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I think this is funny can I be funny?

Introduction

Motivation

The goal of this thesis is to exposit some of the results in [JLZ13]. We aim our exposition at the other masters students in our cohort. To explain the results on poles of Eisenstein series to students in other disciplines there is a fair amount of background to be covered.

Outline of Content

Chapter one deals with the generalities of linear algebraic groups, the objects whose representation theory is the subject of discussion. First we define them and then look at the important subgroups that are used in the study of automorphic forms arising on the adelic points of these groups. We focus on the classical groups.

Chapter two deals with automorphic forms. We define automorphic forms in both the Archimedean and adelic places. Finally we give the details of how to view modular forms as automorphic forms.

Chapter three is a discussion of the concept of the constant term in the Archimedean place. First we define the constant term of an automorphic form (Archimedean) and then we show how it is related to the constant term of the Fourier series of a modular form. Finally we show how the classical Siegel Phi operator can be realised as a constant term.

Chapter four is dedicated to automorphic representations. We define them and specify some important constructions that are needed in the final section.

In chapter five we define adelic Eisenstein series and show how they generalise the classical modular forms also known as Eisenstein series.

Chapter six is dedicated to the constant term in the adelic setting. We first define them and then go through the process of computing them in great detail for

Eisenstein series.

Chapter seven is for defining L-functions, the analytic invariants that are central to the Langlands program. We will give several of the special cases that appear through out history and the literature.

Finally chapter eight contains some exposition of recent work on the poles of residual Eisenstein series.

Contents

Notation

Delete this section later perhaps: Use for reference while writing.

- F is a number field
- ν is a place of F
- F_ν is the completion of F at ν
- $\mathbb{A}$ of $\mathbb{A}_F$ is the Adele ring, $\mathbb{A}_f$ is the finite adeles, and $\mathbb{A}_\infty$ the infinite adeles.
- B, P, M, U: Borel, parabolic, Levi, unipotent

Chapter 1

Classical Groups

We will recall a small amount of the theory of linear algebraic groups to fix conventions, for a more detailed treatment one should consult the litany of sources on this matter: For a full treatment see [Mil17][Milb][Mila][Spr98]. Excellent example computations can also be found in [Bui][Mak][MT][Not]. Or for a brief brush up on the main facts consult [BC79, I.I.1].

The purpose of this section is to define the key examples and properties of algebraic groups. We also define the most important subgroups, attempting to emphasize the role they play in the theory. Throughout we will restrict to the case of the few classical groups that we define explicitly, however the theory of course works much more generally.

1.1 Definition

An **algebraic group** is for us a group scheme that is reduced, of finite type and defined over a field. **A linear algebraic group** (LAG) is simply an affine algebraic group.

Proposition 1.1. *An algebraic group is affine if and only if it is isomorphic to a Zariski closed subgroup of* GL_n.

Proof. The forward implication is [Spr98, 2.3.7(i)]. The converse is the basic fact that closed sub-schemes of affine schemes are affine [Mum99, II.5.T3].

The idea of LAG's is that they are matrix groups defined by polynomial equations, which are the natural combinations of symbols that matrix multiplication will lead to. This means that they come with the technology of algebraic geometry and in particular one must be adept at moving between the following equivalences:

Theorem 1.2 ([Mila], II.6, III.4)**.** *For a field K, then the following categories are equivalent:*

- *Group objects in* Alg_K^{op}
- *Representable (in the category of groups) functors* $\mathrm{Alg}_K \to \mathrm{Group}$
- *Group object in the category of affine schemes over K*
- *Commutative K-Hopf algebras.*

Example 1.3 ($\mathbb{G}_m$)**.** *The first example is the "multiplicative group" denoted $\mathbb{G}_m$ or GL_1 defined over the field K. This is*

$$\mathbb{G}_m \ := \ \mathrm{Spec}\Big(K[x,y]/(xy-1)\Big).$$

As a representable functor this sends a K-algebra R to $\mathrm{Hom}_K(K[x,y]/(xy-1), R)$. These are ring maps that are K-linear, and because $y = x^{-1}$ we know that $f(y) = f(x^{-1}) = f(x)^{-1}$ for $f \in \mathbb{G}_m(R)$. Thus the maps are determined by where they send x, moreover they always send it to a unit, i.e. $Imf \subseteq R^\times$. For each element $r \in R^\times$ we also have a map sending $x \to r$ hence there is an isomorphism (of sets) between $\mathbb{G}_m(R) \cong R^\times$, from which we pull back a group structure.

The other important examples of such groups are the "classical groups". The exact groups that an author might mean by classical may vary, so we define them explicitly here. First let V be a finite dimensional K-vector space with a bilinear form $\langle , \rangle$. An automorphism of this form is a map $\alpha \in \mathrm{Aut}(V)$ such that

$$\langle \alpha(x), \alpha(y) \rangle = \langle x, y \rangle.$$

Therefore we can consider the space of automorphisms of this form $\mathrm{Aut}(V, \langle , \rangle)$. This space, depending on the properties of the bilinear form, will define our classical groups.

If the form is trivial, by which we mean, $\forall x, y \quad \langle x, y \rangle = 0$ then we define the **general linear group**,

$$\mathrm{GL}(V) \ := \ \mathrm{Aut}(V, \langle , \rangle) = \mathrm{Aut}(V).$$

If the form is non-degenerate and symmetric $\forall x, y \quad \langle x, y\rangle = \langle y, x\rangle$ then we define the **orthogonal group**,

$$\mathrm{O}(V) \;\coloneqq\; \mathrm{Aut}(V, \langle , \rangle).$$

Finally if the form is non-degenerate and skew symmetric $\forall x, y \quad \langle x, y\rangle = -\langle y, x\rangle$ then the **symplectic group** is,

$$\mathrm{Sp}(V) \;\coloneqq\; \mathrm{Aut}(V, \langle , \rangle).$$

There are the further classical groups given by the determinant one subgroups, $\mathrm{SL}(V)$ and $\mathrm{SO}(V)$ respectively. The naming of $\mathrm{Sp}(V)$ is somewhat serendipitous as one can show that it is contained in $\mathrm{SL}(V)$. We can make this into a functor from K-algebras to groups, by sending a K-algebra R to $G(V) \otimes_K R$. Thus these define LAG's.

Remark 1.4. Often the unitary groups are considered classical, as is the case in [JLZ13].

1.2 Subgroups

From now on let G be a one of the classical LAG defined above, defined over a number field F with adele ring $\mathbb{A}$.

Remark 1.5. Most everything we say will apply verbatim to so called split reductive groups, however we lose little in restricting to the classical groups.

Subgroups with special properties allow us to reduce and break up problems into smaller ones. Here we will briefly review and compute some examples of special subgroups. The point of these subgroups is two fold. Some of them will help us perform "induction" from smaller simpler groups to larger ones. Others are there essentially as a part of the combinatorial data that classifies the groups we are working with. In particular we need to understand all the pieces of the so called **Langlands-Iwasawa decomposition** [GH24, 2.7],

$$G(\mathbb{A}) = M(\mathbb{A})U(\mathbb{A})K = T(\mathbb{A})U(\mathbb{A})K. \tag{1.2.1}$$

1.2.1 Parabolics, Levis and Unipotents

A subgroup $P \subseteq G$ is called **parabolic** if G/P is a complete variety. Equivalently we can ask for P to contain a Borel (see section 1.2.2).

Completeness is the algebro-geometric analogue of compact, which is always a desirable property. The fact that they contain a Borel gives us an algebraic "parametrisation" of these subgroups, in the case of the classical groups through the use of flags or roots. It is very important to have a parametrisation of the parabolic subgroups when it comes to taking constant terms of Eisenstein series which we will discuss in the later chapter 5.

A matrix m is **unipotent** if for some $n \geq 0$ we have that $(m-1)^n = 0$. A subgroup is **unipotent** if all its elements are unipotent. The **unipotent radical** of G is the maximal closed, connected, unipotent subgroup. A linear algebraic group is **reductive** if its unipotent radical is trivial. Then we have the following fact and definition,

Lemma 1.6 ([Bor91] 11.22)**.** *There is a split exact sequence (of algebraic groups)*

$$0 \to U \to P \to M \to 0,$$

*where U is the unipotent radical of P, and M is a reductive group known as a **Levi** (unique up to conjugacy).*

Thus parabolics and their Levis allows us to induce from a reductive subgroup up to the reductive group. This is the technique of "parabolic induction" [Ber92, Thm. 10] that we wont explicitly talk about here but which is happening secretly in the background in section 3.2.2.

Remark 1.7 (Bad Etymology)**.** *The origin of the name parabolic is a mystery. Borel in his history [Ess, VI.§2] attributes it to R. Godement in [God]. Godement conjectures that the quotient $G(\mathbb{A})/G(\mathbb{Q})$ is compact if and only if every element of $G(\mathbb{Q})$ is semi-simple, as is the case in classical groups. He says that*

is probably known
ow.

> *Lorsque n'est pas compact, il est non moins facile de conjecturer qu'on doit pouvoir définir quelque chose d'analogue aux classiques "pointes paraboliques", lesquelles doivent correspondre à des sous-groupes unipotents non triviaux de $G_{\mathbb{Q}}$*

which roughly (google) translates to that one can also conjecture that non-trivial unipotent elements should correspond to "parabolic points" in a fundamental domain.

In the case of modular forms the fundamental domain is $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R})$ *(for the details see section 2.3). We have the classification of elements of* $\mathrm{SL}_2(\mathbb{R}) - \{\pm 1\}$ *as in [Bor97, 3.5] via their trace*

$$g \text{ is of type } \begin{cases} \textit{Elliptic if} & \frac{1}{2}|tr(g)| < 1 \\ \textit{Parabolic if} & \frac{1}{2}|tr(g)| = 1 \\ \textit{Hyperbolic if} & \frac{1}{2}|tr(g)| > 1 \end{cases}.$$

This classification, it seems, relies entirely on the aesthetic connection with the classification of the sections of conics via eccentricity. Being parabolic is equivalent to having eigenvalue 1 hence by the Jordan decomposition we know that parabolics in SL_2 *are conjugate (over* $\mathbb{C}$*) to*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly the standard parabolic

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \subseteq \mathrm{SL}_2(\mathbb{R}),$$

contains these matrices, and moreover all parabolics are conjugate to this parabolic. Hence all parabolic elements are contained in a parabolic subgroup.

To connect this to Godement's concept we have two facts from classical geometry. Proper parabolic subgroups of $\mathrm{SL}_2(\mathbb{R})$ *can be realised as the stabilisers of lines in* $\mathbb{R}^2$ *under the standard action of* SL_2 *on* $\mathbb{R}^2$ *[Bor97, 2.6] and moreover some an element of* $\mathrm{SL}_2(\mathbb{R})$ *is parabolic if and only if it has one fixed point on* $\partial\bar{\mathcal{H}}$ *and none on* $\mathcal{H}$ *[Bor97, 3.5].*

The take away is that perhaps the folklore of the name being for "para-Borelic", as in kind of a Borel, is probably a better way of thinking of them.

The Example of Sp_{2n}

We collect the following facts as they will be useful in what is to come. Good references are the notes [Con] and the book [Bui, §8].

Let $(V, \langle , \rangle)$ be a symplectic space as above and $Sp(V)$ is the automorphisms preserving the form. A **flag** of V is a sequence of strict inclusions of vector subspaces

$$\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset V.$$

A subspace of V is said to be **isotropic** if the form is constantly zero on it (in both variables). A flag is **isotropic** if the proper subspaces in it are isotropic subspaces. A **maximal isotropic** flag is one with exactly n components. Sp_{2n} acts on a flag by acting on each of the subspaces. This action preserves isotropic flags i.e. it sends an isotropic flag to an isotropic flag. Stabilisers of isotropic flags give parabolics of Sp and moreover all parabolics arise in this way [Spr98, Exercise 3.2.16, 6.2.11].

Example 1.8. *Consider a four dimensional vector space V with a form given by the matrix*

$$\begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix},$$

then a maximal isotropic flag is

$$0 \subset Fe_1 \subset Fe_1 \oplus Fe_2 \subset F^4,$$

where $e_i = (\delta_i^j)_j$. This has stabiliser consisting of matrices in Sp *of the form*

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix}.$$

In particular maximal parabolics of Sp are stabilizers of *minimal* (non-trivial flags), i.e. stabilisers of non-zero isotropic subspaces,

$$0 \subset V_\ell \subset V,$$

where $V_\ell = span_F(e_1, ..., e_\ell)$. Then the stabilizer is

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix},$$

with the sizes of the diagonal blocks being (these numbers square)

$$\begin{pmatrix} \ell & * & * & * \\ 0 & n-\ell & * & * \\ 0 & * & \ell & * \\ 0 & * & * & n-\ell \end{pmatrix}.$$

This has Levi

$$\begin{pmatrix} A & & & \\ & a & & b \\ & & (A^T)^{-1} & \\ & c & & d \end{pmatrix}, \quad A \in \mathrm{GL}_\ell(F), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2(n-\ell)}(F),$$

and unipotent

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & \\ & & 1 & \\ & & * & 1 \end{pmatrix},$$

with relations among the entries.

