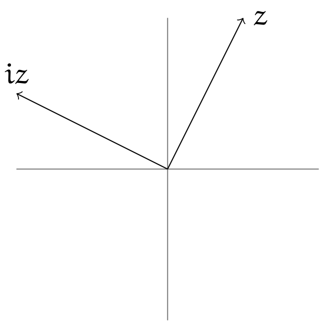
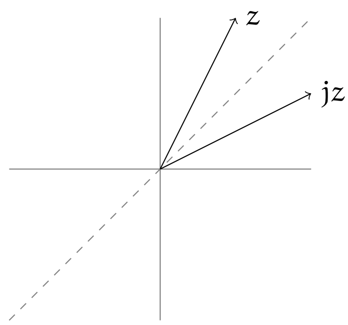
Complex

$$i^2 \coloneqq -1$$
$$i(x + iy) = -y + ix$$



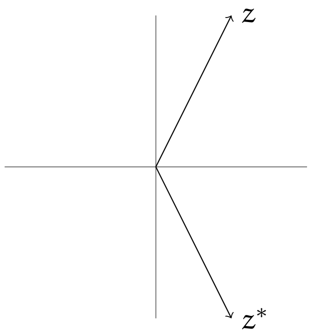
Hyperbolic

$$j^2 \coloneqq +1$$
$$j(x + jy) = y + jx$$

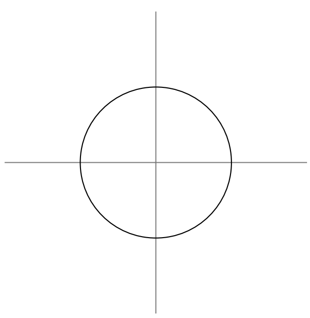


Conjugation and norm

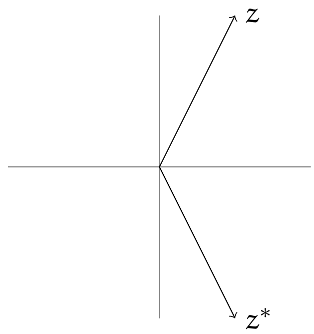
$$(x + iy)^* = x - iy$$



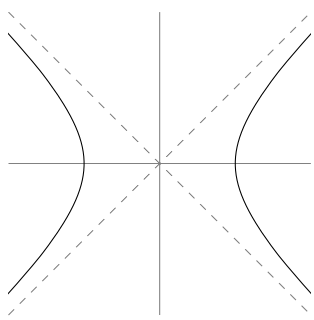
$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x + iy)(x - iy) \\ &= x^2 + y^2 \end{aligned}$$



$$(x + jy)^* = x - jy$$



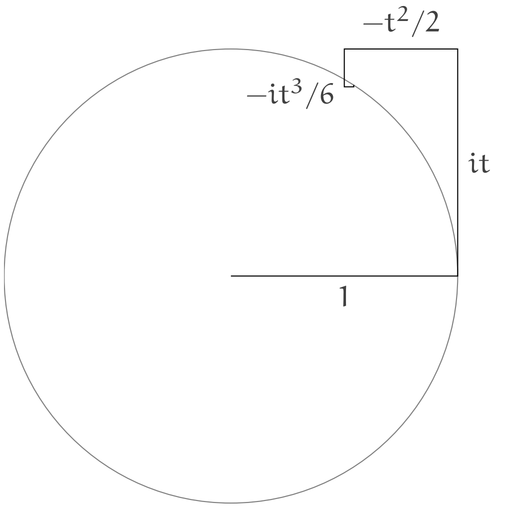
$$\begin{aligned} |z|^2 &\coloneqq zz^* = (x + jy)(x - jy) \\ &= x^2 - y^2 \end{aligned}$$

