Assignment – 1

ME630A

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Q1 The MATLAB code is included in pdf as well as the required scripts are included in the assignment.

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ME630

ASSIGNMENT – 1

Q1: Solve $\frac{d\bar{y}}{dx} = \bar{y}$ for $0 \leq x \leq 1$ with boundary Conditions $\bar{y} = 1$ at $x = 0$.

y = exact solution $= e^x$ —— (1)

For N=1, N=2, N=4, N=5, N=7 and find $\bar{y}$ and compare the result with exact sol^n. Plot the error $(\bar{y} - y)$ Vs x

Sol^n

Let the sol^n by FEM be given by

$$\tilde{y}(x) = \tilde{y}_0(x) + \sum_{j=1}^{N} a_j \phi_j(x) \quad \text{—— (2)}$$

Let us choose $\phi_j(x)$ as x^j

$$\therefore \tilde{y}(x) = \tilde{y}_0(x) + \sum_{j=1}^{N} a_j \phi_j(x)$$

from boundary condition we have

$$\tilde{y}(0) = 1 = \tilde{y}_0 + 0$$

$$\therefore \tilde{y}_0 = 1$$

∴ We now have

$$\tilde{y}(x) = 1 + \sum_{j=1}^{N} a_j x^j \quad \text{——— (3)}$$

We need to find R

we know $\mathcal{L}(\tilde{y}) = R$

$$\therefore \frac{d}{dx}\tilde{y} - \tilde{y} = R$$

$$\therefore \Rightarrow R = \frac{d}{dx}\left(1 + \sum_{j=1}^{N} a_j x^j\right) - 1 - \sum_{j=1}^{N} a_j x^j$$

$$\Rightarrow R = 0 + \sum_{j=1}^{N} a_j \cdot j \cdot x^{j-1} - 1 - \sum_{j=1}^{N} a_j x^j$$

$$\Rightarrow R = -1 + \sum_{j=1}^{N} a_j \left(j x^{j-1} - x^j\right) \quad \text{——— (4)}$$

a_j's will be determined by setting the integral of the weighted residual over the computational domain equal to 0

$$\therefore \int_{x=0}^{x=1} W_m(x) \cdot R \, dx = 0$$

We can take $W_m(x)$ the family of $\phi_j(x)$ and hence here let us take

$$W_m(x) = x^{m-1} \quad \text{with } m = 1, 2 \ldots N$$

$\therefore$ we get $\int_{x=0}^{x=1} W_m(x) \cdot R\, dx = 0$ as

$$\int_0^1 x^{m-1}\left(-1 + \sum_{j=1}^{N} a_j\left(jx^{j-1} - x^j\right)\right)dx = 0 \quad \text{—— (5)}$$

Now

i) $\boxed{N=1}$

we have

$m=1$ and $N=1$

$\Rightarrow$ from $\int_0^1 x^{m-1}\left(-1 + \sum_{j=1}^{N} a_j\left(jx^{j-1} - x^j\right)\right)dx = 0$

$$\Rightarrow \int_0^1 \left(-1 + \sum_{j=1}^{1} a_j\left(jx^{j-1} - x^j\right)\right)dx = 0$$

$$\Rightarrow \int_0^1 \left(-1 + a_1\left(1 \cdot x^0 - x^1\right)\right)dx$$

$$\Rightarrow \int_0^1 \left(-1 + a_1(1 - x)\right)dx$$

$$\Rightarrow \int_0^1 (-1 + a_1 - a_1 x)\, dx = 0$$

$$\Rightarrow \left[-x + a_1 x - \frac{a_1 x^2}{2} \right]_{x=0}^{x=1} = 0$$

$$\Rightarrow -1 + a_1 - \frac{a_1}{2} - 0 = 0$$

$$\Rightarrow \frac{a_1}{2} = 1 \quad \text{—— (ai)}$$

$$\Rightarrow \boxed{a_1 = 2}$$

$$\therefore \tilde{y} = 1 + \sum_{j=1}^{1} a_j x^j$$

$$\tilde{y} = 1 + a_1 x$$

$$\boxed{\tilde{y} = 1 + 2x} \quad \text{Sol}^{n} \text{ by FEM.}$$

$$\boxed{y = e^x} \quad \text{exact sol}^{n}$$

ii) $\boxed{N=2}$

if $m=1$ and $N=2$

from eq ⑤ we have.

$$\int_0^1 x^{m-1}\left(-1+\sum_{j=1}^{N} a_j\left(jx^{j-1}-x^j\right)\right)dx=0$$

replacing for $m=1$ and $N=2$ we get

$$\int_0^1 1\cdot\left(-1+\sum_{j=1}^{2} a_j\left(jx^{j-1}-x^j\right)\right)dx=0$$

$$\Rightarrow \int_0^1\left(-1+a_1(1-x)+a_2(2x-x^2)\right)dx=0$$

$$\Rightarrow \left.\left(-x+a_1\left(x-\frac{x^2}{2}\right)+a_2\left(x^2-\frac{x^3}{3}\right)\right)\right|_0^1=0$$

$$\Rightarrow -1+\frac{a_1}{2}+\frac{2a_2}{3}=0$$

$$\Rightarrow \frac{a_1}{2}+\frac{2a_2}{3}=1 \quad \text{——— (aii)}$$

Similarly if $m=2$, $N=2$ we get from eq (5)

$$\int_0^1 x\left(-1 + a_1(1-x) + a_2(2x-x^2)\right) dx = 0$$

$$\Rightarrow \int_0^1 \left(-x + a_1(x-x^2) + a_2(2x^2-x^3)\right) dx = 0$$

$$\Rightarrow \left(-\frac{x^2}{2} + a_1\left(\frac{x^2}{2}-\frac{x^3}{3}\right) + a_2\left(2\frac{x^3}{3}-\frac{x^4}{4}\right)\right)\Bigg|_0^1 = 0$$

$$\Rightarrow -\frac{1}{2} + a_1 \times \frac{1}{6} + \frac{5}{12}a_2 = 0$$

$$\frac{a_1}{6} + \frac{5a_2}{12} = \frac{1}{2} \quad \text{——— (bii)}$$

$\therefore$ We get

$$\frac{a_1}{2} + \frac{2a_2}{3} = 1$$

$$\frac{a_1}{6} + \frac{5a_2}{12} = \frac{1}{2}$$

$$\Rightarrow \begin{bmatrix} 1/2 & 2/3 \\ 1/6 & 5/12 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

Solving by gaussian elimination

$R_2 \to R_2 - \frac{1}{3}R_1$

R_2 : 2nd Row
R_1 : 1st Row
and so on

we get

$$\begin{bmatrix} 1/2 & 2/3 \\ 0 & 7/36 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/6 \end{bmatrix}$$

$$\Rightarrow \quad \frac{7}{36} a_2 = \frac{1}{6}$$

$$a_2 = 6/7$$

and $\frac{1}{2} a_1 + \frac{2}{\cancel{3}_1} \times \frac{\cancel{6}^2}{7} = 1$

$$\frac{a_1}{2} = 1 - 4/7 = \frac{3}{7}$$

$$a_1 = 6/7$$

$\therefore$ from eq (3)

$$\tilde{y} = 1 + \sum_{j=1}^{N} a_j x^j \qquad N = 2 \text{ here.}$$

$$\Rightarrow \quad \tilde{y} = 1 + a_1 x + a_2 x^2$$

$$\Rightarrow \boxed{\tilde{y} = 1 + \frac{6}{7}x + \frac{6}{7}x^2}$$ Sol^n by FEM for $N=2$

$$\boxed{y = e^x}$$ exact Sol^n.

for higher Values of N it is ~~not~~ tedious to compute ~~for~~ the same integral again and again $\therefore$ We find a general integral form eq-(5) and Substitute for different Values of m and N.

$\therefore$ from eq (5) we have.

$$\int_0^1 x^{m-1}\left(-1+\sum_{j=1}^{N} a_j\left(jx^{j-1}-x^{j}\right)\right)dx = 0$$

performing this ~~ing~~ integral for any general m and N we have.

$$\int_0^1 \left(-x^{m-1}+\sum_{j=1}^{N} a_j\left(jx^{m+j-2}-x^{j+m-1}\right)\right)dx = 0$$

$$\Rightarrow \left[-\frac{x^m}{m}+\sum_{j=1}^{N} a_j\left(j\frac{x^{m+j-1}}{m+j-1}-\frac{x^{j+m}}{j+m}\right)\right]_{x=0}^{x=1} = 0$$

$$\Rightarrow -\frac{1}{m}+\sum_{j=1}^{N} a_j\left(\frac{j}{m+j-1}-\frac{1}{m+j}\right) = 0$$

$$\Rightarrow \sum_{j=1}^{N} a_j\left(\frac{j}{m+j-1} - \frac{1}{m+j}\right) = \frac{1}{m} \quad \text{——— } (6)$$

iii) $\boxed{N=4}$

if $m=1$ and $N=4$ we get from eq (6)

$$\sum_{j=1}^{4} a_j\left(\frac{j}{m+j-1} - \frac{1}{m+j}\right) = \frac{1}{m}$$

$$\Rightarrow a_1\left(\frac{1}{1+1-1} - \frac{1}{1+1}\right) + a_2\left(\frac{2}{1+2-1} - \frac{1}{1+2}\right) + a_3\left(\frac{3}{1+3-1} - \frac{1}{1+3}\right) + a_4\left(\frac{4}{1+4-1} - \frac{1}{1+4}\right) = \frac{1}{1}$$

$$\Rightarrow \frac{a_1}{2} + \frac{2a_2}{3} + \frac{3a_3}{4} + \frac{4a_4}{5} = 1 \quad \text{——— } (a\,iii)$$

Similarly for $m=2$, $N=4$ weget from eq (6)

$$\sum_{j=1}^{4} a_j\left(\frac{j}{m+j-1} - \frac{1}{m+j}\right) = \frac{1}{2}$$

$$\Rightarrow a_1\left(\frac{1}{2+1-1} - \frac{1}{2+1}\right) + a_2\left(\frac{2}{2+2-1} - \frac{1}{2+2}\right) + a_3\left(\frac{3}{2+3-1} - \frac{1}{2+3}\right) + a_4\left(\frac{4}{2+4-1} - \frac{1}{2+4}\right) = \frac{1}{2}$$

$$= \frac{a_1}{6} + \frac{5a_2}{12} + \frac{11}{20}a_3 + \frac{19}{30}a_4 = \frac{1}{2} \quad \text{——— } (b\,iii)$$

Similarly for $m=3$, $N=4$ we get from eq 6

$$\sum_{j=1}^{4} a_j \left(\frac{j}{m+j-1} - \frac{1}{m+j} \right) = \frac{1}{m}$$

$$\Rightarrow a_1\left(\frac{1}{3+1-1} - \frac{1}{3+1}\right) + a_2\left(\frac{2}{3+2-1} - \frac{1}{3+2}\right) + a_3\left(\frac{3}{3+3-1} - \frac{1}{3+3}\right) + a_4\left(\frac{4}{3+4-1} - \frac{1}{3+4}\right) = \frac{1}{3}$$

$$\Rightarrow \frac{a_1}{12} + \frac{3a_2}{10} + \frac{13a_3}{30} + \frac{11a_4}{21} = \frac{1}{3} \quad \text{— (c iii)}$$

Similarly for $m=4$, $N=4$ we get from eq (6)

$$\sum_{j=1}^{4} a_j \left(\frac{j}{m+j-1} - \frac{1}{m+j} \right) = \frac{1}{m}$$

$$\Rightarrow a_1\left(\frac{1}{4+1-1} - \frac{1}{4+1}\right) + a_2\left(\frac{2}{4+2-1} - \frac{1}{4+2}\right) + a_3\left(\frac{3}{4+3-1} - \frac{1}{4+3}\right) + a_4\left(\frac{4}{4+4-1} - \frac{1}{4+4}\right) = \frac{1}{4}$$

$$\Rightarrow \frac{a_1}{20} + \frac{7a_2}{30} + \frac{5a_3}{14} + \frac{25a_4}{56} = \frac{1}{4} \quad \text{(d iii)}$$

Writing all the eq^n together we get :-

$$\frac{a_1}{2} + \frac{2a_3}{3} + \frac{3a_3}{4} + \frac{4a_4}{5} = 1$$

$$\frac{a_1}{6} + \frac{5a_2}{12} + \frac{11a_3}{20} + \frac{19a_4}{30} = \frac{1}{2}$$

$$\frac{a_1}{12} + \frac{3a_2}{10} + \frac{13a_3}{30} + \frac{11a_4}{21} = \frac{1}{3}$$

$$\frac{a_1}{20} + \frac{7a_2}{30} + \frac{5a_3}{14} + \frac{25a_4}{56} = \frac{1}{4}$$

in matrix form

$$\begin{bmatrix} 1/2 & 2/3 & 3/4 & 4/5 \\ 1/6 & 5/12 & 11/20 & 19/30 \\ 1/12 & 3/10 & 13/30 & 11/21 \\ 1/20 & 7/30 & 5/14 & 25/56 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix}$$

Solving the above system of equation we get —

$a_1 = 0.999000999000999$
$a_2 = 0.509490509490526$
$a_3 = 0.139860139860114$
$a_4 = 0.0699300699300 82$

by solving using matlab inversion operator.

(iv) $\boxed{N = 5}$

* if $m=1$ and $N=5$ we get from eq (6)

$$\sum_{j=1}^{5} a_j\left(\frac{j}{m+j-1} - \frac{1}{m+j}\right) = \frac{1}{m}$$

$$\Rightarrow a_1\left(\frac{1}{1+1-1} - \frac{1}{1+1}\right) + a_2\left(\frac{2}{1+2-1} - \frac{1}{1+2}\right) + a_3\left(\frac{3}{1+3-1} - \frac{1}{1+3}\right)$$
$$+ a_4\left(\frac{4}{1+4-1} - \frac{1}{1+4}\right) + a_5\left(\frac{5}{1+5-1} - \frac{1}{1+5}\right) = 1$$

$$\Rightarrow \frac{a_1}{2} + \frac{2a_2}{3} + \frac{3a_3}{4} + \frac{4a_4}{5} + \frac{5a_5}{6} = 1 \quad \text{— } (a_{iv})$$

* if $m=2$ and $N=5$ we get from eq (6)

$$\sum_{j=1}^{5} a_j\left(\frac{j}{m+j-1} - \frac{1}{m+j}\right) = \frac{1}{m}$$

$$\Rightarrow a_1\left(\frac{1}{2+1-1} - \frac{1}{2+1}\right) + a_2\left(\frac{2}{2+2-1} - \frac{1}{2+2}\right) + a_3\left(\frac{3}{2+3-1} - \frac{1}{2+3}\right)$$
$$+ a_4\left(\frac{4}{2+4-1} - \frac{1}{2+4}\right) + a_5\left(\frac{5}{2+5-1} - \frac{1}{2+5}\right) = \frac{1}{2}$$

$$\Rightarrow \frac{a_1}{6} + \frac{5a_2}{12} + \frac{11a_3}{20} + \frac{19a_4}{30} + \frac{29a_5}{42} = \frac{1}{2} \quad \text{— } (b_{iv})$$

* if $m=3$ and $N=5$ we get from eq (6)

$$\sum_{j=1}^{5} a_j\left(\frac{j}{m+j-1}-\frac{1}{m+j}\right)=\frac{1}{m}$$

$$\Rightarrow a_1\left(\frac{1}{3+1-1}-\frac{1}{3+1}\right)+a_2\left(\frac{2}{3+2-1}-\frac{1}{3+2}\right)+a_3\left(\frac{3}{3+3-1}-\frac{1}{3+3}\right)$$
$$+a_4\left(\frac{4}{3+4-1}-\frac{1}{3+4}\right)+a_5\left(\frac{5}{3+5-1}-\frac{1}{3+5}\right)=\frac{1}{3}$$

$$\Rightarrow \frac{a_1}{12}+\frac{3}{10}a_2+\frac{13}{30}a_3+\frac{11a_4}{21}+\frac{33}{56}a_5=\frac{1}{3} \quad \text{— (civ)}$$

* if $m=4$ and $N=5$ we get from eq (6)

$$\Rightarrow a_1\left(\frac{1}{4+1-1}-\frac{1}{4+1}\right)+a_2\left(\frac{2}{4+2-1}-\frac{1}{4+2}\right)+a_3\left(\frac{3}{4+3-1}-\frac{1}{4+3}\right)$$
$$+a_4\left(\frac{4}{4+4-1}-\frac{1}{4+4}\right)+a_5\left(\frac{5}{4+5-1}-\frac{1}{4+5}\right)=\frac{1}{4}$$

$$\Rightarrow \frac{a_1}{20}+\frac{7}{30}a_2+\frac{5}{14}a_3+\frac{25}{56}a_4+\frac{37}{72}a_5=\frac{1}{4} \quad \text{— (div)}$$

* if $m=5$ and $N=5$ we get from eq (6)

$$a_1\left(\frac{1}{5+1-1}-\frac{1}{5+1}\right)+a_2\left(\frac{2}{5+2-1}-\frac{1}{5+2}\right)+a_3\left(\frac{3}{5+3-1}-\frac{1}{5+3}\right)$$
$$+a_4\left(\frac{4}{5+4-1}-\frac{1}{5+4}\right)+a_5\left(\frac{5}{5+5-1}-\frac{1}{5+5}\right)=\frac{1}{5}$$

$$\frac{a_1}{30}+\frac{9a_2}{21}+\frac{17}{56}a_3+\frac{7}{18}a_4+\frac{41}{90}a_5=\frac{1}{5} \quad \text{— (eiv)}$$

Writing all the eq^n together we get

$$\frac{a_1}{2}+\frac{2a_2}{3}+\frac{3a_3}{4}+\frac{4a_4}{5}+\frac{5a_5}{6}=1$$

$$a_1/6+5a_2/12+11a_3/20+19a_4/30+\frac{29a_5}{42}=1/2$$

$$\frac{a_1}{12}+\frac{3a_2}{10}+\frac{13a_3}{30}+\frac{11a_4}{21}+\frac{33a_5}{56}=1/3$$

$$\frac{a_1}{20}+\frac{7a_2}{30}+\frac{5}{14}a_3+\frac{25}{56}a_4+\frac{37a_5}{72}=\frac{1}{4}$$

$$\frac{a_1}{30}+\frac{4a_2}{21}+\frac{17}{56}a_3+\frac{7}{18}a_4+\frac{41}{90}a_5=1/5$$

in matrix form.

$$\begin{bmatrix} 1/2 & 2/3 & 3/4 & 4/5 & 5/6 \\ 1/6 & 5/12 & 11/20 & 19/30 & 29/42 \\ 1/12 & 3/10 & 13/30 & 11/21 & 33/56 \\ 1/20 & 7/30 & 5/14 & 25/56 & 37/72 \\ 1/30 & 4/21 & 17/56 & 7/18 & 41/90 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}=\begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{bmatrix}$$

solving using MATLAB inversion operator we get

$a_1 = 1.00005528221570$

$a_2 = 0.499198407871908$

$a_3 = 0.170269224391334$

$a_4 = 0.034827795897115$

$a_5 = 0.0139311183595 98$

(v) $\boxed{N=7}$

* if $m=1$ and $N=7$ we get from eq (6)

$$\sum_{j=1}^{7} a_j\left(\frac{j}{(m+j-1)} - \frac{1}{m+j}\right) = \frac{1}{m}$$

$$\Rightarrow a_1\left(\frac{1}{1+1-1} - \frac{1}{1+1}\right) + a_2\left(\frac{2}{1+2-1} - \frac{1}{1+2}\right) + a_3\left(\frac{3}{1+3-1} - \frac{1}{1+3}\right) + a_4\left(\frac{4}{1+4-1} - \frac{1}{1+4}\right)$$

$$+ a_5\left(\frac{5}{1+5-1} - \frac{1}{5+1}\right) + a_6\left(\frac{6}{1+6-1} - \frac{1}{1+6}\right) + a_7\left(\frac{7}{1+7-1} - \frac{1}{1+7}\right) = \frac{1}{1}$$

$$\Rightarrow \frac{a_1}{2} + \frac{2}{3}a_2 + \frac{3a_3}{4} + \frac{4a_4}{5} + \frac{5a_5}{6} + \frac{6a_6}{7} + \frac{7a_7}{8} = 1 \quad - (a_v)$$

* if $m=2$ and $N=7$ we get from eq (6)

$$\Rightarrow a_1\left(\frac{1}{2+1-1} - \frac{1}{2+1}\right) + a_2\left(\frac{2}{2+2-1} - \frac{1}{2+2}\right) + a_3\left(\frac{3}{2+3-1} - \frac{1}{2+3}\right)$$

$$+ a_4\left(\frac{4}{2+4-1} - \frac{1}{2+4}\right) + a_5\left(\frac{5}{2+5-1} - \frac{1}{2+5}\right) + a_6\left(\frac{6}{2+6-1} - \frac{1}{2+6}\right)$$

$$+ a_7\left(\frac{7}{2+7-1} - \frac{1}{2+7}\right) = \frac{1}{2} \quad (b_v)$$

$$\Rightarrow \frac{a_1}{6} + \frac{5}{12}a_2 + \frac{11}{20}a_3 + \frac{19}{30}a_4 + \frac{29}{42}a_5 + \frac{41}{56}a_6 + \frac{55}{72}a_7 = \frac{1}{2}$$

* if $m=3$ and $N=7$ we get from eq (6)

$$a_1\left(\frac{1}{3+1-1} - \frac{1}{3+1}\right) + a_2\left(\frac{2}{3+2-1} - \frac{1}{3+2}\right) + a_3\left(\frac{3}{3+3-1} - \frac{1}{3+3}\right) + a_4\left(\frac{4}{3+4-1} - \frac{1}{3+4}\right)$$

$$+ a_5\left(\frac{5}{3+5-1} - \frac{1}{3+5}\right) + a_6\left(\frac{6}{3+6-1} - \frac{1}{3+6}\right) + a_7\left(\frac{7}{3+7-1} - \frac{1}{3+7}\right) = \frac{1}{3}$$

$$— (c_v)$$

$$\Rightarrow \frac{a_1}{12} + \frac{3a_2}{10} + \frac{13}{30}a_3 + \frac{11}{21}a_4 + \frac{33}{56}a_5 + \frac{23}{36}a_6 + \frac{61}{90}a_7 = \frac{1}{3}$$

* if $m=4$ and $N=7$ using eq (6) we get

$$\Rightarrow a_1\left(\frac{1}{4+1-1} - \frac{1}{4+1}\right) + a_2\left(\frac{2}{4+2-1} - \frac{1}{4+2}\right) + a_3\left(\frac{3}{4+3-1} - \frac{1}{4+3}\right) + a_4\left(\frac{4}{4+4-1} - \frac{1}{4+4}\right)$$
$$+ a_5\left(\frac{5}{4+5-1} - \frac{1}{4+5}\right) + a_6\left(\frac{6}{4+6-1} - \frac{1}{4+6}\right) + a_7\left(\frac{7}{4+7-1} - \frac{1}{4+7}\right) = \frac{1}{4}$$

$$\Rightarrow \frac{a_1}{20} + \frac{7}{30}a_2 + \frac{5}{14}a_3 + \frac{25}{56}a_4 + \frac{37}{72}a_5 + \frac{17}{30}a_6 + \frac{67}{110}a_7 = \frac{1}{4}$$

$$— (d_v)$$

* if $m=5$ and $N=7$ using eq (6) we get.

$$\Rightarrow a_1\left(\frac{1}{5+1-1} - \frac{1}{5+1}\right) + a_2\left(\frac{2}{5+2-1} - \frac{1}{5+2}\right) + a_3\left(\frac{3}{5+3-1} - \frac{1}{5+3}\right)$$
$$+ a_4\left(\frac{4}{5+4-1} - \frac{1}{5+4}\right) + a_5\left(\frac{5}{5+5-1} - \frac{1}{5+5}\right) + a_6\left(\frac{6}{5+6-1} - \frac{1}{5+6}\right)$$
$$+ a_7\left(\frac{7}{5+7-1} - \frac{1}{5+7}\right) = \frac{1}{5}$$

$$\Rightarrow \frac{a_1}{30} + \frac{4a_2}{21} + \frac{17}{56}a_3 + \frac{7}{18}a_4 + \frac{41}{90}a_5 + \frac{28a_6}{55} + \frac{73}{132}a_7 = \frac{1}{5}$$

$$— (e_v)$$

* if $m = 6$ and $N = 7$ using eq (6) we get.

$$a_1\left(\frac{1}{6+1-1}-\frac{1}{6+1}\right)+a_2\left(\frac{2}{6+2-1}-\frac{1}{6+2}\right)+a_3\left(\frac{3}{6+3-1}-\frac{1}{6+3}\right)+a_4\left(\frac{4}{6+4-1}-\frac{1}{6+4}\right)$$

$$+a_5\left(\frac{5}{6+5-1}-\frac{1}{6+5}\right)+a_6\left(\frac{6}{6+6-1}-\frac{1}{6+6}\right)+a_7\left(\frac{7}{6+7-1}-\frac{1}{6+7}\right)=\frac{1}{6}$$

$$\frac{a_1}{42}+\frac{9}{56}a_2+\frac{19}{72}a_3+\frac{31}{90}a_4+\frac{9}{22}a_5+\frac{61}{132}a_6+\frac{79}{156}a_7=\frac{1}{6}$$

