UNIT-1

GROUPS - SEM3

LONGS

Theorem No.: 1

Prove that the set of all +ve rational numbers forms an abelian group with respect to the binary operation "o" defined as $a \, o \, b = \frac{ab}{3} \; \forall \; a, b \in Q^+$.

Proof:

Let Q^+ is set of all +ve rational numbers

For $a, b \in Q^+ \Rightarrow a \, o \, b = \frac{ab}{3}$

* **Closure law:**

For $a, b \in Q^+$

$\Rightarrow a \, o \, b = \frac{ab}{3}$

∴ "o" satisfies closure law.

* **Associative law:**

For $a, b, c \in Q^+$

$\Rightarrow a o (b o c) = (a o b) o c \in Q^+$

$$\begin{aligned} L.H.S &= a o (b o c) \\ &= a o \left(\frac{bc}{3}\right) \\ &= \frac{a . \frac{bc}{3}}{3} \\ &= \frac{\frac{ab}{3} . c}{3} \\ &= \frac{(a o b) c}{3} \\ &= (a o b) o c \\ &= R.H.S \end{aligned}$$

∴ L.H.S = R.H.S

∴ "o" satisfies associative law.

* **Existence of Identity law:**

For $a \in Q^+$

$\exists e \in Q^+$ such that $a o e = e o a = a$

$\Rightarrow a o e = a$

$\Rightarrow \frac{ae}{3} = a$

$\Rightarrow ae = 3a$

$\Rightarrow e = 3$

L.H.S = $a \circ e$

$= \frac{ae}{3}$

$= \frac{a.3}{3}$

$= a.$

= R.H.S

Simillarly, we can prove that $e \circ a = a$.

" $e = 3$ " is the identity element of a in Q^+

$\therefore$ "o" Satisfies existence of Identity law in Q^+.

* Existence of Inverse law:

For $a \in Q^+$

$\exists b \in Q^+$ Such that $a \circ b = b \circ a = e$

$\Rightarrow a \circ b = e$

$\Rightarrow \frac{ab}{3} = 3$

$\Rightarrow ab = 9$

$\Rightarrow b = \frac{9}{a}$

L.H.S.= $a \circ b$

$= \frac{ab}{3} \; a \circ \frac{9}{a}$

$= \frac{a \frac{9}{a}}{3}$

$= \frac{9}{3}$

$= 3$

$= e$

= R.H.S

Simillarly, we can prove that $b \circ a = e$.

" $b = \frac{9}{a}$ " is Inverse element of a in Q^+.

$\therefore$ "o" satisfies existence of Inverse in Q^+.

*Abelian law :

For $a, b \in Q^+$

$\Rightarrow a \circ b = b \circ a$

$$\begin{aligned} L.H.S &= a \circ b \\ &= \frac{ab}{3} \\ &= \frac{ba}{3} \\ &= b \circ a \\ &= R.H.S \end{aligned}$$

$\therefore$ "o" satisfies abelian law in Q^+.

(Q^+, o) forms an abelian group.

Theorem: 2

Prove that the set of intezers z forms an abelian group w.r.t the operations defined by $a*b = a+b+2$, for all $a, b \in z$.

Proof:

Let "$\mathbb{Z}$" be the set of all Intezers.

For $a, b \in \mathbb{Z} \Rightarrow a*b = a+b+2$.

* Closure law:

For $a, b \in \mathbb{Z}$

$a*b = a+b+2$.

$\therefore$ "$*$" Satisfies closere law.

* Associative law:

For $a, b, c \in \mathbb{Z}$

$a*(b*c) = (a*b)*c$

$$\begin{aligned} L.H.S &= a*(b*c) \\ &= a*(b+c+2) \\ &= a+b+c+2+2 \\ &= a+b+c+4. \end{aligned}$$

$R.H.S = (a*b)*c$

$= (a+b+2) * c$

$= a+b+c+2+2$

$= a+b+c+4$

$\therefore$ L.H.S = R.H.S

$\therefore$ "$*$" Satisfies associative law.

