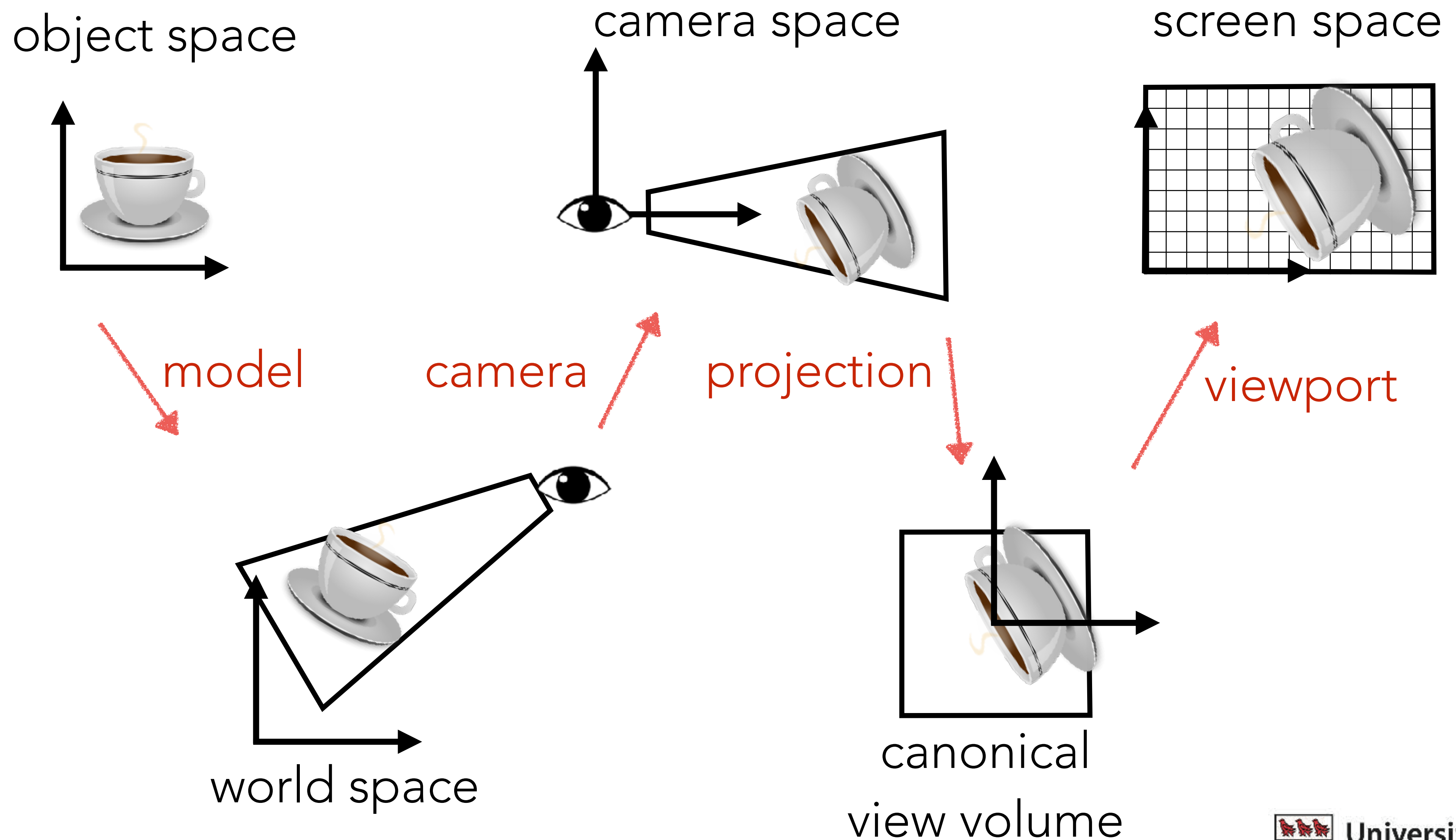
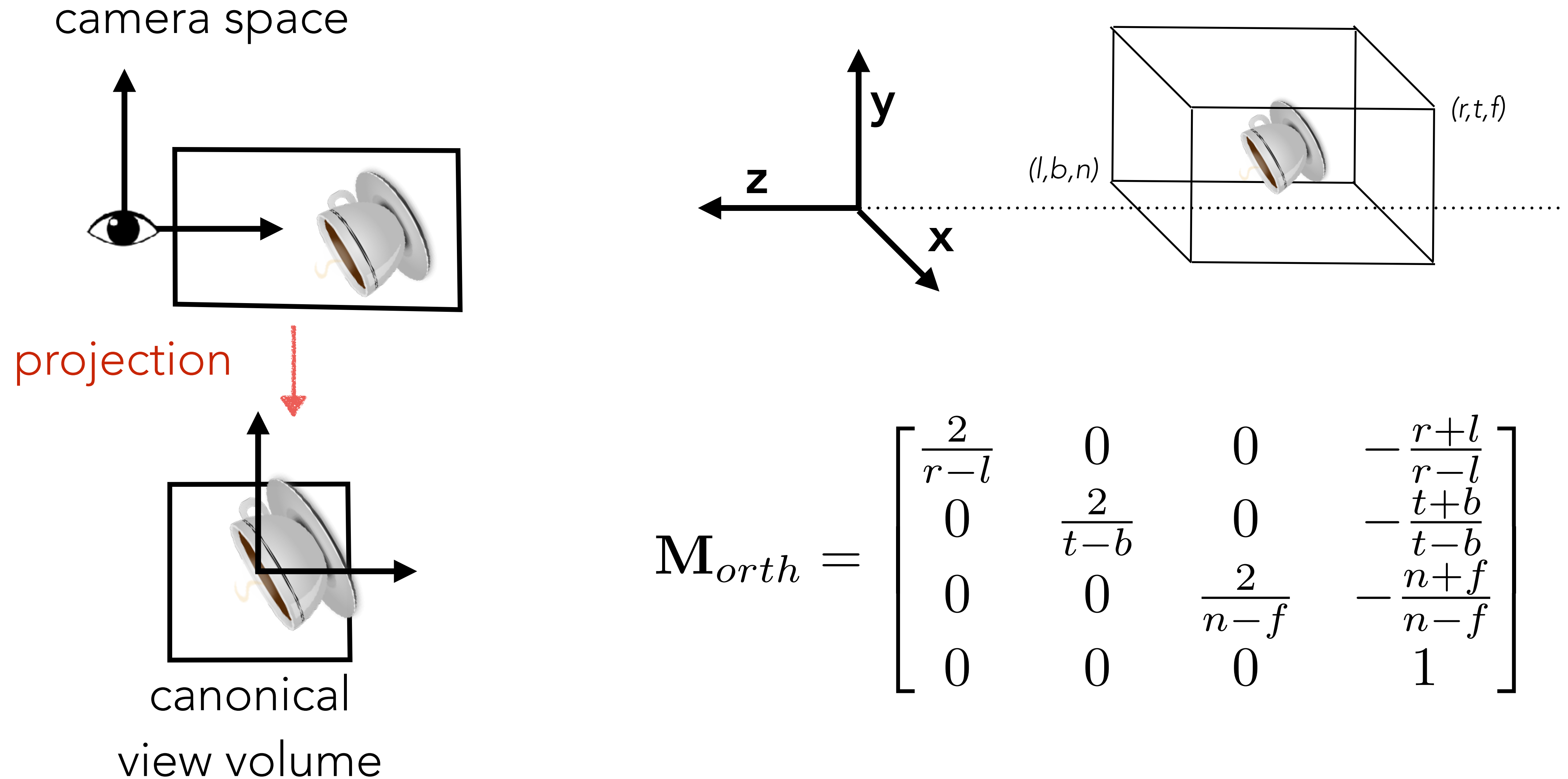


