Getting to algebraic geometry's machinery ASAP but every move is motivated

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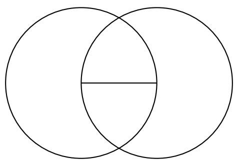
Classical algebraic geometry studies the *solution set* of systems of *polynomial* equations. These are geometric. For example, compass and straightedge constructions can be done through equations of lines and circles. Famously, their intersections can be found by solving a system of equations of circles and lines.

To construct the perpendicular bisector of a segment, draw two circles of equal radius on either side:



$$x^2 + y^2 = 1 \tag{1}$$

$$(x-1)^2 + y^2 = 1 \tag{2}$$

To find their intersections, we need to play the equations off each other. Expand eq. (2) to reveal an $x^2 + y^2$:

$$\begin{aligned}\underbrace{x^2 + y^2}_{1} - 2x + \cancel{1} &= \cancel{1} \\ 1 - 2x &= 0 \\ x &= 1/2. \end{aligned} \tag{3}$$

Our system of equations has grown: equations 1, 2 & 3 are all worth keeping. Further algebraic manipulations yield more equations. Of particular interest for this example is finding the corresponding ys:

$$\begin{aligned} x^2 + y^2 &= 1 \\ x &= 1/2 \\ y^2 &= 3/4 \end{aligned} \tag{4}$$

I suppose I lied. We're not really interested in systems of equations. We're interested in the system of equations *and all its consequences.* The particular starting equations aren't so important. We could have started with

$$\begin{aligned} x^2 + y^2 &= z^2 \\ (x-1)^2 + y^2 &= z^2 \\ z &= 1 \end{aligned}$$

or

$$\begin{aligned} 2x^2 + 2y^2 &= 2 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

and would arrive at the same place.

Modernization 1 Focus on *sets generated by* a system of equations instead of a particular system.

They are ideals

To make this work, we need to take stock of what operations give new equations from old systems. Consider a single equation. If

$$f(x) = g(x),$$

then

$$f(x) + h(x) = g(x) + h(x).$$

Therefore, the placement of the equal sign is arbitrary. The following are equivalent:

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 &= 1 - y^2 \\ x^2 + y^2 - 1 &= 0. \end{aligned} \tag{5}$$

To cut down on possibilities, pick a normal form. Put every equation in that last $= 0$ form:

$$F(x, y, z) = 0.$$

So our equation

$$f(x, y, z) = g(x, y, z)$$

becomes

$$f(x, y, z) - g(x, y, z) = 0.$$

This unifies our view of algebraic manipulations. We no longer need a special set of rules for equations: adding $h(x, y, z)$ to both sides becomes

$$f(x, y, z) + h(x, y, z) - h(x, y, z) - g(x, y, z) = 0,$$

the famous fact $h(x, y, z) - h(x, y, z) = 0$.

What can we do with a system of equations to pull out new equations?

1. Rescaling.

$$f = g \implies k f = kg,$$

2. Additivity.

$$\begin{aligned} f &= g \\ \phi &= \gamma \\ (f+\phi) &= (g+\gamma), \end{aligned}$$

3. Multiplicativity. If

$$\begin{aligned} f &= g \\ \phi &= \gamma \end{aligned}$$

then

$$(f\phi) = (g\gamma).$$

This differs from Rescaling because we're multiplying possibly different things on both sides of the equation.

4. Substitution. Take a function $A(t)$. Since we're doing algebraic geometry, $A(t)$ is a polynomial. If

$$f = g$$

then

$$A(f) = A(g).$$

We want to put these into our normal form. Rescaling and Additivity translate directly:

- Rescaling.

$$F = 0 \implies kF$$

- Additivity.

$$\begin{aligned} F &= 0 \\ G &= 0 \end{aligned}$$

then

$$(F+G) = 0.$$

If we tried to translate multiplicativity directly, we'd get

$$f = 0, g = 0 \implies fg = 0,$$

which is just a special case of rescaling. The real translation would be

$$\begin{gathered} f - g = 0 \\ \phi - \gamma = 0 \\ \Downarrow \\ (f\phi - g\gamma) = 0. \end{gathered}$$

We can actually get this using only Additivity and Rescaling in our normal form. I will start dropping the variables btw.

Theorem 1 (Multiplicativity is redundant) *If*

$$\begin{gathered} \alpha - \beta = 0 \\ \gamma - \delta = 0, \end{gathered}$$

then

$$\alpha\gamma - \beta\delta = 0.$$

Proof · We need to get an $\alpha\gamma - \beta\delta$, so multiply

$$(\alpha \mp \beta)(\gamma \pm \delta) = 0.$$

Then

$$\begin{array}{c|cc} \times & \gamma & -\delta \\ \hline \alpha & \alpha\gamma & -\alpha\delta \\ +\beta & +\beta\gamma & -\beta\delta \end{array} \qquad \underbrace{(\alpha\gamma - \beta\delta)}_{X} + \underbrace{(-\alpha\delta + \beta\gamma)}_{Y} = 0$$

$$\begin{array}{c|cc} \times & \gamma & +\delta \\ \hline \alpha & \alpha\gamma & +\alpha\delta \\ +\beta & -\beta\gamma & -\beta\delta \end{array} \qquad \overbrace{(\alpha\gamma - \beta\delta)}^{X} + \overbrace{(+\alpha\delta - \beta\gamma)}^{-Y} = 0$$

But then

$$X = -Y \qquad \text{and} \qquad X = Y.$$

Conclude $X = 0 = \alpha\gamma - \beta\delta$. $\square$

Substitution, too, is really a consequence of Additivity and Rescaling.

Theorem 2 *If* $P(t)$ *is a polynomial and*

$$f - g = 0,$$

then

$$P(f) - P(g) = 0,$$

and this is a consequence of only Additivity and Rescaling.

Proof · Because P is a polynomial,

$$P(t) = a_0 + a_1 + \dots a_n t^n.$$

By additivity, it suffices to prove the theorem for monomials. By rescaling, it suffices to prove it for exponents only. But if $f - g = 0$, then by theorem 1, $f^n - g^n = 0$. □

Since the right-hand side is just $= 0$, the same for everything now, focus on the left-hand side only. The set of left-hand sides generated by a system of equations must satisfy two properties:

Definition 3 (Ideal) The set $I \subseteq R$ is an *ideal* iff

- Absorbativity (rescaling) under multiplication:

$$\alpha \in I, k \in R \implies k\alpha \in I.$$

- Closure under addition:

$$\alpha, \beta \in I \implies \alpha + \beta \in I.$$

I change to absorbatavity from rescaling to emphasize that it's true for *any* $k \in R$, *i.e.*, I absorbs multiplication. It's not merely closed.

This abstracts the idea of systems of equations and their consequences. Recover a system of equations by saying $f(x, y, z) = 0$ for all $f \in I$.

Explicit reference to equations and even polynomials have been removed!

> Modernization 2 (equations $\to$ ideals) Instead of systems of polynomial equations, work with ideals. Systems of polynomial equations are now an important special case.