Categories for a few specific working physicists

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November 21, 2024

1 Intro & motivation

Theoretical mathematics is, to first order of approximation, the study of objects with a *structure* and the functions that *preserve* this structure, its arrows/morphisms. The prototypical example is the vector space and linear maps. The story of vector spaces is told through linear maps. This is true in general. Here's a list of structures and their associated morphisms, which we'll soon call categories.

- Sets & functions (`set`)
- Vector spaces & linear functions (`vec`)
- Inner product spaces & unitary functions
- Metric spaces & isometries
- Groups & homomorphsims
- Ordered sets & order-preserving (non-decreasing) functions
- Topological spaces & continuous functions (`top`)
- Smooth Manifolds & smooth functions (`smooth`)

- Types & computable functions

Before formalizing this, I want to mention some motivation. First, because familiar objects usually have multiple structures on them, it's important to specify which one we are interested in. As sets $\mathbb{R}$ and $\mathbb{R}^2$ have the same structure because they're bijective, but as vector spaces, they are different because they have different dimensions. We also want to recognize when seemingly-different objects actually have the same structure. Looking for structure-preserving invertible functions provides a general way to do this.

Formalizing this idea is the genesis of category theory.

They're often too big to be sets, but I won't discuss this today.

Definition 1 (Category) A category is a class of objects & a class of arrows between them. Additionally, each object has an identity arrow, and the set of arrows are closed under composition.

If $\mathcal{C}$ is a category, call its set of objects $\mathtt{Ob}(\mathcal{C})$. If $A, B \in \mathtt{Ob}(\mathcal{C})$, the set of arrows $A \to B$, called the hom-set, is denoted

$$\underset{\mathcal{C}}{\hom}(A, B).$$

Arrows are also called homomorphisms, especially in algebra.

If $f \in \hom_{\mathcal{C}} A, B$ and $g \in \hom_{\mathcal{C}}(B, C)$, then

$$g \circ f \in \underset{\mathcal{C}}{\hom}(A, C).$$

so the following diagram *commutes* (following any path shown between two objects gives the same answer)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

And identity arrows have the property

$$f \circ 1_A = f \qquad 1_B \circ f = f$$

$$\begin{tikzcd}
A \arrow[loop left] \arrow[r, "f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[loop right]
\end{tikzcd}$$

for $1_A \in \hom(A, A)$ and $1_B \in \hom(B, B)$

Lemma 2 (Identities are unique) *Suppose* I *is an identity on* A. *Then* $\mathrm{I} = 1_{\mathsf{A}}$.

Proof ·

$$1_{\mathsf{A}} = \mathrm{I} \circ 1_{\mathsf{A}} = \mathrm{I}. \qquad \square$$

Definition 3 (Invertible) An arrow $f : X \to Y$ is invertible, aka an isomorphism, iff there is some $g : Y \to X$ such that

$$1_X = g \circ f \qquad \text{and} \qquad 1_Y = f \circ g$$

i.e., the following diagram *commutes*:

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \arrow[loop right, "1_Y"] \end{tikzcd}$$

Lemma 4 (Inverses are unique) *If* L *and* R *are both inverses of* f, *then* $L = R$.

Actually proves if there's both a left and right inverse, they are equal & two-sided.

Proof ·

$$L = L \circ 1_B = L \circ (f \circ R) = (L \circ f) \circ R = 1_A \circ R = R \qquad \square$$

One-sided inverses need not be unique, for example, a projection has multiple sections.

Groups theory axiomatizes the algebra of symmetry transformations. Category theory generalizes this by axiomatizing the algebra of all structure-preserving functions.

> "This may be regarded as a continuation of the Klein Erlanger Programm, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings" [9, p. 237] (the founding paper)

Our original motivation for arrows were functions, but it's useful to consider other possibilities.

Definition 5 (The category of a partially-ordered set) If $(P, \leqslant)$ is a partially-ordered set, turn it into a category by drawing exactly one arrow $x \to y$ iff $x \leqslant y$.

Existence of identity arrows comes from reflexivity, and existence of compositions comes from transitivity.

For example, $2^{\{0,1,2\}}$, the powerset of $\{0, 1, 2\}$, is partially ordered by $\subseteq$, so it forms the category

$$\begin{tikzcd}
& & \{0,1,2\} \arrow[loop left, gray] & & \\
\{0,1\} \arrow[loop left, gray] \arrow[urr] & & \{0,2\} \arrow[loop left, gray] \arrow[u] & & \{1,2\} \arrow[loop left, gray] \arrow[ull] \\
\{0\} \arrow[loop left, gray] \arrow[u] \arrow[urr] \arrow[uurr, gray] & & \{1\} \arrow[loop left, gray] \arrow[ull] \arrow[urr] & & \{2\} \arrow[loop left, gray] \arrow[ull] \arrow[u] \arrow[uull, gray] \\
& & \varnothing \arrow[loop left, gray] \arrow[ull] \arrow[u] \arrow[urr] \arrow[uuu, gray] & &
\end{tikzcd} .$$

This is usually drawn with identities and compositions omitted:

$$\begin{tikzcd}
& \{0,1,2\} & \\
\{0,1\} \arrow[ur] & \{0,2\} \arrow[u] & \{1,2\} \arrow[ul] \\
\{0\} \arrow[u] \arrow[ur] & \{1\} \arrow[ul] \arrow[ur] & \{2\} \arrow[ul] \arrow[u] \\
& \varnothing \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

In this way, category theory generalizes order theory. Many constructions in category theory correspond to familiar constructions in order theory, like max and sup.

Definition 6 (Proof-arrow category.) The objects are statements and each arrow $A \to B$ is a proof of B given A.

Identity arrows exist because given a conclude a is a tautology, and compositions exist because proofs are transitive (the cut-elimination theorem).

Definition 7 (Path-arrow category) The objects are points in a topological space, and the arrows $x \to y$ are paths from x to y. Identities are the constant path, and composition is path concatenation [2, p. 201].

This hints at the deep relationship between topology, especially homotopy theory, and category theory.

Definition 8 (Monoid) A monoid is a category with a single object.

Every monoid can be realized this way. Make a dummy object and draw an arrow for every monoid element.

If we additionally require invertibility,

Definition 9 (Group) A group is a monoid (one-object category) where every arrow is invertible [1, p. 41].

Here's the trivial group:

$$\begin{tikzcd} \bullet \arrow[loop left, "1"] \end{tikzcd}.$$

Here's the group $(\{-1, 1\}, \cdot)$:

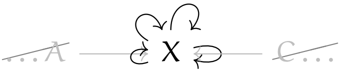
$$\begin{tikzcd} \bullet \arrow[loop left, "1"] \arrow[loop right, "-1"] \end{tikzcd}.$$

The two most important monoids are the endomorphism monoid and the automorphism group.

Definition 10 (End) The endomorphism monoid of an object $X \in \mathtt{Ob}(\mathcal{C})$ is the monoid

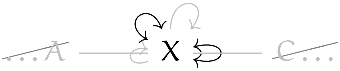
$$\operatorname{End}_{\mathcal{C}} X = \big(\hom(X, X), \circ\big).$$

i.e., restrict focus from the whole category $\mathcal{C}$ to only X and $\hom(X, X)$:



If we throw away the noninvertible arrows,

Definition 11 (Aut) The automorphism group of an object $X \in \mathtt{Ob}(\mathcal{C})$ is the group of invertible arrows from $X \to X$ under composition.



In the category of sets, $\operatorname{Aut}(X)$ is the permutation group of X.

Generalize groups by allowing multiple objects:

Definition 12 (Groupoid) A groupoid is a category where every arrow has an inverse.

You can define groupoids without category theory as a group but the product is not defined everywhere.

Monoids are to categories as groups are to groupoids.

In a groupoid, because all arrows are invertible, they feel a lot like paths in a space. This can be made precise:

Definition 13 (Fundamental groupoid) The fundamental groupoid $\pi(X)$ of a space X is a category whose

- objects are points in X, and

- arrows are homotopy classes of paths in X.

Definition 14 (Opposite category) If you have a category $\mathcal{C}$, flipping all arrows gives a new category $\mathcal{C}^{op}$, called the opposite or dual category.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C & \Longrightarrow & A & B \arrow[l, "f^{op}"'] & C \arrow[l, "g^{op}"'] \arrow[ll, bend left, "f^{op} \circ g^{op}"]
\end{tikzcd}$$

Note

$$(g \circ f)^{op} = f^{op} \circ g^{op}$$

In the case of a partially ordered set, this means flipping the ordering:

$$\begin{tikzcd}
 & \{0,1\} & & & & \varnothing & \\
\{0\} \arrow[ur] & & \{1\} \arrow[ul] & \overset{op}{\Longrightarrow} & \{0\} \arrow[ur] & & \{1\} \arrow[ul] \\
 & \varnothing \arrow[ul] \arrow[ur] & & & & \{0,1\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Cobordisms are important in algebraic topology because homotopy equivilance is intractible, but cobordisms are not. They're shaped like Feynman diagrams [11, p. 8].

Definition 15 (cobord_n) The objects are $n-1$-dimensional compact manifolds with boundary. The arrows are cobordisms between them. A cobordism $X \to Y$ is a compact manifold with boundary M that connects them:

$n-1$ for consistency with [11, p. 8]

$$\partial M = X \sqcup Y \qquad [13].$$

Composition is concatenation/gluing of cobordisms.

An interesting feature of $\mathtt{cobord}_n$ is, since the arrows are also manifolds, they are objects in $\mathtt{cobord}_{n+1}$:

$$\mathtt{Ar}(\mathtt{cobord}_n) \subseteq \mathtt{Ob}(\mathtt{cobord}_{n+1}).$$

It's helpful to keep in mind some examples of things categories generalize or formalize

- Categories generalize the theory of partial orders by allowing multiple arrows between two things
- Categories generalize monoids by relaxing multiplication to a partial function.
- Categories generalize paths in a space by relaxing invertibility (fixing directions).
- Categories formalize Noether's style of focusing on structure-preserving functions.
- Categories clarify similarities and differences across different fields of math.
- Categories give a language for thinking about composition
- Categories suggest heuristics for ways to generalize known results
- Categories suggest heuristics for what to try first when you find a new construction.

