Abstract

The classical Eisenstein series are the first concrete example of a modular form. Their automorphic analogue plays a similarly central role to the theory of automorphic forms and automorphic representations. In fact it was whilst computing coefficients of the Eisenstein series functional equations that Langlands formed his famous conjectures [DS15, Ch VI. Funktorialität in der Theorie der automorphen Formen, Sec. 5 Wieder Princeton]. This thesis is an introduction to the theory of Eisenstein series and automorphic representations with a particular focus on how constant terms can be used to collect information about the zeroes and poles of Eisenstein series.

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Introduction

Motivation

The goal of this thesis is to exposit some of the results in the theory of Eisenstein series and give some example computations. This introduction will attempt to motivate these computations, by contextualising them historically and elaborating on some of the conjectures that form modern work.

There are many surveys and books on the Langlands program, class field theory and modern topics in number theory that this introduction is indebted to. Some exemplars are [FGKP16, BCDS$^+$04] for longer treatments, in particular the statements of the conjectures are most clearly stated in Cogdell's chapters in [BCDS$^+$04]. Shorter surveys are [Gel84, Lan, Lan89, Art81].

0.0.1 From Ancient to Modern

We follow the wonderful exposition in [Wei15] and [Cond]. A problem that Pythagoras could have understood is "which positive integers are the sum of two squares". In 1640 Fermat answered this question, he first reduces the question to when is a prime the sum of two squares. Thus the problem is immediately reformulated as a problem about congruences mod a prime p, "when does there exist a solution to $a^2 + b^2 \equiv 0 \pmod{p}$", or what's the same, by dividing out b^2, "when is there a solution to $x^2 + 1 \equiv 0 \pmod{p}$". Fermat proved:

Theorem 0.1. *Let p be an odd prime. Then $x^2 + 1 \equiv 0 \pmod{p}$ has a solution if and only if $p \equiv 1 \pmod{4}$.*

Recall the Legendre symbol, for p, q odd and non-equal primes we have

$$\left(\frac{q}{p}\right) := \begin{cases} 1, & \text{there is a solution to } x^2 - q \equiv 0 \pmod{p} \\ -1, & \text{else} \end{cases}.$$

Then the theorem of Fermat was generalised by Gauss in 1801 to his reciprocity law:

Theorem 0.2. *For p, q odd and non-equal primes,*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{2}}.$$

Having a solution mod a prime is the same as asking whether the polynomial splits mod that prime. The natural question is then: Given a monic irreducible polynomial with integral coefficients can we determine by congruences whether it splits mod a prime. Gauss's reciprocity is a solution to this problem for polynomials of the form $f(x) = x^2 - q$ for q odd prime.

Remark 0.3. The odd limitation is for brevity here and of course can be lifted.

Recall that if $f(x) \in \mathbb{Z}[x]$ is monic and irreducible then there is a unique minimal field F in which it factors as linear polynomials, called the splitting field. The Galois group of $f(x)$ is then defined to be $\mathrm{Gal}(F/\mathbb{Q})$. Class field theory is a solution to problem above when this Galois group is *Abelian*. To explain we need to introduce the standard algebraic number theory setup.

Let $\mathbb{Q} \subseteq K$ be an extension of number fields, with respective rings of integers $\mathbb{Z} \subseteq \mathcal{O}_K$ and let p be a prime in $\mathbb{Z}$ hence (p) is a prime ideal of $\mathbb{Z}$ and let

$$\mathcal{O}_K(p) = \prod_i \mathfrak{P}_i^{e_i},$$

be the prime decomposition in $\mathcal{O}_K$. Then (p) **splits** in $\mathcal{O}_K$ if for every i we have $e_i = 1$ (this is being unramified) and $\mathcal{O}_K/\mathfrak{P}_i \cong \mathbb{Z}/(p)$. The splitting of primes is related to the splitting of polynomials by the following theorem

Theorem 0.4 ([Lan94], Prop. 26)**.** *If $f \in \mathbb{Z}[x]$ monic and irreducible and for $\alpha \in \bar{\mathbb{Q}}$ we have $f(\alpha) = 0$ then for $K = \mathbb{Q}(\alpha)$ with finitely many exceptions (of p) f is split mod p if and only if (p) splits in $\mathcal{O}_K$.*

Hence to answer when monic irreducible integral polynomials split by congruences it will be necissary to know when prime ideals split over such extensions. Every field extension L/K has a Galois closure, that is an extension L'/L of minimal degree such that L' is Galois over K.

Lemma 0.5 ([Neu02], §8, Ex. 4)**.** *A prime ideal of $\mathcal{O}_K$ is split in $\mathcal{O}_L$ if and only if it is split in $\mathcal{O}_{L'}$.*

Thus we lose nothing by considering only Galois extensions of fields. In 1853 Kronecker constructed an extension of number fields K'/K whose Galois group is isomorphic to the ideal class group of K, $\mathrm{Cl}(K) \cong \mathrm{Gal}(K'/K)$, a so called (by Weber) "class field" for K. Kronecker would go on to make several conjectures that would form the heart of class field theory, for instance, he conjectured that a Galois extension of $\mathbb{Q}$ is determined by the primes of $\mathbb{Z}$ that split over that extension. This was solved by Bauer in 1916:

Theorem 0.6 ([Cond], Thm. 2.6)**.** *Let L_1, L_2 be finite extensions of a number field K, then $L_1 = L_2$ if and only if the primes of $\mathcal{O}_K$ that split in $\mathcal{O}_{L_1}$ is equal to the set of primes that split in $\mathcal{O}_{L_2}$.*

However, there was no systematic way of finding *which* primes split over the extension. Takagi in 1920 made huge progress on this and it was to be made even more explicit finally by Artin in 1927. Together their work proves something similar to "the main theorem" of class field theory:

Theorem 0.7 ([Wei15], Thm. 3.2.1)**.** *Let $K/\mathbb{Q}$ be an Abelian and Galois extension. There is an ideal $\mathfrak{f} = (m) \subseteq \mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$ such that for a prime $p \in \mathbb{Z}$ the ideal (p) is split in $\mathcal{O}_K$ if $p \equiv 1 \pmod{m}$.*

i.e. an explicit solution to the splitting of primes via congruence relations. Thus *global* class field theory was "solved", immediately the natural question was raised, what happens in the *non-Abelian* extensions of number fields. The (global) Langlands conjectures (amongst other things) can be viewed as an attempt to answer this question.

Another direction that people were interested in was the extensions of local fields, as opposed to number fields. It was Hilbert who introduced in 1897 the use of the p-adic numbers, in spirit if not in name, he wrote congruences of arbitrary powers of primes. Let ν be a place of $\mathbb{Q}$, then define the ν-adic Hilbert symbol for $a, b \in \mathbb{Q}^\times$

$$(a,b)_\nu \quad := \quad \begin{cases} 1, & a = x^2 - by^2 \text{ has a solution in } \mathbb{Q}_\nu \\ -1, & \text{else} \end{cases}.$$

Theorem 0.8 (Product Formula)**.** *For all* $a, b \in \mathbb{Q}^\times$

$$\prod_\nu (a,b)_\nu = 1.$$

This is equivalent to Gauss's reciprocity law, however much more uniform to state, treating odd and even primes in the same way, and not requiring any co-prime conditions. This moreover treats finite and infinite places uniformly. Building on this work and using Artin reciprocity, Hasse, after introducing the p-adic numbers in 1927, proved the first versions of *local* class field theory in 1930, that is reciprocity for extensions of the local fields $\mathbb{Q}_\nu$. The statements here are technical, but a key result is

Theorem 0.9 ([Cond], Thm. 7.6)**.** *For a finite Abelian extension of local fields E/F there is a surjection*

$$F^\times \to \mathrm{Gal}(E/F).$$

In particular the Galois group is a quotient of $F^\times$.

Note that the definition and proof of local class field theory *depended logically* on global class field theory. That is the construction of this surjection uses the results of Artin and Takagi. Hasse was able to prove later in 1933 the main results again but without recourse to global class field theory. It lacked the explicit construction of the class fields however which was finally supplied in 1965 by Lubin and Tate.

What remained to do was supply a proof of *global* class field theory from local class field theory. In pursuit of this task the machinery of the ideles and adeles was introduced. In this language (part of) *global* class field theory can be restated as

Theorem 0.10 ([Neu02], Ch. VI, §1 ,Prop. 1.3)**.** *Let the ideal class group of a number field K be denoted* Cl_K*. Then there is a surjection*

$$\mathbb{A}^\times / K^\times \xrightarrow{A} \mathrm{Cl}_K \cong \mathrm{Gal}(K'/K),$$

where K' is the class field of K.

If we think about representations of these groups then this surjection gives a relation between characters χ of $\mathbb{A}^*/K^*$ and characters χ' of $\mathrm{Gal}(K'/K)$ by pulling back along A.

$$\begin{tikzcd}
\mathbb{A}^\times/K^\times \arrow[rr, "A"] \arrow[dr, "\chi"'] & & \mathrm{Gal}(K'/K) \arrow[dl, "\chi'"] \\
& \mathbb{C}^* &
\end{tikzcd}$$

Thus A can be thought of as providing a correspondence

$$\{\text{Maps } \mathbb{A}^\times/K^\times \to \mathbb{C}^*\} \to \{\text{Maps } \mathrm{Gal}(K'/K) \to \mathbb{C}^*\}.$$

One then observes that this can be rewritten as

$$\{\text{Maps } \mathrm{GL}_1(\mathbb{A})/\mathrm{GL}_1(K) \to \mathrm{GL}_1(\mathbb{C})\} \to \{\text{Maps } \mathrm{Gal}(K'/K) \to \mathrm{GL}_1(\mathbb{C})\}.$$

This suggests the generalisation to

$$\{\text{Certain reps of } \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(K)\} \to \{\text{Certain } n\text{-dimensional reps of } \mathrm{Gal}(\bar{K}/K)\}.$$

But according to Langlands [Lan89], who was inspired by the philosophy of Harish-Chandra, we should treat all reductive groups the same, so Langlands conjectures that for any reductive linear algebraic group G there is some correspondence

$$\{\text{Certain reps of } G(\mathbb{A})/G(K)\} \to \{\text{Certain } n\text{-dimensional reps of } \mathrm{Gal}(\bar{K}/K) \text{ that factor through } G\}.$$

These two sides of the correspondence are referred to as the “automorphic side” and the “Galois side” respectively. The content that follows will be almost entirely on the automorphic side.

0.0.2 Harmonic Analysis

As we mentioned the work of Langlands was inspired by the work of Harish-Chandra in harmonic analysis of Lie groups. Here we want to say something about the precursors to Langlands work in this respect. The story starts with the Fourier transform for periodic functions. These of course have ancient precursors in the ideas of the Pythagoreans and were “in the air” of the eighteenth century. Fourier, around 1822, was first to conjecture that *all* functions should be decomposable into elementary periodic functions. The base case is the Fourier transform on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the circle, for every $f \in L^2(\mathbb{T})$ we have that

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}, \quad a_n \in \mathbb{C}.$$

The important properties of the circle as a topological group are that it is Hausdorff, compact and abelian. Locally compact Hausdorff ensures that Haar measures (left and right) exist, abelian ensures that they agree and that the irreducible representations are one dimensional.

The first generalisation appeared in 1927 with the Peter–Weyl theorem. Starting with a locally compact topological group G, then a unitary representation on a Hilbert space $\mathcal{H}$ is a continuous homomorphism

$$\pi : G \to U(\mathcal{H}).$$

We denote the unitary dual group of G by $\hat{G}$, this is defined to be the space of (equivalence classes of) irreducible unitary representations of G.

Theorem 0.11 ([Fol16], Thm. 5.2, Thm. 5.12)**.** *If G is compact then every unitary representation of G is a direct sum of irreducible unitary representations.*

Remark 0.12. For lack of time and space we will need to make this remark several times: The actual content of the Peter–Weyl theorem is not that the representations decompose but *how* they decompose. That is Peter–Weyl tells us how to construct the components of the direct sum, what their dimensions are etc.

Importantly, there is no requirement for finite dimensionality.

Example 0.13. *Consider the regular representation of $\mathbb{T}$ on $L^2(\mathbb{T})$ this decomposes into*

$$L^2(\mathbb{T}) = \bigoplus_{\chi \in \hat{G}} \mathbb{C}\chi.$$

Because $\mathbb{T}$ is compact and Abelian all its irreducible representations are one dimensional, in fact, we have that all characters of G are maps of the form

$$e^{i\theta} \mapsto e^{ni\theta}, \quad n \in \mathbb{Z}.$$

Therefore the decomposition exhibits the exponentials as a basis for functions on the circle.

In the 1940's, Weil worked out the theory for locally compact Abelian groups, proving the general case of Bochner's theorem [Fol16, Thm. 4.18]. The groups that we are interested in however, are neither compact, ${\mathbb{A}_{\mathbb{Q}}}^{\times}$, nor Abelian, GL_n.

A group is **type I** if for every (continuous unitary) representation π such that the centre of $\mathrm{Hom}_{\mathrm{Rep}}(\pi, \pi)$ is trivial, we have a decomposition as a direct sum of irreducible representations.

Example 0.14. *The adelic points of a connected reductive linear algebraic group (LAG) are a type I group. The proof is outside the scope of this thesis but can be found in [Dv17, Thm. 1.7 + Thm. 2.3].*

Example 0.15. *Consider $G(\mathbb{A})$ the adelic points of a connected reductive LAG. This is a second countable group. We don't have a reference for this so we present the argument.*

First consider the adele ring $\mathbb{A}_F$ of F . This has the restricted product topology, where if $\mathcal{O}_\nu$ is the ring of integers of F_ν, then an arbitrary open subset looks like a union of sets of the form

$$U_S \times \prod_{s \notin S} \mathcal{O}_s,$$

where $U_S \subseteq \prod_{s \in S} F_s$ is open in the product topology. Because for any place F_ν is second countable and the product of second countable spaces is second countable it is clear that $\prod_{s \in S} F_s$ is second countable. Moreover there is a countable number of finite subsets of $\mathbb{Z}$, hence there is a bijection between a basis of the restricted product topology and $\aleph_0 \times \aleph_0$ which is countable hence this topology is second countable.

If $G := \operatorname{Spec} F[x_1, ..., x_n]/(f_1, ..., f_m)$ is an affine scheme then the topology on $G(\mathbb{A})$ is the subspace topology of $\mathbb{A}^n$ on which all the $f_1, ..., f_m$ vanish (see section 1.2.3). In particular the *finite product of second countable spaces is second countable and subspaces of second countable spaces are second countable, hence $G(\mathbb{A})$ is second countable.*

Example 0.16. *The adelic points of a connected reductive LAG are a unimodular group. The proof is outside the scope of this thesis but is stated in [Cona, Lem. 2].*

In the 1950s Segal and Mautner proved the Plancherel Theorem which is the Peter–Weyl and Bochner type result for type I, second countable and uni-modular topological groups.

