Spectra

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September 18, 2025

This is a synthesis of the following references [DK01], [Swi75], [DS95] and [BR20]. We will just assume that Top refers to a convenient category for topological spaces (we want the smash to have an adjoint).

1 Spectra in the wild

Mostly following Davis and Kirk. The first stable phenomena appearing is Freudenthals suspension theorem for homotopy groups and its analogue for homology groups. (A coorrolery of) This theorem essentially says that for any X the sequence

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \to \cdots$$

eventually stabilises, that is the maps all become isomorphisms. This leads to the definition of the stable homotopy groups of a space as

$$\pi_i^s(X) \ := \ \mathrm{colim}_k \pi_{k+i}(\Sigma^k X).$$

For $X = S^0$ these are commonly called the stable stems (for other spheres they are just zero). Because suspending is just smashing with spheres and suspending a sphere gives you one higher sphere we can see that the stable homotopy groups are exactly $\pi_i^s(X) = \mathrm{colim}_k[S^{i+k}, S^k \wedge X]$.

The other stable phenomena as we observed was homology. Now we denote $K(\mathbb{Z}, i)$ the Eilenberg-Maclane space, only non-trivial homotopy group is $\mathbb{Z}$ in degree i, then it is also a fact that

$$H_n(X;\mathbb{Z}) = \mathrm{colim}_k[S^{n+k}, X_+ \wedge K(\mathbb{Z}, k)]$$

where $X_+ = X \sqcup *$. Now note that the stable homotopy groups form a generalised homology theory.

There is another stable phenomena that is less obvious. By the Pontryagin-Thom construction we have a bijection

$$\Omega^{\mathrm{fr}}_{k-n,M} \leftrightarrow [M, S^n]$$

where $\Omega^{\mathrm{fr}}_{k-n,M}$ is the space of $k-n$ dimensional framed submanifolds of M up to framed co-bordism. The bijection in the forward direction is given by taking a submanifold with a framing (N, ν) to the so called collapse map, $M \to S^n = \mathbb{R}^n \cup \infty$ given by sending everything outside of N to ∞ and everything in N to its framing (which is a vector in $\mathbb{R}^n$). In the reverse it is given by taking a map $f : M \to S^n$

and simply looking at $f^{-1}(r)$ for r some regular value, this submanifold has a framing by pulling back the standard framing on the sphere. This theorem has a slight refinment to the following

$$\Omega_k^{st+fr}(X) \cong \pi_k^s(X_+)$$

where the group on the right is that of stably framed bordism classes of k dimensional manifolds over X. Other bordism theories, for any stable structure on a vector bundle, are also representable in this way. Note the difference between bordism and stable homotopy groups, they are both homology theories but stable homotopy is *reduced*.

This suggests that stable phenomena, stable invariants should be given by limits of homotopy groups where we smash with some "representing space".

2 An explicit construction of a category

We will now define the category of spectra, with the aim in mind, that we will make precise later, of capturing stable phenomena such as these.

Definition. *The category of sequential spectra, denoted $\mathcal{S}^{\mathbb{N}}$, has objects $E = \{E_n, \sigma_n\}_{n\in\mathbb{N}}$ sequences of spaces with maps $\sigma_n : \Sigma_n E_n \to E_{n+1}$. A morphism $f : E \to F$ is a collection of maps $f_n : E_n \to F_n$ such that the obvious diagram commutes*

$$\begin{tikzcd} S^1 \wedge E_n = \Sigma E_n \arrow[r, "\mathrm{id}\wedge f_n"] \arrow[d, "\sigma"] & \Sigma F_n \arrow[d, "\sigma"] \\ E_{n+1} \arrow[r, "f_{n+1}"] & F_{n+1} \end{tikzcd}$$

This category is not stable in the sense that we will later define, for one thing it has not enough isomorphisms, that is multiple spectra can represent the same homology theory (although all homology theories are represented). We define the homotopy groups of a sequential spectrum as a colimit

$$\pi_i(E) \quad := \quad \mathrm{colim}_k \pi_{i+k}(E_k)$$

Then the notion of weak equivalence between spectra is the same as Top, a map of spectra that induces an isomorphism on all homotopy groups. We then define the so called stable homotopy category as the localisation of sequential spectra at these weak equivalences.

