Chapter 1

Poles of Residual Eisenstein Series

Our goal here is to exposit and survey the work in papers such as [?][?][?] and perhaps give a trivial extension of them. The idea is to locate the poles and zeroes of certain Eisenstein series.

[?] gave an analysis of the residual poles of Eisenstein series attached to Sp_{2n}, there were some minor errors that were corrected in [?] where they give esssentially the same proof; theirs however works for the other classical groups. For our purposes, the case of Sp_{2n}, as a group defined over F a number field, is most relevant, and we shall therefore focus exclusively on this case, however it should be noted that this limitation in the non-covering case is artificial, although it does simplify things a little by removing some casework, and we hope also in the covering case to be able to remove it in future work.

1.1 Residual Eisenstein Series

So for the rest of the chapter we will fix an $n \in \mathbb{N}$, $G_n = \mathrm{Sp}_{2n}$ and the Borel of upper triangular matrices in Sp_{2n}, then we look at partitions of $n = r + m$, where . Then as we saw in **??** there corresponds a maximal standard parabolic of Sp_{2n}, which we denote $P_r = M_r N_r$, such that the Levi component is what are the ranges o

$$\mathrm{GL}_r \times \mathrm{Sp}_{2m}$$

As we saw in **??** the space of characters $X_{M_r}^{\mathrm{Sp}_{2n}}$ is one dimensional by the maximality of P_r. If we look at the divisors of $r = ab$ and fix a τ, an irreducible unitary cuspidal automorphic representation of GL_a, then from **??** we know that $\Delta(\tau, b)$ is a residual representation of $\mathrm{GL}_{ab} = \mathrm{GL}_r$. Now we take an irreducible generic cuspidal automorphic representation σ of Sp_{2m}, and so their tensor product $\Delta(\tau, b) \otimes \sigma$ gives a representation of $\mathrm{GL}_r \times \mathrm{Sp}_{2m}$ and hence of the Levi M_r. We now consider the Eisenstein series attached to this representation, namely if check the exact range pend on the range of

$$\varphi \in \mathcal{A}(N_r(\mathbb{A})M_r(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))_{\Delta(\tau,b)\otimes\sigma}$$

then we have the Eisenstein series

$$E(\varphi, s)(g) = \sum_{\gamma \in P_r(F) \setminus \mathrm{Sp}_{2n}(F)} s.\varphi(\gamma g)$$

for $g \in \mathrm{Sp}_{2n}(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A})$. Becuase it is induced from the residual representation $\Delta(\tau, b)$ we call these residual Eisenstein series.

This is the setup in [**?**], where they prove their results by a sort of induction on b. Here we will focus exclusively on the base case of this induction, leaving the inductive step for future work. So from now on we will fix $b = 1$. Hence $n = a + m$. Then fixing a standard parabolic of Sp_{2n} we have the maximal standard parabolic $P_a = M_a N_a$ where $M_a = \mathrm{GL}_a \times \mathrm{Sp}_{2m}$. Now if τ is irreducible unitary cuspidal automorphic representation of GL_a then by definition

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$$\Delta(\tau, 1)(\varphi)(g) = E(\varphi, s)(g) = s.\varphi(g)$$

where the Eisenstein series is defined via the parabolic induction from the Levi $(\mathrm{GL}_a)^{\times b}$ to GL_{ab}. Thus we have $\Delta(\tau, 1) = \tau$.

1.2 The Constant Term

So far we only know how to do one thing with such Eisenstein series and that is compute their constant term. We will compute the constant term along the maximal parabolic $P_a = MN$.

