1 Newton's lagrangian

$$U(x) = -\frac{M}{r}$$

$$L_N = \frac{1}{2}|\dot{x}|^2 + \frac{M}{r}$$

In polar coordinates, the metric is

$$ds^2 = dr^2 + r^2 d\theta^2$$

and hence

$$L_N = \frac{\dot{r}^2}{2} + \frac{r^2\dot{\theta}^2}{2} + \frac{M}{r}.$$

The equations of motion are

$$\begin{aligned} \frac{d}{dt}\frac{\partial L_N}{\partial \dot{\theta}} &= \frac{\partial L_n}{\partial \theta} \\ \frac{d}{dt}\left(r^2\dot{\theta}\right) &= 0 \\ \dot{\theta} &= \frac{K}{r^2} \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L_N}{\partial \dot{r}} &= \frac{\partial L_N}{\partial r} \\ \ddot{r} &= r\dot{\theta}^2 - \frac{M}{r^2} \\ \ddot{r} &= \frac{K^2}{r^3} - \frac{M}{r^2} \end{aligned}$$

2 Schwarzchild's geodesics lagrangian

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2.$$

Geodesics minimize energy:

$$\begin{aligned} S[\gamma] = \int_\gamma \frac{ds^2}{2} &= \int_\gamma -\left(1 - \frac{2M}{r}\right)\frac{dt^2}{2} + \left(1 - \frac{2M}{r}\right)^{-1}\frac{dr^2}{2} + r^2\frac{d\theta^2}{2} \\ &= \int_\gamma \underbrace{-\left(\cancel{1} - \frac{2M}{r}\right)\frac{\cancel{\dot{t}}^2}{2} + \left(1 - \frac{2M}{r}\right)^{-1}\frac{\dot{r}^2}{2} + r^2\frac{\dot{\theta}^2}{2}}_{\hat{L}} dt \end{aligned}$$

Remove the constant:

$$L = \frac{M}{r} + \left(1 - \frac{2M}{r}\right)^{-1}\frac{\dot{r}^2}{2} + \frac{r^2\dot{\theta}^2}{2}$$

Perturbatively expand:

$$L = L_N + \frac{\dot{r}^2}{2}\sum_{k=1}^{\infty}(2M)^k r^{-k}$$

3 No tangential motion

$$L = \frac{M}{r} + \left(1 - \frac{2M}{r}\right)^{-1} \frac{\dot{r}^2}{2}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{r}} &= \left(1 - \frac{2M}{r}\right)^{-1} \dot{r} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= \left(1 - \frac{2M}{r}\right)^{-1} \ddot{r} - \left(1 - \frac{2M}{r}\right)^{-2} \frac{2M}{r^2}\dot{r}^2 \\
\frac{\partial L}{\partial r} &= -\frac{M}{r^2} - \left(1 - \frac{2M}{r}\right)^{-2} \frac{M}{r^2}\dot{r}^2
\end{aligned}$$

$$\begin{aligned}
\left(1 - \frac{2M}{r}\right)^{-1} \ddot{r} - \left(1 - \frac{2M}{r}\right)^{-2} \frac{2M}{r^2}\dot{r}^2 &= -\frac{M}{r^2} - \left(1 - \frac{2M}{r}\right)^{-2} \frac{M}{r^2}\dot{r}^2 \\
\left(1 - \frac{2M}{r}\right)^{-1} \ddot{r} &= -\frac{M}{r^2} + \left(1 - \frac{2M}{r}\right)^{-2} \frac{M}{r^2}\dot{r}^2 \\
\ddot{r} &= -\left(1 - \frac{2M}{r}\right)\frac{M}{r^2} + \left(1 - \frac{2M}{r}\right)^{-1} \frac{M}{r^2}\dot{r}^2
\end{aligned}$$

4 Slow

$$\ddot{r} = -\frac{M}{r^2} + \frac{2M^2}{r^3}$$

$$V(r) = -\frac{M}{r} + \frac{M^2}{r^2}$$

$$E = \frac{\dot{r}^2}{2} - \frac{M}{r} + \frac{M^2}{r^2}$$

$$\dot{r} = \sqrt{2}\sqrt{E + \frac{M}{r} - \frac{M^2}{r^2}}$$

$$\frac{1}{\sqrt{2}}\int \frac{dr}{\sqrt{E + \frac{M}{r} - \frac{M^2}{r^2}}} = \Delta t$$

$$\frac{1}{\sqrt{2E}}\int \frac{r\,dr}{\sqrt{r^2 + \frac{M}{E}r - \frac{M^2}{E}}} = \Delta t$$

$$\frac{1}{\sqrt{2E}}\int \frac{r\,dr}{\sqrt{\left(r+\frac{M}{2E}\right)^2-\left(\frac{M^2}{4E^2}+\frac{M^2}{E}\right)}}=\Delta t$$

$$\frac{1}{\sqrt{2E}}\int \frac{r\,dr}{\sqrt{R^2-K^2}}=\Delta t$$

$R = K\sec u$, $du = \sec u \tan u$

5 Christoffel symbols

$$E=\int\frac{ds^2}{2}=\int \underbrace{-\left(1-\frac{2M}{r}\right)\frac{\dot{t}^2}{2}+\left(1-\frac{2M}{r}\right)^{-1}\frac{\dot{r}^2}{2}+r^2\frac{\dot{\theta}^2}{2}+r^2\sin^2\theta\frac{\dot{\phi}^2}{2}}_{L}\,d\lambda$$

