Часть 5

11.4.4. $z = x^2+y^2+xy$, $x = a\sin t$, $y = a\cos t$

$\frac{dz}{dt} = ?$

$\frac{\partial z}{\partial x} = (x^2+y^2+yx)'_x = 2x+y$

$\frac{\partial z}{\partial y} = (x^2+y^2+xy)'_y = 2y+x$

$\frac{dx}{d4} = (a\sin t)'_t = a\cos t$, $\frac{dy}{dt} = (a\cos t)'_t = -a\sin t$

$\frac{dz}{dt} = a(2x+y)\cos t - a(2y+x)\sin t$

1.4.5. $z = \cos(2t+4x-y)$, $x = 1/t$, $y = \frac{\sqrt{t}}{\ln t}$

$\frac{dz}{dt} = ?$

$\frac{\partial z}{\partial x} = (\cos(2t+4x^2-y))'_x = -8x\sin(2t+4x^2-y)$

$\frac{\partial z}{\partial y} = (\cos(2t+4x^2-y))'_y = -\sin(2t+4x^2-y)$

$\frac{dx}{dt} = \left(\frac{1}{t}\right)'_t = -\frac{1}{t^2}$

$\frac{dy}{dt} = \left(\frac{\sqrt{t}}{\ln t}\right)'_t = \frac{(\sqrt{t})'\ln t - \sqrt{t}(\ln t)'}{\ln^2 t} = \frac{\ln t - 2}{2\sqrt{t}\ln^2 t}$

$\frac{dz}{dt} = -8x\sin(2t+4x^2-y)\left(-\frac{1}{t^2}\right) - \sin(2t+4x^2-y)\cdot\frac{\ln t - 2}{2\sqrt{t}\ln^2 t}$

$= \frac{8x\sin(2t+4x^2-y)}{t^2} - \frac{\sin(2t+4x^2-y)(\ln t-2)}{2\sqrt{t}\ln^2 t}$

11.4.6.

$z = x^2y^3u,\ x = t,\ y = t^2,\ u = \sin t,\ \frac{dz}{dt} = ?$

$\frac{\partial z}{\partial x} = (x^2y^3u)'_x = 2xy^3u$

$\frac{\partial z}{\partial y} = (x^2y^3u)'_y = 3y^2x^2u$

$\frac{\partial z}{\partial u} = (x^2y^3u)'_u = x^2y^3$

$\frac{dx}{dt} = (t)'_t = 1$

$\frac{dy}{dt} = (t^2)'_t = 2t$

$\frac{du}{dt} = (\sin t)'_t = \cos t$

$\frac{dz}{dt} = 2xy^3u + 6y^2x^2ut + x^2y^3\cos t$

11.4.7. $z = e^{xy}\ln(x+y),\ x = t^3,\ y = 1-t^3$

$\frac{\partial z}{\partial x} = (e^{xy}\ln(x+y))'_x = (e^{xy})'_x\ln(x+y) + e^{xy}(\ln(x+y))'_x =$
$= ye^{xy}\cdot\ln(x+y) + e^{xy}/x+y$

$\frac{\partial z}{\partial y} = (e^{xy}\cdot\ln(x+y))'_y = (e^{xy})'_y\ln(x+y) + e^{xy}(\ln(x+y))'_y =$
$= xe^{xy}\cdot\ln(x+y) + e^{xy}/x+y$

$\frac{dx}{dt} = (t^3)'_t = 3t^2,\ \frac{dy}{dt} = (1-t^3)'_t = -3t^2$

$\frac{dz}{dt} = 3t^2(ye^{xy}\cdot\ln(x+y) + e^{xy}/x+y) - 3t^2(xe^{xy}\ln(x+y) +$
$+ e^{xy}/x+y) = 3t^2(ye^{xy}\ln(x+y) + e^{xy}/x+y - xe^{xy}\cdot\ln$
$(x+y) - e^{xy}/x+y) = 3t^2e^{t^3(1-t^3)}\cdot\ln(t^3+1-t^3)\cdot(1-2t^3) =$
$= 3t^2e^{t^3(1-t^3)}\ln 1\cdot(1-2t^3) = 0$

11.4.8. $z = xy\,\mathrm{arctg}(xy),\ x = t^2+1,\ y = t^3$

$\frac{dz}{dx} = (xy\,\mathrm{arctg}(xy))'_x = (xy)'_x \cdot \mathrm{arctg}(xy) + xy(\mathrm{arctg}(xy))'_x =$

