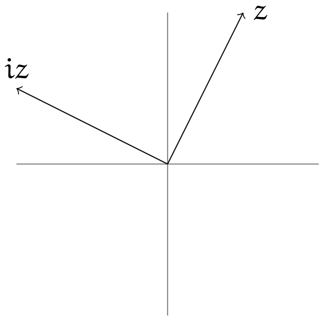
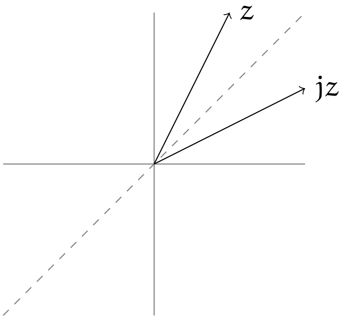
Complex

$$i^2 \coloneqq -1$$
$$i(x+iy) = -y + ix$$



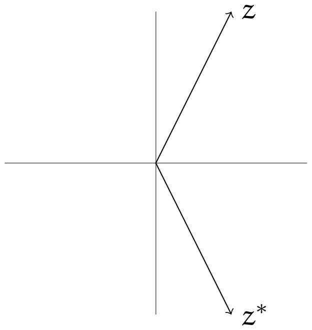
Hyperbolic

$$j^2 \coloneqq +1$$
$$j(x+jy) = y + jx$$

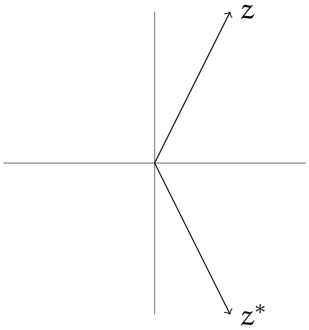


Conjugation and norm

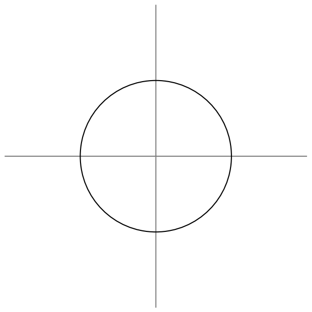
$$(x+iy)^* = x - iy$$



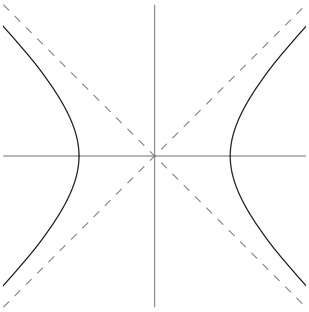
$$(x+jy)^* = x - jy$$



$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x+iy)(x-iy) \\ &= x^2 + y^2 \end{aligned}$$

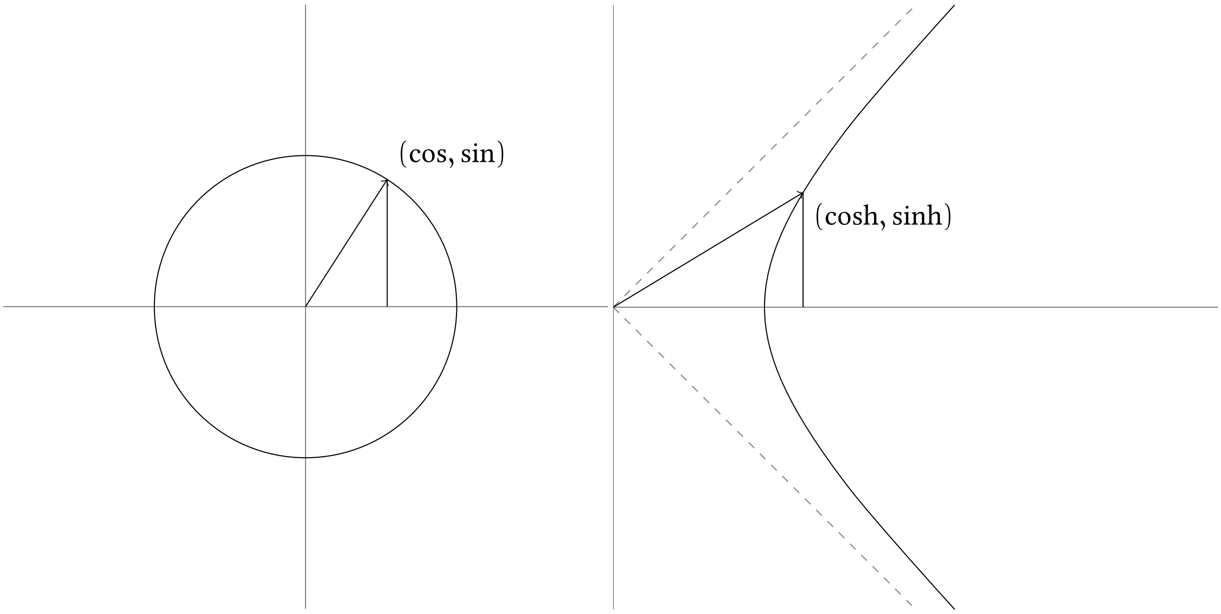


$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x+jy)(x-jy) \\ &= x^2 - y^2 \end{aligned}$$



