INTEGRALI IMPROPRI

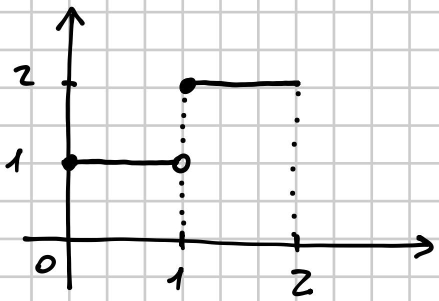
$f:[a,b) \to \mathbb{R}$ continua $\qquad \displaystyle\int_a^b f(x)\,dx = \lim_{x\to b^-} \int_a^x f(t)\,dt$

$f:(a,b] \to \mathbb{R}$ continua $\qquad \displaystyle\int_a^b f(x)\,dx = \lim_{x\to a^+} \int_x^b f(t)\,dt$

$f\,[a,c) \cup (c,b] \to \mathbb{R}$ $\qquad \displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx =$

$$= \lim_{x\to c^-} \int_a^x f(t)\,dt + \lim_{x\to c^+} \int_x^b f(t)\,dt$$

ESEMPIO



$$f(x) = \begin{cases} 1 & \text{se } 0 \le x < 1 \\ 2 & \text{se } 1 \le x \le 2 \end{cases}$$

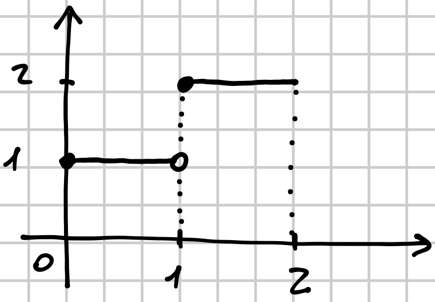
f non è continua

OBIETTIVO: calcolare $\displaystyle\int_0^2 f(x)\,dx = \underbrace{\int_0^1 f(x)\,dx}_{\text{INTEGRALE IMPROPRIO}} + \int_1^2 f(x)\,dx =$

$$= \lim_{x\to 1^-} \int_0^x f(t)\,dt + \int_1^2 f(t)\,dt =$$

$$= \lim_{x\to 1^-} \int_0^x dt + \int_1^2 2\,dt = \lim_{x\to 1^-} [\,t\,]_0^x + 2\cdot(2-1) =$$

$$= \lim_{x\to 1^-} x + 2 = 1 + 2 = 3$$



$$f(x)=\begin{cases}1 & \text{se } 0\le x<1\\ 2 & \text{se } 1\le x\le 2\end{cases}$$

$$F(x)=\int_0^x f(t)\,dt$$

$$F(1)=\int_0^1 f(t)\,dt=1 \qquad F(2)=\int_0^2 f(t)\,dt=3$$

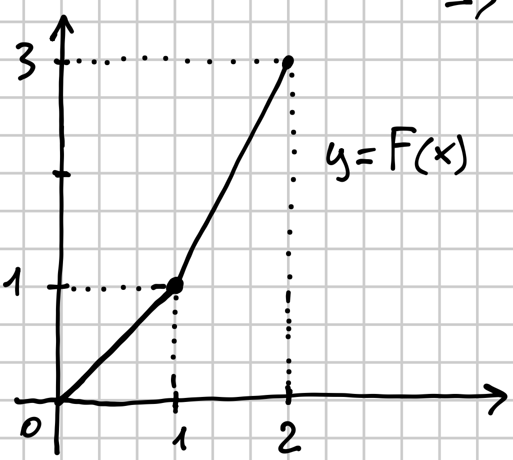
F è "primitiva" di f nel senso che $F'(x)=f(x)\quad \forall x\neq 1$

PUNTO DI DISCONTINUITÀ

Nell'intervallo $[0,1)$ come è fatta F?

$$F(x)=\int_0^x f(t)\,dt \Rightarrow F'(x)=f(x) \text{ in } [0,1)$$

$$\Rightarrow F(x)=x$$



Nell'intervallo $[1,2]$ ho ancora che F deve essere una primitiva di f, quindi

$$F(x)=2x+c$$

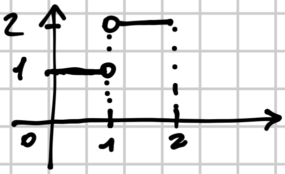
come faccio a trovare c?

