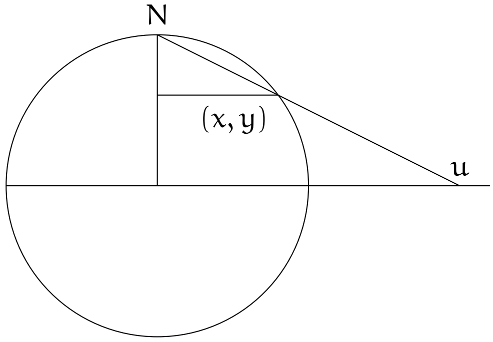
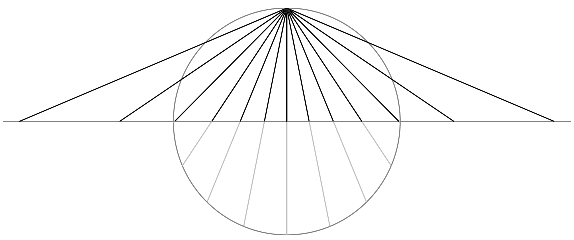
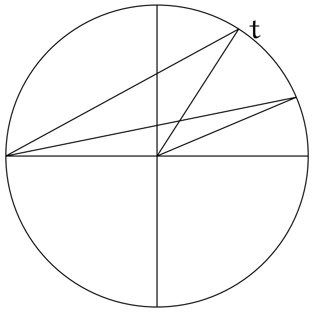
1 Stereographic



$$
\begin{aligned}
N0u &\sim N(0,y)(x,y)\\
\vec{u} &= \frac{\vec{x}}{1-y}\\
k &\coloneqq 1-y\\
x &= ku\\
y^2+x^2 &= 1\\
\underbrace{(1-k)^2}_{1-2k+k^2} + k^2u^2 &= 1\\
(1+u^2)k^2 - 2k\cancel{+1} &= \cancel{1}\\
k &= \frac{2}{1+u^2}\\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{1+u^2}\begin{bmatrix} 2u^2 \\ u^2-1 \end{bmatrix}
\end{aligned}
$$

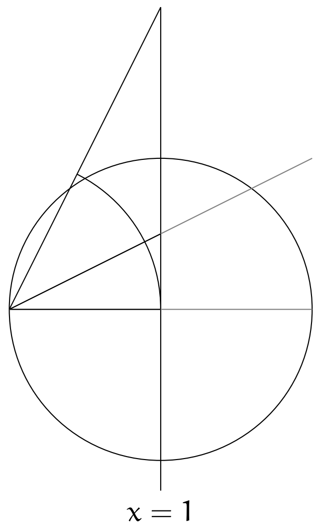


In one dimension



Via projective geometry:

$$
\begin{aligned}
\begin{bmatrix} \cos {}^t/_2 & -\sin {}^t/_2 \\ \sin {}^t/_2 & \cos {}^t/_2 \end{bmatrix}\begin{bmatrix} 1 \\ u \end{bmatrix} &= \begin{bmatrix} -u\sin {}^t/_2 + \cos {}^t/_2 \\ u\cos {}^t/_2 + \sin {}^t/_2 \end{bmatrix}\\
&\sim \begin{bmatrix} 1 \\ \dfrac{u\cos {}^t/_2 + \sin {}^t/_2}{-u\sin {}^t/_2 + \cos {}^t/_2} \end{bmatrix}
\end{aligned}
$$



S^n

$$
\begin{aligned}
\begin{bmatrix} \vec{x} \\ y \end{bmatrix} &\mapsto \frac{1}{u^2+1}\begin{bmatrix} 2\vec{x} \\ 2y \\ u^2-1 \end{bmatrix}\\
&\mapsto \frac{1}{u^2+1}\begin{bmatrix} 1 & & \\ & \gamma & -\sigma \\ & \sigma & \gamma \end{bmatrix}\begin{bmatrix} 2x \\ 2y \\ u^2-1 \end{bmatrix}\\
&\propto \begin{bmatrix} 2x \\ 2y\gamma - \left(u^2-1\right)\sigma \\ 2y\sigma + \left(u^2-1\right)\gamma \end{bmatrix}\\
&\mapsto \frac{\begin{bmatrix} 2\vec{x} \\ 2y\gamma - (u^2-1)\sigma \end{bmatrix}}{1 - 2y\sigma - (u^2-1)\gamma}
\end{aligned}
$$