Remark 0.17. The name Plancherel theorem is overloaded in harmonic analysis. We will give exact references to the precise theorem we are referring to below.

To state it one must be somewhat familiar with direct integrals. The theory is explained in [Fol16, 7.4] and the definitions are stated in B, but some of the basic idea is contained in the example of direct sums.

Example 0.18 (Direct Sums)**.** *Let I be a countable set with the discrete sigma algebra and counting measure μ. Let $(\mathcal{H}_i)_{i\in I}$ be a collection of Hilbert spaces then*

$$\bigoplus_{i\in I}\mathcal{H}_i = \left\{(h_i)_{i\in I}\in\prod_{i\in I}\mathcal{H}_i : \int_I \|h_i\|_i^2 d\mu < \infty\right\}.$$

I.e. the Hilbert space direct sum is by definition square summable sequences, but sums are just discrete integrals.

Then (part of) the Plancherel theorem is

Theorem 0.19 (Plancherel, [Fol16], 7.44)**.** *The regular representation of a type I, second countable and unimodular topological group is a direct integral of the irreducible unitary representations.*

Remark 0.20. Again the Plancherel theorem says much more; it contains details about the topology and measure on the set of unitary irreducible representations, and which representations are associated to them in the direct integral.

0.0.3 The Work of Langlands

It is as a continuation or variation of this tradition that we see the work of Langlands in [Lan76], in which he provides some decomposition of the spectrum of the adelic points of a connected reductive algebraic group over a number field $G(\mathbb{A})$.

Theorem 0.21 ([Art79], MAIN THEOREM (b))**.** *There is an orthogonal decomposition of the representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ into*

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{\mathscr{P}} L^2_{\mathscr{P}}(G(\mathbb{Q})\backslash G(\mathbb{A})),$$

where $\mathscr{P}$ runs over certain “associate classes” of parabolics of G and the summands are the direct integrals of spaces of L^2 automorphic forms.

This construction is very explicit, and the direct integrals are constructed out of the residues of Eisenstein series.

The spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ refers to such a decomposition. In particular, we have some important "pieces" to such a decomposition. We call such decompositions "spectral", alluding to the spectral theorem which provides such a decomposition in terms of the eigenvector of certain operators. Moreover, these decompositions are largely proved in terms of the more general spectral theorems. The piece that decomposes into a direct sum of irreducible is called the **discrete spectrum**. The compliment of the discrete spectrum is called the **continuous spectrum**. One can define cuspidal L^2 functions in the exact same way as cuspidal automorphic forms (see section 5.1) and then it has been shown that the **cuspidal spectrum**, the subspace of L^2 consisting of cusp forms, decomposes as a direct sum [GH24, 9]. Thus the cuspidal spectrum is contained in the discrete spectrum in this case. The **residual spectrum** is defined to be the compliment of the cuspidal spectrum in the discrete spectrum.

It is during this analysis that the ideas expressed in his famous letter [Lan67] would begin to form, as he noticed that certain Euler products of analytic functions were appearing in the constant terms of the Eisenstein series. In particular we will see how the function $M(s)$ appears in the constant term of Eisenstein series. Langlands observed that [Lan71]

$$M(s) = \left(\prod_{\alpha} \frac{\pi^{1/2}\Gamma(\frac{1}{2}\mu_\infty(s)(H_\alpha))}{\Gamma(\frac{1}{2}(\mu_\infty(s)(H_\alpha)+1))}\right) \prod_{p \text{ prime}} \left(\prod_{\alpha} \frac{1-\frac{1}{p^{\mu_p(s)(H_\alpha)+1}}}{1-\frac{1}{p^{\mu_p(s)(H_\alpha)}}}\right).$$

This formula is obviously uninterpretable without further definitions, however we just want to point out some things to notice. First there is a product over the places of $\mathbb{Q}$, namely one item for the infinite place and then a product over the prime numbers. The functions in the product are gamma functions, related intrinsically to the L-function exemplar ζ, the Riemann-Zeta function, and functions of the form $1-p^{-s}$. These facts should be contrasted with the general setting alluded to in appendix A

This led to a general conjecture that there is a holomorphic and non-zero intertwining operator $N(s,w)$ such that

$$M(s,w) = r(s,w)N(s,w),$$

and $r(s,w)$ is a ratio of L-functions, as defined by Langlands in for instance [Lan71].

Note that this is a global statement. There is an analogous set of conjectures for the local pieces, namely $M = \otimes_\nu A$ the tensor over some local functions. Then the conjecture calls for the existence for some normalised local operators, satisfying a long list of properties. This is extensively dealt with in [Sha90]. Shahidi showed some cases of this conjecture in [Sha88]: Let π be a sufficiently nice automorphic representation, let S be a finite set of places such that π_ν is unramified for $\nu \notin S$. We have that there are some finite dimensional complex representations $r_1, ..., r_m$ of ${}^L M$ such that

$$M(s,\pi)f = \bigotimes_{\nu \in S} A(s,\pi_\nu)f_\nu \otimes \bigotimes_{\nu \notin S} \prod_{i=1}^{m} \frac{L_S(is,\pi,\tilde{r_i})}{L_S(1+is,\pi,\tilde{r_i})}\tilde{f}_\nu.$$

Recently it was shown for classical groups that this N indeed has the required properties. In particular, the following theorem is sufficient for the cases dealt with in [JLZ13]:

Theorem 0.22 ([CKPS], 11.1)**.** *Suppose that π_ν is a local component of a globally generic cuspidal representation π of $G_n(\mathbb{A})$. Then for any irreducible admissible unitary generic representation π'_ν*

of $\mathrm{GL}_m(k_\nu)$ *the normalised intertwining operator* $N'(S, \pi'_\nu \times \pi_\nu, w)$ *is holomorphic and non-zero for* $Re(s) \geq 0$

0.0.4 Poles of Residual Eisenstein Series

Consider the group GL_n. We then let $n = ab$ for positive integers a, b. If τ is an irreducible, cuspidal automorphic representation of GL_a then there is a representation of $\mathrm{GL}_{ab} = \mathrm{GL}_n$ called the "Speh representation" denoted

$$\Delta(\tau, b).$$

Moeglin and Waldspurger also achieved a more fine analysis of the spectrum of GL_n by proving that as τ and b vary, these representations span the residual spectrum of $L^2(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}))$ [JLZ13, Thm. 1.1]. The Speh representation is formed by taking iterated residues of Eisenstein series in the sense of [MW95, V], a more concrete explanation can be found in [Bre09, 2.4]. For a nice survey of problems in this area, of residues of Eisenstein series, there is [Jia08].

If we denote G_n one of the classical groups $\mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{U}_n$ etc. then G_{a+b}, have maximal parabolics whose Levis decompose into products $GL_a \times G_b$, and so we can use the representation theory of GL_n on a Levi to induce up to the whole group. One step in this direction is the work of [JLZ13], in which the authors locate the poles of Eisenstein series induced in this manner.

These considerations are supposed to help prove cases of Langlands functorial transfers, that is proving cases of Langlands functoriality for groups by "transferring" the known cases of functoriality from other groups. We quote from the introduction of [JLZ13]:

> "The key ingredient in these constructions is to use certain Fourier coefficients of special types of residues of certain residual Eisenstein series as kernel functions in the corresponding integral transforms"

[Bum11] gives some more detail on how the analytic properties of Eisenstein series and their L-functions imply that the automorphic representations can be lifted to other groups.

Finally, we remark that we spent a good amount of time trying to understand the analogous story for the so called "almost algebraic groups", topological coverings of $G(\mathbb{A})$. In this setting the work of [JLZ13] has also been applied to get similar results on poles of metaplectic Eisenstein series, as in [Kap21]. It was also used to prove certain functoriality results as in [CFK24]. We leave it for future work to understand the full significance of these calculations, but hope we have motivated why they *might* be interesting.

Outline of Content

Chapter one deals with the generalities of linear algebraic groups, the objects whose representation theory is the subject of discussion. First we define them and then look at the important subgroups that are used in the study of automorphic forms arising on the adelic points of these groups. We focus on the classical groups.

Chapter two deals with automorphic forms. We define automorphic forms in both the Archimedean and adelic places. Finally we give the details of how to view modular forms as automorphic forms.

Chapter three is dedicated to automorphic representations. We define them and specify some important constructions that are needed in the final section.

In chapter four we define adelic Eisenstein series and show how they generalise the classical modular forms also known as Eisenstein series.

Chapter five is dedicated to the constant term in the adelic setting. We first define them and then go through the process of computing them in great detail for Eisenstein series.

Chapter six is an example computation of a constant term, this is the base case of an induction formula proven in [JLZ13].

Contents

Chapter 1

Classical Groups

We will recall a small amount of the theory of linear algebraic groups to fix conventions, for a more detailed treatment one should consult the litany of sources on this matter: For a full treatment see [Mil17] [Mil] [Mil12] [Spr98]. Excellent example computations can also be found in [Gar97] [Mak] [MT11]. Or for a brief brush up on the main facts consult Springer's article in [BC79, Part 1, "Reductive Groups"].

The purpose of this section is to define the key examples and properties of algebraic groups. We also define the most important subgroups, attempting to emphasize the role they play in the theory. Throughout we will restrict to the case of the few classical groups that we define explicitly, however, the theory works much more generally.

1.1 Definition

An **algebraic group** is for us a group scheme that is reduced, of finite type and defined over a field. **A linear algebraic group** (LAG) is simply an affine algebraic group.

Proposition 1.1. *An algebraic group is affine if and only if it is isomorphic to a Zariski closed subgroup of* GL_n.

Proof. The forward implication is [Spr98, 2.3.7(i)]. The converse is the basic fact that closed sub-schemes of affine schemes are affine [Mum99, II.5.T3].

The idea of LAG's is that they are matrix groups defined by polynomial equations. This means that they come with the technology of algebraic geometry and in particular one must be adept at moving between the following equivalences:

Theorem 1.2 ([Mil12], II.6, III.4)**.** *For a field* K*, then the following categories are equivalent:*

- *Group objects in* Alg_K^{op}
- *Representable (in the category of groups) functors* $\mathrm{Alg}_K \to \mathrm{Group}$
- *Group object in the category of affine schemes over* K
- *Commutative* K*-Hopf algebras.*

Example 1.3 ($\mathbb{G}_m$)**.** *The first example is the "multiplicative group" denoted $\mathbb{G}_m$ or GL_1 defined over the field K. This is*

$$\mathbb{G}_m \quad := \quad \mathrm{Spec}\Big(K[x,y]/(xy-1)\Big).$$

As a representable functor this sends a K-algebra R to $\mathrm{Hom}_K(K[x,y]/(xy-1), R)$. These are ring maps that are K-linear, and because $y = x^{-1}$ we know that $f(y) = f(x^{-1}) = f(x)^{-1}$ for $f \in \mathrm{Hom}_K(K[x,y]/(xy-1), R)$. Thus the maps are determined by where they send x, moreover they always send it to a unit, i.e. $\mathrm{Im} f \subseteq R^\times$. For each element $r \in R^\times$ we also have a map sending $x \to r$ hence there is an isomorphism (of sets) between $\mathbb{G}_m(R) \cong R^\times$, from which we pull back a group structure.

The other important examples of such groups are the "classical groups". The exact groups that an author might mean by classical may vary, so we define them explicitly here. First let V be a finite dimensional K-vector space with a bilinear form $\langle , \rangle$. An automorphism of this form is a map $\alpha \in \mathrm{Aut}(V)$ such that

$$\langle \alpha(x), \alpha(y) \rangle = \langle x, y \rangle.$$

Therefore we can consider the group of automorphisms of this form $\mathrm{Aut}(V, \langle , \rangle)$. This group, depending on the properties of the bilinear form, will define our classical groups.

If the form is trivial, by which we mean, $\forall x, y \in V \quad \langle x, y \rangle = 0$ then we define the **general linear group**,

$$\mathrm{GL}(V) \quad := \quad \mathrm{Aut}(V, \langle , \rangle) = \mathrm{Aut}(V).$$

If the form is non-degenerate and skew symmetric $\forall x, y \in V \quad \langle x, y \rangle = -\langle y, x \rangle$ then the **symplectic group** is,

$$\mathrm{Sp}(V) \quad := \quad \mathrm{Aut}(V, \langle , \rangle).$$

There are the further classical groups given by the determinant one subgroups, $\mathrm{SL}(V)$. The naming of $\mathrm{Sp}(V)$ is somewhat serendipitous as one can show that it is contained in $\mathrm{SL}(V)$. We can make this into a functor from K-algebras to groups, by sending a K-algebra R to $G(V \otimes_K R)$.

Remark 1.4. Often the unitary and orthogonal groups are considered classical, as is the case in [JLZ13]. These can both be defined in terms of automorphisms of different forms, for instance if the form is non-degenerate and symmetric $\forall x, y \in V \quad \langle x, y \rangle = \langle y, x \rangle$ then we define the **orthogonal group**,

$$\mathrm{O}(V) \quad := \quad \mathrm{Aut}(V, \langle , \rangle).$$

1.2 Subgroups

From now on let G be one of the classical LAG defined above, defined over a number field F with adele ring $\mathbb{A}$.

Remark 1.5. Most everything we say will apply verbatim to so called split reductive groups, however we lose little in restricting to the classical groups we have chosen.

Subgroups with special properties allow us to reduce and break up problems into smaller ones. Here we will briefly review and compute some examples of special subgroups. The point of these subgroups is two fold. Some of them will help us perform "induction" from smaller simpler groups to larger ones. Others are there essentially as a part of the combinatorial data that classifies the groups we are working with. In particular we need to understand all the pieces of the so called **Langlands-Iwasawa decomposition** [GH24, 2.7],

$$G(\mathbb{A}) = M(\mathbb{A})U(\mathbb{A})K = T(\mathbb{A})U(\mathbb{A})K. \tag{1.2.1}$$

1.2.1 Parabolics, Levis and Unipotents

A subgroup $P \subseteq G$ is called **parabolic** if G/P is a complete variety. Equivalently we can ask for P to contain a Borel (see section 1.2.2).

Completeness is the algebro-geometric analogue of compact, always a desirable property. The fact that they contain a Borel gives us an algebraic "parametrisation" of these subgroups, in the case of the classical groups through the use of flags or roots. It is very important to have a parametrisation of the parabolic subgroups when it comes to taking constant terms of Eisenstein series which we will discuss in chapter 5.

A matrix m is **unipotent** if for some $n \geq 0$ we have that $(m-1)^n = 0$. A subgroup is **unipotent** if all its elements are unipotent. The **unipotent radical** of G is the maximal closed, connected, normal, unipotent subgroup. A linear algebraic group is **reductive** if its unipotent radical is trivial. Then we have the following fact and definition,

Lemma 1.6 ([Bor91] 11.22)**.** *There is a split exact sequence (of algebraic groups)*

$$0 \to U \to P \to M \to 0,$$

where U is the unipotent radical of P, and M is a reductive group known as a ***Levi*** *of P (unique up to conjugacy).*

Thus parabolics and their Levis allows us to induce from a reductive subgroup up to the reductive group. This is the technique of "parabolic induction" [Ber92, Thm. 10] that we will not explicitly talk about here but which is happening secretly in the background in section 3.2.2.

