$= L\left[e^{2t}\left(3t^5-2t^4+4e^{-5t}-3\sin 6t+4\cos 4t\right)\right]$$

$$= L\left[e^{2t}3t^5 - 2e^{2t}t^4 + 4e^{-3t} - 3e^{2t}\sin 6t + 4e^{2t}\cos 4t\right]$$

$$= 3L\left(t^5 e^{2t}\right) - 2L\left(t^4 e^{2t}\right) + 4L\left(e^{-3t}\right) - 3L\left(e^{2t}\sin 6t\right) + 4L\left(e^{2t}\cos 4t\right)$$

DP

Now Consider,

$$L\left(t^5 e^{2t}\right) = (-1)^5 \frac{d^5}{dp^5} L\left(e^{2t}\right) \qquad \boxed{e^{at} = \frac{1}{p-a}}$$

$$= -\frac{d^5}{dp^5}\left(\frac{1}{p-2}\right)$$

$$= -\frac{d^5}{dp^5}\left[(p-2)^{-1}\right]$$

$$= -\frac{d^4}{dp^4}\left[(-1)(p-2)^{-1-1}\right]$$

$$= + \frac{d^4}{dp^4}\left[(p-2)^{-2}\right]$$

$$= \frac{d^3}{dp^3}\left[\frac{d}{dp}(p-2)^{-2}\right]$$

$$= \frac{d^3}{dp^3}\left[-2(p-2)^{-2-1}\right]$$

$\star$ $\boxed{x^n = nx^{n-1}}$

$$= -2\frac{d^3}{dp^3}\left[(p-2)^{-3}\right]$$

$$= -2\frac{d^2}{dp^2}\left[(-3)(p-2)^{-3-1}\right]$$

$$= 6\frac{d^2}{dp^2}\left[(p-2)^{-4}\right]$$

$$= 6\frac{d}{dp}\left(-4(p-2)^{-4-1}\right)$$

$$= -24\frac{d}{dp}\left[(p-2)^{-5}\right]$$

$$= -24\left[-5(p-2)^{-5-1}\right]$$

$$= 120(p-2)^{-6}$$

$$= \frac{120}{(p-2)^6}.$$

$$L\left(t^4 e^{2t}\right) = (-1)^4 \frac{d^4}{dp^4}\left(\frac{1}{p-2}\right)$$

$$= \frac{d^4}{dp^4}\left(\frac{1}{p-2}\right)$$

$$= \frac{d^4}{dp^4}\left[(p-2)^{-1}\right]$$

$$= \frac{d^3}{dp^3}\left[-1\,(p-2)^{-1-1}\right]$$

$$= -\frac{d^3}{dp^3}\left[(p-2)^{-2}\right]$$

$$= -\frac{d^2}{dp^2}\left[-2(p-2)^{-2-1}\right]$$

$$= +2\frac{d^2}{dp^2}\left[(p-2)^{-3}\right]$$

$$= 2\frac{d}{dp}\left(-3(p-2)^{-3-1}\right)$$

$$= -6\frac{d}{dp}(p-2)^{-4}$$

$$= -6\left[-4(p-2)^{-4-1}\right]$$

$$= 24\,(p-2)^{-5} = \frac{24}{(p-2)^5}$$

$$L\left(e^{-3t}\right) = \frac{1}{p-(-3)} = \frac{1}{p+3}$$

$$L\left(e^{2t}\sin 6t\right) = ?$$

$$L(\sin 6t) = \frac{6}{p^2+36}$$

$$\therefore \quad L\left(e^{2t}\sin 6t\right) = \frac{6}{(p-2)^2+36}$$

$$= \frac{6}{p^2+4-4p+36}$$

$$= \frac{6}{p^2-4p+40}.$$

Now,

$$L\left(e^{2t}\cos 4t\right) = ?$$

$$L(\cos 4t) = \frac{p}{p^2+16}$$

$$\therefore \quad L\left(e^{2t}\cos 4t\right) = \frac{p-2}{(p-2)^2+16}$$

$$= \frac{p-2}{p^2-4p+20}.$$

$$\therefore 3\,\frac{120}{(p-2)^6} - 2\,\frac{24}{(p-2)^5} + 4\cdot\frac{1}{p+3} - 3\,\frac{1}{p^2-4p+40} + 4\,\frac{1}{p^2-4p+20}$$

$$\therefore \frac{360}{(p-2)^6} - \frac{48}{(p-2)^5} + \frac{4}{p+3} - \frac{3}{p^2-4p+40} + \frac{4}{p^2-4p+20} \;//..$$

