Chapter 2
Properties of electromagnetic waves

- Basic information on electromagnetic waves in free space or other uniform media, but no rigorous development
- Maxwell’s equations (brief reference to them)
- Plane wave properties
- Polarisation

Maxwell’s equations

An electric field is produced by a time-varying magnetic field

A magnetic field is produced by a time-varying electric field or a current

Electric field lines may either start and end on charges, or are continuous

Magnetic field lines are continuous

Will use del-operator next, definitions:

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Maxwell's equations

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

Faraday's law. Time varying magnetic and electrics fields.

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

Ampère's law. Magnetic field and current.

$$\nabla \cdot \left(\varepsilon \mathbf{E} \right) = \rho$$

Gauss' law. Electric flux out of a closed surface surrounding the charge.

$$\nabla \cdot \left(\mu \mathbf{H} \right) = 0$$

No net magnetic flux out or in of a closed surface

E: electric field (V/m)

H: magnetic field (A/m)

J: current density (A/m^2)

ρ: charge density (C/m^3)

ε: permittivity, dielectric constant (F/m)

μ: permeability (H/m)

$\varepsilon = \varepsilon_0 \varepsilon_r$, $\varepsilon_0 = 8.854 \cdot 10^{-12}$ F/m

$\mu = \mu_0 \mu_r$, $\mu_0 = 4\pi 10^{-7}$ H/m

$\varepsilon_r = \mu_r = 1$ in free space

Wave equation in free space

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot (\varepsilon \mathbf{E}) = \rho$$

$$\nabla \cdot (\mu \mathbf{H}) = 0$$

In free space is $\boldsymbol{J} = 0$ and $\rho = 0$.

Take the curl of the first equation

$$\nabla \times \left(\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} \right) = \nabla \times \nabla \times \mathbf{E} + \mu_0 \frac{\partial}{\partial t} \nabla \times \mathbf{H} = 0$$

Use $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$

and second equation $\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

Then $-\nabla^2 \mathbf{E} + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} = 0$ for $\nabla \cdot \mathbf{E} = 0$ (in free space)

Wave equation: $$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} = 0$$

Time-harmonic plane wave equation solution

With time dependency $e^{j\omega t}$ a solution is

$$\mathbf{E} = E_0 \cos(\omega t - kz)\hat{\mathbf{x}}$$

for a wave propagating in the *z*-direction with the field varying in the *x*-direction.

The wave phase angle is $\omega t\text{-}kz$, where ω is angle frequency, $\omega = 2\pi f$ and k the wave number. E_0 is the amplitude.

The wave number expresses the phase angle rate of change as a function of travelled distance z, i.e.

$$k = \frac{2\pi}{\lambda}$$

and

$$\cos(\omega t - kz) = \cos\left(\omega t - \frac{2\pi}{\lambda} z\right)$$

Convenient to use complex variables $\mathbf{E} = E_0 e^{j(\omega t - kz)}\hat{\mathbf{x}}$

Phase front velocity

Given a fixed point on the wave with an arbitrary phase,

$\omega t\text{-}kz$=constant => z= (ωt-constant)/k

then the phase front velocity v

$$v = \frac{dz}{dt} = \frac{\omega}{k} = \lambda f$$

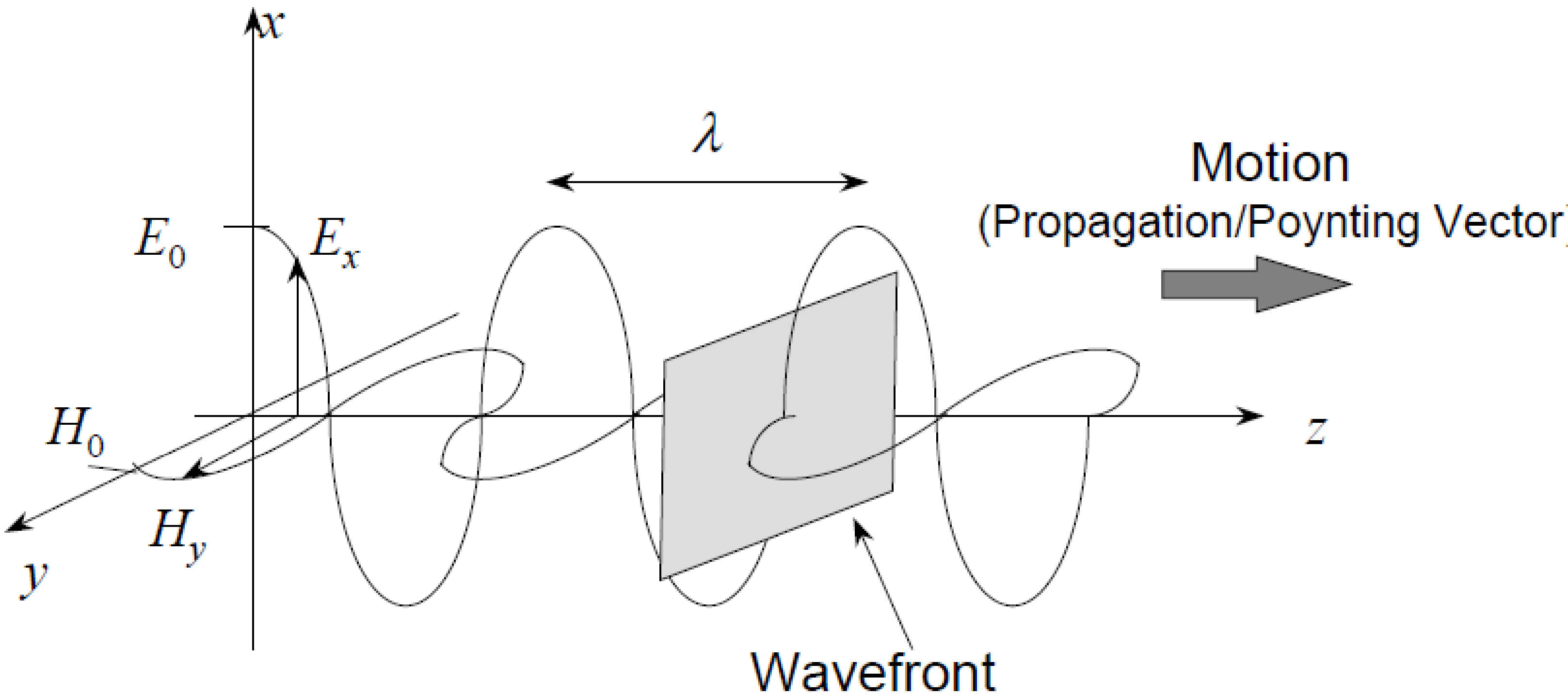
Given $k^2 = \omega^2 \mu_0 \varepsilon_0$ then $\mathbf{E} = E_0 \cos(\omega t - \omega\sqrt{\mu_0 \varepsilon_0}\, z)\hat{\mathbf{x}}$

and

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 3 \cdot 10^8 ms^{-1}$$

The ratio of velocity in free space to the velocity in the medium is the refractive index $n = \frac{c}{v}$

Plane wave propagation



Wave impedance, Z

The solution for **E** and **H** satisfy Maxwell’s equations provided

$$\frac{|\mathbf{E}|}{|\mathbf{H}|} = \frac{E_x}{H_y} = \frac{E_0}{H_0} = \sqrt{\frac{\mu}{\varepsilon}} = Z$$

for a given medium (ε, μ).

In free space (ε_0, μ_0) the impedance becomes

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx \sqrt{4\pi \cdot 10^{-7} \cdot \frac{36\pi}{10^{-9}}} = 120\pi \approx 377\,\Omega$$

Power flow Poynting vector, **S**

From laws of conservation of energy and momentum the instantaneous power flux per unit area **S** is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}^*$$

for a given medium (ε, μ).