2 universal properties, informally

Definition 16 (Initial object) An object $I \in \mathtt{Ob}(\mathcal{C})$ is initial iff for any $X \in \mathtt{Ob}(\mathcal{C})$ there is exactly one morphism $I \to X$.

It's instructive to look at examples.

- In an ordered set, the initial object is the minimum.
- The trivial group is initial in the category of groups.
- The *empty* set is initial in the category of sets.

Dually, there's the notion of final objects.

> Definition 17 (Final object) An object $F \in \mathtt{Ob}(\mathcal{C})$ is final iff for every $X \in \mathtt{Ob}(\mathcal{C})$ there is exactly one morphism $X \to F$.

If $F \in \mathtt{Ob}(\mathcal{C})$ is final, then $F^{op} \in \mathtt{Ob}(\mathcal{C}^{op})$ is initial, and vice versa.

- In an ordered set, the final object is the maximum.
- The trivial group is final in the category of groups.
- "The" *singleton* set is final in the category of sets.

Of course, there's no such thing as *the* singleton set. Any singleton set will do, and they're all basically the same. All singleton sets are *unique up to unique isomorphism*: any two singleton sets are isomorphic, and there is only one isomorphism between them. They have the same structure unambiguously. The structure is the same because they're isomorphic, and since the isomorphism is unique there's no ambiguity in how to identify their structures.

Algebraic completions are unique up to *non*unique isomorphism [6, p. 235]. An algebraic completion's failure to be UU2UI is measured by its Galois group [6, p. 262].

Theorem 18 (Uniqueness of initial objects) *If* $I, I' \in \mathtt{Ob}(\mathcal{C})$ *are both initial objects, then there is a unique isomorphism* $f : I \to I'$.

Proof · Because I' is initial, there is a unique arrow $f : I \to I'$. Because I is initial, there is a unique arrow from $g : I' \to I$. Then their composition $g \circ f : I \to I$. But because I is initial, there is only one arrow $I \to I$, so $g \circ f = 1_I$.

$$\begin{tikzcd} I \arrow[loop left, "1_I"] \arrow[r, shift right, "f"'] & I' \arrow[l, shift right, "g"'] \end{tikzcd}$$

□

By reversing the arrows, the same argument proves

Theorem 19 (Uniqueness of final objects) *If* $F, F' \in \mathtt{Ob}(\mathcal{C})$ *are both final, then there is a unique isomorphism* $f : F \to F'$.

2.1 (co)products

The power of this approach starts to really show with products. Almost every structure has an associated product space. Product sets, product groups, product vector-spaces, product topological-spaces, product manifolds. Why call them all products? In category theory, there's a single definition that unifies them all. Instead of asking what a product is, ask what it *does.*

and, in the darkness, bind them

Definition 20 (product) Suppose $X \sqcap Y$ has arrows $\pi_X : X \sqcap Y \to X$ and $\pi_Y : X \sqcap Y \to Y$ called projections.

$$\begin{tikzcd}
& X \sqcap Y \arrow[ld, "\pi_X"'] \arrow[rd, "\pi_Y"] & \\
X & & Y
\end{tikzcd}$$

This is the product of $X \sqcap Y$ iff for every Z and every pair of arrows $f : Z \to X$ and $g : Z \to Y$, they *factor uniquely* through the projections, meaning there is a unique $f \sqcap g : Z \to X \sqcap Y$ such that $f = \pi_X \circ (f \sqcap g)$ and $g = \pi_Y \circ (f \sqcap g)$.

$$\begin{tikzcd}
& Z \arrow[ldd, bend right, "f"'] \arrow[rdd, bend left, "g"] \arrow[d, dashed, "f \sqcap g"] & \\
& X \sqcap Y \arrow[ld, "\pi_X"] \arrow[rd, "\pi_Y"'] & \\
X & & Y
\end{tikzcd}$$

Like initial objects, products (when they exist) are unique up to unique isomorphism. To abbreviate, say $X \sqcap Y$ is univeral wrt the diagram

$$\begin{tikzcd}
& \square \arrow[ld, "\pi_X"'] \arrow[rd, "\pi_Y"] & \\
X & & Y
\end{tikzcd}$$

Let's check this definition in familiar categories. In the category of sets, it's satisfied by the cartesian product. If $f : Z \to X$ and $g : Z \to Y$, then they factor through $X \times Y$: define

$$f \times g(x) \coloneqq \big(f(x), g(x)\big).$$

Does it factor uniquely? Suppose $h : Z \to X \times Y$ such that $\pi_X \circ h = f$ and $\pi_Y \circ h = g$. This fully defines h: $h(z) = \big(f(z), g(z)\big)$.

In the category of topological spaces, the product of $X \times Y$ is the cartesian product with the coarsest topology τ that makes π_X and π_Y continuous. How does this fit the categorical definition? Certainly, π_X and π_Y must be continuous to be arrows in the category of sets. But why coarsest? Switching to a coarser topology is continuous,

$$x \mapsto x : (X, \tau_{\text{fine}}) \to (X, \tau_{\text{coarse}}),$$

but not the other way around. All finer topologies on $X \times Y$ will factor through the coarsest one.

In the category of an ordered set, the product is the minimum. The projections $\pi_x : x \sqcap y \to x$ and $\pi_y : x \sqcap y \to y$ tell us

$$x \sqcap y \leqslant x, y \implies x \sqcap y \leqslant \min(x, y)$$

On the other hand, any $z \leqslant x, y$ must factor through $x \sqcap y$, so

$$\forall z \quad z \leqslant \min(x, y) \implies z \leqslant x \sqcap y$$

so $x \sqcap y = \min(x, y)$

Reversing all the arrows gives the definition of coproducts.

Definition 21 (Coproduct) Suppose $X \amalg Y$ has arrows $i_X : X \to X \amalg Y$ and $i_Y : Y \to X \amalg Y$ called inclusions.

$$\begin{tikzcd}
X \arrow[dr, "i_X"] & & Y \arrow[dl, "i_Y"'] \\
& X \amalg Y &
\end{tikzcd}$$

This is the coproduct iff for every Z and pair of arrows $f : X \to Z$ and $g : Y \to Z$ factors uniquely through the inclusions as $f \amalg g : X \amalg Y \to Z$

$$\begin{tikzcd}
X \arrow[dr, "i_X"] \arrow[ddr, bend right, "f"'] & & Y \arrow[dl, "i_Y"'] \arrow[ddl, bend left, "g"] \\
& X \amalg Y \arrow[d, dashed, "f \amalg g"] & \\
& Z &
\end{tikzcd}$$

Sometimes coproducts coincide with products, but this is not usually the case. For sets, the coproduct is the *disjoint union*

$$X \sqcup Y \coloneqq \big(X \times \{0\}\big) \cup \big(Y \times \{1\}\big)$$

In an ordered set, the product was min, so, by duality, the coproduct is max. The same argument with the inequalities and arrows flipped works.

The coproduct for vector spaces is just the product space, the same as the product. For groups, the coproduct is the *free* product. The free product $G \star H$ is the group generated by G and H *free* from relations imposed between them, the largest possible group generated by G and H.

This last example shows the importance of the choice of category. Suppose X and Y are vector spaces, hence groups. In the category of vector spaces, $X \amalg_{\mathsf{vec}} Y = X \times Y$, but in the category of groups, $X \amalg_{\mathsf{grp}} Y = X \star Y \neq X \times Y$.

2.2 generalizing

Definition 22 (Universal property, informal) A lot of constructions in category theory work like these: you have some collection of objects and you want to find one all the others factor through uniquely. These are called universal properties. An object defined by a universal property has to be general enough everything factors through it, but specific enough the factorization is unique.

This tension between the general and specific is familiar from inf and sup. The least upper bound is the 'universal' upper bound: every other upper bound factors through it, *i.e.*, is bigger.

Actually, every universal property can be described as an final (or initial) object in a cleverly chosen category. Here's how for products. Given $X, Y \in \mathcal{C}$, define the category $\mathcal{P}$ whose

- objects are product candidates, *i.e.*, objects A with arrows $\pi_X^{(A)} : A \to X$ and $\pi_Y^{(A)} : A \to Y$.

$$\begin{tikzcd}
& A \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X & & Y
\end{tikzcd}$$

- and arrows are arrows from $\mathcal{C}$ that commute with projections:

$$\begin{tikzcd}
& A \arrow[dl, "\pi_X^{(A)}"'] \arrow[dr, "\pi_Y^{(A)}"] \arrow[dd, "f"] & \\
X & & Y \\
& B \arrow[ul, "\pi_X^{(B)}"] \arrow[ur, "\pi_Y^{(B)}"'] &
\end{tikzcd}$$

i.e., the above diagram commutes. Explicitly,

$$\pi_X^{(A)} = \pi_X^{(B)} \circ f \qquad \pi_Y^{(A)} = \pi_Y^{(B)} \circ f$$

A final object, then, is one all product candidates factor though, which is equivalent to the definition of product. As a consequence, this tells us

Lemma 23 (Product's uniqueness) *Products are unique up to unique isomorphism.*

Proof · Every product is a final object somewhere, and final objects are unique up to unique isomorphism. $\square$

Dually,

Corollary 24 (Coproduct's uniqueness) *Coproducts are unique up to unique isomorphism.*

This construction generalizes: make a category out of candidates for the universal property you want. The "best" one is initial or final, depending which way the arrows You can always flip the arrows by moving to the opposite category anyway, so every universal property boils down to finalness.

> Definition 25 (Pushout) Given $f : a \to b$ and $g : a \to c$, the pushout (if it exists) is the universal object $b \amalg_{(f,g)} c$ (also written $b \amalg_a c$ when unambiguous) with inclusions $i_b : b \to P$ and $i_c : c \to P$ for the diagram
>
> $$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[d, "i_b"] \\ c \arrow[r, "i_a"] & \square \end{tikzcd}$$
>
> Explicitly, for any d with $h : b \to d$ and $k : c \to d$, the arrows h and g factor uniquely through i_b, i_c:
>
> $$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[d, "i_b"] \arrow[ddr, bend left, "h"] & \\ c \arrow[r, "i_a"] \arrow[drr, bend right, "k"'] & P \arrow[dr, dashed] & \\ & & d \end{tikzcd}$$

The union of (possibly overlapping) sets is a pushout:

$$\begin{tikzcd} A \cap B \arrow[r, hook, "j_B"] \arrow[d, hook, "j_B"] & B \arrow[d] \\ A \arrow[r] & A \cup B \end{tikzcd}$$

where j_A, j_B are inclusions.

