Fixed Point Theorem

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The rules for LAST can be found in [2].
The proof was expanded on from [1].

Lemmas

y is what in A?

0.1 Properties of Tensor in LAST

Recall the in LAST the tensor of two formulas is defined via an arbitrary (but fixed) closed formula Θ:

$$A \otimes B \equiv \forall x.((A \multimap B \multimap \Theta \in x) \multimap \Theta \in x)$$

We will now prove the standard linear logic deduction rules for these terms in order to use them as derived rules:

Lemma 1

$$\dfrac{\Delta, B_1, B_2 \vdash \Gamma}{\Delta, B_1 \otimes B_2 \vdash \Gamma}\, L\otimes \qquad\qquad \dfrac{\Delta_1 \vdash B_1, \Gamma_1 \qquad \Delta_2 \vdash C, \Gamma_2}{\Delta_1, \Delta_2 \vdash B \otimes C, \Gamma_1, \Gamma_2}\, R\otimes$$

Lemma 2

$$A \otimes B \vdash B \otimes A$$

Lemma 3 *Tensor Cut: Given a proof of $A \vdash B \otimes C$ and a proof of $B \vdash D$ we can construct a proof of of $A \vdash D \otimes C$. We use this as the derived rule*

$$\dfrac{A \vdash B \otimes C \qquad B \vdash D}{A \vdash D \otimes C}\, \textit{tensor cut}$$

Proof.

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ A \vdash B \otimes C\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ B \vdash D\end{array} \qquad \dfrac{}{C \vdash C}\,\text{ax}}{B, C \vdash D \otimes C}\, R\otimes}{B \otimes C \vdash D \otimes C}\, L\otimes}{A \vdash D \otimes C}\,\text{cut}$$

Lemma 4

$$A \vdash B \otimes A$$

0.2 Properties of Equality in LAST

Proof.

Lemma 5

$$x = y \vdash y = x$$

Lemma 6

$$t = u, A[t/x] \vdash A[u/x]$$

0.3 Existence and Consistency

Lemma 7

$$A[t/x] \vdash \exists x.A$$

Lemma 8 *LAST is consistent*

Lemma 9 *If there is a proof of $\exists x.A$ then there is a term t and a proof of $A[t/x]$*

Fixpoint Theorem

Theorem 1 *Fixed-point Theorem: For all formulas, A, in LAST there is a term, f, such that both*

$$x \in f \vdash A[f/y] \quad \textit{and} \quad A[f/y] \vdash x \in f$$

are derivable.

Proof. $\boldsymbol{x \in f \vdash A[f/y]}$**:**
Take fresh variables u, v, w and define the following terms

$$s \equiv \{z : \exists u \exists v (z = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x])\} \equiv \{z : B\}$$

$$f \equiv \{w : \langle w, s\rangle \in s\}$$

Now observe the following proof

$$\dfrac{\dfrac{\overline{\langle x, s\rangle \in s \vdash \langle x, s\rangle \in s}\ \text{(ax)}}{x \in f \equiv x \in \{w : \langle w, s\rangle \in s\} \vdash \langle x, s\rangle \in s}\ (\in L) \qquad \dfrac{\overline{B[\langle x, s\rangle/z] \vdash B[\langle x, s\rangle/z]}\ \text{(ax)}}{\langle x, s\rangle \in s \equiv \langle x, s\rangle \in \{z : B\} \vdash \langle x, s\rangle \in s}\ (\in L)}{x \in f \vdash B[\langle x, s\rangle/z] \equiv \exists u \exists v(\langle x, s\rangle = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x])}\ \text{cut}$$

The consistency of LAST ([2]) tells us that if there is a proof of an existentially quantified LAST term, $\exists x.A$, then there exists a term in LAST, t, and a proof in LAST of the term $A[t/x]$. Thus what we have shown above along with consistency proves that there is a LAST proof π

$$\begin{array}{c} \pi \\ \vdots \\ x \in f \vdash \langle x, s\rangle = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x] \end{array}$$

Then we have the following proof in LAST

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi \\ \vdots \\ x \in f \vdash \langle x, s\rangle = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x]\end{array} \qquad \dfrac{}{\langle x, s\rangle = \langle u, v\rangle \vdash x = u \otimes s = v}\ \text{lemma}}{x \in f \vdash x = u \otimes s = v \otimes A[\{w : \langle w, v\rangle \in v\}/y, u/x]}\ \text{tensor cut} \qquad \dfrac{}{x = u, A[\{w : \langle w, v\rangle \in v\}/y, u/x] \vdash A[\{w : \langle w, v\rangle \in v\}/y, x/x] \equiv A[\{w : \langle w, v\rangle \in v\}/y]}\ \text{lemma}}{x \in f \vdash s = v \otimes A[\{w : \langle w, v\rangle \in v\}/y]}\ \text{tensor cut} \qquad \dfrac{}{s = v, A[\{w : \langle w, v\rangle \in v\}/y] \vdash A[\{w : \langle w, s\rangle \in s\}/y]}\ \text{lemma}}{x \in f \vdash A[\{w : \langle w, s\rangle \in s\}/y] \equiv A[f/y]}\ \text{tensor cut}$$

Note that the second L2 rule is as desired becuase v is not free in A by assumption.

The reverse direction: $\boldsymbol{A[f/y] \vdash x \in f}$**:**

$$\dfrac{\overline{A[f/y] \vdash A[f/y]}\ \text{ax}}{A[f/y] \vdash \langle x, s\rangle = \langle x, s\rangle \otimes A[\{w : \langle w, s\rangle \in s\}/y]}$$

Then by a Lemma we have that $A[f/y] \vdash \exists u \exists v(\langle x, s\rangle = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y])$ is provable. Giving

$$\dfrac{\dfrac{\begin{array}{c}\pi \\ \vdots \\ A[f/y] \vdash \exists u \exists v(\langle x, s\rangle = \langle u, v\rangle \otimes A[\{w : \langle w, v\rangle \in v\}/y])\end{array}}{A[f/y] \vdash \langle x, s\rangle \in s}\ \in R}{A[f/y] \vdash x \in \{w : \langle w, s\rangle \in s\} \equiv x \in f}\ \in R$$

$\square$

References

[1] M. Shirahata. "Fixpoint Theorem in Linear Set Theory". 1999. URL: `https://www.fbc.keio.ac.jp/~sirahata/Research/fixpoint.pdf`.

[2] Kazushige Terui. "Light Affine Set Theory: A Naive Set Theory of Polynomial Time". In: *Studia Logica* 77.1 (June 2004), pp. 9–40. ISSN: 0039-3215. DOI: `10.1023/B:STUD.0000034183.33333.6f`. URL: `http://link.springer.com/10.1023/B:STUD.0000034183.33333.6f` (visited on 10/16/2022).