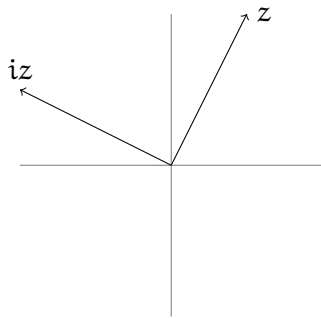


The identity $e^{it} = \cos t + i \sin t$ is a powerful organizing principle for trigonometry. Using *hyperbolic* or *split-complex* numbers, (https://en.wikipedia.org/wiki/Split-complex_number) where $j^2 = +1$ provides an analogous tool for hyperbolic trigonometry.

Complex

$$i^2 := -1$$

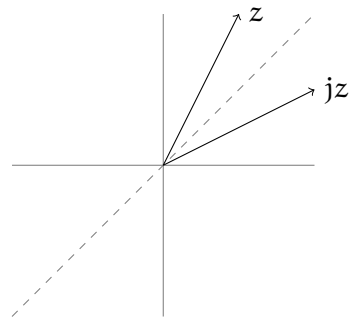
$$i(x + iy) = -y + ix$$



Hyperbolic

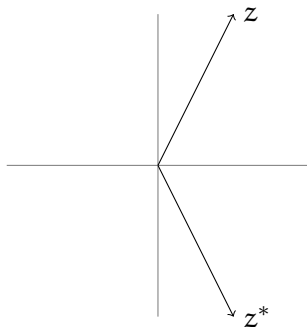
$$j^2 := +1$$

$$j(x + jy) = y + jx$$

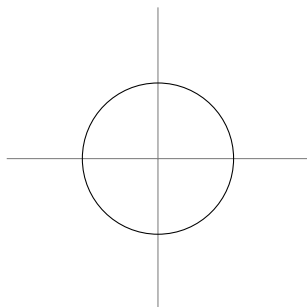


Conjugation and norm

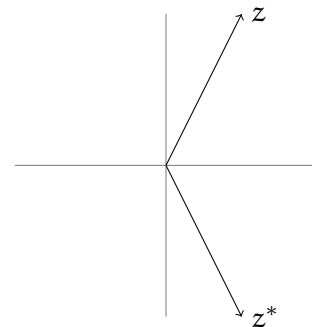
$$(x + iy)^* = x - iy$$



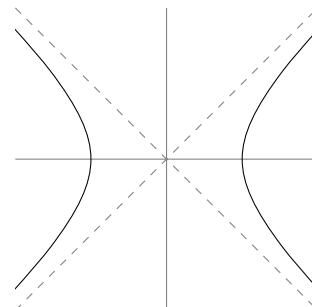
$$\begin{aligned} |z|^2 &:= zz^* = (x + iy)(x - iy) \\ &= x^2 + y^2 \end{aligned}$$



$$(x + jy)^* = x - jy$$



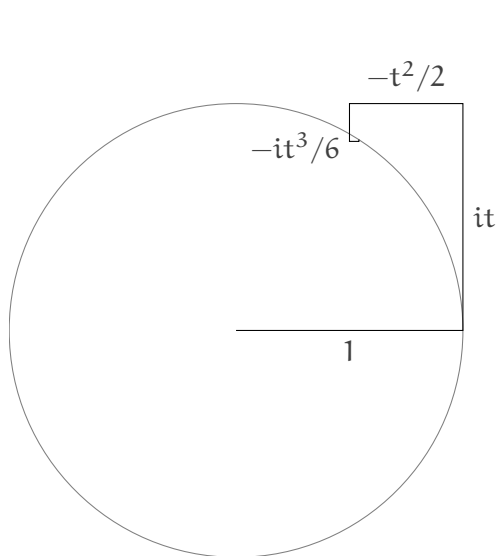
$$\begin{aligned} |z|^2 &:= zz^* = (x + jy)(x - jy) \\ &= x^2 - y^2 \end{aligned}$$



