Algebraic groups vs Lie groups

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There are general theorems by Nash and Tognelli on when smooth real manifolds are the real points of an algebraic variety (see my smoothness notes). They are highly restrictive however. I was thinking about Lie groups and their algebraic properties and was struck by the fact that they are a quite restrictive class of manifolds and realised that similar or better results are surely known for algebraic groups and Lie groups. This is the topic here. [Mil] is the main reference.

1 Three Categories

Let $k = \mathbb{R}$ or $\mathbb{C}$. First we have the category of real Lie groups with smooth maps, we also have the category of complex Lie groups with holomorphic maps.

A Lie group is linear if there exists a faithful representation. E.g. there is a faithful morphism in the relavant category to $\mathrm{GL}(V)$ considered as an object in the same category. A Lie group is reductive if every representaiton is semi-simple. It is not clear to what extent that is a standard definition.

Now we recall some basic algebraic things that I always forget. Denote the category of commutative k algebras by Alg_k, then a linear algebraic group (Milne calls them just algebraic groups) is a functor represented by a finitely generated k algebra

$$\mathrm{Alg}_k \to \mathrm{Group}.$$

Note that linear algebraic groups are closed subschemes of GL_n and (Zariski) smooth. The equivalence of categories below is standard [Mil12, II.6, III.4],

$$\mathrm{LAG} \leftrightarrow \text{finitely generated commutative Hopf algebras}/k$$

We denote the algebra representing a LAG G by $\mathcal{O}(G)$. Then G is connected if $\mathrm{maxSpec}\mathcal{O}(G)$ is (Zariski) connected. G is reductive if the only commutative normal connected subgroups are tori. Note that these definitions are only valid because $\mathrm{char}(k) = 0$.

There is a natural functor

$$L : \mathrm{LAG}/k \to \text{Lie Groups}/k$$

which, because LAG are smooth, can be considered as just taking the k points and giving them the subspace topology (some subtleties here, needing to fix an algebra isomorphism to something of the form $k[x_i]/I$, see my talk on reductive groups to make precise).

Theorem. *This functor is not full, faithful or essentially surjective.*

There exist non-connected LAG's on which this functor is not faithful. The map

$$\mathbb{C} \to \mathbb{C}^{\times}$$

$$x \mapsto e^x$$

does not come from a map $\mathbb{G}_a \to \mathbb{G}_m$, note however that $\mathbb{G}_a$ is not reductive, this exhibits the failure of fullness. Essential surjective fails on many covering Lie groups.

This functor does have one nice property

Theorem. *L preserves Lie algebras.*

2 Complex Groups

Theorem. *L defines an equivalence of categories between complex reductive Lie groups and reductive LAG defined over $\mathbb{C}$.*

3 Real

Theorem. *Given a real reductive Lie group G then there exists a LAG $T(G)$ defined over $\mathbb{R}$ and a morphism $G \to LT(G)$ such that*

- *The assignment $G \to T(G)$ is functorial.*
- *$LT(G)$ is a quotient of G.*
- *The induced functor $\mathrm{Rep}(G) \to \mathrm{Rep}(TG)$ is an equivalence.*

The morphism $G \to LT(G)$ is an iso iff G is linear. Note also that the algebraic group is not guaranteed to be reductive. This does not characterise the category of Lie groups however if all we are interested in is the *representation theory* of real Lie groups then this shows that it is sufficient to study real LAG's.

There is a complimentary result for compact and connected groups

Theorem. *Given a compact connected real Lie group G there exists a semi-simple LAG $T(G)$ such that $LT(G)$ is isomorphic to G. Moreover if G' is a reductive LAG over $\mathbb{C}$ then*

$$\mathrm{Hom}(T(G)_{\mathbb{C}}, G') \cong \mathrm{Hom}(G, L_{\mathbb{C}}(G'))$$

where the subscript $\mathbb{C}$ denotes the base change. Hom is in the category of complex LAG and real Lie groups respectively.

In particular this says that L is essentially surjective from semi-simple LAG's to connected and compact Lie groups, and that there is an adjunction

$$T : \text{Connected Compact Lie Group}/\mathbb{R} \to \text{Reductive LAG}/\mathbb{C}.$$

It would be interesting to see what we can deduce about the full or faithfullness of this functor, as it is not clear.

References

[Mil] J S Milne. Lie Algebras, Algebraic Groups, and Lie Groups.

[Mil12] J S Milne. Basic Theory of Affine Group Schemes. 2012.

Also useful is