Naive definition of a set X:

$$X = \text{an arbitrary collection of objects.}$$

Problem: Russell's Paradox:

$$S = \{X \mid X \text{ doesn't contain itself as an element}\}$$

(1) If S is not an element of S, then S is an element of S

(2) If S is an element of S, then S is not an element of S

Either way, a contradiction.

The fundamental problem is that a set X must always live in an ambient set $T(X \subseteq T)$. But this weakens the naive definition of a set as it implies too much. Therefore, we must add additional axioms known as Zermelo-Fraenkel Axioms:

1. Axiom of Extensionality: if X and Y have the same elements, $X = Y$

$$\forall u(u \in X \equiv u \in Y) \Rightarrow X = Y$$

2. Axiom of Unordered Pair: For any a, b there exists a set $\{a, b\}$

$$\forall a \forall b, \exists c \forall x | (x \in c \equiv (x = a \vee x = b))$$

3. Axiom of Subsets: ϕ property with parameter $Y = \{u \in X | \phi(u, p)\}$
4. Axiom of Sum set: For any set X there exists a set $Y = 0X$
5. Axiom of Power set: $Y = P(X)$ set of all sets.
6. Axiom of Infinity
7. Axiom of Replacement: Image of set under definable function is a set...UNSURE
8. Axiom of Foundation(Regularity)
9. Axiom of Choice

Important Operations and Concepts for naive set theory(works in ZFC):

Union:	$X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$
Intersection:	$X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$
Set Difference (complement):	$X \setminus Y = \{z \mid z \in X \text{ and } z \notin Y\}$
Cartesian Product:	$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$

Additional Ideas:

Elemental membership:

$a \in X$ if a is an element of X

Subset:

$X \subseteq Y$ if every element of X is an element of Y

Note: $\{a\} \subseteq X \iff a \in X$

Overset:

When $S \subseteq X$ and $S \neq X$, we sometimes say that S is an *subset* of X or that X is an *overset* of S.

Powersets :

Let X be a set. The powerset of X denotes the set of all subsets of X.

Example :

$X = \{1, 2, 3\}$

$P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Further additional concepts:

Cardinality :

$\mathrm{Card}(X) = |X| = (X)$ denotes the 'size' of the set X which is defined as the ordinal number measuring the number of elements of X.

Example :

$$X = \{1, 2, 3\}$$
$$\mathrm{Card}(\{1, 2, 3\}) = 3$$
$$|P(X)| = 2^{\mathrm{Card}(X)} \text{ (Question: why?)}$$
$$|P(1, 2, 3)| = 2^3 = 8$$

Inductive set or natural numbers :

$\mathbb{N} = \{1, 2, 3, 4, \ldots, n, n + 1, \ldots\}$

(Key feature): If n_1, n_2 are elements, then $n_1 + 1$ is an element.

Further concepts:

Relation: A relation R is a subset of a Cartesian product $X \times Y$.

Given two sets X and Y, we form the set

$$X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$$

Examples:

1. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$ (INSERT IMAGE)
2. $\mathbb{R}^n := \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^1 \times \ldots \times \mathbb{R}^1$

   $((x_1, \ldots, x_{n-1}), x_n) = (x_1, \ldots, x_{n-1}, x_n)$
3. $\{a\} = X,$ and Y an arbitrary set.

   $X \times Y = \{(a, y) \mid y \in Y\} \neq Y$

   But, $X \times Y \xrightarrow{\text{p}} Y, \quad p((a, y)) = y$ (forgets 1st cordinate) is a bijection. Thus, $\{a\} \approx Y$.
4. (Bad example) Rational numbers $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$
   "can be thought as" $\mathbb{Z} \times (\mathbb{Z}^* \{0\}) \quad (p, q) \to \frac{p}{q}$

Actually, to make precise we need the concept of a relation.

Definition: We say that a subset R of $X \times Y$ is a relation between X and Y. Moreover, we say that R is a function if $(x, y_1), (x, y_2) \in R \implies y_1 = y_2$ (CUTOFF)

Page 5:

We will get to functions in a minute, but lets go back to example 4 above:

$$S' = \mathbb{Z} \times (\mathbb{Z}^* \setminus 0) \to \mathbb{Q}$$

$$(p, q) \to \frac{p}{q}$$

However, the issue is two fractions can reduce:

$$\frac{p}{q} = \frac{p'}{q'} \text{ if } pq' = qp'.$$

Therefore, let

$$R \subseteq S \times S$$
$$R = \{((p, q), (p', q')) \mid pq' - qp' = 0\}$$

This is an example of an equivalence relation, often we take $X = Y$ and have relations on X - i.e., Subsets $R \subseteq X \times X$.

Example 1:

$\Delta_X = \{(x, y) \mid x = y\}$ is called the diagonal of X.

This is also an equivalence relation and corresponds to the regular notion of equality $x = x$.

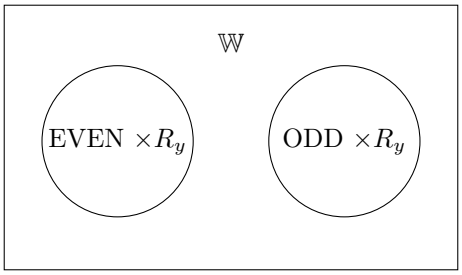
Example 2:

$X = \mathbb{N}\{1, 2, 3, \ldots\} \cup \{0\}$
$R = \{(x, y) \mid x + y$ divisible by $2\}$

Example 2 continued:

Describes an equivalence relation on $\mathbb{W}$ where

$x = y$ iff x and y are even. or $x = y$ iff x and y are odd.



Example 3:

$$R \subseteq \mathbb{R} \times \mathbb{R}$$
$$R = \{(x, y) \mid x < y\}$$

This is a transitive relation as $x < y$ and $y < z \rightarrow x < z$.

