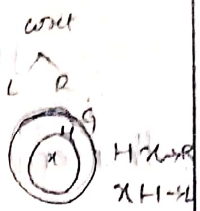
UNIT-3

MATHS - SEM3

LONGS

① A Subgroup H of a group G is Normal iff the each left coset of H in G is a right coset of H in G.



Proof:

Let $(G, \cdot)$ be a group.

Let $(H, \cdot)$ be a Subgroup of G.

The condition is necessary.

Let us suppose that H is Normal.

Now we have to prove that each left coset of H in G is a right coset of H in G.

$$xHx^{-1} = H \quad [\because \text{Theorem ①}]$$

Multiply 'x' on both sides.

$$xHx^{-1}(x) = Hx$$

$$xH(x^{-1}x) = Hx \quad [\because \text{By Associative}]$$

$$xH(e) = Hx$$

$$xH = Hx.$$

The condition is sufficient.

Conversely suppose that each left coset of H in G is a right coset of H in G.

Now we have to prove that H is normal

For $x \in G$,

$$xH = Hy \quad \forall\, y \in G.$$

$$x \in G \Rightarrow x \in H$$

$$x = xe$$

$$x \in xH \quad [\because e \text{ is identity element of } H]$$

$Hx = Hy$ [∵ cosets, theorem 3] [∵ $a \in \cancel{Hy}\ Hb$ then $Ha = Hb$]

$Hx = xH$

Multiply with x^{-1} on both sides.

$$Hxx^{-1} = xHx^{-1}$$

$$H = xHx^{-1}$$ [∵ theorem 4]

∴ H is a Normal.

Theorem : ②

A Subgroup H of group G is normal iff the product of two right cosets of H in G is a right coset of H in G.

Proof :-

Let $(G, \cdot)$ be a group.

Let $(H, \cdot)$ be a subgroup of G.

The condition is necessary.

Let us suppose that H is normal.

Now we have to prove that the product of two right cosets of H in G is also a right coset of H in G.

$$a, b \in G \Rightarrow ab \in G \quad [\because \text{By closure law}]$$

Ha, Hb, Hab are the right cosets of H in G.

$$Ha \cdot Hb = H(aH)b$$

$$= H(Ha)b \quad [\because \text{By theorem ①}]$$

$$= (HH)ab$$

$$= Hab \quad [\because \text{By th. ③ in S.G}]$$

The condition is sufficient.

Scanned with CamScanner

Conversely Suppose that the product of two right cosets of H in G is also a right cosets of H in G is also a right coset of H in G.

Now we have to prove that H is Normal.

$x \in G$, $h \in H$.

$xhx^{-1} = (ex)hx^{-1}$

$\in (Hx)(Hx^{-1})$ [$\because$ e is the identity in H]

$\in H(xx^{-1})$

$\in He$

$xhx^{-1} \in H$.

$\therefore$ H is Normall.

(3) If f is a homomorphic from a group G into G' then Prove that kernal of Homomorphism is a normal Subgroup.

Proof :-

i) kerf $\neq \phi$

ii) kerf → S.G (ab^{-1})

iii) kerf → N.S.G

Let $(G, \cdot)$ $(G', \cdot)$ to be two groups

Let e, e' are to be two identity elements of G, G'.

Let f be Homomorphism from G into G'.

By the definition of kernal f,

$$\text{Kerf} = \{ f(x) = e' \;/ x \in G \}.$$

Now ~~we have~~ to prove that Kerf is a Normal Subgroup.

We know that,

$f(e) = e'$ [$\because$ Th. No. (1)]

$\{e\} \in$ kerf

$\therefore$ kerf $\neq \phi$.

[a, b $\in$ k].

$f(a) = e'$, $f(b) = e'$

$f(ab^{-1}) = f(a) \cdot f(b^{-1})$ [$\because$ f is homo]

$= f(a) \cdot [f(b)]^{-1}$ [$\because$ By th (1)]

$= e' (e')^{-1}$

$f(ab^{-1}) = e'$

$ab^{-1} \in \ker f$.

