$$\begin{tikzcd} & i \arrow[dr, bend left] & \\ k \arrow[ur, bend left] & & j \arrow[ll, bend left] \end{tikzcd}$$

1 Quaternions, conjugation, rotation

Lemma 1 *Group conjugation commutes with hamiltonian conjugation*

Proof. Suppose $|q| = 1$. Then $q^{-1} = q^*$ and

$$\begin{aligned}\left(qpq^{-1}\right)^* &= (qpq^*)^* \\ &= (q^*)^*(p^*)(q)^* \\ &= q(p^*)q^* \\ &= qp^*q^{-1}.\end{aligned}$$

In general, $q = t\hat{q}$ for $t \in \mathbb{R}$. Then

$$\begin{aligned}qpq^{-1} &= (t\hat{q})p(t\hat{q})^{-1} \\ &= (t\hat{q})p\left(t^{-1}\hat{q}^{-1}\right) \\ &= \hat{q}p\hat{q}^{-1}\end{aligned}$$

Conclude

$$\begin{aligned}\left(qpq^{-1}\right)^* &= \left(\hat{q}p\hat{q}^{-1}\right)^* \\ &= \hat{q}p^*\hat{q}^{-1} \\ &= qp^*q^{-1}\end{aligned} \qquad \square$$

Corollary 2

$$\mathcal{R} : q \mapsto q\_\_q^{-1} : S^3 \to O(\mathbb{R}^3)$$

Because S^3 is connected, so is $\mathcal{R}(S^3)$. Because its image contains $\mathcal{R}(1) = 1\_\_1 = \mathrm{id}$, it lies in the identity component:

Corollary 3

$$q \mapsto q\_\_q^{-1} : S^3 \to SO(\mathbb{R}^3).$$

The properties of $\mathcal{R}(q)$ are determined by commutativity. Since we're acting on $\mathbb{R}^3$, decompose:

$$q = r + \vec{v}.$$

In the case $r = 0$,

$$\vec{v}\vec{u}\vec{v}^* = -\vec{v}\vec{x}\vec{v}$$

Split $x = x_{||} + x_{\perp}$

$$\vec{u}x_{||}\vec{u}^* = \vec{u}\vec{u}^*x_{||} = x_{||}$$

$$\vec{u}x_{\perp}\vec{u}^* = -\vec{u}\vec{u}^*x_{\perp} = -x_{\perp}$$

a reflection negating everything but $\operatorname{span} u$ (line symmetry).

Commuting?

$$[q, \vec{x}] = [q + \vec{v}, \vec{x}] = [\vec{v}, \vec{x}]$$

Split $x = x_{||} + x_{\perp}$. Parallels commute and orthogonals anticommute:

$$\begin{aligned}vx_{||} &= x_{||}v \\ vx_{\perp} &= x_{\perp}v\end{aligned}$$

Therefore,

$$qx_{||}q^{-1} = qq^{-1}x_{||} = x_{||}$$

while

$$\begin{aligned}qx_{\perp}q^* &= (r+v)x_{\perp}(r-v) \\ &= (r+v)(r+v)x_{\perp} \\ &= q^2x_{\perp}\end{aligned}$$

so

$$\mathcal{R}(q)x = qxq^{-1} = x_{||} + q^2x_{\perp}$$

$$\begin{aligned}q &= c_\theta + s_\theta\hat{v} \\ q^2 &= c_\theta^2 - s_\theta^2 + 2c_\theta s_\theta\hat{v} \\ &= c_{2\theta} + s_{2\theta}\hat{v} \\ q^2x_{\perp} &= c_{2\theta}x_{\perp} + s_{2\theta}\hat{v}x_{\perp}\end{aligned}$$

In general, I want to show rotations fix subspaces of size 2.

Theorem 4 *Every unit quaternion can be expressed as the product of two pure units*

Proof.

$$\begin{aligned}\vec{u}\vec{v} &= -\vec{u}\cdot\vec{v} + \vec{u}\times\vec{v} \\ &= -\cos\theta + \sin\theta\,\vec{u}\times\vec{v} \\ i(-i\cos\theta - iw\sin\theta) &= \cos\theta + w\sin\theta\end{aligned} \qquad \square$$

Theorem 5 (Cartan-Dieudonné)

Proof. I forgot where I read this.

