Constructing $\mathbb{R}$ via Cauchy completion

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This document details how to define the set of real numbers via rational approximations.

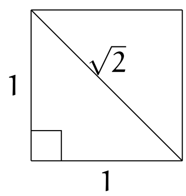
1 Notation

- $:=$ means defined to be.
- $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \dots\}$.
- $\mathbb{Q}$ is the rational numbers (quotient).
- $\mathbb{R}$ is the real numbers.
- $d(x, y) = |x - y|$ is the distance between the points x and y.
- $\mathbb{Q}^{\mathbb{N}}$ is the set of sequences of rational numbers
- $\mathbb{Q}_C^{\mathbb{N}}$ is the set of rational sequences satisfying Cauchy's criterion
- $\mathbb{R}^{\mathbb{N}}$ is the set of sequences of real numbers
- $\mathbb{R}_C^{\mathbb{N}}$ is the set of real sequences satisfying Cauchy's criterion

2 Insufficiency of the rationals

The discovery of irrational numbers radically changed our concept of a number. It is an ancient discovery, credited to Pythagoras, but it is likely this was discovered independently elsewhere in the world because there are surviving documents with approximations to irrational numbers.

Even in pragmatic mathematics, we're forced to accept irrational numbers as soon as we want to calculate the diagonal of a square:



The irrationality of $\sqrt{2}$ would have been understood geometrically at the time, but a modern algebraic proof goes like this:

Theorem 1 *$\sqrt{2}$ is irrational*

Proof. Suppose it is rational. Then there is some fraction a/b in simplest form where a, b are integers, $b \neq 0$, and

$$\left(\frac{a}{b}\right)^2 = 2.$$

Then

$$\frac{a^2}{b^2} = 2$$
$$a^2 = 2b^2.$$

Therefore, a is even. Substitute $a = 2k$. Then

$$(2k)^2 = 2b^2. \tag{1}$$

But,

$$4k^2 = 2b^2, \tag{2}$$
$$2k^2 = b^2, \tag{3}$$

proving b is even too. This contradicts the assumption that a/b was in simplest form. □

2.1 What do now?

How can we work with an irrational number? Relating it back to rational numbers is promising. There are multiple ways to do this. In fact, we defined $\sqrt{2}$ by relating it to 2 algebraically: $(\sqrt{2})^2 = 2$. But for this article, an algebraic relationship is of limited use. There is an ambiguity: $(\sqrt{2})^2 = (-\sqrt{2})^2 = 2$, so which one did we want?

More damning is the existence of *transcendental numbers*, irrational numbers that are not the solution to any algebraic equation we can spelled using rational numbers, the most famous of which are π and e. Instead, consider *rational approximations*. In particular, we want approximations that can be made *arbitrarily precise*.

Definition 2 If x_n is a sequence approximating x_*, it can be made arbitrarily precise when, for any desired error tolerance $\varepsilon > 0$, there is some large enough N_ε that once $n > N_\varepsilon$, the sequence is within the desired tolerance:

$$n > N_\varepsilon \implies d(x_n, x_*) < \varepsilon.$$

Write this

$$\lim_{n\to\infty} x_n = x_*.$$

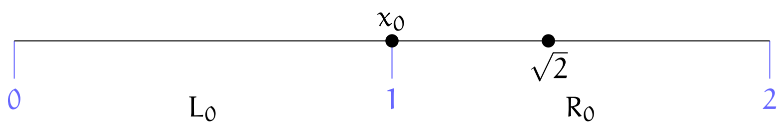
Example 3 (Approximating $\sqrt{2}$ via binary search) Binary search is probably the simplest algorithm to approximate $\sqrt{2}$ with arbitrary precision. The idea is to repeatedly bisect an interval containing $\sqrt{2}$ in order to refine our estimate.

