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Introduction

Outline of Content

The goal of this thesis is to exposit and hopefully provide some small generalisation of some of the results in [JLZ13]. We aim our exposition at the other masters students in our cohort. To explain the results on poles of Eisenstein series to students in other disciplines there is a fair amount of background to be covered. This is the content of the first 5 chapters. Our aim is to aquire similar results as [JLZ13] but for metaplectic coverings. Despite this we largely avoid talking about the metaplectic technicallities in the background sections. We feel that if one is learning from this work those technicallities will only obscure the ideas, moreover we have nothing to add to the exposition that isn't covered in [MW95]. We will however include some details in the final section when they become more integral.

Chapter one deals with the generalities of linear algebraic groups, the objects whose representation theory is the subject of discussion. First we define them and then look at the important subgroups that are used in the study of automorphic forms arising on the adelic points of these groups. We pay specific attention to the classical groups and for this reason, and to maintain the brevity and accessability of this work, assume that all our groups are split. Finally we mention the metaplectic covers of these groups and the appropriate associated subgroups.

Chapter two deals with automorphic forms. We define automorphic forms in both the archemedean and adelic places. Finally we give the details of how to view modular forms as automorphic forms. We feel this is essential intuition for understanding the highly abstract definition of automorphic form.

Chapter three is dedicated to automorphic representations. We define them and specify some important constructions that are needed in the final section.

Chapter four serves two purposes, we introduce the Eisenstein series and then as a motivation for later results we talk about its use in decomposing the regular representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{A}))$. Finally in this section we summarise the theory of automorphic L-functions. We do not go into detail as this theory is vast, confusing and well presented elsewhere. In particular we try to convey briefly how these functions are constructed, give some examples and collect some results that will be needed in the sequel.

Chapter five is a grand look into the constant terms of automorphic forms and Eisenstein series in particular. We present a proof of a well known theorem in great detail that should be helpful to any new comers trying to understand the theory. We also describe some things in the classical setting, including the constant term (in the Fourier sense) of a modular form as well as the Siegel phi operator, as constant terms of certain automorphic forms. We should also make several appologies about this section as we have failed at every step to do our analytic due diligence, naively assuming that we can interchange sums and integrals and apply unfolding techniques without checking the hypothesis.

Finally chapter six contains some exposition of recent work on the poles of residual Eisenstein series. We also retread some ground describing results on L-functions and automorphic representations, now in the case of metaplectic groups.

Motivation

Contents

Notation

Delete this section later perhaps: Use for reference while writing.

- F is a number field
- ν is a place of F
- F_ν is the completion of F at ν
- $\mathbb{A}$ of $\mathbb{A}_F$ is the Adele ring, $\mathbb{A}_f$ is the finite adeles, and $\mathbb{A}_\infty$ the infinite adeles.
- B, P, M, U: Borel, parabolic, Levi, unipotent

Chapter 1

Classical Groups

We will recall a small amount of the theory of linear algebraic groups to fix conventions, for a more detailed treatment one should consult the litany of sources on this matter: For a full treatment see [Mil17][Milb][Mila][Spr98]. Excellent example computations can also be found in [Bui][Mak][MT][Not]. Or for a brief brush up on the main facts consult [BC79, I.I.1]. The purpose of this section is to treat the classical groups and more specifically Sp_{2n} as an example and work out some of the details of the general theory, in order to "get our hands dirty" and have some familiarity with this object, to become fundamental in what follows. Because this theory is made up of simple ideas that can often be obscured by the generality we will make several restrictive assumptions for ease of exposition.

1.1 Definition

An **algebraic group** is for us a group scheme that is reduced, of finite type and defined over a field. **A linear algebraic group** (LAG) is simply an affine algebraic group.

Proposition 1.1. *An algebraic group is affine if and only if it is isomorphic to a Zariski closed subgroup of* GL_n.

Proof. The forward implication is [Spr98, 2.3.7(i)]. The converse is the basic fact that closed sub-schemes of affine schemes are affine [Mum99, II.5.T3].

As Milne points out in [Mil17] these are matrix groups defined by polynomials, which happen to be the natural combinations of symbols that matrix multiplication will lead to. This means that they come with the powerful but cumbersome (for the beginner) technology of algebraic geometry. In particular one must be adept at moving between the following equivalences:

Theorem 1 ([Mila], II.6, III.4)**.** *For F a field then the following categories are equivalent*

- *Group objects in* Alg_F^{op}
- *Representable (in the category of groups) functors* $\mathrm{Alg}_F \to \mathrm{Group}$
- *Group object in the category of affine schemes over F*

- *Commutative F-Hopf algebras.*

Example 1 ($\mathbb{G}_m$)**.** *The first example that we will see again and again is the "multiplicative group" or* GL_1 *defined over the field K. This is*

$$\mathbb{G}_m \quad := \quad \mathrm{Spec}\Big(K[x,y]/(xy-1)\Big).$$

As a representable functor this sends a K-algebra R to $\mathrm{Hom}_K(K[x,y]/(xy-1),R)$*. These are ring maps that are K linear, and because $y = x^{-1}$ we know that $f(y) = f(x^{-1}) = f(x)^{-1}$ for $f \in \mathbb{G}_m(R)$. Thus the maps are determined by where they send x, moreover they always send it to a unit, i.e. $Imf \subseteq R^\times$. For each element $r \in R^\times$ we also have a map sending $x \to r$ hence there is an isomorphism (of sets) that allows us to induce a group structure.*

The other most important examples of such groups are the "classical groups". The exact groups that an author might mean by classical may vary, so we define them explicitly here. First let V be a finite dimensional F vector space with a bilinear form $\langle , \rangle$. An automorphism of this form is a map $\alpha \in \mathrm{Aut}(V)$ such that

$$\langle \alpha(x), \alpha(y) \rangle = \langle x, y \rangle.$$

Therefore we can consider the space of automorphisms of this form $\mathrm{Aut}(V, \langle , \rangle)$. This space, depending on the properties of the bilinear form, will define our classical groups.

If the form is trivial, by which we mean, $\forall x, y \ \ \langle x, y \rangle = 0$ then we define

$$\mathrm{GL}(V) \quad := \quad \mathrm{Aut}(V, \langle , \rangle) = \mathrm{Aut}(V).$$

If the form is non-degenerate and symmetric $\forall x, y \ \ \langle x, y \rangle = \langle y, x \rangle$ then we define

$$\mathrm{O}(V) \quad := \quad \mathrm{Aut}(V, \langle , \rangle).$$

Finally if the form is non-degenerate and skew symmetric $\forall x, y \ \ \langle x, y \rangle = -\langle y, x \rangle$ then

$$\mathrm{Sp}(V) \quad := \quad \mathrm{Aut}(V, \langle , \rangle).$$

There are the further classical groups given by the determinant one subgroups, $\mathrm{SL}(V), \mathrm{SO}(V)$ respectively ($\mathrm{Sp}(V)$ one can show already implies that the determinant is one). We can make this into a functor from F-algebras to groups, by sending an F-algebra R to $G(V) \otimes_F R$. Thus these define LAG's.

Remark 1.2. Often the unitary groups are considered classical, however we don't want to deal with field extensions and so omit them here.

1.2 Subgroups

From now on let G be a (split reductive) LAG defined over a number field F.

Remark 1.3 (For the experts). We restrict to split reductive LAG in what follows. This is justified by the fact that the classical groups are all split reductive over number fields.

Subgroups with special properties allow us to reduce and break up problems into smaller ones. Here we will briefly review and compute some examples of special subgroups. The point of these subgroups is two fold. Some of them will help us perform "induction" from smaller simpler groups to larger ones. Others are there essentially as a part of the combinatorial data that classifies the groups we are working with. In particular we need to understand all the pieces of the so called **Langlands-Iwasawa decomposition**,

$$G(\mathbb{A}) = M(\mathbb{A})U(\mathbb{A})K = T(\mathbb{A})U(\mathbb{A})K. \tag{1.2.1}$$

1.2.1 Parabolics, Levis and Unipotents

Parabolic subgroups have two equivalent formulations, both useful.

Definition 1.4. A subgroup $P \subseteq G$ is called **parabolic** if G/P is a complete variety. Eqquivilently we can ask for P to contain a Borel (see 1.2.2).

Completeness is the algebro-geometric analogue of compact, which is always a desirable property. The fact that they contain a Borel gives us an algebraic "parametrisation" of these subgroups, in the case of the classical groups through the use of flags or roots. It is very important to have a parametrisation of the parabolic subgroups when it comes to taking constant terms of Eisenstein series which we will discuss in the later section 5.

Parabolics also have the nice property that they split into a semi-direct product where one of the factors is a reductive group M. For this recall the definition

Definition 1.5. A matrix m is **unipotent** if for some $n \geq 0$ we have that $(m-1)^n = 0$. A subgroup is **unipotent** if all its elements are unipotent. The **unipotent radical** of $G$is the maximal closed, connected, unipotent subgroup. A linear algebraic group is **reductive** if its unipotent radical is trivial.

Then we have the following fact / definition:

Lemma 1 ([Bor91] 11.22)**.** *There is a split exact sequence (of algebraic groups)*

$$0 \to U \to P \to M \to 0,$$

where U is the unipotent radical of P, and M is a reductive group known as a ***Levi*** *(unique up to conjugacy).*

Thus doing things on a parabolic allows us to induce said actions up to the whole group, whist maintaining the nice property of being reductive. This is the technique of "parabolic induction" [Ber92, Thm. 10] that we wont explicitly talk about here but which is happening secretly in the background in 3.2.2.

Remark 1 (Bad Etymology)**.** *The origin of the name parabolic is a mystery. Borel in his history [Ess, VI.§2] attributes it to R. Godement in [God]. Godement conjectures that the quotient $G(\mathbb{A})/G(\mathbb{Q})$ is compact if and only if every element of $G(\mathbb{Q})$ is semi-simple, as is the case in classical groups.*

this is probably known by now.

He says that

> *Lorsque n'est pas compact, il est non moins facile de conjecturer qu'on doit pouvoir définir quelque chose d'analogue aux classiques "pointes paraboliques", lesquelles doivent correspondre à des sous-groupes unipotents non triviaux de $G_{\mathbb{Q}}$*

which roughly (google) translates to that one can also conjecture that non-trivial unipotent elements should correspond to "parabolic points" in a fundamental domain.

In the case of modular forms the fundamental domain is $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R})$ (using orbit stabiliser theorem). We have the classification of elements of $\mathrm{SL}_2(\mathbb{R}) - \{\pm 1\}$ as in [Bor97, 3.5] via their trace

$$g \text{ is of type } \begin{cases} \textit{Elliptic if} & \frac{1}{2}|tr(g)| < 1 \\ \textit{Parabolic if} & \frac{1}{2}|tr(g)| = 1 \\ \textit{Hyperbolic if} & \frac{1}{2}|tr(g)| > 1 \end{cases}.$$

Being parabolic is equivalent to having eigenvalue 1 hence by the Jordan decomposition we know that parabolics in SL_2 are conjugate (over $\mathbb{C}$) to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly the standard parabolic

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \subseteq \mathrm{SL}_2(\mathbb{R}),$$

contains these matrices, and moreover all parabolics are conjugate to this parabolic. Hence all parabolic elements are contained in a parabolic subgroup. This classification, it seems, relies entirely on the aesthetic connection with the classification of the sections of conics via eccentricity.

To connect this to Godement's concept we have two facts from classical geometry. Proper parabolic subgroups of $\mathrm{SL}_2(\mathbb{R})$ can be realised as the stabilisers of lines in $\mathbb{R}^2$ under the standard action of SL_2 on $\mathbb{R}^2$ [Bor97, 2.6] and moreover some an element of $\mathrm{SL}_2(\mathbb{R})$ is parabolic if and only if it has one fixed point on $\partial\bar{\mathcal{H}}$ and none on $\mathcal{H}$ [Bor97, 3.5].

The take away is that perhaps the folklore of the name being for "para-Borelic", as in kind of a Borel, is probably a better way of thinking of them.

The Example of Sp_{2n}

We collect the following facts as they will be useful in what is to come. Good references are the notes [Con] and the book [Bui, §8].

Let $(V, \langle , \rangle)$ be a symplectic space as above and $Sp(V)$ is the automorphisms preserving the form. A **flag** of V is a sequence of strict inclusions of vector subspaces

$$\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset V.$$

A subspace of V is said to be **isotropic** if the form is constantly zero on it (in both variables). A flag is **isotropic** if the proper subspaces in it are isotropic subspaces. A **maximal isotropic** flag is one with exactly n components. Sp_{2n} acts on a flag by acting on each of the subspaces. This action preserves isotropic flags i.e. it sends an isotropic flag to an isotropic flag. Stabilisers of isotropic flags give parabolics of Sp and moreover all parabolics arise in this way [Spr98, Exercise 3.2.16, 6.2.11].

