1 Rotate and slice

$$C = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}.$$

$$Z(C) \coloneqq \left(x \mapsto x^T C x\right)^{-1}(0)$$

$$MZ(C) = \left(x \mapsto (M^{-1}x)^T C (M^{-1}x)\right)^{-1}(0) = Z\Big(\left(M^{-1}\right)^T C M^{-1}\Big)$$

$$\begin{bmatrix} 1 & \\ & R \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & R^T \end{bmatrix}$$

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 - s^2 & 2cs \\ 2cs & s^2 - c^2 \end{bmatrix} = \begin{bmatrix} \cos(2u) & \sin(2u) \\ \sin(2u) & -\cos(2u) \end{bmatrix}$$

$$R_* C = \begin{bmatrix} 1 & & \\ & \gamma & \sigma \\ & \sigma & -\gamma \end{bmatrix}$$

$$x^2 + \gamma y^2 + 2\sigma yz = \gamma z^2$$

If $\gamma = 0$ this gives the parabola:

$$x^2 + 2\sigma yz = 0, \qquad y = -\frac{x^2}{2\sigma z}$$

Else,

$$\frac{x^2}{\gamma} + y^2 + 2\tau zy = z^2$$

$$\frac{x^2}{\gamma} + (y + \tau z)^2 = z^2\left(1 + \tau^2\right) = z^2/\gamma^2$$

$$\frac{x^2}{z^2/\gamma} + \frac{(y + \tau z)^2}{z^2/\gamma^2} = 1$$

The sign of γ by the x can switch between hyperbola and ellipse. The simple hyperbola:

$$-x^2 + y^2 = 1 \implies y^2 = 1 + x^2 \implies |y| \geqslant 1$$

2 Directrix

Use the x-axis as the directrix so that y is the distance. Put the focus on $(0, f)$.

The directrix/eccentricity property:

$$ey = \big|(x, y) - (0, f)\big| = \big|(x, y - f)\big|.$$

Square:

$$e^2 y^2 = x^2 + (y - f)^2 = x^2 + y^2 - 2yf + f^2$$
$$x^2 + (1 - e^2)y^2 - 2yf + f^2 = 0$$