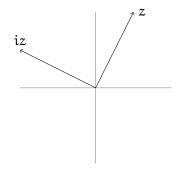
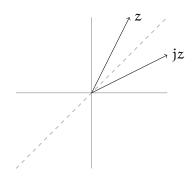
Complex

$$i^2 := -1$$
$$i(x + iy) = -y + ix$$



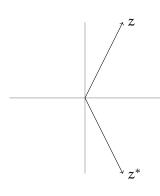
Hyperbolic

$$j^2 := +1$$
$$j(x + jy) = y + jx$$

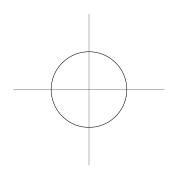


Conjugation and norm

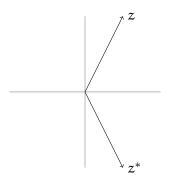
$$(x+iy)^* = x-iy$$



$$|z|^2 := zz^* = (x + iy)(x - iy)$$
$$= x^2 + y^2$$

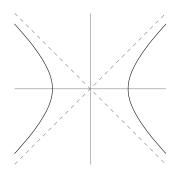


 $(x+jy)^* = x - jy$



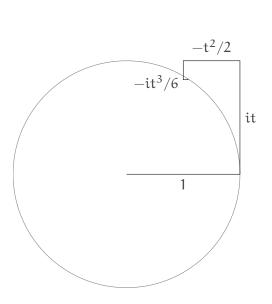
$$|z|^2 := zz^* = (x + jy)(x - jy)$$

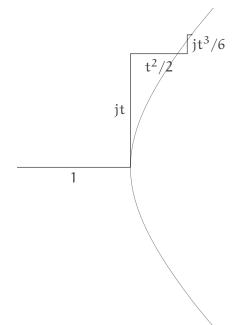
= $x^2 - y^2$

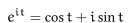


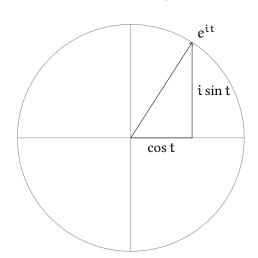
Exponentiation

$$i^{n} = \begin{cases} 1 & n \equiv 0 \pmod{4} \\ i & n \equiv 1 \pmod{4} \\ -1 & n \equiv 2 \pmod{4} \\ -i & n \equiv 3 \pmod{4} \end{cases}$$
$$e^{it} = 1(\dots) + i(\dots)$$
$$-1(\dots) - i(\dots)$$

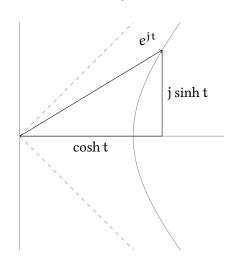








$$e^{jt} = \cosh t + j \sinh t$$

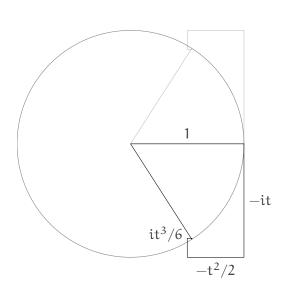


Pythagorean identity

$$e^{-it} = \cos(-t) + i\sin(-t)$$

= $\cos t - i\sin t$

$$\begin{split} e^{-j\,t} &= \cosh(-t) + j \sinh(-t) \\ &= \cosh t - j \sinh t \end{split}$$



$$-jt$$
 $t^{2}/2$
 $-jt^{3}/6$

$$e^{-it} = (e^{it})^*$$
 $|e^{it}|^2 = (e^{it})(e^{it})^*$
 $= e^{it}e^{-it}$
 $= 1$
 $1 = \cos^2 t + \sin^2 t$

$$\begin{split} e^{-jt} &= \left(e^{jt} \right)^* \\ \left| e^{jt} \right|^2 &= \left(e^{jt} \right) \left(e^{jt} \right)^* \\ &= e^{jt} e^{-jt} \\ &= 1 \\ 1 &= \cosh^2 t - \sinh^2 t \end{split}$$

Pythagorean corollaries

$$cos2 t = 1 - sin2 t$$

$$sin2 t = -cos2 t + 1$$

$$sec2 t = 1 + tan2 t$$

$$csc2 t = cot2 t + 1$$

$$cosh2 t = 1 + sinh2 t$$

$$sinh2 t = cosh2 t - 1$$

$$sech2 t = 1 - tanh2 t$$

$$csch2 t = coth2 t - 1$$

Angle sum formulae

Double angle

$$\begin{split} e^{2it} &= \left(e^{it}\right)^2 \\ \cos 2t &= \cos^2 t - \sin^2 t \\ i\sin 2t &= i\,2\cos t\sin t \end{split} \qquad \begin{split} e^{2jt} &= \left(e^{jt}\right)^2 \\ \cosh 2t &= \cosh^2 t + \sinh^2 t \\ j\sinh 2t &= j\,2\cosh t\sinh t \end{split}$$

Double angle + Pythagorean

$$\cos 2t = \cos^2 t - (1 - \cos^2 t) \qquad \qquad \cosh 2t = \cosh^2 t + (\cosh^2 t - 1)$$

