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Known issues: Need to add punctuation to all equations, check spelling, fix some formatting of equations,

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Notation

Chapter 1

Classical Groups

We will recall a small amount of the theory of linear algebraic groups to fix conventions, for a more detailed treatment one should consult the litany of sources on this matter: For a full treatment see [Mil17][Milb][Mila][Spr98]. Excellent example computations can also be found in [Bui][Mak][MT][Not]. Or for a brief brush up on the main facts consult [BC79, I.I.1]. The purpose of this section is to treat the classical groups and more specifically Sp_{2n} as an example and work out some of the details of the general theory, in order to "get our hands dirty" and have some familiarity with this object, to become fundamental in what follows. Because this theory is made up of simple ideas that can often be obscured by the generality we will make several restrictive assumptions for ease of exposition.

1.1 Definition

An algebraic group is for us a group scheme that is reduced, of finite type and defined over a field. A linear algebraic group (LAG) is an affine algebraic group. By [Spr98, 2.3.7(i)] then every LAG is (isomorphic) to a (Zariski) closed subgroup of GL_n. A group scheme is called linear if it is (isomorphic to) a closed subgroup for GL_n. Hence every linear algebraic group is a *linear* algebraic group. Moreover it is a basic fact that a Zariski closed sub-scheme of an affine scheme is affine [Mum99, II.5.T3].

As Milne points out [Mil17, Abstract] these are supposed to be matrix groups defined by polynomials, which are somehow the natural combinations of symbols that matrix multiplication will lead to. This means that they come with the powerful but cumbersome (for the beginner) technology of algebraic geometry. In particular one must be adept at moving between the following equivalences

Theorem ([Mila], II.6, III.4)**.** *For k a field then the following categories are equivalent*

- *Group objects in* Alg_k^{opp}
- *Representable (in the category of groups) functors* $\mathrm{Alg}_k \to \mathrm{Group}$
- *Group object in the category of affine schemes over k*
- *Commutative Hopf algebras*

The exact groups that an author might mean by classical may vary. Here we will follow [Cli, §13] or indeed many other places as defining them as the automorphisms of a vector space with a bilinear form. First let V be a finite dimensional F vector space with a bilinear form $\langle , \rangle$. An automorphism of this form is a map $\alpha \in \mathrm{Aut}(V)$ such that

$$\langle \alpha(x), \alpha(y) \rangle = \langle x, y \rangle$$

Therefore we can consider the space of automorphisms of $\mathrm{Aut}(V, \langle , \rangle)$. This space, depending on the properties of the bilinear form will define our classical groups.

If the form is trivial, by which we mean, $\forall x, y \ \ \langle x, y \rangle = 0$ then we define

$$\mathrm{GL}(V) \ \ := \ \ \mathrm{Aut}(V, \langle , \rangle) = \mathrm{Aut}(V)$$

If the form is non-degenerate and symmetric $\forall x, y \ \ \langle x, y \rangle = \langle y, x \rangle$ then we define

$$\mathrm{O}(V) \ \ := \ \ \mathrm{Aut}(V, \langle , \rangle)$$

Finally if the form is non-degenerate and skew symmetric $\forall x, y \ \ \langle x, y \rangle = -\langle y, x \rangle$ then

$$\mathrm{Sp}(V) \ \ := \ \ \mathrm{Aut}(V, \langle , \rangle)$$

There are the further classical groups given by the determinant one subgroups, $\mathrm{SL}(V), \mathrm{SO}(V)$ respectively ($\mathrm{Sp}(V)$ one can show already implies that the determinant is one). Moreover every F algebra is an F vector space and so we have defined a functor from F-algebras to groups. This is what we will refer to as "classical groups" although we should note the omission of what the unitary groups that would usually be included.

Another way of motivating the importance of these groups is to consider the classification theory of the split reductive groups. Reductive is a representation theoretic condition that means, worse than semi-simple but still manageable, for more detail see [Mil17, 22.138], this is only for motivation. (Split) Reductive groups over fields have a classification in terms of root datum [Mil17, 22.48] which is a strong parallel of the classification of semi-simple Lie algebras, and in fact is proved largely by bootstrapping from that theory to a more general setting. These correspond to Dynkin diagrams that are listed in [Sha10, Appendix A,B]

is this right

and of which there are four infinite families, which correspond to the classical groups (plus the unitary groups). This is in the way of motivating the notion of classical groups as "almost all" reductive groups.

1.2 Subgroups

From now on we restrict to split reductive LAG because these are the natural adjectives for our classical groups over a number field (in particular characteristic 0)

Subgroups with special properties allow us to reduce and break up problems into smaller ones. Here we will briefly review and compute some examples of special subgroups, with a particular eye on those of Sp. The point of these subgroups is two fold. Some of them will help us perform "induction" from smaller simpler groups to larger ones. Others are there essentially as a part of the combinatorial data that classifies the groups we are working with. In particular we will come to understand the following decomposition's (not direct)

$$G(\mathbb{A}) = M(\mathbb{A})U(\mathbb{A})K = T(\mathbb{A})U(\mathbb{A})K$$

1.2.1 Parabolics, Levis and Unipotents

Parabolic subgroups have two equivalent formulations, both useful.

Do I need to give a reference for definitions?

Definition 1.1. A subgroup $P \subseteq G$ is called parabolic if the following equivalent conditions hold

- G/P is a complete variety
- P contains a Borel (see below)

Completeness is the algebro-geometric analogue of compact, which is always a desirable property. The fact that they contain a Borel gives us an algebraic "parametrisation" of these subgroups, in the case of the classical groups through the use of flags or roots. It is very important to have a parametrisation of the parabolic subgroups when it comes to taking constant terms, as we will want to compare the values along some parabolics.

Example (GL_n)**.** *The parabolic subgroups of* GL *are given by stabilisers of flags, [Spr98, Exercise 3.2.16, 6.2.11][Con]. A flag for F^n is a sequence of subspaces*

$$0 \subset W_1 \subset \cdots \subset W_r = F^n$$

note the strict inclusion. When $n = r$ it is a complete flag. $\mathrm{GL}_n(F)$ *acts on a flag component wise i.e.*

$$g.(W_1, ..., W_r) \quad := \quad (g.W_1, ..., g.W_r)$$

If F^n is given the standard basis of $e_i = (\delta_i^j)_j$ then the standard (complete) flag is

$$0 \subset Fe_1 \subset Fe_1 \oplus Fe_2 \subset \cdots \subset \oplus_i Fe_i = F^n$$

Because the stabiliser of a flag is always conjugate to a stabiliser of a sub-flag of the standard flag we allow ourselves from now on to only consider sub-flags of the standard flag.

Remark. *It is worth noting that what we do here by fixing a basis of F is the same as fixing a Borel and then considering only standard parabolics. Thus when we are talking about standard parabolics we are really working up "up to conjugacy".*

Stabilisers of such sub-flags are (possibly upside down) "staircases":

$$\begin{pmatrix} A_{11} & & * \\ & \ddots & \\ & & A_{rr} \end{pmatrix} \quad \uparrow \textit{ gravity } \uparrow$$

Example. *Consider the flag*

$$0 \subset Fe_1 \subset Fe_1 \oplus Fe_2 \oplus Fe_3 \subset F^4$$

lets find the stabiliser under the action of $\mathrm{GL}_4(F)$*. We need to send* Fe_1 *to itself*

$$\forall x \in F, \quad \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{pmatrix} x \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} ax \\ ex \\ ix \\ mx \end{pmatrix} \in F \begin{pmatrix} 1 \\ \\ \\ \\ \end{pmatrix}$$

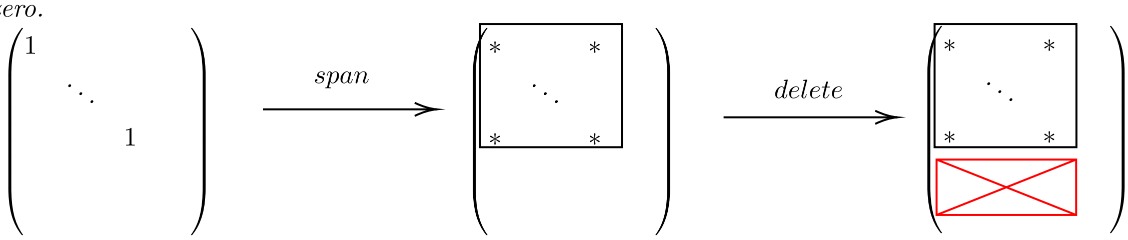
and so $e = i = m = 0$*. Next we need to send* $Fe_1 \oplus Fe_2 \oplus Fe_3$ *to itself (as a subspace)*

$$\forall x, y, z \in F, \quad \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \\ \end{pmatrix} = \begin{pmatrix} ax + by + cz \\ ex + fy + gz \\ iz + jy + kz \\ mx + ny + oz \end{pmatrix} \in Span_F \left\{ \begin{pmatrix} 1 \\ \\ \\ \\ \end{pmatrix}, \begin{pmatrix} \\ 1 \\ \\ \\ \end{pmatrix}, \begin{pmatrix} \\ \\ 1 \\ \\ \end{pmatrix} \right\} = \begin{pmatrix} * \\ * \\ * \\ \\ \end{pmatrix}$$

So we conclude that $m = n = o = 0$*. Thus we are left with elements in* $\mathrm{GL}_4(F)$ *that look like*

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix}$$

So what is happening here, for each piece of the flag you complete the diagonal that the basis vectors are in into the box that they span and you say that everything directly below that box must be zero.



Parabolics also have the nice property that they split into a semi-direct product where one of the factors is a reductive group M. For this recall the definition

Definition 1.2. A subgroup is unipotent if all its elements are unipotent. The maximal closed, connected, unipotent subgroup $U \subseteq G$ is the unipotent radical of G.

Then we have the following fact / definition:

Lemma ([Bor91] 11.22)**.** *There is a split exact sequence*

$$0 \to U \to P \to M \to 0$$

where U *is the unipotent of* P*, and* M *is a reductive group known as a Levi (unique up to conjugacy).*

Example. *The staircase*

$$\begin{pmatrix} A_{11} & & * \\ & \ddots & \\ & & A_{rr} \end{pmatrix}$$

has unipotent N and Levi M given by

$$N = \begin{pmatrix} I_{11} & & * \\ & \ddots & \\ & & I_{rr} \end{pmatrix}, \quad M = \begin{pmatrix} A_{11} & & 0 \\ & \ddots & \\ & & A_{rr} \end{pmatrix}$$

Then

$$P = M \ltimes U$$

Thus doing things on a parabolic allows us to induce said actions up to the whole group, whist maintaining the nice property of being reductive.

