FOURIER TRANSFORM-I

2-01-2020

1.

Theorem :- (Inversion theorem (or) formula for fourier transform)

Statement :-

If $\bar{f}(p)$ is the fourier transform of $f(x)$ if $f(x)$ satisfies the dirichlets condition in every finite interval $(-l, l)$ and further if $\int_{-\alpha}^{\alpha} |f(x)| dx$ is convergent then at every point of continuity of $f(x)$, $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} \bar{f}(p) e^{-ipx} dp$

Proof :-

suppose $\bar{f}(p)$ is the fourier transform of $f(x)$ if $f(x)$ satisfies the dirichlets condition in every finite interval $(-l, l)$ and further if $\int_{-\alpha}^{\alpha} |f(x)| dx$ is convergent.

From fourier integral formula

$$f(x) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} f(v) \left[\int_{-\alpha}^{\alpha} e^{i\omega(x-v)} d\omega \right] dv$$

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\alpha}^{\alpha} f(v) \left[\int_{-\alpha}^{\alpha} e^{i\omega x - i\omega v} d\omega \right] dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(v) \left[\frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{i\omega x} \cdot e^{-i\omega v} d\omega \right] dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(v) e^{-i\omega v} dv \cdot \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{i\omega x} d\omega$$

Put $\omega = -p$.

$d\omega = -dp$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(v) e^{-i\omega v} dv \cdot \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{i\omega x} d\omega$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{-i(-p)v}\, dv \cdot \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{i(-p)x} (-dp)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{ipv}\, dv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx}\, dp.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx}\, dp$$

$$f(x) = \bar{f}(p) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx}\, dp.$$

$$\therefore f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx}\, dp.$$

Non-Symmetrical form of fourier transform

Some authors also defind the fourier transform in the following form $F[f(x)] = \bar{f}(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$

then $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \bar{f}(p)\, dp$

Fourier Sine transform :-

The infinite fourier sine transform of $f(x)$ is denoted by $F_s[f(x)]$ (or) $\bar{f}_s(p)$ (or) $f_s(p)$ and is defind as $F_s[f(x)] = \bar{f}_s(p) = \bar{f}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin px\, dx$

Note :-

The function $f(x)$ is called the inverse fourier sine transform of $\bar{f}(p)$

i.e. $f(x) = F_s^{-1}[\bar{f}_s(p)]$

Theorem :- Inversion formula for fourier sine transform

Statement :-

If $F_s(p)$ is the fourier sine transform of $f(x)$ which satisfies the dichiletes condition in every finite interval $(0, \infty)$ and is such that $\int_0^{\infty} |f(x)|\, dx$ is exist.

Then $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(p) \sin px \, dp$ at every point of continuit

Proof :-

suppose $F_s(p)$ is the fourier sine transform of $f(x)$ which satisfies the dichilets condition in every finite interval $(0, \infty)$ and is such that $\int_0^{\infty} |f(x)| dx$ is exist.

From fourier integral formula.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} f(v) \cos w(x-v)\, dv$$

$$f(x) = \frac{1}{2\pi}\, 2 \int_0^{\infty} dw \int_{-\infty}^{\infty} f(v) (\cos wx \cos wv + \sin wx \sin wv) dv.$$

Put $p = w$
$dp = dw$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} dp \int_{-\infty}^{\infty} f(v) (\cos px \cos pv + \sin px \sin pv)\, dv.$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} dp \int_{-\infty}^{\infty} f(v) \cos px \cos pv\, dv + \frac{1}{\pi} \int_0^{\infty} dp \int_{-\infty}^{\infty} f(v) \sin px \sin pv\, dv.$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cos px\, dp \int_{-\infty}^{\infty} \cos pv\, f(v)\, dv + \frac{1}{\pi} \int_0^{\infty} \sin px\, dp \int_{-\infty}^{\infty} \sin pv\, f(v)\, dv$$

Put $v = x \Rightarrow dv = dx$

$$f(x) = \frac{1}{\pi}\int_0^{\alpha} \cos px\, dp \int_{-\alpha}^{\alpha} \cos px\, f(x)\, dx + \frac{1}{\pi}\int_0^{\alpha} \sin px\, dp \int_{-\alpha}^{\alpha} \sin px\, f(x)\, dx \rightarrow (1)$$

Now we defind $f(x)$ in $(-\alpha, 0)$ such that $f(x)$ is an odd function of x in $(-\alpha, \alpha)$ then obbisously $f(x)$ is an odd function of x. and $f(x)\cos px$ is an odd function of x in $(-\alpha, \alpha)$

$f(x)\sin px$ is an even function

$$\left[\text{formula:- } \int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if odd} \\ 2\int_0^a f(x) & \text{if even} \end{cases}\right]$$

$$\int_{-\alpha}^{\alpha} f(x)\cos px\, dx = 0 \longrightarrow (2)$$

$$\int_{-\alpha}^{\alpha} f(x)\sin px\, dx = 2\int_0^{\alpha} f(x)\sin px\, dx \longrightarrow (3)$$

$eq^n s$. (2) & (3) are subc in eq^n (1)

$$f(x) = \frac{1}{\pi}\int_0^{\alpha} \cos px\, dp\,(0) + \frac{1}{\pi}\int_0^{\alpha} \sin px\, dp \cdot 2\int_0^{\alpha} \sin px\, f(x)\, dx$$

$$f(x) = \left(\sqrt{\frac{2}{\pi}}\right)^2 \int_0^{\alpha} \sin px\, dp \cdot \int_0^{\alpha} f(x)\sin px\, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} \sin px\, dp \cdot \sqrt{\frac{2}{\pi}} \int_0^{\alpha} f(x)\sin px\, dx.$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} \sin px\, dp\ F_s(p)$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} F_s(p)\sin px\, dp.$$

Fourier cosine transform :-

The infinite fourier cosine transform of $f(x)$, $0<x<\alpha$ and is denoted by $F_c[f(x)]$ (or) $\bar{f}_c(p)$ (or) $\bar{f}_c(p)$

i.e $F_c^{*}[f(x)]$ or $\bar{f}_c(p) = \bar{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} \bar{f}_c(p) \cos px \, dp$

Note :-

The function $f(x)$ is called the inversion fourier cosine transform of $\bar{f}_c(p)$ is

i.e $f(x) = F_c^{-1}[\bar{f}_c(p)]$.

Theorem :-

3\. inversion cosine transform.

Statement :-

If $\bar{f}_c(p)$ is the fourier cosine transform of $f(x)$ which satisfies the dirichlets condition in every finite interval $(0,p)$ and is such that $\int_0^{\alpha} |f(x)| \, dx$ exists. Then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} \bar{f}_c(p) \cos px \, dp$$ at every point of continuity

Proof :-

Suppose $\bar{f}_c(p)$ is the fourier cosine transform of $f(x)$ which satisfies the dirichlets condition in every finite interval $(0,p)$ and is such that $\int_0^{\alpha} |f(x)| dx$ exists.

