Notions of K theory

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1 Vector Bundles

For a compact X the set $\mathrm{Vect}(X)$ of complex vector bundles forms a commutative monoid under direct sum. There is an associated abelian group defined by the universal property

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & K(A) \arrow[d, dashed] \\ & G \end{tikzcd}$$

for solid lines being monoid homomorphisms and G a group, then the dotted line is a group homomorphism. There are two explicit constructions of this universal group completion

- Take the free abelian group on all the elements of A and mod out by the subgroup generated by elements of the form $a + a' - (a \oplus a')$, where $+$ is the addition in the free group and $\oplus$ is the addition in A.
- We can also let $K(A) = \{(a, b)\Delta(A) : a, b \in A\}$ be the set of cosets of the image of the diagonal map $\Delta : A \to A \times A$, that is exactly $A \times A/\Delta(A)$. The map $A \to K(A)$ is therefore given by $a \mapsto (a, 0) \mapsto (a, 0)\Delta(A)$.

Lemma. *$[E] = [F]$ iff E and F are stably equivalent.*

Proof. Suppose that $[E] = [F]$. Thus $(E, 0)\Delta(\mathrm{Vect}(X)) = (F, 0)\Delta(\mathrm{Vect}(X))$. That is they have the same orbit under $\mathrm{Vect}(X)$, which means that there is some G such that $E \oplus G \cong F \oplus G$ (their orbits have an intersection). Since every vector bundle over a compact space is a summand in a trivial bundle there exists a bundle G' such that $G \oplus G' \cong \mathbb{C}^n$ the trivial bundle. Adding this to both sides we see that $E \oplus G \oplus G' = E \oplus \mathbb{C}^n = F \oplus \mathbb{C}^n \cong F \oplus G \oplus G'$.

Suppose that E and F are stably equivalent. Then there is a trivial bundle $\mathbb{C}^n$ such that $E \oplus \mathbb{C}^n \cong F \oplus \mathbb{C}^n$. Taking their equivalence classes means that $[E] + [\mathbb{C}^n] = [F] + [\mathbb{C}^n]$ and hence $[E] = [F]$.

Remark. In the proof above we used that every vector bundle over a compact space is a summand in a trivial bundle. Notice that this is *not* the same as being stably trivial! The difference is that we are only assuming it is a summand in a trivial bundle, not that the other summand is itself also trivial. i.e.

$$\mathbb{R}^n = X \oplus Y, \quad \text{not} \quad \mathbb{R}^n = X \oplus \mathbb{R}^m.$$

2 Exact Sequences of Vector Bundles

What we have done above is to consider in some sense sequences of vector bundles

$$0 \to E_1 \to E \to E_2 \to 0$$

and forcibly split them. On the level of fibers we do actually have split ses's. Therefore we might introduce a slight generalisation: Consider finite sequences of vector bundles

$$0 \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

such that the maps square to 0. We call two sequences E, F over X homotopic if there is a sequence G over $X \times I$ such that it restricts on both ends to the respective sequences, $G|_{X \times 0} = E, G|_{X \times 1} = F$. We may form then the semi-group $C_n(X)$ of homotopy classes of length n sequences of bundles (under direct sum), if we then modulo by the sub-semigroup of *fiber wise exact* sequences we get $\mathcal{L}_n(X)$. We can also take C_∞ as the union of all the finite ones. The claim is that

Theorem ([Ati18], Lem 2.6.10)**.** *For all $n, n' \leq \infty$ we have that $\mathcal{L}_n(X) \cong \mathcal{L}_{n'}(X)$, they are all the same.*

and that they also give K theory by taking their "Euler characteristic" (this is also the isomorphism in the above lemma to by the way).

Theorem ([Ati18], Cor 2.6.11)**.** *There is an isomorphism*

$$\mathcal{L}_n(X) \to K(X)$$

$$\{E_i\}_{0 \leq i \leq n} \mapsto \sum_{0 \leq i \leq n} (-1)^i [E_i]$$

where $[E_i]$ is the class in K theory as constructed in the previous section.

So in the end our "generalisation" didnt achieve anything new., however it is useful for some constructions.

Remark. We are mixing the two constructions from [AS68, pg. 489] and [Ati18, pg. 88] and assuming that they play nice. The differences seem to be mainly technical and I think what we have written is fine.

