Constructing $\mathbb{R}$ via Cauchy completion

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This document details how to define the set of real numbers via rational approximations.

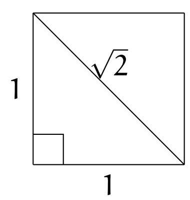
1 Notation

- $:=$ means defined to be.
- $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \dots\}$.
- $\mathbb{Q}$ is the rational numbers (quotient).
- $\mathbb{R}$ is the real numbers.
- $d(x, y) = |x - y|$ is the distance between the points x and y.
- $\mathbb{Q}^{\mathbb{N}}$ is the set of sequences of rational numbers
- $\mathbb{Q}_C^{\mathbb{N}}$ is the set of rational sequences satisfying Cauchy's criterion
- $\mathbb{R}^{\mathbb{N}}$ is the set of sequences of real numbers
- $\mathbb{R}_C^{\mathbb{N}}$ is the set of real sequences satisfying Cauchy's criterion

2 Insufficiency of the rationals

The discovery of irrational numbers radically changed our concept of a number. It is an ancient discovery, credited to Pythagoras, but it is likely this was discovered independently elsewhere in the world because there are surviving documents with approximations to irrational numbers.

Even in pragmatic mathematics, we're forced to accept irrational numbers as soon as we want to calculate the diagonal of a square:



The irrationality of $\sqrt{2}$ would have been understood geometrically at the time, but a modern algebraic proof goes like this:

Theorem 1 *$\sqrt{2}$ is irrational*

Proof. Suppose it is rational. Then there is some fraction a/b in simplest form where a, b are integers, $b \neq 0$, and

$$\left(\frac{a}{b}\right)^2 = 2.$$

Then

$$\begin{aligned} \frac{a^2}{b^2} &= 2 \\ a^2 &= 2b^2. \end{aligned}$$

Therefore, a is even. Substitute $a = 2k$. Then

$$(2k)^2 = 2b^2. \tag{1}$$

But,

$$4k^2 = 2b^2, \tag{2}$$

$$2k^2 = b^2, \tag{3}$$

proving b is even too. This contradicts the assumption that a/b was in simplest form. □

2.1 What do now?

How can we work with an irrational number? Relating it back to rational numbers is promising. There are multiple ways to do this. In fact, we defined $\sqrt{2}$ by relating it to 2 algebraically: $(\sqrt{2})^2 = 2$. But for this article, an algebraic relationship is of limited use. There is an ambiguity: $(\sqrt{2})^2 = (-\sqrt{2})^2 = 2$, so which one did we want?

More damning is the existence of *transcendental numbers*, irrational numbers that are not the solution to any algebraic equation that can be spelled using only rational numbers. The most famous of these are π and e. Instead, let's look at *rational approximations*.

The simplest algorithm to approximate $\sqrt{2}$ is binary search. We know $0 < \sqrt{2} < 2$. Its midpoint, 1 is a rough estimate of $\sqrt{2}$. Call it x_0. Without any further computation, we know $\sqrt{2}$ will lie on either side of $x_0 = 1$. Therefore our error is at most 1:

$$E_0 = d(x_0, \sqrt{2}) \leqslant 1.$$

As $x_0^2 = 1^2 = 1 < 2$, this underestimated $\sqrt{2}$. Now we know $\sqrt{2}$ lies in the interval $(1, 2)$. Taking its midpoint $x_1 = 1.5$ gives an improved estimate. Similarly, because $\sqrt{2}$ must lie in either half of the interval $(1, 2)$, our new estimate has error at most .5:

$$E_1 = d(x_1, \sqrt{2}) = \left|x_1 - \sqrt{2}\right| \leqslant .5.$$

Checking $x_1^2 = (1.5)^2 = 2.25 > 2$ shows we overestimated. Now we know $\sqrt{2}$ lies in the interval $(1, 1.5)$ and we can repeat the process: $x_2 = 1.25$. The error for this new estimate is no more than a quarter:

$$E_2 = d(x_2, \sqrt{2}) = \left|x_2 - \sqrt{2}\right| \leqslant .25$$

This time, we underestimated, so $\sqrt{2}$ lies between $(x_2, x_1) = (1.25, 1.5)$.

Each iteration cuts this upper bound on error in half:

$$E_n = \left|x_n - \sqrt{2}\right| \leqslant \frac{1}{2^n}.$$

With enough iterations, binary search will approximate $\sqrt{2}$ arbitrarily well. Indeed,

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \left|x_n - \sqrt{2}\right| \leqslant \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Generalizing this kind of argument leads to a formal definition for the set of real numbers. However, we must take care to avoid circular reasoning. So far, we found this approximation by assuming that some number called $\sqrt{2}$ exists, is well-defined, and works the way we would expect a number to. We can probably save this specific approximation using the fact that $\sqrt{2}$ solves the equation $x^2 = 2$. But this will not generalize.