Euler lagrange:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{t}} &= -\left(1-\frac{2M}{r}\right)\dot{t} \\
\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{t}} &= -\left(1-\frac{2M}{r}\right)\ddot{t}-\frac{2M}{r^2}\dot{r}\dot{t} \\
0 &= \ddot{t}+2\left(-\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}\right)\dot{r}\dot{t} \\
0 &= \ddot{t}+2\underbrace{\left(-\left(\frac{r^2}{M}-2r\right)^{-1}\right)}_{\Gamma^t{}_{rt}}\dot{r}\dot{t}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial r} &= -\frac{M}{r^2}\dot{t}^2-\left(1-\frac{2M}{r}\right)^{-2}\frac{M}{r^2}\dot{r}^2+r\dot{\theta}^2+r\sin^2\theta\dot{\phi}^2 \\
\frac{\partial L}{\partial \dot{r}} &= \left(1-\frac{2M}{r}\right)^{-1}\dot{r} \\
\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{r}} &= -\left(1-\frac{2M}{r}\right)^{-2}\frac{2M}{r^2}\dot{r}^2+\left(1-\frac{2M}{r}\right)^{-1}\ddot{r} \\
0 &= \left(1-\frac{2M}{r}\right)^{-1}\ddot{r}+\frac{M}{r^2}\dot{t}^2-\left(1-\frac{2M}{r}\right)^{-2}\frac{M}{r^2}\dot{r}^2-r\dot{\theta}^2-r\sin^2\theta\dot{\phi}^2 \\
0 &= \ddot{r}+\left(1-\frac{2M}{r}\right)\frac{M}{r^2}\dot{t}^2-\left(1-\frac{2M}{r}\right)\frac{M}{r^2}\dot{r}^2-\left(1-\frac{2M}{r}\right)r\dot{\theta}^2-\left(1-\frac{2M}{r}\right)r\sin^2\theta\dot{\phi}^2 \\
0 &= \ddot{r}+\underbrace{\left(\frac{M}{r^2}-\frac{2M^2}{r^3}\right)}_{\Gamma^r{}_{tt}}\dot{t}^2\underbrace{-\left(\frac{M}{r^2}-\frac{2M^2}{r^3}\right)}_{\Gamma^r{}_{rr}}\dot{r}^2\underbrace{-(r-2M)}_{\Gamma^r{}_{\theta\theta}}\dot{\theta}^2\underbrace{-(r-2M)\sin^2\theta}_{\Gamma^r{}_{\phi\phi}}\dot{\phi}^2
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} &= 2r^2 \sin\theta\cos\theta\dot{\phi}^2 \\
\frac{\partial L}{\partial \dot{\theta}} &= r^2\dot{\theta} \\
\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\theta}} &= r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} \\
0 &= r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - 2r^2\sin\theta\cos\theta\dot{\phi}^2 \\
0 &= \ddot{\theta} + 2\underbrace{r^{-1}}_{\Gamma^{\theta}{}_{r\theta}}\dot{r}\dot{\theta}\underbrace{-\sin(2\theta)}_{\Gamma^{\theta}{}_{\phi\phi}}\dot{\phi}^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\phi}} &= r^2\sin^2\theta\dot{\phi} \\
\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\phi}} &= r^2\sin^2\theta\ddot{\phi} + 2r\sin^2\theta\dot{r}\dot{\phi} + 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} \\
0 &= \ddot{\phi} + 2\underbrace{r^{-1}}_{\Gamma^{\phi}{}_{r\phi}}\dot{r}\dot{\phi} + 2\underbrace{\cot\theta}_{\Gamma^{\phi}{}_{\theta\phi}}\dot{\theta}\dot{\phi}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{t}{}_{rt,r} &= \left(\frac{r^2}{M} - 2r\right)^{-2}\left(\frac{2r}{M} - 2\right) = \frac{2M}{r^2}\frac{r-M}{(r-2M)^2} \\
\Gamma^{r}{}_{tt,r} &= -\frac{2M}{r^3} + \frac{6M^2}{r^4} \\
\Gamma^{r}{}_{rr,r} &= +\frac{2M}{r^3} - \frac{6M^2}{r^4} \\
\Gamma^{r}{}_{\theta\theta,r} &= -1 \\
\Gamma^{r}{}_{\phi\phi,r} &= -\sin^2\theta \\
\Gamma^{r}{}_{\phi\phi,\theta} &= -(r-2M)\sin 2\theta \\
\Gamma^{\theta}{}_{r\theta,r} &= -\frac{1}{r^2} \\
\Gamma^{\theta}{}_{\phi\phi,\theta} &= -2\cos 2\theta \\
\Gamma^{\phi}{}_{r\phi,r} &= -\frac{1}{r^2} \\
\Gamma^{\phi}{}_{\theta\phi,\theta} &= 2\csc^2\theta
\end{aligned}
$$