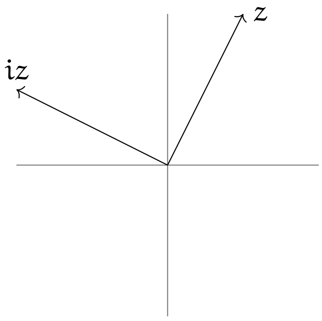
The identity $e^{it} = \cos t + i \sin t$ is a powerful organizing principle for trigonometry. Using *hyperbolic* or *split-complex* numbers (https://en.wikipedia.org/wiki/Split-complex_number) where $j^2 = +1$ provides an analogous tool for hyperbolic trigonometry.

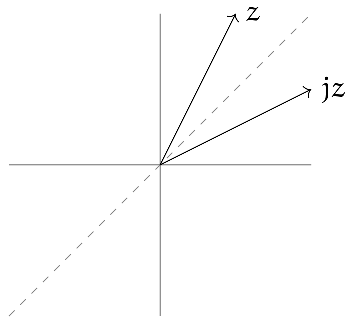
Complex

$$i^2 \coloneqq -1$$
$$i(x + iy) = -y + ix$$



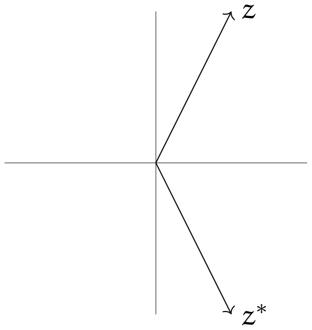
Hyperbolic

$$j^2 \coloneqq +1$$
$$j(x + jy) = y + jx$$

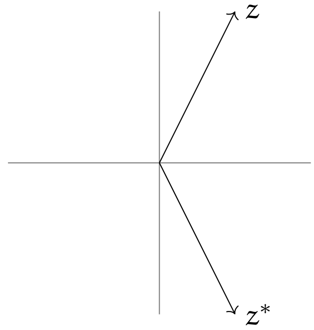


Conjugation and norm

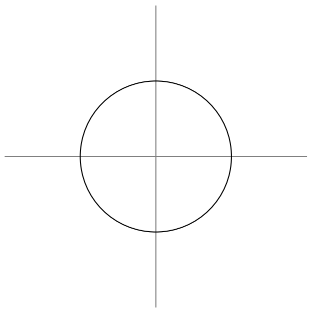
$$(x + iy)^* = x - iy$$



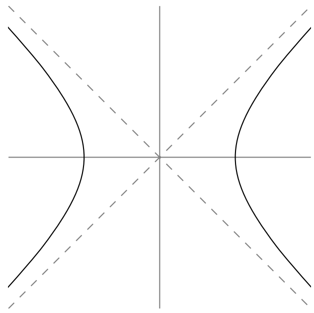
$$(x + jy)^* = x - jy$$



$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x + iy)(x - iy) \\ &= x^2 + y^2 \end{aligned}$$



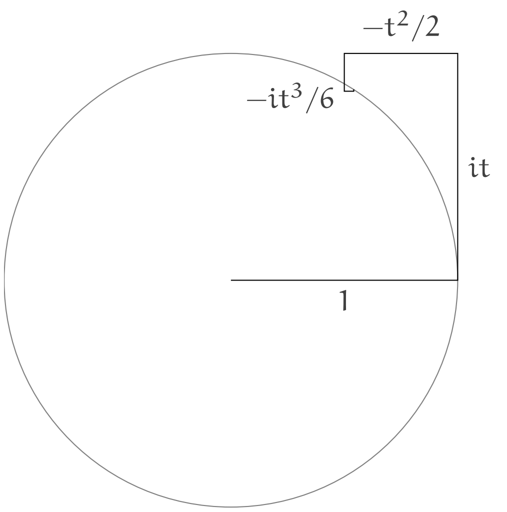
$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x + jy)(x - jy) \\ &= x^2 - y^2 \end{aligned}$$