Remark (Bad Etymology)**.** *The origin of the name parabolic is a mystery. Borel in his history [Ess, VI.§2] attributes it to R. Godement in [God]. Godement conjectures that the quotient $G(\mathbb{A})/G(\mathbb{Q})$ is compact if and only if every element of $G(\mathbb{Q})$ is semi-simple, as is the case in classical groups.*

this is probably known by now.

He says that

> *Lorsque n'est pas compact, il est non moins facile de conjecturer qu'on doit pouvoir définir quelque chose d'analogue aux classiques "pointes paraboliques", lesquelles doivent correspondre à des sous-groupes unipotents non triviaux de $G_{\mathbb{Q}}$*

which roughly (google) translates to that one can also conjecture that non-trivial unipotent elements should correspond to "parabolic points" in a fundamental domain.

In the case of modular forms the fundamental domain is $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R})$ (using orbit stabiliser theorem). We have the classification of elements of $\mathrm{SL}_2(\mathbb{R}) - \{\pm 1\}$ as in [Bor97, 3.5] via their trace

$$g \text{ is of type} \quad \begin{cases} \textit{Elliptic} & \frac{1}{2}|tr(g)| < 1 \\ \textit{Parabolic} & \frac{1}{2}|tr(g)| = 1 \\ \textit{Hyperbolic} & \frac{1}{2}|tr(g)| > 1 \end{cases}$$

Being parabolic is equivalent to having eigenvalue 1 hence by the Jordan decomposition we know that parabolics in SL_2 are conjugate (over $\mathbb{C}$) to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly the standard parabolic

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \subseteq \mathrm{SL}_2(\mathbb{R})$$

contains these matrices, and moreover all parabolics are conjugate to this parabolic. Hence all *parabolic elements are contained in a parabolic subgroup. This classification it seems relies entrirely on the aesthetic connection with the classification of the sections of conics via eccentricity.*

To connect this to Godements concept we have two facts from classical geometry. Proper parabolic subgroups of $\mathrm{SL}_2(\mathbb{R})$ *can be realised as the stabilisers of lines in* $\mathbb{R}^2$ *under the standard action of* SL_2 *on* $\mathbb{R}^2$ *[Bor97, 2.6] and moreover some an element of* $\mathrm{SL}_2(\mathbb{R})$ *is parabolic if and only if it has one fixed point on* $\partial\bar{\mathcal{H}}$ *and none on* $\mathcal{H}$ *[Bor97, 3.5].*

The take away is that perhaps the folklore of the name being for "para-Borelic", as in kind of a Borel, is probably a better way of thinking of them.

Sp_{2n}

The case of Sp_{2n} is very similar to that of GL_n, but we will use this later and so we present the details here. Following [Con] and the very explicit calculations in [Bui, §8]. We let $(V, \langle , \rangle)$ be a symplectic space, hence $Sp(V)$ is the automorphisms preserving the form.

Example.

$$\mathrm{Sp}_{2n} = \{M \in \mathrm{GL}_{2n} : \forall a, b \ \ \langle Ma, Mb \rangle = \langle a, b \rangle\}$$

then fixing a basis of the vector space V*,* e_i *allows us to create a matrix from this form, by setting the entries to be* $A_{ij} = \langle e_i, e_j \rangle$*. Then we get*

$$\forall a, b \ \ \langle Ma, Mb \rangle = (Ma)^T A(Mb) = a^T M^T AMb = a^T Ab$$

and because this is so for all a, b *we get that*

$$M^T AM = A$$

So if we fix a basis of V, where $e_i = (\delta_i^j)_j$*, such that the form is given by the matrix*

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

A subspace is said to be isotropic if the form is constantly zero on it (in both variables). A flag is isotropic if the proper subspaces in it are isotropic subspaces. A maximal isotropic flag is one with exactly n components (we elaborate later).

Example. *Continuing the previous example we can see what the form does on the standard basis vectors as above* $e_i = (\delta_i^j)_j$

$$(e_{n+1})^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} e_{n+1} = 0$$

so the first n *standard basis vectors give a basis for an* n *dimensional isotropic subspace, this is called the standard isotropic subspace. An* n *dimensional isotropic subspace is maximal and any flag of isotropic subspaces can be considered as a sub-flag of a complete flag of a maximal isotropic subspace (by inspecting the matrix of the form, if your space has a dimension of greater than* n *then it cannot be isotropic) [Bui, §8].*

The action of Sp preserves isotropic flags i.e. it sends an isotropic flag to an isotropic flag.

Stabilisers of isotropic flags give parabolics of Sp and moreover all parabolics arise in this way (see the above exercises in Springer).

Example. *Now we have a maximal isotropic flag we can consider sub-flags and find their stabiliser. Lets look at* Sp_4 *and the following flag*

$$0 \subset Fe_1^+ \subset Fe_1^+ \oplus Fe_2^+ \subset F^4$$

Re-use our GL *computations to see that the stabiliser is matrices in* Sp *of the form*

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & x \\ & & * & * \end{pmatrix}$$

Because these are in Sp *we can find more relations among the entries than in the* GL *case, we will pursue this further in the maximal case below, but to placate some we remark that* $x = 0$ *and this emphasises the duality between the upper left block and bottom right block.*

In particular maximal parabolics of Sp are stabilizers of *minimal* (non-trivial flags), i.e. stabilisers of non-zero isotropic subspaces.

$$0 \subset V_\ell \subset V$$

where $V_\ell = span_F(e_1, ..., e_\ell)$. Then the stabilizer is

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$

with the sizes of the diagonal blocks being (these numbers square)

$$\begin{pmatrix} \ell & * & * & * \\ 0 & n-\ell & * & * \\ 0 & * & \ell & * \\ 0 & * & * & n-\ell \end{pmatrix}$$

these sizes clearly determine the sizes of the rest of the matrix. This has Levi

$$\begin{pmatrix} A & & & \\ & a & & b \\ & & (A^T)^{-1} & \\ & c & & d \end{pmatrix}, \quad A \in \mathrm{GL}_\ell(F), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2(n-\ell)}(F)$$

and unipotent

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & \\ & & 1 & \\ & & * & 1 \end{pmatrix}$$

with relations among the entries.

1.2.2 Borel and Torus

One may find it helpful to understand these subgroups to understand the analogous story for Lie groups and their classification [Hal15], however it is not necessary.

Definition 1.3. A split torus is an algebraic group that is isomorphic to GL_1^b for some $b \in \mathbb{N}$.

Example (Bad Etymology)**.** GL_1^2 *is a split torus. Notice that*

$$\mathrm{GL}_1^2(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$$

is isomorphic as abstract groups to $U(1) \times U(1)$ *which when* $U(1)$ *is realised as* $\{z \in \mathbb{C} : |z| = 1\}$ *is topologically equivalent to* $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ *which is a torus. Note that it is clear that*

$$\mathrm{GL}_1^2(\mathbb{C}) \ncong \mathbb{T}^2$$

as topological groups, as the right had side is compact whilst the left is not.

It was pointed out to me that as algebraic groups over $\mathbb{C}$ *we have the following isomorphism*

$$\mathrm{GL}_1 \cong U(1) \otimes_{\mathbb{R}} \mathbb{C}$$

Proof

Perhaps a more compelling reason to call these Tori is that they play the same role in the classification as the genuine tori, $\mathbb{T}^2$, in the theory of Lie groups.

example maybe idk

A subgroup that is isomorphic to a split torus and is maximal in this respect is called a maximal split torus.

Example. *The classic example of a maximal split torus is the diagonal matricies in* GL_n*.*

Definition 1.4. A Borel is a maximal closed solvable connected subgroup of G.

Example. *The standard Borel of* GL_n *is the one given by upper triangular matrices. If* n *is even and one intersects this with* $\mathrm{Sp}_{2(\frac{1}{2}n)}$ *then we get the standard Borel of* $\mathrm{Sp}_{2(\frac{1}{2}n)}$*.*

Lets prove this in GL_2 *and then beleive that the only complication to going to larger* n *is keeping track of indecies. So let*

$$B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$$

we need to show that the derived series terminates for it to be solvable. So let

$$g = \begin{pmatrix} x & y \\ & z \end{pmatrix}, \quad h = \begin{pmatrix} a & b \\ & c \end{pmatrix}$$

be arbitrary in GL_2*, there commutator is then*

$$g^{-1}h^{-1}gh = (hg)^{-1}gh = \begin{pmatrix} ax & ay+cz \\ & cz \end{pmatrix}^{-1} \begin{pmatrix} ax & bx+cz \\ & cz \end{pmatrix} = \begin{pmatrix} 1 & \frac{bx-ay}{ax} \\ & 1 \end{pmatrix}$$

Hence

$$[B,B] = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$

Commutate two arbitrary elements again

$$\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x+a \\ & 1 \end{pmatrix} = 1$$

Hence

$$[[B,B],[B,B]] = 1$$

For the other conditions, it is clear that this is a closed subgroup because it is itself a linear algebraic group, moreover for LAG we have the algebraic criterion of connectedness given by having the only idempotents in the representing algebra being $0,1$ *[GH24, 1.5]. Because* $B = \operatorname{Spec}\mathbb{Z}[x_{i,j} : 1 \le i,j \le 2][y]/(\det(x_{ij})y-1, x_{2,1})$ *we can look for idempotents in* $\mathbb{Z}[x_{i,j} : 1 \le i,j \le 2][y]/(\det(x_{ij})y-1, x_{2,1})$*, clearly* 0 *is idempotent, so assume that* $\begin{pmatrix} a & b \\ & c \end{pmatrix} \neq 0$ *then we have that*

$$\begin{pmatrix} a & b \\ & c \end{pmatrix}^2 = \begin{pmatrix} a^2 & ab+bc \\ & c^2 \end{pmatrix} = \begin{pmatrix} a & b \\ & c \end{pmatrix} \in B(\mathbb{Z})$$

so $a^2 = a, c^2 = c$ *implies that* $a = c = 1$ *and* $ab+bc = 2b = b$ *implies that* $b = 0$*, hence* $\begin{pmatrix} a & b \\ & c \end{pmatrix} = 1$*. Thus this group is connected.*

Finally it is clear that if a subgroup contains this one then it is in fact all of GL_2 *and hence this is maximal. Therefore this is a Borel.*

put a proof of something being a Borel

A Borel can be considered to be a parabolic that is minimal with respect to inclusion. The maximal tori then form the Levis of these parabolics. In particular for a Borel B we have that

$$B = TU$$

for a maximal torus T and unipotent U.

