# 1 Polar coordinate

$$ds^2 = dr^2 + r^2 d\theta^2$$

Lagrangian

$$L = \frac{\dot{r}^2}{2} + r^2 \frac{\dot{\theta}^2}{2}$$

$$\frac{\partial L}{\partial \dot{r}} = \dot{r} \quad \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} \implies L = H$$

Momentum conservation:

$$r^2\dot{\theta} = B \implies \dot{\theta} = B/r^2$$

Energy conservation:

$$2E = A^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + r^2\left(B/r^2\right)^2 = \dot{r}^2 + B^2/r^2$$

$$dr = dt\sqrt{A^2 - B^2/r^2}$$

$$dt = \frac{dr}{\sqrt{A^2 - B^2/r^2}} = \frac{r\,dr}{\sqrt{A^2r^2 - B^2}}$$

$r = (B/A)\cosh u$ and $dr = (B/A)\sinh u\,du$

$$t = \int \frac{B^2/A^2\cosh u \sinh u\,du}{B\sinh u} = \frac{B}{A^2}\sinh u = \frac{B}{A^2}\sqrt{A^2/B^2r^2 - 1} = \frac{\sqrt{A^2r^2 - B^2}}{A^2}$$

$$\left(A^2t\right)^2 = A^2r^2 - B^2 \implies r^2 = \frac{\left(A^2t\right)^2 + B^2}{A^2} = A^2t^2 + (B/A)^2$$

$$r = \sqrt{A^2t^2 + (B/A)^2} = \sqrt{A^2t^2 + {r_0}^2}$$

$$d\theta = \frac{A\,dt}{A^2t^2 + {r_0}^2} \quad t = {r_0}/A\tan u \quad dt = {r_0}/A\sec^2 u\,du$$

$$\Delta\theta = \int \frac{A{r_0}/A\sec^2 u\,du}{\sqrt{{r_0}^2}\sec^2 u} = u = \tan^{-1}\left(\frac{At}{r_0}\right)$$

# 2 sphere

$$ds^2 = d\rho^2 + \sin^2\rho\,d\theta^2$$

$$L = \frac{\dot{\rho}^2}{2} + \sin^2\rho\frac{\dot{\theta}^2}{2}$$

Momentum:

$$\sin^2\rho\,\dot{\theta} = B \implies \dot{\theta} = \frac{B}{\sin^2\rho}$$

Energy conservation:

$$A^2 = 2E = \dot{\rho}^2 + \sin^2\rho\,\dot{\theta}^2 = \dot{\rho}^2 + \frac{B^2}{\sin^2\rho}$$

$$dt = \frac{d\rho}{\sqrt{A^2 - B^2/\sin^2\rho}} = \frac{\sin\rho\,d\rho}{\sqrt{A^2\sin^2\rho - B^2}}$$

$$d\theta = \frac{B}{\sin^2\rho}dt = \frac{B\,d\rho}{\sin\rho\sqrt{A^2\sin^2\rho - B^2}}$$