Remark. Restricting to compact spaces is necessary here to write down the Euler characteristic in this simple form. For non-compact spaces [Ati18, Lem 2.6.7] gives the construction. Given $0 \to E_0 \xrightarrow{\alpha} E_1 \to 0$ where α restricts to an isomorphism $E_0|_A \to E_1|_A$ for $A \subseteq X$ some compact subset (now X is not assumed to be compact), then we can form the glued bundle $[E_1, \alpha, E_0]$ over $X \cup_A X$. This is an element of the K theory of $X \cup_A X$ and we will use the projections to get an element of $K(X, A)$.

First we have a splitting $K(X \cup_A X) = K(Y, X) \oplus K(X)$ and moreover an isomorphism $K(X \cup_A X, X) \cong K(X, A)$ induced by the inclusion. Then we send the class $[E_1, \alpha, E_0]$ to the first component in the splitting and then through this isomorphism to get something in $K(X, A)$.

3 Applications

Above we have given the precise statements for compact spaces, the definitions and theorems need to be altered slightly for non-compact spaces (which we might use below) however it is usually by just saying "compactly supported" so the vibe is the same. See the references for details.

3.1 Thom Class

This second setup makes writing down the Thom class much easier it is just the class in K theory associated to the Exterior algebra $\Lambda^\bullet V$ as a sequence of complex bundles over V.

3.2 The Symbol

Here is a fact from [AS68]

Lemma. *Let V be a real vector bundle over X (not necessarily compact), then every element of $K(V)$ can be represented by a compactly supported homogeneous complex of degree zero of the form*

$$0 \to \pi^*(E_0) \xrightarrow{\alpha} \pi^*(E_1) \to 0$$

where E_i is trivial outside a compact set in X.

Lets go through this a little more carefully, first π here is the projection from $V \to X$. This means that for complex vector bundles over a vector bundle we only need pullbacks of complex vector bundles over the base

$$\begin{tikzcd} \pi^*(E) \arrow[r] \arrow[d] & V \arrow[d, "\pi"] \\ E \arrow[r] & X \end{tikzcd}$$

The condition of being homogeneous of degree m is that

$$\alpha_{\lambda v} = \lambda^m \alpha_v$$

for all $\lambda \in \mathbb{C}$ and $v \in V$, this is the map induced on the fibers by α as a map of bundles over V.

Now consider an order k differential operator between two vector bundles $E, F \to M$, $P \in \mathrm{Diff}^k(M, E, F)$. Then in a local chart U the operator P can be expressed as

$$P = \sum_{|\alpha| \le k} p_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha$$

The coefficients $p_\alpha(x)$ are smooth functions $U \times \mathbb{R}^m \to U \times \mathbb{R}^{m'}$, which are to be thought of as the local trivialisations of E and F respectively and must respect the fibers, i.e. send $(x, v) \mapsto (x, A(x)v)$. In other words $p_\alpha(x, v) = (x, A(x)v)$ preserving the fibers but transforming them linearly, under a matrix A that depends on x, and must be $m' \times m$. That is it takes an $m \times 1$ and gives a $m' \times 1$ vector (multiplying on the right). In other words we see $p_\alpha : U \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^{m'})$ (the second U can be forgoten as it is fixed by the first choice of an element in U).

What does this function do, it eats a section, in local coordinates a map $U \to \mathbb{R}^m$, which can be thought of as a vector $(f_1, ..., f_m)^T$, it takes the partial derivatives in each of the coordinates and returns again an $m \times 1$ vector. Then it multiplies by an $m' \times m$ matrix on the right, leaving an $m' \times 1$ matrix of functions, in other words a section of the fiber of F at x (already we can see here that the matrix might need to be considered having the dual convention to the section vectors and so justifying the cotangent vectors).

We make this a function in two variables by "formally allowing the derivatives to vary", in particular $\sigma(P) \in C^\infty(U)[\xi_i]$ is a polynomaial in the variables ξ_i with smooth section coefficients given by

$$\sigma(P)(x,\xi) = \sum_{|\alpha|\le k} p_\alpha(x)\,(i\xi)^\alpha$$

where ξ^α is the product $\xi_{\alpha_1}\cdots\xi_{\alpha_j}$. Now we have a function that is defined in two variables, one being the local coordinates and the other being a formal variable. We notice that the elements of the tangent and cotangent (or indeed any vector bundle but in a less canonical way) can be written also in the form (x,ν_x) and so it would make sense to plug them into this formula, as the grading on the ξ matches the grading on the basis of these spaces. This is indeed how we started, but now we can consider it as varying over all of the tangent bundle instead of as having fixed the coordinates, i.e.

