1 Ordinals and transfinite induction

Definition 1 (The ordinals, `Ord`) These are defined inductively:

- $0 := \varnothing \in \texttt{Ord}$
- If $n \in \texttt{Ord}$ then
$$\operatorname{succ}(n) := n \cup \{n\} \in \texttt{Ord}$$
- If $\mathcal{X} \subseteq \texttt{Ord}$ is a set, then
$$\sup \mathcal{X} := \bigcup \mathcal{X} \in \texttt{Ord}$$

In other words, an ordinal is the set of all smaller ordinals. In particular, the last definition precludes the ordinals from being a set. `Ord` is bigger than any set. They form a class.

If $\alpha = \operatorname{succ}(n)$ then α is a *successor ordinal*. Otherwise, α is a *limit ordinal*.

Theorem 2 (`Ord` is well-ordered) *Every nonempty class of ordinals contains a least element.*

Proof. Let $\mathcal{X}$ be a nonempty class of ordinals. Then it suffices to show

$$I := \inf \mathcal{X} = \bigcap \mathcal{X} \in \mathcal{X}$$

Suppose $I \notin \mathcal{X}$. Then for any $x \in \mathcal{X}$, $I < x$. Consequently, for any $x \in \mathcal{X}$, $\operatorname{succ}(I) \leq x$. But then $\operatorname{succ}(I) = I$, a contradiction. □

Theorem 3 (Transfinite induction) *For a logical statement ϕ, if*

1. $\phi(0)$ *holds,*
2. $\phi(n) \implies \phi(\operatorname{succ}(n))$ [1]
3. *If, for all $n < \alpha$, $\phi(n)$ is true then $\phi(\alpha)$ is true:*
$$(\forall n < \alpha)\phi(n) \implies \phi(\alpha)$$

Then ϕ holds for all ordinals.

Proof. Let $\mathcal{X}$ be the class where ϕ does not hold. Suppose $\mathcal{X} \neq \varnothing$. Then let for all ordinals $n < \inf \mathcal{X}$, $\phi(n)$ by hypothesis. Hence, by (3), $\phi(\inf \mathcal{X})$ holds. But then $\inf \mathcal{X} \notin \mathcal{X}$, contradicting theorem 2 (`Ord` is well-ordered). Therefore, $\mathcal{X}$ is empty. □

[1] strictly speaking, this follows from (1) and (3) but is usually easier to prove first

Theorem 4 (Ord fixed point theorem) *Let $(X, \leq)$ be a complete ordered space. Suppose $F : X \to X$ is nondecreasing. Define*

$$F^0(x) := x$$
$$F^{\mathrm{succ}(n)}(x) := F(F^n(x))$$

If α is a limit ordinal,

$$F^\alpha(x) := \sup_{\beta<\alpha} F^\beta(x)$$

Then for any $x \in X$, $F^\alpha(x)$ is eventually the smallest fixed point of F greater than or equal to x.

Proof. Fix x. Then $F^\square(x) : \mathtt{Ord} \to X$ is nondecreasing by construction. Because $\mathtt{Ord}$ is bigger than any set, $F^\square(x)$ cannot be injective. Hence there is some smallest $\alpha \in \mathtt{Ord}$ such that $F^\alpha(x) = F^\gamma(x)$ with $\gamma > \alpha$. But $\alpha < \mathrm{succ}(\alpha) \leq \beta$, so by monotonicity,

$$F^\alpha(x) \leq F^{\mathrm{succ}(\alpha)}(x) \leq F^\beta(x) = F^\alpha(x)$$

hence

$$F(F^\alpha(x)) = F^{\mathrm{succ}(\alpha)}(x) = F^\alpha(x)$$

so F fixes $F^\alpha(x)$.

Suppose y is a fixed point and $x \leq y \leq F^\alpha(x)$. Then there is some smallest β such that $y \leq F^\beta(x)$:

- If $\beta = \mathrm{succ}(\gamma)$, then
$$F^\gamma(x) \leq y \leq f(F^\gamma(x)) = F^\beta(x)$$
But by monotonicity, $F(F^\gamma(x)) \leq f(y) = y$ by assumption y is fixed. Thus $y = F^\gamma(x)$ for some $\gamma \leq \alpha$.

- If β is a limit ordinal, then for all $\gamma < \beta$,
$$F^\gamma(x) \leq y \leq F^\beta(x) = \sup_{\hat{\gamma}<\beta} F^{\hat{\gamma}}(x) = y$$
so $F^\beta(x) = y$. □

By characterizing σ-algebras and λ-systems as fixed points of set functions, this result becomes a powerful tool for proving results about generated σ-algebras and λ-systems.

2 π–λ systems and theorem

Special notation:

- I'll use $\amalg$ to represent *disjoint* unions,
- $\bigcup\Uparrow$ to represent *increasing* unions,

- and $\underset{\supseteq}{\setminus}$ to represent *proper* relative complements: $A \setminus B$ when $B \subseteq A$.