Which, for S^1, reduces to

$$
\mapsto \frac{2y\gamma - \left(y^2-1\right)\sigma}{1 - 2y\sigma - (y^2-1)\gamma}
$$

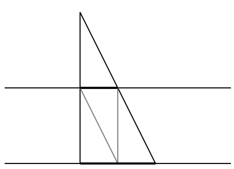
1.1 the metric

$$
\begin{aligned}
F(u) &= \frac{1}{u^2+1}\begin{bmatrix} 2u \\ u^2-1 \end{bmatrix} \\
dF(u) &= -\begin{bmatrix} 2u \\ u^2-1 \end{bmatrix}\frac{2u^\flat}{(u^2+1)^2} + \frac{1}{u^2+1}\begin{bmatrix} 2 \\ 2u^\flat \end{bmatrix} \\
&= 2\frac{\begin{bmatrix} u^2+1 \\ (u^2+1)u^\flat \end{bmatrix} - \begin{bmatrix} 2uu^\flat \\ (u^2-1)u^\flat \end{bmatrix}}{(u^2+1)^2} \\
&= \frac{2}{(u^2+1)^2}\begin{bmatrix} u^2+1-2uu^\flat \\ 2u^\flat \end{bmatrix} \\
dF(u)(X) &= \frac{2}{(u^2+1)^2}\begin{bmatrix} \left(u^2+1\right)X - 2uu^\flat X \\ 2u^\flat X \end{bmatrix} \\
|dF_u(X)|^2 &= \frac{4}{(u^2+1)^4}\begin{pmatrix} (u^2+1)^2|X|^2 \\ \cancel{+4|u|^2(u^\flat X)^2} \\ -4(\cancel{|u|^2+1})\left(X^\flat u\right)^2 \\ \cancel{+4\left(u^\flat X\right)^2} \end{pmatrix} \\
&= \frac{4}{(u^2+1)^2}|X|^2
\end{aligned}
$$

By dimensional analysis,

$$
\begin{aligned}
&= \frac{4R^4}{(u^2+R^2)^2}|X|^2 \\
|dF(u)(X)| &= \frac{2R^2}{u^2+R^2}|X|
\end{aligned}
$$

