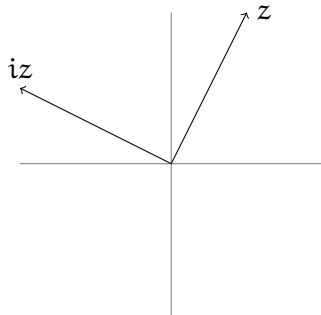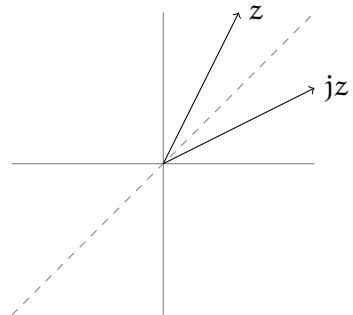Complex

$$i^2 := -1$$
$$i(x + iy) = -y + ix$$

Hyperbolic
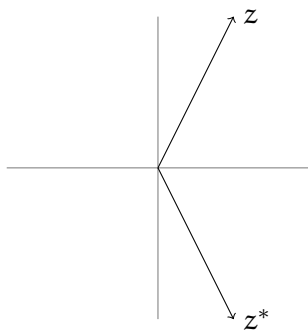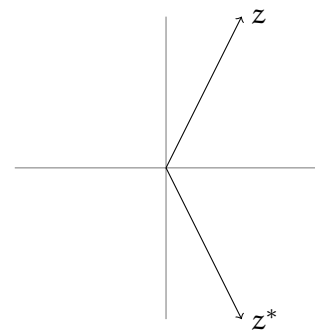
$$j^2 := +1$$
$$j(x + jy) = y + jx$$
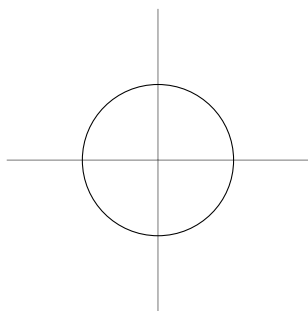

Conjugation and norm
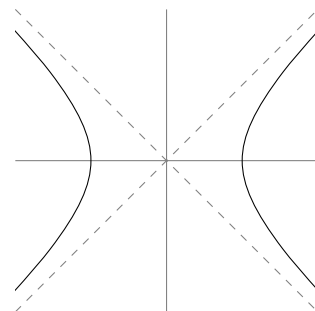
$$(x + iy)^* = x - iy$$