Remark 1.7 (Bad Etymology)**.** *The origin of the name parabolic is a mystery. Borel in his history [Bor01, VI.§2] attributes it to R. Godement in [God61]. Godement conjectures that the quotient $G(\mathbb{A})/G(\mathbb{Q})$ is compact if and only if every element of $G(\mathbb{Q})$ is semi-simple, as is the case in classical groups (this was shortly thereafter proven [MT62]). He says that*

> *"Lorsque n'est pas compact, il est non moins facile de conjecturer qu'on doit pouvoir définir quelque chose d'analogue aux classiques "pointes paraboliques", lesquelles doivent correspondre à des sous-groupes unipotents non triviaux de $G_{\mathbb{Q}}$"*

which roughly (google) translates to that one can also conjecture that non-trivial unipotent elements should correspond to "parabolic points" in a fundamental domain.

In the case of modular forms the fundamental domain is $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ (for the details see section 2.3). We have the classification of elements of $\mathrm{SL}_2(\mathbb{R})\backslash\{\pm 1\}$ as in [Bor97, 3.5] via *their trace*

$$g \textit{ is of type } \begin{cases} \textit{Elliptic if} & \frac{1}{2}|tr(g)| < 1 \\ \textit{Parabolic if} & \frac{1}{2}|tr(g)| = 1 \\ \textit{Hyperbolic if} & \frac{1}{2}|tr(g)| > 1 \end{cases}.$$

This classification, it seems, relies entirely on the aesthetic connection with the classification of the sections of conics via eccentricity. Proper parabolic subgroups of $\mathrm{SL}_2(\mathbb{R})$ *can be realised as the stabilisers of lines in* $\mathbb{R}^2$ *under the standard action of* SL_2 *on* $\mathbb{R}^2$ *[Bor97, 2.6] and moreover an element of* $\mathrm{SL}_2(\mathbb{R})$ *is parabolic if and only if it has one fixed point on* $\partial\bar{\mathcal{H}}$ *and none on* $\mathcal{H}$ *[Bor97, 3.5].*

Being a parabolic element is equivalent to having eigenvalue 1 hence by the Jordan decomposition we know that parabolic elements in SL_2 *are conjugate (over* $\mathbb{C}$*) to*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly the standard parabolic subgroup

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \subseteq \mathrm{SL}_2(\mathbb{R}),$$

contains these matrices, and moreover all parabolics are conjugate to this parabolic. Hence all parabolic elements are contained in a parabolic subgroup.

The take away is that perhaps the folklore of the name being for "para-Borelic", as in kind of a Borel, is probably a better way of thinking of them.

The Example of Sp_{2n}

We collect the following facts as they will be useful in what is to come. Good references are the notes [Conb] and the book [Gar97, §8].

Let $(V, \langle,\rangle)$ be a symplectic space as above and $\mathrm{Sp}(V)$ is the automorphisms preserving the form, when V is fixed or clear we will simply write $\mathrm{Sp} := \mathrm{Sp}(V)$. A **flag** of V is a sequence of strict inclusions of vector subspaces

$$\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset V.$$

A subspace of V is said to be **isotropic** if the form is constantly zero on it (in both variables). A flag is **isotropic** if the proper subspaces in it are isotropic subspaces. A **maximal isotropic** flag is an isotropic flag with exactly n components. Sp_{2n} acts on a flag by acting on each of the subspaces. This action preserves isotropic flags i.e. it sends an isotropic flag to an isotropic flag. Stabilisers of isotropic flags give parabolics of Sp and moreover all parabolics arise in this way [Spr98, Exercise 3.2.16, 6.2.11].

Example 1.8. *Consider a four dimensional vector space* V *with a form given by the matrix*

$$\begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix},$$

then a maximal isotropic flag is

$$0 \subset Fe_1 \subset Fe_1 \oplus Fe_2 \subset F^4,$$

where $e_i = (\delta_{ij})_j$ *(the Kronecker delta). This has stabiliser consisting of matrices in* Sp *of the form*

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix}.$$

In particular maximal standard parabolics of Sp are stabilisers of *minimal* (non-trivial flags), i.e. stabilisers of non-zero isotropic subspaces,

$$0 \subset V_\ell \subset V,$$

where $V_\ell = \mathrm{span}(e_1, ..., e_\ell)$.

Remark 1.9. The fixing of this basis corresponds to fixing a Borel, as in the following section. Hence the name standard.

Then the stabiliser consists of matrices of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix},$$

with the sizes of the diagonal blocks being (these numbers square)

$$\begin{pmatrix} \ell & * & * & * \\ 0 & n-\ell & * & * \\ 0 & * & \ell & * \\ 0 & * & * & n-\ell \end{pmatrix}.$$

This has Levi consisting of matrices of form

$$\begin{pmatrix} A & & & \\ & a & & b \\ & & (A^T)^{-1} & \\ & c & & d \end{pmatrix}, \quad A \in \mathrm{GL}_\ell(F), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2(n-\ell)}(F),$$

and unipotent radical consisting of matrices of form

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & \\ & & 1 & \\ & & * & 1 \end{pmatrix},$$

with relations among the entries. These maximal parabolics are labeled by the size of the Levi's GL block, we label P_r the unique standard maximal parabolic with standard Levi $\mathrm{GL}_r \times \mathrm{Sp}_{2(n-r)}$.

1.2.2 Borel and Torus

A **split torus** is an algebraic group that is isomorphic to GL_1^b for some $b \in \mathbb{N}$.

Example 1.10 (Bad Etymology)**.** $\mathrm{GL}_1/\mathbb{C}$ *is a split torus. Consider the field extension* $\mathbb{C}/\mathbb{R}$*. Then* $\mathbb{C}$ *has the inner product given by*

$$\langle z, z' \rangle \quad := \quad \bar{z}z'.$$

We can look at the elements of $\mathbb{C}$ *that preserve this inner product,*

$$\begin{aligned} U(1) \quad &:= \quad \{c \in \mathrm{GL}_1(\mathbb{C}) : \forall z, z' \in \mathbb{C}, \quad \langle cz, cz' \rangle = \overline{cz}cz' = \bar{z}z'\} \\ &= \{c \in \mathrm{GL}_1(\mathbb{C}) : |c| = 1\}. \end{aligned}$$

Note that this is (locally) a (real) line topologically so we do not expect it to be a complex variety. Indeed this defines a ***real*** *algebraic group given by the zero locus in* $\mathbb{R}^2$ *of the two variable polynomial* $x^2 + y^2 - 1$*. In other words,*

$$U(1) \cong \mathrm{MaxSpec}\big(\mathbb{R}[x, y]/(x^2 + y^2 - 1)\big),$$

the set of maximal ideals. Now if we base change to $\mathbb{C}$ *we have*

$$\begin{aligned} \mathbb{R}[x, y]/(x^2 + y^2 - 1) \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C}[x, y]/\big((x + iy)(x - iy) - 1\big) \\ &\cong \mathbb{C}[s, t]/(st - 1) \\ &\cong \mathbb{C}^*. \end{aligned}$$

Thus $\mathrm{GL}_1/\mathbb{C}$ *is the complexification of the torus* $U(1)$*.*

Remark 1.11. These tori also play the same role in the classification of reductive LAG as the real Lie groups called tori play in the classification of Lie groups [Hal15, Part III].

A subgroup that is isomorphic to a split torus and is maximal in this respect is called a **maximal split torus**.

Example 1.12. *The classic example of a maximal split torus is the maximal split torus of* GL_n*, the group of diagonal matrices in* GL_n*.*

A **Borel** is a maximal, closed, solvable and connected subgroup of G. A Borel can be considered to be a parabolic that is minimal with respect to inclusion. The maximal split tori then form the

Levis of these parabolics. In particular for a Borel B we have that

$$B = TU,$$

for a maximal split torus T and unipotent radical U.

Example 1.13. *The usual Borel of* GL_n *is the group of upper triangular matrices. If* n *is even and one intersects this Borel with* $\mathrm{Sp}_{2(\frac{1}{2}n)}$ *then we get a Borel of* $\mathrm{Sp}_{2(\frac{1}{2}n)}$*.*

Lets prove this in GL_2 *and then believe that the only complication to going to larger* n *is keeping track of indices. So let*

$$B = \begin{pmatrix} * & * \\ & * \end{pmatrix},$$

we need to show that the derived series terminates for it to be solvable. So let

$$g = \begin{pmatrix} x & y \\ & z \end{pmatrix}, \quad h = \begin{pmatrix} a & b \\ & c \end{pmatrix},$$

be arbitrary in GL_2*, their commutator is then*

$$g^{-1}h^{-1}gh = \begin{pmatrix} 1 & \frac{bx-ay}{ax} \\ & 1 \end{pmatrix}.$$

Hence

$$[B, B] = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}.$$

Commutate two arbitrary elements again shows that

$$[[B, B], [B, B]] = 1.$$

It is clear that this is a closed subgroup because it is itself a linear algebraic group, moreover for LAG's we have the algebraic criterion of connectedness given by having the only idempotents in the representing algebra being $0, 1$ *[GH24, 1.5]. Because* $B = \operatorname{Spec} \mathbb{Z}[x_{i,j} : 1 \le i, j \le 2][y]/(\det(x_{ij})y - 1, x_{2,1})$ *it is clear that this group is connected. Finally it is clear that if a subgroup strictly contains the group of upper triangular matrices then it is in fact all of* GL_2 *and hence this is maximal. Therefore this is a Borel.*

If a Borel B is fixed, then a parabolic containing this Borel $B \subseteq P$ is called standard, there is a unique Levi of a standard parabolic containing this Borel called the **standard Levi**.

1.2.3 The Topology on Points

Let F be a number field and $G = \operatorname{Spec} F[x_1, ..., x_n]/(f_1, ..., f_m)$ be a LAG over F. As a locally ringed space this scheme has the Zariski topology, in the theory of automorphic forms however we wish to topologise the local and adelic points in a way which accommodates analysis. In particular the topology should be locally compact and Hausdorff so that we have Haar measures on the groups.

Following [Con12] we think of $G(\mathbb{A})$ as the subset of $\mathbb{A}^n$ on which the functions $f_i : \mathbb{A}^n \to \mathbb{A}$ all vanish. We give it the subspace topology which inherits the local compact and Hausdorff properties from $\mathbb{A}^n$. If ν is a place of F then we have the same definition, $G(F_\nu)$ is the subspace of F_ν^n on which $f_i : F_\nu^n \to F_\nu$ all vanish and it is endowed with the subspace topology. These topologies are referred to as the **Hausdorff topology**.

Remark 1.14. When $F_\nu = \mathbb{C}$ then the Hausdorff topology on $G(\mathbb{C})$ agrees with the topology of the analytification of G, often denoted G^{an}.

1.2.4 Maximal Compact Subgroups

We will often need to fix a maximal compact subgroup $K \subseteq G(\mathbb{A})$ of the Hausdorff topology. These maximal compact subgroups are not unique and as such when fixing one it can be arranged to have many convenient properties [MW95, I.1.4]. In particular if we have a classical group G and a fixed Borel B:

1. First require that
$$K = \prod_\nu K_\nu,$$
where the product is over all places of F and $K_\nu \subseteq G(F_\nu)$ is maximal compact subgroup.

2. If $\mathcal{O}_\nu$ is the ring of integers of F_ν, then for almost all places, $G(\mathcal{O}_\nu)$ is defined and is maximal compact in $G(F_\nu)$ hence we can require $K_\nu = G(\mathcal{O}_\nu)$ at these places.

3. We require
$$G(\mathbb{A}) = B(\mathbb{A})K.$$

4. For every standard parabolic P with Levi decomposition $P = MU$ we have that
$$P(\mathbb{A}) \cap K = \Big(M(\mathbb{A}) \cap K\Big)\Big(U(\mathbb{A}) \cap K\Big),$$
and $M(\mathbb{A}) \cap K$ is a maximal compact subgroup of $M(\mathbb{A})$.

It is in terms of the third property that we like to think of the maximal compact subgroup, it is the complimentary piece of the Borel. Moreover the fourth property should be thought of as a condition that the maximal compact subgroups are well behaved with the way that we are moving between the bigger and smaller reductive groups. Maximal compact groups with all these properties are said to be in **good position** with respect to B.

Chapter 2

Automorphic Forms

The story starts with the classical modular forms, or functions on the upper half plane that satisfy some invariance conditions and differential equations. This evolves into the notions of Maass forms on symmetric spaces and eventually reaches its apotheosis in the concept of automorphic form that we will present here.

We will present two notions of automorphic forms here. In the literature they are both called "automorphic forms" however here we will distinguish those that are defined only on the Archimedean points as "Archimedean automorphic forms" for clarity.

The first natural question is if there is a special case of automorphic forms which yield modular forms. The space of automorphic forms is larger than just modular forms, it gives the space of Maass forms (or modular and Maas forms, depending on convention). This is well covered in the literature [Eme] [Bum97, 3.2] [Boo] [Gar16]. We explain modular forms as Archimedean automorphic forms as we think it is where the connection is clearest. We will give an example of modular forms as adelic automorphic forms when we come to the Eisenstein series in section 4.2.

2.1 Archimedean Automorphic Form

Fix a number field F and a classical group G defined over F. Let ∞ denote the set of Archimedean places. We denote $\mathbb{A}_\infty = F_\infty := \prod_{\nu\in\infty} F_\nu$ and note that $G(F_\infty) \cong \prod_{\nu\in\infty} G(F_\nu)$. We denote $\mathbb{A}_f := \prod'_{\nu\notin\infty} F_\nu$ the finite adeles. Consider $\nu \in \infty$ one such Archimedean place, then F_ν is either $\mathbb{R}$ or $\mathbb{C}$. In particular (the analytification of) $G(F_\nu)$ is a Lie group and we call a function, $\varphi : G(F_\nu) \to \mathbb{C}$, **smooth** if it is smooth in the sense of functions on manifolds. The collection of such smooth functions on $G(F_\infty)$ will be denoted $C^\infty(G(F_\infty))$.

Because $G(F_\infty)$ is a Lie group we know how to define its Lie algebra $\mathfrak{g}$, as the tangent space at the identity, and we now denote $Z(\mathfrak{g})$ the centre of the *universal enveloping algebra* of the *complexification* of $\mathfrak{g}$, it would be more reasonable to use $Z(\mathcal{U}(\mathfrak{g}_\mathbb{C}))$ but that is too cumbersome so we follow the tradition. A vector in a $Z(\mathfrak{g})$-module $\varphi \in V$ is called $Z(\mathfrak{g})$-**finite** if the space $\mathrm{span}\big(Z(\mathfrak{g})\varphi\big)$ is finite dimensional.

Let $K_\infty \subseteq G(F_\infty)$ be a maximal compact subgroup. Then again an element of a K_∞-module is K_∞ **finite** if the span of its orbit is a finite dimensional vector space (we think here of $\mathbb{C}[K_\infty]$-modules).

