TRIG

θ (Rad)	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\pi/6$	$\sqrt{1}/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$\sqrt{1}/2$
$\pi/2$	1	0

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Identities:

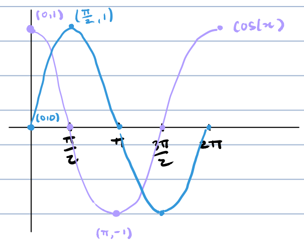
$$\sin(\theta)^2 + \cos(\theta)^2 = 1$$

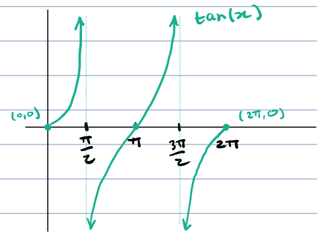
$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

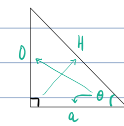
$$\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$



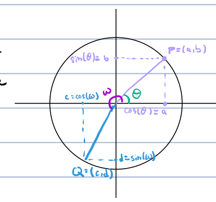




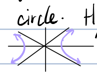
$\sin(\theta) = \frac{O}{H}$

$\cos(\theta) = \frac{a}{H}$

Always take the angle from the positive x axis.



Hyperbolic Trig Functions:

Normal trig functions come from the angle of the unit circle. Hyperbolic ones come from the hyperbola. 

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} = -i\sin(ix)$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} = \cos(ix)$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = -i\tan(ix)$$

SEQENCES & SERIES

A complex sequence $\{z_n\}_{n\geq 1}$ converges iff $\{Re(z_n)\}_{n\geq 1}$ AND $\{Im(z_n)\}_{n\geq 1}$ converge.

A complex series $\sum_{n\geq 1} z_n$ converges iff $\sum_{n\geq 1} Re(z_n)$ AND $\sum_{n\geq 1} Im(z_n)$ converge.

Thus we can, without using ε-N proofs show convergence/divergence using only the corresponding Real sequences/series.

Real Tests of Convergence:

Sequences:

① For a sequence $a_n = \dfrac{f(x)}{g(x)}$ will converge or diverge depending on the growth of f & g.

$$\sqrt{n} < n < n^2 < 2^n < e^n < n! < n^n$$

② For a sequence $\{a_n\}_{n\geq 1} : \mathbb{N} \to \mathbb{R}$ if we have an $f: \mathbb{R} \to \mathbb{R}$ such that $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$

$\Rightarrow \lim_{n\to\infty} a_n = L$

Likewise for divergence.

Note that if f is differentiable we can use L'Hopitals to show its convergence: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$ when the first limit is indeterminate; & $g'(x)$ is not 0 around c.

③ If $\forall n > N \in \mathbb{N}$, $a_n \leq c_n \leq b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \Rightarrow \lim_{n\to\infty} c_n = L$.

④ $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

⑤ If $\{a_n\}_{n\geq 1}$ is bounded & monotonic it converges

⑥ If $\{a_n\}_{n\geq 1}$ has 2 (or more) convergent subsequences $\{a_{n_j}\}_{j\geq 1} \to L$ & $\{a_{n_i}\}_{i\geq 1} \to L'$ where $L \neq L' \Rightarrow \{a_n\}_{n\geq 1}$ diverges

⑦ A sequence converges $\Leftrightarrow$ The sequence is Cauchy.

Key Examples:

$\{r^n\}_{n\geq 1}$ converges $\Leftrightarrow r \in (-1, 1]$

Series:

For $\sum_{n\geq 1} a_n$ & $\sum_{n\geq 1} b_n$

① Divergence Test: $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n\geq 1} a_n$ Diverges

② Integral Test:

f continuous, positive, decreasing on $[1, \infty)$ with $a_n = f(n)$ $\Rightarrow \left[\sum_{n\geq 1} a_n \text{ converges} \Leftrightarrow \int_1^\infty f(x)\,dx \text{ converges}\right]$

③ Comparison: If $a_n, b_n > 0 \ \forall n$:

$\sum b_n$ converges, $(\forall n)(a_n \leq b_n) \Rightarrow \sum a_n$ converges

$\sum b_n$ diverges, $(\forall n)(a_n \geq b_n) \Rightarrow \sum a_n$ diverges

④ Limit Comparison: If $a_n, b_n > 0 \ \forall n$:

$\lim_{n\to\infty} \frac{a_n}{b_n} > 0 \Rightarrow (\sum a_n \text{ conv} \Leftrightarrow \sum b_n \text{ conv})$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, $\sum b_n$ conv. $\Rightarrow \sum a_n$ converges

$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, $\sum b_n$ div. $\Rightarrow \sum a_n$ diverges

⑤ Alternating Test: If $a_n = (-1)^{n+1} b_n$, for $b_n > 0 \ \forall n$ & b_n decreasing then $\left[\lim_{n\to\infty} b_n = 0 \Rightarrow \sum a_n \text{ converges}\right]$

⑥ Absolute Convergence:

$\sum |a_n|$ converges $\Rightarrow \sum a_n$ converges

⑦ Ratio Test:

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum a_n$ converges

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \sum a_n$ diverges

Note if the limit here is 1 the test is indeterminate.