$= y \cdot \mathrm{arctg}(xy) + \frac{xy^2}{y^2x^2+1}$

$\frac{dz}{dy} = (xy \cdot \mathrm{arctg}(xy))'_y = x \cdot \mathrm{arctg}(xy) + x^2y / x^2y^2+1$

$\frac{dx}{dt} = (t^2+1)'_t = 2t,\ \frac{dy}{dt} = (t^3)'_t = 3t^2$

$\frac{dz}{dt} = 2ty(\mathrm{arctg}(xy) + \frac{xy}{x^2y^2+1}) + 3t^2x(\mathrm{arctg}(xy) + \frac{xy}{x^2y^2+1}) =$

$= (\mathrm{arctg}(xy) + \frac{xy}{x^2y^2+1})(2ty + 3t^2x)$

11.4.9. $z = e^{2x-3y},\ x = tg\,t,\ y = t^2-t$

$\frac{dz}{dx} = (e^{2x-3y})'_x = 2e^{2x-3y}$

$\frac{dz}{dy} = (e^{2x-3y})'_y = -3e^{2x-3y}$

$\frac{dx}{dt} = (tg\,t)'_t = \frac{1}{\cos^2 t}$

$\frac{dy}{dt} = (t^2-t)'_t = 2t-1$

$\frac{dz}{dt} = \frac{2e^{2x-3y}}{\cos^2 t} - 3e^{2x-3y}(2t-1) = e^{2x-3y}(\frac{2}{\cos^2 t} - 3(2t-1))$

11.4.10. $z = x^y,\ x = \ln t,\ y = \sin t$

$\frac{dz}{dx} = (x^y)'_x = yx^{y-1}$

$\frac{dz}{dy} = (x^y)'_y = x^y \cdot \ln x$

$\frac{dx}{dt} = (\ln t)'_t = 1/t$

$\frac{dy}{dt} = (\sin t)'_t = -\cos t$

$\frac{dz}{dt} = \frac{yx^{y-1}}{t} - \cos t \cdot x^y \cdot \ln x$

11.4.14. $z = x^3 + y^3,\ x = uv,\ y = \frac{u}{v},\ \frac{dz}{du}, \frac{dz}{dv}, dz = ?$

$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy$

$\frac{dz}{dx} = (x^3+y^3)'_x = 3x^2 + y^3$

$\frac{dz}{dx} = (x^3+y^3)'_x = x^3 + 3y^2$

$dx = \frac{dx}{du}du + \frac{dv}{dv}dv = (uv)'_v = dv + (uv)'_v dv = v \cdot du + u \cdot dv$

$dy = \frac{dy}{du}du + \frac{dy}{dv}dv = \left(\frac{u}{v}\right)'_u du + \left(\frac{u}{v}\right)'_v dv = \frac{1}{v} \cdot du - \frac{y}{v^2}dv$

$dz = (3x^2+y^3)(v \cdot du + u dv) + (x^3 + 3y^2)\left(\frac{du}{v} - u \cdot \frac{dv}{v^2}\right) =$

$= \left(3u^2v^2 + \frac{u^3}{v^3}\right)(v \cdot du + u \cdot dv) + \left(u^3v^3 + \frac{3u^2}{v^2}\right)\left(\frac{du}{v} - \frac{u \cdot dv}{v^2}\right)$

11.4.15. $z = \sqrt{x^2-y^2},\ x = u^v,\ y = u \ln v$

$\frac{dz}{dx} = (\sqrt{x^2-y^2})'_x = \frac{x}{\sqrt{x^2-y^2}},\ \frac{dz}{dy} = (\sqrt{x^2-y^2})'_y = -\frac{y}{\sqrt{x^2-y^2}}$

$dx = \frac{dx}{du}du + \frac{dx}{dv} \cdot du = (u^v)'_u \cdot du + (u^v)'_v \cdot dv = vu^{v-1}du + u^v \ln u \cdot dv$

$dy = \frac{dy}{du} \cdot du + \frac{dy}{du} \cdot dv = (u \ln v)'_u du + (u \cdot \ln v)'_v \cdot dv =$

$= \ln v \cdot du + \frac{u}{v}du$

$dz = \frac{x}{\sqrt{x^2-y^2}} \cdot (vu^{v-1} \cdot du + u^v \cdot \ln u \cdot du) - \frac{y}{\sqrt{x^2-y^2}}\left(\ln v \cdot du + \frac{u}{v}dv\right)$