— (fv)

* if $m = 7$ and $N = 7$ using eq (6) we get

$$a_1\left(\frac{1}{7+1-1}-\frac{1}{7+1}\right)+a_2\left(\frac{2}{7+2-1}-\frac{1}{7+2}\right)+a_3\left(\frac{3}{7+3-1}-\frac{1}{7+3}\right)+a_4\left(\frac{4}{7+4-1}-\frac{1}{7+4}\right)$$

$$+a_5\left(\frac{5}{7+5-1}-\frac{1}{7+5}\right)+a_6\left(\frac{6}{7+6-1}-\frac{1}{7+6}\right)+a_7\left(\frac{7}{7+7-1}-\frac{1}{7+7}\right)=\frac{1}{7}$$

$$\frac{a_1}{56}+\frac{5}{36}a_2+\frac{7}{30}a_3+\frac{17}{55}a_4+\frac{49}{132}a_5+\frac{11}{26}a_6+\frac{85}{182}a_7=\frac{1}{7}$$

— (gv)

Writing all the equations together.

$$\frac{a_1}{2} + \frac{2a_2}{3} + \frac{3a_3}{4} + \frac{4}{5}a_4 + \frac{5a_5}{6} + \frac{6a_6}{7} + \frac{7a_7}{8} = 1$$

$$\frac{a_1}{6} + \frac{5}{12}a_2 + \frac{11}{20}a_3 + \frac{19}{30}a_4 + \frac{29}{42}a_5 + \frac{41}{56}a_6 + \frac{55}{72}a_7 = \frac{1}{2}$$

$$\frac{a_1}{12} + \frac{3a_2}{10} + \frac{13a_3}{30} + \frac{11a_4}{21} + \frac{33}{56}a_5 + \frac{23}{36}a_6 + \frac{61}{90}a_7 = \frac{1}{3}$$

$$\frac{a_1}{20} + \frac{7}{30}a_2 + \frac{5}{14}a_3 + \frac{25}{56}a_4 + \frac{37}{72}a_5 + \frac{17}{30}a_6 + \frac{67}{110}a_7 = \frac{1}{4}$$

$$\frac{a_1}{30} + \frac{4a_2}{21} + \frac{17}{56}a_3 + \frac{7}{18}a_4 + \frac{41}{90}a_5 + \frac{28a_6}{55} + \frac{73}{132}a_7 = \frac{1}{5}$$

$$\frac{a_1}{42} + \frac{9a_2}{56} + \frac{19a_3}{72} + \frac{31a_4}{90} + \frac{9}{22}a_5 + \frac{61a_6}{132} + \frac{79}{156}a_7 = \frac{1}{6}$$

$$\frac{a_1}{56} + \frac{5}{36}a_2 + \frac{7}{30}a_3 + \frac{17a_4}{55} + \frac{49a_5}{132} + \frac{11a_6}{26} + \frac{85}{182}a_7 = \frac{1}{7}$$

in matrix form

$$\begin{bmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \frac{6}{7} & \frac{7}{8} \\ \frac{1}{6} & \frac{5}{12} & \frac{11}{20} & \frac{19}{30} & \frac{29}{42} & \frac{41}{56} & \frac{55}{72} \\ \frac{1}{12} & \frac{3}{10} & \frac{13}{30} & \frac{11}{21} & \frac{33}{56} & \frac{23}{36} & \frac{61}{90} \\ \frac{1}{20} & \frac{7}{30} & \frac{5}{14} & \frac{25}{56} & \frac{37}{72} & \frac{17}{30} & \frac{67}{110} \\ \frac{1}{30} & \frac{4}{21} & \frac{17}{56} & \frac{7}{18} & \frac{41}{90} & \frac{28}{55} & \frac{73}{132} \\ \frac{1}{42} & \frac{9}{56} & \frac{19}{72} & \frac{31}{90} & \frac{9}{22} & \frac{61}{132} & \frac{79}{156} \\ \frac{1}{56} & \frac{5}{36} & \frac{7}{30} & \frac{17}{55} & \frac{49}{132} & \frac{11}{26} & \frac{85}{182} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \\ \frac{1}{7} \end{bmatrix}$$

we get

$a_1 = 1.000000096238539$

$a_2 = 0.499997353451741$

$a_3 = 0.166690036412452$

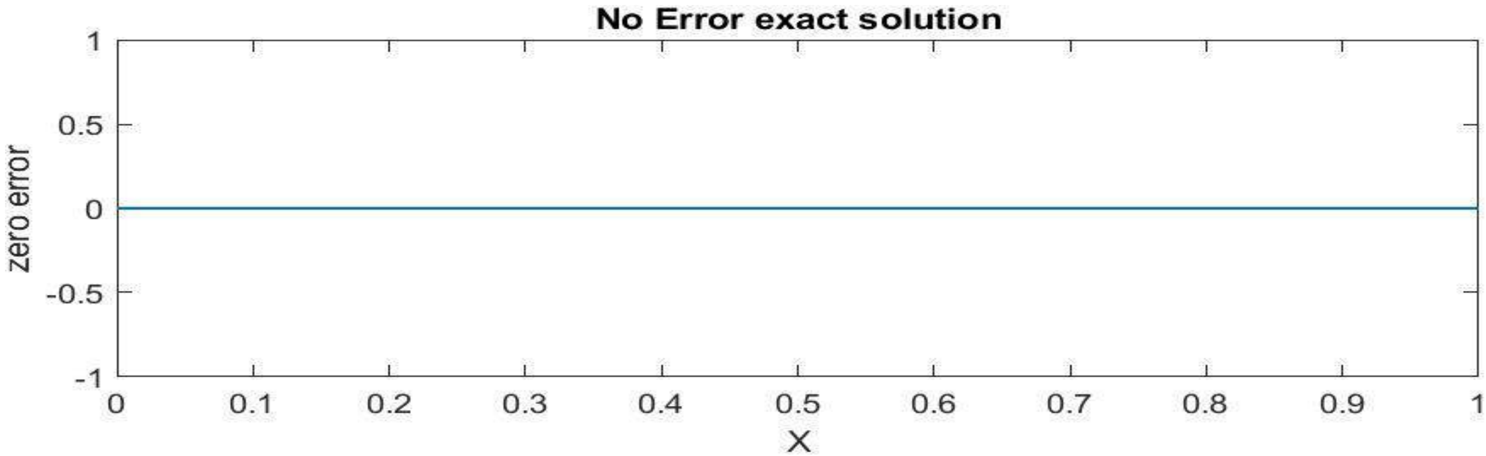
$a_4 = 0.041571459657900$

$a_5 = 0.008536600260243$

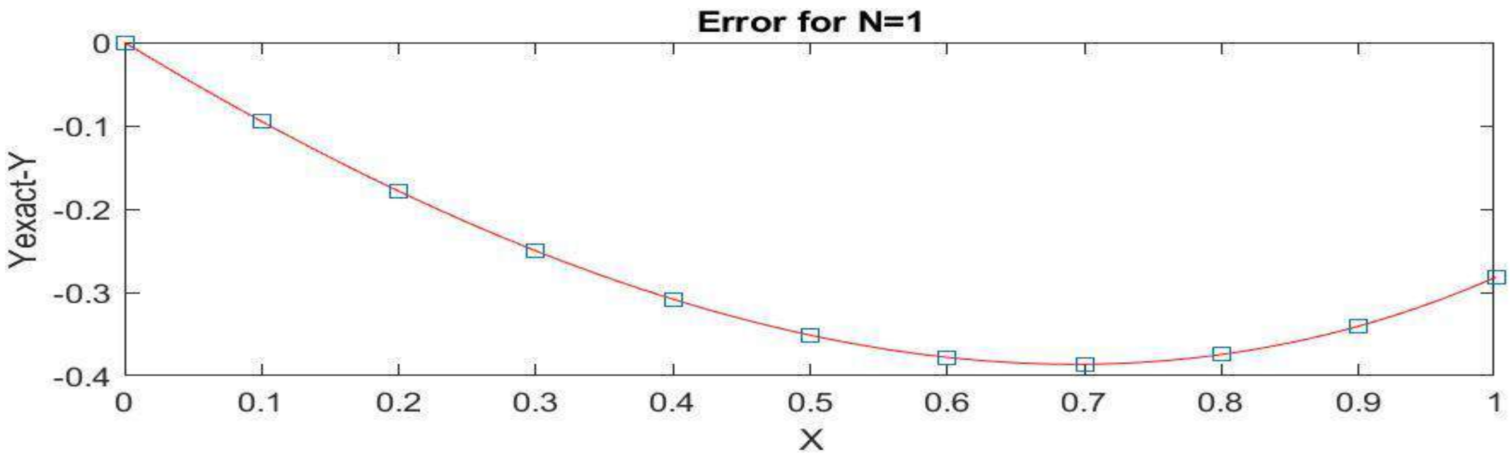
$a_6 = 0.001155997156038$

$a_7 = 0.00033028528133$.

by solving using matlab inversion operator.



Plot for error for N=1 given by above solution



$\therefore$ for N=1

we ~~hay~~ have

$\bar{y} = 1+2x$ for $x \in [0,1]$

$y = e^x$ exact solution

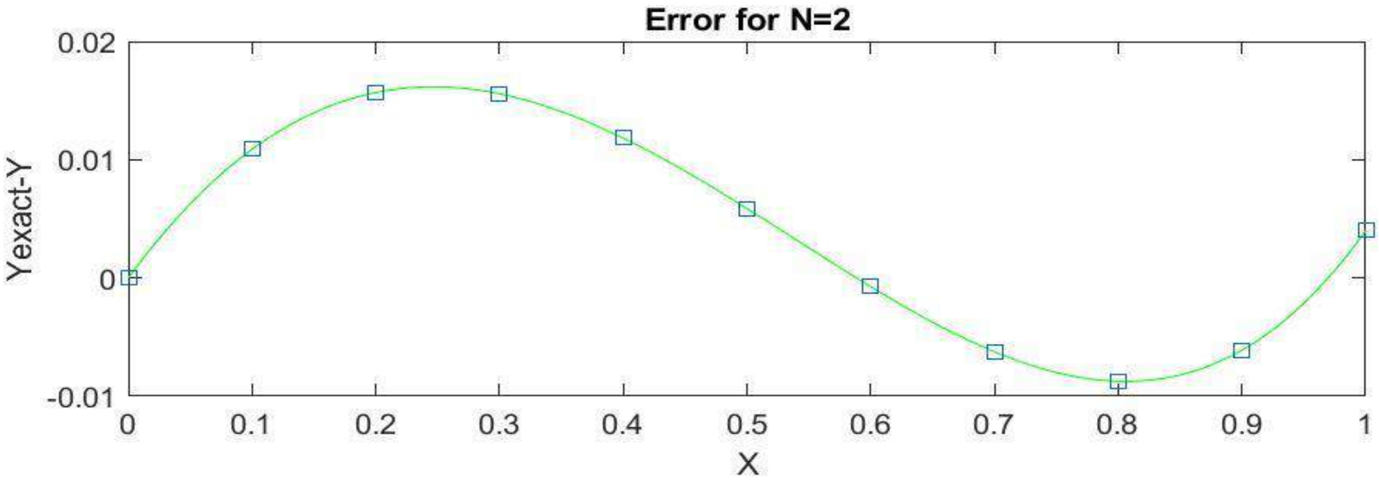
This solution by FEM for N=1 is a linear approximation of the exact solution ~~of~~ over the domain of $x \in [0,1]$

If we plot $(e^x - 1 - 2x)$ vs x, we get the following plot.

The plot shows that linearizing our solution for N=1 has lead to error going as large as "-0.4" for $x \in [0,1]$.

The error is high as it was expected because we have only used N=1.

Plot for error for N=2 given by above solution



$\therefore$ for $N=2$
we can write

$$\tilde{y} = 1 + \frac{6}{7}x + \frac{6x^2}{7}$$

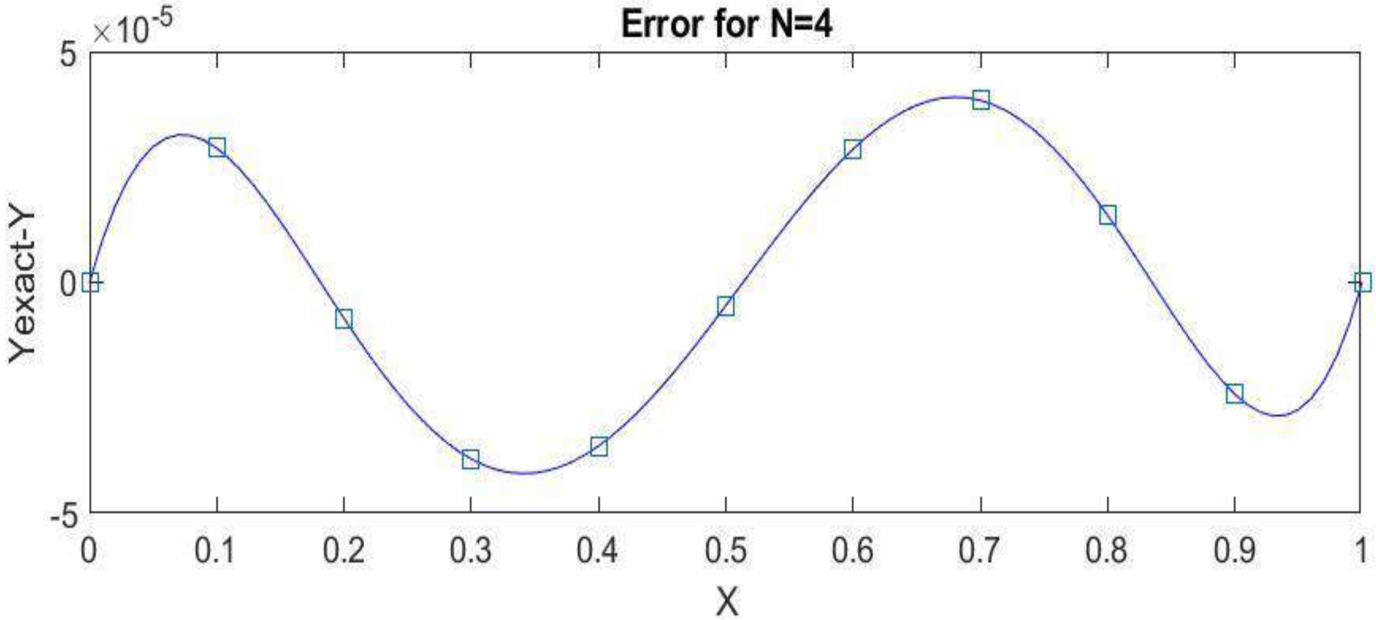
$y = e^x$ exact solution. for $x \in [0,1]$

The plot of $(\underbrace{y - \tilde{y}}_{\text{Error}})$ Vs x is shown below.

~~The~~ From the graph it is clear that the error lies between 0.02 and -0.01 for $x \in [0, 1]$. The error has decreased as expected. Hence solution by FEM ~~by~~ for $N=2$ is a reasonably approximate solution of $\frac{dy}{dx} = y$ for $x \in [0,1]$. However to get more accurate solution the value of N should be increased.

Plot for error for N=4 given by above solution

Plot for error for N=4 given by above solution

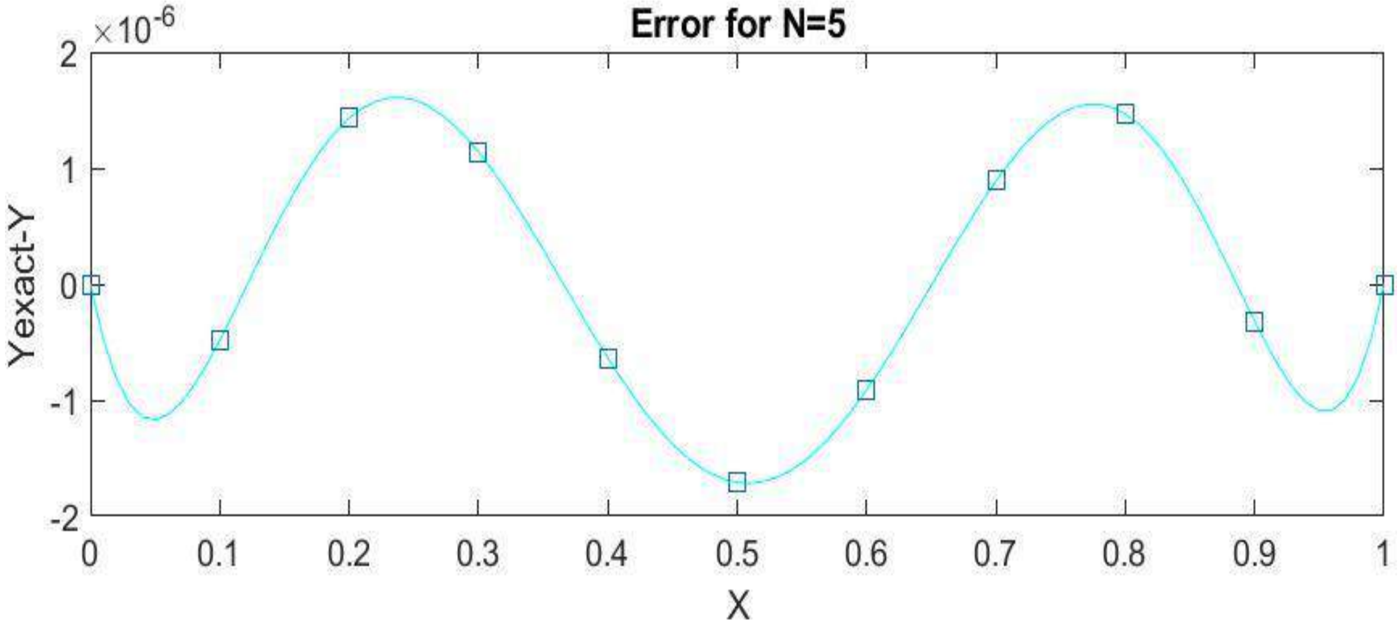


∴ for $N=4$
we can write
$y = e^x$ exact-solution
$$\tilde{y} = 1 + 0.999000999000999x + 0.509490509490x^2 + 0.139860139860114x^3 + 0.069930069930082x^4$$
for $x \in [0,1]$

$(y - \tilde{y})$ vs x is ploted
From the graph we can see that the error further ~~is~~ reduced to lie between $(-5\times10^{-5}, +5\times10^{-5})$ as we increased the value of N. This was expected as taking more number of terms ϕ_j we make our solution by FEM closely agree with exact solution.

The numerical solution for $\frac{dy}{dx} = y$ $\forall x \in [0,1]$ for $N=4$ gave quite ~~good~~ accurate solution with error in $(-5\times10^{-5}, 5\times10^{-5})$ over $x \in [0,1]$.
The solution is expected to become more accurate with increasing value of N.

Plot for error for N=5 given by above solution



For $N=5$

we get—

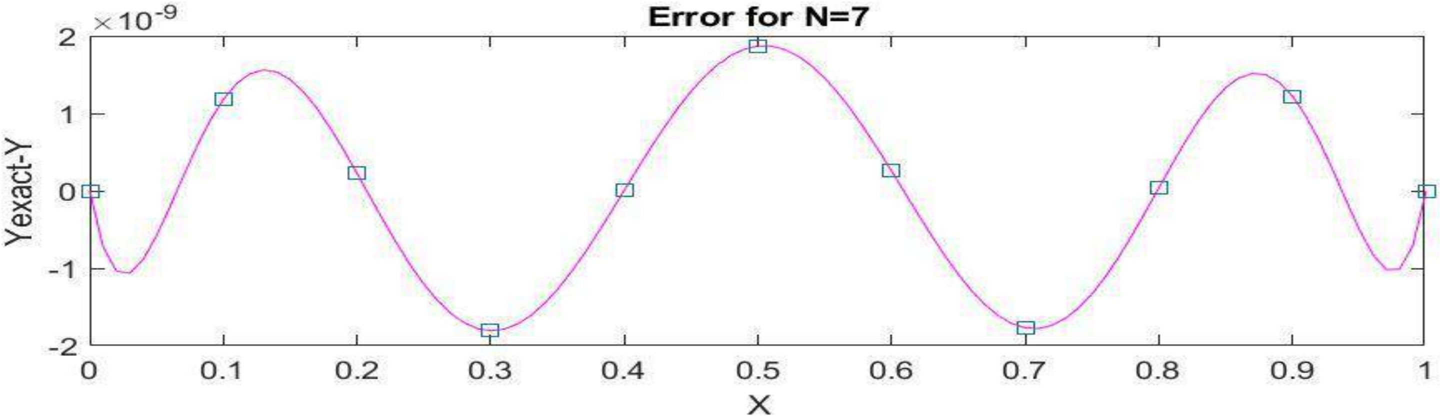
$$\tilde{y} = 1.000055282215740x + 0.499198407871908x^2 + 0.170269224391334x^3 + 0.034827795897115\,x^4 + 0.013931118359598x^5 + 1$$

$y = e^x$ exact solution $\forall x \in [0,1]$

plotting $(\hat{y} - \tilde{y})$ Vs x $\forall x \in [0,1]$ we Can observe that the error lie between 2×10^{-6} to -2×10^{-6} for $x \in [0,1]$

The error has reduced as we increased value of N from $N=4$ to $N=5$.

Plot for error for N=7 given by above solution



for N=7

we get

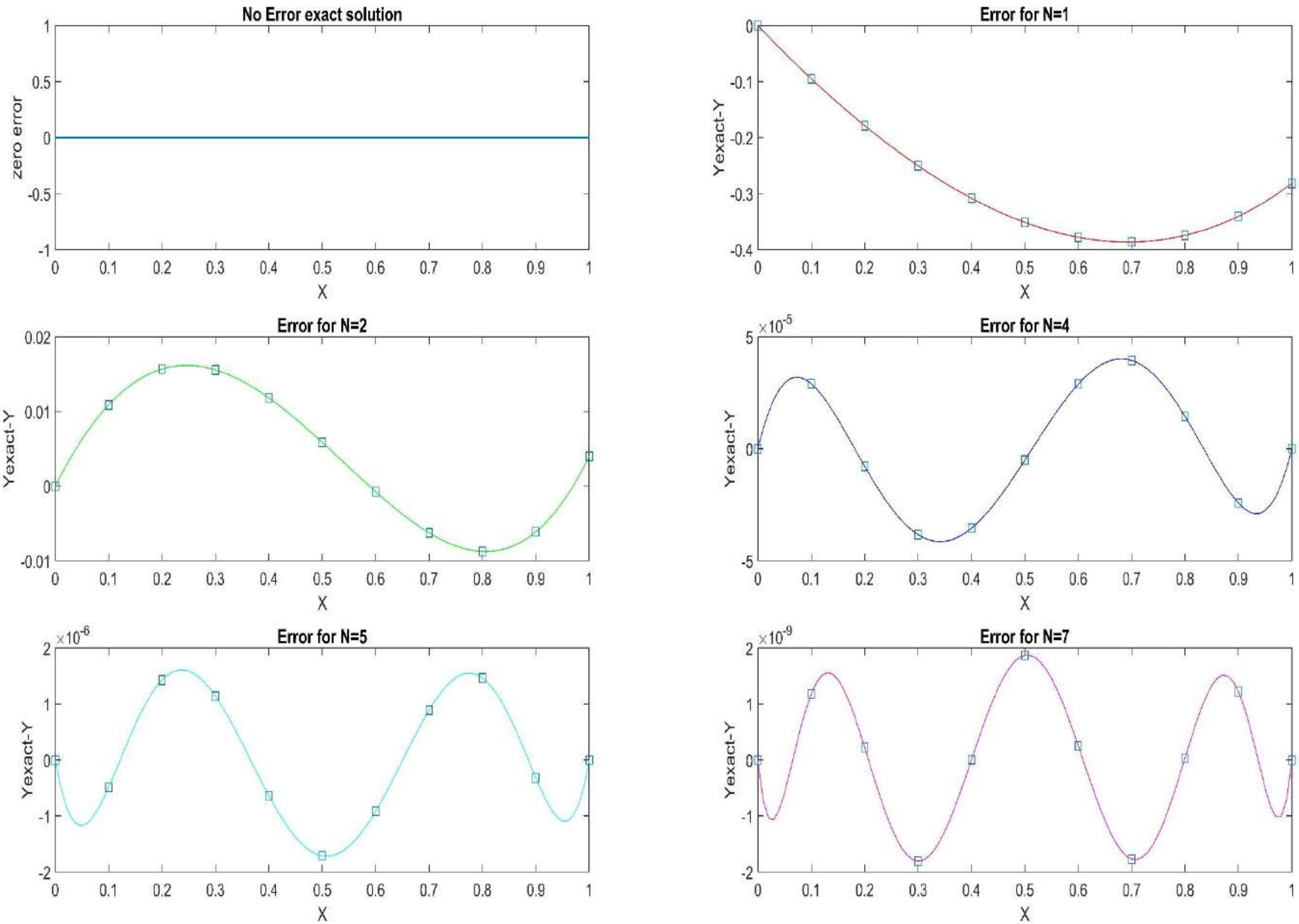
$$\tilde{y} = 1 + 1.000000096238539x + 0.499997353451741x^2 + 0.166690036412452x^3 + 0.041571459657900x^4 + 0.008536600260243x^5 + 0.001155997156038x^6 + 0.00033028528133x^7$$

$y = e^x$ exact solution.

