Smoothness

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1 Ringed Spaces

Quoting from Miles Reid's undergraduate algebraic geometry:

> The specific flavour of algebric geometry comes from the use of only polynomial functions; to explain this, if $U \subseteq \mathbb{R}^2$ is an open interval, on can reasonably consider the following rings of functions on U:
>
> $$C^0(U) = \text{continuous functions } f : U \to \mathbb{R}$$
>
> $$C^\infty(U) = \text{smooth functions } f : U \to \mathbb{R}$$
>
> $$C^\omega(U) = \text{analytic functions } f : U \to \mathbb{R}$$
>
> $$\mathbb{R}[X] = \text{polynomial functions } f : U \to \mathbb{R}$$
>
> These have the inclusions
>
> $$\mathbb{R}[X] \subset C^\omega \subset C^\infty \subset C^0$$
>
> these rings of functions correspond to some of the important categories of geometry: topological manifolds, differentiable manifolds, analytic geometry and algebraic geometry.

A perspective on these spaces, in view on the embedding theorems for manifolds, is that they are the zero sets of functions in these sets that are glued together. Then it is natural to wonder, can these zero loci be compared.

2 Subsets of $\mathbb{R}$

2.1 Smooth Compact Manifolds are Algebraic

Following [Kol16]. A **real algebraic set** is the common zero set of a collection of real polynomials, i.e. $f_i \in \mathbb{R}[x_1, ..., x_n]$ then

$$X = \{x \in \mathbb{R}^n : f_i(x) = 0 \ \ \forall i\}.$$

Note that one can think of these as the real points of an algebraic variety (perhaps defined over $\mathbb{C}$). Note that we refer to these as a set, we do not want to give them the Zariski topology, as this will never relate to manifold topologies. Then we have the following:

Theorem ([Kol16], Thm. 2)**.** *Every smooth, compact manifold is diffeomorphic to a real algebraic set.*

Example. *The circle is the compact manifold, it is also given as the zero locus of the polynomial*

$$x^2 + y^2 - 1.$$

This theorem is "tight" in the following sense.

Example (Compact)**.** $\mathbb{R}^2 \backslash \mathbb{Z}$ *is a smooth manifold that is not a variety.*

Example (Smooth)**.** *Algebraic sets and smooth manifolds are triangulizable. Topological manifolds are not always triangulizable, as exhibited by the counter example* E_8*.*

2.2 Non-Singular Varieties

Following [Har77]. Let k be an algebraically closed field. If $Y \subset \mathbb{A}^n_k$ is an affine variety of dimension r defined by the polynomials $f_1, ..., f_m$ then we say that it is **non-singular** at $y \in Y$ when

$$\left(\frac{\partial f_i}{\partial x_j}(y)\right)_{i,j}$$

is of rank $n - r$. Y is non-singular if it is non-singular at every point.

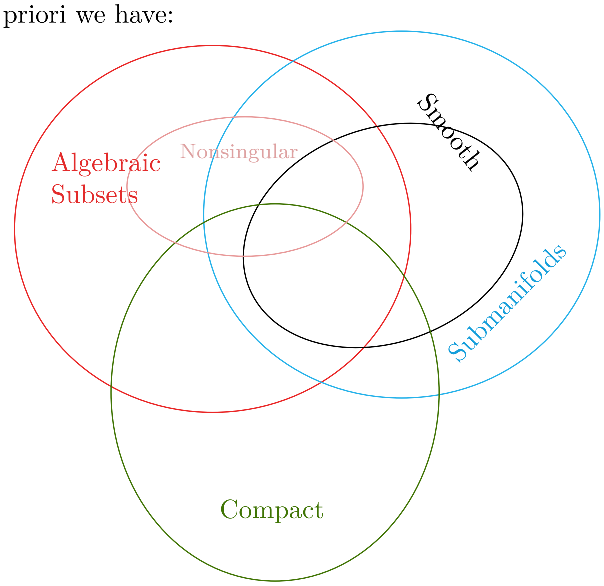
Lemma (Non-singular real varieties)**.** *Let X be a non-singular variety defined over $\mathbb{C}$. Then $X(\mathbb{R})$ can be given the structure of a smooth manifold.*

