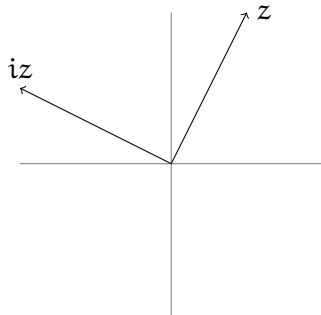


Complex

$$i^2 := -1$$

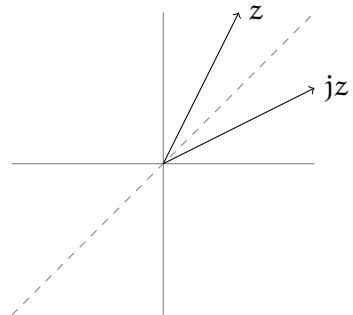
$$i(x + iy) = -y + ix$$



Hyperbolic

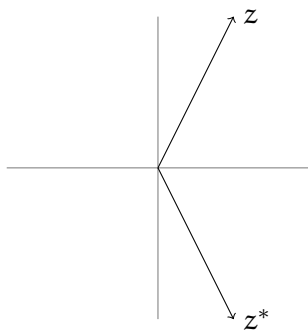
$$j^2 := +1$$

$$j(x + jy) = y + jx$$

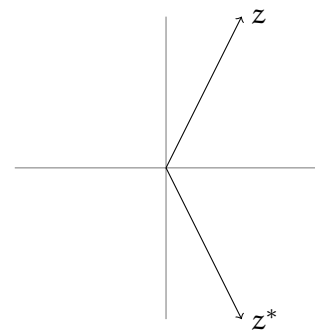


Conjugation and norm

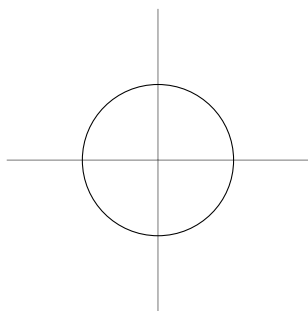
$$(x + iy)^* = x - iy$$



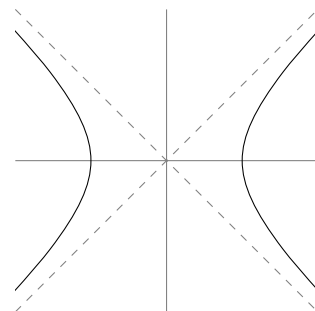
$$(x + jy)^* = x - jy$$



$$|z|^2 := zz^* = (x + iy)(x - iy) \\ = x^2 + y^2$$



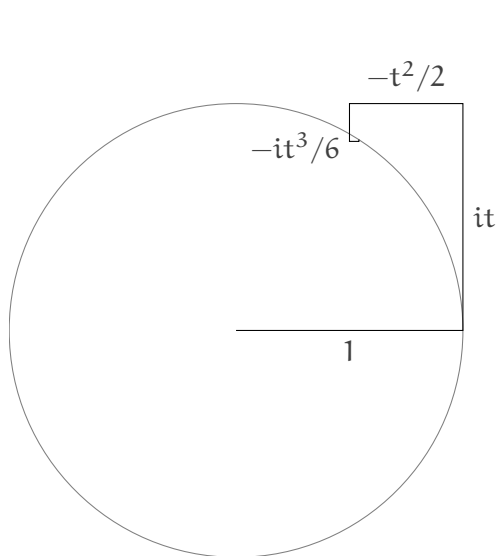
$$|z|^2 := zz^* = (x + jy)(x - jy) \\ = x^2 - y^2$$



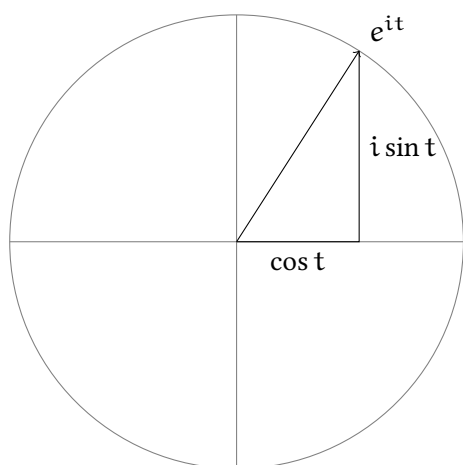
Exponentiation

$$i^n = \begin{cases} 1 & n \equiv 0 \pmod{4} \\ i & n \equiv 1 \pmod{4} \\ -1 & n \equiv 2 \pmod{4} \\ -i & n \equiv 3 \pmod{4} \end{cases}$$