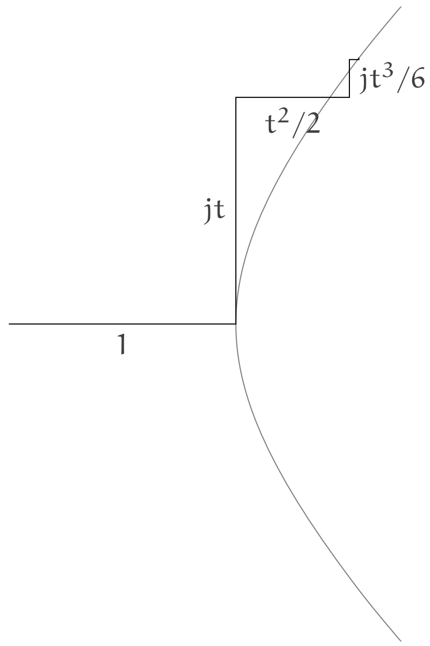


Exponentiation

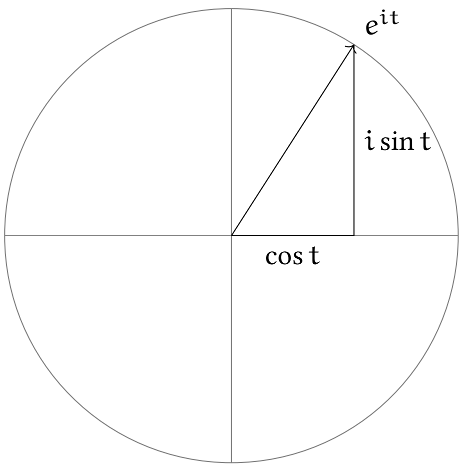
$$i^n = \begin{cases} 1 & n \equiv 0 \pmod 4 \\ i & n \equiv 1 \pmod 4 \\ -1 & n \equiv 2 \pmod 4 \\ -i & n \equiv 3 \pmod 4 \end{cases} \qquad j^n = \begin{cases} 1 & n \equiv 0 \pmod 4 \\ j & n \equiv 1 \pmod 4 \\ 1 & n \equiv 2 \pmod 4 \\ j & n \equiv 3 \pmod 4 \end{cases}$$

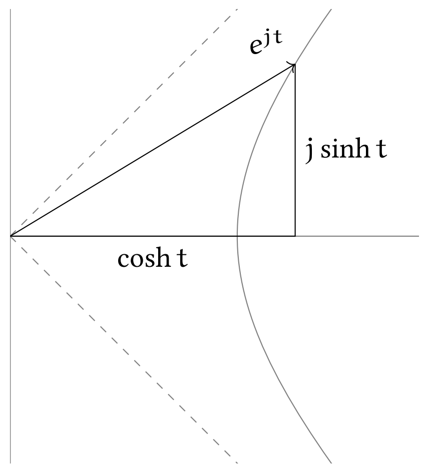
$$\begin{aligned} e^{it} &= 1(\dots) + i(\dots) \\ &\quad - 1(\dots) - i(\dots) \end{aligned} \qquad \begin{aligned} e^{jt} &= 1(\dots) + j(\dots) \\ &\quad + 1(\dots) + j(\dots) \end{aligned}$$