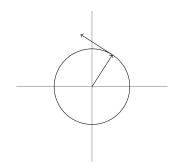
$$= 2\cos^2 t - 1 \qquad \qquad = 2\cosh^2 t - 1$$

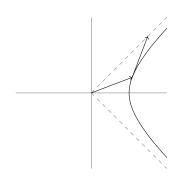
$$= (1 - \sin^2 t) - \sin^2 t \qquad \qquad = (1 + \sinh^2 t) + \sinh^2 t$$

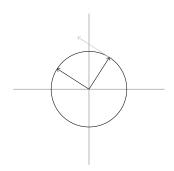
$$= 1 - 2\sin^2 t \qquad \qquad = 1 + 2\sinh^2 t$$

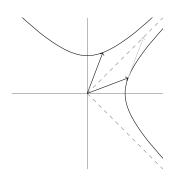
Derivatives

$$\begin{split} f(t) &\coloneqq e^{it} & f(t) \coloneqq e^{jt} \\ f'(t) &= ie^{it} & f'(t) = je^{jt} \\ &= i(\cos t + i\sin t) & = j(\cosh t + j\sinh t) \\ \cos' t + i\sin' t &= -\sin t + i\cos t & \cosh' t + j\sinh' t = \sinh t + j\cosh t \end{split}$$









Dot and cross product

$$u^*v = (\alpha - i\beta)(\gamma + i\delta)$$

$$\frac{|\gamma|}{\alpha} \frac{i\delta}{\alpha |\alpha\gamma|} \frac{i\delta}{i\alpha\delta}$$

$$-i\beta |-i\beta\gamma| + \beta\delta$$

$$= (\alpha\gamma + \beta\delta) + i(\alpha\delta - \beta\gamma)$$

$$u^*v = (u \cdot v) + i(u \times v)$$

$$(iz) \cdot z = \Re(iz^*z) = \Re(i|z|^2) = 0$$

$$u^*v = (\alpha - j\beta)(\gamma + j\delta)$$

$$\frac{|\gamma|}{\alpha} \frac{i\delta}{\alpha + j\alpha\delta}$$

$$-j\beta - j\beta\gamma - \beta\delta$$

$$= (\alpha\gamma - \beta\delta) + j(\alpha\delta - \beta\gamma)$$

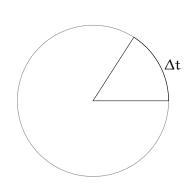
$$u^*v = (u \cdot v) + j(u \times v)$$

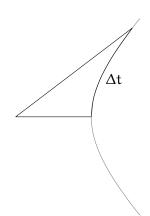
$$(jz) \cdot z = \Re(jz^*z) = \Re(j|z|^2) = 0$$

Arc length

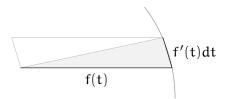
$$\begin{split} f(t) &\coloneqq e^{it} \\ S &= \int \sqrt{|f'(t)|^2} dt \\ &= \int \sqrt{|ie^{it}|^2} dt = \int \sqrt{1} dt \\ &= \Delta t \end{split}$$

$$\begin{split} f(t) &\coloneqq e^{jt} \\ S &= \int \sqrt{-|f'(t)|^2} dt \\ &= \int \sqrt{-|je^{jt}|^2} dt = \int \sqrt{-(-1)} dt \\ &= \Delta t \end{split}$$





Sector area



$$A = \frac{1}{2} \int f(t) \times f'(t) dt$$

$$= \frac{1}{2} \int \Im \left(f^*(t) f'(t) \right) dt$$

$$= \frac{1}{2} \int \Im \left(e^{-it} i e^{it} \right) dt$$

$$= \frac{1}{2} \int 1 dt$$

$$= \frac{\Delta t}{2}$$

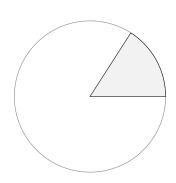
$$A = \frac{1}{2} \int f(t) \times f'(t) dt$$

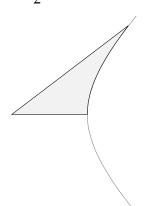
$$= \frac{1}{2} \int \Im(f^*(t)f'(t)) dt$$

$$= \frac{1}{2} \int \Im(e^{-jt}je^{jt}) dt$$

$$= \frac{1}{2} \int 1 dt$$

$$= \frac{\Delta t}{2}$$





SO(2), $SO^+(+,-)$: Multiplying by imaginary exponents preserves inner product and area

$$\begin{split} \left(e^{it}u\right)\cdot\left(e^{it}v\right) \\ + i\left(e^{it}u\right)\times\left(e^{it}v\right) &= \left(e^{it}u\right)^*\left(e^{it}v\right) \\ &= e^{-it}u^*e^{it}v \\ &= u^*v \\ &= u\cdot v + i\,u\times v \end{split}$$

$$(e^{jt}u) \cdot (e^{jt}v)$$

$$+j(e^{jt}u) \times (e^{jt}v) = (e^{jt}u)^*(e^{jt}v)$$

$$= e^{-jt}u^*e^{jt}v$$

$$= u^*v$$

$$= u \cdot v + ju \times v$$

