Calculating Determinants

- The determinant of a 2×2 matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{1}$$

is given by

$$|M| = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}\, m_{22} - m_{12}\, m_{21} \tag{2}$$

- The determinant of a 3×3 matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \tag{3}$$

can be written in terms of the determinants of 2×2 sub-matrices

$$\begin{aligned} |M| &= \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} \\ &= m_{11} \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} - m_{12} \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} + m_{13} \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} \end{aligned} \tag{4}$$

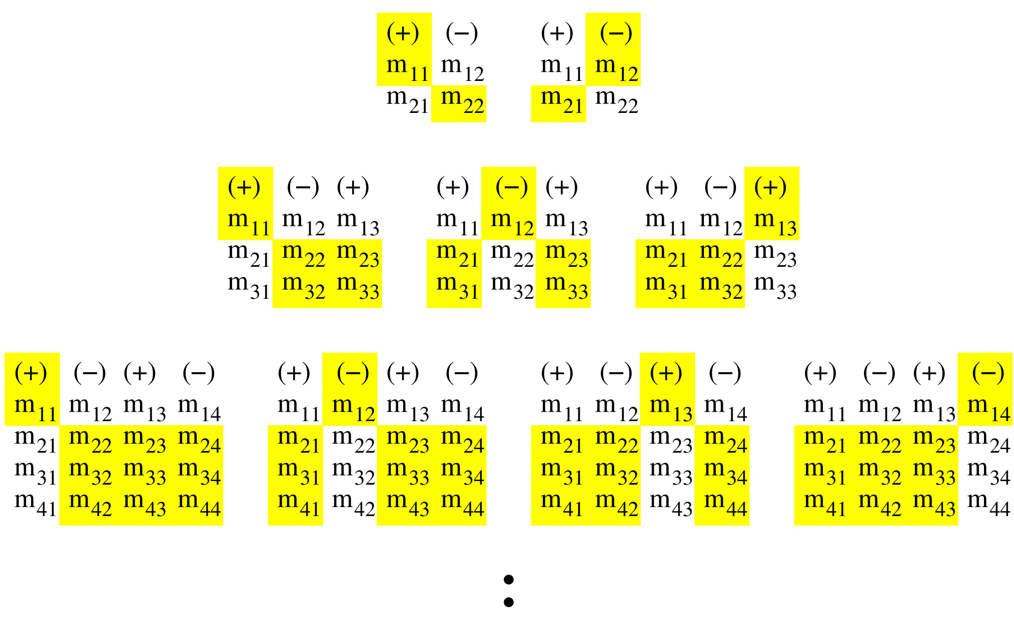

Figure 1: A visual guide to computing the determinants of 2×2, 3×3, and 4×4 matrices.

- In general, each element of the top row of the matrix is multiplied by the determinant of the sub-matrix obtained by removing the row and column containing that element. The results are then added together with alternating sign, starting with a positive m_{11} term. Figure 1 shows the top-row elements and the associated sub-matrices and signs for the 2×2, 3×3, and 4×4 cases.

An $\mathbf{n = 2}$ Example

$$
\begin{aligned}
M &= \begin{pmatrix} 3 & -1 \\ 5 & 7 \end{pmatrix} \\
|M| &= 3 \cdot 7 - (-1) \cdot 5 = 21 + 5 && (5) \\
&= 26 && (6)
\end{aligned}
$$

An $\mathbf{n = 3}$ Example

$$
\begin{aligned}
M &= \begin{pmatrix} 4 & 2 & 5 \\ -1 & 6 & 7 \\ 3 & 1 & 2 \end{pmatrix} \\
|M| &= 4 \begin{vmatrix} 6 & 7 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} -1 & 7 \\ 3 & 2 \end{vmatrix} + 5 \begin{vmatrix} -1 & 6 \\ 3 & 1 \end{vmatrix} \\
&= 4\,(6 \cdot 2 - 7 \cdot 1) - 2\,((-1) \cdot 2 - 7 \cdot 3) + 5\,((-1) \cdot 1 - 6 \cdot 3) \\
&= 20 + 46 - 95 \\
&= -29 && (7)
\end{aligned}
$$

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