Fill

1.2.2 Borel and Torus

A **split torus** is an algebraic group that is isomorphic to GL_1^b for some $b \in \mathbb{N}$.

Example 1.9 (Bad Etymology)**.** $\mathrm{GL}_1/\mathbb{C}$ *is a split torus. Consider the field extension* $\mathbb{C}/\mathbb{R}$*. Then* $\mathbb{C}$ *has the inner product given by*

$$\langle z, z' \rangle := \bar{z}z'.$$

We can look at the elements of $\mathbb{C}$ *that preserve this inner product,*

$$\begin{aligned} U(1) &:= \{c \in \mathrm{GL}_1(\mathbb{C}) : \forall z, z' \in \mathbb{C}, \quad \langle cz, cz' \rangle = \overline{cz}cz' = \bar{z}z'\} \\ &= \{c \in \mathrm{GL}_1(\mathbb{C}) : |c| = 1\}. \end{aligned}$$

Note that this is a (real) line topologically so we dont expect it to be a complex varity. Indeed this defines a ***real*** *algebraic group given by the zero locus in* $\mathbb{R}^2$ *of the two* *variable polynomial* $x^2 + y^2 - 1$*. In other words*

$$U(1) \cong \operatorname{MaxSpec}\big(\mathbb{R}[x,y]/(x^2+y^2-1)\big).$$

Now if we base change to $\mathbb{C}$ *we have*

$$\begin{aligned}\mathbb{R}[x,y]/(x^2+y^2-1)\otimes_{\mathbb{R}}\mathbb{C} &\cong \mathbb{C}[x,y]/\big((x+iy)(x-iy)-1\big)\\ &\cong \mathbb{C}[s,t]/(st-1)\\ &\cong \mathbb{C}^*.\end{aligned}$$

Thus $\mathrm{GL}_1/\mathbb{C}$ *is the complexification of the torus* $U(1)$*.*

Remark 1.10. These tori also play the same role in the classification of reductive LAG as the real Lie groups called tori play in the classification of Lie groups [Hal15, Part III].

A subgroup that is isomorphic to a split torus and is maximal in this respect is called a **maximal split torus**.

Example 1.11. *The classic example of a maximal split torus is the group of diagonal matrices in* GL_n*.*

e detail?

A **Borel** is a maximal, closed, solvable and connected subgroup of G. A Borel can be considered to be a parabolic that is minimal with respect to inclusion. The maximal tori then form the Levis of these parabolics. In particular for a Borel B we have that

$$B = TU,$$

for a maximal torus T and unipotent U.

Example 1.12. *The standard Borel of* GL_n *is the group of upper triangular matrices. If* n *is even and one intersects this Borel with* $\mathrm{Sp}_{2(\frac{1}{2}n)}$ *then we get the standard Borel of* $\mathrm{Sp}_{2(\frac{1}{2}n)}$*.*

Lets prove this in GL_2 *and then believe that the only complication to going to larger* n *is keeping track of indices. So let*

$$B = \begin{pmatrix} * & * \\ & * \end{pmatrix},$$

we need to show that the derived series terminates for it to be solvable. So let

$$g = \begin{pmatrix} x & y \\ & z \end{pmatrix}, \quad h = \begin{pmatrix} a & b \\ & c \end{pmatrix},$$

be arbitrary in GL_2*, their commutator is then*

$$g^{-1}h^{-1}gh = \begin{pmatrix} 1 & \frac{bx-ay}{ax} \\ & 1 \end{pmatrix}.$$

Hence

$$[B, B] = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}.$$

Commutate two arbitrary elements again shows that

$$[[B, B], [B, B]] = 1.$$

It is clear that this is a closed subgroup because it is itself a linear algebraic group, moreover for LAG's we have the algebraic criterion of connectedness given by having the only idempotents in the representing algebra being $0, 1$ *[GH24, 1.5]. Because* $B = \operatorname{Spec} \mathbb{Z}[x_{i,j} : 1 \leq i, j \leq 2][y]/(\det(x_{ij})y - 1, x_{2,1})$ *it is clear that this group is connected. Finally it is clear that if a subgroup strictly contains the group of upper triangular matrices then it is in fact all of* GL_2 *and hence this is maximal. Therefore this is a Borel.*

If a Borel B is fixed, then a parabolic containing this Borel $B \subseteq P$ is called standard, there is a unique Levi of a standard parabolic containing this Borel called the **standard Levi**.

1.2.3 The Topology on Points

Let F be a number field and $G = \operatorname{Spec} F[x_1, ..., x_n]/(f_1, ..., f_m)$ be an LAG over F. As a locally ringed space this scheme has the Zariski topology, in the theory of automorphic forms however we wish to topologise the local and adelic points in a way which accommodates analysis. In particular the topology should be locally compact and Hausdorff so that we have Haar measures on the groups.

Following [Con12] then we think of $G(\mathbb{A})$ as the subset of $\mathbb{A}^n$ on which the functions $f_i : \mathbb{A}^n \to \mathbb{A}$ all vanish. We give it the subspace topology which inherits the local compact and Hausdorff properties from $\mathbb{A}^n$. If ν is a place of F then we have the same definition, $G(F_\nu)$ is the subspace of F_ν^n on which $f_i : F_\nu^n \to F_\nu$ all vanish and it is endowed with the subspace topology. These topologies are refered to as the **Hausdorff topology**.

Remark 1.13. When $F_\nu = \mathbb{C}$ then the Hausdorff topology on $G(\mathbb{C})$ agrees with the topology of the analytification of G, often denoted G^{an}.

1.2.4 Maximal Compact Subgroups

We will often need to fix a maximal compact subgroup $K \subseteq G(\mathbb{A})$ of the Hausdorff topology. These maximal compact subgroups are not unique and as such when fixing one it can be arranged to have many convenient properties [MW95, I.1.4]. In particular if we have a group G and a fixed Borel B:

- First require that
$$K = \prod_\nu K_\nu,$$
where the product is over all places of F and $K_\nu \subseteq F_\nu$ is maximal compact.

- If $\mathcal{O})\nu$ is the ring of integers of F_ν, then for almost all places, $G(\mathcal{O}_\nu)$ is defined and is maximal compact in $G(F_\nu)$ hence we can require $K_\nu = G(\mathcal{O}_\nu)$ at these places.

- We require
$$G(\mathbb{A}) = B(\mathbb{A})K.$$

- For every standard parabolic $P = MU$ we have that
$$P(\mathbb{A}) \cap K = \Big(M(\mathbb{A}) \cap K\Big)\Big(U(\mathbb{A}) \cap K\Big),$$
and $M(\mathbb{A}) \cap K$ is a maximal compact subgroup of $M(\mathbb{A})$.

It is in terms of the third property that we like to think of the maximal compact subgroup, it is the complimentary piece of the Borel. Moreover the fourth property should be thought of as a condition that the maximal compact subgroups are well behaved with the way that we are moving between the bigger and smaller reductive groups. Maximal compact groups with all these properties are said to be in **good position**.

Chapter 2

Automorphic Forms

There are different definitions of the words automorphic forms floating around, here we fix one and then explain how they generalize the classical modular forms.

The story starts with the classical modular forms, or functions on the upper half plane that satisfy some invariance conditions and differential equations. This evolves into the notions of Maas form on symmetric spaces and eventually reaches its apotheosis in the concept of automorphic form that we will present here.

We will present two notions of automorphic form here. In the literature they are both called "automorphic forms" however here we will distinguish those that are defined only on the Archimedean points as "Archimedean automorphic forms" for clarity.

The first natural question is if there is a special case of automorphic forms which yield modular forms. In fact no, the space of automorphic forms is larger than just modular forms, however it gives the space of Maas forms (or modular and Maas forms, depending on convention). This is well covered in the literature [Eme][Bum97, 3.2][Boo][Gar16]. We explain modular forms as Archimedean automorphic forms as we think it is where the connection is clearest. We will give an example of modular forms as adelic automorphic forms when we come to the Eisenstein series in section 4.2.

2.1 Archimedean Automorphic Form

Fix a number field F and a classical group G defined over F. Let ∞ denote the set of Archimedean places. We denote $\mathbb{A}_\infty = F_\infty := \prod_{\nu \in \infty} F_\nu$ and note that $G(F_\infty) \cong \prod_{\nu \in \infty} G(F_\nu)$. Consider $\nu \in \infty$ one such Archimedean place, then F_ν is either $\mathbb{R}$ or $\mathbb{C}$. In particular (the analytification of) $G(F_\nu)$ is a Lie group and we call a function, $\varphi : G(F_\nu) \to \mathbb{C}$, **smooth** if it is smooth in the sense of manifolds. The collection of such smooth functions on $G(F_\infty)$ will be denoted $C^\infty(G(F_\infty))$.

at is the reference
hese facts..

Because $G(F_\infty)$ is a Lie group we know how to define its Lie algebra and we now denote $Z(\mathfrak{g})$ the center of the *universal enveloping algebra* of the *complexification* of $\mathfrak{g}$, it would be more reasonable to use $Z(\mathcal{U}(\mathfrak{g}_\mathbb{C}))$ but that is too cumbersome so we follow the tradition. A vector in a $Z(\mathfrak{g})$-module $\varphi \in V$ is called $Z(\mathfrak{g})$-**finite** if the space $Z(\mathfrak{g})\varphi$ is finite dimensional.

Let $K_\infty \subseteq G(F_\infty)$ be a maximal compact subgroup. Then again an element of a K_∞-module is K_∞ **finite** if its orbit is a finite dimensional vector space space (we think here of $\mathbb{C}[K_\infty]$-modules).

To define automorphic forms we look at the representation $C^\infty(G(F_\infty))$ with the right regular action of K_∞, i.e. $g.f(x) = f(xg)$. In particular the $Z(\mathfrak{g})$ module structure is induced from the action of $\mathfrak{g}$ on $C^\infty(G(F_\infty))$ by

$$z.F(g) = \frac{\partial}{\partial t} F(ge^{tz})|_{t=0}.$$

Finally we want a growth condition. Fix an embedding $\iota : G \to GL_n$ which gives another embedding $G \to SL_{2n}$ via

$$\iota' : g \mapsto \begin{pmatrix} \iota(g) & \\ & (\iota(g))^{-t} \end{pmatrix}.$$

We have denoted the inverse of the transpose $-t$. A function $\varphi : G(F_\infty) \to \mathbb{C}$ is of **moderate growth** if there are constants $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$|\varphi(g)| \leq c\|g\|^r = c \left(\prod_{v \in \infty} \sup_{1 \leq i,j \leq 2n} |\iota'(g)_{i,j,\nu}|_\nu \right)^r.$$

This is taking the maximum of the $2n \times 2n \times |\infty|$ three dimensional matrix.

Remark 2.1. One can define norms on $G(\mathbb{A})$ via the linearisation of such groups, i.e. via their representations. Concretely if σ is a finite dimensional complex representation on some Hilbert space with a K_∞ invariant inner product and $*$ is the adjoint matrix with respect to this Hilbert space structure then a **norm** on $G(\mathbb{A})$ is a function of the form

$$g \mapsto \left(\operatorname{tr} \sigma(g)^* \sigma(g)\right)^{\frac{1}{2}}.$$

This moderate growth condition is then equivalent to some norm $\|-\|$ existing on $G(F_\infty)$ such that

$$|\varphi(x)| \leq C\|x\|^n,$$

for some $C > 0, n \in \mathbb{N}$ and all $x \in G(F_\infty)$. This is also equivalent to all such norms satisfying this condition [BC79, 1.2].

Definition 2.2. *Let $\Gamma \leq G(F_\infty)$ some (arithmetic) subgroup, an* ***automorphic form for*** Γ *is a smooth function of moderate growth*

$$\varphi : G(F_\infty) \to \mathbb{C},$$

that is K_∞ and $Z(\mathfrak{g})$ finite with a (left) Γ invariance. We denote the set of these "Archimedean" automorphic forms by $\mathcal{A}(\Gamma\backslash G(F_\infty))$.