Example 2. *Consider a four dimensional vector space V with a form given by the matrix*

$$\begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix},$$

then a maximal isotropic flag is

$$0 \subset Fe_1 \subset Fe_1 \oplus Fe_2 \subset F^4,$$

where $e_i = (\delta_i^j)_j$. This has stabiliser consisting of matrices in Sp *of the form*

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix}.$$

In particular maximal parabolics of Sp are stabilizers of *minimal* (non-trivial flags), i.e. stabilisers of non-zero isotropic subspaces,

$$0 \subset V_\ell \subset V,$$

where $V_\ell = span_F(e_1, ..., e_\ell)$. Then the stabilizer is

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix},$$

with the sizes of the diagonal blocks being (these numbers square)

$$\begin{pmatrix} \ell & * & * & * \\ 0 & n-\ell & * & * \\ 0 & * & \ell & * \\ 0 & * & * & n-\ell \end{pmatrix}.$$

This has Levi

$$\begin{pmatrix} A & & & \\ & a & & b \\ & & (A^T)^{-1} & \\ & c & & d \end{pmatrix}, \quad A \in \mathrm{GL}_\ell(F), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2(n-\ell)}(F),$$

and unipotent

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & \\ & & 1 & \\ & & * & 1 \end{pmatrix},$$

with relations among the entries.

1.2.2 Borel and Torus

Remark 1.6. It can be very enlightening to to understand the analogous story for Lie groups and their classification [Hal15].

Definition 1.7. A **split torus** is an algebraic group that is isomorphic to GL_1^b for some $b \in \mathbb{N}$.

Example 3 (Bad Etymology)**.** GL_1^2 *is a split torus. Notice that*

$$\mathrm{GL}_1^2(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$$

is isomorphic as abstract groups to $U(1) \times U(1)$ *which when* $U(1)$ *is realised as* $\{z \in \mathbb{C} : |z| = 1\}$ *is topologically equivalent to the torus* $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$*. Note that it is clear that*

$$\mathrm{GL}_1^2(\mathbb{C}) \ncong \mathbb{T}^2$$

as topological groups, as the right hand side is compact whilst the left is not.

It was pointed out to me that as algebraic groups over $\mathbb{C}$ *we have the following isomorphism*

$$\mathrm{GL}_1 \cong U(1) \otimes_{\mathbb{R}} \mathbb{C}$$

Proof

Perhaps a more compelling reason to call these Tori is that they play the same role in the classification as the genuine tori, $\mathbb{T}^2$, in the theory of Lie groups. A subgroup that is isomorphic to a split torus and is maximal in this respect is called a maximal split torus.

Example 4. *The classic example of a maximal split torus is the group of diagonal matrices in* GL_n*.*

Definition 1.8. A **Borel** is a maximal closed solvable connected subgroup of G.

Example 5. *The standard Borel of* GL_n *is the group of upper triangular matrices. If* n *is even and one intersects this Borel with* $\mathrm{Sp}_{2(\frac{1}{2}n)}$ *then we get the standard Borel of* $\mathrm{Sp}_{2(\frac{1}{2}n)}$*.*

Lets prove this in GL_2 *and then believe that the only complication to going to larger* n *is keeping track of indices. So let*

$$B = \begin{pmatrix} * & * \\ & * \end{pmatrix},$$

we need to show that the derived series terminates for it to be solvable. So let

$$g = \begin{pmatrix} x & y \\ & z \end{pmatrix}, \quad h = \begin{pmatrix} a & b \\ & c \end{pmatrix},$$

be arbitrary in GL_2*, their commutator is then*

$$g^{-1}h^{-1}gh = \begin{pmatrix} 1 & \frac{bx-ay}{ax} \\ & 1 \end{pmatrix}.$$

Hence

$$[B,B] = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}.$$

Commutate two arbitrary elements again shows that

$$[[B,B],[B,B]] = 1.$$

It is clear that this is a closed subgroup because it is itself a linear algebraic group, moreover for LAG's we have the algebraic criterion of connectedness given by having the only idempotents in the representing algebra being $0,1$ *[GH24, 1.5]. Because* $B = \operatorname{Spec}\mathbb{Z}[x_{i,j} : 1 \leq i,j \leq 2][y]/(\det(x_{ij})y - 1, x_{2,1})$ *it is clear that this group is connected. Finally it is clear that if a subgroup strictly contains this one then it is in fact all of* GL_2 *and hence this is maximal. Therefore this is a Borel.*

A Borel can be considered to be a parabolic that is minimal with respect to inclusion. The maximal tori then form the Levis of these parabolics. In particular for a Borel B we have that

$$B = TU,$$

for a maximal torus T and unipotent U.

If a Borel B is fixed, then a parabolic containing this Borel $B \subseteq P$ is called standard, the unique Levi of a standard parabolic containing this Borel is called the **standard Levi**.

1.2.3 Maximal Compact Subgroups

We will often need to fix a maximal compact subgroup $K \subseteq G(\mathbb{A})$, note that the topology is not the Zariski topology but the one specified in [Con12], this is sometimes known as the "Hausdorff" topology. These maximal compact subgroups are not unique and as such when fixing one it can be arranged to have many nice properties [MW95, I.1.4]. In particular if we have a group G and a fixed Borel B:

- First require that
$$K = \prod_\nu K_\nu,$$
where the product is over all places of F and $K_\nu \subseteq F_\nu$ is maximal compact.

- For almost all places ν, $G(\mathcal{O}_{F_\nu})$ is defined and is maximal compact in $G(F_\nu)$ hence we can require $K_\nu = G(\mathcal{O}_{F_\nu})$ at these places.

- We require
$$G(\mathbb{A}) = B(\mathbb{A})K.$$

- For every standard parabolic $P = MU$ we have that
$$P(\mathbb{A}) \cap K = \Big(M(\mathbb{A}) \cap K\Big)\Big(U(\mathbb{A}) \cap K\Big),$$
and $M(\mathbb{A}) \cap K$ is a maximal compact subgroup of $M(\mathbb{A})$.

It is in terms of the third property that we like to think of the maximal compact subgroup, it is the complimentary piece of the Borel. Moreover the fourth property should be thought of as a condition that the maximal compact subgroups are well behaved with the way that we are moving between the bigger and smaller reductive groups.

1.3 Metaplectic Covers

METAPLECTIC

We are also be interested in certain covering groups of these LAG's. In particular [MW95, I.1.1] we will be intereseted in $\mathbf{G}$ some topological group given as a finite central cover of $G(\mathbb{A})$. If $\mathrm{pr} : \mathbf{G} \to G(\mathbb{A})$ is the projection then to the subgroups listed above we can associate their "lifts" (preimages under pr).

Example 6 (Metaplectic Group)**.** *There is a rich history and representation theory of this group, which we make no pretense of understanding, however some hints can be found in [Kud] and the references therein.*

There is an analogue of the Langlands program being developed for such groups a nice introduction to which can be found in [GGW17].

perhaps a first natural question is whether or not these things can be given the structure of LAG's. That paper mentions representability. Look into it.

Chapter 2

Automorphic Forms

There are different definitions of the words automorphic forms floating around, here we fix a nice one and then explain how they generalize the classical modular forms. We intend to be terse as this material is somewhat standard.

2.1 Definition and Role

The story starts with the classical modular forms, or functions on the upper half plane that satisfy some invariance conditions and differential equations. This evolves into the notions of Maas form on symmetric spaces and eventually reaches its apotheosis in the concept of automorphic form that we will present here.

We still do not have a good answer as to why the definition below is "the right" definition, from a mathematical perspective, as there are many places in which it could be extended or restricted and we are unable to motivate why one shouldn't consider such things. Indeed there are varying notions of automorphic form that appear for this reason and I think it is important to stress that this is "the right definition" only in so far as people have been able to prove nice theorems about them, and that when functions appear "in nature" this concept has sufficed to encompass and explain their behavior. It is the representation theoretic properties more than anything that suggest the current definition as is mentioned in [BC79, 1.II.3].

We will present two notions of automorphic form here. In the literature they are both called "automorphic forms" however here we will distinguish those that are defined only on the Archimedean points as "Archimedean automorphic forms" for clarity.

2.1.1 Archimedean Automorphic Form

Fix a number field F. Let ν be an Archimedean place and let ∞ denote the set of Archimedean places. Then F_ν is either $\mathbb{R}$ or $\mathbb{C}$. In particular (the analytification of) $G(F_\nu)$ is a Lie group and we call a function, $\varphi : G(F_\nu) \to \mathbb{C}$, **smooth** if it is smooth in the sense of manifolds.

Now we fix an embedding $\imath : G \to GL_n$ which gives another embedding $G \to SL_{2n}$ via

$$g \mapsto \begin{pmatrix} \imath(g) & \\ & (\imath(g))^{-t} \end{pmatrix}.$$

A function $\varphi : G(F_\infty) = G(\prod_{v\in\infty} F_\nu) \cong \prod_{v\in\infty} G(F_\nu) \to \mathbb{C}$ is of **moderate growth** if there are constants $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$|\varphi(g)| \leq c\|g\|^r = c\left(\prod_{v\in\infty} \sup_{1\leq i,j\leq 2n} |\imath(g)_{i,j,\nu}|_\nu\right)^r.$$

This is taking the maximum of the $2n \times 2n \times |\infty|$ three dimensional matrix.

Because $G(F_\infty)$ is a Lie group we know how to define its Lie algebra and we now denote $Z(\mathfrak{g})$ the center of the *universal enveloping algebra* of the *complexification* of $\mathfrak{g}$, it would be more reasonable to use $Z(\mathcal{U}(\mathfrak{g}_\mathbb{C}))$ but that is too cumbersome so we follow the tradition. A vector in a $Z(\mathfrak{g})$-module $\varphi \in V$ is called $Z(\mathfrak{g})$-**finite** if the space $Z(\mathfrak{g})\varphi$ is finite dimensional.

Let $K_\infty \leq G(F_\infty)$ be a maximal compact subgroup. Then again an element of a K_∞-module is K_∞ **finite** if its orbit is a finite dimensional vector space space (we think here of $\mathbb{C}[K_\infty]$-modules).

To define automorphic forms we look at the representation $C^\infty(F_\infty)$ with the right regular action, i.e. $g.f(x) = f(xg)$. In particular the $Z(\mathfrak{g})$ module structure is induced from the action of $\mathfrak{g}$ on $C^\infty(G(F_\infty))$ by

$$z.F(g) = \frac{\partial}{\partial t} F(ge^{tz}).$$

Definition 1. *Let $\Gamma \leq G(F_\infty)$ some (arithmetic) subgroup, an* ***automorphic form for*** Γ *is a smooth function of moderate growth*

$$\varphi : G(F_\infty) \to \mathbb{C},$$

that is K_∞ and $Z(\mathfrak{g})$ finite with a (left) Γ invariance. We denote the set of these "Archimedean" automorphic forms by $\mathcal{A}(\Gamma\backslash G(F_\infty))$.

2.1.2 Adelic Automorphic Form

Here we follow [MW95, I.2.17]. Let G be a reductive group over F, we fix a Borel B and a standard parabolic P with a standard Levi decomposition $P = MU$. We let K be a maximal compact subgroup of $G(\mathbb{A})$ satisfying the conditions laid out in the previous section 1.2.3.

For $v \notin \infty$ a non-Archimedean place then we say that a function $f : G(F_\nu) \to \mathbb{C}$ is smooth if it is locally constant in the induced topology on $G(F_\nu)$, the details of this topology are spelled out in [Con12]. The set of such smooth functions is denoted $C^\infty(G(F_\nu))$. This suggests the definition of smooth functions on the "finite adeles" $\mathbb{A}_f$ as

$$C^\infty(\mathbb{A}_f) \;:=\; \bigotimes_{\nu\notin\infty}{}' C^\infty(G(F_\nu)).$$

Thus for the full adeles we have the notion of smooth as an element of the following,

$$C^\infty(\mathbb{A}_F) \;:=\; C^\infty(G(\mathbb{A}_f)) \otimes C^\infty(G(F_\infty)).$$

Notice that a priori the codomain is an infinite tensor product over $\mathbb{C}$ of copies of $\mathbb{C}$, which is isomorphic to $\mathbb{C}$. Thus we can conflate a smooth function with its composition along this isomorphism, and think of them as functions into $\mathbb{C}$.

We still consider $Z(\mathfrak{g})$ to be the center of the universal enveloping algebra of the Lie algebra at the infinite places, exactly as before. We define an action by linearly extending

$$z.(f \otimes g) = f \otimes (z.g),$$

i.e. it acts on the archimedean places as in the setting of Archimedean automorphic forms.

The definition of moderate growth carries over verbatim, however we change the set of places multiplied over to be all of them now.

Remark 2.1 ([BC79], 1.II.3). The collection of moderate growth functions is independent of the choices of embedding.

Definition 2.2. A function $\varphi : U(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is an **automorphic form** if it is smooth, moderate growth, $Z(\mathfrak{g})$ and K finite. We will denote the set of these automorphic forms by $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$

Remark 2.3. It is important that $M(F)$ is treated as a subgroup of $M(\mathbb{A})$ via the diagonal embedding.

2.2 Modular Forms

One might ask if there is a special case in which automorphic forms yield modular forms. In fact no, the space of automorphic forms is larger than just modular forms, however it gives the space of Maas forms (or modular and Maas forms, depending on convention). This is well covered in the literature [Eme][Bum97, 3.2][Boo][Gar16], but so essential to intuiting automorphic forms that we feel it is necessary to present the details here. To be clear we explain modular forms as Archimedean automorphic forms as we think it is where the connection is clearest.

Recall the definition of a modular form

Definition 2 ([DS05] 1.1.2)**.** *A function*

$$\varphi : \mathcal{H} \to \mathbb{C},$$

where $\mathcal{H}$ is the upper half plane in $\mathbb{C}$, that is holomorphic, satisfies

$$\varphi(\gamma.z) = (cz+d)^k \varphi(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and extends holomorphically to ∞ is called a modular form of weight k.

These are modular forms with trivial character and full level.

