Categories for Cryptographic Composability

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Introduction

To first approximation, cryptography is the *mathematical study of secure computation*. In a computation, we want to use *protocols* to transform *resources*. For a computation to be secure, it must successfully resist *attacks* by *adversaries*. This is an extremely broad scope: cryptography includes secure communication, private data analysis, password-based authentication, distributed consensus-making, fault-tolerance of sensor systems, and many other applications.

In the modern world, computational systems do not run on their own. You may be securely communicating with your bank on one tab, have email open on another tab, and be syncing photos from your phone in the background. After that communication, your bank may want to analyze your data in a way that respects your privacy. All this happens for millions of people simultaneously. As such, we cannot study cryptographic protocols in a vacuum: we need to consider how they behave in concert with other computational systems. This is the *problem of cryptographic composability*.

As we will see, most frameworks for handling composability, including the popular Universal Composability [Can00], rely on precise low-level machine models. Proofs in these frameworks are only technically valid if their protocols can be encoded into the machine model—and if that encoding satisfies certain technical hypotheses which do not generally hold. This state of affairs poses significiant issues for both the feasibility of writing proofs in these frameworks, and—because of the general complexity of the underlying machine models—for the trustworthiness of those proofs.

A natural way out is to give an axiomatization of the properties such a machine model, or a theory of cryptography generated from it, should satisfy. If a composition theorem can be proven for any theory satisfying these axioms, then proofs would not have to use complex machine models except when that complexity is necessary to the development of the protocol.

Cryptography has a composition problem: category theory is a mathematical theory of composition.

In some sense, this thesis has three goals: to give cryptographers the necessary background in category theory to evaluate categorical frameworks for themselves, to give category theorists the enough background in cryptography for them to motivate their work towards the potential applications, and to evaluate the existing literature and pose some barriers and open questions to a successful categorical theory of cryptographic composability. The thesis is divided along those lines.

In Chapter 1, we give an introduction to the foundations of cryptography, focusing on definitions and examples relevant to the study of composability. Cryptographers can

safely skip this chapter, though cryptographers new to questions of composability may be interested in Section 1.3, which presents several of the central difficulties.

In Chapter 2, we give an introduction to category theory oriented towards computer scientists. There are many excellent books with this aim; our narrative distinguishes itself through its focus on monoidal categories, coherence axioms, and in particular string diagrams, which form a powerful graphical language for reasoning about computational objects. As in the previous chapter, the aim is to get to the necessary background for cryptography as quickly as possible, and so we skip several standard topics of substantial interest to computer scientists. Category theorists who are comfortable working with string diagrams can safely skip this chapter, though there are several examples of computational applications that may be of interest.

Chapter 1

Cryptography

[TODO: An introduction to the chapter; cite [KL14; PS10; Ros21].]

1.1 Foundations

1.1.1 One-Way Functions

Many cryptographic protocols rely on *one-way functions*, which are informally functions that are easy to compute, but hard to invert. The former notion is easy to formalize in terms of time complexity, but the latter is more difficult. We typically ask that any "reasonably efficient" algorithm—called the *adversary*—attempting to invert the function has a negligible chance of success.

In computer science, we generally assume that algorithms are efficient if and only if they are polynomial-time; this assumption has been borne out by decades of practice. This motivates our definition of a "negligible" chance: we say that a function f is *negligible* if $f(n) = o(n^{-k})$ for every k; in other words, if it is asymptotically smaller than any rational function. In this case, we write $f = \mathsf{negl}(n)$ or just $f = \mathsf{negl}$. The set of negligible functions has all the nice closure properties we expect; in particular, the sum of negligible functions is negligible.

Notation. We will use PPT as shorthand for probabilistic polynomial-time. When we refer to an *adversary*, *distinguisher*, or *simulator*, we always mean a non-uniform PPT algorithm[1].

Definition 1.1 (one-way function)**.** A function f is *one-way* if:

- (easy to compute) f is PPT-computable;
- (hard to invert) for any adversary $\mathcal{A}$, natural number n, and uniform random choice of input x such that $|x| = n$,

$$\Pr[f(\mathcal{A}(1^n, f(x))) = f(x)] = \mathsf{negl}(n).$$

Note that $|x|$ here is *not* the absolute value, but is instead the length of x as a binary string: if x is a number, then by encoding in binary have that $|x| = \Theta(\log_2 x)$.

[1] Both PPT and non-uniform PPT are defined in Appendix A.

The idea is that, given $y = f(x)$, $\mathcal{A}$ attempts to find some x' such that $f(x') = y$. If some adversary can do this with non-negligible probability, then the function is not one-way. While the probability must be negligible in $|x|$, the adversary is given $f(x)$ and 1^n as an input, and hence must run polynomially only in $|f(x)| + n$. This is a common technique called *padding*, wherein algorithms are given an extra input of 1^n to ensure they have enough time to run.

We do not know that one-way functions exist. In fact, while the existence of one-way functions implies that P $\neq$ NP, the converse is not known[2]. However, as in the following examples, we have excellent candidates under fairly modest assumptions.

Example 1.2 (Factoring [PS10, subsection 2.3])**.** Suppose that for any adversary $\mathcal{A}$ and for uniform random choice of primes $p, q < 2^n$,

$$\Pr[\mathcal{A}(pq) = \{p, q\}] = \mathsf{negl}(n).$$

This is the *factoring hardness assumption*, for which there is substantial evidence. Then $(x, y) \mapsto xy$ is one-way[3].

Example 1.3 (Discrete Logarithm [KL14, subsection 8.3.2])**.** Let $\{G_n\}$ be a fixed sequence of finite groups. The *discrete logarithm hardness assumption* for $\{G_n\}$ is that, for any adversary $\mathcal{A}$ and for uniform random choice of $g \in G_n$ and $h \in \langle g \rangle$ such that $h = g^k$,

$$\Pr[\mathcal{A}(g, h) = k] = \mathsf{negl}(n).$$

Under the discrete logarithm hardness assumption, $(g, k) \mapsto g^k$ is one-way.

The discrete logarithm hardness assumption is known to be false for certain groups, such as the additive groups $\mathbb{Z}_p$ for prime p, in which case $g^k = gk$ and the Euclidean algorithm solves the problem. However, it is believed to hold for groups such as $\mathbb{Z}_p^*$. For a survey of various versions of this assumption, see [SS02].

1.1.2 Proofs by Reduction

Many cryptographic definitions, including Definition 1.1, take the form

> *for any adversary $\mathcal{A}$, natural number n, and uniform random choice of input x such that $|x| = n$, some predicate on the output of $\mathcal{A}$ has negligible probability.*

The basic technique for proving results using these definitions is called *proof by reduction*. The idea is to reduce one problem into another by starting with an arbitrary adversary attacking the second and constructing an adversary attacking the first, such that the probability of their successes is related. If we assume the first problem is hard, then by studying the structure of the reduction we can learn about the hardness of the second problem. As such, we often say that reductions prove *relative hardness results*, so that for instance Example 1.4 below proves the hardness of g relative to f.

More specifically, to prove hardness of a problem Π relative to Π', a proof by reduction generally goes as follows:

[2] [Imp95] gives a classic discussion of the implications of various resolutions to P vs. NP on cryptography, including the case where P $\neq$ NP but one-way functions nevertheless do not exist.

[3] This statement is slightly imprecise: technically, $(x, y) \mapsto xy$ is *weakly one-way*; to get the stronger notion of Definition 1.1 requires a process called *hardness amplification*. See [PS10, Section 2.4] for details.

1. Fix an arbitrary adversary $\mathcal{A}$ attacking a problem Π.

2. Construct an adversary $\mathcal{A}'$ attacking a problem Π' which:

   (a) Receives an input x' to Π'.

   (b) Translates x' into an input x to Π.

   (c) Simulates $\mathcal{A}(x)$, getting back an output y which solves $\Pi(x)$.

   (d) Translates y into an output y' which solve $\Pi(x')$.

3. Analyze the structure of the translations to conclude that $\mathcal{A}'$ solves Π' with probability related to that with which $\mathcal{A}$ solves Π.