To define automorphic forms we look at the representation $C^\infty(G(F_\infty))$ with the right regular action of K_∞, i.e. $g.f(x) = f(xg)$. In particular the $Z(\mathfrak{g})$ module structure is induced from the action of $\mathfrak{g}$ on $C^\infty(G(F_\infty))$ by

$$z.F(g) = \frac{\partial}{\partial t} F(ge^{tz})|_{t=0}, \quad x \in \mathfrak{g}.$$

Finally we want a growth condition. Fix an embedding $\iota : G \to GL_n$ which gives another embedding $G \to SL_{2n}$ via

$$\iota' : g \mapsto \begin{pmatrix} \iota(g) & \\ & (\iota(g))^{-t} \end{pmatrix}.$$

We have denoted the inverse of the transpose by $-t$. A function $\varphi : G(F_\infty) \to \mathbb{C}$ is of **moderate growth** if there are constants $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$|\varphi(g)| \le c\|g\|^r \quad := \quad c\left(\prod_{\nu\in\infty} \sup_{1\le i,j\le 2n} |\iota'(g)_{i,j,\nu}|_\nu\right)^r.$$

Remark 2.1. One can define norms on $G(\mathbb{A})$ via the linearisation of such groups, i.e. their representations. Concretely if σ is a finite dimensional complex representation of $G(\mathbb{A})$ on some Hilbert space with a K_∞ invariant inner product and $*$ is the adjoint matrix with respect to this Hilbert space structure then a **norm** on $G(\mathbb{A})$ is a function of the form

$$g \mapsto \left(\mathrm{tr}\ \sigma(g)^*\sigma(g)\right)^{\frac{1}{2}}.$$

This moderate growth condition is then equivalent to some norm $\|-\|$ existing on $G(F_\infty)$ such that

$$|\varphi(x)| \le C\|x\|^n,$$

for some $C > 0, n \in \mathbb{N}$ and all $x \in G(F_\infty)$. This is also equivalent to all such norms satisfying this condition [BC79, Part 1, "Automorphic Forms and Automorphic Representations", 1.2].

Finally a subgroup $\Gamma \subseteq G(F_\infty) \subseteq \mathrm{GL}_n(F_\infty)$ is called **arithmetic** if $\Gamma \cap \mathrm{GL}_n(\mathcal{O}_\infty)$ is a finite index subgroup in both Γ and $\mathrm{GL}_n(\mathcal{O}_\infty)$.

Definition 2.2. *Let $\Gamma \le G(F_\infty)$ some (arithmetic) subgroup, an **automorphic form for** Γ is a smooth function of moderate growth*

$$\varphi : G(F_\infty) \to \mathbb{C},$$

that is K_∞ and $Z(\mathfrak{g})$ finite with (left) Γ invariance. We denote the set of these "Archimedean" automorphic forms by $\mathcal{A}(\Gamma\backslash G(F_\infty))$.

2.2 Adelic Automorphic Form

Here we follow [MW95, I.2.17] and [BC79, Part 1, "Automorphic Forms and Automorphic Representations", 1.2]. Fix a Borel $B \subseteq G$ and a standard parabolic $B \subseteq P \subseteq G$ with a standard Levi decomposition $P = MU$. We let K be a maximal compact subgroup of $G(\mathbb{A})$ that is in good position as in section 1.2.4.

We say that $f : G(\mathbb{A}_f) \to \mathbb{C}$ is smooth if it is locally constant in the Hausdorff topology and we denote the set of such smooth functions $C^\infty(G(\mathbb{A}_f))$.

Thus for the full adeles we have the notion of smooth as an element of the tensor product,

$$C^\infty(\mathbb{A}_F) \quad := \quad C^\infty(G(\mathbb{A}_f)) \otimes C^\infty(G(F_\infty)).$$

Notice that a priori the codomain is an infinite tensor product over $\mathbb{C}$ of copies of $\mathbb{C}$, this is *canonically* isomorphic to $\mathbb{C}$, thus we can conflate a smooth function with its composition along this isomorphism and think of them as functions into $\mathbb{C}$.

We still consider $Z(\mathfrak{g})$ to be the center of the universal enveloping algebra of the complexified Lie algebra at the infinite places, exactly as before. We define an action by linearly extending

$$z.(f \otimes g) = f \otimes (z.g),$$

i.e. it acts on the Archimedean places as in the setting of Archimedean automorphic forms.

The definition of moderate growth carries over verbatim, however we change the set of places multiplied over to be all of them now. Specifically fix an embedding $\iota : G \to GL_n$ which gives another embedding $G \to SL_{2n}$ via

$$\iota' : g \mapsto \begin{pmatrix} \iota(g) & \\ & (\iota(g))^{-t} \end{pmatrix}.$$

We have denoted the inverse of the transpose by $-t$. A function $\varphi : G(\mathbb{A}) \to \mathbb{C}$ is of **moderate growth** if there are constants $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$|\varphi(g)| \le c\|g\|^r \quad := \quad c\left(\prod_\nu \sup_{1\le i,j\le 2n} |\iota'(g)_{i,j,\nu}|_\nu\right)^r.$$

Remark 2.3 ([BC79], Part 1, “Automorphic Forms and Automorphic Representations”). The collection of moderate growth functions is independent of the choices of embedding.

Definition 2.4. A function $\varphi : U(\mathbb{A})M(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is an **automorphic form** if it is smooth, moderate growth, $Z(\mathfrak{g})$ and K finite. We will denote the set of these automorphic forms by $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$.

Remark 2.5. It is important that $M(F)$ is treated as a subgroup of $M(\mathbb{A})$ via the diagonal embedding.

Remark 2.6. What we have called automorphic forms are sometimes referred to as “smooth K-finite automorphic forms” [Cogc, 2.2].

Remark 2.7. This is a more general setup than in the Archimedean case as we only require $U(\mathbb{A})M(F)$ invariance. By choosing the parabolic to be G itself we get full $G(F)$ invariance as in the Archimedean case.

2.3 Modular Forms

Recall the definition of a **modular form of weight k** (of full level and trivial character) [DS05, 1.1.2] as a function

$$\varphi : \mathcal{H} \to \mathbb{C},$$

where $\mathcal{H}$ is the upper half plane in $\mathbb{C}$, that is holomorphic, satisfies

$$\varphi(\gamma.z) = (cz+d)^k \varphi(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and is of moderate growth, that is sub-exponential growth.

We want to think of the upper half plane as a quotient of the $\mathbb{Q}_\infty = \mathbb{R}$ points of some reductive group. If we have a transitive action of some reductive group then by the orbit stabiliser theorem we would have a bijection of sets.

Theorem 2.8.

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R}),$$

as sets.

Proof. Consider the action

$$\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathcal{H} : \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}.$$

Then look at the orbit of i, namely

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} .i = \frac{ai+b}{d} = a^2 i + ab,$$

which letting $a, b \in \mathbb{R}$ vary is clearly surjective onto the whole upper half plane. So there is one orbit, and hence by the orbit stabiliser we know that

$$\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{stab}(i),$$

so we want to find

$$\mathrm{stab}(i) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : g.i = i \right\},$$

in particular we solve

$$i = g.i = \frac{ai+b}{ci+d} = (c^2+d^2)^{-1}(ac+bd+i\det g).$$

So equating coefficients we have

$$\det g(c^2+d^2)^{-1} = 1 \implies c^2+d^2 = \det g = 1,$$

on the other hand

$$ac + bd = 0.$$

Now the pairs $c^2 + d^2 = \det g = 1$ are parameterized by $\theta \in [0, 2\pi)$ using $c = \sin\theta, d = \cos\theta$ hence subbing this into the above equation

$$\frac{-b}{a} = \tan\theta,$$

and so $b = -k\sin\theta, a = k\cos\theta$ for some $k \in \mathbb{R}$ but the determinant must be 1 so $k = 1$. Hence

$$\operatorname{stab}(i) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\} = \mathrm{SO}_2(\mathbb{R}).$$

Remark 2.9. This can be beefed up to an isomorphism of complex analytic spaces. Sometimes to make the action of certain (Hecke) operators more apparent this is exhibited as

$$\mathcal{H} \cong \mathrm{GL}_2^+(\mathbb{R})/A_{\mathrm{GL}_2}\,\mathrm{SO}_2(\mathbb{R}).$$

This obscures the connection with the reductive group setting however so we avoid it here.

SL_2 is a reductive group and $\mathrm{SO}_2(\mathbb{R})$ is its maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$. The decomposition of the upper half plane in 2.8 suggests that function on the upper half plane might have some invariance along the maximal compact subgroup of the reductive group SL_2. If we define

$$B \;:=\; \left\{ \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} : x, y \in \mathbb{R}, y \neq 0 \right\}$$

which happens to be the real points of a Borel subgroup of SL_2 we have the picture;

$$\begin{tikzcd} B\,\mathrm{SO}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \arrow[r, "\text{project}"] \arrow[dr, "\text{project}"'] & \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}) \arrow[r, "\sim", "g \mapsto g.i"'] & \mathcal{H} \\ & \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R}) & \end{tikzcd}$$

We can lift a function on $\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R})$ to $\mathrm{SL}_2(\mathbb{R})$ by composing with the projection, however this is not $\mathrm{SL}_2(\mathbb{Z})$ invariant, thus we need to add a pre-factor to ensure this in our associated automorphic form. The algebro-geometric perspective in [Eme] can make this seem slightly less ad hoc.

Thus for f a modular form of weight k the following function on $\mathrm{SL}_2(\mathbb{R})$

$$F(g) \;:=\; (ci + d)^{-k} f(g.i),$$

we claim is an Archimedean automorphic form for $\mathrm{SL}_2(\mathbb{Z})$. We take for granted its smoothness. The $\mathrm{SL}_2(\mathbb{Z})$ invariance is obvious from the modularity condition and we consider the moderate growth condition to be tautological. It remains to show the last two properties:

Lemma 2.10. $\mathrm{SO}_2(\mathbb{R})$ *is a maximal compact subgroup inside* $\mathrm{SL}_2(\mathbb{R})$*.* F *is an* $\mathrm{SO}_2(\mathbb{R})$*-finite* *function.*

Proof. Using that $\kappa = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K = \mathrm{SO}_2(\mathbb{R})$ acts trivially on i, an elementary computation shows that for $g \in \mathrm{SL}_2(\mathbb{R})$,

$$F(g\kappa) = e^{-ik\theta}F(g).$$

Hence $F(g)$ is acted on by K via a one dimensional irreducible representation. In particular it is finite dimensional.

Lemma 2.11. *F is a $Z(\mathfrak{sl}_2)$ finite function.*

Proof. Only a sketch.

The center of the universal enveloping algebra of the complexified Lie algebra is generated by the Casimir operator. We have the coordinates on $\mathrm{SL}_2(\mathbb{R})$ from [Bum97, 1.19]

$$\left\{ \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} : x, y \in \mathbb{R}, y \neq 0 \right\} \mathrm{SO}_2(\mathbb{R}),$$

in which the Casimir acts as the differential operator

$$\Delta = y^2 \left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 \right) - y\frac{\partial^2}{\partial x \partial \theta},$$

[Bum97, 1.29,Prop 2.2.5]. Now we claim that F is an eigenfunction for this operator. An element $(x, y, \theta) := \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \kappa_\theta \in \mathrm{SL}_2(\mathbb{R})$ acts on i by sending it to $x+iy$ (elementary computation). The bottom row of the product is $\left(y^{-1/2}\sin\theta; y^{-1/2}\cos\theta\right)$ which results in

$$F(x, y, \theta) = y^{k/2}e^{-ik\theta}f(x + iy).$$

It is then a calculus exercise to apply Δ to this, using the holomorphicity we also get that $f_{xx} - f_{yy} = 0$ and $f_y = if_x$ which cancels away terms and we get that

$$\Delta F(x, y, \theta) = \frac{k}{2}\left(\frac{k}{2} - 1\right) F(x, y, \theta).$$

Therefore the dimension of $Z(\mathfrak{g})F$ is simply one.

This example makes it clear that the two finiteness conditions for automorphic forms are in some sense functional equations that they must satisfy. There is a nice explanation of how to lift this to the adelic setting in several places, the key is essentially the isomorphism

$$\mathbb{Z}\backslash\mathbb{R} \cong \mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}/\hat{\mathbb{Z}}$$

The details are quite clear in [Cogc, 2.1] or [Boo]. We will revisit this perspective through the example of the Eisenstein series in section 4.2.

Chapter 3

Automorphic Representations

The references that will be most helpful are [BC79, Part 1, "Automorphic Forms and Automorphic Representations"] [GH24] for the general theory, we will follow the notation developed in [MW95] as it is somewhat standard. We will discuss some of the details of the representation theory of reductive groups on spaces of automorphic forms. In particular we want to draw attention to some of the quirks of the category of automorphic representations.

3.1 Local Representation Theory

Recall that in the theory of complex representations of finite groups there is really only one important representation, that is the regular representation i.e. for the finite group G, the $\mathbb{C}[G]$ module $\mathbb{C}[G]$. This is important for two reasons, the first is that it is always a priori defined uniformly for all groups. The second is that it decomposes into a direct sum over all irreducible modules [Ser96, Ch. 2.4 Cor. 2].

Let G be a classical group defined over a number field F. As in the finite group case we want to consider the right regular action of the adelic points, $G(\mathbb{A})$, on a space of functions $G(\mathbb{A}) \to \mathbb{C}$, namely

$$g.f(x) = f(xg), \quad g, x \in G(\mathbb{A}), f \in \mathrm{Maps}(G(\mathbb{A}), \mathbb{C}).$$

One can ask if this representation sends an automorphic form to an automorphic form. If $\varphi(x) \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ and $g \in G(\mathbb{A}_f)$ then $\varphi(xg) \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$. Hence $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ is a $G(\mathbb{A}_f)$-module. In particular it is a module for $G(F_\nu)$ for all ν non-Archimedean.

There is a problem with the K-finiteness in the infinite places however which prevents $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ from being a full $G(\mathbb{A})$ module.