From fourier integral formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(v) \left[\int_{-\alpha}^{\alpha} \cos w(x-v) \, dw \right] dv.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} 2 \int_0^{\alpha} dw \int_{-\alpha}^{\alpha} f(v) \cos w(x-v) \, dv$$

Scanned with CamScanner

$$f(x) = \frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{\pi}} \frac{1}{\pi}\int_0^{\alpha} d\omega \int_{-\alpha}^{\alpha} f(v)\,(\cos\omega x \cos\omega v + \sin\omega x \sin\omega v)\, dv$$

$$f(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\pi}\int_0^{\alpha} d\omega \int_{-\alpha}^{\alpha} f(v) \cos\omega x \cos\omega v\, dv + \frac{1}{\pi}\int_0^{\alpha} d\omega \int_{-\alpha}^{\alpha} f(v) \sin\omega x \sin\omega v\, dv$$

Now $\omega = p$

$d\omega = dp$

$$f(x) = \frac{1}{\pi}\int_0^{\alpha} dp \int_{-\alpha}^{\alpha} f(v) \cos px \cos pv\, dv + \frac{1}{\pi}\int_0^{\alpha} dp \int_{-\alpha}^{\alpha} f(v) \sin px \sin pv\, dv$$

$$f(x) = \frac{1}{\pi}\int_0^{\alpha} \cos px\, dp \int_{-\alpha}^{\alpha} f(v) \cos pv\, dv + \frac{1}{\pi}\int_0^{\alpha} \sin px\, dp \int_{-\alpha}^{\alpha} f(v) \sin pv\, dv.$$

Put $v = x \Rightarrow dv = dx$

$$f(x) = \frac{1}{\pi}\int_0^{\alpha} \cos px\, dp \int_{-\alpha}^{\alpha} f(x) \cos px\, dx + \frac{1}{\pi}\int_0^{\alpha} \sin px\, dp \int_{-\alpha}^{\alpha} f(x) \sin px\, dx \quad \rightarrow (A)$$

Now we defind if $(-\alpha, 0)$. such that $f(x)$ is an odd function of x in $(-\alpha, \alpha)$. then obviosly. $f(x)\cos xpx$ is an even function of x and $f(x) \sin px$ is an odd function of x in $(-\alpha, \alpha)$

$$\int_{-\alpha}^{a} f(x) \cos px \cdot dx = 2\int_0^{\alpha} f(x) \cos px\, dx \longrightarrow (B)$$

$$\int_{-\alpha}^{\alpha} f(x) \sin px\, dx = 0 \longrightarrow (C)$$. (B) & (C) are put in eq^n (A).

$$f(x) = \frac{1}{\pi}\int_0^{\alpha} \cos px\, dp \cdot 2\int_0^{\alpha} f(x) \cos px\, dx + \frac{1}{\pi}\int_0^{\alpha} \sin px\, dp\,(0)$$

$$f(x) = \left(\sqrt{\frac{2}{\pi}}\right)^2 \int_0^{\alpha} \cos px\, dp \int_0^{\alpha} f(x) \cos px\, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos px \, dp . \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos px \, dx .$$

w.k.T

$$\bar{F}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos px \, dx$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{F}_c(p) \cos px \, dp .$$

Note :-

fourier sin and cosine integral :-

* $f(x) = \frac{2}{\pi} \int_0^{\infty} \sin px \, dp \int_0^{\infty} f(x) \sin px \, dx$ is called fourier sine integral.

* $f(x) = \frac{2}{\pi} \int_0^{\infty} \cos px \, dp \int_0^{\infty} f(x) \cos px \, dx$ is called the fourier cosine integral.

Therorem : (linear property of fourier transform)

Statement :-

If $\bar{f}(p), \bar{g}(p)$ are fourier transforms of $f(x)$ and $g(x)$ respectively. Then $F[a f(x) + b g(x)] = a \bar{f}(p) + b \bar{g}(p)$ where a and b are constants

Proof :-

Let $\bar{f}(p), \bar{g}(p)$ are fourier transforms of $f(x)$ and $g(x)$ respectively.

we have $\bar{f}(p) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x) \, dx \rightarrow (1)$

$\bar{g}(p) = F[g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} g(x) \, dx \rightarrow (2)$

Let a, b are constants.

consider $F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}[af(x)+bg(x)]dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[e^{ipx}af(x)dx + e^{ipx}bg(x)dx]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}af(x)dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}bg(x)dx$$

$$= a.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}f(x)dx + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}g(x)dx$$

$$F[af(x)+bg(x)] = a\,\bar{f}(p) + b\,\bar{g}(p) \text{ (from ① \& ②)}.$$

Theorem :- (change of scale property for fourier transform)

Statement :-

If $\bar{f}(p)$ is the complex fourier transform of $\bar{f}(x)$ then the complex the fourier transform of $f(x) = \frac{1}{a}\bar{f}(\frac{p}{a})$ (or) If $F[f(x)] = \bar{f}(p)$ then prove that $F[f(ax)] = \frac{1}{a}\bar{f}(p)$.

Proof :-

suppose $\bar{f}(p)$ is the complex fourier transform of $f(x)$ then we have $F[f(x)] = \bar{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}f(x)dx \longrightarrow ①$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}f(ax)dx$$

Let $ax = t \Rightarrow x = t/a$

$adx = dt \Rightarrow dx = \frac{1}{a}dt$

If $x \to -\alpha$ and $t \to -\alpha$

and $x \to \alpha$ then $t \to \alpha$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{ia(t/a)} f(t) \cdot \frac{1}{a} dt.$$

$$F[f(ax)] = \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{i(p/a)t} f(t).dt$$

$$F[f(ax)] = \frac{1}{a} \cdot \tilde{f}(p/a) \qquad (\because \text{from } ①)$$

Theorem :- (Change of scale property for fourier sine transform)

Statement :-

If $\tilde{f}_s(p)$ is the complex fourier sine transform of $f(x)$ then the fourier sine transform of $f(ax)$ is $\frac{1}{a}\tilde{f}_s(p/a)$ (or) $F_s[f(x)] = \tilde{f}_s(p)$. then prove that $F_s[f(ax)] = 1/a \; \tilde{f}_s(p/a)$

Proof :-

Suppose $\tilde{f}_s(p)$ is the complex fourier sine transform. is of $f(x)$.

Then we have $F_s[f(x)] = \tilde{f}_s(p) = \sqrt{2/\pi} \int_0^{\alpha} f(x) \sin px \, dx \to ①$

$$F_s[f(ax)] = \sqrt{2/\pi} \int_0^{\alpha} f(ax) \sin px \, dx$$

put $ax = t \Rightarrow x = t/a$.

$dx = \frac{dt}{a}$

If $x \to \alpha$ then $t \to \alpha$

$x \to 0$ then $t \to 0$

Scanned with CamScanner

$$F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin(p\tfrac{t}{a}) \frac{dt}{a}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin(\tfrac{p}{a})t \cdot \tfrac{1}{a}\, dt$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin(\tfrac{p}{a})t\, dt$$

$$F_s[f(ax)] = \frac{1}{a} \tilde{f}_s(\tfrac{p}{a})$$

Theorem:- (Change of scale property for fourier cosine transform)

Statement:-

If $\tilde{f}_c(p)$ is the complex fourier cosine transform of $f(x)$ then the fourier cosine transform of $f(ax)$ is $\frac{1}{a} \tilde{f}_c(\frac{p}{a})$ if $F_c[f(x)] = \tilde{f}_c(p)$ then prove that $F_c[f(ax)] = \frac{1}{a} \tilde{f}_c(\frac{p}{a})$

Proof:-

Suppose $\tilde{f}_c(p)$ is the complex fourier cosine transform of $f(x)$.

then we have

$$F_c[f(x)] = \tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos px\, dx \longrightarrow (1)$$

$$F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax) \cdot \cos px\, dx.$$

Put $ax = t \Rightarrow x = \frac{t}{a}$.

