$$
\begin{aligned}
K(b,a) &\coloneqq \frac{1}{A} \int\limits_{\gamma : a \rightsquigarrow b} D\gamma \, e^{iS(\gamma)} \\
&= \frac{1}{A} \int\limits_{\gamma : a \rightsquigarrow b} D\gamma \exp\left(i \int_{t_a}^{t_b} L\big(\gamma(t), \dot\gamma(t), t\big) \, dt \right)
\end{aligned}
$$

An infinitesimal Wick rotation probably encourages convergence:

$$
\approx \frac{1}{A} \int\limits_{\gamma : a \rightsquigarrow b} D\gamma \, e^{(i-\alpha)S(\gamma)}
$$

$$
\int : \Big(\big(I \to \mathbb{R}^3 \big) \to \mathbb{C} \Big) \to \mathbb{C}?
$$

What is the structure of the space of curves? It's almost affine, but the action of the vector space isn't free. It fixes endpoints. Given two endpoints, any curves differ by loops through zero, so given endpoints we get a vector space.

1 Feynman's limit

$$
K(b,a) = \lim \frac{1}{A_n} \int \cdots \int dx_1 \ldots dx_{N-1} \, e^{iS(\vec{x})}
$$

with $x_0 = a$ and $x_N = b$

$$
= \lim \frac{1}{A_n} \int \cdots \int dx_1 \ldots dx_{N-1} \exp\left(i\varepsilon \sum L(x_k, v_k, t) \right)
$$

2 Schrödinger's equation

$$
K(x, t+\varepsilon; y, t) = \int\limits_{(y,t) \rightsquigarrow (x, t+\varepsilon)} D\gamma \exp(iS(\gamma))
$$

Mean value theorem:

$$
S(\gamma) = \int_t^{t+\varepsilon} L\big(\gamma(t), \dot\gamma(t)\big) \, dt = \overline{L}(\gamma)\varepsilon \simeq L(\overline{\gamma})\varepsilon \pmod{o(\varepsilon)}?
$$

by continuity of L.

Similarly,

$$
S(\gamma_{\mathrm{cl}} + \delta\gamma) = S(\gamma_{\mathrm{cl}}) + \cancel{DS(\gamma_{\mathrm{cl}})(\delta\gamma)} + o(\delta\gamma) \approx \overline{L}(\gamma_{\mathrm{cl}}) + ??
$$

$$
\begin{aligned}
\psi(x, t+\varepsilon) &= \frac{1}{A}\int \exp\left(i\varepsilon L\left(\frac{x+y}{2}, \frac{x-y}{\varepsilon}\right)\right)\psi(y,t)\,dy \\
&= \frac{1}{A}\int \exp\left(i\varepsilon L\left(x+\frac{\eta}{2}, \frac{\eta}{\varepsilon}\right)\right)\psi(x+\eta,t)\,d\eta \\
&= \frac{1}{A}\int \exp\left(im\frac{\eta^2}{2\varepsilon} - i\varepsilon V\left(x+\frac{\eta}{2}\right)\right)\psi(x+\eta,t)\,d\eta \\
&= \frac{1}{A}\int \exp\left(\frac{-\eta^2}{2\underbrace{(i\varepsilon/m)}_{\sigma^2}}\right)e^{-i\varepsilon V(x+\eta/2)}\psi(x+\eta,t)\,d\eta
\end{aligned}
$$

$$
\psi(x,t) + \frac{\partial\psi}{\partial t}\varepsilon \simeq \int \frac{e^{-\eta^2/2\sigma^2}}{A}(1 - i\varepsilon V(x))\left(\psi + \frac{\partial\psi}{\partial x}\eta + \frac{\partial^2\psi}{\partial x^2}\eta^{\otimes 2}\right) \pmod{o\left(\varepsilon = \eta^2\right)}
$$

A is the normalizing factor for the gaussian.

