1 Quaternions, conjugation, rotation

Lemma 1 *Group conjugation commutes with hamiltonian conjugation*

Proof. Suppose $|q| = 1$. Then $q^{-1} = q^*$ and

$$\left(qpq^{-1}\right)^* = (qpq^*)^*$$
$$= (q^*)^*(p^*)(q)^*$$
$$= q(p^*)q^*$$
$$= qp^*q^{-1}.$$

In general, $q = t\hat{q}$ for $t \in \mathbb{R}$. Then

$$qpq^{-1} = (t\hat{q})p(t\hat{q})^{-1}$$
$$= (t\hat{q})p(t^{-1}\hat{q}^{-1})$$
$$= \hat{q}p\hat{q}^{-1}$$

Conclude

$$\left(qpq^{-1}\right)^* = \left(\hat{q}p\hat{q}^{-1}\right)^*$$
$$= \hat{q}p^*\hat{q}^{-1}$$
$$= qp^*q^{-1} \qquad \square$$

Corollary 2

$$\mathcal{R} : q \mapsto q\_q^{-1} : S^3 \to O(\mathbb{R}^3)$$

Because S^3 is connected, so is $\mathcal{R}(S^3)$. Because its image contains $\mathcal{R}(1) = 1\_1 = \text{id}$, it lies in the identity component:

Corollary 3

$$q \mapsto q\_q^{-1} : S^3 \to SO(\mathbb{R}^3).$$

The properties of $\mathcal{R}(q)$ are determined by commutativity. Since we're acting on $\mathbb{R}^3$, decompose:

$$q = r + \vec{v}.$$

In the case $r = 0$,

$$\vec{v}\vec{u}\vec{v}^* = -\vec{v}\vec{x}\vec{v}$$

Split $x = x_\parallel + x_\perp$

$$\vec{u}x_\parallel\vec{u}^* = \vec{u}\vec{u}^*x_\parallel = x_\parallel$$

$$\vec{u}x_\perp\vec{u}^* = -\vec{u}\vec{u}^*x_\perp = -x_\perp$$

a reflection negating everything but $\text{span}\,u$ (line symmetry).

Commuting?

$$[q, \vec{x}] = [q + \vec{v}, \vec{x}] = [\vec{v}, \vec{x}]$$

Split $x = x_\parallel + x_\perp$. Parallels commute and orthogonals anticommute:

$$vx_\parallel = x_\parallel v$$
$$vx_\perp = x_\perp v$$

Therefore,

$$qx_\parallel q^{-1} = qq^{-1}x_\parallel = x_\parallel$$

while

$$qx_\perp q^* = (r + v)x_\perp(r - v)$$
$$= (r + v)(r + v)x_\perp$$
$$= q^2 x_\perp$$

so

$$\mathcal{R}(q)x = qxq^{-1} = x_\parallel + q^2 x_\perp$$

$$q = c_\theta + s_\theta\hat{v}$$
$$q^2 = c_\theta^2 - s_\theta^2 + 2c_\theta s_\theta\hat{v}$$
$$= c_{2\theta} + s_{2\theta}\hat{v}$$
$$q^2 x_\perp = c_{2\theta}x_\perp + s_{2\theta}\hat{v}x_\perp$$

In general, I want to show rotations fix subspaces of size 2.

Theorem 4 *Every unit quaternion can be expressed as the product of two pure units*

Proof.

$$\vec{u}\vec{v} = -\vec{u} \cdot \vec{v} + \vec{u} \times \vec{v}$$
$$= -\cos\theta + \sin\theta\,\vec{u} \times \vec{v}$$
$$i(-i\cos\theta - iw\sin\theta) = \cos\theta + w\sin\theta \qquad \square$$

Theorem 5 (Cartan-Dieudonné)

Proof. I forgot where I read this.

