Euler

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Translations of Euler are from The Euler Archive, [Eul88] and [Eul90]. In the 1700's there was the begining of the mathematical boom, most recognisable names appear after this time (towards the end of the century even), over the course of this century Euler writes a completely disgusting amount of mathematics. This set the basis of analysis out of which all of modern mathematics would grow.

We will deal mainly with "On the analysis of the infinite" which was published in two volumes in 1748. Written 60 years after Liebniz, it is clear that things have progressed extremely rapidly. This is a text book dedicated to expositing Eulers way of doing mathematics and in particular dealing with functions. The first volume deals with functions *algebraically* while the second volume ties it in with *geometry*. Our understanding is that much of what is to be found in this book is either summarised from Eulers earlier work or indeed original all together. For now we will just summarise how Euler deals with things and in the back of our mind think he probably came up with it, however one day we shall return to find out what others had contributed. Note that Euler gives very little attribution in his book, for he says it would thereby take volumes.

1 On Algebra and Analysis.

Euler states that

> Often I have considered the fact that most of the difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt this more subtle art.

by analysis he means algebra that deals with "the concept of the infinite" or in his words "analysis is concerned with variable quantities and functions of such variables".

2 Numbers.

Euler begins the book with a couple of Euclid style "definitions" (philosophical waxing), that of constant, variable and determining a varible:

> A constant quantity is a determined quantity which always keeps the same value. A variable quantity IS one which IS not determined or IS universal, which can take on any value. A variable quantity is determined when some definite value is assigned to it.

He says that these variables and constants are *numbers*, but states explicitly that he includes amongst these "both positive and negative, integers and rationals, irrationals and transcendentals. Even zero and complex numbers are not excluded". He makes no attempt at defining any of these however. No mention of axioms is made anywhere in the book.

Remark. (Transcendental Values) Despite the Wiki's claim that this was defined by Euler, as usual this is inaccurate. [Pet12] discusses the origins of the term in a much more academic fashion. The result of that discussion is that Euler never defined it, despite using it, he merely understood it as intuitive. The author gives their interpretation of what Euler might have meant, but the bottom line is that Euler and Liebniz were using this term intuitively to mean something like not obviously coming from polynomial equations.

At this time it does seem that they were aware of the decimal expansion of numbers and this could have fashioned them with a definition of an arbitrary number, that is arbitrary infinite strings, but I dont think they made use of this.

Remark. (Axioms) I think that it is possible to still view this in the same axiomatic class as Euclids number theory, but with a few more things tacked on, maybe the existence of certain numbers and the fact that the normal axioms can also be used *infinitely many times*. This is of course a later interpolation by me however, and explicitly Euler does not care.

3 Functions.

Here is the brillance of Euler, the introduction of our modern concept of function. Euler first defines

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.

To this we should add one thing, that by analytic expression he means any sort of written thing with all the usual formulas, roots powers etc. At first glance this definition seems far from our modern one, as it gives a function as an *expression*. This first definition is only a first approximation, he will ammend and clarify his point.

He first clarifies that at function of a variable is itself a variable quantity, and that some expressions may appear to be functions but really they are constants, for example $z - z$. He then adds that the variables and constants must be combined by certain operations,

> "the operations by which the quantities can be arranged and mixed together (are) addition, subtraction, multiplication, division, raising to a power, and extraction of roots. Also the solution of equations have to be considered. Besides there operations, which are usually called algebraic, there are many others which are transcendental, such as exponentials, logarithms, and others which integral calculus supplies in abundance."

Thus his expressions are very general and include not just polynomials but even solutions to equations and "transcendental functions". Euler writes that

> Indeed frequently algebraic functions cannot be expressed explicitly. For example, consider the function Z of z defined by the equation, $Z^5 = az^2Z^3 - bz^4Z^2 + cz^3Z - 1$. Even if this equation cannot be solved, still it remains true that Z is equal to some expression composed of the variable z and constants

thus expressions are not things that we can necissarily write down explicitly but include things that we can write down perhaps implicitly, or described by some other possibly infinite process. He calls such functions "implicit". All this discussion of expressions is then rounded off with the following comment, when he is introducing multi-valued functions, it is here that the modern view point is expressed most clearly

> A single-valued function is one for which, no matter what value is assigned to the variable z, a single value of the function is determined. On the other hand, a multiple-valued function is one such that, for some value substituted for the variable z, the function determines several values.

Hence it becomes clear that really the expression is just a vessel for the idea that a function is a way of assigning to a given input some output(s). The idea of the expression, which Euler again later clarifies should essentially be an expression of the form $\sum_{i\in\mathbb{Z}} a_i x^i$, is more akin to a restriction of continuous of geometric etc, as Descarte restricts from all possible curves to "geometric" curves. The final clue that this is truly what he means is when he states that

> The form of a function is changed, either by introducing a different variable, or if the same variable is kept, the transformation consists in expressing the same function in a different way.

again giving some examples of functions such that "All of these respective expressions differ in form but are truly equivalent", that is the function is the input output pairs not the "form" of the expression!

