Calculus on Manifolds

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1 Defining Integration

Obviously we can define integration more abstractly on an arbitrary measure space but our interest is in manifolds so we will restrict to the Lebesgue measure on the reals. First we define the volume of a cube in $\mathbb{R}^n$ as one would expect, merely as the product of the lengths of its sides.

$$\text{vol}\big([a_1, a_1 + \epsilon] \times \cdots \times [a_n, a_n + \epsilon]\big) = \epsilon^n$$

The Lebesgue (outer) measure is defined for any set $E \subseteq \mathbb{R}^n$ by the infimum over the volume of all coverings of the set by (almost disjoint) cubes. A set is called measurable if for every $\epsilon > 0$ there is an open set such that $\mu(U - E) < \epsilon$.

The other key object is a differential form. Given a manifold then its cotangent bundle is the fiberwise dual of the tangent bundle (linear functions into $\mathbb{R}$). Given local charts on the manifold $(x_1, ..., x_n) : M \to \mathbb{R}^n$ then there is local coordinates on the tangent bundle $\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right) : M \to TM$ and this induces local charts on the cotangent bundle $(dx_1, ..., dx_n) : M \to T^*M$ which are defined as

$$dx_i \frac{\partial}{\partial x_j} = \delta_{ij}$$

A one form is then a smooth section of this cotangent bundle, a general one form can be represented in local coordinates as

$$f_1 dx_1 + \cdots + f_n dx_n$$

where $f_i : M \to \mathbb{R}$ is a smooth function. Given the cotangent bundle then we can construct the bundle of k-forms by taking the fiberwise k-th exterior algebra (defined on any vector space). The exterior algebra is essentially the tensor algebra with some antisymmetry relations, and setting higher powers to zero. In local coordinates there is a basis given then by tensors or tuples of one forms.

1.1 Integration on $\mathbb{R}^n$

To define the integral of a (measurable) function $\mathbb{R}^n \to \mathbb{R}$ we will define it for simple functions and claim that any such function can be approximated by simple functions. A simple function on $\mathbb{R}^n$ is a linear combination of indicator functions on measurable sets.

$$\varphi(x) = \sum_{i=1}^{n} a_i 1(x \in E_i), \quad a_i \in \mathbb{R}$$

We define the integral of such a function as

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} a_i 1(x \in E_i) = \sum_{i=1}^{n} a_i \mu(E_i)$$

If f is a non-negative function then its integral is defined as the infimum of the integrals of simple functions that cover f.

$$\int f = \inf \left\{ \int \varphi : \varphi \text{ simple and } 0 \leq f \leq \varphi \right\}$$

where the function is leq to the other pointwise. For functions that have a negative part then we just break it into two functions, positive and negative (say by taking the max and min with the zero function) and then subtract the integral of the negative of the negative part

$$\int f = \int f_+ - \int f_-$$

The idea is clear, to find the area of an arbitrary function we cover it with step functions and take the smallest such covering. There are a lot of things to show to make all this work, most importantly that measurable functions can be approximated in this way.

1.2 Integration on Manifolds

On manifolds what we integrate are top forms. We can integrate lower degree forms over submanifolds on which they are top forms. Therefore it is sufficient to define the integration of top forms.

First consider a differential form on $\mathbb{R}^n$, we can take global charts and then a top form will in general be of the form

$$f(x) dx_1 \wedge \cdots \wedge dx_n$$

for a smooth $f : \mathbb{R}^n \to \mathbb{R}$. Then the integral of this form is defined as the integral of f

$$\int_{\mathbb{R}^n} f dx_1 \wedge \cdots \wedge dx_n \ := \ \int_{\mathbb{R}^n} f$$

where the right hand side is in the Lebesgue sense. For an abstract manifold M^n then we can define integration in a chart in this way. The final thing is to define it non-locally. We do this by fixing a partition of unity. Then in each local chart we have a well defined (one has to check) integral via the formula above and we define the integral over the total space to be the sum of the integrals over the partition

$$\int_M \omega = \sum_i \int_{U_i, \varphi_i} \varphi^* \omega = \sum_i \int_{\mathbb{R}^n} f_i$$

Remark. The point of the top form machinery is to create another bundle that is locally given by functions into $\mathbb{R}$ in a compatible way, that way we can integrate them. If we were to say take a global function $M \to \mathbb{R}$ and try to integrate it by picking charts and writing it in local coordinates it would not be well defined. See this post for a nice example.