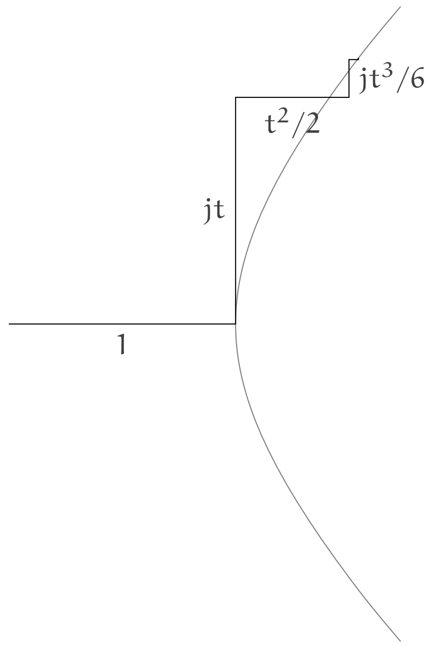


Exponentiation

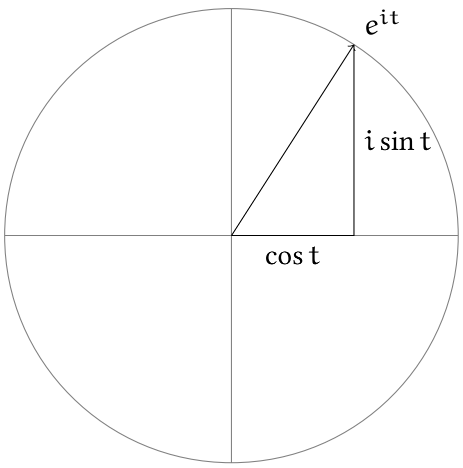
$$i^n = \begin{cases} 1 & n \equiv 0 \pmod 4 \\ i & n \equiv 1 \pmod 4 \\ -1 & n \equiv 2 \pmod 4 \\ -i & n \equiv 3 \pmod 4 \end{cases} \qquad j^n = \begin{cases} 1 & n \equiv 0 \pmod 4 \\ j & n \equiv 1 \pmod 4 \\ 1 & n \equiv 2 \pmod 4 \\ j & n \equiv 3 \pmod 4 \end{cases}$$

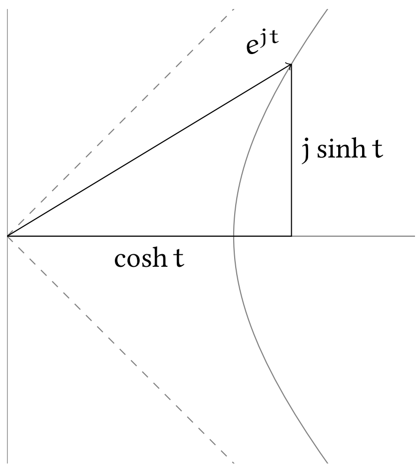
$$\begin{aligned} e^{it} &= 1(\dots) + i(\dots) \\ &\quad - 1(\dots) - i(\dots) \end{aligned} \qquad \begin{aligned} e^{jt} &= 1(\dots) + j(\dots) \\ &\quad + 1(\dots) + j(\dots) \end{aligned}$$





$$e^{it} = \cos t + i \sin t \qquad e^{jt} = \cosh t + j \sinh t$$





Pythagorean identity

$$\begin{aligned}
e^{-it} &= \cos(-t) + i\sin(-t) \\
&= \cos t - i\sin t \\
e^{-it} &= \left(e^{it}\right)^* \\
\left|e^{it}\right|^2 &= \left(e^{it}\right)\left(e^{it}\right)^* \\
&= e^{it}e^{-it} \\
&= 1 \\
1 &= \cos^2 t + \sin^2 t
\end{aligned}$$

$$\begin{aligned}
e^{-jt} &= \cosh(-t) + j\sinh(-t) \\
&= \cosh t - j\sinh t \\
e^{-jt} &= \left(e^{jt}\right)^* \\
\left|e^{jt}\right|^2 &= \left(e^{jt}\right)\left(e^{jt}\right)^* \\
&= e^{jt}e^{-jt} \\
&= 1 \\
1 &= \cosh^2 t - \sinh^2 t
\end{aligned}$$

Pythagorean corollaries

$$\begin{aligned}
\cos^2 t &= 1 - \sin^2 t \\
\sin^2 t &= -\cos^2 t + 1 \\
\sec^2 t &= 1 + \tan^2 t \\
\csc^2 t &= \cot^2 t + 1
\end{aligned}$$

$$\begin{aligned}
\cosh^2 t &= 1 + \sinh^2 t \\
\sinh^2 t &= \cosh^2 t - 1 \\
\operatorname{sech}^2 t &= 1 - \tanh^2 t \\
\operatorname{csch}^2 t &= \coth^2 t - 1
\end{aligned}$$

