Application of laplace transform to Integral $eq^n s$

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[2] Solve the integral eqn $F(t) = 1 + \int_0^t F(u)\sin(t-u)\,du$ and verify it's answer.

Given integral eqn is

$$F(t) = 1 + \int_0^t F(u)\sin(t-u)\,du \longrightarrow (1)$$

Now the given integral is may be expressed

as $F(t) = 1 + [\sin t * F(t)]$

taking laplace transform

$$L(F(t)) = L(1) + L[\sin t * F(t)]$$

$$L[F(t)] = 1/p + L(\sin t) * L[F(t)]$$

$$= 1/p + \frac{1}{p^2+1} L(F(t))$$

$$L(F(t)) - \frac{1}{p^2+1} L(F(t)) = 1/p$$

$$L[F(t)]\left[1 - \frac{1}{p^2+1}\right] = 1/p$$

$$L[F(t)]\left[\frac{p^2+1-1}{\ldots}\right] = 1/p$$

$$L[F(t)] = \frac{1}{p} \times \frac{p^2+1}{p^2}$$

$$L[F(t)] = \frac{p^2+1}{p^3}$$

let $\frac{p^2+1}{p^3} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3}$

$$p^2+1 = Ap^2 + Bp + C$$

p^2 :- $A = 1$ $\quad$ p :- $B = 0$, cons :- $C = 1$

$$\therefore L[F(t)] = \frac{1}{p} + 0 + \frac{1}{p^3}$$

$$F(t) = L^{-1}\left(\frac{1}{p}\right) + L^{-1}\left(\frac{1}{p^3}\right)$$

$$\therefore F(t) = 1 + \frac{1}{2}t^2$$

$$\therefore F(t) = 1 + \frac{t^2}{2}$$

Verification :-

we have $F(t) = 1 + \frac{t^2}{2} \rightarrow (a)$

Now putting in eq (a) R.H.S of given eq^n (1)

we have

$$RHS = 1 + \int_0^t \left(1 + \frac{u^2}{2}\right) \sin(t-u)\,du$$

$U = 1 + \frac{u^2}{2}$, $V = \sin(t-u)$ $\quad$ $\left[\therefore \int UV = U\int V - \int U' \int V\right]$

$$= 1 + \left[\left(1 + \frac{u^2}{2}\right)\int \sin(t-u)\,du - \int u \int \sin(t-u)\,du\right]_0^t$$

$$= 1 + \left[\left(1 + \frac{u^2}{2}\right)\left(\frac{-\cos(t-u)}{-1}\right) - \int u \left(\frac{-\cos(t-u)}{-1}\right) du\right]_0^t$$

$$= 1 + \left[\left(1 + \frac{u^2}{2}\right) \cos(t-u) - \left[u\int \cos(t-u)\,du - \int 1 \int \cos(t-u)\,du\right]\right]_0^t$$

$$= 1+\left[\left(1+\frac{u^2}{2}\right)\cos(t-u) - \left[u\,\frac{\sin(t-u)}{-1} - \int \frac{\sin(t-u)}{-1}\,du\right]\right]_0^t$$

$$= 1+\left[\left(1+\frac{u^2}{2}\right)\cos(t-u) + u\sin(t-u) + \left(\frac{-\cos(t-u)}{+1}\right)\right]_0^t$$

$$= 1+\left[\left(1+\frac{u^2}{2}\right)\cos(t-u) + u\sin(t-u) - \cos(t-u)\right]_0^t$$

$$= 1+\left[\left(1+\frac{t^2}{2}\right) + t\sin(0) - \cos(0) - \cos t - 0 + \cos t\right]$$

$$= 1+1+\frac{t^2}{2}-1$$

$$= 1+\frac{t^2}{2}.$$

[3] Solve the integral eqn $F(t) = a\sin t - 2\int_0^t F(u)\cos(t-u)\,du$

Given integral eqn is.

$$F(t) = a\sin t - 2\int_0^t F(u)\cos(t-u)\,du$$

Now the given integral eqn may be expressed as

$$F(t) = a\sin t - 2\left[\cos t * F(t)\right]$$

taking laplace transform on both sides.