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$\boxed{\text{Unit 1}}$ → Link ✓

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MATHS-7B

UNIT-2

CLASS NO.3

① State and Prove Initial Value theorem?

statement: Let F(t) be Continuous function for all $t \geq 0$, $t \to \infty$ then

$$\underset{t \to 0}{\text{lt}} F(t) = \underset{s \to \infty}{\text{lt}} s \cdot f(s)$$

PROOF: We Know that,

Laplace transforms of Derivatives,

$$L\left[F'(t)\right] = s \cdot f(s) - F(0)$$

$$\int_0^\infty e^{-st} F'(t)\, dt = s f(s) - F(0)$$

Taking limits on both sides,

$$\underset{s \to \infty}{\text{lt}} \int_0^\infty e^{-st} F'(t)\, dt = \underset{s \to \infty}{\text{lt}} \left(s \cdot f(s) - F(0)\right)$$

$$\int_0^\infty e^{-\infty} F'(t)\, dt = \underset{s \to \infty}{\text{lt}} s f(s) - F(0)$$

$$\left[\underset{s \to \infty}{\text{lt}}\, a = a\right]$$

Procedure

→ LT Derivative $L[F'(t)] = s f(s) - F(0)$

* → L.T. $\int_0^\infty e^{-st} f(t)\, dt$

→ $e^{-\infty} = 0$

→ $F(0) = \underset{t \to 0}{\text{lt}} F(t)$

$$0 = \lim_{s\to\infty} s f(s) - F(0)$$

$$\left[\because F(0) = \lim_{t\to 0} F(t)\right.$$

$$0 = \lim_{s\to\infty} s f(s) - \lim_{t\to 0} F(t)$$

$$\therefore \quad \lim_{s\to\infty} s f(s) = \lim_{t\to 0} F(t)$$

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② State and prove final value theorem?

Statement: Let $F(t)$ be a continuous function, for all $t \geq 0$, $t \to \infty$ then

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} s.f(s)$$

PROOF: We know that,

$$L\left[F'(t)\right] = s f(s) - F(0)$$

$$\int_0^\infty e^{-st} F'(t)\, dt = s f(s) - F(0)$$

Taking limits on b.s,

$$\lim_{s\to 0} \int_0^\infty e^{-st} F'(t)\, dt = \lim_{s\to 0} \left(s f(s) - F(0)\right)$$

$$\int_0^\infty e^0 F'(t)\, dt = \lim_{s\to 0} s f(s) - F(0)$$

$$\int_0^{\infty} F'(t)\, dt = \underset{s \to 0}{lt}\ s f(s) - F(0)$$

$$\left[F(t)\right]_0^{\infty} = \underset{s \to 0}{lt}\ s f(s) - F(0)$$

Inte, diff
↓
Resiprocal

$$F(\infty) - \cancel{F(0)} = \underset{s \to 0}{lt}\ s f(s) - \cancel{F(0)}$$

$\cancel{x} \times \frac{1}{\cancel{x}}$

$$\underset{t \to \infty}{lt}\ F(t) = \underset{s \to 0}{lt}\ s f(s)$$

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③ State and prove Laplace transforms of integrals?

Statement: If $L\left[f(t)\right] = \bar{f}(s)$ then

$$L \int_0^t f(t)\, dt = \frac{1}{s} \cdot \bar{f}(s).$$

PROOF: Let $G(t) = \int_0^t f(t)\, dt \longrightarrow ①$

Diff. w.r.t. t on b/s

$$G'(t) = \int_0^t \frac{d}{dt} f(t)\, dt$$

$$G'(t) = \left[f(t)\right]_0^t$$

$$G'(t) = f(t) \longrightarrow ②$$

By the definition of Derivative of Laplace transforms,

$$L\left[G'(t)\right] = s\,L\left[G(t)\right] - G(0)$$

$$L\left[f(t)\right] = s\,L\int_0^t f(t)\,dt - 0$$

$\left[\because \text{from } ① \,\&\, ②\right]$

$$\bar{f}(s) = s\,L\int_0^t f(t)\,dt$$

$$\therefore\ L\int_0^t f(t) = \frac{1}{s}\cdot\bar{f}(s)$$

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