UNIT-V.

VECTOR INTEGRATION APPLICATIONS

INTEGRAL TRANSFORMS

Theorem 1: State and prove Gauss divergence theorem.

Statement:- Let S be a closed surface enclosed a volume V. If F is continuously differentiable vector point function then,

$\int_V \text{div} F \, dv = \int_S F \cdot N \, ds$ where 'N' is the outward drawn ~~unique~~ unit normal vector.

Cartesian form:- Let $F = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$N = \cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k}$

where $\cos\alpha, \cos\beta, \cos\gamma$ are d.c's of N,

$F \cdot N = (F_1\bar{i} + F_2\bar{j} + F_3\bar{k}) \cdot (\cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k})$

$= F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$

$\text{div} F = \nabla \cdot F = \left(i \cdot \frac{\partial}{\partial x} + \bar{j} \cdot \frac{\partial}{\partial y} + k \cdot \frac{\partial}{\partial z}\right) \cdot (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})$

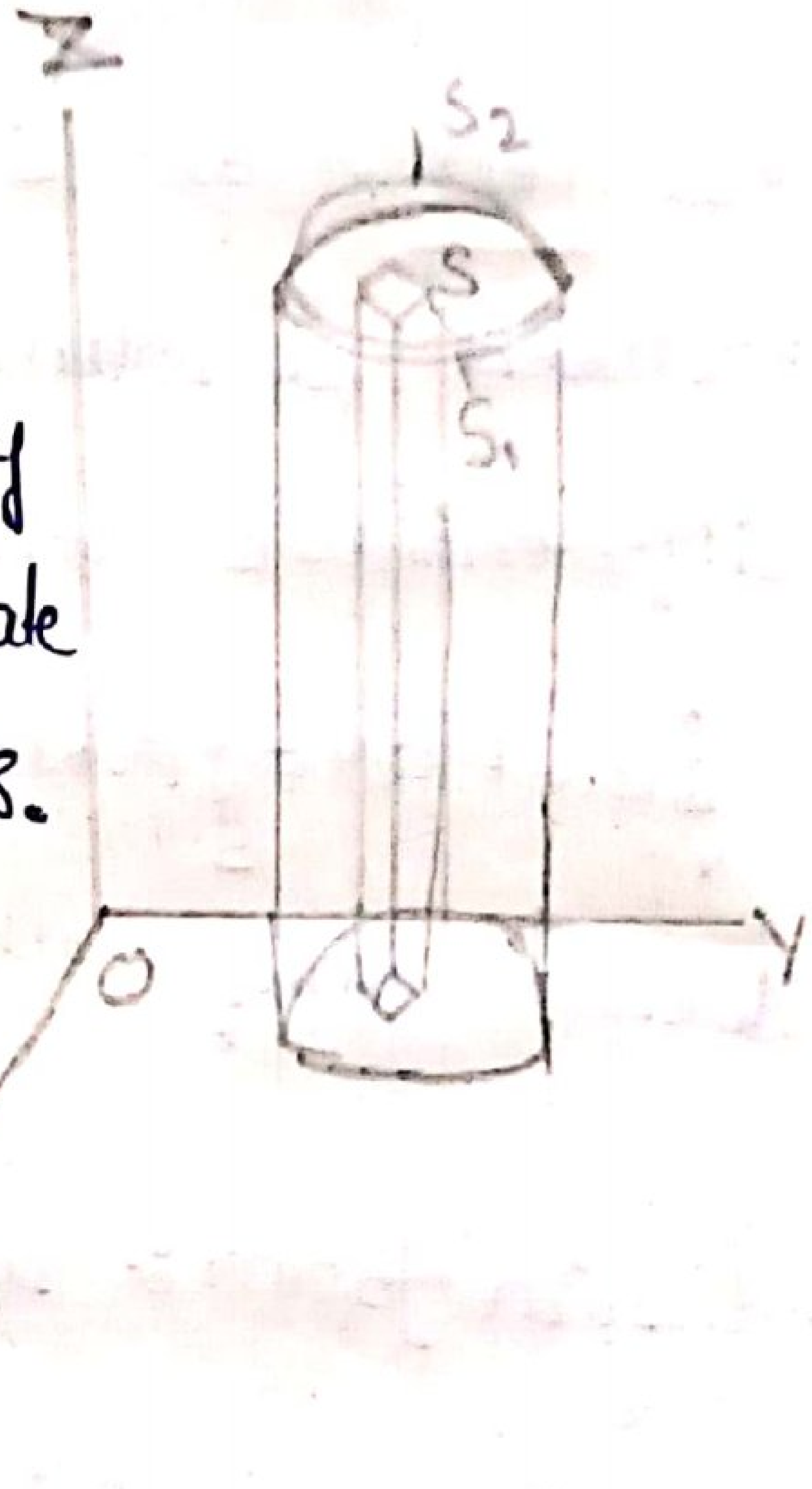
$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$\int_V \operatorname{div} F \, dv = \int_S F \cdot N \, ds$$

$$\iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, ds$$

$$= \iint_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy$$

<u>Proof</u>:- Let 'S' be a closed surface. Let us choose the co-ordinate axes so that any line parallel to the co-ordinate axes cuts S in atmost two points. Let R be the projection of S on XY plane.