Whereas the coproduct $A \amalg B$ is defined by considering universality over *all* pairs of functions out of A, B, the pushout adds a consistency constraint. We want universality over all pairs of functions that *agree* on the intersection. But if you have two functions $f : A \to X$ and $g : B \to X$ that agree on $A \cap B$, you can glue them together into a function $(f \cup g) : A \cup B \to X$. Hence, f factors through $A \cup B$.

In `set`, pushouts are disjoint unions mod something:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[d, "i_b"] \\
c \arrow[r, "i_a"] & P
\end{tikzcd}$$

where

$$P = b \sqcup c/\sim$$

with $\sim$ the equivalence relation generated by $f(t) \sim g(t) \forall t \in a$.

The pushout is the coproduct in the

Definition 26 (under category) Given a category $\mathcal{C}$ and object $A \in \mathrm{Ob}(\mathcal{C})$, define the category of objects under A,

$$A \downarrow \mathcal{C}$$

as the category whose

- objects are objects $X \in \mathcal{C}$ paired with morphisms $f_X \in \hom_{\mathcal{C}}(A, X)$
- morphisms in $\hom_{a\downarrow\mathcal{C}}(X, Y)$ are morphisms $g : X \to Y$ that preserve the image of A, *i.e.*, such that

$$\begin{tikzcd}
 & A \arrow[dl, "f_X"'] \arrow[dr, "f_Y"] & \\
X \arrow[rr, "g"'] & & Y
\end{tikzcd}$$

commutes.

Field extensions live in the category under the base field. But algebraists stand on their heads, so they call it over.

The diagram for a coproduct in $A \downarrow \mathcal{C}$, with the under-ness drawn explicitly, is

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& X \amalg_A Y &
\end{tikzcd}$$

which is the pushout diagram.

Pushouts are important for calculating fundamental groupoids. If you can express a topological space as a pushout between known spaces, often its fundamental groupoid will be the pushout of the groupoids of the other spaces[2, p. xxi].

Theorem 27

$$\pi S^1 = \mathbb{Z}$$

Proof · From [2, p. xxi]. The pushout

$$\begin{tikzcd}
\{0,1\} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\
{[0,1]} \arrow[r] & S^1
\end{tikzcd}$$

builds S^1 by gluing the ends of a line segment. The same construction in groupoids gives

$$\begin{tikzcd}
\{0,1\} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\
I \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

where I is the groupoid

$$\begin{tikzcd}
0 \arrow[r, shift left, "\eta"] & 1 \arrow[l, shift left] .
\end{tikzcd}$$

Taking I and gluing 0 and 1 gives a groupoid that looks like a circle:

$$\begin{tikzcd} 0 \arrow[loop above, "\eta"] \arrow[loop above] \end{tikzcd}$$

But now η loops back on itself. The pushout is a group! Since the pushout must be universal, the group is free, has no relations. Each η^k for $k \in \mathbb{Z}$ is distinct. □

The dual of the under category is the over category.

Definition 28 (over category) If $A \in \mathtt{Ob}(\mathcal{C})$, the over category,

$$\mathcal{C}/A$$

whose

- objects are objects X in $\mathcal{C}$ paired with morphisms $f_X \in \hom_{\mathcal{C}}(X, A)$
- morphisms are morphisms $g : X \to Y$ that preserve the image *in* A, *i.e.*,

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f_X"'] & & Y \arrow[dl, "f_Y"] \\ & A & \end{tikzcd}$$

  commutes.

Every fiber bundle over A is an object in $\mathtt{smooth}/A$, though $\mathtt{smooth}/A$ also contains bundles over subsets.

The product in $\mathcal{C}/A$ is

Definition 29 (pullback) The pullback $X \Pi_{(f_X, f_Y)} Y$ satisfies the universal property

$$\begin{tikzcd}
& \square \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[dr, "f_X"'] & & Y \arrow[dl, "f_Y"] \\
& A &
\end{tikzcd}$$

3 What is a Lie group: group objects

There is another category-theoretic way to define groups. Definition 9 is useful for showing how categories generalize groups, but dropping it into `smooth` gives us smooth group actions (Definition 41 (Smooth group action)), not Lie groups. To get Lie groups, translate the usual group axioms from statements about sets to objects and arrows.

The naming and terminology is from [14] and the pictures are from [5, p. 75]

In preparation, we need some prep work around products. We need the diagonal to articulate xx^{-1} via dots and arrows.

One of the few arrows into $X \Pi X$ guaranteed to exist in general

Definition 30 (the diagonal map) Define

$$\Delta = 1_X \Pi 1_X$$

by factoring $1_X, 1_X$ through the product $X \Pi X$ [10]:

$$\begin{tikzcd}
& X \arrow[d, dashed, "\Delta"] \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\
& X \Pi Y \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
X & & X
\end{tikzcd}$$

In `set` (and its usual subcategories) this is exactly

$$\Delta(x) = (x, x).$$

In an ordered set, the diagonal says $x \geqslant \max(x, x)$

We need 3-ary products to articulate associativity.

The 0-ary product is the final object)

Definition 31 (triple product) The product $X_1 \sqcap X_2 \sqcap X_3$ is the universal object wrt

Products generalize to arbitrarily many factors, even uncountably infinite.

$$\begin{tikzcd}
 & \square \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dr, "\pi_3"] & \\
X & Y & Z
\end{tikzcd}$$

More explicitly, any other Z, $f_i : Z \to X_i$ factors uniquely through the product:

$$\begin{tikzcd}
 & Z \arrow[ddl, bend right, "f_1"'] \arrow[dd, bend right=40, "f_2"'] \arrow[ddr, bend left, "f_3"] \arrow[d, dashed, "\prod f_i"] & \\
 & \prod X_i \arrow[dl, "\pi_1"] \arrow[d, "\pi_2"] \arrow[dr, "\pi_3"] & \\
X_1 & X_2 & X_3
\end{tikzcd}$$

In `set`, this is just $X_1 \times X_2 \times X_3$.

Is $\sqcap$ associative? In `set`, it's associative up to canonical isomorphism:

$$((x, y), z) \mapsto (x, (y, z)) : (X \times Y) \times Z \to X \times (Y \times Z)$$

This generalizes to categories, *i.e.*, without cracking open the objects:

Theorem 32

$$(X \sqcap Y) \sqcap Z \cong X \sqcap Y \sqcap Z \cong X \sqcap (Y \sqcap Z)$$

and the isomorphisms, called associators*, are natural.*

Proof · It suffices to show $(X \sqcap Y) \sqcap Z$ satisfies the universal property for 3-products. Then they are unique up to unique isomorphism.

Start with a 2-ary product:

$$\begin{tikzcd}
& Z \arrow[ddl, bend right, "f"'] \arrow[d, dashed, "f \sqcap g"] \arrow[ddr, bend left, "g"] & \\
& X \sqcap Y \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}$$

then $f \sqcap g$ is unique.

Repeat the process

$$\begin{tikzcd}
& Z \arrow[ddl, bend right, "f \sqcap g"'] \arrow[d, dashed, "(f \sqcap g) \sqcap h"] \arrow[ddr, bend left, "h"] & \\
& (X \sqcap Y) \sqcap Z \arrow[dl] \arrow[dr] & \\
Y & & Z
\end{tikzcd}$$

Again, this factors uniquely. Hence this satisfies the universal property. ☐

Finally, we need the other kind of product of arrows.

Definition 33 If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, there is an induced map

$$f_1 \times f_2 : X_1 \sqcap X_2 \to Y_1 \sqcap Y_2$$

defined by

$$(f_1 \circ \pi_1) \sqcap (f_2 \circ \pi_2)$$

$$\begin{tikzcd}
 & X_1 \Pi X_2 \arrow[ld, "\pi_1"'] \arrow[rd, "\pi_2"] \arrow[dd, dashed, "f_1 \times f_2"] & \\
X_1 \arrow[dd, "f_1"] & & X_2 \arrow[dd, "f_2"] \\
 & Y_1 \Pi Y_2 \arrow[ld, "\pi_1"] \arrow[rd, "\pi_2"'] & \\
Y_1 & & Y_2
\end{tikzcd}$$

Definition 34 (Group object) Suppose $G \in \mathcal{C}$ and $\mathcal{C}$ has all finite products (including a terminal object, F, the 0-ary product). Suppose there's a multiplication

$$m : G \Pi G \to G,$$

a unit

$$e : 1 \to G$$

and an inverse ($n$egation)

$$n : G \to G.$$

Then G, m, e, n in a group if

- m is associative, *i.e.*,

$$\begin{tikzcd}
 & (G \Pi G) \Pi G \arrow[rd, "m \times \mathrm{id}"] & \\
G \Pi (G \Pi G) \arrow[ru, "\alpha"] \arrow[d, "\mathrm{id} \times m"] & & G \Pi G \arrow[d, "m"] \\
G \Pi G \arrow[rr, "m"] & & G
\end{tikzcd}$$

  commutes.

- e is a two-sided unit for m, *i.e.*,

  $$\begin{tikzcd}
  F \sqcap G \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\pi_G"'] & G \sqcap G \arrow[d, "m"] & G \sqcap F \arrow[l, "\mathrm{id} \times m"'] \arrow[dl, "\pi_G"] \\
  & M &
  \end{tikzcd}$$

  commutes.

- n is a two-sided negation, *i.e.*,

  $$\begin{tikzcd}
  & G \sqcap G \arrow[dr, "\mathrm{id} \times n"] & \\
  G \arrow[ur, "\Delta"] \arrow[d] & & G \sqcap G \arrow[d, "m"] \\
  F \arrow[rr, "e"] & & G
  \end{tikzcd}$$

  and

  $$\begin{tikzcd}
  & G \sqcap G \arrow[dr, "n \times \mathrm{id}"] & \\
  G \arrow[ur, "\Delta"] \arrow[d] & & G \sqcap G \arrow[d, "m"] \\
  F \arrow[rr, "e"] & & G
  \end{tikzcd}$$

  where Δ is the the diagonal map.