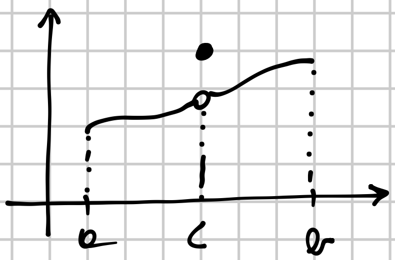
Dal fatto che $F(1)=1$

$$F(1)=2\cdot 1+c=1 \Rightarrow c=-1$$

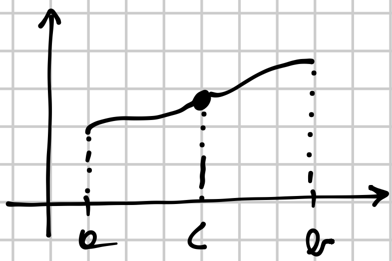
$$F(x)=\int_0^x f(t)\,dt=\begin{cases}x & 0\le x\le 1\\ 2x-1 & 1\le x\le 2\end{cases}$$

F NON è derivabile in 1 (c'è un punto angoloso)

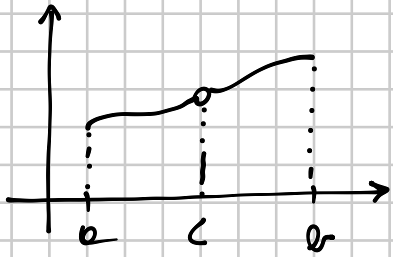
$F'(x)$ è  la derivata di F non è definita in 1, quindi, in realtà, non è f



discontinua in c



continua in c



NON definita in c

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

è lo stesso in tutti e 3 i casi

FUNZIONI DEFINITE IN INTERVALLI NON LIMITATI

$f : [a, +\infty) \to \mathbb{R}$ continua

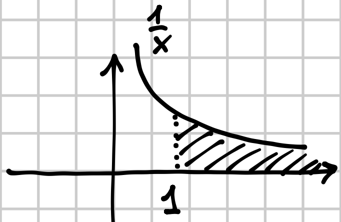
$$\int_a^{+\infty} f(x)\,dx = \lim_{x\to+\infty} \int_a^x f(t)\,dt$$

$g : (-\infty, a] \to \mathbb{R}$ continua

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{x\to-\infty} \int_x^a f(t)\,dt$$

ESEMPI

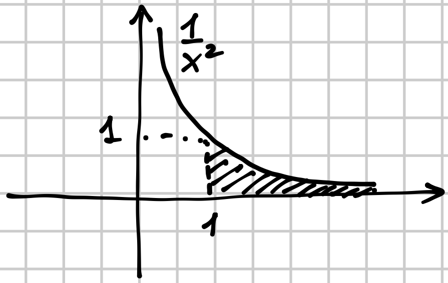
1) $f : [1, +\infty) \to \mathbb{R}$ $\quad f(x) = \frac{1}{x}$



$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{x\to+\infty} \int_1^x \frac{1}{t}\,dt = \lim_{x\to+\infty} \Big[\ln t\Big]_1^x = \lim_{x\to+\infty} \Big[\ln x - \underbrace{\ln 1}_{0}\Big] =$$

$= +\infty$ l'integrale DIVERGE

2) $f:[1,+\infty)\longrightarrow \mathbb{R} \qquad f(x)=\frac{1}{x^2}$

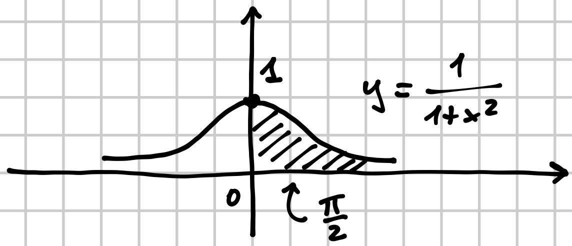


$$\int_1^{+\infty}\frac{1}{x^2}\,dx=\lim_{x\to+\infty}\int_1^x\frac{1}{t^2}\,dt=$$

$$=\lim_{x\to+\infty}\left[-\frac{1}{t}\right]_1^x=\lim_{x\to+\infty}\left[-\frac{1}{x}+1\right]=1$$

L'INTEGRALE CONVERGE

3) $\displaystyle\int_0^{+\infty}\frac{1}{1+x^2}\,dx=$



$$=\lim_{x\to+\infty}\int_0^x\frac{1}{1+t^2}\,dt=\lim_{x\to+\infty}\left[\arctan t\right]_0^x=\lim_{x\to+\infty}(\arctan x-\arctan 0)=$$

$$=\lim_{x\to+\infty}\arctan x=\frac{\pi}{2}$$

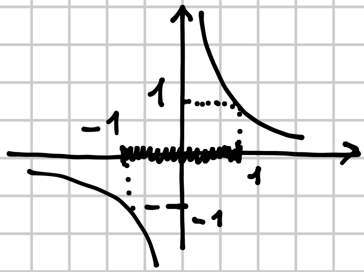
$$\int_{-\infty}^0\frac{1}{1+x^2}\,dx=\lim_{x\to-\infty}\int_x^0\frac{1}{1+t^2}\,dt=\left[\arctan t\right]_{-\infty}^0=$$