We saw that fixing a minimal parabolic is like fixing a basis in the case of classifying parabolics of classical groups. If a Borel B is fixed, then a parabolic containing this Borel $B \subseteq P$ is called standard, the unique Levi of a standard parabolic containing this Borel is called the standard Levi.

1.2.3 Maximal Compact Subgroups

Let our group be defined over the global field k

fix a global convention

. We will often need to fix a maximal compact subgroup $K \subseteq G(\mathbb{A})$. These are not unique and as such when fixing one it can be arranged to have many nice properties [MW95, I.1.4]. In particular if we have a group G and a fixed Borel B:

- First require that
$$K = \prod_{\nu} K_\nu$$
where the product is over all places of k and $K_\nu \subseteq k_\nu$ is maximal compact.

But is the converse true?

- For almost all places ν of k $G(\mathcal{O}_{k_\nu})$ is defined and is maximal compact in $G(k_\nu)$ hence we can require $K_\nu = G(\mathcal{O}_{k_\nu})$ at these places.
- We require
$$G(\mathbb{A}) = B(\mathbb{A})K$$
- For every standard parabolic $P = MU$ we have that
$$P(\mathbb{A}) \cap K = \Big(M(\mathbb{A}) \cap K\Big)\Big(U(\mathbb{A}) \cap K\Big)$$
and $M(\mathbb{A}) \cap K$ is a maximal compact subgroup of $M(\mathbb{A})$.

It is in terms of the third property that we like to think of the maximal compact subgroup, it is the complimentary piece of the Borel. Moreover the fourth property should be thought of as a condition that the maximal compact subgroups are well behaved with the way that we are moving between the bigger an smaller reductive groups.

Example ($\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$)**.** *It is a classical result that the maximal compact subgroup of* $\mathrm{GL}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{Q}_\infty)$ *is the orthogonal group* $O(\mathbb{R})$. *By [Ser92, II.IV.A1] we have that* $\mathrm{GL}_n(\mathbb{Q}_p)$ *for* $p < \infty$ *is* $\mathrm{GL}_n(\mathbb{Z}_p)$ *and hence the maximal compact subgroup of* $\mathrm{GL}_n(\mathbb{A})$ *is the product*

$$K = O(\mathbb{R}) \times \prod_p \mathrm{GL}_n(\mathbb{Z}_p) = O(\mathbb{R}) \times \mathrm{GL}_n(\hat{\mathbb{Z}})$$

1.3 Lie Algebras

- Define the algebraic Lie algebra
- Give as tight a relation between $Lie(G(\mathbb{C}))$ and $\mathfrak{g}_\mathbb{C}$ as we can
- State the examples of the classical groups

1.4 Metaplectic Covers

Chapter 2

Automorphic Forms

There are different definitions of the words automorphic forms floating around, here we fix a nice one and then explain how they generalise the classical modular forms. Although this has been done in several places, none suit my taste to a tee, and so we briefly present our own take on this. We intend to be terse as this material is somewhat standard.

references

2.1 Definition and Role

The story starts with the classical modular forms, or functions on the upper half plane that satisfy some invariance conditions and differential equations. This evolves into the notions of Maas form on symmetric spaces and eventually reaches its apotheosis in the concept of automorphic form that we will present here.

reference or more detail for the history

We still do not have a good answer as to why the definition below is "the right" definition, from a mathematical perspective, as there are many places in which it could be extended or restricted and we are unable to motivate why one shouldnt consider such things. Indeed there are varying notions of automorphic form that appear for this reason and I think it is important to stress that this is "the right definition" only in so far as people have been able to prove nice theorems about them, and that when functions appear "in nature" this concept has sufficed to encompass and explain their behaviour. It is the represntation theoretic properties more than anything that suggest the current definition as is mentioned in [BC79, 1.II.3].

For concreteness we want to provide one, certainly a-historical, reason to be interested in them. It was proved by Jacquet-Langlands that all representations of reductive groups appear as subquotients of the regular represtntation on a space of automorphic forms:

Theorem ([GH24] 4.9, 8.3.5, 10.6.1)**.** *Let G be a connected reductive LAG over a field F.*

actually check hypothesis on G here

All smooth irreducible representations of $G(F)$ are a subquotient of a parabolically induced representation from a Levi. Any irreducible subquotient of such a parabolically induced representation is an automorphic representation and moreover all automorphic representations appear in this way.

Thus if one is interested in reductive group representations then automorphic representations and hence automorphic forms are unavoidable.

We will present three notions of automorphic form here. In the literature they are all called "automorphic forms" however here we will distinguish them with our own terminology for clarity.

2.1.1 Archimedian Automorphic Form

Fix a global (number) field F. Let ν be an archimedian place and let ∞ denote the set of archimedian places. Then F_ν is either $\mathbb{R}$ or $\mathbb{C}$. In particular (the analytification of) $G(F_\nu)$ is a Lie group and we call a function, $\varphi : G(F_\nu) \to \mathbb{C}$, *smooth* if it is smooth in the sense of manifolds.

Now we fix an embedding $\imath : G \to GL_n$ which gives another embedding $G \to SL_{2n}$ via

$$g \mapsto \begin{pmatrix} \imath(g) & \\ & (\imath(g))^{-t} \end{pmatrix}$$

A function $\varphi : G(F_\infty) = G(\prod_{v|\infty} F_\nu) \to \mathbb{C}$ is of *moderate growth* if there are constants $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$|\varphi(g)| \leq c\|g\|^r = c \left(\prod_{v|\infty} \sup_{1 \leq i,j \leq 2n} |\imath(g)_{i,j,\nu}|_\nu \right)^r$$

this is taking the maximum of the $2n \times 2n \times |\infty|$ three dimensional matrix.

there is a nice characterisation over Q for representations and norms as traces of blah in Borel. I like it a lot just need to check its fine for other number fields.

From

reference the Lie algebra section

we know how to define the Lie algebra of G and we now denote $Z(\mathfrak{g})$ the center of the *universal enveloping algebra* of the *complexification* of $\mathfrak{g}$, it would be more reasonable to use $Z(\mathcal{U}(\mathfrak{g}_\mathbb{C}))$ but that is too cumbersome so we follow the tradition. A vector in a $Z(\mathfrak{g})$ module $\varphi \in V$ is called $Z(\mathfrak{g})$-*finite* if the space $Z(\mathfrak{g})\varphi$ is finite dimensional.

Let $K_\infty \leq G(F_\infty)$ be a maximal compact subgroup. Then again an element of a K_∞ module is K_∞ finite if its orbit is a finite dimensional

vector space over what??

space.

To define automorphic forms we look at the representation $C^\infty(F_\infty)$ with the right regular action. In particular the $Z(\mathfrak{g})$ module structure is induced from the action of $\mathfrak{g}$ on $C^\infty(G(F_\infty))$ by

$$z.F(g) = \frac{\partial}{\partial t} F(ge^{tz})$$

Definition. *Let* $\Gamma \leq G(F_\infty)$ *some arithmetic subgroup, an automorphic form for* Γ *is a smooth function of moderate growth*

$$\varphi : G(F_\infty) \to \mathbb{C}$$

that is K_∞ *and* $Z(\mathfrak{g})$ *finite with a (left)* Γ *invariance.*

Borel 1.6 gives a good explination of the growth condition.

2.1.2 Adelic Automorphic Form

Here we follow [MW95, I.2.17]. Let G be a reductive group over F, we fix a Borel B and a standard parabolic P with a standard Levi decomposition $P = MU$. We let K be a maximal compact subgroup of $G(\mathbb{A})$ satisfying the conditions laid out in

reference the previous section

Definition 2.1. A function $\varphi : U(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is an automorphic form if it is smooth, moderate growth, $Z(\mathfrak{g})$ and K finite.

Remark 2.2. It is important that $M(F)$ is treated as a subgroup of $M(\mathbb{A})$ via the diagonal embedding.

For $v \notin \infty$ a non-archimedian place then we say that a function $f : G(F_\nu) \to \mathbb{C}$ is smooth if it is locally constant in the induced topology on $G(F_\nu)$, the details of this topology are spelled out in [Con12]. The set of such smooth functions is denoted $C^\infty(G(F_\nu))$.

For the non-archimedian places we define smooth functions on the "finite adeles" $\mathbb{A}_f$ as

$$C^\infty(\mathbb{A}_f) \quad := \quad \bigotimes_{\nu \notin \infty}{}' C^\infty(G(F_\nu))$$

And for the archimedian places we define

$$C^\infty(G(F_\infty)) \quad := \quad C^\infty\left(\prod_{\nu | \infty} G(F_\nu)\right)$$

For the full Adele we define

$$C^\infty(\mathbb{A}_F) \quad := \quad C^\infty(G(F_\infty)) \otimes C^\infty(G(\mathbb{A}_f))$$

A function on the adeles is smooth if it is in this set. Notice that a priori the codomain is an infinite tensor product over $\mathbb{C}$ of copies of $\mathbb{C}$, which is isomorphic to $\mathbb{C}$. Thus we can conflate a smooth function with its composition along this isomorphism, and think of them as functions into $\mathbb{C}$.

The action of the center of the Lie algebra is given by linearly extending the following to the whole space

$$z.(f \otimes g) = (z.f) \otimes g$$

i.e. it acts on the archimedian places as in the setting of archimedian automorphic forms.

The definition of moderate growth carries over verbatim, however we change the set of places multiplied over to be all of them now.

Remark 2.3 ([BC79], 1.II.3). The collection of moderate growth functions is independent of the choices of embedding.