$*$ **Existence of Identity law:**

For $a \in \mathbb{Z}$

$\exists\, e \in \mathbb{Z}$ Such that $a * e = e * a = a$

L.H.S $= a * e = a$

$a + e + 2 = a$

$e = -2$

R.H.S $= e * a = a$

$e + a + 2 = a$

$e = -2$

"$e = -2$" is the identity element of a in $\mathbb{Z}$

$\therefore$ "$*$" satisfies the existence of Identity Law in $\mathbb{Z}$.

$*$ **Existence of Inverse law:**

For $a \in \mathbb{Z}$

$\exists\, b \in \mathbb{Z}$ Such that $a * b = b * a = e$

$\Rightarrow a * b = e$

$\Rightarrow a + b + 2 = -2$

$\Rightarrow b = -4 - a$

L.H.S $= a * b$

$= a + b + 2$

$= a - 4 - a + 2$

$= -2$

$= e$

$=$ R.H.S

$\therefore$ L.H.S = R.H.S

Similarly, we can prove that $b * a = e$.

$b = -4 - a$ is the inverse of a in $\mathbb{Z}$

$\therefore$ "$*$" Satisfies the existence of Inverse law.

$*$ Abelian law:

For $a, b \in \mathbb{Z}$

$\Rightarrow a * b = b * a$

$L.H.S = a * b$

$= a + b + 2$ [$\because a, b \in \mathbb{Z}$; $a + b = b + a$ By using abelian w.r.t to '+']

$= b + a + 2$

$= b * a$

$= R.H.S$

$L.H.S = R.H.S$

$\therefore$ "$*$" Satisfies abelian law.

$\therefore (\mathbb{Z}, *)$ forms an abelian group.

③ Prove that the set of matrices $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ $\alpha \in \mathbb{R}$ forms a group w.r.t matrix multiplication if $\cos\theta = \cos\phi \Rightarrow \theta = \phi$.

Proof :-

For all $\alpha, \beta, \gamma \in \mathbb{R}$

$$G = \{ A_\alpha \mid \alpha \in \mathbb{R} \}$$

$$\therefore A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$A_\beta = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$A_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}$$

* closure law :

For $A_\alpha, A_\beta \in G$.

$$A_\alpha \cdot A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$\begin{bmatrix} & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \in G \qquad \left[\because \alpha, \beta \in \mathbb{R} ; \text{By using closure law w.r.t '+'} \ \alpha+\beta \in \mathbb{R}\right]$$

$\therefore$ '·' Satisfies closure law in G.

*** Associative law :-**

For $A_\alpha, A_\beta, A_\gamma \in G$

$$\Rightarrow (A_\alpha \cdot A_\beta) \cdot A_\gamma = A_\alpha \cdot (A_\beta \cdot A_\gamma) \in G$$

$$L.H.S = (A_\alpha \cdot A_\beta) A_\gamma$$

$$= \left[\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\right]\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\beta\cos\alpha - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \sin\beta\cos\alpha & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$$

$$= \begin{bmatrix} \cos[(\alpha+\beta)+\gamma] & -\sin[(\alpha+\beta)+\gamma] \\ \sin[(\alpha+\beta)+\gamma] & \cos[(\alpha+\beta)+\gamma] \end{bmatrix} \qquad \left[\because \alpha, \beta, \gamma \in \mathbb{R} \ \text{By using Associative law w.r.t '+'} \ (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)\right]$$

$$= \begin{bmatrix} \cos[\alpha+(\beta+\gamma)] & -\sin[\alpha+(\beta+\gamma)] \\ \sin[\alpha+(\beta+\gamma)] & \cos[\alpha+(\beta+\gamma)] \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos(\beta+\gamma) & -\sin(\beta+\gamma) \\ \sin(\beta+\gamma) & \cos(\beta+\gamma) \end{bmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\left[\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}\right]$$

$$= A_\alpha \cdot (A_\beta \cdot A_\gamma)$$

$$= R.H.S$$

$$L.H.S = R.H.S$$

$\therefore$ '·' Satisfies Associative law in G.

