# Some effects of the Swap Only Repeater Protocol

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## Abstract

The Swap Only Repeater Protocol creates entanglement between remote nodes via a number of intermediate nodes. For practical purposes one often needs an output state fidelity of at least 0.5. In this report we present some results that show how the output state fidelity and the maximal number of repeaters in a chain depend on the fidelity of individual links. The results also show that the average waiting time does not increase much as the number of repeater nodes is increased. The simulations in this report assume that all nodes have perfect quantum memory, as well as perfect (noiseless) gates.

## 1. Introduction

Long distance quantum communication is highly effected by noise. It is almost impossible to transfer a qubit via any distant channel without loss of fidelity or possibly even loss of the entire qubit. Since qubits (quantum bits) cannot be copied by the fundamentals of quantum mechanics, some clever tricks are desirable that make e.g. quantum teleportation possible over long distance, with high probability of success (high fidelity). Instead of sending a qubit over e.g. a glass fiber connection, one uses the fiber to create entanglement, and then uses the entangled output state to teleport the qubit.

The Swap Only Repeater Protocol is a protocol that can be used for such situations. In this report we simulate this use and collect statistics on the probability of successfully teleporting a qubit between the end nodes, as well as on the running time and the number of attempts needed for successfully creating entanglement.

We will vary the number of repeater nodes, as well as the fidelity of the individual links, and the probability of success of creating entanglement between two adjacent links.

In the coming sections we will first introduce the protocol (section 2) and describe how we modeled it (section 3). Then we will formulate the research questions (section 4). We conclude with the numerical results (section 5) and concluding remarks (section 6).

## 2. Protocol

The protocol that is subject of this investigation is the Swap Only Repeater Protocol, and it is designed to create entanglement between remote (distant) nodes (referred to as Alice and Bob) via a chain of connected repeater nodes (referred to as Repeater 0, Repeater 1, etc.). We will give a short description here, for a full description of the protocol we refer the reader to Briegel et al. (1998).

The protocol begins by Alice and Bob agreeing to initialise entanglement between their nodes (this can be done classically), and (one of) them informing the repeater nodes between them. Alice and each repeater will then create an EPR pair of qubits in state $|\Phi_{00}\rangle$ and submit one of the qubits towards the next node. Each node (except Alice) informs the previous node whether they received the qubit as expected. If entanglement creation between two subsequent nodes failed, the first one (counted from Alice) creates another EPR pair and submits a new qubit. All nodes repeat this until they succeeded. (Briegel et al., 1998)

Once a repeater has successfully reached entanglement with both adjacent nodes, the repeater performs an entanglement swapping, by first applying a controlled NOT-gate with the qubit he received as the control, and the qubit he created as the target. The repeater then performs a bell measurement: he measures the received qubit in the Hadamard basis (storing its value as $b$) and the other qubit in the standard basis (storing its value as $a$). Concludingly, the repeater sends a classical message to Bob (possibly via the other repeaters), containing his pair $(a,b)$.[1] Bob adds up all $(a,b)$-pairs he receives modulo 2 to $(A,B)$ and then knows his qubit to be entangled with the qubit of Alice in a state $|\Phi_{AB}\rangle$. He may apply a gate $X^A Z^B$ to bring the pair to a state $|\Phi_{00}\rangle$ if he wishes. (Briegel et al., 1998)

The above works perfectly on noiseless networks. In practice we should assume that on submitting a qubit, the fidelity to the $|\Phi_{00}\rangle$-state of the EPR pair drops. We can model this by using so called Werner states:

$$\rho_W(p) = p\,|\Phi_{00}\rangle\langle\Phi_{00}| + \frac{1-p}{4}\mathbf{1},$$

where $p$ is a parameter in $[0,1]$, which we call the Werner-parameter.[2]

It is an easy consequence that if the perfect Bell states in the protocol are replaced by Werner states of parameter $p$, the input state will have a fidelity[3] of

$$F_{\text{in}} = \frac{3p+1}{4}.$$

The state of the pair shared by Alice and Bob will be

$$\rho_W(p^{N+1}) = p^{N+1}\,|\Phi_{00}\rangle\langle\Phi_{00}| + \frac{1-p^{N+1}}{4}\mathbf{1},$$

1. In fact, what the repeaters do in the swapping operation is exactly applying the Quantum Teleportation Protocol (see Bennett et al. (1993)) to the qubit they received, this way teleporting a qubit to Bob that is fully entangled with the qubit Alice holds.
2. In netqasm, the simulation package used for this paper, the link fidelity can be set by a parameter called `fidelity`, that however does not set the value of the fidelity, but the value of what we refer to here as the Werner parameter.
3. Throughout this paper we use the convention to write $F(\rho) := F(|\Phi_{00}\rangle,\rho) = \langle\Phi_{00}|\rho|\Phi_{00}\rangle$, whereas some authors define this as the square of the fidelity. Unless this gives rise to confusion, we omit the $\rho$ from our notation.

