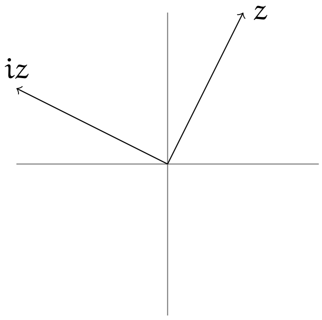
Complex

$$i^2 \coloneqq -1$$

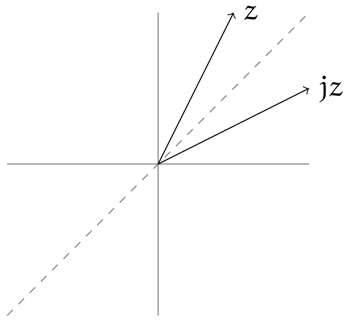
$$i(x + iy) = -y + ix$$



Hyperbolic

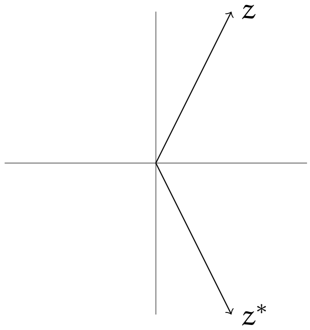
$$j^2 \coloneqq +1$$

$$j(x + jy) = y + jx$$

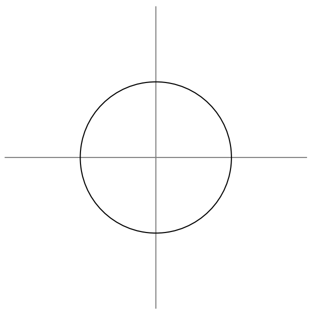


Conjugation and norm

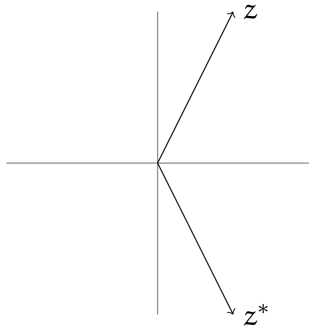
$$(x + iy)^* = x - iy$$



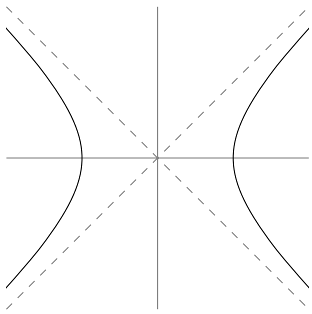
$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x + iy)(x - iy) \\ &= x^2 + y^2 \end{aligned}$$



$$(x + jy)^* = x - jy$$



$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x + jy)(x - jy) \\ &= x^2 - y^2 \end{aligned}$$



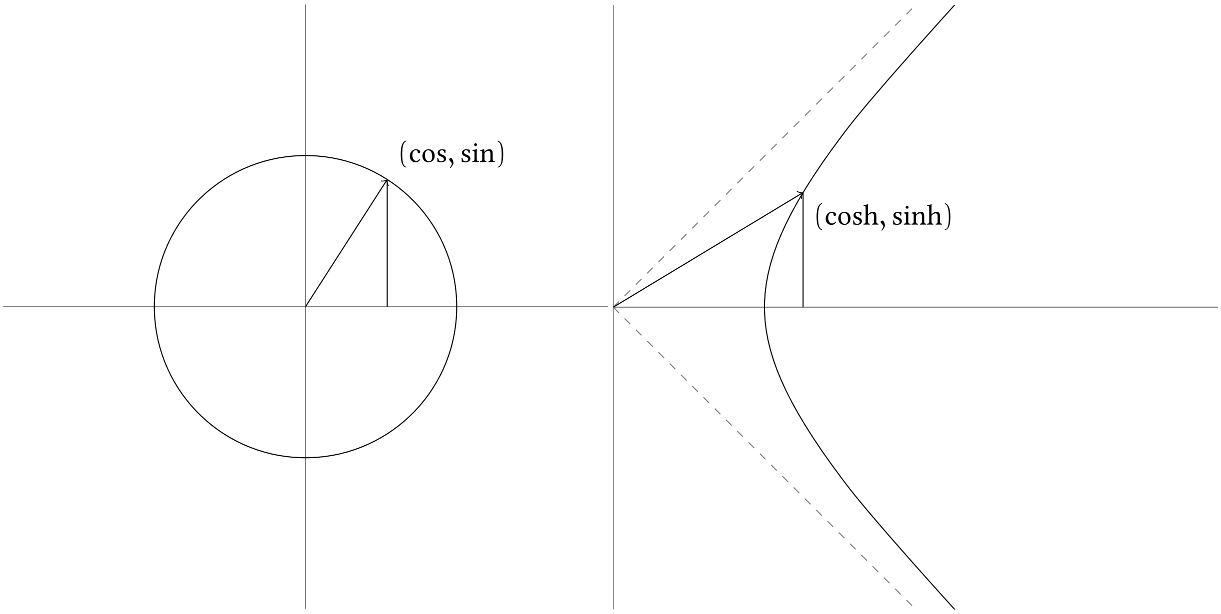
Exponentiation

$$i^n = \begin{cases} 1 \\ i \\ -1 \\ -i \end{cases}$$

$$\begin{aligned} e^{it} &= 1(\dots) + i(\dots) \\ &\quad - 1(\dots) - i(\dots) \\ e^{it} &= \cos t + i \sin t \end{aligned}$$

