Review of Complex Numbers

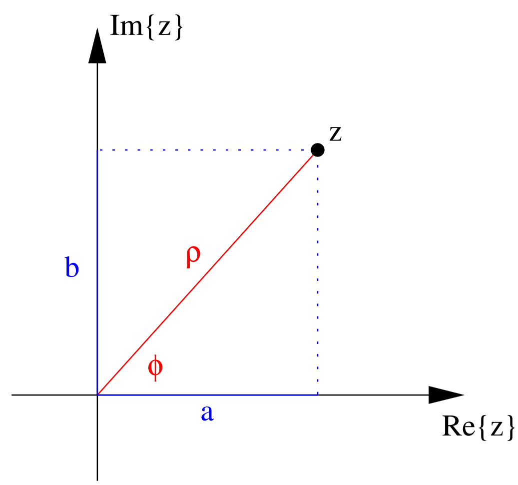
Cartesian Form and the Complex Plane

- Complex numbers and functions contain the number $i = \sqrt{-1}$.
- Any complex number or function z can be written in **Cartesian form**,

$$z = a + ib \tag{1}$$

where a is the **real part** of z and b is the **imaginary part** of z, often denoted $a = Re\{z\}$ and $b = Im\{z\}$, respectively. Note that a and b are both real numbers.

- The form of Eq. 1 is called Cartesian, because if we think of z as a two dimensional vector and $Re\{z\}$ and $Im\{z\}$ as its components, we can represent z as a point on the **complex plane**.



Polar Form

- As with a two dimensional vector, a complex number can be written in a second form, as a magnitude ρ and angle ϕ,

$$\begin{aligned} \rho &= \sqrt{a^2 + b^2} \\ \tan\phi &= \frac{b}{a} \quad (+\pi \text{ if } a < 0) \qquad (2) \\ a &= \rho\cos\phi \\ b &= \rho\sin\phi. \qquad (3) \end{aligned}$$

where ϕ is called the **complex phase** of z.

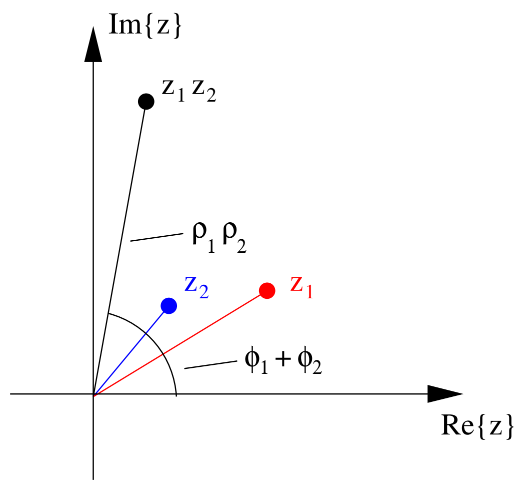
Exponential Form

- Euler's formula relates a complex number on the unit circle expressed in terms of trigonometric functions to the complex exponential function.

$$e^{\pm i\phi} = \cos\phi \pm i\sin\phi. \tag{4}$$

  This can be shown by comparing the Taylor series expansions of $e^{i\phi}$, $\cos\phi$, and $\sin\phi$. It follows that a complex number z can be written in a third form,

$$z = \rho e^{i\phi}. \tag{5}$$



- Eq. 5 provides a useful way of looking at multiplication of complex numbers. The product $z_1 z_2$ is obtained by multiplying magnitudes and adding complex phases,

$$z_1 z_2 = \rho_1 \rho_2 e^{i(\phi_1+\phi_2)}. \tag{6}$$

- Raising complex numbers to powers is also simplified by Eq. 5,

$$(z)^p = \rho^p e^{ip\phi}. \tag{7}$$

  For example, we can evaluate $(i+1)^4$, noting that

$$1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

  and using Eq. 7, we find

$$\begin{aligned}(1+i)^4 &= (\sqrt{2})^4\,(e^{i\frac{\pi}{4}})^4 = 4\,e^{i\pi} \\ &= -4 \end{aligned} \tag{8}$$

Complex Conjugation and the Complex Square

- The **complex conjugate** of $z = a + ib = \rho e^{i\phi}$ is

$$z^* = a - ib = \rho e^{-i\phi}.$$

  It is obtained by changing the sign of i wherever it appears in z.