$$L[F(t)] = a\,L[\sin t] - 2\,L\cos(t)\,L[F(t)]$$

$$L[F(t)] = a\,\frac{1}{p^2+1} - 2\,\frac{p}{p^2+1}\,L\,F(t)$$

$$L[F(t)] + \frac{2p}{p^2+1}\,L[F(t)] = \frac{a}{p^2+1}$$

$$L[F(t)]\left[1+\frac{2p}{p^2+1}\right] = \frac{a}{p^2+1}$$

$$L[F(t)]\left[\frac{p^2+1+2p}{p^2+1}\right] = \frac{a}{p^2+1}$$

$$L[F(t)]\left[\frac{(p+1)^2}{p^2+1}\right] = \frac{a}{p^2+1}$$

$$L[F(t)] = \frac{a}{p^2+1} \times \frac{p^2+1}{(p+1)^2}$$

$$L[F(t)] = \frac{a}{(p+1)^2} \qquad \left[\because \frac{n!}{(p+a)^{n+1}} = e^{-at}t^n\right]$$

$$F(t) = a\, L^{-1} \frac{1}{(p+1)^2}$$

$$F(t) = a\, e^{-t}.t$$

$$\therefore F(t) = a t e^{-t}$$

2.

[4] Solve the integral eqn $\int_0^t F(u)\,F(t-u)\,du = 16 \sin 4t$ (or)

$\int_0^t y(u)\,y(t-u)\,du = 16 \sin 4t$.

Given integral eqn is

$$\int_0^t F(u)\,F(t-u)\,du = 16 \sin 4t$$

Now the given integral eqn may be expressed as

$$F(t) * F(t) = 16 \sin 4t.$$

Taking laplace transforms on both sides.

$$L[F(t) * F(t)] = 16\ L(\sin 4t)$$

$$L[F(t)] * L[F(t)] = 16 \frac{4}{p^2+4^2}$$

$$[L(F(t))]^2 = \frac{64}{p^2+4^2}$$

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$$L[F(t)] = \pm\sqrt{\frac{64}{p^2+4^2}}$$

$$L[F(t)] = \pm\frac{8}{\sqrt{p^2+16}}$$

$$F(t) = \pm 8 L^{-1}\left(\frac{1}{\sqrt{p^2+16}}\right)$$

$F(t) = \pm 8 J_0(4t)$ [$\because$ where $J_0(t)$ is the bessel function of order zero]

$\therefore$ The solution is $F(t) = \pm 8 J_0[4(t)]$

Note :-

* $J_0(at) = L^{-1}\left(\frac{1}{\sqrt{p^2+a^2}}\right)$

[5] Solve the integral eqn

$$\int_0^t \frac{f(u)}{\sqrt{t-u}}\, du = 1+t+t^2$$

Given integral eqn is

$$\int_0^t \frac{f(u)}{\sqrt{t-u}}\, du = 1+t+t^2$$

$$\int_0^t \frac{f(u)}{(t-u)^{1/2}}\, du = 1+t+t^2$$

$$\int_0^t f(u)\,(t-u)^{-1/2}\, du = 1+t+t^2$$

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this eq^n can be expressed as

$$F(t) * t^{-1/2} = 1 + t + t^2$$

taking laplace transform on both sides

$$L[f(t)] \cdot L(t^{-1/2}) = L(1) + L(t) + L(t^2)$$

$$L f(t) \frac{\Gamma -1/2+1}{p^{-1/2+1}} = 1/p + 1/p^2 + 2/p^3$$

$$L f(t) \frac{\Gamma(1/2)}{p^{1/2}} = 1/p + 1/p^2 + 2/p^3$$

$$L f(t) \sqrt{\pi} = \frac{p^{1/2}}{p} + \frac{p^{1/2}}{p^2} + \frac{2p^{1/2}}{p^3}$$

$$L[f(t)] = \frac{1}{\sqrt{\pi}} \left[1/p^{1/2} + \frac{1}{p^{3/2}} + \frac{2}{p^{5/2}} \right]$$

$$F(t) = 1/\sqrt{\pi} \left[L^{-1}(1/p^{1/2}) + L^{-1} \frac{1}{p^{3/2}} + L^{-1} \frac{2}{p^{5/2}} \right] \qquad \left(\because L^{-1}\left[\frac{1}{p^{n+1}}\right] = \frac{t^n}{\sqrt{(n+1)}} \right)$$

