Homology Theory

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1 Of Chains

Consider the category $\mathbb{Z}$ with objects natural numbers and a morphism between successive numbers. This is the diagram

$$\cdots \to \bullet \to \bullet \to \bullet \to \bullet \to \cdots$$

A functor from this category to another $\mathcal{C}$ gives a $\mathbb{Z}$ shaped diagram in $\mathcal{C}$. Thus we have that $\mathrm{Ch}(\mathcal{C})$ is the subcategory of $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ such that the image of successive maps compose to 0. This requires $\mathcal{C}$ to have a zero object. In this language we can formulate the relation between *homology* and *cohomology*. Namely the homology of a chain, element of $\mathrm{Ch}(\mathcal{C})$, is taking the kernel modulo the image of successive maps.

A cochain is a map from $\mathbb{Z}^{\mathrm{op}}$, such that the image of successive maps compose to 0. $\mathbb{Z}^{\mathrm{op}}$ is the diagram

$$\cdots \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$$

The cohomology of a cochain is then the kernel modulo the image of successive maps.

It is clear that $\mathbb{Z}^{\mathrm{op}}$ is equivilent to our diagram $\mathbb{Z}$. This is precisely stated as the equivilence of categories between $\mathbb{Z}$ and $\mathbb{Z}^{\mathrm{op}}$, given by

$$n \mapsto -n.$$

This gives us a bijection

$$\mathrm{Fun}(\mathbb{Z}, \mathcal{C}) \cong \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$$

given by pulling back along this equivilence of categories.

$$\begin{tikzcd}
\mathbb{Z}^{\mathrm{op}} \arrow[r, "\times -1"] \arrow[dr, dashed] & \mathbb{Z} \arrow[d, "F"] \\
& \mathcal{C}
\end{tikzcd}$$

All of this is to say taking the homology of a chain is the same as pulling back and then taking the cohomology of that cochain.

How does this translate into axioms for cohomology.

Coming by Chains: The difference then is not at the level of chains. It is at the level of the *assignment of chains*. Below we will speak of a homology as a functor

$$\mathrm{CW} \to \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$$

we want cohomology to be a functor

$$\mathrm{CW}^{\mathrm{op}} \to \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$$

These are genuinely different. Thus there is an asymmetry, taking cohomology at the level of chains is the same as taking homology, but a cohomology theory is genuinely different to taking homology *of a space*.

Homology of Chains Elsewhere we have discussed that there is an equivilience of categories

$$\mathrm{Gr}_{\mathbb{Z}}\mathrm{Ab} \cong \mathrm{Fun}(\mathbb{Z}, \mathrm{Ab}) \cong \mathrm{Ch}_{\partial=0}(\mathrm{Ab})$$

that is between $\mathbb{Z}$ graded Abelian groups, functors from the set $\mathbb{Z}$ considered as a category into Abelian groups and the subcategory of chain complexes where all the boundary maps are zero.

For singular and cellular homology we start with the relevant chain complex and take the homology of the chain, that is define the groups to be

$$H_n(C_*) = \ker(\partial_n)/\mathrm{Im}(\partial_{n+1}).$$

This gives us another chain

$$\cdots \to H_n \xrightarrow{[\partial_n|_{\ker(\partial_n)}]=0} H_{n-1} \to \cdots$$

with the maps on homology by definition being trivial as we first *restrict to the kernel* and then descend to the quotient. Thus by our equivilence we may as well consider the homology as merely a sequence of Abelian groups or equivilently as a $\mathbb{Z}$ graded abelian group. This is what we will axiomatise.