Proof. Apply the implicit function theorem to the defining polynomials. ∎

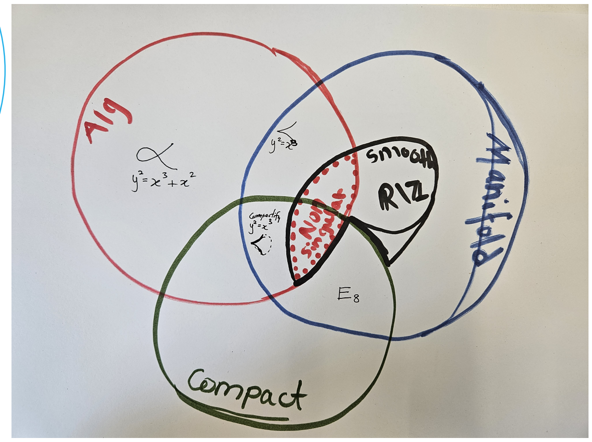
Question: Let M be a smooth manifold, that is the zero set of some real polynomials. Then is it the real points of some non-singular variety?

2.3 Venn Diagrams

A priori we have:



A fortiori we have:



Question: DO ORBIFOLDS COVER BOTH RED AND BLUE? What I now want is a classfication of: 1. Smooth manifolds that are not algebraic. 2. Algebraic varieties that are not topological manifolds. We know that these things are different but somehow they dont feel *very* different. Conjecture 1. Finite genus smooth manifolds are algebraic. Conjecture 2. Algebriac varities without crossings are topological manifolds.

3 Smooth Maps for Schemes

The notion of smooth morphism is a relative version of the notion of non-singular variety.

Definition. *Now we consider finite type schemes over a base field k, no longer required to be algebraically closed. A morphism of such schemes $f : X \to Y$ is **smooth** if*

- *It is flat*
- *There are $X' \subseteq X, Y' \subseteq Y$ irreducible components with $f(X') \subseteq Y'$ such that*
$$\dim X' = \dim Y' + n$$
- *For every $x \in X$ we have*
$$\dim_{k(x)}\left(\Omega_{X/Y} \otimes k(x)\right) = n$$

where $\Omega_{X/Y}$ is the algebraic sheaf of differential forms [Har77, II.8].

Example. *Let $X \to \operatorname{Spec} k$ for k an algebraically closed field field. Then this map is smooth iff X is regular of dimension n. In particular for X irreducible and separated then it is smooth iff it is a non-singular variety.*

So non-singular is a property of schemes, smooth is a property of morphisms, and non-singular should be something like smooth over the base. A scheme is called **equi-dimensional** if all irreducible components have the same dimension. If we denote the residue field at a point $y \in Y$ by $k(y)$ then we have.

Theorem. *$f : X \to Y$ be a morphism of finite type schemes over a field k. f is smooth iff*

- *It is flat*
- *For each point $y \in Y$, $X_{\bar{y}} := f^{-1}(y) \otimes_{k(y)} \bar{k}(y)$ the "fibers" is equi-dimensional and regular.*

Lemma (Wikipedia.)**.** *For a perfect field k, a scheme X is smooth over k iff X is locally of finite type and regular.*

Let $f_1, ..., f_c \in R[x_1, ..., x_n]$ for $n \geq c \geq 0$ then $R[x_1, ..., x_n]/(f_1, ..., f_c)$ is standard smooth over R if

$$\det \left(\frac{\partial f_j}{\partial x_i} \right)_{1 \geq i,j \leq c}$$

is a an invertible element of $R[x_1, ..., x_n]/(f_1, ..., f_c)$. ⚠ Note that we truncate the matrix and assume that $n \geq c$ so that it is square, this **depends on the presentation**. A morphism of affine schemes $f : X \to S$ is standard smooth if ⚠ there exists a presentation $X = \operatorname{Spec} R[x_1, ..., x_n]/(f_1, ..., f_c), S = \operatorname{Spec}(R)$ such that $R \to R[x_1, ..., x_n]/(f_1, ..., f_c)$ is a standard smooth ring map.