2.2 Adelic Automorphic Form

Here we follow [MW95, I.2.17]. Fix a Borel $B \subseteq G$ and a standard parabolic $B \subseteq P \subseteq G$ with a standard Levi decomposition $P = MU$. We let K be a maximal compact subgroup of $G(\mathbb{A})$ that is in good position as in section 1.2.4.

For $v \notin \infty$ a non-Archimedean place then we say that a function $f : G(F_\nu) \to \mathbb{C}$ is smooth if it is locally constant on $G(F_\nu)$, in the Hausdorff topology. The set of such smooth functions is denoted $C^\infty(G(F_\nu))$. This suggests the definition of smooth functions on the "finite adeles" $\mathbb{A}_f$ as

Clarify what this restricted tensor product is

$$C^\infty(\mathbb{A}_f) \;:=\; \bigotimes_{\nu\notin\infty}{}' C^\infty(G(F_\nu)).$$

Thus for the full adeles we have the notion of smooth as an element of the following,

$$C^\infty(\mathbb{A}_F) \;:=\; C^\infty(G(\mathbb{A}_f)) \otimes C^\infty(G(F_\infty)).$$

Notice that a priori the codomain is an infinite tensor product over $\mathbb{C}$ of copies of $\mathbb{C}$, this is *canonically* isomorphic to $\mathbb{C}$, thus we can conflate a smooth function with its composition along this isomorphism and think of them as functions into $\mathbb{C}$.

We still consider $Z(\mathfrak{g})$ to be the center of the universal enveloping algebra of the complexified Lie algebra at the infinite places, exactly as before. We define an action by linearly extending

$$z.(f \otimes g) = f \otimes (z.g),$$

i.e. it acts on the archimedean places as in the setting of Archimedean automorphic forms.

The definition of moderate growth carries over verbatim, however we change the set of places multiplied over to be all of them now.

Remark 2.3 ([BC79], 1.II.3). The collection of moderate growth functions is independent of the choices of embedding.

Definition 2.4. A function $\varphi : U(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is an **automorphic form** if it is smooth, moderate growth, $Z(\mathfrak{g})$ and K finite. We will denote the set of these automorphic forms by $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$.

Remark 2.5. It is important that $M(F)$ is treated as a subgroup of $M(\mathbb{A})$ via the diagonal embedding.

Remark 2.6. What we have called automorphic forms are sometimes referred to as "smooth K-finite automorphic forms" [Coga, 2.2].

2.3 Modular Forms

Recall the definition of a **modular form of weight k** (of full level and trivial character) [DS05, 1.1.2] as a function

$$\varphi : \mathcal{H} \to \mathbb{C},$$

where $\mathcal{H}$ is the upper half plane in $\mathbb{C}$, that is holomorphic, satisfies

$$\varphi(\gamma.z) = (cz + d)^k \varphi(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and extends holomorphically to ∞.

We want to think of the upper half plane as a quotient of the $\mathbb{Q}_\infty = \mathbb{R}$ points of some reductive group. If we have a transitive action of some reductive group then by the orbit stabiliser theorem we would have a bijection of sets.

Theorem 2.7.

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R}),$$

as sets.

Proof. Consider the action

$$\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathcal{H} : \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}.$$

Then look at the orbit of i, namely

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} .i = \frac{ai+b}{d} = a^2 i + ab,$$

which letting $a, b \in \mathbb{R}$ vary is clearly surjective onto the whole upper half plane. So there is one orbit, and hence by the orbit stabiliser we know that

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/stab(i),$$

so we want to find

$$stab(i) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : g.i = i \right\},$$

in particular we solve

$$i = g.i = \frac{ai+b}{ci+d} = (c^2+d^2)^{-1}(ac+bd+i\det g).$$

So equating coefficients we have

$$\det g(c^2+d^2)^{-1} = 1 \implies c^2+d^2 = \det g = 1,$$

on the other hand

$$ac+bd = 0.$$

Now the pairs $c^2+d^2 = \det g = 1$ are parameterized by $\theta \in [0, 2\pi)$ using $c = \sin\theta, d = \cos\theta$ hence subbing this into the above equation

$$\frac{-b}{a} = \tan\theta,$$

and so $b = -k\sin\theta, a = k\cos\theta$ for some $k \in \mathbb{R}$ but the determinant must be 1 so $k = 1$. Hence

$$stab(i) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\} = SO_2(\mathbb{R}).$$

One then has to check that this is all continuous.

Remark 2.8. This can be beefed up to an isomorphism of complex analytic spaces. Sometimes to make the action of certain (Hecke) operators more apparent this is exhibited as

$$\mathcal{H} \cong \mathrm{GL}_2^+(\mathbb{R})/A_{\mathrm{GL}_2}SO_2(\mathbb{R}).$$

This obscures the connection with the reductive group setting however so we avoid it here.

SL_2 is a reductive group and $SO_2(\mathbb{R})$ is its maximal compact subgroup. This decomposition of the upper half plane suggests that function on it might have some invariance along the maximal compact subgroup of the reductive group SL_2.

$$\begin{tikzcd} \left\{ \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} : x, y \in \mathbb{R}, y \neq 0 \right\} SO_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \arrow[r, "\text{project}"] \arrow[dr, "\text{project}"'] & \mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R}) \arrow[r, "\sim", "g \mapsto g.i"'] & \mathcal{H} \\ & \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R}) & \end{tikzcd}$$

We can lift a function on $\mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R})$ to $\mathrm{SL}_2(\mathbb{R})$ by composing with the projection, however this is not $\mathrm{SL}_2(\mathbb{Z})$ invariant, thus we need to add a pre-factor to ensure this in our associated automorphic form. The algebro-geometric perspective in [Eme] can make this seem slightly less ad hoc.

does it? Am I trivialising a line bundle...?

Thus for f a modular form of weight k the following function on $\mathrm{SL}_2(\mathbb{R})$

$$F(g) \ := \ (ci + d)^{-k} f(g.i),$$

we claim is an automorphic form for $\mathrm{SL}_2(\mathbb{Z})$. We take for granted its smoothness. The $\mathrm{SL}_2(\mathbb{Z})$ invariance is obvious from the modularity condition. It remains to show the three other properties:

Lemma 2.9. *$F(g)$ is of moderate growth.*

Proof. Unraveling the definitions we require two constants such that

$$|F(g)| = |ci + d|^{-k}|f(g.i)| \leq c(\sup_{i,j}(g, g^{-1}))^r,$$

A direct computation shows that

$$Im(g.i) = |ci + d|^{-2},$$

hence we require to show

$$\mathrm{Im}(g.i)^{k/2}|f(g.i)| \leq c(\sup_{i,j}(g, g^{-1}))^r.$$

Somehow invoke polynomial growth...?

Fill

but the modularity condition has the growth condition that $\lim_{x\to\infty} f(xi)$ be bounded.

Lemma 2.10. *$SO_2(\mathbb{R})$ is a maximal compact subgroup inside $\mathrm{SL}_2(\mathbb{R})$. F is an $SO_2(\mathbb{R})$ finite function.*

Proof. Using that $\kappa \in K = SO_2(\mathbb{R})$ acts trivially on i, an elementary computation shows that for $g \in \mathrm{SL}_2(\mathbb{R})$,

$$F(g\kappa) = e^{-ik\theta}F(g).$$

Hence $F(g)$ is acted on by K via a one dimensional irreducible representation. In particular it is finite dimensional.

Lemma 2.11. *F is a $Z(\mathfrak{sl}_2)$ finite function.*

Proof. Only a sketch.

The center of the universal enveloping algebra of the complexified Lie algebra is generated by the Casimir operators. From [Gar10] we know that the casimir is

$$\Omega = \frac{1}{2}H^2 + XY + YX.$$

We have the coordinates on $\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} SO_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ from [Bum97][1.19

pg 139] in which the casimir acts as the differential operator

$$\Delta = y^2 \left(\left(\frac{\partial}{\partial x} \right)^2 + \left(\frac{\partial}{\partial y} \right)^2 \right) - y \frac{\partial^2}{\partial x \partial \theta},$$

[Bum97][1.29 pg 143 ,Prop 2.2.5 pg 155]. Now we claim that F is an eigenfunction for this operator. An element $(x, y, \theta) := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \kappa_\theta \in \mathrm{SL}_2(\mathbb{R})$ acts on i by sending it to $x+iy$ (elementary computation). The bottom row of the product is $y^{-1/2}\sin\theta; y^{-1/2}\cos\theta$ which results in

$$F(x, y, \theta) = y^{k/2} e^{-ik\theta} f(x + iy).$$

It is then a calculus exercise to apply Δ to this, using the holomorphicity we also get that $f_{xx} - f_{yy} = 0$ and $f_y = if_x$ which cancels away terms and we get that

$$\Delta F(x, y, \theta) = \frac{k}{2} \left(\frac{k}{2} - 1 \right) F(x, y, \theta).$$

Therefore the dimension of $Z(\mathfrak{g})F$ is simply one.

This example makes it clear that the two finiteness conditions for automorphic forms are in some sense functional equations that they must satisfy. There is a nice explanation of how to lift this to the adelic setting in several places, the key is the isomorphism

$$\mathrm{SL}_2(\mathbb{R}) \cong Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}).$$

The details are quite clear in [Coga, 2.1] or [Boo]. We will revisit this perspective through the example of the Eisenstein series in section 4.2.

Chapter 3

Automorphic Representations

The references that will be most helpful are [BC79, I.II][GH24] for the general theory, we will follow the notation developed in [MW95] as it is somewhat standard. We will discuss some of the details of their representation theory because it is both subtle and essential for the setup in [JLZ13]. In particular we want to draw attention some of the quirks of the category of automorphic representations.

3.1 Local Representation Theory

Recall that in the representation theory of finite groups over $\mathbb{C}$ there is really only one important representation, that is the regular representation i.e. the $\mathbb{C}[G]$ module $\mathbb{C}[G]$. This is important for two reasons, the first is that it is always a priori defined uniformly for all groups, the second is that it decomposes into a direct sum over all irreducible modules [Lin, Ch. 2.4 Cor. 2].

Let G be a classical group defined over a number field F. As in the finite group case we want to consider the right regular action of the adelic points, $G(\mathbb{A})$, on a space of functions $G(\mathbb{A}) \to \mathbb{C}$, namely

$$g.f(x) = f(xg).$$

One can ask if this representation sends an automorphic form to an automorphic form. If $\varphi(x) \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ and $g \in G(\mathbb{A}_f)$ then $\varphi(xg) \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$. Hence $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ is a $G(\mathbb{A}_f)$-module. In particular it is a module for $G(F_\nu)$ for all ν non-Archimedean

There is a problem with the K-finiteness in the infinite places however which prevents $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ from being a full $G(\mathbb{A})$ module.

Example 3.1 ([Coga], 2.3)**.** *If $\varphi \in \mathcal{A}(\Gamma\backslash G(F_\infty))$ is K_∞-finite, then $g.\varphi$ is $gK_\infty g^{-1}$-finite. This is still a maximal compact subgroup, however in the infinite place it will a priori have only the identity in common with the original K.*

For example consider SL_2 where the maximal compact is SO_2, if we conjugate we get gSO_2g^{-1}

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \cos\theta + (db+ca)\sin\theta & -\sin\theta(a^2+b^2) \\ \sin\theta(d^2+c^2) & \cos\theta - (bd+ac)\sin\theta \end{pmatrix}.$$

If we want to find the intersection of SO_2 with gSO_2g^{-1} we need to solve the system

$$\begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix} = \begin{pmatrix} \cos\theta + (db+ca)\sin\theta & -\sin\theta(a^2+b^2) \\ \sin\theta(d^2+c^2) & \cos\theta - (bd+ac)\sin\theta \end{pmatrix}.$$

Where θ might not be θ'. If $\theta = n\pi, n \in \mathbb{Z}$ then the $\sin$ terms on the right vanish and we get the ± 1 as a point of intersection, so consider $\theta \neq n\pi$. Then we require

$$\cos\theta' = \cos\theta - (bd+ac)\sin\theta = \cos\theta + (db+ca)\sin\theta,$$

hence $2(bd+ac)\sin\theta = 0$ and because $\sin\theta$ was assumed to be non-zero this is the same as $bd + ac = 0$. Thus for instance the element $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ conjugates SO_2 to another subgroup that has only trivial intersection.

Finally it is worth noting that this is not an issue at the finite places, namely if $K = K_f K_\infty$ is our maximal compact subgroup of $G(\mathbb{A})$ then K_f is also open and hence $K_f \cap gK_fg^{-1}$ is of finite index in both K_f and gK_fg^{-1} and so their notions of K-finiteness will agree.