$$e^{it} = \cos t + i \sin t \qquad e^{jt} = \cosh t + j \sinh t$$

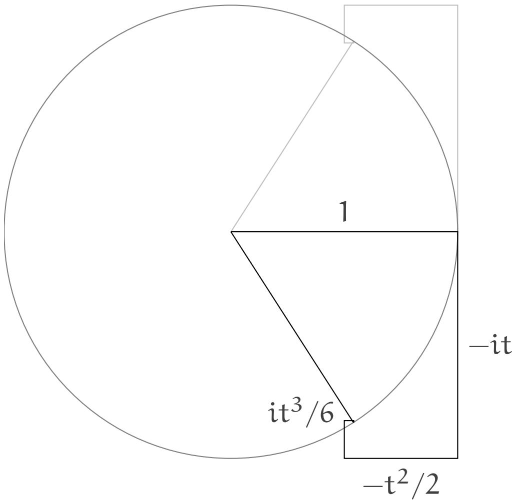


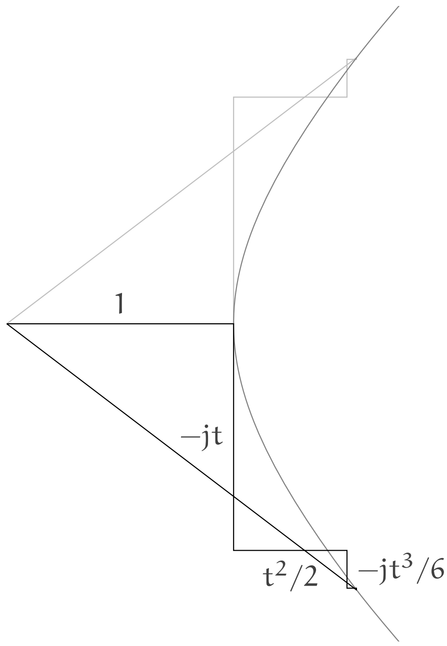


Pythagorean identity

$$\begin{aligned} e^{-it} &= \cos(-t) + i\sin(-t) \\ &= \cos t - i\sin t \end{aligned}$$

$$\begin{aligned} e^{-jt} &= \cosh(-t) + j\sinh(-t) \\ &= \cosh t - j\sinh t \end{aligned}$$





$$\begin{aligned} e^{-it} &= \left(e^{it}\right)^* \\ \left|e^{it}\right|^2 &= \left(e^{it}\right)\left(e^{it}\right)^* \\ &= e^{it}e^{-it} \\ &= 1 \\ 1 &= \cos^2 t + \sin^2 t \end{aligned}$$

$$\begin{aligned} e^{-jt} &= \left(e^{jt}\right)^* \\ \left|e^{jt}\right|^2 &= \left(e^{jt}\right)\left(e^{jt}\right)^* \\ &= e^{jt}e^{-jt} \\ &= 1 \\ 1 &= \cosh^2 t - \sinh^2 t \end{aligned}$$

Pythagorean corollaries

$$\begin{aligned} \cos^2 t &= 1 - \sin^2 t \\ \sin^2 t &= -\cos^2 t + 1 \\ \sec^2 t &= 1 + \tan^2 t \\ \csc^2 t &= \cot^2 t + 1 \end{aligned}$$

$$\begin{aligned} \cosh^2 t &= 1 + \sinh^2 t \\ \sinh^2 t &= \cosh^2 t - 1 \\ \operatorname{sech}^2 t &= 1 - \tanh^2 t \\ \operatorname{csch}^2 t &= \coth^2 t - 1 \end{aligned}$$

Angle sum formulae

$$\begin{aligned} e^{i(A+B)} &= e^{iA}e^{iB} \\ &= (c_A + is_A)(c_B + is_B) \end{aligned}$$

$$\begin{array}{c|cc} & c_B & is_B \\ \hline c_A & c_A c_B & ic_A s_B \\ is_A & ic_B s_A & -s_A s_B \end{array}$$

$$\begin{matrix} \cos(A+B) \\ +i\sin(A+B) \end{matrix} = \begin{matrix} c_A c_B - s_A s_B \\ +i(c_B s_A + c_A s_B) \end{matrix}$$

$$\begin{aligned} e^{j(A+B)} &= e^{jA}e^{jB} \\ &= (c_A + js_A)(c_B + js_B) \end{aligned}$$

$$\begin{array}{c|cc} & c_B & js_B \\ \hline c_A & c_A c_B & jc_A s_B \\ js_A & jc_B s_A & +s_A s_B \end{array}$$

$$\begin{matrix} \cosh(A+B) \\ +j\sinh(A+B) \end{matrix} = \begin{matrix} c_A c_B + s_A s_B \\ +j(c_B s_A + c_A s_B) \end{matrix}$$