11.4.16. $z = \cos xy$, $x = ue^{v}$, $y = v\ln u$

$$\frac{\partial z}{\partial x} = (\cos xy)'_x = -y\sin xy$$

$$\frac{\partial z}{\partial y} = (\cos xy)'_y = -x\sin xy$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv = (u\cdot e^{v})'_u\,du + (u\cdot e^{v})'_v\cdot dv =$$

$$= e^{v}du + ue^{v}\cdot dv$$

$$dy = \frac{\partial y}{\partial u}\cdot du + \frac{\partial y}{\partial v}dv = (v\ln u)'_u du + (v\ln u)'_v\cdot dv =$$

$$= \ln u\cdot du + \frac{v}{u}\cdot dv$$

$$dz = -y\sin xy(e^{v}\cdot du + ue^{v}\cdot dv) - x\sin xy(\ln u\cdot du + \frac{v}{u}\cdot dv)$$

11.4.17. $z = \operatorname{arctg} xy$, $x = \sqrt{u^2+v^2}$, $y = u - v$

$$\frac{\partial z}{\partial x} = (\operatorname{arctg} xy)'_x = \frac{y}{x^2y^2+1}, \quad \frac{\partial z}{\partial y} = (\operatorname{arctg} xy)'_y = \frac{x}{x^2y^2+1}$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv = (\sqrt{u^2+v^2})'_u du + (\sqrt{u^2+v^2})'_v\cdot dv =$$

$$= \frac{u}{\sqrt{u^2+v^2}}\cdot du + \frac{v}{\sqrt{u^2+v^2}}\cdot dv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv = (u-v)'_u du + (u-v)'_v dv = du - dv$$

$$dz = \frac{y}{x^2y^2+1}\cdot\frac{1}{\sqrt{u^2+v^2}}(u\cdot du + v\cdot dv) + \frac{x}{x^2y^2+1}(du - dv) =$$

$$= \frac{1}{x^2y^2+1}\left(\frac{y(u\,du + v\,dv)}{\sqrt{u^2+v^2}} + x(du - dv)\right)$$

11.4.17. $z = \operatorname{arcctg} xy, \; x = \sqrt{u^2+v^2}, \; y = u - v$

$$\frac{\partial z}{\partial x} = (\operatorname{arcctg} xy)'_x = \frac{y}{x^2y^2+1}, \; \frac{\partial z}{\partial y} = (\operatorname{arcctg} xy)'_y = \frac{x}{x^2y^2+1}$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv = (\sqrt{u^2+v^2})'_u du + (\sqrt{u^2+v^2})'_v \cdot dv =$$

$$= \frac{u}{\sqrt{u^2+v^2}} du + \frac{v}{\sqrt{u^2+v^2}} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv = (u-v)'_u du + (u-v)'_v dv = du - dv$$

$$dz = \frac{y}{x^2y^2+1} \cdot \frac{1}{\sqrt{u^2+v^2}} (u\,du + v \cdot dv) + \frac{x}{x^2y^2+1} (du - dv) =$$

$$= \frac{1}{x^2y^2+1} \left(\frac{y(u\,du + v\,dv)}{\sqrt{u^2+v^2}} + x(du - dv) \right)$$

11.4.18. $z = \sqrt{x+y} \quad x = u \cdot \operatorname{tg} v \quad y = u \operatorname{ctg} v$

$$\frac{\partial z}{\partial x} = (\sqrt{x+y})'_x = \frac{1}{2\sqrt{x+y}}, \; \frac{\partial z}{\partial y} = (\sqrt{x+y})'_y = \frac{1}{2\sqrt{x+y}}$$

$$dx = \frac{\partial x}{\partial u} \cdot du + \frac{\partial x}{\partial v} \cdot dv = (u \cdot \operatorname{tg} v)'_u du + (v \operatorname{tg} v)'_v \cdot dv =$$

$$= \operatorname{tg} v \; du + \frac{v}{\cos^2 v} dv$$

$$dy = \frac{\partial y}{\partial u} \cdot du + \frac{\partial y}{\partial v} \cdot dv = (u \operatorname{ctg} v)'_u du + (v \cdot \operatorname{ctg} v)'_v \cdot dx =$$

$$= \operatorname{ctg} v \cdot du - \frac{u}{\sin^2 v} \cdot dv$$

$$dz = \frac{1}{2\sqrt{x+y}} \left(\operatorname{tg} v \; du + \frac{u}{\cos^2 v} dv \right) + \frac{1}{2\sqrt{x+y}} \left(\operatorname{ctg} v \cdot du - \frac{u}{\sin^2 v} \cdot dv \right) =$$

$$= \frac{1}{2\sqrt{x+y}} \left(v\,dv \left(\frac{1}{\cos^2 v} - \frac{1}{\sin^2 v} \right) + du (\operatorname{tg} v + \operatorname{ctg} v) \right)$$