2 Axiomatically

2.1 Relative Homology

The axiomatic notion of a **(unreduced relative) homology** is a pair of functors

$$h_* : \mathrm{pCW} \to \mathrm{Gr}_{\mathbb{Z}}\mathrm{AbGroup}$$

$$\partial : \mathrm{pCW} \to \mathrm{Arr}(\mathrm{AbGroup})$$

the first called the homology and second called the boundary; they must further satisfy the following axioms, where as a shorthand we denote $f_* \;\coloneqq\; h_*(f)$ and also $h_n(X) \;\coloneqq\; h_n(X, \emptyset)$,

- (Homotopy Invariance) If $f \simeq g$ then $f_* = g_*$.
- (Exactness) For each CW pair $A \subseteq X$ the boundary map is an arrow $\partial = \partial(X, A) : h_n(X, A) \to h_{n-1}(A)$ and moreover fits into a long exact sequence

  $$\cdots \xrightarrow{\partial} h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{j_*} h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \xrightarrow{i_*} \cdots$$

  where $i : A \to X$ is the inclusion and $j : (X, \emptyset) \to (X, A)$ is the inclusion.
- (Excision) Given a triple $U \subseteq A \subseteq X$ such that the closure of U is continained in the interior of A then the inclusion

  $$i : (X \backslash U, A \backslash U) \to (X, A)$$

  induces an iso on homology.
- (Additivity) If $i_\alpha : X\alpha \hookrightarrow \sqcup_\alpha X_\alpha$ is the inclusion then

  $$\oplus i_{\alpha *} : \oplus_\alpha h_*(X_\alpha) \to h_*(\sqcup_\alpha X_\alpha)$$

  is an isomorphism.

Note: In this case one can formulate this as a family of functors h_n from pairs to abelian groups and a natural transformation $\partial : h_n \to h_{n-1} \circ \mathrm{res}$ where $\mathrm{res}(X, A) \mapsto (A, \emptyset)$, stands for restriction.

Note: Finite additivity follows from the first two axioms, however we need *infinite additivity.* If you are dealing with finite CW complexes then this additivity axiom may then be omitted. proof?

Note: From a theory as above we can get an "absolute" homology by taking $h_n(X) \;\coloneqq\; h_n(X, \emptyset)$. Such a homology is called an absolute unreduced homology.

2.2 Reduced Homology

A **absolute reduced homology** theory is a pair of functors

$$\tilde{h}_* : \mathrm{CW} \to \mathrm{Gr}_{\mathbb{Z}}\mathrm{AbGroup}$$

$$\partial : \mathrm{pCW} \to \mathrm{Arr}(\mathrm{AbGroup})$$

the first called the homology and second called the boundary; they must further satisfy the following axioms, where as a shorthand we denote $f_* := \tilde{h}_*(f)$,

- (Homotopy Invariance) If $f \simeq g$ then $f_* = g_*$.
- (Exactness) For each CW pair $A \subseteq X$ the boundary map is an arrow $\partial = \partial(X, A) : \tilde{h}_n(X/A) \to \tilde{h}_{n-1}(A)$ and moreover fits into a long exact sequence

  $$\cdots \xrightarrow{\partial} \tilde{h}_n(A) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1}(A) \xrightarrow{i_*} \cdots$$

  where $i : A \to X$ is the inclusion and $q : X \to X/A$ is the quotient.
- (Additivity) If $i_\alpha : X\alpha \hookrightarrow \vee_\alpha X_\alpha$ is the inclusion then

  $$\oplus i_{\alpha *} : \oplus_\alpha \tilde{h}_*(X_\alpha) \to \tilde{h}_*(\vee_\alpha X_\alpha)$$

  is an isomorphism.

track why taking the kernel or whatever introcduces the wedge sum

Note: The axioms we have given here do not include the so called "dimension axiom", that is that $H_n(*) = 0$ for all $n \neq 0$, that is the homology of a point space is only non-trivial in the zeroth degree. What we have defined is sometimes called a generalised or extrodinary homology theory.

Note: A reduced theory is supposed to be a homology *relative to a point*. Thus it is by definition an absolute theory. This is made precise below.