  Following the top of the pentagon sends

  $$g \mapsto gg^{-1}$$

  and following the bottom path throws out g for the unit:

  $$g \mapsto e$$

  Then commutativity means $gg^{-1} = e$.

Now we can define groups for every category that construction makes sense.

Definition 35 (topological group) A topological group is a group object in top, *i.e.*, m and n are continuous.

Definition 36 (Lie group) A Lie group is a group object in smooth, *i.e.*m and n are smooth.

Theorem 37 *Every object in* vec *has a unique group-object structure, the usual vector-space structure.*

Proof · In vec, the null space 0 is both initial and final. Hence, there is only one $e : 0 \to V$, the zero map, $e = 0$. The identity law must take the form

$$m(x, 0) = x = m(0, x). \tag{1}$$

If you use tensors instead of products, you do get new structures

But because $x \in V$ is arbitrary, eq. (1) fully determines m:

$$m(x, y) = x + y. \qquad \square$$

4 Functors

Categories themselves are a structure, so what are the associated structure-preserving functions? It should preserve composition.

Definition 38 (Functor) A functor $F : \mathcal{C} \to \mathcal{D}$ sends objects to objects, and morphisms to morphisms, preserving composition:

$$F(f \circ g) = F(f) \circ F(g).$$

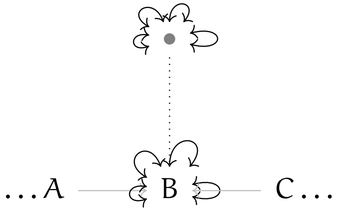
Here's an example:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right] \arrow[d, dotted] & B \arrow[r, "g"] \arrow[d, dotted] & C \arrow[r, "h"] \arrow[d, dotted] & D \arrow[d, dotted] \\
FA \arrow[r, "Ff"] \arrow[rr, bend right, "F(g\circ f)=Fg\circ Ff"'] & FB \arrow[r, "Fg"] & FC \arrow[r, phantom, "="] & FD \arrow[loop right, "Fh"]
\end{tikzcd}$$

Functors are everywhere. If you say the word "induced" and you're not talking about Faraday's law, you have a functor.

Whenever we represent an object's structure with a category directly, like a group as a one-object category, or a poset as a category where arrows are the order, functors are the usual structure-preserving functions. For groups in the sense of Definition 9 (Group), functors are just homomorphisms. Because a group action is a homomorphism into an automorphism group,

> **Definition 39 (Group action)** A group action is a functor from a group to any category.



And if we specialize the target,

> **Definition 40 (Representation)** A representation is a functor from a group into vec.

Definition 41 (Smooth group action) A smooth group action is a functor from a group into `smooth`.

Similarly, functors between individual posets are just non-decreasing functions.

The most famous functor is the derivative. Strictly speaking, it's the tangent bundle and derivative together. The tangent bundle acts on objects, the derivative on arrows:

$$\begin{array}{rccc} d: & \texttt{smooth} & \to & \text{Tangent bundles} \\ \texttt{Ob}: & M & \mapsto & TM \\ \texttt{Ar}: & \Big(f : M \to N\Big) & \mapsto & \Big(df : TM \to T\Big) \end{array}$$

It preserves composition because of the chain rule. Similarly, the derivative at a point, $d_\square$ is a functor from $\texttt{smooth}_*$, the category of *pointed* manifolds—manifolds with a point singled out—, to vector spaces (we need to know what point to put the tangent plane on):

$$\begin{array}{rccc} d_\square: & \texttt{smooth}_* & \to & \texttt{vec} \\ \texttt{Ob}: & (M, x) & \mapsto & T_x M \\ \texttt{Ar}: & \Big(f : (M, x) \to (N, f(x))\Big) & \mapsto & \Big(df_x : T_x M \to T_{f(x)} N\Big) \end{array}$$

An important special case: d_1 is a functor from Lie groups to Lie algebras.

$$\begin{array}{rccc} d_1: & \text{Lie groups} & \to & \text{Lie algebras} \\ \texttt{Ob}: & G & \mapsto & \mathfrak{g} \\ \texttt{Ar}: & \Big(f : G \to H\Big) & \mapsto & \Big(d_1 f : \mathfrak{g} \to \mathfrak{h}\Big) \end{array}$$

Mathematical objects are usually thought of as underlying sets plus a structure. Category theory, on the other hand, pretends objects are atomic. To bring these ideas together, here's a category theory–style way to get at the underlying sets:

Definition 42 (Forgetful functor) If the objects of $\mathcal{C}$ are sets with an additional structure, and the arrows are structure-preserving functions, then there's a functor

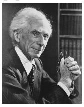
$$\mathtt{forget} : \mathcal{C} \to \mathtt{set}$$

that forgets the additional structure.

Since any structure-preserving function is already a plain-old function, their composition is plain-old composition. Hence `forget` commutes with function composition, so it really is a functor. If you think of, *e.g.*, `vec` as a subcategory of `set`, then $\mathtt{forget} : \mathtt{vec} \to \mathtt{set}$ is just the inclusion functor.

If we're serious about thinking of functions as preserving the category structure,

Don't do it! [4]



Definition 43 (`cat`: the category of categories) The objects of `cat` are "all" categories. The morphisms are the functors between them.

The most famous functor among programmers is the `maybe` functor, also called `optional`. It lets you bail out of computation early.

Definition 44 (`maybe`) Let $\perp$ represent failure. For a type A,

$$\mathtt{maybe}\, A \coloneqq A \cup \perp.$$

A function $f : A \to B$ induces the function that gives up if you pass nothing, otherwise it computes f:

$$(\mathtt{maybe}\, f)\, x = \begin{cases} \perp & \text{if } x = \perp \\ f(x) & \text{if } x \in A \end{cases}$$

This is a special case of the `list` functor, where the lists are only length 0 or 1.

Definition 45 (`list`)

$$\begin{array}{rccc} \mathtt{list} : & \text{types} & \to & \text{types} \\ \mathtt{Ob} : & A & \mapsto & \mathtt{list}\, A \\ \mathtt{Ar} : & \Big(f : A \to B\Big) & \mapsto & \Big([x, y, \ldots] \mapsto [f(x), f(y), \ldots]\Big) \end{array}$$

4.1 Why algebraic topology works

Algebraic topology is the study of functors from `top` to some algebraic category, like `ab` or `grp`. Why does this work? The arrows in and out of a topological space tell us about its topology. Hence, the category of topological spaces remembers topological structure. Because functors preserve the categorical structure, clever use of functors out of `top` present us this topological structure in a way that's useful.

Confronting `top` directly is too hard. There's overwhelmingly many possible continuous functions, and topological structures can be extremely subtle. There's just too much going on in `top`. Using functors to send spaces to simpler objects, like abelian groups, simplifies the structures while retaining information. And, to the extent arrows in `top` capture the structure of spaces (which is fully), functors are the right tool for the job.

Asking if two 4-manifolds have the same fundamental group is already undecidable

- Each degree of homology is a functor from topological spaces to abelian groups. $H_n : \mathtt{top} \to \mathtt{ab}$
- The fundamental groupoid is a functor from topological spaces to groupoids.
- The fundamental group is a functor from *pointed* topological spaces to groups (you need to pick a base point).
- Each degree of *co*homology is a functor $H^n : \mathtt{top}^{op} \to \mathtt{ab}$

Homology and cohomology in particular seem to hit a sweet spot of retaining enough detail to tell us interesting things, but simplifies enough to be usable.

4.2 Contravariance

Definition 46 (Contravariant functor) A contravariant functor is a functor that reverses arrows. Equivalently, a contravaraint functor from $\mathcal{C} \to \mathcal{D}$ is an ordinary (covariant) functor $\mathcal{C}^{op} \to \mathcal{D}$

These show up all over too.

If you consider two partially ordered sets as categories, contravariant functors are order-reversing (non-increasing) functions.

The powerset is a functor set → ordered. Actually, there's two. One's covariant and one's contravariant. They are both familiar. If $f : A \to B$, then f acts on a subset $\alpha \subseteq A$ via

$$\alpha \mapsto f(\alpha) \subseteq B.$$

This is a covariant functor [9, p. 243]. On the other hand, f acts on $\beta \subseteq B$ via

$$\beta \mapsto f^{-1}(\beta) \subseteq A$$

This is going backwards, so it's contravariant [9, p. 243].

Both powerset functors undergird point-set topology and measure theory. A function $f : X \to Y$ is continuous iff f^{-1} preserves open sets. Analogously, a function is measurable iff f^{-1} preserves measureable sets. On the other hand, any continuous f sends compact sets to compact sets, *i.e.*, the induced map on $2^X \to 2^Y$ preserves compactness.

Dualizing a vector space, $\square^*$, is a contravariant functor:

$$\begin{array}{lccc}
\square^* : & \mathtt{vec} & \to & \mathtt{vec} \\
\mathtt{Ob} : & X & \mapsto & X^* = \hom(X, R) \\
\mathtt{Ar} : & \left(f : X \to Y\right) & \mapsto & \left(f^* : Y^* \to X^*\right)
\end{array}$$

where

$$f^*(y) = y \circ f : X \to \mathbb{R}.$$

The cotangent bundle T^* is a contravariant functor (such an unfortunate naming collision) from `smooth` → `vecBundle`. A function $f : M \to N$ induces a function $f^* : T^*N \to T^*M$.

I learned functors before manifolds, so the fact that T^* is a contravariant functor but contains covariant vectors confused me for weeks

The hom functors are important in category theory. Every category has them!