$dx = \frac{dt}{a}$.

If $x \to \infty$ then $t \to \infty$

$x \to 0$ then $t \to 0$

Scanned with CamScanner

$$F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos\left(p \cdot \frac{t}{a}\right) \frac{dt}{a}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos\left(\frac{p}{a}\right) t \cdot \frac{1}{a} dt.$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos\left(\frac{p}{a}\right) t \cdot dt.$$

$$F_c[f(ax)] = \frac{1}{a} \tilde{F}_c\left(\frac{p}{a}\right) \quad (\because \text{from } ①).$$

Theorem :-

If $\phi(p)$ is the fourier sine transform of $f(x)$ for $p>0$ then prove that $F_s[f(x)] = -\phi(-p)$ for $p<0$

Proof :-

Suppose $\phi(p)$ is the fourier sine transform of $f(x)$ for $p>0$. then we have.

$$F_s[f(x)] = \tilde{F}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx$$

$$= \phi(p), \; p>0 \longrightarrow ①$$

For $p<0$

let $p = -s$ where $s>0$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(-s) x \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= -\phi(s), \; s>0$$

$$F_s[f(x)] = -\phi(-p), \; p<0$$

Theorem :- V

If $\tilde{F}_S(P)$ and $\tilde{F}_C(P)$ are fourier sine and cosine transform of $f(x)$ respectively then

[i] $F_S\{(f(x))\cos ax\} = \frac{1}{2}\left[\tilde{f}_S(P+a) + \tilde{f}_S(P-a)\right]$

[ii] $F_C\left\{f(x)\sin ax\right\} = \frac{1}{2}\left[\tilde{f}_S(P+a) - \tilde{f}_S(P-a)\right]$

[iii] $F_S\{f(x)\sin ax\} = \frac{1}{2}\left[\tilde{f}_C(P-a) - \tilde{f}_C(P+a)\right]$

Proof :-

Suppose $\tilde{F}_S(P)$ and $\tilde{f}_C(P)$ are fourier sine and cosine transforms of $f(x)$ respectively.

Then to prove that.

[i] $F_S\{f(x)\cos ax\} = \frac{1}{2}\left[\tilde{f}_S(P+a) + \tilde{f}_S(P-a)\right] \rightarrow ①$

we have.

$$F_S[f(x)] = \tilde{f}_S(P) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin px\,dx \rightarrow ①$$

$$F_S[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{f(x)\,2\sin px\cos ax\,dx}{2}$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\left[\sin(P+a)x + \sin(P-a)x\right]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(P+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(P-a)x\,dx\right]$$

formula
$\left[\therefore \sin(A+B) + \sin(A-B) = \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B\right]$

$$= \frac{1}{2}\left[\tilde{f}_S(P+a) + \tilde{f}_S(P-a)\right]$$

$$\therefore F_S\{f(x)\cos ax\} = \frac{1}{2}\left[\tilde{f}_S(P+a) + \tilde{f}_S(P-a)\right].$$

To prove that

(ii) $F_c\{f(x)\sin ax\} = \frac{1}{2}\left[\tilde{f}_s(p+a) - \tilde{f}_s(p-a)\right]$ —

we have

$$F_c\{f(x)\} = \tilde{f}_c(p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos px\, dx \rightarrow (1)$$

$$F_c\{f(x)\sin ax\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{f(x)\cos px \cdot \sin ax}{2}\, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left[\sin(p+a)x - \sin(p-a)x\right] dx$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(p+a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(p-a)x\,dx\right\}$$

$$= \frac{1}{2}\left[\tilde{f}_s(p+a) - \tilde{f}_s(p-a)\right]$$

$$F_c\{f(x)\sin ax\} = \frac{1}{2}\left[\tilde{f}_s(p+a) - \tilde{f}_s(p-a)\right]$$

To prove that

(iii) $F_s\{f(x)\sin ax\} = \frac{1}{2}\left[\tilde{f}_c(p-a) - \tilde{f}_c(p+a)\right]$

we have

$$F_s[f(x)] = \tilde{f}_s(p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin px\,dx \rightarrow (1)$$

$$F_s\{f(x)\sin ax\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{f(x)\, 2\sin px \cdot \sin ax}{2}\, dx.$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left[\cos(px - ax) - \cos(px + ax)\right] dx$$

formula

$\left[\therefore \cos(A-B) - \cos(A+B) = 2\sin A \sin B\right]$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(p-a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(p+a)x\,dx\right]$$

$$= \frac{1}{2}\left[\tilde{f}_c(p-a) - \tilde{f}_c(p+a)\right]$$

Formulas :-

$\rightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$; $\int_{0}^{-\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{2}}$

$\rightarrow \int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{a}$; $\int_{0}^{-\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a}$

$\rightarrow \int_{0}^{\infty} e^{-ax} x^{n-1} dx = \frac{\sqrt{n}}{a^n}$ where a, n are positive

$\rightarrow \int_{0}^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$

$\rightarrow \int_{0}^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$

$\rightarrow \int_{0}^{\infty} \frac{\sin ax}{x} dx = \pi/2$ where $a > 0$

$\rightarrow \int_{0}^{\infty} \frac{1}{1+x^2} dx = \tan^{-1}x$

$\rightarrow \int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right|$

$\rightarrow \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}(x/a)$

$\rightarrow \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right|$

Leibniz rule :

$\rightarrow \frac{d}{dx}\left[\int_{a}^{b} f(x,t) dt\right] = \int_{a}^{b} \frac{d}{dx} f(x,t) dt$

Scanned with CamScanner

$\rightarrow \Gamma(n+1) = n!$ where $n>0$ is an integer.

$\rightarrow \sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\rightarrow \sin(A+B) - \sin(A-B) = 2\cos A \sin B$

$\rightarrow \cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\rightarrow \cos(A-B) - \cos(A+B) = 2\sin A \sin B$

$\rightarrow \sin 2A = 2\sin A \sin B$

$\rightarrow \cos 2A = 2\cos^2 A - 1 \Rightarrow \cos^2 A = \frac{1+\cos 2A}{2}$

$\rightarrow \cos 2A = 1 - 2\sin^2 A \Rightarrow \sin^2 A = \frac{1-\cos 2A}{2}$

$\rightarrow \cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1-\cos^2 A)$

$= 2\cos^2 A - 1$

$$\rightarrow \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd} \end{cases}$$

$$\rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\rightarrow \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

integrated by parts :-

$$\rightarrow \int_a^b f(x)g(x)\,dx = \left[f\int g(x)\,dx\right]_a^b - \int_a^b f'(x)\int g(x)\,dx$$

$$\rightarrow \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\rightarrow \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\rightarrow \int_0^{\infty} \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}dx = \frac{1}{2}\tan\left(\frac{a}{2}\right)$$

$$\rightarrow \int_0^{\infty} \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}} = \frac{1}{2}\sec\left(\frac{a}{2}\right)$$

Problems on model-I :-

Problems on fourier transform :-

[1] show that fourier transform of $f(x) = e^{-x^2/2}$ is $e^{-p^2/2}$

Given that

$$f(x) = e^{-x^2/2}$$

we have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \cdot f(x) \cdot dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \cdot e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx - x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(2ipx - x^2)/2} dx$$

$$\left[\begin{array}{l}(x-ip)^2 = x^2 - p^2 - 2ipx \\ (x-ip)^2 + p^2 = x^2 - 2ipx\end{array}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x^2-2ipx)}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{[(x-ip)^2+p^2]}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-ip)^2}{2}} \cdot e^{-p^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{-\left[\frac{x-ip}{\sqrt{2}}\right]^2} \cdot e^{-p^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-p^2/2} \int_{-\infty}^{\infty} e^{-\left(\frac{x-ip}{\sqrt{2}}\right)^2} . dx$$