2.3 Comparing Them

From an unreduced homology theory h we get a reduced homology theory by taking

$$\tilde{h}_*(X) \quad := \quad \ker\big(h_*(X \to *)\big)$$

From a reduced homology theory $\tilde{h}_*$ we can produce an unreduced homology theory by taking

$$h_*(X) \quad := \quad \tilde{h}_*(X \sqcup \{pt\})$$

Thus we see something of the form

Lemma. *These operations are inverses to one another.*

Lemma. *If h_* and $\tilde{h}_*$ are related in this way we get that for any $x \in X$*

$$h_*(X) \cong \tilde{h}_*(X) \oplus h_*(x).$$

We say that $h_*(x) = \tilde{h}_*(S^0)$ are called the **coefficients** of the theory.

Note: It is at this stage really not clear what we get from the axiomatic approach. In particular we can prove a Mayer-Vietoris sequence for reduced homology. That seems like the only thing from the other singular that we can "recover". With the dimension axiom we are able to compute the homology of some very simple spaces but without it we cannot.

Ok you can get maybe some homological algebra results, like showing spectral or long exact sequences, that will save you repeating proofs.

It also provides an interesting moduli problem, to study the class of all homology theories, and provide some connective tissue between several functors.

There are more comparison isomorphisms between reduced, absolute and relative *singular* homology too, do these hold axiomatically?

3 Singular Homology

Note: When doing homotopy thoery it is perhaps most natural to consider mapping spheres into a space. When doing homology there are reasons why we want to look at mapping simplexes into a space. One good reason is that we want the boundary of our space to be another version of that space, i.e. the boundary of a simplex is a simplex, the boundary of a disk is not a disk. Although the details probably can be worked out they at present have not.

Because simplicies and disks are homeomorphic there is no "real" difference and the advantages will be purely formal, in how easy the theory is to state and intuite.

Simplex: The standard n-simplex is a "higher dimensional triangle"

$$\Delta^n \;\;:=\;\; \{t \in \mathbb{R}^{n+1} : \sum_i t_i = 1, t_i \geq 0\}$$

The verticies of this simplex are the unit vectors in $\mathbb{R}^{n+1}$, and the simplex is the surface connecting them.

A **singular n-simplex** in X is a map $\sigma : \Delta^n \to X$. Given a chain of abelian groups, a sequence of maps

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

such that $\partial^2 = 0$, that is $\mathrm{Im}(\partial_n) \subseteq \ker(\partial_{n-1})$, then we know how to take the homology of the chain by

$$H_n(C_*) = \ker(\partial_{n+1})/\mathrm{Im}(\partial_n)$$

The singular chain associated to a space is then given by

$$C_n(X) \;\;:=\;\; \mathbb{Z}[\text{singular n-simplex's}]$$

that is the free abelian group on the n-simplexes, that is *finite* formal sums of n-simplexes. We can change $\mathbb{Z}$ to a different abelian group G, called the coefficients.

Elemenents of $C_n(X)$ are called n-chains. *This group is in general huge.* We also need to define the boundary maps.

$$\partial : C_n(X) \to C_{n-1}$$

is defined to be the linear extension of

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_1,...,\hat{v_i},...,v_n]}$$

where $\hat{}$ denotes deleting this vertex. We have to make an identification here between $[v_1, ..., \hat{v_i}, ..., v_n]$ and a the standard $n-1$ simplex, which makes $\sigma|_{[v_1,...,\hat{v_i},...,v_n]}$ an $n-1$ simplex. Moreover this requires us to fix an ordering on the simplexes, essentially an orientation.

The elemenets of the kernel of ∂_n are called cycles and the elements of the image are called boundaries.

This defines a functor because given $f : X \to Y$ we get a chain map

$$C_n(X) \to C_n(Y)$$

$$\sigma \mapsto f \circ \sigma$$

4 Cellular Homology

4.1 Degree

Let $f : S^n \to S^n$ be a continuous map. We know that $\pi_n(S^n) \cong \mathbb{Z}$ (this can be proved using Fruedenthal's suspension theorem), the induced map

$$f_* : \pi_n(S^n) \to \pi_n(S^n)$$

is a (additive) group homomorphism $\mathbb{Z} \to \mathbb{Z}$ and hence has the form

$$1 \mapsto n \in \mathbb{Z}$$

This n is called the degree of $\mathbb{Z}$.