$$F(t) = \frac{1}{\sqrt{\pi}} \left[L^{-1} \frac{1}{p^{-1/2+1}} + L^{-1} \frac{1}{p^{1/2+1}} + L^{-1} \frac{2}{p^{3/2+1}} \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[\frac{t^{-1/2}}{\Gamma -1/2+1} + \frac{t^{1/2}}{\Gamma 1/2+1} + 2 \frac{t^{3/2}}{\Gamma 3/2+1} \right]$$

$$F(t) = 1/\sqrt{\pi} \left[\frac{t^{-1/2}}{\sqrt{\pi}} + \frac{t^{1/2}}{1/2 \sqrt{\pi}} + 2 \frac{t^{3/2}}{3/2 \, \Gamma 3/2} \right]$$

$$F(t) = \frac{t^{-1/2}}{\pi} + \frac{2 t^{1/2}}{\pi} + 4/3 \frac{t^{3/2}}{1/2 \sqrt{\pi} \cdot \sqrt{\pi}}$$

$$F(t) = 1/\pi \left[t^{-1/2} + 2 t^{1/2} + 8/3 \, t^{3/2} \right]$$

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formulas :-

Some important formulas :-

[i] If $n>0$, $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$

[ii] $\Gamma n+1 = n\Gamma n$ if $n>0$ and $\Gamma n = n!$ $\forall n \in Z^+$

[iii] $\Gamma 1 = 1$, $\Gamma(1/2) = \sqrt{\pi}$

$\Gamma 3/2 = \Gamma(1/2+1) = 1/2\,\Gamma 1/2 = 1/2\sqrt{\pi}$

[iv] $\Gamma n\ \cancel{(1-n)}\,\Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)}$ if $0<n<1$ (or) $\Gamma n\,\Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)}$

[v] $L(t^n) = \dfrac{n!}{p^{n+1}} = \dfrac{\Gamma(n+1)}{p^{n+1}}$

[vi] $L^{-1}\left(\dfrac{1}{p^{n+1}}\right) = \dfrac{t^n}{\Gamma(n+1)}$

[vii] $J_0(at) = L^{-1}\dfrac{1}{\sqrt{p^2+a^2}}$

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3.

[6] Solve the integral eqn

$$\int_0^t \frac{f(u)}{(t-u)^{1/3}}\,du = t(1+t)$$

Given integral eqn is

$$\int_0^t \frac{F(u)}{(t-u)^{1/3}}\,du = t(1+t)$$

$$\int_0^t F(u)(t-u)^{-1/3}\,du = t+t^2$$

This eqn can be expressed as

$$F(t) * t^{-1/3} = t+t^2$$

taking laplace transforms on both sides

$$L\,F(t) \cdot L(t^{-1/3}) = L(t) + L(t^2)$$

$$L[F(t)]\,\frac{(-1/3)!}{p^{-1/3+1}} = \frac{1}{p^2} + \frac{2}{p^3} \qquad \left[\because \frac{n!}{p^{n+1}} = \frac{\Gamma(n+1)}{p^{n+1}}\right]$$

$$L\,F(t)\,\frac{\Gamma(-1/3+1)}{p^{-1/3+1}} = \frac{1}{p^2} + \frac{2}{p^3} \qquad \left[\because \Gamma(-1/3+1) = \Gamma(2/3)\right]$$

$$L[F(t)]\,\Gamma(2/3) = \frac{p^{2/3}}{p^2} + \frac{2p^{2/3}}{p^3}$$

$$L[F(t)]\,\Gamma(2/3) = \frac{1}{p^{4/3}} + \frac{2}{p^{7/3}}$$

$$L[F(t)] = \frac{1}{\Gamma(2/3)}\left[\frac{1}{p^{1/3+1}} + \frac{2}{p^{4/3+1}}\right]$$

$$F(t) = \frac{1}{\Gamma(2/3)}\left[L^{-1}\left(\frac{1}{p^{1/3+1}}\right) + 2\,L^{-1}\left(\frac{1}{p^{4/3+1}}\right)\right]$$