Put $\frac{x-ip}{\sqrt{2}} = y$

$$\frac{dx}{\sqrt{2}} = dy$$

$$dx = \sqrt{2}\,dy$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} e^{-p^2/2} \int_{-\infty}^{\infty} e^{-y^2} . \sqrt{2} . dy$$

$$= e^{-p^2/2} \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}} . \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= e^{-p^2/2} . \frac{1}{\cancel{\sqrt{\pi}}} . \cancel{\sqrt{\pi}}$$

$$\therefore F[f(x)] = e^{-p^2/2}$$

5. [2] Find the fourier transform of $f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2\epsilon}, & |x| < \epsilon \\ 0, & |x| > \epsilon \end{cases}$

$$(or) \begin{cases} \frac{\sqrt{2\pi}}{2a}, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Given that

$$f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2\epsilon}, & |x| < \epsilon \\ 0, & |x| > \epsilon \end{cases} \quad (or) \begin{cases} \frac{\sqrt{2\pi}}{2a}, & |x| \le a \\ 0, & |x| > a \end{cases}$$

$$f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2\epsilon}, & -\epsilon < x < \epsilon \\ 0. & x > \epsilon, x < -\epsilon \end{cases}$$

we have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x) dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\alpha}^{-\epsilon} e^{ipx} f(x)\,dx + \int_{-\epsilon}^{\epsilon} e^{ipx} f(x)\,dx + \int_{\epsilon}^{\alpha} e^{ipx} f(x)\,dx \right]$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \left[0 + \int_{-\epsilon}^{\epsilon} e^{ipx} \frac{\sqrt{2\pi}}{2\epsilon} \cdot dx + 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{2\epsilon} \int_{-\epsilon}^{\epsilon} e^{ipx} dx$$

$$= \frac{1}{2\epsilon} \left[\frac{e^{ipx}}{ip} \right]_{-\epsilon}^{\epsilon}$$

$$\left(\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{ip\epsilon} &= \cos p\epsilon + i\sin p\epsilon \\ e^{-ip\epsilon} &= \cos p\epsilon - i\sin p\epsilon \end{aligned} \right)$$

$$= \frac{1}{2\epsilon} \cdot \frac{1}{ip} \left(e^{ip\epsilon} - e^{-ip\epsilon} \right)$$

$$= \frac{1}{\cancel{2}\epsilon} \cdot \frac{1}{\cancel{i}p} \left(\cancel{2i} \sin p\epsilon \right)$$

$$F[f(x)] = \frac{1}{p\epsilon} \sin p\epsilon$$

[3] Find the fourier transform of if $f(x) = \begin{cases} e^{i\omega x}, & a<x<b \\ 0, & x<a, x>b \end{cases}$

Given that

$$f(x) = \begin{cases} e^{i\omega x}, & a<x<b \\ 0, & x<a, x>b \end{cases}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{ipx} \cdot f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\alpha}^{a} e^{ipx} \cdot f(x)\,dx + \int_{a}^{b} e^{ipx} \cdot f(x)\,dx + \int_{b}^{\alpha} e^{ipx} \cdot f(x)\,dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \int_{a}^{b} e^{ipx} \cdot e^{i\omega x} \cdot dx + 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ipx + i\omega x} . dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(p+\omega)x} . dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(p+\omega)x}}{i(p+\omega)} \right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(p+\omega)b} - e^{i(p+\omega)a}}{i(p+\omega)} \right]$$

$$= \frac{-1}{\sqrt{2\pi}} \left[\frac{e^{i(p+\omega)a} - e^{i(p+\omega)b}}{i(p+\omega)} \right]$$

$$= \frac{i^2}{\sqrt{2\pi}} \left[\frac{e^{i(p+\omega)a} - e^{i(p+\omega)b}}{\cancel{i}(p+\omega)} \right]$$

$$\therefore F[f(x)] = \frac{i}{(p+\omega)\sqrt{2\pi}} \left[e^{i(p+\omega)a} - e^{i(p+\omega)b} \right]$$

[4] Find the fourier transform of $f(x) = \begin{cases} x & , |x| \le a \\ 0 & , |x| > a \end{cases}$

Given that

$$f(x) = \begin{cases} x & , |x| \le a \quad \text{i.e } -a < x < a \\ 0 & , |x| > a \quad \text{i.e } x < -a, x > a \end{cases}$$

we have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{ipx} . f(x) dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\alpha}^{-a} e^{ipx} . f(x) dx + \int_{-a}^{a} e^{ipx} . f(x) dx + \int_a^{\alpha} e^{ipx} f(x) dx \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + \int_{-a}^{a} e^{ipx}\, x\, dx + 0\right]$$

$$\left[\int uv = u\int v - \int u'\int v\right] = \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} x\, e^{ipx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[x\int e^{ipx}\, dx - \int 1\int e^{ipx}\, dx\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[x\cdot\frac{e^{ipx}}{ip} - \int\frac{e^{ipx}}{ip}\, dx\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{xe^{ipx}}{ip} - \frac{1}{ip}\cdot\frac{e^{ipx}}{ip}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{xe^{ipx}}{ip} + \frac{e^{ipx}}{p^2}\right]_{-a}^{a}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\left[\frac{ae^{ipa} - (-a)e^{-ipa}}{ip} + \frac{e^{ipa} - e^{-ipa}}{p^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{ip}\left(e^{ipa} + e^{-ipa}\right) + \frac{\left(e^{ipa} - e^{-ipa}\right)}{p^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a}{ip}(2\cos pa) + \frac{1}{p^2}\, 2i\sin pa\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2a\cos pa}{ip} + \frac{2i\sin pa}{p^2}\right]$$

$$= \frac{2i}{\sqrt{2\pi}}\left[\frac{\sin pa}{p^2} + \frac{a\cos pa}{i^2 p}\right]$$

$$= \frac{\sqrt{2}\cdot\sqrt{2}\, i}{\sqrt{2}\cdot\sqrt{\pi}}\left[\frac{\sin pa}{p^2} - \frac{a\cos pa}{p}\right]$$

$$= i\sqrt{\frac{2}{\pi}}\left[\frac{\sin pa - ap\cos pa}{p^2}\right]$$

formula :-

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{ipa} = \cos pa + i\sin pa$$
$$e^{-ipa} = \cos pa - i\sin pa$$

Scanned with CamScanner

$$\therefore F[f(x)] = \frac{i}{p^2}\sqrt{\frac{2}{\pi}}\,(\sin pa - ap\cos pa)$$

(or)

$$F[f(x)] = \frac{-i}{p^2}\sqrt{\frac{2}{\pi}}\,(ap\cos pa - \sin pa)$$

[5] Find the fourier transform of if $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$