Example 3.1 ([Cogc], 2.3)**.** *If $\varphi \in \mathcal{A}(\Gamma\backslash G(F_\infty))$ is K_∞-finite, then $g.\varphi$ is $gK_\infty g^{-1}$-finite. This is still a maximal compact subgroup, however in the infinite place it will a priori have only the identity in common with the original K_∞.*

For example consider SL_2 where the maximal compact is SO_2, if we conjugate we get $g\,\mathrm{SO}_2\,g^{-1}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \cos\theta + (db+ca)\sin\theta & -\sin\theta(a^2+b^2) \\ \sin\theta(d^2+c^2) & \cos\theta - (bd+ac)\sin\theta \end{pmatrix}.$$

If we want to find the intersection of SO_2 *with* $g\,\mathrm{SO}_2\,g^{-1}$ *we need to solve the system*

$$\begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix} = \begin{pmatrix} \cos\theta + (db+ca)\sin\theta & -\sin\theta(a^2+b^2) \\ \sin\theta(d^2+c^2) & \cos\theta - (bd+ac)\sin\theta \end{pmatrix}.$$

Where θ *might not be* θ'*. If* $\theta = n\pi, n \in \mathbb{Z}$ *then the* sin *terms on the right vanish and we get the* ± 1 *as a point of intersection, so consider* $\theta \neq n\pi$*. Then we require*

$$\cos\theta' = \cos\theta - (bd+ac)\sin\theta = \cos\theta + (db+ca)\sin\theta,$$

hence $2(bd+ac)\sin\theta = 0$ *and because* $\sin\theta$ *was assumed to be non-zero this is the same as* $bd+ac=0$*. Thus for instance the element* $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ *conjugates* SO_2 *to another subgroup that has only trivial intersection.*

Finally it is worth noting that this is not an issue at the finite places, namely if $K = K_f K_\infty$ *is our maximal compact subgroup of* $G(\mathbb{A})$ *then* K_f *is also open and hence* $K_f \cap gK_f g^{-1}$ *is of finite index in both* K_f *and* $gK_f g^{-1}$ *and so their notions of* K*-finiteness will agree.*

For this reason we will need to talk about $(\mathfrak{g}, K)$-modules:

Definition 3.2 ([GH24], 4.4.6)**.** Let G be a *real* Lie group (for example the analytification of the real or complex points of our favourite reductive LAG) and K be a maximal compact subgroup of G. Let $\mathfrak{g}_\mathbb{C}$ be the complexification of the real Lie algebra of G and $\mathfrak{k}$ the real Lie algebra of K.

A $(\mathfrak{g}, K)$**-module** is a complex vector space V with two representations

$$\tilde{\pi} : \mathfrak{g}_\mathbb{C} \to End(V), \quad \pi : K \to \mathrm{GL}(V),$$

satisfying the following axioms

1. V decomposes into a countable direct sum of finite dimensional K representations.

2. The representations, $\pi, \tilde{\pi}$, should be compatible: For all $X \in \mathfrak{k}$ and $v \in V$

$$\tilde{\pi}(X)(v) = \frac{\mathrm{d}}{\mathrm{d}t}\pi(e^{tX})(v)|_{t=0} = \lim_{t\to 0} \frac{\pi(e^{tX})(v) - v}{t}.$$

   In particular the right hand limit exists.

3. The representations, $\pi, \tilde{\pi}$, should be compatible with the adjoint representation: For $k \in K$ and $X \in \mathfrak{g}$

$$\pi(k)\tilde{\pi}(X)\pi(k^{-1})(v) = \tilde{\pi}(\mathrm{Ad}(k)(X))(v).$$

Remark 3.3. It is common to use the same symbol for both of these representations in the $(\mathfrak{g}, K)$-module. It is also important to note that these are purely algebraic representations, there is no condition of continuity etc.

If $\mathfrak{g}$ is the Lie algebra of $G(F_\infty)$ and $K_\infty \subseteq G(F_\infty)$ is a maximal compact subgroup in good position we can define a $(\mathfrak{g}, K_\infty)$-module structure on the space of automorphic forms as follows. Recall that by definition we have that

$$\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) \subseteq C^\infty(G(\mathbb{A}_f)) \otimes C^\infty(G(F_\infty)).$$

If $\varphi_f \otimes \varphi_\infty \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ and $(g_f, g_\infty) \in G(\mathbb{A})$ then $k \in K_\infty$ acts by

$$k.(\varphi_f \otimes \varphi_\infty)(g) \quad := \quad \varphi_f(g) \otimes \varphi_\infty(gk),$$

i.e. via the right regular representation on the Archimedean component. We extend this linearly from pure tensors to all tensors. The Lie algebra of $G(F_\infty)$ acts by linearly extending the action we have previously defined in section 2.1, via

$$z.(\varphi_f \otimes \varphi_\infty)(g_f, g_\infty) = \varphi_f(g_f) \otimes \frac{\partial}{\partial t}\varphi_\infty(g_\infty e^{tz})|_{t=0}, \quad z \in \mathfrak{g}.$$

To see that we have really fixed the K_∞ problem we should check that this really defines an action.

Lemma 3.4. *If $\varphi \in \mathcal{A}(\Gamma\backslash G(F_\infty))$ is K_∞-finite and $X \in \mathfrak{g}$ then $X.\varphi$ is K_∞-finite.*

Proof. There is an action of K_∞ on $\mathfrak{g} \otimes C^\infty(\Gamma\backslash G(F_\infty))$. If $k \in K_\infty, \varphi \in C^\infty(\Gamma\backslash G(F_\infty))$ and $X \in \mathfrak{g}$ then the action is given by linearly extending

$$k.(X \otimes \varphi) = \mathrm{Ad}(k)(X) \otimes k.\varphi.$$

The map

$$\mathfrak{g} \otimes C^\infty(\Gamma\backslash G(F_\infty)) \to C^\infty(\Gamma\backslash G(F_\infty)), \quad X \otimes \varphi \mapsto X\varphi, \tag{3.1.1}$$

is K_∞ equivariant by the definition of the adjoint action. Now if $\varphi \in \mathcal{A}(\Gamma\backslash G(F_\infty))$ then the span of φ is a finite dimensional K_∞ module which we will denote M_φ. Then $k.X\varphi$ is in the image of $\mathfrak{g} \otimes M_\varphi$ under the map 3.1.1. But the Lie algebra is finite dimensional and M_φ is finite dimensional so this image is finite dimensional. Therefore the K_∞ span of $X\varphi$ is finite dimensional and so $X\varphi$ is K_∞-finite.

Finally the conditions for these representations to be a $(\mathfrak{g}, K_\infty)$ module can be checked. (1) is [GH24, Thm. 6.3.4]. (2) is immediate from the definitions of the two representations and the fact that automorphic forms are smooth. (3) is immediate from the definition of the adjoint action.

3.2 Automorphic Representations

Recall that if A, B, C are all R modules and we have the inclusions of R modules $C \subseteq B \subseteq A$ then we call B/C a subquotient of A. We now think of $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ as being a $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$ module. An **automorphic representation** is then defined to be a subquotient of this representation. We will say that such an automorphic representation is **induced from** $P = MU$, to indicate that it is a subquotient of the space of $U(\mathbb{A})M(F)$ invariant functions.

Remark 3.5. Some authors will require that an automorphic representation is by definition an *irreducible* subquotient.

Remark 3.6. We really need a set theoretic definition here. The quotient of these modules cannot be considered up to isomorphism of $(\mathfrak{g}, K)$-modules but must be the classical set theoretic realisation of this object, defined as equivalence classes of elements of the module. This is to say if one were to think of the category of automorphic representations it is *much smaller* than the category of $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$-modules (in particular the category of automorphic representations has a cardinality, whilst there is a proper class of $G(\mathbb{A}_f)$-modules). The reason is that we will want to talk about the automorphic forms themselves, and consider their properties.

Remark 3.7. Automorphic representations can also be defined as representations of an algebra $\mathcal{H}$, the global Hecke algebra. This is the approach in [BC79, Part 1, "Automorphic Forms and Automorphic Representations", 4.6], and can be a helpful perspective to simplify definitions. This is also a motivation behind why Harish-Chandra's $(\mathfrak{g}, K)$-modules are the "right" replacement for the regular representation.

Example 3.8. *It is very hard to really write down something explicit. One thing that we can do is take a modular form f. Then we know how to associate a concrete automorphic form to it $\tilde{f}$. To any fixed automorphic form we have an automorphic representation given by taking the span of its orbit*

$$\mathrm{span}_{\mathbb{C}}\left\{\big(G(\mathbb{A}_f) \times (\mathfrak{g}, K)\big).\tilde{f}\right\} \subseteq \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A})).$$

3.2.1 Cuspidal Representations

An automorphic form $\varphi \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ is called **cuspidal** if all its constant terms vanish, see section 5.1 for more detail on constant terms. The space of such automorphic forms is denoted $\mathcal{A}_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$. An automorphic representation is called **cuspidal** if it is an irreducible subquotient of $\mathcal{A}_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))$.

Remark 3.9. Again this is not as a $(\mathfrak{g}, K)$-module.

3.2.2 Isotypic Components

Following the convention of [MW95, II.1] we make two cases: Let π be an irreducible subquotient of the space $\mathcal{A}(M(k)\backslash M(\mathbb{A}))$, that is *not cuspidal*. Then we denote the π isotypic component of $\mathcal{A}(M(k)\backslash M(\mathbb{A}))$ by $\mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi$.

We will also need the space

$$\begin{aligned}&\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi \\ &\quad := \quad \{\varphi \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) : \forall k \in K,\ \varphi_k \in \mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi\},\end{aligned}$$

where $\varphi_k : M(\mathbb{A}) \to \mathbb{C}$ is given by $\varphi_k(x) = -\rho_P(x)\varphi(xk)$.

Now if π is cuspidal, we define $\mathcal{A}(M(k)\backslash M(\mathbb{A}))_\pi$ to be the isotypic component of π in $\mathcal{A}_0(M(k)\backslash M(\mathbb{A}))$ and similarly we have

$$\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi$$

$$:= \{\varphi \in \mathcal{A}_0(U(\mathbb{A})M(F)\backslash G(\mathbb{A})) : \forall k \in K,\ \varphi_k \in \mathcal{A}_0(M(k)\backslash M(\mathbb{A}))_\pi\}.$$

Remark 3.10. We cannot simply take the isotypic components as $(\mathfrak{g}, K)$-modules we need to take the isotypic components of the cuspidal subspace directly. This is to say again that the category of automorphic representations is very explicit.

The point is that we want the isotypic component corresponding to a cuspidal representation to be cuspidal, however this just might not be the case. Yamana in [Yam13, Remark. 7.12] has a counter example when one allows unitary groups over division algebras (non-commutative fields). It could be interesting to investigate this example more closely to see if the example can be pulled back to a unitary group over a field. In [Yam13] there is an automorphic representation of the quarternionic unitary group constructed, $\Pi(V)$, that appears in both the cuspidal and residual spectrum. By that Yamana means that up to isomorphism the representation can been seen in both residual and cuspidal spectrum. In particular if one were to take the component that is in the cuspidal spectrum and look at its isotypic component then the versions in the residual spectrum would also occur and hence by definition of residual spectrum would not be cuspidal.

If we restrict to the cases dealt with in for instance [MW95], namely not dealing with quarternions, then we have been told that this is an open problem whether or not this restriction is superfluous.

Chapter 4

Eisenstein Series

The Eisenstein series is from our perspective the most important tool in the theory of automorphic forms. Some surveys on its role, properties and open problems are [Lap22], [Art79], [Kim] and [Jia08]. To see the relation to the classical Eisenstein series there is [Gar16] which we will also go through in section 4.2. One thing that Eisenstein series do, as in the theory of modular forms, is that they furnish us with quasi-concrete examples of automorphic forms. Another reason that these functions are important is through their normalisation and constant terms, in which products of L-functions appear, we discuss this more in section 5.1. This has been a fruitful method for proving theorems about L-functions as in [Sha10] [Pol] [Art79], or conversely proving theorems about Eisenstein series [JLZ13] using properties of L-functions.

4.1 Eisenstein Series

As usual we fix a classical group G defined over a number field F, with a Borel B and a standard parabolic with standard Levi decomposition $P = MU$.

Following the setup in [MW95, I.1.4] we consider a **character** $\chi \in \mathrm{Rat}(M) := \mathrm{Hom}_{\mathrm{LAG}}(M, \mathbb{G}_m)$, thinking of it below as a natural transformation, and then define

$$|\chi| : M(\mathbb{A}) \to \mathbb{C}, \quad (m_\nu) \mapsto \prod_\nu |\chi(F_\nu)(m_\nu)|_\nu.$$

The intersection of the kernels of these characters is

$$M^1 := \bigcap_{\chi \in \mathrm{Rat}(M)} \ker |\chi|.$$

The collection of characters of $M(\mathbb{A})$ that are trivial on M^1 is denoted

$$X_M := \mathrm{Hom}_{\mathrm{TopGroup}}(M(\mathbb{A})/M^1, \mathbb{C}^*).$$

Remark 4.1. To make it seem less mysterious we comment that this group has some importance in the more general theory. It is one of the pieces in the "Langlands decomposition" of the

Archimedean points of a parabolic $P = MU$, if ν is an Archimedean place then,

$$P(F_\nu) = A_M M^1 U(F_\nu).$$

We will not define A_M. It also has the property that $M(\mathbb{Q})\backslash M(\mathbb{A})^1$ has finite measure [GH24, 4.9], in the induced measure from the Haar measure on $M(\mathbb{A})$.

The set of **complex characters** of M,

$$\mathfrak{a}_M^* \quad := \quad \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{C},$$

is isomorphic as $\mathbb{C}$ vector spaces to X_M [MW95, I.1.4]. If $Z_{G(\mathbb{A})}$ is the center of $G(\mathbb{A})$ then we also have the space

$$X_M^G \quad := \quad \mathrm{Hom}_{\mathrm{TopGroup}}((M(\mathbb{A})/M^1)/Z_{G(\mathbb{A})}, \mathbb{C}^*),$$

which is characters of $M(\mathbb{A})/M^1$ which are also trivial on the center of G.

Example 4.2. *For the maximal standard parabolic P_r with Levi M_r (see section 1.2.1) of Sp_{2n} we have that $X_{M_r}^{\mathrm{Sp}_{2n}}$ is at most a one dimensional $\mathbb{C}$ vector space.*

First of all we have that [MW95, I.1.4]

$$X_{M_r}^{\mathrm{Sp}_{2n}} \subseteq X_{M_r} \cong \mathfrak{a}_{M_r}^* \quad := \quad Rat(M_r) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Thus it is clearly sufficient to bound the dimension of $\mathfrak{a}_{M_r}^*$ as a $\mathbb{C}$ vector space, moreover this dimension agrees with the dimension of $Rat(M_r)$ as a free $\mathbb{Z}$ module.

Thus we compute $\dim_{\mathbb{Z}}(Rat(M_r))$:

$$\begin{aligned} Rat(M_r) &= Rat(\mathrm{GL}_r \times \mathrm{Sp}_{2m}) \\ &= \mathrm{Hom}(\mathrm{GL}_r \times \mathrm{Sp}_{2m}, \mathbb{G}_m) \\ (1) &\cong \mathrm{Hom}(\mathrm{Ab}(\mathrm{GL}_r \times \mathrm{Sp}_{2m}), \mathbb{G}_m) \\ (2) &\cong \mathrm{Hom}(\mathrm{Ab}(\mathrm{GL}_r) \times \mathrm{Ab}(\mathrm{Sp}_{2m}), \mathbb{G}_m) \\ (3) &\cong \mathrm{Hom}(\mathbb{G}_m \times 1, \mathbb{G}_m) \\ &\cong \mathbb{Z}. \end{aligned}$$

In (1) we have used the universal property of the abelianisation $\mathrm{Ab}(G) = \mathcal{D}(G) \setminus G = [G,G] \setminus G$ because $\mathbb{G}_m$ is Abelian. (2) is that the abelianisation commutes with direct products. (3) is because Sp *is a perfect group.*

Using the Langlands-Iwasawa decomposition of equation 1.2.1 we know that $G(\mathbb{A}) = M(\mathbb{A})U(\mathbb{A})K$, hence for $m \in M(\mathbb{A}), u \in U(\mathbb{A}), k \in K$, there is the natural map $m_P : G(\mathbb{A}) \to M^1\backslash M(\mathbb{A})$ sending $umk \mapsto M^1 m$.