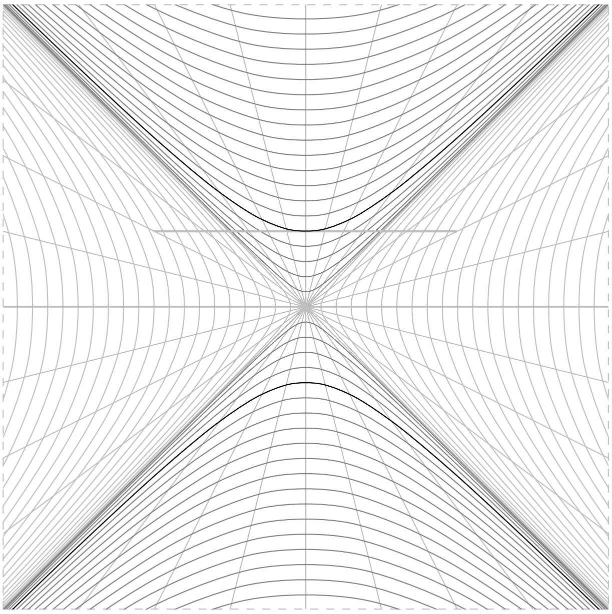
When $R \to \infty$, $|X| \mapsto 2|X|$



Single particle lagranian

$$
\begin{aligned}
S &= \int \frac{1}{2}|\dot{u}|_F^2 \\
&= \int \frac{1}{2}\left(\frac{4}{(u^2+1)^2}\right)|\dot{u}|_{\mathbb{R}^n}^2 \\
&= \int \underbrace{\frac{2\dot{u}^2}{(u^2+1)^2}}_{L} \\
\frac{\partial L}{\partial u} &= \frac{-8\dot{u}^2u^\flat}{(u^2+1)^3} \\
\frac{\partial L}{\partial \dot{u}} &= \frac{4\dot{u}^\flat}{(u^2+1)^2} \\
E &= \frac{4\dot{u}^2}{(u^2+1)^2} - \frac{2\dot{u}^2}{(u^2+1)^2} \\
&= \frac{2\dot{u}^2}{(u^2+1)^2} = L \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} &= \frac{4\ddot{u}^\flat}{(u^2+1)^2} - \frac{16\left(u^\flat\dot{u}\right)\dot{u}^\flat}{(u^2+1)^3} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} &= \frac{4\ddot{u}^\flat}{(u^2+1)^2} + \frac{8\dot{u}^2u^\flat - 16u^\flat\dot{u}\dot{u}^\flat}{(u^2+1)^3} \\
0 &= \ddot{u}^\flat - \frac{2u^\flat}{u^2+1}\left(2\dot{u}\dot{u}^\flat - \dot{u}^2\right) \\
&= \ddot{u} \underbrace{- \frac{2u(2-\delta)_{\alpha\beta}}{u^2+1}}_{\Gamma_{\alpha\beta}}\dot{u}_\alpha\dot{u}_\beta
\end{aligned}
$$

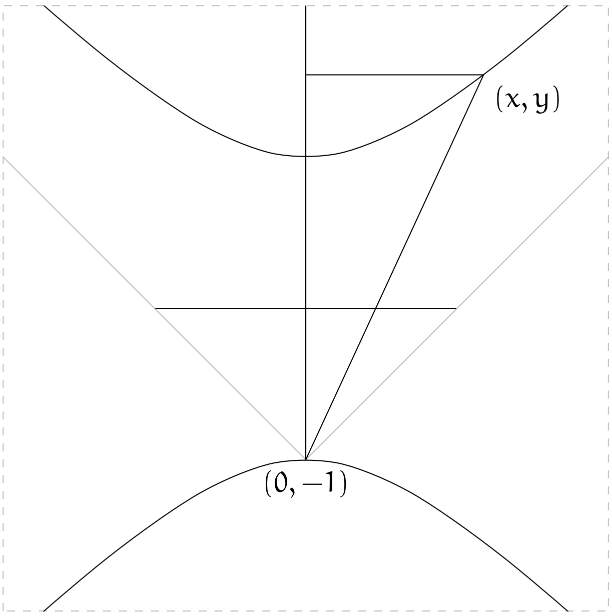
2 Hyperboloid



The proper-length metric is positive definite on H^+:

$$\begin{aligned}
N &= (1, 0) \\
T_N H^+ &= dt^{-1}(0) \\
&= \left\{ \begin{bmatrix} 0 \\ x \end{bmatrix} : x \in \mathbb{R}^n \right\} \\
\begin{bmatrix} 0 \\ x \end{bmatrix}^\flat \begin{bmatrix} 0 \\ x \end{bmatrix} &= \begin{bmatrix} -0 & x^\flat \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} \\
&= |x|^2
\end{aligned}$$

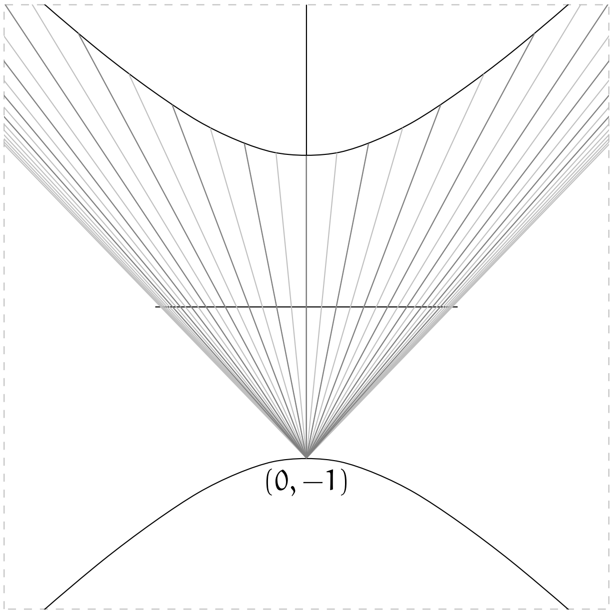
Because $SO^+(1, n)$ acts transitively, this rolls over all of H^+.