S_1 and S_2 be the lower and upper parts of S.

$z = f(x,y)$ and $z = g(x,y)$ be the equations of S_1 and S_2. The relation can be put in the form $f(x,y) \le z \le g(x,y)$.

$$\int_V \frac{\partial F_3}{\partial z}\, dv = \iiint \frac{\partial F_3}{\partial z}\, dx\,dy\,dz$$

$$= \iint_R dx\,dy \int_{z=f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}\, dz$$

$$= \iint_R dx\,dy \left[F_3(x,y,z)\right]_f^g$$

$$= \iint_R dx\,dy \left[F_3(x,y,g) - F_3(x,y,f)\right]$$

$$= \iint_R F_3(x,y,g)\,dx\,dy - \iint_R F_3(x,y,f)\,dx\,dy \rightarrow ①$$

For the upper part S_2 $dx\,dy = ds \cos\gamma = N \cdot \bar{k}\, ds$

Since the normal S_2 makes an acute angle γ with $\bar{k}$.

$$\iint_R F_3(x,y,g)\,dx\,dy = \int_{S_2} F_3 \cdot N \cdot \bar{k}\, ds.$$

For the lower part S_1 $dx\,dy = -\cos\gamma\, ds$

$$= -N \cdot \bar{k}\, ds$$

Since the normal S_1 makes an obtuse angle γ with $\bar{k}$.

$$\iint_R F_3(x,y,f)\,dx\,dy = -\int_{S_1} F_3\, N \cdot \bar{k}\, ds.$$

$$\int_V \frac{\partial F_3}{\partial z}\,dv = \iint_R F_3(x,y,g)\,dx\,dy - \iint_R F_3(x,y,f)\,dx\,dy$$

$$= \int_{S_2} F_3\, N \cdot k\, ds + \int_{S_1} F_3 \cdot N \cdot k\, ds$$

$$\int_V \frac{\partial F_3}{\partial z}\,dv = \int_S F_3\, N \cdot k\, ds \rightarrow ②$$

Similarly, $\int_V \frac{\partial F_2}{\partial y}\, dv = \int_S F_2\, N\cdot \bar{j}\, ds \rightarrow (3)$

$$\int_V \frac{\partial F_1}{\partial x}\, dv = \int_S F_1\, N\cdot \bar{i}\, ds \rightarrow (4)$$

Adding (2), (3), (4),

$$\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv = \int_S (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\cdot N\, ds$$

$$\int_V \nabla\cdot F\, dv = \int_S F\cdot N\, ds$$

$$\int_V \operatorname{div} F\, dv = \int_S F\cdot N\, ds.$$

Theorem 2: State and Prove Greens theorem in a plane.

Statement: Let S be a closed region in xy plane enclosed by a curve c. Let p and q be continuous and scalar functions of x and y then

$$\oint_c P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Proof:- Let any line parallel to either co-ordinate axis cuts 'c' in atmost two points, Let 'S' be between the line $x=a,\ x=b$ & $y=c,\ y=d$

Let $y=f(x)$ be the curve C_1 is AEB.

Let $y=g(x)$ be the curve C_2 is ADB.

where $f(x) \leq g(x)$

Consider $\iint_S \frac{\partial P}{\partial y}\, dx\, dy = \int_{x=a}^{b} dx \int_{y=f(x)}^{y=g(x)} \frac{\partial P}{\partial y}\, dy$

$$= \int_{x=a}^{b} dx \left[P(x,y)\right]_{y=f(x,y)}^{y=g(x,y)}$$

$$= \int_{x=a}^{b} \left[P(x,y) - P(x,f)\right] dx$$

$$= \int_{x=a}^{b} P(x,y)\, dx - \int_{x=a}^{b} P(x,f)\, dx$$

$$= \iint_S \frac{\partial P}{\partial y}\, dx\, dy = -\int_a^b P(x,y)\, dx - \int_a^b P(x,y)\, dy$$

$$= -\int_{C_2} P(x,y)\, dx - \int_{C_1} P(x,y)\, dy$$

$$\Rightarrow \iint_S \frac{\partial P}{\partial y}\, dx\, dy = -\oint_C P\, dx$$

$$= -\iint_S \frac{\partial P}{\partial y}\, dx\, dy = \oint P\, dx \rightarrow (1)$$

$|||^{ly}$ $\iint_S \frac{\partial Q}{\partial y}\, dx\, dy = \oint Q\, dy \rightarrow (2)$