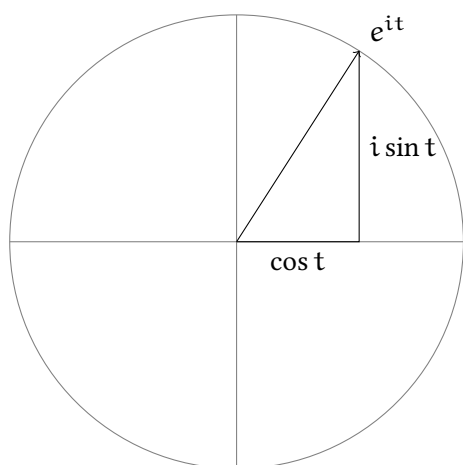
Exponentiation

$$i^n = \begin{cases} 1 & n \equiv 0 \pmod{4} \\ i & n \equiv 1 \pmod{4} \\ -1 & n \equiv 2 \pmod{4} \\ -i & n \equiv 3 \pmod{4} \end{cases}$$

$$e^{it} = 1(\dots) + i(\dots) \\ -1(\dots) - i(\dots)$$

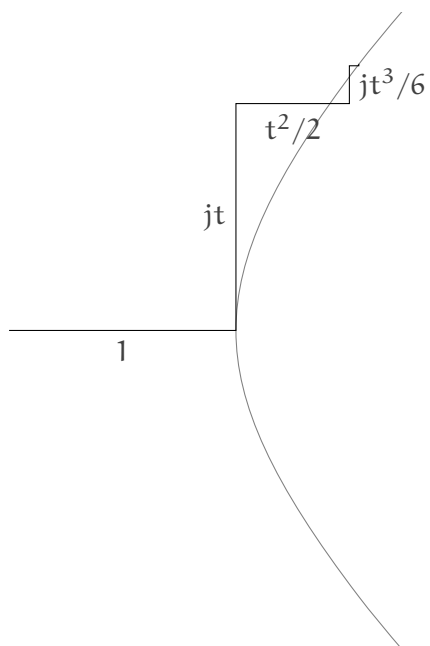


$$e^{it} = \cos t + i \sin t$$

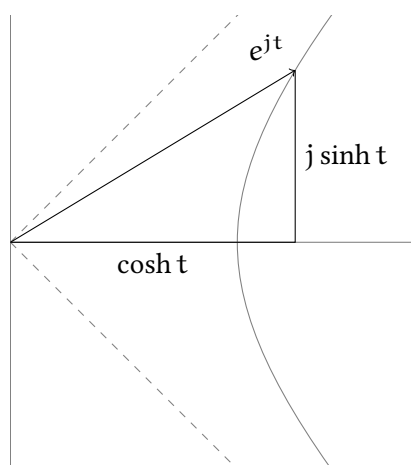


$$j^n = \begin{cases} 1 & n \equiv 0 \pmod{4} \\ j & n \equiv 1 \pmod{4} \\ 1 & n \equiv 2 \pmod{4} \\ j & n \equiv 3 \pmod{4} \end{cases}$$

$$e^{jt} = 1(\dots) + j(\dots) \\ +1(\dots) + j(\dots)$$

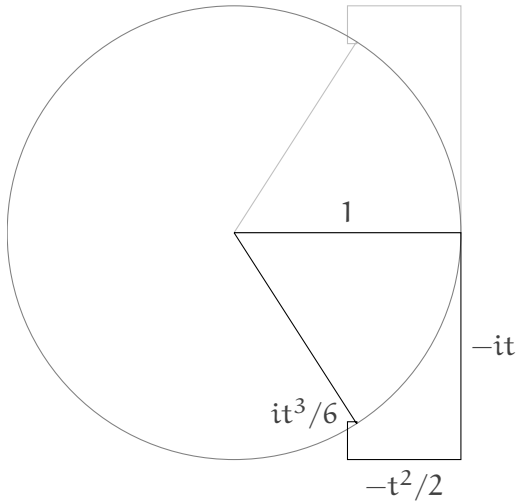


$$e^{jt} = \cosh t + j \sinh t$$



Pythagorean identity

$$\begin{aligned} e^{-it} &= \cos(-t) + i \sin(-t) \\ &= \cos t - i \sin t \end{aligned}$$