Double angle

$$\begin{aligned} e^{2it} &= \left(e^{it}\right)^2 \\ \cos 2t &= \cos^2 t - \sin^2 t \\ i\sin 2t &= i\,2\cos t \sin t \end{aligned}$$

$$\begin{aligned} e^{2jt} &= \left(e^{jt}\right)^2 \\ \cosh 2t &= \cosh^2 t + \sinh^2 t \\ j\sinh 2t &= j\,2\cosh t \sinh t \end{aligned}$$

Double angle + Pythagorean

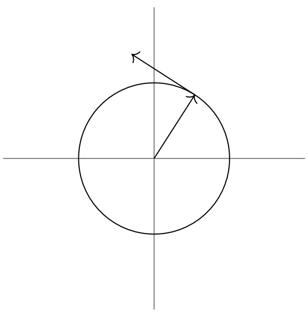
$$\begin{aligned} \cos 2t &= \cos^2 t - (1 - \cos^2 t) \\ &= 2\cos^2 t - 1 \\ &= (1 - \sin^2 t) - \sin^2 t \\ &= 1 - 2\sin^2 t \end{aligned}$$

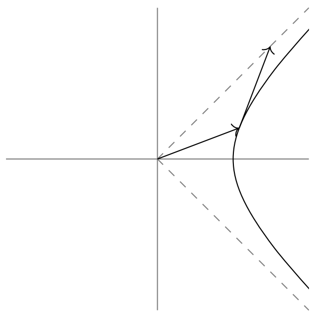
$$\begin{aligned} \cosh 2t &= \cosh^2 t + (\cosh^2 t - 1) \\ &= 2\cosh^2 t - 1 \\ &= (1 + \sinh^2 t) + \sinh^2 t \\ &= 1 + 2\sinh^2 t \end{aligned}$$

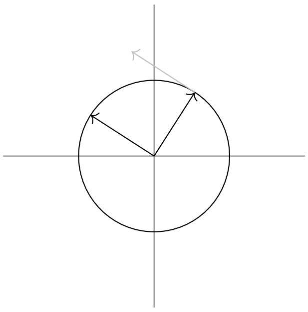
Derivatives

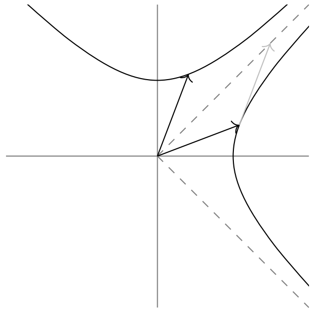
$$\begin{aligned} f(t) &\coloneqq e^{it} \\ f'(t) &= ie^{it} \\ &= i(\cos t + i\sin t) \\ \cos' t + i\sin' t &= -\sin t + i\cos t \end{aligned}$$

$$\begin{aligned} f(t) &\coloneqq e^{jt} \\ f'(t) &= je^{jt} \\ &= j(\cosh t + j\sinh t) \\ \cosh' t + j\sinh' t &= \sinh t + j\cosh t \end{aligned}$$









Dot and cross product

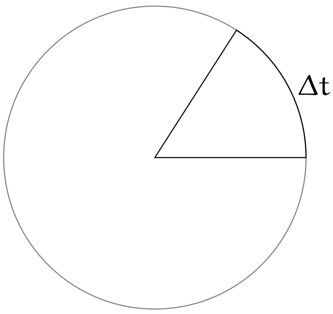
$$
\begin{aligned}
u^* v &= (\alpha - i\beta)(\gamma + i\delta) \\
&\begin{array}{c|cc}
 & \gamma & i\delta \\
\hline
\alpha & \alpha\gamma & i\alpha\delta \\
-i\beta & -i\beta\gamma & +\beta\delta
\end{array} \\
&= (\alpha\gamma + \beta\delta) + i(\alpha\delta - \beta\gamma) \\
u^* v &= (u \cdot v) + i(u \times v) \\
(iz) \cdot z &= \Re(iz^* z) = \Re(i|z|^2) = 0
\end{aligned}
$$