Now give a function on a set X and an action of a group G on X, there is a general way of associating to $\mathrm{Hom}(X, Y)$ a family of maps $\mathrm{Hom}(G, Y)$ indexed by X. This is a manifestation of the tensor-hom adjunction. Effectively if $f : X \to Y$ the we get a map for each $x \in X$ defined on $f_x : G \to Y$ given by $g \mapsto f(g.x)$.

So for our purposes we are trying to take some subset of functions $\mathcal{H} \to \mathbb{C}$ and shift their domain to the $\mathbb{Q}_\infty = \mathbb{R}$ points of some reductive group. In particular it would be sufficient to find a reductive group with a well defined action on the upper half plane and in particular we would want the action to be transitive.

Theorem 2.

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R}),$$

as topological spaces.

Proof. Consider the action

$$\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathcal{H} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}.$$

Then look at the orbit of i, namely

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} .i = \frac{ai+b}{d} = a^2 i + ab,$$

which letting $a, b \in \mathbb{R}$ vary is clearly surjective onto the whole upper half plane. So there is one orbit, and hence by the orbit stabiliser we know that

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/stab(i),$$

so we want to find

$$stab(i) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : g.i = i \right\},$$

in particular we solve

$$i = g.i = \frac{ai+b}{ci+d} = (c^2+d^2)^{-1}(ac+bd+i\det g).$$

So equating coefficients we have

$$\det g(c^2+d^2)^{-1} = 1 \implies c^2+d^2 = \det g = 1,$$

on the other hand

$$ac + bd = 0.$$

Now the pairs $c^2 + d^2 = \det g = 1$ are parameterized by $\theta \in [0, 2\pi)$ using $c = \sin\theta, d = \cos\theta$ hence subbing this into the above equation

$$\frac{-b}{a} = \tan\theta,$$

and so $b = -k\sin\theta, a = k\cos\theta$ for some $k \in \mathbb{R}$ but the determinant must be 1 so $k = 1$. Hence

$$stab(i) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\} = SO_2(\mathbb{R}).$$

One then has to check that this is all continuous.

Remark 2.4. Sometimes for to make the action of certian (Hecke) operators more apparent this is exhibited as

$$\mathcal{H} \cong \mathrm{GL}_2^+(\mathbb{R})/A_{\mathrm{GL}_2} SO_2(\mathbb{R}).$$

This obscures the connection with the reductive group setting however so we avoid it here.

SL_2 is a reductive group and $SO_2(\mathbb{R})$ is its maximal compact subgroup. This decomposition of the upper half plane suggests that function on it might have some invariance along the maximal compact subgroup of the reductive group SL_2. Indeed if we were to push our modular forms along this isomorphism it would, with the construction that we outlined earlier in terms of a group action on a set, exhibit this invariance. This is merely *evidence* that if we were to change our modular forms to functions on the reductive group SL_2 they may preserve *some* of that invariance and indeed be K-finite.

$$\begin{tikzcd} \left\{\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2}\end{pmatrix} : x, y \in \mathbb{R}, y \neq 0\right\} SO_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \arrow[r, "\text{project}"] \arrow[dr, "\text{project}"'] & \mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R}) \arrow[r, "\sim", "g \mapsto g.i"'] & \mathcal{H} \\ & \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R}) & \end{tikzcd}$$

Using something like the universal property of the quotient we can lift a function on $\mathrm{SL}_2(\mathbb{R})/SO_2(\mathbb{R})$ to $\mathrm{SL}_2(\mathbb{R})$ however this is not $\mathrm{SL}_2(\mathbb{Z})$ invariant, thus we need to add a pre-factor to ensure this in our associated automorphic form. The algebro-geometric perspective in [Eme] can make this seem slightly less ad hoc.Thus for f a modular form of weight k the following function on $\mathrm{SL}_2(\mathbb{R})$

$$F(g) \quad := \quad (ci+d)^{-k} f(g.i),$$

we claim is an automorphic form for $\mathrm{SL}_2(\mathbb{Z})$. We take for granted its smoothness. The $\mathrm{SL}_2(\mathbb{Z})$ invariance is obvious from the modularity condition. It remains to show the three other properties:

Lemma 2. *$F(g)$ is of moderate growth.*

Proof. Unraveling the definitions we require two constants such that

$$|F(g)| = |ci+d|^{-k}|f(g.i)| \leq c(\sup_{i,j}(g, g^{-1}))^r,$$

A direct computation shows that

$$Im(g.i) = |ci+d|^{-2},$$

hence we require to show

$$\mathrm{Im}(g.i)^{k/2}|f(g.i)| \leq c(\sup_{i,j}(g, g^{-1}))^r.$$

Somehow invoke polynomial growth...? but the modularity condition has the growth condition that $\lim_{x\to\infty} f(xi)$ be bounded.

Lemma 3. *$SO_2(\mathbb{R})$ is a maximal compact subgroup inside $\mathrm{SL}_2(\mathbb{R})$. F is an $SO_2(\mathbb{R})$ finite function.*

Proof. Using that $\kappa \in K = SO_2(\mathbb{R})$ acts trivially on i, an elementary computation shows that for $g \in \mathrm{SL}_2(\mathbb{R})$,

$$F(g\kappa) = e^{-ik\theta}F(g).$$

Hence $F(g)$ is acted on by K via a one dimensional irreducible representation. In particular it is finite dimensional.

Lemma 4. *F is a $Z(\mathfrak{sl}_2)$ finite function.*

Proof. Only a sketch.

The center of the universal enveloping algebra of the complexified Lie algebra is generated by the Casimir operators. From [Gar10] we know that the casimir is

$$\Omega = \frac{1}{2}H^2 + XY + YX.$$

We have the coordinates on $\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} SO_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ from [Bum97][1.19 pg 139] in which the casimir acts as the differential operator

$$\Delta = y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) - y\frac{\partial^2}{\partial x \partial \theta},$$

[Bum97][1.29 pg 143 ,Prop 2.2.5 pg 155]. Now we claim that F is an eigenfunction for this operator. An element $(x, y, \theta) := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \kappa_\theta \in \mathrm{SL}_2(\mathbb{R})$ acts on i by sending it to $x + iy$ (elementary computation). The bottom row of the product is $y^{-1/2}\sin\theta; y^{-1/2}\cos\theta$ which results in

$$F(x, y, \theta) = y^{k/2}e^{-ik\theta}f(x + iy).$$

It is then a calculus exercise to apply Δ to this, using the holomorphicity we also get that $f_{xx} - f_{yy} = 0$ and $f_y = if_x$ which cancels away terms and we get that

$$\Delta F(x, y, \theta) = \frac{k}{2}\left(\frac{k}{2} - 1\right)F(x, y, \theta).$$

Therefore the dimension of $Z(\mathfrak{g})F$ is simply one.

This example makes it clear that the two finiteness conditions for automorphic forms are in some sense functional equations that they must satisfy. There is a nice explanation of how to lift this to the adelic setting in several places, however it is stated quite clearly in [Coga, 2.1]

Chapter 3

Automorphic Representations

The references that will be most helpful are [BC79, I.II][GH24] for the general theory, we will follow the notation developed in [MW95] as it is somewhat standard. We will discuss some of the details of their representation theory because it is both subtle and essential for the setup in [JLZ13]. In particular we want to draw attention to the failure of this theory to be “categorical” or “algebraic” in any nice sense.

3.1 Local Representation Theory

Recall that in the representation theory of finite groups over $\mathbb{C}$ there is really only one important representation, that is the regular representation i.e. the $\mathbb{C}[G]$ module $\mathbb{C}[G]$. This is important for two reasons, the first is that it is always a priori defined uniformly for all groups, the second is that it decomposes into a direct sum over all irreducible modules, we will have more to say in section 4.2.1.

3.1.1 At the Archimedean Places

Remark 3.1. What we have called automorphic forms are sometimes referred to as “smooth K-finite automorphic forms” [Coga, 2.2].

As in the finite group case we want to consider the right regular action of the group $G(\mathbb{A})$ on a space of functions, namely the representation

$$g.f(x) = f(xg).$$

One can ask if this representation sends an automorphic form to an automorphic form. The answer is no, the problem is in the K-finiteness in the infinite places.

Example 7 ([Coga], 2.3)**.** *If $\varphi \in \mathcal{A}(\Gamma\backslash G(F_\infty))$ is K_∞-finite, then $g.\varphi$ is $gK_\infty g^{-1}$-finite. This is still a maximal compact subgroup, however in the infinite place it will a priori have only the identity in common with the original K.*

For example consider SL_2 *where the maximal compact is* SO_2*, if we conjugate we get* gSO_2g^{-1}

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \cos\theta + (db+ca)\sin\theta & -\sin\theta(a^2+b^2) \\ \sin\theta(d^2+c^2) & \cos\theta - (bd+ac)\sin\theta \end{pmatrix}.$$

If we want to find the intersection of SO_2 *with* gSO_2g^{-1} *we need to solve the system*

$$\begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix} = \begin{pmatrix} \cos\theta + (db+ca)\sin\theta & -\sin\theta(a^2+b^2) \\ \sin\theta(d^2+c^2) & \cos\theta - (bd+ac)\sin\theta \end{pmatrix}.$$

Where θ *might not be* θ'*. If* $\theta = n\pi, n \in \mathbb{Z}$ *then the* $\sin$ *terms on the right vanish and we get the* ± 1 *as a point of intersection, so consider* $\theta \neq n\pi$*. Then we require*

$$\cos\theta' = \cos\theta - (bd+ac)\sin\theta = \cos\theta + (db+ca)\sin\theta,$$

hence $2(bd+ac)\sin\theta = 0$ *and because* $\sin\theta$ *was assumed to be non-zero this is the same as* $bd+ac = 0$*. Thus for instance the element* $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ *conjugates* SO_2 *to another subgroup that has only trivial intersection.*

Finally it is worth noting that this is not an issue at the finite places, namely if $K = K_f K_\infty$ *is our maximal compact subgroup of* $G(\mathbb{A})$ *then* K_f *is also open and hence* $K_f \cap gK_fg^{-1}$ *is of finnite index in both* K_f *and* gK_fg^{-1} *and so their notions of* K*-finiteness will agree.*

For this reason we will need to talk about $(\mathfrak{g}, K)$-modules. This is the solution of Harish-Chandra that we do not yet understand the full significance of.

Definition 3.2 ([GH24], 4.4.6)**.** Let G be a Lie group (for example the analytification of the real or complex points of our favourite reductive LAG) and K be a maximal compact subgroup of G. Let $\mathfrak{g}$ be the complexified Lie algebra of G and $\mathfrak{k}$ the real Lie algebra of K.

A $(\mathfrak{g}, K)$**-module** is a complex vector space V with two representations

$$\tilde{\pi} : \mathfrak{g} \to End(V), \quad \pi : K \to \mathrm{GL}(V),$$

satisfying the following axioms

- V decomposes into a countable direct sum of finite dimensional K representations.
- The representations should be compatible: For all $X \in \mathfrak{k}$ and $v \in V$

$$\tilde{\pi}(X)(v) = \frac{\mathrm{d}}{\mathrm{d}t}\pi(e^{tX})(v)|_{t=0} = \lim_{t\to 0}\frac{\pi(e^{tX})(v) - v}{t}.$$

  In particular the right hand limit exists

- And compatible with the adjoint representation: For $k \in K$ and $X \in \mathfrak{g}$

$$\pi(k)\tilde{\pi}(X)\pi(k^{-1})(v) = \tilde{\pi}(Ad(k)(X))(v).$$

Remark 3.3. It is common to use the same symbol for both of these representations in the $(\mathfrak{g}, K)$-module.

Now one can check that the space of Archimedean automorphic forms is in fact an (admissible) $(\mathfrak{g}, K_\infty)$-module, under the representations that we have already specified; namely the regular action given by K and the representation that we defined 2.1.1 when talking about the center of the enveloping algebra [GH24, Thm. 6.2.6].

rewrite it here?

3.1.2 At the Non-Archimedean Places

As we noted above the right regular representation on the space of automorphic forms is well defined for the finite places i.e. if $\varphi(x) \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ and $g \in G(\mathbb{A}_f)$ then $\varphi(xg) \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$. Hence $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ is a $G(\mathbb{A}_f)$-module. In particular it is a module for $G(F_\nu)$ for all ν non-Archimedean.

3.2 Automorphic Representations

Recall that if A, B, C are all R modules and we have the inclusions of R modules $C \subseteq B \subseteq A$ then we call B/C a subquotient of A. We now think of $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ as being a $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$ module. An automorphic representation is then a subquotient of this representation.

Remark 3.4. Some authors will require that representation is by definition an *irreducible* subquotient.

Remark 3.5. We really need a set theoretic definition here. The quotient of these modules cannot be considered up to isomorphism of $(\mathfrak{g}, K)$-modules but must be the classical set theoretic realisation of this object, defined as equivalence classes of elements of the module. This is to say if one were to think of the category of automorphic representations it is quite small. The reason is that we will want to talk about the automorphic forms themselves, and consider their properties.

Remark 3.6. Automorphic representations can also be defined as representations of an algebra $\mathcal{H}$, the global Hecke algebra. This is the approach in [BC79, I.II(4.6)], and can be a helpful perspective to simplify definitions. We believe it is also the motivation behind Harish-Chandras $(\mathfrak{g}, K)$-modules.