Remark. Most of the section is then dedicated to defining certain classes of expressions (odd, even, rational, etc) and showing that some cute property holds, or restating the definitions for multivalued functions etc. He also spends a lot of time on partial fraction decompositions, which leads to the above characteristation of his class of functions, as those with Laurent series.

He also dedicates a lot to the logarithm and exponential and finding an infinite series expression. This is done by sort of assuming that these functions are defined and exist and have some properties (being inverse) and then rearanging equations and performing approximations with "infinitely large values" to get some series expression, which I beleive was the common definition at the time.

Similarly for sin and cosine. The idea I beleive is that as we have discussed these were originally operations that were geometric, but Descarte opened the door to assigning to them algebraic expressions. Soon after people found that these expressions required infinite sums. This allowed totally algebraic definitions of these operations which Euler then employs in this section.

4 Geometry.

Euler claims that

> I have proposed a theory of curves with enough generality that it can advantageously be applied to an examination of the nature of any curve whatsoever. I use only an equation by which the nature of every curve is expressed, and I show how to derive from this both the shape and its primary characteristics... Until this time they have ordinarily been treated only from the geometric viewpoint, or if by analysis, in an awkward and unnatural way.

Indeed from what we have seen his assesment is mostly accurate, although his method is not *entirely* new, it seems to be stated here for the first time in absolutely clear and general terms. This takes place in Book II Chapter I and is for some reason oddly repetitive. We quote at length his discussion, with bolding added

> A variable quantity is a **magnitude** considered in general and for this reason, it contains all determined quantities. Likewise in geometry a variable quantity is most conveniantly represented by a straight line RS, of indefinite lenghth. Since in a line of indefinite length **we can cut off any determined magnitude, the line can be associated in the mind with the vairbale quanity**. First we choose a point A in the line RS and associate with any **determined quantity** an interval of that magnitude which begin at A. Thus a determined portion of the line, AP, represents the determined value contained in the variable quantity.

> Let x be a variable quantity which is represented by the line RS, then it is clear that **any determined value of x which is real can be represented by an interval of the line RS.** The Interval AP ... manifest the determined values of x.
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> Since the line RS extends indefinitely in either direction from A, both negative and positive values of x can be represented. We will represent positive values by cutting off intervals to the right of A and negative values by intervals to the left of A. The farther to the right right the point P is from A, the greater the value of x is represented by the interval AP. If P is identical with A, the value of x is 0.
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> Since the indefinite straight line representes the variable x, we would like to see how a function of x can be most conveniently represented. Let y be any function of x, **so that y takes on a determined value when a determined value is assigned to x**. After having taken a straight line RAS to denote the values of x, for any determined value of x we take the corresponding interval AP and then **erect a perpendicular interval PM** corresponding to the value of y. If the value of y is positive, then PM is above the line RS, while if y is negative, then PM is perpendicular below the line...
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> **All of the extremeties, M, of the perpendiculars form a line which may be straight or curved. In this way the line is determined by the function y.** Thus any function of x is translated into geometry and determines a line, whose nature is dependent on the nature of the function.

Some things to remark about this. First is that this is different to what Descartes and other were doing for some time which I beleive to be more similar to what we do now in algebraic geometry, that is they determined a subset not as the graph of some equation but as the solution set of some equation, although it is really hard to tell because their methods are ad hoc and confusing. Note that Euler also defines functions as the points satisfying some equation so this is indeed one possible way in which the y values would be determined for Euler. We also see here the nature of functions that we have argued for above reaffirmed. What we have here is a way of assigning to every point on the x axis a given point in the plane, and the line is merely then made up of these points. Euler states

> Although many diffent curves can be described mechanically as a continuously moving point ... still we will consider these curves as having their origins in functions... Any function of x gives a curve or a straight line and conversely a curve can define a function.

This is an interesting assertion or perhaps restriction, as it is not clear at all and he provides no proof. He also immediately contradicts it by defining a continuous curve to be one "expressed by a single function" while discontinuous curves are those that require different functions at different parts. Clearly then discontinuous curves do not have an associated function.

Finally it is again interesting to note that he states that different equations can result in the same curve, and he gives a method of comparing them.

References

[Eul88] Euler. *Introduction to Analysis of the Infinite*. Springer, New York, NY, 1988.

[Eul90] Leonhard Euler. Introduction to analysis of the infinite. 2. Springer, New York Berlin Heidelberg London Paris Tokio, 1990. Num Pages: 504.

[Pet12] Bruce J. Petrie. Leonhard Euler's use and understanding of mathematical transcendence. *Historia Mathematica*, 39(3):280–291, August 2012.