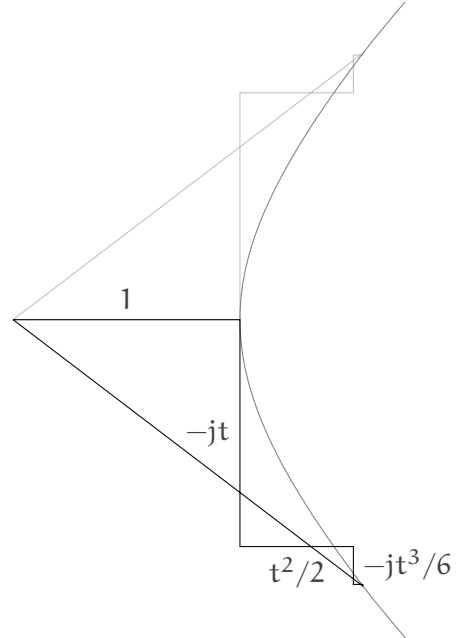


$$\begin{aligned} e^{-it} &= (e^{it})^* \\ |e^{it}|^2 &= (e^{it})(e^{it})^* \\ &= e^{it}e^{-it} \\ &= 1 \\ 1 &= \cos^2 t + \sin^2 t \end{aligned}$$

Pythagorean corollaries

$$\begin{aligned} \cos^2 t &= 1 - \sin^2 t \\ \sin^2 t &= -\cos^2 t + 1 \\ \sec^2 t &= 1 + \tan^2 t \\ \csc^2 t &= \cot^2 t + 1 \end{aligned}$$

$$\begin{aligned} e^{-jt} &= \cosh(-t) + j \sinh(-t) \\ &= \cosh t - j \sinh t \end{aligned}$$



$$\begin{aligned} e^{-jt} &= (e^{jt})^* \\ |e^{jt}|^2 &= (e^{jt})(e^{jt})^* \\ &= e^{jt}e^{-jt} \\ &= 1 \\ 1 &= \cosh^2 t - \sinh^2 t \end{aligned}$$

$$\begin{aligned} \cosh^2 t &= 1 + \sinh^2 t \\ \sinh^2 t &= \cosh^2 t - 1 \\ \operatorname{sech}^2 t &= 1 - \tanh^2 t \\ \operatorname{csch}^2 t &= \coth^2 t - 1 \end{aligned}$$

Angle sum formulae

$$\begin{aligned}
 e^{i(A+B)} &= e^{iA} e^{iB} \\
 &= (c_A + i s_A)(c_B + i s_B) \\
 &\quad \begin{array}{c|cc} & c_B & i s_B \\ \hline c_A & c_A c_B & i c_A s_B \\ i s_A & i c_B s_A & -s_A s_B \end{array} \\
 \cos(A+B) &= c_A c_B - s_A s_B \\
 +i \sin(A+B) &= +i(c_B s_A + c_A s_B)
 \end{aligned}$$

$$\begin{aligned}
 e^{j(A+B)} &= e^{jA} e^{jB} \\
 &= (c_A + j s_A)(c_B + j s_B) \\
 &\quad \begin{array}{c|cc} & c_B & j s_B \\ \hline c_A & c_A c_B & j c_A s_B \\ j s_A & j c_B s_A & +s_A s_B \end{array} \\
 \cosh(A+B) &= c_A c_B + s_A s_B \\
 +j \sinh(A+B) &= +j(c_B s_A + c_A s_B)
 \end{aligned}$$

Double angle

$$\begin{aligned}
 e^{2it} &= (e^{it})^2 \\
 \cos 2t &= \cos^2 t - \sin^2 t \\
 i \sin 2t &= i 2 \cos t \sin t
 \end{aligned}$$

$$\begin{aligned}
 e^{2jt} &= (e^{jt})^2 \\
 \cosh 2t &= \cosh^2 t + \sinh^2 t \\
 j \sinh 2t &= j 2 \cosh t \sinh t
 \end{aligned}$$