Now if we take the collection of irreducible automorphic representations of $M(\mathbb{A})$,

$$\hat{\mathcal{M}(\mathbb{A})} \quad := \quad \{(\pi, V) : \pi \text{ is an irreducible automorphic representation of } M(\mathbb{A})\},$$

then we can think of X_M^G as being one dimensional automorphic representations (with some extra invariance) and so there is a natural action on $\hat{\mathcal{M}}$ given by tensoring, i.e. if $\lambda \in X_M^G$ and $(\pi, V) \in \hat{\mathcal{M}}$

then

$$\lambda.\pi \quad := \quad \lambda \otimes \pi.$$

Then $\hat{\mathcal{M}}$ decomposes as a disjoint union of its orbits. The orbit $\mathfrak{P}$ of a cuspidal representation π_0 is called a **cuspidal datum**. By definition X_M^G acts transitively on any cuspidal datum $\mathfrak{P}$ but by [MW95, II.1] it also acts freely. Thus $\mathfrak{P}$ is in bijection with X_M^G. Through this bijection we transmit the complex structure on $\mathfrak{a}_M^*$ to X_M then to the subspace X_M^G and finally to $\mathfrak{P}$.

Let $\mathfrak{P}$ be a cuspidal datum with a complex structure as above. Let $\pi \in \mathfrak{P}$ and $\varphi_\pi \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi$, then $\lambda \in X_M^G$ acts on φ_π by

$$\lambda.\varphi_\pi(g) = (\lambda \circ m_P)(g)\varphi_\pi(g),$$

which is then an element of $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_{\pi\otimes\lambda}$. Finally we have the **Eisenstein series** which is defined by the following sum for $g \in G(\mathbb{A})$,

$$E(\varphi_\pi, \lambda, g) = \sum_{\gamma \in P(F)\backslash G(F)} \lambda.\varphi_\pi(\gamma g),$$

whenever it is absolutely convergent. The first thing to note is that for a fixed φ_π there is an open set in X_M^G and a compact subset of $G(F)\backslash G(\mathbb{A})$ such that the Eisenstein series converges (normally) [MW95, II.1.5].

If $P = MU$ and $P' = M'U'$ are two standard parabolics of G such that their Levi's are conjugate, i.e. such that for $w \in G(k)$ we have $wMw^{-1} = M'$, then w maps $\mathfrak{P}$ to $w\mathfrak{P}$, an orbit of an irreducible representations of M to an orbit of irreducible representations of M'. The Eisenstein series is closely related through its constant terms (as discussed in section 5.2.3) to the following

$$M(w,\pi)(\varphi_\pi)(g) = \int_{(U'(F)\cap wU(F)w^{-1})\backslash U'(\mathbb{A})} \varphi_\pi(w^{-1}ug)du,$$

where $\pi \in \mathfrak{P}$, $g \in G(\mathbb{A})$ and $\varphi_\pi \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_\pi$.

The Eisenstein series has three inputs and can be considered as a function in different variables. If π is a cuspidal automorphic representation of $G(\mathbb{A})$ induced from P, then for a fixed $\varphi \in \mathcal{A}_0(U(\mathbb{A})M(F) \setminus G(\mathbb{A}))_\pi$ the Eisenstein series $E(\varphi)$ can be thought of as a function from some open subset of the cuspidal datum $\mathfrak{P}$ into $L^2_{\mathrm{loc}}(G)$, the set of locally square integrable complex valued functions on $G(\mathbb{A})$, given by

$$E(\varphi)(\lambda)(g) = \sum_{\gamma \in P(F)\backslash G(F)} \lambda.\varphi(\gamma g), \quad \lambda \in \mathfrak{P},\ g \in G(\mathbb{A}),$$

where it converges. The space $L^2_{\mathrm{loc}}(G(\mathbb{A}))$ can be endowed with a Frechet space structure coming from the semi-norms associated to compact sets $C \subseteq G(\mathbb{A})$ given by

$$\varphi \mapsto \|\varphi|_C\|_{L^2} = \int_C |\varphi(x)|^2 \mathrm{d}x.$$

Then it makes sense to talk about the holomorphicity of $E(\varphi)$ in this sense (see [MW95, I.4.9] for details). The key properties of both the Eisenstein series and the operator $M(w,\pi)$ can be found in [MW95, IV.1.8, IV.1.9, IV.1.10, IV.1.11]. Most importantly as a function of $\mathfrak{P}$ it can be shown that they both have a meromorphic continuation to all of $\mathfrak{P}$. This was also given a second "soft proof" more recently in [BL23], with the spectral decomposition that follows from it also being worked out in [Del21]. Moreover an Eisenstein series attached to an automorphic form, at a point $p \in \mathfrak{P}$ at which it is holomorphic, is also an automorphic form.

4.2 Classical Eisenstein Series

We will follow the excellent exposition in [Gar16], the section [BVDGHZ08, 1.2] on classical Eisenstein series. The typical example of a classical Eisenstein series is that defined on $s \in \mathbb{C}$ by the meromorphic continuation of the sum

$$\mathbf{E}(z,s) \quad := \quad \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\},\ \text{coprime}} \frac{\operatorname{Im}(z)^s}{|mz+n|^{2s}}, \quad z \in \mathcal{H},$$

which converges absolutely for $\operatorname{Re}(s) > \frac{1}{2}$. Consider the algebraic group SL_2 with the parabolic of upper triangular matrices P.

First we want to look at the index of the sum, we aim to define a map

$$\omega : P(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z}) \to \{(m,n) \in \mathbb{Z}^2\backslash\{(0,0)\} : m,n \text{ are co-prime}\}.$$

The cosets of $P(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})$ look like

$$P(\mathbb{Z})\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{ \pm \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : n \in \mathbb{Z} \right\} = \left\{ \pm \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Moreover because $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have by Bezout's lemma (applied to the determinant expression) that c and d are co-prime. Therefore there is a well defined map

$$P(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z}) \to \{(m,n) \in \mathbb{Z}^2\backslash\{(0,0)\} : m,n \text{ are co-prime}\},$$

if we denote the indicator function $\mathbf{1}(c<0)$ then it is given by

$$\omega : \left\{ \pm \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix} : n \in \mathbb{Z} \right\} \mapsto (|c|, (-1)^{\mathbf{1}(c<0)}d).$$

The point is that $|mz+n| = |(-m)z + (-n)|$ and so the sum in the Eisenstein series, having a prefactor of a half is really just the sum over $\{(m,n) \in \mathbb{Z}^2\backslash\{(0,0)\} : m,n$ are co-prime **and** $m \geq 0\}$, which by our argument is in bijection with $P(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})$ via ω.

Lemma 4.3 ([Gar16], 3.5)**.**

$$P(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z}) \cong P(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q}).$$

Proof. The bijection is explicitly

$$P(\mathbb{Z})g \mapsto P(\mathbb{Q})g.$$

Recall that $\mathrm{SL}_2(\mathbb{Z})$ acts via the Mobius transformations on the upper half plane. If $z = x+iy \in \mathcal{H}$, $s \in \mathbb{C}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ then an elementary computation shows that,

$$\mathrm{Im}(\gamma.z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Hence the classical Eisenstein series is

$$\mathbf{E}(z,s) \quad := \quad \frac{1}{2} \sum_{m,n\in\mathbb{Z}\setminus\{(0,0)\},\ \text{coprime}} \frac{\mathrm{Im}(z)^s}{|mz+n|^{2s}} = \sum_{\gamma\in P(\mathbb{Q})\setminus \mathrm{SL}_2(\mathbb{Q})} \mathrm{Im}(\gamma.z)^s,$$

Where these sum are notation for their own meromorphic continuation to the complex plane.

We want to realize this as the Eisenstein series associated to an automorphic form so first we must design a function on $\mathrm{SL}_2(\mathbb{A})$. For any place ν of $\mathbb{Q}$ we have the local Iwasawa decomposition $\mathrm{SL}_2(\mathbb{Q}_\nu) = P(\mathbb{Q}_\nu)K_\nu$ where

$$K_\nu \quad := \quad \begin{cases} \mathrm{SL}_2(\mathbb{Z}_\nu), & \nu \text{ non-Archimedean} \\ \mathrm{SO}_2(\mathbb{R}), & \nu \text{ Archimedean} \end{cases},$$

are the local maximal compact subgroups. We define a function on the adeles by defining it on the local pieces,

$$\varphi_{\nu,s}\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} k\right) \quad := \quad \left|\frac{a}{d}\right|_\nu^s.$$

Finally we define φ_s as the map

$$(g_\nu)_\nu \mapsto \prod_\nu \varphi_{\nu,s}(g_\nu).$$

Lemma 4.4. *φ_s is an automorphic form on $P(\mathbb{Q})\setminus \mathrm{SL}_2(\mathbb{A})$.*

Proof. Smooth, moderate growth and K-finiteness are obvious from the definition. Using the product formula, i.e. for all $x \in \mathbb{Q}^\times$ we have that $\prod_\nu |x|_\nu = 1$, we get that φ_s is left $P(\mathbb{Q})$ invariant. $Z(\mathfrak{g})$ finiteness can be checked using the known Casimir of the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$, which we again omit.

To this we have an Eisenstein series associate as in the adelic setting by

$$E(\varphi,g,s) \quad := \sum_{\gamma\in P(\mathbb{Q})\setminus \mathrm{SL}_2(\mathbb{Q})} \varphi_s(\gamma g).$$

Lemma 4.5. *Let $g \in \mathrm{SL}_2(\mathbb{R})$ then we consider it as an element of $\mathrm{SL}_2(\mathbb{A})$, denoted by $\iota(g)$, by setting all other entries to 1. Then*

$$E(\varphi_s, \iota(g)) = \mathbf{E}(g.i, s)$$

Proof. First the left hand side,

$$\begin{aligned} E(\varphi_s, \iota(g)) &= \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{Q})} \varphi_s(\gamma \iota(g)) \\ &= \sum_{\gamma \in P(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{Z})} \prod_{\nu} \varphi_{\nu,s}(\gamma g_\nu) \\ &= \sum_{\gamma \in P(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{Z})} \varphi_{\infty,s}(\gamma g) \prod_{\nu<\infty} \varphi_{s,\nu}(\gamma) \\ &= \sum_{\gamma \in P(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{Z})} \varphi_{\infty,s}(\gamma g). \end{aligned}$$

Because $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{SL}_2(\mathbb{Z}_\nu)$ for each place ν and so $\varphi_{s,\nu}$ is by definition trivial on these. The final step is then to show that

$$\varphi_{\infty,s}(g) = |\mathrm{Im}(g.i)|^s.$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ then for some $k \in \mathrm{SO}_2(\mathbb{R})$ we have that [Conc],

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (a^2+c^2)^{\frac{1}{2}} & * \\ & (a^2+c^2)^{-\frac{1}{2}} \end{pmatrix} k.$$

With this explicit Iwasawa decomposition the proof is finished with some elementary matrix manipulation, this is done very explicitly in [Gar16, 3.3].

Chapter 5

Constant Terms of Eisenstein Series

This section is a discussion of the adelic constant term, especially its application to the Eisenstein series.

Through constant terms we can define cusp forms which play a central role in the theory of automorphic forms. They appear historically as interesting examples such as the Ramanujan tau function, by a theorem of Ribet [SZS77, T2.3] the Galois representation associated to a cusp form is irreducible and they form the "base case" for proofs, for example the spectral decomposition in [MW95].

Constant terms preserve analytic properties whilst sometimes reducing the functions to more tractable forms..

5.1 Definition and Role

Consider $P = MU$ a standard parabolic of a classical group G defined over a number field F and $\varphi : U(F) \setminus G(\mathbb{A}) \to \mathbb{C}$ a measurable and locally L^1 function then its **constant term** along P is defined to be [MW95, I.2.6],

$$\varphi_P : U(\mathbb{A}) \setminus G(\mathbb{A}) \to \mathbb{C},$$

$$\varphi_P(g) \quad := \quad \int_{U(k)\setminus U(\mathbb{A})} \varphi(ug)\mathrm{d}u.$$

Example 5.1. *Consider f a modular form of full level and weight k, which has a Fourier expansion given by*

$$f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}.$$

In section 2.3 we verified that

$$\tilde{f}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ci+d)^{-k} f\Big(\frac{ai+b}{ci+d}\Big),$$

is an Archimedean automorphic form on Sp_2. The only non-trivial standard parabolic P is the one *of upper triangular matrices, with Levi and unipotant given respectively*

$$M = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \cong \mathrm{GL}_1, \quad N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cong \mathbb{G}_a,$$

along which we can then compute the constant term, defined analogously in the Archimedean setting

$$\begin{aligned} \tilde{f}_P(m) &= \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \tilde{f}(mb)\mathrm{d}b \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} \tilde{f}\begin{pmatrix} m & mb \\ 0 & m^{-1} \end{pmatrix} \mathrm{d}b \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} m^k f(m^2 i + m^2 b)\mathrm{d}b \\ &= m^k a_0. \end{aligned}$$

We have chosen normalisation to remove the usual factor of $1/2\pi$ in the constant term of the Fourier series.

If φ is smooth or moderate growth then so is its constant term [MW95, I.2.6]. Moreover if φ is an automorphic form on $G(\mathbb{A})$ then its constant term along P is an automorphic form on $M(\mathbb{A})$ [GH24, 6.5].

Let φ be an automorphic form on $U(\mathbb{A})M(k) \setminus G(\mathbb{A})$ for $P = MU$ a standard parabolic. Then φ is **cuspidal** if for all standard parabolics $P' \subset P$ we have that $\varphi_{P'}$ is identically zero.

Theorem 5.2 ([MW95], I.4.10)**.** *Let $P = MU$ be a standard parabolic of G. If π is a cuspidal automorphic representation induced from P (see section 3.2), then for a fixed $\varphi \in \mathcal{A}_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\pi$ the Eisenstein series E can be thought of as a function from some open subset of the cuspidal datum $\mathfrak{P}$, the orbit of π (see section 4.1), into $L^2_{\mathrm{loc}}(G(\mathbb{A}))$ given by*

$$E(p)(g) = \sum_{\gamma \in P(k)\backslash G(k)} \lambda.\varphi(\gamma g), \quad p \in \mathfrak{P}, \ g \in G(\mathbb{A}),$$

where it converges. If $D \subseteq \mathfrak{P}$, is an open subset minus a finite number of points on which E is holomorphic then E has a holomorphic continuation to the finite number of points if and only if the constant term of E_Q has a holomorphic continuation to these finite number of points for all standard parabolics Q.