Exponentiation

$$
\begin{aligned}
i^n &= \begin{cases} 1 \\ i \\ -1 \\ -i \end{cases} \\
e^{it} &= 1(\dots) + i(\dots) \\
&\quad - 1(\dots) - i(\dots) \\
e^{it} &= \cos t + i \sin t
\end{aligned}
\qquad
\begin{aligned}
j^n &= \begin{cases} 1 \\ j \\ 1 \\ j \end{cases} \\
e^{jt} &= 1(\dots) + j(\dots) \\
&\quad + 1(\dots) + j(\dots) \\
e^{jt} &= \cosh t + j \sinh t
\end{aligned}
$$



Pythagorean identity

$$
\begin{aligned}
e^{-it} &= \cos(-t) + i \sin(-t) \\
&= \cos t - i \sin t \\
e^{-it} &= \left(e^{it}\right)^* \\
\left|e^{it}\right|^2 &= e^{it-it} \\
&= 1 \\
1 &= \cos^2 t + \sin^2 t
\end{aligned}
\qquad
\begin{aligned}
e^{-jt} &= \cosh(-t) + j \sinh(-t) \\
&= \cosh t - j \sinh t \\
e^{-jt} &= \left(e^{jt}\right)^* \\
\left|e^{jt}\right|^2 &= e^{jt-jt} \\
&= 1 \\
1 &= \cosh^2 t - \sinh^2 t
\end{aligned}
$$

Pythagorean corollaries

$$\begin{aligned} \cos^2 t &= 1 - \sin^2 t \\ \sin^2 t &= 1 - \cos^2 t \\ \sec^2 t &= 1 + \tan^2 t \\ \csc^2 t &= \cot^2 t + 1 \end{aligned}$$

$$\begin{aligned} \cosh^2 t &= 1 + \sinh^2 t \\ \sinh^2 t &= \cosh^2 t - 1 \\ \operatorname{sech}^2 t &= 1 - \tanh^2 t \\ \operatorname{csch}^2 t &= \coth^2 t - 1 \end{aligned}$$

Angle sum formulae

$$\begin{aligned} e^{i(A+B)} &= e^{iA}e^{iB} \\ &= (c_A + is_A)(c_B + is_B) \end{aligned}$$

$$\begin{array}{c|cc} & c_B & is_B \\ \hline c_A & c_Ac_B & ic_As_B \\ is_A & ic_Bs_A & -s_As_B \end{array}$$

$$\begin{matrix} \cos(A+B) \\ +i\sin(A+B) \end{matrix} = \begin{matrix} c_Ac_B - s_As_B \\ +i(c_Bs_A + c_As_B) \end{matrix}$$

$$\begin{aligned} e^{j(A+B)} &= e^{jA}e^{jB} \\ &= (c_A + js_A)(c_B + js_B) \end{aligned}$$

$$\begin{array}{c|cc} & c_B & js_B \\ \hline c_A & c_Ac_B & jc_As_B \\ js_A & jc_Bs_A & +s_As_B \end{array}$$

$$\begin{matrix} \cosh(A+B) \\ +j\sinh(A+B) \end{matrix} = \begin{matrix} c_Ac_B + s_As_B \\ +j(c_Bs_A + c_As_B) \end{matrix}$$

Double angle

$$\begin{aligned} e^{2it} &= \left(e^{it}\right)^2 \\ \cos 2t &= \cos^2 t - \sin^2 t \\ i\sin 2t &= i\,2\cos t\sin t \end{aligned}$$

$$\begin{aligned} e^{2jt} &= \left(e^{jt}\right)^2 \\ \cosh 2t &= \cosh^2 t + \sinh^2 t \\ j\sinh 2t &= j\,2\cosh t\sinh t \end{aligned}$$

Double angle + Pythagorean

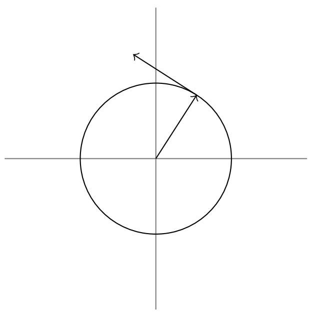
$$\begin{aligned} \cos 2t &= \cos^2 t - (1 - \cos^2 t) \\ &= 2\cos^2 t - 1 \\ &= (1 - \sin^2 t) - \sin^2 t \\ &= 1 - 2\sin^2 t \end{aligned}$$

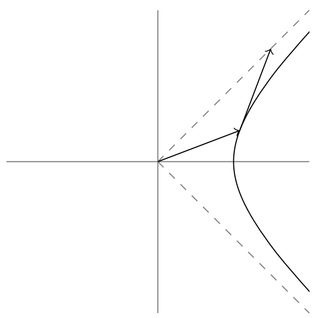
$$\begin{aligned} \cosh 2t &= \cosh^2 t + (\cosh^2 t - 1) \\ &= 2\cosh^2 t - 1 \\ &= (1 + \sinh^2 t) + \sinh^2 t \\ &= 1 + 2\sinh^2 t \end{aligned}$$