# Projective Transformations

# Viewing Transformation

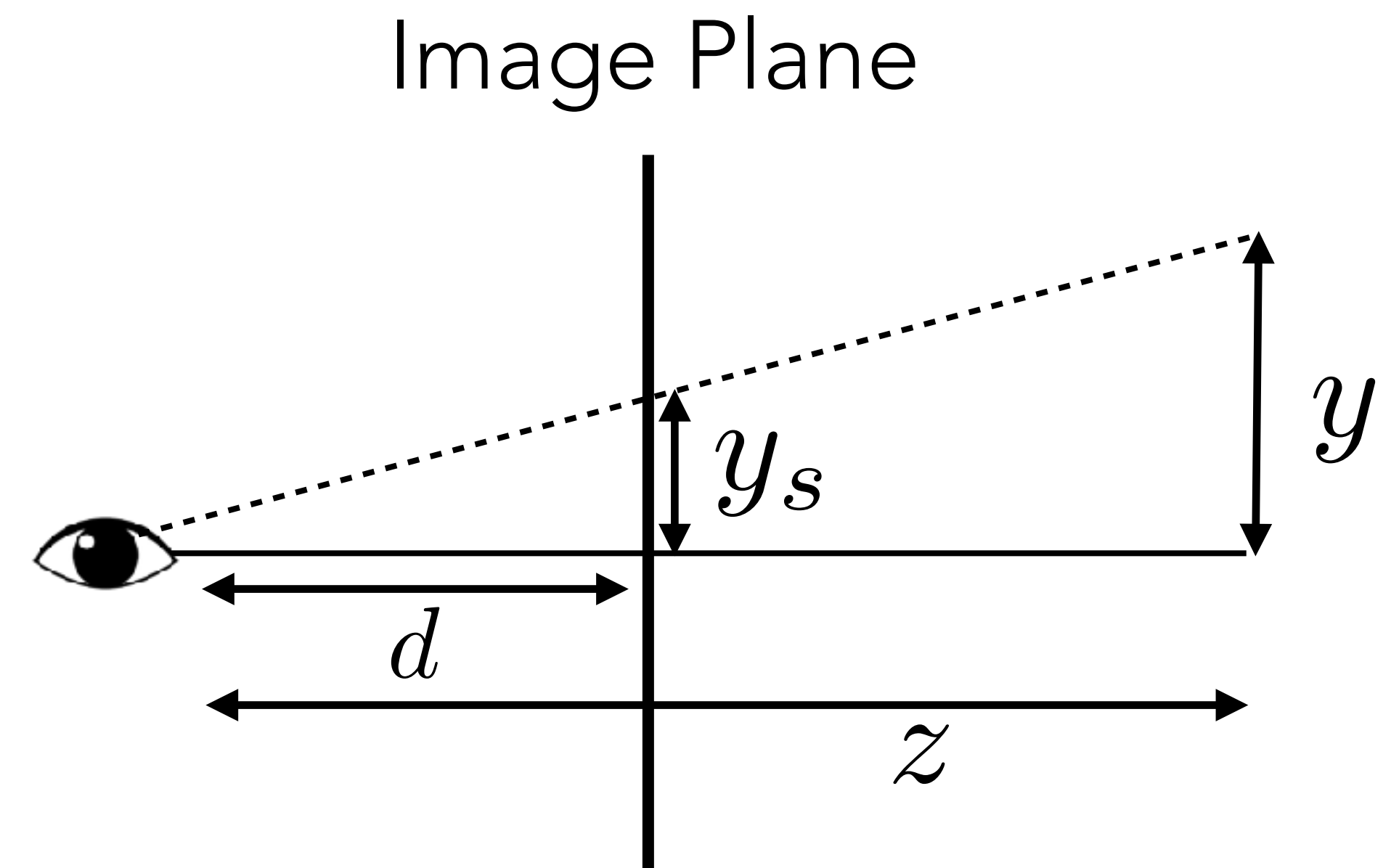
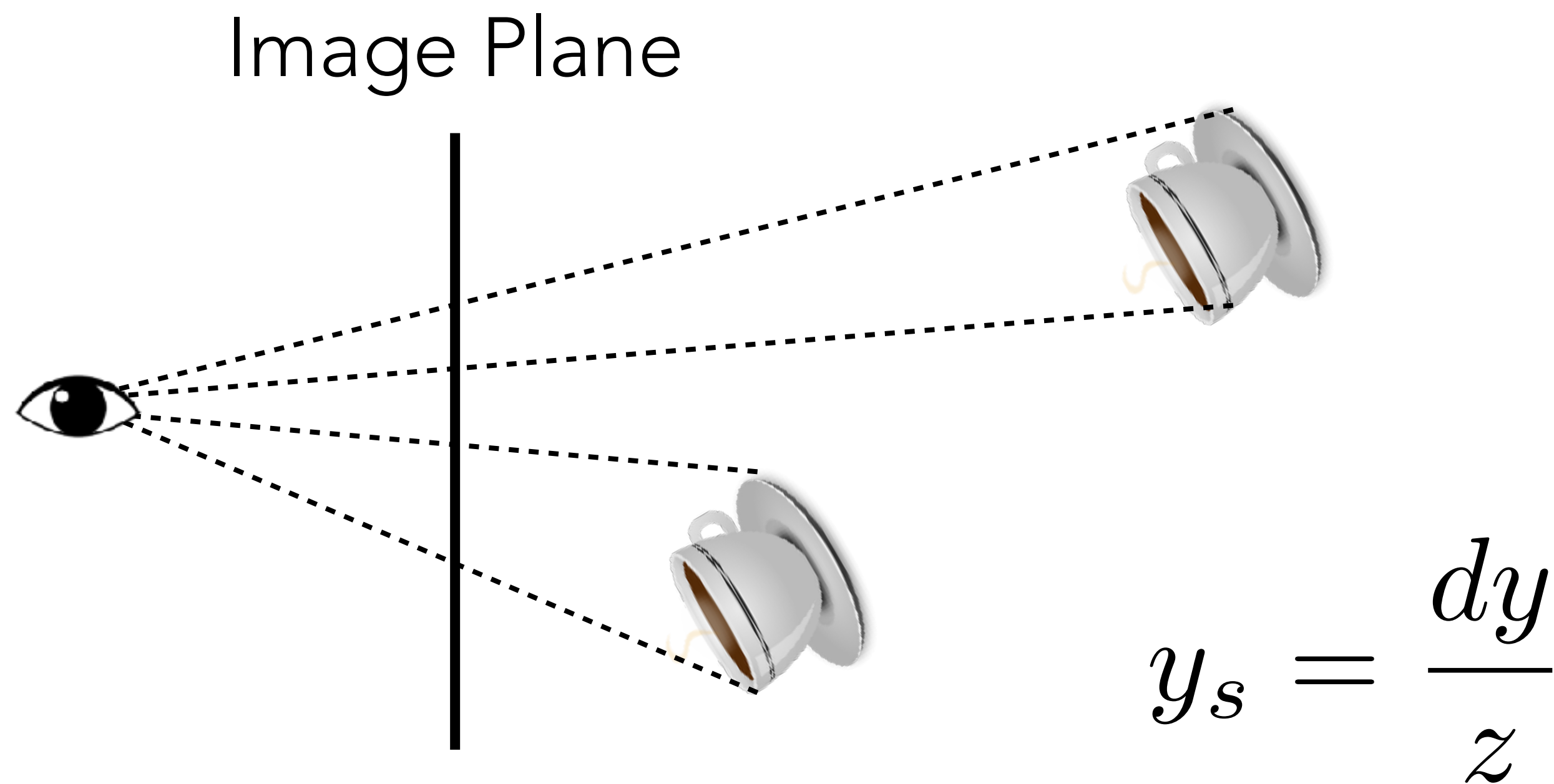