Let 'S' be between the line $x=a, x=b$ & $y=c, y=d$

let $y=f(x)$ be the curve C_1 is AEB.

let $y=g(x)$ be the curve C_2 is ADB.

where $f(x) \leq g(x)$

Consider $\iint_S \frac{\partial P}{\partial y} dx\,dy = \int_{x=a}^{b} dx \int_{y=f(x)}^{y=g(x)} \frac{\partial P}{\partial y} dy$

$= \int_{x=a}^{b} dx \left[P(x,y)\right]_{y=f(x,y)}^{y=g(x,y)}$

$= \int_{x=a}^{b} \left[P(x,y) - P(x,f)\right] dx$

$= \int_{x=a}^{b} P(x,y)\,dx - \int_{x=a}^{b} P(x,f)\,dx$

$= \iint_S \frac{\partial P}{\partial y} dx\,dy = -\int_a^b P(x,y)\,dx - \int_a^b P(x,y)\,dy$

$= -\int_{C_2} P(x,y)\,dx - \int_{C_1} P(x,y)\,dy$

$\Rightarrow \iint_S \frac{\partial P}{\partial y} dx\,dy = -\oint_C P\,dx$

$= -\iint_S \frac{\partial P}{\partial y} dx\,dy = \oint P\,dx \rightarrow ①$

Illy $\iint_S \frac{\partial Q}{\partial y} dx\,dy = \oint Q\,dy \rightarrow ②$

Adding ① + ②,

$$\oint_C (P\,dx + Q\,dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Theorem 3: State and Prove Stokes theorem

Statement:-

let 'S' be a surface bounded by 'A' closed non-intersecting curve 'c'. If 'f' is any differentiable vector point function then,

$\oint_c F \cdot dr = \int_S curl\ f \cdot N ds$ where 'c' is traversed in the positive direction. The direction of 'c' is called the boundary of S in this direction with his head pointing in the direction of outward dracon normal 'N' to 'S' as the surface on its left.

Cartesian form :-

let $F = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$

$N = \cos\alpha \bar{i} + \cos\beta \bar{j} + \cos\gamma \bar{k}$

$d\bar{r} = dx \bar{i} + dy \bar{j} + dz \bar{k}$

$$F \cdot dr = (F_1\bar{i} + F_2\bar{j} + F_3\bar{k}) \cdot (dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$= F_1\,dx + F_2\,dy + F_3\,dz$$

curl $F = \nabla \times F$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \bar{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \bar{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

curl $F \cdot N\,ds = (\nabla \times F) \cdot N\,ds$

$$= \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] ds$$

Stokes theorem is equivalent to $\int_C F \cdot dr = \int_S \text{curl } F \cdot N\,ds$

$$\int_C F_1\,dx + F_2\,dy + F_3\,dz = \int_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] ds$$

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Proof:- Let 'S' be a surface which is such that its projections on xy, yz, zx planes are regions bounded by the simple closed curves.

Let 'S' have equations $z = f(x,y)$, or $x = g(y,z)$ or $y = h(z,x)$ where f, g, h are simple values continuous and differentiable functions.

Now to prove $\int_S \text{curl } F \cdot N\,ds = \int_S (\nabla \times F) \cdot N\,ds$

$$= \int_S [\nabla \times (F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k})] \cdot N\,ds$$

Let us consider,

$$\int_S (\nabla \times F_1 \bar{i}) \cdot N\,ds$$

$$\nabla \times F_1 \bar{i} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix}$$

$$= -\bar{j}\left[0 - \frac{\partial F_1}{\partial z}\right] + \bar{k}\left[0 - \frac{\partial F_1}{\partial y}\right]$$

$$= \frac{\partial F_1}{\partial z}\bar{j} - \frac{\partial F_1}{\partial y}\bar{k}$$

$$(\nabla\times F_1 i)\cdot N\,ds = \left(\frac{\partial F_1}{\partial z}j - \frac{\partial F_1}{\partial y}k\right)\cdot N\,ds$$

$$(\nabla\times F_1 i)\cdot N\,ds = \left[\frac{\partial F_1}{\partial z}N\cdot j - \frac{\partial F_1}{\partial y}N\cdot \bar{k}\right]ds \rightarrow (1)$$

Let $\bar{z} = f(x,y)$ be the eq^n of S for any point s,

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\bar{r} = x\bar{i} + y\bar{j} + f(x,y)\bar{k}$$

$$\frac{\partial \bar{r}}{\partial y} = \bar{j} + \frac{\partial z}{\partial y}\bar{k}$$