Example 8. *It is very hard to really write down something explicit. One thing that we can do is take a modular form f. Then we know how to associate a concrete automorphic form $\tilde{f}$. To this (or any fixed automorphic form) we have an automorphic representation given by acting on this vector:*

$$\operatorname{span}_{\mathbb{C}}\left\{\big(G(\mathbb{A}_f) \times (\mathfrak{g}, K)\big).\tilde{f}\right\} \subseteq \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$$

3.2.1 Cuspidal Representations

Recall that an automorphic form $\varphi \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ is called **cuspidal** if all its constant terms vanish, see section 5.1 for more detail on constant terms. The space of such automorphic forms is denoted $\mathcal{A}_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$. An automorphic representation is called **cuspidal** if it is an irreducible subquotient of $\mathcal{A}_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$.

Remark 3.7. Again this is not as a $(\mathfrak{g}, K)$-module.

3.2.2 Isotypic Components

Following the convention of [MW95, II.1] we make two cases: Let π be an irreducible subquotient of the space $\mathcal{A}(M(k)\backslash M(\mathbb{A}))$, that is *not cuspidal*. Then we denote the π isotypic component of $\mathcal{A}(M(k)\backslash M(\mathbb{A}))$ by $\mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi$.

We will also need the space

$$\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi \quad := \quad \{\varphi \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) : \forall k \in K,\ \varphi_k \in \mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi\}$$

where $\varphi_k : M(\mathbb{A}) \to \mathbb{C}$ is given by $\varphi_k(x) = \varphi(xk)$.

Now if π is cuspidal, we define $\mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi$ to be the isotypic component of π in $\mathcal{A}_0(M(k)\backslash M(\mathbb{A}))$ and similarly we have

$$\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi \quad := \quad \{\varphi \in \mathcal{A}_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) : \forall k \in K,\ \varphi_k \in \mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi\}$$

AHHHHHHHHH BUT DO I TAKE THE CUSPIDAL PART OF INSIDE HERE OR NOT AHHHHHH DEFINE YOU TERMSSSSSS AHHHH

Remark 3.8. We cannot simply take the isotypic components as $(\mathfrak{g}, K)$-modules we need to take the isotypic components after explicitly restricting the spaces. This is to say again that the category of automorphic reps is very explicit.

The point is that we want the isotypic component corresponding to a cuspidal representation to be cuspidal, however this just might not be the case. Yamana in [Yam13, Rm. 7.12] has a counter example when one allows unitary groups over division algebras (non-commutative fields). It could be interesting to investigate this example more closely to see if the example can be pulled back to a unitary group over a field. In [Yam13] there is an automorphic representation of the quarternionic unitary group constructed, $\Pi(V)$, that appears in both the cuspidal and residual spectrum (see 4.2 for more detail). By that Yamana means that up to isomorphism the representation can been seen in both residual and cuspidal spectrum. In particular if one were to take the component that is in the cuspidal spectrum and look at its isotypic component then the versions in the residual spectrum would also occur and hence by definition of residual spectrum would not be cuspidal.

If we restrict to the cases dealt with in for instance [MW95], namely not dealing with quarternions, then we have been told that this is an open problem whether or not this restriction is superfluous.

Chapter 4

Spectral Decomposition and Eisenstein Series

4.1 Eisenstein Series

As usual we fix a connected reductive group G defined over a number field F, with a Borel B, a standard parabolic with Levi decomposition $P = MU$.

Following the setup in [MW95, I.1.4] we consider a **character** $\chi \in \mathrm{Rat}(M) \;:=\; \mathrm{Hom}_{\mathrm{LAG}}(M, \mathbb{G}_m)$, thinking of it below as a natural transformation, and then define

$$|\chi| : M(\mathbb{A}) \to \mathbb{C}, \quad (m_\nu) \mapsto \prod_\nu |\chi(F_\nu)(m_\nu)|_\nu.$$

The intersection of the kernels of these characters is

$$M^1 \;:=\; \bigcap_{\chi \in \mathrm{Rat}(M)} \ker|\chi|.$$

Thus we can define

$$X_M \;:=\; \mathrm{Hom}_{\mathrm{TopGroup}}(M(\mathbb{A})/M^1, \mathbb{C}^*).$$

i.e. the collection of characters of $M(\mathbb{A})$ that are trivial on M^1.

Remark 4.1. To make it seem less mysterious this group has some importance in the more general theory, in particular it is one of the pieces in the "Langlands decomposition" (1.2.1) of the Archimedean points of a parabolic and it has the property that $M(\mathbb{Q})\backslash M(\mathbb{A})^1$ has finite measure [GH24, 4.9].

The set of **complex characters** of M,

$$\mathfrak{a}_M^* \;:=\; \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{C},$$

is isomorphic as $\mathbb{C}$ vector spaces to X_M. If $Z_{G(\mathbb{A})}$ is the center of $G(\mathbb{A})$ then we also have the space

$$X_M^G \;:=\; \mathrm{Hom}_{\mathrm{TopGroup}}((M(\mathbb{A})/M^1)/Z_G, \mathbb{C}^*)$$

which is characters of $M(\mathbb{A})/M^1$ which are also trivial on the center of G.

Example 9. *For the maximal parabolic P_r with Levi M_r of Sp_{2n} we have that $X_{M_r}^{\mathrm{Sp}_{2n}}$ is at most a one dimensional $\mathbb{C}$ vector space.*

First of all we have that [MW95, I.1.4]

$$X_{M_r}^{\mathrm{Sp}_{2n}} \subseteq X_{M_r} \cong \mathfrak{a}_{M_r}^* \quad := \quad Rat(M_r) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Thus it is clearly sufficient to bound the dimension of $\mathfrak{a}_{M_r}^*$ as a $\mathbb{C}$ vector space, moreover this dimension agrees with the dimension of $Rat(M_r)$ as a free $\mathbb{Z}$ module.

Thus we compute $\dim_{\mathbb{Z}}(Rat(M_r))$:

$$\begin{aligned} Rat(M_r) &= Rat(\mathrm{GL}_r \times \mathrm{Sp}_{2m}) \\ &= \mathrm{Hom}(\mathrm{GL}_r \times \mathrm{Sp}_{2m}, \mathbb{G}_m) \\ (2) &\cong \mathrm{Hom}(\mathrm{Ab}(\mathrm{GL}_r \times \mathrm{Sp}_{2m}), \mathbb{G}_m) \\ (1) &\cong \mathrm{Hom}(\mathrm{Ab}(\mathrm{GL}_r) \times \mathrm{Ab}(\mathrm{Sp}_{2m}), \mathbb{G}_m) \\ (3) &\cong \mathrm{Hom}(\mathbb{G}_m \times 1, \mathbb{G}_m) \\ &\cong \mathbb{Z}. \end{aligned}$$

In (2) we have used the universal property of the abelianization $\mathrm{Ab}(G) = \mathcal{D}(G) \setminus G = [G,G] \setminus G$ because $\mathbb{G}_m$ is abelian. (1) is that the abelianization commutes with direct products. (3) is because Sp is a perfect group.

Remark 4.2. This generalises to the metaplectic covers immediately as $X_{M_r}^{\mathrm{Mp}_{2n}(\mathbb{A})} \subseteq X_{M_r}$.

There is the natural map $m_P : G(\mathbb{A}) \to M^1 \backslash M(\mathbb{A})$ sending $umk \mapsto M^1 m$, where $g = umk$ using the Langlands-Iwasawa decomposition 1.2.1.

Now if we take the collection of irreducible automorphic representations of M,

$$\hat{\mathcal{A}} \quad := \quad \{(\pi, V) : \pi \text{ is an irreducible automorphic representation of } M\}$$

then we can think of X_M^G as being one dimensional automorphic representations (with some extra invariance) and so there is a natural action on $\hat{\mathcal{A}}$ given by tensoring, i.e. if $\lambda \in X_M^G$ and $(\pi, V) \in \hat{\mathcal{A}}$ then

$$\lambda.\pi \quad := \quad \lambda \otimes \pi$$

Then $\hat{\mathcal{A}}$ decomposes as a disjoint union of its orbits. Consider the orbit $\mathfrak{P}$ of a cuspidal representation π_0, then by definition X_M^G acts transitively but it also acts freely [MW95, II.1]. Thus $\mathfrak{P}$ is in bijection with X_M^G. Through this bijection we transmit the complex structure on $\mathfrak{a}_M^*$ to X_M then to the quotient X_M^G and finally to $\mathfrak{P}$.

Now we will define an Eisenstein series: Let $\mathfrak{P}$ be as above, the orbit of a cuspidal automorphic representation endowed with a complex structure. Let $\pi \in \mathfrak{P}$ and $\varphi_\pi \in \mathcal{A}(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_\pi$, then $\lambda \in X_M^G$ acts on φ_π by

$$\lambda.\varphi_\pi(g) = (\lambda \circ m_P)(g)\varphi_\pi(g).$$

which is then an element of $\mathcal{A}(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_{\pi\otimes\lambda}$. Finally we have the **Eisenstein series** which is defined by the following sum

$$E(\varphi_\pi, \lambda, g) = \sum_{\gamma \in P(k)\backslash G(k)} \lambda.\varphi_\pi(\gamma g)$$

whenever it is convergent. The first thing to note is that for a fixed φ there is an open set in X_M^G and a compact subset of $G(k)\backslash G(\mathbb{A})$ such that the Eisenstein series converges (normally) [MW95, II.1.5].

If $P = MU, P' = M'U'$ are two standard parabolics of G that are conjugate, i.e. such that for $w \in G(k)$ we have $wMw^{-1} = M'$ Then w maps $\mathfrak{P}$ to $w\mathfrak{P}$, an orbit of an irreducible representations of M to an orbit of irreducible representations of M'.

Then the Eisenstein series is closely related (through its constant terms as discussed in 5.3.3) to the operator

$$M(w,\pi)(\varphi_\pi)(g) = \int_{(U'(k)\cap wU(k)w^{-1})\backslash U'(\mathbb{A})} \varphi_\pi(w^{-1}ug)du$$

where $\pi \in \mathfrak{P}$, $g \in G(\mathbb{A})$ and $\varphi_\pi \in \mathcal{A}(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_\pi$.

The key properties of both the Eisenstein series and this operator can be found in [MW95, IV.1.8, IV.1.9, IV.1.10, IV.1.11]. Most importantly as a function of $\mathfrak{P}$ it can be shown that (in the sense of Frechet spaces) they both have a meromorphic continuation to all of $\mathfrak{P}$. This was also given a second "soft proof" more recently in [BL22], with the spectral decomposition that follows from it also being worked out in [Del21]. Moreover for the Eisenstein series at a point in $p \in \mathfrak{P}$ at which it is holomorphic then $E(\varphi, p, g)$ is an automorphic form.

We are not really in a position to convey the true importance of these objects in the theory of automorphic forms, however we will make some comments. First some surveys are [Lap22], [Art79], [Kim], [Jia08]. To see the relation to the classical Eisenstein series there is [Gar16]. One thing that Eisenstein series do, as in the theory of modular forms, is that they furnish us with quasi-concrete examples. A we mentioned above [MW95, IV.1.9.(b).i] tells us that at the holomorphic points the Eisenstein series takes an automorphic form and returns an automorphic form, thus we can use them to multiply our examples. Another reason that these functions are important is through their normalisation and constant terms, in which products of L functions appear, we discuss this more in section

ref later

. This has been a fruitful method for proving theorems about L-functions as in [Sha10][Pol][Art79].

4.2 Spectral Decomposition

This is a short explanation of some terms that frequently appear as well as some motivation for the later results. The results contained here-in are proved using the Eisenstein series as an essential component.

4.2.1 The Decomposition of the Spectrum In General

For this section let H be a locally compact topological group. It is a classical theorem that for representations of finite groups over an algebraically closed field the regular representation decomposes into a direct sum, where ever irreducible representation appears [Lin, Ch. 2.4 Cor. 2]. This still holds for compact topological groups, when one considers continuous unitary representations [Fol16, 5.1].

Remark 4.3. This is a strict generalisation of the finite groups case, when we give the finite group the discrete topology then all its linear representations are continuous and unitary.

There is one final more general incarnation of this line of investigation in the Plancherel theorem. A group is **type I** if for every (continuous unitary) representation π such that the center of $\mathrm{Hom}_{\mathrm{Rep}}(\pi, \pi)$ is trivial we have a decomposition as a direct sum of irreducible representations.

Example 10. *Consider $G(\mathbb{A})$ the adelic points of a connected reductive LAG. This is a type one group.*

Example 11. *Consider $G(\mathbb{A})$ the adelic points of a connected reductive LAG. This is a seecond countable group.*

Example 12. *Consider $G(\mathbb{A})$ the adelic points of a connected reductive LAG. This is a unimodule group.*

fill

The idea of a direct integral is review in A to get a quick idea consider the following example:

Example 13 (Direct Sums)**.** *Let I be a countable set with the discrete sigma algebra and counting measure μ. Let $(\mathcal{H}_i)_{i\in I}$ be a collection of Hilbert spaces then*

$$\bigoplus_{i\in I} \mathcal{H}_i = \left\{ (h_i)_{i\in I} \in \prod_{i\in I} \mathcal{H}_i : \int_I \|h_i\|_i^2 d\mu < \infty \right\}.$$

I.e. the Hilbert space direct sum is by definition square summable sequences, but sums are just discrete integrals.