$$e^{it} = 1(\dots) + i(\dots) \\ -1(\dots) - i(\dots)$$

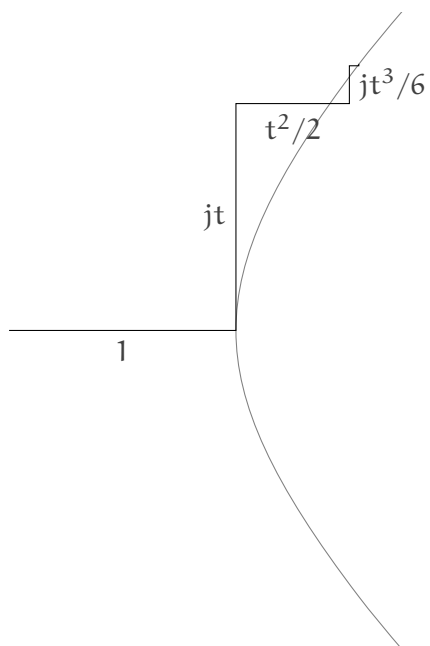


$$e^{it} = \cos t + i \sin t$$

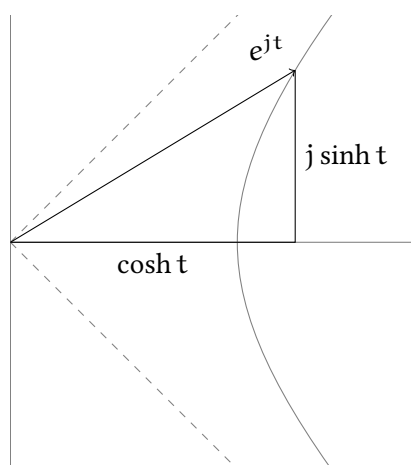


$$j^n = \begin{cases} 1 & n \equiv 0 \pmod{4} \\ j & n \equiv 1 \pmod{4} \\ 1 & n \equiv 2 \pmod{4} \\ j & n \equiv 3 \pmod{4} \end{cases}$$

$$e^{jt} = 1(\dots) + j(\dots) \\ +1(\dots) + j(\dots)$$



$$e^{jt} = \cosh t + j \sinh t$$



Pythagorean identity

$$\begin{aligned} e^{-it} &= \cos(-t) + i \sin(-t) \\ &= \cos t - i \sin t \end{aligned}$$

$$e^{-it} = (e^{it})^*$$

$$\begin{aligned} |e^{it}|^2 &= (e^{it})(e^{it})^* \\ &= e^{it}e^{-it} \\ &= 1 \end{aligned}$$

$$1 = \cos^2 t + \sin^2 t$$

$$\begin{aligned} e^{-jt} &= \cosh(-t) + j \sinh(-t) \\ &= \cosh t - j \sinh t \end{aligned}$$

$$e^{-jt} = (e^{jt})^*$$

$$\begin{aligned} |e^{jt}|^2 &= (e^{jt})(e^{jt})^* \\ &= e^{jt}e^{-jt} \\ &= 1 \end{aligned}$$

$$1 = \cosh^2 t - \sinh^2 t$$

Pythagorean corollaries

$$\cos^2 t = 1 - \sin^2 t$$

$$\sin^2 t = 1 - \cos^2 t$$

$$\sec^2 t = 1 + \tan^2 t$$

$$\csc^2 t = \cot^2 t + 1$$

$$\cosh^2 t = 1 + \sinh^2 t$$

$$\sinh^2 t = \cosh^2 t - 1$$

$$\operatorname{sech}^2 t = 1 - \tanh^2 t$$

$$\operatorname{csch}^2 t = \coth^2 t - 1$$

Angle sum formulae

$$\begin{aligned} e^{i(A+B)} &= e^{iA}e^{iB} \\ &= (c_A + is_A)(c_B + is_B) \end{aligned}$$

	c_B	is_B
c_A	$c_A c_B$	$i c_A s_B$
is_A	$i c_B s_A$	$-s_A s_B$

$$\begin{aligned} \cos(A+B) &= c_A c_B - s_A s_B \\ +i \sin(A+B) &= +i(c_B s_A + c_A s_B) \end{aligned}$$

$$\begin{aligned} e^{j(A+B)} &= e^{jA}e^{jB} \\ &= (c_A + js_A)(c_B + js_B) \end{aligned}$$

	c_B	js_B
c_A	$c_A c_B$	$j c_A s_B$
js_A	$j c_B s_A$	$+s_A s_B$

$$\begin{aligned} \cosh(A+B) &= c_A c_B + s_A s_B \\ +j \sinh(A+B) &= +j(c_B s_A + c_A s_B) \end{aligned}$$