Definition 47 (Covariant hom) For an object $A \in \mathcal{C}$,

$$\hom(A, \square)$$

is the covariant hom functor at A. It's defined by

$$\begin{array}{rccc}
\hom(A,\Box): & \mathcal{C} & \to & \mathtt{set} \\
\mathtt{Ob}: & B & \mapsto & \hom(A,B) \\
\mathtt{Ar}: & \big(f : B \to C\big) & \mapsto & \big(f \circ \Box : \hom(A,B) \to \hom(A,C)\big)
\end{array}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[dr, dashed] & B \arrow[d, "f"] \\
 & C
\end{tikzcd}$$

Definition 48 (Contravariant hom) For an object $Z \in \mathcal{C}$,

$$\hom(\Box, Z)$$

is the contravariant hom functor at Z.

$$\begin{array}{rccc}
\hom(\Box,Z): & \mathcal{C} & \to & \mathtt{set} \\
\mathtt{Ob}: & A & \mapsto & \hom(A,Z) \\
\mathtt{Ar}: & \big(f : B \to C\big) & \mapsto & \big(\Box \circ f : \hom(C,Z) \to \hom(B,Z)\big)
\end{array}$$

$$\begin{tikzcd}
B \arrow[d, "f"'] \arrow[dr, dashed] & \\
C \arrow[r] & Z
\end{tikzcd}$$

Dualizing vector spaces is a special case of Contravariant hom:

$$\Box^* = \hom(\Box, \mathbb{R})$$

5 Natural transformations

The original motivation of category theory was to pin down what it means for a construction to be *natural*. In the paper [9] that loosed category theory onto the world, Eilenberg & Mac Lane open with

> "The subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its 'dual' or 'conjugate' space L^*...Since L^* is in its turn a real vector space with the same dimensions as L, it is clear that L and L^* are isomorphic. But such an isomorphism *cannot be exhibited* until one *chooses* a definite set of basis vectors for L, and, furthermore the *isomorphism which reults will differ for different choices* of this basis.
>
> For the iterated conjugate space L^{**}, on the other hand, it is well known that one can exhibit an isomorphism between L and L^{**} *without using any special basis* in L. This exhibition of the isomorphism $L \cong L^{**}$ is 'natural' in that is given simultaneously for all finite-dimensional vector spaces L" [9, p. 231].

italics mine, notation changed for consistency

This idea of *naturality* is closely related to the idea of *canonicity*, requiring no arbitrary choices. This is a generalization of uniqueness. Usually, when there's a canonical construction, there's only one thing you can do.

The isomorphism from L to L^{**} is canonical in this sense because, with no other information available, there's only one (mod rescaling) nontrivial way to turn an $x \in L$ into an element of $L^{**} = \hom_{\mathbb{R}\text{-linear}}(L^*, \mathbb{R})$: pass it to $\phi : L \to \mathbb{R}$.

The trivial way is $\_ \mapsto 0$.

$$\phi \mapsto \phi(x).$$

There's nothing else you can do because x and ϕ are totally arbitrary. There's not much all vectors, vector spaces, & their duals have in common.

The goal of [9], and, in turn, us, is to capture this. First, for something to be natural, we need *totality*: one construction that works for everything. Second, we need *path-independence*: if the construction is canonical, the specific way you get to it shouldn't matter.

The key insight of [9] is that the correct setting to define naturality is *between functors.*

For the case of $L \cong L^{**}$, we are looking at two functors: the identity functor, 1,

$$\begin{array}{rccc} 1: & \mathsf{vec} & \to & \mathsf{vec} \\ \mathsf{Ob}: & X & \mapsto & X \\ \mathsf{Ar}: & \left(f : X \to Y\right) & \mapsto & \left(f : X \to Y\right) \end{array},$$

and the double-dual functor, $\square^{**}$,

$$\begin{array}{rccc} \square^{**}: & \mathtt{vec} & \to & \mathtt{vec} \\ \mathtt{Ob}: & X & \mapsto & X^{**} \\ \mathtt{Ar}: & \left(f : X \to Y\right) & \mapsto & \left(f^{**} : X^{**} \to Y^{**}\right) \end{array}.$$

For a space L, the map 1 to $\square^{**}$ is

$$D_L \coloneqq x \mapsto \square(x) : L \to L^{**}.$$

Every finite dimensional vector L space gets its own D_L. We have totality. Drawing only two cases looks like

$$\begin{tikzcd} X \arrow[r, "D_X"] \arrow[d, "f"'] & X^{**} \arrow[d, "f^{**}"] \\ Y \arrow[r, "D_Y"'] & Y^{**} \end{tikzcd}$$

To capture *path-independence*, require the above square commutes:

$$D_y \circ f = f^{**} \circ D_x.$$

In summary,

> **Definition 49 (Natural transformation)** Suppose $F, G : \mathcal{C} \to \mathcal{D}$ are two functors. A natural transformation τ from F to G,
>
> $$\tau : F \Rightarrow G,$$
>
> is an arrow τ_X for each $x \in \mathtt{Ob}(\mathcal{C})$ where, for all $X, Y \in \mathtt{Ob}(\mathcal{C})$,
>
> $$\begin{tikzcd} X \arrow[d, "f"'] & F(X) \arrow[r, "\tau_X"] \arrow[d, "Ff"'] & G(X) \arrow[d, "Gf"] \\ Y & F(Y) \arrow[r, "\tau_Y"'] & G(Y) \end{tikzcd}$$
>
> A natural *equivalence*, also called a natural *isomorphism*, is an invertible natural transformation.

Theorem 50 *There is no natural isomorphism* $1 \Rightarrow \square^*$.

Proof · First, they go in opposite directions. $1 : \mathsf{vec} \to \mathsf{vec}$ is covariant, but $\square^* : \mathsf{vec}^{\mathrm{op}} \to \mathsf{vec}$ is contravariant. We can adapt the naturality condition, but it ends up rather restrictive [9, p. 234]:

$$\begin{tikzcd}
X \arrow[r, hook, two heads, "\tau_X"] \arrow[d, "f"'] & X^* \\
Y \arrow[r, hook, two heads, "\tau_Y"'] & Y^* \arrow[u, "f^*"']
\end{tikzcd} \tag{2}$$

must commute, but this is impossible unless f is injective, for every f. $\square$

This gives another perspective on *why* orthogonal matrices are invertable. In the category of finite-dimensional *inner* product spaces, whose arrows are orthogonal matrices, there really *is* a natural equivalence $X \cong X^*$ given by

$$x \mapsto \langle x, \square \rangle,$$

so eq. (2) does commute! Hence, every arrow in the category must be an injective function. An analogous result works for unitary matrices if you're careful with anti-linearity.

If you're wondering why the naturality square looks familiar,

Definition 51 (Equivariant map) Let G be a group. Suppose $A : G \to \mathrm{Aut}(X)$ and $B : G \to \mathrm{Aut}(Y)$ are group actions (functors). Then an equivariant map is a natural transformation $f : A \Rightarrow B$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "A(g)"'] & Y \arrow[d, "B(g)"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Natural transformations resemble homotopies between paths. If you think of Ff and Gf as paths, then the natural transformation deforms $F \Rightarrow G$. We can make this insight precise as follows.

Definition 52 (homotopies = natural transformations) Let $I = \pi[0,1]$ be the fundamental groupoid of $[0,1]$. It looks like this:

$$0 \xrightarrow{\eta} 1$$

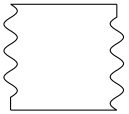
with the inverse of η and identity arrows omitted. This plays the role of time. Let X be a topological space. Take two paths

$$f, g : [0,1] \to X.$$

Passing to the fundamental groupoid turns them into functors from I to πX

$$\begin{tikzcd}
\text{in } I & \text{in } X & \\
& \pi f & \pi g : \\
0 \arrow[d, "\eta"] & f_0 \arrow[d, "(\pi f)\eta"'] & g_0 \arrow[d, "(\pi g)\eta"] \\
1 & f_1 & g_1
\end{tikzcd}$$

which send $0, 1$ to endpoints of the curve and send η to the interior of the curve. Then a natural transformation τ really is a homotopy between them.



$$\Downarrow$$

$$\begin{tikzcd}
0 \arrow[d, "\eta"] & f_0 \arrow[r, "\tau"] \arrow[d, "(\pi f)\eta"'] & g_0 \arrow[d, "(\pi g)\eta"] \\
1 & f_1 \arrow[r, "\tau"] & g_1
\end{tikzcd}$$

The requirement the square commutes says you can actually fill in the square. At this level of abstraction, this commutavity is the *only* difference between an honest-to-god homotopy and any four arrows that connect four dots. [2, p. 228]

In categories that aren't groupoids, natural transformations need not be invertable. In that sense, a natural transformation is a homotopy with an associated direction.

Theorem 53 *There is a natural isomorphism between a vector space V and the tangent space T_aV [8, p. 59]. Consider vec_*, the category of pointed vector spaces. Let $F : \mathsf{vec}_* \to \mathsf{vec}$ be the functor that forgets the distinguished point. Let $T_{\square} : \mathsf{vec}_* \to \mathsf{vec}$ be the tangent space functor. Then the isomorphism*

$$\tau_{(V,a)} = (\partial_{\square})_a : V \to T_aV$$

is natural.

Proof ·

$$\begin{tikzcd}
(V,a) \arrow[d, "L"] & V \arrow[r, "\tau_{(V,a)}"] \arrow[d, "L"'] & T_aV \arrow[d, "dL_a"] \\
(W,La) & W \arrow[r, "\tau_{(W,La)}"'] & T_{La}W
\end{tikzcd}$$

Because L is linear,

$$\begin{aligned} dL_a(\partial_v)_a &= (\partial_{Lv})_{La} \\ dL_a\tau_{(V,a)}(v) = dL_a(\partial_v)_a &= (\partial_{Lv})_{La} = \tau_{(W,La)}Lv \end{aligned}$$

so the diagram commutes. ☐

Because bundles are often given by functors, their relationships are often natural transformations. For example, for any manifold M there is an inclusion

$$i_M : TM \hookrightarrow T^{\otimes}M$$

where $T^{\otimes}M = \bigoplus (TM)^{\otimes n}$ is the the tensor-bundle functor. It is natural: the square

$$\begin{tikzcd}
TM \arrow[r, "i_M"] \arrow[d, "df"'] & T^{\otimes}M \arrow[d, "df^*"] \\
TN \arrow[r, "i_N"'] & T^{\otimes}N
\end{tikzcd}$$

commutes because the induced map on tensor bundles agrees with the induced map on tangent bundles.

Because natural transformations go between functors, they're morphisms in some category

> Definition 54 (Functor category) Let $\mathcal{C}, \mathcal{D}$ be two categories. Define $\mathtt{functors}(\mathcal{C}, \mathcal{D})$ as the category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are natural transformations between them.

6 Adjoints

> Definition 55 (Adjoint) The functor $F : \mathcal{C} \to \mathcal{D}$ is a left-adjoint to $G : \mathcal{D} \to \mathcal{C}$, written
>
> $$F \dashv G$$
>
> if
>
> $$\hom(FX, Y) \cong \hom(X, GY)$$
>
> and the bijection is natural.