Given that

$$f(x) = \begin{cases} x^2, & |x| < a \quad \text{i.e } -a < x < a \\ 0, & |x| > a \quad \text{i.e } x < -a,\ x > a \end{cases}$$

we have

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}\cdot f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} e^{ipx}\cdot f(x)\,dx + \int_{-a}^{a} e^{ipx}\cdot f(x)\,dx + \int_{a}^{\infty} e^{ipx}\cdot f(x)\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + \int_{-a}^{a} e^{ipx}\cdot x^2\,dx + 0\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ipx}\cdot x^2\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[x^2\int e^{ipx}\,dx - \int 2x\int e^{ipx}\,dx\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[x^2\,\frac{e^{ipx}}{ip} - 2\int x\cdot\frac{e^{ipx}}{ip}\,dx\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{x^2 e^{ipx}}{ip} - \frac{2}{ip}\left[x\int e^{ipx}\,dx - \int 1\int e^{ipx}\,dx\right]\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{x^2 e^{ipx}}{ip} - \frac{2x}{ip}\cdot\frac{e^{ipx}}{ip} + \frac{2}{ip}\int\frac{e^{ipx}}{ip}\,dx\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{x^2 e^{ipx}}{ip} + \frac{2x e^{ipx}}{p^2} - \frac{2}{p^2}\cdot\frac{e^{ipx}}{ip}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{x^2 e^{ipx}}{ip} + \frac{2xe^{ipx}}{p^2} - \frac{2}{ip^3} e^{ipx}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a^2e^{ipa} - a^2e^{-ipa}}{ip} + \frac{2ae^{ipa} + 2ae^{-ipa}}{p^2} - \frac{2}{ip^3}\left(e^{ipa} - e^{-ipa}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a^2}{ip}\left(e^{ipa} - e^{-ipa}\right) + \frac{2a}{p^2}\left(e^{ipa} + e^{-ipa}\right) - \frac{2}{ip^3}\left(e^{ipa} - e^{-ipa}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a^2}{\cancel{i}p} 2\cancel{i} \sin pa + \frac{2a}{p^2} 2\cos pa - \frac{2}{\cancel{i}p^3} 2\cancel{i} \sin pa\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{a^2 \sin pa}{p} + \frac{2a\cos pa}{p^2} - \frac{2\sin pa}{p^3}\right]$$

$$= \frac{\sqrt{2}.\sqrt{2}}{\sqrt{2\pi}}\left[\frac{a^2p^2 \sin pa + 2ap\cos pa - 2\sin pa}{p^3}\right]$$

$$\therefore F[f(x)] = \frac{1}{p^3}\sqrt{\frac{2}{\pi}}\left[a^2p^2 \sin pa + 2ap\cos pa - 2\sin pa\right]$$

7.

[8] Find the fourier transform of $f(x)$ if

$$f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$ and hence evaluate $\int_0^{\alpha} \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\left(\frac{x}{2}\right) dx$

Given that:

$$f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} = \begin{cases} 1-x^2, & -1 \leq x \leq 1 \\ 0, & x < -1, x > 1 \end{cases}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{ipx} \cdot f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\alpha}^{-1} e^{ipx} \cdot f(x)\, dx + \int_{-1}^{1} e^{ipx} f(x)\, dx + \int_{1}^{\alpha} e^{ipx} \cdot f(x)\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \int_{-1}^{1} e^{ipx} (1-x^2)\, dx + 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{ipx} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\int e^{ipx}dx - \int(-2x)\int e^{ipx}dx\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{ipx}}{ip} + 2\int x\frac{e^{ipx}}{ip}dx\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{(1-x^2)e^{ipx}}{ip} + \frac{2}{ip}\left[x\int e^{ipx}dx - \int 1\int e^{ipx}.dx\right]\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{(1-x^2)e^{ipx}}{ip} + \frac{2x}{ip}\cdot\frac{e^{ipx}}{ip} - \frac{2}{ip}\int\frac{e^{ipx}}{ip}dx\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{(1-x^2)e^{ipx}}{ip} - \frac{2xe^{ipx}}{p^2} + \frac{2}{p^2}\cdot\frac{e^{ipx}}{ip}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - \frac{2}{p^2}(e^{ip}+e^{-ip}) + \frac{2}{ip^3}(e^{ip}-e^{-ip})\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{-2}{p^2}2\cos p + \frac{2}{\cancel{i}p^3}2\cancel{i}\sin p\right)$$

$$= \frac{4}{p^3}\cdot\frac{1}{\sqrt{2\pi}}\left[\sin p - p\cos p\right]$$

$$= \frac{-2.\sqrt{2}.\sqrt{2}}{p^3.\sqrt{2}.\sqrt{\pi}}(p\cos p - \sin p)$$

$$F\left[f(x)\right] = \tilde{F}(p) = \frac{-2}{p^3}\sqrt{\frac{2}{\pi}}\,(p\cos p - \sin p)$$

By inversion theorem we have.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\alpha}^{\alpha}\tilde{F}(p)\,e^{-ipx}dp.$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\alpha}^{\alpha}\frac{-2}{p^3}\sqrt{\frac{2}{\pi}}\,(p\cos p - \sin p)(\cos px - i\sin px)\,dp.$$

$$f(x) = \frac{-2}{p^3}\cdot\frac{\sqrt{2}}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{2}.\sqrt{\pi}}\left[\int_{-\alpha}^{\alpha}(p\cos p - \sin p)\cos px\,dp - (p\cos p - \sin p)\,i\sin px\,dp\right]$$

$$f(x) = \frac{-2}{\pi p^3} \int_{-\infty}^{\infty} (p\cos p - \sin p)\cos px\,dp - i(0)]$$

$$\frac{-2}{\pi} \int_{-\infty}^{\infty} \frac{(p\cos p - \sin p)}{p^3} \cos px\,dp = \begin{cases} 1-x^2 & , |x| \le 1 \\ 0 & , |x| > 1 \end{cases}$$

$$-\int_{-\infty}^{\infty} \left(\frac{p\cos p - \sin p}{p^3}\right) \cos px\,dp = \frac{\pi}{2} \begin{cases} (1-x^2) & , |x| \le 1 \\ 0 & , |x| > 1 \end{cases}$$

$$-\int_{-\infty}^{\infty} \left(\frac{p\cos p - \sin p}{p^3}\right) \cos px\,dp = \begin{cases} \pi/2\,(1-x^2) & , |x| \le 1 \\ 0 & , |x| > 1 \end{cases}$$

take $x = 1/2 \, (\le 1)$

$$-\int_{-\infty}^{\infty} \left(\frac{p\cos p - \sin p}{p^3}\right) \cos p(1/2)\,dp = \pi/2\left[1 - (1/2)^2\right]$$

$$= \pi/2\left[1 - 1/4\right]$$

$$= \pi/2\left[3/4\right]$$

$$-\int_{-\infty}^{\infty} \left(\frac{p\cos p - \sin p}{p^3}\right) \cos (p/2)\,dp = \frac{3\pi}{8}$$

$$\therefore \int_{-\infty}^{\infty} \left(\frac{p\cos p - \sin p}{p^3}\right) \cos (p/2)\,dp = \frac{-3\pi}{8}$$

$$2\int_{0}^{\infty} \left(\frac{p\cos p - \sin p}{p^3}\right) \cos (p/2)\,dp = \frac{-3\pi}{8}$$

$$\int_{0}^{\infty} \left(\frac{p\cos p - \sin p}{p^3}\right) \cos (p/2)\,dp = \frac{-3\pi}{16}$$

and hence $\int^{\infty} \left(\frac{x\cos x - \sin x}{x^3}\right) \cos (x/2)\,dx = \frac{-3\pi}{16}$

Scanned with CamScanner

model - II :-

Problems on fourier sine and cosine transform :-

8.