4. Given the hardness assumptions on Π', conclude relative hardness of Π.

The point is that $\mathcal{A}'$'s job is to "simulate" the problem Π to $\mathcal{A}$, using the data it gets from Π' to construct an input to Π. We illustrate this concept now.

Example 1.4 (a straightforward proof by reduction [PS10, subsection 2.4.1])**.** Let f be a one-way function. Then we claim $g : (x, y) \mapsto (f(x), f(y))$ is a one-way function. We can compute g in polynomial time by computing f twice, so it remains to show that g is hard to invert.

Let $\mathcal{A}$ be any adversary. We will construct an adversary $\mathcal{A}'$ such that, if $\mathcal{A}$ can non-negligibly invert g, then $\mathcal{A}'$ can non-negligibly invert f.

The adversary $\mathcal{A}'$ takes input 1^n and y. It then uniformly randomly chooses u of length n and computes $v = f(u)$, which is possible because f is easy to compute. Now $\mathcal{A}'$ computes $(u', x') := \mathcal{A}(1^{2n}, (v, y))$ and outputs x'.

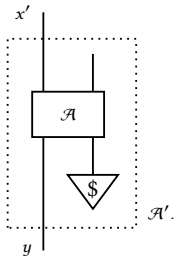
When $\mathcal{A}'$ simulates $\mathcal{A}$, it passes v, which is $f(u)$ for a uniform random u, and y, which is (on well-formed inputs) $f(x)$ for a uniform random x. Thus, this looks like exactly the input that $\mathcal{A}$ would "expect" to receive if it is attempting to break g. As such, whenever $\mathcal{A}$ successfully inverts g, $\mathcal{A}'$ successfully inverts f. Since everything is uniform we may pass to probabilities, and so:

$$
\begin{aligned}
&\Pr[g(\mathcal{A}(1^{2n}, g(u, x))) = g(u, x)] \\
&\quad = \Pr[g(\mathcal{A}(1^{2n}, (f(u), f(x)))) = (f(u), f(x))] && \text{by definition of } g \\
&\quad \leq \Pr[f(\mathcal{A}'(1^n, f(x))) = f(x)] && \text{by the above argument} \\
&\quad = \mathsf{negl}(n) && \text{by the hardness assumption for } f.
\end{aligned}
$$

Thus g is one-way.

Comparing this example to the above schema, we see that the problem Π' is to invert f, while the problem Π is to invert g. The input x' to Π' is y, while the computed input x to Π is (v, y). The output y of $\mathcal{A}$ is (x', u'), while the computed output y' is x'.

Diagramatically, we can represent the algorithm $\mathcal{A}'$ as follows:



While this is not standard notation in cryptography, it will be useful for our future purposes. We read these diagrams—called *circuit* or *string diagrams*—from bottom to top. This diagram says that $\mathcal{A}'$ is an algorithm which takes y, uniformly randomly generates another input (this is what the \$ means), calls $\mathcal{A}$, and returns its first output.

1.1.3 Computational Indistinguishability

Computational indistinguishability formalizes the notion of two probability distributions which "look the same" to adversarial processes. We begin with probability distributions, but because we want to do asymptotic analysis, we will eventually need to switch to working with sequences of probability distributions.

Definition 1.5 (computational advantage)**.** Let X and Y be probability distributions over a set A. The *computational advantage* of an adversary $\mathcal{D}$, called the *distinguisher*, over X and Y is

$$\mathrm{ca}_{\mathcal{D}}(X, Y) = \left| \Pr_{x \leftarrow X}[\mathcal{D}(x) = 1] - \Pr_{y \leftarrow Y}[\mathcal{D}(y) = 1] \right|.$$

The idea is that the distinguisher $\mathcal{D}$ is trying to guess whether its input was drawn from X or Y; the computational advantage is how often it can do so.

Proposition 1.6. *Let $\mathcal{D}$ be a fixed distinguisher. Then $\mathrm{ca}_{\mathcal{D}}$ is a pseudometric*[4] *on the space of probability distributions over an underlying set A.*

Proof. Symmetry and non-negativity are immediate from the definition, while the triangle inequality follows from the triangle inequality for real numbers. □

We now turn to the asymptotic case.

Definition 1.7 (probability ensemble)**.** A *probability ensemble* is a sequence $\{X_n\}$ of probability distributions over sets $\{A_n\}$.

We say that two ensembles are computationally indistinguishable if there is no efficient way to tell between them. Formally:

[4] A *pseudometric* on a space X is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ which is zero on identical points, symmetric, and satisfies the triangle inequality; in other words, it is a metric which does not necessarily differentiate distinct points.

Definition 1.8 (computational indistinguishability)**.** Two probability ensembles $\{X_n\}$ and $\{Y_n\}$ over a set A are *computationally indistinguishable* if for any (non-uniform PPT) distinguisher $\mathcal{D}$ and any natural number n,

$$\mathrm{ca}_{\mathcal{D}}(X_n, Y_n) = \mathsf{negl}(n).$$

In this case, we write $\{X_n\} \overset{c}{\equiv} \{Y_n\}$.

Remark 1.9. A natural thought is to define a metric on probability distributions by

$$\mathrm{ca}(X, Y) = \sup_{\mathcal{D}} \mathrm{ca}_{\mathcal{D}}(X, Y),$$

and extend to ensembles by asking that $\mathrm{ca}(X_n, Y_n) = \mathsf{negl}(n)$. Unfortunately, this does not quite yield the correct notion, as there exist ensembles which are computationally indistinguishable, but have sequences of distinguishers whose advantages for any fixed n converge to 1.

Proposition 1.10. *Computational indistinguishability is an equivalence relation on the space of probability ensembles over a fixed set A.*

Proof. Reflexivity and symmetry follow from the case of distributions. To show transitivity, let $\{X_n\} \overset{c}{\equiv} \{Y_n\}$ and $\{Y_n\} \overset{c}{\equiv} \{Z_n\}$. Let $\mathcal{D}$ be any distinguisher. Then for any n,

$$\begin{aligned} \mathrm{ca}_{\mathcal{D}}(X_n, Z_n) &\leq \mathrm{ca}_{\mathcal{D}}(X_n, Y_n) + \mathrm{ca}_{\mathcal{D}}(Y_n, Z_n) && \text{by the triangle inequality} \\ &= \mathsf{negl}(n) + \mathsf{negl}(n) && \text{by assumption} \\ &= \mathsf{negl}(n). \end{aligned}$$

□

It is necessary to be precise about what is being claimed here. Transitivity states that for any *constant, finite sequence* of probability ensembles, if each is computationally indistinguishable from its neighbors, then the two ends of the sequence are computationally indistinguishable. In cryptography, we sometimes want to consider the more general case of a countable sequence of probability ensembles. We can do slightly better than the previous result:

Proposition 1.11. *Let $\{X^k\}$ be a sequence of probability ensembles, so that each $X^k = \{X^k_n\}$ is itself a sequence of probability distributions, each over the underlying same sequence of sets $\{A_n\}$. Let $\{X^i\} \overset{c}{\equiv} \{X^{i+1}\}$ for each i. Let $\{Y_n = X^{K(n)}_n\}$ for some polynomial K. Then $\{X^1_n\} \overset{c}{\equiv} \{Y_n\}$.*

Proof. Let $\mathcal{D}$ be any distinguisher. Then for any n,

$$\begin{aligned} \mathrm{ca}_{\mathcal{D}}(X^1_n, Y_n) &= \mathrm{ca}_{\mathcal{D}}(X^1_n, X^{K(n)}_n) \\ &\leq \mathrm{ca}_{\mathcal{D}}(X^1_n, X^2_n) + \cdots + \mathrm{ca}_{\mathcal{D}}(X^{K(n)-1}_n, X^{K(n)}_n) \\ &= K(n)\mathsf{negl}(n) \\ &= \mathsf{negl}(n). \end{aligned}$$

In particular, the last equality follows because K is polynomial. □

On the other hand, the result does not hold for arbitrary K. As we will see, this is a fundamental limitation for cryptographic composition: we only expect composition to work up to polynomial bounds.

One more closure result is valuable:

Proposition 1.12. *Let* $\{X_n\} \overset{c}{\equiv} \{Y_n\}$*, and let* $\mathcal{M}$ *be a non-uniform PPT algorithm. Then* $\{\mathcal{M}(X_n)\} \overset{c}{\equiv} \{\mathcal{M}(Y_n)\}$.