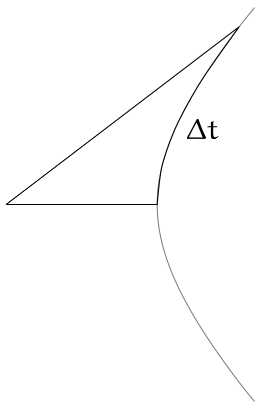
$$
\begin{aligned}
u^* v &= (\alpha - j\beta)(\gamma + j\delta) \\
&\begin{array}{c|cc}
 & \gamma & i\delta \\
\hline
\alpha & \alpha\gamma & j\alpha\delta \\
-j\beta & -j\beta\gamma & -\beta\delta
\end{array} \\
&= (\alpha\gamma - \beta\delta) + j(\alpha\delta - \beta\gamma) \\
u^* v &= (u \cdot v) + j(u \times v) \\
(jz) \cdot z &= \Re(jz^* z) = \Re(j|z|^2) = 0
\end{aligned}
$$

Arc length

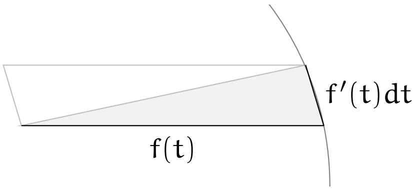
$$
\begin{aligned}
f(t) &\coloneqq e^{it} \\
S &= \int \sqrt{|f'(t)|^2} dt \\
&= \int \sqrt{|ie^{it}|^2} dt = \int \sqrt{1} dt \\
&= \Delta t
\end{aligned}
$$

$$
\begin{aligned}
f(t) &\coloneqq e^{jt} \\
S &= \int \sqrt{-|f'(t)|^2} dt \\
&= \int \sqrt{-|je^{jt}|^2} dt = \int \sqrt{-(-1)} dt \\
&= \Delta t
\end{aligned}
$$



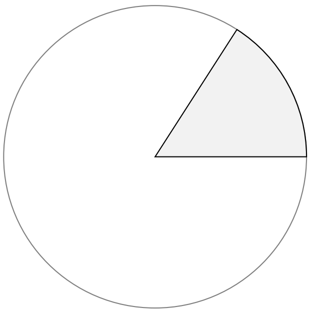


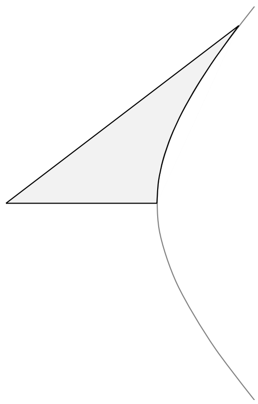
Sector area



$$
\begin{aligned}
A &= \frac{1}{2}\int f(t) \times f'(t)dt \\
&= \frac{1}{2}\int \Im\big(f^*(t)f'(t)\big)dt \\
&= \frac{1}{2}\int \Im\big(e^{-it}ie^{it}\big)dt \\
&= \frac{1}{2}\int 1\,dt \\
&= \frac{\Delta t}{2}
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{1}{2}\int f(t) \times f'(t)dt \\
&= \frac{1}{2}\int \Im\big(f^*(t)f'(t)\big)dt \\
&= \frac{1}{2}\int \Im\big(e^{-jt}je^{jt}\big)dt \\
&= \frac{1}{2}\int 1\,dt \\
&= \frac{\Delta t}{2}
\end{aligned}
$$





$SO(2), SO^+(+,-)$: Multiplying by imaginary exponents preserves inner product and area

$$
\begin{aligned}
&\big(e^{it}u\big)\cdot\big(e^{it}v\big) \\
+i\big(e^{it}u\big)\times\big(e^{it}v\big) &= \big(e^{it}u\big)^*\big(e^{it}v\big) \\
&= e^{-it}u^*e^{it}v \\
&= u^*v \\
&= u\cdot v + i\,u\times v
\end{aligned}
$$

$$
\begin{aligned}
&\big(e^{jt}u\big)\cdot\big(e^{jt}v\big) \\
+j\big(e^{jt}u\big)\times\big(e^{jt}v\big) &= \big(e^{jt}u\big)^*\big(e^{jt}v\big) \\
&= e^{-jt}u^*e^{jt}v \\
&= u^*v \\
&= u\cdot v + j\,u\times v
\end{aligned}
$$