2.2 Modular Forms

One might ask if there is a special case in which automorphic forms yeild modular forms. In fact no, the space of automorphic forms is larger than just modular forms, however it gives the space of Maas forms (or modular and Maas forms, depending on convention). This is well covered in the litterature [Eme][Bum97, 3.2][Boo][Gar16], but so essential to intuiting automorphic forms that we feel it is necissary to present the details here. To be clear we explain modular forms as archimedian automorphic forms as we think it is where the connection is clearest.

Recall the definition of a modular form

Definition ([DS05] 1.1.2)**.** *A function*

$$\varphi : \mathcal{H} \to \mathbb{C},$$

where $\mathcal{H}$ is the upper half plane in $\mathbb{C}$, that is holomorphic, satisfies

$$\varphi(\gamma.z) = (cz+d)^k \varphi(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and extends holomorphically to ∞ is called a modular form of weight k.

These are modular forms with trivial character and full level.

Now give a function on a set X and an action of a group G on X, there is a general way of associating to $\mathrm{Hom}(X, Y)$ a family of maps $\mathrm{Hom}(G, Y)$ indexed by X. This is a manifestation of the tensor-hom adjunction. Effectively if $f : X \to Y$ the we get a map for each $x \in X$ defined on $f_x : G \to Y$ given by $g \mapsto f(g.x)$.

So for our purposes we are trying to take some subset of functions $\mathcal{H} \to \mathbb{C}$ and shift their domain to the $\mathbb{Q}_\infty = \mathbb{R}$ points of some reductive group. In particular it would be sufficient to find a reductive group with a well defined action on the upper half plane and in particular we would want to quotient to be transitive.

Theorem.

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R})$$

as topological spaces.

Proof. Consider the action

$$\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathcal{H} : \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$$

Then look at the orbit of i, namely

$$\begin{pmatrix} a & b \\ & d \end{pmatrix}.i = \frac{ai+b}{d} = a^2 i + ab$$

which letting $a, b \in \mathbb{R}$ vary is clearly surjective onto the whole upper half plane. So there is one orbit, and hence by the orbit stabiliser we know that

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/stab(i)$$

so we want to find

$$stab(i) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : g.i = i \right\}$$

in particular we solve

$$\begin{aligned} i &= g.i \\ &= \frac{ai+b}{ci+d} \\ &= (c^2+d^2)^{-1}(ai+b)(d-ci) \\ &= (c^2+d^2)^{-1}(ac+bd+i\det(g)) \end{aligned}$$

So equating coefficients we have

$$\det g(c^2+d^2)^{-1} = 1 \implies c^2+d^2 = \det g = 1$$

on the other hand

$$ac + bd = 0$$

Now the pairs $c^2+d^2 = \det g$ are parametrised by $\theta \in [0, 2\pi)$ using $c = \sin\theta, d = \cos\theta$ hence subbing this into the above equation

$$\frac{-b}{a} = \tan\theta$$

and so $b = -k\sin\theta, a = k\cos\theta$ for some $k \in \mathbb{R}$ but the determinant must be 1 so $k = 1$. Hence

$$stab(i) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\} = SO_2(\mathbb{R})$$

One then has to check that this is all continuous.

Remark 2.4. Sometimes for

what reasons...

this is exhibited as

$$\mathcal{H} \cong \mathrm{GL}_2^+(\mathbb{R})/A_{\mathrm{GL}_2} SO_2(\mathbb{R})$$

this obscures the connection with the reductive group setting however becuase it is not obvious that

and probably not even true

Lemma. GL_2^+ *is a reductive group over* $\mathbb{Q}$ *.*

Proof. It is the connected component of the identity and therefore a closed subgroup.

references for these bold claims?

Therefore by Matsushima's criterion ([Arz05] for references) we have that $\mathrm{GL}_2 / \mathrm{GL}_2^+$ is affine iff GL_2^+ is reductive. But the thing on the left is the constant group scheme $\mathbb{Z}/2\mathbb{Z}$ which is affine.

SL_2 is a reductive group and $SO_2(\mathbb{R})$ is its maximal compact subgroup. This decomposition of the upperhalf plane suggests that function on it might have some invariance along the maximal compact subgroup of the reductive group SL_2. Indeed if we were to push our modular forms along this isomorphism it would, with the construction that we outlined earlier in terms of a group action on a set, excibit this invariance. This is merely *evidence* that if we were to change our modular forms to functions on the reductive group SL_2 they may preserve *some* of that invariance and indeed be K-finite.

$$\begin{tikzcd}
\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} SO_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \arrow[r, "\text{project}"] \arrow[dr, "\text{descend}"'] & \mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R}) \arrow[r, "\sim", "x \mapsto x.i"'] & \mathcal{H} \\
& \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R}) &
\end{tikzcd}$$

Using something like the universal property of the quotient we can lift a function on $\mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R})$ to $\mathrm{SL}_2(\mathbb{R})$ however this is not $\mathrm{SL}_2(\mathbb{Z})$ invariant, thus we need to add a prefactor to ensure this in our associated automorphic form. The algebro-geometric perspective in [Eme] can make this seem slightly less ad hoc.

Finally for f be a modular form of weight k then we associate the following function on $\mathrm{SL}_2(\mathbb{R})$

$$F(g) \quad := \quad (ci+d)^{-k} f(g.i)$$

We take for granted its smoothness. The $\mathrm{SL}_2(\mathbb{Z})$ invariance is obvious from the modularity condition.

It remains to show the three other properties:

Lemma. *$F(g)$ is of moderate growth.*

Proof. Unraveling the definitions we require two constants such that

$$|F(g)| = |ci+d|^{-k} |f(g.i)| \leq c(\sup_{i,j}(g, g^{-1}))^r$$

A direct computation shows that

$$Im(g.i) = |ci+d|^{-2}$$

hence we require to show

$$\mathrm{Im}(g.i)^{k/2} |f(g.i)| \leq c(\sup_{i,j}(g, g^{-1}))^r$$

Somehow invoke polynomial growth...? but the modularity condition has the growth condition that $\lim_{x \to \infty} f(xi)$ be bounded.

Lemma. *$SO_2(\mathbb{R})$ is a maximal compact subgroup inside $\mathrm{SL}_2(\mathbb{R})$. F is an $SO_2(\mathbb{R})$ finite function.*

Proof. First take $\kappa = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K = SO_2(\mathbb{R})$ then

$$\kappa.i = \frac{i\cos\theta - \sin\theta}{i\sin\theta + \cos\theta} = \frac{-i(-\cos\theta - i\sin\theta)}{e^{i\theta}} = i$$

hence for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ we have that

$$\begin{aligned}
F(g\kappa) &= ((c\cos\theta + d\sin\theta)i - c\sin\theta + d\cos\theta)^{-k} f(g.\kappa.i) \\
&= ((c\cos\theta + d\sin\theta)i - c\sin\theta + d\cos\theta)^{-k} f(gi) \\
&= (ci\cos\theta - c\sin\theta + d\cos\theta + di\sin\theta)^{-k} f(gi) \\
&= (-i^2(ci\cos\theta - c\sin\theta) + de^{i\theta})^{-k} f(gi) \\
&= (ice^{i\theta} + de^{i\theta})^{-k} f(gi) \\
&= (ic + d)^{-k} e^{-ik\theta} f(gi) \\
&= e^{-ik\theta} F(g)
\end{aligned}$$

Now this shows that $F(g)$ is acted on by K via a one dimensional irreducible representation. In particular it is finite dimensional.

Lemma. *F is a $Z(\mathfrak{sl}_2)$ finite function.*

Proof. Only a sketch.

The center of the universal enveloping algebra of the complexified Lie algebra is generated by the Casimir operators. From [Gar10] we know that the casimir is

$$\Omega = \frac{1}{2}H^2 + XY + YX$$

we have the coordinates on $\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} SO_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ from [Bum97][1.19 pg 139] in which the casimir acts as the differential operator

$$\Delta = y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) - y\frac{\partial^2}{\partial x \partial\theta}$$

[Bum97][1.29 pg 143 ,Prop 2.2.5 pg 155]. Now we claim that F is an eigenfunction for this operator. An element $(x, y, \theta) := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \kappa_\theta \in \mathrm{SL}_2(\mathbb{R})$ acts on i by sending it to $x + iy$ (elementary computation). The bottom row of the product is $y^{-1/2}\sin\theta; y^{-1/2}\cos\theta$ which results in

$$F(x, y, \theta) = y^{k/2} e^{-ik\theta} f(x + iy)$$

It is then a calculus exercise to apply Δ to this, using the holomorphicity we also get that

$f_{xx} - f_{yy} = 0$ and $f_y = i f_x$ which cancels away terms and we get that

$$\Delta F(x, y, \theta) = \frac{k}{2}\left(\frac{k}{2} - 1\right) F(x, y, \theta)$$

Therefore the dimension of $Z(\mathfrak{g})F$ is simply one.

this example makes it clear that the two finiteness conditions for automorphic forms are in some sense functional equations that they must satisfy.

Chapter 3

Automorphic Representations

The references that will be most helpful are [BC79, I.II][GH24] for the general theory, we will follow the notation developed in [MW95] as it is somewhat standard. We will discuss some of the details of their representation theory because it is both subtle and needed later. In particular we want to draw attention to what we think of as the "non-algebraic" nature of the representation theory.

3.1 $(\mathfrak{g}, K)$-Modules

3.2 Hecke Algebra

3.3 Automorphic Representations

3.3.1 Cuspidal Representations

e definition in the

We will recal the definition of a cusp form in the next chapter.

xample of isotypic

3.3.2 Tensor Products of Representations

Boxed Tensors

3.4 Eisenstein Series

[Lap22], [Art79]

3.5 Spectral Decomposition

3.5.1 Definition and Role

This is another one of the tools that can be used to compartmentalise problems in automorphic forms, by dealing with representations that appear in different parts of the spectrum.

give shahidis conjecture on plancherel measures some time. Make sure to talk about his proof based on a reasonable hypothesis.

3.5.2 The Decomposition of the Spectrum

3.5.3 Residual Representations of GL_n

3.6 L-Functions

3.6.1 In General

3.6.2 Standard L-Functions for Classical Groups

3.6.3 L-Functions of Covering Groups

Chapter 4

Constant Terms

Here we will explain the role of the constant term in our calculation of poles.