⑧ Root test:

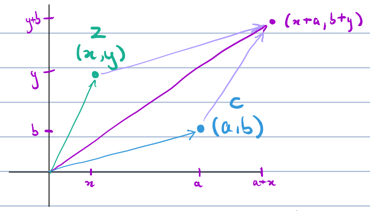
$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1 \Rightarrow \sum a_n$ converges

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} > 1 \Rightarrow \sum a_n$ diverges

Key Examples:

- Taylor series of functions
- $\sum_{n\geq 0} x^n = \frac{1}{1-x}$, $|x| < 1$ (Geometric)
- $\sum_{n\geq 0} (-1)^n x^n = \frac{1}{1+x}$, $|x| < 1$
- $\sum_{n\geq 1} \frac{1}{n} = \infty$ Diverges (Harmonic)

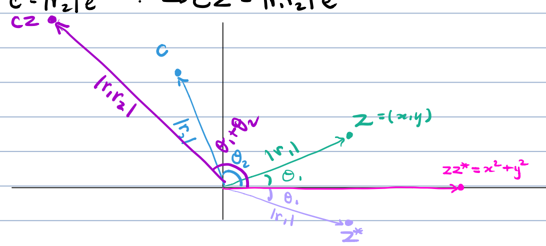
BASIC ℂ GEOMETRY



Complex numbers add / subtract like normed vectors in $\mathbb{R}^2$.

Now consider $z = |r_1| e^{i\theta_1}$ and

$c = |r_2| e^{i\theta_2}$. $\Rightarrow cz = |r_1 r_2| e^{i(\theta_1+\theta_2)}$



DIFFERENTIATION:

A complex function $f: S \to \mathbb{C}$ where $S \subset \mathbb{C}$ open is complex differentiable at $c \in S$ iff the limit exists

$$f'(c) = \lim_{z \to c} \frac{f(z) - f(c)}{z - c}$$

Common Functions:

Polynomials $\sum c_j z^j$, $c_j \in \mathbb{C}$ \
Exponential e^z } Entire functions $\longrightarrow$ Finite sums, products & compositions are also entire.

Integrals & derivatives are entire.

$\downarrow$

cos, sin, cosh, sinh $\longleftarrow$ As linear combinations of e^z

Square Root $\sqrt{z} = e^{\frac{1}{2}\log(z)}$ $\Rightarrow$ Holomorphic where $\log(z)$ is.

Rational functions $P(z)/Q(z)$ Holomorphic on $\mathbb{C}$ whenever $Q(z) \neq 0$
(special case is $1/z$)

INTEGRATION:

INTEGRATION

- Proof of regular arcs having length given by $\int |T'(t)| dt$.
- Algorithm for parametrisation. Common parametrizations.
- Define $\int$ on $\mathbb{C}$. What Riemann integration is formally. Then common complex integrals & simplifying rules.
  - ↳ linearity ↳ commuting a constant
    - proof of these.

↳ List of antiderivatives over $\mathbb{C}$.

↳ compare with Real counterpart.

What is antidiffereventiatien over $\mathbb{C}$? When is it possible? → Necissary & sufficient conditions

contour integrals as an application of complex integration.
Need to be able to
- Parametrise a contour
- Evaluate arbitrary complex integral

- Integration techniques in $\mathbb{R}$
- Def of Rieman integration.

POLYNOMIAL LONG DIVISION

$P(x) = \sum_{i=0}^{n} a_i x^i$, $n > m$

$Q(x) = \sum_{i=0}^{m} b_i x^i$

Write it down as $Q(x)\,\overline{)\;P(x)}$ and follow this algorithm...

① Divide the first term of the numerator by the first term of the denominator & put that in the answer $A(x)$

② Multiply the denominator by that answer to get $\lambda(x)$

③ Subtract $\lambda(x)$ from the numerator $P(x)$ & then let $P'(x) = P(x) - \lambda(x)$

④ Repeat using $P'(x)$

$$\begin{array}{r} x \\ x-2\;\overline{)\;x^2+2x-7} \\ \frac{x^2}{x} \end{array}$$

$$\begin{array}{r} x(x-2) \\ x-2\;\overline{)\;x^2+2x-7} \\ x^2-2x \end{array}$$

$$\begin{array}{r} x \\ x-2\;\overline{)\;x^2+2x-7} \\ \underline{x^2-2x+0} \\ 0 \quad 4x-7 \end{array}$$

Repeat...

$$\begin{array}{r} x+4(x-2)\;② \\ x-2\;\overline{)\;x^2+2x-7} \\ x^2-2x \\ 4x-7 \\ ①\;\frac{4x}{x} \\ 4x-8 \\ ③\;\overline{0+1} \end{array}$$

Thus

$$\frac{x^2+2x-7}{x-2} = x+4+\frac{1}{x-2}$$

PARTIAL FRACTIONS

For arbitrary $\frac{P(x)}{Q(x)}$

- First ensure that the degree of the numerator $P(x)$ is less than the degree of $Q(x)$.

If NOT perform polynomial long division to get $P'(x)$ such that this is the case.

Next factorise $Q(x)$. Because we are factoring over $\mathbb{C}$ there should only be linear factors.

$Q(x)$ will be composed of factors of form $(x-a)^n$ $n \in \mathbb{N}$.

$$\frac{P'(x)}{Q(x)} = \frac{P'(x)}{\prod_{i=0} (x-a_i)^{n_i}}$$

Now each factor in the denominator of $(x-a_i)^{n_i}$ will expand to...

$$\sum_{j=1}^{n_i} \frac{A_j}{(x-a_i)^j}$$

Not just the power but all below too.

Thus

$$P'(x) = Q(x)\left[\sum_{i=0} \left(\sum_{j=1}^{n_i} \frac{A_j}{(x-a_i)^j}\right)\right]$$

Then expand the RHS & compare coefficients with $P(x)$ to get a system of equations to solve for the A_j.