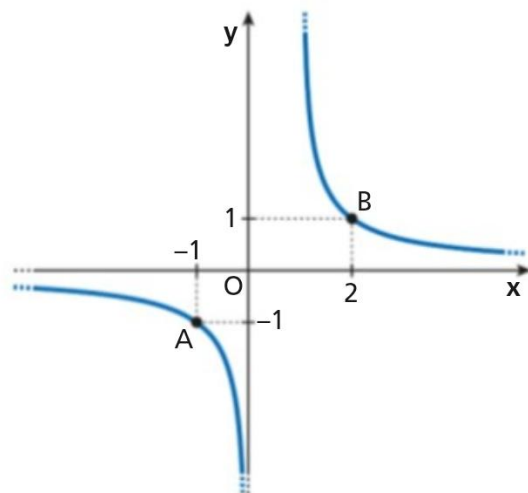In figura è rappresentato il grafico della funzione $f(x) = \log_2\left(\dfrac{x+a}{x+b}\right)$.

a. Determina l'espressione analitica della funzione e trovane il dominio.

b. Scrivi l'espressione della funzione inversa $f^{-1}(x)$.

c. Trova le soluzioni della disequazione $|f(x)| < 2$.

$$\left[\text{a) } f(x) = \log_2\left(\frac{x}{x-1}\right); D: x < 0 \lor x > 1; \text{ b) } f^{-1}(x) = \frac{2^x}{2^x-1};\right.$$

$$\left.\text{c) } x < -\frac{1}{3} \lor x > \frac{4}{3}\right]$$

a) $B(2,1)$ $\qquad y = \log_2\left(\dfrac{x+a}{x+b}\right)$

$$\begin{cases} 1 = \log_2\left(\dfrac{2+a}{2+b}\right) \\ -1 = \log_2\left(\dfrac{-1+a}{-1+b}\right) \end{cases}$$

$A(-1,-1) \rightsquigarrow$

$$\begin{cases} \dfrac{2+a}{2+b} = 2 \\ \dfrac{-1+a}{-1+b} = \dfrac{1}{2} \quad \leftarrow 2^{-1} \end{cases}$$

$$\begin{cases} 2+a = 4+2b \\ -2+2a = -1+b \end{cases} \qquad \begin{cases} a = 2+2b \\ -2+2(2+2b) = -1+b \end{cases} \qquad \begin{cases} \text{//} \\ -2+4+4b = -1+b \end{cases}$$

$$\begin{cases} \text{//} \\ 3b = -3 \end{cases} \qquad \begin{cases} a = 0 \\ b = -1 \end{cases} \implies f(x) = \log_2 \dfrac{x}{x-1}$$

DOMINIO: $\dfrac{x}{x-1} > 0$

$x > 0$

$x > 1$

	0		1	
	−	$\vert$ +		+
	−	$\vert$ −		+
	⊕	$\vert$ −		⊕

$x < 0 \lor x > 1$

DOMINIO $D = (-\infty, 0) \cup (1, +\infty)$

$f: D \to \mathbb{R}$ $\qquad \text{im}(f) = \mathbb{R} \setminus \{0\}$

INSIEME IMMAGINE

b) $f(x) = \log_2 \frac{x}{x-1}$

$y = \log_2 \frac{x}{x-1}$

$2^y = \frac{x}{x-1}$

$(x-1)2^y = x$

$x \cdot 2^y - 2^y = x$

$x \cdot 2^y - x = 2^y$

$x(2^y - 1) = 2^y \implies x = \frac{2^y}{2^y - 1} \rightsquigarrow y = \frac{2^x}{2^x - 1}$

$$\boxed{f^{-1}(x) = \frac{2^x}{2^x - 1}}$$

c) $\left| \log_2 \left(\frac{x}{x-1} \right) \right| < 2$
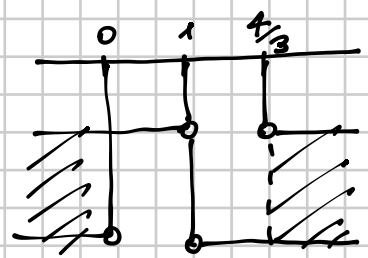
$\Updownarrow$

$-2 < \log_2 \frac{x}{x-1} < 2$

$\begin{cases} ① \log_2 \frac{x}{x-1} < 2 \\ \\ ② \log_2 \frac{x}{x-1} > -2 \end{cases}$

① $\begin{cases} \frac{x}{x-1} < 4 \\ x<0 \lor x>1 \end{cases}$ $\begin{cases} \frac{x}{x-1} - 4 < 0 \\ x<0 \lor x>1 \end{cases}$ $\begin{cases} \frac{x-4x+4}{x-1} < 0 \\ x<0 \lor x>1 \end{cases}$ $\begin{cases} \frac{4-3x}{x-1} < 0 \\ x<0 \lor x>1 \end{cases}$

$$\begin{cases} \dfrac{4-3x}{x-1} < 0 \\ x<0 \ \lor \ x>1 \end{cases} \qquad \begin{cases} \dfrac{3x-4}{x-1} > 0 \\ x<0 \ \lor \ x>1 \end{cases} \qquad \begin{cases} x<1 \ \lor \ x>\dfrac{4}{3} \\ x<0 \ \lor \ x>1 \end{cases}$$

$$x<0 \ \lor \ x>\dfrac{4}{3}$$

② $$\begin{cases} \log_2 \dfrac{x}{x-1} > -2 \\ x<0 \ \lor \ x>1 \end{cases} \qquad \begin{cases} \dfrac{x}{x-1} > \dfrac{1}{4} \\ x<0 \ \lor \ x>1 \end{cases} \qquad \begin{cases} \dfrac{x}{x-1} - \dfrac{1}{4} > 0 \\ x<0 \ \lor \ x>1 \end{cases}$$

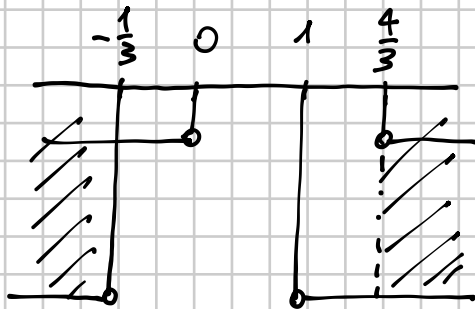$$\begin{cases} \dfrac{4x-(x-1)}{4(x-1)} > 0 \\ x<0 \ \lor \ x>1 \end{cases} \qquad \begin{cases} \dfrac{3x+1}{4(x-1)} > 0 \\ x<0 \ \lor \ x>1 \end{cases} \qquad \begin{cases} x<-\dfrac{1}{3} \ \lor \ x>1 \\ x<0 \ \lor \ x>1 \end{cases}$$

$$x<-\dfrac{1}{3} \ \lor \ x>1$$

① $\begin{cases} x<0 \ \lor \ x>\dfrac{4}{3} \\[2mm] ② \ x<-\dfrac{1}{3} \ \lor \ x>1 \end{cases}$

$$\boxed{x<-\dfrac{1}{3} \ \lor \ x>\dfrac{4}{3}}$$