Consider $\mathbb{R}$ as an ordered set. Then `floor` and `ceil` are functors $\mathbb{R} \to \mathbb{R}$ and

$$\mathtt{floor} \dashv \mathtt{ceil}$$

because

$$\begin{aligned} \mathtt{floor}\, x \leqslant y \quad &\iff \quad x \leqslant \mathtt{ceil}\, y \\ \hom(\mathtt{floor}\, x, y) \quad &\cong \quad \hom(x, \mathtt{ceil}\, y) \end{aligned}$$

The most famous adjoint pair is the free/forgetful functors. A free functor from $\mathcal{C} \to \mathcal{D}$ sends an object $X \in \mathcal{C}$ to the largest object of $\mathcal{D}$ generated by X.

Definition 56 $\mathtt{free}_{\mathcal{C}}$ is defined as the left adjoint to the forgetful functor:

$$\underset{\mathcal{C}}{\mathtt{free}} \dashv \underset{\mathcal{C}}{\mathtt{forget}}\,.$$

Explicitly,

$$\underset{\mathcal{C}}{\hom}\left(\underset{\mathcal{C}}{\mathtt{free}}\,X, Y\right) \cong \underset{\mathrm{set}}{\hom}\left(X, \underset{\mathcal{C}}{\mathtt{forget}}\,Y\right)$$

naturally.

Theorem 57

$$\underset{\mathrm{vec}}{\mathtt{free}} = \mathrm{span} \qquad \textit{[5, p. 79]}.$$

Proof · Suppose $X = \{x_i\}$. Then

$$\underset{\mathcal{C}}{\hom}\left(\underbrace{\underset{\mathrm{vec}}{\mathtt{free}}\,\{x_i\}}_{?}, Y\right) \cong \underset{\mathrm{set}}{\hom}\left(\{x_i\}, \underset{\mathrm{vec}}{\mathtt{forget}}\,Y\right)$$

The left-hand side is the set of *linear* functions from the (unknown) $\mathtt{free}\,\{x_i\}$. The right-hand side is the set of *all* functions between $\{x_i\} \to Y$. Using the indexing, an arbitrary function $f : \{x_i\} \to Y$ is of the form $x_i \mapsto f(x_i) = y_i$.

We want to extend $\{x_i\}$ to a vector space while extending f to a linear map $\bar{f} : \mathtt{free}\,X \to Y$. Use Y as a point of contact between f and $\bar{f}$: identity x_i with the vector $\overline{x_i}$ defined by $\bar{f}(\overline{x}_i) = y_i$. Therefore, I'll drop the overline on $\overline{x_i}$. Extend $\bar{f}$ by stipulating linearity:

$$\bar{f} = \sum_i y_i \otimes x^i = u \mapsto \sum_i y_i x^i(u). \tag{3}$$

As long as $\mathtt{free}\,X$ has no extraneous vectors, $\bar{f}$ is uniquely determined by eq. (3). In other words,

$$\mathtt{free}\,X = \mathrm{span}\,X.$$

To summarize,

$$\begin{array}{rccc} \mathtt{free}: & \mathtt{set} & \to & \mathtt{vec} \\ \mathtt{Ob}: & \{x_i\} & \mapsto & \operatorname{span}\{x_i\} \\ \mathtt{Ar}: & f & \mapsto & \overline{f} = \sum f(x_i) \otimes x^i \end{array}$$

Because $\overline{f}$ is uniquely determined by its values on each x_i, we have naturality in both X and Y.

You just feel in your heart it's natural

Here's the details. Pick an arbitrary $g \in \hom_{\mathtt{set}}(\{x_i\}, \{g(x_i)\})$. By linearity, g induces

$$\mathtt{free}\, g : \operatorname{span}\{x_i\} \to \operatorname{span}\{g(x_i)\}$$

Because $\{x_i\}$ is a spanning set, $\mathtt{free}\, g$ is completely determined. Therefore,

$$\begin{tikzcd}
\underset{\mathtt{vec}}{\hom}(\mathtt{free}\{x_i\}, Y) \arrow[r, leftrightarrow, "\overline{\square}"] \arrow[d, "\mathtt{free}\, g"'] & \underset{\mathtt{set}}{\hom}(\{x_i\}, \mathtt{forget}\, Y) \arrow[d, "g"] \\
\underset{\mathtt{vec}}{\hom}(\mathtt{free}\{g(x_i)\}, Y) \arrow[r, leftrightarrow, "\overline{\square}"] & \underset{\mathtt{set}}{\hom}(\{g(x_i)\}, \mathtt{forget}\, Y)
\end{tikzcd}$$

commutes. The equivalence is natural in X.

On the other hand, suppose $h \in \underset{\mathtt{vec}}{\hom}(Y, h(Y))$. Then

$$\begin{tikzcd}
\underset{\mathtt{vec}}{\hom}(\mathtt{free}\{x_i\}, Y) \arrow[r, leftrightarrow, "\overline{\square}"] \arrow[d, "h \circ \square"'] & \underset{\mathtt{set}}{\hom}(\{x_i\}, \mathtt{forget}\, Y) \arrow[d, "(\mathtt{forget}\, h) \circ \square"] \\
\underset{\mathtt{vec}}{\hom}(\mathtt{free}\{x_i\}, h(Y)) \arrow[r, leftrightarrow, "\overline{\square}"] & \underset{\mathtt{set}}{\hom}(\{x_i\}, \mathtt{forget}\, h(Y))
\end{tikzcd}$$

Suppose $f : \{x_i\} \to Y$. Along either route, you get a map that sends $x_i \mapsto h \circ f(x_i)$. Because this uniquely determines the induced linear map,

$$\overline{h \circ f} = h \circ \overline{f},$$

and the diagram commutes. $\square$

6.1 Internal homs and tensors

Usually hom-sets have additional structure. Sometimes, they're even objects in the same category. For example, in `set`, hom-sets are sets, hence objects. In `vec`, pointwise addition & multiplication make

$$\hom_{\texttt{vec}}(X, Y)$$

into a vector space. It's even finite-dimensional when both X and Y are. In `top`,

$$\hom_{\texttt{top}}(X, Y) \subseteq X^Y$$

and X^Y has the product topology. Then $\hom_{\texttt{top}}(X, Y)$ inherits the subset topology, the topology of pointwise convergence.

> Definition 58 (in/external hom) In these cases, it's useful to distinguish between the *external* homs, the set of functions, and the *internal* homs, the object in the category. Denote the *internal* hom as
>
> $$[X, Y] \in \texttt{Ob}(\mathcal{C}) \quad \text{where } X, Y \in \texttt{Ob}(\mathcal{C})$$
>
> A category with internal homs is called *closed.*

With internal homs, we can now have arrows into hom-sets. Consider `vec`.

$$A \xrightarrow{f} [B, C]$$

Choosing an $a \in A$ gives

$$B \xrightarrow{f(a)} C$$

Sometimes I draw this like

$$A \xrightarrow{f} \left[B \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} C \right]$$

but I've never seen that anywhere else.

The map

$$(a, b) \mapsto f(a)(b)$$

is bilinear:

$$(f(a + t\alpha))b = (f(a) + tf(\alpha))b = f(a)(b) + tf(\alpha)(b)$$

and

$$f(a)(b + t\beta) = f(a)(b) + tf(a)(\beta)$$

Internal homs give us one way to define multilinear maps from linear maps. The other way is with the tensor product. A bilinear map can be written

$$f : A \otimes B \to C$$

or

$$f : A \to [B, C].$$

In other words, there's a natural equivalence

$$[A \otimes B, C] \cong [A, [B, C]]$$

i.e., they're adjoints:

> **Definition 59 (Tensor-hom adjunction)** The tensor-hom adjunction is
>
> $$\square \otimes B \dashv [B, \square]$$
>
> It holds in `vec` for example.

In categories with both a tensor product and internal homs, Definition 59 (Tensor-hom adjunction) is usually satisfied. In categories with internal homs only, it suggests an approach to define a tensor product.

In `set`, the tensor product (in this sense) is just the cartesian product. In `set`,

$$[Y, Z] = Z^Y.$$

Then

$$[X, [Y, Z]] = \left(Z^Y\right)^X \cong \left(Z^{X \times Y}\right) = [X \times Y, Z].$$

The `set`-isomorphism (bijection) is natural and given by Currying, *i.e.*, send $f \in [X, [Y, Z]]$

$$(x, y) \mapsto f(x)(y)$$

7 Sheaves

This doesn't really belong here, but it's here anyway.

Before defining sheaves, I'll explain why geometers working in `smooth` don't need them. In `smooth`, we have smooth bump functions, smooth cutoff functions, and smooth partitions of unity. Therefore, we can extend local constructions globally.

Suppose $f : U \to \mathbb{R}$ for $U \subseteq M$. If ϕ is a bump function on U and $\phi(V) = 1$ for some $V \subset U$. Then

$$(\phi f)(x) = \begin{cases} f(x) & \text{if } x \in V \\ 0 & \text{if } x \notin U. \end{cases}$$

It contains essentially the same data as f, but it's defined on all of M. In `smooth`, we can be sloppy with domains.

This is not true anymore for analytic functions. Analytic functions are rigid. Take $f(x) = e^x$ for example. Just from its values on the infinitesimal neighborhood $[-\varepsilon, \varepsilon]$, we can already get its Taylor series:

$$f^{(n+1)} = \lim_{h \to 0} \frac{f^{(n)}(h) - f^{(n)}(0)}{h} = 1$$

fits entirely within $[-\varepsilon, \varepsilon]$. But now

$$f(x) = \sum \frac{f^{(n)}(0)x^n}{n!} = e^x$$

converges on all of $\mathbb{C}$. Its values on an *infinitesimal* neighborhood fix the values *everywhere* in $\mathbb{C}$.

The case e^x is an ideal one, e^x is entire, but the point stands in general:

Theorem 60 (Liouville) *Every holomorphic function $\mathbb{C} \to \mathbb{C}$ is either constant or unbounded [15].*

This eliminates the hope of analytic bump functions. Similarly, there's

Theorem 61 (Maximum modulus principle) *A holomorphic function $U \to \mathbb{C}$ is either unbounded or locally constant [16].*

and the especially frustrating

Corollary 62 *"Any holomorphic function on a compact Riemann surface is necessarily constant" [15].*

The situation is even worse when we look at polynomials and rational functions. A polynomial can't have more zeroes than its degree, so any polynomial that's zero infinitely often is zero constantly. Additionally, rational functions are interesting, but they have poles (`s/rational/meromorphic`). Sometimes the poles are the interesting part. If we want to study rational functions as *functions* instead of formal symbolic expressions, we are forced to be cautious with their domains.