More useful with time averaged over one period

$$S_{av} = \frac{1}{2} E_0 H_0 \hat{\mathbf{z}}$$

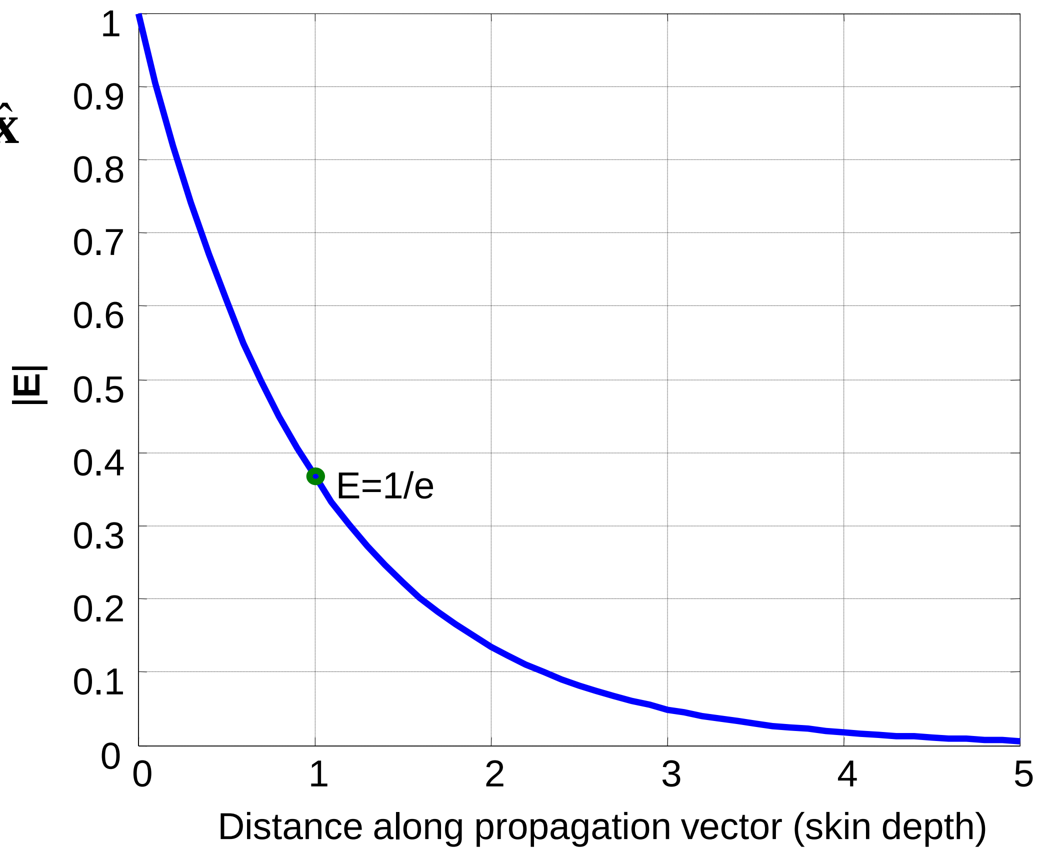
Lossy media

Write $n = n' - jn''$ in a lossy medium. Note $k = \omega n / c$

$$\mathbf{E} = E_0 \exp\left[j(\omega \mathbf{t} - k'z) - \alpha z\right]\hat{\mathbf{x}}$$

Energy is removed from the wave and converted to heat. The attenuation constant is α.

Skin depth δ: travelled distance where the field is reduced to 1/*e* is $\delta = 1/\alpha$



Attenuation constant, wave number, wave impedance, wavelength and phase velocity (Table 2.1)

$n = ck/\omega$ in all cases	Exact expression	Good dielectric (insulator) $(\sigma/\omega\varepsilon)^2 \ll 1$	Good conductor $(\sigma/\omega\varepsilon)^2 \gg 1$
Attenuation constant α [m^{-1}]	$\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}$	$\approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\approx \sqrt{\frac{\omega\mu\sigma}{2}}$
Wavenumber k [m^{-1}]	$\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}$	$\approx \omega\sqrt{\mu\varepsilon}$	$\approx \sqrt{\frac{\omega\mu\sigma}{2}}$
Wave impedance Z [Ω]	$\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$	$\approx \sqrt{\frac{\mu}{\varepsilon}}$	$\approx \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$
Wavelength λ [m]	$\frac{2\pi}{k}$	$\approx \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$	$\approx 2\pi\sqrt{\frac{2}{\omega\mu\sigma}}$
Phase velocity v [m s^{-1}]	$\frac{\omega}{k}$	$\approx \frac{1}{\sqrt{\mu\varepsilon}}$	$\approx \sqrt{\frac{2\omega}{\mu\sigma}}$

Example

A linear polarised 900 MHz wave travels in the z-direction in a medium with constitutive parameters $\mu_r = 1$, $\varepsilon_r = 3$, and $\sigma = 0.01\ \mathrm{Sm^{-1}}$. The electric field at $z = 0$ is $1\ \mathrm{Vm^{-1}}$.

Calculate:

a) the wave impedance
b) the magnitude of the magnetic field at z = 0
c) the average power available in a $0.5\ \mathrm{m^2}$ area perpendicular to the direction of propagation at $z = 0$
d) the time taken for the wave to travel 10 cm
e) the distance travelled by the wave before its field strength drops to one tenth its value at $z = 0$

Example solution

A linear polarised 900 MHz wave travels in the *z*-direction in a medium with constitutive parameters $\mu_r = 1$, $\varepsilon_r = 3$, and $\sigma = 0.01\ \mathrm{Sm^{-1}}$.
The electric field at $z = 0$ is 1 $\mathrm{Vm^{-1}}$.

$$\frac{\sigma}{\omega\varepsilon} = \frac{0.01}{2\pi \cdot 900 \cdot 10^{6} \cdot 3 \cdot \dfrac{10^{-9}}{36\pi}} \approx 0.07 << 1$$

From Table 2.1: good insulator

a) the wave impedance

$$Z \approx \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_r \varepsilon_0}} = \frac{Z_0}{\sqrt{\varepsilon_r}} \approx 218\Omega$$

b) the magnitude of the magnetic field at $z = 0$

$$H = \frac{E}{Z} = \frac{1}{218} \approx 4.6\mathrm{Am}^{-1}$$

Example solution continued

A linear polarised 900 MHz wave travels in the *z*-direction in a medium with constitutive parameters $\mu_r = 1$, $\varepsilon_r = 3$, and $\sigma = 0.01$ Sm^{-1}.
The electric field at $z = 0$ is 1 Vm^{-1}.

c) the average power available in a 0.5 m^2 area perpendicular to the direction of propagation at $z = 0$

$$P = SA = \frac{EH}{2} A = \frac{1 \cdot 0.005}{2} = 1.25\text{mW}$$

d) the time taken for the wave to travel 10 cm

$$t = \frac{d}{v} = d\sqrt{\mu\varepsilon} = d\sqrt{\mu_0 \varepsilon_r \varepsilon_0} = \frac{d}{c}\sqrt{\varepsilon_r} \approx \frac{0.1}{3 \cdot 10^8}\sqrt{3} \approx 0.6\text{n}s$$

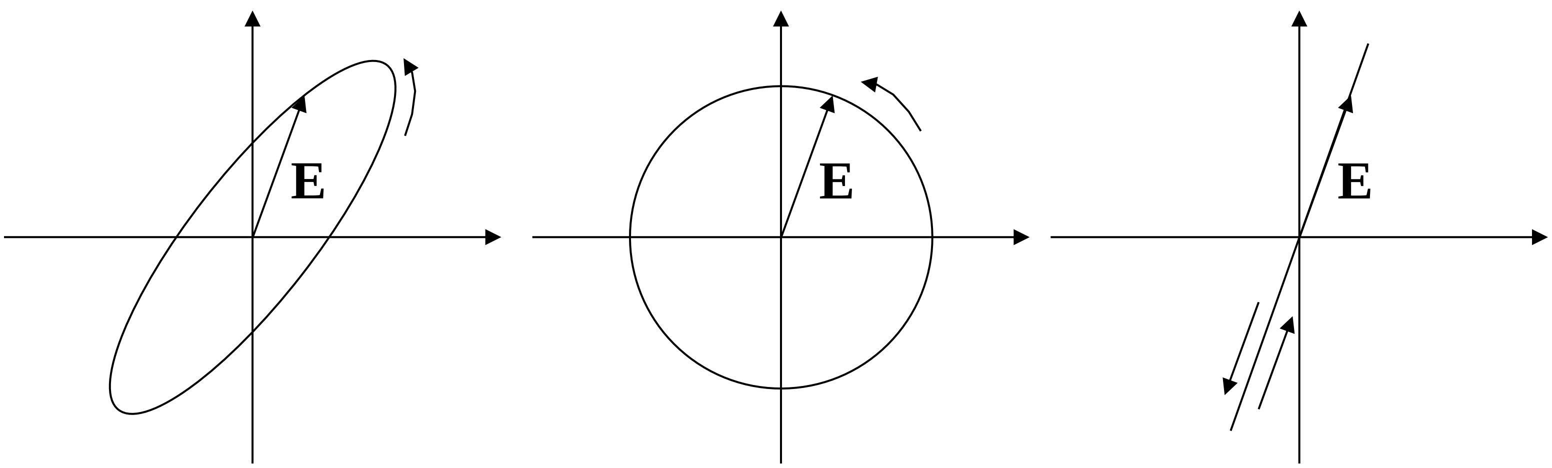
e) the distance travelled by the wave before its field strength drops to one tenth its value at z = 0

$$E(z) = E(0)e^{-z/\delta} \Rightarrow z = -\delta \ln\frac{E(z)}{E(0)} \quad \text{where} \quad \delta = \frac{1}{\alpha} \approx \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}} = \frac{2}{\sigma Z}$$