ABBREVIAZIONE (SIGNIFICA $\lim\limits_{x\to-\infty}$)

$$=\arctan(0)-\underbrace{\arctan(-\infty)}_{\text{significa limite}}=0-\left(-\frac{\pi}{2}\right)=\frac{\pi}{2}$$

$$\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx=\int_{-\infty}^0\frac{1}{1+x^2}\,dx+\int_0^{+\infty}\frac{1}{1+x^2}\,dx=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

4) $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$



$$\int_{-1}^{1} \frac{1}{x}\,dx = \underbrace{\int_{-1}^{0} \frac{1}{x}\,dx}_{\text{IMPROPRIO}} + \underbrace{\int_{0}^{1} \frac{1}{x}\,dx}_{\text{IMPROPRIO}} = \Big[\ln|x|\Big]_{-1}^{0} + \Big[\ln x\Big]_{0}^{1}$$

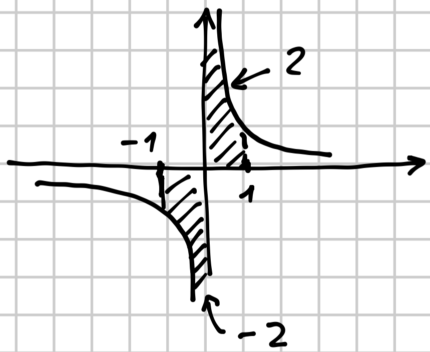
$= -\infty + \infty$ l'integrale NON CONVERGE

$\frac{1}{x}$ non è integrabile in $[-1,1]$

5) $$\int_{-1}^{1} f(x)\,dx = \underbrace{\int_{-1}^{0} f(x)\,dx}_{-2} + \underbrace{\int_{0}^{1} f(x)\,dx}_{2} = -2 + 2 = 0$$

l'integrale CONVERGE e vale 0

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{se } x > 0 \\ -\frac{1}{\sqrt{-x}} & \text{se } x < 0 \end{cases}$$



Si può dimostrare che ($\alpha \in \mathbb{R}$)

$\displaystyle\int_{1}^{+\infty} \frac{1}{x^\alpha}\,dx$ CONVERGE sse $\alpha > 1$, DIVERGE sse $\alpha \leq 1$

$\displaystyle\int_{0}^{1} \frac{1}{x^\alpha}\,dx$ CONVERGE sse $\alpha < 1$, DIVERGE sse $\alpha \geq 1$

$\Rightarrow$ IN PARTICOLARE $\displaystyle\int_{0}^{+\infty} \frac{1}{x^\alpha}\,dx$ NON CONVERGE PER NESSUN α

$$\left(\int_a^x f(t)\,dt\right)' = f(x)$$

1° TH. FOND. CALCOLO

230 $\lim_{x \to 1} \dfrac{\int_1^x (3t+1)\,dt}{e^{1-x}-1} = \dfrac{0}{0}$ F.I.

$$\overset{H}{=} \lim_{x \to 1} \frac{3x+1}{-e^{1-x}} = \frac{4}{-1} = -4$$

232 $\lim_{x \to 0} \dfrac{\int_0^{x^3} e^{t^2}\,dt}{6x^3} = \dfrac{0}{0}$ F.I. (*)

ATTENZIONE!

$$\left(\int_0^{x^3} e^{t^2}\,dt\right)' = 3x^2 e^{x^6}$$

$$F(x) = \int_0^x e^{t^2}\,dt \qquad g(x) = x^3 \qquad F(g(x)) = \int_0^{x^3} e^{t^2}\,dt$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$F'(x) = e^{x^2} \qquad g'(x) = 3x^2 \qquad [F(g(x))]' = F'(g(x)) \cdot g'(x) =$$

$$= e^{(x^3)^2} \cdot 3x^2 = 3x^2 e^{x^6}$$

$$(*) \overset{H}{=} \lim_{x \to 0} \frac{\overset{1}{\cancel{3}}\cancel{x^2} e^{x^6}}{\underset{6}{\cancel{18}}\cancel{x^2}} = \frac{1}{6}$$