4.1 Definition and Role

The constant term is an operation defined on a large class of functions and is supposed to generalise the constant term of a fourier expansion, we will see this later, although one may consult [Bum97, 1.6] for some examples as well. In particular [MW95, I.2.6] give the definition as follows: We consider $P = MU$ a standard parabolic of G and $\varphi : U(k) \setminus \mathbf{G} \to \mathbb{C}$ a measurable and locally L^1 function then its constant term along P is

$$\varphi_P : U(\mathbb{A}) \setminus \mathbf{G} \to \mathbb{C}$$

$$\varphi_P(g) \quad := \quad \int_{U(k)\setminus U(\mathbb{A})} \varphi(ug)\mathrm{d}u$$

which inherits many of the properties of φ such as smoothness and moderate growth. For instance if φ is an automorphic form on $\mathbf{G}$ then its constant term is an automorphic form on M [GH24, 6.5].

This allows us to recal the definition of cuspidal automorphic forms or "cusp forms". Let φ be a an automorphic form on $U(\mathbb{A})M(k) \setminus \mathbf{G}$ for $P = MU$ a standard parabolic. Then φ is cuspidal if for all standard parabolics $P' \subset P$ we have that

$$\varphi_{P'} = 0$$

Cusp forms have a central role in the theory of automorphic forms, this is for several reasons. They appear historically as interesting examples such as the Ramanujan tau function, by a theorem of Ribet [SZS77, T2.3] the Galois representation associated to a cusp form is irreducible and finally they just make formulas easier to work with.

The constant term itself is of central importance. One reason for this is that it controls the growth of the automorphic forms. More precisely we have the following theorem [MW95, I.4.10]

Theorem. *Let $P = MU$ be a standard parabolic of G, $V_P \subseteq \mathcal{A}_0(U(\mathbb{A})M(k) \setminus G)$ a finite dimensional subspace, $\Gamma_P \subseteq X_M$ a compact subset and N_P an integer. Let $n \in \mathbb{Z}$, $D \subseteq \mathbb{C}^n$ open and* *connected, $f : D \to \mathbb{C}$ holomorphic and not identically zero. Let $D' = \{x \in D : f(x) \neq 0\}$.*

If $\varphi : D' \to L^2_{loc}(G)$ is a function such that for all $z \in D', \varphi(z) \in \mathcal{A}((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G})$ and for all P

$$\psi_P : D' \to L^2_{loc}(G)$$

$$z \mapsto \psi_P(z) = \varphi_P^{cusp}(z)$$

is a holomorphic function.

Then φ can be analytically continued to a holomorphic function on D iff ψ_P can be continued to a holomorphic function on D.

Unpacking this a bit we see that an Eisenstein series satisfies the hypothesis on φ by [MW95, IV.1.9]

make this more precise

, we think of D' as the positive open cone on which the Eisenstein series conveges, and then $D = \mathbb{C}^n \setminus S$ as the rest of the cuspidal datum $\mathfrak{P}$ on which the Eisenstein series is holomorphic (S is its set of singularities). Therefore the domain on which the Eisenstein series is holomorphic is the same as the domain on which (roughly) its constant term is holomorphic.

4.2 Integration Lemmas

Theorem. *If G is a locally compact Hausdorff group with a left Haar measure μ and if $\chi\colon G \to \mathbf{C}^\times$ is a non-trivial character on G, then*

$$\int_G \chi(g)\, d\mu(g) = 0.$$

Proof. Pick an element h of G such that $\chi(h) \neq 1$. The equation above then follows from

$$\int_G \chi(g)\, d\mu(g) = \int_G \chi(hg)\, d\mu(g) = \int_G \chi(h)\chi(g)\, d\mu(g) = \chi(h) \int_G \chi(g)\, d\mu(g).\square$$

Integrating trivial characters gives the volume of the measure space which we typically normalize to be one.

Theorem ([Gar] 5.2, [Fol16] Thm 2.49)**.** *Let $H \leq G$ be a closed subgroup. If $H \setminus G$ has a right G invariant measure (iff their modular functions agree on H)then the integral is unique up to scalar, namely for a given Haar measures dh on H and dg on G there is a unique invariant measure dq on $H \setminus G$ such that for all $f \in C^0_c(G)$*

$$\int_{H\setminus G} \int_H f(hq) dh dq = \int_G f(g) dg$$

Note that this quotient may not be a group, becuase H is not required to be normal.

4.3 Constant Terms of Eisenstein Series

This computation forms the heart of a well known theorem, [GH24, Prop 10.4.2][MW95, II.1.7][Sha10, 6.2]. I give more detail here then I could find anywhere else.

Notice that the Eisenstein series has a full $G(k)$ invariance and so we can take its constant terms along *any* standard parabolic.

is it true

4.3.1 In General

We will use the following Lemmas to give a simplified expression of the constant term of an Eisenstein series. First fix $P = MN$ and $P' = M'N'$ two standard parabolics of suitable group G over F, with $E(x, \varphi, \lambda)$ defined via parabolic induction from P.

Lemma.
$$P(F) \setminus G(F) \cong \coprod_{w \in W_{M'} \setminus W_G / W_M} P'(F) \cap wP(F)w^{-1} \setminus P'(F)$$

Proof. Consider the Bruhat decomposition:

$$G(F) = \coprod_{w \in W_{M'} \setminus W_G / W_M} P(F)w^{-1}P'(F)$$

then becuase the action of $P(F)$ keeps the disjoint sets disjoint we can move the quotient through and get
$$P(F) \setminus G(F) = \coprod_w P(F) \setminus P(F)w^{-1}P'(F)$$

so we analyse the summands, by the second isomorphism theorem we have a bijection

$$P(F) \setminus P(F)w^{-1}P'(F) \cong P(F) \cap P'(F) \setminus w^{-1}P'(F)$$

now if $[w^{-1}p] \in P(F) \cap P'(F) \setminus w^{-1}P'(F)$ then its $pw^{-1}p'$ for some $p \in P(F) \cap P'(F)$ and hence multiplying by w, in particular an isomorphism, gives $wpw^{-1}p' \in wP(F)w^{-1} \times P'(F)$ and so

$$w(P(F) \cap P'(F) \setminus w^{-1}P'(F)) \cong wP(F)w^{-1} \cap P'(F) \setminus P'(F)$$

Lemma. *Let $m', n' \in M'(F) \times N'(F)$ then*

$$m'n' \in wP(F)w^{-1} \iff m' \in wP(F)w^{-1} \text{ and } n' \in (m')^{-1}wP(F)w^{-1}m'$$

Proof. The forward implication is stated in [GH24], the converse follows from some algebra: First let $m' = wp_1w^{-1}$ and $n' = (m')^{-1}wp_2w^{-1}m'$ then

$$\begin{aligned} m'n' &= (wp_1w^{-1})^{-1}wp_2w^{-1}wp_1w^{-1} \\ &= wp_1^{-1}w^{-1}wp_2w^{-1}wp_1w^{-1} \\ &= wp_1^{-1}p_2p_1w^{-1} \in wP(F)w^{-1} \end{aligned}$$

Taking the contrapositive of this lemma will be used below. This is becuase our sums will be over quotients like $A \setminus B$ and therefore summing over the "elements" in B that are not in A; by our lemma would be the same as summing over two different such quotients. Now consider the computation:

I have lost the lambda in the below computation

$$\begin{aligned}
E_{P'}(x,\varphi,\lambda) &= \int_{N'(F)\setminus N'(\mathbb{A})} E(nx,\varphi,\lambda)dn \\
([N'] := N'(F)\setminus N'(\mathbb{A})) \quad &= \int_{[N']} \sum_{\delta\in P(F)\setminus G(F)} \varphi(\delta nx)dn \\
\text{(Lemma 1)} \quad &= \int_{[N']} \sum_{\delta\in\coprod_{w\in W_{M'}\setminus W_G/W_M} P'(F)\cap wP(F)w^{-1}\setminus P'(F)} \varphi(\delta nx)dn \\
&= \sum_{w\in W_{M'}\setminus W_G/W_M} \int_{[N']} \sum_{p'\in P'(F)\cap wP(F)w^{-1}\setminus P'(F)} \varphi(w^{-1}p'nx)dn \\
\text{(Lemma 2)} \quad &= \sum_{w} \sum_{m'\in M'(F)\cap wP(F)w^{-1}\setminus M'(F)} \int_{[N']} \sum_{n'\in N'(F)\cap(m')^{-1}wP(F)w^{-1}m'\setminus N'(F)} \varphi(w^{-1}m'n'nx)dn \\
\text{(Change Var)} \quad &= \sum_{w}\sum_{m'} \int_{[N']} \sum_{n'\in N'(F)\cap wP(F)w^{-1}\setminus N'(F)} \varphi(w^{-1}n'nm'x)dn \\
\text{(Unfold)} \quad &= \sum_{w}\sum_{m'} \int_{N'(F)\cap wP(F)w^{-1}\setminus N'(\mathbb{A})} \varphi(w^{-1}nm'x)dn
\end{aligned}$$

The change of variables is $(m',n') \mapsto ((m')^{-1}n'm', (m')^{-1}n'm')$. Again we assume that our x is sufficiently large so all the integrals converge.

4.3.2 Constant Terms of Cuspidal Eisenstein Series

Lemma (4)**.** *For $w \in W_{M'} \setminus W_G/W_M$ we have that $w^{-1}P'w \cap M$ is a standard parabolic of M with Levi $w^{-1}M'w \cap M$ and unipotent $w^{-1}N'w \cap M$.*

Proof. This is [GH24, 10.4.1] stated without proof. They give the reference [RS, V.4.6] which is in French. We will return to this if we have time / energy.

Lemma (5)**.**

$$w^{-1}U'w \cap P = (w^{-1}U'w \cap M)(w^{-1}U'w \cap U)$$

Proof. [GH24, 10.4.1] has some decompositions, as well as the standard decomposition of $P = MU$ I think I could prove this...