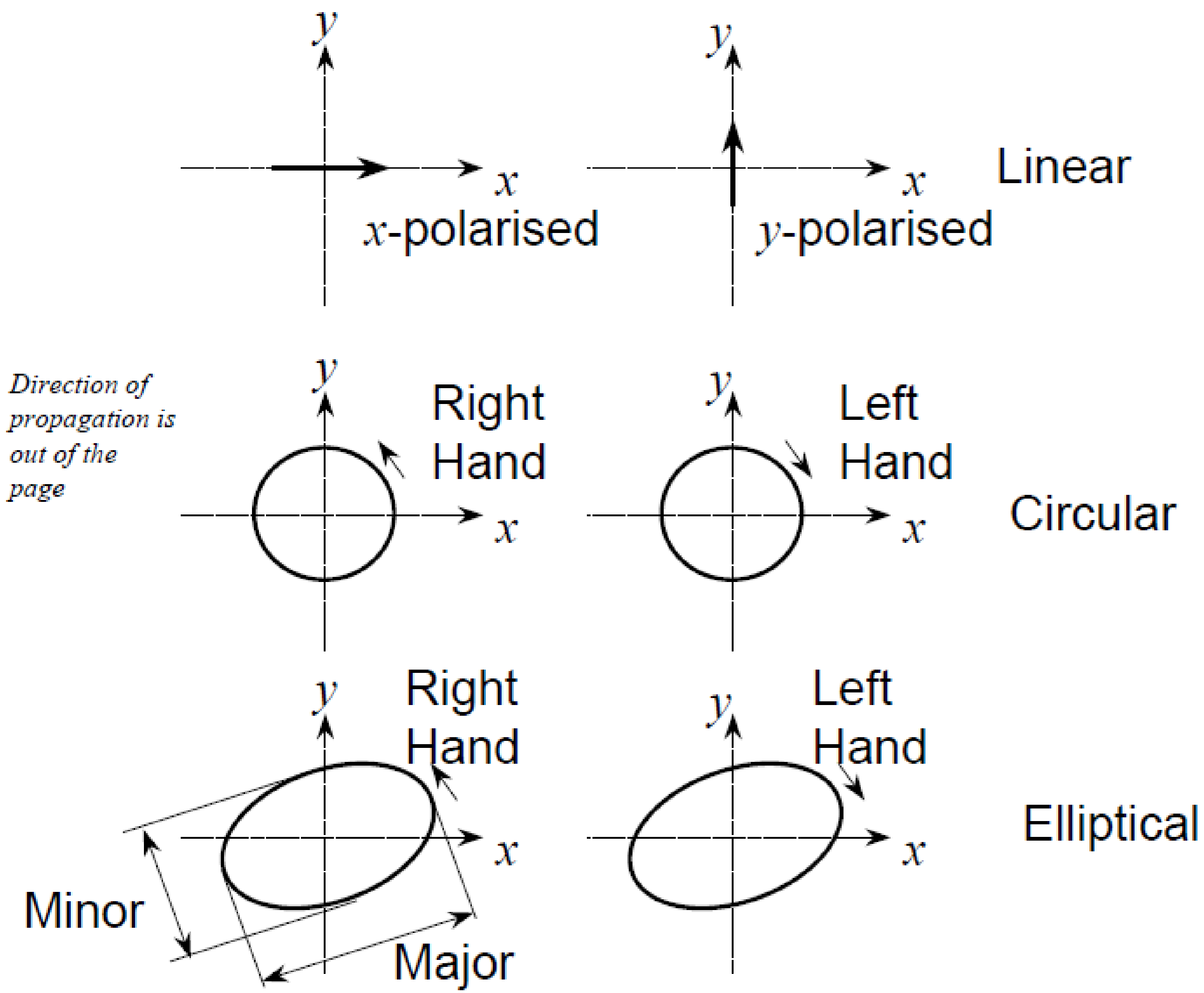
$$z = -\frac{2}{\sigma Z}\ln\frac{E(z)}{E(0)} = -\frac{2}{0.01 \cdot 218}\ln\frac{1}{10} = 2.11\text{m}$$

Polarisation

Generally, along the propagation direction the tip of the time-varying electric field vector will form an ellipse, where circle or line is degenerated special cases



Polarisation states



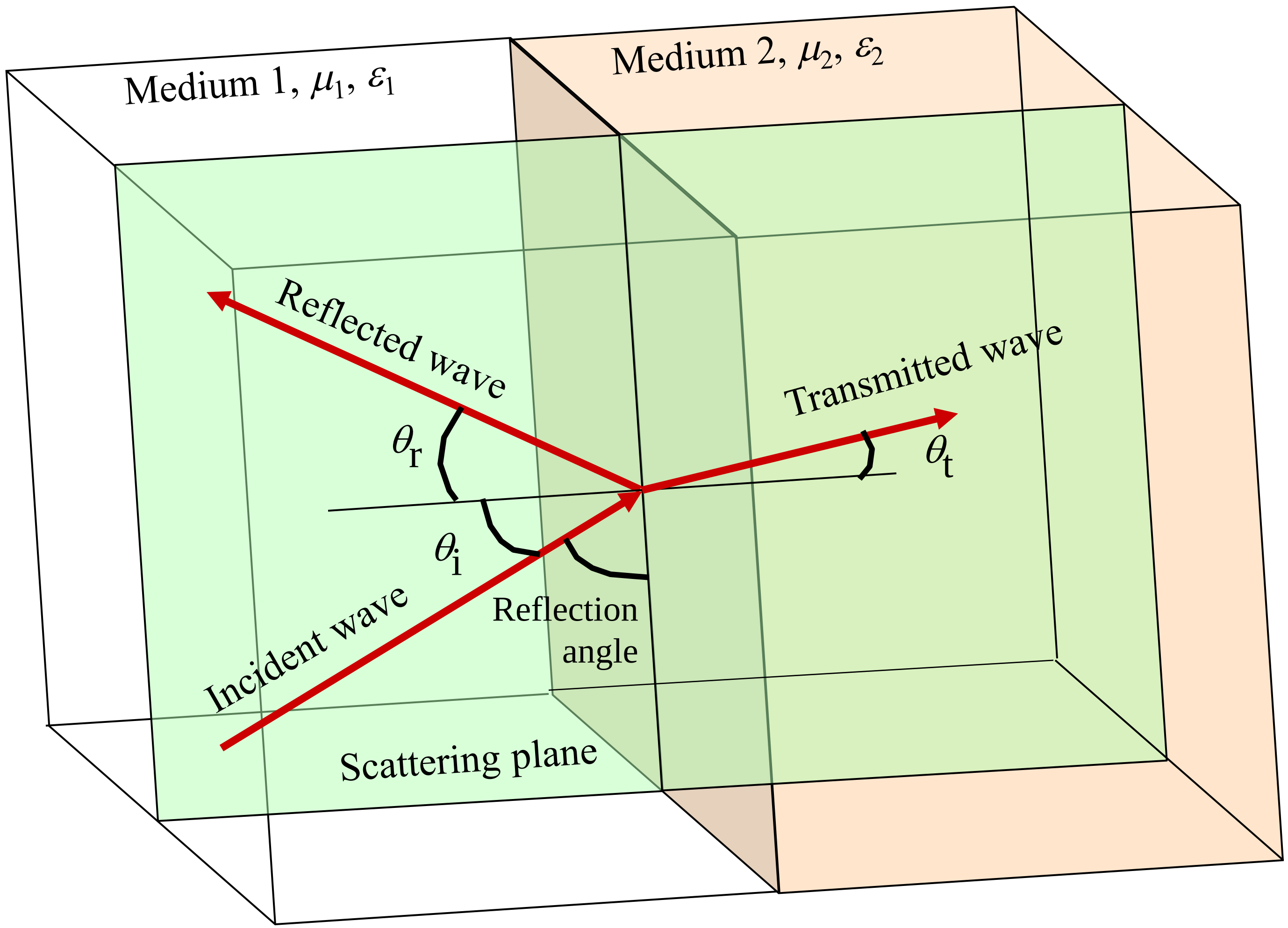
Conclusion

- Plane wave in uniform media
- Plane wave properties and interaction with the medium
- Poynting vector
- Polarisation