For this reason we will need to talk about $(\mathfrak{g}, K)$-modules:

Definition 3.2 ([GH24], 4.4.6)**.** Let G be a *real* Lie group (for example the analytification of the real or complex points of our favourite reductive LAG) and K be a maximal compact subgroup of G. Let $\mathfrak{g}_\mathbb{C}$ be the complexification of the real Lie algebra of G and $\mathfrak{k}$ the real Lie algebra of K.

A $(\mathfrak{g}, K)$**-module** is a complex vector space V with two representations

$$\tilde{\pi} : \mathfrak{g} \to End(V), \quad \pi : K \to \mathrm{GL}(V),$$

satisfying the following axioms

1. V decomposes into a countable direct sum of finite dimensional K representations.

2. The representations should be compatible: For all $X \in \mathfrak{k}$ and $v \in V$

$$\tilde{\pi}(X)(v) = \frac{\mathrm{d}}{\mathrm{d}t}\pi(e^{tX})(v)|_{t=0} = \lim_{t\to 0}\frac{\pi(e^{tX})(v) - v}{t}.$$

   In particular the right hand limit exists

3. And compatible with the adjoint representation: For $k \in K$ and $X \in \mathfrak{g}$

$$\pi(k)\tilde{\pi}(X)\pi(k^{-1})(v) = \tilde{\pi}(Ad(k)(X))(v).$$

Remark 3.3. It is common to use the same symbol for both of these representations in the $(\mathfrak{g}, K)$-module. It is also important to note that these are purely algebraic representations, there is no condition of continuity etc.

If $\mathfrak{g}$ is the Lie algebra of $G(F_\infty)$ and $K_\infty \subseteq G(F_\infty)$ is a maximal compact subgroup in good position we can define a $(\mathfrak{g}, K_\infty)$-module structure on the space of automorphic forms as follows. Recall that by definition we have that

$$\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) \subseteq C^\infty(G(\mathbb{A}_f)) \otimes C^\infty(G(F_\infty)).$$

If $\varphi_f \otimes \varphi_\infty \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ then K_∞ acts by

$$k.(\varphi_f \otimes \varphi_\infty)(g) \;:=\; \varphi_f(g) \otimes \varphi_\infty(gk),$$

i.e. via the right regular representation on the archimedean component. The lie algebra of $G(F_\infty)$ acts as we have previously defined in section 2.1 via

$$z.(\varphi_f \otimes \varphi_\infty)(g) = \varphi_f(g) \otimes \frac{\partial}{\partial t}\varphi_\infty(ge^{tz})|_{t=0}.$$

Lets check the three conditions. (2) is immediate from the definitions of the two representations and the fact that automorphic forms are smooth.

Lemma 3.4. *$\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ decomposes into a countable direct sum of finite dimensional K_∞ representations.*

Lemma 3.5. *For all $k \in K_\infty$ and $X \in \mathfrak{g}$ we have that*

$$\varphi_f(g) \otimes \frac{\partial}{\partial t}\varphi_\infty(gk^{-1}e^{tX}k)|_{t=0} = \varphi_f(g) \otimes \frac{\partial}{\partial t}\varphi_\infty(g^{-1}e^{tkXk^{-1}})|_{t=0}.$$

Is this right...

3.2 Automorphic Representations

Recall that if A, B, C are all R modules and we have the inclusions of R modules $C \subseteq B \subseteq A$ then we call B/C a subquotient of A. We now think of $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ as being a $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$ module. An automorphic representation is then a subquotient of this representation.

Remark 3.6. Some authors will require that representation is by definition an *irreducible* subquotient.

Remark 3.7. We really need a set theoretic definition here. The quotient of these modules cannot be considered up to isomorphism of $(\mathfrak{g}, K)$-modules but must be the classical set theoretic realisation of this object, defined as equivalence classes of elements of the module. This is to say if one were to think of the category of automorphic representations it is *much smaller* than the category of $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$-modules (in particular the cardinality of the category of automorphic representations is bounded, whilst there is a proper class of $G(\mathbb{A}_f)$ -modules). The reason is that we will want to talk about the automorphic forms themselves, and consider their properties.

Remark 3.8. Automorphic representations can also be defined as representations of an algebra $\mathcal{H}$, the global Hecke algebra. This is the approach in [BC79, I.II(4.6)], and can be a helpful perspective to simplify definitions. This is also a motivation behind why Harish-Chandras $(\mathfrak{g}, K)$-modules are the “right” replacement for the regular representation.

Example 3.9. *It is very hard to really write down something explicit. One thing that we can do is take a modular form f. Then we know how to associate a concrete automorphic form $\tilde{f}$. To this (or any fixed automorphic form) we have an* *automorphic representation given by acting on this vector:*

$$\mathrm{span}_{\mathbb{C}}\left\{ \big(G(\mathbb{A}_f) \times (\mathfrak{g}, K)\big).\tilde{f} \right\} \subseteq \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$$

3.2.1 Cuspidal Representations

Recall that an automorphic form $\varphi \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ is called **cuspidal** if all its constant terms vanish, see section 5.1 for more detail on constant terms. The space of such automorphic forms is denoted $\mathcal{A}_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$. An automorphic representation is called **cuspidal** if it is an irreducible subquotient of $\mathcal{A}_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$.

Remark 3.10. Again this is not as a $(\mathfrak{g}, K)$-module.

3.2.2 Isotypic Components

Following the convention of [MW95, II.1] we make two cases: Let π be an irreducible subquotient of the space $\mathcal{A}(M(k)\backslash M(\mathbb{A}))$, that is *not cuspidal*. Then we denote the π isotypic component of $\mathcal{A}(M(k)\backslash M(\mathbb{A}))$ by $\mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi$.

We will also need the space

$$\begin{aligned} &\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi \\ &\quad := \{\varphi \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) : \forall k \in K,\ \varphi_k \in \mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi\} \end{aligned}$$

where $\varphi_k : M(\mathbb{A}) \to \mathbb{C}$ is given by $\varphi_k(x) = \varphi(xk)$.

Now if π is cuspidal, we define $\mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi$ to be the isotypic component of π in $\mathcal{A}_0(M(k)\backslash M(\mathbb{A}))$ and similarly we have

$$\begin{aligned} &\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi \\ &\quad := \{\varphi \in \mathcal{A}_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) : \forall k \in K,\ \varphi_k \in \mathcal{A}_0(M(k)\backslash M(\mathbb{A}))_\pi\} \end{aligned}$$

Remark 3.11. We cannot simply take the isotypic components as $(\mathfrak{g}, K)$-modules we need to take the isotypic components after explicitly restricting the spaces. This is to say again that the category of automorphic reps is very explicit.

The point is that we want the isotypic component corresponding to a cuspidal representation to be cuspidal, however this just might not be the case. Yamana in [Yam13, Rm. 7.12] has a counter example when one allows unitary groups over division algebras (non-commutative fields). It could be interesting to investigate this example more closely to see if the example can be pulled back to a unitary group over a field. In [Yam13] there is an automorphic representation of the quarternionic unitary group constructed, $\Pi(V)$, that appears in both the cuspidal and residual spectrum. By that Yamana means that up to isomorphism the representation can been seen in both residual and cuspidal spectrum. In particular if one were to take the component that is in the cuspidal spectrum and look at its isotypic component then the versions in the residual spectrum would also occur and hence by definition of residual spectrum would not be cuspidal.

If we restrict to the cases dealt with in for instance [MW95], namely not dealing with quarternions, then we have been told that this is an open problem whether or not this restriction is superfluous.

Chapter 4

Eisenstein Series

The Eisenstein series is from our perspective the most important tool in the theory of automorphic forms. Some surveys on its role, properties and open problems are [Lap22], [Art79], [Kim] and [Jia08]. To see the relation to the classical Eisenstein series there is [Gar16] which we will also go through in section 4.2. One thing that Eisenstein series do, as in the theory of modular forms, is that they furnish us with quasi-concrete examples of automorphic forms. Another reason that these functions are important is through their normalisation and constant terms, in which products of L functions appear, we discuss this more in section 5.1. This has been a fruitful method for proving theorems about L-functions as in [Sha10][Pol][Art79], or conversely proving theorems about Eisenstein series [JLZ13].

4.1 Eisenstein Series

As usual we fix a classical group G defined over a number field F, with a Borel B and a standard parabolic with Levi decomposition $P = MU$.

Following the setup in [MW95, I.1.4] we consider a **character** $\chi \in \mathrm{Rat}(M) := \mathrm{Hom}_{\mathrm{LAG}}(M, \mathbb{G}_m)$, thinking of it below as a natural transformation, and then define

$$|\chi| : M(\mathbb{A}) \to \mathbb{C}, \quad (m_\nu) \mapsto \prod_\nu |\chi(F_\nu)(m_\nu)|_\nu.$$

The intersection of the kernels of these characters is

$$M^1 := \bigcap_{\chi \in \mathrm{Rat}(M)} \ker |\chi|.$$

Thus we can define

$$X_M \;:=\; \mathrm{Hom}_{\mathrm{TopGroup}}(M(\mathbb{A})/M^1, \mathbb{C}^*).$$

i.e. the collection of characters of $M(\mathbb{A})$ that are trivial on M^1.

Remark 4.1. To make it seem less mysterious we comment that this group has some importance in the more general theory. It is one of the pieces in the "Langlands decomposition" of the Archimedean points of a parabolic $P = MU$, if ν is an archimedean place then,

$$P(F_\nu) = A_M M(F_\nu) U(F_\nu).$$

We will not define A_M. It also has the property that $M(\mathbb{Q})\backslash M(\mathbb{A})^1$ has finite measure [GH24, 4.9].

The set of **complex characters** of M,

$$\mathfrak{a}_M^* \;:=\; \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{C},$$

is isomorphic as $\mathbb{C}$ vector spaces to X_M. If $Z_{G(\mathbb{A})}$ is the center of $G(\mathbb{A})$ then we also have the space

$$X_M^G \;:=\; \mathrm{Hom}_{\mathrm{TopGroup}}((M(\mathbb{A})/M^1)/Z_G, \mathbb{C}^*)$$

which is characters of $M(\mathbb{A})/M^1$ which are also trivial on the center of G.

Example 4.2. *For the maximal parabolic P_r with Levi M_r of Sp_{2n} we have that $X_{M_r}^{\mathrm{Sp}_{2n}}$ is at most a one dimensional $\mathbb{C}$ vector space.*

First of all we have that [MW95, I.1.4]

$$X_{M_r}^{\mathrm{Sp}_{2n}} \subseteq X_{M_r} \cong \mathfrak{a}_{M_r}^* \;:=\; Rat(M_r) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Thus it is clearly sufficient to bound the dimension of $\mathfrak{a}_{M_r}^*$ as a $\mathbb{C}$ vector space, moreover this dimension agrees with the dimension of $Rat(M_r)$ as a free $\mathbb{Z}$ module.

Thus we compute $\dim_{\mathbb{Z}}(Rat(M_r))$*:*

$$\begin{aligned}
Rat(M_r) &= Rat(\mathrm{GL}_r \times \mathrm{Sp}_{2m}) \\
&= \mathrm{Hom}(\mathrm{GL}_r \times \mathrm{Sp}_{2m}, \mathbb{G}_m) \\
(1) &\cong \mathrm{Hom}(\mathrm{Ab}(\mathrm{GL}_r \times \mathrm{Sp}_{2m}), \mathbb{G}_m) \\
(2) &\cong \mathrm{Hom}(\mathrm{Ab}(\mathrm{GL}_r) \times \mathrm{Ab}(\mathrm{Sp}_{2m}), \mathbb{G}_m) \\
(3) &\cong \mathrm{Hom}(\mathbb{G}_m \times 1, \mathbb{G}_m) \\
&\cong \mathbb{Z}.
\end{aligned}$$

In (1) we have used the universal property of the abelianization $\mathrm{Ab}(G) = \mathcal{D}(G)\backslash G = [G,G]\backslash G$ *because* $\mathbb{G}_m$ *is abelian. (2) is that the abelianization commutes with direct products. (3) is because* Sp *is a perfect group.*

There is the natural map $m_P : G(\mathbb{A}) \to M^1\backslash M(\mathbb{A})$ sending $umk \mapsto M^1m$, where $g = umk$ using the Langlands-Iwasawa decomposition of equation 1.2.1.