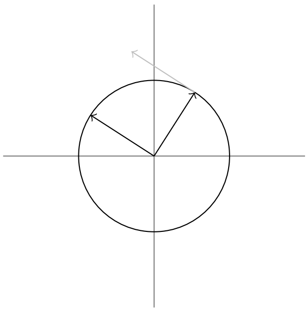
Derivatives

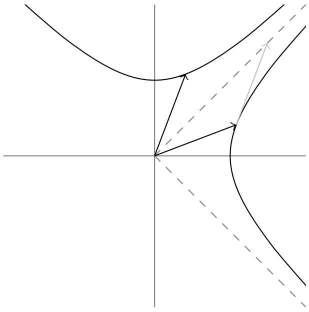
$$\begin{aligned} f(t) &\coloneqq e^{it} \\ f'(t) &= ie^{it} \\ &= i(\cos t + i\sin t) \\ \cos' t + i\sin' t &= -\sin t + i\cos t \end{aligned}$$

$$\begin{aligned} f(t) &\coloneqq e^{jt} \\ f'(t) &= je^{jt} \\ &= j(\cosh t + j\sinh t) \\ \cosh' t + j\sinh' t &= \sinh t + j\cosh t \end{aligned}$$









Dot and cross product

$$\begin{aligned}
u^*v &= (\alpha - i\beta)(\gamma + i\delta) \\
& \begin{array}{c|cc}
 & \gamma & i\delta \\ \hline
\alpha & \alpha\gamma & i\alpha\delta \\
-i\beta & -i\beta\gamma & +\beta\delta
\end{array} \\
&= (\alpha\gamma + \beta\delta) + i(\alpha\delta - \beta\gamma) \\
u^*v &= (u \cdot v) + i(u \times v) \\
(iz) \cdot z &= \mathfrak{R}(iz^*z) = \mathfrak{R}(i|z|^2) = 0
\end{aligned}$$

$$\begin{aligned}
u^*v &= (\alpha - j\beta)(\gamma + j\delta) \\
& \begin{array}{c|cc}
 & \gamma & i\delta \\ \hline
\alpha & \alpha\gamma & +j\alpha\delta \\
-j\beta & -j\beta\gamma & -\beta\delta
\end{array} \\
&= (\alpha\gamma - \beta\delta) + j(\alpha\delta - \beta\gamma) \\
u^*v &= (u \cdot v) + j(u \times v) \\
(jz) \cdot z &= \mathfrak{R}(jz^*z) = \mathfrak{R}(j|z|^2) = 0
\end{aligned}$$

Arc length

$$\begin{aligned}
f(t) &\coloneqq e^{it} \\
S &= \int \sqrt{|f'(t)|^2} dt \\
&= \int \sqrt{|ie^{it}|^2} dt = \int \sqrt{1} dt \\
&= \Delta t
\end{aligned}$$

$$\begin{aligned}
f(t) &\coloneqq e^{jt} \\
S &= \int \sqrt{-|f'(t)|^2} dt \\
&= \int \sqrt{|je^{jt}|^2} dt = \int \sqrt{-(-1)} dt \\
&= \Delta t
\end{aligned}$$

Sector area

$$\begin{aligned}
A &= \frac{1}{2} \int f(t) \times f'(t) dt \\
&= \frac{1}{2} \int \mathfrak{I}\Big(f^*(t) f'(t) \Big) dt \\
&= \frac{1}{2} \int \mathfrak{I}\big(e^{-it} i e^{it} \big) dt \\
&= \frac{1}{2} \int 1 \, dt \\
&= \frac{\Delta t}{2}
\end{aligned}$$

$$\begin{aligned}
A &= \frac{1}{2} \int f(t) \times f'(t) dt \\
&= = \frac{1}{2} \int \mathfrak{I}\big(f^*(t) f'(t) \big) dt \\
&= = \frac{1}{2} \int \mathfrak{I}\big(e^{-jt} j e^{jt} \big) dt \\
&= \frac{1}{2} \int 1 \, dt \\
&= \frac{\Delta t}{2}
\end{aligned}$$

$SO(2), SO^{+}(+,-)$: Multiplying by imaginary exponents preserves inner product and area

$$
\begin{aligned}
&(e^{it}u)\cdot(e^{it}v) \\
+i(e^{it}u)\times(e^{it}v) &= (e^{it}u)^{*}(e^{it}v) \\
&= e^{-it}u^{*}e^{it}v \\
&= u^{*}v \\
&= u\cdot v + i\,u\times v
\end{aligned}
\qquad
\begin{aligned}
&(e^{jt}u)\cdot(e^{jt}v) \\
j(e^{jt}u)\times(e^{jt}v) &= (e^{jt}u)^{*}(e^{jt}v) \\
&= e^{-jt}u^{*}e^{jt}v \\
&= u^{*}v \\
&= u\cdot v + j\,u\times v
\end{aligned}
$$