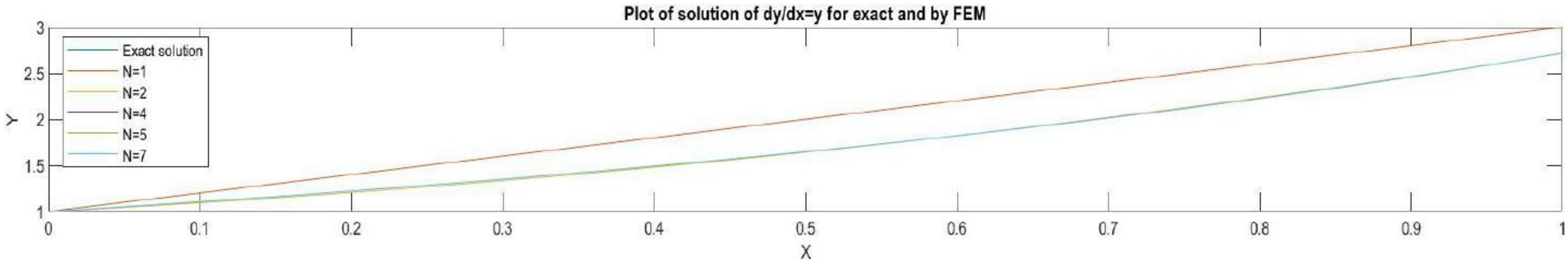
ploting the error $(y-\tilde{y})$ Vs x $\forall x \in [0,1]$ we can observe that the error Oscillates between $(2\times10^{-9}$ to $-2\times10^{-9})$ very close to zero for $x \in [0,1]$

The error is ~~very~~ very small and hence for N=7 ~~is very~~ FEM gives very good approximation of exact solution.

The trend is decreasing error as we increase Value of N can be verified by the these above plots.



No Error exact solution
zero error
X
Error for N=1
Yexact-Y
X
Error for N=2
Yexact-Y
X
Error for N=4
Yexact-Y
X
Error for N=5
Yexact-Y
X
Error for N=7
Yexact-Y
X



Hence we can say that for $x \in [0, 1]$

i) for $N=1$ the exact solution is linearized however error are large.

ii) for $N=2$ the ~~error~~ error are low however observable.

iii) As we increase the value of N from $N=2$ to $N=4$ to $N=5$ to $N=7$ the error continuously decreases to a very small number. hence larger the value of N smaller is the error.

iv) the error oscillates between ~~to~~ two ~~to~~ values or an interval and this interval is reduced as N increases.

For the sake of completeness We can show all the curve (error function) on the same plot below as.

MATLAB code:

```
function plotResult()

    %Exact Solution
    subplot(4,2,1);
    x=0:0.01:1;
    y_exact=exp(x);
    error=y_exact-y_exact;
    plot(x,error,'linewidth',1);
    title('No Error exact solution');
    xlabel('X');
    ylabel('zero error');

    hold on

    %for N=1
    subplot(4,2,2);
    A=1/2;
    b=1;
    a=A\b;
    x_vec=x;
    y1=1+a*x_vec;
    error=y_exact-y1;
    plot(x,error,'-
rs','MarkerIndices',1:10:length(x),'MarkerEdgeColor',[0, 0.4470, 0.7410]
);
    title('Error for N=1');
    xlabel('X');
    ylabel('Yexact-Y');

    disp(a);

    %for N=2
    subplot(4,2,3);
    A=[1/2, 2/3;
        1/6, 5/12];
    b=[1;1/2];
    a=A\b;
    x_vec=[x;x.*x];
    y2=1+a'*x_vec;
    error=y_exact-y2;
    plot(x,error,'-
gs','MarkerIndices',1:10:length(x),'MarkerEdgeColor',[0, 0.4470, 0.7410]
);
    title('Error for N=2');
    xlabel('X');
    ylabel('Yexact-Y');

    disp(a);
```

```
    %for N=4
    subplot(4,2,4);
    A = [ 1/2,  2/3,  3/4,   4/5;
          1/6,  5/12, 11/20, 19/30;
          1/12, 3/10, 13/30, 11/21;
          1/20, 7/30, 5/14,  25/56];
    b = [1; 1/2; 1/3; 1/4];
    a = A\b;
    x_vec = [x; x.*x; x.*x.*x; x.*x.*x.*x];
    y3 = 1+a'*x_vec;
    error=y_exact-y3;
    plot(x,error,'-
bs','MarkerIndices',1:10:length(x),'MarkerEdgeColor',[0, 0.4470, 0.7410]
);
    title('Error for N=4');
    xlabel('X');
    ylabel('Yexact-Y');

    disp(a);

    %for N=5
    subplot(4,2,5);
    A = [ 1/2,  2/3,  3/4,   4/5,   5/6;
          1/6,  5/12, 11/20, 19/30, 29/42;
          1/12, 3/10, 13/30, 11/21, 33/56;
          1/20, 7/30, 5/14,  25/56, 37/72;
          1/30, 4/21, 17/56, 7/18,  41/90];
    b = [1; 1/2; 1/3; 1/4; 1/5];
    a = A\b;
    x_vec = [x; x.*x; x.*x.*x; x.*x.*x.*x; x.*x.*x.*x.*x];
    y4 = 1+a'*x_vec;
    error=y_exact-y4;
    plot(x,error,'-
cs','MarkerIndices',1:10:length(x),'MarkerEdgeColor',[0, 0.4470, 0.7410]
);
    title('Error for N=5');
    xlabel('X');
    ylabel('Yexact-Y');

    disp(a);

    %for N=7
    subplot(4,2,6);
    A = [ 1/2,  2/3,  3/4,   4/5,   5/6,    6/7,    7/8;
          1/6,  5/12, 11/20, 19/30, 29/42,  41/56,  55/72;
          1/12, 3/10, 13/30, 11/21, 33/56,  23/36,  61/90;
          1/20, 7/30, 5/14,  25/56, 37/72,  17/30,  67/110;
          1/30, 4/21, 17/56, 7/18,  41/90,  28/55,  73/132;
          1/42, 9/56, 19/72, 31/90, 9/22,   61/132, 79/156;
          1/56, 5/36, 7/30,  17/55, 49/132, 11/26,  85/182];
    b = [1; 1/2; 1/3; 1/4; 1/5; 1/6; 1/7];
```

```
    a = A\b;
    x_vec = [x; x.*x; x.*x.*x; x.*x.*x.*x; x.*x.*x.*x.*x;
x.*x.*x.*x.*x.*x; x.*x.*x.*x.*x.*x.*x];
    y5 = 1+a'*x_vec;
    error=y_exact-y5;
    plot(x,error,'-
ms','MarkerIndices',1:10:length(x),'MarkerEdgeColor',[0, 0.4470, 0.7410]
);
    title('Error for N=7');
    xlabel('X');
    ylabel('Yexact-Y');

    disp(a);

    %plot of all functions
    subplot(4,2,7:8)
    plot(x,y_exact,x,y1,x,y2,x,y3,x,y4,x,y5);
    title('Plot of solution of dy/dx=y for exact and by FEM');
    xlabel('X');
    ylabel('Y');
    legend({'Exact
solution','N=1','N=2','N=4','N=5','N=7'},'Location','northwest');


end
```

Q 2a.)

Rikesh Sharma
180606

2.a) Given

$$\frac{d\phi}{dt} = \lambda\phi$$

where $\lambda = -1$ given

$\phi(t=0) = 1$ given

Exact sol^n

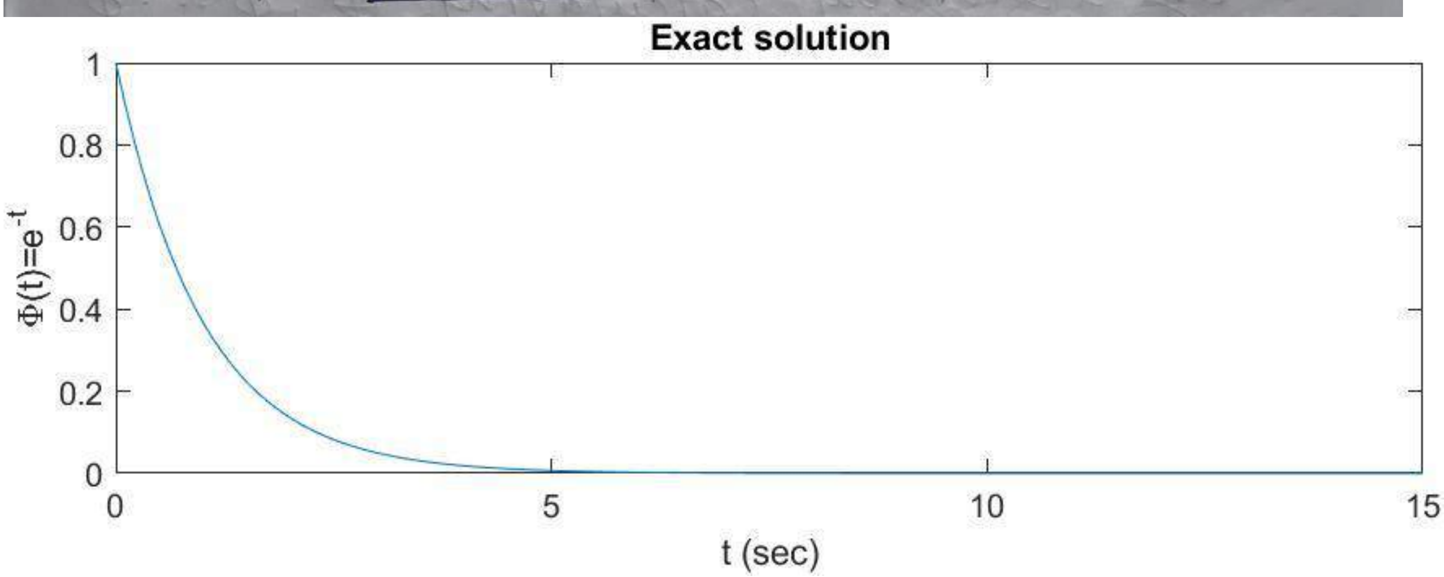
$$\frac{d\phi}{dt} = \lambda\phi$$

$$\Rightarrow \int_{\phi(t=0)}^{\phi(t)} \frac{d\phi}{\phi} = -\int_0^t 1 \cdot dt$$

$$\Rightarrow \ln\left(\phi(t)/\phi(t=0)\right) = -t + 0$$

$$\Rightarrow \phi(t) = \phi(t=0)\, e^{-t} \quad \Big| \text{ Since } \phi(t=0) = 1$$

$$\Rightarrow \phi(t) = e^{-t} \text{ exact solution}$$



According to question we need to solve using Explicit Euler, Implicit Euler and Crank Nicolson for

$\Delta t = 2.1$ sec
$\Delta t = 0.6$ sec
$\Delta t = 0.1$ sec

Explicit Euler

Using Explicit Euler

we can write the given equation as

$$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt} dt = \int_{t_n}^{t_{n+1}} \lambda \phi \, dt$$

$\phi^{n+1} = \phi(t_{n+1})$ and so on

$$\Rightarrow \phi_{n+1} - \phi^n = \lambda \phi^n \Delta t$$

$$\Rightarrow \phi^{n+1} = \phi^n + \lambda \phi^n \Delta t$$

$$\Rightarrow \boxed{\phi^{n+1} = (1 + \lambda \Delta t)\phi^n} \quad \text{with } \phi^0 = 1$$

We can solve this for different values of Δt $\to$ 0.1, 0.6, 2.1 sec.

by iteratively calculating $\phi^1, \phi^2 \ldots \phi^n$ and so on.

We may write a computer program for the same.

```
function z=explicitEuler()

    deltat=0.1;
    lambda=-1;
    tmax=15;
    t=linspace(0,tmax,tmax/deltat);
    phi=exp(-t);
    z0=phi;
    subplot(3,2,1);
    plot(t,z0);
    title('Exact solution');
    xlabel('t (sec)');
    ylabel('\Phi(t)=e^{-t}');
    hold on

    phi=zeros(2,1);
    phi(1)=1;
    nmax=tmax/deltat;
    for n=2:nmax
        phi(n)=(1+lambda*deltat)*phi(n-1);
    end
    subplot(3,2,2);
    t=linspace(0,tmax,tmax/deltat);
    z1=phi;
    plot(t,z1,'Color',[0.8500, 0.3250, 0.0980]);
    title('Explicit Euler for \Deltat=0.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.1 sec');

    deltat=0.6;
    phi=zeros(2,1);
    phi(1)=1;
    nmax= tmax/deltat;
    for n=2:nmax
        phi(n)=(1+lambda*deltat)*phi(n-1);
    end
    subplot(3,2,3);
    z2=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z2,'Color',[0.9290, 0.6940, 0.1250]);
    title('Explicit Euler for \Deltat=0.6 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.6 sec');

     deltat=2.1;
     phi=zeros(2,1);
     phi(1)=1;
     nmax=tmax/deltat;
     for n=2:nmax
         phi(n)=(1+lambda*deltat)*phi(n-1);
     end
     subplot(3,2,4);
```

```
     z3=phi;
     t=linspace(0,tmax,tmax/deltat);
     plot(t,z3,'color',[0.4940, 0.1840, 0.5560]);
     title('Explicit Euler for \Deltat=2.1 sec');
     xlabel('t (sec)');
     ylabel('\Phi(t) for \Deltat=2.1 sec');

    subplot(3,2,5:6);
    deltat1=0.1; deltat2=0.6; deltat3=2.1;
    t0=linspace(0,tmax,tmax/deltat1);
    t1=linspace(0,tmax,tmax/deltat1);
    t2=linspace(0,tmax,tmax/deltat2);
    t3=linspace(0,tmax,tmax/deltat3);
    plot(t0,z0,t1,z1,t2,z2,t3,z3);
    title('Plots of solution for Exact and Explicit Euler for different
value of \Deltat');
    xlabel('t (sec)');
    ylabel('\Phi(t)');
    legend({'\phi(t)=e^{-t}','\Deltat=0.1 sec','\Deltat=0.6
sec','\Deltat=2.1 sec'},'Location','southwest');


     %z=[z1(1:20),z2(1:20),z3(1:20)];
end
```

The Exact solution being $\phi(t) = e^{-t}$ exponentially decreases starting at $\phi(t=0) = 1$ to 0 as t varies from $[0, \infty]$.
This can be ~~cr~~ clearly observed from the plot.

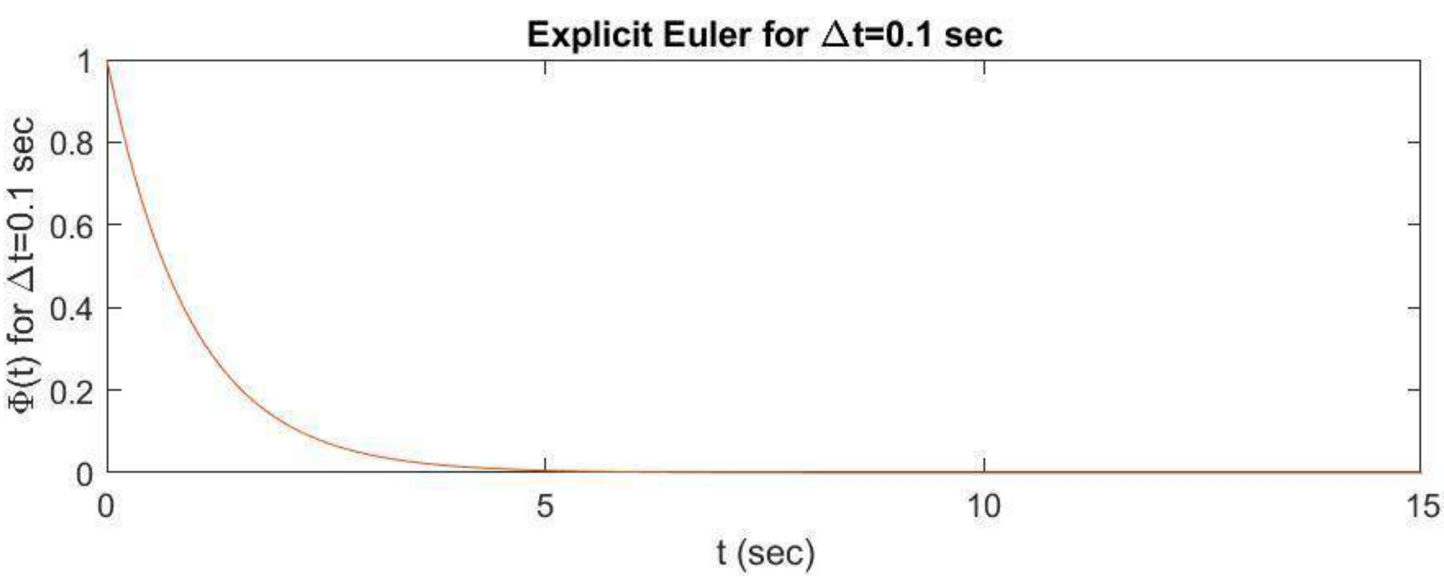
Explicit Euler

for $\Delta t = 0.1$

The solution gives $\phi(t)$, which very closely relates to exact solution and has similar behaivor, exponential decrease and values at particular.

The solution by explicit Euler for $\Delta t = 0.1$ is quite accurate. and value of $\phi(t)$ Closely resembles exact solution.

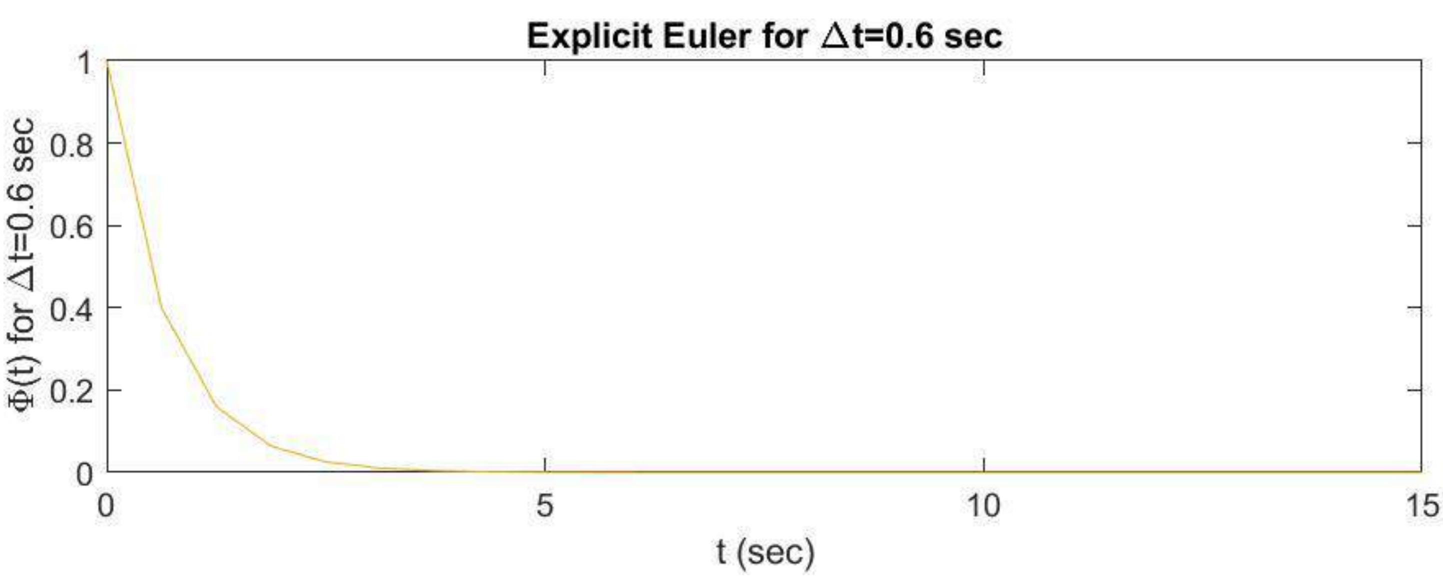
The solution is also stable and ~~re~~ decreases to 0 as t increases.



for $\Delta t = 0.6$

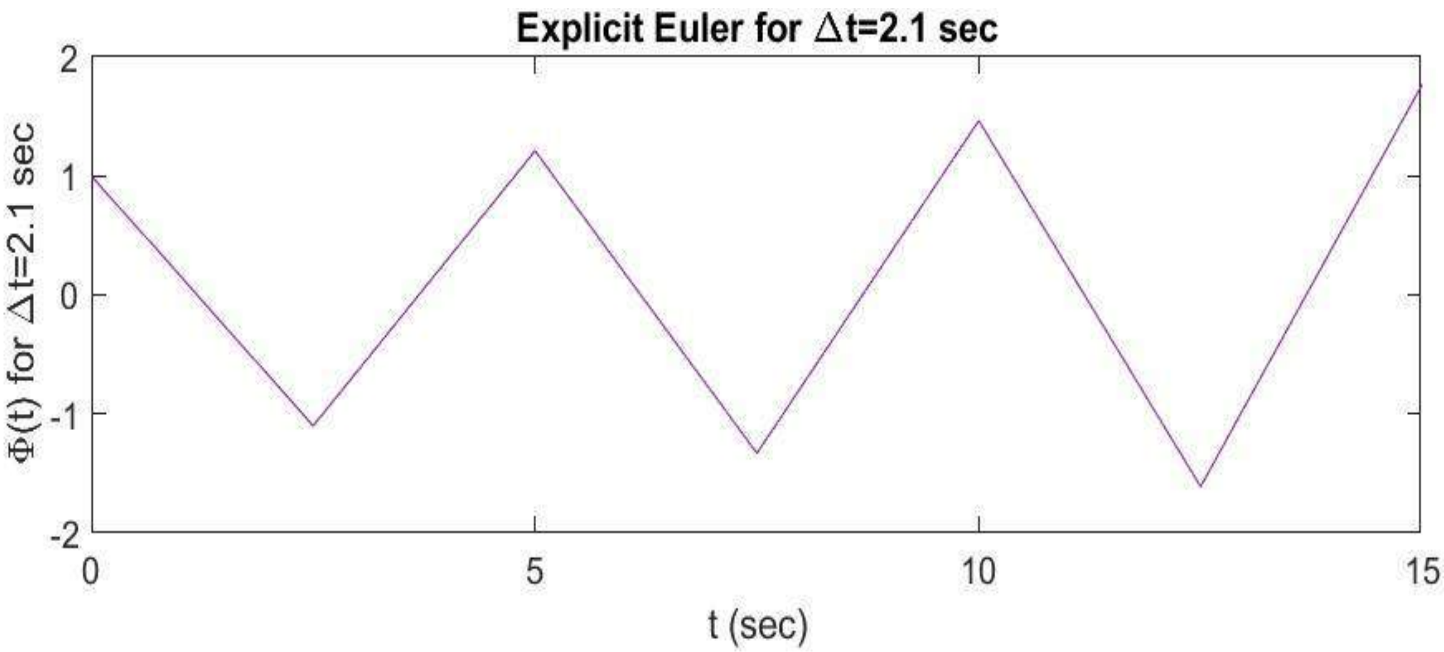
The solution is stable and it decreases to zero.

However the error for $\Delta t = 0.6$ (solution by explicit Euler) has increased. This error is large for lower values of t and relatively smaller for larger value of t.



for $\Delta t = 2.1$

The solution by explicit Euler for $\Delta t = 2.1$ sec becomes unstable and it oscillates and increase without bound.
The solution is errorneous and diverges to infinity.