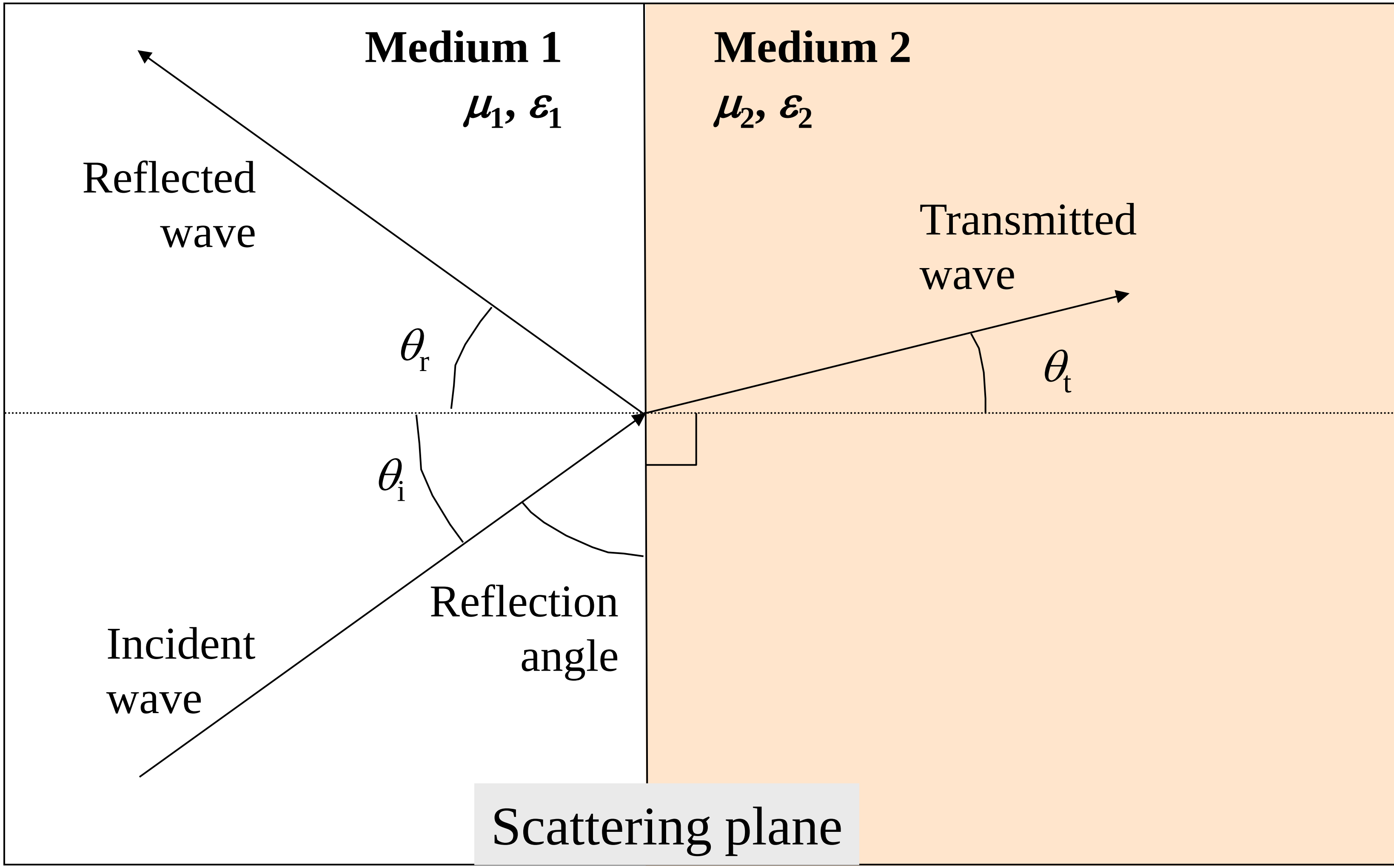
Chapter 3
Propagation mechanisms

- Reflection and refraction: reflected and transmitted wave
- Rough surfaces
- Diffraction

Reflection and refraction



Reflection and refraction



Snell's law of refraction

Wave number k and Fermat's principle minimising the electrical path length

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{k_2}{k_1}$$

Refractive index n, the ratio of free space phase velocity c to the phase velocity in the medium

$$n = \frac{c}{v} = \frac{ck}{\omega}$$

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1}$$

Fresnel reflection and transmission coefficients

$$R_{\|} = \frac{E_{r\|}}{E_{i\|}} = \frac{Z_1\cos\theta_i - Z_2\cos\theta_t}{Z_2\cos\theta_t + Z_1\cos\theta_i} \qquad R_{\perp} = \frac{E_{r\perp}}{E_{i\perp}} = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$

$$T_{\|} = \frac{E_{t\|}}{E_{i\|}} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i} \qquad T_{\perp} = \frac{E_{t\perp}}{E_{i\perp}} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$

Total reflected field:

$$\mathbf{E}_r = E_{r\|}\mathbf{a}_{\|} + E_{r\perp}\mathbf{a}_{\perp} = E_{i\|}R_{\|}\mathbf{a}_{\|} + E_{i\perp}R_{\perp}\mathbf{a}_{\perp}$$

$$\mathbf{E}_i = E_{i\|}\mathbf{a}_{\|} + E_{i\perp}\mathbf{a}_{\perp}$$

where $a_{\|}$ and $a_{\perp}$ are unit vectors parallel and normal to the scattering plane, respectively

Reflected and transmitted field

$$\mathbf{E}_r = \mathbf{R}\mathbf{E}_i$$

where

$$\mathbf{E}_r = \begin{bmatrix} E_{r\|} \\ E_{r\perp} \end{bmatrix}, \mathbf{R} = \begin{bmatrix} R_{\|} & 0 \\ 0 & R_{\perp} \end{bmatrix}, \mathbf{E}_i = \begin{bmatrix} E_{i\|} \\ E_{i\perp} \end{bmatrix}$$

similarly

$$\mathbf{E}_r = \mathbf{T}\mathbf{E}_i$$

where

$$\mathbf{E}_t = \begin{bmatrix} E_{t\|} \\ E_{t\perp} \end{bmatrix}, \mathbf{T} = \begin{bmatrix} T_{\|} & 0 \\ 0 & T_{\perp} \end{bmatrix}, \mathbf{E}_i = \begin{bmatrix} E_{i\|} \\ E_{i\perp} \end{bmatrix}$$

Loss media

Fresnel reflection and transmission coefficients still hold given

$$Z = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

Example

A linear polarised plane wave in free space with an electric field amplitude 30 V/m is incident with angle 30º onto a plane boundary of a lossless non-magnetic medium with ε_r=2. The electric field is parallel to the plane of incidence.