formulas

$$\therefore L^{-1}\left(\frac{1}{p^{n+1}}\right) = \frac{t^n}{\Gamma(n+1)}$$

$$\therefore \Gamma(n+1) = n\Gamma n$$

$$= \frac{1}{\Gamma(2/3)}\left[\frac{t^{1/3}}{\Gamma(1/3+1)} + 2\cdot\frac{t^{4/3}}{\Gamma(4/3+1)}\right]$$

$$= \frac{1}{\Gamma(2/3)}\left[\frac{t^{1/3}}{1/3\,\Gamma(1/3)} + 2\frac{t^{4/3}}{4/3\,\Gamma 4/3}\right]$$

$$= \frac{1}{\Gamma(2/3)}\left[\frac{t^{1/3}}{1/3\,\Gamma 1/3} + \frac{2t^{4/3}}{4/3\,\Gamma 1/3+1}\right]$$

$$= \frac{1}{\Gamma(2/3)}\left[\frac{t^{1/3}}{1/3\,\Gamma 1/3} + \frac{2t^{4/3}}{4/3\cdot 1/3\,\Gamma 1/3}\right]$$

$$= \frac{1}{\Gamma(2/3)\,\Gamma(1/3)}\left[3t^{1/3} + \frac{9t^{4/3}}{2}\right]$$

$$F(t) = \frac{1}{\Gamma(1/3)\,\Gamma(1-1/3)}\left[3t^{1/3} + 9/2\, t^{4/3}\right] \qquad \left[\therefore \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1/\pi}{\sin \pi/3}\; 3\left[t^{1/3} + \frac{3t^{4/3}}{2}\right]$$

$$= \frac{3\sin 60^\circ}{\pi}\left[t^{1/3} + 3/2\, t^{4/3}\right]$$

$$= \frac{3\sqrt{3}}{2\pi}\left[t^{1/3} + \frac{3t^{4/3}}{2}\right]$$

$$F(t) = \frac{3\sqrt{3}}{4\pi}\left[2t^{1/3} + 3t^{4/3}\right]$$

$$\therefore F(t) = \frac{3\sqrt{3}\,t^{1/3}}{4\pi}(2+3t)$$

which is the required solution.

4.

[9] Solve $y(t) = t + \frac{1}{6}\int_0^t (t-u)^3 y(u)\,du$

Given integral eqⁿ is

$$y(t) = t + \frac{1}{6}\int_0^t (t-u)^3 y(u)\,du$$

this can be written as

$$y(t) = t + \frac{1}{6}\left(t^3 * y(t)\right)$$

taking laplace transforms on both sides

$$L[y(t)] = L(t) + \frac{1}{6}L(t^3) \cdot L[y(t)].$$

$$L[y(t)] = \frac{1}{p^2} + \frac{1}{6}\cdot\left(\frac{6}{p^4}\right) L[y(t)] \qquad \left[\because \frac{n!}{p^{n+1}} = t^n\right]$$

$$L[y(t)] = \frac{1}{p^2} + L[y(t)]\frac{1}{p^4}.$$

$$L[y(t)] - \frac{L[y(t)]}{p^4} = \frac{1}{p^2}$$

$$L[y(t)]\left[1 - \frac{1}{p^4}\right] = \frac{1}{p^2}.$$

$$L[y(t)]\left[\frac{p^4-1}{p^4}\right] = \frac{1}{p^2}.$$

$$L[y(t)] = \frac{1}{p^2} \times \frac{p^4}{p^4-1}$$

$$L[y(t)] = \frac{p^2}{(p^2)^2 - 1^2}$$

$$= \frac{p^2}{(p^2-1)(p^2+1)}$$

$$L[y(t)] = \frac{p^2-1+1}{(p^2-1)(p^2+1)} = \frac{\cancel{p^2-1}}{(\cancel{p^2-1})(p^2+1)} + \frac{1}{(p^2-1)(p^2+1)}$$

$$L[y(t)] = \frac{1}{p^2+1} + \frac{1}{2}\left[\frac{1}{p^2-1} - \frac{1}{p^2+1}\right].$$

$$= \frac{1}{p^2+1}\left[1 - \frac{1}{2}\right] + \frac{1}{2} \cdot \frac{1}{p^2-1}$$