[Q] Find the fourier sine transform of $f(x) = 1/x$

Given that

$f(x) = 1/x$

we have $\tilde{f}_S(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx$

$\tilde{f}_S(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin px \, dx \rightarrow ①$

we have $\int_0^\infty e^{-\alpha x} \sin px \, dx = \frac{p}{\alpha^2+p^2} \rightarrow ②$

integrating o.b.s b/w the limits α_1 to α_2.

formula.

$$\int_{\alpha_1}^{\alpha_2} \int_0^\infty e^{-\alpha x} \sin px \, dx \, d\alpha = \int_{\alpha_1}^{\alpha_2} \frac{p}{\alpha^2+p^2} d\alpha \qquad \left[\because \int \frac{1}{x^2+a^2} dx = \tan^{-1}(x/a)\right]$$

$$\Rightarrow \int_0^\infty \sin px \, dx \left(\frac{e^{-\alpha x}}{-x}\right)_{\alpha_1}^{\alpha_2} = \cancel{p} \left[\tan^{-1}\left(\frac{\alpha}{p}\right)\right]_{\alpha_1}^{\alpha_2} \cancel{1/p}$$

$$\Rightarrow \int_0^\infty \sin px \, dx \left[-\frac{1}{x}\left[e^{-\alpha_2 x} - e^{-\alpha_1 x}\right]\right] = \cancel{p}\left[\tan^{-1}\left(\frac{\alpha_2}{p}\right) - \tan^{-1}\left(\frac{\alpha_1}{p}\right)\right]$$

$$\Rightarrow \int_0^\infty \sin px \, dx \left[-\frac{e^{-\alpha_2 x}}{x} + \frac{e^{-\alpha_1 x}}{x}\right] = \cancel{p}\left[\tan^{-1}\left(\frac{\alpha_2}{p}\right) - \tan^{-1}\left(\frac{\alpha_1}{p}\right)\right]$$

$$\Rightarrow \int_0^\infty \sin px \, dx \left[\frac{e^{-\alpha_1 x} - e^{-\alpha_2 x}}{x}\right] = \cancel{p}\left[\tan^{-1}\left(\frac{\alpha_2}{p}\right) - \tan^{-1}\left(\frac{\alpha_1}{p}\right)\right] \rightarrow ③$$

Now $\alpha_1 = 0$ and $\alpha_2 \rightarrow \infty$ in eq^n ③

$$\int_0^{\infty} \sin px \, dx \left[\frac{e^{0} - e^{-\alpha}}{x}\right] = p\left[\tan^{-1}\left(\frac{\alpha}{p}\right) - \tan^{-1}\left(\frac{0}{p}\right)\right]$$

$$\int_0^{\infty} \frac{\sin px \, dx}{x} (1-0) = p\left[\tan^{-1}(\tan(90^\circ) - \tan^{-1}(\tan(0))\right]$$

$$= p(90^\circ - 0)$$

$$\int_0^{\infty} \frac{\sin px}{x} dx = p \,\pi/2 \quad \text{sub}_c \text{ in eq}^n \text{ (1)}$$

$$\tilde{f}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin px}{x} dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \frac{\sqrt{2}\cdot\sqrt{\pi}\cdot\sqrt{\pi}}{\sqrt{\pi}\cdot\sqrt{2}\cdot\sqrt{2}} = \sqrt{\pi/2}$$

$$\therefore \tilde{f}_s(p) = \sqrt{\pi/2}.$$

[10] Find the cosine transform of function $f(x)$ if

$$f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$$

Given that

$$f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$$

we have $\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos px \, dx$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^{a} f(x) \cos px \, dx + \int_a^{\infty} f(x) \cos px \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^{a} \cos x \cos px \, dx + 0\right]$$

$[\because 2\cos A \cos B = \cos(A+B) + \cos(A-B)]$ $$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right) \int_0^{a} 2\cos x \cos px \, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}\,\sqrt{2}\ldots} \int_0^{a} \left[\cos(x+px) + \cos(x-px)\right] dx$$

$$= \sqrt{\frac{2}{\pi}} \quad \frac{2\cos p}{p^2}(1-\cos p)$$

$$\therefore \tilde{f}_c(p) = 2 \cdot \sqrt{\frac{2}{\pi}} \quad \frac{\cos p}{p^2}(1-\cos p)$$

14] Find the fourier sine and cosine transform of the function $f(x) = x^{m-1}$

Given that

$$f(x) = x^{m-1}$$

we have

$$\tilde{f}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x^{m-1} \cdot \sin px \, dx$$

and

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x^{m-1} \cdot \cos px \, dx.$$

$$\int_0^\infty x^{m-1} \cos px \, dx - i\int_0^\infty x^{m-1} \sin px \, dx = \int_0^\infty x^{m-1} dx \, (\cos px - i \sin px)$$

$$= \int_0^\infty x^{m-1} \cdot e^{-ipx} \cdot dx$$

$$\left[\therefore \int_0^\infty e^{-ax} \cdot x^{n-1} dx = \frac{\Gamma n}{a^n} \right]$$

$$= \frac{\Gamma m}{(ip)^m}$$

$$= \frac{\Gamma m}{i^m p^m}$$

$$= \frac{\Gamma m}{p^m} i^{-m}$$

$$= \frac{\Gamma m}{p^m} \left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right]^{-m}$$

$$= \frac{\Gamma m}{p^m} \left[e^{i\frac{\pi}{2}}\right]^{-m}$$

Scanned with CamScanner

$$= \frac{\Gamma m}{p^m}\left[e^{-im\pi/2}\right]$$

$$= \frac{\Gamma m}{p^m}\left[\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right]$$

$$= \frac{\Gamma m}{p^m}\cos\frac{m\pi}{2} - \frac{\Gamma m}{p^m}\, i\sin\frac{m\pi}{2}$$

$$\therefore \int_0^\infty x^{m-1}\cos px\,dx - i\int_0^\infty x^{m-1}\sin px\,dx = \frac{\Gamma m}{p^m}\cos\frac{m\pi}{2} - i\frac{\Gamma m}{p^m}\sin\frac{m\pi}{2}$$

Equiting real and imaginary parts.

$$\int_0^\infty x^{m-1}\cos px\,dx = \frac{\Gamma m}{p^m}\cos\frac{m\pi}{2}$$

$$\int_0^\infty x^{m-1}\sin px\,dx = \frac{\Gamma m}{p^m}\sin\frac{m\pi}{2}$$

$$\therefore \tilde{f}_c(p) = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma m}{p^m}\cos\frac{m\pi}{2}$$

$$\therefore \tilde{f}_s(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{m-1}\sin px\,dx$$

$$\tilde{f}_s(p) = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma m}{p^m}\sin\frac{m\pi}{2}.$$

[15] Find the fourier sine and cosine transform of $f(x)$ if

$f(x) = x$

Given that

$f(x) = x$

we know that:

$$\tilde{f}_s(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin px\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty x\sin px\,dx \rightarrow (1)$$

Scanned with CamScanner

$$\tilde{F}_c(p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos px\,dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} x\cos px\,dx \to (2)$$