$$\begin{aligned}
u &= \frac{x}{1+y} \\
k &\coloneqq 1 + y \\
x &= ku \\
y &= k - 1 \\
y^2 - x^2 &= 1 \\
\underbrace{(k-1)^2}_{k^2-2k+1} - k^2u^2 &= 1 \\
(1-u^2)k^2 - 2k\cancel{+1} &= \cancel{1} \\
k &= \frac{2}{1-u^2} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{1-u^2} \begin{bmatrix} 2u \\ u^2+1 \end{bmatrix}
\end{aligned}$$

Combine with the sphere's equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1 \pm u^2} \begin{bmatrix} 2u \\ u^2 \mp 1 \end{bmatrix}$$



2.1 Hyperbolic angles

Let $j^2 = 1$. Because $(t + jx)^* = t - jx$,

$$|t + jx|^2 \coloneqq (t + jx)^*(t - jx) = t^2 - x^2$$

which is proper time.

The main trig identities:

$$
\begin{aligned}
\left|e^{jt}\right| &= 1 \\
\cosh^2 t - \sinh^2 t &= 1 \\
e^{j(\alpha+\beta)} &= e^{j\alpha}e^{j\beta} \\
c_{\alpha+\beta} + js_{\alpha+\beta} &= (c_\alpha + js_\alpha)\big(c_\beta + js_\beta\big) \\
&= c_\alpha c_\beta + s_\alpha s_\beta + j\big(c_\alpha s_\beta + s_\alpha c_\beta\big) \\
c_2 + js_2 &= \left(c^2 + s^2\right) + j(2cs) \\
\cosh 2t &= 2\cosh^2 t - 1
\end{aligned}
$$

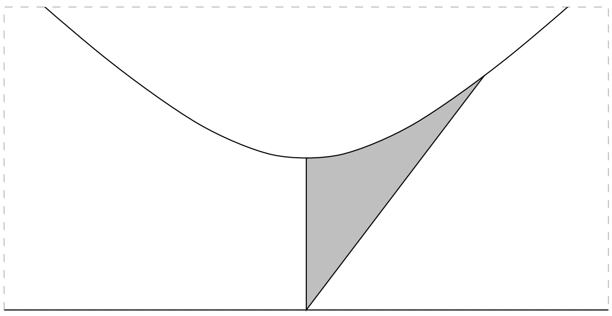
Angle, arclength, area

$$
\begin{aligned}
\gamma(t) &\coloneqq e^{jt} \\
|\gamma(t)|^2 &= e^{jt}e^{-jt} = 1 \\
\dot\gamma(t) &= je^{jt}
\end{aligned}
$$

gives the identities

has arclength

$$
\begin{aligned}
\int \sqrt{-\left|\dot\gamma(t)\right|}\,dt &= \int dt \\
&= \Delta t
\end{aligned}
$$



Using cross products:

$$
\begin{array}{c|cc}
\cdot & \gamma & j\delta \\
\hline
\alpha & \alpha\gamma & j\alpha\delta \\
-j\beta & -j\beta\gamma & -\beta\delta
\end{array}
\implies u^* v = u \cdot v + ju \times v
$$

$$
\begin{aligned}
A &= \int \frac{e^{jt} \times je^{jt}}{2} = \Im \int \frac{je^{-jt}e^{jt}}{2} \\
&= \int \frac{1}{2}dt = \frac{\Delta t}{2}
\end{aligned}
$$