$$= \frac{1}{2}\left[\frac{1}{p^2+1} + \frac{1}{p^2-1}\right]$$

taking inverse laplace transform

$$y(t) = \frac{1}{2}\left[L^{-1}\left(\frac{1}{p^2+1}\right) + L^{-1}\left(\frac{1}{p^2-1}\right)\right]$$

$$y(t) = \frac{1}{2}\left[\sin t + \sinh t\right]$$

$$y(t) = \frac{1}{2}\sin t + \frac{1}{2}\left(\frac{e^t - e^{-t}}{2}\right)$$

$$y(t) = \frac{1}{2}\sin t + \frac{1}{4}\left(e^t - e^{-t}\right)$$

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[12] using laplace transform to solve $y(t) = 1-e^{-t} + \int_0^t y(t-u)\sin u \, du$

Given integral eqn is

$$y(t) = 1-e^{-t} + \int_0^t y(t-u)\sin u\,du$$

we can written as

$$y(t) = 1-e^{-t} + y(t) * \sin t$$

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taking laplace transforms on both sides

$$L[y(t)] = L(1) - L(e^{-t}) + L[y(t)]\, L(\sin t)$$

$$L[y(t)] = 1/p - \frac{1}{p+1} + L[y(t)]\, \frac{1}{p^2+1^2}$$

$$L[y(t)] - L\, y(t) \frac{1}{p^2+1} = 1/p - \frac{1}{p^2+1}$$

$$L[y(t)]\left[1 - \frac{1}{p^2+1}\right] = \frac{p^2+1-p}{p(p+1)}$$

$$L[(y(t))]\left[\frac{p^2+1-1}{p^2+1}\right] = \frac{1}{p(p+1)}$$

$$L[y(t)] = \frac{1}{p(p+1)} \times \frac{p^2+1}{p^2}$$

$$L[y(t)] = \frac{p^2+1}{p^3(p+1)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{p+1} \longrightarrow (1)$$

$$p^2+1 = Ap^2(p+1) + Bp(p+1) + C(p+1) + Dp^3.$$

$$p^2+1 = Ap^3 + Ap^2 + Bp^2 + Bp + Cp + C + Dp^3.$$

p^3 :- $A + D = 0$, p^2 :- $A + B = 1$, p° :- $B + C = 0$, con :- $C = 1$

$2 + D = 0$ $A - 1 = 1$ $B = -1$

$D = -2$ $A = 2$

$$L[y(t)] = \frac{2}{p} + \frac{-1}{p^2} + \frac{1}{p^3} + \frac{-2}{p+1}$$

$$L[y(t)] = 2/p - 1/p^2 + 1/p^3 - \frac{2}{p+1}$$

$$y(t) = 2L^{-1}(1/p) - L^{-1}(1/p^2) + \tfrac{1}{2} L^{-1}\left(\frac{2}{p^3}\right) - 2L^{-1}\left(\frac{1}{p+1}\right)$$

$$y(t) = 2(1) - t + 1/2\, t^2 - 2e^{-t}.$$

$$y(t) = 2(1 - e^{-t}) - t + \frac{t^2}{2}$$

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Problems related to integro differential eq^n :-

[13] Solve the following eq^n $F(t)$ with $F(0)=0$,

$$F'(t) = \sin t + \int_0^t F(t-u)\cos u \, du$$

Given differential eq^n is

$$F'(t) = \sin t + \int_0^t F(t-u)\cos u \, du \quad \text{where } F(0)=0.$$

It can be epressed as

$$F'(t) = \sin t + [F(t) * \cos t]$$

taking laplace transforms on both sides

$$L[F'(t)] = L(\sin t) + L[F(t)] \cdot L(\cos t)$$

$$p[F(p)] - F(0) = \frac{1}{p^2+1} + L[F(t)] \cdot \frac{p}{p^2+1} \qquad [\therefore L[f(t)] = f(p)]$$

$$p(L(F(t))) - F(0) = \frac{1}{p^2+1} + L(F(t)) \frac{p}{p^2+1}$$

$$L(F(t))\, p - L[F(t)] \frac{p}{p^2+1} = \frac{1}{p^2+1}$$

$$L[F(t)] \left[p - \frac{p}{p^2+1}\right] = \frac{1}{p^2+1}$$

$$L[F(t)] \left[\frac{p^3+\cancel{p}-\cancel{p}}{p^2+1}\right] = \frac{1}{p^2+1}$$

$$L[F(t)] = \frac{1}{p^2+1} \times \frac{p^2+1}{p^3}$$

$$L[F(t)] = 1/p^3.$$

inverse laplace transform.