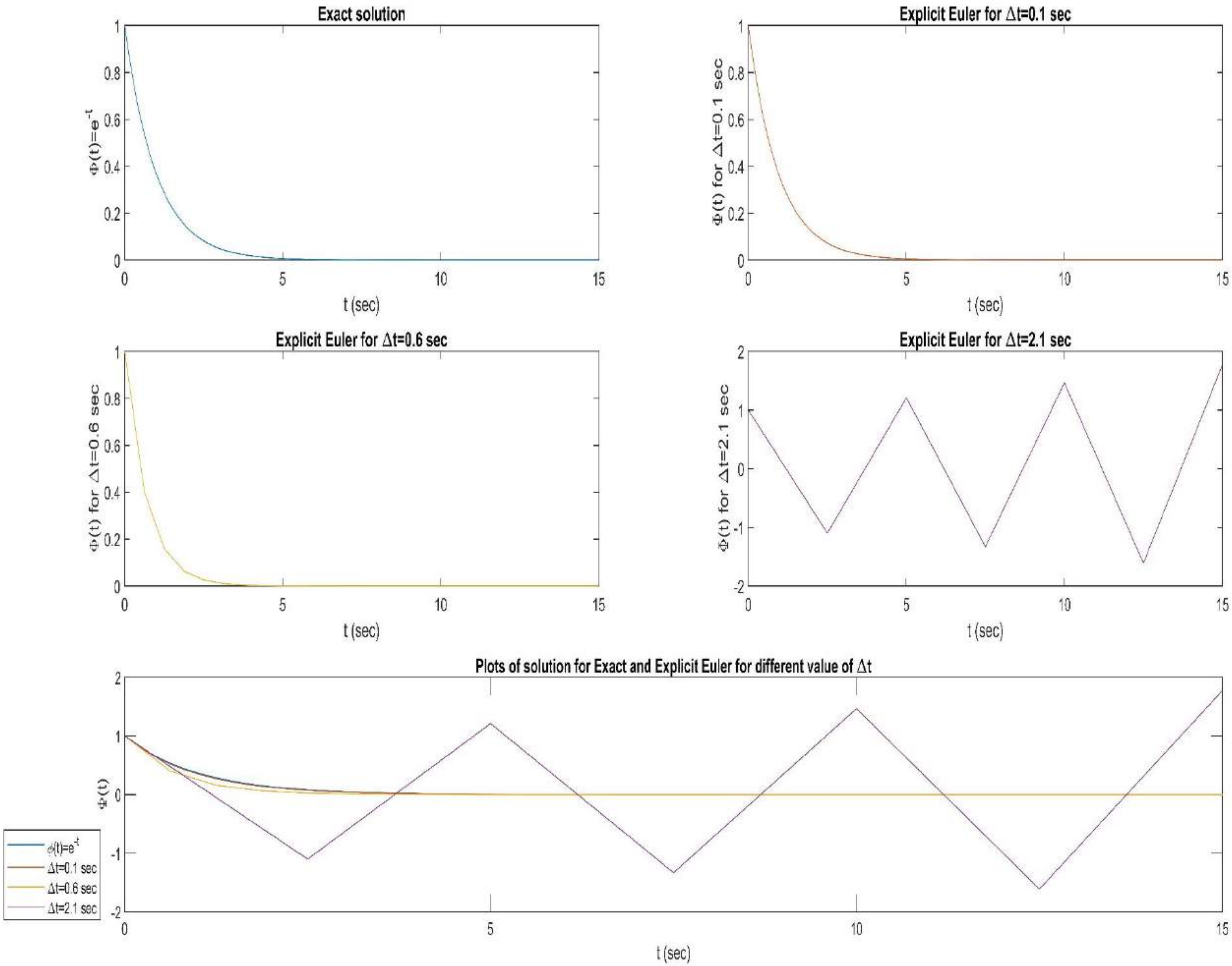

Exact solution
$\Phi(t)=e^{-t}$
t (sec)
Explicit Euler for Δt=0.1 sec
Φ(t) for Δt=0.1 sec
t (sec)
Explicit Euler for Δt=0.6 sec
Φ(t) for Δt=0.6 sec
t (sec)
Explicit Euler for Δt=2.1 sec
Φ(t) for Δt=2.1 sec
t (sec)
Plots of solution for Exact and Explicit Euler for different value of Δt
Φ(t)
t (sec)
$\phi(t)=e^{-t}$
Δt=0.1 sec
Δt=0.6 sec
Δt=2.1 sec

Table for the Φ(t) by Explicit Euler

Nth time step	Φ(t) for Δt =0.1sec	Φ(t) for Δt =0.6sec	Φ(t) for Δt =2.1sec
1	1	1	1
2	0.900000000000000	0.400000000000000	-1.10000000000000
3	0.810000000000000	0.160000000000000	1.21000000000000
4	0.729000000000000	0.0640000000000000	-1.33100000000000
5	0.656100000000000	0.0256000000000000	1.46410000000000
6	0.590490000000000	0.0102400000000000	-1.61051000000000
7	0.531441000000000	0.00409600000000000	1.77156100000000
8	0.478296900000000	0.00163840000000000	-1.94871710000000
9	0.430467210000000	0.000655360000000000	2.14358881000000
10	0.387420489000000	0.000262144000000000	-2.35794769100000
11	0.348678440100000	0.000104857600000000	2.59374246010000
12	0.313810596090000	4.19430400000000e-05	-2.85311670611000
13	0.282429536481000	1.67772160000000e-05	3.13842837672100
14	0.254186582832900	6.71088640000000e-06	-3.45227121439310
15	0.228767924549610	2.68435456000000e-06	3.79749833583242
16	0.205891132094649	1.07374182400000e-06	-4.17724816941566
17	0.185302018885184	4.29496729600000e-07	4.59497298635722
18	0.166771816996666	1.71798691840000e-07	-5.05447028499294
19	0.150094635296999	6.87194767360001e-08	5.55991731349224
20	0.135085171767299	2.74877906944000e-08	-6.11590904484146

Implicit Euler

If we use implicit Euler

we have to write

$$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt} dt = \int_{t_n}^{t_{n+1}} \lambda \phi \, dt \quad \text{as}$$

note

$\phi^n = \phi(t = t_n)$

$$\phi^{n+1} - \phi^n = \lambda \phi^{n+1} \Delta t$$

$$\Rightarrow \quad \phi^{n+1}(1 - \lambda \Delta t) = \phi^n$$

$$\Rightarrow \quad \boxed{\phi^{n+1} = \frac{1}{1 - \lambda \Delta t} \phi^n}$$

with $\phi^0 = \phi(t=0) = 1$

We can solve this eq^n to calculate ϕ for all t.

We can use $\Delta t = 0.1$ sec
$= 0.6$ sec
$= 2.1$ sec

for different cases.

This can be done by iteratively computing

$$\phi^1 = \frac{1}{1 - \lambda \Delta t} \phi^0 \qquad \phi^0 = 1$$

$$\phi^2 = \frac{1}{1 - \lambda \Delta t} \phi^1 \quad \text{and so on}$$

We may write a computer program for the same.

```matlab
function z=implicitEuler()

    deltat=0.1;
    lambda=-1;
    tmax=10;
    t=linspace(0,tmax,tmax/deltat);
    phi=exp(-t);
    z0=phi;
    subplot(3,2,1);
    plot(t,z0);
    title('Exact solution');
    xlabel('t (sec)');
    ylabel('\Phi(t)=e^{-t}');
    hold on

    phi=zeros(2,1);
    phi(1)=1;
    nmax=tmax/deltat;

    for n=2:nmax
        phi(n)=phi(n-1)/(1-lambda*deltat);
    end
    subplot(3,2,2);
    t=linspace(0,tmax,tmax/deltat);
    z1=phi;
    plot(t,z1,'Color',[0.8500, 0.3250, 0.0980]);
    title('Implicit Euler for \Deltat=0.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.1 sec');


    deltat=0.6;
    phi=zeros(2,1);
    phi(1)=1;
    nmax= tmax/deltat;
    for n=2:nmax
        phi(n)=phi(n-1)/(1-lambda*deltat);
    end
    subplot(3,2,3);
    z2=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z2,'Color',[0.9290, 0.6940, 0.1250]);
    title('Implicit Euler for \Deltat=0.6 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.6 sec');


     deltat=2.1;
     phi=zeros(2,1);
     phi(1)=1;
     nmax=tmax/deltat;
     for n=2:nmax
```

```
        phi(n)=phi(n-1)/(1-lambda*deltat);
    end
    subplot(3,2,4);
    z3=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z3,'color',[0.4940, 0.1840, 0.5560]);
    title('Implicit Euler for \Deltat=2.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=2.1 sec');


    subplot(3,2,5:6);
    deltat1=0.1; deltat2=0.6; deltat3=2.1;
    t0=linspace(0,tmax,tmax/deltat1);
    t1=linspace(0,tmax,tmax/deltat1);
    t2=linspace(0,tmax,tmax/deltat2);
    t3=linspace(0,tmax,tmax/deltat3);
    plot(t0,z0,t1,z1,t2,z2,t3,z3);
    title('Plots of solution for Exact and Implicit Euler for different
value of \Deltat');
    xlabel('t (sec)');
    ylabel('\Phi(t)');
    legend({'\phi(t)=e^{-t}','\Deltat=0.1 sec','\Deltat=0.6
sec','\Deltat=2.1 sec'},'Location','northeast');

     %z=[z1(1:4),z2(1:4),z3(1:4)];
end
```

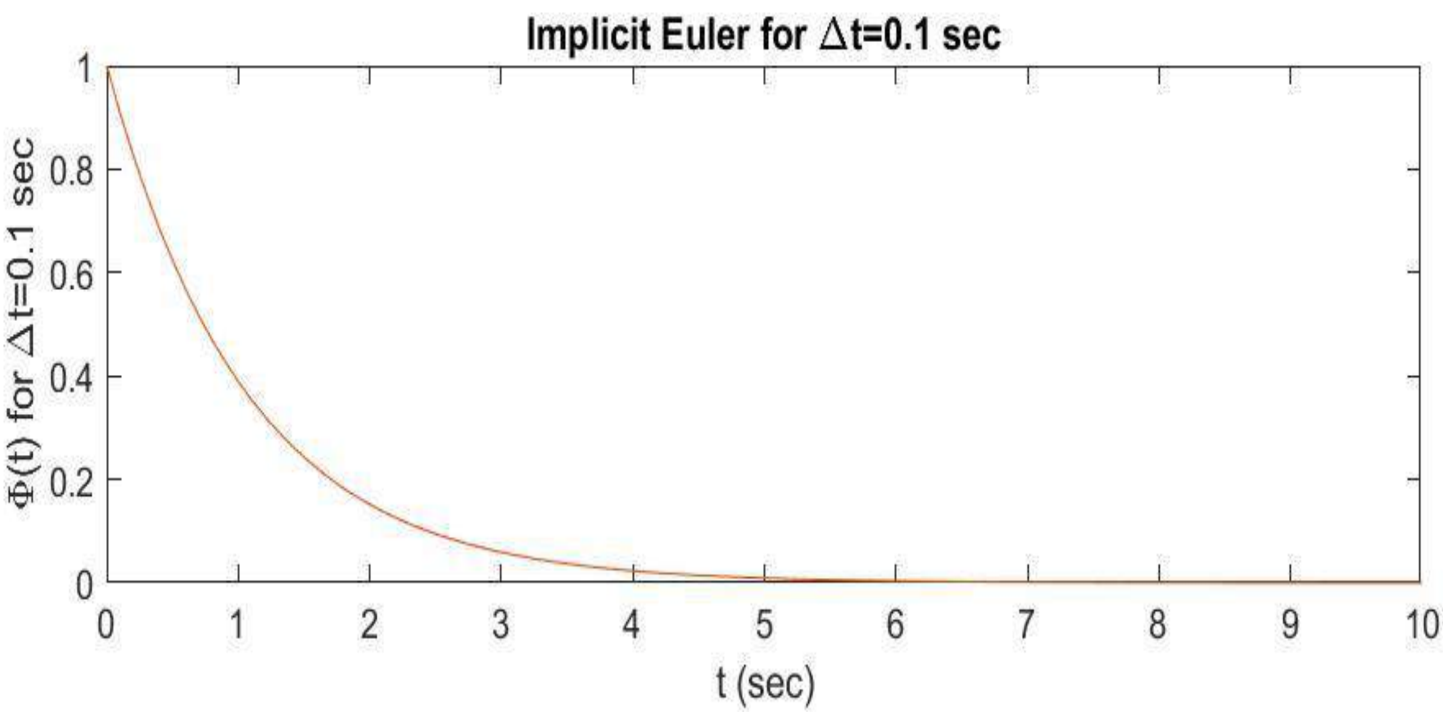
Implicit Euler

for $\Delta t = 0.1$

The solution given by implicit Euler for $\phi(t)$ with $\Delta t = 0.1$ sec, very closely resembles with exact solution,

The solution is accurate.

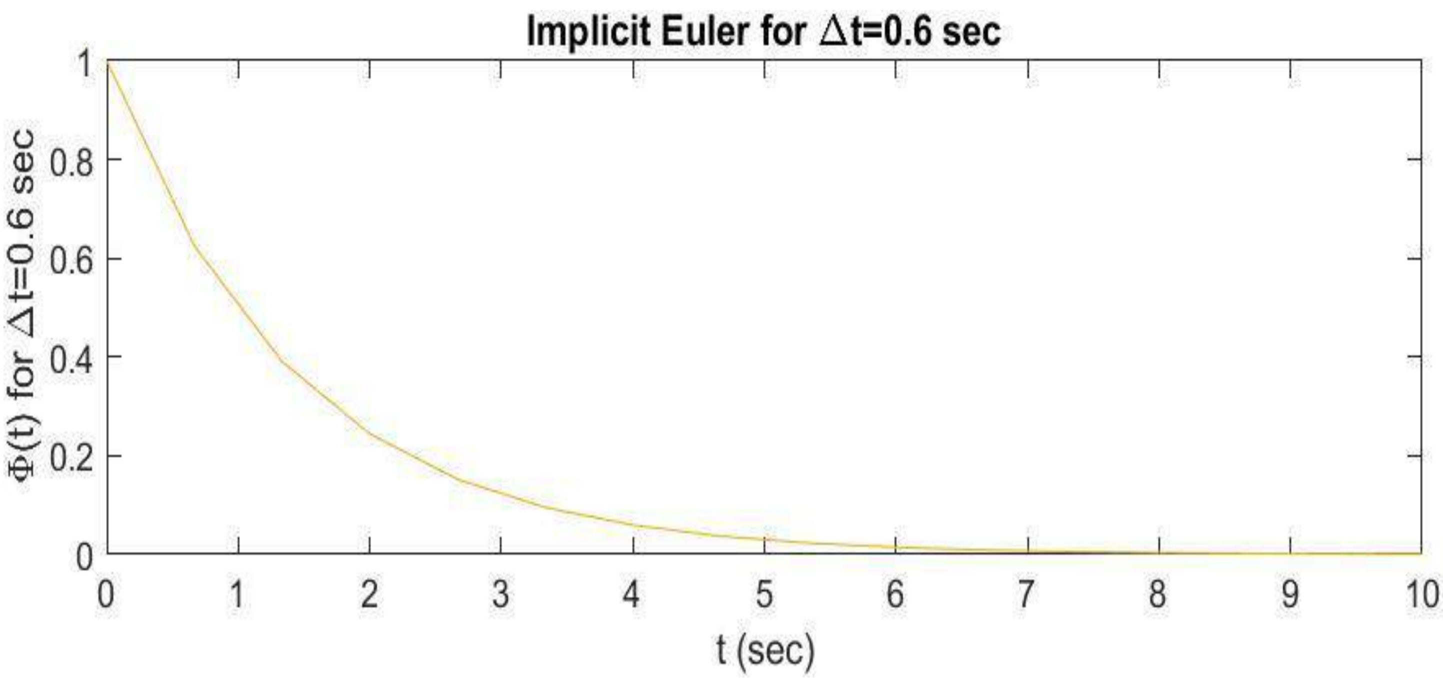
The solution is also stable and approaches to "0" as t increases.



for $\Delta t = 0.6$

The solution is Stable.

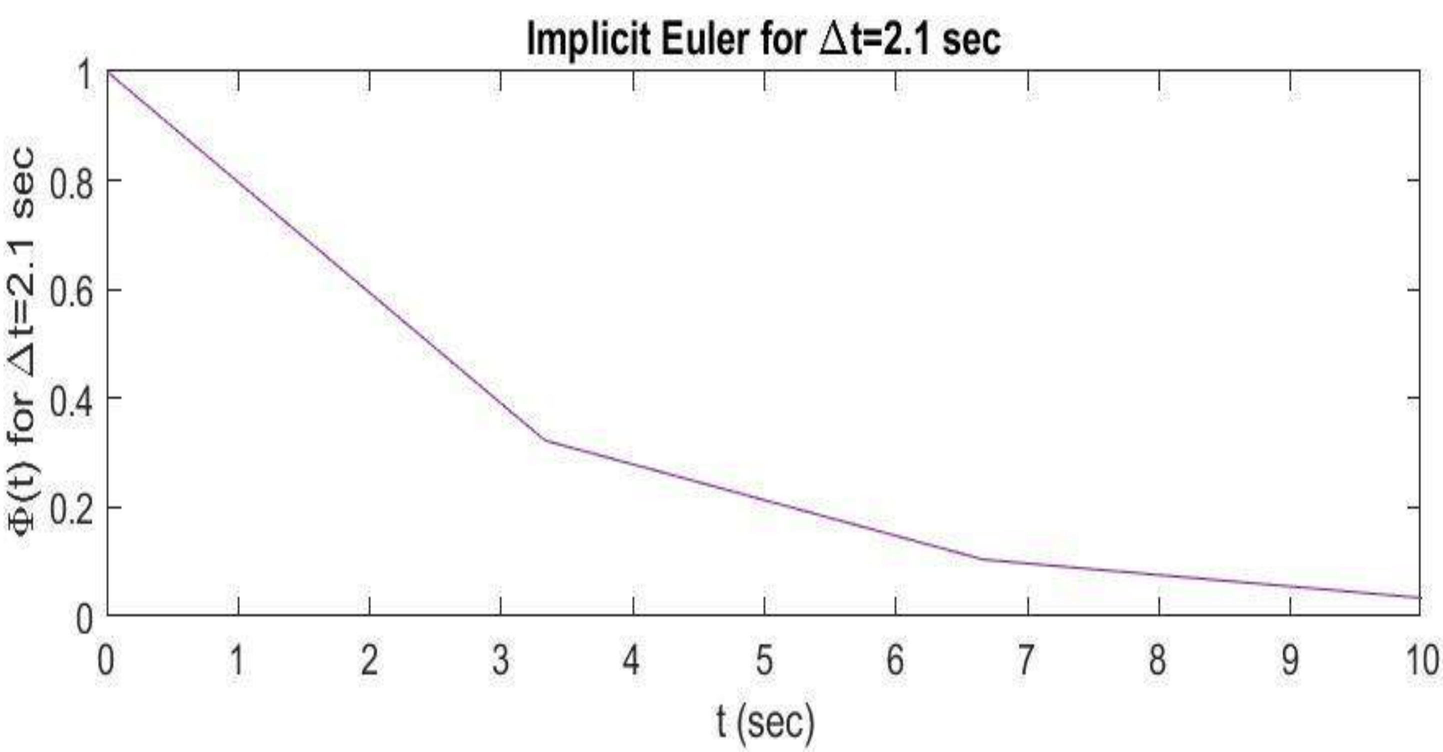
Error are larger for lower value of t but Error decreases as t increases.

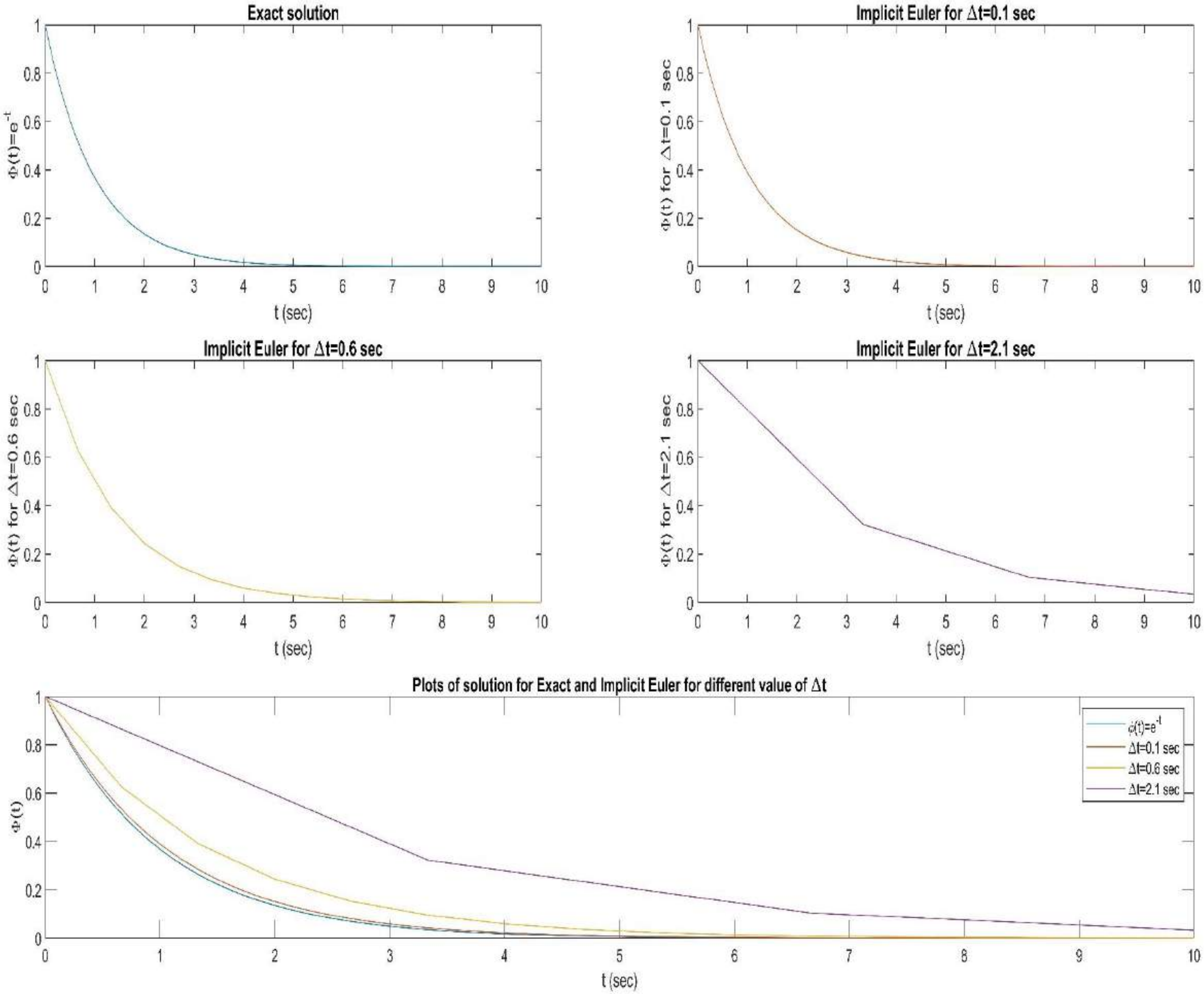


for $\Delta t = 2.1$

The solution by implicit Euler is stable in contrast to Explicit Euler for $\Delta t = 2.1$

The error is larger as compared to $\Delta t = 0.1$ or $\Delta t = 0.6$.





