# Why Are Lie Alegbras Everywhere

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This will be a quick answer to the question "why is all the theory of algebraic groups about Lie Algebras". The main reference is [**?**, text].

A lot of what I have seen so far for the represenation theory of algebraic groups was done along the lines of:

- Know what happens when you just look at the Lie algebra
- Guess what the result for the group should "look like" based on this

So specifically I want to understand how the two are related, and how this "guessing" is actually being made precise.

## 1 Fully Faithful Philosophy

It is an easy exercise to show that if $F : \mathcal{C} \to \mathcal{D}$ is a fully faithful functor then

$$FX \cong FY \implies X \cong Y$$

Thus a fully-faithful functor is "injective up to isomorphism". In particular they define equivilences of categories onto the subcategory that is their image. Thus the domain of a fully-faithful functor can be thought of as a subcategory of the codomain.

## 2 Algebraic Groups are Hopf Algebras

If we consider affine group schemes as group objects in the the category of affine schemes (as representable functors) then this amounts to the extra structure of natural transformations

$$m : G \times G \to G$$

$$e : * \to G$$

$$(-)^{-1} : G \to G$$

making some diagrams commute. When we apply the coordinate ring functor (contravariant) to this we get algebra morphisms

$$\Delta(G) : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$$

$$\epsilon(G) : \mathcal{O}(G) \to \mathcal{O}(*) = k$$

$$S(G) : \mathcal{O}(G) \to \mathcal{O}(G)$$

The diagrams and these functions on $\mathcal{O}(G)$ ensure that $\mathcal{O}(G)$ is a Hopf algebra, with respectively co-multiplication, co-unit and antipode.

# 3 The Functor Lie

We give two constructions of the Lie algebra of an algebraic group (everything is done over a field of charcteristic 0), this is because they parrallel two different constructions of the tangent space of a lie group; one as derivations and one as curves.

Let G be an algebraic group over the field k.

## 3.1 Infinitesimals

Given a k-algebra R define

$$R[\epsilon] \quad := \quad R[X]/(X^2) \cong R \oplus R\epsilon$$

for $\epsilon^2$, where the decomposition is as R modules. This makes the following morphisms clear

$$R \xrightarrow{i} R[\epsilon] \xrightarrow{\pi} R$$

$$a \mapsto a + \epsilon 0, a + \epsilon b \mapsto a$$

Moreover their composition is the identity. This induces maps on the points of G

$$G(R) \to G(R[\epsilon]) \xrightarrow{\pi'} G(R)$$

and we define

$$\mathfrak{g}(R) \quad := \quad ker(\pi')$$

This has a cannonical $R-$module structure and in fact we have that

$$\mathfrak{g}(R) \cong R \otimes \mathfrak{g}(k)$$

Thus knowing $\mathfrak{g}(k)$ is sufficient to know all $\mathfrak{g}(R)$.

**Theorem.** *There is a unique functor, Lie, from the category of algebraic groups over k to the category of Lie algebras such that*

over k?

- $Lie(G) = \mathfrak{g}(k)$ *as vector spaces*
- *The bracket on* $Lie(GL_n)$ *is* $[X, Y] = XY - YX$

We can also give a direct construction of the Lie algebra structure using the adjoint action.

Not illuminating to me

## 3.2 Derivations

Given a k-algebra A and an A-module M a derivation is a k-linear map $D : A \to M$ such that

$$D(fg) = f.D(g) + g.D(f)$$

Let $Der_k(A)$ be the space of k-derivations of $A \to A$. This space is a lie algebra using the bracket

$$[D, D'] = D \circ D' - D' \circ D$$

A derivation $D : \mathcal{O}(G) \to \mathcal{O}(G)$ is said to be left invariant if

$$\Delta \circ D = (id \otimes D) \circ \Delta$$

where these are the operations of the Hopf-algebra structure on $\mathcal{O}(G)$. The left invariant derivations form a Lie subalgebra of $Der_k(\mathcal{O}(G))$. Thus we define $Lie(A)$ to be the left invariant Lie subalgebra of $Der_k(A)$. One can check that it satisfies the conditions of 3.1 and so the definitions agree.

## 4 This is a Nice Functor

**Theorem.** *Lie is fully-faithful.*

So algebraic groups are nothing more than special Lie algebras. More than this the theory for algebraic groups can be entirely recast in this light. We provide a summary table:

Historical Question: Which came first (algebras I think). Was the theory of alg groups developed with this in mind or is it a coincidence? (former I think...)

For a connected algebraic group G, Lie commutes with

- Fiber products
- Intersections
- Kernels
- Derived subgroup / algebra
- Radicals

An ***isogeny*** is a surjective homomorphism with finite kernel.
It enjoys the further properties that

- The map $H \mapsto Lie(H)$ from connected algebraic subgroups of G to Lie subalgebras of $LieG$ is injective and inclusion preserving.
- It is exact
- A morphism of connected algebraic groups is an isogeny iff its image under Lie is an isomorphism
- $H \leq G$ is normal iff $LieH \leq LieG$ is an ideal
- $H \leq Cent(G)$ iff $LieH \leq Cent(LieG)$
- G is commutatuve iff $LieG$ is

Prop 3.18, 3.25, 3.26

## 5 Representations

The functor Lie defines a fully-faithful functor

$$Rep(G) \to Rep(Lie(G))$$

When $\mathfrak{g}$ is semi-simple then there exists a semi-simple G such that $Rep(G) = Rep(\mathfrak{g})$.