# Orthographic Projection



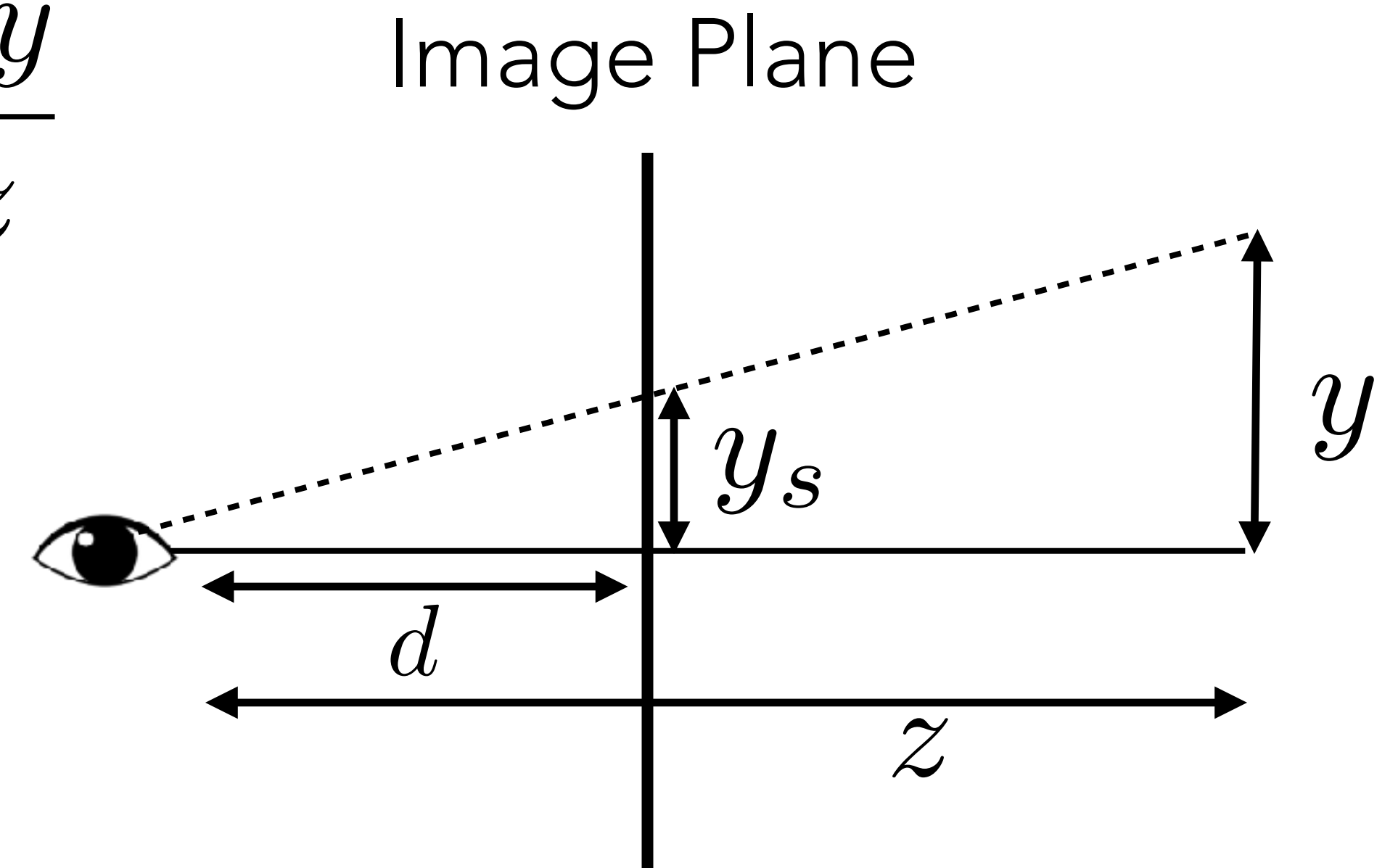
# Perspective Projection

- In Orthographic projection, the size of the objects does not change with distance
- In Perspective projection, the objects that are far away look smaller



# Divisions in Matrix Form

- We would like to reuse the matrix machinery that we built in the previous lectures
- How do we encode divisions?  $y_s = \frac{dy}{z}$
- We extend homogeneous coordinates





# Until now...

- What do we have left?

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_1x + b_1y + c_1 \\ a_2x + b_2y + c_2 \\ 1 \end{pmatrix}$$

- We can use the last row of the transformation:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ e & f & g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_1x + b_1y + c_1 \\ a_2x + b_2y + c_2 \\ ex + fy + g \end{pmatrix} \sim \begin{pmatrix} \frac{a_1x + b_1y + c_1}{ex + fy + g} \\ \frac{a_2x + b_2y + c_2}{ex + fy + g} \\ 1 \end{pmatrix}$$