Angle sum formulae

$$\begin{aligned}
e^{i(A+B)} &= e^{iA}e^{iB} \\
&= (c_A + is_A)(c_B + is_B) \\
&\begin{array}{c|cc}
 & c_B & is_B \\ \hline
c_A & c_Ac_B & ic_As_B \\
is_A & ic_Bs_A & -s_As_B
\end{array} \\
\begin{array}{r}\cos(A+B) \\ +i\sin(A+B)\end{array} &= \begin{array}{c}c_Ac_B - s_As_B \\ +i(c_Bs_A + c_As_B)\end{array}
\end{aligned}$$

$$\begin{aligned}
e^{j(A+B)} &= e^{jA}e^{jB} \\
&= (c_A + js_A)(c_B + js_B) \\
&\begin{array}{c|cc}
 & c_B & js_B \\ \hline
c_A & c_Ac_B & jc_As_B \\
js_A & jc_Bs_A & +s_As_B
\end{array} \\
\begin{array}{r}\cosh(A+B) \\ +j\sinh(A+B)\end{array} &= \begin{array}{c}c_Ac_B + s_As_B \\ +j(c_Bs_A + c_As_B)\end{array}
\end{aligned}$$

Double angle

$$\begin{aligned}
e^{2it} &= \left(e^{it}\right)^2 \\
\cos 2t &= \cos^2 t - \sin^2 t \\
i\sin 2t &= i\,2\cos t\sin t
\end{aligned}$$

$$\begin{aligned}
e^{2jt} &= \left(e^{jt}\right)^2 \\
\cosh 2t &= \cosh^2 t + \sinh^2 t \\
j\sinh 2t &= j\,2\cosh t\sinh t
\end{aligned}$$

Double angle + Pythagorean

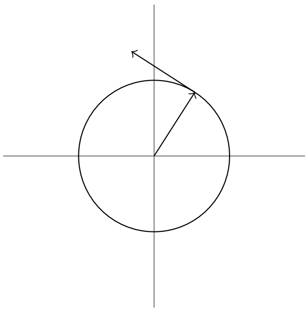
$$\begin{aligned}
\cos 2t &= \cos^2 t - (1 - \cos^2 t) \\
&= 2\cos^2 t - 1 \\
&= (1 - \sin^2 t) - \sin^2 t \\
&= 1 - 2\sin^2 t
\end{aligned}$$

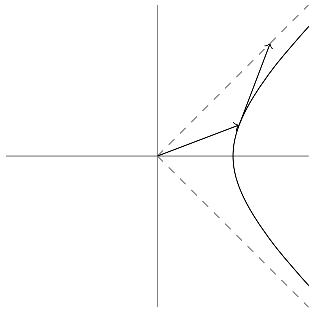
$$\begin{aligned}
\cosh 2t &= \cosh^2 t + (\cosh^2 t - 1) \\
&= 2\cosh^2 t - 1 \\
&= (1 + \sinh^2 t) + \sinh^2 t \\
&= 1 + 2\sinh^2 t
\end{aligned}$$

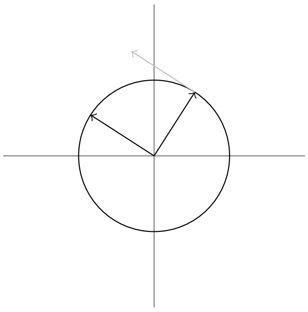
Derivatives

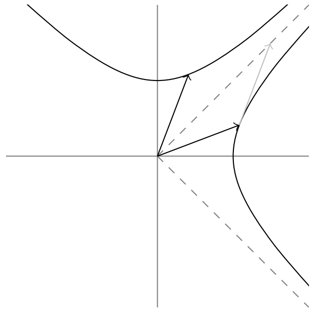
$$\begin{aligned} f(t) &\coloneqq e^{it} \\ f'(t) &= ie^{it} \\ &= i(\cos t + i \sin t) \\ \cos' t + i \sin' t &= -\sin t + i \cos t \end{aligned}$$

$$\begin{aligned} f(t) &\coloneqq e^{jt} \\ f'(t) &= je^{jt} \\ &= j(\cosh t + j \sinh t) \\ \cosh' t + j \sinh' t &= \sinh t + j \cosh t \end{aligned}$$









