1 Polar coordinate

$$ds^2 = dr^2 + r^2 d\theta^2$$

Lagrangian

$$L = \frac{\dot{r}^2}{2} + r^2 \frac{\dot{\theta}^2}{2}$$

$$\frac{\partial L}{\partial \dot{r}} = \dot{r} \quad \frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta} \implies L = H$$

Momentum conservation:

$$r^2 \dot{\theta} = B \implies \dot{\theta} = B/r^2$$

Energy conservation:

$$2E = A^2 = \dot{r}^2 + r^2 \dot{\theta}^2 = \dot{r}^2 + r^2 \left(B/r^2\right)^2 = \dot{r}^2 + B^2/r^2$$

$$dr = dt \sqrt{A^2 - B^2/r^2}$$

$$dt = \frac{dr}{\sqrt{A^2 - B^2/r^2}} = \frac{r\,dr}{\sqrt{A^2 r^2 - B^2}}$$

$r = (B/A) \cosh u$ and $dr = (B/A) \sinh u \, du$

$$t = \int \frac{B^2/A^2 \cosh u \sinh u \, du}{B \sinh u} = \frac{B}{A^2} \sinh u = \frac{B}{A^2} \sqrt{A^2/B^2 r^2 - 1} = \frac{\sqrt{A^2 r^2 - B^2}}{A^2}$$

$$\left(A^2 t\right)^2 = A^2 r^2 - B^2 \implies r^2 = \frac{\left(A^2 t\right)^2 + B^2}{A^2} = A^2 t^2 + (B/A)^2$$

$$r = \sqrt{A^2 t^2 + (B/A)^2} = \sqrt{A^2 t^2 + {r_0}^2}$$

$$d\theta = \frac{A\,dt}{A^2 t^2 + {r_0}^2} \quad t = r_0/A \tan u \quad dt = r_0/A \sec^2 u \, du$$

$$\Delta\theta = \int \frac{A r_0/A \sec^2 u \, du}{\sqrt{{r_0}^2} \sec^2 u} = u = \tan^{-1}\left(\frac{At}{r_0}\right)$$

2 sphere

$$ds^2 = d\rho^2 + \sin^2 \rho \, d\theta^2$$

$$L = \frac{\dot{\rho}^2}{2} + \sin^2 \rho \frac{\dot{\theta}^2}{2}$$

Momentum:

$$\sin^2\rho\,\dot\theta = B \implies \dot\theta = \frac{B}{\sin^2\rho}$$

Energy conservation:

$$A^2 = 2E = \dot\rho^2 + \sin^2\rho\,\dot\theta^2 = \dot\rho^2 + \frac{B^2}{\sin^2\rho}$$

$$dt = \frac{d\rho}{\sqrt{A^2 - B^2/\sin^2\rho}} = \frac{\sin\rho\,d\rho}{\sqrt{A^2\sin^2\rho - B^2}}$$

$$d\theta = \frac{B}{\sin^2\rho}dt = \frac{B\,d\rho}{\sin\rho\sqrt{A^2\sin^2\rho - B^2}}$$

3 light cone coordinates

$$C = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$
$$C^2 = \frac{1}{2}\begin{bmatrix}2 & \\ & 2\end{bmatrix} = I$$

so it's idempotent.

Metric

$$\begin{aligned}\eta(x,y) &= \eta_s\left(C^{-1}x, C^{-1}y\right)\\ &= \eta_s(Cx, Cy)\\ &= x^T C^T \eta_s C y\end{aligned}$$

$$C\eta_s C = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\underbrace{\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}}_{\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}}$$

$$\eta = \begin{bmatrix} & 1\\ 1 & \end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix}a\\ b\end{bmatrix}\cdot\begin{bmatrix}c\\ d\end{bmatrix} &= \begin{bmatrix}a & b\end{bmatrix}\begin{bmatrix} & 1\\ 1 & \end{bmatrix}\begin{bmatrix}c\\ d\end{bmatrix}\\ &= ad + bc\end{aligned}$$

$$\left|\left|\begin{bmatrix}x\\ y\end{bmatrix}\right|\right|_\eta^2 = 2xy$$

Boosts

$$\begin{aligned}CBC^{-1} &= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\underbrace{\begin{bmatrix}c & s\\ s & c\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}}_{\begin{bmatrix}c+s & c-s\\ c+s & s-c\end{bmatrix}}\\ &= \begin{bmatrix}c+s & 0\\ 0 & c-s\end{bmatrix}\\ &= \begin{bmatrix}e^t & \\ & e^{-t}\end{bmatrix}\end{aligned}$$