Theorem 3 (Plancherel, [Fol16], 7.44)**.** *The regular represntation of a type I, second countable and unimodular topological group is a direct integral of the irreducible unitary representations.*

Remark 4.4. The Plancherel theorem says much more in fact. Like the Peter-Weyl theorem for compact groups it doesnt just give you that some direct integral decomposition exists, it contains many more details about the topology and measure on the set of unitary irreducible representations, and which representations are associated to them in the direct integral. We are being breif as this is motivational.

Thus what one wants to do is find a decomposition of the regular representation $G(\mathbb{A}) \curvearrowright \mathrm{L}^2(G(\mathbb{A}))$. We call such decompositions "spectral", alluding to the spectral theorem which provides such a decomposition in terms of the eigenvector of certain operators. Moreover these decompositions are largely proved in terms of the more general spectral theorems. So once accomplished this is another one of the tools that can be used to compartmentalise problems in automorphic forms, by dealing with representations that appear in different parts of the spectrum.

4.2.2 Langlands Decomposition of the Spectrum

We have the Plancherel theorem but Langlands also provides a fine analysis of the spectrum using automorphic forms. The key result in this theory is the following decomposition,

Theorem 4 ([Art79], MAIN THEOREM (b))**.** *There is an orthogonal decomposition of the representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ into*

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{\mathscr{P}} L^2_{\mathscr{P}}(G(\mathbb{Q})\backslash G(\mathbb{A})),$$

where $\mathscr{P}$ runs over certain "associate classes" of parabolics of G and the summands are the direct integrals of spaces of L^2 automorphic forms.

These direct integrals are in fact constructed out of subspaces generated by Eisenstein series.

The spectrum of $L^2(G(\mathbb{A}))$ refers to such a decomposition. In particular we have some important "pieces" to such a decomposition. The piece that decomposes into a direct sum of irreducible is called the **discrete spectrum**. The compliment of the discrete spectrum is called the **continuous spectrum**. One can define cuspidal L^2 functions in the exact same way as cuspidal automorphic forms 5.1 and then it has been shown that the **cuspidal spectrum**, the subspace of L^2 consisting of cusp forms, decomposes as a direct sum [GH24, 9]. Thus the cuspidal spectrum is contained in the discrete spectrum in this case. The **residual spectrum** is defined to be the compliment of the cuspidal spectrum in the discrete spectrum.

4.2.3 Residual Spectrum

Moeglin and Waldspurger also acheived a more fine analysis of the spectrum of GL_n in terms of residues of Eisenstein series. First consider the group GL_n. We then let $n = ab$ for positive integers a, b. If τ is an irreducible, cuspidal automorphic rep of GL_a then Moeglin and Waldspurger construct a representation of $\mathrm{GL}_{ab} = \mathrm{GL}_n$ called the "Speh representation" and denote it

$$\Delta(\tau, b).$$

They go on to prove that as τ and b vary these representations span the residual spectrum of $L^2(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}))$ [JLZ13, Thm. 1.1].

This representation is formed by taking iterated residues of Eisenstein series in the sense of [MW95, V]. For a nice survey of problems in this area, of residues of Eisenstein series, there is [Jia08].

4.3 Automorphic L-Functions

We don't intent to define in great detail automorphic L-functions, as there are many other better sources to learn from [BC79, Part 2.III.2][Sha10][Cogb][BCDS$^+$04, 9, 10, 11][**?**], we will recall the idea and then discuss some of the properties and relations with Eisenstein series and interwining operators that we will need later.

The first thing is to recall the classification of connected reductive groups defined over an algebraically closed field via root datum. A root datum is a tuple $(X, \Phi, \check{X}, \check{\Phi})$ where X and $\check{X}$ are two free abelian groups of finite type, $\Phi, \check{\Phi}$ are subgroups that are in duality via a perfect pairing on $X, \check{X}$. Then each reductive group G over a number field F has associated the root datum that is associated to its base change to $\mathbb{C}$. Thus to a connected reductive group over a number field we associate a connected reductive group over $\mathbb{C}$, given by the dual root datum. We call this the **dual group** of G and denote it $\hat{G}$. The **Langlands dual group** is then the dual group producted with the $\mathrm{Gal}(\bar{k}/k)$

$$^LG \;\; := \;\; \hat{G} \rtimes \mathrm{Gal}(\bar{k}/k).$$

Example 14 (Classical Groups, [BCDS+04], 11.1)**.** *We have the following table*

G	$\hat{G}$
GL_n	GL_n
SO_{2n+1}	Sp_{2n}
SO_{2n}	SO_{2n}
Sp_{2n}	SO_{2n+1}

Then, using the Satake isomorphism [Sha10, 2.2], to each unramified representation of $G(F_\nu)$ we can associate a conjugacy class of ^LG, via some map call it c, and hence there is a way to apply a complex representation $r :^L G \to \mathrm{GL}_n(\mathbb{C})$ to representations of $G(F_\nu)$. Thus the automorphic L-functions are defined as follows: Let ρ be a representation of $G(\mathbb{A})$, let r be a complex representation of ^LG and $s \in \mathbb{C}$ then

$$L(s, \rho, r) \;\; := \;\; \prod_\nu L_\nu(s, \rho_\nu, r) = \prod_\nu \frac{1}{\det\big(I - r(c(\rho_\nu))q^{-s}\big)},$$

where ν runs over the unramified places. It is a part of the grand Langlands philosophy that there should be suitable L-functions for the ramified places satisfying very nice properties.

Remark 4.5. The global L-functions have been defined for many groups at this point and indeed [JLZ13] uses known properties to prove their results. One should note that the questions that we are interested in are still tractable even though the L-functions might not be defined (for instance for the metaplectic group). This is because only finitely many places will ramify, and so as long as those places are neither zero or poles we can transfer questions about zeros and poles from the full global L-functions to L-functions at almost all places.

Example 15 (Standard Representations / Classical Groups)**.** *In the case of classical groups it is common to see L-functions with only two entries e.g. if ρ is a representation of $G = \mathrm{Sp}\,2n$ then you may see $L(s, \rho)$. The reason is that there is a standard representation of the dual groups of classical groups. Namely the standard representation of a matrix group inside GL_n is the one that sends $g \mapsto g$. It is this representation that is to be taken for the dual group in this setting.*

Example 16 (Rankin-Selberg, [?], 1.2, [?], Ch. 2 Example. 2)**.**

fill

Let ν be a finite place of $\mathbb{Q}$ and π, π' be two unramified generic representations of $\mathrm{GL}_n(\mathbb{Q}_\nu)$ and $\mathrm{GL}_m(\mathbb{Q}_\nu)$ respectively. Let B_n be the standard Borel of upper triangular matricies in GL_n. Such representations have been classified

reference

in terms of characters of $\mathbb{Q}_\nu^\times$*, in particular for* π *there are* $\mu_1, ..., \mu_n$ *unramified characters such that*

$$\pi \cong \operatorname{Ind}_{B(\mathbb{Q}_\nu)}^{\mathrm{GL}_n(\mathbb{Q}_\nu)} \big(\mu_1 \otimes \cdots \otimes \mu_n\big).$$

If we fix a uniformizer ϖ *of* $\mathbb{Q}_\nu$ *then we have the so called "Satake parameters"* $\mu_i(\varpi)$ *which determines* π *uniquely. Of course the same is true for* π'*, with say characters* $\mu'_1, ..., \mu'_m$*. We then define*

$$L(s, \pi \times \pi') \quad := \quad \prod_{i,j} \frac{1}{1 - \mu_i(\varpi)\mu'_j(\varpi)q^{-s}}.$$

Consider the group $G = \mathrm{GL}_n \times \mathrm{GL}_m$ *which has dual* $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$*, then there is a canonical representation*

$$r : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_{nm}(\mathbb{C}).$$

Moreover any automorphic representation π *of* G *will be a tensor product of a representation of* GL_n *and of* GL_m

$$\pi \cong \pi \otimes \pi'.$$

Then

$$L(s, \pi, r) = L(s, \pi \otimes \pi', r) = L(s, \pi \times \tilde{\pi}'),$$

where the tilde denotes the contragradient.

Example 17 (Dirichlet L-functions)**.** *Recall that a Dirichlet character* χ *is a character of the group* $(\mathbb{Z}/N\mathbb{Z})^*$*. Through the series of maps*

$$A^\times \cong \mathbb{Q}^\times \times \mathbb{R}_{>0}^\times \times \hat{\mathbb{Z}}^\times \to (\lim \mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C},$$

one get a bijection between Dirichlet characters and finite-order Grossencharacters, i.e. characters of $\mathbb{A}_F^\times/F^\times$*. Grossencharacters have the associated L-function as they are just automorphic forms of* GL_1*, which generate automorphic representations. These give us the classical Dirichlet L-functions.* reference? More deta

Although this might seem unrelated to the current section on spectral decomposition and Eisenstein series we will see later that the two are inextricably linked

ref

Chapter 5

Constant Terms

Here we will explain the role of the constant term in our calculation of poles.

5.1 Definition and Role

The constant term is an operation defined on a large class of functions and is supposed to generalise the constant term of a Fourier expansion, we will see this later, although one may consult [Bum97, 1.6] for some examples as well. In particular [MW95, I.2.6] give the definition as follows: We consider $P = MU$ a standard parabolic of G and $\varphi : U(k) \setminus \mathbf{G} \to \mathbb{C}$ a measurable and locally L^1 function then its constant term along P is

$$\varphi_P : U(\mathbb{A}) \setminus \mathbf{G} \to \mathbb{C}$$
$$\varphi_P(g) \quad := \quad \int_{U(k)\setminus U(\mathbb{A})} \varphi(ug)\mathrm{d}u$$

which inherits many of the properties of φ such as smoothness and moderate growth. For instance if φ is an automorphic form on $\mathbf{G}$ then its constant term is an automorphic form on M [GH24, 6.5].

This allows us to recall the definition of cuspidal automorphic forms or "cusp forms". Let φ be a an automorphic form on $U(\mathbb{A})M(k) \setminus \mathbf{G}$ for $P = MU$ a standard parabolic. Then φ is cuspidal if for all standard parabolics $P' \subset P$ we have that

$$\varphi_{P'} = 0$$

Cusp forms have a central role in the theory of automorphic forms, this is for several reasons. They appear historically as interesting examples such as the Ramanujan tau function, by a theorem of Ribet [SZS77, T2.3] the Galois representation associated to a cusp form is irreducible, they just make formulas easier to work with as things will become zero and they form the "base case" for the proof of the spectral decomposition 4.2.

The constant term itself is of central importance. One reason for this is that it controls the growth of the automorphic forms. More precisely we have the following theorem [MW95, I.4.10]

Theorem 5. *Let $P = MU$ be a standard parabolic of G, $V_P \subseteq \mathcal{A}_0(U(\mathbb{A})M(k) \setminus G)$ a finite* *dimensional subspace, $\Gamma_P \subseteq X_M$ a compact subset and N_P an integer. Let $n \in \mathbb{Z}$, $D \subseteq \mathbb{C}^n$ open and connected, $f : D \to \mathbb{C}$ holomorphic and not identically zero. Let $D' = \{x \in D : f(x) \neq 0\}$.*

If $\varphi : D' \to L^2_{loc}(G)$ is a function such that for all $z \in D', \varphi(z) \in \mathcal{A}((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G})$ and for all P

$$\psi_P : D' \to L^2_{loc}(G)$$

$$z \mapsto \psi_P(z) = \varphi_P^{cusp}(z)$$

is a holomorphic function.

Then φ can be analytically continued to a holomorphic function on D iff ψ_P can be continued to a holomorphic function on D.

Unpacking this a bit we see that an Eisenstein series satisfies the hypothesis on φ by [MW95, IV.1.9]

make this more precise

, we think of D' as the positive open cone on which the Eisenstein series converges, and then $D = \mathbb{C}^n \setminus S$ as the rest of the cuspidal datum $\mathfrak{P}$ on which the Eisenstein series is holomorphic (S is its set of singularities). Therefore the domain on which the Eisenstein series is holomorphic is the same as the domain on which (roughly) its constant term is holomorphic.

5.2 Integration Lemmas

Theorem 6. *If G is a locally compact Hausdorff group with a left Haar measure μ and if $\chi\colon G \to \mathbf{C}^\times$ is a non-trivial character on G, then*

$$\int_G \chi(g)\, d\mu(g) = 0.$$

Proof. Pick an element h of G such that $\chi(h) \neq 1$. The equation above then follows from

$$\int_G \chi(g)\, d\mu(g) = \int_G \chi(hg)\, d\mu(g) = \int_G \chi(h)\chi(g)\, d\mu(g) = \chi(h) \int_G \chi(g)\, d\mu(g).\square$$

Integrating trivial characters gives the volume of the measure space which we typically normalize to be one.

Theorem 7 ([Gar] 5.2, [Fol16] Thm 2.49)**.** *Let $H \leq G$ be a closed subgroup. If $H \backslash G$ has a right G invariant measure (iff their modular functions agree on H)then the integral is unique up to scalar, namely for a given Haar measures dh on H and dg on G there is a unique invariant measure dq on $H \setminus G$ such that for all $f \in C_c^0(G)$*

$$\int_{H\backslash G} \int_H f(hq) dh dq = \int_G f(g) dg$$

Note that this quotient may not be a group, because H is not required to be normal.

5.3 Constant Terms of Eisenstein Series

This computation forms the heart of a well known theorem, [GH24, Prop 10.4.2][MW95, II.1.7][Sha10, 6.2]. I give more detail here then I could find anywhere else.

Notice that the Eisenstein series has a full $G(k)$ invariance and so we can take its constant terms along *any* standard parabolic.

