Setting Up K & A Theory

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References are [Kup], [Wal85], [Hat78], [Wei].

Here we will give the definition of Waldhausen K-theory. The defintion of K_0 is at this point classical, for a ring at least it is

$$K_0(R) = (\{\text{finitely gen projective R modules}\}/\text{iso}, \oplus)^+$$

where $+$ denotes the group completion functor sending a commutative monoid to its completion. There were definitions of K_1 given for a ring which arrise when talking about the Whitehead torsion in the s-corbordism theorem given by

$$K_1(R) = \text{colim}_n H_1(\text{GL}_n(R))$$

and it was noticed that there was an exact sequence [Wei, II.6.4.1, III.3.2]

$$K_1(\mathbb{F}_p) \to K_1(\mathbb{Z}_{(p)}) \to K_1(\mathbb{Q}) \to K_0(\mathbb{F}_p) \to K_0(\mathbb{Z}_{(p)}) \to K_0(\mathbb{Q})$$

and it was conjectured that this should be the begining of the long exact sequence in homotopy groups for a fibration of spaces

$$K(\mathbb{F}_p) \to K(\mathbb{Z}_{(p)}) \to K(\mathbb{Q})$$

More generally it was seen that we have an exact sequence for any ring

$$K_1(R) \to K_1(R[1/s]) \to K_0(R/sR) \to K_0(R) \to K_0(R[1/s])$$

Thus the goal is to define a topological space whose homotopy groups give the K groups.

The way that this is done is to construct a certain simplicial set and then take the geometric realisation. In Quillens original construction he started with a category, with the extra condition that it was “exact”. This is sort of a weakening of the notion of abelian. Note that this means that we cant take the K theory of the category of topological spaces or subcategories as Top is the furthest thing from abelian (although we can still take the K theory of rings associated to topological *spaces*). The innovation of Waldhausen then is to introduce a much weaker categorical structure, minimal in the extreme, that allows a construction to go through.

Remark. To put a very fine point on it, exactness means that if we want to take K theory for a category associated to a topological space we have to go through an algebraic category, Waldhausen means we can stay in the realm of topological spaces (retractions over a space to be exact). Note that Top itself is not Waldhausen as it is not pointed, however certain over categories of topological spaces will be.

Remark. Exact categories are just categories with classes of maps that behave like exact sequences, in particular they have "admissable monomorphisms" which one can think of as the first of two maps $A \to B \to C$ which are declared to be admissible sequences.

Remark. (Liors Perspective) In normal K theory the moral is that we want to group complete some commutative monoid. If we instead considered a commutative monoid object in the category of spaces (say Top or homotopy types) then it would make sense to weaken everything so that it was only up to homotopy. In particular we could consider a homotopy coherent commutative monoid, that is a monoid only up to homotopies of homotopies of homotopies etc. If we go to complete this monoid we would want to do it in a homotopy coherent way. So first we need to split the short exact sequences $X \to Y \to Z$ but then we also need to keep track of the homotopies between these splitings and split them in a coherent way. These are forming the simplicies of the $S_\bullet$ construction. More concretely we have that for a category there is an associated groupoid $\mathcal{C}^{\simeq}$ which is all objects and morphisms are just isomorphisms. Then this gives us a homotopy coherent monoid in spaces simply by taking $|N\mathcal{C}^{\simeq}|$, the realization of the nerve. Now we see that the construction is group completing this homotopy coherent monoid in a homotopy coherent way. Looking closer at the geometric realisation in the $S_\bullet$ K theory construction we see that the one cells are the objects of the category $\mathcal{C}$, therefore we get a loop for each object (all connected to a single 0 cell). Then the two cells are the relations generated by the splitting of the sequences of maps in $\mathcal{C}$, the three cells are given by homotopies between these splittings etc. Thus *at the level of spaces* we are just getting the group completion of the monoid of the category! Finally we take the loop of this space just to correct the indexing, that is we put loops in for objects, but we want K_0 to be the group completion of the objects so we shift it down by looping it. Notice that we are just building a space by first putting a one cell for objects, then adding relations to split sequences, i.e. to quotienting out relations on the loops; this is just building a space whos π_1 is the group completion of the objects! And so it goes with the higher groups.

1 Waldhausen Categories

A category is called pointed if it has a zero object, that is an object that is both initial and terminal.