Decompose the space. Consider the map U : $(x, \vec{u}) \mapsto (y, \vec{v})$. There's a reflection F sending the first coordinate to the correct location:

$$F(x, \vec{u}) = (y, F'\vec{u})$$

so

$$U = (UF)F$$

and $UF(x, w) = (x, w')$ which is orthogonal on a codimension 1 space, so go by induction. On a dimension 0 space, nothing is necessary to do (base case is vacuously true). $\square$

2 Representations of $\mathbb{H}^\times$

$$q = \alpha + \beta i + \gamma j + \delta k$$
$$= (\alpha + \beta i) + (\gamma + \delta i)j$$
$$= u + vj$$
$$= (\alpha + \beta i) + j(\gamma - \delta i)$$
$$= u + v^* j$$

$$\begin{array}{ccc} & w & xj \\ u & uw & uxj \\ vj & vw^*j & vx^*j^2 \end{array},$$

so we get the action

$$\begin{bmatrix} u \\ v \end{bmatrix}(w + xj) = \begin{bmatrix} uw - vx^* \\ vw^* + ux \end{bmatrix}$$
$$= \begin{bmatrix} w & -x^* \\ x & w^* \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$
$$\_\_(w + xj) \mapsto \begin{bmatrix} w & -x^* \\ x & w^* \end{bmatrix}$$

Or to $\mathrm{SO}\mathbb{R}^4$

$$i\_\_ \mapsto \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}$$

$$j\_\_ \mapsto \begin{bmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{bmatrix}$$

$$k\_\_ \mapsto \begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$\_\_i \mapsto \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}$$

$$\_\_j \mapsto \begin{bmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{bmatrix}$$

$$\_\_k \mapsto \begin{bmatrix} & & & -1 \\ & 1 & & \\ & & -1 & \\ 1 & & & \end{bmatrix}$$

$$q\_\_ \mapsto \begin{bmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & -\delta & \gamma \\ \gamma & \delta & \alpha & -\beta \\ \delta & -\gamma & \beta & \alpha \end{bmatrix}$$

$$\_\_q \mapsto \begin{bmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{bmatrix}$$

For pure vectors,

$$vq = v\pi_1(q) - v \cdot \pi_{ijk}q + v \times \pi_{ijk}q$$
$$v\_\_ = v\_\_\mathbb{R} - v \cdot \_\_ + v \times \_\_$$
$$v\_\_ = \begin{bmatrix} 0 & -v^\mathsf{T} \\ v & v \times \_\_ \end{bmatrix} \quad \_\_v = \begin{bmatrix} 0 & -v^\mathsf{T} \\ v & \_\_ \times v \end{bmatrix}$$

$$v\_\_v = \begin{bmatrix} -v \cdot v & -v \cdot (\_\_ \times v) \\ v \times v & -vv^\mathsf{T} + v \times (\_\_ \times v) \end{bmatrix}$$
$$v \times (x \times v) = -(v \cdot x)v + x(v \cdot v)$$
$$= (-vv^\mathsf{T} + vv^\mathsf{T})x$$
$$v\_\_v = \begin{bmatrix} -v \cdot v & \\ & -2vv^\mathsf{T} + v^\mathsf{T}v \end{bmatrix}$$

Hopf map things

$$H(q) := qkq^{-1} = {}^q k$$

$$H(pq) = {}^{pq}k$$
$$= {}^p({}^q k)$$
$$= {}^p H(q)$$
$$H(qr) = H(rq^r)$$
$$= {}^r H(q^r)$$

3 Winding and linking number

There's too many Hs in this field. Winding number:

$$S^1 \xrightarrow{\gamma} S^1$$

$$\mathbb{Z} \xrightarrow{\mathcal{H}_1(\gamma)} \mathbb{Z}$$

If you homotope one curve to a standard circle, the linking number is the winding number of the other curve.

Linking number of $s^1 \to \underline{}$ is only nontrivial in dimension 3 because you can homotope through the fourth dimension to separate.

Gauss's linking number (via wikipedia). The Gauss map

$$\Gamma(\alpha, \beta)(s, t) = \text{sgn}\left(\alpha(s) - \beta(t)\right)$$

"Pick a point in the unit sphere, v, *so that orthogonal projection* of the *link* to the plane perpendicular to v *gives a link diagram.* Observe that a point (s, t) that *goes to v* under the Gauss map *corresponds to a crossing* in the link diagram *where [α] is over [β]*. Also, a neighborhood of (s, t) is mapped under the Gauss map to a neighborhood of v preserving or reversing *orientation* depending on the *sign* of the crossing. Thus in order to compute the linking number of the diagram corresponding to v it suffices to *count the signed number of times the Gauss map covers v*. Since v is a *regular value*, this is precisely the *degree of the Gauss map* (i.e. the signed number of times that the image of $[(\alpha, \beta)]$ covers the sphere). Isotopy invariance of the linking number is automatically obtained as the *degree is invariant* under homotopic maps. Any other regular value would give the same number, so the linking number doesn't depend on any particular link diagram."

