Affine Group Scheme Summary

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A List of Properties

- Linear
- Finite type
- Algebraic
- Connected
- Reduced
- Irreducible
- Integral
- Smooth
- Unipotent
- Solvable
- Reductive
- Semi-simple
- Algebraic Torus
- Split Algebraic torus
- Split
- Quasi-split
- Isotropic

A List of Distinguished Subgroups

- Derived Subgroup – (solvable)
- Unipotent Radical – (reductive)
- Solvable Radical – (semisimple)
- Maximal Torus – (split)
- Borel – (quasi-split)
- Neutral component
- Levi Subgroup
- Normalizer of torus
- Centralizer of torus
- Weyl group
- Parabolic
- Standard parabolic

We take the functor of points approach given in [**?**]; it would proabably be a good idea to also go through [**?**]. We treat only of groups schemes over a field k. We denote the algebraic closre of k by $\bar{k}$ and the seperable closure k^{sep}.

An ***affine group scheme*** over k is a representable functor

$$G : \text{Alg}_k \to \text{Groups}$$

$$A \mapsto Hom_{Alg_k}(R, A)$$

The algebra that represents G is denoted $\mathcal{O}(G)$

The first example is the group scheme that sends an algebra to its multiplicative units. We denote this $\mathbb{G}_m$ this is represented by the k algebra

$$k[x, y]/(xy - 1)$$

Another key example is GL_V sending an algebra A to the A-module automorphisms $V \otimes_k A \to V \otimes_k A$, or simply the $n \times n$ matricies with A entries. This is represented by

$$k[\{x_{ij} : 1 \leq i, j \leq n\}][y]/(ydet(x_{ij}) - 1)$$

A ***representation*** of an affine group scheme (from now on GS) G is a morphism (natural transformation) of GS

$$G \to GL_V$$

We say that a representation is faithful if the associated map

$$\mathcal{O}(GL_V) \to \mathcal{O}(G)$$

is surjective

would usually make more sense to be injective. Is it becuase we are in the opposite cat or something

A GS is ***linear*** if there exists some faithful representation.

It is of ***finite type*** if it is represented by a finitely generated algebra.

An ***affine algebraic group*** (AAG) is an GS of finite type over a field.

Theorem. *All AAG are linear.*

Let G be a GS then G is

- ***connected*** iff the only idempotents of $\mathcal{O}(G)$ are 0 and 1
- ***reduced*** iff $\mathcal{O}(G)$ has no (non-zero) nilpotent elements
- ***irreducible*** iff the nilradical (collection of nilpotent elements) of $\mathcal{O}(G)$ is a prime ideal
- ***integral*** iff $\mathcal{O}(G)$ is an integral domain
- ***smooth*** iff $\mathcal{O}(G)$ is formally smooth

Lemma. *If $k \subseteq \mathbb{C}$ then G is connected iff $G(\mathbb{C})$ is connected in the analytic topology.*

A ***subgroup scheme*** $H \subseteq G$ is a subscheme (not defined here) such that on points $H(A) \subseteq G(A)$ we have subgroups.

If G is a AAG then there is a subgroup scheme G° such that G° is

- normal
- contains the identity
- maximally connected

G° is called the ***neutral component***

is it unique or are they conjugate?

Recall that an element $x \in M_n(k)$ is called

- ***semisimple*** if there is some $g \in M_n(\bar{k})$ such that $g^{-1}xg$ is diagonal
- ***nilpotent*** if there is some $n \in \mathbb{N}$ such that $x^n = 0$
- ***unipotent*** if $x - I$ is nilpotent

Theorem. *If G is a linear GS then an element $r \in G(R)$ is —-one of the things above—- if there exists a faithful representation of G, ρ such that $\rho(r)$ —-is that thing—-.*

An AAG is called ***unipotent*** if every representation has a fixed vector.

The ***derived subgroup*** of an AAG G is the intersection of all normal subgroups (normal on points) $N \subseteq G$ such that G/N is commutative. We denote this G^{det} or $\mathcal{D}G$.

We say G is ***solvable*** if there is some $n \in \mathbb{N}$ such that $\mathcal{D}^n(G)$ is trivial.

The ***unipotent radical*** of G, denoted $R_U(G)$, is the maximal connected, normal and unipotent subgroup. The ***solvable radical*** of G, denoted $R(G)$, is the maximal connected, normal and solvable subgroup.

A smooth, connected AAG G is ***reductive*** iff $R_U(G_{\bar{k}}) = \{1\}$ and semi-simple iff $R(G_{\bar{k}}) = \{1\}$.

A subgroup $M \subseteq G$ of an AAG is a ***Levi subgroup*** iff the following is exact

$$1 \to M_{\bar{k}} \hookrightarrow G_{\bar{k}} \xrightarrow{\pi} G_{\bar{k}}/R_U(G_{\bar{k}}) \to 1$$

An ***algebraic torus*** is an AAG T such that for some $n \in \mathbb{N}$ $T_{\bar{k}} \cong \mathbb{G}_m^n$. We say that T ***splits*** (as a torus) if $T \cong \mathbb{G}_m^n$. T is a ***maximul torus*** in G if $T \subseteq G$ and $T_{\bar{k}}$ is a maximal element of the set of tori (ordered by inclusion).

If $T \subseteq G$ is a torus inside a reductive group then $N_G(T)$ is the normalizer and $C_G(T)$ is its centralizer. The Weyl group is $W(G,T) = N_G(T)/C_G(T)$.

Lemma. *T is a maximal torus iff $C_G(T) = T$.*

A reductive group is ***split*** if there exists a split maximal torus.

If G is reductive then $B \subseteq G$ is called a Borel iff $B_{\bar{k}}$ is a maxima, smooth, connected and solvable subgroup of $G_{\bar{k}}$. A smooth subgroup $P \subseteq G$ is parabolic if $P_{\bar{k}}$ contains a borel subgroup of $G_{\bar{k}}$.

A reductive group is ***quasi-split*** if it contains a Borel. If G contains a split torus it is called isotropic.

Given a minimal parabolic subgroup $P_0 \subseteq G$ then the parabolic subgroups that contain P_0 are called standard.

References