# UNIT - 5

## MATHS - SEM 3

## LONGS

The ① : If R is a Boolean ring then

(i) $a+a=0 \;\forall\; a\in R$ (ii) $a+b=0 \Rightarrow a=b$

(iii) R is commutative under multiplication

or)

Every boolean ring is abelian.

Proof:- (i) $a\in R,\ a\in R \Rightarrow a+a\in R$

$a\in R \Rightarrow a^2=a$ (R is a boolean ring)

$a+a\in R \Rightarrow (a+a)^2=a+a$

$\Rightarrow (a+a)(a+a)=(a+a)$

$\Rightarrow a(a+a)+a(a+a)=(a+a)$

$\Rightarrow (a^2+a^2)+(a^2+a^2)=(a+a)+0$

(∵ R is a ring 1st prop)

$\Rightarrow (a+a)+(a+a)=(a+a)+0$

$a+a=0$

ii) let $a,b\in R$ $a+b=0 \Rightarrow a=b$

$a+b=0$

$a+b=a+a$ (∵ by (i))

$b=a$

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(iii) $a, b \in R \Rightarrow a+b \in R$

R is a boolean ring $a \in R \Rightarrow a^2 = a$

$a+b \in R \Rightarrow (a+b)^2 = (a+b)$

$\Rightarrow (a+b)(a+b) = (a+b)$

$\Rightarrow a(a+b) + b(a+b) = a+b$

$\Rightarrow (a^2+ab) + (ba+b^2) = (a+b)+0$

$\Rightarrow (a^2+b^2) + (ab+ba) = (a+b)+0$

$\Rightarrow (a+b) + (ab+ba) = (a+b)+0$

by cancellation law,

$ab+ba = 0$

$ab = ba$ [by condition (i)]

$\therefore$ R is commutative under multiplication

The ③: Every field is an integral domain.

Proof:- Let $(F,+,\cdot)$ be a field. Then the ring $F$ is a commutative ring with unity and having every non-zero element as unit.

But integral domain is a commutative ring with unity and having no zero divisors

So we have to p.T $F$ has no zero divisors.

Let $(F,+,\circ)$ be a field. let $a,b \in F$

$a \neq 0 \in F$. $F$ is a field $\Rightarrow \exists\, a^{-1} \in F$ such that

$aa^{-1} = a^{-1}a = 1$.

$ab = 0 \Rightarrow a^{-1}(ab) = a^{-1}(0)$

$\Rightarrow (a^{-1}a)\,b = 0$

$\Rightarrow (1)\,b = 0$

$b = 0$

$a,b \in R$, $a \neq 0$ $ab = 0 \Rightarrow b = 0$

lly we can p.T $a,b \in R$, $b \neq 0$. $ab = 0 \Rightarrow a = 0$

$\therefore$ $F$ has no zero divisors.

The ④ : Every finite integral domain is a field.

**Proof:-** Let $0, 1, a_1, a_2, \ldots a_n$ be all the elements of the integral domain D.

Then D has $(n+2)$ elements which is finite.

Integral domain D is a commutative ring with unity and no zero divisors.

So we have to p.T every non-zero element. D has multiplicative inverse.

let $a \in D$ and $a \neq 0$

Now consider $(n+1)$ products $a1, aa_1, aa_2, \ldots aa_n$

If suppose possible $aa_i = aa_j$ for $i \neq j$

Since $a \neq 0$ by cancellation law we have

$a_i = a_j$

This is contradiction & $i \neq j$

$\therefore$ $a1, aa_1, aa_2, \ldots aa_n$ are $(n+1)$ distinct elements in D.

Since D has no zero divisors none of these $(n+1)$ elements is zero element. Hence by counting

$a_1, aa_1, aa_2, \cdots aa_n$ are the $(n+1)$ elements

$1, a_1, a_2, \cdots a_n$

let $a_1 = 1$ (or) $a = 1$ (or) $aa_i = 1$

$\exists\, a_i = b \Rightarrow ab = 1$

D has multiplicative inverse

$\therefore$ D is a field.

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The ⑤ : The characteristic of an integral domain is either prime (or) zero.

Proof :- Let $(R, +, \cdot)$ be an integral domain

Let the characteristic of $R = P$.

If possible suppose that $P$ is not a prime

Then $p = mn$ when $1 < m, n < p$

$a \neq 0 \in R \Rightarrow a \cdot a = a^2 \in R$

$a^2 \in R \Rightarrow pa^2 = 0$

$\Rightarrow mna^2 = 0$

$\Rightarrow (ma)(na) = 0$

$\Rightarrow ma = 0$ (or) $na = 0$

Let $ma = 0$ for any $x \in R$

$(ma)(x) = 0(x)$

$\Rightarrow (ma)(x) = 0$

$\Rightarrow a(mx) = 0$

$mx = 0 \quad (a \neq 0)$

This is absurd $1 < m < p$ and characteristic of $R = P$ . $ma \neq 0$

Ily we can p.T $na \neq 0$

$\therefore$ This is contradiction

$\therefore$ Hence $p$ is prime.