$\frac{\partial \bar{r}}{\partial y}$ is tangent vector of S, $N\cdot\frac{\partial \bar{r}}{\partial y} = 0$

$$N\cdot\frac{\partial \bar{r}}{\partial y} = N\cdot\bar{j} + \frac{\partial z}{\partial y}N\cdot\bar{k}$$

$$0 = N\cdot\bar{j} + \frac{\partial z}{\partial y}N\cdot\bar{k}$$

$$N\cdot\bar{j} = -\frac{\partial z}{\partial y}N\cdot k$$

Substitute $N\cdot\bar{j}$ in eq^n (1),

$$\Rightarrow (\nabla\times F_1 i)\cdot N\,ds = \left[\frac{\partial F_1}{\partial z}\left(-\frac{\partial z}{\partial y}\right)N\cdot\bar{k} - \frac{\partial F_1}{\partial y}(N\cdot k)\right]ds$$

$$\Rightarrow (\nabla\times F_1 i)\cdot N\,ds = -\left[\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y}\right](N\cdot k)\,ds$$

$$(\nabla \times F_1 \bar{i}) \cdot N\, ds = -\frac{\partial F_1}{\partial y} N \cdot k\, dS = \frac{-\partial F_1}{\partial y} \cos\gamma\, ds$$

$$= \frac{\partial F_1}{\partial y} dx\, dy$$

Let 'R' be the projection of S on xy-plane and σ be the boundary of R.

$$\therefore \int_S (\nabla \times F_1 \bar{i}) \cdot N\, ds = \iint_R -\frac{\partial F_1}{\partial y} N \cdot k\, dx\, dy$$

By Greens theorem in xy-plane,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$\boxed{P = F_1, \; Q = 0}$

$$= \iint_R \left(0 - \frac{\partial F_1}{\partial y}\right) dx\, dy = \int_C F_1\, dx + 0\, dy$$

$$\int_S (\nabla \times F_1 \bar{i}) \cdot N\, ds = \oint_C F_1\, dx \rightarrow (2)$$

Similarly,

$$\int_S (\nabla \times F_2 j) \cdot N\, ds = \oint_C F_2\, dy \rightarrow (3)$$

$$\int_S (\nabla \times F_3 k) \cdot N\, ds = \oint_C F_3\, dz \rightarrow (4)$$

Adding eq^n's (2), (3), (4) we get,

$$\int_S \nabla \times (F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}) \cdot N\, ds = \oint_C F_1\, dx + F_2\, dy + F_3\, dz$$

$$\int_S (\nabla \times F) \cdot N\,ds = \oint_C F \cdot dr$$

$$\therefore \int_S \text{curl}\, F \cdot N\,ds = \oint_C F \cdot dr.$$

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2) Verify gauss divergence theorem to evaluate $\int_S (x^3-yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}) \cdot N ds$ over the surface of a cube bounded by the co-ordinate planes $x=y=z=a$

Sol:- By gauss theorem $\int_S F.N ds = \int_V \text{div } F dv$

$$F=(x^3-yz)\bar{i}-2x^2y\bar{j}+z\bar{k}$$

$$F=F_1\bar{i}+F_2\bar{j}+F_3\bar{k}$$

$F_1 = x^3-yz$, $F_2=-2x^2y$, $F_3=z$

$$RHS = \int_V \text{div } F.dv = \int_V \nabla . F dv$$

$$= \int_V \left(\frac{\partial}{\partial x}\bar{i}+\frac{\partial}{\partial y}\bar{j}+\frac{\partial}{\partial z}\bar{k}\right).\left((x^2-yz)\bar{i}-2x^2y\bar{j}+z\bar{k}\right)dv$$

$$= \int_V (3x^2-2x^2+1)dv$$

$$= \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a} (x^2+1)dv$$

$$= \left(\frac{x^3}{3}+x\right)_0^a (y)_0^a (z)_0^a$$

$$= \left(\frac{a^3}{3}+a\right)a^2 = \frac{a^5}{3}+a^3$$

$$= \cancel{\frac{a^5}{3}} + \cancel{6a^5} \quad \cancel{\frac{a^3}{3}+a^5}$$

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* Let us calculate directly the value of $\int_S F\cdot N\, dS$ over the ~~surface~~ six faces of the cube,

