Cramer's Rule

Cramer's rule is a method of solving n simultaneous equations for n unknowns.

- Any system of equations of this kind can be written in the form

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1n}\,x_n &= b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2n}\,x_n &= b_2 \\ &\vdots \\ a_{n1}\,x_1 + a_{n2}\,x_n + \ldots + a_{nn}\,x_n &= b_n \end{aligned} \tag{1}$$

- The coefficients on the left side of Eq.'s 1 can be written as a matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \tag{2}$$

- The value of any of the x_i can be found via

$$x_i = \frac{|B_i|}{|A|} \tag{3}$$

where the notation $|A|$ and $|B_i|$ denote the determinants of matrices A and B_i, and the matrix B_i is obtained by replacing the i^{th} column of matrix A with the coefficients on the left side of Eq.'s 1. For example,

$$B_2 = \begin{pmatrix} a_{11} & b_1 & \ldots & a_{1n} \\ a_{21} & b_2 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \ldots & a_{nn} \end{pmatrix} \tag{4}$$

The $\mathbf{n = 2}$ Case

- Two equations involving two unknowns can be written in the form

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 &= b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 &= b_2 \end{aligned} \tag{5}$$

which yields

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{6}$$

- Eq. 3 gives

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1\,a_{22} - a_{12}\,b_2}{a_{11}\,a_{22} - a_{12}\,a_{21}} \tag{7}$$

and

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11}\, b_2 - b_1\, a_{21}}{a_{11}\, a_{22} - a_{12}\, a_{21}} \tag{8}$$

An $\mathbf{n = 2}$ Example

- The system of equations

$$\begin{aligned} 2\, x_1 - x_2 &= 1 \\ 5\, x_1 + 3\, x_2 &= 2 \end{aligned} \tag{9}$$

has solutions

$$x_1 = \frac{\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix}} = \frac{5}{11} \tag{10}$$

and

$$x_2 = \frac{\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix}} = -\frac{1}{11} \tag{11}$$

- The results can be verified by substituting them into either of Eq.'s 9,

$$\frac{10}{11} + \frac{1}{11} = \frac{11}{11} = 1 \qquad \checkmark \tag{12}$$

An $\mathbf{n = 3}$ Example

- The system of equations

$$\begin{aligned} 2\, x_1 - x_2 + 4\, x_3 &= 2 \\ 5\, x_1 + 3\, x_2 + 2\, x_3 &= 1 \\ x_1 + 6\, x_2 + x_3 &= -3 \end{aligned} \tag{13}$$

has solutions

$$\begin{aligned} x_1 &= \frac{\begin{vmatrix} 2 & -1 & 4 \\ 1 & 3 & 2 \\ -3 & 6 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 4 \\ 5 & 3 & 2 \\ 1 & 6 & 1 \end{vmatrix}} \\ &= \frac{2(3 \cdot 1 - 2 \cdot 6) - (-1)(1 \cdot 1 - 2 \cdot -3) + 4(1 \cdot 6 - 3 \cdot -3)}{2(3 \cdot 1 - 2 \cdot 6) - (-1)(5 \cdot 1 - 2 \cdot 1) + 4(5 \cdot 6 - 3 \cdot 1)} \\ &= \frac{49}{93} \end{aligned} \tag{14}$$

$$
\begin{aligned}
x_2 &= \frac{\begin{vmatrix} 2 & 2 & 4 \\ 5 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 4 \\ 5 & 3 & 2 \\ 1 & 6 & 1 \end{vmatrix}} \\
&= \frac{2(1 \cdot 1 - 2 \cdot -3) - 2(5 \cdot 1 - 2 \cdot 1) + 4(5 \cdot -3 - 1 \cdot 1)}{93} \\
&= -\frac{56}{93} \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
x_3 &= \frac{\begin{vmatrix} 2 & -1 & 2 \\ 5 & 3 & 1 \\ 1 & 6 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 4 \\ 5 & 3 & 2 \\ 1 & 6 & 1 \end{vmatrix}} \\
&= \frac{2(3 \cdot -3 - 1 \cdot 6) - (-1)(5 \cdot -3 - 1 \cdot 1) + 2(5 \cdot 6 - 3 \cdot 1)}{93} \\
&= \frac{8}{93} \qquad (16)
\end{aligned}
$$

- The results can be verified by substituting them into any of Eq.'s 13,

$$
\frac{2 \cdot 49 - (-56) + 4 \cdot 8}{93} = \frac{186}{93} = 2 \qquad \checkmark \qquad (17)
$$

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