where $N$ is the number of repeaters between Alice and Bob. This state has a fidelity of

$$F_{\text{out}} = \frac{3p^{N+1}+1}{4}. \tag{1}$$

Whenever $p < 1$, the fidelity is seen to drop to $\frac{1}{4}$ exponentially in the number of repeater nodes. Briegel et al. propose to merge the repeater protocol with some entanglement purification protocol. Many such protocols require the fidelity for the states they start on to be larger then for example $\frac{1}{2}$, as in the purification protocol described in Bennett et al. (1996). If $F_{\text{out}}$ fails to be larger than $\frac{1}{2}$, one could in principle apply this protocol in some intermediate way to create high fidelity pairs on certain parts of the network and continue with those.

In this paper we model a chain of repeater nodes with perfect memory. The advantage of (perfect) memory is if that the creation of two EPR pairs between some nodes failed, the other nodes can just wait and keep the pairs that were successfully transmitted. In a setup without memory, the success rate of each iteration drops exponentially in the number of nodes.

To simulate large networks, we wrote a script that can implement any number of nodes, creates entanglement between every pair of subsequent nodes (succeeding with certain probability), and after successfully doing so performs the entanglement swapping and correction. Finally, the first node (Alice) uses the resulting entangled pair to teleport a qubit in state $|0\rangle$ to the last node (Bob). Bob then measures the qubit in the standard basis and proclaims a 1 (success) if he measured $|0\rangle$ and a 0 (failure) if he measured $|1\rangle$.

Using the Python package `multiprocessing`, we run the script many times simultaneously and collect running time, the number of attempts and the success rate of Bob measuring the state $|0\rangle$.

## 3. Model

To model the protocol in a quantum network, we use the software package netqasm, which runs on Python. In this simulator we simulate a network of a variable amount of repeaters, that use the Swap Only Repeater Protocol between the simulated nodes of Alice and Bob. This simulator actually allows us to retrieve the exact fidelity of the simulated qubits. In practice however, one cannot simply measure the fidelity of a qubit, but one measures only the state of a qubit. Running a protocol many times, one could approximate the measurement results to retrieve the fidelity.

Remember that using the protocol over $N$ repeaters, we will end up with an entangled pair in the state $\rho_W(p^{N+1})$. If we now use this pair to teleport a qubit in state $|0\rangle$ from Alice to Bob, the probability that Bob will measure his qubit in state $|0\rangle$ is easily calculated to be

$$p_{|0\rangle} = \frac{1}{2} + \frac{1}{2}p^{N+1},$$

where $p$ denotes the Werner parameter of the network links.[4] If we retrieve $p_{|0\rangle}$ by measuring the rate of finding $|0\rangle$, we can theoretically use the above expression and (1) to find the output fidelity $F_{\text{out}}$ of the entangled pair that Alice and Bob share:

$$F_{\text{out}} = \frac{3p_{|0\rangle}-1}{2}.$$

Let us denote by $T_i$ ($i \in \{0,\dots,N\}$) the number of attempts needed to create entanglement between the $i$-th node and the $(i+1)$-th node (so $T_0$ denotes the number of attempts Alice and Repeater 0 needed to establish entanglement). If this process is assumed to succeed deterministically with probability $q$, we have $\mathbb{E}(T_i) = \frac{1}{q}$ (from geometric distribution). The number $T := \max_i T_i$ seems to be an important factor, that determines, next to some global initialisation and processing time, the amount of time needed to create entanglement. We can derive that[5]

$$\mathbb{E}(T) = \sum_{k=0}^{\infty}\left[1-\left(1-(1-q)^k\right)^{N+1}\right].$$

It is easily argued that $\mathbb{E}(T)$ is monotonically increasing in the number of repaeter nodes $N$. However, this increase is observed (see the results) to be less than linear, whereas the success probability of the creation of a remote EPR pair is known to drop exponentially over distance.

## 4. Research question

We use our simulator to test for the waiting time, and our first question is how the waiting time of the protocol varies for the number of repeater nodes used. We will provide a two-way estimate of the waiting time: we will measure both the average value of $T$, as well as the average time the script needs before it terminates. It must be noted that this latter estimate may depend highly on limits of our simulation machine, and also includes the time needed to initialise all the nodes (which in practice could happen simultaneously on each node).

