Three Nerves, Simplicies and Models

Riley Moriss

May 5, 2025

1 Nerve **1**

2 Duskin Nerve **2**

3 Coherent Nerve **2**

The most important aspects of infinity category theory can be summarised in the following three nerve constructions. Here we will denote

- sSet the 1-cat of simplicial sets
- Cat the strict 1-cat of 1-categories
- $\text{Cat}_{(2,1)}$ the strict one category of weak (2,1) categories, with quasi-functors
- $\text{Cat}^{\text{su}}_{(2,1)}$ is the subcategory of $\text{Cat}_{(2,1)}$ that is composed of strictly unital categories and strictly unital quasi-functor.
- Cat_{Δ} is the strict 1-category of simplicially enriched categories with simplicial functors.

We recall here that

$$\text{sSet} \quad := \quad \text{Fun}\big(\Delta^{\text{op}}, \text{Set}\big)$$

where Δ is the simplex cateogry.

The references are secret notes and the pages at `https://kerodon.net/`.

The other most important constructions are that of "spaces" and model categories.

1 Nerve

The nerve is a functor

$$N : \text{Cat} \to \text{sSet}$$

defined by

$$N(\mathscr{C})_n \quad := \quad \text{Hom}_{\text{Cat}}\big([n], \mathscr{C}\big)$$

Lemma. *The nerve is fully faithful.*

Intuitively the zero cells are objects, the one cells are morphisms, two cells are compositions and three cells witness associativity, the higher cells all have no "new information". Thus in this precise sense normal categories are "just" simplicial sets.

2 Duskin Nerve

The Duskin nerve is a functor

$$N^D : \mathrm{Cat}_{(2,1)} \to \mathrm{sSet}$$

given by

$$N^D(\mathscr{C})_n \quad := \quad \mathrm{suLax}\big([n], \mathscr{C}\big)$$

where suLax are strictly unital lax functors considering $[n]$ as a (degenerate or trivial) weak 2-category. Recall that if psuedo-functors preserve unity and composition only up to natural isomorphisms then lax functors preserve them only up to natural transformation (much weaker).

Lemma. *The Duskin nerve is fully faithful when restricted to* $\mathrm{Cat}^{\mathrm{su}}_{(2,1)}$.

Thus at least strictly unital weak 2-categories are "just" simplicial sets.

3 Coherent Nerve

The coherent nerve is a functor

$$N^\Delta : \mathrm{Cat}_\Delta \to \mathrm{sSet}$$

defined by

$$N^\Delta(\mathscr{C})_n \quad := \quad \mathrm{Hom}_{\mathrm{Cat}_\Delta}\big([n], \mathscr{C}\big)$$

for this to make sense however we need to simplicially enrich $[n]$. Note that there is an obvious way of doing this, that is composing the fully faithful functors

$$\Delta \to \mathrm{Cat} \to \mathrm{Cat}_\Delta$$

given by the embedding $\mathrm{Set} \to \mathrm{sSet}$, that is considering the hom sets as "trivial" simplicial sets. This construction however does not capture the higher coherences, that is if we have $\mathscr{C} \in \mathrm{Cat}_\Delta$ then a simplicially enriched functor $[2] \to \mathscr{C}$ where $[2]$ is enriched in the above trivial way is merely three objects and three morphisms (0-cells of the simplicial hom sets)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "h"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

such that $f \circ g = h$. What we really want is that there is a *homotopy* from $f \circ g$ to h. Essentially this structure does not use the higher simplicies of the hom sets.

Instead we define $\mathfrak{C} : \Delta \to \mathrm{Cat}_\Delta$ as follows

- $\mathfrak{C}[n]$ has the same objects as $[n]$
- The simplicial set $\mathrm{Hom}_{\mathrm{Cat}_\Delta}\big([n], [m]\big)$ is the nerve of the poset (ordered by inclusion) of $\mathcal{P}[n, m] = \mathcal{P}\{n, ..., m\}$ the power set, considered as a one category, in particular if $m < n$ this is empty.

And thus we consider $[n]$ as a simplicial category by looking at its image under $\mathfrak{C}$, which defines our coherent nerve.

Lemma. *For a strict 1-cateogy* $\mathscr{C}$ *there are isomorphisms of simplicial sets*

$$N(\mathcal{C}) \cong N^D(\mathcal{C}) \cong N^\Delta(\mathcal{C})$$

for a strict 2-category we have

$$N^D(\mathcal{C}) \cong N^\Delta(\mathcal{C}).$$

The coherent nerve is *not* fully faithful as a functor between 1-categories. We can see that it restricts to a fully faithful functor however. Moreover if the two categories are given appropriate model structures the coherent nerve exhibits a Quillen equivilence between the two. It therefore presents a fully faithful functor on the underlying $(\infty, 1)$-categories in the ∞-categorical sense. See `https://mathoverflow.net/q/491460/528175` and the posts linked therein.