3 Avoiding circularity via Cauchy's criterion

Our goal is to define a real number as something we can approximate using rational numbers. But how can you tell if you are correctly approximating something that hasn't been defined yet? Without knowing what $\sqrt{2}$ is, how can we make sense of $|x_n - \sqrt{2}|$?

We need a way to tell if a sequence of rational numbers looks enough like an approximation that we can work with it, without needing to know how to take its limit (or even if its limit exists). That is the purpose of the following test:

Definition 2 (Cauchy's criterion) A sequence x_n satisfies Cauchy's criterion ("is Cauchy") when its elements eventually become arbitrarily close:

$$\lim_{i,j\to\infty} d(x_i, x_j) = \lim_{i,j\to\infty} \left|x_i - x_j\right| = 0$$

Theorem 3 *Every convergent sequence satisfies Cauchy's criterion.*

Proof. Recall the triangle inequality:

$$\begin{aligned} d(x, z) &\leqslant d(x, y) + d(y, z), \\ |x - z| &\leqslant |x - y| + |y - z|. \end{aligned}$$

Suppose the sequence $x_\square$ has limit x_*. Then

$$\lim_{i\to\infty} d(x_i, x_*) = \lim_{j\to\infty} d(x_j, x_*) = 0,$$

hence

$$\lim_{i,j\to\infty} d(x_i, x_*) + d(x_j, x_*) = 0,$$

but

$$\begin{aligned} d(x_i, x_j) &\leqslant \qquad d(x_i, x_*) + d(x_j, x_*) \\ \lim_{i,j\to\infty} d(x_i, x_j) &\leqslant \lim_{i,j\to\infty} d(x_i, x_*) + d(x_j, x_*) = 0 \end{aligned}$$

□

The converse is not always true. There are some Cauchy sequences that have no rational limit. The bisection approximation of $\sqrt{2}$ is one such sequence is an example. I would love to tell you it is Cauchy because it converges—to $\sqrt{2}$. But that is circular reasoning. Instead, I will carefully redefine it and prove it is Cauchy.

Definition 4 (Bisection approximation to (REDACTED)) Let $I_0 \coloneqq (0, 2)$ be the starting interval. Pick x_n to be the midpoint of I_n (in particular, $x_0 = 1$). Let

$$H_n^- \coloneqq I_n \cap (0, x_n)$$

be the left half of I_n and

$$H_n^+ \coloneqq I_n \cap (x_n, 2)$$

be the right half. Define I_{n+1} by ~~choosing the half containing $\sqrt{2}$~~

$$I_{n+1} \coloneqq \begin{cases} H_n^- & \text{if } x_n^2 < 2 \\ H_n^+ & \text{if } x_n^2 > 2 \end{cases}$$

The sequence $n \mapsto x_n$ this generates is the desired approximation.

Theorem 5 *The sequence $n \mapsto x_n$ defined above satisfies Cauchy's criterion.*

Proof. As H_n^- and H_n^+ are each halves of I_n, their lengths are each half the length of I_n:

$$\text{length}(I_{n+1}) = \frac{1}{2}\text{length}(I_n).$$

By induction,

$$\text{length}(I_n) = \frac{2}{2^n}.$$

By construction, whenever $i, j > n$ both x_i and x_j are contained in I_n; therefore,

$$d(x_i, x_j) \leqslant \text{length}(I_n) = \frac{2}{2^n}$$

Taking limits gives

$$\lim_{i,j\to\infty} d(x_i, x_j) \leqslant \lim \frac{2}{2^n} = 0$$

as desired. □

Theorem 6 *The sequence* $n \mapsto x_n$ *has no limit within the set of rational numbers.*

Proof. Observe that the intervals $I_\square$ are nested:

$$I_0 \supset I_1 \supset I_2 \supset \ldots I_n \supset I_{n+1} \supset \ldots$$

Fix N. For any $i > N$, $x_i \in I_i \subset I_N$. Therefore $x_* \in I_N$. But N was arbitrary, so x_* has to be contained in I_n for all n.

Write $I_n = (a_n, b_n)$. Then $a_n < x_* < b_n$ and

$$a_n^2 < x_*^2 < b_n^2.$$

Because of how each I_n was chosen ~~to contain $\sqrt{2}$~~, we are guaranteed that $a_n^2 < 2 < b_n^2$. Because the length of I_n approaches 0, we find that $a_n - b_n \to 0$. Therefore $\cancel{x_* = \sqrt{2}}$ $\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} b_n^2 = 2$, forcing

$$x_*^2 = 2.$$

There is no such rational number. □

4 Constructing $\mathbb{R}$ via equivalence classes of Cauchy sequences

Multiple sequences can approximate the same number. We need a way to check this that avoids circular reasoning, again without direct reference to the sequences' limit.

