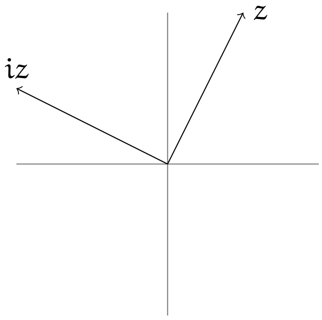
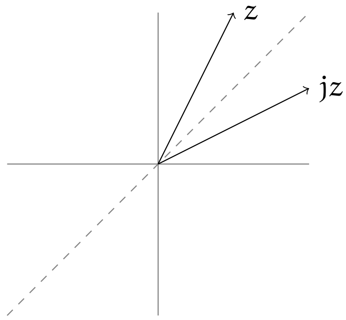
Complex

$$i^2 \coloneqq -1$$
$$i(x + iy) = -y + ix$$

Hyperbolic

$$j^2 \coloneqq +1$$
$$j(x + jy) = y + jx$$

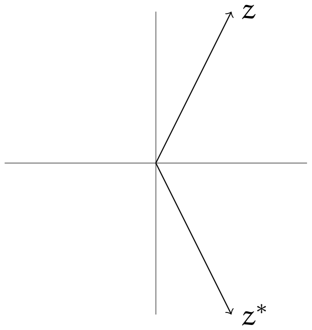


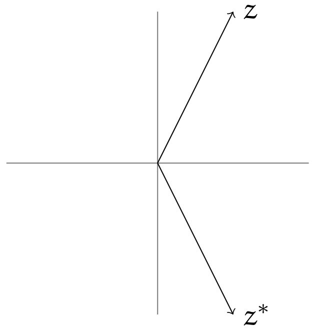


Conjugation and norm

$$(x + iy)^* = x - iy$$

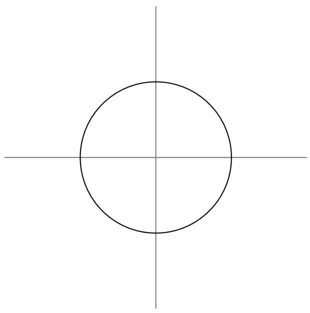
$$(x + jy)^* = x - jy$$

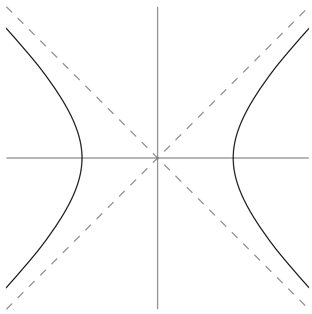




$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x + iy)(x - iy) \\ &= x^2 + y^2 \end{aligned}$$

$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x + jy)(x - jy) \\ &= x^2 - y^2 \end{aligned}$$





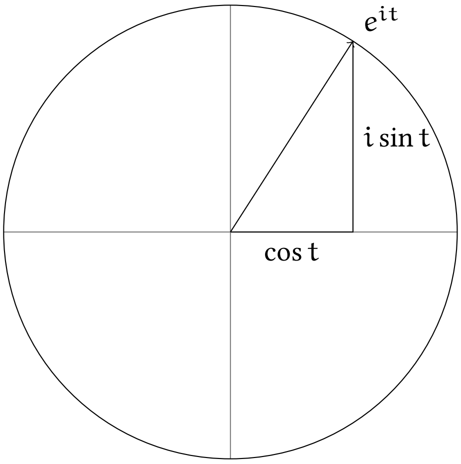
Exponentiation

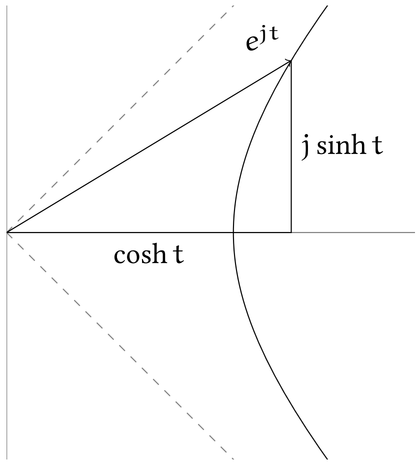
$$i^n = \begin{cases} 1 \\ i \\ -1 \\ -i \end{cases}$$

$$\begin{aligned} e^{it} &= 1(\dots) + i(\dots) \\ &\quad - 1(\dots) - i(\dots) \\ e^{it} &= \cos t + i \sin t \end{aligned}$$

$$j^n = \begin{cases} 1 \\ j \\ 1 \\ j \end{cases}$$

$$\begin{aligned} e^{jt} &= 1(\dots) + j(\dots) \\ &\quad + 1(\dots) + j(\dots) \\ e^{jt} &= \cosh t + j \sinh t \end{aligned}$$