$$(x + jy)^* = x - jy$$
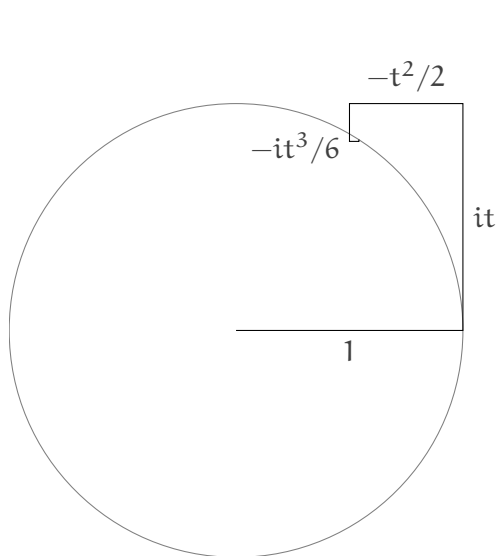

$$|z|^2 := zz^* = (x + iy)(x - iy)$$
$$= x^2 + y^2$$

$$|z|^2 := zz^* = (x + jy)(x - jy)$$
$$= x^2 - y^2$$

Exponentiation

$$i^n = \begin{cases} 1 & n \equiv 0 \pmod 4 \\ i & n \equiv 1 \pmod 4 \\ -1 & n \equiv 2 \pmod 4 \\ -i & n \equiv 3 \pmod 4 \end{cases} \qquad j^n = \begin{cases} 1 & n \equiv 0 \pmod 4 \\ j & n \equiv 1 \pmod 4 \\ 1 & n \equiv 2 \pmod 4 \\ j & n \equiv 3 \pmod 4 \end{cases}$$
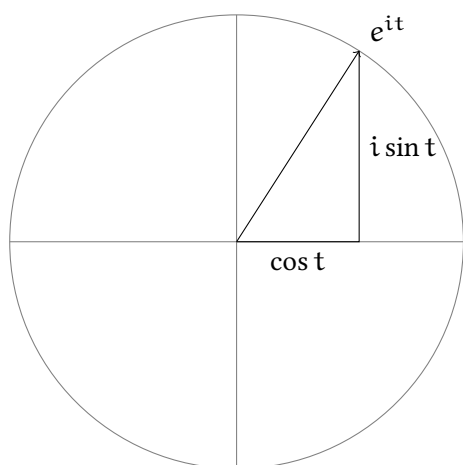
$$e^{it} = 1(\ldots) + i(\ldots) \qquad\qquad e^{jt} = 1(\ldots) + j(\ldots)$$
$$-1(\ldots) - i(\ldots) \qquad\qquad +1(\ldots) + j(\ldots)$$
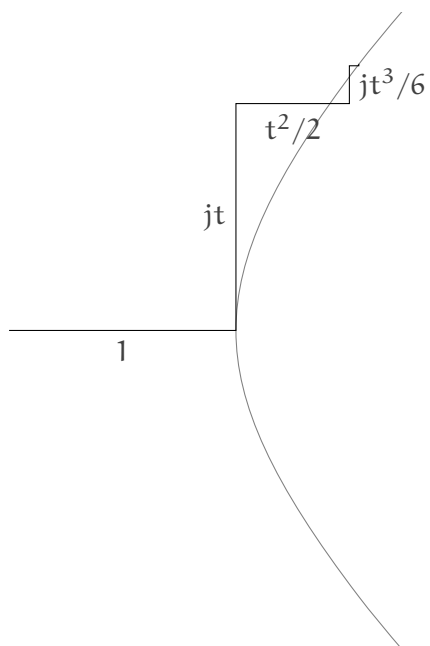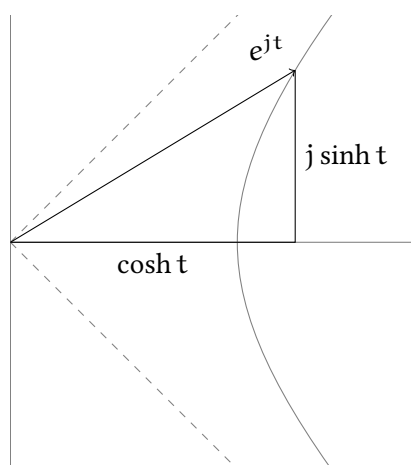
$-t^2/2$ $jt^3/6$

$-it^3/6$ $t^2/2$

it jt

1 1

$$e^{it} = \cos t + i\sin t \qquad\qquad e^{jt} = \cosh t + j\sinh t$$

e^{it} e^{jt}

$i\sin t$ $j\sinh t$

$\cos t$ $\cosh t$

Pythagorean identity

$$e^{-it} = \cos(-t) + i\sin(-t)$$
$$= \cos t - i\sin t$$
$$e^{-it} = \left(e^{it}\right)^*$$
$$\left|e^{it}\right|^2 = \left(e^{it}\right)\left(e^{it}\right)^*$$
$$= e^{it}e^{-it}$$
$$= 1$$
$$1 = \cos^2 t + \sin^2 t$$

$$e^{-jt} = \cosh(-t) + j\sinh(-t)$$
$$= \cosh t - j\sinh t$$
$$e^{-jt} = \left(e^{jt}\right)^*$$
$$\left|e^{jt}\right|^2 = \left(e^{jt}\right)\left(e^{jt}\right)^*$$
$$= e^{jt}e^{-jt}$$
$$= 1$$
$$1 = \cosh^2 t - \sinh^2 t$$

Pythagorean corollaries

$$\cos^2 t = 1 - \sin^2 t$$
$$\sin^2 t = 1 - \cos^2 t$$
$$\sec^2 t = 1 + \tan^2 t$$
$$\csc^2 t = \cot^2 t + 1$$

$$\cosh^2 t = 1 + \sinh^2 t$$
$$\sinh^2 t = \cosh^2 t - 1$$
$$\operatorname{sech}^2 t = 1 - \tanh^2 t$$
$$\operatorname{csch}^2 t = \coth^2 t - 1$$

