

Categorical Phase Semantics for Linear Logic

Riley Shahar
Steve Zdancewic

 REPL



- Categorical semantics, an introduction

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- Linear logic and phase semantics

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- Our work: categorical phase semantics

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- Linear logic and phase semantics
- Our work: categorical phase semantics
- Conclusion & future work

PLs

Logics

Categories

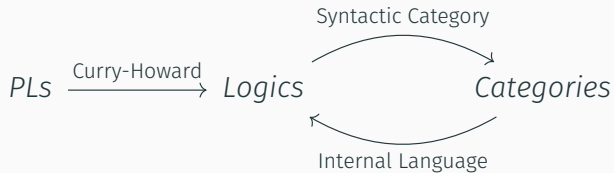
PLs $\xrightarrow{\text{Curry-Howard}}$ *Logics*

Categories

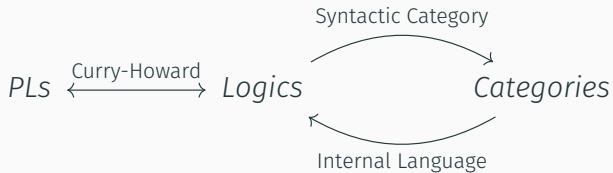
The Story



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These form a *signature (language, syntax)*,

$$\mathcal{L} = \{X, \neg Y, (X \wedge Y) \rightarrow (T \vee X), \dots\}.$$

Inference Rules

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We can use these to write *proof trees*:

$$\frac{\frac{A \quad A \rightarrow B}{B} \rightarrow E \quad B \rightarrow C}{C} \rightarrow E.$$

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Provability Semantics

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Any logic with these rules has an associated preorder, its *provability semantics*.

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You can keep going: you'll get a *Heyting algebra* or a *Boolean algebra*.

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Alternate notation:

$$\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} x \quad \begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \vdots \end{array} x$$

$A \quad , \quad A$

The Curry-Howard Isomorphism

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$$\frac{\frac{\begin{array}{c} \vdots \\ \vdots \\ x \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ y \end{array}}{A \quad B}}{A \wedge B}$$

$$A$$

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This is the *Curry-Howard isomorphism*:

Types are propositions; programs are proofs.

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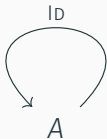
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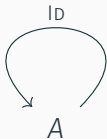
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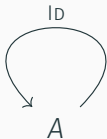
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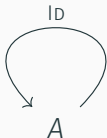
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These is a *category*! Any logic (or language) has its *syntactic category* as its *proof semantics*.

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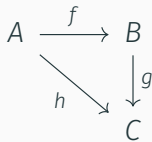
B

C

A *category* consists of:

- Objects

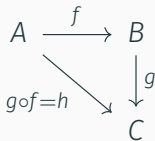
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A *category* consists of:

- Objects
- Morphisms

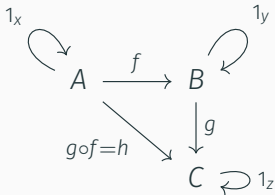
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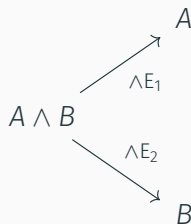
A

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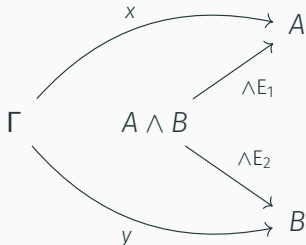
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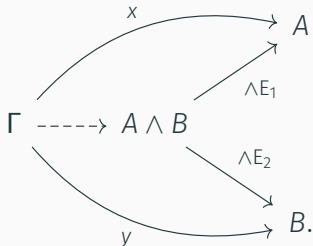
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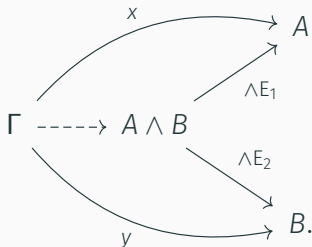
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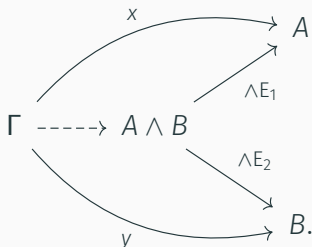
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Again, you can keep going: you'll get (something like) a *catesian closed category*.

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How can we frame this categorically?

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If you pick \mathcal{M} to be *monomorphisms*, you get the normal notion of subsets, subgroups, subspaces, etc.

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With sets, the power set is a Heyting (Boolean) algebra. Such categories are called *Heyting (Boolean) categories*. These categories form the *predicate semantics* of intuitionistic (classical) logic.

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This is a predicate! So we probably want our type system to be a Heyting category. Category theory can tell us what we need to do to get that.

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$$\begin{aligned}\llbracket A \rrbracket &\subseteq M \\ \llbracket A \otimes B \rrbracket &= \text{cl}(\llbracket A \rrbracket \llbracket B \rrbracket) \\ &= \text{cl}\{ab : a \in \llbracket A \rrbracket, b \in \llbracket B \rrbracket\}\end{aligned}$$

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Example Applications of Phase Semantics

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Takeaway: phase semantics are *tunable*.

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Yetter (1990) observed that you don't need elements to do phase semantics.

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Quantale semantics are the provability semantics of linear logic.

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Intuitionistic Logic

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Intuitionistic Logic

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Our Contribution

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To make subobjects closed under these operations, we need to take *images*.

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Generalizing, $\mathbf{List}_{2\mathbb{N}} \otimes \mathbf{Maybe} = \mathbf{List}$.

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Completely characterize predicate semantics for linear logic

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Thanks! :)