Example 4:

$$R \subseteq \mathbb{R} \times \mathbb{R}$$
$$R = \{(x, y) \mid x \leq y\}$$

This is both a transitive and reflexive relation since $x \leq x$.

Example 5:

Consider $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (or $\mathbb{W}$ or $\mathbb{N}$).

Let m be a natural number $m = 1, 2, 3, \ldots$

Then for any $z \in Z$, there exists a unique q called a quotient such that

$$\boxed{z = mq + r} \quad \text{where} \quad r = 0, 1, 2, \ldots, m-1,$$
$$\frac{z}{m} = q + \frac{r}{m}$$

If $m = 2$, then either z is even in which case $z = 2q$ or odd and then $z = 2q' + 1$

We place an equivalence relation on $\mathbb{Z}$ called mod_m or $\equiv_m$ by

$$z \equiv_m z' \iff z = mq + r \text{ and } z = mq' + r'$$
$$z \equiv z' \mod m \implies r = r'$$

$$z \equiv 0 \mod 0 \leftrightarrow \square$$
$$z \equiv 0 \mod 1 \leftrightarrow \bigcirc$$
$$z \equiv 0 \mod 2 \leftrightarrow \diamondsuit$$

$\{\boxed{0}, \textcircled{1}, \diamondsuit 2 \diamondsuit, \boxed{3}, \textcircled{4}, \diamondsuit 5 \diamondsuit, \boxed{6}, \textcircled{7}, \diamondsuit 8 \diamondsuit, \boxed{9}, \textcircled{10}, \diamondsuit 11 \diamondsuit, \ldots,\}$

Example :

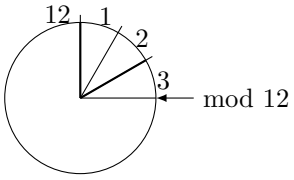
$$17 \equiv 7 \pmod{10}$$
$$27 \equiv 7 \pmod{10} \qquad \text{Yet } 17 \not\equiv 27 \mod 3$$

$231 \equiv 1111 \mod 4$? To check:

$$\begin{array}{r} 57 \\ 4\overline{)231} \\ \underline{20} \\ 31 \\ \underline{28} \\ 3 \end{array} \quad r = \boxed{3}, \text{ so } 231 \equiv 3 \mod 4, \qquad \begin{array}{r} 277 \\ 4\overline{)1111} \\ \underline{8} \\ 31 \\ \underline{28} \\ 31 \\ \underline{28} \\ 3 \end{array} \quad r = \boxed{3}, \text{ so } 1111 \equiv 3 \mod 4$$

$\therefore$ Yes, $231 \equiv 1111 \mod 4$

Example



$15h \equiv 3 \mod 12$
$147 \mod 12 \equiv 147$ hours after midnight, which is 3 o'clock.

Review: Modular Arithmetic

On the set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, we place an equivalence relation $\equiv \mod n$ defined by

$$x \equiv y \mod n \iff x = mq + r \text{ and } y = mq' + r'$$

x and y are equal provided they have the same remainder.

For each n, this relation behaves differently:

$$17 \equiv 27 \mod 10 \quad \text{but} \quad 17 \not\equiv 27 \mod 3.$$

Question

Find all positive integers x such that

$$x \equiv 1 \mod 3.$$

$$\{1, 4, 7, 10, 13, \ldots, 3q + 1, \ldots\}$$

As discussed, arithmetic operations will carry over. What this means is that adding two numbers goes to the right set.

We define:

$$a + b := (a + b) \mod n$$

$$a \cdot b := (a \cdot b) \mod n$$

Example

$$(17 + 3) \mod 3 = 20 \mod 3 = 2 \mod 3$$

(CUTOFF)

When (x, y) is an element of R, we write xRy or $x \leq_R y$.
A relation can have a lot of properties. Assume $X = Y$:

- **Reflexive** iff $x \leq_R x$ for all $x \in X$
- **Symmetric** iff $x \leq_R y \implies y \leq_R x$ for all $x, y \in X$
- **Transitive** iff $x \leq_R y$ and $y \leq_R z \implies x \leq_R z$ for all $x, y, z \in X$

If R has all of these properties, then it is an **equivalence relation** on X. In this case, we write $=_R$ or just $=$ (Example: $=$ on rational numbers).
More on this topic later.
Def: A function from X to Y is a relation with the following: if for all $x \in X$ and any $y_1, y_2 \in Y$,

$$x \leq_R y_1 \text{ and } x \leq_R y_2 \implies y_1 = y_2.$$

In this case, we write $y = f(x)$ for $x \leq_R y$.

We say a function $f : X \to Y$ is **injective** if

$$[f(x_1) = f(x_2) \implies x_1 = x_2]$$

$$A \hookrightarrow B$$

And we say $f : X \to Y$ is **surjective** if

$$\forall y \in Y, \text{ there exists an } x \text{ such that } f(x) = y$$

Provide examples:

$$X = \mathbb{R}$$
$$Y = \mathbb{R}$$
$$R = \text{real numbers}$$

$y = f(x), \quad f(x) = 3x + 1$ injective and surjective
$y = f(x), \quad f(x) = \sqrt{x}$ injective but not surjective
$y = f(x), \quad f(x) = x^3 - x$ surjective but not injective

If a function is both injective and surjective, we say that it is **bijective** (or a bijection).

Thm: There exists a bijection $f : X \to Y$ if and only if $\text{Card}(X) = \text{Card}(Y)$.

Question: Are $\mathbb{N}$ and $\mathbb{Q}$ of the same size?

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Let $X = Y$ and consider all bijections $f : X \to X$. This is a set usually denoted

$$S(X), \quad \mathrm{Sym}(X), \quad \mathrm{Biject}(X), \quad \mathrm{Bi}(X).$$

This is a prototypical example of a group.

$$G := \mathrm{Sym}(X)$$

An element is a bijective map $f : X \to Y$.