4.2 Cellular Chain

Hatchers book is not such a good reference for this. Most of the other texts I consult setup singular / simplicial and then *apply* it to CW complexes. I will consult them for proofs but try to set things up intrinsically for CW complexes.

The only thing we need to define here is the cellular chain associated to a CW complex. So start with a CW complex X where X_n is the n-skeleton. We define $X_{-1} \quad := \quad \emptyset$. The abelian group

$$C_n(X_n, X_{n-1}) \quad := \quad \mathbb{Z}[X_n \backslash X_{n-1}]$$

that is the free abelian group on the n-cells. The other thing to define is the boundary maps ∂_n.

We will also use ∂ to denote taking the boundary of a space. Let e_n^α be an n-cell. There is an attaching map $\chi_n^\alpha : \partial e_n^\alpha \to X_{n-1}$, from the boundary of the n-cell to the $(n-1)$-skeleton. Let

$$\chi_n^{\alpha\beta} : \partial e_n^\alpha \xrightarrow{\chi_n^\alpha} X_{n-1} \xrightarrow{q} X_{n-1}/\big(X_{n-1} \backslash e_{n-1}^\beta\big)$$

composing the inclusion of the n-cell with the quotient of the $(n-1)$-skeleton onto an $(n-1)$-cell e_{n-1}^β. Note that this is a map between two spaces isomorphic to S^{n-1}, so we know what the degree of such a map would mean.

Finally we define the boundary by linearly extending

$$\partial_n(e_n^\alpha) \quad := \quad \sum_\beta \deg\big(\chi_n^{\alpha\beta}\big) e_{n-1}^\beta$$

This sum is in $C_{n-1}(X_{n-1}, X_{n-2})$ over all the $n-1$ cells, that is the generators of the group.

Lemma. *The boundary maps square to zero.*

prove

Note: This defines a linear map from C_n to C_{n-1}. If there are a n-cells and b $(n-1)$-cells then a linear map between them has a matrix representation, given by a $b \times a$ matrix, with columns given by $\partial_n(e_n^\alpha)$ as α ranges over the n-cells. From our formula the explicit matrix can be written down

$$\big(\partial_n\big)^{\beta,\alpha} = \deg\big(\chi_n^{\alpha\beta}\big)$$

This reduces questions about kernel and image to computing linear algebra row spaces and stuff.

4.3 Coefficients

Cellular homology was given by taking the free $\mathbb{Z}$ module on the cells of the CW complex. Equally we could have taken the free G module where G is any abelian group. This defines a chain we denote $C_n(X;G)$ and a relative version as well $C_n(X,A;G) = C_n(X;G)/C_n(A;G)$.

The claim is that this also defines a chain with the same maps and therefore taking homology also defines a homology theory. Indeed all of the theory we will work out goes through identically. We will do it just for $\mathbb{Z}$ however for expedience of notation.

4.4 Functoriality

We would also like a way to take a map of topological spaces and induce a map of the chain complexes, so also induce a map on homology. So $f : X \to Y$ be a *cellular map* then we define

$$f_\sharp : C_n^{\text{sing}}(X) \to C_n^{\text{sing}}(Y)$$

given by extending linearly

$$f_\sharp(e_X^i) \;\; := \;\; \sum_j \deg(\bar{f}_{i,j}) e_Y^j$$

where $\bar{f}_{i,j}$ is defined as the composition

$$e_X^i \xrightarrow{\varphi_X^i} X \xrightarrow{f} Y \xrightarrow{q_j} Y/\big(Y - e_Y^j\big)$$

Where the map on the left is the inclusion and on the right is the quotient. Notice the similarity with the boundary map. We will denote the map induced on homology by

$$f_* : H_n(X) \to H_n(Y).$$

Note that this definition relies on the map being cellular, as two homotopic maps can induce very differenet maps on the *chain groups*. What we then need to do is descend to the homology and check that this is well defined for any map up to homotopy. We should also want them to be chain maps, that is commute with the boundary.