Now

$$\int_0^{\infty} x\cos px\,dx - i\int_0^{\infty} x\sin px\,dx = \int_0^{\infty} x\,dx\,(\cos px - i\sin px)$$

$$= \int_0^{\infty} x\,e^{-ipx}\,dx$$

$$\left[\therefore \int_0^{\infty} x^{n-1}\cdot e^{-ax}\,dx = \frac{\Gamma n}{a^n}\right]$$

$$= \frac{\Gamma 2}{(ip)^2}$$

$$= \frac{\Gamma 2}{i^2 p^2}$$

$$= \frac{\Gamma 2}{p^2}\, i^{-2}$$

$$= \frac{\Gamma 2}{p^2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-2}$$

$$= \frac{\Gamma 2}{p^2}\left[e^{i\pi/2}\right]^{-2}$$

$$= \frac{\Gamma 2}{p^2}\, e^{-i\frac{2\pi}{2}}$$

$$= \frac{\Gamma 2}{p^2}\, e^{-i\pi}$$

$$= \frac{\Gamma 2}{p^2}(\cos\pi - i\sin\pi)$$

$$= \frac{\Gamma 2}{p^2}\cos\pi - i\frac{\Gamma 2}{p^2}\sin\pi$$

$$= \frac{\Gamma 2}{p^2}(-1) - i(0)$$

$$\therefore \int_0^{\infty} x\cos px\,dx - i\int_0^{\infty} x\sin px\,dx = -\frac{\Gamma 2}{p^2} - i(0)$$

$$= \frac{-\Gamma(1+1)}{p^2} - i(0)$$

$$= \frac{-1}{p^2} - i(0) \longrightarrow (A)$$

Equating real and imaginary parts in (A)

$$\int_0^{\alpha} x\cos px\,dx = \frac{-1}{p^2} \quad \& \quad \int_0^{\alpha} x\sin px\,dx = 0$$ this values are put in ① & ②

$$\therefore \tilde{f}_s(p) = \sqrt{\frac{2}{\pi}}\,(0) = 0.$$

$$\therefore \tilde{f}_c(p) = \sqrt{\frac{2}{\pi}}\left(\frac{-1}{p^2}\right) = \frac{-1}{p^2}\sqrt{\frac{2}{\pi}}$$

Scanned with CamScanner

V

(18) Find fourier cosine transform of $\frac{e^{-ax}}{x}$ and deduce that

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} \cos px \, dx = \log \frac{\sqrt{p^2+b^2}}{\sqrt{p^2+a^2}}$$

Given that

$$f(x) = \frac{e^{-ax}}{x}$$

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx$$

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos px \, dx \longrightarrow (1)$$

differensating on both sides w.r.to p. we have

$$\frac{d}{dp}[\tilde{f}_c(p)] = \sqrt{\frac{2}{\pi}} \frac{d}{dp} \int_0^\infty \frac{e^{-ax}}{x} \cos px \, dx$$

$$\frac{d}{dp}[\tilde{f}_c(p)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \frac{d}{dp} \cos px \, dx$$

Scanned with CamScanner

$$= \sqrt{\frac{2}{\pi}} \int_0^{\alpha} \frac{e^{-ax}}{x} (-\sin px) . x dx.$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^{\alpha} e^{-ax} \sin px \, dx$$

$$\frac{d}{dp}[\tilde{f}_c(p)] = -\sqrt{\frac{2}{\pi}} \left[\frac{p}{a^2+p^2}\right] \longrightarrow (2)$$

Integrating on both sides w.r. to 'p' we have

$$\int \frac{d}{dp}[\tilde{f}_c(p)] = -\sqrt{\frac{2}{\pi}} \int \left(\frac{p}{a^2+p^2}\right) + c$$

$$\tilde{f}_c(p) = -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int \frac{2p}{a^2+p^2} dp + c.$$

$$\tilde{f}_c(p) = -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \cdot \log(a^2+p^2) + c \longrightarrow (3)$$

where c is any orbitary constant.

Put $p=0$ in eqn (1) & (3)

$$(1) \Rightarrow \tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} \frac{e^{-ax}}{x} (\cos 0) dx$$

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} \frac{e^{-ax}}{x} dx \longrightarrow (4)$$

Now we find the value $I = \int_0^{\alpha} \frac{e^{-ax}}{x} dx \longrightarrow (A)$

Differensate o.b.s eqn w.r.to 'a' o.b.s eqn (1)

$$\frac{dI}{da} = \int_0^{\alpha} \frac{d}{da}(e^{-ax}) . \frac{dx}{x}$$

$$\frac{dI}{da} = \int_0^{\alpha} e^{-ax}(-x) . \frac{dx}{x}$$

$$\frac{dI}{dt} = -\int_0^{\alpha} e^{-ax}\, dx.$$

$$\frac{dI}{dt} = -\left(\frac{e^{-ax}}{-a}\right)_0^{\alpha}$$

$$= \frac{1}{a}(e^{-\alpha} - e^{0})$$

$$= \frac{1}{a}(0-1)$$

$$\frac{dI}{dt} = -\frac{1}{a}$$

variables on separation

$$dI = -\frac{da}{a}$$

I.O.B.S

$$\int dI = -\int \frac{da}{a}$$

$I = -\log a$ subc in eq^n (A)

$$\int_0^{\alpha} \frac{e^{-ax}}{x}\, dx = -\log a \text{ put in eq}^n \text{ (4)}$$

$$\tilde{f}_c(P) = -\sqrt{\frac{2}{\pi}} \log a \longrightarrow \text{(B)}$$

(3) $\Rightarrow \tilde{f}_c(P) = -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \cdot \log(a^2+0) + C$

$$\Rightarrow \tilde{f}_c(P) = -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \log a^2 + C$$

$$\tilde{f}_c(P) = -\sqrt{\frac{2}{\pi}} \cdot \frac{2 \log a}{2} + C$$

$$\tilde{f}_c(P) = -\sqrt{\frac{2}{\pi}} \log a + C \longrightarrow \text{(C)}$$

$\Rightarrow$ (B) = (C)

$$-\sqrt{\frac{2}{\pi}} \log a = -\sqrt{\frac{2}{\pi}} \log a + C$$

$C = 0$ ς in eq^n (3)

$\therefore \tilde{f}_c(p) = -\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \log(a^2+p^2) \longrightarrow (5)$

$\therefore (1) = (5)$

$\therefore \tilde{f}_c(p) = \tilde{f}_c(p)$

$\Rightarrow \sqrt{\frac{\cancel{2}}{\cancel{\pi}}} \int_0^\infty \frac{e^{-ax}}{x} \cos px \, dx = -\sqrt{\frac{\cancel{2}}{\cancel{\pi}}} \cdot \frac{1}{2} \log(a^2+p^2)$

$\Rightarrow \int_0^\infty \frac{e^{-ax}}{x} \cos px \, dx = -\frac{1}{2} \log(a^2+p^2).$

Now consider

$$\int_0^\infty \left(\frac{e^{-ax}-e^{-bx}}{x}\right) \cos px \, dx = \int_0^\infty \frac{e^{-ax}}{x} \cos px \, dx - \int_0^\infty \frac{e^{-bx}}{x} \cos px \, dx$$

$$= -\frac{1}{2} \log(a^2+p^2) + \frac{1}{2} \log(b^2+p^2)$$

$$= \frac{1}{2}\left[\log(b^2+p^2) - \log(a^2+p^2)\right]$$

$$= \frac{1}{2} \log\left(\frac{p^2+b^2}{p^2+a^2}\right)$$

$$= \log\left(\frac{p^2+b^2}{p^2+a^2}\right)^{1/2}$$

$$= \log\sqrt{\frac{p^2+b^2}{p^2+a^2}}$$

Scanned with CamScanner

[23] Find the sine and cosine transform of $f(x) = \dfrac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}$