Pythagorean identity

$$\begin{aligned} e^{-it} &= \cos(-t) + i \sin(-t) \\ &= \cos t - i \sin t \\ e^{-it} &= \left(e^{it}\right)^* \\ \left|e^{it}\right|^2 &= e^{it-it} \\ &= 1 \\ 1 &= \cos^2 t + \sin^2 t \end{aligned}$$

$$\begin{aligned} e^{-jt} &= \cosh(-t) + j \sinh(-t) \\ &= \cosh t - j \sinh t \\ e^{-jt} &= \left(e^{jt}\right)^* \\ \left|e^{jt}\right|^2 &= e^{jt-jt} \\ &= 1 \\ 1 &= \cosh^2 t - \sinh^2 t \end{aligned}$$

Pythagorean corollaries

$$\begin{aligned} \cos^2 t &= 1 - \sin^2 t \\ \sin^2 t &= 1 - \cos^2 t \\ \sec^2 t &= 1 + \tan^2 t \\ \csc^2 t &= \cot^2 t + 1 \end{aligned}$$

$$\begin{aligned} \cosh^2 t &= 1 + \sinh^2 t \\ \sinh^2 t &= \cosh^2 t - 1 \\ \operatorname{sech}^2 t &= 1 - \tanh^2 t \\ \operatorname{csch}^2 t &= \coth^2 t - 1 \end{aligned}$$

Angle sum formulae

$$
\begin{aligned}
e^{i(A+B)} &= e^{iA}e^{iB} \\
&= (c_A + is_A)(c_B + is_B)
\end{aligned}
$$

$$
\begin{array}{c|cc}
 & c_B & is_B \\
\hline
c_A & c_A c_B & ic_A s_B \\
is_A & ic_B s_A & -s_A s_B
\end{array}
$$

$$
\begin{array}{r}
\cos(A+B) \\
+i\sin(A+B)
\end{array}
=
\begin{array}{c}
c_A c_B - s_A s_B \\
+i(c_B s_A + c_A s_B)
\end{array}
$$

$$
\begin{aligned}
e^{j(A+B)} &= e^{jA}e^{jB} \\
&= (c_A + js_A)(c_B + js_B)
\end{aligned}
$$

$$
\begin{array}{c|cc}
 & c_B & js_B \\
\hline
c_A & c_A c_B & jc_A s_B \\
js_A & jc_B s_A & +s_A s_B
\end{array}
$$

$$
\begin{array}{r}
\cosh(A+B) \\
+j\sinh(A+B)
\end{array}
=
\begin{array}{c}
c_A c_B + s_A s_B \\
+j(c_B s_A + c_A s_B)
\end{array}
$$

Double angle

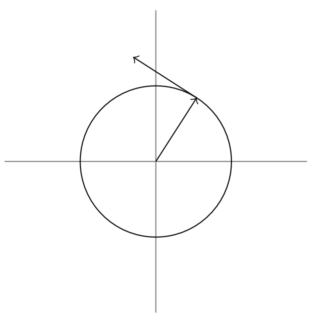
$$
\begin{aligned}
e^{2it} &= \left(e^{it}\right)^2 \\
\cos 2t &= \cos^2 t - \sin^2 t \\
i\sin 2t &= i\,2\cos t\sin t
\end{aligned}
\qquad
\begin{aligned}
e^{2jt} &= \left(e^{jt}\right)^2 \\
\cosh 2t &= \cosh^2 t + \sinh^2 t \\
j\sinh 2t &= j\,2\cosh t\sinh t
\end{aligned}
$$

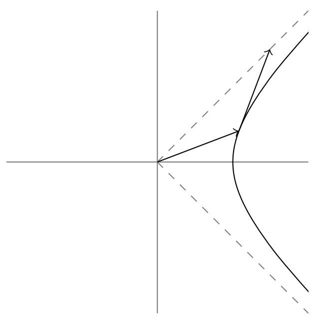
Double angle + Pythagorean

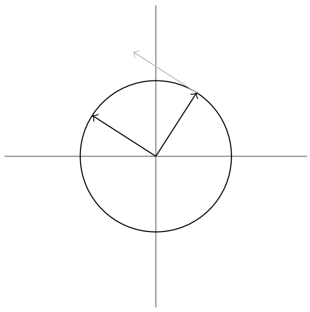
$$
\begin{aligned}
\cos 2t &= \cos^2 t - (1 - \cos^2 t) \\
&= 2\cos^2 t - 1 \\
&= (1 - \sin^2 t) - \sin^2 t \\
&= 1 - 2\sin^2 t
\end{aligned}
\qquad
\begin{aligned}
\cosh 2t &= \cosh^2 t + (\cosh^2 t - 1) \\
&= 2\cosh^2 t - 1 \\
&= (1 + \sinh^2 t) + \sinh^2 t \\
&= 1 + 2\sinh^2 t
\end{aligned}
$$