1.) Prove that $Q[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b \in Q\}$ is a field with respect to ordinary addition and multiplication of numbers.

Sol:- Let $x, y, z \in Q[\sqrt{2}]$

$x = a_1+b_1\sqrt{2}$ $\quad y = a_2+b_2\sqrt{2}$ $\quad z = a_3+b_3\sqrt{2}$

1) closure law:

$x, y \in Q[\sqrt{2}] \Rightarrow x+y \in Q[\sqrt{2}]$

$x+y = (a_1+b_1\sqrt{2}) + (a_2+b_2\sqrt{2})$

$= (a_1+a_2) + (b_1+b_2)\sqrt{2} \in Q[\sqrt{2}]$

$x+y \in Q[\sqrt{2}]$

closure law satisfied

(ii) Associative law :

$x+y, z \in Q[\sqrt{2}] \Rightarrow (x+y)+z = x+(y+z)$

L.H.S $= x+y = (a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2})$

$= (a_1+a_2)+(b_1+b_2)\sqrt{2}$

$(x+y)+z = (a_1+a_2)+(b_1+b_2)\sqrt{2} + (a_3+b_3\sqrt{2})$

$= (a_1+a_2+a_3)+(b_1+b_2+b_3)\sqrt{2}$

RHS :- $y+z = (a_2+b_2\sqrt{2})+(a_3+b_3\sqrt{2})$

$= (a_2+a_3)+(b_2+b_3)\sqrt{2}$

$(y+z)+x = (a_2+a_3)+(b_2+b_3)\sqrt{2} + a_1+b_1\sqrt{2}$

$= (a_1+a_2+a_3)+(b_1+b_2+b_3)\sqrt{2}$

L.H.S = R.H.S

Associative law satisfied.

(iii) Identity :

$0 \in Q[\sqrt{2}]$ so we have $0+0\sqrt{2} = 0 \in Q[\sqrt{2}]$

$x+0 = 0+x = x$

$x+0 = a_1+b_1\sqrt{2}+0+0\sqrt{2}$

$= a_1+b_1\sqrt{2}$

$= x$

Identity element $0 \in Q[\sqrt{2}]$ exists

(iv) Inverse :

For $x = a_1 + b_1\sqrt{2} \in Q[\sqrt{2}]$

$(-x) = -a_1 - b_1\sqrt{2} \in Q[\sqrt{2}]$

$x + (-x) = x - x = 0$

$x + (-x) = \cancel{a_1} + \cancel{b_1\sqrt{2}} - \cancel{a_1} - \cancel{b_1\sqrt{2}}$

$= 0$

$\therefore$ Additive inverse exists.

v) $x, y \in Q\sqrt{2} \Rightarrow x + y = y + x$

LHS

$x + y = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2}$

$= (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$

RHS

$y + x = a_2 + b_2\sqrt{2} + a_1 + b_1\sqrt{2}$

$= (a_2 + a_1) + (b_1 + b_2)\sqrt{2}$

LHS = RHS

$(Q(\sqrt{2}), +)$ is a commutative group.

vi) Multiplication is commutative :

$x, y \in Q[\sqrt{2}] \Rightarrow x \cdot y = y \cdot x$

L.H.S

$x \cdot y = (a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2})$

$= a_1a_2 + a_1b_2\sqrt{2} + a_2b_1\sqrt{2} + b_1b_2 \cdot 2$

$= a_1a_2 + 2b_1b_2 + (a_1b_2 + a_2b_1)\sqrt{2}$

R.H.S:- $y \cdot x$

$= (a_2+b_2\sqrt{2})(a_1+b_1\sqrt{2})$

$= (a_2a_1 + a_2b_1\sqrt{2} + b_2a_1\sqrt{2} + b_1b_2 \cdot 2$

$= a_1a_2 + 2b_1b_2 + (a_1b_2 + a_2b_1)\sqrt{2}$

LHS = RHS

satisfies multiplication is commutative.

vii) Multiplication is associative :

$x, y, z \in Q[\sqrt{2}] \Rightarrow (xy)z = x(yz)$

$(x \cdot y) = a_1a_2 + 2b_1b_2 + (a_1b_2 + a_2b_1)\sqrt{2}$

$(x \cdot y)z = [a_1a_2 + 2b_1b_2 + (a_1b_2 + a_2b_1)\sqrt{2}]\,[a_3 + b_3\sqrt{2}]$

$= a_1a_2a_3 + a_1a_2b_3\sqrt{2} + 2b_1b_2a_3 + 2\sqrt{2}\,b_1b_2b_3$

$\cancel{+\sqrt{2}a_1b_1a_3 + a_1b_1b_3\sqrt{2} + a_2b_1a_3\sqrt{2} + 2a_2b_1b_3}$