To begin, pick an interval I_0 containing $\sqrt{2}$, so $I_0 = (0, 2)$. For our initial estimate, take the midpoint of I_0, so $x_0 = 1$. This cuts I_0 into its left half, $L_0 = I_0 \cap (-\infty, x_0)$, and the right half $R_0 = I_0 \cap (x_0, \infty)$. Then

$$\mathrm{length}(L_0) = \mathrm{length}(R_0) = \frac{1}{2}\mathrm{length}(I_0) = 1.$$

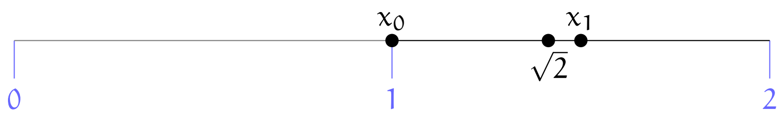
Observing that $\sqrt{2}$ must lie in one of these halves leads to an error bound:

$$E_0 \coloneqq d(x_0, \sqrt{2}) \leqslant 1.$$



Now check if we under or over–estimated. Because ${x_0}^2 = (1)^2 = 1$ is less than 2, $x_0 < \sqrt{2}$; therefore, $\sqrt{2} \in R_0$.

We can refine this estimate by picking $I_1 = R_0$ and repeating this process. Take x_1 to be its midpoint, 1.5. Again, call the left half $L_1 = I_1 \cap (-\infty, x_1)$ and the right half $R_1 = I_1 \cap (x_1, \infty)$.



The same argument will also give new error bounds. Observe

$$\mathrm{length}(L_1) = \mathrm{length}(R_1) = \frac{1}{2}\mathrm{length}(I_1) = \frac{1}{2}.$$

Because $\sqrt{2}$ must lie in one of these halves, we get the error bound

$$E_1 \coloneqq d(x_1, \sqrt{2}) \leqslant \frac{1}{2}.$$

Checking ${x_1}^2 = 2.25 > 2$ shows this time we overestimated, so $\sqrt{2} \in I_2 \coloneqq L_0$.

In general, given I_n, pick x_n to be its midpoint. Let $L_n = I_n \cap (-\infty, x_n)$ be the left half and $R_n = I_n \cap (x_n, \infty)$ be the right half. The lengths of these halves gives an error bound:

$$E_n \coloneqq d(x_n, \sqrt{2}) \leqslant \mathrm{length}(L_n) = \mathrm{length}(R_n) = \frac{1}{2}I_n.$$

Induction on n gives

$$E_n \leqslant \frac{1}{2^n}. \tag{4}$$

Setting

$$I_{n+1} = \begin{cases} L_n & \text{if } {x_n}^2 > 2 \text{ (overestimated)} \\ R_n & \text{if } {x_n}^2 < 2 \text{ (underestimated)} \end{cases}$$

sets us up for the next iteration.

Equation (4) is particularly interesting because it shows error can be made arbitrarily small:

$$\lim_{n\to\infty} E_n \leqslant \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

3 The plan, the hitch

This analysis of the binary search approximation suggests a plan. We can define a real number by its rational approximation. The information we want about $\sqrt{2}$ for example can be extracted from a sufficiently accurate approximation, or even by studying the approximation algorithm itself. This matches the way we carry out calculations with irrational numbers in practice: approximate it with a rational number that is close enough for our needs, like using 1.414 for $\sqrt{2}$ or 3.14159 for π.

But to turn "represent real numbers by their approximations" into an exact definition requires care. Binary search is by no means the only algorithm to approximate $\sqrt{2}$. Should we pick one and declare it *the* definition? If so, how do we choose? If not, how do we consider the infinite posibilities a coherent whole?

There is also the problem of circular reasoning. We derived Approximating $\sqrt{2}$ via binary search by starting with $\sqrt{2}$ and working backwards; we had to already know, in some sense, what $\sqrt{2}$ was. But we don't—we can't until we have a definition for the real numbers. Sure, for $\sqrt{2}$ in particular we can find a way around that. But what about the other irrational numbers? Will we need to invent an ad-hoc definition for each irrational we stumble upon?

