

1 Polar coordinate

$$ds^2 = dr^2 + r^2 d\theta^2$$

Lagrangian

$$L = \frac{\dot{r}^2}{2} + r^2 \frac{\dot{\theta}^2}{2}$$
$$\frac{\partial L}{\partial \dot{r}} = \dot{r} \quad \frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta} \implies L = H$$

Momentum conservation:

$$r^2 \dot{\theta} = B \implies \dot{\theta} = B/r^2$$

Energy conservation:

$$2E = A^2 = \dot{r}^2 + r^2 \dot{\theta}^2 = \dot{r}^2 + r^2 (B/r^2)^2 = \dot{r}^2 + B^2/r^2$$

$$dr = dt \sqrt{A^2 - B^2/r^2}$$

$$dt = \frac{dr}{\sqrt{A^2 - B^2/r^2}} = \frac{r dr}{\sqrt{A^2 r^2 - B^2}}$$

$r = (B/A) \cosh u$ and $dr = (B/A) \sinh u du$

$$t = \int \frac{B^2/A^2 \cosh u \sinh u du}{B \sinh u} = \frac{B}{A^2} \sinh u = \frac{B}{A^2} \sqrt{A^2/B^2 r^2 - 1} = \frac{\sqrt{A^2 r^2 - B^2}}{A^2}$$

$$(A^2 t)^2 = A^2 r^2 - B^2 \implies r^2 = \frac{(A^2 t)^2 + B^2}{A^2} = A^2 t^2 + (B/A)^2$$

$$r = \sqrt{A^2 t^2 + (B/A)^2} = \sqrt{A^2 t^2 + r_0^2}$$

$$d\theta = \frac{A dt}{A^2 t^2 + r_0^2} \quad t = r_0/A \tan u \quad dt = r_0/A \sec^2 u du$$

$$\Delta\theta = \int \frac{A r_0/A \sec^2 u du}{\sqrt{r_0^2} \sec^2 u} = u = \tan^{-1} \left(\frac{At}{r_0} \right)$$

2 sphere

$$ds^2 = d\rho^2 + \sin^2 \rho d\theta^2$$

$$L = \frac{\dot{\rho}^2}{2} + \sin^2 \rho \frac{\dot{\theta}^2}{2}$$

Momentum:

$$\sin^2 \rho \dot{\theta} = B \implies \dot{\theta} = \frac{B}{\sin^2 \rho}$$

Energy conservation:

$$A^2 = 2E = \dot{\rho}^2 + \sin^2 \rho \dot{\theta}^2 = \dot{\rho}^2 + \frac{B^2}{\sin^2 \rho}$$

$$dt = \frac{d\rho}{\sqrt{A^2 - B^2/\sin^2 \rho}} = \frac{\sin \rho d\rho}{\sqrt{A^2 \sin^2 \rho - B^2}}$$

$$d\theta = \frac{B}{\sin^2 \rho} dt = \frac{B d\rho}{\sin \rho \sqrt{A^2 \sin^2 \rho - B^2}}$$

3 light cone coordinates

Boosts

$$C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$C^2 = \frac{1}{2} \begin{bmatrix} 2 & \\ & 2 \end{bmatrix} = I$$

$$\begin{aligned} CBC^{-1} &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} c & s \\ s & c \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\begin{bmatrix} c+s & c-s \\ c+s & s-c \end{bmatrix}} \\ &= \begin{bmatrix} c+s & 0 \\ 0 & c-s \end{bmatrix} \\ &= \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} \end{aligned}$$

so it's idempotent.

Metric

$$\begin{aligned} \eta(x, y) &= \eta_s(C^{-1}x, C^{-1}y) \\ &= \eta_s(Cx, Cy) \\ &= x^T C^T \eta_s Cy \\ C \eta_s C &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}} \\ \eta &= \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \end{aligned}$$

