Definition 1 (nowhere dense) A nowhere dense set's closure has empty interior.

Definition 2 (meager) A meager set is countable union of nowhere dense sets

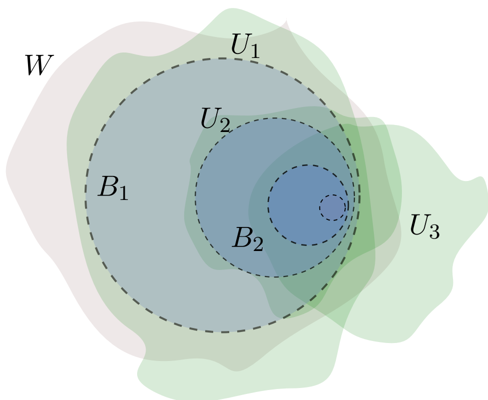
Theorem 3 (Baire category) *If X is a* completely metrizable *space,*

1. *countable intersections of* open *dense subsets are dense,*
2. *X is not meager.*

Proof.

1. Consider U_i, a sequence of open dense sets. It suffices to show a nonempty open set W meets $\bigcap U_i$. The proof proceeds by inductively[1] nesting open balls inside successive $W \cap U_i$. By assumption of density, $W \cap U_1$ is nonempty. It is also open, so there is some $B_1 = B(x_1, r_1)$ with $\overline{B_1} \subseteq W \cap U_0$. Observe $B_1 \cap U_1$ is also open; therefore, find a $B_2 = B(x_2, r_2)$ where $\overline{B_2} \subseteq B_1 \cap U_1$ and proceed inductively: $B_{n+1} = B(x_n, r_n)$ where $\overline{B_{n+1}} \subseteq B_n \cap U_n$. These are nested and

$$x_n \in \underbrace{B_n}_{\subseteq U_n} \subseteq \underbrace{B_{n-1}}_{\subseteq U_{n-1}} \subseteq \underbrace{B_2}_{\subseteq U_2} \subseteq \underbrace{B_1}_{\subseteq U_1} \subseteq W$$



If r_n are selected such that $r_n \to 0$, *e.g.*, by requiring $r_n < 2^{-n}$ at each step of the induction, then x_n is Cauchy. By completeness, there is a limit; call it x. But because x_i is eventuall in B_n, $x \in \overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1})$ for all n. Then

$$x \in \ldots \underbrace{B_n}_{\subseteq U_n} \subseteq \underbrace{B_{n-1}}_{\subseteq U_{n-1}} \subseteq \underbrace{B_2}_{\subseteq U_2} \subseteq \underbrace{B_1}_{\subseteq U_1} \subseteq W.$$

Therefore, $x \in \bigcap U_n \cap W$.

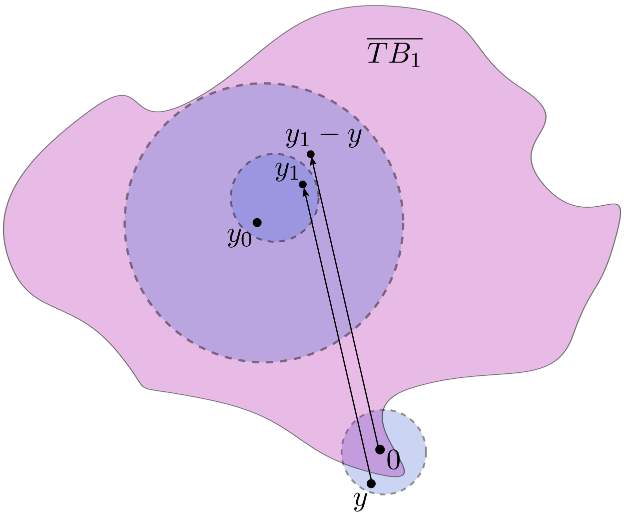
[1] In a general (nonseparable) metric space, this requires the axiom of dependent choice.

2. By the previous result, it suffices to show the complement of any nowhere dense set contains an open, dense set. Suppose N is nowhere dense. Then N^C is dense; hence, $(N^C)^\circ$ is open and dense. $\square$

Theorem 4 (Open mapping theorem) *Surjective Banach-morphisms are open.*

Proof. Call the morphism $T : X \to Y$. Let $B_r = B(0, r) \in X$. Because linear maps commute with translations and dilations, it suffices to show that 0 is an interior point of $T(B_1)$. By surjectivity, $Y = \bigcup_{n=1}^{\infty} TB_n$. The map $n\square : (Y, TB_1) \to (Y, TB_n)$ is a homeomorphism for $n > 0$. Therefore, B_1 cannot be nowhere dense: if it were, B_n would be too, making Y meager, which contradicts theorem 3 (Baire category).

The closure $\overline{TB_1}$ contains an interior point y_0. For some r, $B(y_0, 4r) \subseteq \overline{TB_1}$. Then there is some $y_1 = Tx_1 \in TB_1$ within $2r$ of y_0. In particular, its neighborhood $B(y_1, 2r) \subseteq B(y_0, 4r) \subseteq \overline{TB_1}$



If $\|y\| < 2r$, then $y_1 - y \in \overline{TB_1}$ so

$$y = y_1 - (y_1 - y) = Tx_1 - (y_1 - y) \in \overline{T(x + B_1)} \subseteq \overline{TB_2}.$$

Halving this shows $\|y\| < r \implies y \in \overline{TB_1}$; repeatedly halving shows

$$\|y\| < r2^{-n} \implies y \in \overline{TB_{2^{-n}}}. \tag{1}$$

Now, chase an arbitrary y in $\{y : \|y\| < r/2\}$ with a sequence of Tx_i. Equation (1) guarantees an $x_1 \in B_1$ such that $\|y - Tx_1\| < r/4$, and, by induction, a sequence such that $\|y - \sum_1^n Tx_i\| < r2^{n-1}$ with $x_i \in B_i$. The series $\sum x_i$ is hence absolutely convergent; by completeness, it has some limit x. But

$$\|x\| \leq \sum_i \|x_i\| = \sum_i 2^{-i} = 1$$

with $Tx = \lim T(\sum x_i) = y$ by construction. Therefore $y \in TB_1$ itself, not just its closure. Hence

$$\underbrace{\{y : \|y\| < r/2\}}_{\text{open nhood of } 0} \subseteq TB_1$$

$\square$

Because the inverse of an open map is continuous,

Corollary 5 *A bijective Banach-morphism is an isomorphism.*