4.5 Variations

We define a relative chain complex. For $A \subseteq X$ we define

$$C_n(X,A) \;\; := \;\; C_n(X)/C_n(A),$$

with the chain map being the one induced on the quotient (must check well definied).

explicitly what is the reduced homology doing in this case?

5 Properties

The miraculous thing is that these homology theories agree. We can therefore check the properties they satisfy simultaneously. Moreover we will check that they satisfy the axioms.

5.1 Homotopy Invariance

Theorem. *If $f \simeq g$ then $f_* = g_*$.*

Theorem. *If $X \simeq Y$ then $H_n(X) \cong H_n(Y)$.*

5.2 Excision and Subspaces

Let H_n be cellular homology and $\tilde{H}_n$ be the associated *reduced* homology, that is $\tilde{H}(X) = \ker(H(X \to *))$.

Theorem. *Consider $A \xrightarrow{i} X \xrightarrow{j} X/A$, where i is the inclusion of a subspace and j is the quotient and A is a non-empty, closed subspace that is a deformation retract of a neighbourhood in X, then we have a LES of groups*

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots \to \tilde{H}_0(X/A) \to 0$$

Note that the ∂ map is not the map from the chain which we also denoted ∂.

define it in theis context....

Now consider $A \xrightarrow{i} X \xrightarrow{j} X/A$ to be the inclusion and quotient of X by an arbitrary subspace.

Theorem. *There is a SES of chain complexes*

$$0 \to C_*(A) \xrightarrow{i_\sharp} C_*(X) \to C_*(X, A) \to 0$$

Theorem. *There is a LES in homology*

$$\cdots \to H_n(A) \xrightarrow{i_*}\to H_n(X) \xrightarrow{j_*} H_n(X, A) \to \cdots \to H_0(X, A) \to 0$$

Again define boundary operator

Note: The first theorem says we have a LES for quotients *before* taking homology, the second says we have a LES for quotients *after* taking homology. The difference is between *reduced* and *un-reduced* homology.

is there more to be said here? Is this a general phenomena?

Theorem. *Let $Z \subseteq A \subseteq X$ such that the closure of Z is contained in the interior of A. Then the inclusion*

$$(X - Z, A - Z) \hookrightarrow (X, A)$$

induces an iso on

$$H_n(X - Z, A - Z) \to H_n(X, A)$$

Theorem. *Let $A, B \subseteq X$ such that their interiors cover X, i.e. $\mathring{A} \cup \mathring{B} = X$. Then the inclusion*

$$(B, B \cap A) \hookrightarrow (X, A)$$

induces an iso on

$$H_n(B, A \cap B) \to H_n(X, A)$$

Note: These iso's can be put into the LES to aid in computations.

Note: This can be used to show additivity of the homology theory.

5.3 Mayer-Vietoris

For $A, B \subseteq X$, denote by $C_n(A+B)$ the subgroup of $C_n(X)$ consisting of chains that are sums of chains in A and B.

Theorem. *Let $A, B \subseteq X$ such that their interiors cover X. There is a short exact sequence of complexes*

$$0 \to C_*(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) = x + y$.

Note that in particular $C_n(A+B) \hookrightarrow C_n(X)$ induces an iso on homology groups (this is intuitively obvious and shown in Prop. 2.21 Hatcher, it is essential to the proof of excision). The intuition is that cells in X can be broken into cells in A and cells in B, as they cover X. So on homology we get

Theorem. *Let $A, B \subseteq X$ such that their interiors cover X. There is a long exact sequence*

$$H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0$$

specify the maps explicitly.

References