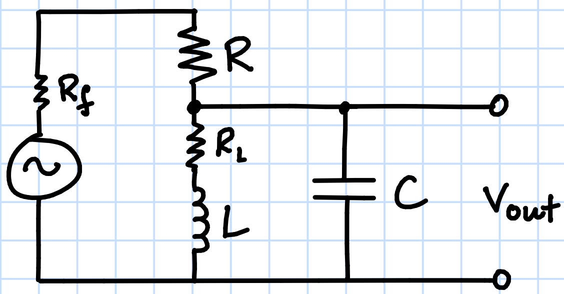

$R_f \approx 50\,\Omega$ and $R \approx 1\,M\Omega$

$\Rightarrow R_f \approx 0$ is a very good approximation.

R_L is small, but X_L can also be small at low frequency, so we need to include R_L.

$Z_1 = R$

$$Z_2 = \left[\frac{1}{j\omega L + R_L} - \frac{\omega C}{j}\right]^{-1} = \left[\frac{j - \omega C(j\omega L + R_L)}{-\omega L + jR_L}\right]^{-1}$$

$$= \frac{-\omega L + jR_L}{-\omega R_L C + j(1-\omega^2 LC)}$$

Since Z_1 is real, let's go ahead and put Z_2 in $A + jB$ form.

$$Z_2 = \frac{(-\omega L + jR_L)[-\omega R_L C - j(1-\omega^2 LC)]}{(1-\omega^2 LC)^2 + (\omega R_L C)^2}$$

$$= \frac{-\cancel{\omega^2 LCR_L} + R_L(1-\cancel{\omega^2 LC}) - j\omega R_L^2 C + j\omega L(1-\omega^2 LC)}{(1-\omega^2 LC)^2 + (\omega R_L C)^2}$$

$$= \frac{R_L + j[\omega L(1-\omega^2 LC) - \omega R_L^2 C]}{(1-\omega^2 LC)^2 + (\omega R_L C)^2}$$

$$\boxed{A = \frac{R_L}{(1-\omega^2 LC)^2 + (\omega R_L C)^2} \qquad B = \frac{[\omega L(1-\omega^2 LC) - \omega R_L^2 C]}{(1-\omega^2 LC)^2 + (\omega R_L C)^2} \qquad \frac{Z_2}{Z_1 + Z_2} = \frac{A + jB}{R + A + jB}}$$

$$\boxed{A_v = \sqrt{\frac{A^2 + B^2}{(R+A)^2 + B^2}}}$$

(If $R_L = 0$, $A = 0$, $B = \frac{\omega L}{1-\omega^2 LC}$

$$A_v = \sqrt{\frac{\frac{\omega^2 L^2}{(1-\omega^2 LC)^2}}{R^2 + \frac{\omega^2 L^2}{(1-\omega^2 LC)^2}}} = \frac{1}{\sqrt{1 + \frac{R^2}{\omega^2 L^2}(1-\omega^2 LC)}} \quad \checkmark$$
)

Now, we need to do the trick we use to find the phase angle:

$$\frac{Z_2}{Z_1 + Z_2} = \frac{A + jB}{R + A + jB} = \frac{(A + jB)(R + A - jB)}{\text{whatever}}$$

$$= \frac{A(R+A) + B^2 + j[B(R+\cancel{A}) - \cancel{AB}]}{\text{whatever}}$$

$$\tan\phi = \frac{\mathrm{Im}\left\{\frac{Z_2}{Z_1+Z_2}\right\}}{\mathrm{Re}\left\{\frac{Z_2}{Z_1+Z_2}\right\}}$$

The trick here is that we only need the imaginary and real parts of the numerator. The whatever cancels out.

$$\boxed{\tan\phi = \frac{BR}{A(R+A) + B^2}}$$

$$\left(\begin{array}{l} \text{If } R_L = 0,\ A = 0,\ B = \frac{\omega L}{1 - \omega^2 LC} \\ \quad \tan\phi = \frac{BR}{B^2} = \frac{R}{B} = \frac{R}{\omega L}(1 - \omega^2 LC) \end{array} \right)$$

Define A, B, Av, and phi Python functions to facilitate plotting.

Find ω_0 and the 3dB points numerically, with a cursor on an interactive plot.