Decompose the space. Consider the map $U : (x, \vec{u}) \mapsto (y, \vec{v})$. There's a reflection F sending the first coordinate to the correct location:

$$F(x, \vec{u}) = (y, F'\vec{u})$$

so

$$U = (UF)F$$

and $UF(x, w) = (x, w')$ which is orthogonal on a codimension 1 space, so go by induction. On a dimension 0 space, nothing is necessary to do (base case is vacuously true). □

2 Representations of $\mathbb{H}^\times$

$$\begin{aligned} q &= \alpha + \beta i + \gamma j + \delta k \\ &= (\alpha + \beta i) + (\gamma + \delta i)j \\ &= u + vj \\ &= (\alpha + \beta i) + j(\gamma - \delta i) \\ &= u + v^* j \end{aligned}$$

$$\begin{array}{ccc} & w & xj \\ u & uw & uxj \\ vj & vw^*j & vx^*j^2 \end{array},$$

so we get the action

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} (w + xj) &= \begin{bmatrix} uw - vx^* \\ vw^* + ux \end{bmatrix} \\ &= \begin{bmatrix} w & -x^* \\ x & w^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

$$\_\_(w + xj) \mapsto \begin{bmatrix} w & -x^* \\ x & w^* \end{bmatrix}$$

Or to $SO\mathbb{R}^4$

$$i\_\_ \mapsto \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}$$

$$j\_\_ \mapsto \begin{bmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{bmatrix}$$

$$k\_\_ \mapsto \begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\_\_i \mapsto \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}$$

$$\_\_j \mapsto \begin{bmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{bmatrix}$$

$$\_\_k \mapsto \begin{bmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}$$

$$q\_\_ \mapsto \begin{bmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & -\delta & \gamma \\ \gamma & \delta & \alpha & -\beta \\ \delta & -\gamma & \beta & \alpha \end{bmatrix}$$

$$\_\_q \mapsto \begin{bmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{bmatrix}$$

For pure vectors,

$$\begin{aligned} vq &= v\pi_1(q) - v \cdot \pi_{ijk} q + v \times \pi_{ijk} q \\ v\_\_ &= v\_\_{}_{\mathbb{R}} - v \cdot \_\_ + v \times \_\_ \end{aligned}$$

$$v\_\_ = \begin{bmatrix} 0 & -v^T \\ v & v \times \_\_ \end{bmatrix} \qquad \_\_v = \begin{bmatrix} 0 & -v^T \\ v & \_\_ \times v \end{bmatrix}$$

$$v\_\_v = \begin{bmatrix} -v \cdot v & -v \cdot (\_\_ \times v) \\ v \times v & -vv^T + v \times (\_\_ \times v) \end{bmatrix}$$

$$\begin{aligned} v \times (x \times v) &= -(v \cdot x)v + x(v \cdot v) \\ &= \left(-vv^T + vv^T\right)x \end{aligned}$$

$$v\_\_v = \begin{bmatrix} -v \cdot v & \\ & -2vv^T + v^T v \end{bmatrix}$$

Hopf map things

$$H(q) \coloneqq qkq^{-1} = {}^q k$$

$$\begin{aligned} H(pq) &= {}^{pq}k \\ &= {}^p({}^q k) \\ &= {}^p H(q) \\ H(qr) &= H(rq^r) \\ &= {}^r H(q^r) \end{aligned}$$

3 Winding and linking number

https://en.wikipedia.org/wiki/Linking_number

There's too many Hs in this field. Winding number:

$$S^1 \xrightarrow{\gamma} S^1$$

$$\mathbb{Z} \xrightarrow{\mathcal{H}_1(\gamma)} \mathbb{Z}$$

If you homotope one curve to a standard circle, the linking number is the winding number of the other curve.

Linking number of $s^1 \to$ __ is only nontrivial in dimension 3 because you can homotope through the fourth dimension to separate.

Gauss's linking number (via wikipedia). The Gauss map

$$\Gamma(\alpha,\beta)(s,t) = \operatorname{sgn}\left(\alpha(s) - \beta(t)\right)$$

> "Pick a point in the unit sphere, v, *so that orthogonal projection* of the *link* to the plane perpendicular to v *gives a link diagram.* Observe that a point (s,t) that *goes to* v under the Gauss map *corresponds to a crossing* in the link diagram *where [α] is over [β].* Also, a neighborhood of (s,t) is mapped under the Gauss map to a neighborhood of v preserving or reversing *orientation* depending on the *sign* of the crossing. Thus in order to compute the linking number of the diagram corresponding to v it suffices to *count the signed number of times the Gauss map covers* v. Since v is a *regular value*, this is precisely the *degree of the Gauss map* (i.e. the signed number of times that the image of $[(\alpha,\beta)]$ covers the sphere). Isotopy invariance of the linking number is automatically obtained as the *degree is invariant* under homotopic maps. Any other regular value would give the same number, so the linking number doesn't depend on any particular link diagram."

$$\deg \Gamma_{(\alpha,\beta)} \int_{S^2} 1 = \int_{T^2} \Gamma(\alpha,\beta)$$

$$\deg \Gamma_{(\alpha,\beta)} = (2\tau)^{-1} \int_{T^2} \Gamma_{(\alpha,\beta)}(ds \wedge dt)$$

Because slotting a unit vector into a volume form gives area

$$= \int (\wedge^2 d\Gamma) \wedge \Gamma$$

With the added benefit of discarding the normal components of $\wedge^2 d\Gamma$:

$$= \int \frac{d\alpha}{|\alpha-\beta|} \wedge \frac{-d\beta}{|\alpha-\beta|} \wedge \frac{\alpha-\beta}{|\alpha-\beta|}$$

But I think I lost a minus sign.