1. $f \circ g : X \to X$ is bijective. $f(g(x)) = (f \circ g)(x)$ is also bijective.
2. $(f \circ g) \circ h = f \circ (g \circ h)$.
3. $1(x) = x$ for all x and $(f \circ 1) = f$, $(1 \circ g) = g$.
4. If $f : X \to X$ is a bijection, then $f^{-1} : X \to X$ exists and is also a bijection.

These are currently just statements which may be true or not. We need to prove these statements and the question is, how?

Cyclic Groups

Definition for x in G: In general, $x^0 = e$ and $x^n = x \cdot x \cdot \ldots \cdot x$ (n times).

$$(x^n)^{-1} = (x^{-1})^n$$

$$x^{-n} = (x^{-1})^n$$

Thm. *Let G be a group and let $x \in G$. Let m, n be integers. Then:*

1. $x^m x^n = x^{m+n}$
2. $(x^m)^{-1} = x^{-m}$
3. $(x^m)^n = x^{mn} = (x^n)^m$

Proof. For m and n positive, $x^m x^n = x^{m+n}$ (m times x).

If $m, n < 0$, say $m = -r$, $n = -s$ then

$$x^m x^n = x^{-r} x^{-s} = (x^{-1})^r (x^{-1})^s = (x^{-1})^{r+s} = x^{-(r+s)} = x^{m+n}$$

If $m < 0$ (say $m = -r$) and $n > 0$

$$x^m x^n = (x^{-1})^r x^n = (x^{-1})^r x^n = x^{r-n} = x^{m+n}$$

Part (2) and (3) are easy. □

Definition. *Let x be an element of a group G. We say x has order n if $x^n = e$ (called finite order).*

If x doesn't have finite order, then we say x has infinite order.

Example 1.

$$(\mathbb{Z}_3, \oplus) \quad x = [1] \quad [1] + [1] + [1] = [0] \quad \textit{order of } [1] = 3$$

What is the order of $[2]$? Of $[0]$?

Example 2.

$$(U(\mathbb{Z}_5), \otimes) \quad \textit{Calculate the order of all elements}$$

$$4^1 \not\equiv 1 \mod 5 \quad 4^2 = 16 \equiv 1 \mod 5 \quad \textit{order is } 2$$

$$3 \cdot 3 = 9 \equiv 4 \mod 5 \quad 3^2 = 9 \equiv 4 \mod 5$$

$$3 \cdot 3 \cdot 3 = 4 \cdot 3 \equiv 12 \equiv 2 \mod 5 \quad \textit{order of 3 is 4}$$

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 \equiv 16 \equiv 1 \mod 5 \quad \textit{order of 2 is 4}$$

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Def. Let x be an element of a group G. We say x has order n if $x^n = e$ (called *finite order*). If x doesn't have finite order, then we say x has *infinite order*.

Example: $(\mathbb{Z}_3, \oplus)$

$$\begin{aligned} x &= [1] \\ x^2 &= [1] \oplus [1] = [2] \neq [0] \\ x^3 &= [1] \oplus [1] \oplus [1] = [2] \oplus [1] = [0] = e \end{aligned}$$

[1] has order 3. What is the order of [2]? of [0]?

Example: $(\mathbb{U}(\mathbb{Z}_5), \otimes)$ Calculate the order of all elements.

$$\begin{aligned} 4^1 &\neq 1 \\ 4^2 &= 16 \equiv 1 \pmod 5 \quad \text{(order is 2)} \\ 3 \cdot 3 &= 9 \equiv 4 \pmod 5 \\ 3 \cdot 3 \cdot 3 &= 4 \cdot 3 \equiv 12 \equiv 2 \pmod 5 \\ 3^4 &= 3 \cdot 3 \cdot 3 \cdot 3 = 2 \cdot 3 \equiv 6 \equiv 1 \pmod 5 \quad \text{(order of 3 is 4)} \end{aligned}$$

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Example: $G = GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0 \right\}$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, this has order 2.

Example: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ Order of 1 is ∞.

Def. A group G is called *cyclic* if there is an element x in G such that $G = \{x^n \mid n \in \mathbb{Z}\}$. In this case, x is said to be a *generator* of G.

Presentation of a group, compact notation: In additive notation, $\langle x \rangle = \{nx \mid n \in \mathbb{Z}\}$.

Examples:

$$\mathbb{Z} = \langle 1 \rangle$$
$$\mathbb{Z}_n = \langle 1 \rangle$$
$$\mathbb{Z} = \langle 1 \rangle \text{ (in additive notation)}$$

$\mathbb{Q}$ is not cyclic since $\frac{1}{2} \notin \langle q \rangle$.

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Thm Let x be an element of G.

1. The order of x is the order of x^{-1}.
2. If the order of $x = n$ and $x^m = e$, then n divides m.
3. If the order of $x = n$ and $\gcd(m, n) = d$, then the order of $x^{m/d} = \frac{n}{d}$.

Proof. Assume $x^n = e$. Write $m = qn + r$. $0 \leq r < n$

$$\begin{aligned} x^m &= e \\ x^{qn+r} &= e \\ x^{qn}x^r &= e \\ (e)^q x^r &= e \\ x^r &= e \Rightarrow r = 0 \end{aligned}$$

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Thm. If $G = \langle x \rangle$ and the order of $x = \infty$, then $x^j \neq x^k$ for all $j \neq k$. If the order of $x = n$, then $x^j = x^k$ iff $j \equiv k \pmod{n}$.

Def. Number of elements of G is called the order of G.

For finite group with $G = \langle x \rangle$, it must be the case that $G = \{e, x, x^2, x^3, \ldots, x^{n-1}\}$.

Cyclic subgroups

Let x be an element of order 18. Calculate orders of $x^2, x^3, x^4, x^5, x^{12}$.

