Definition 1 (nowhere dense) A nowhere dense set's closure has empty interior.

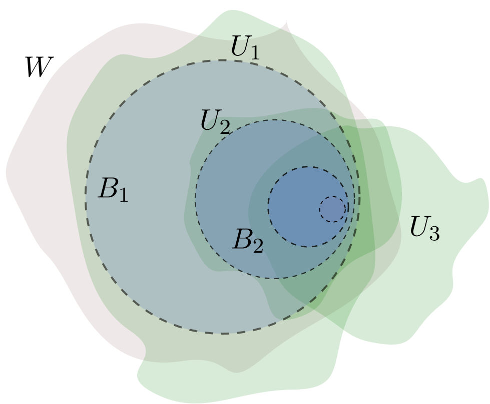
Definition 2 (meager) A meager set is countable union of nowhere dense sets

Theorem 3 (Baire category) *If X is an* completely metrizable *space,*

1. *countable intersections of* open *dense subsets are dense,*
2. *X is not meager.*

Proof. 1. Consider U_i, a sequence of open dense sets. It suffices to show a nonempty open set W meets $\bigcap U_i$. The proof proceeds by inductively[1] nesting open balls inside successive $W \cap U_i$. By assumption of density, $W \cap U_1$ is nonempty. It is also open, so there is some $B_1 = B(x_1, r_1)$ with $\overline{B_1} \subseteq W \cap U_0$. Observe $B_1 \cap U_1$ is also open; therefore, find a $B_2 = B(x_2, r_2)$ where $\overline{B_2} \subseteq B_1 \cap U_1$ and proceed inductively: $B_{n+1} = B(x_n, r_n)$ where $\overline{B_{n+1}} \subseteq B_n \cap U_n$. These are nested and

$$x_n \in \underbrace{B_n}_{\subseteq U_n} \subseteq \underbrace{B_{n-1}}_{\subseteq U_{n-1}} \subseteq \underbrace{B_2}_{\subseteq U_2} \subseteq \underbrace{B_1}_{\subseteq U_1} \subseteq W$$



If r_n are selected such that $r_n \to 0$, *e.g.*, by requiring $r_n < 2^{-n}$ at each step of the induction, then x_n is Cauchy. By completeness, there is a limit; call it x. But because x_i is eventuall in B_n, $x \in \overline{B(x_n, r_n)}$. But

$$x \in \overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \subseteq U_{n-1} \quad \text{for all } n.$$

Therefore, $x \in \bigcap U_n$. But $x \in W$ as well because every $\overline{B(x_n, r_n)} \in W$; hence $\bigcap U_n$ meets W.

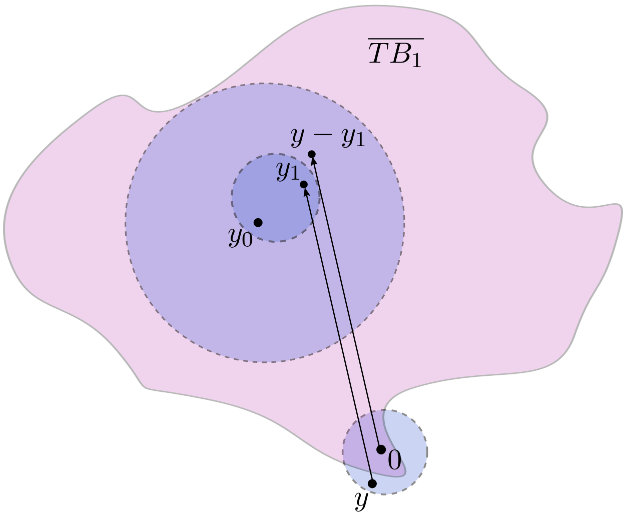
2. By the previous result, it suffices to show the complement of any nowhere dense set contains an open, dense set. Suppose N is nowhere dense. Then N^C is dense; hence, $(N^C)^{\circ}$ is open and dense. □

[1] In a general (nonseparable) metric space, this requires the axiom of dependent choice.

Theorem 4 (Open mapping theorem) *Surjective Banach-morphisms are open.*

Proof. Call the morphism $T : X \to Y$. Let $B_r = B(0,r) \in X$. Because linear maps commute with translations and dilations, it suffices to show that 0 is an interior point of $T(B_1)$. By surjectivity, $Y = \bigcup_{n=1}^{\infty} TB_n$. The map $n\square : (Y, TB_1) \to (Y, TB_n)$ is a homeomorphism for $n > 0$. Therefore, B_1 cannot be nowhere dense: if it were, B_n would be too, making Y meager, which contradicts theorem 3 (Baire category).

The closure $\overline{TB_1}$ contains an interior point y_0. For some r, $B(y_0, 4r) \subseteq \overline{TB_1}$. Then there is some $y_1 = Tx_1 \in TB_1$ within $2r$ of y_0. In particular, its neighborhood $B(y_1, 2r) \subseteq B(y_0, 4r) \subseteq \overline{TB_1}$



If $\|y\| < 2r$, then

$$y = y_1 + y - y_1 = Tx_1 + (y - y_1) \in \overline{T(x + B_1)} \subseteq \overline{TB_2}.$$

Halving this shows $\|y\| < r \implies y \in \overline{TB_1}$. In general, linearity tells us

$$\|y\| < r2^{-n} \implies y \in \overline{TB_{2^{-n}}} \tag{1}$$

Now, chase an arbitrary point y where $\|y\| < r/2$ with a sequence of Tx_i: eq. (1) guarantees an $x_1 \in B_1$ such that $\|y - Tx_1\| < r/4$, and, by induction, a sequence such that $\|y - \sum_1^n Tx_i\| < r2^{n-1}$ with $x_i \in B_i$. The series $\sum x_i$ is hence absolutely convergent; by completeness, it has some limit x. But

$$\|x\| \le \sum_i \|x_i\| = \sum_i 2^{-i} = 1$$

with $Tx = \lim T(\sum x_i) = y$ by construction. Therefore $y \in TB_1$ itself, not just its closure. Hence

$$\underbrace{\{y : \|y\| < r/2\}}_{\text{open nhood of } 0} \subseteq TB_1$$

$\square$