5.3.1 In General

We will use the following Lemmas to give a simplified expression of the constant term of an Eisenstein series. First fix $P = MN$ and $P' = M'N'$ two standard parabolics of suitable group G over F, with $E(x, \varphi, \lambda)$ defined via parabolic induction from P.

Lemma 5.

$$P(F) \setminus G(F) \cong \coprod_{w \in W_{M'} \setminus W_G / W_M} P'(F) \cap wP(F)w^{-1} \setminus P'(F)$$

Proof. Consider the Bruhat decomposition:

$$G(F) = \coprod_{w \in W_{M'} \setminus W_G / W_M} P(F)w^{-1}P'(F)$$

then becuase the action of $P(F)$ keeps the disjoint sets disjoint we can move the quotient through and get

$$P(F) \setminus G(F) = \coprod_{w} P(F) \setminus P(F)w^{-1}P'(F)$$

so we analyse the summands, by the second isomorphism theorem we have a bijection

$$P(F) \setminus P(F)w^{-1}P'(F) \cong P(F) \cap P'(F) \setminus w^{-1}P'(F)$$

now if $[w^{-1}p] \in P(F) \cap P'(F) \setminus w^{-1}P'(F)$ then its $pw^{-1}p'$ for some $p \in P(F) \cap P'(F)$ and hence multiplying by w, in particular an isomorphism, gives $wpw^{-1}p' \in wP(F)w^{-1} \times P'(F)$ and so

$$w(P(F) \cap P'(F) \setminus w^{-1}P'(F)) \cong wP(F)w^{-1} \cap P'(F) \setminus P'(F)$$

Lemma 6. *Let $m', n' \in M'(F) \times N'(F)$ then*

$$m'n' \in wP(F)w^{-1} \iff m' \in wP(F)w^{-1} \text{ and } n' \in (m')^{-1}wP(F)w^{-1}m'$$

Proof. The forward implication is stated in [GH24], the converse follows from some algebra: First let $m' = wp_1w^{-1}$ and $n' = (m')^{-1}wp_2w^{-1}m'$ then

$$\begin{aligned} m'n' &= (wp_1w^{-1})^{-1}wp_2w^{-1}wp_1w^{-1} \\ &= wp_1^{-1}w^{-1}wp_2w^{-1}wp_1w^{-1} \\ &= wp_1^{-1}p_2p_1w^{-1} \in wP(F)w^{-1} \end{aligned}$$

Taking the contrapositive of this lemma will be used below. This is because our sums will be over quotients like $A \setminus B$ and therefore summing over the "elements" in B that are not in A; by our lemma would be the same as summing over two different such quotients. Now consider the computation:

$$\begin{aligned}
E_{P'}(\varphi, \lambda, x) &= \int_{N'(F)\setminus N'(\mathbb{A})} E(\varphi, \lambda, nx)dn \\
([N'] \;:=\; N'(F) \setminus N'(\mathbb{A})) \quad &= \int_{[N']} \sum_{\delta \in P(F)\setminus G(F)} \lambda.\varphi(\delta nx)dn \\
(\text{Lemma 1}) \quad &= \int_{[N']} \sum_{\delta \in \coprod_{w \in W_{M'}\setminus W_G/W_M} P'(F)\cap wP(F)w^{-1}\setminus P'(F)} \lambda.\varphi(\delta nx)dn \\
&= \sum_{w \in W_{M'}\setminus W_G/W_M} \int_{[N']} \sum_{p' \in P'(F)\cap wP(F)w^{-1}\setminus P'(F)} \lambda.\varphi(w^{-1}p'nx)dn \\
(\text{Lemma 2}) \quad &= \sum_{w} \sum_{m' \in M'(F)\cap wP(F)w^{-1}\setminus M'(F)} \int_{[N']} \sum_{n' \in N'(F)\cap (m')^{-1}wP(F)w^{-1}m'\setminus N'(F)} \lambda.\varphi(w^{-1}m'n'nx)d \\
(\text{Change Var}) \quad &= \sum_{w}\sum_{m'} \int_{[N']} \sum_{n' \in N'(F)\cap wP(F)w^{-1}\setminus N'(F)} \lambda.\varphi(w^{-1}n'nm'x)dn \\
(\text{Unfold}) \quad &= \sum_{w}\sum_{m'} \int_{N'(F)\cap wP(F)w^{-1}\setminus N'(\mathbb{A})} \lambda.\varphi(w^{-1}nm'x)dn.
\end{aligned}$$

The change of variables is $(m', n') \mapsto ((m')^{-1}n'm', (m')^{-1}n'm')$. Again we assume that our x is sufficiently large so all the integrals converge.

Maybe appologise for doing integrals naivelly lol.. clarify this.... is it even the x that I need to worry about here?

5.3.2 Constant Terms of Cuspidal Eisenstein Series

Lemma 7 (4)**.** *For $w \in W_{M'} \setminus W_G/W_M$ we have that $w^{-1}P'w \cap M$ is a standard parabolic of M with Levi $w^{-1}M'w \cap M$ and unipotent $w^{-1}N'w \cap M$.*

Proof. This is [GH24, 10.4.1] stated without proof. They give the reference [RS, V.4.6] which is in French..

Lemma 8 (5)**.**

$$w^{-1}U'w \cap P = (w^{-1}U'w \cap M)(w^{-1}U'w \cap U).$$

Proof. [GH24, 10.4.1] has some decompositions, as well as the standard decomposition of $P = MU$ I think I could prove this...

Lemma 9 (6)**.**

$$c \setminus (b \setminus a) = (bc) \setminus a$$

Continuing the computation of the constant term above, we will focus purely on the inner integral now

$$\begin{aligned}\int_{N'(F)\cap wP(F)w^{-1}\backslash N'(\mathbb{A})} \lambda.\varphi(w^{-1}nm'x)dn &= \int_{w^{-1}N'(F)w\cap P(F)\backslash w^{-1}N'(\mathbb{A})w} \lambda.\varphi(nw^{-1}m'x)dn \\ \text{(Lemma 5)} &= \int_{(w^{-1}U'w\cap M)(w^{-1}U'w\cap U)(F)\backslash w^{-1}N'(\mathbb{A})w} \lambda.\varphi(nw^{-1}m'x)dn \\ \text{(Unfold + Lemma 6)} &= \int_{(w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A}))\backslash A} \int_{w^{-1}U'(F)w\cap M(F)\backslash w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A})} \lambda.\varphi(n_1n_2w^{-1}m'x)d\end{aligned}$$

the first equality is the change of variables $w^{-1}nw \mapsto n$ and $A = (w^{-1}U'(F)w\cap U(F))\backslash w^{-1}N'(\mathbb{A})w$. Now look at the inner integral here more closely

$$\int_{w^{-1}U'(F)w\cap M(F)\backslash w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A})} \lambda.\varphi(n_1n_2w^{-1}m'x)dn_1dn_2,$$

applying Lemma (6) we see that this is a constant term for a parabolic of M, of the function $m \mapsto \varphi(mn_2w^{-1}m'x)$.

Lemma 10. *$n_2w^{-1}m'x \in K$ with variables as above.*

This was all in complete generality as well. If we now assume further that the Eisenstein series was induced from a *cuspidal* automorphic representation, then $m \mapsto \varphi(mk)$ is a cusp form and therefore this last integral will vanish whenever $w^{-1}U'w \cap M \neq \{1\}$, because in that case the inner integral doesn't exist (its over a point).

5.3.3 Constant Term Of Eisenstein Series for Conjugate Levis

If we now assume that $M' = wMw^{-1}$ and recall the definition of our intertwining operator **??** we can use the following

Lemma 11 ([MW95] II.1.7 (6))**.**

$$U'(k) \cap wP(k)w^{-1} = U'(k) \cap wU(k)w^{-1},$$

to see that

$$\begin{aligned}E_{P'}(\varphi, \lambda, x) &= \sum_w \sum_{m'} \int_{N'(F)\cap wP(F)w^{-1}\backslash N'(\mathbb{A})} \lambda.\varphi(w^{-1}nm'x)dn \\ &= \sum_w \sum_{m'} \int_{N'(k)\cap wN(k)w^{-1}\backslash N'(\mathbb{A})} \lambda.\varphi(w^{-1}nm'x)dn \\ &= \sum_w \sum_{m'} M(w, \pi)(\lambda.\varphi)(x)\end{aligned}$$

I have mixed up my N's and U's too much...

In particular we can combine the conjugate and cuspidal cases to get a much simpler expression for some constant terms of some Eisenstein series.

5.4 Siegel Phi Operator

Here we give an example of the constant term which connects it to the classical picture. We thank Chengjing Zhang for showing us this example, and present it here because we cannot find it in the literature.

give references for the other interpretations of the constant term, there must be some

We deal only with the classical Siegel modular forms of full level and moreover are less explicit with the steps as they should be clear after exposure to the previous arguments.

Because we are trying to connect this to the classical picture it is most convenient to think of thinks in the Archimedean places, recall the way that modular forms are automorphic forms most naturally in the archimedian sense ([GH24, 6.2]) [Eme][Bum97][Boo]. So for this section alone, by automorphic form we will mean automorphic forms on the Archimedean places, and the constant term will be taken only on the Archimedean part: i.e. for $f : G(\mathbb{R}) \to \mathbb{C}$ and automorphic its constant term along a parabolic of G, call it $P = MN$, is [GH24, 8.6]

$$f(x)_P = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} f(xn)\mathrm{d}n.$$

We assume here for simplicity (and because it will apply to the examples below) that our groups are unimodular.

is that true....

5.4.1 Siegel Modular Forms

We collect some definitions from [BVDGHZ08] to fix notation. Let the Siegal upper half plane be defined as

$$\begin{aligned} \mathcal{H}_g &:= \{\tau \in \mathrm{M}_{g\times g}(\mathbb{C}) : \tau \text{ is symmetric and has positive definite imaginary part}\} \\ &\cong \mathrm{Sp}_{2g}(\mathbb{R})/U(g) \end{aligned}$$

where the isomorphism is as analytic manifolds and

$$U(g) \quad := \quad \left\{ \begin{pmatrix} A & B \\ -B & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}) : AA^t + BB^t = 1 \right\}$$

For every $\gamma = (A\ B;\ C\ D) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\tau \in \mathcal{H}_g$ we have the action

$$\gamma.\tau = (A\tau + B)(C\tau + D)^{-1}$$

We say that a holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ is a (classical) Siegel modular form of weight k if

$$f(\gamma.\tau) = \det(C\tau + D)^k f(\tau)$$

with the extra condition that if $g = 1$ it must be holomorphic at ∞. Because $\mathrm{Sp}_2 = \mathrm{SL}_2$ this is a strict generalisation of an (elliptic) modular form.

The space of Siegel modular forms of weight k and genus g is denoted $\mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$. There is a useful operator know as the Siegel Phi Operator which allows you to lift known modular forms from lower genus to higher genus [BVDGHZ08, 5]

$$\mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z})) \xrightarrow{\Phi} \mathcal{M}_k(\mathrm{Sp}_{2(g-1)}(\mathbb{Z}))$$

defined by the limit for $\tau \in \mathcal{H}_{g-1}$

$$\Phi(f)(\tau) \quad := \quad \lim_{t\to\infty} f \begin{pmatrix} \tau & \\ & it \end{pmatrix}$$

in this context a cusp form is defined to be a Siegel modular form in the kernel of the Siegel Φ operator and so it is natural to wonder if there is a constant term that is being taken here.

5.4.2 Automorphising

Given a Siegel modular form $f \in \mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ we can associate an automorphic form

$$\tilde{f} : \mathrm{Sp}_{2g}(\mathbb{R}) \to \mathbb{C}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \det(ci+d)^{-k} f\Big((ai+b)(ci+d)^{-1}\Big),$$

where a, b, c, d are $g \times g$ matrices such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{R})$.Fix the Borel of upper triangular matrices. Now for $1 \leq r \leq g-1$ an integer we have the standard maximal parabolic of Sp_{2g}, $P_r = M_r N_r$ such that

$$M_r \cong \mathrm{GL}_r \times \mathrm{Sp}_{2(g-r)}$$

Theorem 8 (Zhang)**.** *If f is a classical Siegel modular form of weight k and degree g, then*

$$\tilde{f}_{P_r}(u\gamma) = \det u^k \cdot (\Phi^r f)^{\sim}(\gamma) \tag{5.4.1}$$

for every element γ of $\mathrm{Sp}_{2(g-r)}(\mathbb{R})$ and every element u of $\mathrm{GL}_r(\mathbb{R})$.

In particular

$$\tilde{f}_{P_{g-1}} \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\Phi f)^{\sim} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This shows that perhaps the correct generalisation of the Siegel Phi function is just the constant term that we all know and love. We could also attempt to expand this to Siegel modular forms that are vector valued or not of full level.

The only other work on generalising the Siegel Φ operator that we could find appears in [Gre24]. Grenier formulates the Φ operator in the language of symmetric spaces [Ter16, Ch. 2] and then shows that the analogous definition in the case of "automorphic forms" in the sense of the symmetric space $\mathscr{P}_n/\,\mathrm{GL}_n(\mathbb{Z})$ of symmetric positive definite real matrices [Ter16, 1.5.1] behaves in the same way. Namely his [Gre92, Thm. 2] shows that it sends an automorphic form for $GL_n(\mathbb{Z})$ to an automorphic form for $\mathrm{GL}_{n-1}(\mathbb{Z})$. The point is that the Φ operator can be defined in the generality of symmetric spaces and Grenier shows that at least in one other case it still preserves the relevant notion of automorphic form. This suggests two things that would be interesting to investigate; using the classification of symmetric spaces is it possible to give a uniform definition of the Φ

operator following Grenier and does this definition agree with the constant term in the way that the Siegel Φ operator does. With my limited knowledge of symmetric spaces this seems to be very tractable.