Exercises:

1. $\mathbb{Z}_{15}$ and list all elements of order 15.

2. $\mathbb{Z}_{24}$ (cyclic group of order 24). List all elements of order 12.

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Let X be a set (finite or infinite). $S_X = S(X) = \mathrm{Sym}(X) = \{f : X \to X \mid f \text{ is a bijection}\}$

When X is finite, say $X = \{1, 2, 3, \ldots, n\}$, we call $S_n := \mathrm{Sym}(X)$ the *symmetric group of degree* n.

Note: $|S_n| = n!$

For $f \in S_X$, f *shuffles the elements* $\{1, 2, 3, \ldots, n\}$. Therefore we can represent f explicitly by writing:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ f(1) & f(2) & f(3) & \cdots & f(n) \end{pmatrix}$$

For example, consider:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

$h = f \circ g$ Find where each element goes.

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Let $X_1, X_2, \ldots, X_r$ $(1 \leq r \leq n)$ be r distinct elements of $\{1, 2, \ldots, n\}$. The r-cycle $(X_1, X_2, \ldots, X_r)$ is the element of S_n such that

$$\begin{aligned} X_1 &\to X_2 \\ X_2 &\to X_3 \\ &\vdots \\ X_{r-1} &\to X_r \\ X_r &\to X_1 \end{aligned}$$

For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (2, 4)$$

The identity permutation can be written as $(1)(2)(3)(4)$.

Two cycles are disjoint if their elements as a set are disjoint: $\{x_1, \ldots, x_r\} \cap \{y_1, \ldots, y_s\} = \emptyset$.

Theorem: For any f in S_n, there exist disjoint cycles $\sigma_1, \sigma_2, \ldots, \sigma_m$ such that $f = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_m$.

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Proof. Choose some $x_1 \in \{1, \ldots, n\}$.

$$\begin{aligned} x_2 &= f(x_1) \\ x_3 &= f(x_2) \\ &\vdots \end{aligned}$$

There must be a *first element* x_k which is the same as a previous element $x_j \to$ say $x_k = x_j$. $j < k$. In this case $j = 1$ since $x_k = x_j$ implies $x_k = x_{j-1}$ which contradicts the minimality of k because f is one-to-one.

Thus, the first $k-1$ elements $x_1, x_2, \ldots, x_{k-1}$ are distinct with $x_k = x_1$. Thus $f = f_1 \circ h_1$ with $f_1 = (x_1, x_2, \ldots, x_{k-1})$ with h_1 permutes elements other than those above.

We can continue this argument until

$$\begin{aligned} f &= f_1 \circ f_2 \circ h_2 \\ f &= f_1 \circ f_2 \circ f_3 \circ h_3 \\ &\vdots \\ f &= f_1 \circ f_2 \circ f_3 \circ \ldots \circ f_m \circ h_m \end{aligned}$$ where h_m has nothing left to permute.

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\3&5&7&4&2&8&1&6\end{pmatrix} = (1\ 3\ 7)\circ\begin{pmatrix}1&2&3&4&5&6&7&8\\1&5&3&4&2&6&7&8\end{pmatrix}$$

$$= (1\ 3\ 7)\circ(2\ 5)$$

$$= (1\ 3\ 7)\circ(2\ 5)\circ(6\ 8)$$

This is similar to factorization of integers into primes.

Note that disjoint cycles commute.

Theorem: For $n \geq 2$, any permutation in S_n factors as a product of transpositions $(a\ b)$.

Proof:

$$(1) = (1\ 2)\circ(2\ 1)$$

For a general r-cycle with $r \geq 2$,

$$(x_1\ x_2\ \ldots\ x_r) = (x_1\ x_r)\circ(x_1\ x_{r-1})\circ\ldots\circ(x_1\ x_2)$$

Example:

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\3&5&7&4&2&8&1&6\end{pmatrix} = (1\ 3\ 7)\circ(2\ 5)\circ(6\ 8)$$

$$= (1\ 7)\circ(1\ 3)\circ(2\ 5)\circ(6\ 8)$$

Def. A permutation is *even* if it can be written as an even number of transpositions and *odd* if it can be written as an odd number of transpositions.

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Theorem: No permutation is both even and odd. Why?

$A_n = \{f \in S_n \mid f \text{ is even}\}$ for $n \geq 2$.

Theorem: A_n is a subgroup of S_n and $|A_n| = \frac{n!}{2}$. Why?

Investigate S_3:

$$f = (1\ 2\ 3), \quad g = (1\ 3\ 2)$$

$$S_3 = \{e, f, f^2, g, fg, f^2g\} = D_3$$

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Review Exam. Quiz 6.

Computations with A_3

$$|A_3| = \frac{3!}{2} = 3$$

$$A_3 = \{(1), (1\ 3\ 2), (1\ 2\ 3)\} = (1\ 2\ 3)$$

Claim: A_3 is normal in S_3.

Def. We say a subgroup $N < G$ is *normal* (written $N \triangleleft G$) if $gN = Ng$ for all g in G.

Note:

$$gN = \{g \cdot x \mid x \in N\}$$

$$Ng = \{x \cdot g \mid x \in N\}$$

When $N \triangleleft G$, then the left cosets are equal to right cosets.

Remark: In general, $H < G$ general subgroup

$$|H| \cdot [G : H] = |G| \quad \text{equal to the number of left cosets}$$

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We know $S_n \xrightarrow{\sigma} \{-1, 1\}$ where $\sigma(\varphi) = (-1)^{\text{sign of } \varphi}$ is a group homomorphism and $\ker(\sigma) = A_n$, which provides an automatic proof of normality.