Double angle + Pythagorean

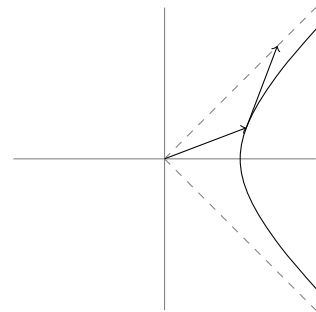
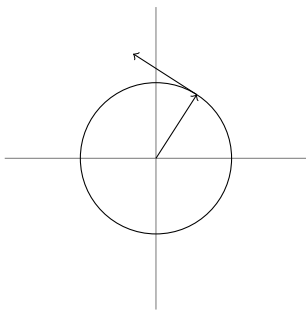
$$\begin{aligned}
 \cos 2t &= \cos^2 t - (1 - \cos^2 t) \\
 &= 2 \cos^2 t - 1 \\
 &= (1 - \sin^2 t) - \sin^2 t \\
 &= 1 - 2 \sin^2 t
 \end{aligned}$$

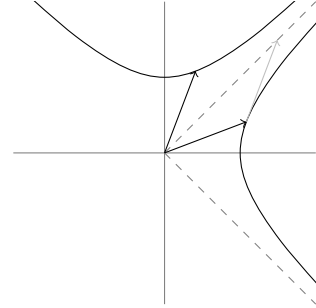
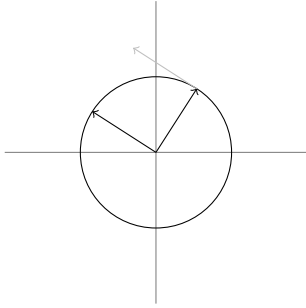
$$\begin{aligned}
 \cosh 2t &= \cosh^2 t + (\cosh^2 t - 1) \\
 &= 2 \cosh^2 t - 1 \\
 &= (1 + \sinh^2 t) + \sinh^2 t \\
 &= 1 + 2 \sinh^2 t
 \end{aligned}$$

Derivatives

$$\begin{aligned}
 f(t) &:= e^{it} \\
 f'(t) &= i e^{it} \\
 &= i(\cos t + i \sin t) \\
 \cos' t + i \sin' t &= -\sin t + i \cos t
 \end{aligned}$$

$$\begin{aligned}
 f(t) &:= e^{jt} \\
 f'(t) &= j e^{jt} \\
 &= j(\cosh t + j \sinh t) \\
 \cosh' t + j \sinh' t &= \sinh t + j \cosh t
 \end{aligned}$$





Dot and cross product

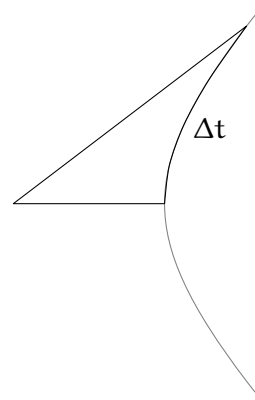
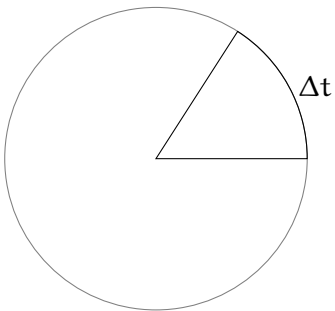
$$\begin{aligned} \mathbf{u}^* \mathbf{v} &= (\alpha - i\beta)(\gamma + i\delta) \\ &= \begin{array}{c|cc} & \gamma & i\delta \\ \hline \alpha & \alpha\gamma & i\alpha\delta \\ -i\beta & -i\beta\gamma & +\beta\delta \end{array} \\ &= (\alpha\gamma + \beta\delta) + i(\alpha\delta - \beta\gamma) \\ \mathbf{u}^* \mathbf{v} &= (\mathbf{u} \cdot \mathbf{v}) + i(\mathbf{u} \times \mathbf{v}) \\ (iz) \cdot z &= \Re(iz^*z) = \Re(i|z|^2) = 0 \end{aligned}$$