Proof. The proof is by reduction. Let $\mathcal{D}$ be a distinguisher. Then construct $\mathcal{D}'$ which, on input x, simulates $\mathcal{D}(\mathcal{M}(x))$. Then $\mathcal{D}'$ outputs 1 on x if and only if $\mathcal{D}$ outputs 1 on $\mathcal{M}(x)$, so

$$\mathrm{ca}_{\mathcal{D}}(\mathcal{M}(X_n), \mathcal{M}(Y_n)) = \mathrm{ca}_{\mathcal{D}'}(X_n, Y_n) = \mathsf{negl}(n)$$

by the computational indistinguishability assumption. □

Example 1.13 (pseudorandom generators [PS10, Sections 3.2-3.3])**.** We can use computational indistinguishability to formalize the notion of pseudorandomness.

Let $\{\mathcal{X}_n\}$ be a sequence of spaces, and let $\{X_n\}$ be a sequence of probability distributions over $\bigcup \mathcal{X}_n$. We say that $\{X_n\}$ is *pseudorandom for* $\mathcal{X}$ if there exists a polynomial p such that

$$\{X_n\} \overset{c}{\equiv} \{\mathcal{X}_{p(n)}\},$$

where the latter is equipped with the uniform distribution. In other words, pseudorandom ensembles look uniformly random to distinguishers.

For simplicity, we now work over $\mathcal{X}_n = \mathbb{Z}_2^n$. Let $G : \mathbb{Z}_2^* \to \mathbb{Z}_2^*$ be a *deterministic* function. We say G is a *pseudorandom generator* if

- G is polynomial-time computable;
- for any x, $|G(x)| > |x|$;
- $\{G(\mathbb{Z}_2^n)\}$ is pseudorandom.

The idea is that G gets some input $x \in \mathbb{Z}_2^n$ and produces an output in $\mathbb{Z}_2^{p(n)}$ which looks uniformly random to distinguishers if x is chosen uniformly at random. The polynomial $p(n)$ in the definition of pseudorandomness is now called the *expansion factor*. It this sense, pseudorandom generators allow us to "bootstrap" randomness from random draws even on very small inputs.

As usual, while we have excellent candidates, we have no proof that pseudorandom generators exist. However, there is a known procedure, due to Håstad et al. [Hås+99], to turn any one-way function into a pseudorandom generator.

1.1.4 Interactive and Zero-Knowledge Computation

Cryptographic protocols do not occur in a vacuum; instead, they rely on computations involving multiple parties. We call such situations *interactive computations*. In general, a model of interaction depends on the underlying model of computation; this is for instance the case with the popular notion of interactive Turing machines [Gol01, Definition 4.2.1]. As our approach in this chapter has been model-independent, we can only give an informal discussion of interaction.

An *interactive computation* consists of a finite number of *parties*, which we think of as algorithms $\mathcal{A}_i$, who may potentially communicate by sending messages to each other, and whose behavior may change in response to messages they receive. An *interactive protocol* just consists of descriptions of some interactive algorithms $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle$.

We often think of interactive computations as being indexed by a *security parameter* $n \in \mathbb{N}$. Instead of asking each algorithm to be polynomial-time in its inputs, we ask it to be polynomial in n, with the stipulation that the inputs themselves are no more than polynomial in n, so that each algorithm has time to read its own inputs. Intuitively, the security parameter represents a "tuning" of the security of the system, so that a bigger n incurs greater computational cost but gives stronger security guarantees. Often, the security parameter is formalized by ensuring that all parties get an extra input of 1^n at the start of the computation, as we did in Definition 1.1; we assume this formalization in every protocol we give here.

At the start of an interactive computation, there is a *global input* x known to all parties, and each party $\mathcal{A}_i$ may have a *private* or *auxiliary input* x_i known only to itself. We generally assume that there are known sequences of *input spaces* $\mathcal{X}_n$ and $\mathcal{X}_n^i$, such that when the security parameter is n, $x \in \mathcal{X}_n$ and $x_i \in \mathcal{X}_n^i$. At the end of the computation, each party may make some output, the sequence of which we denote $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle (x, x_1, \ldots, x_N)$, so that party i's output is $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle (x, x_1, \ldots, x_N)_i$. When any of these algorithms are potentially probabilistic, we think of this value as a distribution over possible outputs, and we always assume that the internal randomness of the parties is independent.

Example 1.14. Here is a simple interactive protocol. We have two algorithms, $\mathcal{A}$ and $\mathcal{B}$. The input spaces are $\mathcal{X}_n^{\mathcal{A}}$ and $\mathcal{X}_n^{\mathcal{B}}$; there is no global input (which means the global input is just the security parameter 1^n.) The algorithm $\mathcal{A}$ takes its input $x \in \mathcal{X}_n^{\mathcal{A}}$, sends it to $\mathcal{B}$, and outputs the first message it recieves from $\mathcal{B}$. The algorithm $\mathcal{B}$ takes its input $y \in \mathcal{X}_n^{\mathcal{B}}$, sends it to $\mathcal{A}$, and outputs the first message it recieves from $\mathcal{A}$. Then we have that $\langle \mathcal{A}, \mathcal{B} \rangle (x, y) = (y, x)$.

The *view* of a party is roughly all of the information it has available to it over the course of the computation. This includes the global input, its private input, any random bits it uses, and all the messages it receives. We denote the view of party i by $\text{view}_i^{\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle}(x, x_1, \ldots, x_N)$. When the algorithms are clear from context, we may omit the superscript. Importantly, while each private input x_k is a parameter of each view view_i, the view does not necessarily include each of these inputs; they are parameters merely because they may affect the messages received by party i.

Example 1.15. In the protocol of Example 1.14,

$$\text{view}_{\mathcal{A}}^{\langle \mathcal{A}, \mathcal{B} \rangle}(1^n, x, y) = \text{view}_{\mathcal{B}}^{\langle \mathcal{A}, \mathcal{B} \rangle}(1^n, x, y) = \{1^n, x, y\}.$$

Suppose that $\mathcal{B}'$ always sends the string 0^n to $\mathcal{A}$, instead of its input. Then

$$\text{view}_{\mathcal{A}}^{\langle \mathcal{A}, \mathcal{B}' \rangle}(1^n, x, y) = \{1^n, x, 0\}, \qquad \text{view}_{\mathcal{B}'}^{\langle \mathcal{A}, \mathcal{B}' \rangle}(1^n, x, y) = \{1^n, x, y\}.$$

Notice that $\text{view}_{\mathcal{B}'}$ does not include the messages which it sends $\mathcal{A}$.

The *running time* of an interactive algorithm $\mathcal{A}$ is now the function $T_{\mathcal{A}} : \mathbb{N} \to \mathbb{N}$ which, for any n, gives the maximum number of "steps" it takes $\mathcal{A}$ to halt over any choice of:

- global input x and private input y of total length $n = |x| + |y|$;
- other algorithms involved in the computation;
- internal randomness of $\mathcal{A}$ and of any other algorithms involved in the computation.

Essentially, when we say an algorithm is polynomial-time, we mean it is *always* polynomial-time, no matter what. We sometimes assume that each algorithm has a "clock" that it uses to count the number of steps it has taken and ensure it halts in some fixed polynomial number of steps.

We can now formalize the idea of a party "learning something" from an interaction. We say that an interactive protocol $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle$ is *zero-knowledge for party i* if there exists a non-uniform PPT algorithm $\mathcal{S}$ such that for any choice of inputs $(x, x_1, \ldots, x_N)$,

$$\mathcal{S}(x, x_i) \stackrel{c}{\equiv} \text{view}_i^{\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle}(x, x_1, \ldots, x_N).$$

The idea is that the "simulator" $\mathcal{S}$ gets only the inputs to $\mathcal{A}_i$ and is responsible for producing a distribution that is indistinguishable from the actual view of $\mathcal{A}_i$. If they can do this, then $\mathcal{A}_i$ must not have learned anything that they could not have computed directly from their inputs.

More often, we want to consider the situation where $\mathcal{A}_i$ is supposed to learn *something* from the computation, but should not learn anything *extra*.