$$\deg \Gamma_{(\alpha, \beta)} \int_{S^2} 1 = \int_{T^2} \Gamma(\alpha, \beta)$$

$$\deg \Gamma_{(\alpha, \beta)} = (2\tau)^{-1} \int_{T^2} \Gamma_{(\alpha, \beta)}(ds \wedge dt)$$

Because slotting a unit vector into a volume form gives area

$$= \int (\wedge^2 d\Gamma) \wedge \Gamma$$

With the added benefit of discarding the normal components of $\wedge^2 d\Gamma$:

$$= \int \frac{d\alpha}{|\alpha - \beta|} \wedge \frac{-d\beta}{|\alpha - \beta|} \wedge \frac{\alpha - \beta}{|\alpha - \beta|}$$

But I think I lost a minus sign.

4 stereographic projection of hopf links

Foliate S^3 with tori:

$$F(\alpha)(s, t) := \cos \alpha \, e^{sk} + \sin \alpha \, ie^{-tk}$$
$$: [0, \tau/4] \to T^2 \to S^3$$

Spin

$$Fe^{uk} = \cos \alpha \, e^{(s+u)k} + \sin \alpha \, ie^{(u-t)k}$$

It spins within leaves not across. Project

$$P(Fe^{uk}) = \frac{\cos \alpha \cos(s + u) + \sin \alpha \, ie^{(u-t)k}}{1 - \cos \alpha \sin(s + u)}$$

Project

5 $S^3 \times S^3$ and $SO(4)$

$$z(u + vj) = (zu) + (zv)j \qquad (u + vj)z = uz + vjz$$
$$j(u + vj) = ju + jvj \qquad\qquad = uz + vz^*j$$
$$= -v^* + u^*j \qquad (u + vj)j = uj - v$$

$$z\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} zu \\ zv \end{bmatrix} \qquad\qquad \begin{bmatrix} u \\ v \end{bmatrix} z = \begin{bmatrix} zu \\ z^*v \end{bmatrix}$$

$$j\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -v^* \\ u^* \end{bmatrix} \qquad\qquad \begin{bmatrix} u \\ v \end{bmatrix} j = \begin{bmatrix} -v \\ u \end{bmatrix}$$

We can rotate $\mathbb{H}$ fixing 1 to make an arbitrary unit vector into i. In that case, the effect of exponential-rotation is

$$e^{is}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} e^{is}u \\ e^{is}v \end{bmatrix} \qquad \begin{bmatrix} u \\ v \end{bmatrix} e^{it} = \begin{bmatrix} e^{it}u \\ e^{-it}v \end{bmatrix}$$

$$e^{is}\begin{bmatrix} u \\ v \end{bmatrix} e^{it} = \begin{bmatrix} e^{i(s+t)}u \\ e^{i(s-t)}v \end{bmatrix}$$

So we can rotate the two complex numbers independently.

$$\exp\left(i\frac{A + B}{2}\right)\begin{bmatrix} u \\ v \end{bmatrix}\exp\left(i\frac{A - B}{2}\right) = \begin{bmatrix} e^{iA}u \\ e^{iB}v \end{bmatrix}$$

Especially

$$e^{iA/2}\begin{bmatrix} u \\ v \end{bmatrix} e^{iA/2} = \begin{bmatrix} e^{iA}u \\ v \end{bmatrix}$$

So now we can (1) rotate the vector parts, (2) rotate between a vector and scalar.

$e^{tx/2}\_e^{-tx/2}$ rotates vectors around the axle x, fixing 1 and x.

$e^{tx/2}\_e^{tx/2}$ rotates the $1, x$-plane, fixing vectors orthogonal to x.