Lemma (6)**.**

$$c \setminus (b \setminus a) = (bc) \setminus a$$

Continuing the computation of the constant term above, we will focus purely on the inner integral now

$$\begin{aligned}\int_{N'(F)\cap wP(F)w^{-1}\backslash N'(\mathbb{A})}\varphi(w^{-1}nm'x)dn &= \int_{w^{-1}N'(F)w\cap P(F)\backslash w^{-1}N'(\mathbb{A})w}\varphi(nw^{-1}m'x)dn\\ \text{(Lemma 5)} &= \int_{(w^{-1}U'w\cap M)(w^{-1}U'w\cap U)(F)\backslash w^{-1}N'(\mathbb{A})w}\varphi(nw^{-1}m'x)dn\\ \text{(Unfold + Lemma 6)} &= \int_{(w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A}))\backslash A}\int_{w^{-1}U'(F)w\cap M(F)\backslash w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A})}\varphi(n_1n_2w^{-1}m'x)dn_1dn_2\end{aligned}$$

the first equality is the change of variables $w^{-1}nw \mapsto n$ and $A = (w^{-1}U'(F)w\cap U(F))\backslash w^{-1}N'(\mathbb{A})w$. Now look at the inner integral here more closely

$$\int_{w^{-1}U'(F)w\cap M(F)\backslash w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A})}\varphi(n_1n_2w^{-1}m'x)dn_1dn_2$$

applying Lemma (6) we see that this is a constant term for a parabolic of M, of the function $m \mapsto \varphi(mn_2w^{-1}m'x)$.

Lemma. *$n_2w^{-1}m'x \in K$ with variables as above.*

This was all in complete generality as well. If we now assume further that the Eisenstein series was induced from a *cuspidal* automorphic representation, then $m \mapsto \varphi(mk)$ is a cusp form and therefore this last integral will vanish whenever $w^{-1}U'w\cap M \neq \{1\}$, because in that case the inner integral doesnt exist (its over a point).

4.3.3 Constant Term Of Eisenstein Series for Conjugate Levis

If we now assume that $M' = wMw^{-1}$ and recall the definition of our intertwining operator

put this in the Eisenstein series section

we can use the following

Lemma ([MW95] II.1.7 (6))**.**

$$U'(k) \cap wP(k)w^{-1} = U'(k) \cap wU(k)w^{-1}$$

to see that

$$\begin{aligned}E_{P'}(x,\varphi,\lambda) &= \sum_w\sum_{m'}\int_{N'(F)\cap wP(F)w^{-1}\backslash N'(\mathbb{A})}\varphi(w^{-1}nm'x)dn\\ &= \sum_w\sum_{m'}\int_{N'(k)\cap wN(k)w^{-1}\backslash N'(\mathbb{A})}\varphi(w^{-1}nm'x)dn\\ &= \sum_w\sum_{m'}M(w,\pi)(\varphi)(x)\end{aligned}$$

I have mixed up my N's and U's too much...

In particular we can combine the conjugate and cuspidal cases to get a much simpler expression for some constant terms of some Eisenstein series.

4.4 Siegel Phi Operator

Here we give an example of the constant term which connects it to the classical picture. We thank Chengjing Zhang for showing us this example, and present it here because we cannot find it in the literature.

give references for the other interpretations of the constant term, there must be some

We deal only with the classical Siegel modular forms of full level and moreover are less explicit with the steps as they should be clear after exposure to the previous arguments.

Because we are trying to connect this to the classical picture it is most convenient to think of thinks in the archimedian places, recall the way that modular forms are automorphic forms most naturally in the archimedian sense ([GH24, 6.2]) [Eme][Bum97][Boo]. So for this section alone, by automorphic form we will mean automorphic forms on the archimedian places, and the constant term will be taken only on the archemedian part: i.e. for $f : G(\mathbb{R}) \to \mathbb{C}$ and automorphic its constant term along a parabolic of G, call it $P = MN$, is [GH24, 8.6]

$$f(x)_P = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} f(xn)\mathrm{d}n$$

We assume here for simplicity (and becuase it will apply to the examples below) that our groups are unimodular.

4.4.1 Siegel Modular Forms

We collect some definitions from [BVDGHZ08] to fix notation. Let the Siegal upper half plane be defined as

$$\begin{aligned} \mathcal{H}_g &:= \{\tau \in \mathrm{M}_{g\times g}(\mathbb{C}) : \tau \text{ is symetric and has positive definite imaginary part}\} \\ &\cong \mathrm{Sp}_{2g}(\mathbb{R})/U(g) \end{aligned}$$

where the isomorphism is as analytic manifolds and

$$U(g) \;\; := \;\; \left\{ \begin{pmatrix} A & B \\ -B & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}) : AA^t + BB^t = 1 \right\}$$

For every $\gamma = (A\ B;\ C\ D) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\tau \in \mathcal{H}_g$ we have the action

$$\gamma.\tau = (A\tau + B)(C\tau + D)^{-1}$$

We say that a holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ is a (classical) Siegel modular form of weight k if

$$f(\gamma.\tau) = \det(C\tau + D)^k f(\tau)$$

with the extra condition that if $g = 1$ it must be holomorphic at ∞. Becuase $\mathrm{Sp}_2 = \mathrm{SL}_2$ this is a strict generalisation of an (elliptic) modular form.

The space of Siegel modular forms of weight k and genus g is denoted $\mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$. There is a useful operator know as the Siegel Phi Operator which allows you to lift known modular forms from lower genus to higher genus [BVDGHZ08, 5]

$$\mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z})) \xrightarrow{\Phi} \mathcal{M}_k(\mathrm{Sp}_{2(g-1)}(\mathbb{Z}))$$

defined by the limit for $\tau \in \mathcal{H}_{g-1}$

$$\Phi(f)(\tau) \quad := \quad \lim_{t\to\infty} f\begin{pmatrix} \tau & \\ & it \end{pmatrix}$$

in this context a cusp form is defined to be a Siegel modular form in the kernel of the Siegel Φ operator and so it is natural to wonder if there is a constant term that is being taken here.

4.4.2 Automorphising

Given a Siegel modular form $f \in \mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ we can associate an automorphic form

$$\tilde{f} : \mathrm{Sp}_{2g}(\mathbb{R}) \to \mathbb{C}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \det(ci+d)^{-k} f\Big((ai+b)(ci+d)^{-1}\Big),$$

where a, b, c, d are $g \times g$ matrices such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{R})$.Fiz the Borel of upper triangular matricies. Now for $1 \le r \le g-1$ an integer we have the standard maximal parabolic of Sp_{2g}, $P_r = M_r N_r$ such that

$$M_r \cong \mathrm{GL}_r \times \mathrm{Sp}_{2(g-r)}$$

Theorem (Zhang). *If f is a classical Siegel modular form of weight k and degree g, then*

$$\tilde{f}_{P_r}(u\gamma) = \det u^k \cdot (\Phi^r f)^{\sim}(\gamma) \tag{4.4.1}$$

for every element γ of $\mathrm{Sp}_{2(g-r)}(\mathbb{R})$ and every element u of $\mathrm{GL}_r(\mathbb{R})$.

In particular

$$\tilde{f}_{P_{g-1}} \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\Phi f)^{\sim} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This shows that perhaps the correct generalisation of the Siegel Phi function is just the constant term that we all know and love. We could also attempt to expand this to Siegel modular forms that are vector valued or not of full level.

The only other work on generalising the Siegel Φ operator that we could find appears in [Gre24]. Grenier formulates the Φ operator in the language of symmetric spaces [**?**, Ch. 2] and then shows that the analogous definition in the case of "automorphic forms" in the sense of the symmetric space $\mathscr{P}_n/\,\mathrm{GL}_n(\mathbb{Z})$ of symmetric positive definitie real matricies [**?**, 1.5.1] behaves in the same way. Namely his [Gre92, Thm. 2] shows that it sends an automorphic form for $GL_n(\mathbb{Z})$ to an automorphic form for $\mathrm{GL}_{n-1}(\mathbb{Z})$. The point is that the Φ operator can be defined in the generality of symetric spaces and Grenier shows that at least in one other case it still preserves the relevant notion of automorphic form. This suggests two things that would be interesting to investigate; using the classification of symetric spaces is it possible to give a uniform definition of the Φ operator following Grenier and does this definition agree with the constant term in the way that the Siegel Φ operator does. With my limited knowledge of symmetric spaces this seems to be very tractable.

4.4.3 Base Case

The base case is very instructive, it deals with modular forms. So consider f a (elliptic) modular form of full level and weight k, which has a fourier expansion given by

$$f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$$

Then one can verify that

$$\tilde{f}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ci+d)^{-k} f\Big(\frac{ai+b}{ci+d}\Big)$$

is an automorphic form on Sp_2. The only non-trivial parabolic P is the one of upper triangular matricies, with Levi and unipotant given respectively

$$M = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \cong \mathrm{GL}_1, \quad N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cong \mathbb{G}_a$$

along which we can now compute the constant term

$$\begin{aligned} \tilde{f}_P(m) &= \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \tilde{f}(mb)\mathrm{d}b \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} \tilde{f}\begin{pmatrix} m & mb \\ 0 & m^{-1} \end{pmatrix} \mathrm{d}b \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} m^k f(m^2 i + m^2 b)\mathrm{d}b \\ &= m^k a_0 \end{aligned}$$

We have chosen normalisation to remove the usual factor of $1/2\pi$ in the constant term of the Fourier series. Moreover we see that

$$\Phi(f) = \lim_{t\to\infty} f(it) = \lim_{t\to\infty} \sum_{n\geq 0} a_n e^{-2\pi n t} = a_0$$

4.4.4 Simplifying the Constant Term

As we saw in

reference the maximal parabolic section

for $1 \leq r \leq g-1$ an integer we have the standard maximal parabolic of Sp_{2g}, $P_r = M_r N_r$ such that

$$M_r \cong \mathrm{GL}_r \times \mathrm{Sp}_{2(g-r)}$$

which can be given the explicit matrix representations

$$m(\gamma, A) \quad := \quad \begin{pmatrix} A & & & \\ & a & & b \\ & & (A^t)^{-1} & \\ & c & & d \end{pmatrix}, \quad A \in \mathrm{GL}_r(F),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2(g-r)}(F)$$

and unipotent

$$n(s; h, k) \quad := \quad \begin{pmatrix} 1 & 0 & 0 & h \\ -k^t & 1 & h^t & s + h^t k \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h, k \in \mathrm{Mat}_{(g-r)\times r}(\mathbb{R})\ s \in \mathrm{Sym}_r(\mathbb{R})$$