4 stereographic projection of hopf links

Foliate S^3 with tori:

$$F(\alpha)(s,t) \coloneqq \cos\alpha\, e^{sk} + \sin\alpha\, ie^{-tk}$$
$$: [0,\tau/4] \to T^2 \to S^3$$

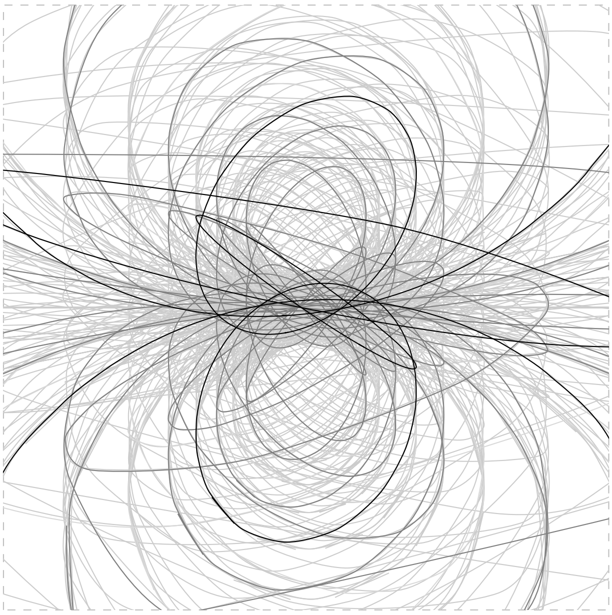
Spin

$$Fe^{uk} = \cos\alpha\, e^{(s+u)k} + \sin\alpha\, ie^{(u-t)k}$$

It spins within leaves not across. Project

$$P(Fe^{uk}) = \frac{\cos\alpha\cos(s+u) + \sin\alpha\, ie^{(u-t)k}}{1 - \cos\alpha\sin(s+u)}$$

Project



5 $S^3 \times S^3$ and $SO(4)$

$$\begin{aligned} z(u+vj) &= (zu) + (zv)j & (u+vj)z &= uz + vjz \\ j(u+vj) &= ju + jvj & &= uz + vz^*j \\ &= -v^* + u^*j & (u+vj)j &= uj - v \end{aligned}$$

$$z\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}zu\\zv\end{bmatrix} \qquad \begin{bmatrix}u\\v\end{bmatrix}z = \begin{bmatrix}zu\\z^*v\end{bmatrix}$$

$$j\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}-v^*\\u^*\end{bmatrix} \qquad \begin{bmatrix}u\\v\end{bmatrix}j = \begin{bmatrix}-v\\u\end{bmatrix}$$

We can rotate $\mathbb{H}$ fixing 1 to make an arbitrary unit vector into i. In that case, the effect of exponential-rotation is

$$e^{is}\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}e^{is}u\\e^{is}v\end{bmatrix} \qquad \begin{bmatrix}u\\v\end{bmatrix}e^{it} = \begin{bmatrix}e^{it}u\\e^{-it}v\end{bmatrix}$$

$$e^{is}\begin{bmatrix}u\\v\end{bmatrix}e^{it} = \begin{bmatrix}e^{i(s+t)}u\\e^{i(s-t)}v\end{bmatrix}$$

So we can rotate the two complex numbers independently.

$$\exp\left(i\frac{A+B}{2}\right)\begin{bmatrix}u\\v\end{bmatrix}\exp\left(i\frac{A-B}{2}\right) = \begin{bmatrix}e^{iA}u\\e^{iB}v\end{bmatrix}$$

Especially

$$e^{iA/2}\begin{bmatrix}u\\v\end{bmatrix}e^{iA/2} = \begin{bmatrix}e^{iA}u\\v\end{bmatrix}$$

So now we can (1) rotate the vector parts, (2) rotate between a vector and scalar.

$e^{tx/2}$__$e^{-tx/2}$ rotates vectors around the axle x, fixing 1 and x.

$e^{tx/2}$__$e^{tx/2}$ rotates the $1, x$-plane, fixing vectors orthogonal to x.