$\therefore \ker f$ is a Subgroup.

$f(xax^{-1}) = f(x) f(a) f(x^{-1})$

$= f(x) . e' [f(x)]^{-1}$

$= f(x) . [f(x)]^{-1}$

$f(xax^{-1}) = e'$.

$xax^{-1} \in \ker f$

$\therefore \ker f$ is a Normal Sub group.

Theorem : 7

(4) State and prove Fundamental theorem of Homomorphism of Groups.

Statement :- Every Homomorphic Image of a group G is Isomorphic to its quotient group.

Proof :

Let $(G,\cdot)$, $(G',\cdot)$ are to be two groups.

Let e, e' are two identity elements of G, G'.

Let f is a Homomorphism from G into G'

By the definition of Homomorphic Image,

$f(G) = Im(f) = \{ f(a) = a' / a \in G \}$; $f(G) \subseteq G'$.

We know that Homomorphic Image of a group is a group.

By the definition of ker f,

$\ker f = K = \{ f(x) = e' / x \in G \}$

We know that ker f is a Normal subgroup.

$\therefore \left(\frac{G}{K}, \cdot\right)$ is a quotient group

Define : $\phi : \frac{G}{K} \to G'$ such that $\phi(Ka) = f(a); \forall a, b \in G$.

Now we have to Prove that $\frac{G}{K} \cong G'$.

To prove ϕ is well-defined :

For $a, b \in G$, $Ka, Kb \in \frac{G}{K}$.

$$Ka = Kb.$$

$\Rightarrow ab^{-1} \in K$ [$\because$ By th ② on cosets] $Ha = Hb \Leftrightarrow ab^{-1} \in H$.

$f(ab^{-1}) = e'$

$f(a) \cdot f(b^{-1}) = e'$

Multiply with $f(b)$ on both sides

$f(a)[f(b)]^{-1} f(b) = e' f(b)$

$f(a) = f(b)$

$\phi(Ka) = \phi(Kb)$

$\therefore$ ϕ is well defined

To prove ϕ is one - one :

For $a, b \in G \Rightarrow Ka, Kb \in \frac{G}{K}$.

$\phi(Ka) = \phi(Kb)$

$f(a) = f(b)$.

Multiply with $[f(b)]^{-1}$ on both sides

$f(a) \cdot [f(b)]^{-1} = f(b)[f(b)]^{-1}$

$f(a) \cdot f(b^{-1}) = e'$

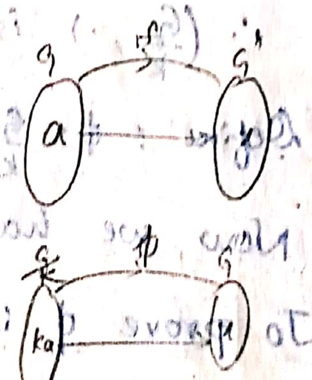
$f(ab^{-1}) = e'$

$ab^{-1} \in K$

$Ka = Kb$

$\therefore$ ϕ is one - one.

To prove ϕ is onto:



Let $x \in G'$

$\exists\, a \in G$ such that $f(a) = x$

$\exists\, Ka \in \frac{G}{K}$ such that $\phi(Ka) = f(a)$.

$\therefore$ ϕ is onto.

To prove ϕ is Homomorphism:

$[Ka, Kb \in \frac{G}{K}]$

$\phi(Ka \cdot Kb) = \phi(Kab)$ $[\because$ Coset Multi. defined in $\frac{G}{K}]$

$= f(ab)$

$= f(a) \cdot f(b)$ $[\because$ Since f is Homo$]$

$\phi(Ka \cdot Kb) = \phi(Ka) \cdot \phi(Kb)$

$\therefore$ ϕ is Homomorphism

$\therefore$ ϕ is Isomorphism