UNIT- IV

VECTOR INTEGRATION

2) Evaluate $\oint_C F \cdot dr$ where $F = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the straight line C from (0,0,0) to (2,1,3).

(Or)

Sol:

Find the work done in moving a particle force field $F = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the straight line from (0,0,0) to (2,1,3).

Sol: The eqn to the line joining (0,0,0) and (2,1,3)

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$x = 2t, \ y = t, \ z = 3t$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{r} = 2t\bar{i} + t\bar{j} + 3t\bar{k}$

$\frac{d\bar{r}}{dt} = 2\bar{i} + \bar{j} + 3\bar{k}$

$F = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$

$= 3(2t)^2\bar{i} + (2(2t)(3t) - t)\bar{j} + 3t\bar{k}$

$F = 12t^2\bar{i} + (12t^2 - t)\bar{j} + 3t\bar{k}$

$F \cdot \frac{d\bar{r}}{dt} = (12t^2\bar{i} + (12t^2 - t)\bar{j} + 3t\bar{k}) \cdot (2\bar{i} + \bar{j} + 3\bar{k})$

$= 24t^2 + 12t^2 - t + 9t$

$= 36t^2 + 8t$

At (0,0,0) and (2,1,3)

t varying 0 to 1,

$\oint_c F \cdot dr = \oint_c \left(F \cdot \frac{d\bar{r}}{dt}\right) dt = \int_0^1 (36t^2 + 8t)\, dt$

$= \left(\frac{\cancel{36}^{12}\, t^3}{\cancel{3}} + \frac{\cancel{8}^{4}\, t^2}{\cancel{2}}\right)_0^1$

$= (12t^3 + 4t^2)_0^1$

$= (12 + 4) - (0 + 0)$

$= 16$

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5.) If $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where the curve c is the rectangle in the xy plane bounded by $y=0, y=b$; $x=0, x=a$

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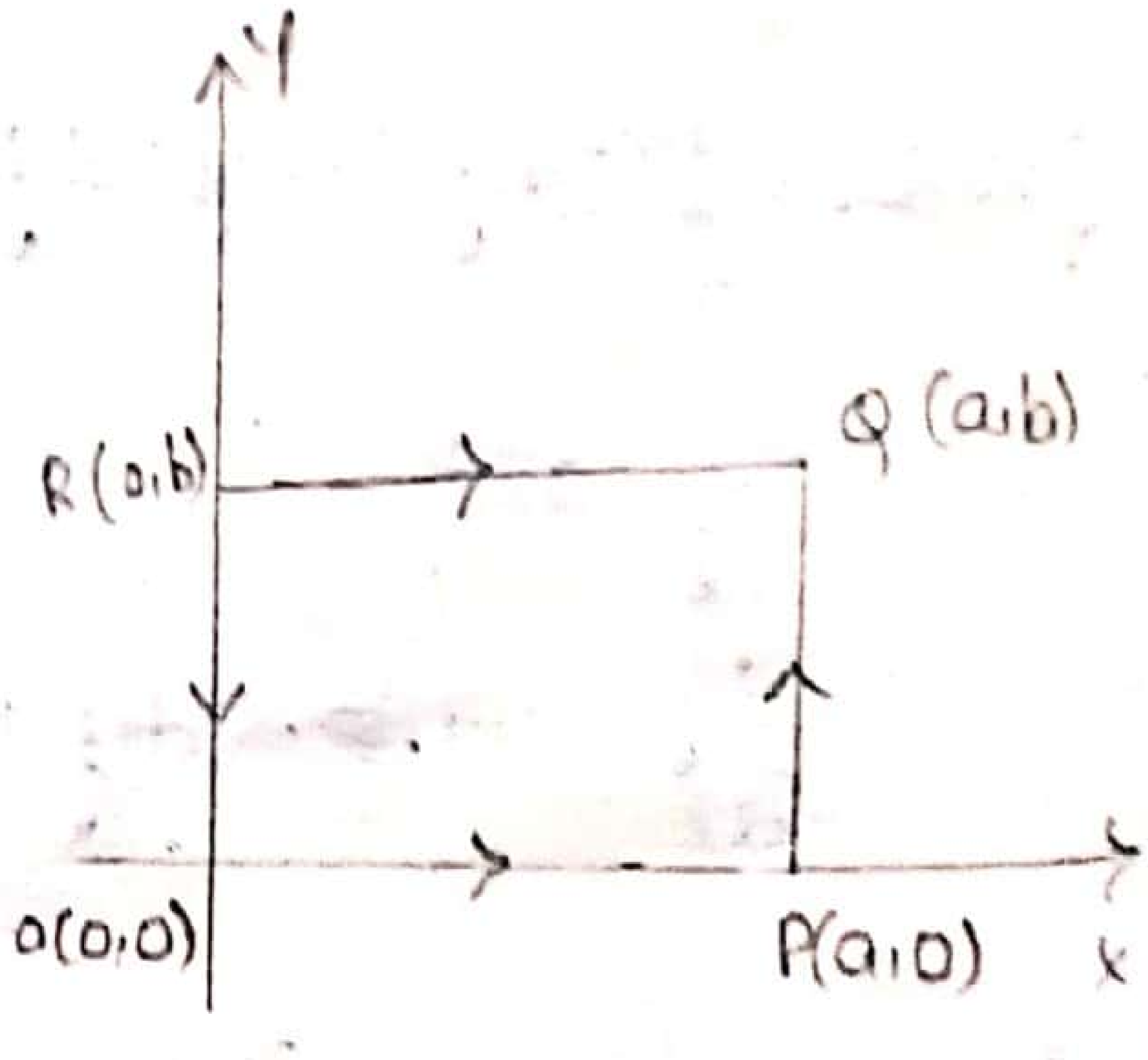
Sol:- $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j}$