Lemma (Stacks Project Lemma 29.34.11)**.** *A morphism is smooth at a point iff there is an affine open such that the restriction is standard smooth.*

Definition ([Mum99], III. Def 3)**.** $\operatorname{Spec} R[x_1, ..., x_{n+k}]/(f_1, ..., f_n) \to \operatorname{Spec}(R)$ *is smooth at a point if*

$$\operatorname{rank} \left(\frac{\partial f_i}{\partial x_j}(y) \right)_{i,j} = n$$

Definition (Smooth)**.** *A scheme X over S is smooth if the morphism $X \to S$ is smooth.*

Definition (Regular)**.** *If A is a noetherian local ring with maximal ideal $\mathfrak{m}$ then A is* ***regular*** *if* $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A)$.

A scheme is regular if around every point there is an affine open such that the structure sheaf applied to the open is regular.

Lemma. *If k is a perfect field X is smooth over* $\operatorname{Spec} k$ *iff it is regular.*

[Stacks Lemma 33.25.8], `https://math.stackexchange.com/questions/3218502/problem-with-jacobian-criterion-and-regular-local-ring`

4 Examples

Example (Reduction)**.** *Consider the scheme over* $\operatorname{Spec} \mathbb{Z}$ *given by*

$$\operatorname{Spec} \mathbb{Z}[x, y]/(y^2 - x^3 - 13) \to \operatorname{Spec} \mathbb{Z}.$$

We can base change to $\mathbb{F}_p$ for $p \neq 2, 3, 13$ and we get

$$\operatorname{Spec} \mathbb{F}_p[x, y]/(y^2 - x^3 - 13) \to \operatorname{Spec} \mathbb{F}_p,$$

Look at the Jacobian

$$(-3x^2, 2y)$$

which to be rank less than 1 will have to be the zero vector (if we were in characteristic 2 or 3 then this would not be so). If $-3x^2 = 2y = 0$ then becuase we are in a field $x = y = 0$ but this does not lie on the curve as $-13 \neq 0$. I AM BEING NAIVE HERE AND JUST EVALUATING THE POLYNOMIAL ON THE ELEMENTS OF FP IS THAT VALID? ALG CLOSED?

If we base change to $\mathbb{F}_{13}$ we get

$$\operatorname{Spec} \mathbb{F}_{13}[x, y]/(y^2 - x^3) \to \operatorname{Spec} \mathbb{F}_{13}$$

which we claim is not smooth because at $(0, 0)$ the jacobian is rank 0. Note that what has changed is that now $(0, 0)$ is on the curve, as before it was not.

Note that this same reasoning can show that the Weirstrass equation $y^2 = x^3 + Ax + B$ defines an elliptic curve so long as $4A^3 + 27B^2 \neq 0$ (for characteristic not $2, 3$). i.e. the equation is smooth as long as the discriminant is non-zero. The proof is in [Sil09, Prop 1.4].

Because smoothness is invariant under base change we can see here that this is not a smooth scheme over $\operatorname{Spec} \mathbb{Z}$. So being smooth over $\mathbb{Z}$ is a *very* strong condition, it would require being smooth over *every field / ring*.

This scheme is defined over $\mathbb{Z}$ and so we can also take its real points. Using the Jacobian criterion for smoothness and the fact that $\mathbb{R}$ is a field we see that again the curve is smooth.

Question: WHAT IS THE RELATION BETWEEN THE SMOOTHNESS OF THE REAL POINTS AND SMOOTHNESS OF THE REDUCTION, can we say anything more than on a case by case basis.

Jacobian ideal `https://math.stackexchange.com/questions/1550652/relationship-between-discriminants-and-smoothness-of-curves`

References

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