1 Rotation matrix facts

Start small

$$R_\theta \coloneqq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Eigenvalues

$$\begin{aligned} 0 &= \det(R_\theta - t) \\ &= \begin{bmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{bmatrix} \\ 0 &= (\cos\theta - t)^2 + \sin^2\theta \\ (\cos\theta - t)^2 &= -\sin^2\theta \\ \cos\theta - t &= \pm i\sin\theta \\ t &= \cos\theta \pm i\sin\theta \\ &= e^{i\theta} \end{aligned}$$

Eigenvectors

$$\begin{aligned} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} &= \begin{bmatrix} ic & -s \\ is & c \end{bmatrix} = \begin{bmatrix} i(c+is) \\ 1(c+is) \end{bmatrix} \\ &= e^{i\theta} \begin{bmatrix} i \\ 1 \end{bmatrix} \\ \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} &= \begin{bmatrix} c - si \\ s + ci \end{bmatrix} = \begin{bmatrix} 1(c-si) \\ i(c-si) \end{bmatrix} \\ &= e^{-i\theta} \begin{bmatrix} 1 \\ i \end{bmatrix} \end{aligned}$$

Diagonalized

$$\begin{aligned} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} &= \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^* \\ &= \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \end{aligned}$$

In general, by Jordan normal form,

$$\begin{aligned} R &\sim \begin{bmatrix} e^{\pm i\theta_0} & & \\ & e^{\pm i\theta_1} & \\ & & \ddots \end{bmatrix} \\ &\sim \begin{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} e^{-i\theta_0} & \\ & e^{i\theta_0} \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} & \\ & \ddots \end{bmatrix} \end{aligned}$$

I think I'm missing an idea to close this off

2 Rotor

$$R = u_0 u_1$$

What does it do on span u_0, u_1? The map $\square R$ rotates counterclockwise (+):

$$\begin{aligned} u_0 R &= u_0 u_0 u_1 = u_1 \\ u_1 R &= u_1 u_0 u_1 = -u_0 \end{aligned}$$

And $R\square$ rotates clockwise (−) as R anticommutes with u_0, u_1:

$$\begin{aligned} R u_0 &= u_0 u_1 u_0 = -u_1 \\ R u_1 &= u_0 u_1 u_1 = u_0 \end{aligned}$$

Because R commutes with the other u_n, this rotates the u_0, u_1 plane and fixes the rest:

$$e^{-tR} \square e^{tR} : \begin{cases} u_0 & \mapsto u_0 e^{2tR} \\ u_1 & \mapsto u_1 e^{2tR} \\ u_n & \mapsto u_n \end{cases}$$

Doran, Lasenby Geometric algebra for physicists 2.7 Rotations

Lemma 1 *Unitary equivalence restricts to orthogonal equivalence*

Proof. ? □