In xy plane $z=0 \Rightarrow dz=0$

$d\bar{r} = dx\bar{i} + dy\bar{j}$

$\int_C F\cdot dr = \int_C (x^2+y^2)dx - 2xy\,dy$



(i) Along OP (0,0) and (a,0)

$y=0$ x varies 0 to a

$\int_C F\cdot dr = \int_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$

(ii) Along PQ (a,0) and (a,b)

$x=a \Rightarrow dx=0$ y varies ~~fr~~ 0 to b

$\int_C F\cdot dr = \int_0^b -2ay\,dy$

$= -2a\left(y^2/2\right) dy$

$= -ab^2$

(iii) Along QR (a,b) and (0,b)

$y=b$ x varies a to 0

$dy=0$

$$\int_C F\cdot dr = \int_a^0 (x^2+b^2)dx = \left(\frac{x^3}{3}+b^2x\right)_a^0$$

$$=\left(0-\frac{a^3}{3}+ab^2\right)$$

$$=-\frac{a^3}{3}-ab^2$$

(iv) Along RO (0,b) (0,0)

$x=0$ y varies b to 0

$dx=0$

$$\int_C F\cdot dr = \int_b^0 0\, dy = 0$$

$$\oint_C F\cdot dr = \cancel{\frac{a^3}{3}}-ab^2-\cancel{\frac{a^3}{3}}-ab^2$$

$$= -2ab^2$$

1) Evaluate $\int_S F.N\, ds$ where $F = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Sol:- Let R be the projection of YZ plane

then $\int_S F.N \frac{dy\,dz}{|N\cdot i|}$

$\phi = x^2+y^2=16$

$\nabla\phi = i\cdot\frac{\partial\phi}{\partial x} + \bar{j}\cdot\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z}$

$= 2x\bar{i} + 2y\bar{j}$

$|\nabla\phi| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2}$

$N = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\cancel{2}(x\bar{i}+y\bar{j})}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x\bar{i}+y\bar{j}}{\sqrt{16}} = \frac{x\bar{i}+y\bar{j}}{4}$

$N\cdot i = \left(\frac{x\bar{i}+y\bar{j}}{4}\right) i = \frac{x}{4}$

$|N\cdot i| = \left|\frac{x}{4}\right| = x/4$

$F.N = (z\bar{i}+x\bar{j}-3y^2z\bar{k})\cdot\left(\frac{x\bar{i}+y\bar{j}}{4}\right)$