Now if we take the collection of irreducible automorphic representations of M,

$$\hat{\mathcal{M}} \;:=\; \{(\pi, V) : \pi \text{ is an irreducible automorphic representation of } M\}$$

then we can think of X_M^G as being one dimensional automorphic representations (with some extra invariance) and so there is a natural action on $\hat{\mathcal{M}}$ given by tensoring, i.e. if $\lambda \in X_M^G$ and $(\pi, V) \in \hat{\mathcal{M}}$ then

$$\lambda.\pi \;:=\; \lambda \otimes \pi$$

Then $\hat{\mathcal{M}}$ decomposes as a disjoint union of its orbits. The orbit $\mathfrak{P}$ of a cuspidal representation π_0 is called a **cuspidal datum**. By definition X_M^G acts transitively on any cuspidal datum $\mathfrak{P}$ but by [MW95, II.1] it also acts freely. Thus $\mathfrak{P}$ is in bijection with X_M^G. Through this bijection we transmit the complex structure on $\mathfrak{a}_M^*$ to X_M then to the quotient X_M^G and finally to $\mathfrak{P}$.

Let $\mathfrak{P}$ be a cuspidal datum with a complex structure as above. Let $\pi \in \mathfrak{P}$ and $\varphi_\pi \in \mathcal{A}(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_\pi$, then $\lambda \in X_M^G$ acts on φ_π by

$$\lambda.\varphi_\pi(g) = (\lambda \circ m_P)(g)\varphi_\pi(g).$$

which is then an element of $\mathcal{A}(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_{\pi\otimes\lambda}$. Finally we have the **Eisen-**

stein series which is defined by the following sum

$$E(\varphi_\pi, \lambda, g) = \sum_{\gamma \in P(k)\backslash G(k)} \lambda.\varphi_\pi(\gamma g)$$

whenever it is convergent. The first thing to note is that for a fixed φ there is an open set in X_M^G and a compact subset of $G(k)\backslash G(\mathbb{A})$ such that the Eisenstein series converges (normally) [MW95, II.1.5].

If $P = MU, P' = M'U'$ are two standard parabolics of G that are conjugate, i.e. such that for $w \in G(k)$ we have $wMw^{-1} = M'$. Then w maps $\mathfrak{P}$ to $w\mathfrak{P}$, an orbit of an irreducible representations of M to an orbit of irreducible representations of M'. The Eisenstein series is closely related through its constant terms (as discussed in section 5.2.3) to the operator

$$M(w, \pi)(\varphi_\pi)(g) = \int_{(U'(k)\cap wU(k)w^{-1})\backslash U'(\mathbb{A})} \varphi_\pi(w^{-1}ug)du$$

where $\pi \in \mathfrak{P}$, $g \in G(\mathbb{A})$ and $\varphi_\pi \in \mathcal{A}(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_\pi$.

The Eisenstein series has three inputs and can be considered as a function in different variables, it can be a tedious task to specify the correct domain and codomains for these maps however. If π is a cuspidal automorphic representation induced from P, then for a fixed $\varphi \in \mathcal{A}_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\pi$ the Eisenstein series $E(\varphi)$ can be thought of as a function from some open subset of the cuspidal datum $\mathfrak{P}$ into $L^2_{\text{loc}}(G)$, the set of locally square integrable complex valued functions on $G(\mathbb{A})$, given by

$$E(\varphi)(\lambda)(g) = \sum_{\gamma \in P(k)\backslash G(k)} \lambda.\varphi(\gamma g), \quad \lambda \in \mathfrak{P},\ g \in G(\mathbb{A}),$$

where it converges. The space $L^2_{\text{loc}}(G(\mathbb{A}))$ can be endowed with a Frechet space structure coming from the semi-norms associated to compact sets $C \subseteq G(\mathbb{A})$ given by

$$\varphi \mapsto \|\varphi|_C\|_{L^2} = \int_C |\varphi(x)|^2 \mathrm{d}x.$$

Then it makes sense to talk about the holomorphicity of $E(\varphi)$ in this sense [MW95, I.4.9]. The key properties of both the Eisenstein series and this operator can be found in [MW95, IV.1.8, IV.1.9, IV.1.10, IV.1.11]. Most importantly as a function of $\mathfrak{P}$ it can be shown that they both have a meromorphic continuation to all of $\mathfrak{P}$. This was also given a second "soft proof" more recently in [BL22], with the spectral decomposition that follows from it also being worked out in [Del21]. Moreover an Eisenstein series attached to an automorphic form, at a point $p \in \mathfrak{P}$ at which it is holomorphic, is also an automorphic form.

4.2 Classical Eisenstein Series

The typical example of a classical Eisenstein series is that introduced by Maas in 1949 given by the sum [Lap22]

$$\mathbf{E}(z,s) \quad := \quad \frac{1}{2} \sum_{m,n \in \mathbb{Z},\ \text{coprime}} \frac{\operatorname{Im}(z)^{s+\frac{1}{2}}}{|mz+n|^{2s+1}}, \qquad z \in \mathcal{H}, \quad \operatorname{Re}(s) > \frac{1}{2},$$

which converges absolutely. Consider the algebraic group SL_2 with the parabolic of upper triangular matrices P.

Lemma 4.3.

$$P(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{Z}) \cong \{(m,n) \in \mathbb{Z}^2 : m, n \ \textit{are co-prime}\}$$

Lemma 4.4. *Recall that* $\mathrm{SL}_2(\mathbb{Z})$ *acts via Mobius transformations on the upper half plane. If* $z = x + iy \in \mathcal{H}$*,* $s \in \mathbb{C}$ *and* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *then we have,*

$$\operatorname{Im}(\gamma.z)^s = \frac{y^s}{2|cz+d|^{2s}}$$

Lemma 4.5 ([Gar16], 3.5)**.**

$$P(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{Z}) \cong P(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{Q}).$$

Its stated for GL

Hence the classical Eisenstein series is

$$\mathbf{E}(z,s) \quad := \quad \frac{1}{2} \sum_{m,n \in \mathbb{Z},\ \text{coprime}} \frac{\operatorname{Im}(z)^{s+\frac{1}{2}}}{|mz+n|^{2s+1}} = \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{Q})} \operatorname{Im}(\gamma.z)^{s+\frac{1}{2}}.$$

We want to realize this as the Eisenstein series associated to an automorphic form so first we must design a function on $\mathrm{SL}_2(\mathbb{A})$. For any place ν of $\mathbb{Q}$ we have the local Iwasawa decomposition $\mathrm{SL}_2(\mathbb{Q}_\nu) = P(\mathbb{Q}_\nu)K_\nu$ where

$$K_\nu \;:=\; \begin{cases} \mathrm{SL}_2(\mathbb{Z}_\nu), & \nu \text{ non-Archimedean} \\ SO_2(\mathbb{R}), & \nu \text{ Archimedean} \end{cases}$$

are the local maximal compact subgroups. We define a function on the adeles by defining it on the local pieces,

$$\varphi_{\nu,s}\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} k\right) \;:=\; \left|\frac{a}{d}\right|_\nu^s.$$

Finally we define φ_s as the map

$$(g_\nu)_\nu \mapsto \prod_\nu \varphi_{\nu,s}(g_\nu.)$$

Lemma 4.6. *φ_s is an automorphic form on $\mathrm{SL}_2(\mathbb{A})$.*

To this we have an Eisenstein series associate as in the adelic setting by

$$E(\varphi, g) \;:=\; \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} \varphi_s(\gamma g)$$

Lemma 4.7. *Let $g \in \mathrm{SL}_2(\mathbb{R})$ then we consider it as an element of $\mathrm{SL}_2(\mathbb{A})$, denoted by $\iota(g)$, by setting all other entries to 1. Then*

$$E(\varphi_s, \iota(g)) = \mathbf{E}(g.i, s)$$

Chapter 5

Constant Terms of Eisenstein Series

This section is a discussion of the adelic constant term, especially its application to the Eisenstein series.

Through constant terms we can define cusp forms which play a central role in the theory of automorphic forms. They appear historically as interesting examples such as the Ramanujan tau function, by a theorem of Ribet [SZS77, T2.3] the Galois representation associated to a cusp form is irreducible and they form the "base case" for the proof of the spectral decomposition in [MW95].

Constant terms preserve analytic properties whilst sometimes reducing the functions to more tractable forms. This is how they will be used in our calculation of poles of Eisenstein series.

5.1 Definition and Role

Consider $P = MU$ a standard parabolic of a classical group G and $\varphi : U(k)\backslash G(\mathbb{A}) \to \mathbb{C}$ a measurable and locally L^1 function then its **constant term** along P is defined to be [MW95, I.2.6],

$$\varphi_P : U(\mathbb{A}) \setminus G(\mathbb{A}) \to \mathbb{C},$$
$$\varphi_P(g) \;\; := \;\; \int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\mathrm{d}u.$$

We have dedicated the next chapter (6) to showing how this is related to classical notions of constant terms. If φ is smooth or moderate growth then so is its constant term. Moreover if φ is an automorphic form on $G(\mathbb{A})$ then its constant term is an automorphic form on $M(\mathbb{A})$ [GH24, 6.5].

Let φ be an automorphic form on $U(\mathbb{A})M(k) \setminus G(\mathbb{A})$ for $P = MU$ a standard parabolic. Then φ is **cuspidal** if for all standard parabolics $P' \subset P$ we have that $\varphi_{P'}$ is identically zero.

Theorem 5.1 ([MW95], I.4.10). *Let $P = MU$ be a standard parabolic of G. If π is a cuspidal automorphic representation induced from P, then for a fixed $\varphi \in \mathcal{A}_0(U(\mathbb{A})M(k)\setminus G(\mathbb{A}))_\pi$ the Eisenstein series E can be thought of as a function from some open subset of the cuspidal datum $\mathfrak{P}$ into $L^2_{\text{loc}}(G(\mathbb{A}))$ given by*

$$E(p)(g) = \sum_{\gamma \in P(k)\setminus G(k)} \lambda.\varphi(\gamma g), \quad p \in \mathfrak{P},\ g \in G(\mathbb{A}),$$

where it converges. If $D \subseteq \mathfrak{P}$, is an open subset minus a finite number of points on which E is holomorphic then E has a holomorphic continuation to the finite number of points if and only if the constant term of E_Q has a holomorphic continuation to these finite number of points for all standard parabolics Q.

Remark 5.2. The theorem in Moeglin and Waldspurger is proved in much more generality, however after sufficient symbol pushing this is the essence.

So one can say that the poles of an Eisenstein series are controlled by its constant terms. We can say more:

Theorem 5.3 ([MW95], II.1.7). *The constant term of an Eisenstein series induced from a standard maximal parabolic P is zero along any other standard parabolic P' unless $P = P'$.*

Putting these two theorems together we see that for an Eisenstein series induced from a maximal parabolic P, has a holomorphic continuation to a point if and only if its constant term along P has a holomorphic continuation.

5.2 Constant Terms of Eisenstein Series

This computation forms the heart of a well known theorem, [GH24, Prop 10.4.2][MW95, II.1.7][Sha10, 6.2], although for an amateur the detail is lacking in other presentations. Notice that the Eisenstein series has a full $G(k)$ invariance and so we can take its constant terms along *any* standard parabolic.