Definition 1.16 (zero-knowledge)**.** Let f be a function. An interactive protocol $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ is *zero-knowledge for party i relative to f* if there exists a (non-uniform PPT) simulator $\mathcal{S}$ such that for any choice of inputs $(x, x_1, \ldots, x_N)$,

$$\mathcal{S}(x, x_i, f(x, x_1, \ldots, x_N)) \stackrel{c}{\equiv} \text{view}_i^{\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle}(x, x_1, \ldots, x_N).$$

In the above definition, we are asking that the simulator produces a distribution which is negligibly close, in the sense of computational indistinguishability, to the actual view. While this is all that is possible in many situations in practice, we could ask for the stronger condition that the produced distribution is *identical* to the view. We call this notion *perfect* or *information-theoretic* zero-knowledge, and refer to Definition 1.16 as *computational* zero-knowledge when we wish to emphasize the distinction.

Example 1.17. We show that the *trivial protocol*, in which two algorithms $\mathcal{A}$ and $\mathcal{B}$ do nothing, is zero-knowledge for $\mathcal{B}$. Our goal is to give a simulator $\mathcal{S}$ such that for any choice of security parameter n,

$$\mathcal{S}(1^n) \stackrel{c}{\equiv} \text{view}_{\mathcal{B}}^{\langle \mathcal{A}, \mathcal{B} \rangle}(1^n).$$

Since $\mathcal{B}$ never gets sent any messages, its view is just the input 1^n. We therefore let $\mathcal{S}$ compute the identity, so that the two distributions are both constantly $\{1^n\}$. This shows that the trivial protocol is perfect zero-knowledge.

Example 1.18. Consider the following protocol: $\mathcal{A}$ gets input $x \in \mathbb{Z}_2^n$, which it then sends to $\mathcal{B}$. To show this is not zero-knowledge for $\mathcal{B}$, we must show that for any simulator $\mathcal{S}$, there exists a choice of input x and a distinguisher $\mathcal{D}$ which distinguishes $\mathcal{S}(1^n)$ from $\text{view}_{\mathcal{B}}(1^n, x) = \{1^n, x\}$ with non-negligible probability.

Let $\mathcal{S}$ be fixed. If $\mathcal{S}$ does anything other than outputting 1^n and some $y \in \mathbb{Z}_2^n$, then we will be able to distinguish it syntactically. We may therefore safely assume that $\mathcal{S}$ outputs $(1^n, y)$ for some (potentially random) choice of y. For each n, now choose x such that $\Pr[y = x] \leq 2^{-n}$, which is possible by the pigeonhole principle. Let $\mathcal{D}$ output 1 on input $\{1^n, x\}$, and 0 otherwise. Then

$$\Pr[\mathcal{D}(\mathcal{S}(1^n)) = 1] = \Pr[y = x] \leq 2^{-n},$$

while

$$\Pr[\mathcal{D}(\text{view}_{\mathcal{B}}(1^n, x)) = 1] = 1.$$

Since $|1 - 2^{-n}|$ is not negligible, the protocol is not zero-knowledge for $\mathcal{B}$.

1.1.5 Adversaries and the Real-Ideal Paradigm

Zero-knowledge is a surprisingly general tool for formalizing security definitions, but in some cases it is not enough. For instance, we may want to verify that protocols for electronic coin-flips are fair: in this case, the issue is not that the parties may learn something extra, but that they may be able to unduly influence the outcome of the computation. The general approach taken in the literature is to define security on an ad-hoc basis for each such task by enumerating the properties we want the protocol to have and formalizing them as adversarial games. We will take a more systematic approach, sometimes called the *real-ideal paradigm*.

The idea is to define an *ideal protocol*—also called an *ideal functionality*—which represents the desired behavior of the cryptosystem. The protocol under study—the *real protocol*—is then supposed to emulate the ideal protocol, in the sense that its outcomes should be computationally indistinguishable from the outcomes of the ideal protocol. The subtelty here is our use of the term *outcome*; which necessarily looks different for different protocols: it may just be the information learned by a party, in which case we recover zero-knowledge, it may be some function of the party's outputs, or it may be something else altogether. We will see several different examples of this in Section 1.2, but the important point is that to define security in this paradigm requires both a definition of the ideal protocol and of the data to be compared.

Unlike in the case of zero-knowledge, where we only cared about what parties could learn from the protocol assuming it was executed correctly, in this setting we also want to discuss security against *malicious behavior*, in which one or more parties deliberately try to sabotage the outcome of the protocol. We often refer to these parties as the *adversaries*, and allow them to be non-uniform even when ordinary parties in the protocol are uniform. Since a protocol is not secure if even one possible attack is likely to succeed, we say that a protocol is *secure in the presence of malicious adversaries* if for any choice of algorithms taking over some fixed number of parties, the outcomes of the protocol are

indistinguishable from the ideal. Again, we will see examples of this notion in the next section.

In contrast, sometimes we do want to talk about a protocol being zero-knowledge without dealing with arbitrary adversarial behavior. In this case, we use the term *semi-honest adversaries*, which informally refers to adversaries which follow the prescriptions of the protocol, but attempt to learn as much as they can within those bounds. There are several other notions of adversarial strength considered in the literature—for instance, adaptive vs. static adversaries [Cra+99], Byzantine adversaries [LLR04], and "coercible" parties [CGP15]. We do not explore all these models here.

1.2 Cryptographic Problems

1.2.1 Encryption

Much of the machinery defined in the previous section was originally developed in the 1970s and 80s for the purpose of analyzing *encryption problems*, culminating in the work of Goldwasser and Micali [GM82]. The idea of an encryption problem is that a party Alice has a message m in the *message space* $\mathcal{M}_n$ which they want to send to Bob, but any message they send to Bob must also be sent to the eavesdropping Eve. In the simpler *shared-key encryption problem*, which we consider here, Alice and Bob share some secret key k from the *key space* $\mathcal{K}_n$, which is unknown to Eve.

Definition 1.19 (shared-key encryption scheme)**.** Let $\{\mathcal{M}_n\}$ and $\{\mathcal{K}_n\}$ be sequences of sets. An $(\mathcal{M}, \mathcal{K})$-shared-key encryption scheme is an interactive protocol consisting of three interactive algorithms $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{E}$, where:

- the global input is 1^n, the security parameter;
- $\mathcal{A}$ gets a uniform random key $k \in \mathcal{K}_n$ and a message $m \in \mathcal{M}_n$ as private input;
- $\mathcal{B}$ gets the same[5] key k as private input;
- $\mathcal{E}$ gets no private input;
- $\mathcal{A}$ and $\mathcal{B}$ only send messages to each other if they also send the message to $\mathcal{E}$.

A shared-key encryption scheme is *correct* if $\mathcal{B}$ outputs m at the end of the computation. A shared-key encryption scheme is *secure* if it is zero-knowledge for $\mathcal{E}$; explicitly, if there exists a simulator $\mathcal{S}$ such that for any choice of security parameter n and message m,

$$\mathcal{S}(1^n) \overset{c}{\equiv} \text{view}_{\mathcal{E}}^{\langle \mathcal{A}, \mathcal{B}, \mathcal{E} \rangle}(1^n, k, m),$$

where the randomness of the second distribution is over both the randomness of the algorithms and uniform random choice of k.

[5] Notice that this definition includes the stipulation that $\mathcal{A}$ and $\mathcal{B}$ share a uniform random key as input, which is not directly possible using the machinery of Section 1.1.4. One way to formalize this notion is to add a fourth machine $\mathcal{G}$ (the "generator"), which can message $\mathcal{A}$ and $\mathcal{B}$ freely (but not recieve messages from them), whose job is to generate the key and send it to both parties. For our purposes, the important point is that while the input m is seen as a parameter of the system which can be controlled in indistinguishability proofs, the input k is instead always randomly generated.

The point is that the eavesdropper should learn nothing from the interaction, while the intended recipient should learn the message.

Example 1.20. We can construct both a secure-but-not-correct and a correct-but-not-secure encryption scheme using work already done.