Double angle

$$e^{2it} = (e^{it})^2$$

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$i \sin 2t = i 2 \cos t \sin t$$

$$e^{2jt} = (e^{jt})^2$$

$$\cosh 2t = \cosh^2 t + \sinh^2 t$$

$$j \sinh 2t = j 2 \cosh t \sinh t$$

Double angle + Pythagorean

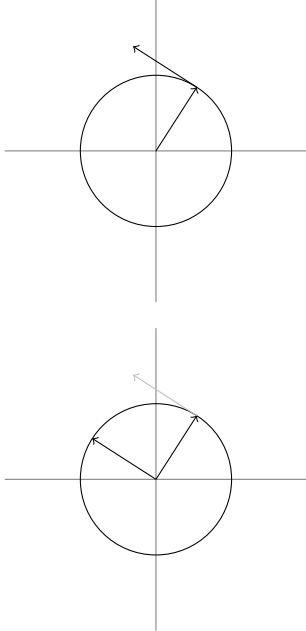
$$\begin{aligned} \cos 2t &= \cos^2 t - (1 - \cos^2 t) \\ &= 2 \cos^2 t - 1 \\ &= (1 - \sin^2 t) - \sin^2 t \\ &= 1 - 2 \sin^2 t \end{aligned}$$

$$\begin{aligned} \cosh 2t &= \cosh^2 t + (\cosh^2 t - 1) \\ &= 2 \cosh^2 t - 1 \\ &= (1 + \sinh^2 t) + \sinh^2 t \\ &= 1 + 2 \sinh^2 t \end{aligned}$$

Derivatives

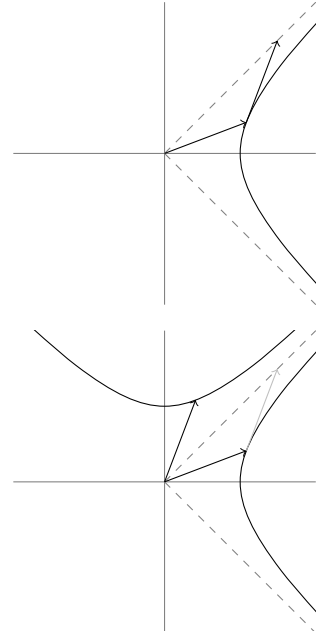
$$\begin{aligned} f(t) &:= e^{it} \\ f'(t) &= ie^{it} \\ &= i(\cos t + i \sin t) \end{aligned}$$

$$\cos' t + i \sin' t = -\sin t + i \cos t$$



$$\begin{aligned} f(t) &:= e^{jt} \\ f'(t) &= je^{jt} \\ &= j(\cosh t + j \sinh t) \end{aligned}$$

$$\cosh' t + j \sinh' t = \sinh t + j \cosh t$$



Dot and cross product

$$\begin{aligned} \mathbf{u}^* \mathbf{v} &= (\alpha - i\beta)(\gamma + i\delta) \\ &= \begin{vmatrix} & \gamma & i\delta \\ \alpha & \alpha\gamma & i\alpha\delta \\ -i\beta & -i\beta\gamma & +\beta\delta \end{vmatrix} \\ &= (\alpha\gamma + \beta\delta) + i(\alpha\delta - \beta\gamma) \end{aligned}$$

$$\mathbf{u}^* \mathbf{v} = (\mathbf{u} \cdot \mathbf{v}) + i(\mathbf{u} \times \mathbf{v})$$

$$(iz) \cdot z = \Re(iz^*z) = \Re(i|z|^2) = 0$$

$$\begin{aligned} \mathbf{u}^* \mathbf{v} &= (\alpha - j\beta)(\gamma + j\delta) \\ &= \begin{vmatrix} & \gamma & i\delta \\ \alpha & \alpha\gamma & +j\alpha\delta \\ -j\beta & -j\beta\gamma & -\beta\delta \end{vmatrix} \\ &= (\alpha\gamma - \beta\delta) + j(\alpha\delta - \beta\gamma) \end{aligned}$$