5.2.1 In General

We will use the following Lemmas to give a simplified expression of the constant term of an Eisenstein series. First fix $P = MU$ and $P' = M'U'$ two standard parabolics of a suitable group G over a number field F, with $E(x, \varphi, \lambda)$ defined from P as in section 4.1.

define the Weyl group defined here

Lemma 5.4.

$$P(F) \setminus G(F) \cong \coprod_{w \in W_{M'} \setminus W_G / W_M} P'(F) \cap wP(F)w^{-1} \setminus P'(F)$$

Proof. Consider the Bruhat decomposition:

$$G(F) = \coprod_{w \in W_{M'} \setminus W_G / W_M} P(F)w^{-1}P'(F)$$

then because the action of $P(F)$ keeps the disjoint sets disjoint we can move the quotient through and get

$$P(F) \setminus G(F) = \coprod_{w} P(F) \setminus P(F)w^{-1}P'(F)$$

so we analyse the summands, by the second isomorphism theorem we have a bijection

$$P(F) \setminus P(F)w^{-1}P'(F) \cong P(F) \cap P'(F) \setminus w^{-1}P'(F)$$

now if $[w^{-1}p] \in P(F) \cap P'(F) \setminus w^{-1}P'(F)$ then its represented by some $pw^{-1}p'$ where $p \in P(F) \cap P'(F)$ and hence multiplying by w, in particular an isomorphism, gives $wpw^{-1}p' \in wP(F)w^{-1} \times P'(F)$ and so

$$w(P(F) \cap P'(F) \setminus w^{-1}P'(F)) \cong wP(F)w^{-1} \cap P'(F) \setminus P'(F)$$

Lemma 5.5. *Let $m' \in M'(F), u' \in U'(F)$ then*

$$m'u' \in wP(F)w^{-1} \iff m' \in wP(F)w^{-1} \textit{ and } \quad u' \in (m')^{-1}wP(F)w^{-1}m'$$

Proof. The forward implication is stated in [GH24], the converse follows from some algebra: First let $m' = wp_1w^{-1}$ and $u' = (m')^{-1}wp_2w^{-1}m'$ then

$$\begin{aligned} m'u' &= (wp_1w^{-1})^{-1}wp_2w^{-1}wp_1w^{-1} \\ &= wp_1^{-1}w^{-1}wp_2w^{-1}wp_1w^{-1} \\ &= wp_1^{-1}p_2p_1w^{-1} \in wP(F)w^{-1} \end{aligned}$$

Taking the contrapositive of this lemma will be used below. This is because our sums will be over quotients like $A \setminus B$ and therefore summing over the "elements" in B that are not in A; by our lemma would be the same as summing over two different such quotients. Now we will apply our lemmas to simplify and make more explicit the constant term of an Eisenstein series. Denote $[U'] := U'(F) \setminus U'(\mathbb{A})$

$$\begin{aligned} E_{P'}(\varphi, \lambda, x) &= \int_{U'(F)\setminus U'(\mathbb{A})} E(\varphi, \lambda, nx)du \\ &= \int_{[U']} \sum_{\delta \in P(F)\setminus G(F)} \lambda.\varphi(\delta nx)du \\ \text{(Lemma 5.4)} \quad &= \int_{[U']} \sum_{\delta \in \coprod_{w\in W_{M'}\setminus W_G/W_M} P'(F)\cap wP(F)w^{-1}\setminus P'(F)} \lambda.\varphi(\delta ux)du \\ &= \sum_{w\in W_{M'}\setminus W_G/W_M} \int_{[U']} \sum_{p'\in P'(F)\cap wP(F)w^{-1}\setminus P'(F)} \lambda.\varphi(w^{-1}p'ux)du \end{aligned}$$

Now apply Lemma 5.5 to the above sum and we get the equality

$$\begin{aligned} &= \sum_w \sum_{m'\in M'(F)\cap wP(F)w^{-1}\setminus M'(F)} \int_{[U']} \sum_{u'\in U'(F)\cap(m')^{-1}wP(F)w^{-1}m'\setminus U'(F)} \lambda.\varphi(w^{-1}m'u'ux)du \\ \text{(Change Var)} \quad &= \sum_w \sum_{m'} \int_{[U']} \sum_{n'\in U'(F)\cap wP(F)w^{-1}\setminus U'(F)} \lambda.\varphi(w^{-1}u'um'x)du \\ \text{(Unfold)} \quad &= \sum_w \sum_{m'} \int_{U'(F)\cap wP(F)w^{-1}\setminus U'(\mathbb{A})} \lambda.\varphi(w^{-1}um'x)du. \end{aligned}$$

The change of variables is $(m', u') \mapsto ((m')^{-1}u'm', (m')^{-1}u'm')$. Again we assume that our x is sufficiently large so all the integrals converge.

Ok I need to expliain how this analysis is working. The eisenstein series is only defined bby the meromorphic continuation of this sum. So if we calculate the constant term and then meromorphiccally continue its the same I guess is the claim?

5.2.2 Constant Terms of Cuspidal Eisenstein Series

Lemma 5.6. *For $w \in W_{M'} \setminus W_G / W_M$ we have that $w^{-1}P'w \cap M$ is a standard parabolic of M with Levi $w^{-1}M'w \cap M$ and unipotent $w^{-1}U'w \cap M$.*

Proof. This is [GH24, 10.4.1] stated without proof. They give the reference [RS, V.4.6] which is in French..

Lemma 5.7.

$$w^{-1}U'w \cap P = (w^{-1}U'w \cap M)(w^{-1}U'w \cap U).$$

Proof. [GH24, 10.4.1] has some decompositions, as well as the standard decomposition of $P = MU$ I think I could prove this...

Lemma 5.8.

$$c \setminus (b \setminus a) = (bc) \setminus a$$

need to fill in these lemmas

Continuing the computation of the constant term above, we will focus purely on the inner integral now

$$\begin{aligned}
&\int_{U'(F)\cap wP(F)w^{-1}\setminus U'(\mathbb{A})} \lambda.\varphi(w^{-1}um'x)du \\
&\qquad = \int_{w^{-1}U'(F)w\cap P(F)\setminus w^{-1}U'(\mathbb{A})w} \lambda.\varphi(uw^{-1}m'x)du \\
&\text{(Lemma 5.7)} = \int_{(w^{-1}U'w\cap M)(w^{-1}U'w\cap U)(F)\setminus w^{-1}U'(\mathbb{A})w} \lambda.\varphi(uw^{-1}m'x)du.
\end{aligned}$$

where the first equality is the change of variables $w^{-1}uw \mapsto u$. Denote $A = (w^{-1}U'(F)w \cap U(F)) \setminus w^{-1}U'(\mathbb{A})w$. If we apply Lemma 5.8 and unfold we get the equality

$$= \int_{(w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A}))\setminus A} \int_{w^{-1}U'(F)w\cap M(F)\setminus w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A})} \lambda.\varphi(u_1u_2w^{-1}m'x)du_1du_2.$$

Now look at the inner integral here more closely

$$\int_{w^{-1}U'(F)w\cap M(F)\setminus w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A})} \lambda.\varphi(u_1u_2w^{-1}m'x)du_1du_2,$$

applying Lemma 5.8 we see that this is a constant term for a parabolic of M, of the function $m \mapsto \varphi(mu_2w^{-1}m'x)$.

Lemma 5.9. *$u_2w^{-1}m'x \in K$ with variables as above.*

This was in complete generality. If we now assume further that the Eisenstein series was induced from a *cuspidal* automorphic representation, then $m \mapsto \varphi(mk)$ is a cusp form and therefore this last integral will vanish whenever $w^{-1}U'w \cap M \neq \{1\}$, because in that case the inner integral doesn't exist (its over a point).

5.2.3 Constant Term Of Eisenstein Series for Conjugate Levis

If we now assume that $M' = wMw^{-1}$ and recall the definition of our intertwining operator from section 4.1 we can use the following

Lemma 5.10 ([MW95] II.1.7 (6))**.**

$$U'(k) \cap wP(k)w^{-1} = U'(k) \cap wU(k)w^{-1},$$

to see that

$$\begin{aligned} E_{P'}(\varphi, \lambda, x) &= \sum_w \sum_{m'} \int_{U'(F)\cap wP(F)w^{-1}\backslash U'(\mathbb{A})} \lambda.\varphi(w^{-1}um'x)du \\ &= \sum_w \sum_{m'} \int_{U'(k)\cap wU(k)w^{-1}\backslash U'(\mathbb{A})} \lambda.\varphi(w^{-1}um'x)du \\ &= \sum_w \sum_{m'} M(w,\pi)(\lambda.\varphi)(x) \end{aligned}$$

In particular we can combine the conjugate and cuspidal cases to get a much simpler expression for some constant terms of some Eisenstein series, we will go through a detailed example in the final chapter 8.

Chapter 6

Siegel Phi Function

In the last chapter we saw some general computations around constant terms of automorphic forms. This chapter we continue with more computations, however we attempt to relate the constant term to the classical setting. This should be understood as a continuation of the example of modular forms as archimedean automorphic forms, as here we investigate the constant term in this setting in order to gain similar intuition. Specifically we will relate the constant term to the Fourier series constant term and the Siegel Phi operator.

We thank Chengjing Zhang for showing us this example, and present it here in detail because we cannot find it in the literature.

6.1 Constant Terms

Let G be a classical group over a number field $\mathbb{Q}$. For an archimedean automorphic form $f : G(\mathbb{R}) \to \mathbb{C}$ its **constant term** along a parabolic of G, $P = MN \subseteq G$, is defined to be [GH24, 8.6]

$$f(x)_P = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} f(xn)\mathrm{d}n.$$

To effectively compute constant terms we will routinely use the following two lemmas.

Theorem 6.1. *If G is a locally compact Hausdorff group with a left Haar measure μ and if $\chi\colon G \to \mathbf{C}^\times$ is a non-trivial character on G, then*

$$\int_G \chi(g)\, d\mu(g) = 0.$$

Proof. Pick an element h of G such that $\chi(h) \neq 1$. The equation above then follows from

$$\int_G \chi(g)\, d\mu(g) = \int_G \chi(hg)\, d\mu(g) = \int_G \chi(h)\chi(g)\, d\mu(g) = \chi(h) \int_G \chi(g)\, d\mu(g).\square$$

Integrating trivial characters gives the volume of the measure space which we typically normalise to be one.

Theorem 6.2 (Unfolding, [Gar] 5.2, [Fol16] Thm 2.49)**.** *Let $H \leq G$ be a closed subgroup. If $H \setminus G$ has a right G invariant measure (iff their modular functions agree on H) then the integral is unique up to scalar, namely for a given Haar measures dh on H and dg on G there is a unique invariant measure dq on $H \setminus G$ such that for all $f \in C_c^0(G)$*

$$\int_{H\setminus G} \int_H f(hq)dhdq = \int_G f(g)dg.$$

Note that this quotient may not be a group, because H is not required to be normal. The use of this lemma is called **unfolding** the integral.

6.2 Siegel Modular Forms

Following [BVDGHZ08]. Recall the **Siegal upper half plane** of "genus" $g \in \mathbb{N}$ is

$$\begin{aligned}\mathcal{H}_g &:= \{\tau \in \mathrm{M}_{g\times g}(\mathbb{C}) : \tau \text{ is symmetric and has positive definite imaginary part}\} \\ &\cong \mathrm{Sp}_{2g}(\mathbb{R})/U(g).\end{aligned}$$

where the isomorphism is as analytic manifolds and

$$U(g) := \left\{ \begin{pmatrix} A & B \\ -B & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}) : AA^t + BB^t = 1 \right\}.$$

For every $\gamma = (A\ B;\ C\ D) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\tau \in \mathcal{H}_g$ we have the action

$$\gamma.\tau = (A\tau + B)(C\tau + D)^{-1}.$$

We say that a holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ is a (classical) **Siegel modular**

form of weight k if

$$f(\gamma.\tau) = \det(C\tau + D)^k f(\tau),$$

with the extra condition that if $g = 1$ it must be holomorphic at ∞. Because $\mathrm{Sp}_2 = \mathrm{SL}_2$ this is a strict generalisation of an (elliptic) modular form.

The space of Siegel modular forms of weight k and genus g is denoted $\mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$. There is a useful operator know as the **Siegel Phi Operator** which allows you to lift known modular forms from lower genus to higher genus [BVDGHZ08, 5]

$$\mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z})) \xrightarrow{\Phi} \mathcal{M}_k(\mathrm{Sp}_{2(g-1)}(\mathbb{Z})),$$

defined by the limit for $\tau \in \mathcal{H}_{g-1}$

$$\Phi(f)(\tau) \ := \ \lim_{t\to\infty} f\begin{pmatrix} \tau & \\ & it \end{pmatrix}.$$

In this context a cusp form is defined to be a Siegel modular form in the kernel of the Siegel Φ operator and so it is natural to wonder if there is a constant term that is being taken here.

6.2.1 Automorphising

Just as in the case of modular forms, given a Siegel modular form $f \in \mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ we can associate an automorphic form

$$\tilde{f} : \mathrm{Sp}_{2g}(\mathbb{R}) \to \mathbb{C}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \det(ci+d)^{-k} f\Big((ai+b)(ci+d)^{-1}\Big),$$

where a, b, c, d are $g \times g$ matrices such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{R})$. Fix the Borel of upper triangular matrices. Now for $1 \leq r \leq g-1$ an integer we have the standard maximal parabolic of Sp_{2g}, $P_r = M_r N_r$ such that

$$M_r \cong \mathrm{GL}_r \times \mathrm{Sp}_{2(g-r)}$$

Theorem 6.3 (Zhang). *If f is a classical Siegel modular form of weight k and degree g, then*

$$\tilde{f}_{P_r}(u\gamma) = \det u^k \cdot (\Phi^r f)^{\sim}(\gamma) \tag{6.2.1}$$

for every element γ of $\mathrm{Sp}_{2(g-r)}(\mathbb{R})$ and every element u of $\mathrm{GL}_r(\mathbb{R})$.

In particular

$$\tilde{f}_{P_{g-1}}\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\Phi f)^{\sim}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This shows that perhaps the correct generalisation of the Siegel Phi function is just the constant term that we all know and love. We could also attempt to expand this to Siegel modular forms that are vector valued or not of full level.