- For a secure-but-not-correct scheme, simply have each machine do nothing. The security proof is exactly the same as in Example 1.17.
- For a correct-but-not-secure scheme, have $\mathcal{A}$ send m to $\mathcal{B}$ (and therefore also to $\mathcal{E}$) as a message, and have $\mathcal{B}$ output that message. The insecurity proof is exactly the same as in Example 1.18.

Example 1.21 (the one-time pad)**.** We now give a secure and correct shared key encryption scheme, called the *one-time pad*. Let $\{G_n\}$ be a sequence of finite additive groups[6] such that $|G_n| = \Omega(2^n)$, for instance $G_n = \mathbb{Z}_2^n$. We work over $\mathcal{M}_n = \mathcal{K}_n = G_n$. Given a message m and key k, $\mathcal{A}$ computes $c = m + k$, which it sends to $\mathcal{B}$ (and $\mathcal{E}$). $\mathcal{B}$ then computes $c - k$, which it outputs.

Correctness of this scheme is immediate, as $\mathcal{B}$ outputs $c - k = m + k - k = m$. To prove security, our goal is to construct a simulator $\mathcal{S}$ such that $\mathcal{S}(1^n)$ is indistinguishable from $\text{view}_{\mathcal{E}}(1^n, k, m) = \{1^n, m + k\}$. Because addition by m is a bijection, and k is chosen uniformly at random, the distribution $\{m + k\}$ is just a uniform random sample from G_n. As such, we simply let $\mathcal{S}(1^n)$ draw g uniformly at random from G_n and output $\{1^n, g\}$. This is again a perfectly-secure encryption scheme, since the two distributions are identical.

Because the security is perfect, we don't need the asymptotics, so the one-time pad is more commonly defined on a fixed group G, usually $\mathbb{Z}_2^m$ for some fixed m.

Example 1.22 (the bootstrap one-time-pad)**.** One disadvantage of the one-time pad is that the key must be drawn from the same space as the message. We now show how to rectify this, assuming a pseudorandom generator (Example 1.13) $\mathcal{G}$ with expansion factor p is available. The idea is to use the pseudorandom generator to expand a short key into a longer one.

The protocol is as follows. We let $\mathcal{K}_n = \mathbb{Z}_2^n$ and $\mathcal{M}_n = \mathbb{Z}_2^{p(n)}$. Given a message $m \in \mathbb{Z}_2^{p(n)}$ and a key $k \in \mathbb{Z}_2^n$, $\mathcal{A}$ first computes $\mathcal{G}(k)$ to get a key $k' \in \mathbb{Z}_2^{p(n)}$. It then sends $c = m + k$ to $\mathcal{B}$. Similarly, $\mathcal{B}$ computes k' and then $c - k'$, which it outputs. Since $\mathcal{G}$ is deterministic, both $\mathcal{A}$ and $\mathcal{B}$ get the same value for k', and so the protocol is correct.

Security can be shown by a reduction to the hardness assumption entailed by pseudorandomness of $\mathcal{G}$, but an easier route is available. By definition of pseudorandomness, the distributions $\{\mathcal{G}(\mathbb{Z}_2^n)\}$ and $\{\mathbb{Z}_2^{p(n)}\}$ are computationally indistinguishable. To obtain c in this protocol and in the one-time pad, we perform the same computation on these distributions—merely adding the fixed message m. As such, by Proposition 1.12 the view of $\mathcal{E}$ in this protocol is indistinguishable from the view of $\mathcal{E}$ in the one-time pad. Now we obtain the desired result by security of the one-time pad and transitivity of indistinguishability.

[6] We also want that $\{G_n\}$ is *efficiently sampleable*, so that it is possible to generate an element from it uniformly at random in polynomial time.

1.2.2 Interactive Function Computation

Suppose we are given a (potentially stochastic) series of functions

$$f : X_1 \times \cdots \times X_N \to Y_1 \times \cdots \times Y_N.$$

Each such function yields the following cryptographic problem[7]:

> Can N parties, each given a private input $x_i \in X_i$, work together so that the ith party outputs the value $f_i(x_1, \ldots, x_n)$?

Here, f_i is the projection $\mathrm{pr}_i \circ f$. In particular:

Definition 1.23. An interactive protocol $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle$ *computes the function* f if for any choice of inputs $x_1, \ldots, x_N$,

$$\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle(x_1, \ldots, x_N) = f(x_1, \ldots, x_N),$$

where if f is stochastic the equality should be interpreted in the distributional sense.

Example 1.24. The protocol of Example 1.14 computes the function $f(x, y) = (y, x)$.

On the other hand, there are several possible notions of security in this setting, roughly following the lines discussed in Section 1.1.5. We first consider the more straightforward semi-honest case, in which we ask that no party should learn anything from the computation other than the value they are intended to output.

Definition 1.25. A protocol $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle$ *securely computes the function* f *in the presence of semi-honest adversaries* if it computes f and it is zero-knowledge for each ith party relative to f_i.

Example 1.26. Consider the following protocol for $f(x, y, *) = (*, *, xy)$: the first two algorithms each send their inputs two the third, which computes and outputs the product. This protocol is not secure in the presence of semi-honest adversaries, because the third party learns the two factors, not just the product; this insecurity can be proved very similarly to Example 1.18.

On the other hand, consider the following protocol for $f(x, y) = (*, xy)$: the first algorithm sends x to the second, which computes and outputs the product. This protocol is secure in the presence of semi-honest adversaries. The simulator for the second algorithm, which gets its input y and output xy, is responsible for producing a distribution indistinguishable from $\{x, y\}$; it can do this by computing xy/y.

Example 1.27 (oblivious bit-transfer)**.** The *oblivious bit-transfer problem* is as follows. Alice has two bits $b_1, b_2 \in \mathbb{Z}_2$, and Bob has a query $\sigma \in \{1, 2\}$. The goal is for Bob to learn the appropriate bit from Alice, without revealing which bit they asked for. We can formalize this using Definition 1.25: the problem is to securely compute

$$f : \mathbb{Z}_2^2 \times \{1, 2\} \to \{*\} \times \mathbb{Z}_2, \qquad ((b_1, b_2), \sigma) \mapsto (*, b_\sigma)$$

[7] Here we need to assume that there is a canonical grading on each input set, for instance given by the length of a bitstring, so that each X_i can also be thought of as a sequence of sets X_n^i in the style of Section 1.1.4.

in the presence of semi-honest adversaries. While the solution is outside our scope, this is possible (under standard complexity-theoretic assumptions) via a protocol originally due to [EGL85].

The situation with malicious adversaries is much more complicated, but we will not need all the details here; they can be found in, for instance, [Lin17, Section 4].

1.2.3 Zero-Knowledge Proof

In an *interactive proof*, one party—the *prover*—tries to convince the other—the *verifier*—that some statement is true. We generally consider proofs of membership predicates over fixed sets (called *languages* in this context), so that the prover is trying to convince the verifier that some global input x is in a fixed set $\mathcal{L}$. The key point is that the prover generally has some computational advantage over the verifier, so that the verifier cannot simply reproduce all the steps taken by the prover.

Informally, there are two correctness properties we want an interactive proof system to have. It should be *sound*: when given an input $x \in \mathcal{L}$, the verifier should except with high probability. It should also be *complete*, which informally means that no adversarial machine should be able to convince the verifier to accept an input not in $\mathcal{L}$ except with low probability. Formally:

Definition 1.28 (interactive proof system)**.** Let $\mathcal{L}$ be a fixed language. An *interactive proof system for* $\mathcal{L}$ consists of an algorithm $\mathcal{P}$ and a PPT algorithm $\mathcal{V}$, such that:

- (soundness) for each $x \in \mathcal{L}$,

$$\Pr[\langle \mathcal{P}, \mathcal{V} \rangle(x)_{\mathcal{V}} = 1] > \frac{2}{3};$$

- (completeness) for each $x \notin \mathcal{L}$ and each algorithm $\mathcal{P}'$,

$$\Pr[\langle \mathcal{P}', \mathcal{V} \rangle(x)_{\mathcal{V}} = 1] < \frac{1}{3}.$$

Once we achieve any probability of success greater than $\frac{1}{2}$, we can repeat the protocol to achieve any constant probability of success. It is standard to choose $\frac{2}{3}$ as the desired bound for such cases.