Example. *The sphere spectrum is $\mathbb{S} = \{S^n\}_n$ where the structure maps are the homeomorphisms $\Sigma S^n \to S^{n+1}$.*

Example. $\mathbb{K}(\mathbb{Z}) = \{K(\mathbb{Z}, n)\}_n$ *whose structure maps are given by the adjoint of the homotopy equivalence $\Omega K(\mathbb{Z}, n+1) \simeq K(\mathbb{Z}, n)$, to be specific*

$$h \in [K(\mathbb{Z}, n), \Omega K(\mathbb{Z}, n+1)] \cong [\Sigma K(\mathbb{Z}, n), K(\mathbb{Z}, n+1)]$$

by the adjunction.

This category has an explicit description of the morphisms given by so called "cofinal" maps. This is a little clunky so we will just sketch it. For two CW spectrum, spectrum where every space is CW complex and the structure maps are inclusions of subcomplexes (they are the fibrant objects) a subspectrum $F \subseteq E$ is called cofinal if each cell of the spectrum (a cell in some space), under the inclusion into the higher levels of the spectrum, eventually ends up in F. Morally E is sort of becoming F in the limit. We can now describe the morphisms $E \to F$, let

$$S = \left\{(E', f') : E' \subseteq E \text{ cofinal }, f' : E' \to F \text{ a function}\right\}$$

be maps from cofinal subcomplexes into F. Then we say that two such maps are equivalent $(E', f') \sim (E'', f'')$ iff there is an (E''', f''') cofinal in $E' \cap E''$ such that f', f'' and f''' agree on E'''. (something like the germ of the functions). Composition can just be defined by picking representatives that make sense to compose.

Remark. A spectra is called an Ω spectrum if the adjoint maps to the structure maps are all (weak) homotopy equivalences, these are the fibrant objects. It is called CW if every space in the spectrum is a CW complex and the structure maps are inclusions of subcomplexes, these are the cofibrant objects in the unstated model structure. A spectra with only finitely many negative homotopy groups is called connective, with no negative homotopy groups is called connected. Every loop space defines a connected Ω spectrum by applying Ω^∞ defined below.

Remark. It is a corrolerary of the Brown representability theorem that every reduced cohomology theory is represented by a spectrum. Given a spectrum $\{E_n\}$ then we have a cohomology theory given by

$$h^n(X) = \mathrm{colim}_k[X \wedge S^{n+k}, E_{n+k}] = \mathrm{colim}_k[X, E_n]$$

(equality given by adjunction) or a reduced homology theory given by

$$h_n(X) = \mathrm{colim}_k[S^{n+k}, X_+ \wedge E_n]$$

One thing to note is that to get the cohomology theory we really need the structure maps; even though we can get the groups as above to construct the long exact sequences we need the maps. Also notive that we can make the theories unreduced by just precomposing with $X \mapsto X_+$.

Example. *We have already seen that $\Omega_k^{fr+st}(-)$ is the unreduced theory associated to the sphere spectrum, while $\pi_*^s(-)$ is the reduced theory. There is also $\mathbb{K}(\mathbb{Z})$ which represents ordinary cohomology and homology. Finally topological K theory is represented by $\mathbb{Z} \times BU$ in even degrees and $\Omega(\mathbb{Z} \times BU)$ in odd degrees.*

Remark. Sequential spectra do not have a nice notion of product defined on them, this is where things like symmetric and orthogonal spectra come in. By rigidity we can go between them at the level of the homotopy category.