Exact solution
Implicit Euler for Δt=0.1 sec
Implicit Euler for Δt=0.6 sec
Implicit Euler for Δt=2.1 sec
Plots of solution for Exact and Implicit Euler for different value of Δt
Φ(t)=e^-t
Φ(t) for Δt=0.1 sec
Φ(t) for Δt=0.6 sec
Φ(t) for Δt=2.1 sec
Φ(t)
t (sec)
φ(t)=e^-t
Δt=0.1 sec
Δt=0.6 sec
Δt=2.1 sec

Table for the Φ(t) by Implicit Euler

Nth time step	Φ(t) for Δt =0.1sec	Φ(t) for Δt =0.6sec	Φ(t) for Δt =2.1sec
1	1	1	1
2	0.909090909090909	0.625000000000000	0.322580645161290
3	0.826446280991735	0.390625000000000	0.104058272632674
4	0.751314800901578	0.244140625000000	0.0335671847202175
5	0.683013455365071	0.152587890625000	0.0108281241032960
6	0.620921323059155	0.0953674316406250	0.00349294325912773
7	0.564473930053777	0.0596046447753906	0.00112675589004120
8	0.513158118230707	0.0372529029846191	0.000363469641948776
9	0.466507380209733	0.0232830643653870	0.000117248271596379
10	0.424097618372485	0.0145519152283669	3.78220230956062e-05
11	0.385543289429531	0.00909494701772928	1.22006526114859e-05
12	0.350493899481392	0.00568434188608080	3.93569439080189e-06
13	0.318630817710356	0.00355271367880050	1.26957883574255e-06
14	0.289664379736688	0.00222044604925031	4.09541559916950e-07
15	0.263331254306080	0.00138777878078145	1.32110180618371e-07
16	0.239392049369163	0.000867361737988404	4.26161872962487e-08
17	0.217629135790148	0.000542101086242752	1.37471571923383e-08
18	0.197844668900135	0.000338813178901720	4.43456683623816e-09
19	0.179858789909214	0.000211758236813575	1.43050543104457e-09
20	0.163507990826558	0.000132348898008484	4.61453364853086e-10

Crank Nicolson

Now using Crank Nicolson we can write

$$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt}.dt = \int_{t_n}^{t_{n+1}} \lambda\phi\, dt$$

as $\phi^{n+1} - \phi^n = \frac{\lambda}{2}(\phi^{n+1} + \phi^n)\Delta t$

$\Rightarrow \phi^{n+1} - \frac{\Delta t \lambda}{2}\phi^{n+1} = \phi^n + \frac{\lambda\Delta t}{2}\phi^n$

$$\Rightarrow \boxed{\phi^{n+1} = \frac{(1 + \lambda\Delta t/2)}{(1 - \lambda\Delta t/2)}\phi^n}$$

with $\phi^0 = 1$

the above eqn can now be solved iteratively using a computer program for different values of $\Delta t = 0.1$

$= 0.6$

$= 2.1$

We may write a computer program for the same.

```
function z=crankNicolson()

    deltat=0.1;
    lambda=-1;
    tmax=10;
    t=linspace(0,tmax,tmax/deltat);
    phi=exp(-t);
    z0=phi;
    subplot(3,2,1);
    plot(t,z0);
    title('Exact solution');
    xlabel('t (sec)');
    ylabel('\Phi(t)=e^{-t}');
    hold on


    phi=zeros(2,1);
    phi(1)=1;
    nmax=tmax/deltat;
    for n=2:nmax
        phi(n)=phi(n-1)*((1+lambda*0.5*deltat)/(1-lambda*0.5*deltat));
    end
    subplot(3,2,2);
    t=linspace(0,tmax,tmax/deltat);
    z1=phi;
    plot(t,z1,'Color',[0.8500, 0.3250, 0.0980]);
    title('Crank Nicolson for \Deltat=0.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.1 sec');

    deltat=0.6;
    phi=zeros(2,1);
    phi(1)=1;
    nmax= tmax/deltat;
    for n=2:nmax
        phi(n)=phi(n-1)*((1+lambda*0.5*deltat)/(1-lambda*0.5*deltat));
    end
    subplot(3,2,3);
    z2=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z2,'Color',[0.9290, 0.6940, 0.1250]);
    title('Crank Nicolson for \Deltat=0.6 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.6 sec');


    deltat=2.1;
    phi=zeros(2,1);
    phi(1)=1;
    nmax=tmax/deltat;
    for n=2:nmax
        phi(n)=phi(n-1)*((1+lambda*0.5*deltat)/(1-lambda*0.5*deltat));
```

```
    end
    subplot(3,2,4);
    z3=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z3,'color',[0.4940, 0.1840, 0.5560]);
    title('Crank Nicolson for \Deltat=2.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=2.1 sec');


    subplot(3,2,5:6);
    deltat1=0.1; deltat2=0.6; deltat3=2.1;
    t0=linspace(0,tmax,tmax/deltat1);
    t1=linspace(0,tmax,tmax/deltat1);
    t2=linspace(0,tmax,tmax/deltat2);
    t3=linspace(0,tmax,tmax/deltat3);
    plot(t0,z0,t1,z1,t2,z2,t3,z3);
    title('Plots of solution for Exact and Crank Nicolson for different
value of \Deltat');
    xlabel('t (sec)');
    ylabel('\Phi(t)');
    legend({'\phi(t)=e^{-t}','\Deltat=0.1 sec','\Deltat=0.6
sec','\Deltat=2.1 sec'},'Location','northeast');
   %z=[z1(1:4),z2(1:4),z3(1:4)];
end
```

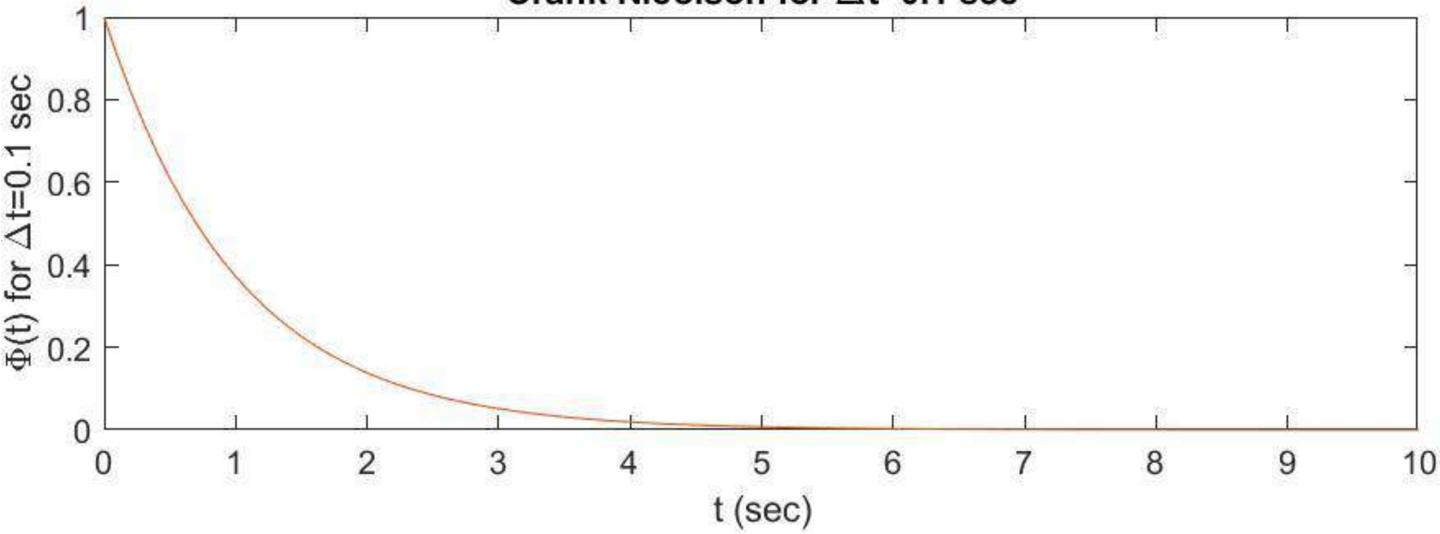
Crank Nicolson

for $\Delta t = 0.1$

The solution is accurate with acceptable range.

The solution is stable.

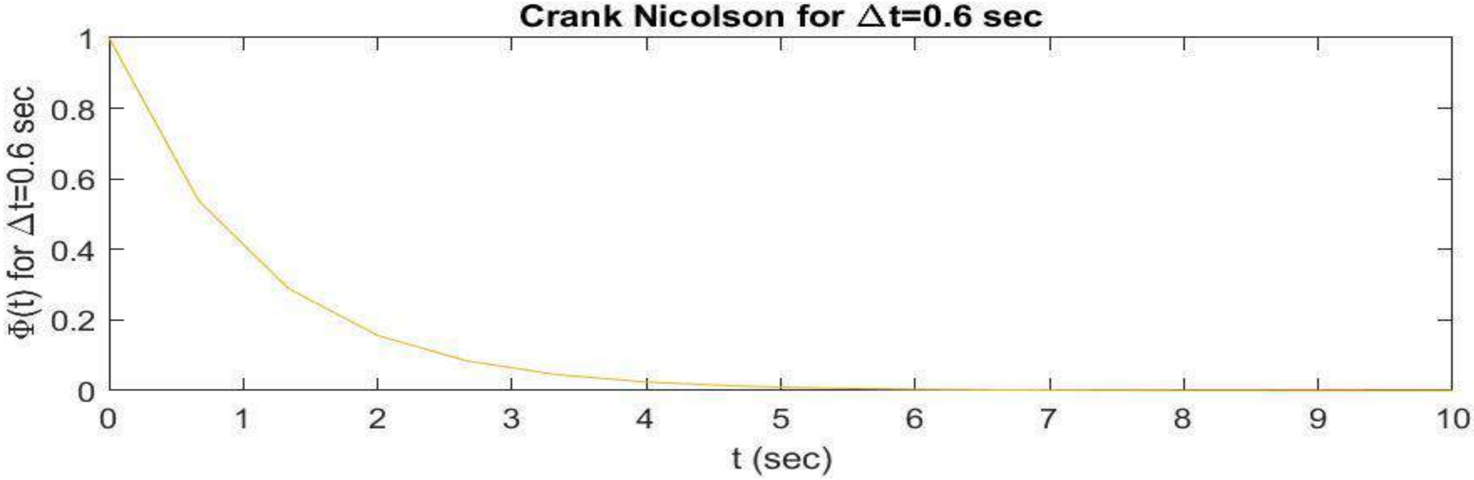
Crank Nicolson for Δt=0.1 sec



for $\Delta t = 0.6$

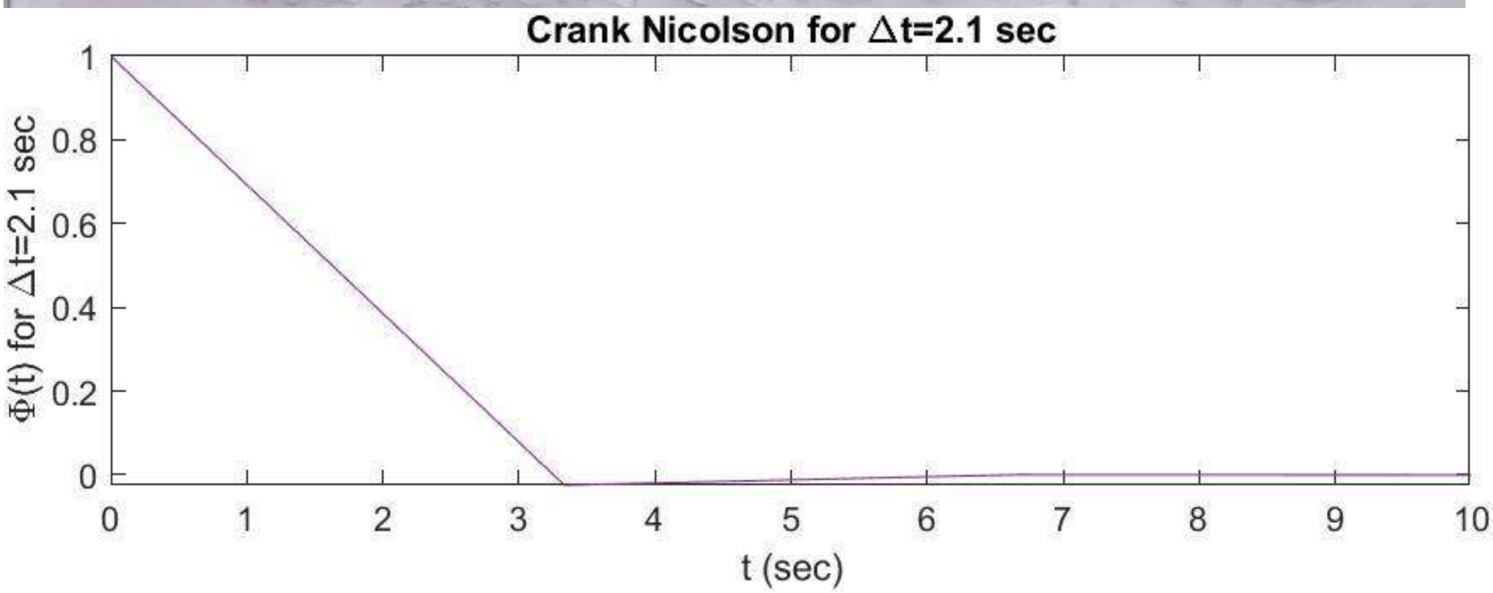
The solution is stable but less accurate in comparison to $\Delta t = 0.1$.

Crank Nicolson for Δt=0.6 sec



for $\Delta t=2.1$

The solution remains stable for $\Delta t = 2.1$ sec also in comparison to Explicit Euler for $\Delta t = 2.1$ sec. However there is a large error.



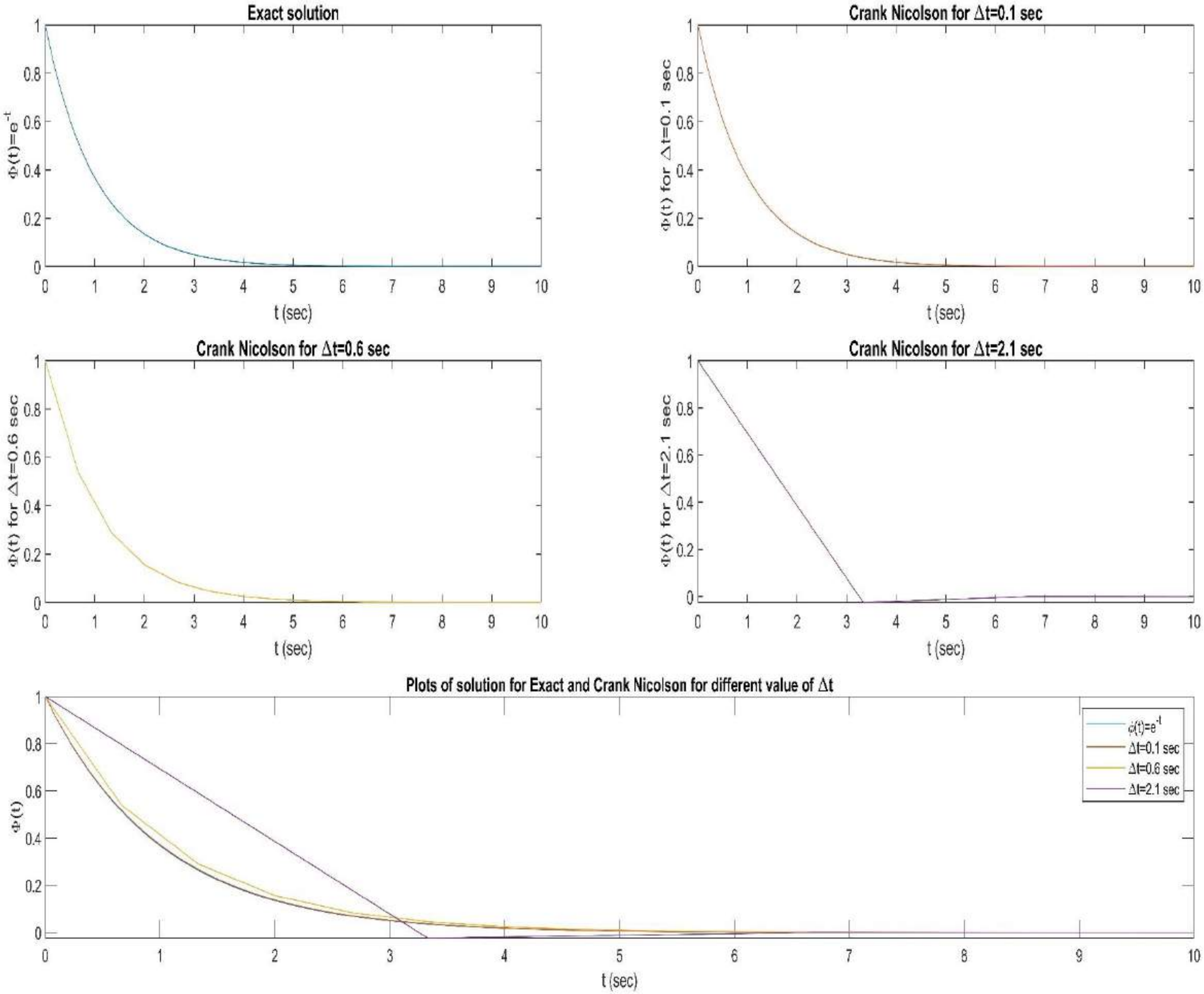

Exact solution
$\Phi(t)=e^{-t}$
t (sec)
Crank Nicolson for Δt=0.1 sec
Φ(t) for Δt=0.1 sec
t (sec)
Crank Nicolson for Δt=0.6 sec
Φ(t) for Δt=0.6 sec
t (sec)
Crank Nicolson for Δt=2.1 sec
Φ(t) for Δt=2.1 sec
t (sec)
Plots of solution for Exact and Crank Nicolson for different value of Δt
$\varphi(t)=e^{-t}$
Δt=0.1 sec
Δt=0.6 sec
Δt=2.1 sec
Φ(t)
t (sec)

Table for the $\Phi(t)$ by Crank Nicolson

Nth time step	$\Phi(t)$ for Δt =0.1sec	$\Phi(t)$ for Δt =0.6sec	$\Phi(t)$ for Δt =2.1sec
1	1	1	1
2	0.904761904761905	0.538461538461538	-0.0243902439024391
3	0.818594104308390	0.289940828402367	0.000594883997620465
4	0.740632761040924	0.156121984524351	-1.45093657956211e-05
5	0.670096307608455	0.0840656839746507	3.53886970624906e-07
6	0.606277611645745	0.0452661375248119	-8.63138952743673e-09
7	0.548536886727103	0.0243740740518218	2.10521695791140e-10
8	0.496295278467378	0.0131245014125194	-5.13467550710098e-12
9	0.449029061470485	0.00706703922212585	1.25235987978073e-13
10	0.406264388949487	0.00380532881191392	-3.05453629214812e-15
11	0.367572542382869	0.00204902320641519	7.45008851743445e-17
12	0.332565633584500	0.00110332018806972	-1.81709476034987e-18
13	0.300892716100262	0.000594095485883693	4.43193843987773e-20
14	0.272236266947856	0.000319897569321989	-1.08096059509213e-21
15	0.246309003429013	0.000172252537327225	2.63648925632227e-23
16	0.222851003102440	9.27513662531210e-05	-6.43046160078604e-25
17	0.201627098045065	4.99430433670651e-05	1.56840526848440e-26
18	0.182424517278868	2.68924079668812e-05	-3.82537870362049e-28
19	0.165050753728500	1.44805273667822e-05	9.33019196004999e-30
20	0.149331634325785	7.79720704365195e-06	-2.27565657562195e-31

RK-3 Low Storage

Using Low storage RK3 method

$\frac{d\phi}{dt} = \lambda\phi \qquad \therefore f(t, \phi(t)) = \lambda\phi$

$\therefore$ RK3 low storage states.

1st RK $\quad \phi^* = \phi^n$

$K_1 = \lambda\phi^*$

2nd RK $\quad \phi^* = \phi^* + \frac{\Delta t}{3} K_1$

$K_1 = -\frac{5}{9} K_1 + \lambda\phi^*$

3rd RK $\quad \phi^* = \phi^* + \frac{15}{16} \Delta t K_1$

$K_1 = \frac{-153}{128} K_1 + \lambda\phi^*$

and $\phi^{n+1} = \phi^* + \frac{8}{15}\Delta t K_1$

as n varies from 0-t sec

This can be solved iteratively for all values of t

We can also change $\Delta t = 0.1$, 0.6, 2.1 sec.

to find ϕ in these different cases.

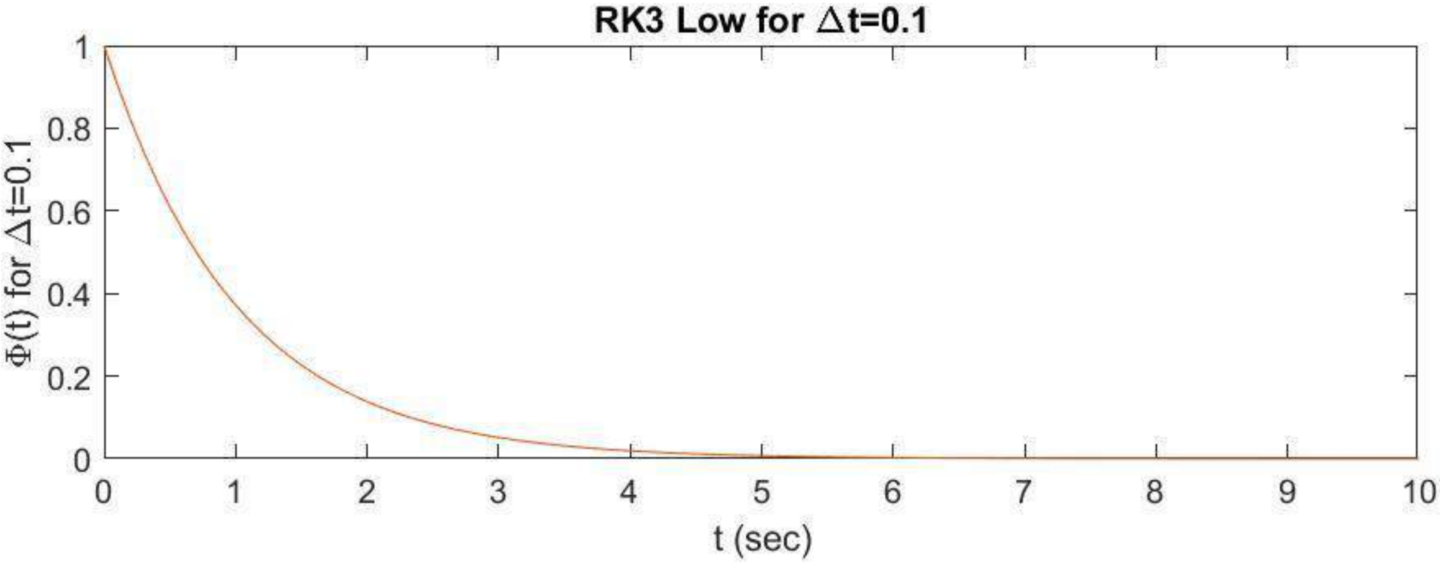
We have to write a computer program for the same.

```
function z=rk3Low()

    deltat=0.1;
    lambda=-1;
    tmax=10;
    t=linspace(0,tmax,tmax/deltat);
    phi=exp(-t);
    z0=phi;
    subplot(3,2,1);
    plot(t,z0);
    title('Exact solution');
    xlabel('t (sec)');
    ylabel('\Phi(t)=e^{-t}');
    hold on

    phi=zeros(2,1);
    phi(1)=1;
    nmax=tmax/deltat;
    for n=2:nmax
        phi_t=phi(n-1);
        k1=lambda*phi_t;
        phi_t=phi_t+deltat*k1/3;
        k1=-5*k1/9 + lambda*phi_t;
        phi_t=phi_t+15*deltat*k1/16;
        k1=-153*k1/128 +lambda*phi_t;
        phi(n)= phi_t + 8*deltat*k1/15;
    end
    subplot(3,2,2);
    z1=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z1,'Color',[0.8500, 0.3250, 0.0980]);
    title('RK3 Low for \Deltat=0.1');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.1');



    phi=zeros(2,1);
    deltat = 0.6;
    phi(1)=1;
    nmax=tmax/deltat;

    for n=2:nmax
        phi_t=phi(n-1);
        k1=lambda*phi_t;
        phi_t=phi_t+deltat*k1/3;
        k1=-5*k1/9 + lambda*phi_t;
        phi_t=phi_t+15*deltat*k1/16;
        k1=-153*k1/128 +lambda*phi_t;
        phi(n)= phi_t + 8*deltat*k1/15;
    end
    subplot(3,2,3);
```

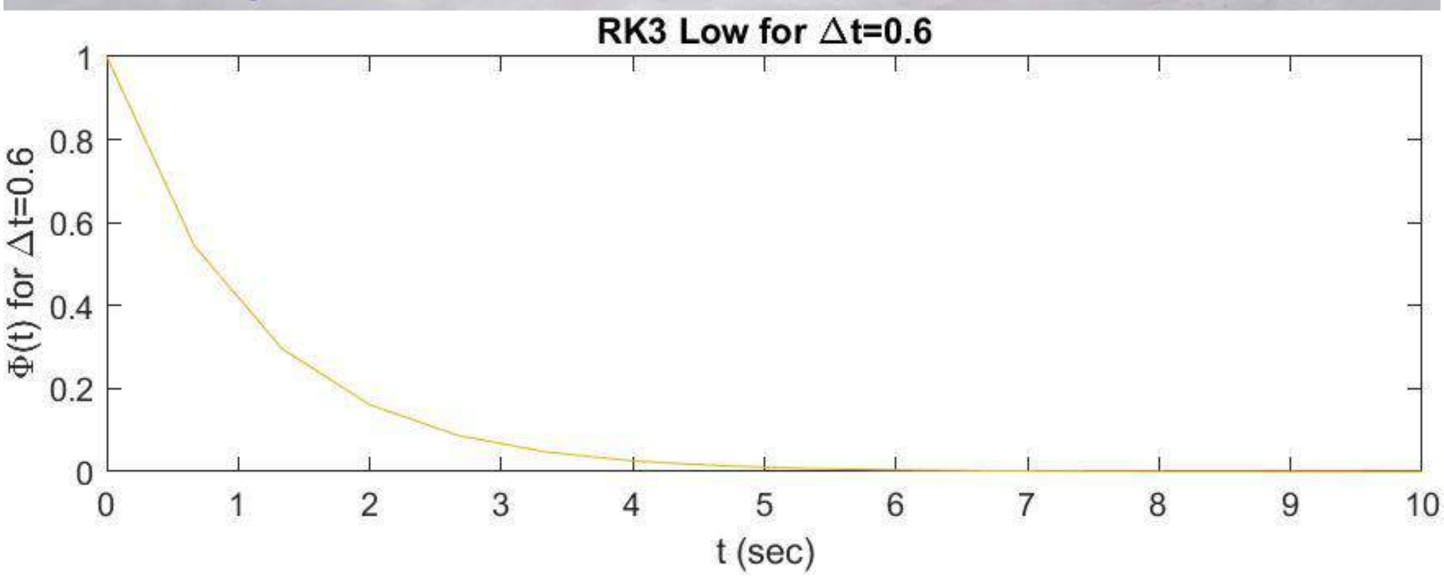
```
    z2=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z2,'Color',[0.9290, 0.6940, 0.1250]);
    title('RK3 Low for \Deltat=0.6');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.6');


    phi=zeros(2,1);
    deltat = 2.1;
    phi(1)=1;
    nmax=tmax/deltat;

    for n=2:nmax
        phi_t=phi(n-1);
        k1=lambda*phi_t;
        phi_t=phi_t+deltat*k1/3;
        k1=-5*k1/9 + lambda*phi_t;
        phi_t=phi_t+15*deltat*k1/16;
        k1=-153*k1/128 +lambda*phi_t;
        phi(n)= phi_t + 8*deltat*k1/15;
    end
    subplot(3,2,4);
    z3=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z3,'color',[0.4940, 0.1840, 0.5560]);
    title('RK3 Low for \Deltat=2.1');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=2.1');

    subplot(3,2,5:6);
    deltat1=0.1; deltat2=0.6; deltat3=2.1;
    t0=linspace(0,tmax,tmax/deltat1);
    t1=linspace(0,tmax,tmax/deltat1);
    t2=linspace(0,tmax,tmax/deltat2);
    t3=linspace(0,tmax,tmax/deltat3);
    plot(t0,z0,t1,z1,t2,z2,t3,z3);
    title('Plots of solution for Exact and RK3 Low for different value of
\Deltat');
    xlabel('t (sec)');
    ylabel('\Phi(t)');
    legend({'\phi(t)=e^{-
t}','\Deltat=0.1','\Deltat=0.6','\Deltat=2.1'},'Location','northeast');
end
```

RK-3 Low Storage

for $\Delta t = 0.1$
The solution is accurate and stable.



for $\Delta t = 0.6$
The error in the solution is higher but within reasonable approximation.
The solution is stable.