Calculate

a) angle of reflection
b) angle of refraction
c) amplitude of transmitted electric and magnetic fields
d) amplitude of electric field

Example solution

a) Wave and boundary are plane. Snell's law gives reflection angle of 30º.

b) Use Snell's law

$$\theta_t = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right) = \sin^{-1}\left(\frac{\sqrt{\mu_{r1}\varepsilon_{r1}}}{\sqrt{\mu_{r2}\varepsilon_{r2}}}\sin\theta_i\right) = \sin^{-1}\left(\frac{\sqrt{1}}{\sqrt{2}}\sin 30\right) = 20.7^\circ$$

c) Find $T_{\|}$ with $Z_1 = Z_0 = 377\Omega$ and $Z_2 = Z_0/\mathrm{sqrt}(\varepsilon_{r2}) = 377/\mathrm{sqrt}(2)\Omega$

$$T_{\|} = \frac{2\cos\theta_i}{\cos\theta_t + \frac{Z_1}{Z_2}\cos\theta_i} = \frac{2\cos 30^\circ}{\cos 20.7^\circ + \sqrt{2}\cos 30^\circ} \approx 0.8$$

$$E_t = E_i T_{\|} = 0.8\cdot 30 = 24Vm^{-1} \quad H_t = \frac{E_t}{Z_2} = \frac{24\sqrt{2}}{377} \approx 0.09\,Am^{-1}$$

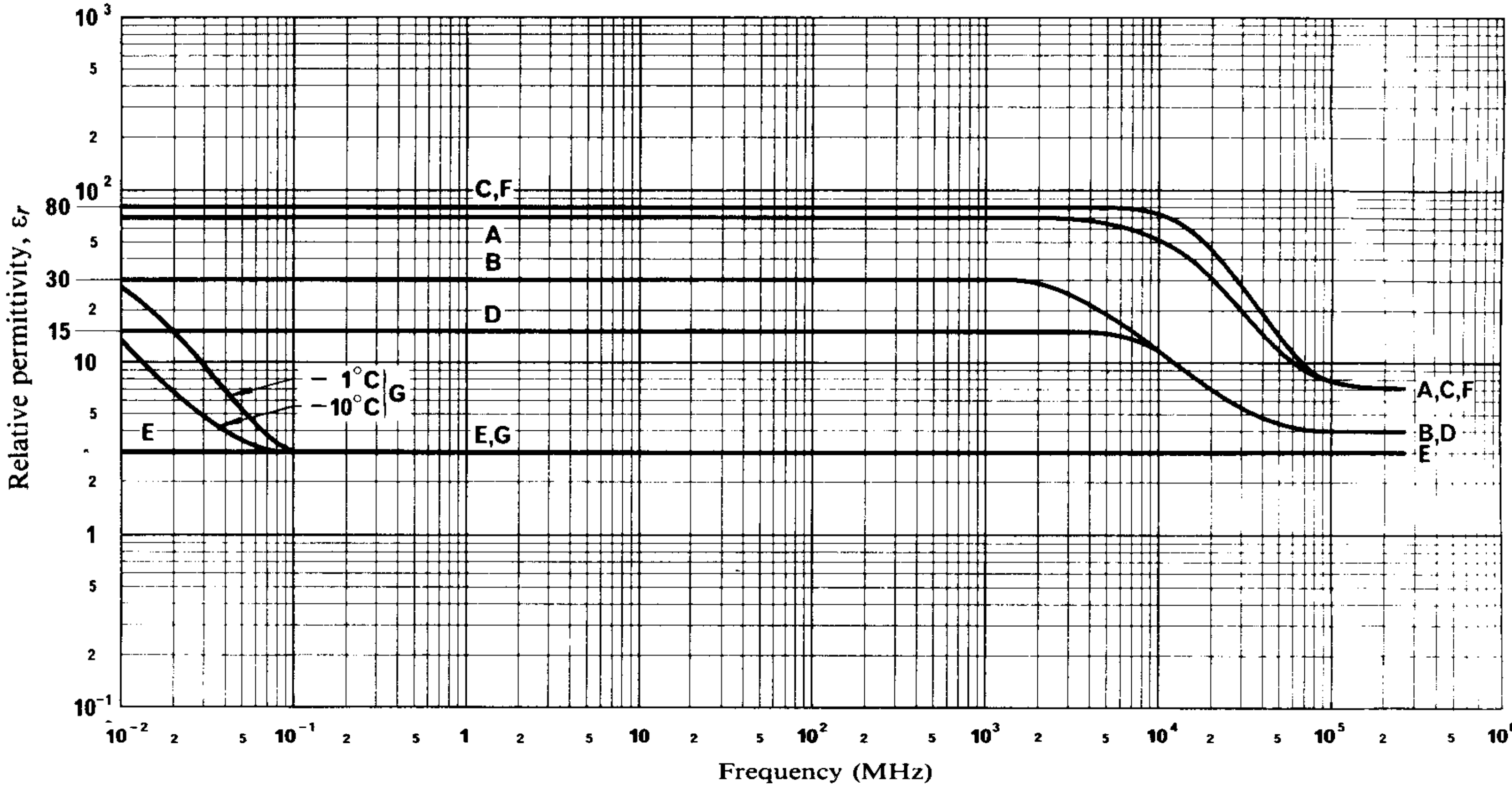
d) Reflected electric field

$$E_r = R_{\|}E_i = \frac{\frac{Z_1}{Z_2}\cos\theta_i - \cos\theta_t}{\cos\theta_t + \frac{Z_1}{Z_2}\cos\theta_i}E_i \approx -8.54\cdot 30 = -256.2Vm^{-1}$$

Typical transmission and reflection coefficients

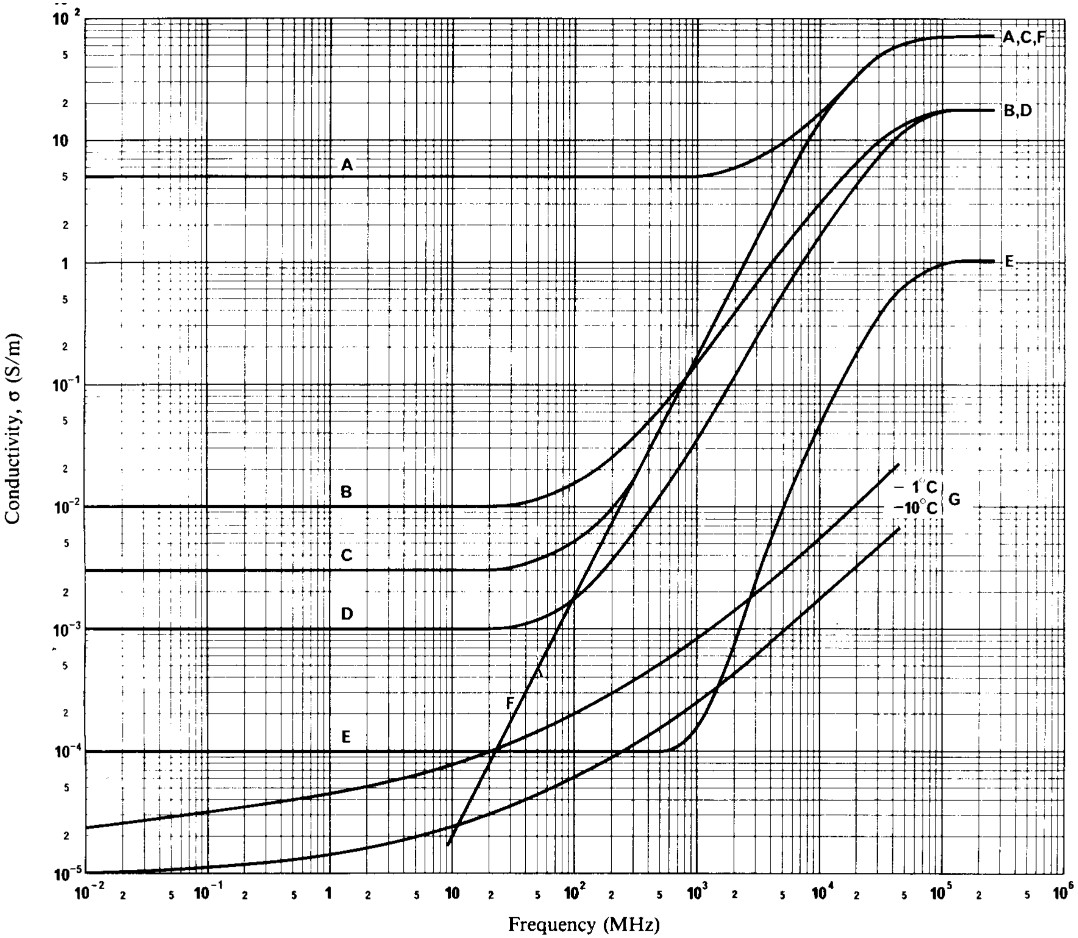
Surface	Conductivity, σ [S m^{-1}]	Relative dielectric constant, ε_r
Dry ground	0.001	4-7
Average ground	0.005	15
Wet ground	0.02	25-30
Sea water	5	81
Fresh water	0.01	81