This is no mere formality. We know $\sqrt{2}$ is irrational, but how many other numbers are? Is it common or rare? If you throw a dart at a number line, what is the probability it hits an irrational number? It is impossible to ask these questions in a meaningful way, let alone try to answer them, without a definition of $\mathbb{R}$.

4 Avoiding circularity via Cauchy's criterion

To define $\mathbb{R}$ using rational approximations, we need to distinguish a divergent sequence like

$$n \mapsto n \text{ \textit{i.e.} } 0, 1, 2, 3, \ldots$$

from an approximation. This is easy to check when a sequence converges to a rational number; just check it gets arbitrarily close to its limit. For example,

Example 4 the sequence

$$n \mapsto x_n = \frac{1}{n}$$

approaches 0:

$$d(x_n, 0) = |x_n - 0| = \left|\frac{1}{n}\right|.$$

For an error tolerance ε, we want to know when $|x_n| < \varepsilon$. And

$$\frac{1}{n} < \varepsilon$$

exactly when

$$n < \frac{1}{\varepsilon}.$$

Not only does this tell us exactly how far down the series we need to look to land within any desired error tolerance, but it ensures it is always possible.

Unfortunately, this style of reasoning will not carry over for sequences that converge to an irrational number, because we can't know what "converge to an irrational number" means until after we define $\mathbb{R}$. We need a test that lets us take a sequence and decide if it ought to converge without referring to its limit at all.

Definition 5 (Cauchy's criterion) A sequence x_n satisfies Cauchy's criterion ("is Cauchy") when its elements eventually become arbitrarily close:

$$\lim_{i,j\to\infty} d(x_i, x_j) = \lim_{i,j\to\infty} \left|x_i - x_j\right| = 0$$

A sequence satisfying Cauchy's criterion bunches up within smaller and smaller regions exactly like a convergent sequence would:

Theorem 6 *Every convergent sequence satisfies Cauchy's criterion.*

We will need this tool to prove it:

Lemma 7 (the triangle inequality) *Recall that*

$$\begin{aligned} d(x,z) &\leqslant d(x,y) + d(y,z), \\ |x - z| &\leqslant |x - y| + |y - z|. \end{aligned}$$

The shortest distance between any two points is the straight line between them.

Proof. Suppose the sequence $x_\square$ has limit x_*. Then

$$\lim_{i\to\infty} d(x_i, x_*) = \lim_{j\to\infty} d(x_j, x_*) = 0,$$

hence

$$\lim_{i,j\to\infty} \quad d(x_i, x_*) + d(x_j, x_*) = 0,$$

but by the triangle inequality

$$\begin{aligned} d(x_i, x_j) \leqslant& \quad d(x_i, x_*) + d(x_j, x_*), \\ \lim_{i,j\to\infty} d(x_i, x_j) \leqslant& \lim_{i,j\to\infty} d(x_i, x_*) + d(x_j, x_*) = 0. \end{aligned}$$

□

The converse is not always true. There are some Cauchy sequences that have no rational limit, namely the ones that converge to irr[redacted]s .

Definition 8 (Binary search approximation to [redacted]) Let $I_0 \coloneqq (0, 2)$ be the starting interval. Pick x_n to be the midpoint of I_n (in particular, $x_0 = 1$). Let

$$L_n \coloneqq I_n \cap (-\infty, x_n)$$

be the left half of I_n and

$$R_n \coloneqq I_n \cap (x_n, \infty)$$

be the right half. Define I_{n+1} by ~~choosing the half containing $\sqrt{2}$~~

$$I_{n+1} \coloneqq \begin{cases} L_n & \text{if } {x_n}^2 < 2 \\ R_n & \text{if } {x_n}^2 > 2 \end{cases}$$

The sequence $n \mapsto x_n$ this generates is the desired approximation.