$= \frac{zx+xy}{4} = \frac{x}{4}(y+z)$

$$\int_S F\cdot N \frac{dy\,dz}{|N\cdot i|} = \int_S \frac{\frac{x}{4}(y+z)}{x/4}\,dy\,dz$$

$$= \int_S (y+z)\,dy\,dz$$

z limits 0 to 5

in yz plane $x=0$

$$x^2+y^2=16$$

$$y^2=16$$

$$y=4$$

y limits 0 to 4

$$\int_S F\cdot N\,ds = \int_{y=0}^{4}\int_{z=0}^{5} (y+z)\,dy\,dz$$

$$\int_S F\cdot N\,ds = \int_{y=0}^{4}\int_{z=0}^{5} y\,dy\,dz + \int_{y=0}^{4}\int_{z=0}^{5} z\,dy\,dz$$

$$= \int_{y=0}^{4} y\,dy \int_{z=0}^{5} 1\,dz + \int_{y=0}^{4} 1\,dy \int_{z=0}^{5} z\,dz$$

$$= \left(\frac{y^2}{2}\right)_0^4 (z)_0^5 + (y)_0^4 \left(\frac{z^2}{2}\right)_0^5$$

$$= \left(\frac{\cancel{16}^{\,8}}{\cancel{2}}\right)(5) + (\cancel{4}^{\,2})\left(\frac{25}{\cancel{2}}\right)$$

$$= 40+50$$

$$= 90$$

2) Evaluate $\int_S F.N\,ds$ where $F = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the part of the plane $2x+3y+6z=12$ located in the first octant.

Sol: Let R be the projection of xy plane then

$$\int_S F.N\,ds = \int_S F.N \frac{dx\,dy}{|N.K|}$$

$$\phi = 2x+3y+6z-12$$

$$\nabla\phi = \bar{i}.\frac{\partial\phi}{\partial x} + \bar{j}.\frac{\partial\phi}{\partial y} + \bar{k}.\frac{\partial\phi}{\partial z}$$

$$= 2\bar{i} + 3\bar{j} + 6\bar{k}$$

$$|\nabla\phi| = \sqrt{2^2+3^2+6^2} = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$N = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}$$

$$N.k = \frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k}).k$$

$$= \frac{6}{7} \qquad \Rightarrow |N.K| = \left|\frac{6}{7}\right| = \frac{6}{7}$$

$$F.N = (18z\bar{i} - 12\bar{j} + 3y\bar{k}) . \frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})$$

$$= \frac{1}{7}(36z - 36 + 18y)$$

$$= \frac{6}{7}(6z - 6 + 3y)$$

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$$\int_S F\cdot N\, ds = \int_S F\cdot N \frac{dx\,dy}{|N\cdot k|}$$

$$= \int_S \frac{\frac{\cancel{6}}{\cancel{7}}(6z-6+3y)\,dx\,dy}{\cancel{6}/\cancel{7}}$$

$$= \int_S (6z-6+3y)\,dx\,dy$$

$$2x+3y+6z=12$$

$$6z = 12-2x-3y$$

$$= \int_S (12-2x-\cancel{3y}-6+\cancel{3y})\,dx\,dy$$

$$= \int_S (6-2x)\,dx\,dy$$

$$= 2\int_S (3-x)\,dx\,dy$$

In xy plane $z=0$

$$2x+3y+6z=12$$

$$2x+3y=12$$

$$3y=12-2x$$

$$y=\frac{12-2x}{3}$$

y limits 0 to $\frac{12-2x}{3}$

$y=0 \Rightarrow 2x=12 \quad x=6$ x limits 0 to 6

$$\int_S F\cdot N\,dS = 2\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (3-x)\,dx\,dy$$

$$= 2\int_{x=0}^{6} (3-x)\,dx \int_{y=0}^{\frac{12-2x}{3}} 1\,dy$$

$$= 2\int_{x=0}^{6} (3-x)\,dx\,(y)_0^{\frac{12-2x}{3}}$$

$$= 2\int_{x=0}^{6} (3-x)\,dx\left(\frac{12-2x}{3}\right)$$

$$= \frac{2}{3}\int_{x=0}^{6} (3-x)(12-2x)\,dx$$

$$= \frac{4}{3}\int_{x=0}^{6} (3-x)(6-x)\,dx$$

$$= \frac{4}{3}\int_{x=0}^{6} (18-3x-6x+x^2)\,dx$$

$$= \frac{4}{3}\int_{x=0}^{6} (x^2-9x+18)\,dx$$

$$= \frac{4}{3}\left[\left(\frac{x^3}{3}\right)_0^6 - 9\left(\frac{x^2}{2}\right)_0^6 + 18(x)_0^6\right]$$