, all others are

By our earlier calculations **??** and the cuspidality of the tensor **??** and [**?**] we know that

$$E(\varphi, s)_P = \sum_{w} \sum_{m'} \int_{(w^{-1}N(\mathbb{A})w \cap M(\mathbb{A})) \setminus A} \int_{w^{-1}N(F)w \cap M(F) \setminus w^{-1}N(\mathbb{A})w \cap M(\mathbb{A})} s.\varphi(n_1 n_2 w^{-1} m' x) dn_1 dn_2,$$

and the inner integral vanishes for all $w \neq id, \omega$ (ω as in [**?**]). Hence the first sum becomes over two elements and we have

$$E(\varphi, s)_P = E(\varphi, s)_{P,id} + E(\varphi, s)_{P,\omega}.$$

where

$$E(\varphi, s)_{P,w}(x) = \sum_{m' \in M(F) \cap wP(F)w^{-1} \setminus M(F)} \int_{N(F) \cap wP(F)w^{-1} \setminus N(\mathbb{A})} s.\varphi(w^{-1} n m' x) dn.$$

First the identity term simplifies

$$\begin{aligned} E(\varphi, s)_{P',id}(x) &= \sum_{m' \in M(F) \cap P(F) \setminus M(F)} \int_{N(F) \cap P(F) \setminus N(\mathbb{A})} s.\varphi(n m' x) dn \\ &= \sum_{m' \in M(F) \setminus M(F)} \int_{N(F) \setminus N(\mathbb{A})} s.\varphi(n m' x) dn \\ &= \int_{N(F) \setminus N(\mathbb{A})} s.\varphi(n x) dn \\ &= s.\varphi(x)_P. \end{aligned}$$

Considering now the ω term

$$E(\varphi, s)_{P,\omega}(x) = \sum_{m' \in M(F) \cap \omega P(F)\omega^{-1} \setminus M(F)} \int_{N(F) \cap \omega P(F)\omega^{-1} \setminus N(\mathbb{A})} s.\varphi(\omega^{-1} n m' x) dn.$$

By [**?**, 2C] $M(F) \cap \omega P(F)\omega^{-1} \setminus M(F)$ is isomorphic to $P_0 \setminus \mathrm{Sp}_{2(n-a)}$, but P_0 has Levi $M_0 = \mathrm{Sp}_{2(n-a)}$ by definition and hence is itself $\mathrm{Sp}_{2(n-a)}$. Thus the sum is over $\mathrm{Sp}_{2(n-a)}(F)\setminus\mathrm{Sp}_{2(n-a)}(F)$ and hence is over a point. Therefore we get by definintion of the intertwining operator

$$E(\varphi, s)_{P,\omega}(x) = \int_{N(F)\cap\omega P(F)\omega^{-1}\setminus N(\mathbb{A})} \varphi(\omega^{-1}nx)dn = M(\omega, s)(\varphi)(x)$$

becuase we took the constant term along the same parabolic as the definition of the Eisenstein series we know that the Levis are (the same) conjugate. Thus we have shown that

$$E(\varphi, s)_P = s.\varphi_P + M(\omega, s)(\varphi)$$

cite the relevant parts and brenner..

Remark 1.1. This work was entirely at the level of the terminals and hence generalises verbatim to covering groups.

Because φ is an automorphic form it has no poles and so we have shown the following:

Lemma 1. *The poles of $E(\varphi, s)$ are exactly the poles of $E(\varphi, s)_{P_a}$ which are exactly the poles of $M(\omega, s)$.*

1.3 Analysing the Intertwining Operator

Recall from **??** that we know in this case that there is some ratio of L-functions $r(s, \omega)$ such that $M(s, \omega) = r(s, \omega)N(s, \omega)$ where $N(s, \omega)$ is both holomorphic and non-zero. In the case we are considering the normalising factor r is given by the equation [**?**, 4A]

$$r(w, s) = \frac{L(s, \tau \times \sigma)L(2s, \tau, \wedge^2)}{L(s+1, \tau \times \sigma)L(2s+1, \tau, \wedge^2)},$$

where $\wedge^2$ denotes the exterior second power of the standard representation of $\mathrm{GL}_a(\mathbb{C})$. Thus

Lemma 2. *The Eisenstein series above has pole at s if and only if $r(w, s)$ has a pole.*

The final step is then to use the known properties of L-functions to conclude when our r-factor will have poles and of what order those poles will be.

