A short note on Wei Wangs short note on $\pi_1\mathrm{Diff}_\partial(D^{4k})$

Riley Moriss

January 30, 2026

Recall from previous discussions we have the following diagram exact in the horrozontal and verticle directions, but not arround corners:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & \vdots \arrow[d] & & \\
 & \pi_1\mathrm{Diff}_\partial(D^{n+2}) \arrow[d] & & & \pi_0\mathrm{Diff}_\partial(D^{n+2}) \arrow[d] & & \\
 & \pi_1\mathscr{C}(D^{n+1}) \arrow[d, "r"] \arrow[dr, dashed, "or"] & & & \pi_0\mathscr{C}(D^{n+1}) \arrow[d] & & \\
\cdots \arrow[r] & \pi_1\mathrm{Diff}_\partial(D^{n+1}) \arrow[r, "o"] & \pi_1\mathscr{C}(D^n) \arrow[r] & \pi_1\mathrm{Diff}_\partial(D^n) \arrow[r] & \pi_0\mathrm{Diff}_\partial(D^{n+1}) \arrow[r] & \pi_0\mathscr{C}(D^n) \arrow[r] & \pi_0\mathrm{Diff}_\partial(D^n)
\end{tikzcd}$$

where the verticle sequences continue beneath the horrozontal as well. Assuming we are in the stable range for the concordance groups we can put in the known values

$$\begin{tikzcd}
 & & \vdots \arrow[d] & & & & \\
 & & \pi_1\mathrm{Diff}_\partial(D^{n+2}) \arrow[d] & & & & \\
 & & \mathbb{Z}_2 \arrow[d, "r"] \arrow[dr, dashed, "or"] & & & 0 \arrow[d] & \\
\cdots \arrow[r] & \pi_2\mathrm{Diff}_\partial(D^n) \arrow[r] & \pi_1\mathrm{Diff}_\partial(D^{n+1}) \arrow[r, "o"] \arrow[d, two heads] & \mathbb{Z}_2 \arrow[r] & \pi_1\mathrm{Diff}_\partial(D^n) \arrow[r, two heads] & \pi_0\mathrm{Diff}_\partial(D^{n+1}) \arrow[r] & 0 \\
 & & \pi_0\mathrm{Diff}_\partial(D^{n+2}) & & & &
\end{tikzcd}$$

and our goal is to study this first non-trivial crossing.

Lemma ([Wan], §1.2)**.** *The following diagram commutes*

$$\begin{tikzcd}
\pi_1\mathrm{Diff}_\partial(D^n) \arrow[r, "\iota_{block}"] \arrow[d, "\partial_{\mathscr{C}}"'] & \pi_1\widetilde{\mathrm{Diff}}_\partial(D^n) \arrow[d, "\sim"] \\
\pi_0\mathrm{Diff}_\partial(D^{n+1}) \arrow[r, "\sim"'] & \pi_0\widetilde{\mathrm{Diff}}_\partial(D^{n+1})
\end{tikzcd}$$

Where $\partial_{\mathscr{C}}$ is the boundary coming from the LES of the fibration $\mathrm{Diff}_\partial(D^{n+1}) \to \mathscr{C}(D^n) \to \mathrm{Diff}_\partial(D^n)$, *which is equivalently given by the Gromoll assembly map, and ι_{block} is the map induced in the LES* *from the fibration* $\mathrm{Diff}_\partial(D^n) \to \widetilde{\mathrm{Diff}}_\partial(D^n) \to \widetilde{\mathrm{Diff}}_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$*. The bottom map is also the map induced from that fibration, the right verticle map is described in [Kup, 25.2] or [ABK72, 2.3.2 Le,] where it is also shown to be an iso.*

According to [Wan24] when $n = 4k, k \geq 3$ then the boundary map from $\pi_1\mathrm{Diff}_\partial$ to $\pi_1\widetilde{\mathrm{Diff}}_\partial$ is an injection, thus the boundary map $\partial_{\mathscr{C}}$ is also an injection. This implies the map out of the horrizontal $\mathbb{Z}_2$ is the zero map. Thus we have a diagram

$$\begin{tikzcd}
 & & \vdots \arrow[d] & & \\
 & & \pi_1\mathrm{Diff}_\partial(D^{4k+2}) \arrow[d] & & \\
 & & \mathbb{Z}_2 \arrow[d, "r"] \arrow[dr, dashed, "or"] & & \\
0 \arrow[r] & \pi_2\mathrm{Diff}_\partial(D^{4k}) \arrow[r] & \pi_1\mathrm{Diff}_\partial(D^{4k+1}) \arrow[r, "o"] \arrow[d, two heads] & \mathbb{Z}_2 \arrow[r] & 0 \\
 & & \pi_0\mathrm{Diff}_\partial(D^{4k+2}) \arrow[d, leftrightarrow, "\cong"] & & \\
0 \arrow[r] & bP_{4(k+1)} \arrow[r] & \Theta_{4k+3} \arrow[r] & \mathrm{coker}J_{4k+3} \arrow[r] & 0
\end{tikzcd}$$

Where the purple vertical arrow is not exact and all horizontal arrows are. The 0 on the left of $\pi_2\mathrm{Diff}_\partial$ is $\pi_2\mathscr{C}(D^n) = 0$ in the stable range. Now recall that bP_{4m} is a finite cyclic group of order $2^{2m-2}(2^{2m-1}-1)\text{numerator}\left(\frac{4B_m}{m}\right)$. In this case a result of Brumfiel implies that there is a section from the cokernel of J to the group of homotopy spheres and so this bottom horrozontal ses is split. The cokernel is a quotient of the stable stems and is not known. Now the *or* map forms the differential in the Hatcher spectral sequence and we would like to compute it, Kupers and hatcher give a formula for the differential in terms of the involution and Wang shows that it is zero in this case.

