Illustrated companion for Folland's "The Baire Category Theorem and its Consequences"

Riley Levy

December 23, 2019

These notes are a companion for Section 5.3: "The Baire Category Theorem and its Consequences", in Folland's *Real Analysis: Modern Techniques and their Applications*.

Definition 1 (nowhere dense) A nowhere dense set's closure has empty interior.

Definition 2 (meager) A meager set is countable union of nowhere dense sets. [1, pg. 161]

Theorem 3 (Baire category) *If X is a* completely metrizable *space,*

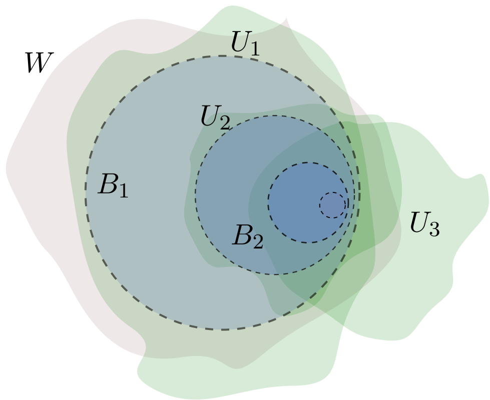
1. *countable intersections of* open *dense subsets are dense,*
2. *X is not meager.*

Proof.

1. Consider U_i, a sequence of dense open sets. It suffices to show a nonempty open set W meets $\bigcap U_i$. The proof proceeds by inductively[1] nesting open balls inside successive $W \cap U_i$. By assumption of density, $W \cap U_1$ is nonempty. It is also open, so there is some $B_1 = B(x_1, r_1)$ with $\overline{B_1} \subseteq W \cap U_0$. Observe $B_1 \cap U_1$ is also open; therefore, find a $B_2 = B(x_2, r_2)$ where $\overline{B_2} \subseteq B_1 \cap U_1$ and proceed inductively: $B_{n+1} = B(x_n, r_n)$ where $\overline{B_{n+1}} \subseteq B_n \cap U_n$. These are nested and

$$x_n \in \underbrace{B_n}_{\subseteq U_n} \subseteq \underbrace{B_{n-1}}_{\subseteq U_{n-1}} \subseteq \cdots \subseteq \underbrace{B_2}_{\subseteq U_2} \subseteq \underbrace{B_1}_{\subseteq U_1} \subseteq W$$

[1] In a general (nonseparable) metric space, this requires the axiom of dependent choice.



If r_n are selected such that $r_n \to 0$—*e.g.*, by requiring $r_n < 2^{-n}$ at each step of the induction—then x_n is Cauchy. By completeness, there is a limit; call it x. But because x_i is eventually in B_n, $x \in \overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1})$ for all n. Then

$$x \in \dots \underbrace{B_n}_{\subseteq U_n} \subseteq \underbrace{B_{n-1}}_{\subseteq U_{n-1}} \subseteq \dots \subseteq \underbrace{B_2}_{\subseteq U_2} \subseteq \underbrace{B_1}_{\subseteq U_1} \subseteq W;$$

therefore, x is in the intersection $\bigcap U_n \cap W$, proving it's nonempty.

2. By the previous result, it suffices to show the complement of any nowhere dense set contains an dense open set. Suppose N is nowhere dense. Then N^C is dense; hence, $(N^C)^{\circ}$ is open and dense. □

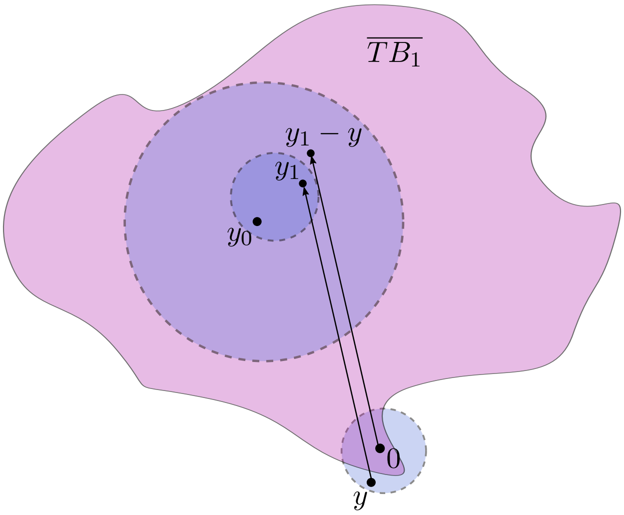
The proofs of theorem 4 (open mapping) and theorem 7 (uniform boundedness principle) both share a technique: find a bound around an arbitrary point (given by a density argument) and then translate this to a bound around the origin (via the triangle inequality and linearity).

Theorem 4 (open mapping) *Surjective Banach-morphisms are open.*

Proof. Call the morphism $T : X \to Y$. Let $B_r = B(0, r) \in X$. Because linear maps commute with translations and dilations, it suffices to show that 0 is an interior point of $T(B_1)$.