$$j^n = \begin{cases} 1 \\ j \\ 1 \\ j \end{cases}$$

$$\begin{aligned} e^{jt} &= 1(\dots) + j(\dots) \\ &\quad + 1(\dots) + j(\dots) \\ e^{jt} &= \cosh t + j \sinh t \end{aligned}$$



Pythagorean identity

$$\begin{aligned} e^{-it} &= \cos(-t) + i \sin(-t) \\ &= \cos t - i \sin t \\ e^{-it} &= \left(e^{it}\right)^* \\ \left|e^{it}\right|^2 &= e^{it - it} \\ &= 1 \\ 1 &= \cos^2 t + \sin^2 t \end{aligned}$$

$$\begin{aligned} e^{-jt} &= \cosh(-t) + j \sinh(-t) \\ &= \cosh t - j \sinh t \\ e^{-jt} &= \left(e^{jt}\right)^* \\ \left|e^{jt}\right|^2 &= e^{jt - jt} \\ &= 1 \\ 1 &= \cosh^2 t - \sinh^2 t \end{aligned}$$

Pythagorean corollaries

$$\begin{aligned}\cos^2 t &= 1 - \sin^2 t \\ \sin^2 t &= 1 - \cos^2 t \\ \sec^2 t &= 1 + \tan^2 t \\ \csc^2 t &= \cot^2 t + 1\end{aligned}$$

$$\begin{aligned}\cosh^2 t &= 1 + \sinh^2 t \\ \sinh^2 t &= \cosh^2 t - 1 \\ \operatorname{sech}^2 t &= 1 - \tanh^2 t \\ \operatorname{csch}^2 t &= \coth^2 t - 1\end{aligned}$$

Angle sum formulae

$$\begin{aligned}e^{i(A+B)} &= e^{iA}e^{iB} \\ &= (c_A + is_A)(c_B + is_B)\end{aligned}$$

$$\begin{array}{c|cc} & c_B & is_B \\ \hline c_A & c_A c_B & ic_A s_B \\ is_A & ic_B s_A & -s_A s_B \end{array}$$

$$\begin{matrix}\cos(A+B) \\ +i\sin(A+B)\end{matrix} = \begin{matrix}c_A c_B - s_A s_B \\ +i(c_B s_A + c_A s_B)\end{matrix}$$

$$\begin{aligned}e^{j(A+B)} &= e^{jA}e^{jB} \\ &= (c_A + js_A)(c_B + js_B)\end{aligned}$$

$$\begin{array}{c|cc} & c_B & js_B \\ \hline c_A & c_A c_B & jc_A s_B \\ js_A & jc_B s_A & +s_A s_B \end{array}$$

$$\begin{matrix}\cosh(A+B) \\ +j\sinh(A+B)\end{matrix} = \begin{matrix}c_A c_B + s_A s_B \\ +j(c_B s_A + c_A s_B)\end{matrix}$$

Double angle

$$\begin{aligned}e^{2it} &= \left(e^{it}\right)^2 \\ \cos 2t &= \cos^2 t - \sin^2 t \\ i\sin 2t &= i\,2\cos t\sin t\end{aligned}$$

$$\begin{aligned}e^{2jt} &= \left(e^{jt}\right)^2 \\ \cosh 2t &= \cosh^2 t + \sinh^2 t \\ j\sinh 2t &= j\,2\cosh t\sinh t\end{aligned}$$

Double angle + Pythagorean

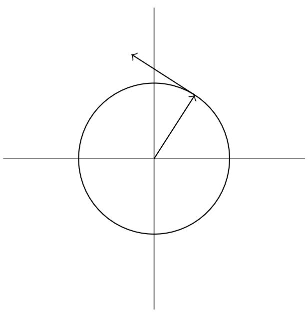
$$\begin{aligned}\cos 2t &= \cos^2 t - (1 - \cos^2 t) \\ &= 2\cos^2 t - 1 \\ &= (1 - \sin^2 t) - \sin^2 t \\ &1 - 2\sin^2 t\end{aligned}$$