$$
\begin{aligned}
\psi(x,t) + \frac{\partial\psi}{\partial t}\varepsilon &= \mathbb{E}\left((1 - i\varepsilon V)\left(\psi + \frac{\partial\psi}{\partial x}\eta + \frac{\partial^2\psi}{\partial x^2}\frac{\eta^{\otimes 2}}{2}\right)\right) \\
\cancel{\psi} + \frac{\partial\psi}{\partial t}\varepsilon &= \cancel{\psi} - i\varepsilon V\psi + \frac{1}{2}\underbrace{\frac{i\varepsilon}{m}}_{\sigma^2}\nabla^2\psi \\
\frac{\partial\psi}{\partial t} &= -i\left(\frac{-\nabla^2}{2m} + V\right)\psi
\end{aligned}
$$

What about magnetism?

$$
\begin{aligned}
\psi(x,t) + \frac{\partial\psi}{\partial t}\varepsilon &= \frac{1}{A}\int \exp\left(im\frac{\eta^2}{2\varepsilon} - i\varepsilon V\left(x+\frac{\eta}{2}, \frac{\eta}{\varepsilon}\right)\right)\psi(x+\eta,t)\,d\eta \\
&= \mathbb{E}\left(e^{-i\varepsilon V(x+\eta/2, \eta/\varepsilon)}\psi(x+\eta,t)\right) \\
V(x,\dot{x}) &= -\dot{x}\cdot A(x) + \phi(x) \\
V\left(x+\frac{\eta}{2}, \frac{\eta}{\varepsilon}\right) &= -\frac{\eta}{\varepsilon}\cdot A\left(x+\frac{\eta}{2}\right) + \phi\left(x+\frac{\eta}{2}\right) \\
\varepsilon V\left(x+\frac{\eta}{2}, \frac{\eta}{\varepsilon}\right) &= -\eta\cdot A\left(x+\frac{\eta}{2}\right) + \varepsilon\phi\left(x+\cancel{\frac{\eta}{2}}\right) \\
&= -\eta\cdot\left(A + \frac{A'\eta}{2}\right) + \varepsilon\phi \\
\psi(x,t) + \frac{\partial\psi}{\partial t}\varepsilon &= \mathbb{E}\left(\left(1 + i\eta\cdot A + i\frac{\eta\cdot A'\eta}{2} - i\varepsilon\phi\right)\left(\psi + \frac{\partial\psi}{\partial x}\eta + \frac{\partial^2\psi}{\partial x^2}\frac{\eta^{\otimes 2}}{2}\right)\right)
\end{aligned}
$$

odd terms in η cancel by symmetry

$$
= \psi + \mathbb{E}\left(i\eta\cdot A\frac{\partial\psi}{\partial x}\eta\right) + \mathbb{E}\left(i\frac{\eta\cdot A'\eta}{2}\psi\right) - i\varepsilon\phi\psi + \mathbb{E}\left(\frac{\partial^2\psi}{\partial x^2}\frac{\eta^{\otimes 2}}{2}\right)
$$

Expectationss of quadratics is trace!

$$\begin{aligned}
\varepsilon\frac{\partial\psi}{\partial t} &= i\mathbb{E}\,\mathrm{tr}\left(\eta\cdot A\frac{\partial\psi}{\partial x}\eta\right)+\frac{i\psi}{2}\mathbb{E}\,\mathrm{tr}\left(\eta\cdot A'\eta\right)-i\varepsilon\phi\psi+\frac{1}{2}\mathbb{E}\,\mathrm{tr}\left(\frac{\partial^2\psi}{\partial x^2}\frac{\eta^{\otimes 2}}{2}\right)\\
&= i\left(\frac{i\varepsilon}{m}\right)\underbrace{\mathrm{tr}\left(A\otimes\frac{\partial\psi}{\partial x}\right)}_{A\cdot\nabla\psi}+\frac{i\psi}{2}\left(\frac{i\varepsilon}{m}\right)\underbrace{\mathrm{tr}\left(A'\right)}_{\nabla\cdot A}-i\varepsilon\phi\psi+\frac{1}{2}\left(\frac{i\varepsilon}{m}\right)\nabla^2\psi?
\end{aligned}$$

I'm missing the A^2 term.