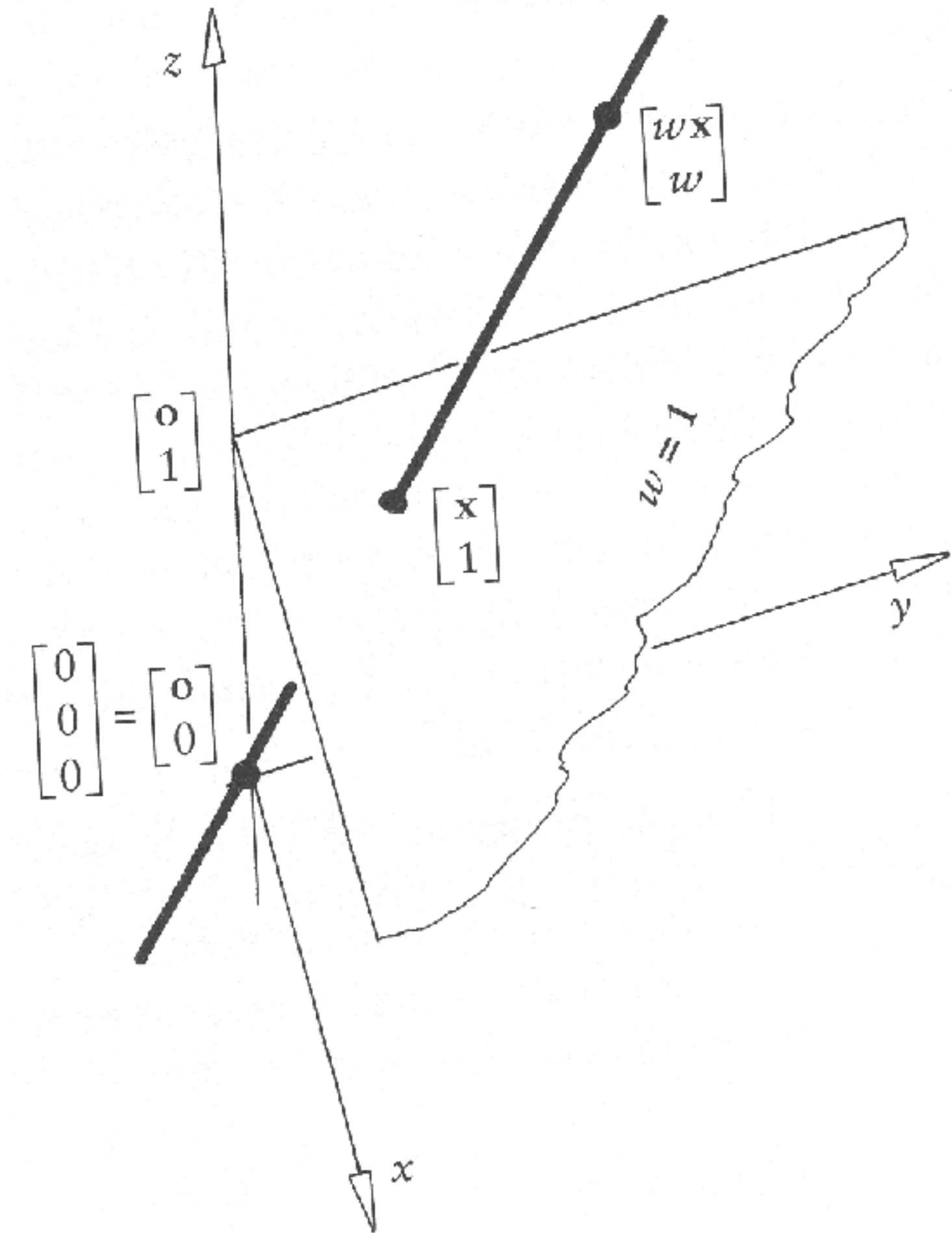


# Intuition

- Purely algebraic:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \sim \begin{pmatrix} x/w \\ y/w \\ 1 \end{pmatrix}$$

- Or as a projection, where each line is identified by a point on the plane  $z=1$
- Note that in this case, you can think of it as a transformation in a space with one more dimension



# Projective Transformation

- A transformation of this form is called a *projective transformation* (or a homography)
- The points are represented in *homogeneous coordinates*

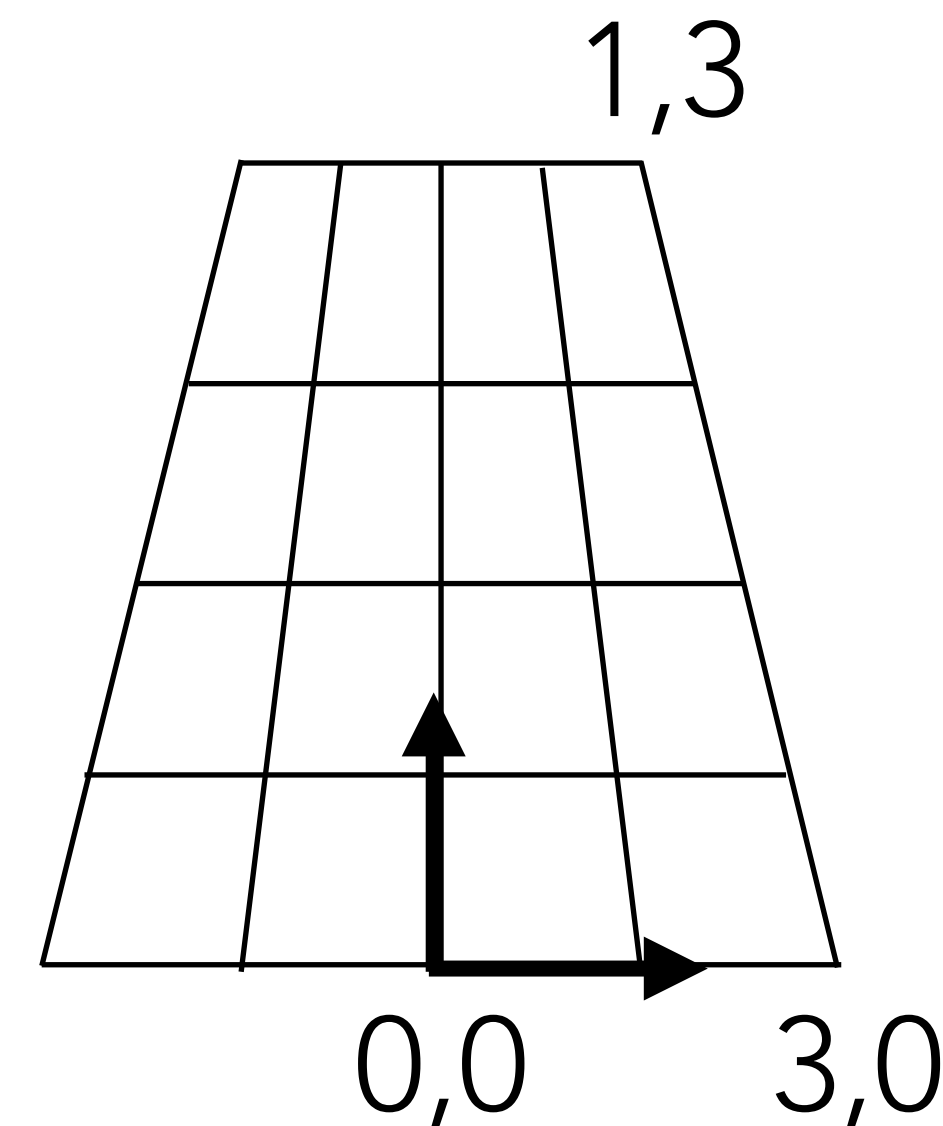
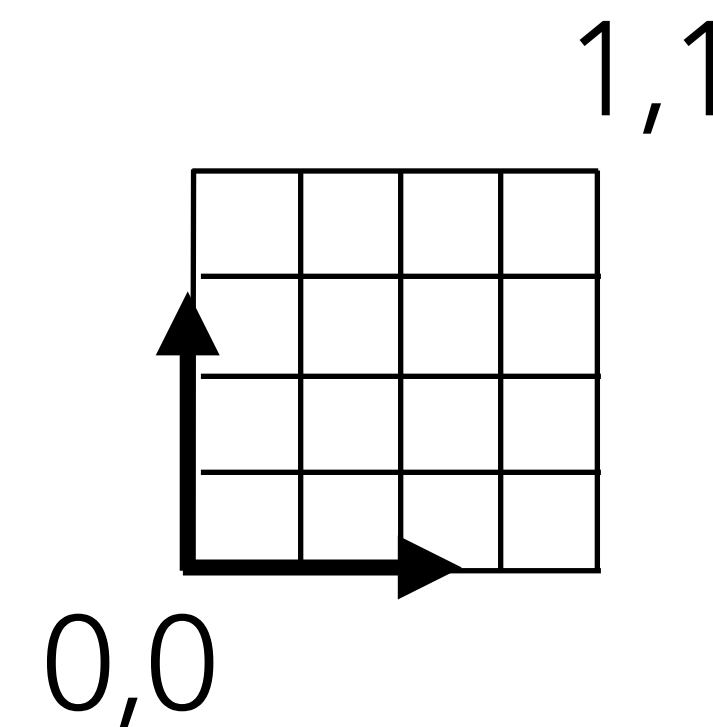
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ e & f & g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_1x + b_1y + c_1 \\ a_2x + b_2y + c_2 \\ ex + fy + g \end{pmatrix} \sim \begin{pmatrix} \frac{a_1x + b_1y + c_1}{ex + fy + g} \\ \frac{a_2x + b_2y + c_2}{ex + fy + g} \\ 1 \end{pmatrix}$$