The second question in this research is what value of $F_{\text{in}}$ is needed to ensure that $F_{\text{out}} > \frac{1}{2}$ (so to ensure that purification of the output states is possible), depending on the number of repeater nodes. Or, which amounts to the same, given a certain link fidelity, over how many repeater nodes at most can one apply the Swap Only Repeater Protocol to maintain a fidelity larger than $\frac{1}{2}$?

We will perform measurements for 21 different values of the Werner parameter for link fidelity, ranging from 0 to 1, with 400 measurements per value. Moreover, we will vary the network size between 1 and 10 repeater nodes to capture the time needed for the protocol, and the output fidelity as function of the number of repeaters.

In our simulations we do not vary the distance between adjacent nodes, that is, the probability of successfully creating an EPR pair between two nodes, as well as the link fidelity remain fixed. An increase in the number of nodes thus corresponds linearly to an increase of the distance between Alice and Bob. We assume (and set) the probability of successfully creating entanglement between two adjacent node on each single attempt to be 0.5, and repeat the simulations once more for a probability 0.8.

4. If the links in the network have mutually differing Werner parameters, the expression still holds when we replace $p^{N+1}$ by the product of the Werner parameters of the individual links.

5. Observe that $\mathbb{P}(T_i \le k) = 1-(1-q)^k$, so under assumption of independence $\mathbb{P}(T \le k) = (1-(1-q)^k)^{N+1}$. The result then follows from $\mathbb{E}(T) = \sum_{k=0}^{\infty}\mathbb{P}(T > k)$. In this regard, a special thanks goes to Riley Badenbroek for helping me out with this elementary computation.

## 5. Numerical results

We will briefly discuss the numerical results we derived from the model. For brevity, we did not include plots for all ranges. The reader may however reproduce these plots with the python code we attached to this article.

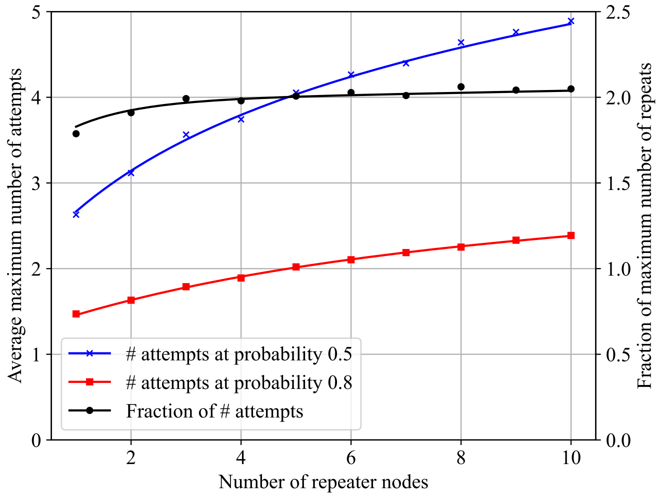


**Figure 1:** A plot of the average of $T := \max_i T_i$, the maximum of the amount of attempts needed to create entanglement between any two adjacent nodes, as function of the number of repeater nodes. The average is taken over roughly 2000 simulations on each number of repeater nodes, and with success probabilities of 0.5 (blue crosses) and 0.8. In black we printed the fraction of the number of attempts at 0.5 (red squares) over the number of attempts at 0.8 (right axis), we observe that the number of attempts of roughly doubled.

In Figure 1 we plotted the average maximum number of attempts, which seems to agree well to the predicted values as function of the number of repeaters. We did so for two different scenarios: one is where the probability of creating entanglement is simulated to be 0.5, in the other scenario this probability is 0.8. One observes that the plot has a decreasing slope, indicating that the expected waiting time does not rapidly increase as the number of repeater nodes in a chain grows. If the probability of successfull entanglement between any two adjacent nodes drops from 0.8 to 0.5, the expected maximal number of attempts is roughly doubled.

From Figure 2 we observe however that the running time of the simulation does increase significantly over the number of repeater nodes, in a more than linear fashion. It seems that the simulation time is not a good indicator for the average waiting time, and we suspect the observed increase to be due to machine limits. A larger network requires more memory and processor time, whereas the capacities of the simulation computer of course does not increase. This presumption is further supported by the observation that decreasing the success probability from 0.8 to 0.5 increases the running time by only 20%, whereas we observed that the expected maximal number of attempts is doubled.