*** Existence of Identity law :**

For $A_\alpha \in G$

$\exists\, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = A_0 \in G \ \forall\ 0 \in \mathbb{R}$ Such

that $A_\alpha \cdot A_0 = A_0 \cdot A_\alpha = A_\alpha$.

$L.H.S = A_\alpha \cdot A_0$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha \cos 0 + \sin\alpha \sin 0 & -\sin 0 \cos\alpha - \sin\alpha \cos 0 \\ \sin\alpha \cos 0 + \sin 0 \cos\alpha & -\sin 0 \sin\alpha + \cos\alpha \cos 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+0) & -\sin(\alpha+0) \\ \sin(\alpha+0) & \cos(\alpha+0) \end{bmatrix} \quad \left[\because 0, \alpha \in \mathbb{R} \text{ By using closure law w.r.t '+' } \alpha+0 \in \mathbb{R}\right]$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$= A_\alpha$

$= R.H.S$

$L.H.S = R.H.S$

∴ Simillarly, we can prove that $A_0 \cdot A_\alpha = A_\alpha$

∴ A_0 is the multiplicative identity.

∴ " " Satisfies Existence of Identity law in G.

*** Existence of Inverse law :**

For $A_\alpha \in G$

$$A_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$|A_\alpha| = \cos^2\alpha + \sin^2\alpha$

$= 1 \neq 0$

A_α is non singular matrix.

A_α^{-1} exists.

$$A_\alpha^{-1} = \frac{Adj\,A}{|A|}$$

$$Adj\,A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$\exists\, A_\alpha^{-1} \in G$ such that $A_\alpha \cdot A_\alpha^{-1} = A_\alpha^{-1} \cdot A_\alpha = A_0$

$$L.H.S = A_\alpha \cdot A_\alpha^{-1}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \sin\alpha\cos\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \sin\alpha\cos\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= A_0$$

$$= R.H.S$$

$L.H.S = R.H.S$

$\therefore$ Simillarly, we can prove that $A_\alpha^{-1} \cdot A_\alpha = A_0$

$\therefore$ A_α^{-1} is the multiplicative inverse of A_α in G

$\therefore$ "." Satisfies Existence of Inverse law in G.

$\therefore$ (G, .) forms a Group.

Theorem. 4

In a group G, for $a, b, x, y \in G$ prove that the equations $ax = b$; $ya = b$ have unique solutions.

Proof :

Let $(G, \cdot)$ be a group.

Let 'e' be the identity element in G.

For $a \in G \Rightarrow a^{-1} \in G$ [∵ By Inverse law.]

$a^{-1} \in G$, $b \in G \Rightarrow a^{-1}b \in G$ [∵ By closure law]

Also, $ax = b$

Multiply with a^{-1} on both sides

$$a^{-1}(ax) = a^{-1}b$$

$$(a^{-1}a)x = a^{-1}b$$

$$ex = a^{-1}b$$

$$x = a^{-1}b$$

$$\begin{aligned} L.H.S &= ax \\ &= a(a^{-1}b) \\ &= (aa^{-1})b \\ &= e.b \\ &= b \\ &= R.H.S \end{aligned}$$

$L.H.S = R.H.S$

$x = a^{-1}b$ is the solution of $ax = b$ Now we have to prove that $x = a^{-1}b$ is the unique solution.

If possible, let x_1, x_2 are the solution sets of $ax = b$.

$$ax_1 = b \; ; ax_2 = b$$

$$ax_1 = ax_2$$

By left Cancellation law

$x_1 = x_2$

$\therefore$ $x = a^{-1}b$ is the unique Solution of $ax = b$

Similarly, we can prove that $y = ba^{-1}$ is the Solution of $ya = b$.

Now we have to prove that $y = ba^{-1}$ is the unique solution sets of $ya = b$

If possible, let y_1, y_2 are the Solution sets of $ya = b$

$y_1 a = b$; $y_2 a = b$

$\Rightarrow y_1 a = y_2 a$

By right cancellation law

$y_1 = y_2$

$\therefore$ $y = ba^{-1}$ is the unique solution of $ya = b$.

Theorem : ⑤

Prove that n^{th} roots of unity forms an abelian group.