Example 1.29 (graph non-isomorphism [Gol01, Section 4.2.2])**.** We give an interactive proof system for the problem of *graph non-isomorphism*[8], which is formalized as the language

$$\text{GNI} = \{(G_1, G_2) : G_1 \text{ and } G_2 \text{ are non-isomorphic finite graphs}\}.$$

[8]Since we do not know whether graph non-isomorphism is in NP, the reader may now wonder what the computational strength of interactive proof is. It turns out that IP, the class of languages with interactive proof, equals PSPACE, the class of languages whose membership predicates can be computed in polynomial space. This result was first proven by [Sha92].

The idea is that the verifier will construct a random graph isomorphic to one of the chosen graphs, and the prover will have to guess which one. If the graphs are non-isomorphic, then it should always be able to do so; if they are, it can only do so with probability $\frac{1}{2}$.

The algorithms get a pair of graphs (G_1, G_2) as shared input; we let $G_1 = (V_1, E_1)$ and similarly for G_2. The verifier uniform-randomly chooses $\sigma \in \{1, 2\}$ and a relabeling of G_σ. In particular, it uniform-randomly chooses a bijection $\pi : V_\sigma \to \{1, \dots, |V_\sigma|\}$ and generates the graph

$$G'_\sigma = (\{1, \dots, |V_\sigma|\}, \{(\pi(v), \pi(w)) : (v, w) \in E_\sigma\}),$$

which it sends to the prover. The prover, which is unbounded, can then check whether $G'_\sigma \cong G_1$ or $G'_\sigma \cong G_2$ and send the result to the verifier; if both hold, then it sends a random bit. The verifier repeats the experiment again, outputting 1 if the prover guessed right both times, and 0 otherwise.

To show soundness, note that if $G_1 \ncong G_2$, then the prover will always guess correctly, in which case the verifier outs 1 with probability 1.

To show completeness, note that G'_σ is a uniform random draw from

$$\{G = (V, E) : G \cong G_\sigma, V = \{1, \dots, |V_\sigma|\},$$

i.e. from the isomorphism class of G_σ with vertex set $\{1, \dots, |V_\sigma|\}$. Since G_1 and G_2 have identical isomorphism classes and the same number of vertices, this implies that the distribution of G'_σ is independent of σ. Now let $\mathcal{P}'$ be any algorithm which is given G_1, G_2, and G'_σ and must guess σ. By independence, we have that

$$\begin{aligned}
\Pr[\mathcal{P}'(G'_\sigma) = \sigma] &= \sum_{\tau \in \{1,2\}} \Pr[\mathcal{P}'(G'_\sigma) = \tau \text{ and } \sigma = \tau] \\
&= \sum_{\tau \in \{1,2\}} \Pr[\mathcal{P}'(G'_\sigma) = \tau] \Pr[\sigma = \tau] \\
&= \frac{\Pr[\mathcal{P}'(G'_\sigma \in \{1, 2\})]}{2} \\
&\leq \frac{1}{2};
\end{aligned}$$

since the experiment repeats twice, we get a probability of $\frac{1}{4} < \frac{1}{3}$, as desired.

In cryptography, we are especially concerned with *zero-knowledge proofs*, which are meant to reveal no information other than the truth of the statement under proof. We will need something slightly stronger than for multi-party computation: because this protocol involves the prover giving information in response to queries from the verifier (such as the query graph G'_σ in Example 1.29), we will need to ensure that no verifier can learn anything extra from the prover, even if they give different queries than the protocol prescribes.

Definition 1.30 (zero-knowledge proof)**.** An interactive proof system $(\mathcal{P}, \mathcal{V})$ is *honest-verifier zero-knowledge* it is zero-knowledge for $\mathcal{V}$. It is *semi-honest-verifier zero-knowledge*,

or just *zero-knowledge*, if for each non-uniform PPT $\mathcal{V}'$, the protocol $\langle \mathcal{P}, \mathcal{V}' \rangle$ is zero-knowledge for $\mathcal{V}'$. It is *black-box zero-knowledge* if the simulator can query $\mathcal{V}'$ but must be defined independently of it[9].

If one-way functions exist, then every problem in NP has a zero-knowledge proof [GMW91]. In fact, rather remarkably, under the same assumption *any* language that admits an interactive proof admits a zero-knowledge proof [Ben+90].

Example 1.31. The protocol of Example 1.29 is an honest-verifier zero-knowledge proof.

1.3 Composition

1.3.1 The Issues at Hand

We now wish to consider whether our security definitions are closed under composition of protocols. There are many different issues to consider in stating such a composition theorem, including:

1. What kinds of protocols are being composed? Our security definitions do not capture security of arbitrary interactive processes, so either we will need a substantially more general definition or we will need to limit our composition theorem to a specific class of protocols.

2. What does it mean to compose these protocols? It is not immediately clear how to compose arbitary interactive algorithms.

3. What kind of composition is allowed? In particular, we can consider *sequential composition*, in which only one protocol is "running" simultaneously, or *parallel composition*, in which many protocols may be running simultaneously. To formally state a parallel composition theorem, we need to either specify a *scheduling model* and deal with low-level issues like atomicity, or find some way to abstract over these details.

4. What kind of security is being preserved? Given a security definition for the component protocols, we need some say to derive the security definition for the composite protocol.

5. What kinds of adversaries does the theorem handle? Composition theorems may look very different for security against uniform and non-uniform adversaries, for instance—these subtle issues can lead to very different results.

6. What protocols are we allowed to compose with? We could be allowed to compose with arbitrary protocols, which might not even be secure, or only with other protocols we already know are secure.

[9] There appear to be several distinct choices in the literature for naming conventions, for the computational strength of the prover, and for the level of honesty of the verifier. Our naming convention follows [Vad07]; a good survey of various definitions is [GO94, Section 3].

7. How many times we can compose—for instance, must it be constant in the security parameter?

As these questions demonstrate, composition theorems are really quite difficult to state. However, even without a formal theorem in mind, we can make a few concrete observations.

In regards to Question 6, we need to have some kind of idea of "independence of state" between two protocols before we can compose them. For instance, we certainly should not be allowed to compose the one-time pad with a protocol that publicly broadcasts the key.

In regards to Question 7, we should at best expect to be able to compose polynomially many protocols in the security parameter. This is in part for complexity reasons—a polytime turing machine cannot simulate super-polynomially many protocols—but there is also a security explanation: as a consequence of Proposition 1.11, we should only expect to be able to compose polynomially many computationally indistinguishable distributions before losing the indistinguishability.

We do not make attempt to give a comprehensive review of the composition theorems or counterexamples in the literature. Instead, our goal is to chart a motivating path towards the theory of *universal composability*, and then give a sufficiently detailed overview of that theory to equip the reader to compare it to the cryptographic models we will study in Chapter 3.

1.3.2 Composing Interactive Function Computations

Possibly the easiest non-trivial composition theorem to state is for sequential composition of interactive function computations, but already in this case we will run into several fundamental issues.

Suppose we have two N-party protocols $\langle \mathcal{A}_1, \dots, \mathcal{A}_N \rangle$ and $\langle \mathcal{B}_1, \dots, \mathcal{B}_N \rangle$ which securely compute the N-party functions $f : X_1 \times \cdots \times X_N \to Y_1 \times \cdots \times Y_N$ and $g : Y_1 \times \cdots \times Y_N \to Z_1 \times \cdots \times Z_N$ in the presence of semi-honest adversries. We wish to use this data to construct a composite protocol for the composite function $g \circ f$. The natural choice is to have the parties first run the protocol for f, then run the protocol for g. Thus, we can state the following claim:

Claim 1.32. *Suppose $\{f^i : X_1^i \times \cdots \times X_N^i \to X_1^{i+1} \times \cdots \times X_N^{i+1}, 1 \le i \le K\}$ is a finite sequence of functions which are securely computable in the presence of semi-honest adversaries. Then the composite $f = f^K \circ \cdots \circ f^1$ is securely computable in the presence of semi-honest adversaries.*

Proof attempt. For each i, let $\Pi_i = \langle \mathcal{A}_1^i, \dots, \mathcal{A}_N^i \rangle$ be a protocol which securely computes f^i in the presence of semi-honest adversaries. Form a new protocol Π which performs each of the Π_i in sequence. Correctness is immediate, since the correctness of each Π_i implies that Π computes each step of the composite in turn.