Relative permittivity, ε_r (ITU-R Rec.P.527)



A: sea water (average salinity), 20° C
B: wet ground
C: fresh water, 20° C
D: medium dry ground
E: very dry ground
F: pure water, 20° C
G: ice (fresh water)

Conductivity, σ (ITU-R Rec.P.527)



A: sea water (average salinity), 20° C
B: wet ground
C: fresh water, 20° C
D: medium dry ground
E: very dry ground
F: pure water, 20° C
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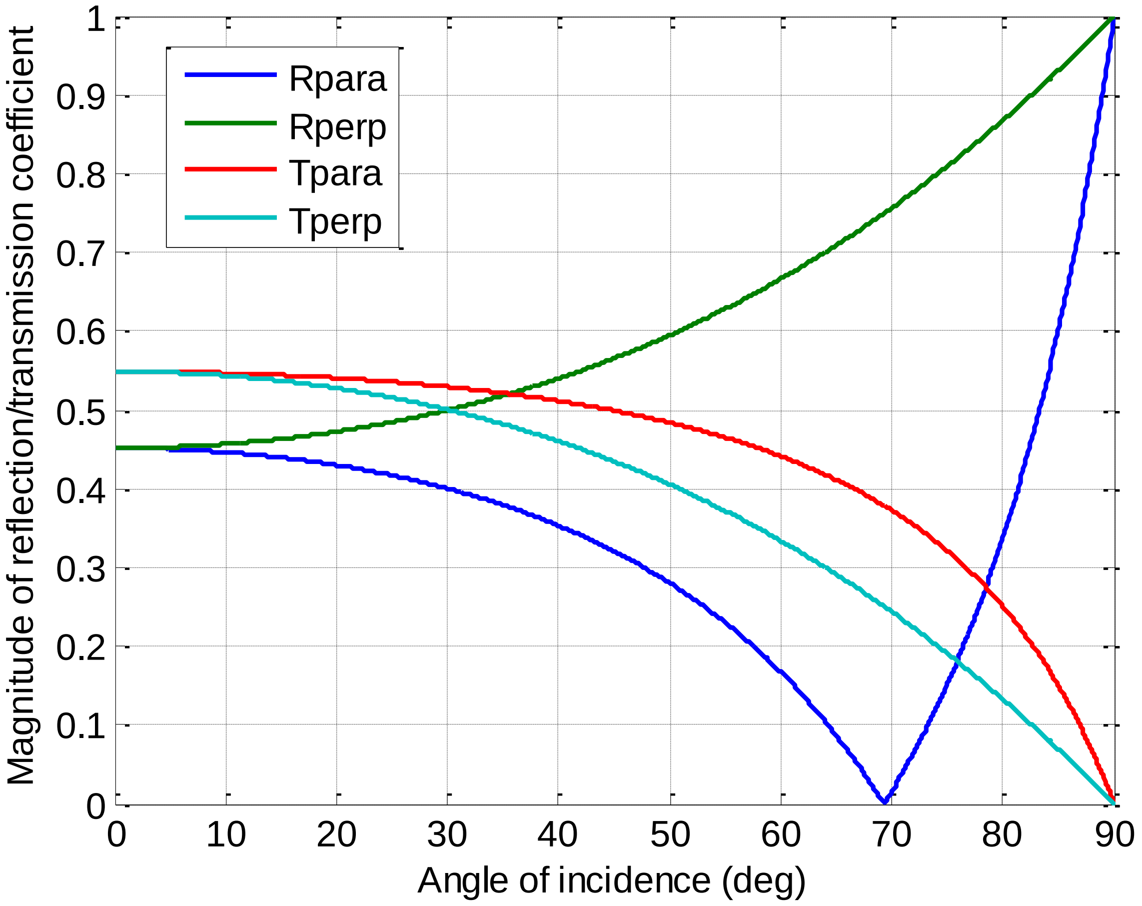
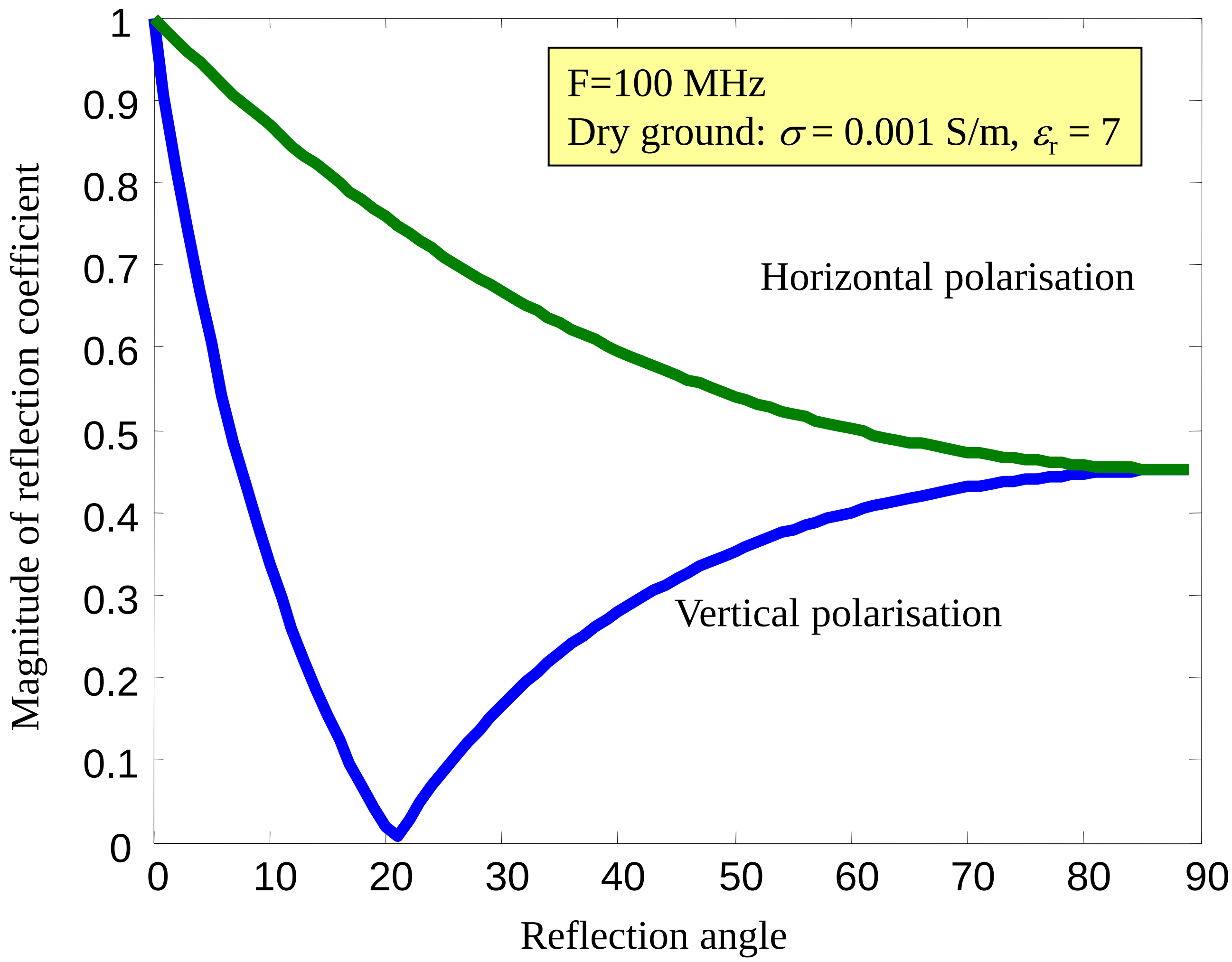
Reflection and transmission coefficients