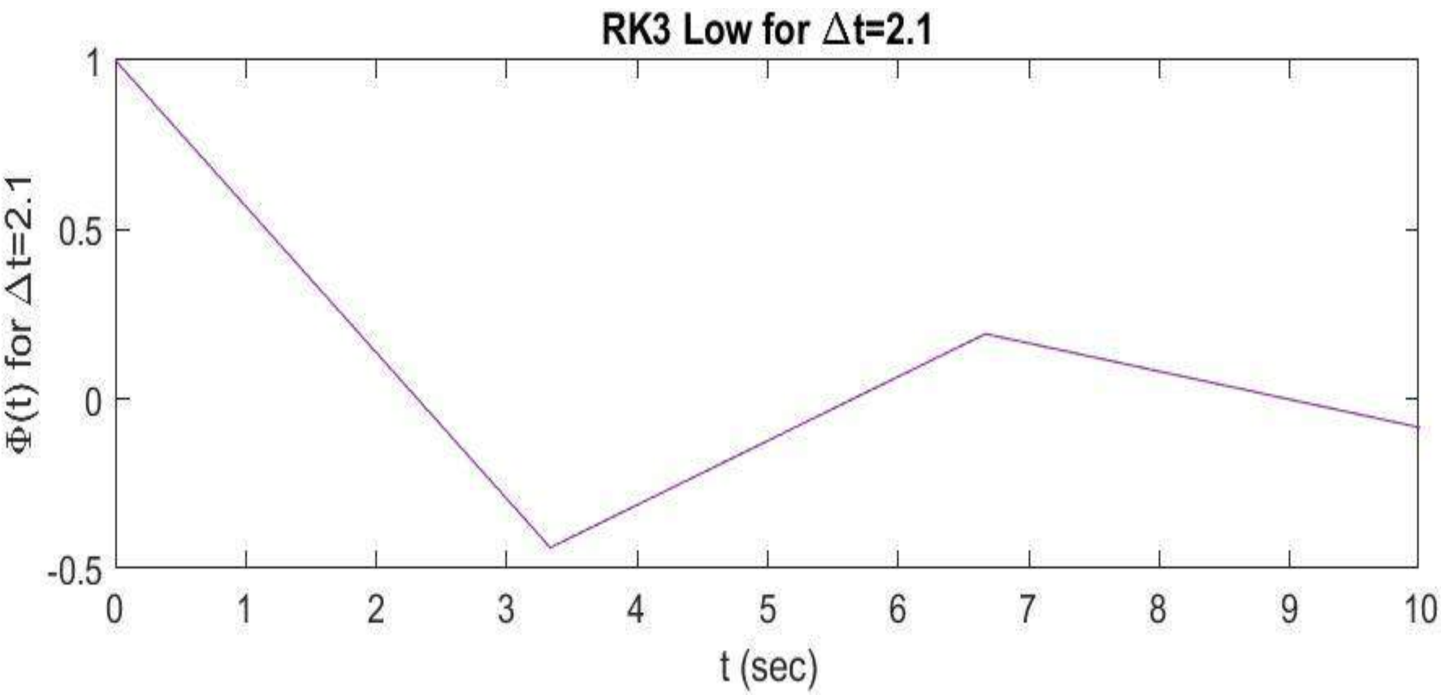


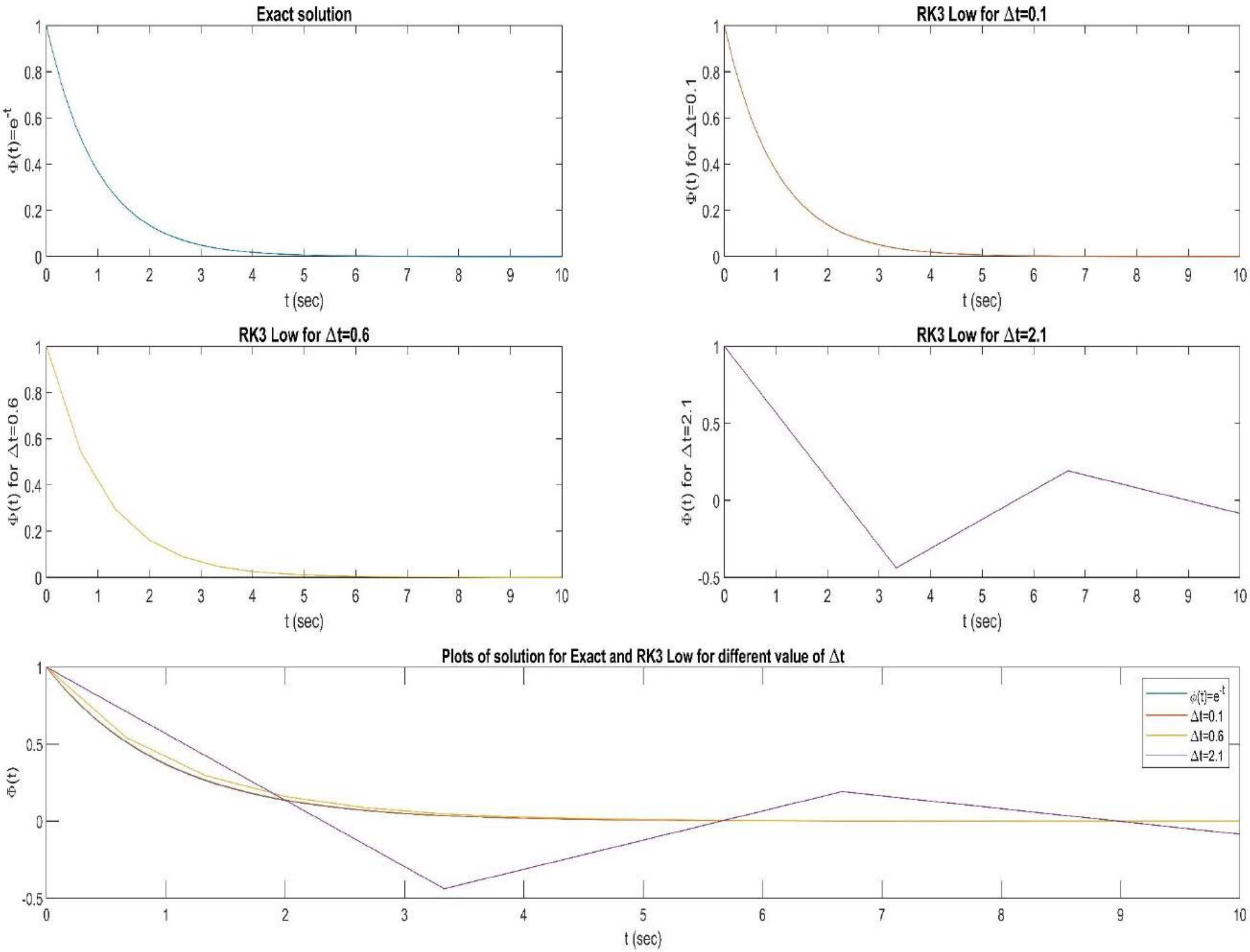
for $\Delta t = 2.1$

There is large error.

The solution osscilates about zero.

However solution do not diverge but converges to zero hence stable.




Exact solution
RK3 Low for Δt=0.1
RK3 Low for Δt=0.6
RK3 Low for Δt=2.1
Plots of solution for Exact and RK3 Low for different value of Δt
Φ(t)=e^-t
Φ(t) for Δt=0.1
Φ(t) for Δt=0.6
Φ(t) for Δt=2.1
Φ(t)
t (sec)
φ(t)=e^-t
Δt=0.1
Δt=0.6
Δt=2.1

<u>Table for the Φ(t) by RK3 Low Storage</u>

Nth time step	Φ(t) for Δt =0.1sec	Φ(t) for Δt =0.6sec	Φ(t) for Δt =2.1sec
1	1	1	1
2	0.904833333333333	0.544000000000000	-0.438500000000000
3	0.818723361111111	0.295936000000000	0.192282250000000
4	0.740808187912037	0.160989184000000	-0.0843157666250000
5	0.670307942029075	0.0875781160960000	0.0369724636650625
6	0.606516969545975	0.0476424951562240	-0.0162124253171299
7	0.548796771277516	0.0259175173649859	0.00710914850156147
8	0.496569611877606	0.0140991294465523	-0.00311736161793471
9	0.449312737147254	0.00766992641892446	0.00136696306946437
10	0.406553141662073	0.00417243997189490	-0.000599413305960126
11	0.367862834347233	0.00226980734471083	0.000262842734663515
12	0.332854554611854	0.00123477519552269	-0.000115256539149951
13	0.301177896164626	0.000671717706364344	5.05399924172537e-05
14	0.272515799712959	0.000365414432262203	-2.21617866749658e-05
15	0.246581379440276	0.000198785451150638	9.71794345697249e-06
16	0.223115051496876	0.000108139285425947	-4.26131820588244e-06
17	0.201881935762757	5.88277712717153e-05	1.86858803327945e-06
18	0.182669504876001	3.20023075718131e-05	-8.19375852593039e-07
19	0.165285456995302	1.74092553190663e-05	3.59296311362047e-07
20	0.149555791004582	9.47063489357209e-06	-1.57551432532258e-07

The solution for $\frac{d\phi}{dt} = \lambda\phi$ $\lambda = -1$ by Explicit Euler, Implicit Euler, Crank-Nicolson and RK3 Low Storage closely resembles in their value for $\Delta t = 0.1$. The error increase with increasing Δt value. We see that solution becomes unstable for Explicit Euler for $\Delta t = 2.1$ sec ~~inco~~ in contrast to Implicit Euler, Crank Nicolson and RK3 Low Storage where the solution is stable for $\Delta t = 2.1$

Q2b.)

2b) $\frac{d^2\phi}{dt^2} = -\omega^2\phi \qquad t>0$

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given $\omega^2 = 1$

we know that general solution is of the form

$\cdot\ \phi = \phi_0 \cos(\omega(t-t_0))$

with initial condition

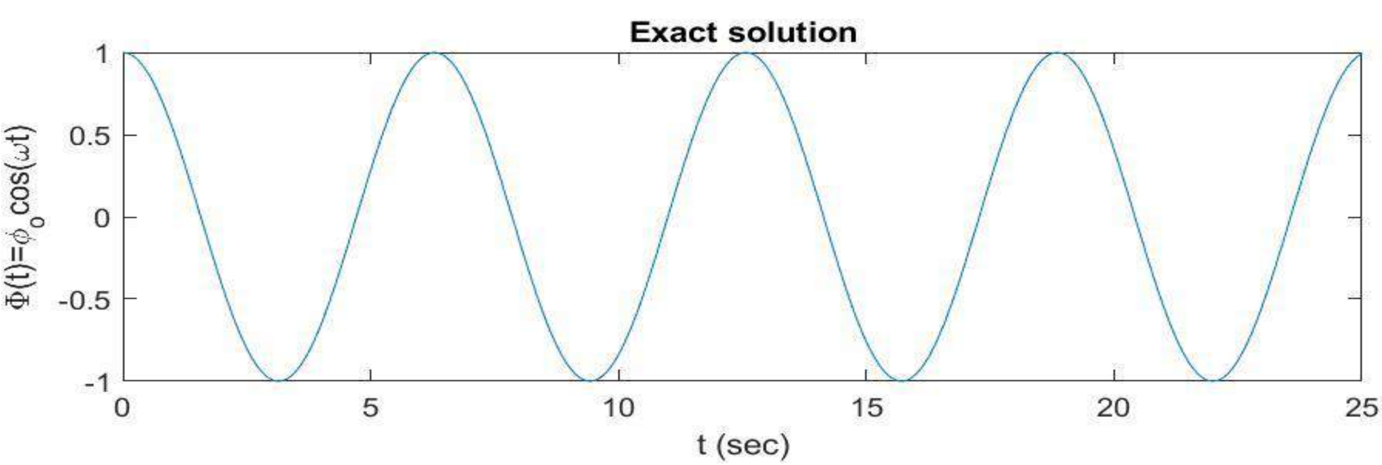
$\phi(t=0) = 1$

$\left.\frac{d\phi}{dt}\right|_{t=0} = 0$

$\therefore \Rightarrow \phi_0 = 1$ and $t_0 = 0$

$\therefore$ Exact solution $\Rightarrow$ $\boxed{\phi = \cos(\omega t)}$

The Exact solution being $\phi(t) = \cos(\omega t)$ $\omega = 1$
sinusoidally oscillates between
-1 and 1 as t varies in $[0, \infty)$
This can be clearly seen from the plot.



Explicit Euler

Explicit Euler

$$\frac{d^2\phi}{dt^2} = -\omega^2\phi$$

Let $\frac{d\phi}{dt} = u$ $\therefore$ we can rewrite the above equation as

$$\frac{du}{dt} = -\omega^2\phi \quad \text{and} \quad \frac{d\phi}{dt} = u$$

applying explicit Euler

$u^n = u(t_n)$

I) $$\int_{t_n}^{t_{n+1}} \frac{du}{dt}\,dt = -\int_{t_n}^{t_{n+1}} \omega^2\phi\,dt$$

$$u^{n+1} - u^n = -\omega^2\phi^n\Delta t$$

$$\boxed{u^{n+1} = u^n - \omega^2\phi^n\Delta t} \quad \text{— (a)}$$

II) $$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt}\,dt = \int_{t_n}^{t_{n+1}} u\,dt$$

$$\phi^{n+1} - \phi^n = u^n\Delta t$$

$$\boxed{\phi^{n+1} = \phi^n + u^n\Delta t} \quad \text{— (b)}$$

Therefore, we have (a) and (b) eqns to march in time,

with $\phi(t=0) = 1$ and $u(t=0) = 0$

$\omega^2 = 1$

$$\boxed{\begin{array}{l} \phi^{n+1} = \phi^n + u^n \Delta t \\ u^{n+1} = u^n - \omega^2 \phi^n \Delta t \end{array}}$$

$\omega^2 = 1$

$\phi^0 = 1 \quad u^0 = 0$

The following piece of code is used to evaluate $\phi(t)$ .. for different value of Δt
$\Delta t = 0.1,\ 0.6$ or 2.6

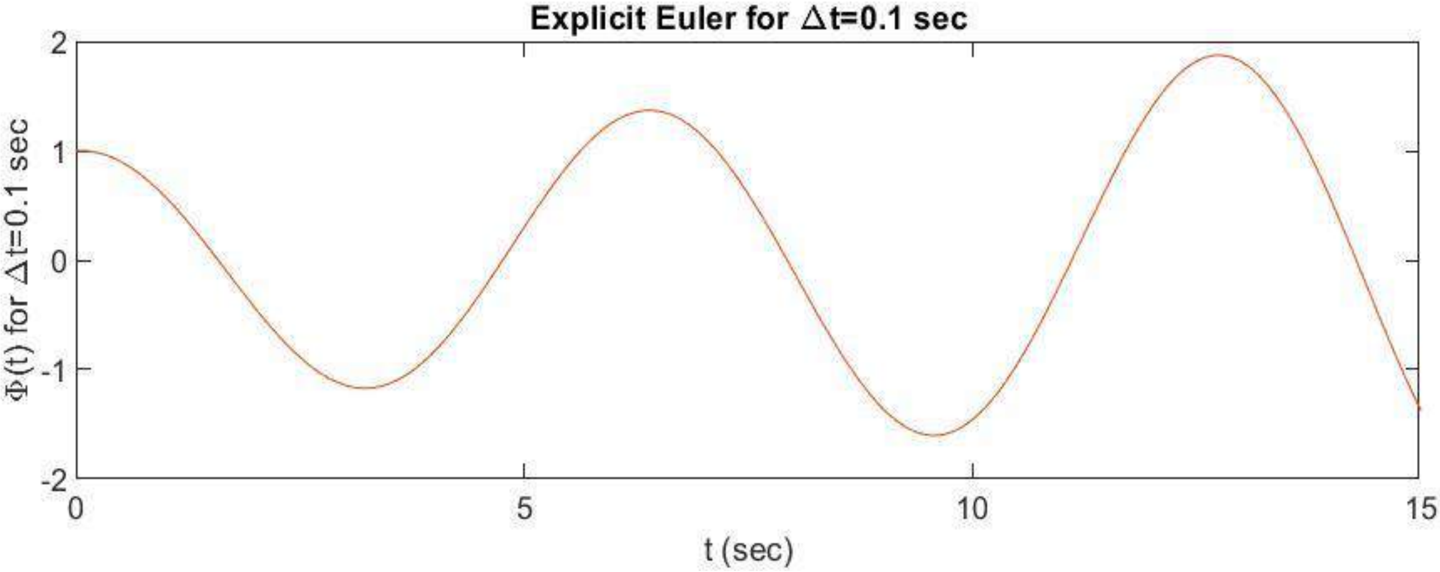
```
function z=explicitEuler()
    %Exact solution
    w2=1;
    w=1;
    phi0=1;
    deltat=0.1;
    tmax=15;
    t=linspace(0,tmax,tmax/deltat);
    phi=phi0*cos(w*t);
    z0=phi;
    subplot(3,2,1);
    plot(t,z0);
    title('Exact solution');
    xlabel('t (sec)');
    ylabel('\Phi(t)=\phi_ocos(\omegat)');
    hold on

    % for delat t=0.1
    phi=zeros(2,1);
    phid=zeros(2,1);
    phi(1)=1;
    phid(1)=0;
    nmax=tmax/deltat;
    for n=2:nmax
        phid(n)=phid(n-1)-w2*phi(n-1)*deltat;
        phi(n)=phi(n-1)+phid(n-1)*deltat;
    end
    subplot(3,2,2);
    z1=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z1,'Color',[0.8500, 0.3250, 0.0980]);
    title('Explicit Euler for \Deltat=0.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.1 sec');

    % for delat t=0.6
    deltat=0.6;
    phi=zeros(2,1);
    phid=zeros(2,1);
    phi(1)=1;
    phid(1)=0;
```

```
    nmax=tmax/deltat;
    for n=2:nmax
        phid(n)=phid(n-1)-w2*phi(n-1)*deltat;
        phi(n)=phi(n-1)+phid(n-1)*deltat;
    end
    subplot(3,2,3);
    z2=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z2,'Color',[0.9290, 0.6940, 0.1250]);
    title('Explicit Euler for \Deltat=0.6 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.6 sec');

    % for delat t=2.1
    deltat=2.1;
    phi=zeros(2,1);
    phid=zeros(2,1);
    phi(1)=1;
    phid(1)=0;
    nmax=tmax/deltat;
    for n=2:nmax
        phid(n)=phid(n-1)-w2*phi(n-1)*deltat;
        phi(n)=phi(n-1)+phid(n-1)*deltat;
    end
    subplot(3,2,4);
    z3=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z3,'color',[0.4940, 0.1840, 0.5560]);
    title('Explicit Euler for \Deltat=2.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=2.1 sec');

    %all plots in one graph
    subplot(3,2,5:6);
    deltat1=0.1; deltat2=0.6; deltat3=2.1;
    t0=linspace(0,tmax,tmax/deltat1);
    t1=linspace(0,tmax,tmax/deltat1);
    t2=linspace(0,tmax,tmax/deltat2);
    t3=linspace(0,tmax,tmax/deltat3);
    plot(t0,z0,t1,z1,t2,z2,t3,z3);
    title('Plots of solution for Exact and Explicit Euler for different
value of \Deltat');
    xlabel('t (sec)');
    ylabel('\Phi(t)');
    legend({'\Phi(t)=\phi_ocos(\omegat)','\Deltat=0.1 sec','\Deltat=0.6
sec','\Deltat=2.1 sec'},'Location','northwest');


end
```

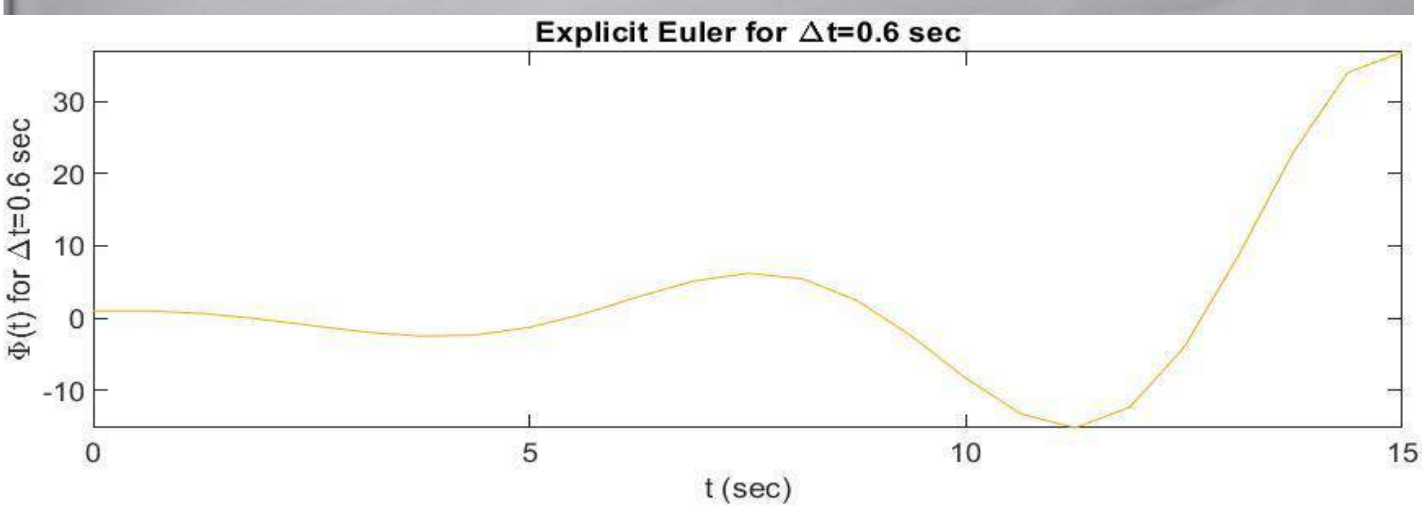
Explicit Euler.

For $\Delta t = 0.1$
The solution look similar to exact solution initially but is unstable and diverges ~~as t tend~~ to infinity as t increases.

The error also increases as t increases and hence solution is not accurate.