3 Why *this* category?

In [BR20, 1.1.4] gives the following list of desiderata for a stable homotopy category $\mathcal{SHC}$, that is a category that captures all of our stable phenomena. The properties are as follows

1. There should be a natural adjunction

$$\Sigma^\infty : \mathrm{HoTop} \rightleftharpoons \mathcal{SHC} : \Omega^\infty$$

2. Given two spaces A, B there should be a bijection

$$[\Sigma^\infty A, \Sigma^\infty B] \cong [A, B]^{stab} \quad := \quad \mathrm{colim}_k[\Sigma^k A, \Sigma^k B]$$

   between homotopy classes of maps of their spectra and *stable* homotopy classes of maps of the spaces.

3. For every (co)homology theory there is a unique up to isomorphism object that represents theory.

4. A map in $\mathcal{SHC}$ should induce a natural transformation on the (co)homology theory.

5. Various nice categorical properties, such as being monoidal, cartesian closed and being enriched over graded abelian groups.

These are obviously all very reasonable and desirable, moreover encapsulate stability. Either explicitly requiring hom sets to be stable hom sets for spaces, or by requiring it to be representing cohomologies, which by our examples are somehow given by stable homotopy groups of smashing with spaces. Now the theorem is that the category of sequential spectrum localised at weak equivalences satisfies all of these properties. There is also a type of uniqueness, called rigidity

Theorem ([BR20], 5.7.1)**.** *If $\mathcal{C}$ is a stable model category and there is an equivalence of triangulated categories*

$$\mathcal{SHC} \to \mathrm{Ho}\mathcal{C}$$

then $\mathcal{C}$ is Quillen equivalent to sequential spectra.

We wont define any of this here but the point is that if two model categories present the stable homotopy category as its homotopy category then they are equivalent. Note that it doesnt say that if you satisfy *these* axioms you are equivalent to the stable homotopy category as defined above, merely that the stable homotopy category is one category satisfying these things and any model category representing *this* category are equivalent.

4 Some functors just lying around

Now we have defined our category of spectra as well as our stable homotopy category. Theyre are many functors to and from these categories that are useful to know.

There is an (Quillen) adjunction called the shifted suspension / evaluation

$$F_d : \mathrm{Top} \rightleftharpoons \mathcal{S}^{\mathbb{N}} : \mathrm{Ev}_d$$

$$X \mapsto \begin{cases} \Sigma^{n-d}, & n \geq d \\ *, & \text{else} \end{cases}$$

$$K_d \leftarrow\!\shortmid \{K_n\}_n$$

The left adjoint is faithful, the derived functors, those induced on the homotopy category are not faithful, as some maps may be identified after repeated suspension. We denote $\Sigma^\infty = F_0$, this can also be thought of as the left derived functor of the above. The right derived functor is denoted Ω^∞, note that it indeed always produces an infinite loop space. Moreover it is clear that

$$\pi_n(F_0(X)) = \pi_n^s(X)$$

the stable homotopy groups.

Unrelated to this is the functor given by wedging and homing to a topological space. Let A be an ordinary space and E be a spectrum, then

$$\begin{aligned} E \wedge A &:= \{E_n \wedge A\}_n \\ \mathrm{Hom}(A, E) &:= \{\mathrm{Hom}(A, E_n)\}_n \end{aligned}$$

which gives the loop and suspension of spectra $\Sigma E = E \wedge S^1$ and $\Omega E = \mathrm{Hom}(S^1, E)$. These two form a Quillen adjunction

$$\Sigma : \mathcal{S}^{\mathbb{N}} \rightleftharpoons \mathcal{S}^{\mathbb{N}} : \Omega$$

that is moreover an equivalence.

We might have indexed by $\mathbb{Z}$, however there is again a Quillen equivalence

$$\mathcal{S}^{\mathbb{N}} \rightleftharpoons \mathcal{S}^{\mathbb{Z}}$$

given by truncating in the backward direction and extending into the negative degrees by the point in the forward direction.

References

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