The proof of security is by reduction to the security of the Π_i. By induction, it suffices to consider the case $K = 2$. Suppose that the composite Π does not securely compute $f^2 \circ f^1$. Without loss of generality, suppose that the composite is not zero-knowledge for

party 1. Then for any $\mathcal{S}$, there exists some non-uniform PPT $\mathcal{D}$ which distinguishes, for any choice of inputs $x_i \in X_i^1$,

$$\mathcal{S}(x_1, f_1^2(f^1(x_1, \ldots, x_N))) \qquad \text{and} \qquad \text{view}_1^{\Pi}(x_1, \ldots, x_2). \tag{1.1}$$

All this works, but we run into issues when we attempt to construct a simulator which contradicts this assumption. Let $\mathcal{S}_i^1$ and $\mathcal{S}_i^2$ be the simulators which witness security of Π_1 and Π_2 with respect to party i. We would like to construct a simulator $\mathcal{S}$, which combines the simulated views of party i in both parts of the computation. The issue is that, in order to fit the form of (1.1), $\mathcal{S}$ should get only $(x_1, f_1^2(f^1(x_1, \ldots, x_N)))$ as input. However, to be able to run $\mathcal{S}_1^1$, it needs $f_1^1(x_1, \ldots, x_N)$, which we have no way to obtain.

I am not aware of a direct way to repair this proof. Instead, to avoid the issue, we need the structure of the composite protocol to mirror the structure of the reduction: there should be some "outermost" protocol which handles calls to the sub-protocols, just as we need to make an outermost simulator which calls the sub-simulators. The right tool for the job is the *oracle algorithm*.

Definition 1.33 (oracle algorithms; oracle protocols)**.** An *oracle algorithm* is an algorithm $\mathcal{A}$ equipped with a "slot" for an *oracle* O, to which it can make *queries* x and receive *responses* $O(x)$. We write $\mathcal{A}^O$ to refer to the oracle-algorithm $\mathcal{A}$ equipped with the specific oracle O.

An *oracle protocol* is a protocol $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle$ with a slot for an oracle O, where each $\mathcal{A}_i$ may write queries x_i, and if each machine does so, they each receive outputs $O(x_1, \ldots, x_N)_i$. We write $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle^O$.

Let $f : X_1 \times \cdots \times X_N \to Y_1 \times \cdots \times Y_N$ be a function. An *f-oracle* is an oracle which, when queried with $x_1, \ldots, x_n$, responds with $f(x_1, \ldots, x_N)$. We overloadingly write f to refer to an f-oracle.

The idea is to make an oracle protocol Π which, when instantiated with an f-oracle, securely computes g. Then if we have a protocol for securely computing f, we will show that we can substitute it for the oracle in Π. In particular, while we will not be precise about what secure oracle computation means, it is important that the view of an oracle algorithm includes its queries and responses.

Definition 1.34 (oracle reduction)**.** Let f and g be functions. Then *g securely oracle-reduces to f in the presence of semi-honest adversaries* if there exists an oracle protocol $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle$ such that $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle^f$ securely computes g in the presence of semi-honest adversaries.

Theorem 1.35 ([Gol01, Theorem 7.3.3])**.** *Suppose g securely reduces to f in the presence of semi-honest adversaries, and f is securely computable in the presence of semi-honest adversaries. Then g is securely computable in the presence of semi-honest adversaries.*

Proof sketch. Let $\Pi_g = \langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle$ be an oracle protocol such that $\langle \mathcal{A}_1, \ldots, \mathcal{A}_N \rangle^f$ securely computes g in the presence of semi-honest adversaries, and let $\Pi_f = \langle \mathcal{B}_1, \ldots, \mathcal{B}_N \rangle$ be a protocol which securely computes f in the presence of semi-honest adversaries. We

construct a new protocol Π which runs Π_g, but whenever the oracle is queried, it instead replaces the oracle with a complete run of Π_f. Correctness of Π is immediate from its construction; we show security by a reduction.

Let $\mathcal{S}_i^g$ simulate the view of party i in Π_g^f, and let $\mathcal{S}_i^f$ simulate the view of party i in Π_f. We construct a simulator $\mathcal{S}_i$ for the view of party i in Π. Since the inputs to Π_g^f are the same as to Π, we can immediately run $\mathcal{S}_i^g$ to obtain the view of party i in the computation Π_g^f, which in particular includes the queries and responses of the oracle. We can then pass these queries and responses to Π_f, obtaining a view of each of the invocations of the sub-protocol.

To show the this simulator achieves the necessary indistinguishability result, we work in two steps. First, the output of $\mathcal{S}_i$ is indistinguishable from the view of party i in Π with the sub-protocol invocations are replaced with the simulated views; we prove this by the security of Π_f, since otherwise appending the view from Π_g^f would distinguish the view from Π_f with its simulated counterpart. Similarly, this latter distribution is indistinguishable from the view of party i in Π without any such replacement, this time by security of Π_g^f. Filling in these details completes the proof. □

A few remarks on this proof are warranted. First, it fundamentally relies on the semi-honesty of the adversaries, since it means that the execution of the sub-protocols is completely independent from anything other than their inputs. If the adversaries could behave maliciously, and thus use information learned in the larger protocol to affect run the sub-protocols, then the main simulator would be unable to properly invoke the simulator for the sub-protocols.

Second, these techniques seem completely unable to handle parallel composition. The issue is that oracle queries are in some sense immediate—the assumption is that everything else pauses while the oracle does its work. It is much more difficult to handle parallelism, or even more strongly, to handle concurrency, which may in particular be asynchronous.

Third, if we want to handle protocols other than function computations, we will need a more robust notion of composition of ideal functionalities—this proof is tied to the easy-to-understand structure of function computation.

The goal of universal composability is to resolve these issues, but first we will explicitly give an example wherein parallel composition fails.

1.3.3 A Counterexample to Parallel Composition

Before giving the counterexample, we briefly discuss sequential composition of zero-knowledge proofs, which is quite subtle. When the definitions are as given in Definition 1.30, black-box zero knowledge proofs do compose in sequence [GO94]. On the other hand, if we change the definitions so that both the simulator of Definition 1.16 and the adversarial verifier of Definition 1.30 are required to be *uniform* PPT, then computational zero-knowledge does not compose [GK96]. However, if under the same definition we require that the prover is in NP, then uniform computational zero-knowledge proofs do compose in sequence up to a constant number of times; but if the distinguishers in the definition of computational indistinguishability are required to be uniform, then sequen-

tial composition again fails [BV10]. There are a huge number of variations to consider and I believe there are many definitions for which the question of sequential composition is still open.

For parallel composition, the situation is much simpler: we give an argument due to [GK96] that zero-knowledge proofs do not compose in parallel. The idea of the counterexample is as follows. In proof A, the prover poses a randomly chosen computationally intractable challenge to the verifier, and then gives the verifier knowledge if and only if the verifier can solve the challenge. This proof is zero-knowledge because PPT verifiers can solve the challenge only negligibly often. In proof B, the verifier poses a challenge to the prover, which the prover answers. The trick is to choose a class of challenges whose answers are pseudorandom, so that the answer to the challenge on its own does not carry knowledge. However, when proof A and proof B are run together in parallel, the verifier can take the challenge it gets in proof A, get an answer in proof B, and that use that to get knowledge from the prover in proof A.

The difficulty in formalizing this is to find a class of challenges which are computationally intractable, in that PPT algorithms solve them with negligible probability; zero-knowledge, in that answers look pseudorandom; and decidable, in that unbounded algorithms can answer challenges and check answers. Our challenges will be phrased as sets: for each security parameter n, we will agree to some set of sets $S_1^n, \ldots, S_{2^n}^n \subseteq \{0,1\}^{Q(n)}$, where Q is a polynomial. A challenge looks like a value $i \in 1, \ldots, Q(n)$, and a solution is a value $s \in S_i^n$. We now formalize our desiderata as follows.

