Categorical Phase Semantics for Linear Logic

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Outline

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- Categorical semantics, an introduction

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- Linear logic and phase semantics

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- Conclusion & future work

The Story

PLs

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$$PLs \xrightarrow{\text{Curry-Howard}} Logics \qquad\qquad Categories$$

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These form a *signature* (*language*, *syntax*),

$$\mathcal{L} = \{X, \neg Y, (X \wedge Y) \rightarrow (T \vee X), \ldots\}.$$

Inference Rules

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We can use these to write *proof trees:*

$$\frac{\dfrac{A \qquad A \to B}{B}\ {\to}E \qquad B \to C}{C}\ {\to}E.$$

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Any logic with these rules has an associated preorder, its *provability semantics*.

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You can keep going: you'll get a *Heyting algebra* or a *Boolean algebra*.

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Alternate notation:

$$\begin{array}{c} \\ \vdots\, x \\ A \end{array} , \quad \begin{array}{c} \Gamma \\ \vdots\, x \\ A \end{array}$$

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This is the *Curry-Howard isomorphism:*

Types are propositions; programs are proofs.

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These is a *category*! Any logic (or language) has its *syntactic category* as its *proof semantics*.

Categories

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A B

C

A *category* consists of:

- Objects

Categories

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

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\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f = h"'] \arrow[loop above, "1_x"] & B \arrow[d, "g"] \arrow[loop above, "1_y"] \\
& C \arrow[loop right, "1_z"]
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Again, you can keep going: you'll get (something like) a *catesian closed category*.

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How can we frame this categorically?

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If you pick $\mathcal{M}$ to be *monomorphisms*, you get the normal notion of subsets, subgroups, subspaces, etc.

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With sets, the power set is a Heyting (Boolean) algebra. Such categories are called *Heyting (Boolean) categories*. These categories form the *predicate semantics* of intuitionistic (clasical) logic.

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This is a predicate! So we probably want our type system to be a Heyting category. Category theory can tell us what we need to do to get that.

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$$\begin{aligned} [\![A]\!] &\subseteq M \\ [\![A \otimes B]\!] &= \mathrm{cl}([\![A]\!][\![B]\!]) \\ &= \mathrm{cl}\{ab : a \in [\![A]\!], b \in [\![B]\!]\} \end{aligned}$$

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&= \bigcup \{C \subseteq M : [\![A]\!]C \subseteq [\![B]\!]\}
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Example Applications of Phase Semantics

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Technical lemmas about linear logic [Girard 1987, Okada 1999].

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Takeaway: phase semantics are *tunable*.

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Yetter (1990) observed that you don't need elements to do phase semantics.

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Quantale semantics are the provability semantics of linear logic.

Categorical Semantics of Linear Logic

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To make subobjects closed under these operations, we need to take *images*.

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Generalizing, $\mathtt{List}_{2\mathbb{N}} \otimes \mathtt{Maybe} = \mathtt{List}$.

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Completely characterize predicate semantics for linear logic

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Thanks! :)