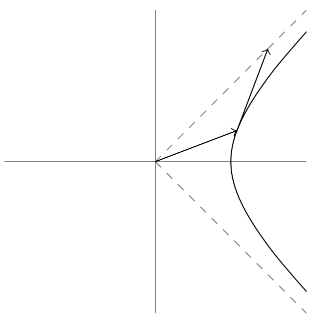
$$\begin{aligned}\cosh 2t &= \cosh^2 t + (\cosh^2 t - 1) \\ &= 2\cosh^2 t - 1 \\ &= (1 + \sinh^2 t) + \sinh^2 t \\ &= 1 + 2\sinh^2 t\end{aligned}$$

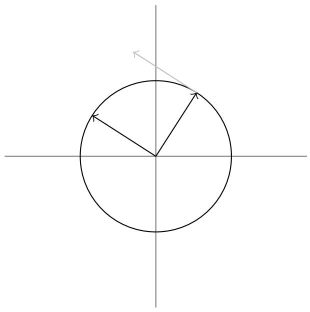
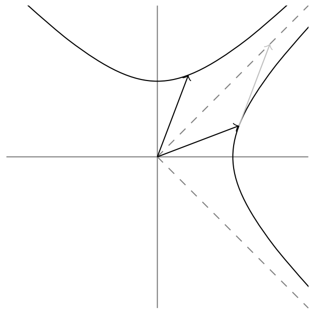
Derivatives

$$\begin{aligned}f(t) &\coloneqq e^{it} \\ f'(t) &= ie^{it} \\ &= i(\cos t + i\sin t) \\ \cos' t + i\sin' t &= -\sin t + i\cos t\end{aligned}$$

$$\begin{aligned}f(t) &\coloneqq e^{jt} \\ f'(t) &= je^{jt} \\ &= j(\cosh t + j\sinh t) \\ \cosh' t + j\sinh' t &= \sinh t + j\cosh t\end{aligned}$$





Dot and cross product

$$u^*v = (\alpha - i\beta)(\gamma + i\delta)$$

	γ	$i\delta$
α	$\alpha\gamma$	$i\alpha\delta$
$-i\beta$	$-i\beta\gamma$	$+\beta\delta$

$$= (\alpha\gamma + \beta\delta) + i(\alpha\delta - \beta\gamma)$$

$$u^*v = (u \cdot v) + i(u \times v)$$

$$(iz) \cdot z = \Re(iz^*z) = \Re(i|z|^2) = 0$$

$$u^*v = (\alpha - j\beta)(\gamma + j\delta)$$

	γ	$i\delta$
α	$\alpha\gamma$	$+j\alpha\delta$
$-j\beta$	$-j\beta\gamma$	$-\beta\delta$

$$= (\alpha\gamma - \beta\delta) + j(\alpha\delta - \beta\gamma)$$

$$u^*v = (u \cdot v) + j(u \times v)$$

$$(jz) \cdot z = \Re(jz^*z) = \Re(j|z|^2) = 0$$

Arc length

$$\begin{aligned} f(t) &\coloneqq e^{it} \\ S &= \int \sqrt{|f'(t)|^2}\,dt \\ &= \int \sqrt{|ie^{it}|^2}\,dt = \int \sqrt{1}\,dt \\ &= \Delta t \end{aligned}$$

$$\begin{aligned} f(t) &\coloneqq e^{jt} \\ S &= \int \sqrt{-|f'(t)|^2}\,dt \\ &= \int \sqrt{|je^{jt}|^2}\,dt = \int \sqrt{-(-1)}\,dt \\ &= \Delta t \end{aligned}$$

Sector area

$$\begin{aligned} A &= \frac{1}{2}\int f(t) \times f'(t)\,dt \\ &= \frac{1}{2}\int \Im\Big(f^*(t)f'(t)\Big)\,dt \\ &= \frac{1}{2}\int \Im\big(e^{-it}ie^{it}\big)\,dt \\ &= \frac{1}{2}\int 1\ dt \\ &= \frac{\Delta t}{2} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int f(t) \times f'(t)\,dt \\ &= = \frac{1}{2}\int \Im\big(f^*(t)f'(t)\big)\,dt \\ &= = \frac{1}{2}\int \Im\big(e^{-jt}je^{jt}\big)\,dt \\ &= \frac{1}{2}\int 1\ dt \\ &= \frac{\Delta t}{2} \end{aligned}$$

$SO(2), SO^+(+,-)$: Multiplying by imaginary exponents preserves inner product and area!

$$\begin{aligned} &(e^{it}u)\cdot(e^{it}v) \\ +&(e^{it}u)\times(e^{it}v) = (e^{it}u)^*(e^{it}v) \\ &= e^{-it}u^*e^{it}v \\ &= u^*v \\ &= u\cdot v + i\,u\times v \end{aligned}$$

$$\begin{aligned} &(e^{jt}u)\cdot(e^{jt}v) \\ &(e^{jt}u)\times(e^{jt}v) = (e^{jt}u)^*(e^{jt}v) \\ &= e^{-jt}u^*e^{jt}v \\ &= u^*v \\ &= u\cdot v + j\,u\times v \end{aligned}$$