We have the following short exact sequence

prove it

$$1 \to \mathrm{Sym}_r(\mathbb{R}) \to N_r(\mathbb{R}) \to \mathrm{Mat}_{(g-r)\times r}(\mathbb{R}) \times \mathrm{Mat}_{(g-r)\times r}(\mathbb{R}) \to 1.$$

which we will use to unfold our integral below, for compacness we define $H_r \ := \ \mathrm{Mat}_{(g-r)\times r}$. We will now denote $[G] \ := \ G(\mathbb{Z})\backslash G(\mathbb{R})$ and compute the constant term

$$\begin{aligned} \tilde{f}_{P_r}\big(m(\gamma, A)\big) &= \int_{[N_r]} \tilde{f}\big(nm(\gamma, A)\big)\, dn \\ &= \int_{[H_r \times H_r]} \int_{[\mathrm{Sym}_{g-r}]} \tilde{f}\big(n(s; h, k)m(\gamma, A)\big)\, ds\, d(h, k) \\ &= \int_{[H_r]} \int_{[H_r]} \int_{[\mathrm{Sym}_{g-r}]} \tilde{f}\big(n(s; h, k)m(\gamma, A)\big)\, ds\, dh\, dk. \end{aligned} \tag{4.4.2}$$

Now we focus on simplyfing the integrand. We want an explicit form of the matrix so we can relate it back to the value of the un-lifted Siegel modular form f; simply multiply the matricies gives, where (all rings are commutative) $A^{-t} \ := \ (A^t)^{-1}$

$$n(s; h, k)m(\gamma, A) = \begin{pmatrix} a & 0 & b & hA^{-t} \\ -k^t a + h^t c & A & -k^t b + h^t d & sA^{-t} + h^t k A^{-t} \\ c & 0 & d & kA^{-t} \\ 0 & 0 & 0 & A^{-t} \end{pmatrix}.$$

because $a, b, c, d \in \mathrm{Mat}_{(g-r)\times(g-r)}, A \in \mathrm{Mat}_{r\times r}$ we see that the $g \times g$ blocks that we now need to take the determinant of are the 4×4 conrners of this picture, hence the matricies below should all be in $\mathcal{H}_g \subseteq \mathrm{Mat}_{g\times g}$

$$\tilde{f}(n(s; h, k)m(\gamma, A)) = \det\left(\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} i + \begin{pmatrix} d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}\right)^{-k}.$$

$$f\left(\left(\begin{pmatrix} a & 0 \\ -k^t a + h^t c & A \end{pmatrix} i + \begin{pmatrix} b & hA^{-t} \\ -k^t b + h^t d & sA^{-t} + h^t k A^{-t} \end{pmatrix}\right)\left(\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} i + \begin{pmatrix} d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}\right)^{-1}\right)$$
$$= \det\left(\begin{pmatrix} ic+d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}\right)^{-k}.$$
$$f\left(\begin{pmatrix} ia+b & hA^{-t} \\ -k^t(ia+b) + h^t(d+ic) & iA + sA^{-t} + h^t k A^{-t} \end{pmatrix}\begin{pmatrix} ic+d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}^{-1}\right)$$
$$= \left(\frac{\det(ic+d)}{\det(A)}\right)^{-k}.$$
$$f\left(\begin{pmatrix} ia+b & hA^{-t} \\ -k^t(ia+b) + h^t(d+ic) & iA + sA^{-t} + h^t k A^{-t} \end{pmatrix}\begin{pmatrix} (ci+d)^{-1} & -(ci+d)^{-1}k \\ 0 & A^t \end{pmatrix}\right)$$
$$= \left(\frac{\det(A)}{\det(ic+d)}\right)^{k} f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}, \quad \tau := (ai+b)(ci+d)^{-1}$$

So we have shown that

$$\tilde{f}_{P_r}(m(\gamma, A)) = \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} \left(\frac{\det(A)}{\det(ic+d)}\right)^{k} f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} ds\, dh\, dk$$
$$= \left(\frac{\det(A)}{\det(ic+d)}\right)^{k} \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} ds\, dh\, dk$$

Again lets focus on this integrand $f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}$ and compute its Fourier expansion, see [BVDGHZ08, 3.4]. Recall that a symmetric matrix $n \in \mathrm{GL}_g(\mathbb{Q})$ is called half integral if $2n$ is integral with even diagonal entries, then a Siegel modular form has a Fourier expansion of the form

$$f(z) = \sum_{n \text{ half integral}} a(n) e^{2\pi i \mathrm{Tr}(nz)}$$

First the space of half integral $g \times g$ matricies, HI_g, decomposes as a direct sum via the (additive) group isomorphism

prove it

$$\mathrm{HI}_{g-r} \oplus \tfrac{1}{2}\mathrm{Mat}_{r\times(g-r)}(\mathbb{Z}) \oplus \mathrm{HI}_r \to \mathrm{HI}_g, \qquad (n, m, l) \mapsto \begin{pmatrix} n & m \\ m^t & l \end{pmatrix},$$

thus unfolding the (discrete) integral we get

$$f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} = \sum_{n\in \mathrm{HI}_{g-r}} \sum_{m \in \frac{1}{2}\mathrm{Mat}_{r\times(g-r)}(\mathbb{Z})} \sum_{l \in \mathrm{HI}_r} a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}$$

$$\exp\left(2\pi i \mathrm{Tr}\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}\right)$$

because all the block sizes are compatible we can "block multiply" the inner matricies and because we are taking the trace we can forget about off diagonal entries

$$\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} = \begin{pmatrix} n\tau + m(-k^t\tau + h^t) & * \\ * & m^t(-\tau k + h) + l(k^t\tau k + AA^t i + s) \end{pmatrix}$$

putting this into our Fourier expansion

$$f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} = \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}$$
$$\exp\left(2\pi i(\mathrm{Tr}(n\tau) + \mathrm{Tr}(m(-k^t\tau + h^t)) + \mathrm{Tr}(m^t(-\tau k + h)) + \mathrm{Tr}(l(k^t\tau k + AA^t i + s))\right)$$

If we denote $T_l := \mathrm{Tr}(l(k^t\tau k + AA^t i + s))$ and

my Tm differs from Chengjing

$$T_m := \mathrm{Tr}(m(-k^t\tau+h^t))+\mathrm{Tr}(m^t(-\tau k+h)) = \mathrm{Tr}(-mk^t\tau-m^t\tau k)+\mathrm{Tr}(mh^t+m^t h) := T_{m,k}+T_{m,h}$$

we can sub this back into our constant term

Converges uniformly a priori on compact sets, well I dont know if I can swap all these sums haha

$$\begin{aligned}
\tilde{f}_{P_r}(m(\gamma, A)) &= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix} \exp\left(2\pi i(\mathrm{Tr}(n\tau) + T_m + T_l)\right) ds\, dh\, dk \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)} \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i(T_m+T_l)}\, ds\, dh\, dk \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)} \int_{[H_r]}\int_{[H_r]} e^{2\pi i T_m} \int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i T_l}\, ds\, dh\, dk \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)} \int_{[H_r]} e^{2\pi i T_{m,k}} \int_{[H_r]} e^{2\pi i T_{m,h}} \int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i T_l}\, ds\, dh\, dk
\end{aligned}$$

Now we use that integration unitary characters lemma

ref the lemma

and the fact that

$$s \mapsto e^{2\pi i T_l}$$

is a non-trivial unitary character of Sym_{g-r} whenerver $l \neq 0$ to get that

$$\int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i T_l}\, ds = \begin{cases} 1, & l = 0 \\ 0, & l \neq 0 \end{cases}$$

we repeat this trick with the second integral, which enforces that $m = 0$ and end up with

$$\tilde{f}_{P_r}\big(m(\gamma, A)\big) = \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_{n \in \mathrm{HI}_{g-r}} a\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}$$

but by [BVDGHZ08, 3.5] we know that the Fourier expansion of the Siegel Phi operator is

$$(\Phi^r f)(\tau) = \sum_{n \in \mathrm{HI}_{g-r}} a\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}.$$

hence

$$\begin{aligned} \tilde{f}_{P_r}\big(m(\gamma, A)\big) &= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \Phi^r(f)(\tau) \\ &= \det(A)^k (\Phi^r(f))^{\sim}(\gamma) \end{aligned}$$

which concludes the proof.

$\square$

Chapter 5

Poles of Residual Eisenstein Series

Our goal here is to exposit and survey the work in papers such as [Bre09][JLZ13], we were going to generalise some of these results to metaplectic covers however we found these results in [GS21].

5.1 Their Results

[Bre09] gave an analysis of the residual poles of Eisenstein series attached to Sp_{2n}, there were some minor errors that were corrected in [JLZ13] where they give esssentially the same proof; theirs however works for the other classical groups. For our purposes, the case of Sp_{2n}, as a group defined over F a number field, is most relevant, and we shall therefore focus exclusively on this case, however it should be noted that this limitation in the non-covering case is artificial, although it does simplify things a little by removing some casework, and we hope also in the covering case to be able to remove it in future work.

We fix an $n \in \mathbb{N}$ and the Borel of upper triangular matricies in Sp_{2n}, then we look at partitions of $n = r + m$, where

what are the ranges of these

. Then as we saw in

write this up in the classical group section

—- there corresponds a maximal standard parabolic of Sp_{2n}, which we denote $P_r = M_r N_r$, such that the Levi component is

$$\mathrm{GL}_r \times \mathrm{Sp}_{2m}$$

As we saw in

explain this in the Eisenstein series section

—- the space of characters $X_{M_r}^{\mathrm{Sp}_{2n}}$ is one dimensional by the maximality of P_r. If we look at the divisors of $r = ab$

check the exact ranges, will depend on the range of r

and fix a τ, an irreducible unitary cuspidal automorphic representation of GL_a, then from

link to the residual representation section

—- we know that $\Delta(\tau, b)$ is a residual representation of $\mathrm{GL}_{ab} = \mathrm{GL}_r$. Now we take an irreducible generic cuspidal automorphic representation σ of Sp_{2m}, and so their tensor product $\Delta(\tau, b) \otimes \sigma$

gives a representation of $\mathrm{GL}_r \times \mathrm{Sp}_{2m}$ and hence of the Levi M_r. We now consider the Eisenstein series attached to this representation, namely if

$$\varphi \in \mathcal{A}(N_r(\mathbb{A})M_r(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))_{\Delta(\tau,b)\otimes\sigma}$$

then we have the Eisenstein series

$$E(\varphi, s)(g) = \sum_{\gamma \in P_r(F)\setminus \mathrm{Sp}_{2n}(F)} s.\varphi(\gamma g)$$

explain the s action in Eisenstein series section.

for $g \in \mathrm{Sp}_{2n}(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A})$. Becuase it is induced from the residual representation $\Delta(\tau, b)$ we call these residual Eisenstein series. The main theorem is then a statement that after some normalisation the poles of this Eisenstein series are limited to a particular set and that they are simple. Details will be given later.