For example, the only rational functions defined on all of $\mathbb{C}$ are plain-old polynomials (`s/rational/meromorphic`, `s/polynomial/holomorphic`). The only rational functions defined on $\mathbb{C} - \{0\}$ are

$$\left\{ \frac{p(x)}{x^n} : n \in \mathbb{N}, p \text{ is a polynomial} \right\}.$$

On $\mathbb{C} - \{a, b\}$, the only rational functions are

$$\left\{ \frac{p(x)}{(x-a)^m (x-b)^n} \right\}$$

Shrinking the domain grows the set of functions, which is backwards from `set`.

To organize the situation, organize the relationships between the functions defined on various open sets. Additionally, we want to ensure the copy of $f(x)$ defined on the set U and the copy defined on V really are the same thing. Enter sheaves.

Well, first, presheaves

Definition 63 (Category of opens) Let τ be the category of open sets on your space ordered by $\subseteq$.

i.e., draw one arrow $A \to B$ iff $A \subseteq B$.

Definition 64 (Presheaves) A presheaf is a functor $F : \tau^{\mathrm{op}} \to$ `set`.

The arrows in τ^{op} are reverse-inclusions, also called *restrictions*. When you restrict from a bigger set to a smaller one, the presheaf follows suit.

- The identity functor is a presheaf, trivially.
- $C^\infty : \tau^{op} \to \mathtt{set}$ is a presheaf on a smooth manifold. Each open set U gets its set of functions $C^\infty(U)$. Restrictions are the usual restrictions of a function's domain.
- $\mathtt{Rational} : \tau^{op} \to \mathtt{set}$ is a presheaf on $\mathbb{C}$. Each open set gets its set of rational functions. Restrictions are still usual restriction of domain, but something interesting happens: restrictions *are not* surjective. $\mathtt{Rational}(\mathbb{C})$ are just polynomials, but $\mathtt{Rational}(\mathbb{C} - \mathbb{Z})$ are every rational function with singularites on $\mathbb{Z}$. Additionally, because polynomials, hence rational functions, are fully determined by their values on finitely many points, restrictions *are* injective.

We're not quite there, though. We want to make sure, in some sense, functions are the same entity after restriction. Wikipedia calls elements in the sheaf sections [17], which I don't love. But I'm not gonna call it wheat fields, so section it is.

Definition 65 (Sheaf) A sheaf is a presheaf $\mathcal{S} : \tau^{op} \to \mathtt{set}$ satisfying locality and gluing. Let U be an open set, covered by U_i,

$$U = \bigcup U_i.$$

- Locality. Sections on U, aka elements in $\mathcal{S}(U)$, are completely determined by their values locally. Two sections are the same if they're the same locally everywhere, *i.e.*, on each U_i. Suppose $s, t \in \mathcal{S}(U)$. Then locality says

$$(\forall i \quad s|_{U_i} = t|_{U_i}) \implies s = t$$

- Gluing. If a collection of sections agree on overlaps, you can glue them together. Suppose there is a family $s_i \in \mathcal{S}(U_i)$ that agrees on overlaps,

  $$\forall i, j \quad s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

  then there is an $s \in \mathcal{S}(U)$ formed by gluing them together:

  $$s_i = S|_{U_i}$$

[17]

Most things on manifolds are sheaves. C^∞ is a sheaf, but so are the space of vector fields, tensor fields, and differential forms. You can restrict them, they're fully defined by their local behavior, and you can glue them when they agree. In `smooth`, this machinery can be overkill, but when you lose partitions of unity, it's crucial.

8 Algebra

This *really* doesn't belong here.

The most important kinds of algebraic spaces roughly mirror the way we teach kids numbers, which makes a good mnemonic

- monoids ~ $(\mathbb{N}, +)$
  There's a binary operation (+) and an identity element (0)

You don't see monoids in abstract algebra 1, but they're all over category theory

- groups ~ $(\mathbb{Z}, +)$ integers
  Now inverses (negatives)

- rings ~ $(\mathbb{Z}, +, \times)$
  Now there's multiplication. It distributes over addition

- fields ~ $(\mathbb{Q}, +, \times)$
  And fractions!

in fields, multiplication *does* need to commute

Addition in rings is commutative. Some authors require multiplication in rings be commutative, but some do not because then $n \times n$ matrices are a ring. Either way, this is a good example to keep in mind, because square matrices show you lots of ways multiplication can fail to be invertable.

Definition 66 (ring) A ring $(R, +, \cdot, 0, 1)$ is

- An abelian group under $(+, 0)$
- A monoid under $(\cdot, 1)$
- And distributive:

$$\begin{aligned} a(b + c) &= ab + ac. \\ (a + b)c &= ac + bc \end{aligned}$$

using group and monoid in the definition lets us omit writing all the associativity laws.

Sometimes people drop the requirement that 1, a multiplicative identity, exists. Sometimes this is called a rng, a ring without *i*dentity. Similarly, sometimes people drop the requirement additive inverses exist and call it a rig, a ring without *n*egatives. I propose the term *rin*, a ring without *g*ommutativity. In other words, I'm assuming rings commute.

Definition 67 (Polynomial ring) Given a ring R, the polynomials with coefficients in $\mathbb{R}$ and formal variable x, written

$$R[x],$$

is a ring.

Similarly, $R[x, y, z]$ is the ring of polynomials in all three variables.

Definition 68 The tensor algebra $V^{\otimes}$ of a vector space V is a ring.

You can't quotient by any old group, only *normal* subgroups. Analogously, you can't quotient by any subrng, only *ideals*. Recall

$$0 + 0 = 0$$

so 0 is closed under addition, and

$$0x = 0,$$

0 is absorbative under multiplication. Any set that can be collapsed to zero must have these properties as well

Definition 69 (Ideal) A subset $I \subseteq R$ is an ideal iff

- I is closed under addition, *i.e.*

  $$\forall x, y \in I \quad x + y \in I$$

  or, more concisely, $I + I = I$.

- I is absorbative under multiplication, *i.e.*

  $$\forall r \in R \, \forall x \in I \quad rx \in I$$

  or, more concisely, $RI = I$.

The ideal generated by x is written (x).

In a *rin*, you have to distinguish left-ideals from right-ideals, so I'm skipping them (I'm committed to the bit).

This is already enough to do something interesting:

Definition 70 (Dual numbers) The dual numbers are

$$\mathbb{R}[x]/\left(x^2\right)$$

Dual numbers contain nilpotent elements. After taking quotients, $x^2 = 0$. As a consequence, x is non-invertable.

Definition 71 (Complex numbers)

$$\mathbb{C} = \mathbb{R}[x]/\left(x^2 + 1\right)$$

Interestingly, $\mathbb{C}$ is a field: every nonzero complex number is invertable, even though the definition looks almost the same as dual numbers. The reason for this is $x^2 + 1$ is an *irreducible* polynomial over $\mathbb{R}$. We see the same thing happen in $\mathbb{Z}/(4)$ vs $\mathbb{Z}/(5)$. In $\mathbb{Z}/(4)$, $2^2 = 4 \equiv 0 \pmod 4$, but every nonzero element is invertable in $\mathbb{Z}/(5)$ because 5 is prime.

Definition 72 (Hyperbolic numbers)

$$\mathbb{R}[x]/\left(x^2 - 1\right)$$

Because $x^2 - 1 = (x+1)(x-1)$, this isn't a field. In fact, after taking mods, we get

$$(x+1)(x-1) = 0.$$

Definition 73

$$\mathbb{Q}[\sqrt{2}] = \mathbb{Q}[x]/\left(x^2 - 2\right)$$

Because $x^2 - 2$ is irreducible over $\mathbb{Q}$, we get a field again.

Theorem 74 *In $\mathbb{Q}$, every ideal trivial—either $\{0\}$ or $\{\mathbb{Q}\}$.*

Proof · Suppose I is an ideal. If $I = \{0\}$, we're done. Otherwise, there is some $x \in I$ such that $x \neq 0$. But then $x^{-1} \in \mathbb{Q}$ exists. By hypothesis I is absorbative, so

$$Ix^{-1} = I$$

But then $xx^{-1} = 1 \in I$. Applying absorbativity again, $y = y1 \in yI = I$ for any $y \in \mathbb{Q}$. Hence $I = \mathbb{Q}$. □

Actually, this proof generalizes. Ideals measure the failure of a ring to be a field.

Theorem 75 *A ring R is a field iff its only ideals are R and 0.*

Proof ·

- If ring, the only ideals are 0 and R.
  Take an ideal $I \neq 0$. Pick a nonzero $x \in I$. If x has an inverse, then by absorbativity
  $$xx^{-1} = 1 \in I.$$
  But then, by absorbativity, $I = R$.
- If the only ideals are R and 0, it's a ring.
  Pick a nonzero $x \in R$. Then (x), the set of multiples of x, is an ideal. By hypothesis, it contains 1. Therefore, there is some y such that
  $$xy = 1.$$
  □

Ideals generalize the idea of multiples of a number:

ideals like (a) are called *principal*

Theorem 76 *Every ideal in $\mathbb{Z}$ is of the form (a), the set of multiples of a.*

Proof · Suppose I is an ideal in $\mathbb{Z}$. If $I = \{0\}$, then $I = (0)$, so we're done.

Suppose $I \neq 0$. Let a be the smallest positive element of I. Let $b > a$ be an element of I. By repeated subtraction (Euclid's algorithm), find $\mathsf{gcd}(a, b)$. Because I is closed under subtraction, $\mathsf{gcd}(a, b) \in I$. By construction,

$$\mathsf{gcd}(a, b) \leqslant a$$

But by assumption a is minimal,

$$a \leqslant \mathsf{gcd}(a, b)$$

so $a = \mathsf{gcd}(a, b)$. Then b is a multiple of a.

