Mergable Priority Queues

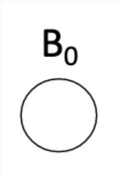
- what if we wanted to merge two priority queues?
- normal min-heap implementation does not support this in a smart way
  - with min-heap data structure, must take all elements from both and construct new heap from scratch

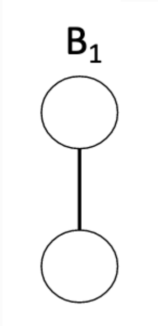
Operations

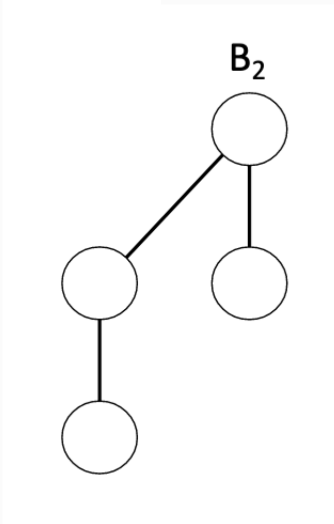
- `insert(A, x)`
  - add an element `x`
- `min(A)`
  - return the highest priority element
- `extract_min(A)`
  - remove and return the highest priority element
- **`union(A, B)`**
  - create a new heap with all the elements of both `A` and `B`
- **`decrease_key(A, x, k)`**
  - decrease the key of `x` to `k`
- **`remove(A, x)`**
  - remove `x` from the heap entirely

Binomial Trees

B_k Trees defined recursively:

0. 

1. 

2. 

In general, create B_{n+1} tree by taking two B_n trees, A and B and making the root of B the first child of the root of A

Properties of B_k

- number of nodes is 2^k
- height of tree is k
- number of nodes with height h is binomial coefficient $\binom{k}{h}$

Binomial Forest of size n (F_n)

A sequence of B_k trees with k strictly decreasing with n total nodes

- this is always possiblle because we can always represent $n > 0$ in binary, so we can always write it as a sum of unique powers of 2
- if $\alpha(n)$ is the number of 1s in the binary representation of n

- F_n has $\alpha(n)$ trees
- F_n has $n - \alpha(n)$ edges
- $\alpha(n) \in \mathcal{O}(\log(n))$

Min Binomial Heap

A min binomial heap is a binomial forest where:

- each node of F_n stores one element
- each tree in F_n is min-heap ordered

Implementation

Storage

- edges as drawn are not exactly stored as pointers
- each node stores pointers to:
    - `parent`
    - `left_child`
    - `right_sibling`
- tree stores pointer to `head`, aka rightmost top level node

Merging min binomial heaps of same size P_k and Q_k

- if `Pk.root < Qk.root` then make `Pk.left_child = Qk.root`, otherwise `Qk.left_child = Pk.root`
    - change parent pointers similarly

Union of min binomial forests (`union(A, B)`)

(works exactly like algorithm for addition of binary numbers)

1. start at smallest B_k tree in both
2. if tree is only in one forest (or is carry), then keep as-is in the union forest
3. if there are B_k trees with same size in both, merge them to get a

   B_{k+1} tree
4. carry the new B_{k+1} tree, repeat from step 2

for $|A| \leq n$ and $|B| \leq n$, the complexity of `union(A, B)` is $\mathcal{O}(\log(n))$

`insert(A, x)`

Make a new binomial forest with a single B_0 tree which stores `x`, then "add" it to `A`

```
def insert(A, x):
    b = new B_0 Tree
    b.insert(x)
    B = new BinomialHeap(b)
    A = union(A, B)
```

`decrease_key(A, x, k)`