(i) for the face PQAR

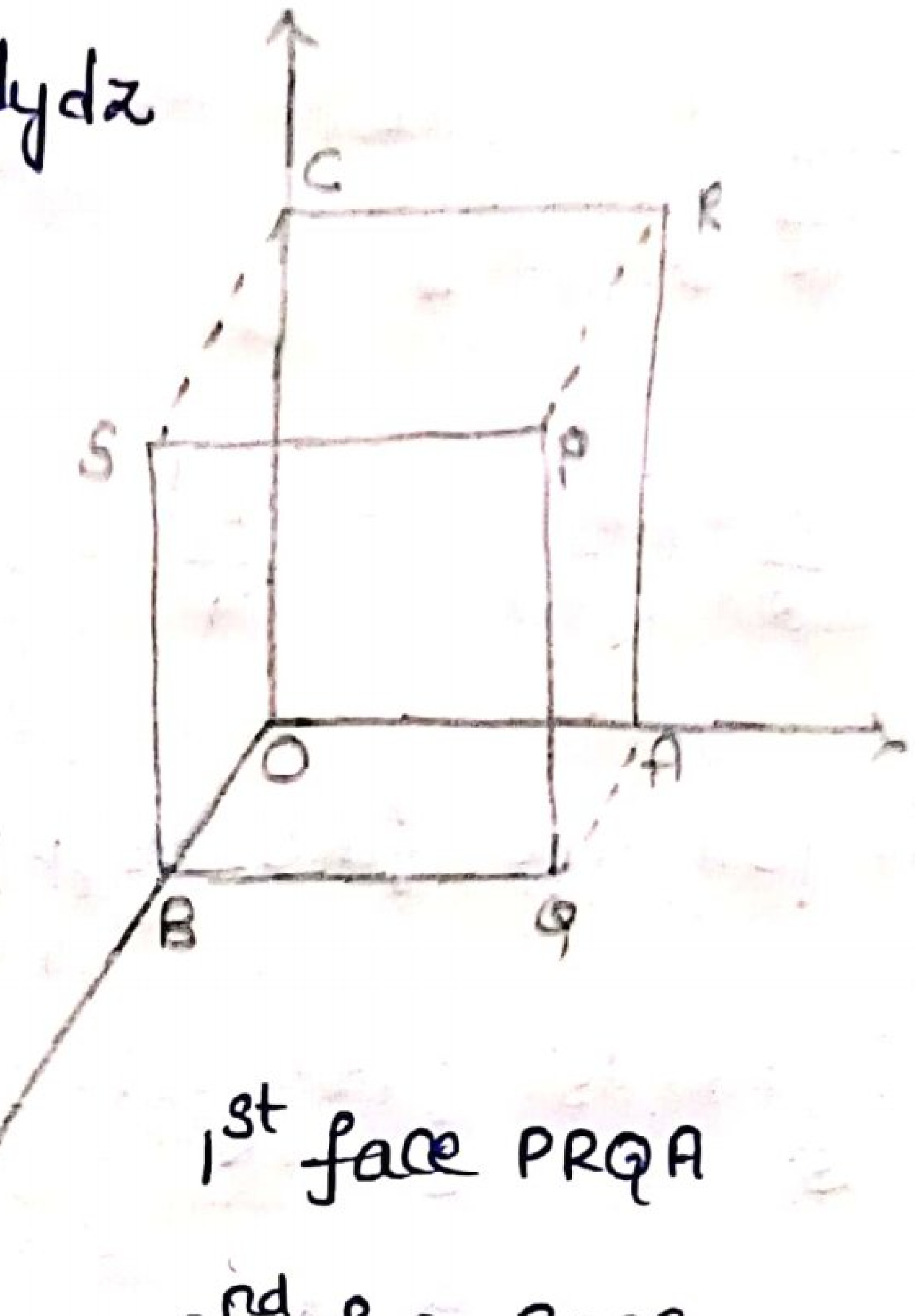
$N = i \quad dS = dy\,dz \quad x = a$

$$\int_S F\cdot N\,dS = \int_S \left((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right)\cdot i\, dy\,dz$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} (x^3 - yz)\,dy\,dz$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} (a^3 - yz)\,dy\,dz$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} a^3\,dy\,dz - \int_{y=0}^{a}\int_{z=0}^{a} yz\,dy\,dz$$



$$= a^3 (y)_0^a (z)_0^a - \left(\frac{y^2}{2}\right)_0^a \left(\frac{z^2}{2}\right)_0^a$$

$$= a^3(a)(a) - \left(\frac{a^2}{2}\right)\left(\frac{a^2}{2}\right)$$

$$= a^5 - \frac{a^4}{4}$$

1^{st} face PRQA

2^{nd} face BOCS

3^{rd} face SBQP

4^{th} face OARC

5^{th} face SPRC

6^{th} face BQOA

(ii) for the face BOCS $\quad N = -i \quad dS = dy\,dz \quad x = 0$

$$\int_S F\cdot N\,dS = \int_S \left((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right)(-i)\,dy\,dz$$

$$= -\int_{y=0}^{a}\int_{z=0}^{a} (x^3 - yz)\,dy\,dz$$

$$= -\int_{y=0}^{a}\int_{z=0}^{a} -yz\,dy\,dz$$

$$= \int_{y=0}^{a} y\,dy \int_{z=0}^{a} z\,dz$$

$$= \left(\frac{y^2}{2}\right)_0^a \left(\frac{z^2}{2}\right)_0^a$$

$$= \left(\frac{a^2}{2}\right)\left(\frac{a^2}{2}\right) = \frac{a^4}{4}$$

(iii) for the face SBQP $N = \hat{j}$, $ds = dx\,dz$, $y = a$

$$\int_S F\cdot N\,ds = \int_S \left((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right)\cdot(\hat{j})\,dx\,dz$$