The simplest "generic" rotation would be in the e^{it}, j plane:

$$(c1 + is) \wedge j = c1 \wedge j + si \wedge j$$

6 $\mathbb{H}$ and geometric algebra

$$\mathbb{H} = \mathbb{R}^3/(v^2 = -1)$$

whereas your standard geometric algebra is

$$\mathbb{R}^3/(v^2 = 1).$$

But they Hodge$*$ into each other. Let x, y, z be the geometric algebra's basis. Then

$$\begin{aligned} I &= xyz \\ I^2 &= (xyz)(xyz) = -1 \\ xI &= x^2yz = xyzx = Ix \end{aligned}$$

Define

$$\phi : \begin{cases} i & \mapsto & xI & = & xxyz & = yz \\ j & \mapsto & -yI & = & -yxyz & = xz \\ k & \mapsto & zI & = & zxyz & = xy \end{cases}$$

This has the right squares:

$$\phi(\vec{x})^2 = (xI)(xI) = x^2I^2 = -1.$$

And products?

$$\begin{aligned} (yz)(xz) &= -yx & &= xy \\ (xz)(xy) &= -zy & &= yz \\ (xy)(yz) &= & &= xz \end{aligned}$$

$$\begin{tikzcd} & i \arrow[dr, bend left] & \\ k \arrow[ur, bend left] & & j \arrow[ll, bend left] \end{tikzcd} \overset{\phi}{\Longrightarrow} \begin{tikzcd} & yz \arrow[dr, bend left] & \\ xy \arrow[ur, bend left] & & xz \arrow[ll, bend left] \end{tikzcd} = \begin{tikzcd} & xI \arrow[dr, bend left] & \\ zI \arrow[ur, bend left] & & -yI \arrow[ll, bend left] \end{tikzcd}$$

Punning between vectors and spinors depends on planes and lines being dual in $\mathbb{R}^3$:

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3 \xrightarrow{1\oplus\phi} \mathbb{R} \oplus \wedge^2\mathbb{R}^3$$

7 rotation tricks

Log of a rotor/product of unit vectors

$$\begin{aligned} uv &= \underbrace{u \cdot v}_{\cos\theta} + u \wedge v \\ &= \cos\theta + \sin\theta \frac{u \wedge v}{|u \wedge v|} \\ &= \exp\left(\cos^{-1}(u \cdot v)\frac{u \wedge v}{|u \wedge v|}\right) \end{aligned}$$

Product of 2-rotors. If they commute:

$$e^{\alpha xy}e^{\beta zt} = e^{\alpha xy + \beta zt} \quad \text{as } [xy, zt] = 0$$

If they don't commute,

$$\begin{aligned} e^{\alpha xy}e^{\beta yz} &= \overbrace{(e^{\alpha xy}y)}^{\deg=1}\overbrace{(ye^{\beta yz})}^{\deg=1} \\ &= \underbrace{(e^{\alpha xy}y) \cdot (ye^{\beta yz})}_{\cos\theta = c_\alpha c_\beta} + (e^{\alpha xy}y) \wedge (ye^{\beta yz}) \\ &= \exp\left(\cos^{-1}(c_\alpha c_\beta)\frac{(e^{\alpha xy}y) \wedge (ye^{\beta yz})}{\sqrt{1 - {c_\alpha}^2{c_\beta}^2}}\right) \end{aligned}$$

Note $xy + zt$ is not a blade means it is a double rotation

8 Rotor representation?

I'm writing these so rotors conjugate on the left

$$i\_\_ = \begin{cases} 1 & \mapsto i \\ i & \mapsto -1 \\ j & \mapsto k \\ k & \mapsto -j \end{cases}$$

Double rotation:

$$i\_\_ \mapsto \left(\frac{1 + x_1 x_0}{\sqrt{2}}\right)\left(\frac{1 + x_3 x_2}{\sqrt{2}}\right)$$

$$\_\_i = \begin{cases} 1 & \mapsto i \\ i & \mapsto -1 \\ j & \mapsto -k \\ k & \mapsto j \end{cases}$$

$$\_\_i \mapsto \left(\frac{1 + x_1 x_0}{\sqrt{2}}\right)\left(\frac{1 + x_2 x_3}{\sqrt{2}}\right)$$

$$j\_\_ = \begin{cases} 1 & \mapsto j \\ j & \mapsto -1 \\ i & \mapsto k \\ k & \mapsto -i \end{cases}$$

$$\_\_j = \begin{cases} 1 & \mapsto j \\ j & \mapsto -1 \\ i & \mapsto -k \\ k & \mapsto i \end{cases}$$

$$k\_\_ = \begin{cases} 1 \mapsto k \\ k \mapsto -1 \\ i \mapsto j \\ j \mapsto -i \end{cases}$$

$$\_\_k = \begin{cases} 1 \mapsto k \\ k \mapsto -1 \\ i \mapsto -j \\ j \mapsto i \end{cases}$$