Ramification in Number Theory and Geometry

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Based on [Lor97, 3.5].

$$\begin{tikzcd}
\mathcal{O}_L \arrow[r, phantom, "\subseteq"] \arrow[d, dash, dashed] & L \arrow[d, dash] & \mathfrak{p}\mathcal{O}_L = \prod_i \mathfrak{P}_i^{e_i} \arrow[d, dash] \\
\mathcal{O}_K \arrow[r, phantom, "\subseteq"] & K & \mathfrak{p}
\end{tikzcd}$$

The usual setup, L/K is a field extension of degree n, K is the fraction field of $\mathcal{O}_K$ and $\mathcal{O}_{\mathcal{L}}$ is the integral closure of $\mathcal{O}_K$ in L, we also require B to be a finitely generated A module and A to be a Dedkind domain. Then we consider a prime of $\mathcal{O}_K$, $\mathfrak{p}$ and its decomposition in $\mathcal{O}_L$, $\prod_i \mathfrak{P}_i^{e_i}$.

Then a prime upstairs $\mathfrak{P}_i$ is **unramified** if $e_i = 1$, the condition on the seperability of the residue fields can be ignored if all finite extensions of the residue fields are seperable. In particular if the residue field of $\mathcal{O}_K$ at every prime is *perfect*. This is the case for anything that I would know of, in particular finite fields, characteristic 0 fields and algebraically closed fields are perfect. So if $\mathcal{O}_K$ is a number field or $\mathcal{O}_K = k[x, y]/(f)$ for f irreducible and k algebraically closed this condition is satisfied.

A prime downstairs $\mathfrak{p}$ is **unramified** if all the primes above it are unramified.

Now let k be an algebraically closed field and $f \in k[x, y]$ irreducible with $\deg_y(f) = n > 0$ and monic in y and $\operatorname{Spec} k[x, y]/(f)$ is non-singular. These two conditions ensure that $k[x, y]/(f)$ is the integral closure of $k[x]$ below. Then we have the following diagram,

$$\begin{tikzcd}
k[x,y]/(f) \arrow[r, phantom, "\subseteq"] & \mathrm{Frac}(k[x,y]/(f)) \arrow[d, dash] & (x-a, y-b) \arrow[d, dash] \\
k[x] \arrow[u, hook, dashed] \arrow[r, phantom, "\subseteq"] & k(x) & (x-a)
\end{tikzcd}$$

satisfying the conditions above, we really are using that the bottom left is $k[x]$ other wise it wouldnt be dimension 1 hence not a Dedekind domain its not clear to me why $\mathrm{Frac}(k[x, y]/(f))$ is a degree n extension of $k(x)$. non-zero Primes in $k[x]$ are of the form $(x - a)$ for some $a \in k$, and primes in $k[x, y]$ are of the form $(x - a, y - b)$ for $a, b \in k$, in particular if $(x - a, y - b)$ are the primes above $(x - a)$ this is not 100% clear to me. Finally because the residue field at any prime above and below are isomorphic, all inertial degrees are 1,

$$k[x, y](f)(x - a, y - b) \cong k \cong k[x]/[x - a]$$

we have by the fundamental identity that

$$\sum_i e_i = n.$$

Thus $(x-a)$ ramifies if there are less than n distinct primes above it, ramification in this case agrees with totally split! So ramification is the principle quantifier and totally split is like an addendum to ramified, like calling it mega-unramified. If we spec this inclusion we get the following map

$$\begin{tikzcd} \operatorname{Spec}\big(k[x,y]/(f)\big) \arrow[d, "\pi"] & (x-a, y-b) \arrow[d, mapsto] & (a,b) \arrow[d, mapsto] \\ \operatorname{Spec}\big(k[x]\big) & (x-a) & a \end{tikzcd}$$

and look at the fibres $\pi^{-1}(a)$. Then it is clear that $(x-a)$ is ramified iff $\pi^{-1}(a)$ has less than n points in it.

Remark. *So really its just a linguistic thing. You do your $\mathbb{C}$ algebraic geometry and you call the map unramified, then you look for an algebraic criteria, its exactly what you think it is. Great. Then you apply that criteria to more general rings and all of a sudden you have inertial degrees and the picture is a bit more blurry, so instead of changing the definition of ramified which you already had you make a new definition, totally split.*

THUS WHAT REMAINS TO BE DONE IS A HISTORICAL INVESTIGATION INTO WHETHER OR NOT THAT TRULY WAS THE PROCESS BY WHICH THESE NAMES CAME TO BE...

References

[Lor97] Dino Lorenzini. *An Invitation to Arithmetic Geometry.* Number 9 in Graduate Studies in Mathematics. American Mathematical Society, Providence, Rhode Island, repr. with corr edition, 1997.