$$= \int_{x=0}^{a}\int_{z=0}^{a} -2x^2y\,dx\,dz$$

$$= \int_{x=0}^{a}\int_{z=0}^{a} -2x^2a\,dx\,dz$$

$$= -2a\int_{x=0}^{a} x^2 \int_{z=a}^{a} dz$$

$$= -2a\left(\frac{x^3}{3}\right)_0^a (z)_0^a$$

$$= -2a\left(\frac{a^3}{3}\right)(a) = -\frac{2}{3}a^5$$

(iv) for the face OARC $N = -j$, $ds = dx\,dz$, $y = 0$

$$\int_S F \cdot N\,dS = \int_S \left((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right) \cdot (-\bar{j})\,dx\,dz$$

$$= \int_S 2x^2y\,dx\,dz = \int_{x=0}^{a}\int_{z=0}^{a} 2x^2y\,dx\,dz$$

$$= \int_{x=0}^{a}\int_{z=0}^{a} 2x^2(0)\,dx\,dz$$

$$= 0.$$

(v) for the face SPRQ $N = k$, $ds = dx\,dy$, $z = a$

$$\int_S F \cdot N\,dS = \int_S \left((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right)(k)\,dx\,dy$$

$$= \int_S z\,dx\,dy$$

$$= \int_S a\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{a} a\,dx\,dy$$

$$= a\int_{x=0}^{a} dx \int_{y=0}^{a} dy$$

$$= a(x)_0^a\,(y)_0^a$$

$$= a(a)(a) = a^3$$

(vi) for the face BQOA $N = -k$, $ds = dx\,dy$, $z = 0$

$$\int_S F \cdot N\,ds = \int_S \left((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right) \cdot (-\bar{k})\,dx\,dy$$

$$= \int_S -z\,dx\,dy$$

$$= \int 0\,dx\,dy$$

$$= 0$$

$$\int_S F \cdot N\,ds = a^5 - \cancel{\frac{a^4}{4}} + \cancel{\frac{a^4}{4}} - \frac{2}{3}a^5 + 0 + a^3 + 0$$

$$= a^5 - \frac{2}{3}a^5 + a^3$$

$$= \frac{3a^5 - 2a^5}{3} + a^3$$

$$= \frac{a^5}{3} + a^3$$

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2/11/21 Problems on Greens Theorem :

$$\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

1) Show that the area bounded by a simple closed curve 'C' given by $\frac{1}{2}\oint_C x\,dy - y\,dx$ and hence find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ $0 \leq \theta \leq 2\pi$.

Sol:- $\frac{1}{2}\oint_C x\,dy - y\,dx$

By greens theorem $\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$P = -y$ $\qquad Q = x$

$\frac{\partial P}{\partial x} = -1$ $\qquad \frac{\partial Q}{\partial x} = 1$

$$\oint_C (-y\,dx + x\,dy) = \iint (1+1)\,dx\,dy$$

$$= 2\iint_S dx\,dy$$

$= 2A$ where A is the area of the surface S.

$$\frac{1}{2}\oint_C x\,dy - y\,dx = A.$$

$$\text{Area} = \frac{1}{2}\oint x\,dy - y\,dx$$

$x = a\cos\theta$ $\quad\quad$ $y = b\sin\theta$

$dx = -a\sin\theta\,d\theta$ $\quad\quad$ $dy = b\cos\theta\,d\theta$

$$= \frac{1}{2}\int_0^{2\pi} a\cos\theta\, b\cos\theta\,d\theta + b\sin\theta\, a\sin\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,d\theta \quad = \frac{1}{2}ab\int 1\,d\theta$$