Remark. For this setup to actually work the manifold must be orientable. Otherwise the integral is only defined up to a sign.

Remark. In the case of manifolds we integrate top forms which at least in local charts are given by a single smooth function. Notice that at least in a single chart this is measure theoretically a very strong condition to be smooth.

Remark. So far we have exclusively seen how to integrate functions whose codomain is the real numbers or in the case of manifolds the cotangent bundle, which at least locally is given by a function into the real numbers. Given a curve in a manifold $[0, 1] \to M$ then we can find its so called arc length. Note here that the integral we take is also of a real valued function *constructed from the curve*. Thus this is not anthing different. The same is true for line integrals.

Remark. If we have a function $\mathbb{R}^n \to \mathbb{R}^m$ then it is given by m functions $\mathbb{R}^n \to \mathbb{R}$. We may then take the integral of the vector valued function by just integrating each of the functions and collecting them into a vector. Thus the integral of a vector valued function in this sense is another vector. A special case of this is complex valued functions which are essentially $\mathbb{R}^2$ and we treat them as such by integrating the imaginary and real parts seperately.

Remark. In the context of automorphic forms mostly we integrate $\mathbb{C}$ valued functions. On occasion we have need of integrals over functions valued in infinite dimensional vector spaces. The Gelfand–Pettis integral is used in such cases (or Bochner integral). However even in this case the it is merely a construction to give a real valued function (via a pairing) that can then be Lebesgue integrated.

2 Total and Partial Derivatives

2.1 Derivatives on $\mathbb{R}^n$

The idea of the derivative is to find "the best linear approximation" to a function. Here we will consider all functions to be smooth. First consider a function $\mathbb{R} \to \mathbb{R}$ then we know that its derivative is given by the function
$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
This derivative function defines the slope of the tangent line at any given point, and so the *equation* of this tangent line at any given point x_0 is given by (using slope intercept form)
$$df(x) = \left(\frac{d}{dx} f(x_0) \right) (x - x_0) + f(x_0)$$

Which is the linear approximation of f at the point x_0. Alternitively by centering the function at the origin we can approximate it not just be a line but by a *linear function* $\mathbb{R} \to \mathbb{R}$ given by $\frac{d}{dx} f(x_0) x$. Thus there are two senses in which we can think of this as a linear approximation, as a tangent line or as a linear function.

If we have a function $\mathbb{R} \to \mathbb{R}^n$ then its derivative is just the derivative of the component functions, as it can be expressed as n functions $\mathbb{R} \to \mathbb{R}$. Using the vector addition and multiplication by scalars this is seen to be exactly the same formula above where $f(x)$ is now some vector. This gives us a line in $\mathbb{R}^n$ by taking the exact same equation as above too.

Functions $\mathbb{R}^n \to \mathbb{R}$ again define functions from $\mathbb{R} \to \mathbb{R}$ in a less canonical way, by fixing $n - 1$ of the variables. So at a point $(x_0, ..., x_n)$ the derivative in the x_i direction is denoted
$$\frac{\partial}{\partial x_i} f(x_0, ..., x_n) = \lim_{h \to 0} \frac{f((x_0, ..., x_n) + he_i) - f(x_0, ..., x_n)}{h}$$

where $e_i = (0, ..., 0, 1, 0, ..., 0)$ is the $i-th$ standard basis vector. We again have a function into $\mathbb{R}$ and so our linear approximation should be a line in $\mathbb{R}$. It will be the tangent to the hypersurface in the x_i direction. Our equation for df above still gives the equation for this line by fixing $n-1$ of the entries it is the equation of a line $\mathbb{R} \to \mathbb{R}$. Alternitively by taking the span of these lines we get a subspace of $\mathbb{R}^{n+1}$ that can be considered as a hyperplane to the *graph* of the function.