# Example

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

- It transforms a square into a quadrilateral — note that straight lines are preserved, but parallel lines are not!
- Note that you can use homogeneous coordinates for as many transformations as you want, only when you need the cartesian representation you have to normalize

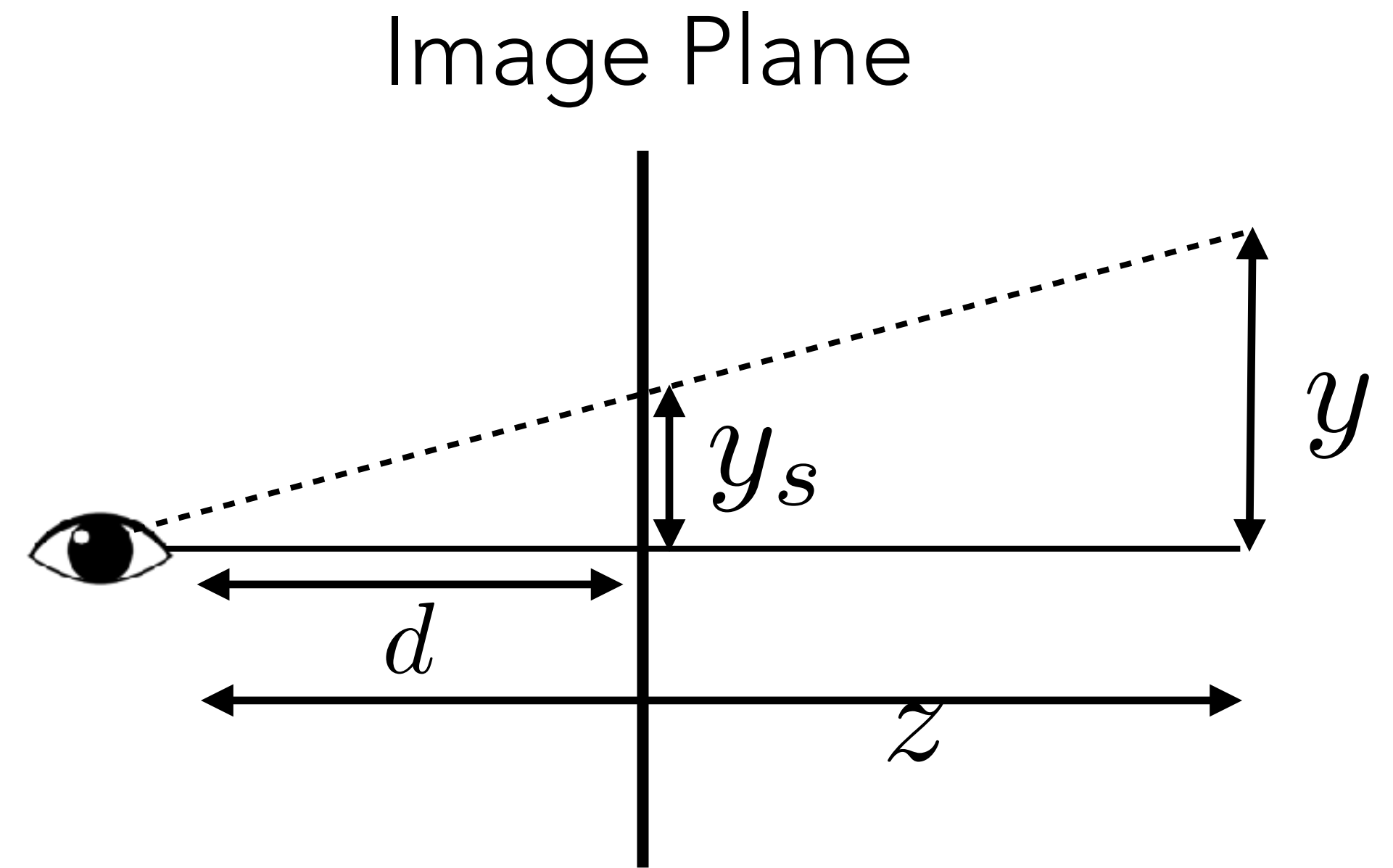


# Perspective Projection

- Perspective projection is easily implementable using this machinery

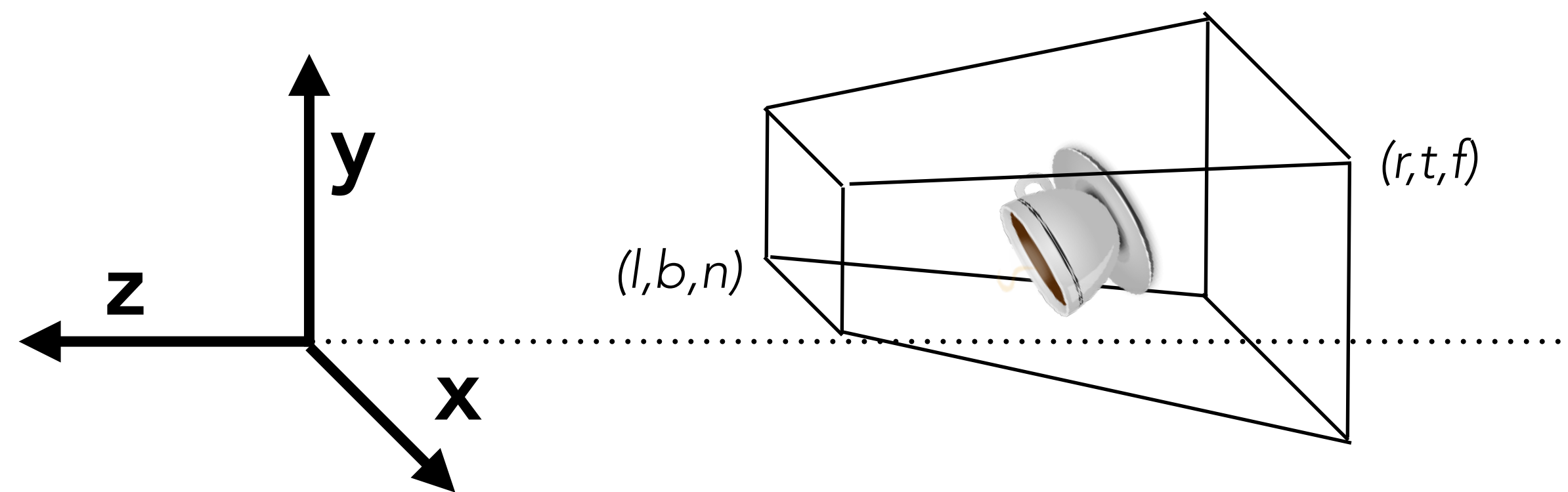
$$y_s = \frac{dy}{z}$$

$$\begin{pmatrix} y_s \\ 1 \end{pmatrix} \sim \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \\ 1 \end{pmatrix}$$



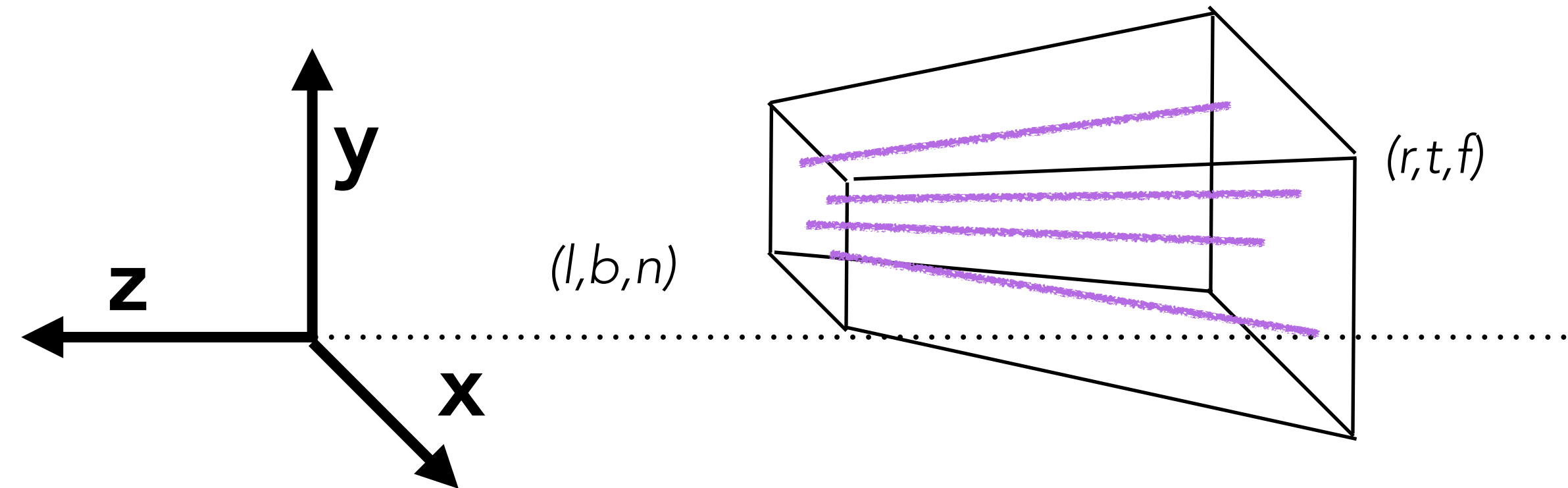