Let $\mathcal{C}$ be a pointed category with two subcategories $c(\mathcal{C}), w(\mathcal{C})$ (subcategories implies closed under composition). We call the former cofibrations and the latter weak equivalences. We denote cofibrations with $\rightarrowtail$ in diagrams. Such a category is called *Waldhausen* if the following are satisfied:

- All isomorphisms are cofibrations and weak equivalences.
- For all $A \in \mathcal{C}$ we have that the map $* \to A$ is a cofibration.
- (Base change)

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d, dashed] \\ C \arrow[r, tail, dashed] & B \cup_A C \end{tikzcd}$$

  So if we have $A \rightarrowtail B$ a cofibration and $A \to B$ any morphism then the pushout exists and the canonical map $C \rightarrowtail C \cup_A B$ is a cofibration.

- (Gluing)

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd, tail] \arrow[dr, "\sim"] & & C \arrow[dd, tail, dashed] \arrow[dr, "\sim"] & \\
& A' \arrow[rr] \arrow[dd, tail] & & C' \arrow[dd, tail, dashed] \\
B \arrow[rr, dashed] \arrow[dr, "\sim"] & & B \cup_A C \arrow[dr, dotted, "\sim"] & \\
& B' \arrow[rr, dashed] & & B' \cup_{A'} C'
\end{tikzcd}$$

  That is if the maps coming out of the page are weak equivalences and the maps vertically down are cofibrations then the map between the pushouts is a weak equivalence. Note the map along the bottom is that induced by the universal property of the pushout

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd, tail] & & C \arrow[dd, tail, dashed] \arrow[dr] & \\
& & & \bullet \arrow[dd, tail, dashed] \\
B \arrow[rr, dashed] \arrow[dr] & & B \cup_A C \arrow[dr, dotted, "\sim"] & \\
& \bullet \arrow[rr, dashed] & & B' \cup_{A'} C'
\end{tikzcd}$$

Example. *A category that has a zero object and admits all pushouts can be given a cofibration structure by declaring all morphisms to be cofibrations.*

Example. *Given a category with cofibrations we can always take two sets of weak equivalences. The minimal set is the collection of isomorphisms. The maximal set is all morphisms.*

Example. *Given an exact category in the sense of Quillen then we can define the cofibrations as the admissible monomorphisms.*

Example (Abelian Categories / Modules)**.** *Given a (commutative unital) ring R we have the category of finitely generated R-modules, which is abelian. An abelian category is Waldhausen by taking cofibrations to be monomorphisms and weak equivalences to be isomorphisms. Note that this is just taking the exact category associated to the abelian category and then the associated Waldhausen category.*

Example (Finite Sets)**.** *The category of finite sets is a Waldhausen category when cofibrations are injections and weak equivalences are bijections.*

Example (Top)**.** *Top is not pointed! The initial is the empty set and the final is the one point and these are not isomorphic.*

2 The Simplicial Set

We start with the category $[n]$ given by $0 \to 1 \to \cdots \to n$, plus compositions of maps and identities. Then there is an arrow category $\mathrm{Ar}[n]$, whose morphisms are commutative squares. Then for any category $\mathcal{C}$ we can form the functor category

$$\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{C})$$

If we further assume that $\mathcal{C}$ is Waldhausen we can define $S_n(\mathcal{C})$ as a full subcategory on objects that satisfy the following

1. $F(\mathrm{id_i}) = 0$ the zero object in the Waldhausen category.
2. $F((i,j) \to (i,k))$ is a cofibration for every $i \leq j \leq k$.
3. For $i \leq j \leq k$ we have the following diagram

$$\begin{tikzcd}
F(i,j) \arrow[r, "F((i,j)\to(i,k))"] \arrow[d, "F((i,j)\to(j,j))"'] & F(i,k) \arrow[d, "F((i,k)\to(j,k))"] \\
F(j,j) \arrow[r, "F((j,j)\to(jk))"'] & F(j,k)
\end{tikzcd}$$

   is a pushout.

This defines a category for each n and we claim then that

$$S_\bullet(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathrm{Cat}$$

is a simplicial category (a simplicial set that lands in the category of categories). There is a sub (functor) simplicial category given by $wS_n(\mathcal{C})$ given by taking the same objects and only natural transformations that are given by weak equivalences. Then $wS_\bullet(\mathcal{C})$ is also a simplicial category.

Remark. This simplicial set can also be described as the nerve of some gross diagram category, which however is simply making all the stuff here more explicit.

The last two are not clear, are they morphisms in the image of the functor or are they arbitrary morphisms between the things in the image of F? Waldhausen says "the" maps and so they should be somehow uniquely the ones under the functor so this is my guess, however its not clear.