The simplest "generic" rotation would be in the e^{it}, j plane:

$$(c1 + is) \wedge j = c1 \wedge j + si \wedge j$$

6 $\mathbb{H}$ and geometric algebra

$$\mathbb{H} = \mathbb{R}^3/\left(v^2 = -1\right)$$

whereas your standard geometric algebra is

$$\mathbb{R}^3/\left(v^2 = 1\right).$$

But they Hodge$*$ into each other. Let x, y, z be the geometric algebra's basis. Then

$$I = xyz$$
$$I^2 = (xyz)(xyz) = -1$$
$$xI = x^2yz = xyzx = Ix$$

Define

$$\phi : \begin{cases} i & \mapsto & xI & = & xxyz & = yz \\ j & \mapsto & -yI & = & -yxyz & = xz \\ k & \mapsto & zI & = & zxyz & = xy \end{cases}$$

This has the right squares:

$$\phi(\vec{x})^2 = (xI)(xI) = x^2I^2 = -1.$$

And products?

$$\begin{array}{lll} (yz)(xz) = -yx & = xy \\ (xz)(xy) = -zy & = yz \\ (xy)(yz) = & = xz \end{array}$$
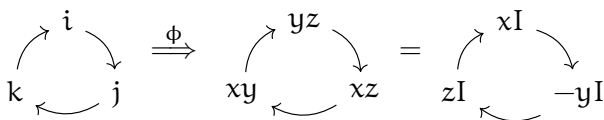
Punning between vectors and spinors depends on planes and lines being dual in $\mathbb{R}^3$:

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3 \xrightarrow{1 \oplus \phi} \mathbb{R} \oplus \wedge^2 \mathbb{R}^3$$

7 rotation tricks

Log of a rotor/product of unit vectors

$$uv = \underbrace{u \cdot v}_{\cos \theta} + u \wedge v$$
$$= \cos \theta + \sin \theta \frac{u \wedge v}{|u \wedge v|}$$
$$= \exp\left(\cos^{-1}(u \cdot v)\frac{u \wedge v}{|u \wedge v|}\right)$$

Product of 2-rotors. If they commute:

$$e^{\alpha xy}e^{\beta zt} = e^{\alpha xy + \beta zt} \quad \text{as } [xy, zt] = 0$$

If they don't commute,

$$e^{\alpha xy}e^{\beta yz} = \overbrace{(e^{\alpha xy}y)}^{\deg=1}\overbrace{(ye^{\beta yz})}^{\deg=1}$$
$$= \underbrace{(e^{\alpha xy}y) \cdot (ye^{\beta yz})}_{\cos \theta = c_\alpha c_\beta} + (e^{\alpha xy}y) \wedge (ye^{\beta yz})$$
$$= \exp\left(\cos^{-1}(c_\alpha c_\beta)\frac{(e^{\alpha xy}y) \wedge (ye^{\beta yz})}{\sqrt{1 - c_\alpha^2 c_\beta^2}}\right)$$

Note $xy + zt$ is not a blade means it is a double rotation

8 Rotor representation?

I'm writing these so rotors conjugate on the left

$$i\_ = \begin{cases} 1 & \mapsto i \\ i & \mapsto -1 \\ j & \mapsto k \\ k & \mapsto -j \end{cases}$$

Double rotation:

$$i\_ \mapsto \left(\frac{1 + x_1 x_0}{\sqrt{2}}\right)\left(\frac{1 + x_3 x_2}{\sqrt{2}}\right)$$

$$\_i = \begin{cases} 1 & \mapsto i \\ i & \mapsto -1 \\ j & \mapsto -k \\ k & \mapsto j \end{cases}$$

$$\_i \mapsto \left(\frac{1 + x_1 x_0}{\sqrt{2}}\right)\left(\frac{1 + x_2 x_3}{\sqrt{2}}\right)$$

$$j\_ = \begin{cases} 1 & \mapsto j \\ j & \mapsto -1 \\ i & \mapsto k \\ k & \mapsto -i \end{cases}$$

$$\_j = \begin{cases} 1 & \mapsto j \\ j & \mapsto -1 \\ i & \mapsto -k \\ k & \mapsto i \end{cases}$$

$$k\_ = \begin{cases} 1 \mapsto k \\ k \mapsto -1 \\ i \mapsto j \\ j \mapsto -i \end{cases}$$

$$\_k = \begin{cases} 1 \mapsto k \\ k \mapsto -1 \\ i \mapsto -j \\ j \mapsto i \end{cases}$$