Angle sum formulae

$$e^{i(A+B)} = e^{iA}e^{iB}$$
$$= (c_A + is_A)(c_B + is_B)$$

	c_B	is_B
c_A	$c_A c_B$	$ic_A s_B$
is_A	$ic_B s_A$	$-s_A s_B$

$$\begin{aligned}\cos(A+B)\\+i\sin(A+B)\end{aligned} = \begin{aligned}c_A c_B - s_A s_B\\+i(c_B s_A + c_A s_B)\end{aligned}$$

$$e^{j(A+B)} = e^{jA}e^{jB}$$
$$= (c_A + js_A)(c_B + js_B)$$

	c_B	js_B
c_A	$c_A c_B$	$jc_A s_B$
js_A	$jc_B s_A$	$+s_A s_B$

$$\begin{aligned}\cosh(A+B)\\+j\sinh(A+B)\end{aligned} = \begin{aligned}c_A c_B + s_A s_B\\+j(c_B s_A + c_A s_B)\end{aligned}$$

Double angle

$$e^{2it} = \left(e^{it}\right)^2$$
$$\cos 2t = \cos^2 t - \sin^2 t$$
$$i\sin 2t = i\,2\cos t\sin t$$

$$e^{2jt} = \left(e^{jt}\right)^2$$
$$\cosh 2t = \cosh^2 t + \sinh^2 t$$
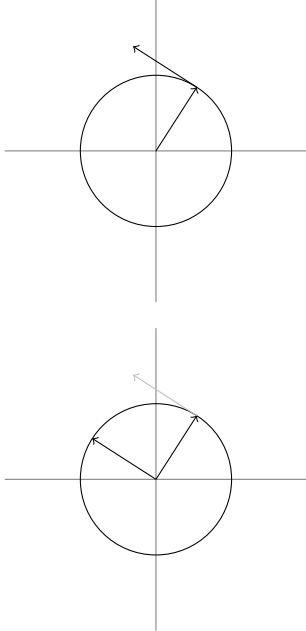$$j\sinh 2t = j\,2\cosh t\sinh t$$

Double angle + Pythagorean

$$\cos 2t = \cos^2 t - (1 - \cos^2 t)$$
$$= 2\cos^2 t - 1$$
$$= (1 - \sin^2 t) - \sin^2 t$$
$$= 1 - 2\sin^2 t$$

$$\cosh 2t = \cosh^2 t + (\cosh^2 t - 1)$$
$$= 2\cosh^2 t - 1$$
$$= (1 + \sinh^2 t) + \sinh^2 t$$
$$= 1 + 2\sinh^2 t$$

Derivatives

$$f(t) \coloneqq e^{it}$$
$$f'(t) = ie^{it}$$
$$= i(\cos t + i \sin t)$$
$$\cos' t + i \sin' t = -\sin t + i \cos t$$

$$f(t) \coloneqq e^{jt}$$
$$f'(t) = je^{jt}$$
$$= j(\cosh t + j \sinh t)$$
$$\cosh' t + j \sinh' t = \sinh t + j \cosh t$$

Dot and cross product

$$u^*v = (\alpha - i\beta)(\gamma + i\delta)$$

	γ	$i\delta$
α	$\alpha\gamma$	$i\alpha\delta$
$-i\beta$	$-i\beta\gamma$	$+\beta\delta$

$$= (\alpha\gamma + \beta\delta) + i(\alpha\delta - \beta\gamma)$$
$$u^*v = (u \cdot v) + i(u \times v)$$
$$(iz) \cdot z = \mathfrak{R}(iz^*z) = \mathfrak{R}(i|z|^2) = 0$$

$$u^*v = (\alpha - j\beta)(\gamma + j\delta)$$

	γ	$i\delta$
α	$\alpha\gamma$	$+j\alpha\delta$
$-j\beta$	$-j\beta\gamma$	$-\beta\delta$

$$= (\alpha\gamma - \beta\delta) + j(\alpha\delta - \beta\gamma)$$
$$u^*v = (u \cdot v) + j(u \times v)$$
$$(jz) \cdot z = \mathfrak{R}(jz^*z) = \mathfrak{R}(j|z|^2) = 0$$

