Disjoint Sets

Operations

- `union(S1, S2)`
  - return union of S_1 and S_2
- `find(S, x)`
  - find the "representative member" (determined by some rule) of the set that x belongs to

Let σ be the procedure of $n-1$ `union`s and $m \geq n$ `find`s

Naive implementations

Linked List

- one list per set
- for each list, store pointers to head and tail
  - this way, `union` is fast
- finding representative takes $\mathcal{O}(n)$ time

Augmented Linked List

- each element stores pointer to its representative
  - this way, `find` is fast
- `union` is slow because it must now change $\mathcal{O}(min(|S_1|, |S_2|))$ pointers to the new representative

Forest (tree-like)

- each set is a tree
- root of a tree is representative of set

- non-root nodes point to a parent node
  - `find` works by ascending the tree until it reaches the root
  - it takes $\mathcal{O}(h)$ time where h is height of tree
- `union` is fast because we must only change one parent pointer - make one tree's root point to an element of other tree
  - to support this, record size (number of nodes) for each tree

Weighted Union

To do `union`, make root of smaller tree point to root of larger tree

Lemma

With weighted union, any tree T of height h created during the execution of σ has $|T| \geq 2^h$

Proof.

Base case:

If $h = 0$, then the tree has $1 = 2^0$ node

Inductive step:

Suppose lemma holds for some $h \geq 0$.

To construct a tree of height $h + 1$, we make the root of a tree A of height h point to the root of a bigger tree B (because we are using weighted union)

By IH, $|A| \geq 2^h$

By WU rule, $|B| \geq |A| \geq 2^h$

Then resulting tree has number of nodes $|A| + |B| \geq 2(2^h) = 2^{h+1}$

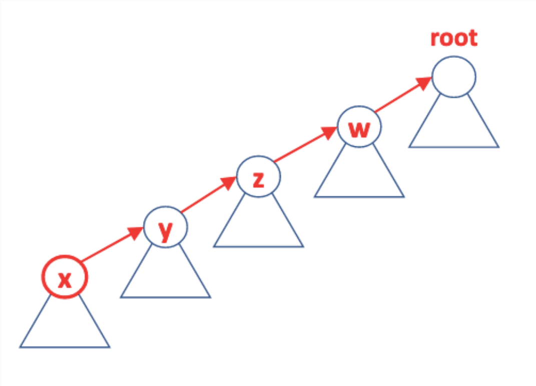
Path compression

- `find(S, x)` ascends tree until root is reached
- intermediate nodes $n_1, \ldots, n_m$ on way to root do not matter, so after we find `root` we can change `n.parent = root` for each n

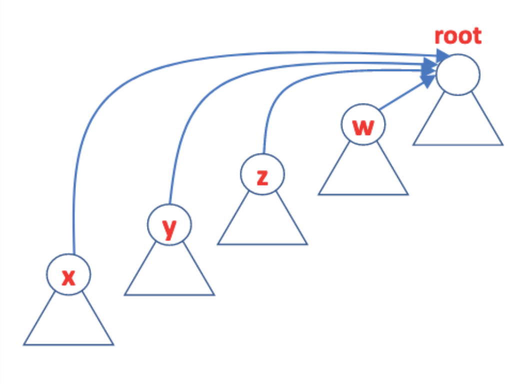
on the way to the root (as well as `x.parent = root`

Before:

`find(S, x)` traverses from `x` to `root`



After:



- this does not increase complexity, as same nodes must be operated on twice, aka a constant factor difference
- speeds up subsequent calls to `find` on `x` or any of its ancestors - becomes constant time
  - does not affect time for `find` on other parts of the tree, does not affect `union` at all

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History: 25 years after publication of weighted union + path compression implementation, proof that σ takes $\Theta(m \cdot \alpha(m, n))$ time using WU+PC was finally completed (where α is inverse Ackerman function).

The Ackerman function grows ridiculously quickly so α grows ridiculously slowly, so for all reasonable (read: physically possible) intents, σ runs in linear time.