5.2 Our Results

We consider an almost identical setup but we deal with the metaplectic cover of Sp_{2n}, again over a number field F, Mp_{2n}

reference the section I discuss this in.

. We again fix the Borel of upper triangular matricies, consider partitions of $n = r + m$ and look at maximal standard parabolics of Sp_{2n}, $P_r = M_r N_r$ such that

$$M_r = \mathrm{GL}_r \times \mathrm{Sp}_{2m}$$

then if $r = ab$ we still have that $\Delta(\tau, b) \otimes \sigma$, for τ irreducible unitary cuspidal automorphic representation of GL_a and σ irreducible generic cuspidal automorphic representation of Sp_{2m}, is a representation of M_r. The difference is in the parabolic induction as we now consider

$$\varphi \in \mathcal{A}(N_r(\mathbb{A})M_r(F) \setminus \mathrm{Mp}_{2n}(\mathbb{A}))_{\Delta(\tau,b)\otimes\sigma}$$

and then the Eisenstein series is defined in the same way

$$E(\varphi, s)(g) = \sum_{\gamma \in P_r(F)\setminus \mathrm{Sp}_{2n}(F)} s.\varphi(\gamma g)$$

for $g \in \mathrm{Sp}_{2n}(F) \setminus \mathrm{Mp}_{2n}(\mathbb{A})$ and $s \in \mathbb{C} \cong X_{M_r}^{\mathrm{Mp}_{2n}}$.

Lemma. *When $b = 1$ we have the constant term*

$$E(\varphi, s)(g)_{P_a} = \varphi(g)_{P_a} + M(\omega, \tau \otimes \sigma)(\varphi)(g)$$

fill in here as theorems or whatever anything that I end up actually checking....

5.3 Setup

In the setup we used that $s \in \mathbb{C} \cong X_{M_r}^{\mathrm{Mp}_{2n}}$ the first step is to make sure that this is actually true

Lemma. *$X_{M_r}^{\mathrm{Mp}_{2n}(\mathbb{A})}$ is at most a one dimensional $\mathbb{C}$ vector space.*

Proof. First of all we have that [MW95, I.1.4]

$$X_{M_r}^{\mathrm{Mp}_{2n}(\mathbb{A})} \subseteq X_{M_r} \cong \mathfrak{a}_{M_r}^* \quad := \quad Rat(M_r) \otimes_{\mathbb{Z}} \mathbb{C}$$

thus it is clearly sufficent to bound the dimension of $\mathfrak{a}_{M_r}^*$ as a $\mathbb{C}$ vector space, moreover this dimension agrees with the dimension of $Rat(M_r)$ as a free $\mathbb{Z}$ module.

Thus we compute $\dim_{\mathbb{Z}}(Rat(M_r))$:

$$\begin{aligned} Rat(M_r) &= Rat(\mathrm{GL}_r \times \mathrm{Sp}_{2m}) \\ &= \mathrm{Hom}(\mathrm{GL}_r \times \mathrm{Sp}_{2m}, \mathbb{G}_m) \\ (2) &\cong \mathrm{Hom}(\mathrm{Ab}(\mathrm{GL}_r \times \mathrm{Sp}_{2m}), \mathbb{G}_m) \\ (1) &\cong \mathrm{Hom}(\mathrm{Ab}(\mathrm{GL}_r) \times \mathrm{Ab}(\mathrm{Sp}_{2m}), \mathbb{G}_m) \\ (3) &\cong \mathrm{Hom}(\mathbb{G}_m \times 1, \mathbb{G}_m) \\ &\cong \mathbb{Z} \end{aligned}$$

in (2) we have used the universal property of the abelianization $\mathrm{Ab}(G) = \mathcal{D}(G) \setminus G = [G, G] \setminus G$ because $\mathbb{G}_m$ is abelian. (1) is that the abelianization commutes with direct products (citation as comment in Tex). (3) is because Sp is a perfect group.

I havent shown that it is not trivial...

5.4 Lemma 1

The representation is supposed to be of the covering of the Levi///? need to fix this

We here consider the case that $b = 1$, hence $n = a + m$. Then fixing a standard parabolic of Sp_{2n} we have the maximal standard parabolic $P_a = M_a N_a$ where $M_a = \mathrm{GL}_a \times \mathrm{Sp}_{2m}$. Now if τ is irreducible unitary cuspidal automorphic representation of GL_a then by definition

Brenner..

$$\Delta(\tau, 1)(\varphi)(g) = E(\varphi, s)(g) = s.\varphi(g)$$

where the Eisenstein series is defined via the parabolic induction from the Levi $(\mathrm{GL}_a)^{\times b}$ to GL_{ab}. Thus we have $\Delta(\tau, 1) = \tau$. So for the appropriate σ a rep of Sp_{2m} we get a rep of the Levi of Sp_{2n}, $M_r = M_a = \mathrm{GL}_a \times \mathrm{Sp}_{2m}$ given by $\tau \otimes \sigma$. To this we associate the Eisenstein series for $\varphi \in \mathcal{A}(N_r(\mathbb{A})M_r(F) \setminus \mathrm{Mp}_{2n}(\mathbb{A}))_{\tau\otimes\sigma}$ $E(\varphi, s)$ as usual. Now we proceed to calculate the constant term of this Eisenstein series along the parabolic $P_a = MN$.

M+W II.1.7, all others are zero..?

By our calculations

red

, the cuspidality of the tensor

ref

and [JLZ13] we know that

$$E(\varphi,s)_P=\sum_w\sum_{m'}\int_{(w^{-1}N(\mathbb{A})w\cap M(\mathbb{A}))\backslash A}\int_{w^{-1}N(F)w\cap M(F)\backslash w^{-1}N(\mathbb{A})w\cap M(\mathbb{A})}\varphi(n_1n_2w^{-1}m'x)dn_1dn_2$$

and the inner integral vanishes for all $w\neq id,\omega$ (ω as in [JLZ13]). Hence the first sum becomes over two elements and we have

$$E(\varphi,s)_P=E(\varphi,s)_{P,id}+E(\varphi,s)_{P,\omega}$$

where

$$E(\varphi,s)_{P,w}=\sum_{m'\in M(F)\cap wP(F)w^{-1}\backslash M(F)}\int_{N(F)\cap wP(F)w^{-1}\backslash N(\mathbb{A})}\varphi(w^{-1}nm'x)dn$$

First the identity term simplifies

$$\begin{aligned}E(\varphi,s)(x)_{P',id}&=\sum_{m'\in M(F)\cap P(F)\backslash M(F)}\int_{N(F)\cap P(F)\backslash N(\mathbb{A})}\varphi(nm'x)dn\\&=\sum_{m'\in M(F)\backslash M(F)}\int_{N(F)\backslash N(\mathbb{A})}\varphi(nm'x)dn\\&=\int_{N(F)\backslash N(\mathbb{A})}\varphi(nx)dn\\&=\varphi(x)_P\end{aligned}$$

I really need to fix this s thing that I dropped in the constant term computations.

Considering now the ω term

$$E(\varphi,s)_{P,\omega}=\sum_{m'\in M(F)\cap\omega P(F)\omega^{-1}\backslash M(F)}\int_{N(F)\cap\omega P(F)\omega^{-1}\backslash N(\mathbb{A})}\varphi(\omega^{-1}nm'x)dn$$

by [JLZ13, 2C] $M(F)\cap\omega P(F)\omega^{-1}\setminus M(F)$ is isomorphic to

this is not clear in their paper im just guessing this is what they mean

$P_0\setminus\mathrm{Sp}_{2(n-a)}$ where $P_0\setminus\mathrm{Sp}_{2(n-a)}$ by defintion, but P_0 has Levi $M_0=\mathrm{Sp}_{2(n-a)}$ by definition and hence is itself $\mathrm{Sp}_{2(n-a)}$. Thus the sum is over $\mathrm{Sp}_{2(n-a)}(F)\setminus\mathrm{Sp}_{2(n-a)}(F)$ and hence is over a point. Therefore we get by definintion of the intertwining operator

$$E(\varphi,s)_{P,\omega}=\int_{N(F)\cap\omega P(F)\omega^{-1}\backslash N(\mathbb{A})}\varphi(\omega^{-1}nx)dn=M(\omega,-)(\varphi)(x)$$

becuase we took the constant term along the same parabolic as the definition of the Eisenstein series we know that the Levis are (the same) conjugate. Thus we have shown that

$$E(\varphi,s)_P(x)=\varphi(x)_P+M(\omega,-)(\varphi)(x)$$

Notice that the computation takes place completely at the level of the terminals which are indupendent of the fact that we have taken a covering group, hence we have really only reused work from [JLZ13].

Appendix A

Motivation

We have tried to motivate the individual objects as we went, how they would help get us to the result that we were looking for. Here we try to provide a birds eye veiw of whats going on.

We do have to preface that much like the motivation throughout we have only understood a small piece of the story here, however we still feel obligated to communicate what we can.

A.0.1 Reminders —- Remove Later

Paul Garrett said to me something along the lines of it being arguably more important to understand the *role* of the Eisenstein series in the theory of automorphic forms rather than the technical details of its proof of meromorphicity. I think to some extent he is correct, not to say that the proof isnt important, only that its importance is given to it by its role and is hence subservient to it.

Try to elaborate the role of each introduced item in the theory, not just its technical properties.

Todo list

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