2.1 The Shape of the Diagram

Just to make things a bit more explicit. Lets start by investigating the structure of this arrow category a bit more carefully, as functors out of it can be considered as subcategories in the target "of the same shape" as the the arrow category.

The objects of $\mathrm{Ar}[n]$ are things of the type $0 \leq i \xrightarrow{\leq} j \leq n$. Arrows between such pairs are commuting diagrams of the shape

$$\begin{tikzcd}
i \arrow[r, "\leq"] \arrow[d, "\leq"'] & j \arrow[d, "\leq"] \\
k \arrow[r, "\leq"'] & l
\end{tikzcd}$$

We can think of an object as an interval on a number line and then an arrow will only exist between this interval and another that is shifted to the right:

$$\begin{tikzcd} 0 \arrow[rrrr, dash] & & & & n \end{tikzcd}$$

$$\begin{tikzcd} i \arrow[rr] \arrow[dr, dashed] & & j \arrow[dr, dashed] & \\ & k \arrow[rr] & & l \end{tikzcd}$$

What we need is that the head of the second arrow is after the head of the first and the tail is after the tail of the first. The lengths ar not related.

Now $\mathrm{Ar}[n]$ has exactly $\binom{n+2}{2} = \frac{(n+1)(n+2)}{2}$ objects, that is pairs of numbers where the second is larger than the first.

For instance for $n = 0$ $\mathrm{Ar}[0]$ is just the singleton category with only the identity. For $\mathrm{Ar}[1]$ is is the following category

$$\begin{tikzcd} 0 \to 0 \arrow[d, Rightarrow] \\ 0 \to 1 \arrow[d, Rightarrow] \\ 1 \to 1 \end{tikzcd}$$

along with the identity morphisms. And $\mathrm{Ar}[2]$ is

$$\begin{tikzcd} 0 \to 0 \arrow[d, Rightarrow] & 1 \to 1 \arrow[d, Rightarrow] \\ 0 \to 1 \arrow[d, Rightarrow] \arrow[ur, Rightarrow] & 1 \to 2 \arrow[d, Rightarrow] \\ 0 \to 2 \arrow[ur, Rightarrow] & 2 \to 2 \end{tikzcd}$$

We can see the pattern for $\mathrm{Ar}[3]$

$$\begin{tikzcd} 0 \to 0 \arrow[d, Rightarrow] & 1 \to 1 \arrow[d, Rightarrow] & 2 \to 2 \arrow[d, Rightarrow] & 3 \to 3 \\ 0 \to 1 \arrow[d, Rightarrow] \arrow[ur, Rightarrow] & 1 \to 2 \arrow[d, Rightarrow] \arrow[ur, Rightarrow] & 2 \to 3 \arrow[ur, Rightarrow] & \\ 0 \to 2 \arrow[d, Rightarrow] \arrow[ur, Rightarrow] & 1 \to 3 \arrow[ur, Rightarrow] & & \\ 0 \to 3 \arrow[ur, Rightarrow] & & & \end{tikzcd}$$

If we relable the objects as bullets then this is the same diagram as

$$\begin{tikzcd} \bullet \arrow[d] & & & \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & & \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

and so we can see that a functor out of $\mathrm{Ar}[n]$ is a diagram of this shape in our category (going up means adding another diagonal of dots and stiching them into the diagram. Keeping this in mind we can see further that if we are requiring our diagrams to satisfy the $S_\bullet$ conditions then we have that the diagram is of the form

$$\begin{tikzcd}
0 \arrow[d, tail] \\
\bullet \arrow[r] \arrow[d, tail] & 0 \arrow[d, tail] \\
\bullet \arrow[r] \arrow[d, tail] & \bullet \arrow[r] \arrow[d, tail] \arrow[ul, phantom, "\ulcorner", very near start] & 0 \arrow[d, tail] \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & \bullet \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & 0
\end{tikzcd}$$

the diagonal has to be the zero object by (1), the verticle arrows are cofibrations by (2) and all the squares are pushouts by (3).