$$F(t) = \tfrac{1}{2} L^{-1}\, {}^{2}/_{p^3}$$

$$F(t) = \tfrac{1}{2} t^2$$

[14] Solve $F'(t) = t + \int_0^t F(t-u)\cos u\, du$, $F(0) = 4$

Given differential eq^n is

$$F'(t) = t + \int_0^t F(t-u)\cos u\, du, \quad \text{where } F(0) = 4.$$

we can be written as

$$F'(t) = t + [F(t) * \cos t].$$

Taking laplace transform on both sides.

$$L[F'(t)] = L(t) + L[F(t)] \cdot L(\cos t)$$

$$p[\bar{F}(p)] - F(0) = 1/p^2 + L[F(t)]\frac{p}{p^2+1}$$

$$p\,L[F(t)] - 4 = 1/p^2 + L[F(t)]\frac{p}{p^2+1}$$

$$p\,L[F(t)] - L\,F(t)\cdot\frac{p}{p^2+1} = 1/p^2 + 4$$

$$L[F(t)]\left[p - \frac{p}{p^2+1}\right] = \left[\frac{1+4p^2}{p^2}\right]$$

$$L[F(t)]\left[\frac{p^3+p-p}{p^2+1}\right] = \frac{4p^2+1}{p^2}$$

$$L(F(t)) = \frac{4p^2+1}{p^2} \times \frac{p^2+1}{p^3}.$$

$$L[F(t)] = \frac{4p^4+5p^2+1}{p^5} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{p^4} + \frac{E}{p^5} \rightarrow (1)$$

$$4p^4+5p^2+1 = Ap^4 + Bp^3 + Cp^2 + Dp + E$$

p^4:- $A = 4$, p^3:- $B = 0$, p^2:- $C = 5$, p:- $D = 0$, cons:- $E = 1$

$$L[F(t)] = 4/p + 0/p^2 + 5/p^3 + 0/p^4 + 1/p^5$$

$$L[F(t)] = 4/p + 5/p^3 + 1/p^5$$

$$L[F(t)] = \frac{4}{p} + \frac{5}{2}\cdot\left(\frac{2}{p^3}\right) + \frac{1}{24}\left(\frac{24}{p^5}\right)$$

inverse laplace transform.

$$F(t) = 4L^{-1}\left(\frac{1}{p}\right) + \frac{5}{2}L^{-1}\left(\frac{2!}{p^3}\right) + \frac{1}{24}L^{-1}\frac{4!}{p^5}$$

$$F(t) = 4(1) + \frac{5}{2}t^2 + \frac{1}{24}t^4$$

$$F(t) = 4 + \frac{5t^2}{2} + \frac{t^4}{24}.$$

Convert the differential equations into integral equations with constant coefficients :-

[15] convert the differential eq^n

$F''(t) + 2F'(t) - 8F(t) = 5t^2 - 3t$, $F(0) = -2$, $F'(0) = 3$ into an integral eq^n.

(or)

$y''(t) + 2y'(t) - 8y(t) = 5t^2 - 3t$, $y(0) = -2$, $y'(0) = 3$.

Given that

$$F''(t) + 2F'(t) - 8F(t) = 5t^2 - 3t \longrightarrow (1)$$

Now integrating eq^n (1) b/w the limits '0' to 't' so, we get

$$\int_0^t F''(u)\,du + 2\int_0^t F'(u)\,du - 8\int_0^t F(u)\,du = 5\int_0^t u^2\,du - 3\int_0^t u\,du$$

$$\Rightarrow [F'(u)]_0^t + 2[F(u)]_0^t - 8\int_0^t F(u)\,du = 5\left(\frac{u^3}{3}\right)_0^t - 3\left(\frac{u^2}{2}\right)_0^t$$

$$\Rightarrow F'(t) - F'(0) + 2[F(t) - F(0)] - 8\int_0^t F(u)\,du = \frac{5}{3}(t^3 - 0) - \frac{3}{2}(t^2 - 0)$$

$$\Rightarrow F'(t) - 3 + 2F(t) - 2(-2) - 8\int_0^t F(u)\,du = \frac{5t^3}{3} - \frac{3t^2}{2}$$

$$\Rightarrow F'(t) + 2F(t) + 1 - 8\int_0^t F(u)\,du = \frac{5t^3}{3} - \frac{3t^2}{2}$$