By negating everything, the same argument applies to the negative side of I. Conclude

$$I = (a),$$

the ideal generated by a. □

In "easy" rings, ideals are just multiples. If a ring has a gcd algorithm, then ideals are of this form. Unfortunately, life is not so simple.

In a *principal ideal domain*, every ideal is principal.

Theorem 77 $\mathbb{C}[x, y]$ *has an ideal that's not merely a set of all multiples. It's given by*

$$(x, y)$$

Proof · Consider

$$(x, y) = \{\text{polynomials with } 0 \text{ constant term}\}.$$

Certainly, we can quotient by (x, y), in effect, setting $x = y = 0$.

However, there is no $\gcd(x, y)$ because both x and y are indeterminate, hence irreducible. ☐

Theorem 78 (Unique factorization fails) *Unique factorization fails in*

$$\mathbb{Z}\left[\sqrt{-5}\right].$$

There are two distinct irreducible factorizations of 6*,*

$$\left(1 + \sqrt{-5}\right)\left(1 - \sqrt{-5}\right) = 6 = (2)(3).$$

irreducible $\neq$ prime, but never mind. They're the same in $\mathbb{Z}$, but not everywhere.

Proof · The trick is to use the norm

$$N\left(a + b\sqrt{-5}\right) := \left(a + b\sqrt{-5}\right)\left(a - b\sqrt{-5}\right) = a^2 + 5b^2.$$

It's multiplicative,

$$N(xy) = N(x)N(y),$$

so a necessary condition for x divides z is $N(x)$ divides $N(z)$. This screens out possible factors, so we can check irreducibility.

- 2 is irreducible.
  Suppose $xy = 2$. Then

  $$N(x)N(y) = N(2) = 4 = 1 \cdot 4 = 2 \cdot 2.$$

  Without loss of generality, either $N(x) = 1$ or $N(x) = 2$. Write $x = a + b\sqrt{-5}$. Suppose

  $$N(x) = a^2 + 5b^2 = 1 \implies a = \pm 1, b = 0. \tag{4}$$

so $x = \pm 1$. Alternately, suppose

$$N(x) = a^2 + 5b^2 = 2 \implies a^2 = 2 \implies \text{no solution over } \mathbb{Z}$$

- 3 is irreducible.
  By the same argument,

  $$N(3) = 9 = 1 \cdot 9 = 3 \cdot 3$$

  so either $N(x) = 1$ or $N(x) = 3$. From eq. (4), we know the only solution for $N(x) = 1$ is $x = \pm 1$. So consider $N(x) = 3$:

  $$N(x) = a^2 + 5b^2 = 3 \implies a^2 = 3 \implies \text{no solution over } \mathbb{Z}$$

- $1 + \sqrt{-5}$ is irreducible.

  $$N\left(1 + \sqrt{-5}\right) = 1 + 5 = 6 = 1 \cdot 6 = 2 \cdot 3$$

  So if $xy = 6$, then either $N(x) = 1$ or $N(x) = 2$. But we already checked the only solution is $x = \pm 1$.

- $1 - \sqrt{-5}$ is irreducible.

  $$N\left(1 - \sqrt{-5}\right) = 1 + 5 = 6 = 1 \cdot 6 = 2 \cdot 3,$$

  so the same argument works. $\square$

Algebraists, unwilling to give up unique factorization, defined maximal and prime ideals.

Definition 79 (Maximal ideal) An ideal I over ring R is maximal if the only bigger ideal is all of R.

In $\mathbb{Z}$, maximal ideals are exactly (p) when p is prime.

An analogous result holds in $\mathbb{C}[x]$. Because of the fundamental theorem of algebra, every polynomial splits into linear factors, uniquely. Thus, the maximal ideals are exactly $(x - a)$.

the fundamental theorem is necessary. This only works for polynomials over *algebraically complete* fields

Every maximal ideal is also prime:

Definition 80 (Prime ideal) An ideal P is prime iff

$$P \ni ab \implies P \ni a \text{ or } P \ni b$$

in analogy with the fact

"$|$" means divides

$$p \mid ab \implies p \mid a \text{ or } p \mid b$$

for p a prime integer.

Prime ideals need not be maximal. In fact, the minimal (zero) ideal is often prime

Lemma 81 *In a ring with no zero-divisors, the ideal* (0) *is prime.*

The converse is true too.

$\mathbb{Z}/(4)$ has zero-divisors: $2^2 = 0$.

Dual numbers have zero-divisors, $\varepsilon^2 = 0$.

Proof · If R has no zero-divisors, then $0 = ab$ only if $a = 0$ or $b = 0$. Suppose

$$(0) \ni ab.$$

Because $(0) = \{0\}$, conclude $0 = ab$. By hypothesis, either $a = 0$ or $b = 0$. Suppose $a = 0$. Then

$$(0) \ni a.$$

By the same argument, if $b = 0$ then $(0) \ni b$. □

The prime ideals of $\mathbb{Z}$ are the maximal ideals *and* (0). This is typical of "easy" rings. When you have a $\gcd$ algorithm, every maximal ideal is generated by an irreducible element, and the prime ideals are maximal or (0).

8.1 Algebraic geometry

In algebraic algebraic geometry, ideals are systems of equations, more or less. Consider the ring of polynomials

$$\mathbb{C}[x, y, z].$$

Any system of polynomial equations generates an ideal. For example,

$$\left.\begin{aligned} x &= z^2 \\ y &= 3z \end{aligned}\right\} \implies \left(x - z^2, y - 3z\right)$$

Quotienting out by this ideal amounts to setting

$$\begin{aligned} x - z^2 &= 0 \\ y - 3z &= 0 \end{aligned}$$

The basic manipulations to solve or simplify systems of equations, like Gaussian elimination, stay within the ideal. For example, squaring $y = 3z$ translates to multiplying $y - 3z$ by $y + 3z$:

$$(y - 3z)(y + 3z) = y^2 - 9z^2$$

so that, too, is in the ideal. Continuing down that road will reveal

$$9x - y^2 \in \left(x - z^2, y - 3z\right)$$

You can encode points in the base space as systems of equations, hence ideals. The point (a, b, c) becomes the system $x = a, y = b, z = c$ becomes the ideal

$$(x - a, y - b, z - c).$$

This observation will become Hilbert's Weak Nullenstellensatz.

This classical result about polynomial division tells us a lot about ideals in $\mathbb{C}[x]$:

Theorem 82 (Remainder-root theorem) *Suppose* $p(x)$ *is a polynomial. Then*

$$p(x) \equiv p(a) \pmod{x - a}.$$

Explicitly,

$$p(x) = q(x)(x - a) + p(a).$$

Polynomial evaluation is a special case of quotienting by ideals. This characterizes the ideals generated by linear factors:

$$(x - a) = \{p(x) : p(a) = 0\}.$$

Theorem 83 (Hilbert's Weak Nullenstellensatz) *The maximal ideals of*

$$\mathbb{C}[x_1, \dots x_n]$$

are exactly

$$(x_1 - a_1, \dots x_n - a_n) \qquad \textit{[12, p. 107].}$$

If we think of polynomials as *functions* over the space $\mathbb{C}^n$, taking maximal ideals brings us back to the base space. Taking quotients by maximal ideals *evaluates* the function at the corresponding point.

[7, p. 56] has an analogous theorem for ideals of continuous functions. It needs convergence. (Algebraists doing analysis)

This hints at a *geometric* interpretation of ideals. Suppose I is an ideal, not maximal, over $\mathbb{C}[x_1 \dots x_n]$. Then I can be refined to some maximal ideal:

$$I \subseteq (x_1 - a_1, \dots x_n - a_n).$$

Geometrically, this means the subset of $\mathbb{C}^n$ associated to I contains the point $(a_1, \dots a_n)$.

What about solution sets of systems of equations? Consider the solution set of $y = x^2$ over $\mathbb{C}^2$. But instead of carving the solution out of the complex plane, use ideals in $\mathbb{C}[x, y]$:

$$\mathbb{C}[x, y]/(y - x^2)$$

What does this quotient look like? It just sets $y = x^2$

$$p(x, y) \mapsto p(x, x^2) : \mathbb{C}[x, y] \to \mathbb{C}[x]$$

We've restricted the functions in $\mathbb{C}[x, y]$ to $y = x^2$ by taking this quotient.

What are the maximal ideals containing $(y - x^2)$? The points on the parabola! For example, if $a \in \mathbb{C}$, the point (a, a^2) lies on the parabola, and the ideal

$$(x - a, y - a^2) \supseteq (y - x^2)$$

Proof · Here's the details. To be in the ideal, $y - x^2$ must be a linear combination of $x - a$ and $y - a^2$. You can start squaring and solving, but ordinary polynomial division with remainder does the trick.

Because the division algorithm only works for one variable at a time, treat y as a constant.

$$\begin{array}{rllll}
 & & x & +a & \\ \hline
x - a \big) & x^2 & & & -y \\
 & x^2 & -ax & & \\ \hline
 & & ax & & \\
 & & ax & -a^2 & \\ \hline
 & & & a^2 & -y
\end{array}$$

so the remainder is $-y + a^2$ and, all together,

$$-y + x^2 = (x-a)(x+a) - (y - a^2) \in \left(x - a, y - a^2\right) \qquad \square$$

9 Reading list

- Eilenberg & Mac Lane *Categories for the working mathematician* [5] is the canonical reference for category theory. They are founders. But the book is hard.
- The Catsters have a series of short youtube lectures on category theory. That's where I learned universal properties `https://www.youtube.com/user/TheCatsters/`.
- Baez & Stay "Physics, topology, logic and computation: a Rosetta stone" [11] looks at correspondences in categories, particularly monoidal categories and braidings, of interest in all these subjects.
- Brown's *Topology and Groupoids* [2]. If you want to learn category theory through the fundamental groupoid, read chapter 6.
- Aluffi's *Algebra Chapter 0* is an introduction to algebra with category theory (category theory is traditionally a grad-school topic).

- Dummit & Foote [3] covers a ton of undergrad algebra, but no category theory.
- Vakil's *Rising Sea* is an introduction to the machinery of modern algebraic geometry. From what I've read so far, its writing's clear and stresses intuition.
- Lang's *Algebra*. I brought up a Lang book to a professor once, who said "he's' a fallen Bourbaki-ist". Not a good first algebra book, but it has a lot of info, generality, and a sense of taste, if not too Bourbaki.

References

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