Dot and cross product

$$u^* v = (\alpha - i\beta)(\gamma + i\delta)$$

	γ	$i\delta$
α	$\alpha\gamma$	$i\alpha\delta$
$-i\beta$	$-i\beta\gamma$	$+\beta\delta$

$$\begin{aligned} &= (\alpha\gamma + \beta\delta) + i(\alpha\delta - \beta\gamma) \\ u^* v &= (u \cdot v) + i(u \times v) \\ (iz) \cdot z &= \Re(iz^* z) = \Re(i|z|^2) = 0 \end{aligned}$$

$$u^* v = (\alpha - j\beta)(\gamma + j\delta)$$

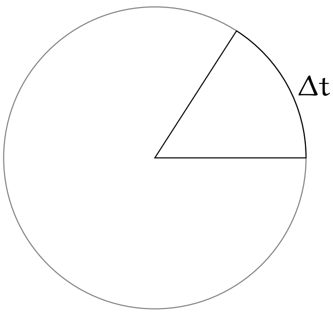
	γ	$i\delta$
α	$\alpha\gamma$	$+j\alpha\delta$
$-j\beta$	$-j\beta\gamma$	$-\beta\delta$

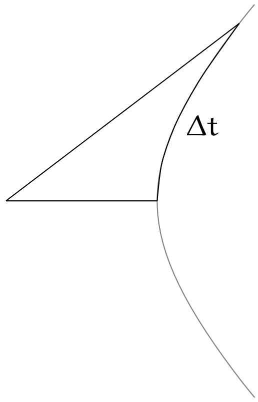
$$\begin{aligned} &= (\alpha\gamma - \beta\delta) + j(\alpha\delta - \beta\gamma) \\ u^* v &= (u \cdot v) + j(u \times v) \\ (jz) \cdot z &= \Re(jz^* z) = \Re(j|z|^2) = 0 \end{aligned}$$

Arc length

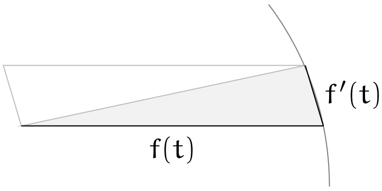
$$\begin{aligned} f(t) &\coloneqq e^{it} \\ S &= \int \sqrt{|f'(t)|^2} dt \\ &= \int \sqrt{|ie^{it}|^2} dt = \int \sqrt{1} dt \\ &= \Delta t \end{aligned}$$

$$\begin{aligned} f(t) &\coloneqq e^{jt} \\ S &= \int \sqrt{-|f'(t)|^2} dt \\ &= \int \sqrt{-|je^{jt}|^2} dt = \int \sqrt{-(-1)} dt \\ &= \Delta t \end{aligned}$$



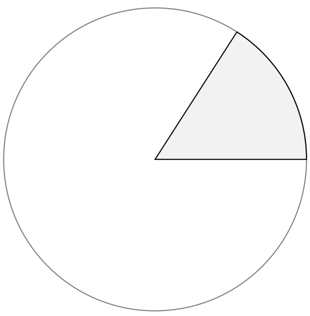


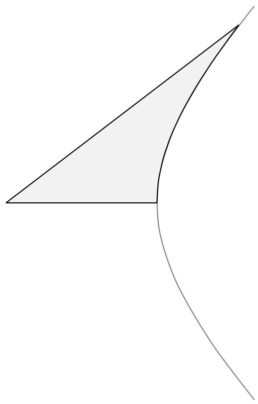
Sector area



$$\begin{aligned} A &= \frac{1}{2}\int f(t) \times f'(t) dt \\ &= \frac{1}{2}\int \Im\Big(f^*(t) f'(t)\Big) dt \\ &= \frac{1}{2}\int \Im\big(e^{-it} i e^{it}\big) dt \\ &= \frac{1}{2}\int 1 \, dt \\ &= \frac{\Delta t}{2} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int f(t) \times f'(t) dt \\ &= \frac{1}{2}\int \Im\big(f^*(t) f'(t)\big) dt \\ &= \frac{1}{2}\int \Im\big(e^{-jt} j e^{jt}\big) dt \\ &= \frac{1}{2}\int 1 \, dt \\ &= \frac{\Delta t}{2} \end{aligned}$$





$SO(2), SO^+(+,-)$: Multiplying by imaginary exponents preserves inner product and area

$$\begin{aligned}
&(e^{it}u)\cdot(e^{it}v) \\
+i&(e^{it}u)\times(e^{it}v) = (e^{it}u)^*(e^{it}v) \\
&= e^{-it}u^*e^{it}v \\
&= u^*v \\
&= u\cdot v + i\,u\times v
\end{aligned}
\qquad
\begin{aligned}
&(e^{jt}u)\cdot(e^{jt}v) \\
+j&(e^{jt}u)\times(e^{jt}v) = (e^{jt}u)^*(e^{jt}v) \\
&= e^{-jt}u^*e^{jt}v \\
&= u^*v \\
&= u\cdot v + j\,u\times v
\end{aligned}$$