Definition 5 (λ-system aka Dynkin system) $\mathcal{D}$ is a Dynkin system on Ω iff

- $\Omega \in \mathcal{D}$
- Closure under *proper* relative complements:

$$A \subseteq B \implies B \underset{\supseteq}{\setminus} A \in \mathcal{D}$$

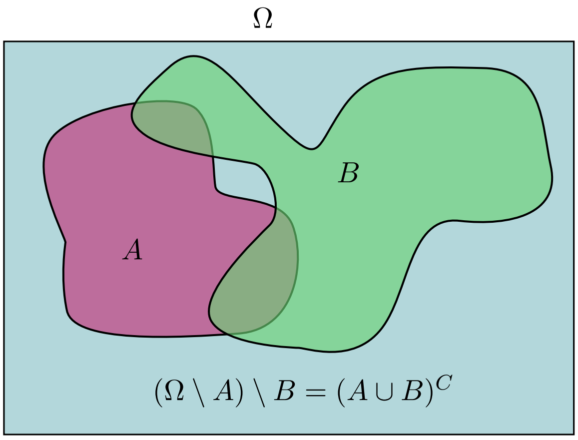
- Closure under *increasing* σ-unions:

$$A_1 \subseteq A_2 \subseteq \cdots \in \mathcal{D} \implies \bigcup\Uparrow A_n \in \mathcal{D}$$

Allowing arbitrary complements would give a σ-algebra as

$$(\Omega \setminus A) \setminus B = (A \cup B)^C = A^C \cap B^C$$

allowing arbitrary σ–unions and intersections.



Theorem 6 (Alternative definition of λ-system) *Equivalently,*

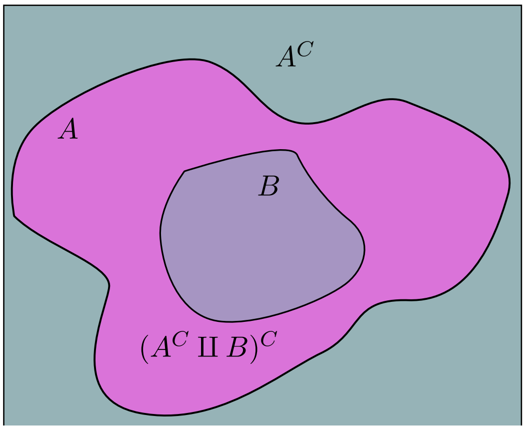
- $\Omega \in \mathcal{D}$
- *Closure under* global *complements:*

$$A \in \mathcal{D} \implies A^C \in \mathcal{D}$$

- *Closure under* disjoint *σ-unions:*

$$\{A_i \in \mathcal{D}\}, A_i \cap A_j = \varnothing \implies \coprod A_i \in \mathcal{D}$$

Proof. Suppose $\mathcal{D}$ meets the criteria of theorem 6 (Alternative definition of λ-system). Then $\mathcal{D}$ contains proper relative complements as $A \mathbin{\underset{\supseteq}{\setminus}} B = (A^C \cup B)^C$:



But by taking proper relative complements, we can convert an ascending σ-union into a disjoint σ-union: if $A_n \uparrow A$, let

$$\begin{aligned} B_1 &= A_1 \in \mathcal{D} \\ B_n &= A_n \mathbin{\underset{\supseteq}{\setminus}} \bigcup_{i<n} A_i \in \mathcal{D} \end{aligned}$$

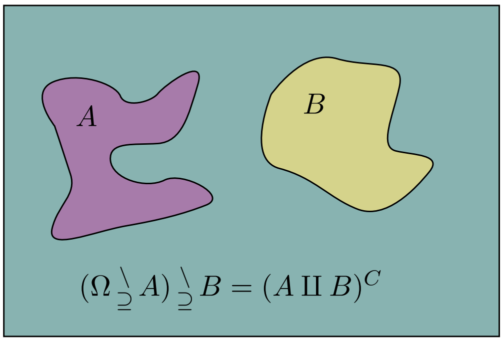
Then $\{B_n\}$ are pairwise-disjoint and

$$A = \bigcup\Uparrow A_n = \coprod_n B_n \in \mathcal{D}$$

proving sufficiency.

For necessity, suppose $\mathcal{D}$ satisfies the criteria of definition 5 (λ-system aka Dynkin system). As $A^C = \Omega \setminus A$, a proper complement, it suffices to show closure under disjoint unions. First, note that it is closed under finite disjoint unions:

$$A \amalg B = ((\Omega \mathbin{\underset{\supseteq}{\setminus}} A) \mathbin{\underset{\supseteq}{\setminus}} B)^C \in \mathcal{D}$$



because the complements are proper by disjointness. Suppose B_i are pairwise disjoint. Then $B = \coprod B_i \in \mathcal{D}$ by hypothesis.

$$\begin{aligned} A_1 &= B_1 \in \mathcal{D} \\ A_n &= B_n \amalg A_{n-1} \in \mathcal{D} \end{aligned}$$

therefore

$$B = \coprod B_n = \bigcup\Uparrow A_n \in \mathcal{D} \qquad \square$$

Definition 7 (π-system) A π-system is closed under finite intersections.