Now in Wangs proof of the injectivity he proves the surjectivity of the previous map in the block fibration, that is he proves the surjectivity of a map

$$\Theta_{n+3} \cong \pi_2\widetilde{\mathrm{Diff}}_\partial(D^{4k}) \to \pi_2\frac{\widetilde{\mathrm{Diff}}_\partial(D^{4k})}{\mathrm{Diff}_\partial(D^{4k})} \cong \mathbb{Z}_2$$

he does this by showing that an element of Θ_{4k+3} is mapped to 1 in $\mathbb{Z}_2$. This element is the generator of bP_{4k}, or what is called the "Kervaire sphere". To show that the Kervaire sphere doesnt hit zero they rely on result [WW, 6.6]. What this result says is

Lemma. *Let* $x \in \pi_0\widetilde{\mathrm{Diff}}_\partial(D^{i-1})$ *and* $i+1 \equiv 0$ *mod* 4 *and bound an s-parallelizable manifold* M*. If* x *has disc of origin* $\geq j$ *then the signature of* M *is divisable by* $2a_{j+1}$*.*

Here $a_k = 1, 2, 4, 4, 8, 8, 8, 8$ for $k = 1, 2, 3, 4, 5, 6, 7, 8$ respectively, and $a_{k+8} = 16a_k$. Im making some educated guesses actually what they say is that x is in the image $\bar{x} \in L_{i+1}(\mathbb{Z})$ then the signature of $\bar{x}$ is divisible by $2a_{j+1}$. Im pretty sure that this is the same however. Note that the generator of bP_{4k} always has signature 8, by definition since the map to the finite cyclic group is given by the signature divided by 8 see [KM63]. The contrapositive of this lemma then says that x doesnt pullback if its bounded manifold has signature not divisible by $2a_{j+1}$.

For Wang he wants to show that there is an element that cant be pulled back to the third step and so he needs a sphere which bounds something with a signature that does not divide $2a_4 = 8$, and since the generator has signature 1 it doesnt pull back. Doesnt the generator never pull back then in dimension 4?

This suggests in this case that the same Kervaire sphere pulled back along the purple arrow and then the verticle surjection is responsible for the o map being a surjection. I dont think that this is immediate or obvious however.

Actually since one can do the same calculations as Wang did for $\pi_2 \frac{\tilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}$ to $\pi_3 \frac{\tilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}$ to show that it is also $\mathbb{Z}_2$ and since $\Gamma_4 \subseteq \Gamma_3$ we known that there is also an element that doesnt pull back a further step we get that at the third homotopy groups we also have a surjection

$$\pi_3 \tilde{\mathrm{Diff}}_\partial(D^{4k}) \to \pi_3 \frac{\tilde{\mathrm{Diff}}_\partial(D^{4k})}{\mathrm{Diff}_\partial(D^{4k})}$$

and hence the boundary to the π_2 section of the LES is zero and we end up with a ses

$$0 \to \pi_2 \mathrm{Diff}_\partial(D^{4k}) \to \pi_2 \tilde{\mathrm{Diff}}_\partial(D^{4k}) \to \pi_2 \frac{\tilde{\mathrm{Diff}}_\partial(D^{4k})}{\mathrm{Diff}_\partial(D^{4k})} \to 0$$

Moreover because the image of the first map is Γ_3^{4k-1} we get that $\pi_2 \mathrm{Diff}_\partial(D^{4k}) \cong \Gamma_3^{4k-1}$.

References

[ABK72] P. L. Antonelli, D. Burghelea, and P. J. Kahn. The non-finite homotopy type of some diffeomorphism groups. *Topology*, 11(1):1–49, January 1972.

[KM63] Michel A. Kervaire and John W. Milnor. Groups of Homotopy Spheres: I. *Annals of Mathematics*, 77(3):504–537, 1963.

[Kup] Alexander Kupers. Lectures on diffeomorphism groups of manifolds, version February 22, 2019.

[Wan] Wei Wang. Note on some Gromoll filtration groups and fundamental groups of $\mathrm{Diff}_{\partial}(D^n)$.

[Wan24] Wei Wang. A short note on $_1\mathrm{Diff}_{\partial} (D^{4k})$ for $k\geq 3$, April 2024. arXiv:2307.03417 [math].

[WW] Michael S Weiss and Bruce Williams. AUTOMORPHISMS OF MANIFOLDS.