Let's check directly:

$$(12)(132)(12)^{-1} = (12)(132)(12)$$
$$(12)(123)(12)^{-1} = (12)(123)(12)$$
$$(13)(132)(13)^{-1} = (13)(132)(13)$$
$$(23)(132)(23)^{-1} = (23)(123)(23)$$

Groups up to order 16:

1	trivial group
2	$\mathbb{Z}/2\mathbb{Z}$
3	$\mathbb{Z}/3\mathbb{Z},\ A_3$
4	$\mathbb{Z}/4\mathbb{Z}$ and $K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
5	$\mathbb{Z}/5\mathbb{Z}$
6	$\mathbb{Z}/6\mathbb{Z}$ non-abelian, D_3
7	$\mathbb{Z}/7\mathbb{Z}$
8	$\mathbb{Z}/8\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$ non-abelian, D_4
9	$\mathbb{Z}/9\mathbb{Z}$
10	$\mathbb{Z}/10\mathbb{Z},$ non-abelian D_5
12	$\mathbb{Z}/12\mathbb{Z},$ abelian, D_6
16	$\mathbb{Z}/16\mathbb{Z},\ \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

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$A_4 = \langle a, b \rangle$ where $a = (12)(34)$ and $b = (123)$.

A_4 does not have a subgroup of order 6 as its index would be 2.

Proof: Any subgroup of index 2 must contain all elements of odd order.

Let H be finite, $H \triangleleft G$.

$$|G : H| = 2$$

Note that index 2 implies normality.

Let g be any element of G. If $g \notin H$, then $gH \neq Hg$. If $g \in H$, cosets are H and gH. Since left cosets are disjoint, $gH = G - H$. But right cosets are also disjoint, $Hg = G - H$. Thus, H is normal.

Thus, if $H \triangleleft A_4$, consider coset {3-cycle} H and cosets H, xH, x^2H. A_4 implies $x^2H = gH$ since if $x \cdot H = H \Rightarrow x^{-1} \cdot e \in H \Rightarrow x \cdot H = H$ or $x \cdot H = x^2H \Rightarrow x = e$.

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Next: Applications, solvable groups, and extra groups by page 30 of supplementary text after mid Isomorphism Theorems.

Cayley's Theorem: Every group G can be identified with a subgroup of $\mathrm{Sym}(G)$.

Proof: This is established by left group action $G \times G \to G$.

In general, a group homomorphism

$$\varphi : G \to \mathrm{Sym}(X) \text{ is equivalent to a group action } G \times X \to X.$$

1. When $\ker(\varphi) = \{e\}$, the group action is called *faithful*. 2. When φ is surjective, we say the group action is *transitive*.

Theorem: $G \times X$ and $|G| \,/\, |X| \Rightarrow \ker(\varphi) = \{e\}$, the action is not faithful.

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The Orbit-Stabilizer Theorem: $G \times X \to G \times X$ finite.

For any $x \in X$, $|G : G_x| = \#(G/G_x)$.

Proof: Consider the mapping $\varphi : G/G_x \to O_x$, given by $\varphi(hG_x) = h \cdot x$. 1. φ is well-defined: $gG_x = hG_x \Rightarrow h^{-1}g \in G_x$ and so $\varphi(h^{-1}g \cdot x) = \varphi(h \cdot x)$. 2. φ is surjective: $\varphi(gG_x) = \varphi(hG_x)$ same element of orbit. 3. φ is injective: $\varphi(gG_x) = \varphi(hG_x) \Rightarrow \varphi(h \cdot x) = \varphi(g \cdot x)$.

Corollary: $|G : G_x| = \#O_x$ since $O_x = |G : G_x|$ for all $x \Rightarrow G \times X \to X$ transitive.

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As we saw before, the conjugation action $\varphi : G \to \mathrm{Sym}(G)$ has $\ker(\varphi) = Z(G)$, the center of G.

Proof demonstrated last class.

Burnside's Orbit Counting Theorem: Let G be a finite group,

$$\frac{1}{|G|}\sum_{g\in G}|\mathrm{Stab}_G(g)| = \text{number of orbits of } G \text{ on } X$$

In the case that the group action is conjugation,

$$\frac{1}{|G|}\sum_{g\in G}|C_G(g)| = \text{number of conjugacy classes in } G$$

where $C_G(g) = \{h \in G \mid hg = gh\}$, the number of elements that commute with g.

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Thus, for $h \in G$, the orbit of the conjugation action is given by

$$O_h = \{ghg^{-1} \mid g \in G\} = [h] \text{ (conjugacy class of } h)$$

The stabilizer subgroup of h is the centralizer of h in G,

$$\text{Stab}(h) = \{g \in G \mid ghg^{-1} = h\} = C_G(h)$$

and

$$|O_h| = |[h]| = [G : C_G(h)] \quad \text{(number of cosets)}$$

Because $X = G$ for the conjugation action, we can write

$$|G| = |O_{g_1}| + |O_{g_2}| + \ldots + |O_{g_m}| \text{ for some } g_1, g_2, \ldots, g_m$$

Note $|O_h| = 1$ iff $h \in Z(G)$ (center).

The Class Equation:

$$|G| = |Z(G)| + \sum_i [G : C_G(y_i)]$$

This equation reveals the deeper structure of the group.

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Def. A p-group P is any group of order $|P| = p^n$ with p a prime.

Lemma: Let P be a p-group acting on a set X. If $|X|$ is not divisible by p, then there is at least one fixed element in X.

Proof: Generally, $|X| = |O_{x_1}| + \ldots + |O_{x_n}|$.

Since P is a p-group, $|O_{x_i}|$ is a power of p. Thus, $|O_{x_i}| = 1$ for some i. Otherwise, if $|O_{x_i}| > 1$ for all i, p divides $|X|$, a contradiction.

But if $|O_{x_i}| = 1$, then $O_{x_i} = \{x_i\}$.

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Proposition: For a prime p and a p-group P, the center of P is not the trivial subgroup.

