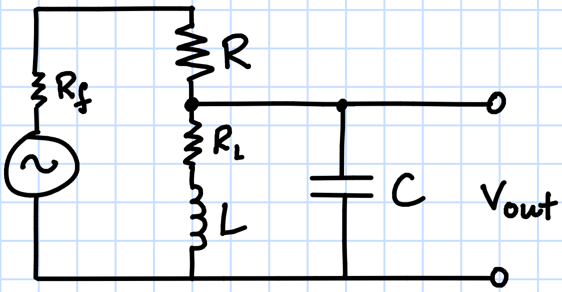

$R_f \approx 50\,\Omega$ and $R \approx 1\,M\Omega$

$\Rightarrow R_f \approx 0$ is a very good approximation.

R_L is small, but X_L can also be small at low frequency, so we need to include R_L.

$Z_1 = R$

$$Z_2 = \left[\frac{1}{j\omega L + R_L} - \frac{\omega C}{j}\right]^{-1} = j\left[\frac{1 - \omega C(\omega L - jR_L)}{\omega L - jR_L}\right]^{-1}$$

$$= j\,\frac{\omega L - jR_L}{1 - (\omega^2 LC - j\omega R_L C)} = \frac{R_L - j\omega L}{(1 - \omega^2 LC) - j\omega R_L C}$$

Since Z_1 is real, let's go ahead and put Z_2 in $A + jB$ form.

$$Z_2 = \frac{(R_L - j\omega L)[1 - \omega^2 LC + j\omega R_L C]}{(1 - \omega^2 LC)^2 + (\omega R_L C)^2}$$

$$= \frac{R_L(1 - \cancel{\omega^2 LC}) + \cancel{\omega^2 LC} R_L - j\omega R_L^2 C - j\omega L(1 - \omega^2 LC)}{(1 - \omega^2 LC)^2 + (\omega R_L C)^2}$$

$$= \frac{R_L - j[\omega C R_L^2 + \omega L(1 - \omega^2 LC)]}{(1 - \omega^2 LC)^2 + (\omega C R_L)^2}$$

$$\boxed{A = \frac{R_L}{(1 - \omega^2 LC)^2 + (\omega C R_L)^2} \qquad B = -\frac{[\omega C R_L^2 + \omega L(1 - \omega^2 LC)]}{(1 - \omega^2 LC)^2 + (\omega C R_L)^2} \qquad \frac{Z_2}{Z_1 + Z_2} = \frac{A + jB}{R + A + jB}}$$

$$\boxed{A_v = \sqrt{\frac{A^2 + B^2}{(R + A)^2 + B^2}}}$$

If $R_L = 0$, $A = 0$, $B = \dfrac{\omega L}{1 - \omega^2 LC}$

$$A_v = \sqrt{\frac{\frac{\omega^2 L^2}{(1 - \omega^2 LC)^2}}{R^2 + \frac{\omega^2 L^2}{(1 - \omega^2 LC)^2}}} = \frac{1}{\sqrt{1 + \frac{R^2}{\omega^2 L^2}(1 - \omega^2 LC)}} \quad ✓$$

Now, we need to do the trick we use to find the phase angle:

$$\frac{Z_2}{Z_1+Z_2} = \frac{A+jB}{R+A+jB} = \frac{(A+jB)(R+A-jB)}{\text{whatever}}$$

$$= \frac{A(R+A)+B^2 + j[B(R+\cancel{A})-\cancel{AB}]}{\text{whatever}}$$

$$\tan\phi = \frac{Im\left\{\frac{Z_2}{Z_1+Z_2}\right\}}{Re\left\{\frac{Z_2}{Z_1+Z_2}\right\}}$$

The trick here is that we only need the imaginary and real parts of the numerator. The whatever cancels out.

$$\boxed{\tan\phi = \frac{BR}{A(R+A)+B^2}}$$

$$\left(\begin{array}{l} \text{If } R_L=0,\ A=0,\ B=\frac{\omega L}{1-\omega^2 LC} \\ \tan\phi = \frac{BR}{B^2} = \frac{R}{B} = \frac{R}{\omega L}(1-\omega^2 LC) \end{array}\right)$$

Define A, B, Av, and phi Python functions to facilitate plotting.

Find ω_0 and the 3dB points numerically, with a cursor on an interactive plot.