Given that

$$f(x) = \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}$$

$$\left[\begin{array}{l} e^{ipx} = \cos px + i\sin px \\ e^{-ipx} = \cos px - i\sin px \\ e^{ipx} - e^{-ipx} = 2i\sin px \end{array}\right]$$

$$\tilde{f}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin px\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}} \sin px\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(e^{ax}+e^{-ax})}{e^{\pi x}-e^{-\pi x}} \frac{(e^{ipx}-e^{-ipx})}{2i}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(e^{ax}.e^{ipx} - e^{ax}e^{-ipx} + e^{-ax}.e^{ipx} - e^{-ax}.e^{-ipx})}{2i(e^{\pi x}-e^{-\pi x})}\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{(a+ip)x} - e^{(a-ip)x} + e^{(-a+ip)x} - e^{-(a+ip)x}}{2i(e^{\pi x}-e^{-\pi x})}\, dx$$

Scanned with CamScanner

$$= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\int_0 \left[\frac{e^{(a+ip)x} - e^{-(a+ip)x}}{e^{\pi x}-e^{-\pi x}} - \frac{e^{(a-ip)x}-e^{-(a-ip)x}}{e^{\pi x}-e^{-\pi x}}\right]dx$$

$$\tilde{f}_s(p) = \frac{1}{2i}\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{(a+ip)x} - e^{-(a+ip)x}}{e^{\pi x}-e^{-\pi x}}\,dx - \frac{1}{2i}\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{(a-ip)x} - e^{-(a-ip)x}}{e^{\pi x}-e^{-\pi x}}\,dx$$

$$\left[\because \int_0^{\infty} \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\,dx = \frac{1}{2}\tan\left(\frac{a}{2}\right)\right]$$

$$= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\,\frac{1}{2}\tan\left(\frac{a+ip}{2}\right) - \frac{1}{2i}\sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\tan\left(\frac{a-ip}{2}\right)$$

$$= \frac{1}{4i}\sqrt{\frac{2}{\pi}}\left[\frac{\sin\left(\frac{a+ip}{2}\right)}{\cos\left(\frac{a+ip}{2}\right)} - \frac{\sin\left(\frac{a-ip}{2}\right)}{\cos\left(\frac{a-ip}{2}\right)}\right]$$

$$= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\left[\frac{\sin\left(\frac{a+ip}{2}\right)\cos\left(\frac{a-ip}{2}\right) - \cos\left(\frac{a+ip}{2}\right)\sin\left(\frac{a-ip}{2}\right)}{2\cos\left(\frac{a+ip}{2}\right)\cos\left(\frac{a-ip}{2}\right)}\right]$$

$$= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\;\frac{\sin\left(\frac{a+ip}{2} - \frac{a-ip}{2}\right)}{\cos\left(\frac{a+ip}{2} + \frac{a-ip}{2}\right) + \cos\left(\frac{a+ip}{2} - \frac{a-ip}{2}\right)}$$

$$= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\left[\frac{\sin\left(\frac{a+ip-a+ip}{2}\right)}{\cos\left(\frac{a+ip+a-ip}{2}\right) + \cos\left(\frac{a+ip-a+ip}{2}\right)}\right]$$

$$= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\left[\frac{\sin\left(\frac{2ip}{2}\right)}{\cos\left(\frac{2a}{2}\right) + \cos\left(\frac{2ip}{2}\right)}\right]$$

$$= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\left[\frac{\sin(ip)}{\cos a + \cos ip}\right] \qquad \left[\begin{array}{l}\because \sin(ip) = i\sinh p \\ \because \cos(ip) = \cosh p\end{array}\right]$$

$$\tilde{f}_s(p) = \frac{1}{2\cancel{i}}\sqrt{\frac{2}{\pi}}\left[\frac{\cancel{i}\sinh p}{\cos a + \cosh p}\right] \qquad \left[\begin{array}{l}\because \sinh p = \frac{e^{p}-e^{-p}}{2} \\ \because \cosh p = \frac{e^{p}+e^{-p}}{2}\end{array}\right]$$

$$\tilde{f}_s(p) = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\frac{\frac{e^{p}-e^{-p}}{2}}{\cos a + \frac{e^{p}-e^{-p}}{2}}\right]$$

$$\tilde{f}_s(p) = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{p}-e^{-p}}{2\cos a + e^{p}-e^{-p}}\right]$$

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos px \, dx$$

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}} \cdot \cos px \, dx$$

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}} \left(\frac{e^{ipx}+e^{-ipx}}{2}\right) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{ax}e^{ipx}+e^{ax}\cdot e^{-ipx}+e^{-ax}\cdot e^{ipx}+e^{-ax}\cdot e^{-ipx}}{2(e^{\pi x}-e^{-\pi x})} \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left[\frac{e^{(a+ip)x}+e^{(a-ip)x}+e^{(-a+ip)x}+e^{-(a+ip)x}}{2(e^{\pi x}-e^{-\pi x})}\right] dx$$

$$\left[\therefore \int_0^{\infty} \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}} dx = \frac{1}{2}\sec\left(\frac{a}{2}\right)\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left[\frac{e^{(a+ip)x}+e^{-(a+ip)x}}{2(e^{\pi x}-e^{-\pi x})} + \frac{e^{(a-ip)x}+e^{-(a-ip)x}}{2(e^{\pi x}-e^{-\pi x})}\right] dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{2}\sec\left(\frac{a+ip}{2}\right) + \frac{1}{2}\sec\left(\frac{a-ip}{2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \left[\frac{1}{\cos\left(\frac{a+ip}{2}\right)} + \frac{1}{\cos\left(\frac{a-ip}{2}\right)}\right]$$

$$\left[\therefore 2\cos A \cos B = \cos(A+B) + \cos(A-B)\right] = \sqrt{\frac{2}{\pi}} \cdot \left[\frac{\cos\left(\frac{a-ip}{2}\right)+\cos\left(\frac{a+ip}{2}\right)}{2\cos\left(\frac{a+ip}{2}\right)\cos\left(\frac{a-ip}{2}\right)}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\cos\left(\frac{a}{2}-\frac{ip}{2}\right)+\cos\left(\frac{a}{2}+\frac{ip}{2}\right)}{\cos\left(\frac{a-ip}{2}+\frac{a+ip}{2}\right)+\cos\left(\frac{a-ip}{2}-\frac{a+ip}{2}\right)}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\cos\left(\frac{a}{2}\right)\cos\frac{ip}{2}}{\cos\left(\frac{a-ip+a+ip}{2}\right)+\cos\left(\frac{a-ip-a-ip}{2}\right)}\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{2\cos\left(\frac{a}{2}\right)\cos\left(\frac{ip}{2}\right)}{\cos\left(\frac{2a}{2}\right) + \cos\left(\frac{2ip}{2}\right)} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\cos\left(\frac{a}{2}\right)\cos\left(\frac{ip}{2}\right)}{\cos a + \cos ip} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2\cos\left(\frac{a}{2}\right)\cosh\left(\frac{p}{2}\right)}{\cos a + \cosh p} \right)$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\cos\left(\frac{a}{2}\right)\frac{e^{p/2}+e^{-p/2}}{2}}{\cos a + \left(\frac{e^{p}+e^{-p}}{2}\right)} \right]$$

$$\tilde{f}_c(p) = 2\sqrt{\frac{2}{\pi}} \left[\frac{\cos\left(\frac{a}{2}\right)\left(e^{p/2}+e^{-p/2}\right)}{2\cos a + e^{p} + e^{-p}} \right]$$

Scanned with CamScanner