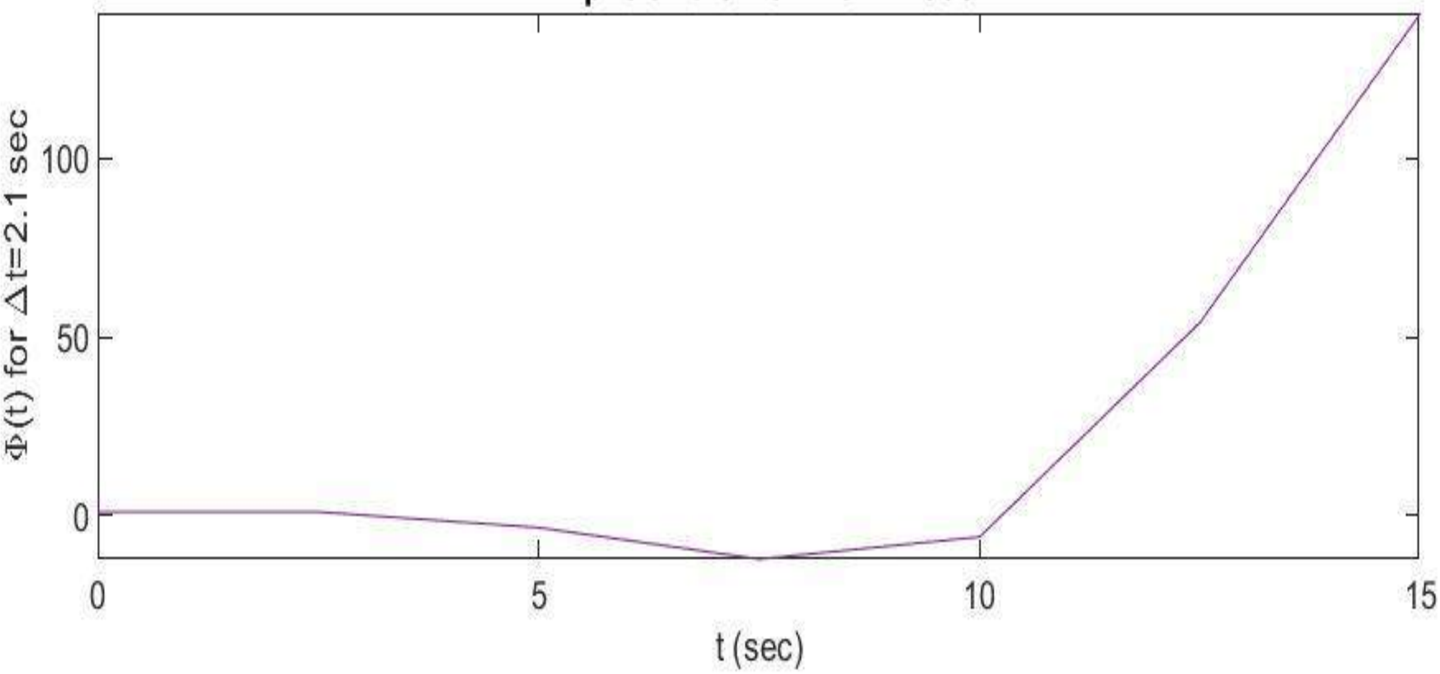
For $\Delta t = 0.6$
The solution osscialates about 0 and very quickly diverges as t increases.
The solution is unstable.

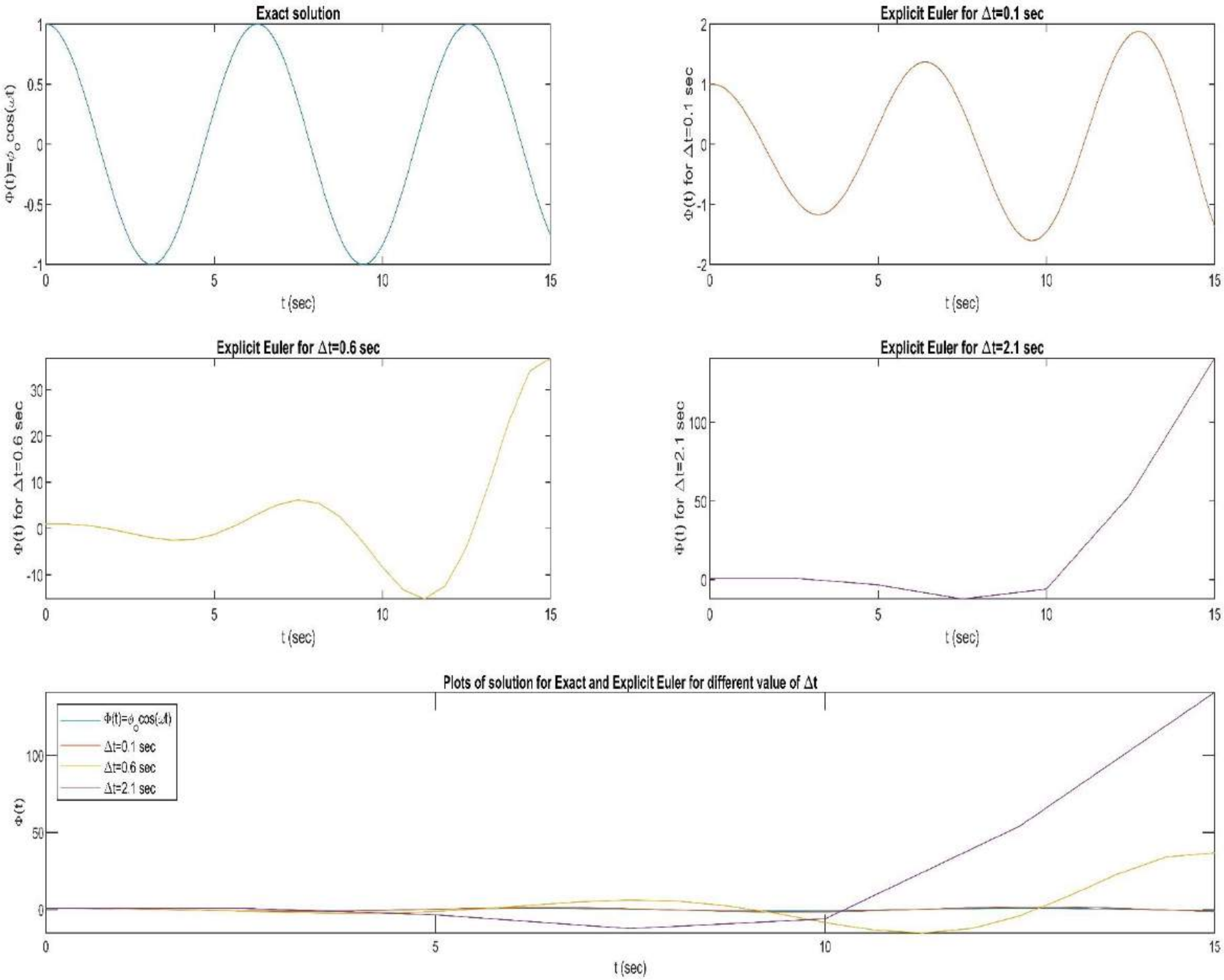


For $\Delta t = 2.1$

The solution is neither accurate nor-
-stable.

Explicit Euler for Δt=2.1 sec




Exact solution
Φ(t)=φo cos(ωt)
1
0.5
0
-0.5
-1
0
5
10
15
t (sec)
Explicit Euler for Δt=0.1 sec
Φ(t) for Δt=0.1 sec
2
1
0
-1
-2
0
5
10
15
t (sec)
Explicit Euler for Δt=0.6 sec
Φ(t) for Δt=0.6 sec
30
20
10
0
-10
0
5
10
15
t (sec)
Explicit Euler for Δt=2.1 sec
Φ(t) for Δt=2.1 sec
100
50
0
0
5
10
15
t (sec)
Plots of solution for Exact and Explicit Euler for different value of Δt
Φ(t)=φo cos(ωt)
Δt=0.1 sec
Δt=0.6 sec
Δt=2.1 sec
Φ(t)
100
50
0
0
5
10
15
t (sec)

<u>Table for the Φ(t) by Explicit Euler</u>

Nth time step	Φ(t) for Δt =0.1sec	Φ(t) for Δt =0.6sec	Φ(t) for Δt =2.1sec
1	1	1	1
2	1	1	1
3	0.990000000000000	0.640000000000000	-3.41000000000000
4	0.970000000000000	-0.0800000000000000	-12.2300000000000
5	0.940100000000000	-1.03040000000000	-6.01190000000000
6	0.900500000000000	-1.95200000000000	54.1405000000000
7	0.851499000000000	-2.50265600000000	140.805379000000
8	0.793493000000000	-2.35059200000000	-11.2893470000001
9	0.726972010000000	-1.29757184000000	-784.335794390000
10	0.652516090000000	0.601661440000000	-1507.59622151000
11	0.570790449900000	2.96802058240000	1228.06420462990
12	0.482539648900000	5.11778160640000	10612.2239676289
13	0.388580943401000	6.19905522073600	14580.6205882100
14	0.289796841413000	5.43792745676800	-28250.8904884523
15	0.187126929990990	2.44513981333504	-135382.938359121
16	0.0815590501548499	-2.50530171453440	-117928.559175715
17	-0.0258800989812000	-8.33599357520445	496564.578171414
18	-0.134134838618798	-13.2647768186421	1631122.66148345
19	-0.242130777266585	-15.1926023750062	575830.955059542
20	-0.348785367528183	-12.3451082766591	-7672711.68850637

Implicit Euler

Implicit Euler

we have $\frac{d^2\phi}{dt^2} = -\omega^2\phi$

can be written as

$$\frac{d\phi}{dt} = u$$

and $\frac{du}{dt} = -\omega^2\phi$

applying implicit Euler

1) $\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt}\,dt = \int_{t_n}^{t_{n+1}} u\,dt$

$$\phi^{n+1} - \phi^n = u^{n+1}\Delta t$$

$$\boxed{\phi^{n+1} = \phi^n + u^{n+1}\Delta t} \quad \text{— (c)}$$

ii) $$\int_{t_n}^{t_{n+1}} \frac{du}{dt}\,dt = \int_{t_n}^{t_{n+1}} -\omega^2 \phi\, dt$$

$$u^{n+1} - u^n = -\omega^2 \phi^{n+1} \Delta t$$

$$\boxed{u^{n+1} = u^n - \omega^2 \cdot \phi^{n+1} \Delta t} \quad \text{— (d)}$$

from (c) and (d)

$$\phi^{n+1} = \phi^n + u^{n+1} \Delta t$$

$$u^{n+1} = u^n - \omega^2 \phi^{n+1} \Delta t$$

replacing u^{n+1} into eq (c)

$$\phi^{n+1} = \phi^n + (u^n - \omega^2 \phi^{n+1} \Delta t) \Delta t$$

$$\phi^{n+1} = \phi^n + u^n \Delta t - \omega^2 \phi^{n+1} \Delta t^2$$

$$\phi^{n+1}(1 + \omega^2 \Delta t^2) = \phi^n + u^n \Delta t$$

$$\Rightarrow \boxed{\phi^{n+1} = \frac{(\phi^n + u^n \Delta t)}{(1 + \omega^2 \Delta t^2)}} \quad (e)$$

Replacing eq^n (e) ~~into~~ for ϕ^{n+1} into eq (d) we have.

$$u^{n+1} = u^n - \omega^2 \left(\frac{\phi^n + u^n \Delta t}{1+\omega^2 \Delta t^2} \right) \Delta t$$

$$u^{n+1} = \frac{u^n (1+\omega^2 \Delta t^2) - \Delta t \omega^2 \phi^n - \omega^2 \Delta t^2 u^n}{(1+\omega^2 \Delta t^2)}$$

$$\boxed{u^{n+1} = \frac{u^n - \omega^2 \Delta t \phi^n}{1+\omega^2 \Delta t^2}} \quad (f)$$

we can use eq^n (e) and eq^n (f) along with

$$\boxed{\begin{aligned} \phi^{n+1} &= \frac{\phi^n + u^n \Delta t}{1+\omega^2 \Delta t^2} \\ u^{n+1} &= \frac{u^n - \omega^2 \Delta t \phi^n}{1+\omega^2 \Delta t^2} \end{aligned}}$$

with $\phi^0 = 1$
$u^0 = 0$
$\omega^2 = 1$

to march in time and find the value of $\phi(t)$ for different value of Δt ($\Delta t = 0.1$, 0.6 and 2.1)

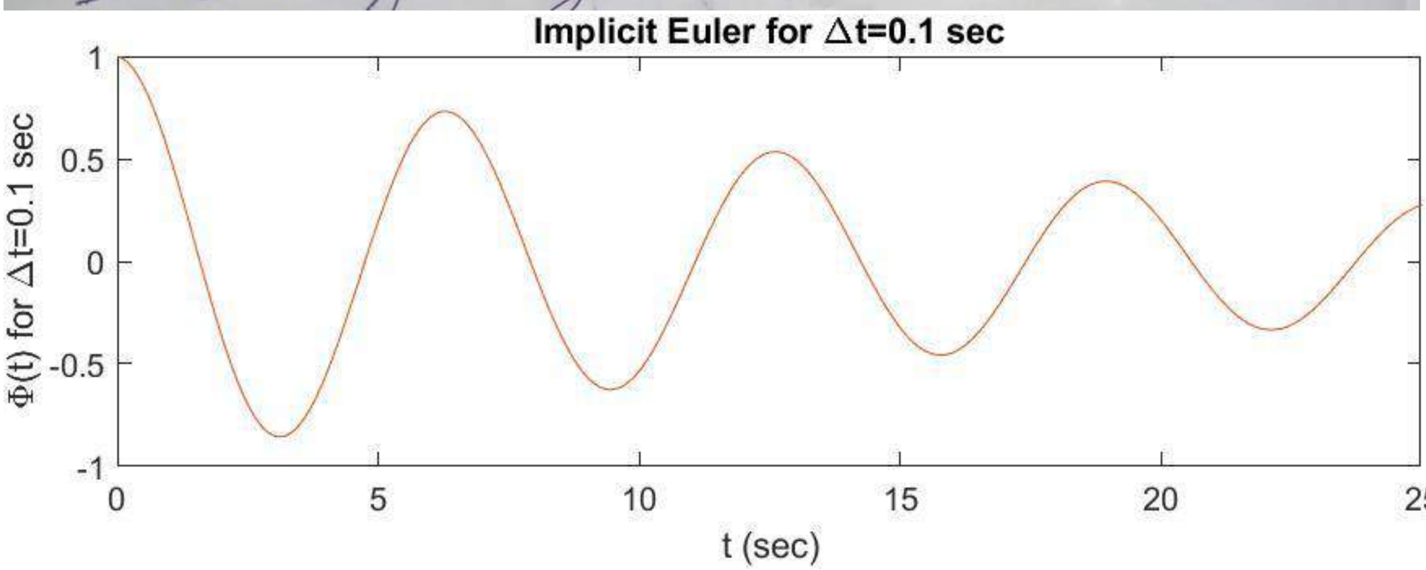
```
function z=implicitEuler()
    %Exact solution
    w2=1;
    w=1;
    phi0=1;
    deltat=0.1;
    tmax=25;
    t=linspace(0,tmax,tmax/deltat);
    phi=phi0*cos(w*t);
    z0=phi;
    subplot(3,2,1);
    plot(t,z0);
    title('Exact solution');
    xlabel('t (sec)');
    ylabel('\Phi(t)=\phi_ocos(\omegat)');
    hold on

    % for delat t=0.1
    phi=zeros(2,1);
    phid=zeros(2,1);
    phi(1)=1;
    phid(1)=0;
    nmax=tmax/deltat;
    for n=2:nmax
        phi(n)=(phi(n-1)+phid(n-1)*deltat)/(1+w2*deltat*deltat);
        phid(n)=(phid(n-1)-w2*phi(n-1)*deltat)/(1+w2*deltat*deltat);

    end
    subplot(3,2,2);
    z1=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z1,'Color',[0.8500, 0.3250, 0.0980]);
    title('Implicit Euler for \Deltat=0.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.1 sec');

    % for delat t=0.6
    deltat=0.6;
    phi=zeros(2,1);
    phid=zeros(2,1);
    phi(1)=1;
    phid(1)=0;
    nmax=tmax/deltat;
    for n=2:nmax
        phi(n)=(phi(n-1)+phid(n-1)*deltat)/(1+w2*deltat*deltat);
        phid(n)=(phid(n-1)-w2*phi(n-1)*deltat)/(1+w2*deltat*deltat);
    end
    subplot(3,2,3);
    z2=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z2,'Color',[0.9290, 0.6940, 0.1250]);
    title('Implicit Euler for \Deltat=0.6 sec');
```

```
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.6 sec');


    % for delat t=2.1
    deltat=2.1;
    phi=zeros(2,1);
    phid=zeros(2,1);
    phi(1)=1;
    phid(1)=0;
    nmax=tmax/deltat;
    for n=2:nmax
        phi(n)=(phi(n-1)+phid(n-1)*deltat)/(1+w2*deltat*deltat);
        phid(n)=(phid(n-1)-w2*phi(n-1)*deltat)/(1+w2*deltat*deltat);
    end
    subplot(3,2,4);
    z3=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z3,'color',[0.4940, 0.1840, 0.5560]);
    title('Implicit Euler for \Deltat=2.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=2.1 sec');

    %all plots in one graph
    subplot(3,2,5:6);
    deltat1=0.1; deltat2=0.6; deltat3=2.1;
    t0=linspace(0,tmax,tmax/deltat1);
    t1=linspace(0,tmax,tmax/deltat1);
    t2=linspace(0,tmax,tmax/deltat2);
    t3=linspace(0,tmax,tmax/deltat3);
    plot(t0,z0,t1,z1,t2,z2,t3,z3);
    title('Plots of solution for Exact and Implicit Euler for different
value of \Deltat');
    xlabel('t (sec)');
    ylabel('\Phi(t)');
    legend({'\Phi(t)=\phi_ocos(\omegat)','\Deltat=0.1 sec','\Deltat=0.6
sec','\Deltat=2.1 sec'},'Location','northwest');



end
```

Implicit Euler

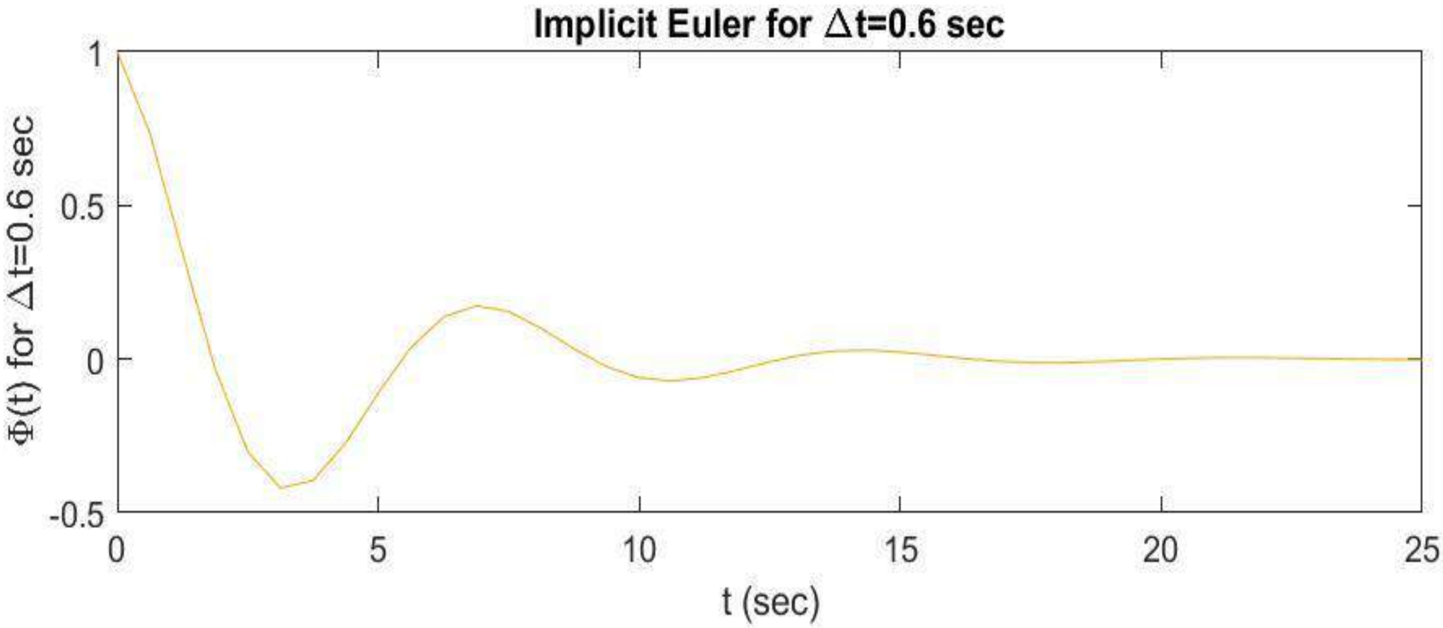
for $\Delta t = 0.1$

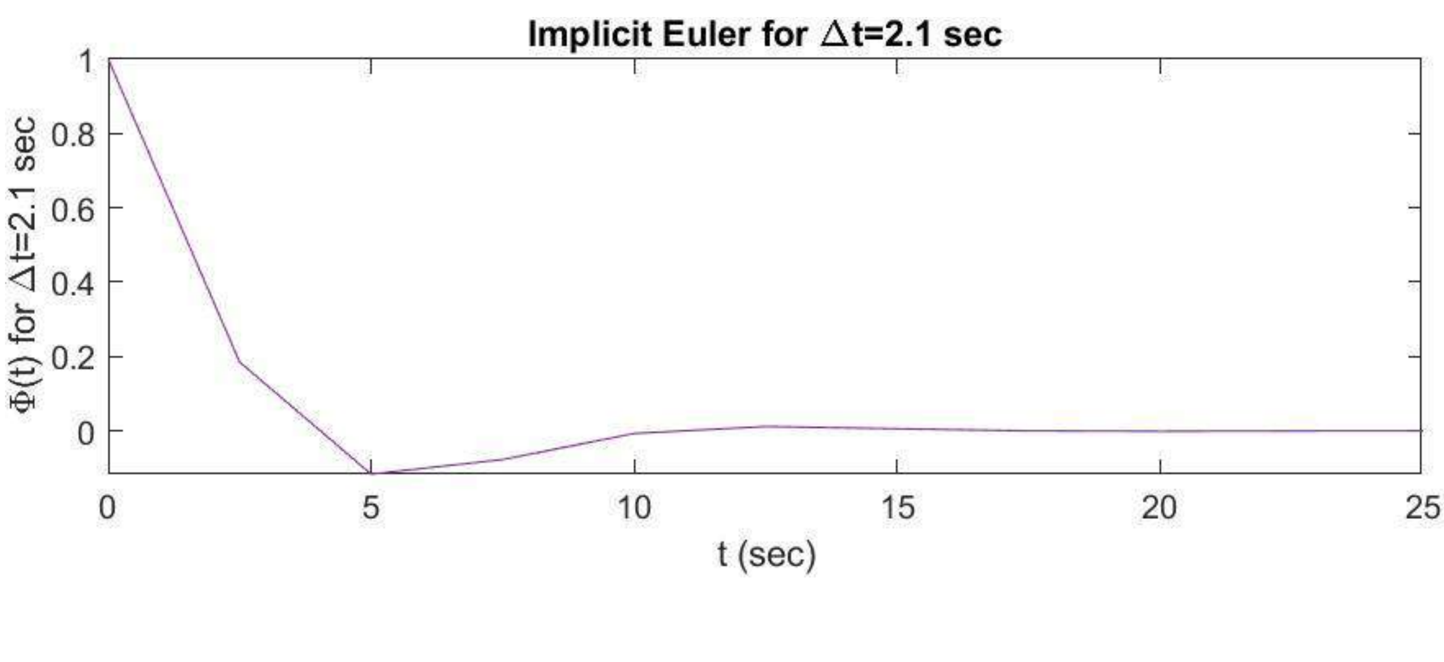
The solution is stable but not accurate. Initially the plot of the solution has similar nature as exact solution. but soon solution obtained is ~~damped~~ looks like damped and error increases in comparison to exact solution. It converges to zero as $t \to \infty$.

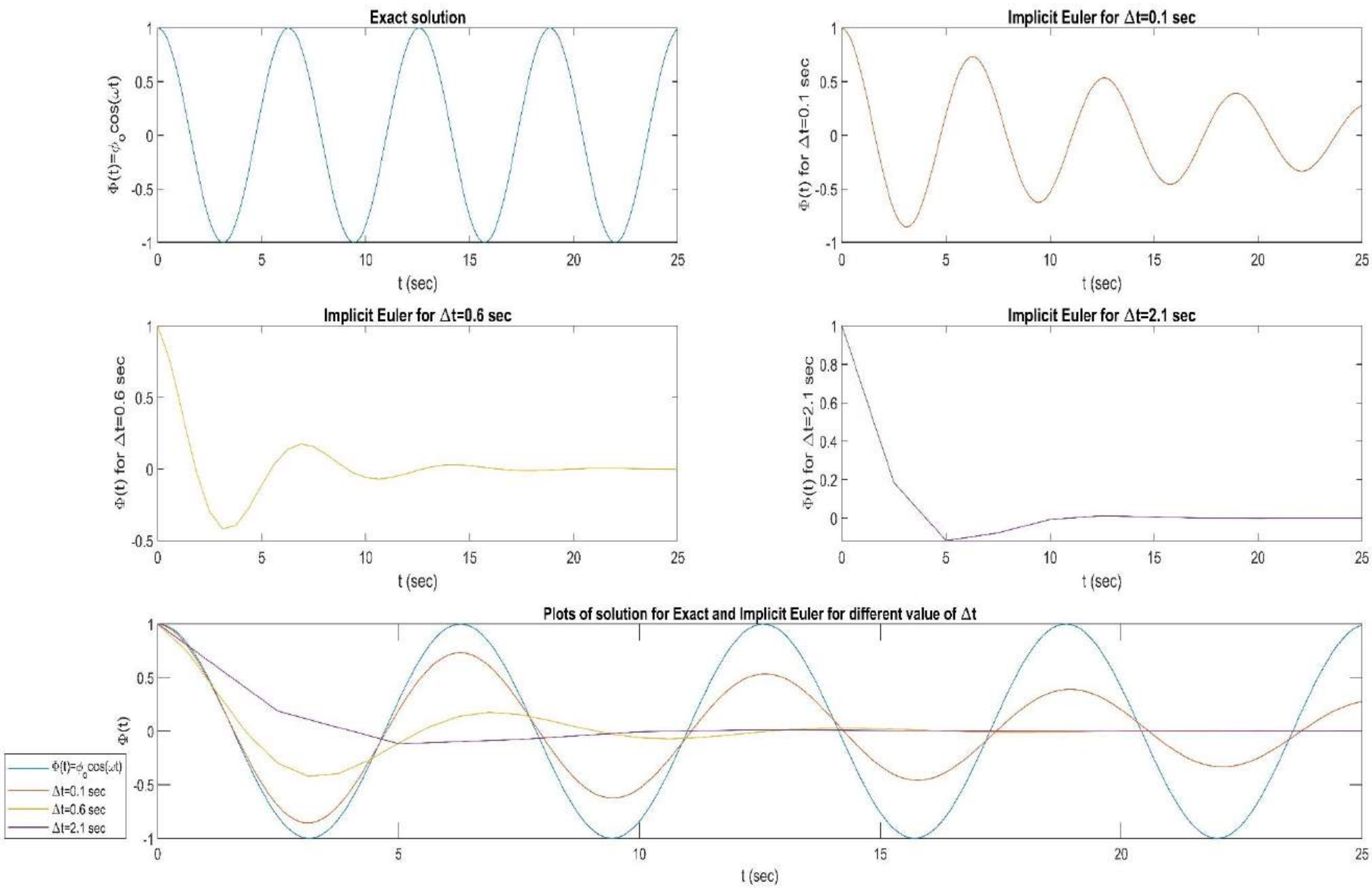


for $\Delta t = 0.6$ and $\Delta t = 2.1$

They have similar nature. Both are stable but very inaccurate solution. The solution stabilizes to zero in contrast to exact solution.