Proof :

$$\sqrt[n]{1} = (1)^{1/n}$$

$$= [\cos 0 + i\sin 0]^{1/n}$$

$$= [\cos 2k\pi + i\sin 2k\pi]^{1/n} \quad ; k = 0,1,2,\cdots n-1$$

$$= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \quad ; k = 0,1,2,\cdots n-1$$

[∵ euler's formula]

$$G = e^{\frac{2k\pi i}{n}} \quad ; k = 0,1,2,\cdots n-1$$

Let $G = e^{\frac{2k\pi i}{n}}$; $k = 0,1,2,\cdots n-1$

* Closure law :

For $a, b \in G$

$$a = (1)^{1/n}$$

$$a^n = 1$$

$$b = (1)^{1/n}$$

$$b^n = 1$$

Consider,

$$(ab)^n = a^n \cdot b^n$$

$$ab = (1)^{1/n}$$

For $a, b \in G \Rightarrow ab \in G$.

$\therefore$ "." Satisfies closure law in G.

*Associative Law :-

Complex < A.S
A.H

Since the elements of G are Complex.

We know that,

The complex no's are associative with respect to multiplication.

$\therefore$ "." is Satisfies Associative law in G.

* Existence Of Identity law :-

For $e^{\frac{2k\pi i}{n}} \in G$.

$\exists\, 1 = e^{\frac{2(0)\pi i}{n}} \in G$ such that

$$e^{\frac{2k\pi i}{n}} \cdot e^{\frac{2(0)\pi i}{n}} = e^{\frac{2(0)\pi i}{n}} e^{\frac{2k\pi i}{n}} = e^{\frac{2k\pi i}{n}}$$

$$\begin{aligned} L.H.S &= e^{\frac{2k\pi i}{n}} \cdot e^{\frac{2(0)\pi i}{n}} \\ &= e^{\frac{2k\pi i}{n} + \frac{2(0)\pi i}{n}} \\ &= e^{\frac{2k\pi i + 2(0)\pi i}{n}} \\ &= e^{\frac{2\pi i(k+0)}{n}} \\ &= e^{\frac{2k\pi i}{n}} \\ &= R.H.S \end{aligned}$$

$\therefore$ Simillarly we can prove that $e^{\frac{2(0)\pi i}{n}} \cdot e^{\frac{2k\pi i}{n}} = e^{\frac{2k\pi i}{n}}$

$\therefore e^{\frac{2(0)\pi i}{n}}$ is the multipliative identity of $e^{\frac{2k\pi i}{n}}$ in G

$\therefore$ "." Satisfies existence of identity law.

* Existence of Inverse law :

For $e^{\frac{2r\pi i}{n}} \in G \quad \forall \ 0 \le r \le n-1$

$$e^{\frac{2(0)\pi i}{n}} \in G \quad \forall \ r=0$$

$$e^{\frac{2(n-r)\pi i}{n}} \in G \quad \forall \ 0 < r \le n-1$$

$$e^{\frac{2r\pi i}{n}} \cdot e^{\frac{2(0)\pi i}{n}} = e^{\frac{2(0)\pi i}{n} + \frac{2(0)\pi i}{n}}$$

$$= e^{\frac{2(0)\pi i + 2(0)\pi i}{n}}$$

$$= e^{0}$$

$$= 1$$

$$e^{\frac{2r\pi i}{n}} \cdot e^{\frac{2(n-r)\pi i}{n}} = e^{\frac{2r\pi i}{n} + \frac{2(n-r)\pi i}{n}}$$

$$= e^{\frac{2r\pi i + 2n\pi i - 2r\pi i}{n}}$$

$$= e^{\frac{2n\pi i}{n}}$$

$$= e^{2\pi i}$$

$$= \cos 2\pi + i \sin 2\pi$$

$$= 1 + i(0)$$

$$= 1$$

∴ Every element of G are invertible.

∴ " Satisfies existence of Inverse law in G.

* Abelian law :-

Since the elements of G are complex. We know that, The complex no's are associative with respect to multiplication.

∴ " Satisfies abelian law in G.