From Figure 3 we observe that with one repeater node, an input state fidelity of approximately 0.7 suffices for the network to create entanglement with fidelity higher than 0.5. When the chain of nodes is extended to include four repeater nodes, we observe from Figure 4 that an input state fidelity of approximately 0.85 or higher is needed to obtain a sufficiently high output fidelity. With ten repeater nodes, the minimum input fidelity again increases, to roughly 0.94 (cf. Figure 5.

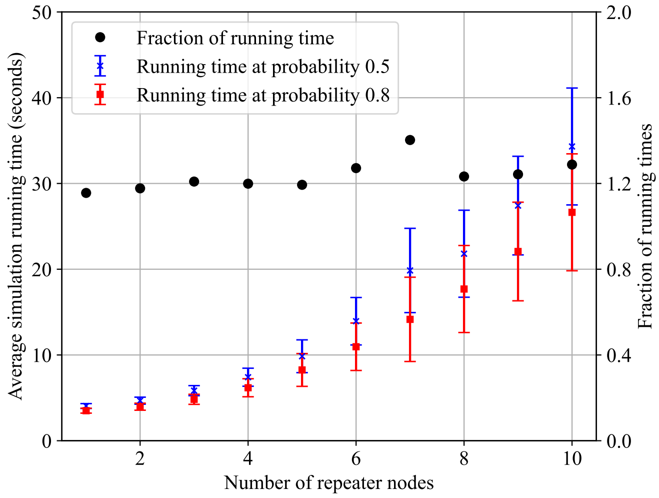


**Figure 2:** A plot of the average simulation time as function of the number of repeater nodes, with error bars of two standard deviations length. The average is taken over roughly 2000 simulations on each number of repeater nodes, at success probabilities of 0.5 (blue crosses) and 0.8 (red squares). In black we printed the fraction of the running time at 0.5 over the running time at 0.8 (right axis), we observe that the running time increases with rougly 20%.

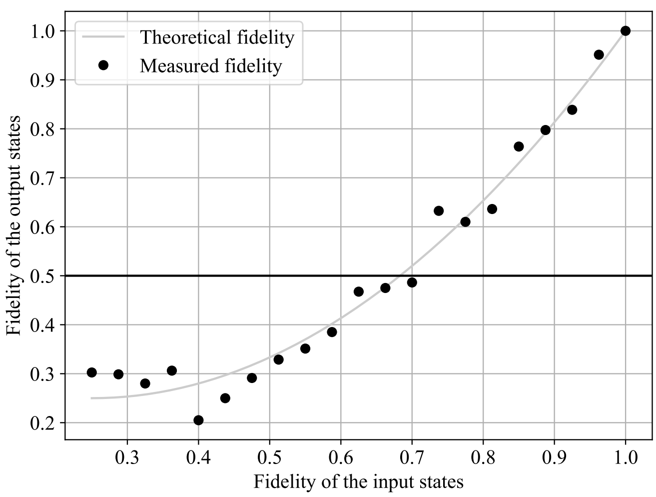


**Figure 3:** A plot of the measured output fidelity with one repeater node (400 simulations for each setting).

In Figure 6 and Figure 7 the measured and predicted output fidelities are used to construct a contour plot. From the contours one sees that for an output fidelity larger than $\frac{1}{2}$, the minimum input state fidelity rapidly increases in the number of repeater nodes. If the link fidelity of individual links is approximately 0.9, no more than six repeaters can be used in a row whenever one wants to apply a distillation protocol to the output state.

## 6. Concluding remarks

We observed that the maximum number of attempts (over all of the links) is dependent on the probability of success over the links, but only slightly increases in the number of repeaters. The number of repeaters thus has only limited influence on the total waiting time. We argue the waiting time is not accurately predicted by the simulation running time.