Theorem 1 ([**?**], 4.1)**.**

1.4 The Metaplectic Generalisation

We need to restate the setup now in the metaplectic case. Using the same notation as above, ref we now also denote $\mathbf{M}_a$ the pre-image of M_a in Mp_{2n}, in particular if $\mathrm{pr} : \mathrm{Mp}_{2n} \to \mathrm{Sp}_{2n}$ is the defining projection we have that $\mathbf{M}_a := \mathrm{pr}^{-1}(M_a)$. As we remarked we still have that $X^{\mathrm{Mp}_{2n}}_{\mathbf{M}_r}$ is ref one dimensional. Now we want to consider representations of

$$\mathbf{M}_a = \mathrm{pr}^{-1}(\mathrm{GL}_a \times \mathrm{Sp}_{2m}) \cong \big(\mathrm{pr}^{-1}(\mathrm{GL}_a) \times \mathrm{pr}^{-1}(\mathrm{Sp}_{2m})\big)/\Delta\mu_2,$$

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where $\mu_2 = \{\pm 1\}$ acts on the product via the diagonal embeding . So now we let $\tilde{\tau}$ be generic cuspidal irreducible representation of $\mathrm{pr}^{-1}(\mathrm{GL}_a)$ and $\tilde{\sigma}$ a generic cuspidal irreducible representation of $\mathrm{pr}^{-1}(\mathrm{Sp}_{2m})$.

Remark 1.2. If neither are *genuine* representations of the covers then they both factor through a representation of the Levi of Sp_{2n} and hence we can reduce to the case of the classical group itself (instead of its cover).

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Because we may we assume that $\tilde{\tau} \cong \chi \otimes \mathrm{pr}^*(\tau)$ and $\tilde{\sigma} \cong \chi' \otimes \mathrm{pr}^*(\sigma)$, for some characters χ , where σ, τ are as in the setup above, irreducible generric cuspidal representations of $\mathrm{Sp}_{2n}, \mathrm{GL}_{2m}$ respectively. Finally we can form the Eisenstein series associate to an automorphic form $\varphi \in \mathcal{A}(N_a(\mathbb{A})M_a(F) \setminus \mathrm{Mp}_{2n}(\mathbb{A}))_{\tilde{\tau}\otimes\tilde{\sigma}}$ defined in the same way as before,

$$E(\varphi, s)(g) = \sum_{\gamma \in P_r(F)\setminus \mathrm{Sp}_{2n}(F)} s.\varphi(\gamma g),$$

for $g \in \mathrm{Sp}_{2n}(F) \setminus \mathrm{Mp}_{2n}(\mathbb{A})$ and $s \in \mathbb{C} \cong X_{M_r}^{\mathrm{Mp}_{2n}}$.

As we remarked earlier the constant term computation applies immediately to this case as well hence we still have:

Lemma 3. *The poles of $E(\varphi, s)$ are exactly the poles of $E(\varphi, s)_{P_a}$ which are exactly the poles of $M(\omega, s)$.*

This is where the general story ends for the metaplectic groups. The results that we used in the classical group setting have not yet been proven (and are far beyond my scope).

1.4.1 Siegel Parabolic

Kaplan in a series of recent works with collaborators [**?**][**?**][**?**] has supplied some of the key peices for the normalisation of the metaplectic intertwining operators in the case of the Siegel parabolic. Namely

Theorem 2 ([**?**], 4.2)**.**

$$M(s, w) = r(s, w)N(s, w),$$

Where N is a non-zero and holomorphic function and r is a ratio of L-functions.

In this case the relvant normalisation is

$$r(s, \omega) =$$

Lemma 4. *The Eisenstein series above has pole at s if and only if $r(w, s)$ has a pole.*

Finally we need to once again see what properties of these L-fuctions have been proven.