Remark 5.3. The theorem in Moeglin and Waldspurger is proved in much more generality, however after sufficient symbol pushing this is the essence.

So one can say that the poles of an Eisenstein series are controlled by its constant terms. We can say more:

Theorem 5.4 ([MW95], II.1.7)**.** *The constant term of an Eisenstein series induced from a standard maximal parabolic P is zero along any other standard parabolic P' i.e. if $P \neq P'$ then the constant term of the Eisenstein series along P' is zero.*

Putting these two theorems together we see that for an Eisenstein series induced from a maximal parabolic P, it has a holomorphic continuation around a point if and only if its constant term along P has a holomorphic continuation around that point.

5.2 Constant Terms of Eisenstein Series

This computation forms the heart of a well known theorem, [GH24, Prop 10.4.2] [MW95, II.1.7] [Sha10, 6.2]. Notice that the Eisenstein series has full $G(F)$-invariance and so we can take its constant terms along *any* standard parabolic.

Also note that we assume the computations are taking place in the domain of $\mathfrak{P}$ on which the Eisenstein series is given by the sum formula. By the uniqueness of meromorphic continuation taking constant terms commutes with meromorphic continuation.

5.2.1 In General

We will use the following Lemmas to give a simplified expression of the constant term of an Eisenstein series. Let G be a classical group over a number field F, fix a Borel B and fix $P = MU$ and $P' = M'U'$ two standard *maximal* parabolics. Let $E(x, \varphi, \lambda)$ be defined from P as in section 4.1.

If H is an algebraic group of the form $H = G_1 \times G_2$ where both G_1 and G_2 are classical in our sense then H is also split. Thus we can take any maximal split torus $T \subseteq H$ and defined the **Weyl group** of H

$$W_H \quad := \quad \mathrm{Norm}_{H(F)} T(F)/\mathrm{Cent}_{H(F)} T(F),$$

where $H(F)$ acts on $T(F)$ by conjugation. Note that this is independent of the choice of Torus. Now as we have seen in section 1.2.1 the Levi subgroups of maximal parabolics of classical groups are of the form $G_1 \times G_2$ for G_i classical groups. Thus we know how to talk about the Weyl groups of Levis of maximal parabolics of classical groups.

Finally we will use the common technique of unfolding

Theorem 5.5 (Unfolding, [Gar18] 5.2, [Fol16] Thm 2.49). *Let $H \leq G$ be a closed subgroup. If $H \setminus G$ has a right G invariant measure then the integral is unique up to scalar, namely for a given Haar measures dh on H and dg on G there is a unique invariant measure dq on $H \setminus G$ such that for all $f \in C_c^0(G)$*

$$\int_{H\setminus G} \int_H f(hq) dh dq = \int_G f(g) dg.$$

Note that this quotient may not be a group, because H is not required to be normal. The use of this lemma is called **unfolding** the integral.

Lemma 5.6.

$$P(F) \setminus G(F) \cong \coprod_{w \in W_{M'} \setminus W_G / W_M} P'(F) \cap wP(F)w^{-1} \setminus P'(F).$$

Proof. Consider the Bruhat decomposition:

$$G(F) = \coprod_{w \in W_{M'} \setminus W_G / W_M} P(F) w^{-1} P'(F).$$

Then because the action of $P(F)$ keeps the disjoint sets disjoint we can move the quotient through and get

$$P(F) \setminus G(F) = \coprod_w P(F) \setminus P(F) w^{-1} P'(F).$$

Analysing the summands, by the proof of the second isomorphism theorem for groups we have a bijection
$$P(F) \setminus P(F)w^{-1}P'(F) \cong P(F) \cap w^{-1}P'(F) \setminus w^{-1}P'(F).$$
Note that the group in the denominator still acts on $w^{-1}P'(F)$ *on the left*. Multiplication by w is a bijection that sends for $p' \in P'(F)$
$$[w^{-1}p'] = \{pw^{-1}p' : p \in P(F) \cap w^{-1}P'(F)\} \in P(F) \cap w^{-1}P'(F) \setminus w^{-1}P'(F),$$
to
$$\{wpw^{-1}p' : p \in P(F) \cap w^{-1}P'(F)\} \in wP(F)w^{-1} \cap P'(F) \backslash P'(F).$$

Thus we have that
$$w(P(F) \cap P'(F) \setminus w^{-1}P'(F)) \cong wP(F)w^{-1} \cap P'(F) \setminus P'(F).$$

Lemma 5.7 ([GH24], 10.4.1)**.** *There exists a set of representatives R for $W_{M'} \setminus W_G/W_M$ such that for $w \in R$ we have $w^{-1}P'w \cap M$ is a standard parabolic of M with Levi $w^{-1}M'w \cap M$ and unipotent $w^{-1}U'w \cap M$.*

Lemma 5.8 ([GH24], 10.4.1)**.** *For $w \in R$ as above*
$$w^{-1}U'w \cap P = (w^{-1}U'w \cap M)(w^{-1}U'w \cap U).$$

Lemma 5.9. *Let $m' \in M'(F), u' \in U'(F)$ then for $w \in R$ as above,*
$$m'u' \in wP(F)w^{-1} \iff m' \in wP(F)w^{-1} \text{ and } u' \in (m')^{-1}wP(F)w^{-1}m'.$$

Proof. The forward implication is stated in [GH24, Lem. 10.4.1], the converse follows from some algebra: First let $m' = wp_1w^{-1}$ and $u' = (m')^{-1}wp_2w^{-1}m'$ for $p_1, p_2 \in P(F)$ then
$$\begin{aligned} m'u' &= m'(m')^{-1}wp_2w^{-1}wp_1w^{-1} \\ &= wp_2p_1w^{-1} \in wP(F)w^{-1}. \end{aligned}$$

Taking the contrapositive of this lemma will be used below. This is because our sums will be over quotients like $A \setminus B$ and therefore summing over the “elements” in B that are not in A; by our lemma would be the same as summing over two different such quotients. Now we will apply our lemmas to simplify and make more explicit the constant term of an Eisenstein series. Denote

$[U'] := U'(F) \setminus U'(\mathbb{A})$ and fix a set of representatives R for $W_{M'} \setminus W_G / W_M$ as in lemma 5.7

$$
\begin{aligned}
E_{P'}(\varphi, \lambda, x) &= \int_{U'(F)\setminus U'(\mathbb{A})} E(\varphi, \lambda, nx)du \\
&= \int_{[U']} \sum_{\delta \in P(F)\setminus G(F)} \lambda.\varphi(\delta n x)du \\
\text{(Lemma 5.6)} \quad &= \int_{[U']} \sum_{\delta \in \coprod_{w\in R} P'(F)\cap wP(F)w^{-1}\setminus P'(F)} \lambda.\varphi(\delta u x)du \\
&= \sum_{w\in R} \int_{[U']} \sum_{p'\in P'(F)\cap wP(F)w^{-1}\setminus P'(F)} \lambda.\varphi(w^{-1}p'ux)du.
\end{aligned}
$$

Now apply Lemma 5.9 to the above sum to see that it is equal to

$$
\begin{aligned}
&\sum_{w} \sum_{m'\in M'(F)\cap wP(F)w^{-1}\setminus M'(F)} \int_{[U']} \sum_{u'\in U'(F)\cap (m')^{-1}wP(F)w^{-1}m'\setminus U'(F)} \lambda.\varphi(w^{-1}m'u'ux)du \\
\text{(Change Var)} \quad &= \sum_{w}\sum_{m'} \int_{[U']} \sum_{n'\in U'(F)\cap wP(F)w^{-1}\setminus U'(F)} \lambda.\varphi(w^{-1}u'um'x)du \\
\text{(Unfold)} \quad &= \sum_{w}\sum_{m'} \int_{U'(F)\cap wP(F)w^{-1}\setminus U'(\mathbb{A})} \lambda.\varphi(w^{-1}um'x)du.
\end{aligned}
$$

The change of variables is $(m', u') \mapsto ((m')^{-1}u'm', (m')^{-1}u'm')$.

5.2.2 Constant Terms of Cuspidal Eisenstein Series

So if R is a set of representatives for $W_{M'} \setminus W_G/W_M$ as given by lemma 5.7 we have seen so far that

$$
E_{P'}(\varphi, \lambda, x) = \sum_{w\in R} \sum_{m'\in M'(F)\cap wP(F)w^{-1}\setminus M'(F)} \int_{U'(F)\cap wP(F)w^{-1}\setminus U'(\mathbb{A})} \lambda.\varphi(w^{-1}um'x)du.
$$

Continuing the computation of the constant term above, we will focus purely on the inner integral now

$$
\begin{aligned}
&\int_{U'(F)\cap wP(F)w^{-1}\setminus U'(\mathbb{A})} \lambda.\varphi(w^{-1}um'x)du \\
&\qquad = \int_{w^{-1}U'(F)w\cap P(F)\setminus w^{-1}U'(\mathbb{A})w} \lambda.\varphi(uw^{-1}m'x)du \\
&\text{(Lemma 5.8)} = \int_{(w^{-1}U'w\cap M)(w^{-1}U'w\cap U)(F)\setminus w^{-1}U'(\mathbb{A})w} \lambda.\varphi(uw^{-1}m'x)du.
\end{aligned}
$$

where the first equality is the change of variables $w^{-1}uw \mapsto u$. Denote $A = (w^{-1}U'(F)w \cap U(F)) \setminus w^{-1}U'(\mathbb{A})w$. Unfolding we get the equality

$$
= \int_{(w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A}))\setminus A} \int_{w^{-1}U'(F)w\cap M(F)\setminus w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A})} \lambda.\varphi(u_1u_2w^{-1}m'x)du_1du_2.
$$

Now look at the inner integral here more closely

$$\int_{w^{-1}U'(F)w\cap M(F)\backslash w^{-1}U'(\mathbb{A})w\cap M(\mathbb{A})} \lambda.\varphi(u_1u_2w^{-1}m'x)du_1du_2,$$

applying Lemma 5.7 we see that this is a constant term for a parabolic of M, of the function $m \mapsto \varphi(mu_2w^{-1}m'x)$. This was in complete generality. If we now assume further that the Eisenstein series was induced from a *cuspidal* automorphic representation, then $m \mapsto \varphi(mk)$ is a cusp form and therefore this last integral will vanish whenever $w^{-1}U'w \cap M \neq \{1\}$, because in that case the inner integral doesn't exist (its over a point).

5.2.3 Constant Term Of Eisenstein Series for Conjugate Levis

If we now assume that $M' = wMw^{-1}$ for $w \in R$ and recall the definition of our intertwining operator from section 4.1 we can use the following

Lemma 5.10 ([MW95] II.1.7 (6)). *For $w \in R$ as above*

$$U'(k) \cap wP(k)w^{-1} = U'(k) \cap wU(k)w^{-1},$$

to see that

$$\begin{aligned} E_{P'}(\varphi, \lambda, x) &= \sum_w \sum_{m'} \int_{U'(F)\cap wP(F)w^{-1}\backslash U'(\mathbb{A})} \lambda.\varphi(w^{-1}um'x)du \\ &= \sum_w \sum_{m'} \int_{U'(k)\cap wU(k)w^{-1}\backslash U'(\mathbb{A})} \lambda.\varphi(w^{-1}um'x)du \\ &= \sum_w \sum_{m'} M(w,\pi)(\lambda.\varphi)(x). \end{aligned}$$

Recall that M was defined in section 4.1. In particular we can combine the conjugate and cuspidal cases to get a much simpler expression for some constant terms of some Eisenstein series, we will go through a detailed example in the final chapter 6.

Chapter 6

Poles of Residual Eisenstein Series

Our goal here is to exposit a small example that forms the heart of the work in papers such as [Bre09] [JLZ13].

[Bre09] gave an analysis of the residual poles of Eisenstein series attached to Sp_{2n}, there were some minor errors that were corrected in [JLZ13] where they give essentially the same proof; theirs however works for the other classical groups. To show the pattern we will focus on the case of Sp_{2n}, as an algebraic group defined over F a number field.

Theirs is a proof by induction and we will try to give the details of the base case, which is simply an explicit computation of a constant term.

6.1 Residual Eisenstein Series

So for the rest of the chapter we will fix an $n \in \mathbb{N}$ and $G_n = \mathrm{Sp}_{2n}$, then we look at partitions of $n = r + m$, where $1 \leq r, m \leq n$ and $r, m \in \mathbb{Z}$. Then as we saw in section 1.2.1 there corresponds a maximal standard (proper) parabolic of Sp_{2n}, which we denote $P_r = M_r U_r$, such that the Levi component is

$$\mathrm{GL}_r \times \mathrm{Sp}_{2m}.$$

As we saw in section 4.2 the space of characters $X_{M_r}^{\mathrm{Sp}_{2n}}$ is one dimensional by the maximality of P_r. We fix a τ, an irreducible unitary cuspidal automorphic representation of GL_r. Now we take an irreducible cuspidal automorphic representation σ of Sp_{2m}, then the tensor product $\tau \otimes \sigma$ gives a representation of $\mathrm{GL}_r \times \mathrm{Sp}_{2m}$ and hence of the Levi M_r. We now consider the Eisenstein series attached to this representation, namely if

$$\varphi \in \mathcal{A}(U_r(\mathbb{A})M_r(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))_{\tau\otimes\sigma},$$

then we have the Eisenstein series

$$E(\varphi, s)(g) = \sum_{\gamma \in P_r(F)\setminus \mathrm{Sp}_{2n}(F)} s.\varphi(\gamma g),$$

for $g \in \mathrm{Sp}_{2n}(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A})$. This is the base case of the setup in [JLZ13].

6.2 The Constant Term

So far we only know how to do one thing with such Eisenstein series and that is compute their constant term. We will compute the constant term along the maximal parabolic $P_r = M_rU_r$ because by [MW95, II.1.7 (ii)] the others are zero (see theorem 5.4).