$$P = \sigma(P)\left(x, \frac{\partial}{\partial x}\right)$$

At this point we can make the choice to put in cotangent vectors if we wanted, essentially by using the isomorphism between tangent and cotangent bundles (at least fiberwise). Why might we want to? Well there is another setting in which the symbol arises and this is when we take the Fourier transform

$$\mathcal{F}^{-1}\mathcal{F}[Pu] = \sigma(P)\mathcal{F}(u)$$

which implies that

$$Pu = \int \sigma(P)(x,\xi)\mathcal{F}(u)e^{i\langle\xi,x\rangle}d\xi$$

in which we can see ξ is acting as a covector, that is as a linear functional on x, which we can think of as living in the tangent space. Well this begs the question why we should thing of x as in the tangent space? Well again it is acting like a vector and we need then to consider the linearisation of our manifold, which is its tangent space. Notice that at a purely formal level we are in a chart, and so are already in $\mathbb{R}^n$ so might as well think of this as the tangent space at the point around which we take the chart.

Thus we now think of the symbol of an operator as having domain the cotangent bundle, where does it land? Initially we were an operator on the sections of E to the sections of F, but since we have pulled back the operator in local coordiantes by the isomorphism between the cotangent and tangent bundle we also want to pull back the vector bundles to the cotangent bundle, regardless we now have a map

$$T^*M \to \mathrm{Hom}(\pi^*E, \pi^*F)$$

which in local coordinates sends

$$(x,\xi) \mapsto \big((\xi,v) \mapsto (\xi, p_\alpha(x)v)\big)$$

Note that we have used a non-trivial fact here that the map in local coordinates given above does in fact glue to a globally well defined map. The final peice of the puzzle is to now realise that this is *itself a section* of the Hom bundle over the cotangent bundle, and therefore coresponds to a vector bundle map

$$\sigma(P) : \pi^*(E) \to \pi^*(F)$$

where π is the projection from the tangent or cotangent bundles.

Since we originally required that the symbol was homogeneous it is fiberwise homogeneous still, and therefore by our discussion above defines a homogeneous complex over the tangent / cotangent bundle of X and hence an element of $K(TX)$! To wrap up let us make a comment on how this works with the analytic index of [AS68].

Lemma. *The operators we have considered above are Fredholm.*

This means that they have closed image, and the dimension of both their kernel and cokernel is finite (on every fiber). What this means to us is that they have an associated Fredholm index

$$i(P) = \dim\ker P - \dim \mathrm{coker} P \in \mathbb{Z}$$

Moreover

Lemma. *If $\sigma(P) = \sigma(Q)$ then $i(P) = i(Q)$*

Thus we have shown that there is a surjective map σ : nice operators $\to K(TX)$ and that there is at least a well defined map

$$K(TX) \to \mathbb{Z}$$

given by picking some inverse and taking the Fredholm index (different opperators will have the same symbol, but once we take the index it doesnt matter anymore). [AS68] then proves that this is not only well defined but coincides with the topological index.

Remark. Notice that smooth functions $U \to \mathbb{R}$ is just smooth sections of a line bundle over U, i.e. an operator $\mathcal{D}(U;\mathbb{R}) \to \mathcal{E}(U;\mathbb{R})$. If we are in the global case then locally we have that

$$\mathcal{D}(U;\mathbb{R}^n) \to \mathcal{E}(U;\mathbb{R}^m)$$

which is the same as

$$\mathcal{D}(U;\mathbb{R})^n \to \mathcal{E}(U;\mathbb{R})^n$$

which can then be dealt with in the way we set up for line bundles over U.

Remark. Is the definition of kernel and cokernel as the fiberwise kernel and cokernel or is it a theorem that we can take a global definition and reduce it to the fibers for nice operators, either way in this application its the same (in Atiyah-Singer) but not totally clear on what is the definition otherwise.

Remark. Sections of the Hom bundle are in bijection with vector bundle maps.

References

[AS68] M. F. Atiyah and I. M. Singer. The Index of Elliptic Operators: I. *Annals of Mathematics*, 87(3):484–530, 1968. Publisher: [Annals of Mathematics, Trustees of Princeton University on Behalf of the Annals of Mathematics, Mathematics Department, Princeton University].

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