$+ (a_1b_2 + a_2b_1)\sqrt{2}a_3 + (a_1b_2 + a_2b_1)\sqrt{2} \cdot b_3\sqrt{2}$

$= a_1a_2a_3 + 2b_1b_2a_3 + 2a_1b_2b_3 + 2a_1b_1b_3 + \sqrt{2}$

$(a_1a_2b_3 + 2b_1b_2b_3 + a_1b_2a_3 + a_2b_1a_3)$

$y \cdot z = (a_2+b_2\sqrt{2}) \cdot (a_3+b_3\sqrt{2})$

$= a_2a_3 + a_2b_3\sqrt{2} + a_3b_2\sqrt{2} + 2b_2b_3$

$(y \cdot z)x = [(a_2a_3 + 2b_3b_2) + (a_2b_3 + a_3b_2)\sqrt{2}]\,(a_1 + b_1\sqrt{2})$

$= a_1a_2a_3 + a_2a_3b_1\sqrt{2} + 2b_3b_2a_1 + 2\sqrt{2}\,b_1b_2b_3 + a_1a_2b_3\sqrt{2}$

$+ 2a_2b_3b_1 + \sqrt{2}\,a_1a_3b_2 + 2b_1b_2a_3$

$= a_1a_2a_3 + 2b_1b_2a_3 + 2a_2b_3b_1 + 2a_1b_2b_3 + \sqrt{2}$

$(a_1a_2b_3 + 2b_1b_2b_3 + a_1b_2a_3 + a_2b_1a_3)$

LHS = RHS

$\therefore$ Multiplicative is associative

viii) Multiplicative is distributive

$x, y, z \in Q[\sqrt{2}] \Rightarrow x(y+z) = xy + x \cdot z$ (or)

$(y+z)x = y \cdot x + z \cdot x$

$(y+z) = a_2a_3 + a_2b_3\sqrt{2} + a_3b_2\sqrt{2} + 2b_3b_2$

$y+z = a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2}$

$= (a_2+a_3) + \sqrt{2}(b_2+b_3)$

LHS :

$x \cdot (y+z) = (a_1+b_1\sqrt{2})\,[(a_2+a_3) + (b_2+b_3)\sqrt{2}]$

$= a_1a_2 + a_1a_3 + a_1b_2\sqrt{2} + a_1b_3\sqrt{2} + a_2b_1\sqrt{2} + a_3b_1\sqrt{2}$

$+ 2b_2b_1 + 2b_3b_1$

$x \cdot y = (a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) = a_1a_2 + 2b_1b_2 + (a_1b_2 + a_2b_1)\sqrt{2}$

$x \cdot z = (a_1+b_1\sqrt{2})(a_3+b_3\sqrt{2}) = a_1a_3 + 2b_1b_3 + (a_1b_3 + a_3b_1)\sqrt{2}$

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$x \cdot y + x \cdot z = a_1a_2 + 2b_1b_2 + \sqrt{2}a_1b_2 + \sqrt{2}a_2b_1 + a_1a_3 + 2b_1b_3$
$+ \sqrt{2}a_1b_3 + \sqrt{2}a_3b_1$

$\therefore$ LHS = RHS

$\therefore$ multiplicative is distributive

Similarly we can prove for $(y+z) \cdot x = y \cdot x + z \cdot x$

(ix) Multiplicative inverse:

$x, y \in Q[\sqrt{2}] \Rightarrow xy = yx = 1$

To show that $Q[\sqrt{2}]$ is a field we have to prove further every non-zero element $Q[\sqrt{2}]$ has multiplicative inverse.

$xy = 1$

$y = \frac{1}{x}$

$x = a + b\sqrt{2}$

$$\frac{1}{x} = \frac{1}{a+b\sqrt{2}} \times \frac{a-b\sqrt{2}}{a-b\sqrt{2}}$$

$$= \frac{a-b\sqrt{2}}{a^2-2b^2}$$

$$= \left(\frac{a}{a^2-2b^2}\right) + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2}$$

$$x \cdot y = x \cdot \frac{1}{x}$$

$$= (a+b\sqrt{2}) \left[\left(\frac{a}{a^2-2b^2}\right) + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2}\right]$$

$$= \frac{(a+b\sqrt{2})a}{a^2-2b^2} - \frac{(a+b\sqrt{2})\, b\sqrt{2}}{a^2-2b^2}$$

$$= \frac{a^2 + \cancel{ab\sqrt{2}} - \cancel{ab\sqrt{2}} - 2b^2}{a^2-2b^2}$$

$$= \frac{\cancel{a^2-2b^2}}{\cancel{a^2-2b^2}} = 1$$

$$= 1 + 0\sqrt{2}$$

$\therefore xy = 1 \in Q[\sqrt{2}]$

x) unity element:

$1+0\sqrt{2} \in Q[\sqrt{2}]$ so that

$x \cdot 1 = (a_1 + b_1\sqrt{2})$

$= (1+0\sqrt{2}) = x \; \forall \; x \in Q[\sqrt{2}]$

$\therefore Q[\sqrt{2}]$ is commutative ring with unity element.

Hence $Q[\sqrt{2})$ is a field.