  - To calculate the magnitude ρ directly from z written in any form, we use the **complex square**,

$$|z|^2 = z^* z$$

    The complex square in terms of a and b is

$$\begin{aligned} |z|^2 &= (a + ib)(a - ib) = a^2 + iba - iab - (i^2)b^2 \\ &= a^2 + b^2 = \rho^2 \end{aligned} \tag{9}$$

    and in terms of ρ and ϕ

$$|z|^2 = \rho e^{-i\phi} \rho e^{i\phi} = \rho^2.$$

    Hence,

$$\rho = \sqrt{|z|^2}. \tag{10}$$

  - We can also use complex conjugation to separate the real and imaginary parts of z.

$$z + z^* = a + ib + a - ib = 2a$$

    so

$$Re\{z\} = \frac{z + z^*}{2} \tag{11}$$

    similarly

$$Im\{z\} = \frac{z - z^*}{2i} \tag{12}$$

    For example, it follows from Eq.'s 11 and 12 together with Eq. 4 that

$$\begin{aligned} Re\{e^{i\phi}\} &= \cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2} \\ Im\{e^{i\phi}\} &= \sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i} \end{aligned} \tag{13}$$

Finding Roots

- $\sqrt[n]{z}$ has n unique values for integer n. For example, $\sqrt{4} = +2, -2$. In general, some or all of the n roots are complex numbers.

- The cyclical nature of angles means that

$$z = \rho e^{i\phi},\ \rho e^{i(\phi+2\pi)},\ \rho e^{i(\phi+4\pi)},\ \rho e^{i(\phi+6\pi)}, \ldots$$

  all represent the same number.

- However, if we take the nth root of these representations of z, we find that there are n unique results with complex phase angles less than 2π.

- **Example 1**

  - The first 6 representations of $z = 8$ are

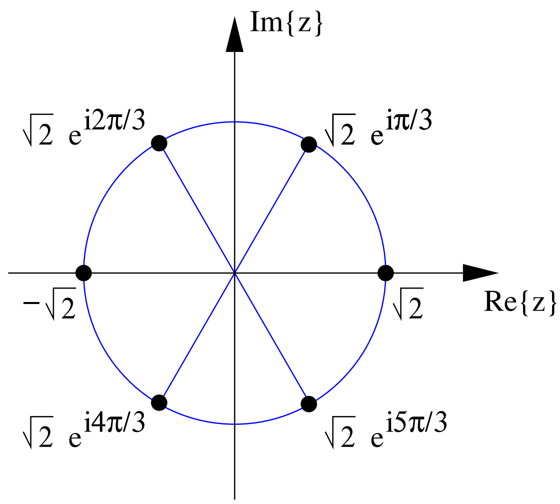
    $$\begin{aligned} 8 &= 8,\, 8e^{i2\pi},\, 8e^{i4\pi}, \\ &\quad 8e^{i6\pi},\, 8e^{i8\pi},\, 8e^{i10\pi}. \end{aligned} \tag{14}$$

    Taking the 6th root, we obtain

    $$\begin{aligned} \sqrt[6]{8} &= \sqrt{2},\, \sqrt{2}e^{i\pi/3},\, \sqrt{2}e^{i2\pi/3}, \\ &\quad \sqrt{2}e^{i\pi},\, \sqrt{2}e^{i4\pi/3},\, \sqrt{2}e^{i5\pi/3} \end{aligned} \tag{15}$$

    The rest of the roots have complex phase $\geq 2\pi$ and all of them are alternate representations of the six roots above.

  - Graphically,



- In general, to find the n roots of a number $z = \rho e^{i\phi}$, start with $\sqrt[n]{\rho}e^{i\phi/n}$. The remaining roots lie, along with the first, on a circle of radius $\sqrt[n]{\rho}$ in the complex plane at an equal spacing of $2\pi/n$ in phase angle.

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