$$= \frac{4}{3}\left[\frac{6\times 6\times \cancel{6}^{\,2}}{\cancel{3}} - 9\left(\frac{6\times\cancel{6}^{\,3}}{\cancel{2}}\right) + 18(6)\right]$$

$$= \frac{4}{3}\left[72-162+108\right] = \frac{4}{\cancel{3}}(\cancel{18}^{\,6}) = 24$$

1) If $F = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$. Evaluate $\int_V F dv$ where V is the region bounded by the surfaces $x=0, x=2$; $y=0; y=6$, $z=x^2$; $z=4$.

Sol:- $\int_V F dv = \int_V (2xz\bar{i} - x\bar{j} + y^2\bar{k}) dx\, dy\, dz$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\, dx\, dy\, dz\, \bar{i} - \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\, dx\, dy\, dz\, \bar{j}$$

$$+ \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\, dx\, dy\, dz\, \bar{k}$$

$$= \int_{x=0}^{2} 2x\, dx \int_{y=0}^{6} dy \int_{z=x^2}^{4} z\, dz\, \bar{i} - \int_{x=0}^{2} x\, dx \int_{y=0}^{6} dy \int_{z=x^2}^{4} dz\, \bar{j}$$

$$+ \int_{x=0}^{2} dx \int_{y=0}^{6} y^2\, dy \int_{z=x^2}^{4} dz\, \bar{k}$$

$$= \int_{x=0}^{2} 2x\, dx \int_{y=0}^{6} dy \left(\frac{z^2}{2}\right)_{x^2}^{4} \bar{i} - \int_{x=0}^{2} x\, dx \int_{y=0}^{6} dy\, (z)_{x^2}^{4}\, \bar{j}$$

$$+ \int_{x=0}^{2} dx \int_{y=0}^{6} y^2 \cdot dy\, (z)_{x^2}^{4}\, \bar{k}$$

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$$= \int_{x=0}^{2} 2x\,dx \int_{y=0}^{6} dy \left(\frac{8-x^4}{2}\right)\bar{i} - \int_{x=0}^{2} x\,dx \int_{y=0}^{6} dy\,(4-x^2)\bar{j} + \int_{x=0}^{2} dx \int_{y=0}^{6} y^2\,dy\,(4-x^2)\bar{k}$$

$$= \int_{x=0}^{2} x(16-x^4)\,dx \int_{y=0}^{6} dy\,\bar{i} - \int_{x=0}^{2} x(4-x^2)\,dx \int_{y=0}^{6} dy\,\bar{j} + \int_{x=0}^{2} (4-x^2)\,dx \int_{y=0}^{6} y^2\,dy$$

$$= \int_{x=0}^{2} (16x-x^5)\,dx \int_{y=0}^{6} dy\,\bar{i} - \int_{x=0}^{2} (4x-x^3)\,dx \int_{y=0}^{6} dy\,\bar{j} + \int_{x=0}^{2} (4-x^2)\,dx \int_{y=0}^{6} y^2\,dy\,\bar{k}$$

$$= \left(16\frac{x^2}{2} - \frac{x^6}{6}\right)_0^2 (y)_0^6\,\bar{i} - \left(4\frac{x^2}{2} - \frac{x^4}{4}\right)_0^2 (y)_0^6\,\bar{j} + \left(4x - \frac{x^3}{3}\right)_0^2 \left(\frac{y^3}{3}\right)_0^6 \bar{k}$$

$$= \left(32 - \frac{32}{3}\right)(6)\,\bar{i} - (8-4)(6-0)\bar{j} + \left(8 - \frac{8}{3}\right)\left(\frac{6\times6\times\cancel{6}^{2}}{\cancel{3}}\right)\bar{k}$$

$$= (96-32)(2)\bar{i} - 24\bar{j} + \left(\frac{24-8}{\cancel{3}}\right)\left(\cancel{6}^{2}\times6\times2\right)\bar{k}$$

$$= 128\bar{i} - 24\bar{j} + 384\bar{k} \quad //.$$

SHORTS

1) Find $\oint_C F \cdot dr$ where $F = xy\bar{i} + yz\bar{j} + zx\bar{k}$ and the curve C is $\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$ t varying from -1 to 1.