Theorem 9 *The sequence $n \mapsto x_n$ defined above satisfies Cauchy's criterion.*

Proof. As L_n and R_n are each halves of I_n, their lengths are each half the length of I_n:

$$\operatorname{length}(I_{n+1}) = \frac{1}{2}\operatorname{length}(I_n).$$

By induction,

$$\operatorname{length}(I_n) = \frac{2}{2^n}.$$

By construction, whenever $i, j > n$ both x_i and x_j are contained in I_n; therefore,

$$d(x_i, x_j) \leqslant \operatorname{length}(I_n) = \frac{2}{2^n}$$

Taking limits gives

$$\lim_{i,j\to\infty} d(x_i, x_j) \leqslant \lim \frac{2}{2^n} = 0$$

as desired. □

Theorem 10 *The sequence $n \mapsto x_n$ has no limit within the set of rational numbers.*

Proof. Observe that the intervals $I_\square$ are nested:

$$I_0 \supset I_1 \supset I_2 \supset \ldots I_n \supset I_{n+1} \supset \ldots$$

Fix N. For any $i > N$, $x_i \in I_i \subset I_N$. Therefore $x_* \in I_N$. But N was arbitrary, so x_* has to be contained in I_n for all n.

Write $I_n = (a_n, b_n)$. Then $a_n < x_* < b_n$ and

$${a_n}^2 < {x_*}^2 < {b_n}^2.$$

Because of how each I_n was chosen ~~to contain $\sqrt{2}$~~, we are guaranteed that ${a_n}^2 < 2 < {b_n}^2$. Because the length of I_n approaches 0, we find that $a_n - b_n \to 0$. Therefore $\cancel{x_* = \sqrt{2}}$ $\lim_{n\to\infty} {a_n}^2 = \lim_{n\to\infty} {b_n}^2 = 2$, forcing

$$ {x_*}^2 = 2. $$

There is no such rational number. $\square$

One might say it converges to (REDACTED).

5 Taming the infinite with equivalence classes

Multiple sequences can approximate the same number. We need a way to check this that without needing to know about its limit, as some Cauchy sequences ~~converge to irrational numbers~~ have no defined limit.

Definition 11 ($\approx$) We say $(n \mapsto x_n) \approx (n \mapsto y_n)$ (or briefly $x_n \approx y_n$) whenever

$$ \lim_{n\to\infty} d(x_n, y_n) = 0. $$

The goal is to use $\approx$ to collect together equivalent approximations into buckets. These buckets will become our construction of the real numbers. But how do we know they will split cleanly? The necessary property is that $\approx$ be an *equivalence relation.*

Definition 12 (Equivalence relation) An equivalence relation $\sim$ must satisfy:

1. reflexivity: $x \sim x$,
2. symmetry: $x \sim y$ is equivalent to $y \sim x$,
3. transitivity: if $x \sim y$ and $y \sim z$ then $x \sim z$.

Theorem 13 $\approx$ *is an equivalence relation.*

Proof.

1. Reflexivity follows from noting $d(x, x) = 0$. Therefore, if $n \mapsto x_n$ is a Cauchy sequence, $d(x_n, x_n) = 0$.
2. Symmetry follows from observing d is symmetric: $d(x, y) = d(y, x)$.
3. Transitivity is a consequence of the triangle inequality. If $x_n \approx y_n$, then $d(x_n, y_n) \to 0$. If $y_n \approx z_n$, then $d(y_n, z_n) \to 0$. Therefore

$$ d(x_n, z_n) \leqslant d(x_n, y_n) + d(y_n, z_n) \to 0. \qquad \square $$

Now for the buckets:

Definition 14 (Equivalence class) If $n \mapsto x_n$ is a cauchy sequence, define its *equivalence class* to be the set of all cauchy sequences equivalent to $n \mapsto x_n$:

$$[n \mapsto x_n] \coloneqq \big\{n \mapsto y_n : (n \mapsto x_n) \approx (n \mapsto y_n)\big\}.$$

The set of all such equivalence classes is written $\mathbb{Q}_C^{\mathbb{N}}/\approx$.