Proof: If $Z(P) = \{e\}$ and the class equation implies that the other conjugacy classes $[y_i]$ have cardinality p^k for some $k \geq 1$,

$$|P| = p^n = 1 + p^m \quad \text{which is impossible}$$

Corollary: For a prime p, a group of order p^2 is abelian.

Proof: We know by the previous lemma $|Z(P)| = p$ or p^2. If $|Z(P)| = p^2$, then $Z(P) = P$, so P is abelian. Suppose $|Z(P)| = p$ and $x \in P$ and $x \notin Z(P)$. The centralizer of x,

$$C_P(x) = \{g \in P \mid gx = xg\} \supseteq Z(P)$$

Thus, $Z(P) \subset C_P(x) \subset P$, but then $|C_P(x)| \neq p^2$, and $C_P(x) = P$. Thus, x commutes with every element of P, and so $x \in Z(P)$, a contradiction.

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Cauchy's Theorem: For a finite group G, if p is a prime that divides $|G|$, then there is an element $g \in G$ of order p.

Ludwig Sylow extended the above result.

Def. Let p be a prime number and let G be a finite group. Suppose $|G| = p^e m$ where $p \nmid m$. A *Sylow p-subgroup* of G is a subgroup P that is a p-group of order p^e.

The Sylow Theorems: Suppose p is a prime and G is a finite group for which p divides $|G|$. Then:

1. G contains a Sylow p-subgroup.
2. The Sylow p-subgroups of G are conjugates of one another.
3. If P and Q are Sylow p-subgroups of G, then $\exists x \in G$ such that $P = xQx^{-1}$.
4. If $|G| = p^e m$ and n_p denotes the number of Sylow p-subgroups of G, then n_p divides m and $n_p \equiv 1 \pmod{p}$.

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Proposition: If $|G| = p^k m$, $p \nmid m$, then G is not a simple group.

For example,

$$\begin{aligned} 30 &= 2 \cdot 3 \cdot 5, \\ 56 &= 2^3 \cdot 7, \\ 60 &= 2^2 \cdot 3 \cdot 5, \\ 63 &= 3^2 \cdot 7, \\ 72 &= 2^3 \cdot 3^2, \\ 90 &= 2 \cdot 3^2 \cdot 5, \\ 105 &= 3 \cdot 5 \cdot 7, \\ 108 &= 2^2 \cdot 3^3, \\ 120 &= 2^3 \cdot 3 \cdot 5. \end{aligned}$$

None of these orders can form simple groups, as they all have non-trivial Sylow p-subgroups.

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A brief note:

$$PSL_2(\mathbb{F}_7) = PSL_2(7)$$

is a simple group of order 168. This explains why 168 was stubborn.

This is an example of a matrix group over a finite field. As we will see later, $\mathbb{Z}/n\mathbb{Z}$ is actually a ring, but when $n = p^e$ with p a prime, it is actually a field.

Thus, we can talk about vector spaces V over a field and in particular over $\mathbb{Z}/p\mathbb{Z}$.

Note:

$$\begin{aligned} GL_n(k) &= \{n \times n \text{ matrices with entries in } k \text{ with } \det(A) \neq 0\}, \\ SL_n(k) &= \{A \in GL_n(k) \mid \det(A) = 1\}, \\ PGL_n(k) &= GL_n(k)/Z(GL_n), \\ PSL_n(k) &= SL_n(k)/Z(SL_n). \end{aligned}$$

The center of the group is identified with the n-th roots of unity in k.

Theorem: $PGL_n(k) \cong PSL_n(k)$ iff every element of k has an n-th root in k.

For example, $PGL_n(\mathbb{C}) \cong PSL_n(\mathbb{C})$ but $PGL_n(\mathbb{R}) > PSL_n(\mathbb{R})$.

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This can be defined over a ring R with the most important example being the modular group $\Gamma = PSL_2(\mathbb{Z})$, linear fractional transformations of upper half-plane.

This group is actually generated by two elements:

$$S : z \mapsto -\frac{1}{z} \text{ (reflection)}, \quad T : z \mapsto z + 1 \text{ (translation)}.$$

In the 1830's, after finding the alternating group A_n, Galois discovered another family of simple groups as $PSL_2(q)$ where q is a power of a prime.

We have the so-called exceptional isomorphisms:

$$PSL_2(2) \cong S_3,$$
$$PSL_2(3) \cong A_4,$$
$$PGL_2(3) \cong S_4.$$

Also,

$$PSL_2(4) \cong A_5,$$
$$PSL_2(5) \cong A_5,$$
$$PSL_2(9) \cong A_6 \quad \text{unless } (n, q) = (2, 2) \text{ or } (2, 3).$$

Finally,

$$PSL_2(7) \cong PSL_3(2) \text{ (second smallest non-abelian simple group)}.$$

There are a few other exceptional isomorphisms.

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Question: Why does $PSL_2(7)$ have 168 elements?

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Total possibilities for the first column is $7^2 - 1 = 49 - 1 = 48$.

Then we must solve the equation $ad - bc = 1$. $ay - bx \neq 0$.

Continue to count: Column 2 must not be a multiple of Column 1. So we have 49 - 1 possibilities like before, except we have to account for:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, 49 - 7 possibilities for column 2 = 42.

Now $\det(A) = 1, 2, 3, 4, 5, 6 = 7!$. We must divide by 6 to reduce down to $SL_2(7)$. Thus, $|PSL_2(7)| = 48 \cdot 42/7 \cdot 24/7 = 168$.

On another note: 3^{rd} $\frac{|S_5|}{|A_5|} = 5$. Thus, $S_n/A_n \cong S/S_5N$.

