Cut Elimination on Russells Paradox

October 13, 2022

We will first represent set theory with unrestricted comprehension in sequent calculus. Then we can write a proof of a contradiction (using Russells paradox). Finally we will attempt cut elimination and observe that it does not terminate.

Representing the Paradox

Consider classical sequent calculus with $\neg, \wedge, \vee$, contraction and weakening[3]. Onto this we will add the following deduction rules that correspond to unrestricted comprehension [1].

$$\dfrac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, t \in \{x : A(x)\}}\ \text{(comp1)} \qquad\qquad \dfrac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, t \in \{x : A(x)\} \vdash \Delta}\ \text{(comp2)}$$

Now consider the following wff in naive set theory

$$A := x \notin x, \qquad R := \{x : A(x)\} = \{x : x \notin x\}$$

Combining these with our deduction rules from comprehension we get the following

$$\dfrac{\Gamma \vdash \Delta, A[R/x] \quad (= R \notin R)}{\Gamma \vdash \Delta, R \in \{x : A(x)\} \quad (= R \in R)}\ \text{(comp1)} \qquad \dfrac{\Gamma, A[R/x] \quad (= R \notin R) \vdash \Delta}{\Gamma, R \in \{x : A(x)\} \quad (= R \in R) \vdash \Delta}\ \text{(comp2)}$$

In other words we can deduce from $R \notin R$ that $R \in R$ on either side of the turnstile. This allows us to write a proof of a contradiction as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{R \in R \vdash R \in R}\ \text{(ax)}}{\vdash R \in R, R \notin R}\ (\neg)}{\vdash R \in R, R \in R}\ \text{(comp1)}}{\vdash R \in R}\ \text{(con)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{R \notin R \vdash R \notin R}\ \text{(ax)}}{R \notin R, R \in R \vdash}\ (\neg)}{R \in R, R \in R \vdash}\ \text{(comp2)}}{R \in R \vdash}\ \text{(con)}}{\vdash}\ \text{(cut)}}{\vdash B}\ \text{(weak)}$$

Performing Cut Elimination

The idea of cut elimination is to iteratively cut earlier in the proof until you are cutting against only axiom rules and the cut disappears [2]. In the usual case this is a terminating process.

In particular there is an algorithm with deep connections to the normalisation of lambda terms for the elimination of the cut from sequent calculus. The rules for this cut elimination algorithm are given by [5]. We will denote an equivalent proof under the cut elimination equality (defined in [5]) by $\sim$.

For simplicity denote

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Then we use the first cut elimination rule on the right contraction giving

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We have introduced an extra logical rule not considered in [5] and as such we need to think about how we might edit our cut elimination algorithm to include cutting against the new logical rules comp1 and comp2.

By divine inspiration we will just use the following rule

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\UnaryInfC{$\Gamma, A[t/x] \vdash \Delta$}
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These rules are effectively the rules for cut elimination on (universal) quantifiers given in [3], however it is always hard to justify an incorrect choice.

Given this rule however we are now able to proceed with our cut elimination process.

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But this is precisely the original proof. Thus it becomes clear that the rules of cut elimination do not work in this setting.

Curry Howard Correspondence

Cut elimination corresponds to β reduction under a Curry-Howard type map [4]. It has been shown that for certain sequent calculus that the correspondence implies that cut elimination is strongly normalisable [6]. i.e. every reduction strategy following some set of rules will terminate.

The strong normalisation property is good reason (as well as the symmetry of the proof) that we needn't look at cut eliminating the other side.

It is clear then that this is not such a system because we have exhibited a reduction strategy that can be carried out indefinitely.

References

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