Again integrating b/w the limits '0' to 't'

$$\int_0^t F'(u)\,du + 2\int_0^t F(u)\,du - 8\int_0^t (t-u)\,F(u)\,du = \frac{5}{3}\int_0^t u^3\,du - \frac{3}{2}\int_0^t u^2\,du - \int_0^t du$$

$$[F(u)]_0^t + \int_0^t F(u)\,du\,[2-8t+8u] = \frac{5}{3}\left(\frac{u^4}{4}\right)_0^t - \frac{3}{2}\left(\frac{u^3}{3}\right)_0^t - (u)_0^t$$

$$\Rightarrow F(t) - F(0) + \int_0^t F(u)\,(2-8t+8u)\,du = \frac{5t^4}{12} - \frac{t^3}{2} - t$$

$$\Rightarrow F(t) + \int_0^t F(u)\,(2-8t+8u)\,du = \frac{5t^4}{12} - \frac{t^3}{2} - t - 2$$

[16] convert the differential eq^n

$F''(t) - 3F'(t) + 2F(t) = 4\sin t$, $F(0) = 1$, $F'(0) = -2$ into

integral eq^n.

Given that

$F''(t) - 3F'(t) + 2F(t) = 4\sin t \rightarrow (1)$ $F(0) = 1$, $F'(0) = -2$

Now integrating eq^n (1) b/w the limits '0' to 't'

So, we get

$$\int_0^t F''(u)\,du - 3\int_0^t F'(u)\,du + 2\int_0^t F(u)\,du = 4\int_0^t \sin u\,du$$

$$[F'(u)]_0^t - 3[F(u)]_0^t + 2\int_0^t F(u)\,du = 4(-\cos t)_0^t$$

$$F'(t) - F'(0) - 3[F(t) - F(0)] + 2\int_0^t F(u)\,du = -4(\cos t - \cos 0)$$

$$F'(t) - (-2) - 3F(t) + 3(1) + 2\int_0^t F(u)\,du = -4(\cos t - 1)$$

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$$F'(t) - 3F(t) + 2\int_0^t F(u)du = -4\cos t + 4 - 5$$

again integraing b/w the limits '0' to 't'.

$$\int_0^t F'(u)du - 3\int_0^t F(u)du + 2\int_0^t (t-u)F(u)du = -4\int_0^t \cos u\, du - \int_0^t du$$

$$[F(u)]_0^t - \int_0^t F(u)du\,[3-2t+2u] = -4(\sin u)_0^t - (u)_0^t$$

$$F(t) - F(0) - \int_0^t (3-2t+2u)F(u)du = -4(\sin t - \sin 0) - t$$

$$F(t) - \int_0^t (3-2t+2u)F(u)du = -4\sin t - t + 1.$$

[17] Express $2F''(t) - 3F'(t) - 2F(t) = 4e^{-t} + 2\cos t$, $F(0) = 4$, $F'(0) = -1$.

in an integral eq^n.

Given differential eq^n is

$$2F''(t) - 3F'(t) - 2F(t) = 4e^{-t} + 2\cos t \rightarrow ①$$

Now integrating eq^n ① b/w the limits '0' to 't'

so, we get

$$2\int_0^t F''(u)du - 3\int_0^t F'(u)du - 2\int_0^t F(u)du = 4\int_0^t e^{-u}du + 2\int_0^t \cos u\, du.$$

$$2[F'(u)]_0^t - 3[F(u)]_0^t - 2\int_0^t F(u)du = 4\left(\frac{e^{-u}}{-1}\right)_0^t + 2(\sin u)_0^t$$

$$2[F'(t) - F'(0)] - 3[F(t) - F(0)] - 2\int_0^t F(u)du = -4(e^{-t} - e^{-0}) + 2(\sin t - \sin 0)$$

$$2F'(t) - 2(-1) - 3F(t) + 3(4) - 2\int_0^t F(u)du = -4(e^{-t} - 1) + 2\sin t$$

$$2F'(t) - 3F(t) - 2\int_0^t F(u)du = -14 - 4e^{-t} + 4 + 2\sin t$$

Again integrating b/w the limits '0' to 't'.

