UNIT-2
MATHS-SEM 3
LONGS

(10M)

Theorem: (1)

If H_1 and H_2 are two subgroups of group G then $H_1 \cup H_2$ is a subgroup iff $H_1 \subseteq H_2$ (or) $H_2 \subseteq H_1$.

Proof:

Let $(G, \cdot)$ be a group.

Let $(H_1, \cdot)$ and $(H_2, \cdot)$ is a subgroups of G.

Let us suppose that $H_1 \cup H_2$ is a subgroup

Now we have to prove that $H_1 \subseteq H_2$ (or) $H_2 \subseteq H_1$

If possible, suppose that $H_1 \nsubseteq H_2$ (or) $H_2 \nsubseteq H_1$

Let $H_1 \nsubseteq H_2$,

For $a \in H_1 \Rightarrow a \notin H_2$ —— (1)

Let $H_2 \nsubseteq H_1$,

For $b \in H_2 \Rightarrow b \notin H_1$ —— (2)

But $a, b \in H_1 \cup H_2$

Since $H_1 \cup H_2$ is a subgroup.

$\therefore ab \in H_1 \cup H_2$ [$\because$ By closure law]

$\Rightarrow ab \in H_1, ab \in H_2, ab \in H_1 \cap H_2$

Let $ab \in H_1, a^{-1} \in H_1$ [$\because$ By Inverse law]

$a^{-1}(ab) \in H_1$ [$\because$ By closure law]

$(a^{-1}a)b \in H_1$

$(e) b \in H_1$

$b \in H_1$

Which is contradiction to (2)

Let $ab \in H_2, b \in H_2 \Rightarrow b^{-1} \in H_2$ [$\because$ By Inverse law]

$(ab) b^{-1} \in H_2$ [$\because$ By closure law]

$a(bb^{-1}) \in H_2$

$a(e) \in H_2$

$a \in H_2$

Which is contradiction to ①

$ab \notin H_1$, $ab \notin H_2$, $ab \notin H_1 \cap H_2$

$\therefore H_1 \subseteq H_2$ (or) $H_2 \subseteq H_1$.

Conversely suppose that $H_1 \subseteq H_2$ (or) $H_2 \subseteq H_1$

Now we have to prove that $H_1 \cup H_2$ is a Subgroup

Since, $H_1 \subseteq H_2$,

$\Rightarrow H_1 \cup H_2 = H_2$

Since, H_2 is a Subgroup,

$\Rightarrow H_1 \cup H_2$ is a subgroup

Since, $H_2 \subseteq H_1$

$H_1 \cup H_2 = H_1$

Since H_1 is a subgroup

$\therefore H_1 \cup H_2$ is also a subgroup.

Theorem : (2)

A non empty complex H of a group G is subgroup if and only if

i) $a \in H, b \in H \Rightarrow ab \in H$ ii) $a \in H, a^{-1} \in H$.

Proof :

Let $(G, \cdot)$ be a group.

Let H be a non empty complex of G

The condition is necessary

Let us suppose that H is a subgroup.

Now we have to prove that i) $a \in H, b \in H \Rightarrow ab \in H$.

ii) $a \in H, a^{-1} \in H$

Since H is a subgroup,

By closure law,

$a \in H, b \in H \Rightarrow ab \in H$

By Inverse law,

$a \in H \Rightarrow a^{-1} \in H$

The condition is sufficient

Conversely suppose that i) $a \in H, b \in H \Rightarrow ab \in H$.

ii) $a \in H \Rightarrow a^{-1} \in H$.

Now we have to prove that H is a subgroup.

By (i) $\Rightarrow$

The binary composition is closure in

H.

Since the elements of H are in G.

The Binary Composition '.' is closure, associative in H.

By ii)

"a^{-1}" is the inverse Element of a in H.

For $a \in H$, $a^{-1} \in H$

$\Rightarrow aa^{-1} \in H$. [$\because$ By (i)]

$\Rightarrow e \in H$

"e" is the identity element in H.

