Analytic vs Smooth Lie groups

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There is an equivalence of categories given by the forgetful functor between

$$\big(\text{Real analytic groups, analytic maps}\big) \to \big(\text{Real Lie groups, smooth maps}\big).$$

This map is clearly functorial, as analytic maps are smooth ([Ser92, Part II, Chapter II pg 73] for the faithless), and it is immediate that **the forgetful functor is faithful.**

The definition of smooth Lie group is in any geomtry book, and the definition of analytic groups can be found in [Ser92].

Theorem (Serre, Part II, Chapter V.9, Thm 2 + Remark)**.** *The category of analytic groups over $\mathbb{R}$ or $\mathbb{Q}_p$ is a full subcategory of of all locally compact topological groups.*

In particular real analytic groups are a full subcategory of real Lie groups, this category is by definition the essential image of the forgetful functor. **Thus the forgetful functor is full.**

Now Whitneys classic result [Whi36] shows that every smooth manifold can be *smoothly* embedded into $\mathbb{R}^n$ as a *analytic* submanifold. First for compact manifolds [Mor58] and then for non-compact manifolds less than a year later [Gra58] showed that every analytic manifold can be *analytically* embedded into R^n. The point is that

Theorem (Morrey-Grauert Theorem)**.** *Every smooth manifold can be made into an analytic manifold that is unique up to analytic isomorphism.*

The proof is that any two embeddings are analytically isomorphic to the original space and therefore to one another. This shows that **the forgetful functor is essentially surjective.** Thus *the forgetful functor is an equivalence.*

References

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