# Perspective Projection

- We will use the same conventions that we used for orthographic:
- Camera at the origin, pointing negative z
- We scale x, y and “bring along” the z



$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n + f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

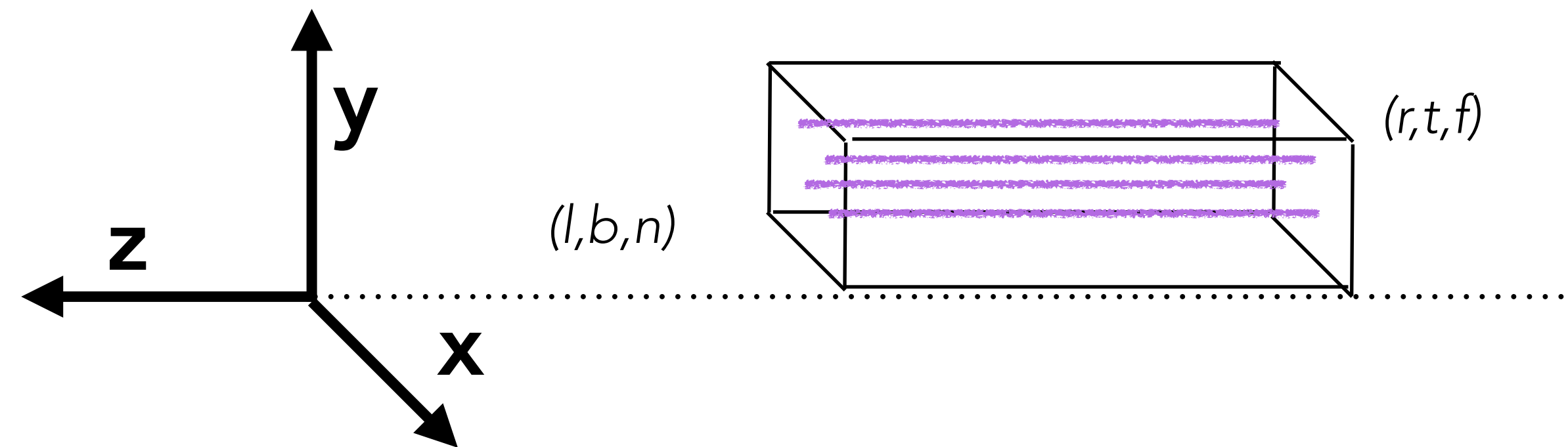
# Effect on the points



$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n + f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{P} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ (n + f)z - fn \\ z \end{pmatrix} \sim \begin{pmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n + f - \frac{fn}{z} \\ 1 \end{pmatrix}$$