Implicit Euler for Δt=0.6 sec
Φ(t) for Δt=0.6 sec
t (sec)
1
0.5
0
-0.5
0
5
10
15
20
25


Implicit Euler for Δt=2.1 sec
Φ(t) for Δt=2.1 sec
t (sec)
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25


Exact solution
Φ(t)=φ_o cos(ωt)
t (sec)
Implicit Euler for Δt=0.1 sec
Φ(t) for Δt=0.1 sec
t (sec)
Implicit Euler for Δt=0.6 sec
Φ(t) for Δt=0.6 sec
t (sec)
Implicit Euler for Δt=2.1 sec
Φ(t) for Δt=2.1 sec
t (sec)
Plots of solution for Exact and Implicit Euler for different value of Δt
Φ(t)
t (sec)
Φ(t)=φ_o cos(ωt)
Δt=0.1 sec
Δt=0.6 sec
Δt=2.1 sec

<u>Table for the Φ(t) by Implicit Euler</u>

Nth time step	Φ(t) for Δt =0.1sec	Φ(t) for Δt =0.6sec	Φ(t) for Δt =2.1sec
1	1	1	1
2	0.990099009900990	0.735294117647059	0.184842883548983
3	0.970493088912852	0.346020761245675	-0.116509100351577
4	0.941472443489815	-0.0318033787909628	-0.0772386477360699
5	0.903417621848295	-0.301196705020294	-0.00701815066923520
6	0.856794851689877	-0.419551493565902	0.0116825039551940
7	0.802150575773722	-0.395519325081992	0.00561611064318358
8	0.740105247383729	-0.273152321028001	-8.32315469180828e-05
9	0.671346454449244	-0.110871556598538	-0.00106886760388535
10	0.596621447044315	0.0378008881109753	-0.000379760380933941
11	0.516729148157809	0.137112744720947	5.71805622952809e-05
12	0.432511731951785	0.173841618625676	9.13348439047139e-05
13	0.344845857174020	0.154831244507650	2.31957718140752e-05
14	0.254633645936887	0.0998682870511947	-8.30744921932782e-06
15	0.162793499702726	0.0330186247020141	-7.35871908553250e-06
16	0.0702508450183816	-0.0248757629758578	-1.18484084135623e-06
17	-0.0220710986791711	-0.0608604048924483	9.22188059670989e-07
18	-0.113260437996756	-0.0712095932419403	5.59929197910944e-07
19	-0.202425522093407	-0.0599696923466414	3.65379549262289e-08
20	-0.288703570485206	-0.0358307290083401	-8.99913656300343e-08

Crank - Nicolson

The two equation

$$\frac{d\phi}{dt} = u$$

and $$\frac{du}{dt} = -\omega^2 \phi$$

integrating

1) $$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt} dt = \int_{t_n}^{t_{n+1}} u dt$$

$$\boxed{\phi^{n+1} = \phi^n + \frac{1}{2}\left(u^{n+1} + u^n\right)\Delta t} \quad (g)$$

ii) $$\int_{t_n}^{t_{n+1}} \frac{du}{dt} dt = -\int_{t_n}^{t_{n+1}} \omega^2 \phi \, dt$$

$$\boxed{u^{n+1} = u^n - \frac{\omega^2}{2}\left(\phi^{n+1} + \phi^n\right)\Delta t} \quad (h)$$

replacing for u^{n+1} in eq (h) into eq (g) we may write

$$\phi^{n+1} = \phi^n + \frac{1}{2}\left(u^n - \frac{\omega^2}{2}(\phi^{n+1}+\phi^n)\Delta t + u^n\right)\Delta t$$

$$\Rightarrow \phi^{n+1} = \phi^n + \frac{1}{2}\left(2u^n \Delta t - \frac{\omega^2}{2}\phi^{n+1}\Delta t^2 - \frac{\omega^2}{2}\phi^n \Delta t^2\right)$$

$$\Rightarrow \phi^{n+1} = \phi^n + u^n \Delta t - \frac{\omega^2}{4}\phi^{n+1}\Delta t^2 - \frac{\omega^2}{4}\phi^n \Delta t^2$$

$$\Rightarrow \boxed{\phi^{n+1} = \frac{1}{\left\{1+\frac{\omega^2 \Delta t^2}{4}\right\}}\left\{\phi^n\left(1-\frac{\omega^2\Delta t^2}{4}\right) + u^n \Delta t\right\}} \quad (i)$$

$\therefore$ We can use eq (i) and (h) in following sequence (first i) then h)) to march in time and find $\phi(t)$

$$\boxed{\phi^{n+1} = \frac{1}{\left(1+\frac{\omega^2\Delta t^2}{4}\right)}\left\{\phi^n\left(1-\frac{\omega^2\Delta t^2}{4}\right) + u^n \Delta t\right\}}$$

$$\boxed{u^{n+1} = u^n - \frac{\omega^2}{2}\left(\phi^{n+1}+\phi^n\right)\Delta t}$$

with $\phi^0 = 1 \quad u^0 = 0 \quad \omega^2 = 1$

with different Δt $(\Delta t = 0.1, 0.6, 2.1$

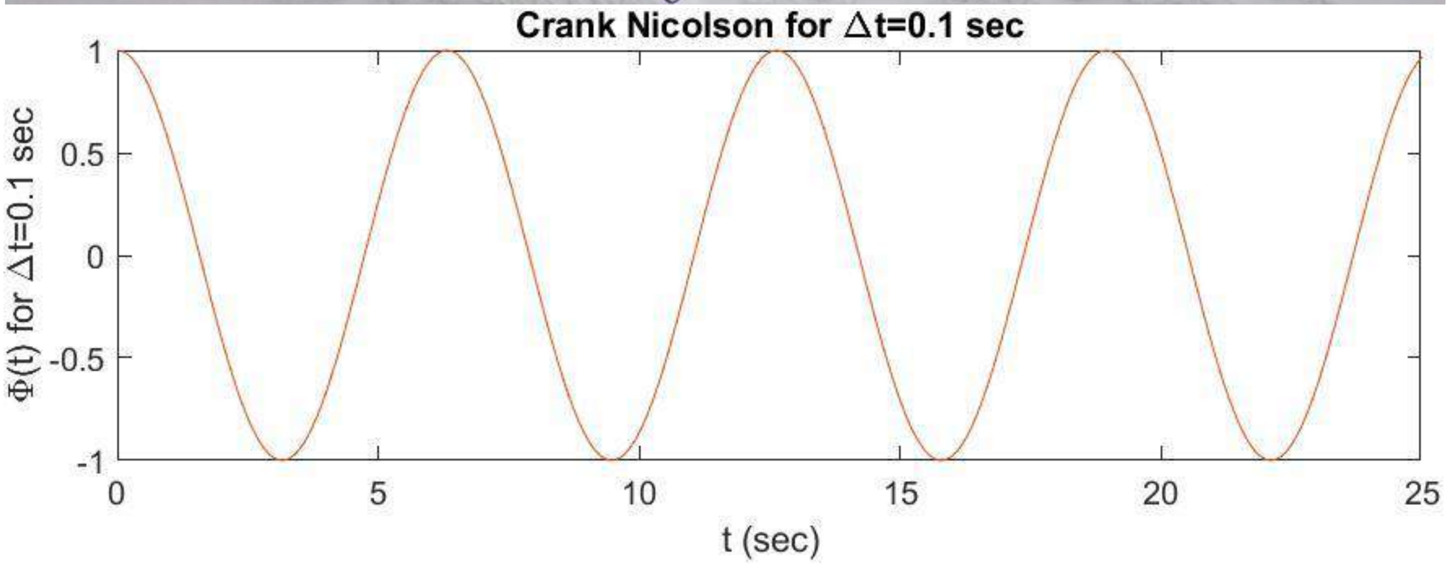
```
function z=crankNicolson()
    %Exact solution
    w2=1;
    w=1;
    phi0=1;
    deltat=0.1;
    tmax=25;
    t=linspace(0,tmax,tmax/deltat);
    phi=phi0*cos(w*t);
    z0=phi;
    subplot(3,2,1);
    plot(t,z0);
    title('Exact solution');
    xlabel('t (sec)');
    ylabel('\Phi(t)=\phi_ocos(\omegat)');
    hold on

    % for delat t=0.1
    phi=zeros(2,1);
    phid=zeros(2,1);
    phi(1)=1;
    phid(1)=0;
    nmax=tmax/deltat;
    for n=2:nmax
       phi(n)=(phi(n-1)+phid(n-1)*deltat-w2*deltat*deltat*phi(n-
1)/4)/(1+w2*deltat*deltat/4);
       phid(n)=phid(n-1)-w2*(phi(n-1)+phi(n))*deltat/2;
    end
    subplot(3,2,2);
    z1=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z1,'Color',[0.8500, 0.3250, 0.0980]);
    title('Crank Nicolson for \Deltat=0.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.1 sec');

    % for delat t=0.6
    deltat=0.6;
    phi=zeros(2,1);
    phid=zeros(2,1);
    phi(1)=1;
    phid(1)=0;
    nmax=tmax/deltat;
    for n=2:nmax
       phi(n)=(phi(n-1)+phid(n-1)*deltat-w2*deltat*deltat*phi(n-
1)/4)/(1+w2*deltat*deltat/4);
       phid(n)=phid(n-1)-w2*(phi(n-1)+phi(n))*deltat/2;
    end
    subplot(3,2,3);
    z2=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z2,'Color',[0.9290, 0.6940, 0.1250]);
```

```
    title('Crank Nicolson for \Deltat=0.6 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=0.6 sec');

    % for delat t=2.1
    deltat=2.1;
    phi=zeros(2,1);
    phid=zeros(2,1);
    phi(1)=1;
    phid(1)=0;
    nmax=tmax/deltat;
    for n=2:nmax
       phi(n)=(phi(n-1)+phid(n-1)*deltat-w2*deltat*deltat*phi(n-
1)/4)/(1+w2*deltat*deltat/4);
       phid(n)=phid(n-1)-w2*(phi(n-1)+phi(n))*deltat/2;
    end
    subplot(3,2,4);
    z3=phi;
    t=linspace(0,tmax,tmax/deltat);
    plot(t,z3,'color',[0.4940, 0.1840, 0.5560]);
    title('Crank Nicolson for \Deltat=2.1 sec');
    xlabel('t (sec)');
    ylabel('\Phi(t) for \Deltat=2.1 sec');

    %all plots in one graph
    subplot(3,2,5:6);
    deltat1=0.1; deltat2=0.6; deltat3=2.1;
    t0=linspace(0,tmax,tmax/deltat1);
    t1=linspace(0,tmax,tmax/deltat1);
    t2=linspace(0,tmax,tmax/deltat2);
    t3=linspace(0,tmax,tmax/deltat3);
    plot(t0,z0,t1,z1,t2,z2,t3,z3);
    title('Plots of solution for Exact and Crank Nicolson for different
value of \Deltat');
    xlabel('t (sec)');
    ylabel('\Phi(t)');
    legend({'\Phi(t)=\phi_ocos(\omegat)','\Deltat=0.1 sec','\Deltat=0.6
sec','\Deltat=2.1 sec'},'Location','northwest');
      % z=[z1(1:20),z2(1:20),z3(1:20)];

end
```

Crank Nicolson

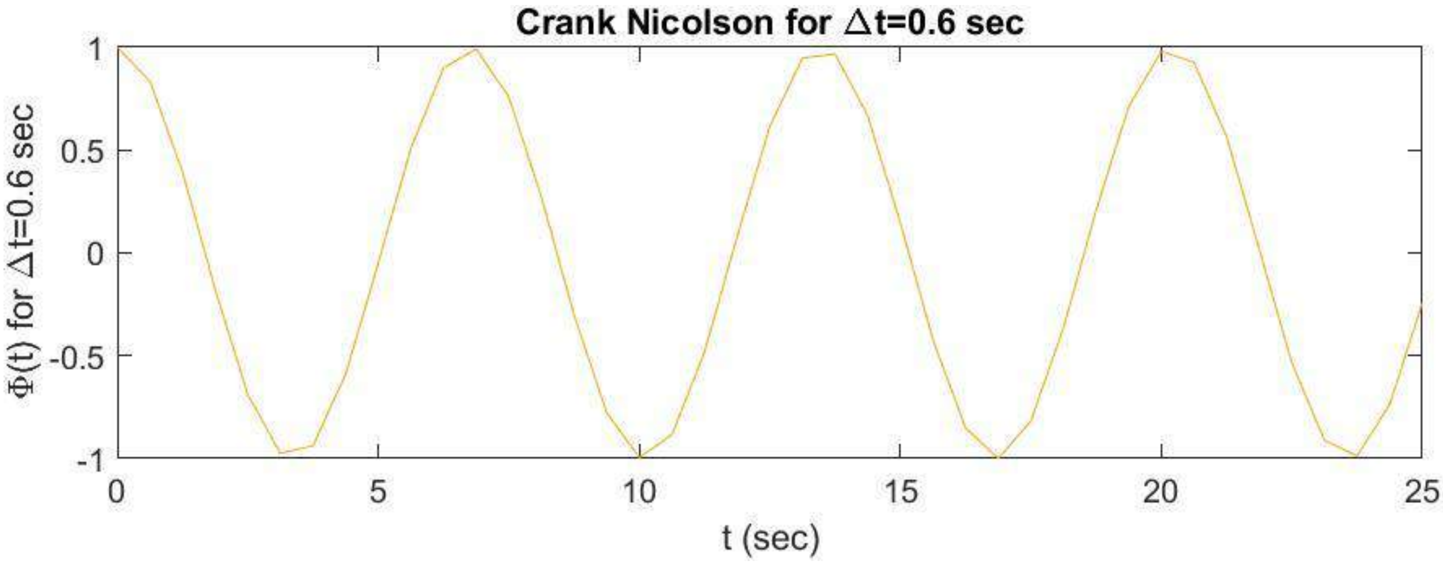
for $\Delta t = 0.1$

The solution is quite accurate.
The solution is stable.
The solution closely resembles exact solution



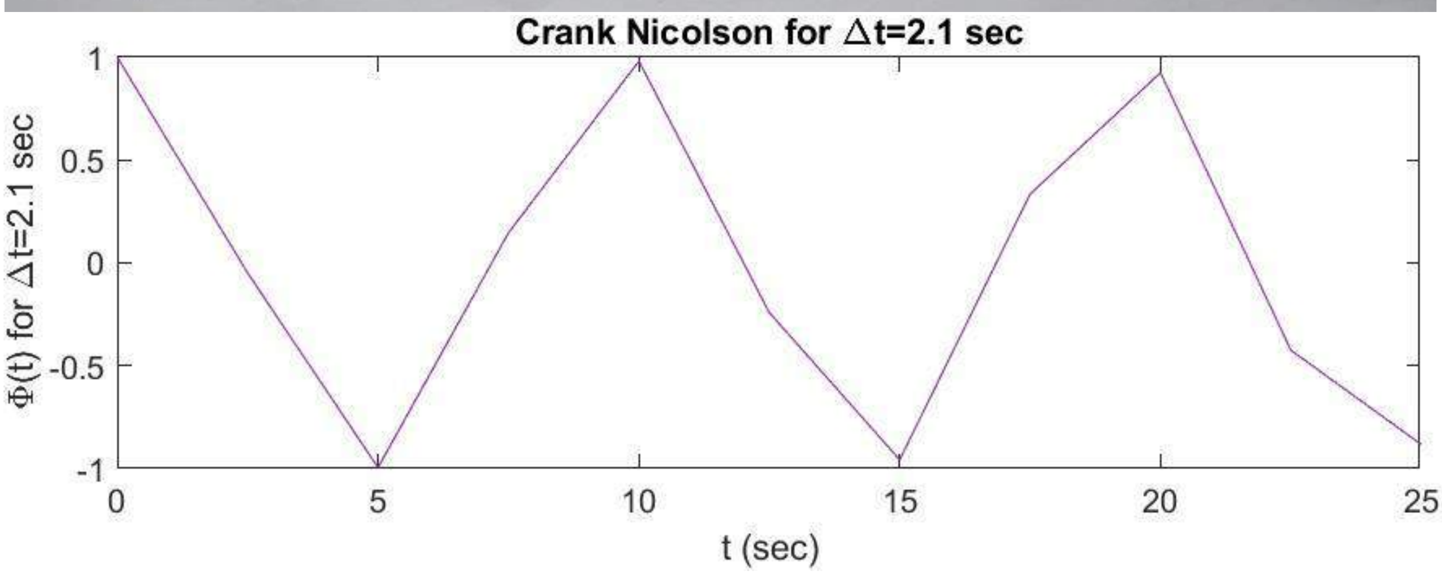
for $\Delta t = 0.6$

The solution is reasonably accurate but ~~more~~ lesser than $\Delta t = 0.1$.
The solution is stable.
The solution has similar nature as exact solution



<u>for $\Delta t = 2.1$</u>

The error ~~o~~ ~~in~~ is large in comparison to solution by $\Delta t = 0.1$ and $\Delta t = 0.6$ by Crank Nicolson. But the solution is stable.



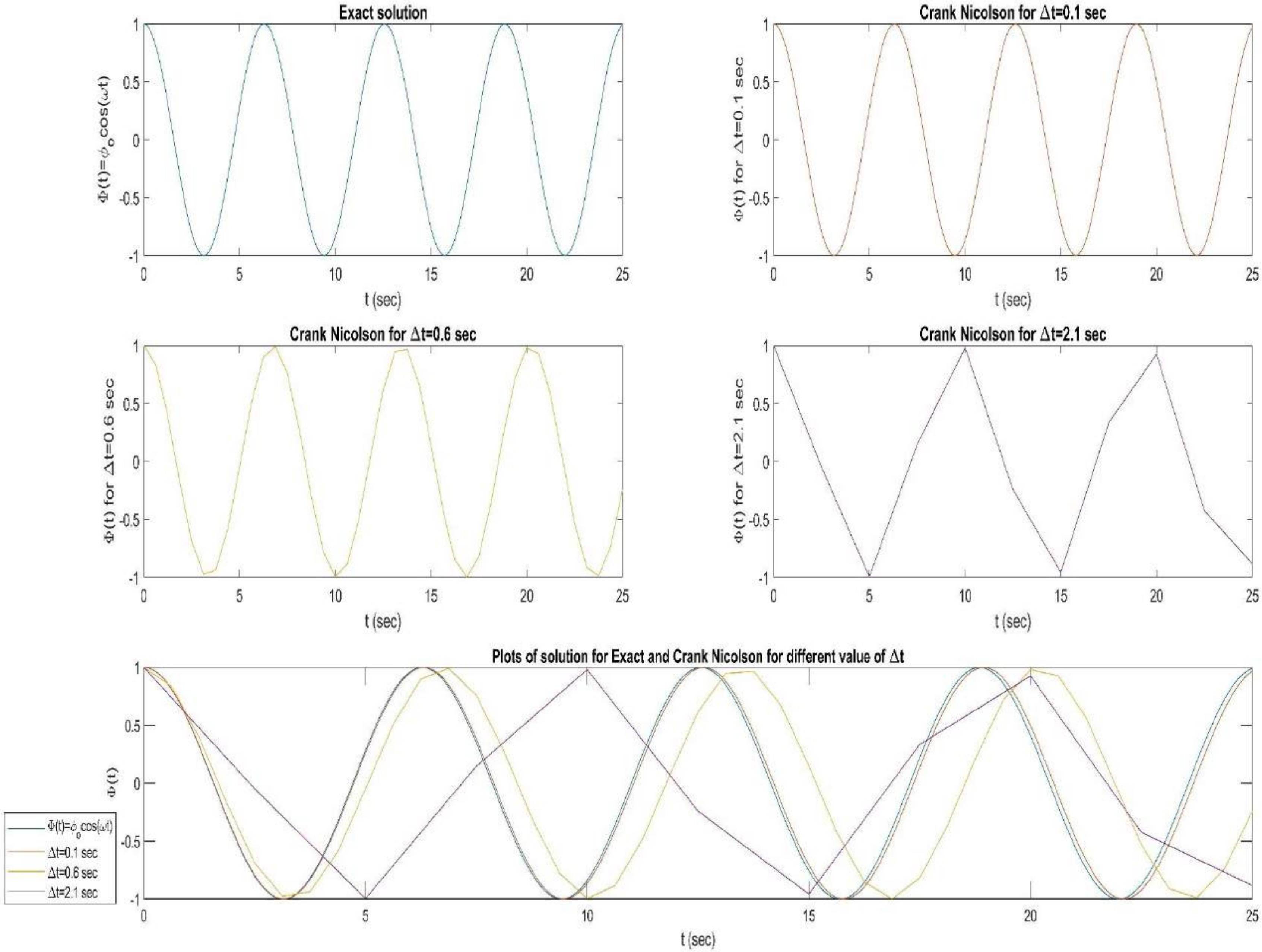

Exact solution
Φ(t)=φ_o cos(ωt)
t (sec)
Crank Nicolson for Δt=0.1 sec
Φ(t) for Δt=0.1 sec
t (sec)
Crank Nicolson for Δt=0.6 sec
Φ(t) for Δt=0.6 sec
t (sec)
Crank Nicolson for Δt=2.1 sec
Φ(t) for Δt=2.1 sec
t (sec)
Plots of solution for Exact and Crank Nicolson for different value of Δt
Φ(t)
t (sec)
Φ(t)=φ_o cos(ωt)
Δt=0.1 sec
Δt=0.6 sec
Δt=2.1 sec

Table for the Φ(t) by Crank Nicolson

Nth time step	Φ(t) for Δt =0.1sec	Φ(t) for Δt =0.6sec	Φ(t) for Δt =2.1sec
1	1	1	1
2	0.995012468827930	0.834862385321101	-0.0487514863258026
3	0.980099626246106	0.393990404848077	-0.995246585162050
4	0.955410228789008	-0.177006846950918	0.145790986900461
5	0.921190554735511	-0.689543121775297	0.981031530553444
6	0.877781947467695	-0.974340383903248	-0.241444477394377
7	0.825617410549329	-0.937337152264989	-0.957489976277179
8	0.765217288488326	-0.590754677676825	0.334802596365517
9	0.697184076267838	-0.0490605664247557	0.924845727880066
10	0.622196409421237	0.508837034655673	-0.424977804077961
11	0.541002294600359	0.898678367409458	-0.883409128671513
12	0.454411648162521	0.991708496248192	0.511112820190998
13	0.363288217204357	0.757201874032478	0.833574109342545
14	0.268540963630689	0.272610229200532	-0.592388773777310
15	0.171114997202850	-0.302017821605901	-0.775814442933818
16	0.0719821480098947	-0.776896867111303	0.668032988189407
17	-0.0278687275971243	-0.995186121644164	0.710679240756016
18	-0.127441610908910	-0.884790051597302	-0.737326326764904
19	-0.225743256206642	-0.482169744325642	-0.638787732082149
20	-0.321793098449944	0.0796992858425599	0.799610029536291

The solution by Explicit Euler is untable for $\Delta t = 0.1$, 0.6 and 2.1 and they diverges to infinity with increasing t. The solution by Implicit Euler is stable but very inaccurate as solution converges to zero as t increase. Crank Nicolson gives a solution that is both accurate enough and stable for given $\frac{d^2\phi}{dt^2} = -\omega^2\phi$ $\omega^2 = 1$.
The accuracy of the solution decreases as Δt increases from 0.1 to 2.1 sec.