Related to Quiz 11:

$$PGL_2(\mathbb{C}) = GL_2(\mathbb{C})/(\mathbb{Z}_2 \times N) = PSL_2(\mathbb{C}) \cong SL_2(\mathbb{C})/E.$$

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Notes on Rings:

There are slightly different definitions of rings. For us, the general idea is that a ring is a set R with two operations $(R, +, \cdot)$ where $(R, +)$ is an abelian group with additive identity 0.

Then $(R, \cdot)$ can either be one of the three:

1. Semigroup
2. Monoid
3. Commutative Monoid

Since I like commutative algebra, I choose option B and then later specialize to option C. In general, we mainly focus on option C.

Therefore, a ring is a set R with set theoretic maps $+ : R \times R \to R$ and $\cdot : R \times R \to R$ which satisfy:

1. $\forall a, b, c \in R, a + (b + c) = (a + b) + c$ (associativity)
2. $\forall a, b \in R, a + b = b + a$ (commutativity)
3. $\exists 0 \in R$ such that $a + 0 = a$ (additive identity)
4. $\forall a \in R, \exists -a \in R$ such that $a + (-a) = 0$ (additive inverse)
5. $\forall a, b, c \in R, a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity of multiplication)
6. $\forall a, b, c \in R, a \cdot (b + c) = a \cdot b + a \cdot c$ (left distributivity)
7. $\forall a, b, c \in R, (a + b) \cdot c = a \cdot c + b \cdot c$ (right distributivity)

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If, in addition, we have property 9

$$\forall a, b \in R, a \cdot b = b \cdot a$$

then we say that R is a commutative ring. When the context is clear, $CR \equiv$ Commutative Ring = Ring.

If, in addition, we have property 10 (inverses), then we say R is a field.

$$\forall a \in R \setminus \{0\}, \exists b \in R \text{ such that } a \cdot b = b \cdot a = 1$$

An aside:

If we have property 10 but not property 9, then we say that R is a skew-field or a division ring.

We can use other notations such as a semi-ring (basically $(R, +)$ semigroup and $(R, \cdot)$ semigroup with 0 additive identity).

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Next: Ring homomorphism, subrings, ideals, isomorphism theorem.

Examples:

1. Prototypical example $(\mathbb{Z}, +, \cdot)$
2. Another great example $(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$
3. When $n = p$ a prime, a field (finite field) $(\mathbb{Z}/p\mathbb{Z}, +, \cdot)$ is usually denoted $\mathbb{F}_p$.
4. Non-commutative example: $M_n(\mathbb{R}) = n \times n$ matrices with matrix addition and matrix multiplication.

Endomorphism of a Group: $G \to R$
For these, go into non-commutative stuff.

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In general, given a group G, there is a such thing as a group ring:

$$R[G] = \{f : G \to R \mid f \text{ finite support}\}$$

Thus, an element can be written as:

$$f = \sum_{g \in G} f(g)g \quad \text{(example of free construction)}$$

If you take an abelian group, you get commutative rings (example polynomial rings). Otherwise, you get non-commutative rings.

$$R[\mathbb{Q}] \quad \text{elements are } x^2 + x(g) + \ldots$$

These are the so-called quaternions over $\mathbb{R}$, or otherwise the so-called ring of differential operators.

Note: Actually for a monoid M, we can form $R[M]$.

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Free Groups:

Given a set S, we define F_S the free group generated over S by all finitely supported maps $f : \mathbb{Z} \to S$.

$$\sum f(x)e_x \quad \text{(finite support)}$$

Addition notation because it is abelian. A similar construction can occur if we want a non-commutative group.

$$\langle S \rangle = \{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \mid x_i \in S, a_i \in \mathbb{Z} \setminus \{1\}\}$$

with identification

$$x_i^m x_i^n = x_i^{m+n}, \quad x_i^{-m} x_i^m = 1 \quad \text{(identity)}$$

Universal Properties:

$$F_S \to G \quad \text{abelian}$$

For general free group:

$$S \to \langle S \rangle \to G$$

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Further examples of rings: 1. $R[x]$: polynomial ring over R, R ring.

$$R[x] = \{a_n x^n + \ldots + a_0 \mid a_i \in R\}$$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$$

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0$$

$$p(x) \cdot q(x) = \sum a_i b_j x^{i+j}$$

Generally speaking, polynomial rings have additional properties (ID, UFD, PID).

2. $R[x_1, \ldots, x_n]$: multivariate polynomial ring over R.

In general, given a polynomial $p(x_1, \ldots, x_n) \in S[x_1, \ldots, x_n]$ and $s_1, \ldots, s_n \in R$:

$$\text{evaluation} \quad p(x_1, \ldots, x_n) \mapsto p(s_1, \ldots, s_n)$$

This is a ring homomorphism.

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Let S and R be two rings. We say that $f : S \to R$ is a ring homomorphism if:

1. $\forall x, y \in S, \quad f(x + y) = f(x) + f(y)$
2. $\forall x, y \in S, \quad f(x \cdot y) = f(x) \cdot f(y)$
3. $f(1_S) = 1_R$

Condition 3 is necessary if option A definition is allowed.

Some basic examples:

1. $R \to R[x]$, R is a subring of $R[x]$.
2. $R \to \mathbb{Q}$ (actually two fields have only injective morphisms).
3. $\mathbb{C}[x] \to \mathbb{C}$, $p(x) \mapsto$ constant term (surjective).
4. $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z} \ncong \mathbb{Z}$ not a ring homomorphism.

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When I is just a subring:

1. $I = \{f \mid f : I \to I\}$.
2. (Absorption) $\forall I \in I, \quad \text{ideal}(I) \Rightarrow I$.

Let I be a subset of R closed under $+$ and $\cdot$. For all subsets S of a ring R, I is the ideal of S if I satisfies:

1. If $a \in I$, $b \in R \Rightarrow ab \in I$ (left ideal, ideal is absorbed).
2. (I) ideal is absorbed if $b \in I$.

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CRU:

Def. Let R be a ring. We trust R is an *integral domain* if it has the zero property:

$$\forall a, b \in R, \quad ab = 0 \Rightarrow a = 0 \text{ or } b = 0$$

Examples: Fields, $\mathbb{Z}$, $R[x, \ldots, x_n]$.