$\therefore (H, \cdot)$ is itself a group.

$\therefore$ H is a Subgroup.

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Theorem : 11

If H and K are two subgroups of a group G then HK is a subgroup iff $HK = KH$.

Proof:-

Let $(G, \cdot)$ be a group.

Let H and K are two subgroups of G.

The condition is sufficient.

Let us suppose that $HK = KH$.

Now we have to prove that HK is a subgroup.

It is sufficient to prove $(HK)(HK^{-1}) = HK$

$L.H.S = (HK)(HK)^{-1} = (HK)(K^{-1}H^{-1})$

$= H\left[K(K^{-1}H^{-1})\right]$ [$\because$ By Associative]

$= H\left[(KK^{-1})H^{-1}\right]$ [$\because$ By Associative]

$= H\left[KH^{-1}\right]$ [$\because$ By theorem (10)]

$= (HK)H^{-1}$ [$\because$ By Associative]

$= (KH)H^{-1}$

$= K(HH^{-1})$ [$\because$ By Associative]

$= KH$ [$\because$ By theorem (10)]

$= HK = R.H.S$

$\therefore$ HK is a Subgroup.

The condition is necessary.

Conversely suppose that HK is a Subgroup.

Now we have to prove that $HK = KH$

$\Rightarrow (HK)^{-1} = HK$ [$\because$ By theorem (9)]

$\Rightarrow K^{-1}H^{-1} = HK$

$\Rightarrow KH = HK.$

Theorem : 4 (10M) & (5M)

Any two left cosets (right) of a subgroup of a group are disjoint or identical.

Proof :

Let $(G, \cdot)$ be a group.

Let $(H, \cdot)$ be a subgroup.

Let aH, bH are two left cosets of H in G.

Let aH, bH are disjoint.

Then there is nothing to prove.

Let $aH \cap bH \neq \emptyset$.

Let there exist a common element

$c \in aH \cap bH$

$c \in aH$ and $c \in bH$

$c = ah_1$, $c = bh_2$

Multiply with h_1^{-1} on b.s.

$(ah_1)h_1^{-1} = (bh_2)h_1^{-1}$

$ae = bh_3$

Multiply with H on b.s.

$aH = b(h_3 H)$ $\quad [\because h_3 H = H]$.

$aH = bH$.

Theorem: 5

State and prove Lagrange's theorem on groups [or] [Pdf unit - II - long Q.No.5].

Statement : The order of a subgroup of a finite group divides the order of a group.

Proof :

i) Every right coset have same no. of elements
ii) No. of right cosets are finite
iii) Ha, Hb, Hc.... are right
iv) Ha is also right coset
v) O(Ha) = ... = O(H)
vi) No of right cosets be 'k'
vii) All right cosets are disjoint and induce a partition in G.

Let (G, ·) be a finite group.

Let (H, ·) be a subgroup of G.

Since H is a subgroup of G.

$\Rightarrow$ H is also finite

Case (i)

If $H = G$

$\Rightarrow O(H) = O(G)$

Case (ii)

If $H \neq G$

Let $O(G) = n$

$O(H) = m$.

Let every right coset of H in G. has same no. of elements and the no. of right cosets of H in G is finite.

Ha, Hb, Hc, ... are the right cosets of H in G.

He = H is also a right coset of H in G.

$O(Ha) = O(Hb) = O(Hc) \cdot \cdots\cdots = O(H)$

Let the no. of right cosets of H in G be 'k'

All the right cosets are disjoint and induce a partition in G,

$O(Ha) + O(Hb) + \cdots\cdots + O(H) = O(G)$

$m + m + \cdots\cdots + m\,(k\text{ times}) = n$

$km = n$

$k = \dfrac{n}{m} = \dfrac{O(G)}{O(H)}$

$\therefore$ Hence proved.