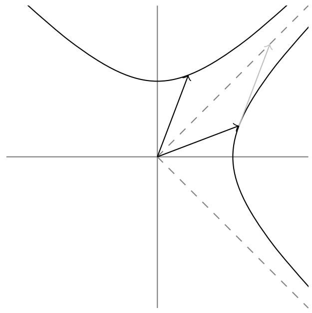
Derivatives

$$
\begin{aligned}
f(t) &\coloneqq e^{it} \\
f'(t) &= ie^{it} \\
&= i(\cos t + i\sin t) \\
\cos' t + i\sin' t &= -\sin t + i\cos t
\end{aligned}
\qquad
\begin{aligned}
f(t) &\coloneqq e^{jt} \\
f'(t) &= je^{jt} \\
&= j(\cosh t + j\sinh t) \\
\cosh' t + j\sinh' t &= \sinh t + j\cosh t
\end{aligned}
$$









Dot and cross product

$$
\begin{aligned}
u^* v &= (\alpha - i\beta)(\gamma + i\delta) \\
&\begin{array}{c|cc}
 & \gamma & i\delta \\
\hline
\alpha & \alpha\gamma & i\alpha\delta \\
-i\beta & -i\beta\gamma & +\beta\delta
\end{array} \\
&= (\alpha\gamma + \beta\delta) + i(\alpha\delta - \beta\gamma) \\
u^* v &= (u \cdot v) + i(u \times v) \\
(iz) \cdot z &= \Re(iz^* z) = \Re(i|z|^2) = 0
\end{aligned}
$$

$$
\begin{aligned}
u^* v &= (\alpha - j\beta)(\gamma + j\delta) \\
&\begin{array}{c|cc}
 & \gamma & i\delta \\
\hline
\alpha & \alpha\gamma & +j\alpha\delta \\
-j\beta & -j\beta\gamma & -\beta\delta
\end{array} \\
&= (\alpha\gamma - \beta\delta) + j(\alpha\delta - \beta\gamma) \\
u^* v &= (u \cdot v) + j(u \times v) \\
(jz) \cdot z &= \Re(jz^* z) = \Re(j|z|^2) = 0
\end{aligned}
$$

Arc length

$$
\begin{aligned}
f(t) &\coloneqq e^{it} \\
S &= \int \sqrt{|f'(t)|^2} dt \\
&= \int \sqrt{|ie^{it}|^2} dt = \int \sqrt{1} dt \\
&= \Delta t
\end{aligned}
$$

$$
\begin{aligned}
f(t) &\coloneqq e^{jt} \\
S &= \int \sqrt{-|f'(t)|^2} dt \\
&= \int \sqrt{|je^{jt}|^2} dt = \int \sqrt{-(-1)} dt \\
&= \Delta t
\end{aligned}
$$

Sector area

$$
\begin{aligned}
A &= \frac{1}{2} \int f(t) \times f'(t) dt \\
&= \frac{1}{2} \int \Im\Big(f^*(t) f'(t)\Big) dt \\
&= \frac{1}{2} \int \Im\big(e^{-it} i e^{it}\big) dt \\
&= \frac{1}{2} \int 1 \, dt \\
&= \frac{\Delta t}{2}
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{1}{2} \int f(t) \times f'(t) dt \\
&= = \frac{1}{2} \int \Im\big(f^*(t) f'(t)\big) dt \\
&= = \frac{1}{2} \int \Im\big(e^{-jt} j e^{jt}\big) dt \\
&= \frac{1}{2} \int 1 \, dt \\
&= \frac{\Delta t}{2}
\end{aligned}
$$

$\mathrm{SO}(2), \mathrm{SO}^{+}(+,-)$: Multiplying by imaginary exponents preserves inner product and area

$$\begin{aligned}(e^{it}u)\cdot(e^{it}v)&\\ +i(e^{it}u)\times(e^{it}v) &= (e^{it}u)^{*}(e^{it}v)\\ &= e^{-it}u^{*}e^{it}v\\ &= u^{*}v\\ &= u\cdot v + i\,u\times v\end{aligned}$$

$$\begin{aligned}(e^{jt}u)\cdot(e^{jt}v)&\\ +j(e^{jt}u)\times(e^{jt}v) &= (e^{jt}u)^{*}(e^{jt}v)\\ &= e^{-jt}u^{*}e^{jt}v\\ &= u^{*}v\\ &= u\cdot v + j\,u\times v\end{aligned}$$