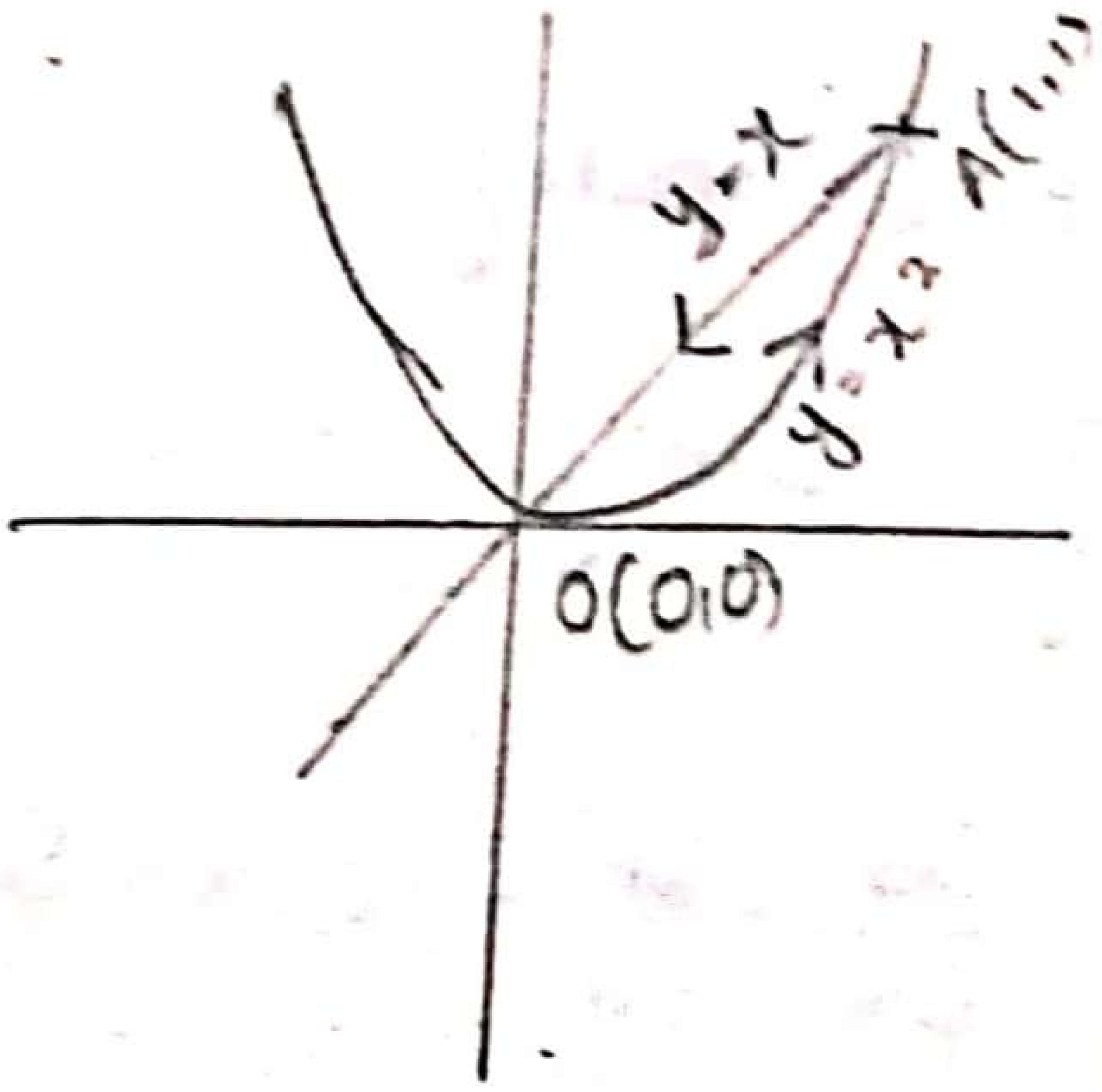
$$= \frac{1}{2}ab(\theta)_0^{2\pi}$$

$$= \frac{1}{\cancel{2}}ab(\cancel{2}\pi)$$

$$= \pi ab \;\;//$$

) Verify greens theorem in the plane for $\oint_C (xy+y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Sol:- $\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$



$P = xy + y^2$ $\qquad$ $Q = x^2$

$\frac{\partial P}{\partial y} = x + 2y$ $\qquad$ $\frac{\partial Q}{\partial x} = 2x$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - x - 2y = x - 2y$$

$y = x \qquad y = x^2$

$x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$x = 0,\ x = 1$

if $x=0 \Rightarrow y=0$

if $x=1 \Rightarrow y=1$

x limits 0 to 1

y limits x^2 to x

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x-2y)\,dx\,dy$$

$$= \int_{x=0}^{1} x\,dx \int_{y=x^2}^{x} dy - 2\int_{x=0}^{1} dx \int_{y=x^2}^{x} y\,dy$$

$$= \int_{x=0}^{1} x\,dx\,(y)_{x^2}^{x} - 2\int_{x=0}^{1} dx \left(\frac{y^2}{2}\right)_{x^2}^{x}$$

$$= \int_{x=0}^{1} x(x-x^2)\,dx - \int (x^2 - x^4)\,dx$$

$$= \int_{x=0}^{1} (\cancel{x^2} - x^3 - \cancel{x^2} + x^4)\,dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx$$

$$= \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = \frac{-1}{20}$$

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verification:-

The line integral C = The line integral $y=x^2$ i.e (0 to A) + The line integral $y=x$ i.e (A to 0)

$y=x^2$ from 0 to A

$\frac{dy}{dx}=2x \Rightarrow dy=2x\,dx$

$\int_0^1 (xx^2+x^4)\,dx+x^2 2x\,dx$

$=\int_0^1 (x^3+x^4+2x^3)\,dx = \int_0^1 (3x^3+x^4)\,dx$

$=\left(\frac{3x^4}{4}+\frac{x^5}{5}\right)_0^1$

$=\frac{3}{4}+\frac{1}{5} = \frac{15+4}{20} = \frac{19}{20}.$

$y=x$ from (A to 0)

$\frac{dy}{dx}=1 \Rightarrow dy=dx$

$\int_1^0 (x\cdot x+x^2)\,dx+x^2\,dx = \int_1^0 3x^2\,dx$

$=\left(\frac{3x^3}{3}\right)_1^0$

$=0-1=-1$

The line integral $C=\frac{19}{20}-1=\frac{19-20}{20}=-1/20$.