5.4.3 Base Case

The base case is very instructive, it deals with modular forms. So consider f a (elliptic) modular form of full level and weight k, which has a Fourier expansion given by

$$f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$$

Then one can verify that

$$\tilde{f}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ci+d)^{-k} f\Big(\frac{ai+b}{ci+d}\Big)$$

is an automorphic form on Sp_2. The only non-trivial parabolic P is the one of upper triangular matricies, with Levi and unipotant given respectively

$$M = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \cong \mathrm{GL}_1, \quad N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cong \mathbb{G}_a$$

along which we can now compute the constant term

$$\begin{aligned} \tilde{f}_P(m) &= \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \tilde{f}(mb)\mathrm{d}b \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} \tilde{f}\begin{pmatrix} m & mb \\ 0 & m^{-1} \end{pmatrix} \mathrm{d}b \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} m^k f(m^2 i + m^2 b)\mathrm{d}b \\ &= m^k a_0 \end{aligned}$$

We have chosen normalisation to remove the usual factor of $1/2\pi$ in the constant term of the Fourier series. Moreover we see that

$$\Phi(f) = \lim_{t\to\infty} f(it) = \lim_{t\to\infty} \sum_{n\geq 0} a_n e^{-2\pi n t} = a_0$$

5.4.4 Simplifying the Constant Term

As we saw in 1.2.1 for $1 \leq r \leq g-1$ an integer we have the standard maximal parabolic of Sp_{2g}, $P_r = M_r N_r$ such that

$$M_r \cong \mathrm{GL}_r \times \mathrm{Sp}_{2(g-r)}$$

which can be given the explicit matrix representations

$$m(\gamma, A) \quad := \quad \begin{pmatrix} A & & & \\ & a & & b \\ & & (A^t)^{-1} & \\ & c & & d \end{pmatrix}, \quad A \in \mathrm{GL}_r(F),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2(g-r)}(F)$$

and unipotent

$$n(s; h, k) \quad := \quad \begin{pmatrix} 1 & 0 & 0 & h \\ -k^t & 1 & h^t & s + h^t k \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h, k \in \mathrm{Mat}_{(g-r)\times r}(\mathbb{R})\ s \in \mathrm{Sym}_r(\mathbb{R})$$

We have the following short exact sequence

prove it

$$1 \to \mathrm{Sym}_r(\mathbb{R}) \to N_r(\mathbb{R}) \to \mathrm{Mat}_{(g-r)\times r}(\mathbb{R}) \times \mathrm{Mat}_{(g-r)\times r}(\mathbb{R}) \to 1.$$

which we will use to unfold our integral below, for compactness we define $H_r \ := \ \mathrm{Mat}_{(g-r)\times r}$. We will now denote $[G] \ := \ G(\mathbb{Z})\backslash G(\mathbb{R})$ and compute the constant term

$$\begin{aligned} \tilde{f}_{P_r}\big(m(\gamma, A)\big) &= \int_{[N_r]} \tilde{f}\big(nm(\gamma, A)\big)\, dn \\ &= \int_{[H_r \times H_r]} \int_{[\mathrm{Sym}_{g-r}]} \tilde{f}\big(n(s; h, k)m(\gamma, A)\big)\, ds\, d(h, k) \\ &= \int_{[H_r]} \int_{[H_r]} \int_{[\mathrm{Sym}_{g-r}]} \tilde{f}\big(n(s; h, k)m(\gamma, A)\big)\, ds\, dh\, dk. \end{aligned} \tag{5.4.2}$$

Now we focus on simplifying the integrand. We want an explicit form of the matrix so we can relate it back to the value of the un-lifted Siegel modular form f; simply multiply the matrices gives, where (all rings are commutative) $A^{-t} \ := \ (A^t)^{-1}$

$$n(s; h, k)m(\gamma, A) = \begin{pmatrix} a & 0 & b & hA^{-t} \\ -k^t a + h^t c & A & -k^t b + h^t d & sA^{-t} + h^t k A^{-t} \\ c & 0 & d & kA^{-t} \\ 0 & 0 & 0 & A^{-t} \end{pmatrix}.$$

because $a, b, c, d \in \mathrm{Mat}_{(g-r)\times(g-r)}, A \in \mathrm{Mat}_{r\times r}$ we see that the $g \times g$ blocks that we now need to take the determinant of are the 4×4 corners of this picture, hence the matrices below should all be in $\mathcal{H}_g \subseteq \mathrm{Mat}_{g\times g}$

$$\tilde{f}(n(s; h, k)m(\gamma, A)) = \det\left(\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} i + \begin{pmatrix} d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}\right)^{-k}.$$

$$f\left(\left(\begin{pmatrix} a & 0 \\ -k^t a + h^t c & A \end{pmatrix} i + \begin{pmatrix} b & hA^{-t} \\ -k^t b + h^t d & sA^{-t} + h^t k A^{-t} \end{pmatrix}\right)\left(\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} i + \begin{pmatrix} d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}\right)^{-1}\right)$$
$$= \det\left(\begin{pmatrix} ic + d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}\right)^{-k}.$$
$$f\left(\begin{pmatrix} ia + b & hA^{-t} \\ -k^t(ia+b) + h^t(d+ic) & iA + sA^{-t} + h^t k A^{-t} \end{pmatrix}\begin{pmatrix} ic + d & kA^{-t} \\ 0 & A^{-t} \end{pmatrix}^{-1}\right)$$
$$= \left(\frac{\det(ic+d)}{\det(A)}\right)^{-k}.$$
$$f\left(\begin{pmatrix} ia + b & hA^{-t} \\ -k^t(ia+b) + h^t(d+ic) & iA + sA^{-t} + h^t k A^{-t} \end{pmatrix}\begin{pmatrix} (ci+d)^{-1} & -(ci+d)^{-1}k \\ 0 & A^t \end{pmatrix}\right)$$
$$= \left(\frac{\det(A)}{\det(ic+d)}\right)^{k} f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}, \quad \tau := (ai+b)(ci+d)^{-1}$$

So we have shown that

$$\tilde{f}_{P_r}(m(\gamma, A)) = \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} \left(\frac{\det(A)}{\det(ic+d)}\right)^{k} f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} ds\, dh\, dk$$
$$= \left(\frac{\det(A)}{\det(ic+d)}\right)^{k} \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} ds\, dh\, dk$$

Again lets focus on this integrand $f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}$ and compute its Fourier expansion, see [BVDGHZ08, 3.4]. Recall that a symmetric matrix $n \in \mathrm{GL}_g(\mathbb{Q})$ is called half integral if $2n$ is integral with even diagonal entries, then a Siegel modular form has a Fourier expansion of the form

$$f(z) = \sum_{n \text{ half integral}} a(n) e^{2\pi i \mathrm{Tr}(nz)}$$

First the space of half integral $g \times g$ matrices, HI_g, decomposes as a direct sum via the (additive) group isomorphism

prove it

$$\mathrm{HI}_{g-r} \oplus \tfrac{1}{2}\mathrm{Mat}_{r\times(g-r)}(\mathbb{Z}) \oplus \mathrm{HI}_r \to \mathrm{HI}_g, \qquad (n, m, l) \mapsto \begin{pmatrix} n & m \\ m^t & l \end{pmatrix},$$

thus unfolding the (discrete) integral we get

$$f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} = \sum_{n\in\mathrm{HI}_{g-r}} \sum_{m\in\frac{1}{2}\mathrm{Mat}_{r\times(g-r)}(\mathbb{Z})} \sum_{l\in\mathrm{HI}_r} a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}$$

$$\exp\left(2\pi i \mathrm{Tr}\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix}\right)$$

because all the block sizes are compatible we can "block multiply" the inner matrices and because we are taking the trace we can forget about off diagonal entries

$$\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} = \begin{pmatrix} n\tau + m(-k^t\tau + h^t) & * \\ * & m^t(-\tau k + h) + l(k^t\tau k + AA^t i + s) \end{pmatrix}$$

putting this into our Fourier expansion

$$f\begin{pmatrix} \tau & -\tau k + h \\ -k^t\tau + h^t & k^t\tau k + AA^t i + s \end{pmatrix} = \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}$$
$$\exp\left(2\pi i(\mathrm{Tr}(n\tau) + \mathrm{Tr}(m(-k^t\tau + h^t)) + \mathrm{Tr}(m^t(-\tau k + h)) + \mathrm{Tr}(l(k^t\tau k + AA^t i + s))\right)$$

If we denote $T_l := \mathrm{Tr}(l(k^t\tau k + AA^t i + s))$ and

my Tm differs from Chengjing

$$T_m := \mathrm{Tr}(m(-k^t\tau+h^t))+\mathrm{Tr}(m^t(-\tau k+h)) = \mathrm{Tr}(-mk^t\tau-m^t\tau k)+\mathrm{Tr}(mh^t+m^th) := T_{m,k}+T_{m,h}$$

we can sub this back into our constant term

Converges uniformly a priori on compact sets, well I don't know if I can swap all these sums haha

$$\begin{aligned}
\tilde{f}_{P_r}(m(\gamma, A)) &= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]}\sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix}\exp\left(2\pi i(\mathrm{Tr}(n\tau) + T_m + T_l)\right) ds\, dh\, dk \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}\int_{[H_r]}\int_{[H_r]}\int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i(T_m+T_l)}\, ds\, dh\, dk \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}\int_{[H_r]}\int_{[H_r]} e^{2\pi i T_m}\int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i T_l}\, ds\, dh\, dk \\
&= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_n\sum_m\sum_l a\begin{pmatrix} n & m \\ m^t & l \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}\int_{[H_r]} e^{2\pi i T_{m,k}}\int_{[H_r]} e^{2\pi i T_{m,h}}\int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i T_l}\, ds\, dh\, dk
\end{aligned}$$

Now we use that the integration of unitary characters is very simple 6 and the fact that

$$s \mapsto e^{2\pi i T_l}$$

is a non-trivial unitary character of Sym_{g-r} whenever $l \neq 0$ to get that

$$\int_{[\mathrm{Sym}_{g-r}]} e^{2\pi i T_l}\, ds = \begin{cases} 1, & l = 0 \\ 0, & l \neq 0 \end{cases}$$

we repeat this trick with the second integral, which enforces that $m = 0$ and end up with

$$\tilde{f}_{P_r}\big(m(\gamma, A)\big) = \left(\frac{\det(A)}{\det(ic+d)}\right)^k \sum_{n \in \mathrm{HI}_{g-r}} a\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}$$

but by [BVDGHZ08, 3.5] we know that the Fourier expansion of the Siegel Phi operator is

$$(\Phi^r f)(\tau) = \sum_{n \in \mathrm{HI}_{g-r}} a\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}.$$

hence

$$\begin{aligned} \tilde{f}_{P_r}\big(m(\gamma, A)\big) &= \left(\frac{\det(A)}{\det(ic+d)}\right)^k \Phi^r(f)(\tau) \\ &= \det(A)^k (\Phi^r(f))^{\sim}(\gamma) \end{aligned}$$

which concludes the proof.

□

Chapter 6

Poles of Residual Eisenstein Series

Our goal here is to exposit and survey the work in papers such as [Bre09][JLZ13][GS21] and perhaps give a trivial extension of them.

6.1 Residual Eisenstein Series

[Bre09] gave an analysis of the residual poles of Eisenstein series attached to Sp_{2n}, there were some minor errors that were corrected in [JLZ13] where they give esssentially the same proof; theirs however works for the other classical groups. For our purposes, the case of Sp_{2n}, as a group defined over F a number field, is most relevant, and we shall therefore focus exclusively on this case, however it should be noted that this limitation in the non-covering case is artificial, although it does simplify things a little by removing some casework, and we hope also in the covering case to be able to remove it in future work.

We fix an $n \in \mathbb{N}$ and the Borel of upper triangular matricies in Sp_{2n}, then we look at partitions of $n = r + m$, where

what are the ranges of these

. Then as we saw in 1.2.1 there corresponds a maximal standard parabolic of Sp_{2n}, which we denote $P_r = M_r N_r$, such that the Levi component is

$$\mathrm{GL}_r \times \mathrm{Sp}_{2m}$$

As we saw in 9 the space of characters $X_{M_r}^{\mathrm{Sp}_{2n}}$ is one dimensional by the maximality of P_r. If we look at the divisors of $r = ab$

check the exact ranges, will depend on the range of r

and fix a τ, an irreducible unitary cuspidal automorphic representation of GL_a, then from 4.2.3 we know that $\Delta(\tau, b)$ is a residual representation of $\mathrm{GL}_{ab} = \mathrm{GL}_r$. Now we take an irreducible generic cuspidal automorphic representation σ of Sp_{2m}, and so their tensor product $\Delta(\tau, b) \otimes \sigma$ gives a representation of $\mathrm{GL}_r \times \mathrm{Sp}_{2m}$ and hence of the Levi M_r. We now consider the Eisenstein series attached to this representation, namely if

$$\varphi \in \mathcal{A}(N_r(\mathbb{A})M_r(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))_{\Delta(\tau,b)\otimes\sigma}$$

then we have the Eisenstein series

$$E(\varphi, s)(g) = \sum_{\gamma \in P_r(F) \backslash \mathrm{Sp}_{2n}(F)} s.\varphi(\gamma g)$$

for $g \in \mathrm{Sp}_{2n}(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A})$. Becuase it is induced from the residual representation $\Delta(\tau, b)$ we call these residual Eisenstein series.

state theorem 4.1 and 4.2

Theorem 9.