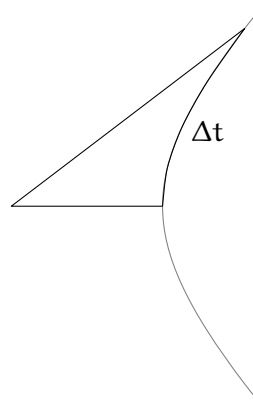
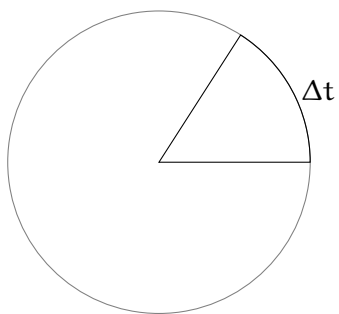
$$\mathbf{u}^* \mathbf{v} = (\mathbf{u} \cdot \mathbf{v}) + j(\mathbf{u} \times \mathbf{v})$$

$$(jz) \cdot z = \Re(jz^*z) = \Re(j|z|^2) = 0$$

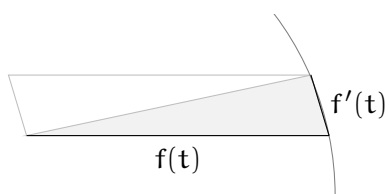
Arc length

$$\begin{aligned} f(t) &:= e^{it} \\ S &= \int \sqrt{|f'(t)|^2} dt \\ &= \int \sqrt{|ie^{it}|^2} dt = \int \sqrt{1} dt \\ &= \Delta t \end{aligned}$$

$$\begin{aligned} f(t) &:= e^{jt} \\ S &= \int \sqrt{|f'(t)|^2} dt \\ &= \int \sqrt{|je^{jt}|^2} dt = \int \sqrt{-(-1)} dt \\ &= \Delta t \end{aligned}$$

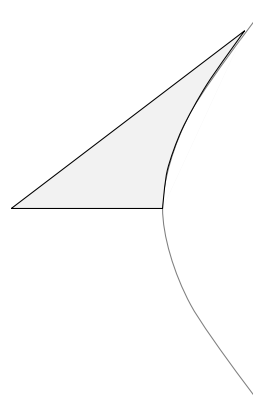
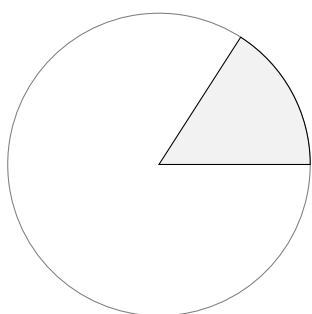


Sector area



$$\begin{aligned}
 A &= \frac{1}{2} \int f(t) \times f'(t) dt \\
 &= \frac{1}{2} \int \Im(f^*(t)f'(t)) dt \\
 &= \frac{1}{2} \int \Im(e^{-it}ie^{it}) dt \\
 &= \frac{1}{2} \int 1 dt \\
 &= \frac{\Delta t}{2}
 \end{aligned}$$

$$\begin{aligned}
 A &= \frac{1}{2} \int f(t) \times f'(t) dt \\
 &= \frac{1}{2} \int \Im(f^*(t)f'(t)) dt \\
 &= \frac{1}{2} \int \Im(e^{-jt}je^{jt}) dt \\
 &= \frac{1}{2} \int 1 dt \\
 &= \frac{\Delta t}{2}
 \end{aligned}$$



$SO(2), SO^+(+, -)$: Multiplying by imaginary exponents preserves inner product and area

$$\begin{aligned}
 & (e^{it}u) \cdot (e^{it}v) \\
 +i(e^{it}u) \times (e^{it}v) &= (e^{it}u)^*(e^{it}v) \\
 &= e^{-it}u^*e^{it}v \\
 &= u^*v \\
 &= u \cdot v + i u \times v
 \end{aligned}$$

$$\begin{aligned}
 & (e^{jt}u) \cdot (e^{jt}v) \\
 +j(e^{jt}u) \times (e^{jt}v) &= (e^{jt}u)^*(e^{jt}v) \\
 &= e^{-jt}u^*e^{jt}v \\
 &= u^*v \\
 &= u \cdot v + j u \times v
 \end{aligned}$$