11.4.19. $z = \ln \sqrt[7]{x^2+3y^5}$ $x = u \cdot \cos v$, $y = v \cdot \sin v$

$\frac{\partial z}{\partial x} = \frac{7x}{x^2+3y^5}$, $\frac{\partial z}{\partial y} = \frac{105y^4}{6y^5+2x^2}$

$dx = (u\cos v)'_u du + (u \cdot \cos v)'_v dv = \cos v \cdot u d - u \cdot \sin v du$

$dy = (u\sin v)'_u du + (u \cdot \sin v)'_v dv = \sin v du + u \cdot \cos v \cdot dv$

$dz = \frac{7x}{x^2+3y^5}(\cos v du - u\sin v dv) + \frac{105y^4}{6y^5+2x^2}(\sin v du + \cos v dv)$

11.4.22. $xe^{2y} - y\ln x = 8$, $F(x;y) = xe^{2y} - y\ln x - 8$

$F'_x = e^{2y} - \frac{y}{x}$

$F'_y = 2xe^{2y} - \ln x$

$\Rightarrow y'_x = \frac{e^{2y} - \frac{y}{x}}{2xe^{2y} - \ln x}$

11.4.23. $e^y + 9x^2e^{-y} - 26x = 0$, $F(x;y) = e^y + 9x^2e^{-y} - 26x$

$F'_x = 18xe^{-y} - 26$

$F'_y = e^y - 9x^2e^{-y}$

$y'_x = \frac{18xe^{-y} - 26}{e^y - 9x^2e^{-y}}$

11.4.24. $\ln \frac{\sqrt{x^2+y^2}}{2} = \operatorname{arctg} \frac{y}{x}$, $F(x;y) = \ln \frac{\sqrt{x^2+y^2}}{2} - \operatorname{arctg} \frac{x}{y}$

$F'_x = \frac{x-y}{x^2+y^2}$

$F'_y = \frac{y}{y^2+x^2} + \frac{x}{y^2+x^2} = \frac{x+y}{x^2+y^2}$

$y'_x = \frac{y-x}{x+y}$

11.4.25 $x^2 \ln y - y^2 \ln x = 0$

$F(x;y) = x^2 \ln y - y^2 \ln x$

$F'_x = 2x \ln y - \frac{y^2}{x}$

$F'_y = (x^2 \ln y - y^2 \ln x)'_y = \frac{x^2}{y} - 2y \ln x$

$y'_x = \frac{-2x \ln y - \frac{y^2}{x}}{\frac{x^2}{y} - 2y \cdot \ln x}$

11.4.26. $1 + xy - \ln(e^{xy} + e^{-xy}) = 0$

$F(x;y) = 1 + xy - \ln(e^{xy} + e^{-xy})$

$F'_x = y - \frac{ye^{2xy} - y}{e^{2yx} + 1}$, $F'_y = x - \frac{xe^{2xy} - x}{e^{2xy} + 1}$

$y'_x = -\frac{2y}{2x} = -\frac{y}{x}$

11.4.35. $z^3 - 3xyz = R^2$, $F(x;y;z) = z^3 - 3yzx - R^2$

$F'_x = -3yz$ $\qquad \frac{dz}{dx} = \frac{yz}{z^2 - xy}$

$F'_y = -3xz$

$F'_z = 3z^2 - 3xy$ $\qquad \frac{dz}{dy} = \frac{xz}{z^2 - xy}$

$dz = \frac{yz\,dx + xz\,dy}{z^2 - xy}$

11.4.36. $x + y + z = e^z$, $F(x;y;z) = x + y + z e^z$

$F'_x = 1$ $\qquad \frac{dz}{dx} = \frac{-1}{1 - e^z}$

$F'_y = 1$

$F'_z = 1 - e^z$ $\qquad \frac{dz}{dy} = -\frac{1}{1 - e^z}$

$dz = \frac{-dx - dy}{1 - e^z}$

11.4.37. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$F(x; y; z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$F'_x = \frac{2x}{a^2}$ $\qquad$ $\frac{dz}{dx} = -\frac{c^2 x}{a^2 z}$

$F'_y = \frac{2y}{b^2}$ $\qquad$ $\frac{dz}{dy} = \frac{-c^2 y}{b^2 z}$

$F'_z = \frac{2z}{c^2}$

$dz = -\frac{c^2 x}{a^2 z} dx - \frac{c^2 y}{b^2 z} dy.$