A Special Case of Van Kampen

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Theorem ([Hat02], Thm 1.20)**.** *Let X be the union of path-connected open sets A_α such that each A_α contains the base point. If each intersection $A_\alpha \cap A_\beta$ is path-connected then the homomorphism*

$$\varphi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

is surjective. In particular

$$\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/ker\varphi.$$

If in addition each triple intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected then the kernel of φ is the normal subgroup generated by elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$.

Recal that the free product of groups is simply the categorical co-product and therefore categorised by maps out of the space, i.e. Given X_1, X_2 then their coproduct is an object $X_1 \sqcup X_2$ along with two maps $X_i \to X_1 \sqcup X_2$ such that for any other object Y with maps $X_i \to Y$ we have a unique map out of the coproduct $X_1 \sqcup X_2 \to Y$ making the diagram commute:

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"'] \arrow[dr] & X_1 \sqcup X_2 \arrow[d, dashed, "\exists!"] & X_2 \arrow[l, "i_2"] \arrow[dl] \\
& Y &
\end{tikzcd}$$

Now φ is the induced map given by the inclusions of $\pi_1(A_\alpha) \to \pi_1(X)$ and $i_{\alpha\beta}$ is the inclusion $\pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$. It is intuitively clear that elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$ are always in the kernel. This is a product and inverse in the free product, the domain of φ.This is just saying that if we consider a loop in the intersection as a loop in either one of the peices then they should be the same when considered as loops in the total space.

Lets look at successivly special cases. First imagine that X is the union of *two* path-connected open sets, then the triple intersection condition is vacuous and we get that

$$\pi_1(X) \cong \big(\pi_1(A_1) * \pi_1(A_2)\big)/ker\varphi$$

which is the amalgamated product of the two subgroups, or categorically a *pushout*

$$\pi(X) \cong \pi_1(A_1) \bigsqcup_{\pi_1(A_1 \cap A_2)} \pi_1(A_2).$$

Finally lets look at a very special case. Consider $M \cup_\varphi D^2$ for $\psi \in \pi_1(M)$ (injective), that is the pushout of

$$\begin{tikzcd}
S^1 \arrow[r, hook, "i"'] \arrow[d, hook, "\psi"] & D^2 \arrow[d, dashed] \\
M \arrow[r, dashed] & M \cup_{\varphi} D^2
\end{tikzcd}$$

Then the fundamental group is given by

$$\pi_1(M \cup_{\psi} D^2) = \pi_1(M) * \pi_1(D^2)/ker\varphi = \pi_1(M)/ker\varphi$$

We claim that $ker(\varphi) = \langle\psi\rangle$, that is the kernel is given by the group generated by the homotopy class of the attaching map. So consider an element $\omega \in \pi_1(M \cap D^2) = \pi_1(S^1) = \mathbb{Z}[\psi]$ we want to show that $i_{D^2M}(\omega)i_{MD^2}(\omega)^{-1} = \psi^n$ for some n. But $i_{D^2M}(\omega)i_{MD^2}(\omega)^{-1} = i_{D^2M}(\omega)$ and is thus an element of $\pi_1(M)$ comming from an element of $\pi_1(S^1)$ and hence from $\mathbb{Z}[\psi]$ as required. Thus Van Kampen implies that

$$\pi_1(M \cup_{\psi} D^2) = \pi_1(M)/\langle\psi\rangle.$$

In this way attaching disks can kill homotopy classes in the fundamental group of M.

References

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge ; New York, 2002.