$$\begin{aligned} \mathbf{u}^* \mathbf{v} &= (\alpha - j\beta)(\gamma + j\delta) \\ &= \begin{array}{c|cc} & \gamma & i\delta \\ \hline \alpha & \alpha\gamma & j\alpha\delta \\ -j\beta & -j\beta\gamma & -\beta\delta \end{array} \\ &= (\alpha\gamma - \beta\delta) + j(\alpha\delta - \beta\gamma) \\ \mathbf{u}^* \mathbf{v} &= (\mathbf{u} \cdot \mathbf{v}) + j(\mathbf{u} \times \mathbf{v}) \\ (jz) \cdot z &= \Re(jz^*z) = \Re(j|z|^2) = 0 \end{aligned}$$

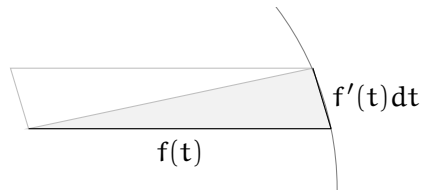
Arc length

$$\begin{aligned} f(t) &:= e^{it} \\ s &= \int \sqrt{|f'(t)|^2} dt \\ &= \int \sqrt{|ie^{it}|^2} dt = \int \sqrt{1} dt \\ &= \Delta t \end{aligned}$$

$$\begin{aligned} f(t) &:= e^{jt} \\ s &= \int \sqrt{|f'(t)|^2} dt \\ &= \int \sqrt{|je^{jt}|^2} dt = \int \sqrt{-(-1)} dt \\ &= \Delta t \end{aligned}$$

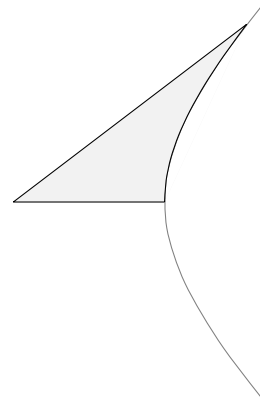
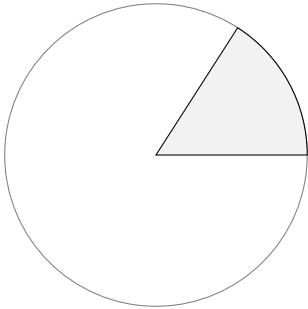


Sector area



$$\begin{aligned}
 A &= \frac{1}{2} \int f(t) \times f'(t) dt \\
 &= \frac{1}{2} \int \Im(f^*(t)f'(t)) dt \\
 &= \frac{1}{2} \int \Im(e^{-it}ie^{it}) dt \\
 &= \frac{1}{2} \int 1 dt \\
 &= \frac{\Delta t}{2}
 \end{aligned}$$

$$\begin{aligned}
 A &= \frac{1}{2} \int f(t) \times f'(t) dt \\
 &= \frac{1}{2} \int \Im(f^*(t)f'(t)) dt \\
 &= \frac{1}{2} \int \Im(e^{-jt}je^{jt}) dt \\
 &= \frac{1}{2} \int 1 dt \\
 &= \frac{\Delta t}{2}
 \end{aligned}$$



$SO(2), SO^+(+,-)$: Multiplying by imaginary exponents preserves inner product and area

$$\begin{aligned}
 (e^{it}u) \cdot (e^{it}v) &= (e^{it}u)^* (e^{it}v) \\
 +i(e^{it}u) \times (e^{it}v) &= (e^{it}u)^* (e^{it}v) \\
 &= e^{-it}u^* e^{it}v \\
 &= u^*v \\
 &= u \cdot v + i u \times v
 \end{aligned}$$

$$\begin{aligned}
 (e^{jt}u) \cdot (e^{jt}v) &= (e^{jt}u)^* (e^{jt}v) \\
 +j(e^{jt}u) \times (e^{jt}v) &= (e^{jt}u)^* (e^{jt}v) \\
 &= e^{-jt}u^* e^{jt}v \\
 &= u^*v \\
 &= u \cdot v + j u \times v
 \end{aligned}$$