The only other work on generalising the Siegel Φ operator that we could find appears in [Gre24]. Grenier formulates the Φ operator in the language of symmetric spaces [Ter16, Ch. 2] and then shows that the analogous definition in the case of "automorphic forms" in the sense of the symmetric space $\mathscr{P}_n/\operatorname{GL}_n(\mathbb{Z})$ of symmetric positive definite real matrices [Ter16, 1.5.1] behaves in the same way. Namely his [Gre92, Thm. 2] shows that it sends an automorphic form for $GL_n(\mathbb{Z})$ to an automorphic form for $\operatorname{GL}_{n-1}(\mathbb{Z})$. The point is that the Φ operator can be defined in the generality of symmetric spaces and Grenier shows that at least in one other case it still preserves the relevant notion of automorphic form. This suggests two things that would be interesting to investigate; using the classification of symmetric spaces is it possible to give a uniform definition of the Φ operator following Grenier and does this definition agree with the constant term in the way that the Siegel Φ operator does. With my limited knowledge of symmetric spaces this seems to be very tractable.

6.2.2 Modular Form Case

The base case is very instructive, it deals with modular forms. So consider f a (elliptic) modular form of full level and weight k, which has a Fourier expansion given by

$$f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$$

Then one can verify that

$$\tilde{f}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ci+d)^{-k} f\Big(\frac{ai+b}{ci+d}\Big)$$

is an automorphic form on Sp_2. The only non-trivial parabolic P is the one of upper triangular matricies, with Levi and unipotant given respectively

$$M = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \cong \mathrm{GL}_1, \quad N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cong \mathbb{G}_a$$

along which we can now compute the constant term

$$\begin{aligned} \tilde{f}_P(m) &= \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \tilde{f}(mb)\mathrm{d}b \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} \tilde{f}\begin{pmatrix} m & mb \\ 0 & m^{-1} \end{pmatrix} \mathrm{d}b \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} m^k f(m^2 i + m^2 b)\mathrm{d}b \\ &= m^k a_0 \end{aligned}$$

We have chosen normalisation to remove the usual factor of $1/2\pi$ in the constant term of the Fourier series. Moreover we see that

$$\Phi(f) = \lim_{t\to\infty} f(it) = \lim_{t\to\infty} \sum_{n\geq 0} a_n e^{-2\pi n t} = a_0$$

6.2.3 Simplifying the Constant Term

As we saw in 1.2.1 for $1 \leq r \leq g-1$ an integer we have the standard maximal parabolic of Sp_{2g}, $P_r = M_r N_r$ such that

$$M_r \cong \mathrm{GL}_r \times \mathrm{Sp}_{2(g-r)}$$

which can be given the explicit matrix representations

$$m(\gamma, A) := \begin{pmatrix} A & & & \\ & a & & b \\ & & (A^t)^{-1} & \\ & c & & d \end{pmatrix}, \quad A \in \mathrm{GL}_r(F),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2(g-r)}(F)$$

and unipotent

$$n(s;h,k) \;\; := \;\; \begin{pmatrix} 1 & 0 & 0 & h \\ -k^t & 1 & h^t & s+h^tk \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h,k \in \mathrm{Mat}_{(g-r)\times r}(\mathbb{R}),\; s \in \mathrm{Sym}_r(\mathbb{R})$$

We have the following short exact sequence

prove it

$$1 \to \mathrm{Sym}_r(\mathbb{R}) \to N_r(\mathbb{R}) \to \mathrm{Mat}_{(g-r)\times r}(\mathbb{R}) \times \mathrm{Mat}_{(g-r)\times r}(\mathbb{R}) \to 1.$$

which we will use to unfold our integral below, for compactness we define $H_r := \mathrm{Mat}_{(g-r)\times r}$. We will now denote $[G] := G(\mathbb{Z})\backslash G(\mathbb{R})$ and compute the constant term

$$\begin{aligned} \tilde{f}_{P_r}\big(m(\gamma,A)\big) &= \int_{[N_r]} \tilde{f}\big(nm(\gamma,A)\big)\,dn \\ &= \int_{[H_r\times H_r]} \int_{[\mathrm{Sym}_{g-r}]} \tilde{f}\big(n(s;h,k)m(\gamma,A)\big)\,ds\,d(h,k) \\ &= \int_{[H_r]} \int_{[H_r]} \int_{[\mathrm{Sym}_{g-r}]} \tilde{f}\big(n(s;h,k)m(\gamma,A)\big)\,ds\,dh\,dk. \end{aligned} \tag{6.2.2}$$

Now we focus on simplifying the integrand. We want an explicit form of the matrix so we can relate it back to the value of the un-lifted Siegel modular form f; simply multiply the matrices gives, where (all rings are commutative) $A^{-t} := (A^t)^{-1}$

$$n(s;h,k)m(\gamma,A) = \begin{pmatrix} a & 0 & b & hA^{-t} \\ -k^ta+h^tc & A & -k^tb+h^td & sA^{-t}+h^tkA^{-t} \\ c & 0 & d & kA^{-t} \\ 0 & 0 & 0 & A^{-t} \end{pmatrix}.$$

because $a,b,c,d \in \mathrm{Mat}_{(g-r)\times(g-r)}$, $A \in \mathrm{Mat}_{r\times r}$ we see that the $g\times g$ blocks that we now need to take the determinant of are the 4×4 corners of this picture, hence the matrices below should all be in $\mathcal{H}_g \subseteq \mathrm{Mat}_{g\times g}$

$\tilde{f}(n(s;h,k)m(\gamma,A))$

$$= \det\left(\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} i + \begin{pmatrix} d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}\right)^{-k} \cdot$$

$$f\left(\left(\begin{pmatrix} a & 0 \\ -k^t a + h^t c & A \end{pmatrix} i + \begin{pmatrix} b & hA^{-t} \\ -k^t b + h^t d & sA^{-t} + h^t k A^{-t} \end{pmatrix}\right)\left(\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} i + \begin{pmatrix} d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}\right)^{-1}\right)$$

$$= \det\left(\begin{pmatrix} ic + d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}\right)^{-k} \cdot$$

$$f\left(\begin{pmatrix} ia + b & hA^{-t} \\ -k^t(ia+b) + h^t(d+ic) & iA + sA^{-t} + h^t k A^{-t} \end{pmatrix}\begin{pmatrix} ic + d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}^{-1}\right)$$

$$= \left(\frac{\det(ic+d)}{\det(A)}\right)^{-k} \cdot$$

$$f\left(\begin{pmatrix} ia + b & hA^{-t} \\ -k^t(ia+b) + h^t(d+ic) & iA + sA^{-t} + h^t k A^{-t} \end{pmatrix}\begin{pmatrix} (ci+d)^{-1} & -(ci+d)^{-1}k \\ 0 & A^t \end{pmatrix}\right)$$

$$= \left(\frac{\det(A)}{\det(ic+d)}\right)^{k} f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}, \quad \tau := (ai+b)(ci+d)^{-1}$$

So we have shown that

$$\begin{aligned}
&\tilde{f}_{P_r}(m(\gamma, A)) \\
&= \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} \left(\frac{\det(A)}{\det(ic+d)}\right)^{k} f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} ds\, dh\, dk \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^{k} \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} ds\, dh\, dk
\end{aligned}$$

Again lets focus on this integrand $f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}$ and compute its Fourier expansion, see [BVDGHZ08, 3.4]. Recall that a symmetric matrix $n \in \mathrm{GL}_g(\mathbb{Q})$ is called half integral if $2n$ is integral with even diagonal entries, then a Siegel modular form has a Fourier expansion of the form

$$f(z) = \sum_{n \text{ half integral}} a(n) e^{2\pi i \mathrm{Tr}(nz)}$$

First the space of half integral $g \times g$ matrices, HI_g, decomposes as a direct sum via the (additive) group isomorphism

prove it

$$\mathrm{HI}_{g-r} \oplus \tfrac{1}{2}\mathrm{Mat}_{r\times(g-r)}(\mathbb{Z}) \oplus \mathrm{HI}_r \to \mathrm{HI}_g, \qquad (n,m,l) \mapsto \begin{pmatrix} n & m \\ m^t & l \end{pmatrix},$$

thus unfolding the (discrete) integral we get

$$f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} = \sum_{n\in\mathrm{HI}_{g-r}} \sum_{m\in\frac{1}{2}\mathrm{Mat}_{r\times(g-r)}(\mathbb{Z})} \sum_{l\in\mathrm{HI}_r} a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}$$
$$\exp\left(2\pi i \mathrm{Tr}\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}\right)$$

because all the block sizes are compatible we can "block multiply" the inner matrices and because we are taking the trace we can forget about off diagonal entries

$$\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}$$
$$= \begin{pmatrix} n\tau + m(-k^t\tau + h^t) & * \\ * & m^t(-\tau k + h) + l(k^t\tau k + AA^t i + s) \end{pmatrix}$$

putting this into our Fourier expansion

$$f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}$$
$$= \sum_n \sum_m \sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix} \exp\Big(2\pi i\big(\quad \mathrm{Tr}(n\tau) + \mathrm{Tr}(m(-k^t\tau + h^t)) + \mathrm{Tr}(m^t(-\tau k + h))$$
$$+\mathrm{Tr}(l(k^t\tau k + AA^t i + s))\big)\Big)$$

Tm differs from
ngjing

If we denote $T_l \;:=\; \mathrm{Tr}(l(k^t\tau k + AA^t i + s))$,

$$T_{m,h} \;:=\; \mathrm{Tr}(mh^t + m^t h), \;\; T_{m,k} \;:=\; \mathrm{Tr}(-mk^t\tau - m^t\tau k),$$

and $T_m := T_{m,h} + T_{m,k}$ then we can substitute this back into our constant term

Converges uniformly a priori on compact sets, well I don't know if I can swap all these sums haha

$$
\begin{aligned}
&\tilde{f}_{P_r}\big(m(\gamma, A)\big) \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l\end{pmatrix} \exp\left(2\pi i(\mathrm{Tr}(n\tau) + T_m + T_l)\right) \, ds\, dh\, dk \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l\end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)} \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i(T_m+T_l)} \, ds\, dh\, dk \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l\end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)} \int_{[H_r]} e^{2\pi i T_{m,k}} \int_{[H_r]} e^{2\pi i T_{m,h}} \int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i T_l} \, ds\, dh\, dk
\end{aligned}
$$

Now we use that the integration of unitary characters is very simple 6.1 and the fact that

$$s \mapsto e^{2\pi i T_l}$$

is a non-trivial unitary character of Sym_{g-r} whenever $l \neq 0$ to get that

$$\int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i T_l} \, ds = \begin{cases} 1, & l = 0 \\ 0, & l \neq 0 \end{cases}$$

we repeat this trick with the second integral, which enforces that $m = 0$ and end up with

$$\tilde{f}_{P_r}\big(m(\gamma, A)\big) = \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_{n \in \mathrm{HI}_{g-r}} a\begin{pmatrix} n & 0 \\ 0 & 0\end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}$$

but by [BVDGHZ08, 3.5] we know that the Fourier expansion of the Siegel Phi operator is

$$(\Phi^r f)(\tau) = \sum_{n \in \mathrm{HI}_{g-r}} a\begin{pmatrix} n & 0 \\ 0 & 0\end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}.$$

hence

$$
\begin{aligned}
\tilde{f}_{P_r}\big(m(\gamma, A)\big) &= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \Phi^r(f)(\tau) \\
&= \det(A)^k (\Phi^r(f))^{\sim}(\gamma)
\end{aligned}
$$

which concludes the proof.

$\square$

Chapter 7

L-Functions

7.1 Automorphic L-Functions

We don't intent to define in great detail automorphic L-functions, as there are many other better sources to learn from [BC79, Part 2.III.2][Sha10][Cogb][BCDS$^+$04, 9, 10, 11][**?**], we will recall the idea and then discuss some of the properties and relations with Eisenstein series and interwining operators that we will need later.