$$2\int_0^t F'(u)\,du - 3\int_0^t F(u)\,du - 2\int_0^t (t-u)F(u)\,du = -10\int_0^t du - 4\int_0^t e^{-u}\,du + 2\int_0^t \sin u\,du$$

$$2\left[F(u)\right]_0^t - \int_0^t F(u)\,du\,(3+2t-2u) = -10(u)_0^t - 4\left(\frac{e^{-u}}{-1}\right)_0^t + 2(-\cos u)_0^t$$

$$2[F(t)-F(0)] - \int_0^t (3+2t-2u)F(u)\,du = -10t + 4(e^{-t}-e^{0}) - 2(\cos t - \cos 0)$$

$$2F(t) - 2(4) - \int_0^t (3+2t-2u)F(u)\,du = -10t + 4e^{-t} - 4 - 2\cos t + 2$$

$$2F(t) - \int_0^t (3+2t-2u)F(u)\,du = -10t + 4e^{-t} - 2\cos t + 6.$$

Convert the differential eqⁿs into integral eqⁿs with varible coefficients :-

[08] convert $F''(t) - tF'(t) + t^2F(t) = 1+t$ and $F(0)=4$, $F'(0)=2$ into an integral eqⁿ.

Given differential eqⁿ is

$$F''(t) - tF'(t) + t^2F(t) = 1+t \longrightarrow (1)$$

$$\text{and } F(0)=4,\ F'(0)=2 \longrightarrow (2)$$

$$\text{Let } F''(t) = G(t) \longrightarrow (3)$$

Now integrating eqⁿ (3) w.r.to 't'.

$$\int_0^t F''(u)\,du = \int_0^t G(u)\,du + C_1 \longrightarrow (A)$$

$$F'(u) = \int_0^t G(u)\,du + C_1$$

$$F'(t) = \int_0^t G(u)\,du + c_1 \rightarrow (4)$$

Now Let $t=0 \Rightarrow F'(0) = c_1$

$\Rightarrow c_1 = 2$ subc in (4)

$$\int_0^t F''(u)\quad F'(t) = \int_0^t G(u)\,du + 2 \rightarrow (5)$$

Again integrating eqn (5) w.r.to 't'

$$\int F'(u)\,du = \int_0^t (t-u)\,G(u)\,du + 2\int_0^t du + c_2$$

$$F(t) = \int_0^t (t-u)\,G(u)\,du + 2t + c_2 \rightarrow (6)$$

Put $t=0$ in eqn (6)

$F(0) = 2(0) + c_2$

$c_2 = F(0)$

$c_2 = 4$ in eqn (6)

$$F(t) = \int_0^t (t-u)\,G(u)\,du + 2t + 4 \rightarrow (7)$$

subc in eqn (3), eqn (5), eqn (7) in eqn (1)

$$G(t) - t\left[\int_0^t G(u)\,du + 2\right] + t^2\left[\int_0^t (t-u)\,G(u)\,du + 2t + 4\right] = 1 + t$$

$$G(t) - t\int_0^t G(u)\,du + 2t + t^2\int_0^t (t-u)\,G(u)\,du + 2t^3 + 4t^2 = 1 + t$$

$$G(t) + \int_0^t G(u)\,du\,(t^3 - t^2u - t) = 1 + t + 2t - 2t^3 - 4t^2$$

$$G(t) + \int_0^t (t^3 - t^2u - t)\,G(u)\,du = 1 + 3t - 4t^2 - 2t^3$$

$$\therefore G(t) = 1 + 3t - 4t^2 - 2t^3 - \int_0^t (t^3 - ut^2 - t)\,G(u)\,du.$$