Automorphic Forms

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Our references are to the relevant definitions in [Int].
Let G be a reductive group over a global field F.

Definition (6.5)**.** *If F is a number field then a function*

$$\phi : G(\mathbb{A}_F) \to \mathbb{C}$$

is an automorphic form on G if it is

- *Smooth*
- *Moderate growth*
- *$G(F)$ left invariant*
- *K-finite*
- *$Z(\mathfrak{g})$-finite*

Definition (6.7)**.** *If F is a function field then*

$$\phi : G(\mathbb{A}_F) \to \mathbb{C}$$

is an automorphic form on G if it is

- *$G(F)$ left invariant*
- *Invariant on the right under the action of some compact open subgroup of $G(\mathbb{A}_F)$*
- *The $\mathbb{C}$ span of*
$$x \mapsto \phi(xg) : g \in G(\mathbb{A}_F)$$
is an admissable representation

So thats a lot of words that we should define now

1 Smooth

1.1 Archimedian Local Field

Let H be an AAG over an archimedian local field k. It is a fact that every archimedian local field is either $\mathbb{R}$ or $\mathbb{C}$. H is AAG and hence linear so we can embed it in GL_n, thus a smooth function from $H(k) \to \mathbb{C}$ is a smooth function in the ordinary differential topology sense from the manifold $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$.

1.2 Non-Archimedian Local Field

If H is an affine algebraic group (AAG) over a non-archimedian local field k then $H(k)$ is totally disconnected and we say that

$$f : H(k) \to \mathbb{C}$$

is smooth if it is locally constant in the induced topology on $H(k)$ from the topology on k.

1.3 Global Field

Let H be an AAG over a global field k and ν a place

Theorem.

$$H(k_\nu) \cong H_{k_\nu}(k_\nu)$$

Therefore a function $f : H(k_\nu) \to \mathbb{C}$ is smooth if it is smooth as a function $f : H_{k_\nu}(k_\nu) \to \mathbb{C}$ as defined for archimedian and non-archimedian local fields above.

1.4 Adelic Smoothness

Recalling that G is a reductive group over a global field F we make the following definitions:

For the non-archimedian places we define

$$C^\infty(\mathbb{A}_F^\infty) \quad := \quad \bigotimes_{\nu \nmid \infty}{}' C^\infty(G(F_\nu))$$

And for the archimedian places we define

$$C^\infty(G(F_\infty)) \quad := \quad C^\infty\left(\prod_{\nu \mid \infty} G(F_\nu)\right)$$

For the full Adele we define

$$C^\infty(\mathbb{A}_F) \quad := \quad C^\infty(G(F_\infty)) \otimes C^\infty(G(\mathbb{A}_F^\infty))$$

A function is ***smooth*** if it is in one of these sets for the appropriate domain. Note that this gives functions with codomain being the tensor product of a bunch of $\mathbb{C}$'s over $\mathbb{C}$ which is isomorphic to $\mathbb{C}$, so we are justified in making this identification.

Another remark is that in our notation ∞ simply stands for the collection of archimedian places.

2 The Rest

Invariance: A function
$$\phi : G(\mathbb{A}_F) \to \mathbb{C}$$
is (left) ***invariant*** under the action of a subgroup $H \leq G(\mathbb{A}_F)$ when $\forall \gamma \in H$ we have that
$$\phi(\gamma g) = g \quad \forall g \in G(\mathbb{A}_F)$$
For the above definitions we view $G(F) \leq G(\mathbb{A}_F)$ via the diagonal map.

2.1 Adelic Number Field

- **Moderate growth:** First we define a norm on $G(\mathbb{A}_F)$. Becuase G is reductive it is in particular linear, we therefore fix a closed embedding $\iota' : G \to GL_n$, which gives another closed embedding $\iota : G \to SL_{2n}$ by
$$g \mapsto \begin{pmatrix} \iota'(g) & \\ & \iota'(g^{-1})^t \end{pmatrix}$$
and the norm is
$$\|g\| = \prod_{\nu} \sup_{1 \leq i,j \leq 2n} |\iota(g)_{ij}|_\nu$$
There is an abuse of notation here $\iota(g)_{ij}$ should actually be the projection onto the ν place and then take the norm. Note that we have made some choices of embeddings here however the class of functions that is of moderate growth is actually independent of the embedding. Then a function $f : G(\mathbb{A}_F) \to \mathbb{C}$ is of ***moderate growth*** if there exists some $c, r \in \mathbb{R}_{>0}$ such that for every $g \in G(\mathbb{A}_F)$
$$|f(g)| \leq c\|g\|^r$$

- **K-finite:** We choose two subgroups this time; $K_\infty \leq G(F_\infty), K^\infty \leq G(\mathbb{A}_F^\infty)$ where as before K_∞ is a maximal compact subgroup, and K^∞ is some compact open subgroup. We then define $K = K_\infty K^\infty$ the direct product. We then say that a function $f : G(\mathbb{A}_F) \to \mathbb{C}$ is ***K-finite*** if
$$\dim[span_{\mathbb{C}}\{x \mapsto f(xk) : k \in K\}] < \infty$$
I have been assured that this is infact independent of the choice made.

- **$Z(\mathfrak{g})$-finite:** $Z(\mathfrak{g})$ is the center of the Lie algebra associated to $G(F_\infty)$ and we say that a vector $f \in V$ is $Z(\mathfrak{g})$***-finite*** if $Z(\mathfrak{g})f$ is finite dimensional.

2.2 Adelic Function Field

- **The $\mathbb{C}$ span of**
$$\{x \mapsto \phi(xg) : g \in G(\mathbb{A}_F)\}$$
is an admissable representation: Recall that an ***admissable representation*** of a topological group (actual group) H is a representation (π, V) such that for every $v \in V$ the stabilizer $\mathrm{stab}_H(v)$ is open in G and for every open subgroup $K \subseteq H$ $dimV^K < \infty$.

Remark. In the archimedain subcase [Int] gives explicitly that the functions are invariant under some arithmentic subgroup. The general definition of automorphic form does not have this restriction. Moreover the choice of K does not effect the collection of automorphic forms. The correct analogie is that if we required the functinos to be K_∞ invariant functions. Then we recover the more familar notion, in particular modular forms etc.

References

[Int] *An Introduction to Automorphic Representations.*