Non-example: $\mathbb{Z}/n\mathbb{Z}$, $n = pq$ composite.

In any ring, we have the ideal generated by the zero element 0:

$$(0) = \{0\}$$

Definition of prime ideal:

$$\text{Prove } R \text{ a Domain} \Rightarrow (0) \text{ a prime ideal}$$

Another concept: principal ideal and principal ideal domain.

Definition of maximal ideal:

$$R/I \text{ domain} \Leftrightarrow I \text{ prime}$$

$$R/I \text{ field} \Leftrightarrow I \text{ field}$$

Field Theory: Simple Extensions.

$F \subseteq k$, $\alpha \in k$, fields.

$F(\alpha)$ = smallest subfield of k containing α and F = all possible applications of $f \mapsto x$, $f \mapsto \alpha$ and F.

$$F[x] = F(\alpha) \quad \text{(Remark: function field of an integral domain)}$$

Rings of integers of a number field

Evaluation Map:

$$F[x] = \text{Univariate polynomials over } F$$

F a field $\Rightarrow$

1. Euclidean Domain
2. PID
3. UFD

Ex: $\mathbb{E}(x) \to F[x] \subseteq K \subseteq \mathbb{C}$ calc. closed field π irreducible elements

ev_x is a ring homomorphism.

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There are only two possibilities:

1. $\ker(ev_\alpha) = \{0\} \Rightarrow ev_\alpha$ is injective.
2. $\ker(ev_\alpha) = (f_\alpha)$ because it's a PID.

In the case of 0, we can construct an inverse, so:

$$F[x] \cong F(\alpha)$$

In this case, we say α is transcendental over F.

Example: $\mathbb{Q}[x] \cong \mathbb{Q}[\pi]$, e, π, e^π transcendental over rationals.

Note: For any $\alpha, \beta \in k$, $F(\alpha) \cong F(\beta)$ as fields $\cong F(x)$.

But this doesn't mean $F(\alpha) = F(\beta)$ in k.

$$\mathbb{Q}(x, y) \neq \mathbb{Q}(e^\pi) \quad \text{(non-canonical } \mathbb{Q}(x) \text{ in } \mathbb{R})$$

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2nd case:

$$\ker(ev_\alpha) = (f_\alpha) \quad f_\alpha \neq 0$$

In a Euclidean Domain, we find f_α by minimizing deg f up to a unit u (leading coefficient).

$\Rightarrow f_\alpha$ is a monic polynomial of minimal degree d such that $f_\alpha(\alpha) = 0$.

Thm: f_α is irreducible (hence (f_α) is prime).

Proof: If $f_\alpha = gh$ then $0 \leq \deg(g), \deg(h) < \deg(f_\alpha) = d$ then $g(\alpha), h(\alpha) \neq 0$ by minimizing property.

Then $g(\alpha)h(\alpha) = 0 \Rightarrow$ zero divisors in $F[x] \subseteq k$ but k is a field so it is integral domain. Contradiction.

So, f_α irreducible / $\Rightarrow f_\alpha$ irreducible in $F[x] \Rightarrow (f_\alpha)$ maximal ideal (prime = max for $F[x]$).

Theory:

$$ev_\alpha : F[x]/(f_\alpha) \to F(\alpha) \subseteq k$$

is a field homomorphism.

Theorem: $F(\alpha)$ is a subfield of k with $[F(\alpha) : F] = \deg(f_\alpha)$.

Basis:

$$F[x]/(f_\alpha) \quad 1, x, x^2, \ldots, x^{d-1}$$

Basis:

$$F(\alpha) \quad 1, \alpha, \alpha^2, \ldots, \alpha^{d-1}$$

Recall: $F[x]/(f_\alpha)$ field with cosets as elements.
$F[u]$ s.t. $f_\alpha(u) = 0$ (factors all roots).

$$f(x) = x^d + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

$$f_\alpha(u) = 0 \Rightarrow u^d = -a_{n-1}u^{d-1} - \ldots - au - a$$

Alternatively, $F[x]$ ED $\Rightarrow$ division algorithm:

$$g = qf_\alpha + r \quad r = 0$$

$$\deg(r) < \deg(f_\alpha) = d$$

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Now, $g + (f_\alpha) = r + (f_\alpha) \Rightarrow r(\alpha) \in F(\alpha)$.

Thus, every non-zero coset is represented by r with $\deg(r) < \deg(f_\alpha)$.

That is, all elements in $F[x]$ are of the form $a + bx + \ldots + c\alpha^{d-1}$ (dim = d).

Basically obvious: $\mathbb{Q} \subseteq r(\alpha)^{-1}$ in $F(\alpha)$.

$F[x]$ ED $\Rightarrow$ Bezout's Identity.

f_α irreducible $\deg r < \deg f_\alpha$, so $\gcd(r, f_\alpha) = 1$.

$\exists p, q$ s.t. $pr + qf_\alpha = 1$ Bezout's Identity.

Evaluate at α: $p(\alpha)r(\alpha) = 1$.

Example:

$$\mathbb{Q}[\sqrt{2}] \quad (a + b\sqrt{2})^{-1}$$

$$(a + b\sqrt{2})(c + d\sqrt{2}) = 1 \Rightarrow ac + 2bd = 1 \Rightarrow bc + ad = 0$$

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Def. $\alpha \in k$ is called algebraic over F if there exists $a_i \in F$ (not all 0) such that:

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0 = 0$$

If so, $\exists!$ monic polynomial f_α of smallest degree s.t. $f_\alpha(\alpha) = 0$.

f_α is called the minimal polynomial of α/F (irreducible over F).