Sol:- $\oint_C F \cdot dr = \oint_C \left(F \cdot \frac{d\bar{r}}{dt}\right) dt$

$\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$

$\frac{d\bar{r}}{dt} = \bar{i} + 2t\bar{j} + 3t^2\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$x = t, \; y = t^2, \; z = t^3$

$F = xy\bar{i} + yz\bar{j} + zx\bar{k}$

$F = t^3\bar{i} + t^5\bar{j} + t^4\bar{k}$

$F \cdot \frac{d\bar{r}}{dt} = (t^3\bar{i} + t^5\bar{j} + t^4\bar{k}) \cdot (\bar{i} + 2t\bar{j} + 3t^2\bar{k})$

$= t^3 + 2t^6 + 3t^6 \quad = t^3 + 5t^6.$

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$$\oint_c F \cdot dr = \oint_c \left(F \cdot \frac{d\bar{r}}{dt}\right) dt$$

$$= \int_{-1}^{1} (t^3 + 5t^6)\, dt$$

$$= \left(\frac{t^4}{4} + \frac{5t^7}{7}\right)\Bigg|_{-1}^{1} \qquad = \left(\frac{1}{4} + \frac{5}{7}\right) - \left(\frac{1}{4} - \frac{5}{7}\right)$$

$$= \cancel{\frac{1}{4}} + \frac{5}{7} - \cancel{\frac{1}{4}} + \frac{5}{7}$$

$$= \frac{10}{7} \,//$$

3) If $F=(3x^2+6y)i-(14yz)\bar{j}+(20xz^2)\bar{k}$ calculate $\oint_C F \cdot dr$ along the lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$.

Sol:- $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$$\int_C F \cdot dr = \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

i) along $(0,0,0)$ $(1,0,0)$

$y=0$ $\quad z=0$

$dy=0$ $\quad dz=0$ $\quad$ x varies from 0 to 1

$$\int_C F \cdot dr = \int_0^1 3x^2\,dx = \left(3\frac{x^3}{3}\right)_0^1 = 1$$

ii) Along $(1,0,0)$ $(1,1,0)$

$x=1$ $\quad z=0$

$dx=0$ $\quad dz=0$ $\quad$ y varies from 0 to 1

$$\int_C F \cdot dr = \int_0^1 0 \cdot dy = 0$$

iii) Along $(1,1,0)$ $(1,1,1)$

$x=1$ $\quad y=1$

$dx=0$ $\quad dy=0$ $\quad$ z varies from 0 to 1

$$\int_c F\cdot dr = \int_0^1 20z^2\,dz = \left(20\frac{z^3}{3}\right)_0^1 = \frac{20}{3}$$

$$\oint_c F\cdot dr = 1+0+\frac{20}{3} = \frac{23}{3}$$

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4) If $\bar{F} = 3xy\,\bar{i} - y^2\bar{j}$ Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where c is the curve $y = 2x^2$ in xy plane from (0,0) to (1,2).

Sol:- $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

In xy plane $\Rightarrow z = 0 \Rightarrow dz = 0$

$d\bar{r} = dx\bar{i} + dy\bar{j}$

$\int_C F\cdot dr = \int_C (3xy\,dx - y^2\,dy)$

$y = 2x^2$

$\frac{dy}{dx} = 4x \Rightarrow dy = 4x\cdot dx$

$\int_C F\cdot dr = \int_C 3x\,2x^2\,dx - 4x^4\,4x\,dx$

$= \int_0^1 6x^3\,dx - 16x^5\,dx$

$= \left(6\frac{x^4}{4} - 16\frac{x^6}{6}\right)_0^1$

$= \left(\frac{\cancel{6}^3}{\cancel{4}_2} - \frac{\cancel{16}^8}{\cancel{6}_3}\right)_0^1$

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$$= \frac{3}{2} - \frac{8}{3} \Rightarrow \frac{9-16}{6} = -7/6$$ //

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