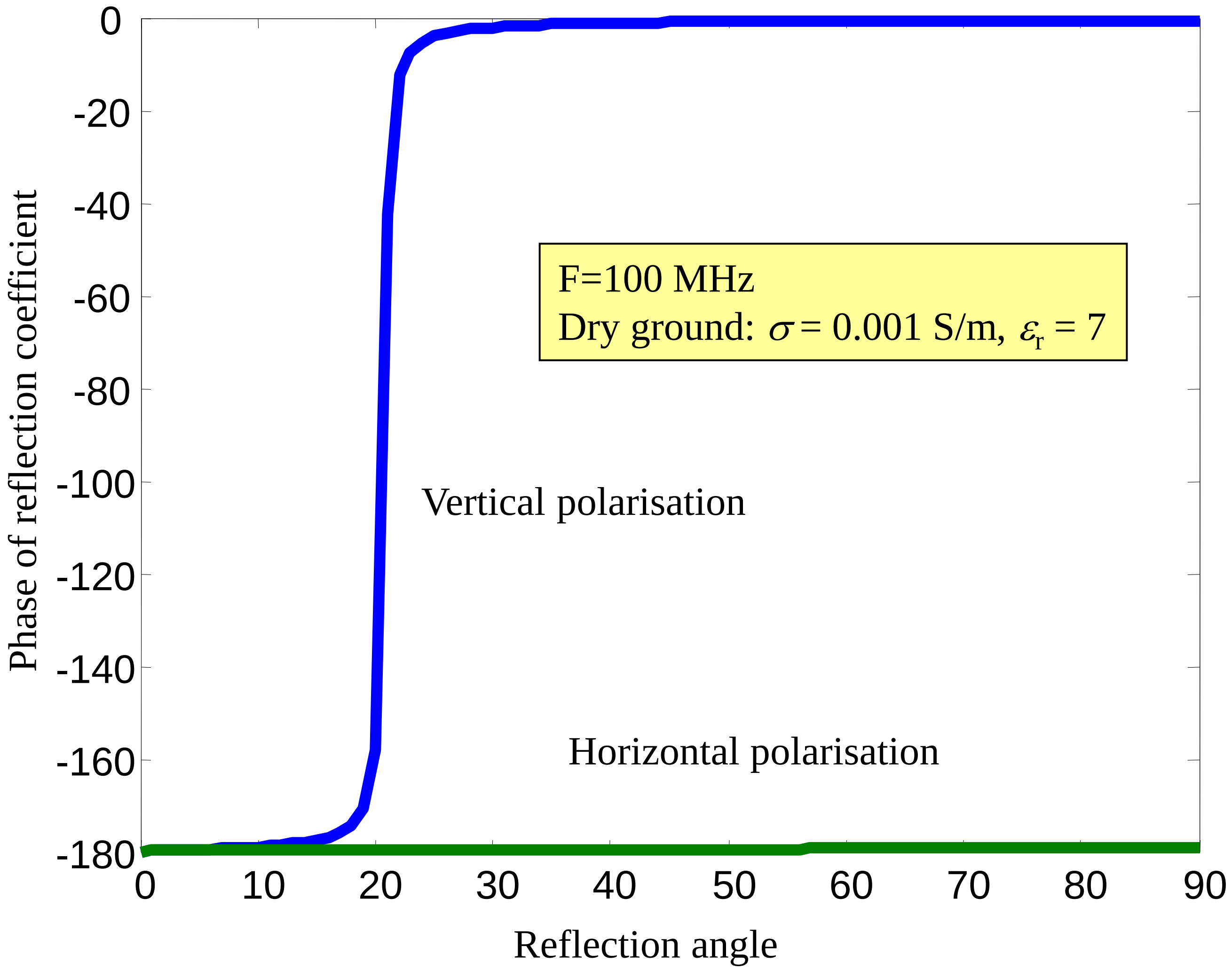
F=100 MHz
Dry ground: $\sigma = 0.001$ S/m, $\varepsilon_r = 7$



Example absolute value of reflection coefficient



Example phase of reflection coefficient



Brewster's angle

- The reflection example:
  $\parallel$ represents the vertical polarisation
  $\perp$ represents the horizontal polarisation
- Vertical polarisation reflected amplitude goes to zero at angle θ_B, Brewster's angle

$$\theta_B = \tan^{-1}\frac{n_2}{n_1}$$

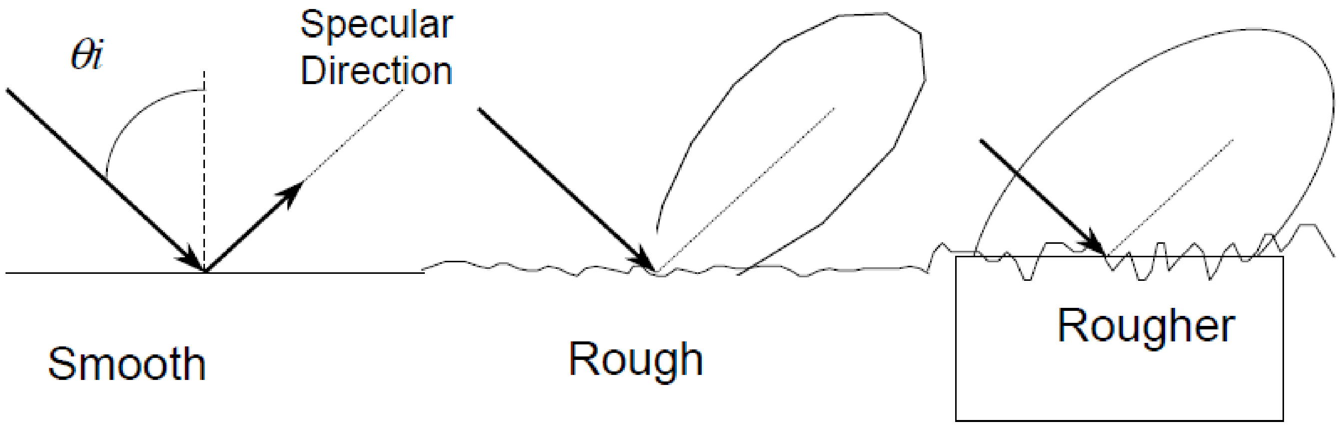
- At reflection angle of 0, i.e., incidence angle of 90, the reflected amplitude is -1. This is the grazing angle
- Note that vertical polarisation undergoes a phase shift from 0 to -180

Change of polarisation state

$$R_{co} = \frac{1}{2}\left(R_{\parallel} + R_{\perp}\right) \quad \text{and} \quad R_{cx} = \frac{1}{2}\left(R_{\parallel} - R_{\perp}\right)$$

	$\theta_i < \theta_B$	$\theta_i > \theta_B$
Right-hand circular	Left-hand elliptical	Right-hand elliptical
Left-hand circular	Right-hand elliptical	Left-hand elliptical
Right -hand elliptical	Left-hand elliptical (axial ratio changed)	Right-hand elliptical (axial ratio changed)
Left-hand elliptical	Right-hand elliptical (axial ratio changed)	Left-hand elliptical (axial ratio changed)

Rough surface



Roughness depends on :