Arc length

$$f(t) \coloneqq e^{it}$$
$$S = \int \sqrt{|f'(t)|^2} dt$$
$$= \int \sqrt{|ie^{it}|^2} dt = \int \sqrt{1} dt$$
$$= \Delta t$$

$$f(t) \coloneqq e^{jt}$$
$$S = \int \sqrt{-|f'(t)|^2} dt$$
$$= \int \sqrt{-|je^{jt}|^2} dt = \int \sqrt{-(-1)} dt$$
$$= \Delta t$$

4
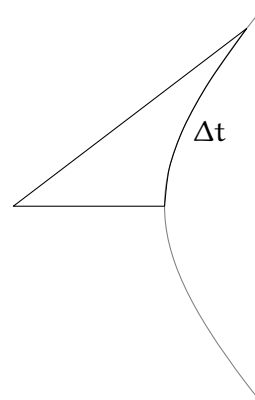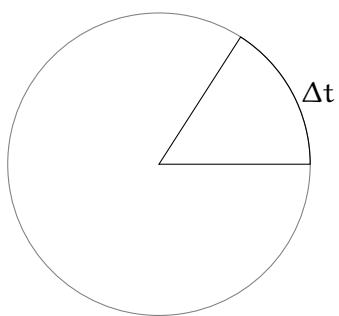
Sector area
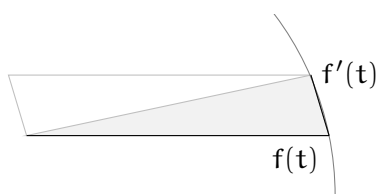
$$A = \frac{1}{2}\int f(t) \times f'(t)\,dt$$
$$= \frac{1}{2}\int \Im\Big(f^*(t)f'(t)\Big)\,dt$$
$$= \frac{1}{2}\int \Im\big(e^{-it}ie^{it}\big)\,dt$$
$$= \frac{1}{2}\int 1\,dt$$
$$= \frac{\Delta t}{2}$$

$$A = \frac{1}{2}\int f(t) \times f'(t)\,dt$$
$$= \frac{1}{2}\int \Im\big(f^*(t)f'(t)\big)\,dt$$
$$= \frac{1}{2}\int \Im\big(e^{-jt}je^{jt}\big)\,dt$$
$$= \frac{1}{2}\int 1\,dt$$
$$= \frac{\Delta t}{2}$$
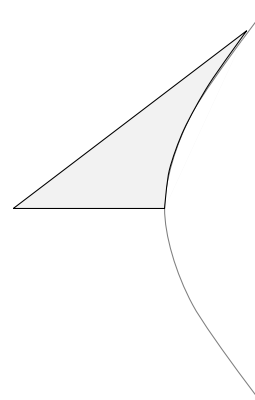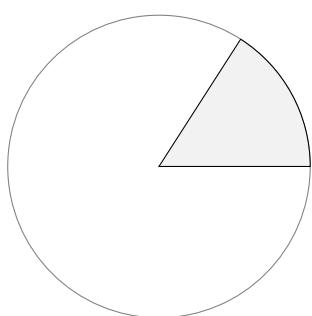
$\mathrm{SO}(2), \mathrm{SO}^+(+, -)$: Multiplying by imaginary exponents preserves inner product and area

$$\left(e^{it}u\right) \cdot \left(e^{it}v\right)$$
$$+i\left(e^{it}u\right) \times \left(e^{it}v\right) = \left(e^{it}u\right)^*\left(e^{it}v\right)$$
$$= e^{-it}u^*e^{it}v$$
$$= u^*v$$
$$= u \cdot v + i\,u \times v$$

$$\left(e^{jt}u\right) \cdot \left(e^{jt}v\right)$$
$$+j\left(e^{jt}u\right) \times \left(e^{jt}v\right) = \left(e^{jt}u\right)^*\left(e^{jt}v\right)$$
$$= e^{-jt}u^*e^{jt}v$$
$$= u^*v$$
$$= u \cdot v + j\,u \times v$$