A similar result was given for the Siegel parabolic and the metaplectic cover of Sp in [GS21]

Theorem 10.

6.2 Computing the Constant Term

The representation is supposed to be of the covering of the Levi///? need to fix this

We want to give some of the details of the proof in the $b = 1$ case, the base case for the induction. This will then be mirrored in the metaplectic case, when we extend the result in [GS21] to non-siegel parabolics. The first step is to compute a constant term

We here consider the case that $b = 1$, hence $n = a + m$. Then fixing a standard parabolic of Sp_{2n} we have the maximal standard parabolic $P_a = M_a N_a$ where $M_a = \mathrm{GL}_a \times \mathrm{Sp}_{2m}$. Now if τ is irreducible unitary cuspidal automorphic representation of GL_a then by definition

Brenner..

$$\Delta(\tau, 1)(\varphi)(g) = E(\varphi, s)(g) = s.\varphi(g)$$

where the Eisenstein series is defined via the parabolic induction from the Levi $(\mathrm{GL}_a)^{\times b}$ to GL_{ab}. Thus we have $\Delta(\tau, 1) = \tau$. So for the appropriate σ a rep of Sp_{2m} we get a rep of the Levi of Sp_{2n}, $M_r = M_a = \mathrm{GL}_a \times \mathrm{Sp}_{2m}$ given by $\tau \otimes \sigma$. To this we associate the Eisenstein series for $\varphi \in \mathcal{A}(N_r(\mathbb{A})M_r(F) \setminus \mathrm{Mp}_{2n}(\mathbb{A}))_{\tau\otimes\sigma}$ $E(\varphi, s)$ as usual. Now we proceed to calculate the constant term of this Eisenstein series along the parabolic $P_a = MN$.

M+W II.1.7, all others are zero..?

By our earlier calculations 5.3 and the cuspidality of the tensor **??** and [JLZ13] we know that

$$E(\varphi, s)_P = \sum_w \sum_{m'} \int_{(w^{-1}N(\mathbb{A})w \cap M(\mathbb{A}))\backslash A} \int_{w^{-1}N(F)w\cap M(F)\backslash w^{-1}N(\mathbb{A})w\cap M(\mathbb{A})} \varphi(n_1 n_2 w^{-1} m' x) dn_1 dn_2$$

and the inner integral vanishes for all $w \neq id, \omega$ (ω as in [JLZ13]). Hence the first sum becomes over two elements and we have

$$E(\varphi, s)_P = E(\varphi, s)_{P,id} + E(\varphi, s)_{P,\omega}$$

where

$$E(\varphi, s)_{P,w} = \sum_{m' \in M(F)\cap wP(F)w^{-1}\backslash M(F)} \int_{N(F)\cap wP(F)w^{-1}\backslash N(\mathbb{A})} \varphi(w^{-1} n m' x) dn$$

First the identity term simplifies

$$\begin{aligned} E(\varphi, s)(x)_{P',id} &= \sum_{m' \in M(F) \cap P(F) \backslash M(F)} \int_{N(F) \cap P(F) \backslash N(\mathbb{A})} \varphi(nm'x) dn \\ &= \sum_{m' \in M(F) \backslash M(F)} \int_{N(F) \backslash N(\mathbb{A})} \varphi(nm'x) dn \\ &= \int_{N(F) \backslash N(\mathbb{A})} \varphi(nx) dn \\ &= \varphi(x)_P \end{aligned}$$

I really need to fix this s thing that I dropped in the constant term computations.

Considering now the ω term

$$E(\varphi, s)_{P,\omega} = \sum_{m' \in M(F) \cap \omega P(F) \omega^{-1} \backslash M(F)} \int_{N(F) \cap \omega P(F) \omega^{-1} \backslash N(\mathbb{A})} \varphi(\omega^{-1} nm'x) dn$$

by [JLZ13, 2C] $M(F) \cap \omega P(F) \omega^{-1} \backslash M(F)$ is isomorphic to $P_0 \backslash \mathrm{Sp}_{2(n-a)}$, but P_0 has Levi $M_0 = \mathrm{Sp}_{2(n-a)}$ by definition and hence is itself $\mathrm{Sp}_{2(n-a)}$. Thus the sum is over $\mathrm{Sp}_{2(n-a)}(F) \backslash \mathrm{Sp}_{2(n-a)}(F)$ and hence is over a point. Therefore we get by definintion of the intertwining operator

$$E(\varphi, s)_{P,\omega} = \int_{N(F) \cap \omega P(F) \omega^{-1} \backslash N(\mathbb{A})} \varphi(\omega^{-1} nx) dn = M(\omega, -)(\varphi)(x)$$

becuase we took the constant term along the same parabolic as the definition of the Eisenstein series we know that the Levis are (the same) conjugate. Thus we have shown that

$$E(\varphi, s)_P(x) = \varphi(x)_P + M(\omega, -)(\varphi)(x)$$

Notice that the computation takes place completely at the level of the terminals which are indupendent of the fact that we have taken a covering group, hence we have really only reused work from [JLZ13].

6.3 Analysing the Intertwining Operator

For classical groups it has been known for a while that the intertwining operator has a normalization in terms of ratios of L-functions. The point is then that the normalised operator is holomorphic and so the poles of the constant term depend entirely on the poles of the L-functions. In particular we will look at incarnations of the following statement: There is a holomorphic and non-zero intertwining operator $N(s, w)$ such that

$$M(s, w) = r(s, w) N(s, w),$$

and $r(s, w)$ is a ratio of L-functions.

Note that this is the global statement. There is an analogous set of conjectures for the local pieces, namely $M = \otimes_\nu A$ the tensor over local intertwiners. Then one wants a normalisation of the local operators $\mathscr{A}$ satisfying a long list of properties. This is extensively dealt with in [Sha90].

It has been known for a long time that there was some normalisation $M = rN$ where r is a ratio of L-functions, for instance Shahidi gives the following [**?**]: Let π be an automorphic representation, let S be a finite set of places such that π_ν is unramified for $\nu \notin S$. We have that there are some finite dimensional complex representations $r_1, ..., r_m$ of ${}^L M$ such that

$$M(s,\pi)f = \bigotimes_{\nu \in S} A(s,\pi_\nu,w)f_\nu \otimes \bigotimes_{\nu \notin S} \prod_{i=1}^{m} \frac{L_S(is,\pi,\tilde{r_i})}{L_S(1+is,\pi,\tilde{r_i})} \tilde{f}_\nu.$$

For example for a group over $\mathbb{Q}$ we have the following from [**?**]

$$M(s) = \left(\prod_\alpha \frac{\pi^{1/2}\Gamma(\frac{1}{2}\mu_\infty(s)(H_\alpha))}{\Gamma(\frac{1}{2}(\mu_\infty(s)(H_\alpha)+1))} \right) \prod_{p \text{ prime}} \left(\prod_\alpha \frac{\frac{1}{1-p^{\mu_p(s)(H_\alpha)+1}}}{1-\frac{1}{p^{\mu_p(s)(H_\alpha)}}} \right).$$

However it was not shown until recently, and only for classical groups that this N indeed has the required properties. In particular the following theorem is sufficient for the cases dealt with in [JLZ13]:

Theorem 11 ([CKPS], 11.1)**.** *Suppose that π_ν is a local component of a globally generic cuspidal representation π of $G_n(\mathbb{A})$. Then for any irreducible admissible unitary generic representation π'_ν of $\mathrm{GL}_m(k_\nu)$ the normalized intertwining operator $N'(S, \pi'_\nu \times \pi_\nu, w)$ is holomorphic and non-zero for $Re(s) \geq 0$*

In the case we are considering the normalising factor r is given by the equation [JLZ13, 4A]

$$r(w,s) = \frac{L(s,\tau\times\sigma)L(2s,\tau,\rho)}{L(s+1,\tau\times\sigma)L(2s+1,\tau,\rho)}$$

and this proves that

Lemma 12. *The Eisenstein series above has pole at s if and only if $r(w,s)$ has a pole. The Eisenstein series has a zero at s if and only if $r(w,s)$ has a zero.*

The final step is then to use the known properties of L-functions to conclude when our r-factor will have poles and zeroes.

6.4 The Metaplectic Generalisation

In [GS21,] this setup is also carried out for the Siegel parabolic induced up to the metaplectic group. Here we investigate the literature to hopefully conclude the same result for all maximal parabolics. The steps will be the same we simply need for the Langlands conjectures to be proven in certain cases.

Thankfully Kaplan in a series of recent works with collaborators [Kap21][Kap20][CFK24] has supplied some of the key peices.

First we have that

$$M(s,w)f = \ f$$

note that these are metaplectic L-functions as defined in that paper.

Thus we get

Lemma 13. *The Eisenstein series above has pole at s if and only if $r(w,s)$ has a pole. The Eisenstein series has a zero at s if and only if $r(w,s)$ has a zero.*

Finally we need to once again see what properties of these L-fuctions have been proven.

Appendix A

Direct Integrals

A.0.1 Of Spaces

Consider a countable collection of Hilbert spaces $(\mathcal{H}_\alpha)_{\alpha\in A}$ then their direct sum is defined to be

$$\bigoplus_\alpha \mathcal{H}_\alpha \quad := \quad \left\{(h_\alpha) \in \prod_\alpha \mathcal{H}_\alpha : \sum_\alpha \|h_\alpha\|_\alpha^2 < \infty\right\}$$

i.e. square summable sequences from the product. This is to ensure that the resulting space is still complete. If we recall that summing over a countable set is the same as *integrating* over that countable set when we equip it with the counting measure and discrete sigma algebra then this can be re-written as

$$\bigoplus_\alpha \mathcal{H}_\alpha = \left\{(h_\alpha) \in \prod_\alpha \mathcal{H}_\alpha : \int_A \|h_\alpha\|_\alpha^2 d\alpha < \infty\right\}$$

This definition can be obviously generalised to an indexing set that is now an arbitrary measure space, $(A, \mathcal{M}, \mu)$. We need to make some technical arrangment to accompany this change, namely ensureing everything agrees with the measure structure, if we're to integrate we better only integrate *measurable* things. So now a collection $(\mathcal{H}_\alpha)_{\alpha\in A}$ along with a countable set of elements $e_j \in \prod_\alpha \mathcal{H}_\alpha, j \geq 1$ is called a measurable field over A if

$$\forall j, k \geq 1 \;\; \alpha \mapsto \langle e_j(\alpha), e_k(\alpha)\rangle$$

is measurable and for each $\alpha \in A$

$$span\{e_j(\alpha)\}_{j=1}^{\infty} \subseteq \mathcal{H}_\alpha$$

is dense; fixing an α and varying the j form a basis of each of the hilbert spaces, fixing the indecies and variying the α is measurable. An element $f \in \prod_\alpha \mathcal{H}_\alpha$ is called a measurable vector field if

$$\forall j \;\; \alpha \mapsto \langle f(\alpha), e_j(\alpha)\rangle_\alpha$$

is a measurable function. Note that we consider elements of the potentially uncountable product as functions from the indexing set into the relevant space (functions into the union of the hilbert spaces satisfying the property that $f(\alpha) \in \mathcal{H}_\alpha$). Now we define

$$\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha) \;\; := \;\; \left\{ f \in \prod_\alpha \mathcal{H}_\alpha : f \text{ is measurable and } \int_A \|f(\alpha)\|_\alpha^2 < \infty \right\}$$

Indeed this forms a Hilbert space. Note that a priori this construction depended on the basis (e_j) that we picked but up to isomorphism the basis doesnt matter.

A.0.2 Of Operators

We want to decompose representations and so we should look at how operators fit into this picture. We call an element

$$T \in \prod_\alpha \mathcal{L}(\mathcal{H}_\alpha)$$

a field of operators on A. It defines a linear map from $\prod_\alpha \mathcal{H}_\alpha$ to itself via

$$\left(\int^{\oplus} T \right) (f)(\alpha) \;\; := \;\; T(\alpha)(f(\alpha))$$

We say that it is measurable if for all measurable vector fields f the function

$$\alpha \mapsto \left(\int^{\oplus} T \right) (f)(\alpha)$$

is measurable. If moreover $\operatorname{ess\,sup}_\alpha \|T(\alpha)\| < \infty$ then $\int^{\oplus} T$ defines a bounded operator on $\int^{\oplus} \mathcal{H}_\alpha$.

A.0.3 Of Representations

Now we consider a group G and a collection of unitary represntations π_α on $\mathcal{H}_\alpha$ such that for every α and every $x \in G$

$$\alpha \mapsto \pi_\alpha(x)$$

is a measurable field of operators. We call such a collection a measurable field of representations; a G indexed collection of measurable fields of operators. From a measurable field of representations we get a unitary representation

$$\pi(x) \;\; := \;\; \int^{\oplus} \pi_\alpha(x)$$

of G on $\int^{\oplus} \mathcal{H}_\alpha$ which we call the direct integral of representations.

Todo list

Disambiguate all teh L2 automorphic form stuff. What isi teh action of the Lie algebra then....?

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