Finally functions $\mathbb{R}^n \to \mathbb{R}^m$ combine the last two generalisations. First we consider it as a colletion of m functions $\mathbb{R}^n \to \mathbb{R}$ and then we consider the partial derivatives of these functions in all directions. There is no clear way of interpreting this as a tangent line because there are multiple parametrers, it is more like the approximation of the m dimensional surface by an n dimensional hyperplane; taking the span of the n partial derivatives gives a plane for each function into $\mathbb{R}$ and there are m such planes approximating the function in each direction. We could have considered the function as n functions $\mathbb{R} \to \mathbb{R}^m$ howevever we claim that this should give the same matrix. In the end we have lots of derivatives and the only *clean* way of expressing it now is as df that is the linear (in the sense of vector spaces) approximation, if $f = (f_1, ..., f_m)$ then

$$df = \left(\frac{\partial}{\partial x_i} f_j \right)_{ij}$$

This is a matrix that summarises all the partial derivatives, however it also defines a linear function $\mathbb{R}^n \to \mathbb{R}^m$ which approximates the original in the same way as the other linear functions did. If we were to use it as the gradient of a hyper plane and then translate that plane to the relevant point we would get the required tangent plane thing. Note this is sometimes called the "total derivative" of the function.

2.2   Differentials on Manifolds

Over $\mathbb{R}$ we saw that the derivative gives us the linear approximation in many senses, and we can go between them using basic point line geometry. The easiest to write down however was always the linear approximation in the sense of linear algebra, that is a linear function $\mathbb{R}^n \to \mathbb{R}^m$. In the context of differential geometry it is no different, by talking about embeddings of submanifolds one can regain the explit equations of the tangent lines as linear subspaces, however this is cumbersome. Much easier to work with is often the embedding free definition.

Thus if we have a map between two manifolds $f : N \to M$ then we define its derivative as the best linear approximation in the sense of linear algebra as the smooth map on the only vector spaces around $df : TN \to TM$. It has a natural definition for whatever definition of the tangent bundle you choose, however the key point is that in local charts f can be written as a function $\mathbb{R}^n \to \mathbb{R}^m$ and its derivative is the matrix given above.

Remark.   Another notion of derivative is given by distributions. One can in this setting differentiate more than just smooth functions but only "in the sense of distributions".

3   Fundamental Results of Calculus

3.1   Stokes Theorem

Consider A manifold with boundary M^n and a differential $n-1$ form on M call it ω. Then $d\omega$ the exterior derivative is a top form on M and we have

$$\int_M d\omega = \int_{\partial M} \omega$$

A special case of this is if we consider a differential top form on $\mathbb{R}$, $f(x)dx$ then it restricts to a top form on a submanifolds with boundary given by closed intervals $[a, b]$ Stokes theorem then implies that

$$\int_{[a,b]} f(x)dx = \int_{\{a,b\}} f(x) = f(b) - f(a)$$

where we have used the fact that f is a zero form and that $d(f) = fdx$.

3.2   Laplacians

Given a function $\mathbb{R}^n \to \mathbb{R}$ then it has several partial derivatives. These can be summarised by the ∇ operator given by

$$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right)$$

By considering this as the degenerate $m = 1$ case of a map $\mathbb{R}^n \to \mathbb{R}^m$ then we see that

$$\nabla f = df$$

as we have defined above. That is applying ∇ is exactly getting its total derivative. Note that this is now a vector field $\mathbb{R}^n \to \mathbb{R}^n$!

Taking $\Delta := \nabla^2 = \nabla \cdot \nabla$ we get the Laplacian of a scalar function. This operator is informing how much nearby points are influcing the behaviour at a given point. An important point of view is via the divergence of a vector field, that is a function $F : \mathbb{R}^n \to \mathbb{R}^n$,

$$\mathrm{div}(F) = \nabla \cdot F$$

and so we see that $\Delta(f) = \mathrm{div}(\nabla f)$.

This definition generalises to function between Riemannian manifolds (manifolds with a nice metric) via the Riemannian divergence and gradient operations. The idea is to reformulate the definition of the divergence and grad operator in terms of the inner product on $\mathbb{R}^n$, this then generalises by just taking the inner product given by the metric.