3 Defining K Theory

The Waldhausen K theory of a Waldhausen category $\mathcal{C}$ is defined as

$$K : \text{Waldhausen - Categories} \to \text{Top}$$

$$K(\mathcal{C}) \ := \ \Omega|wS_\bullet(\mathcal{C})|.$$

The K groups are then given by the homotopy groups of this space. Note that the various K theory constructions all produce spaces, these spaces are only the same *up to weak equivalence*. This means that the space produced is different but the K groups are all the same. Notice also that for $n \geq 0$ we have that

$$\pi_n K(\mathcal{C}) = \pi_{n+1}|wS_\bullet(\mathcal{C})|.$$

Example (Rings)**.** *As was previously discussed for a ring R we call its K theory the K theory of its category of finitely generated projective modules, if we dont require projective then [Wei] refers to this as G.*

We want to check that the K_0 given by this definition agrees with our expectations. This is clear from the following lemma: The Waldhausen K_0 of $\mathcal{C}$ is the abelian group on weak equivalence classes of objects in $\mathcal{C}$ subject to the relations that for every cofibration $A \rightarrowtail B$ we declare $[B] = [A] + [B \cup_A 0]$ (this is to be thought of as splitting the ses $A \to B \to B/A$). This is sketched in [Wei, IV.8.4]. The idea is that for a simplicial space (simplicial set that factors through Top) we know something about the homotopy groups of the geometric realisation, in particular

$$\pi_1|X_\bullet| = \langle \pi_0 X_1 \rangle / \langle \partial_1(x) = \partial_2(x)\partial_0(x),\ x \in \pi_0(X_2) \rangle$$

so the first homotopy group is just the free group on the connected components of the one simplicies modulo the things coming from the two simplicies. It is then a matter of understanding the one and two simplicies of the $S_\bullet$ construction. The one simplicies are $0 \to A \to 0$, just objects, but with the w we require our natural transformations to be weak equivalences, so the connected components of this space is the weak equivalence classes of objects. Then the two boundary maps are just sending sequences like $A \to B \to B/A$ to each of the three objects (indexed 0,1,2). Thus we are just splitting them with the relations.

Then we can just see that for the cagtegory of projective modules this is the group complete, as all sequences of projective modules split (universal property of projective).

Example. *//Are there any familar spaces that can be constructed this way? How about up to weak homomtopy... Are all weak homotopy types attainable this way?*

Remark. ([Wei, IV.8.5.5]) The space given by this construction is an infinite loop space and can therefore be thought of as a connected spectrum.

Remark. This is not an easy thing to calculuate despite all the tools we have. For instance $K(\mathbb{Z})$ is still unknown. Put the details here and references.....

4 Defining A Theory

Clearly defined in [Wei, EII.9.1, pg 170, Ex 8.7.1, pg 338]. A theory is refered to as "the K theory of spaces". We will start with a topological space, construct a Waldhausen category out of it and then take the K theory.

Let X be a CW complex, then the category $\mathcal{R}(X)$ has as objects CW complexes formed from X by attaching finitely many cells and such that there is a retraction to X (a map $r : Y \to X$ such that $r|_X = \mathrm{id}_X$). Morphisms are cellular maps that are compatible with the retraction (this ensures it is pointed). This forms a Waldhausen category where cofibrations are (cellular) inclusions and weak equivalences are topological weak equivalences (homotopy equivalances because we are in CW). Then the A theory of X is given by

$$A(X) \ := \ K(\mathcal{R}(X)).$$

Example (Of a Point)**.** *The category of finite retractive spaces over the point is just the category of finite pointed CW complexes that have a single 0 cell, (Weibel doesnt include this condition, why is it superfluous?) with cellular maps (everything retracts onto the point). Note that it is not all CW complexes, for instance not the line, but up to weak homotopy it is path connected CW complexes. The question is then what is the difference between $wS_n(\mathcal{C})$ and $S_n(Ho(\mathcal{C}))$?*

[Wei, II.9.1.5] calculates $K_0(*) = \mathbb{Z}$.// include.

The higher ones are rationally equivalent to $K(\mathbb{Z})$ (see [Kup, §25]) which is still an open problem to compute and so I dont expect that it is well known.

Remark. The zero object in this category of retractive spaces is X. It is clearly terminal because any map to X has to be the retraction for it to be compatible with the retraction. It is initial again because the retraction ensures a unique inclusion, if you include as another subspace then the retraction would not fix that subspace.

References

[Hat78] A. E. Hatcher. Concordance spaces, higher simple-homotopy theory, and applications. In R. Milgram, editor, *Proceedings of Symposia in Pure Mathematics*, volume 32.1, pages 3–21. American Mathematical Society, Providence, Rhode Island, 1978.

[Kup] Alexander Kupers. Lectures on diffeomorphism groups of manifolds, version February 22, 2019.

[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In Andrew Ranicki, Norman Levitt, and Frank Quinn, editors, *Algebraic and Geometric Topology*, volume 1126, pages 318–419. Springer Berlin Heidelberg, Berlin, Heidelberg, 1985. Series Title: Lecture Notes in Mathematics.

[Wei] Charles A Weibel. The K-book an introduction to Algebraic K-theory.