By surjectivity, $Y = \bigcup_{n=1}^{\infty} TB_n$. The map $n\square : (Y, TB_1) \to (Y, TB_n)$ is a homeomorphism for $n > 0$. Therefore, B_1 cannot be nowhere dense: if it were, B_n would be too, making Y meager, contradicting theorem 3 (Baire category).

The closure $\overline{TB_1}$ contains an interior point y_0. For some r, $B(y_0, 4r) \subseteq \overline{TB_1}$. Then there is some $y_1 = Tx_1 \in TB_1$ within $2r$ of y_0. In particular, its neighborhood $B(y_1, 2r) \subseteq B(y_0, 4r) \subseteq \overline{TB_1}$.



If $\|y\| < 2r$, then $y_1 - y \in \overline{TB_1}$ so

$$y = y_1 - (y_1 - y) = Tx_1 - (y_1 - y) \in \overline{T(x + B_1)} \subseteq \overline{TB_2}.$$

Halving this shows $\|y\| < r \implies y \in \overline{TB_1}$; repeatedly halving shows

$$\|y\| < r2^{-n} \implies y \in \overline{TB_{2^{-n}}}. \tag{1}$$

Now, chase an arbitrary y in $\{y : \|y\| < {}^{r}\!/\!{}_{2}\}$ with a sequence of Tx_i. Equation (1) guarantees an $x_1 \in B_1$ such that $\|y - Tx_1\| < {}^{r}\!/\!{}_{4}$, and, by induction, a sequence such that $\|y - \sum_1^n Tx_i\| < r2^{n-1}$ with $x_i \in B_i$. The series $\sum x_i$ is hence absolutely convergent; by completeness, it has some limit x. But

$$\|x\| \leq \sum_i \|x_i\| = \sum_i 2^{-i} = 1$$

with $Tx = \lim T(\sum x_i) = y$ by construction. Therefore $y \in TB_1$ itself, not just its closure. Conclude

$$\underbrace{\{y : \|y\| < {}^{r}\!/\!{}_{2}\}}_{\text{open nhood of } 0} \subseteq TB_1.$$

□

Because the inverse of an open map is continuous,

Corollary 5 (bounded inverse) *A bijective Banach-morphism is an isomorphism.*

Theorem 6 (Closed graphs) *A linear map between Banach spaces is continuous iff its graph is closed.*

Proof. Let $T : X \to Y$. Denote its graph by $\Gamma := \{(x, Tx) : x \in X\}$. Let π_X, π_Y indicate the projection maps from Γ to X and Y respectively.

Suppose T is continuous. Suppose (x_n, Tx_n) is a convergent sequence in Γ. Then there is some $x = \lim x_n$. But by continuity, $Tx_n \to Tx$ hence $(x_n, Tx_n) \to (x, Tx) \in \Gamma$, so Γ is closed.

Suppose Γ is closed. Then, as $X \times Y$ is complete, being a closed subset, Γ is complete as well. Then π_X is a bijective Banach-morphism so it is actually an isomorphism by corollary 5 (bounded inverse). Hence $T = \pi_Y \circ {\pi_X}^{-1} : X \to Y$ is a continuous. □

Theorem 7 (uniform boundedness principle)

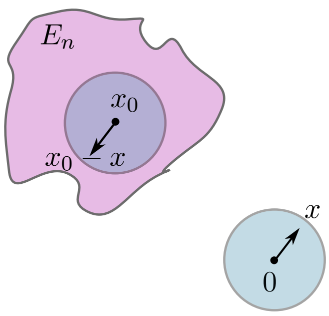
1. *If* $\sup_{T \in A} \|Tx\| < \infty$ *for all* x *in a nonmeager subset of* X*, then* $\sup_{T \in A} \|T\| < \infty$.
2. *If* X *is a Banach space and* $\sup_{T \in A} \|Tx\| < \infty$ *for all* $x \in X$*, then* $\sup_{T \in A} \|T\| < \infty$.

Proof. By theorem 3 (Baire category), a Banach space is nonmeager, hence (2) follows from (1). So it suffices to prove (1).

Let

$$E_n := \left\{ x \in X : \sup_{T \in A} \|Tx\| \leq n \right\} = \bigcap_{T \in A} \{x \in X : \|Tx\| \leq n\}.$$

The E_n are closed. By hypothesis of (1), some E_n is nonmeager; therefore it contains a nontrivial closed ball $\overline{B(r, x_0)}$. Now, we use this fact to construct a ball around 0 contained in E_{2n}. If $\|x\| \leq r$ then $x_0 - x \in E_n$:



hence

$$\|Tx\| \leq \|T(x - x_0)\| + \|Tx_0\| = \|T(x_0 - x)\| + \|Tx_0\| \leq 2n.$$

Therefore, $\|Tx\| \leq 2n$ whenever $T \in A$ and $\|x\| \leq r$. Therefore $\sup_{T \in A} \|T\| \leq {}^{2n}/_{r}$. □

References

[1] Gerald B. Folland. *Real analysis: modern techniques and their applications.* 2nd ed. Wiley-Interscience, 1999. ISBN: 0-471-31716-0.