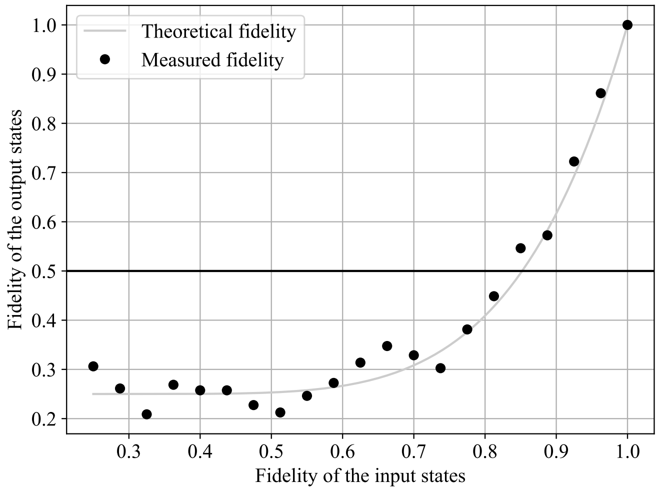


**Figure 4:** A plot of the measured output fidelity with four repeater nodes (400 simulations for each setting).

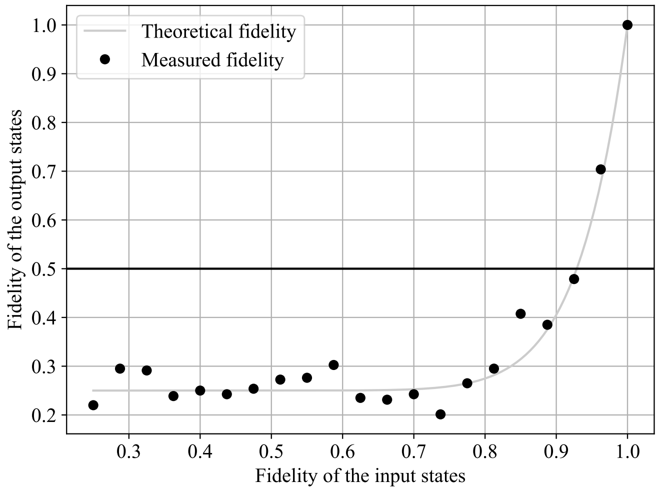


**Figure 5:** A plot of the measured output fidelity with ten repeater nodes (400 simulations for each setting). The contours indicate the levels of the output state fidelity, the bold curve represents a fidelity of 0.5.

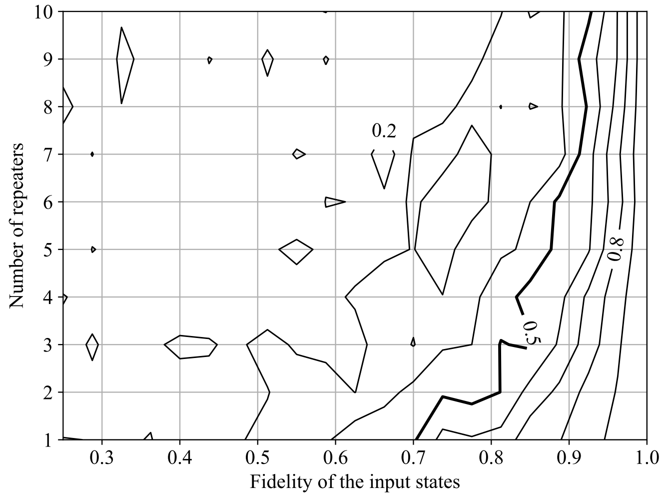


**Figure 6:** A contour plot of the measured output fidelity as function of input fidelity and number of repeater nodes. (400 simulations for each setting.) The contours indicate the levels of the output state fidelity, the bold curve represents a fidelity of 0.5.

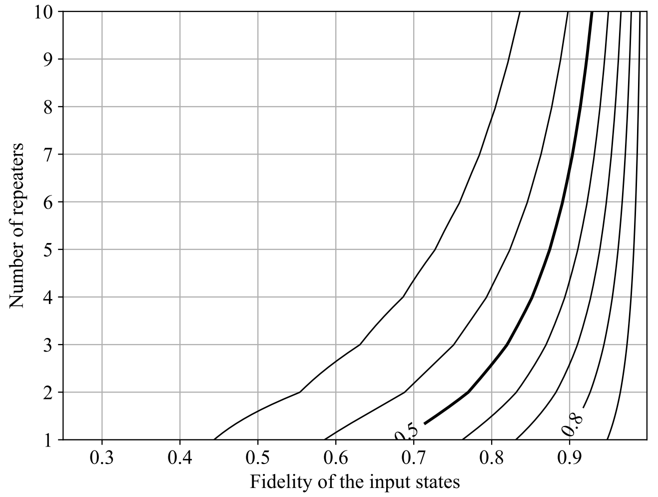


**Figure 7:** A contour plot of the theoretically predicted output fidelity as function of input fidelity and number of repeater nodes.

It is found that to ensure an output fidelity bigger than 0.5, the maximal number of repeaters is highly dependent on the fidelity of the individual links, where one can read the measured and theoretical values from Figure 6 and Figure 7 respectively.

We must however remark that we simulated all gates to be perfect/noiseless, i.e. the gate fidelity equals 1, and also we simulated all nodes to have perfect memory, so that the protocol always succeeds. Different simulation setups may be needed to study the effect of imperfect memory and imperfect gates.

## References

Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, 70:1895–1899, Mar 1993. doi: 10.1103/PhysRevLett.70.1895.

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