By our earlier calculations in section 5.2, the fact that the tensor of cuspidal representations is cuspidal (elementary) and [JLZ13] we know that

$$E(\varphi, s)_{P_r} = \sum_{w \in W_{M'} \backslash W_G / W_M} \sum_{m' \in M_r(F) \cap w P_r(F) w^{-1} \backslash M_r(F)} \int_{U_r(F) \cap w P_r(F) w^{-1} \backslash U_r(\mathbb{A})} \lambda.\varphi(w^{-1} u m' x) du.$$

By [JLZ13] the inner integral vanishes for all $w \neq id, \omega$ where $\omega \in W_{\mathrm{Sp}_{2n}}$, this element is computed explicitly in [GRS11] and is

$$\omega \;\; := \;\; (-1)^r \begin{pmatrix} & & I_r \\ & I & \\ \pm I_r & & \end{pmatrix},$$

where I_a is the $a \times a$ identity matrix. Note that the $\pm$ is there to make sure the matrix is in Sp_{2n} and will in general depend on n and r. Hence the first sum becomes over two elements and we have

$$E(\varphi, s)_{P_r} = E(\varphi, s)_{P_r, id} + E(\varphi, s)_{P_r, \omega},$$

where

$$E(\varphi, s)_{P_r, w}(x) = \sum_{m' \in M_r(F) \cap w P_r(F) w^{-1} \backslash M_r(F)} \int_{U_r(F) \cap w P_r(F) w^{-1} \backslash U_r(\mathbb{A})} s.\varphi(w^{-1} u m' x) du.$$

First the identity term simplifies

$$\begin{aligned} E(\varphi, s)_{P_r, id}(x) &= \sum_{m' \in M_r(F) \cap P_r(F) \backslash M_r(F)} \int_{U_r(F) \cap P_r(F) \backslash U_r(\mathbb{A})} s.\varphi(u m' x) du \\ &= \sum_{m' \in M_r(F) \backslash M_r(F)} \int_{U_r(F) \backslash U_r(\mathbb{A})} s.\varphi(u m' x) du \\ &= \int_{U_r(F) \backslash U_r(\mathbb{A})} s.\varphi(u x) du \\ &= s.\varphi(x)_{P_r}. \end{aligned}$$

Note that because φ was an automorphic form that is $U(\mathbb{A})$ invariant we have in particular that

$$s.\varphi(x)_{P_r} = s.\varphi(x).$$

Considering now the ω term

$$E(\varphi, s)_{P_r, \omega}(x) = \sum_{m' \in M_r(F) \cap \omega P_r(F) \omega^{-1} \backslash M_r(F)} \int_{U_r(F) \cap \omega P_r(F) \omega^{-1} \backslash U_r(\mathbb{A})} s.\varphi(\omega^{-1} u m' x) du.$$

By [JLZ13, 2C] $M_r(F) \cap \omega P_r(F) \omega^{-1} \backslash M_r(F)$ is isomorphic to $P_0 \backslash \mathrm{Sp}_{2(n-a)}$, but P_0 has Levi $M_0 =$

$\mathrm{Sp}_{2(n-a)}$ by definition and hence is itself $\mathrm{Sp}_{2(n-a)}$. Thus the sum is over $\mathrm{Sp}_{2(n-a)}(F)\backslash\mathrm{Sp}_{2(n-a)}(F)$ and hence is over a point. Therefore we get by definition of the intertwining operator

$$E(\varphi,s)_{P_r,\omega}(x)=\int_{U_r(F)\cap\omega P_r(F)\omega^{-1}\backslash U_r(\mathbb{A})}\varphi(\omega^{-1}ux)du=M(\omega,s)(\varphi)(x),$$

because we took the constant term along the same parabolic as the definition of the Eisenstein series we know that the Levis are (the same) conjugate. Thus we have shown that

$$E(\varphi,s)_{P_r}=s.\varphi+M(\omega,s)(\varphi).$$

Because φ is an automorphic form it has no poles and so we have shown the following:

Lemma 6.1 (Base case of [JLZ13], 2.1)**.** *The poles of $E(\varphi,s)$ are exactly the poles of $E(\varphi,s)_{P_a}$ (see section 5.1) which are exactly the poles of $M(\omega,s)$.*

The poles of the $M(w,s)$ function of of great interest due to their relation to L–functions. We leave this to future work.

Appendix A

L-Functions

The theory of L-functions is not yet systematic; Langlands has provided a conjectural framework, however it is still under construction. In the mean time there are two major "paradigms" for constructing and proving theorems about L-functions, those are the Langlands-Shahidi type constructions and the Rankin-Selberg type constructions. General surveys can be found in [BC79, Part 2.III.2] [Sha10] [Cogd] [BCDS+04, 9, 10, 11] [Art].

The Rankin-Selberg type functions are surveyed in [Bum11]. The $\mathrm{GL}_n \times \mathrm{GL}_m$ case is dealt with in [Cogb]. For Rankin-Selberg L-functions of the form $\mathrm{Sp}_{2n} \times \mathrm{GL}_m$ the theory (for generic cuspidal representations) is worked out in [GRS98].

The Langlands-Shahidi paradigm is explained in [Sha90, Sha10].

We have by [Cogd] some properties uniquely determining L-functions for *tempered* representations. It is a conjecture that all generic representations are tempered, some work in this direction is in [Sha11], under this hypothesis we can apply the theory of Rankin-Selberg and Ginzburg-Ralis to explicitly construct global L-functions and prove theorems about them. In particular their analytic properties are well understood in these cases from [Grb11, Coga]. Note that [Grb11] is conditional on the unfinished work of Arthur [Art13].

A.1 The Langlands Framework

We follow closely Borels exposition in [BC79, Part 2, "Automorphic L-functions"] and [Sha10]. Given a reductive LAG G defined over $\mathbb{C}$ there is an associated root datum $(X, \Phi, \hat{X}, \hat{\Phi})$, where for any choice of maximal torus we have $X = \mathrm{Hom}(T, \mathbb{G}_m)$, $\hat{X} = \mathrm{Hom}(\mathbb{G}_m, T)$, and $\Phi, \hat{\Phi}$ are the roots and coroots of G with respect to T [Spr98, 7.4.3]. Then each reductive LAG G over a number field F has the root datum that is associated to the base change of G to $\mathbb{C}$, $(X, \Phi, \hat{X}, \hat{\Phi})$. By the existence theorem [Spr98, 10] to the dual root datum $(\hat{X}, \hat{\Phi}, X, \Phi)$ there is a LAG defined over $\mathbb{C}$ that corresponds, we call this the **dual group** of G and we denote it $\hat{G}$. It is possible through the use of the root datum to specify a "cannonical" action of $\mathrm{Gal}(\bar{F}/F)$ on $\hat{G}$ as in loc. cit. The **Langlands dual group** is then the dual group semi-direct producted with the $\mathrm{Gal}(\bar{F}/F)$ via this action, which we omit

$$^{L}G \quad := \quad \hat{G} \rtimes \mathrm{Gal}(\bar{F}/F).$$

Example A.1 (Classical Groups, [BCDS+04], 11.1)**.** *We have the following table*

G	$\hat{G}$
GL_n	GL_n
SO_{2n+1}	Sp_{2n}
SO_{2n}	SO_{2n}
Sp_{2n}	SO_{2n+1}

If ν is a non-archimedean place of F, then $\mathcal{O}_\nu$ is a local ring and we denote q_ν the cardinality of the residue field i.e. if $\mathfrak{p}_\nu$ is the unique maximal ideal of $\mathcal{O}_\nu$ then $q_\nu := [\mathcal{O}_\nu : \mathfrak{p}_\nu]$. Using the Satake isomorphism, to each unramified representation of $G(F_\nu)$ we can associate a conjugacy class of ${}^LG$, via some map call it c, and hence there is a way to apply a complex representation $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$ to unramified representations of $G(F_\nu)$, details in [Sha10, 2]. Given such an unramified representation of $G(F_\nu)$, call it π_ν, the local automorphic L-function is then

$$L_\nu(s, \pi_\nu, r) \quad := \quad \frac{1}{\det\bigl(I - r(c(\pi_\nu))q_\nu^{-s}\bigr)}, \qquad s \in \mathbb{C}.$$

In the global case we consider an irreducible automorphic representation $\pi = \otimes_\nu \pi_\nu$ of $G(\mathbb{A})$, and a finite set of places of F, call it S, such that S contains all infinite places and for all $\nu \notin S$ π_ν is unramified. Recall that we denoted the Langlands dual of G defined over F by ${}^LG$. We denote the Langlands dual of G defined over F_ν for $\nu \notin S$ by ${}^LG_{F_\nu}$. If r is a finite dimensional complex representation of ${}^LG$ then the embedding of Galois groups $\mathrm{Gal}(\bar{F}_\nu/F_\nu) \hookrightarrow \mathrm{Gal}(\bar{F}/F)$ induces a map ${}^LG_{F_\nu} \to^L G$ along which we can pull r back, giving a representation r_ν of ${}^LG_{F_\nu}$. Then the partial global L-functions are defined to be

$$L_S(s, \pi, r) \quad := \quad \prod_{\nu \notin S} L(s, \pi_\nu, r_\nu), \qquad s \in \mathbb{C}.$$

Example A.2 (Standard Representations / Classical Groups)**.** *In the case of classical groups it is common to see L-functions with only two entries e.g. if ρ is a representation of $G = \mathrm{Sp}_{2n}$ then you may see $L(s, \rho)$. The reason is that there is a standard representation of the dual groups of classical groups. Namely the standard representation of a matrix group inside GL_m is the one that sends $g \mapsto g$. It is this representation that is to be taken for the dual group in this setting.*

Example A.3 (Rankin-Selberg, [Cogb], 1.2, [AG91], Ch. 2 Example. 2)**.** *Let ν be a finite place of $\mathbb{Q}$ and π, π' be two unramified generic representations of $\mathrm{GL}_n(\mathbb{Q}_\nu)$ and $\mathrm{GL}_m(\mathbb{Q}_\nu)$ respectively. Let B_n be the standard Borel of upper triangular matricies in GL_n. Such representations have been classified in terms of characters of $\mathbb{Q}_\nu^\times$, in particular for π there are $\mu_1, ..., \mu_n$ unramified characters such that*

$$\pi \cong \mathrm{Ind}_{B(\mathbb{Q}_\nu)}^{\mathrm{GL}_n(\mathbb{Q}_\nu)} \bigl(\mu_1 \otimes \cdots \otimes \mu_n\bigr).$$

If we fix a uniformizer ϖ of $\mathbb{Q}_\nu$ then we have the so called "Satake parameters" $\mu_i(\varpi)$ which determines π uniquely. Of course the same is true for π', with say characters $\mu'_1, ..., \mu'_m$. We then define

$$L(s, \pi \times \pi') \quad := \quad \prod_{i,j} \frac{1}{1 - \mu_i(\varpi)\mu'_j(\varpi)q^{-s}}.$$

Consider the group $G = \mathrm{GL}_n \times \mathrm{GL}_m$ which has dual $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$, then there is a *cannonical representation*

$$\otimes : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_{nm}(\mathbb{C}).$$

Then

$$L(s, \pi \otimes \pi', \otimes) = L(s, \pi \times \tilde{\pi}'),$$

where the tilde denotes the contragradient.

Appendix B

Direct Integrals

B.1 Of Spaces

Consider a countable collection of Hilbert spaces $(\mathcal{H}_\alpha)_{\alpha\in A}$ then their direct sum is defined to be

$$\bigoplus_\alpha \mathcal{H}_\alpha \quad := \quad \left\{ (h_\alpha) \in \prod_\alpha \mathcal{H}_\alpha : \sum_\alpha \|h_\alpha\|_\alpha^2 < \infty \right\}$$

i.e. square summable sequences from the product. This is to ensure that the resulting space is still complete. If we recall that summing over a countable set is the same as *integrating* over that countable set when we equip it with the counting measure and discrete sigma algebra then this can be re-written as

$$\bigoplus_\alpha \mathcal{H}_\alpha = \left\{ (h_\alpha) \in \prod_\alpha \mathcal{H}_\alpha : \int_A \|h_\alpha\|_\alpha^2 d\alpha < \infty \right\}$$

This definition can be obviously generalised to an indexing set that is now an arbitrary measure space, $(A, \mathcal{M}, \mu)$. We need to make some technical arrangment to accompany this change, namely ensureing everything agrees with the measure structure, if we're to integrate we better only integrate *measurable* things. So now a collection $(\mathcal{H}_\alpha)_{\alpha\in A}$ along with a countable set of elements $e_j \in \prod_\alpha \mathcal{H}_\alpha, j \geq 1$ is called a measurable field over A if

$$\forall j, k \geq 1 \;\; \alpha \mapsto \langle e_j(\alpha), e_k(\alpha) \rangle$$

is measurable and for each $\alpha \in A$

$$span\{e_j(\alpha)\}_{j=1}^\infty \subseteq \mathcal{H}_\alpha$$

is dense; fixing an α and varying the j form a basis of each of the hilbert spaces, fixing the indecies and variying the α is measurable. An element $f \in \prod_\alpha \mathcal{H}_\alpha$ is called a measurable vector field if

$$\forall j \;\; \alpha \mapsto \langle f(\alpha), e_j(\alpha) \rangle_\alpha$$

is a measurable function. Note that we consider elements of the potentially uncountable product as functions from the indexing set into the relevant space (functions into the union of the hilbert spaces satisfying the property that $f(\alpha) \in \mathcal{H}_\alpha$). Now we define

$$\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha) \quad := \quad \left\{ f \in \prod_\alpha \mathcal{H}_\alpha : f \text{ is measurable and } \int_A \|f(\alpha)\|_\alpha^2 < \infty \right\}$$

Indeed this forms a Hilbert space. Note that a priori this construction depended on the basis (e_j) that we picked but up to isomorphism the basis doesnt matter.

B.2 Of Operators

We want to decompose representations and so we should look at how operators fit into this picture. We call an element

$$T \in \prod_\alpha \mathcal{L}(\mathcal{H}_\alpha)$$

a field of operators on A. It defines a linear map from $\prod_\alpha \mathcal{H}_\alpha$ to itself via

$$\left(\int^{\oplus} T \right) (f)(\alpha) \quad := \quad T(\alpha)(f(\alpha))$$

We say that it is measurable if for all measurable vector fields f the function

$$\alpha \mapsto \left(\int^{\oplus} T \right) (f)(\alpha)$$

is measurable. If moreover $\operatorname{ess\,sup}_\alpha \|T(\alpha)\| < \infty$ then $\int^{\oplus} T$ defines a bounded operator on $\int^{\oplus} \mathcal{H}_\alpha$.

B.3 Of Representations

Now we consider a group G and a collection of unitary represntations π_α on $\mathcal{H}_\alpha$ such that for every α and every $x \in G$

$$\alpha \mapsto \pi_\alpha(x)$$

is a measurable field of operators. We call such a collection a measurable field of representations; a G indexed collection of measurable fields of operators. From a measurable field of representations we get a unitary representation

$$\pi(x) \quad := \quad \int^{\oplus} \pi_\alpha(x)$$

of G on $\int^{\oplus} \mathcal{H}_\alpha$ which we call the direct integral of representations.

Theorem B.1. *If G is a second countable LCH group and π is a unitary rep of G on a seperable Hilbert space $\mathcal{H}$ and $\mathcal{B}$ is some (weakly closed) C^* subalgebra of $\mathrm{Hom}(\pi, \pi)$ then π is unitarily equivilent to the direct integral of some representations, which moreover act diagonally on the conjugation of $\mathcal{B}$ by the unitary isomorphism.*

So there is some measure space and measurable field of representations that can be combined to get almost any unitary representation, however we would like to know what these measure spaces are, how they interact with the irreps etc.

Note that one of the major peices of the proof is indeed **??** and hence one is justified in calling such a decomposition of the representation spectral, this direct integral is indeed a generalisation of our direct sum after all.

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