# Effect on the points

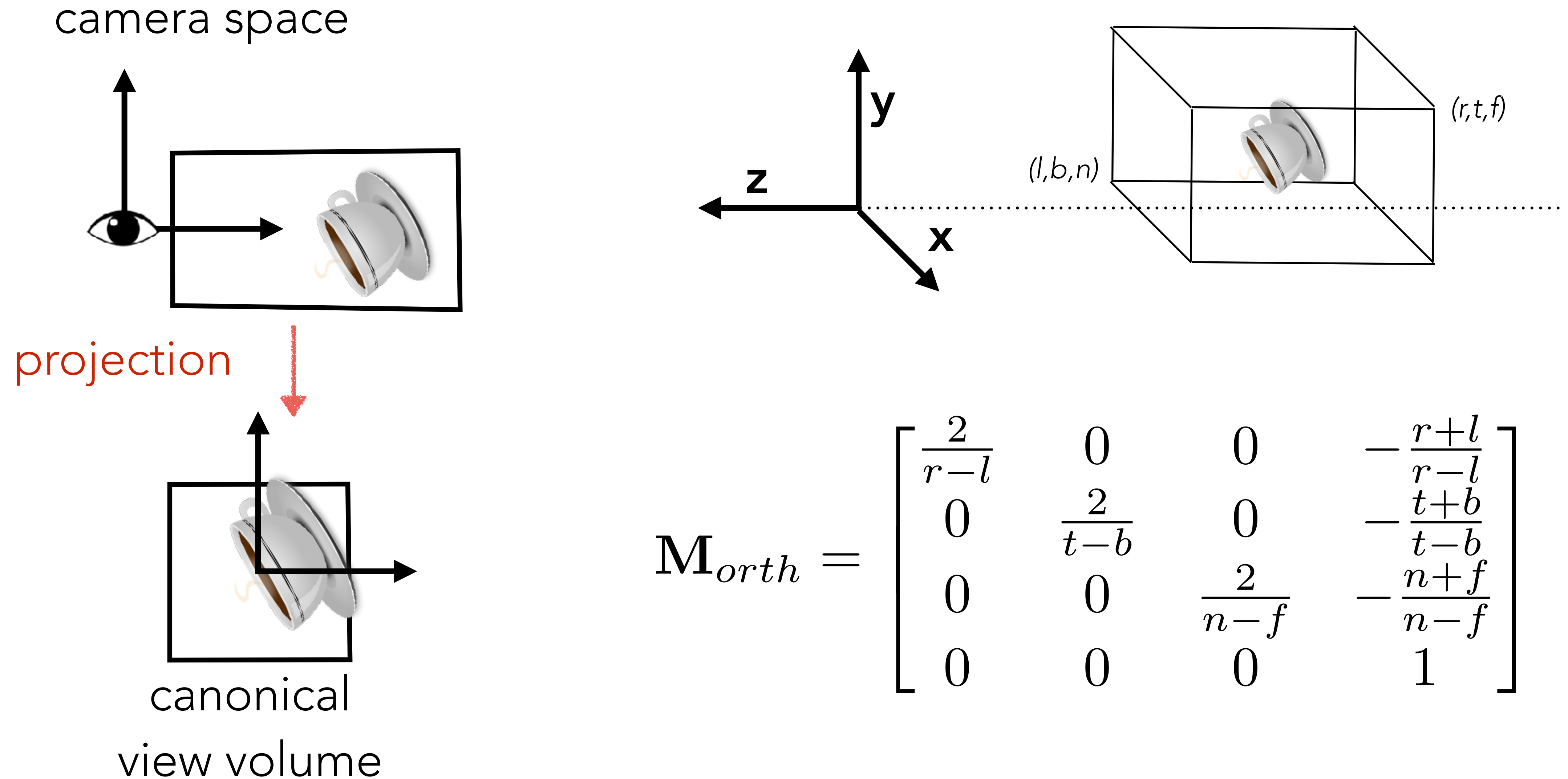


$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n + f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{P} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ (n + f)z - fn \\ z \end{pmatrix} \sim \begin{pmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n + f - \frac{fn}{z} \\ 1 \end{pmatrix}$$