SHORTS

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1) Show that $\int_S (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot N\,ds = \frac{4\pi}{3}(a+b+c)$ where S is the surface of the sphere $x^2+y^2+z^2=1$

Sol:- By Gauss theorem $\int_S F\cdot N\,ds = \int_V \mathrm{div}\cdot F$

$$F = ax\bar{i}+by\bar{j}+cz\bar{k}$$

$$\int_S (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot N\,ds = \int_V \nabla\cdot F\,dv$$

$$= \int_V \left(\bar{i}\cdot\frac{\partial}{\partial x}+\bar{j}\cdot\frac{\partial}{\partial y}+\bar{k}\cdot\frac{\partial}{\partial z}\right)\cdot(ax\bar{i}+by\bar{j}+cz\bar{k})\,dv$$

$$= \int_V (a+b+c)\,dv$$

$$= a+b+c\int_V 1\,dv$$

$$= (a+b+c)v$$

For the given sphere $v = \frac{4\pi}{3}$

$$\therefore \int (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot N\,ds = \frac{4\pi}{3}(a+b+c)$$

Compute $\int_S (ax^2+by^2+cz^2)\,ds$ over the sphere $x^2+y^2+z^2=1$

By Gauss theorem $\int_S F\cdot N\,ds = \int_V \operatorname{div} F\,dv$

$F\cdot N = ax^2+by^2+cz^2$

$\phi = x^2+y^2+z^2-1$

$\nabla\phi = \bar{i}\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$

$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$|\nabla\phi| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2}$

$|\nabla\phi| = 2.$

$N = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\cancel{2}(x\bar{i}+y\bar{j}+z\bar{k})}{\cancel{2}} = x\bar{i}+y\bar{j}+z\bar{k}$

$F \cdot N = ax^2+by^2+cz^2$

$F\cdot(x\bar{i}+y\bar{j}+z\bar{k}) = ax^2+by^2+cz^2$

$F = ax\bar{i}+by\bar{j}+cz\bar{k}$

$$\int_S (ax^2+by^2+cz^2)\,dS = \int_V \nabla\cdot F\,dv$$

$$= \int_V \left(\bar{i}\cdot\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\cdot\frac{\partial}{\partial z}\right)\cdot(ax\bar{i}+by\bar{j}+cz\bar{k})$$

$$= \int_V (a+b+c)\,dv$$

$$= (a+b+c)\int_V 1\,dv$$

$= (a+b+c)v$ $\left[\because \text{ for the given sphere } V = \frac{4\pi}{3}\right]$

$= \frac{4\pi}{3}(a+b+c)$ //

3) Evaluate $\oint_C (\cos x \sin y - xy)dx + (\sin x \cos y)dy$ by greens theorem where c is the circle $x^2+y^2=1$.

Sol: By greens theorem $\oint_C P dx + Q dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$

$P = \cos x \sin y - xy$ $\quad\quad$ $Q = \sin x \cos y$

$\frac{\partial P}{\partial y} = \cos x \cos y - x$ $\quad\quad$ $\frac{\partial Q}{\partial x} = \cos x \cos y$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cancel{\cos x \cos y} - \cancel{\cos x \cos y} + x$$

$$= x.$$

$$\oint_C (\cos x \sin y - xy)dx + \sin x \cos y\, dy = \iint_S x\, dx\, dy$$

for the circle $x^2+y^2=1$

change into polar coordinates,

$x = r\cos\theta$ $\quad\quad$ $y = r\sin\theta$ $\quad\quad$ $dx\,dy = r\,dr\,d\theta$

r limits 0 to 1 ; θ limits 0 to 2π

$x = r\cos\theta$ $\quad$ | $\quad$ $y = r\sin\theta$

$dx = -r\sin\theta\, d\theta$ $\quad$ | $\quad$ $dy = r\cos\theta\, d\theta$

$$\oint_C (\cos x \sin y - xy)dx + \sin x \cos y\, dy = \iint_S x\, dx\, dy$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r\cos\theta\; r\,dr\,d\theta$$

$$= \int_{r=0}^{1} r^2 \, dr \int_{\theta=0}^{2\pi} \cos\theta \, d\theta$$

$$= \left(\frac{r^3}{3}\right)_0^1 (\sin\theta)_0^{2\pi}$$

$$= \left(\frac{1}{3} - 0\right)(\sin 2\pi - 0)$$

$$= \frac{1}{3}(0)$$

$$= 0$$