We can finally construct the reals!

Definition 15 The set of real numbers is the set of equivalence classes of rational cauchy sequences:

$$\mathbb{R} \coloneqq \mathbb{Q}_C^{\mathbb{N}}/\approx .$$

We would like to think of the real numbers as containing the rational numbers, so we need to find a copy of $\mathbb{Q}$ in $\mathbb{R}$.

Definition 16 Let $i : \mathbb{Q} \to \mathbb{R}$ via $i(q) = [\_ \mapsto q]$, the sequence that is constantly q.

The basic operations $+, \times, -, \div, |\cdot|$ extend to $\mathbb{R}$ similarly:

$$[n \mapsto x_n] + [n \mapsto y_n] \coloneqq [n \mapsto x_n + y_n].$$

Theorem 17 *The function* $i : \mathbb{Q} \mapsto \mathbb{R}$ *is an* isometry *(preserves distance).*

Proof. Consider rational numbers x and y. Then

$$d\Big(i(x), i(y)\Big) = \Big|[\_ \mapsto x] - [\_ \mapsto y]\Big| = \Big[\_ \mapsto |x - y|\Big] = i\Big(d(x, y)\Big) \qquad \square$$

Now we can define $\sqrt{2}$ by the equivalence class $[n \mapsto x_n]$ of the sequence $n \mapsto x_n$ found by binary search. We would expect in some sense

$$\lim_{n\to\infty} x_n = \sqrt{2}.$$

We can state this more precisely using $i : \mathbb{Q} \to \mathbb{R}$ as

$$\lim_{n\to\infty} i(x_n) = \sqrt{2}.$$

We intuitively interpret the construction of $\mathbb{R}$ as representing a real number r by the rational approximations *that converge to* r. However, until now we haven't formally investigated the convergence of these sequences. The following theorem establishes a formalized version of this intuition as fact.

Theorem 18 *If* $n \mapsto x_n \in \mathbb{R} = \mathbb{Q}_C^{\mathbb{N}}$, *then*

$$\lim_{n\to\infty} i(x_n) = [n \mapsto x_n],$$

i.e. *rational Cauchy sequences converge (in* $\mathbb{R}$*) to the real number they represent.*

Proof. Let $x_* = [n \mapsto x_n]$. It suffices to show

$$\lim_{k\to\infty} x_* - i(x_k) = 0.$$

Note that the 0 in the right hand side is the *real number* 0, *i.e.* $[\_ \mapsto 0]$. But

$$x_* - i(x_k) = [n \mapsto x_n] - [\_ \mapsto x_k] = [n \mapsto x_n - x_k];$$

therefore,

$$\lim_{k\to\infty} x_* - i(x_k) = \lim_{k\to\infty} [n \mapsto x_n - x_k].$$

Now it suffices to show

$$\lim_{k\to\infty} (n \mapsto x_n - x_k) \approx \_ \mapsto 0,$$

which follows from observing

$$\lim_{n\to\infty} \lim_{k\to\infty} x_n - x_k = 0$$

by hypothesis that $n \mapsto x_n$ is Cauchy. □

Theorem 19 *Every Cauchy sequence of real numbers converges to a real number.*

Proof. Suppose there is an $n \mapsto x^{(n)} \in \mathbb{R}_C^{\mathbb{N}}$. Each $x^{(n)} \in \mathbb{R}$ so it is represented by some rational cauchy sequence

$$x^{(n)} = \left[m \mapsto x_m^{(n)}\right].$$

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$$x^{(n)} = \left[m \mapsto x_m^{(n)}\right].$$

Consider the diagonal sequence

$$D \coloneqq m \mapsto x_m^{(m)}.$$

It suffices to show that

$$\lim_{n\to\infty} x^{(n)} - [D] = 0,$$

or

$$\lim_{m\to\infty} \lim_{n\to\infty} x_m^{(n)} - x_m^{(m)} = 0.$$

□

to be continued ???