# Orthographic Projection

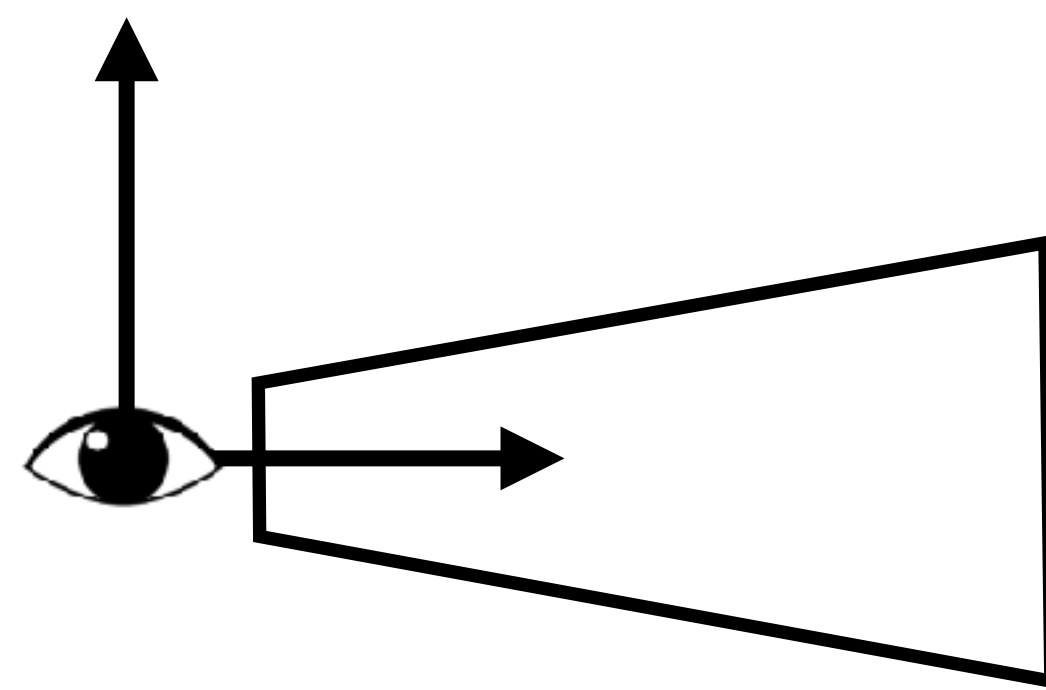


# Complete Perspective Transformation

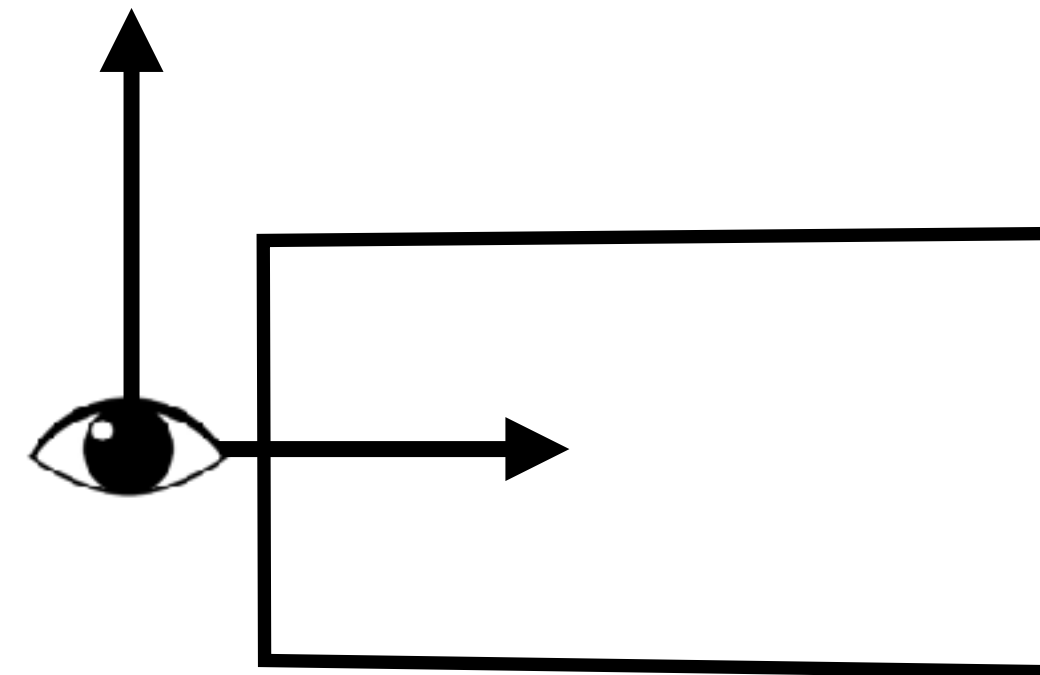
$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

camera space



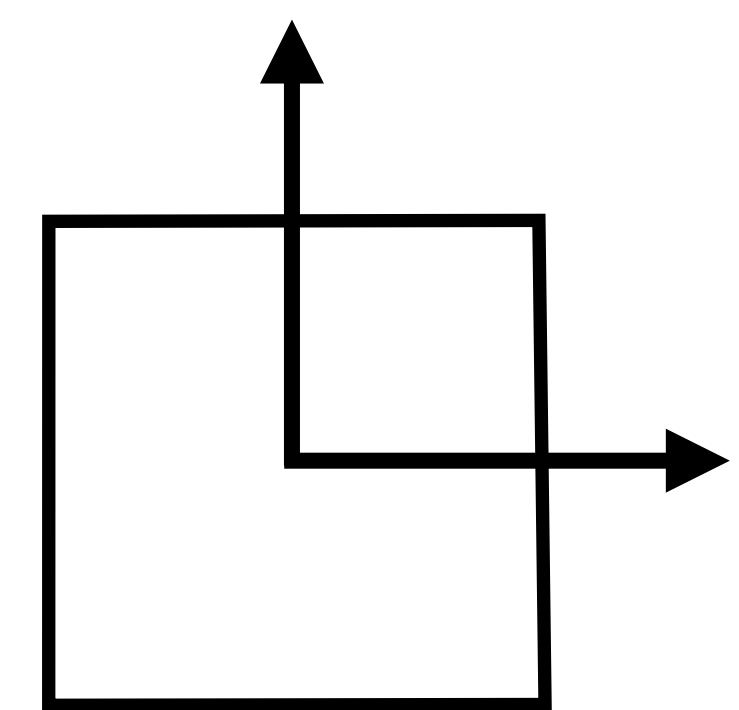
$\mathbf{P}$



$\mathbf{M}_{orth}$



canonical  
view volume

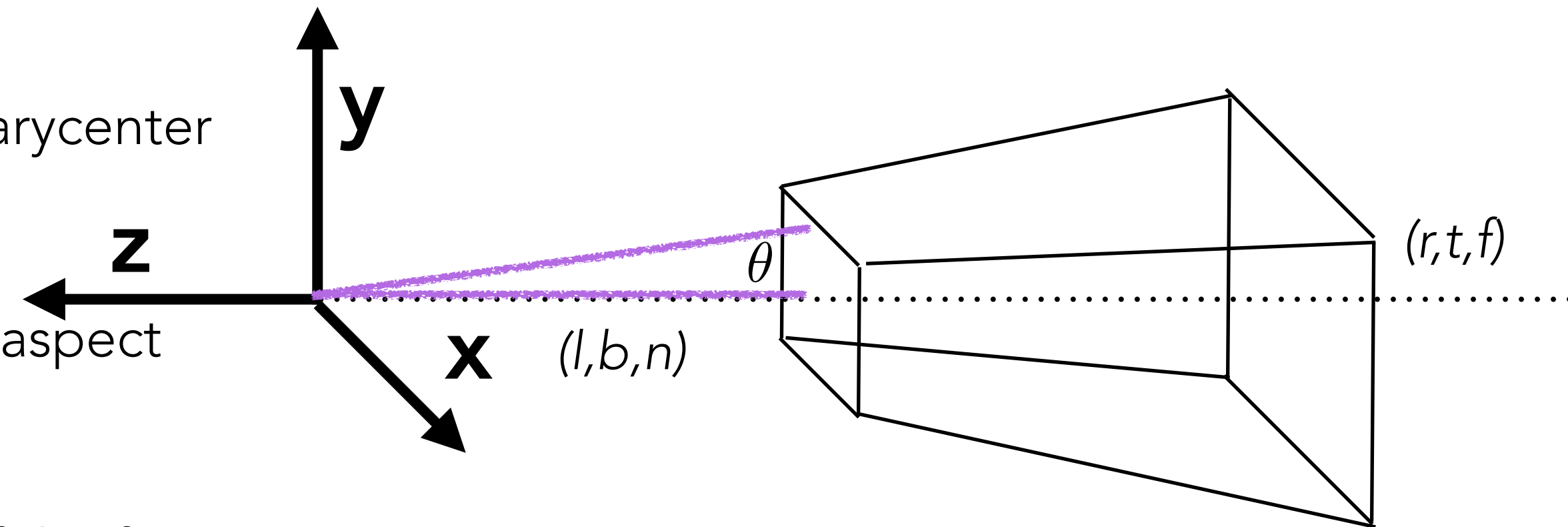


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# Parameters?

- How to set the parameters of the transformation?
- If we look at the center of the center of the window then the barycenter of the front back should be at  $(0,0,f)$
- If we want no distortion on the image we need to keep a fixed aspect ratio:
  - width/height =  $r/t$  (width and height are the size in pixels of the final image)
- There is only one degree of freedom left, the field of view angle  $\theta$ :
  - $\tan \frac{\theta}{2} = \frac{t}{|n|}$
- The parameters can thus be found by fixing  $n$  and  $\theta$ . You can then compute  $t$  and consequently all the other parameters needed to construct the transformation



# References

**Fundamentals of Computer Graphics, Fourth Edition**

4th Edition **by Steve Marschner, Peter Shirley**

Chapter 7