- Surface height range
- Angle of incidence
- Wavelength

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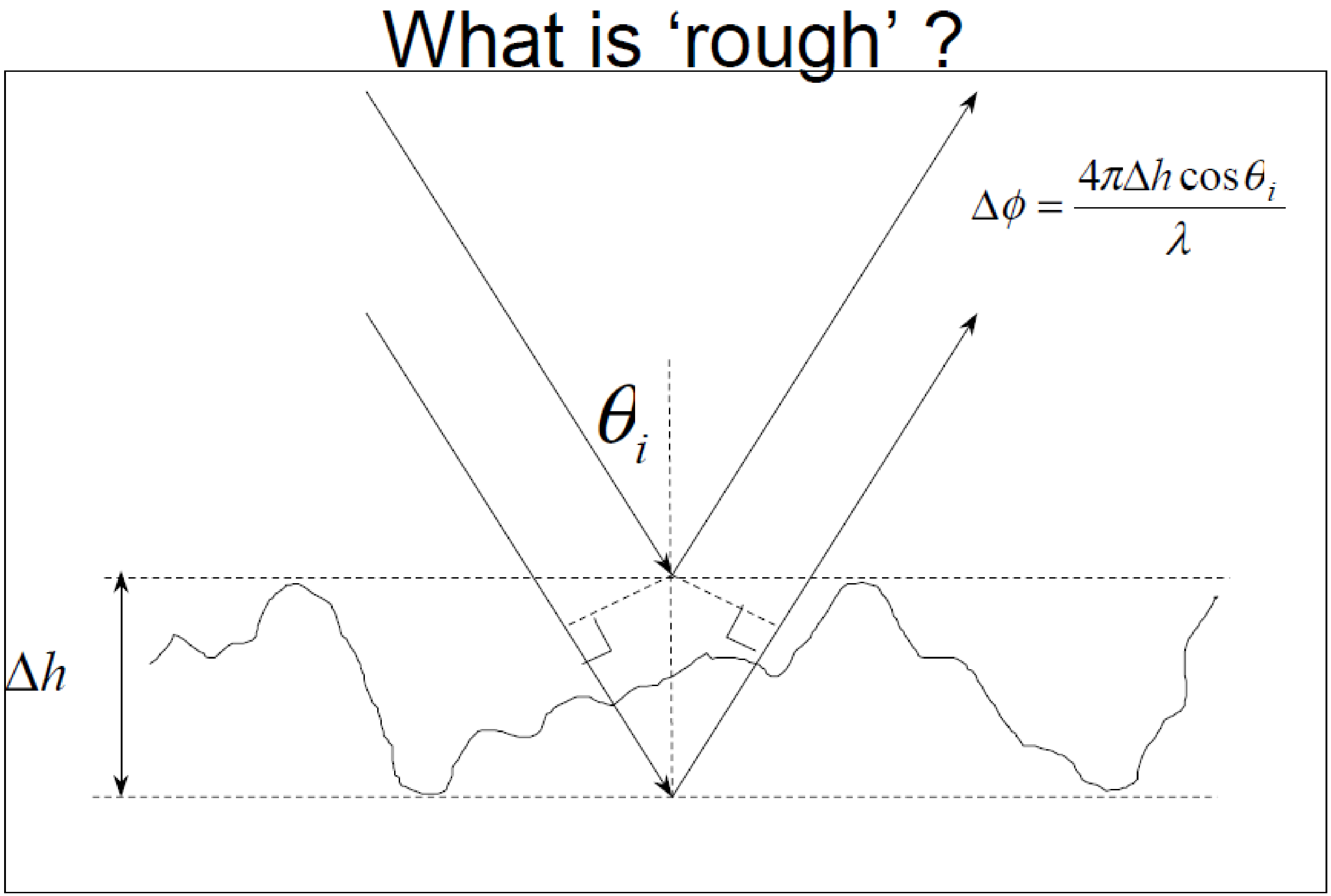
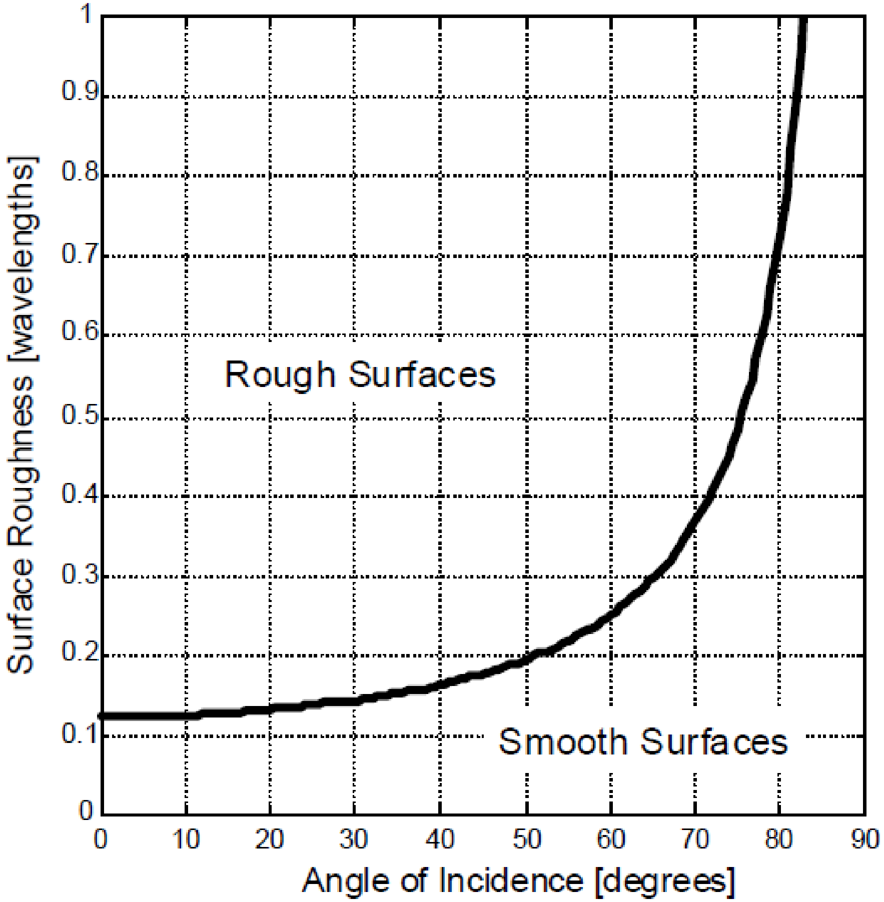
Definition of rough surface



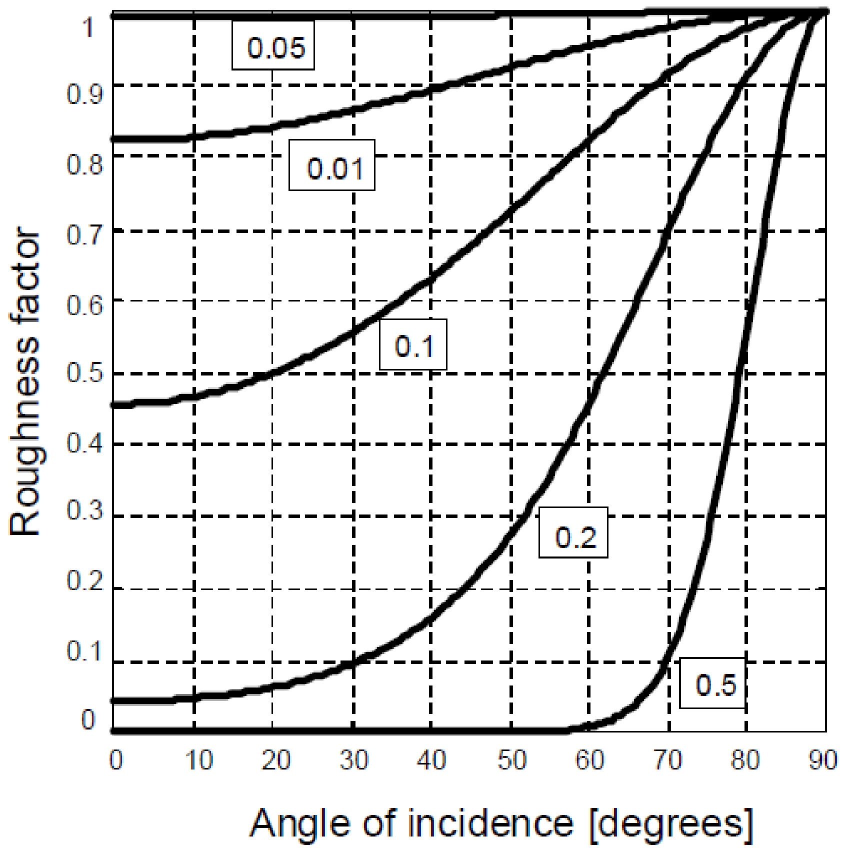
Rayleigh criterion



$$\Delta\phi < \frac{\pi}{2}$$

$$\Delta h < \frac{\lambda}{8\cos\theta_i}$$

Roughness factor *f*



$$R_{eff} = Rf(\sigma_s)$$

$$f(\sigma_s) = \exp\left[-\frac{1}{2}\left(\frac{4\pi\sigma_s \cos\theta}{\lambda}\right)^2\right]$$

Conclusion

- Reflection and refraction
- Transmission
- Rough surface scattering