Definition 7 ($\approx$) We say $(n \mapsto x_n) \approx (n \mapsto y_n)$ (or briefly $x_n \approx y_n$) whenever

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

The goal is to use $\approx$ to collect together equivalent approximations into buckets. These buckets will become our construction of the real numbers. But how do we know they will split cleanly? The necessary property is that $\approx$ be an *equivalence relation.*

Definition 8 (Equivalence relation) An equivalence relation $\sim$ must satisfy:

1. reflexivity: $x \sim x$,
2. symmetry: $x \sim y$ is equivalent to $y \sim x$,
3. transitivity: if $x \sim y$ and $y \sim z$ then $x \sim z$.

Theorem 9 $\approx$ *is an equivalence relation.*

Proof.

1. Reflexivity follows from noting $d(x, x) = 0$. Therefore, if $n \mapsto x_n$ is a Cauchy sequence, $d(x_n, x_n) = 0$.

2. Symmetry follows from observing d is symmetric: $d(x, y) = d(y, x)$.
3. Transitivity is a consequence of the triangle inequality. If $x_n \approx y_n$, then $d(x_n, y_n) \to 0$. If $y_n \approx z_n$, then $d(y_n, z_n) \to 0$. Therefore

$$d(x_n, z_n) \leqslant d(x_n, y_n) + d(y_n, z_n) \to 0. \qquad \square$$

Now for the buckets:

Definition 10 (Equivalence class) If $n \mapsto x_n$ is a cauchy sequence, define its *equivalence class*

$$[n \mapsto x_n] \coloneqq \big\{ n \mapsto y_n : (n \mapsto x_n) \approx (n \mapsto y_n) \big\}.$$

The set of all such equivalence classes is written $\mathbb{Q}_C^{\mathbb{N}} / \approx$.

We can finally construct the reals!

Definition 11 The set of real numbers is the set of equivalence classes of rational cauchy sequences:

$$\mathbb{R} \coloneqq \mathbb{Q}_C^{\mathbb{N}} / \approx .$$

We would like to think of the real numbers as containing the rational numbers, so we need to find a copy of $\mathbb{Q}$ in $\mathbb{R}$.

Definition 12 Let $i : \mathbb{Q} \to \mathbb{R}$ via $i(q) = [\_ \mapsto q]$, the sequence that is constantly q.

The basic operations $+, \times, -, \div, |\cdot|$ extend to $\mathbb{R}$ similarly:

$$[n \mapsto x_n] + [n \mapsto y_n] \coloneqq [n \mapsto x_n + y_n].$$

Theorem 13 *The function* $i : \mathbb{Q} \mapsto \mathbb{R}$ *is an* isometry *(preserves distance).*

Proof. Consider rational numbers x and y. Then

$$d\Big(i(x), i(y)\Big) = \Big| [\_ \mapsto x] - [\_ \mapsto y] \Big| = \Big[\_ \mapsto |x - y| \Big] = i\Big(d(x, y)\Big) \qquad \square$$

Now we can define $\sqrt{2}$ by the equivalence class $[n \mapsto x_n]$ of the sequence $n \mapsto x_n$ found by bisection. We would expect in some sense

$$\lim_{n \to \infty} x_n = \sqrt{2}.$$

We can state this more precisely using $i : \mathbb{Q} \to \mathbb{R}$ as

$$\lim_{n \to \infty} i(x_n) = \sqrt{2}.$$

In general we expect

Theorem 14 *If* $n \mapsto x_n \in \mathbb{R} = \mathbb{Q}_C^{\mathbb{N}}$, *then*

$$\lim_{n\to\infty} i(x_n) = [n \mapsto x_n],$$

i.e. *Cauchy sequences converge to the real number they represent.*

Proof. Let $x_* = [n \mapsto x_n]$. It suffices to show

$$\lim_{k\to\infty} x_* - i(x_k) = 0.$$

Note that the 0 in the right hand side is the *real number* 0, *i.e.* $[\_ \mapsto 0]$. But

$$x_* - i(x_k) = [n \mapsto x_n] - [\_ \mapsto x_k] = [n \mapsto x_n - x_k];$$

therefore,

$$\lim_{k\to\infty} x_* - i(x_k) = \lim_{k\to\infty} [n \mapsto x_n - x_k].$$

Now it suffices to show

$$\lim_{k\to\infty} (n \mapsto x_n - x_k) \approx \_ \mapsto 0,$$

which follows from observing

$$\lim_{n\to\infty} \lim_{k\to\infty} x_n - x_k = 0$$

by hypothesis that $n \mapsto x_n$ is Cauchy. □

Theorem 15 *Every Cauchy sequence of real numbers converges to a real number.*

Proof. This follows by a diagonal argument. (To be continued) □