The Quadratic Formula

- Equations of the form
$$ax^2 + bx + c = 0 \tag{1}$$
are called **quadratic equations**. The **quadratic formula**
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{2}$$
gives the solutions.

- The $\pm$ in Eq. 2 is an important detail. In general, there are two solutions to a quadratic equation. The two solutions are also called the **roots of the equation**.

- **Example**

  A ball is tossed directly upward from a height of 2.0 m above the ground with an initial velocity of 5.0 m/s. It is subject only to the force of gravity while in flight, so it has an acceleration of -9.8 m/s^2. When does the ball reach a height of 3.0 m? The position of the ball as a function of time is given by the equation
$$y = y_o + v_o t + \frac{1}{2} a t^2.$$

  Solution

  The equation with the given information is
$$3.0 \text{ m} = 2.0 \text{ m} + (5.0 \text{ m/s})t + (-4.9 \text{ m/s}^2)t^2$$
which in the form of Eq. 1 becomes
$$(-4.9 \text{ m/s}^2)t^2 + (5.0 \text{ m/s})t + (-1.0 \text{ m}) = 0. \tag{3}$$
We can identify $a = -4.9 \text{ m/s}^2$, $b = 5.0 \text{ m/s}$, and $c = -1.0 \text{ m}$ and apply Eq. 2,
$$\begin{aligned} t &= -\frac{5.0 \text{ m/s}}{2(-4.9 \text{ m/s}^2)} \pm \frac{\sqrt{(5.0 \text{ m/s})^2 - 4(-4.9 \text{ m/s}^2)(-1.0 \text{ m})}}{2(-4.9 \text{ m/s}^2)} && (4) \\ &= 0.51\ s \mp 0.24\ s = \boxed{\text{0.27 s, 0.75 s}}. && (5) \end{aligned}$$

  - To understand the two solutions, the graph of the left side of Eq. 3 shown in Fig. 1 is helpful. The extreme value of the quadratic occurs at $-\frac{b}{2a}$. In the example, the extreme value is at 0.51 s.
  - The roots of the quadratic lie $\frac{\sqrt{b^2-4ac}}{2a}$ to either side of $-\frac{b}{2a}$. In the example, the two roots of the equation lie 0.24 s to either side of 0.51 s.
  - A graph like this is often helpful in choosing appropriate solutions. In the example, both roots of the equation are appropriate. The ball reaches a height of 3.0 m twice, once going up, and once coming down.

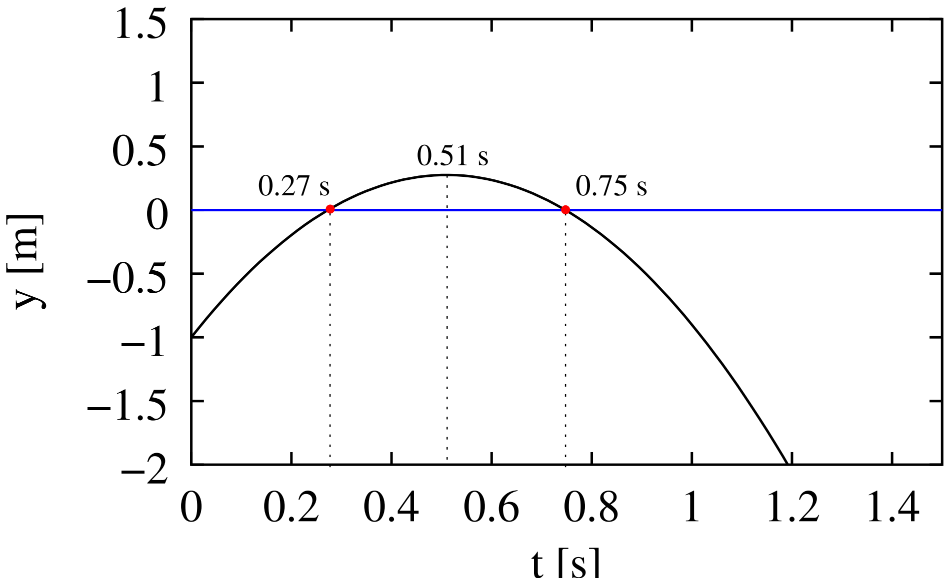

Figure 1: The left side of Eq. 3 plotted vs. t.

- **Special Cases**
  - **Single solution**
    If $4ac = b^2$, then $\sqrt{b^2 - 4ac} = 0$, and there is only one solution, $x = -\frac{b}{2a}$. In the above example, this corresponds to 0.51 s, the time at which the ball reaches its maximum height.
  - **Complex solutions**
    If $4ac > b^2$, then $\sqrt{b^2 - 4ac}$ is imaginary (involves $i = \sqrt{-1}$). The above example does not have a physically reasonable solution corresponding to this situation. (This situation corresponds to heights never reached by the ball.)

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