The first thing is to recall the classification of connected reductive groups defined over an algebraically closed field via root datum. A root datum is a tuple $(X, \Phi, \check{X}, \check{\Phi})$ where X and $\check{X}$ are two free abelian groups of finite type, $\Phi, \check{\Phi}$ are subgroups that are in duality via a perfect pairing on $X, \check{X}$. Then each reductive group G over a number field F has associated the root datum that is associated to its base change to $\mathbb{C}$. Thus to a connected reductive group over a number field we associate a connected reductive group over $\mathbb{C}$, given by the dual root datum. We call this the **dual group** of G and denote it $\hat{G}$. The **Langlands dual group** is then the dual group producted with the $\mathrm{Gal}(\bar{k}/k)$

$$ {}^{L}G \;:=\; \hat{G} \rtimes \mathrm{Gal}(\bar{k}/k). $$

Example 7.1 (Classical Groups, [BCDS$^+$04], 11.1)**.** *We have the following table*

G	$\hat{G}$
GL_n	GL_n
SO_{2n+1}	Sp_{2n}
SO_{2n}	SO_{2n}
Sp_{2n}	SO_{2n+1}

Then, using the Satake isomorphism [Sha10, 2.2], to each unramified representation of $G(F_\nu)$ we can associate a conjugacy class of ${}^LG$, via some map call it c, and hence there is a way to apply a complex representation $r :^L G \to \mathrm{GL}_n(\mathbb{C})$ to representations of $G(F_\nu)$. Thus the automorphic L-functions are defined as follows: Let ρ be a representation of $G(\mathbb{A})$, let r be a complex representation of ${}^LG$ and $s \in \mathbb{C}$ then

$$L(s, \rho, r) \ := \ \prod_\nu L_\nu(s, \rho_\nu, r) = \prod_\nu \frac{1}{\det\big(I - r(c(\rho_\nu))q^{-s}\big)},$$

where ν runs over the unramified places. It is a part of the grand Langlands philosophy that there should be suitable L-functions for the ramified places satisfying very nice properties.

Remark 7.2. The global L-functions have been defined for many groups at this point and indeed [JLZ13] uses known properties to prove their results. One should note that the questions that we are interested in are still tractable even though the L-functions might not be defined (for instance for the metaplectic group). This is because only finitely many places will ramify, and so as long as those places are neither zero or poles we can transfer questions about zeros and poles from the full global L-functions to L-functions at almost all places.

Example 7.3 (Standard Representations / Classical Groups)**.** *In the case of classical groups it is common to see L-functions with only two entries e.g. if ρ is a representation of $G = \mathrm{Sp}\,2n$ then you may see $L(s, \rho)$. The reason is that there is a standard representation of the dual groups of classical groups. Namely the standard representation of a matrix group inside GL_n is the one that sends $g \mapsto g$. It is this representation that is to be taken for the dual group in this setting.*

Example 7.4 (Rankin-Selberg, [?], 1.2, [?], Ch. 2 Example. 2)**.**

fill

Let ν be a finite place of $\mathbb{Q}$ and π, π' be two unramified generic representations of $\mathrm{GL}_n(\mathbb{Q}_\nu)$ and $\mathrm{GL}_m(\mathbb{Q}_\nu)$ respectively. Let B_n be the standard Borel of upper triangular matricies in GL_n. Such representations have been classified

reference

in terms of characters of $\mathbb{Q}_\nu^\times$, in particular for π there are $\mu_1, ..., \mu_n$ unramified characters such that

$$\pi \cong \mathrm{Ind}_{B(\mathbb{Q}_\nu)}^{\mathrm{GL}_n(\mathbb{Q}_\nu)} \big(\mu_1 \otimes \cdots \otimes \mu_n\big).$$

If we fix a uniformizer ϖ *of* $\mathbb{Q}_\nu$ *then we have the so called "Satake parameters"* $\mu_i(\varpi)$ *which determines* π *uniquely. Of course the same is true for* π'*, with say characters* $\mu'_1, ..., \mu'_m$*. We then define*

$$L(s, \pi \times \pi') := \prod_{i,j} \frac{1}{1 - \mu_i(\varpi)\mu'_j(\varpi)q^{-s}}.$$

Consider the group $G = \mathrm{GL}_n \times \mathrm{GL}_m$ *which has dual* $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$*, then there is a cannonical representation*

$$r : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_{nm}(\mathbb{C}).$$

Then

$$L(s, \pi \otimes \pi', r) = L(s, \pi \times \tilde{\pi}'),$$

where the tilde denotes the contragradient.

Example 7.5 (Dirichlet L-functions)**.** *Recall that a Dirichlet character* χ *is a character of the group* $(\mathbb{Z}/N\mathbb{Z})^*$*. Through the series of maps*

$$A^\times \cong \mathbb{Q}^\times \times \mathbb{R}^\times_{>0} \times \hat{\mathbb{Z}}^\times \to (\lim \mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C},$$

one get a bijection between Dirichlet characters and finite-order Grossencharacters, i.e. characters of $\mathbb{A}_F^\times/F^\times$*. Grossencharacters have the associated L-function as they are just automorphic forms of* GL_1*, which generate automorphic representations. These give us the classical Dirichlet L-functions.*

reference? More details?

Chapter 8

Poles of Residual Eisenstein Series

Our goal here is to exposit and survey the work in papers such as [Bre09][JLZ13][GS21] and perhaps give a trivial extension of them. The idea is to locate the poles and zeroes of certain Eisenstein series.

[Bre09] gave an analysis of the residual poles of Eisenstein series attached to Sp_{2n}, there were some minor errors that were corrected in [JLZ13] where they give esssentially the same proof; theirs however works for the other classical groups. For our purposes, the case of Sp_{2n}, as a group defined over F a number field, is most relevant, and we shall therefore focus exclusively on this case, however it should be noted that this limitation in the non-covering case is artificial, although it does simplify things a little by removing some casework, and we hope also in the covering case to be able to remove it in future work.

8.1 Residual Eisenstein Series

So for the rest of the chapter we will fix an $n \in \mathbb{N}$, $G_n = \mathrm{Sp}_{2n}$ and the Borel of upper triangular matrices in Sp_{2n}, then we look at partitions of $n = r + m$, where $1 \leq r, m \leq n$ and $r, m \in \mathbb{Z}$. Then as we saw in 1.2.1 there corresponds a maximal standard (proper) parabolic of Sp_{2n}, which we denote $P_r = M_r N_r$, such that the Levi component is

$$\mathrm{GL}_r \times \mathrm{Sp}_{2m}$$

As we saw in 4.2 the space of characters $X_{M_r}^{\mathrm{Sp}_{2n}}$ is one dimensional by the maximality of P_r. If we look at the divisors of $r = ab$ and fix a τ, an irreducible unitary cuspidal automorphic representation of GL_a, then from **??** we know that $\Delta(\tau, b)$ is a residual representation of $\mathrm{GL}_{ab} = \mathrm{GL}_r$. Now we take an irreducible generic cuspidal automorphic representation σ of Sp_{2m}, and so their tensor product $\Delta(\tau, b) \otimes \sigma$ gives a representation of $\mathrm{GL}_r \times \mathrm{Sp}_{2m}$ and hence of the Levi M_r. We now consider the Eisenstein series attached to this representation, namely if

$$\varphi \in \mathcal{A}(N_r(\mathbb{A})M_r(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))_{\Delta(\tau,b)\otimes\sigma}$$

then we have the Eisenstein series

$$E(\varphi, s)(g) = \sum_{\gamma \in P_r(F)\setminus \mathrm{Sp}_{2n}(F)} s.\varphi(\gamma g)$$

for $g \in \mathrm{Sp}_{2n}(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A})$. Becuase it is induced from the residual representation $\Delta(\tau, b)$ we call these residual Eisenstein series.

This is the setup in [JLZ13], where they prove their results by a sort of induction on b. Here we will focus exclusively on the base case of this induction, leaving the inductive step for future work. So from now on we will fix $b = 1$. Hence $n = a + m$. Then fixing a standard parabolic of Sp_{2n} we have the maximal standard parabolic $P_a = M_a N_a$ where $M_a = \mathrm{GL}_a \times \mathrm{Sp}_{2m}$. Now if τ is irreducible unitary cuspidal automorphic representation of GL_a then by definition **??**

$$\Delta(\tau, 1)(\varphi)(g) = E(\varphi, s)(g) = s.\varphi(g)$$

where the Eisenstein series is defined via the parabolic induction from the Levi $(\mathrm{GL}_a)^{\times b}$ to GL_{ab}. Thus we have $\Delta(\tau, 1) = \tau$.

8.2 The Constant Term

So far we only know how to do one thing with such Eisenstein series and that is compute their constant term. We will compute the constant term along the maximal parabolic $P_a = MN$ becuase by [MW95, II.1.7 (ii)] the others are zero.

By our earlier calculations 5.2, the fact that the tensor of cuspidal representations is cuspidal and [JLZ13] we know that

$$E(\varphi, s)_P = \sum_{w} \sum_{m'} \int_{(w^{-1}N(\mathbb{A})w \cap M(\mathbb{A}))\setminus A} \int_{w^{-1}N(F)w\cap M(F)\setminus w^{-1}N(\mathbb{A})w\cap M(\mathbb{A})} s.\varphi(n_1 n_2 w^{-1} m' x) dn_1 dn_2,$$

and the inner integral vanishes for all $w \neq id, \omega$ (ω as in [JLZ13]). Hence the first sum becomes over two elements and we have

I would like to expa
on this if there is tir

$$E(\varphi, s)_P = E(\varphi, s)_{P,id} + E(\varphi, s)_{P,\omega}.$$

where

$$E(\varphi, s)_{P,w}(x) = \sum_{m' \in M(F) \cap wP(F)w^{-1} \backslash M(F)} \int_{N(F) \cap wP(F)w^{-1} \backslash N(\mathbb{A})} s.\varphi(w^{-1}nm'x)dn.$$

First the identity term simplifies

$$\begin{aligned} E(\varphi, s)_{P',id}(x) &= \sum_{m' \in M(F) \cap P(F) \backslash M(F)} \int_{N(F) \cap P(F) \backslash N(\mathbb{A})} s.\varphi(nm'x)dn \\ &= \sum_{m' \in M(F) \backslash M(F)} \int_{N(F) \backslash N(\mathbb{A})} s.\varphi(nm'x)dn \\ &= \int_{N(F) \backslash N(\mathbb{A})} s.\varphi(nx)dn \\ &= s.\varphi(x)_P. \end{aligned}$$

Considering now the ω term

$$E(\varphi, s)_{P,\omega}(x) = \sum_{m' \in M(F) \cap \omega P(F)\omega^{-1} \backslash M(F)} \int_{N(F) \cap \omega P(F)\omega^{-1} \backslash N(\mathbb{A})} s.\varphi(\omega^{-1}nm'x)dn.$$

By [JLZ13, 2C] $M(F) \cap \omega P(F)\omega^{-1} \setminus M(F)$ is isomorphic to $P_0 \setminus \mathrm{Sp}_{2(n-a)}$, but P_0 has Levi $M_0 = \mathrm{Sp}_{2(n-a)}$ by definition and hence is itself $\mathrm{Sp}_{2(n-a)}$. Thus the sum is over $\mathrm{Sp}_{2(n-a)}(F) \setminus \mathrm{Sp}_{2(n-a)}(F)$ and hence is over a point. Therefore we get by definition of the intertwining operator

$$E(\varphi, s)_{P,\omega}(x) = \int_{N(F) \cap \omega P(F)\omega^{-1} \backslash N(\mathbb{A})} \varphi(\omega^{-1}nx)dn = M(\omega, s)(\varphi)(x)$$

because we took the constant term along the same parabolic as the definition of the Eisenstein series we know that the Levis are (the same) conjugate. Thus we have shown that

$$E(\varphi, s)_P = s.\varphi_P + M(\omega, s)(\varphi)$$

Remark 8.1. This work was entirely at the level of the terminals and hence generalises verbatim to covering groups.

Because φ is an automorphic form it has no poles and so we have shown the following:

Lemma 8.2 (Base case of [JLZ13], 2.1). *The poles of $E(\varphi, s)$ are exactly the poles of $E(\varphi, s)_{P_a}$ which are exactly the poles of $M(\omega, s)$.*

8.3 Analysing the Intertwining Operator

Recall from **??** that we know in this case that there is some ratio of L-functions $r(s, \omega)$ such that $M(s, \omega) = r(s, \omega)N(s, \omega)$ where $N(s, \omega)$ is both holomorphic and non-zero. In the case we are considering the normalising factor r is given by the equation [JLZ13, 4A]

$$r(w, s) = \frac{L(s, \tau \times \sigma)L(2s, \tau, \wedge^2)}{L(s+1, \tau \times \sigma)L(2s+1, \tau, \wedge^2)},$$

where $\wedge^2$ denotes the exterior second power of the standard representation of $\mathrm{GL}_a(\mathbb{C})$. Thus

Lemma 8.3. *The Eisenstein series above has pole at s if and only if $r(w, s)$ has a pole.*

The final step is then to use the known properties of L-functions to conclude when our r-factor will have poles and of what order those poles will be.

———

Theorem 8.4 ([JLZ13], 4.1).

Appendix A

Test

Todo list

Fix bibliography

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