Theorem 8 *If $\mathcal{D}$ is a π-system and λ-system then it is a σ-algebra.*

Proof. First, prove $\mathcal{D}$ is an algebra. Note that $\mathcal{D}$ is closed under binary union as $A \cup B = (A^C \cap B^C)^C$ and $A^C, B^C \in \mathcal{D}$ because $\mathcal{D}$ is a λ-system, so $A^C \cap B^C \in \mathcal{D}$ because it is a π-system, and its complement is in $\mathcal{D}$ by closure under complements. In particular, $\mathcal{D}$ must be closed under *arbitrary* relative complements as

$$A \setminus B = A \underset{\supseteq}{\setminus} (A \cap B)$$

Suppose A_i is a countable sequence of sets in $\mathcal{D}$. Then let

$$\begin{aligned} B_1 &= A_1 \\ B_n &= A_n \setminus \coprod_{i<n} B_i \end{aligned}$$

so B_i are pairwise-disjoint and

$$\bigcup A_n = \coprod B_n \in \mathcal{D}$$

proving closure under σ-unions. $\square$

Theorem 9 (Dynkin's π–λ theorem) *If $P \subseteq \mathcal{D}$ where π is a π-system and $\mathcal{D}$ is a λ-system, then*

$$\sigma(P) \subseteq \mathcal{D}$$

Proof. As $\lambda(P)$ is the minimal λ-system containing P,

$$\lambda(P) \subseteq \mathcal{D}$$

It suffices to show $\lambda(P)$ is a π-system because then it must be a σ-algebra.

Let $U(X)$ indicate the set of disjoint σ-unions in X:

$$U(X) \coloneqq \left\{ \coprod_{i\in\mathbb{N}} A_i : A_i \in X, A_i \cap A_j = \varnothing \right\}$$

Let $C(X)$ indicate the set of complements in X:

$$C(X) \coloneqq \left\{ A^C : A \in X \right\}$$

And define

$$\Lambda(X) \coloneqq X \cup U(X) \cup C(X) = U(X) \cup C(X) : (2^{2^\Omega}, \subseteq) \to (2^{2^\Omega}, \subseteq)$$

By construction, $\Lambda(X) = X$ iff X is a λ-system. By monotonicity ($X \subseteq \Lambda(X)$), the minimum fixed point of Λ containing X is the λ-system generated by X. By theorem 4 (`Ord` fixed point theorem),

$$\Lambda^{\omega}(P) = \lambda(P)$$

for some $\omega \in \mathtt{Ord}$.

We will show by induction that for any $\alpha \in \mathtt{Ord}$, the intersection of any two elements in $\Lambda^{\alpha}(P)$ is in $\Lambda^{3\alpha}(P)$.

1. This is trivially true for $\alpha = 0$ by the hypothesis that P is a π-system.

2. Any element in $\Lambda^{\mathrm{succ}(\alpha)}(P)$ can be written as A, A^C, or $\coprod A_i$ with $A, A_i \in \Lambda^{\alpha}$. Consider writing an intersection of elements in $\Lambda^{\mathrm{succ}(\alpha)}$ in terms of proper complements and disjoint unions of intersections of elements in $\Lambda^{\alpha}(P)$:

$$\begin{aligned}
&A \cap B^C && = A \mathbin{\underset{\supseteq}{\setminus}} \overbrace{(B \cap A)}^{\in \Lambda^{3\alpha}(P)} && \in \Lambda^{3\alpha+1}(P) \\
&A^C \cap B^C && = (A \cup B)^C = \Big((A \cap B^C) \amalg B\Big)^C && \in \Lambda^{3\alpha+3}(P) \\
&\left(\coprod_i A_i\right) \cap \left(\coprod_j B_j\right) && = \coprod_{i,j} \overbrace{A_i \cap B_j}^{\in \Lambda^{3\alpha}(P)} && \in \Lambda^{3\alpha+1}(P) \\
&A^C \cap \coprod B_j && = \coprod \underbrace{A^C \cap B_j}_{\in \Lambda^{3\alpha+1}(P)} && \in \Lambda^{3\alpha+2}(P)
\end{aligned}$$

   which are all in $\Lambda^{3\alpha+3}(P) = \Lambda^{3\,\mathrm{succ}(\alpha)}(P)$.

3. If α is a limit ordinal, then set

$$3\alpha \geq 3 \sup_{\beta<\alpha} \beta = \sup_{\beta<\alpha} 3\beta$$

   For any two $A, B \in \Lambda^{\alpha}$ there must be some $n_A, n_B < \alpha$ such that $A \in \Lambda^{n_A}(P)$ and $B \in \Lambda^{n_B}(P)$. Hence, if $n = \max(n_A, n_B)$, then $A \cap B \in \Lambda^{3n}(P) \subseteq \Lambda^{3\alpha}(P)$.

But this process terminates at ω, hence $\Lambda^{3\omega}(P) = \Lambda^{\omega}(P) = \lambda(P)$ and $\lambda(P)$ is closed under intersections. $\square$