Definition 1.36. A *non-uniform ensemble* is a sequence $\{S^n\}$, where for each n, $S^n = \{S_1^n, \ldots, S_{2^n}^n\}$ is a set of 2^n sets. A non-uniform ensemble is:

- *polynomially-sized* if there exists a polynomial Q such that for each n and i, $S_i^n \subseteq \{0,1\}^{Q(n)}$;
- *pseudorandom* if for each n and i, S_i^n is pseudorandom (in the sense of Example 1.13);
- *decidable* if there exists an algorithm which, on input (n, i), outputs the elements of S_i^n and then halts;
- *evasive* if for any non-uniform PPT algorithm $\mathcal{A}$,

$$\Pr_{i \in \{1,\ldots,2^n\}}[\mathcal{A}(i) \in S_i^n] = \mathsf{negl}(n).$$

Theorem 1.37 ([GK96, Theorem 3.2])**.** *There exists a polynomially-sized, pseudorandom, decidable, evasive non-uniform ensemble.*

Now the actual construction of the counterexample is relatively straightforward. Let S be the ensemble from Theorem 1.37 and Q the size polynomial. Let K be a computable predicate which is known not to be in BPP; such things exist by the time hierarchy theorem from complexity theory. We will give two zero-knowledge proofs for the language $\{0,1\}^*$.

Proof A goes as follows: on input $x \in \{0,1\}^n$, $\mathcal{P}_A$ chooses $i \in \{1, \ldots, 2^n\}$ uniformly at random, which it sends to $\mathcal{V}_A$. Next, $\mathcal{V}_A$ chooses $s \in \{0,1\}^{Q(n)}$ uniformly at random, which it sends to $\mathcal{P}_A$. If $s \in S_i^n$, $\mathcal{P}_A$ sends $K(x)$ to $\mathcal{V}_A$. Then $\mathcal{V}_A$ outputs 1. This proof is zero-knowledge because the probability that any cheating $\mathcal{V}_A'$ sends an $s \in S_i^n$ is negligible, so the probability it learns $K(x)$ is negligible.

Proof B goes as follows: on input $x \in \{0,1\}^n$, $\mathcal{V}_B$ chooses $i \in \{1,\dots,2^n\}$ uniformly at random, which it sends to $\mathcal{P}_B$. Next, $\mathcal{P}_B$ sends $s \in S_i^n$ to $\mathcal{V}_B$. Then $\mathcal{V}_B$ outputs 1. This proof is zero-knowledge because S_i^n is pseudorandom, so the simulator can just output a uniform random value and by definition of pseudorandomness no distinguisher can tell whether it is seeing that random value or the value from S_i^n.

However, in the parallel composition, the verifier can wait to send i until it receives the i from $\mathcal{P}_A$. Then, when it gets back $s \in S_i^n$ from $\mathcal{P}_B$, it can send s to $\mathcal{P}_A$ and get back $K(x)$, and hence it learns $K(x)$ with probability 1.

Under computational hardness assumptions, Theorem 1.37 can be extended to, for instance, evasive ensembles which are decidable in NP, and hence under such assumptions zero-knowledge proofs with NP provers also do not compose in parallel.

As a consequence of this construction, it is clear that we need more technology to handle parallel composition in any reasonable way. By far the most popular attempt to do so is universal composability.

1.3.4 Universal Composability

In Section 1.3.2, we identified three critical issues with generalizing the proof of Theorem 1.35: we need to handle malicious adversaries which can forward their views to each other; we need to handle parallel composition during which other protocols can run; and we need a very general way to describe composition and security of a large class of computational protocols. Universal composability (UC), due to Ran Canetti [Can00; Can20], resolves all of these.

The single big idea of UC is to strengthen the notion of emulation required in the real-ideal paradigm. Whereas our formulations of simulation security require only that *by the end of the computation* the real protocol produces an output indistinguishable from the ideal protocol, in UC, this indistinguishability must hold *throughout* the computation. Intuitively, we should expect this to be strong enough to allow us to prove a parallel composition theorem, because in particular the other computation we are composing with will be unable to distinguish our protocol from the ideal. In this section, we will sketch some of the key tools used to make this formal, without attempting to be fully formal ourselves.

In a UC proof, an algorithm called the *environment* represents all the other computational processes happening along with the protocol under study. The environment is also responsible for giving inputs to parties in a protocol. Ultimately, the environment is also the distinguisher, responsible for attempting to determine between the real or ideal protocol. If it cannot do so, then no other protocol is able to do so, so in particular the security guarantees will hold no matter what else is going on. In this way, environments abstract over a lot of the complexity in ordinary cryptographic definitions.

Adversarial behavior is somewhat complicated in UC. In addition to the environment, there is an algorithm called the *adversary* which may write to the *backdoor tapes* of all the parties involved in the protocol. The protocol is responsible for specifying how the parties should behave in respnose to messages writen on their backdoor tapes.

When π is a protocol and $\mathcal{A}$ and $\mathcal{E}$ are algorithms, we write $\text{exec}_{\pi,\mathcal{A},\mathcal{E}}$ for the output of $\mathcal{E}$ after an interactive computation with all the algorithms in π and with the adversary

$\mathcal{A}$. A protocol π *UC-emulates* another protocol ϕ if for any adversary $\mathcal{A}$ there exists a simulator $\mathcal{S}$ such that, for any environment $\mathcal{E}$,

$$\text{exec}_{\pi,\mathcal{A},\mathcal{E}} \overset{c}{\equiv} \text{exec}_{\phi,\mathcal{S},\mathcal{E}}\,.$$

The idea is that $\mathcal{E}$ is trying to output a guess of whether it is interacting with π or ϕ; it should be able to guess right only negligibly better than half of the time.

We need a way to express ideal protocols for general cryptographic resources; UC calls these *ideal functionalities*. An ideal protocol for a functionality essentially consists of an algorithm $\mathcal{F}$, which receives its inputs from some "dummy parties" and then sends them the correct outputs back. In order to work with more than just function computation, $\mathcal{F}$ is also constantly talking to the adversary, and so can simulate functionalities like public commitments that do not show up in a function signature. A protocol *UC-realizes* the functionality $\mathcal{F}$ if it UC-emulates the ideal protocol for $\mathcal{F}$.

The next issue is a formal notion of composition. This should look like the "oracle substitution" construction from the proof of Theorem 1.35, but handle more general forms of composition than the sequential composition implied by the oracle reductions. Pick a protocol π and a subset ρ of the parties involved in π. Pick another protocol ϕ such that there is an injective map from the parties in ρ to those in ϕ. Then the protocol $\pi^{\rho\to\phi}$ replaces ρ with ϕ, wiring up the communication with the rest of π according to the injection (note that ϕ may have more machines than ρ; these do not talk to the other parties in π). This is called the *universal composition operation.* It takes rather a lot of work to make this substitution operation precise—for instance, there is a very technical compatibility condition which essentially asks that the interfaces of ϕ and ρ are "functional", in the intuitive sense that they only interact with the rest of the protocol via their inputs and outputs (and maybe as a result of corruption). In [Can20], such protocols are called *subroutine respecting*.

Theorem 1.38 (the univeral composition theorem [Can20, Theorem 22])**.** *Under technical assumptions, if ρ is a subroutine of π and ϕ UC-emulates ρ, then $\pi^{\rho\to\phi}$ UC-emulates π. In particular, if $\mathcal{F}$ and $\mathcal{G}$ are ideal functionalities such that π UC-realizes $\mathcal{G}$ and ϕ UC-realizes $\mathcal{F}$, then $\pi^{\mathcal{F}\to\phi}$ UC-realizes $\mathcal{G}$.*

All this is just a sketch; there are of course many technical details to work out. For instance, UC comes with its own entire low-level model of distributed computation. The underlying machine model is a specific kind of *interactive Turing machine*, which has seven different tapes with differing semantics and read/write permissions, and the network is mediated by a *control function*, which can modify or block messages between machines involved in a computation. There is no formal notion of a "secure channel" in UC, nor of other basic cryptographic resources; instead every resource is meant to be represented as an auxiliary machine which implements some ideal functionality.

To summarize and begin to evaluate the framework, we now return to the questions from Section 1.3.1.

In Question 1, we asked what kinds of protocols are being composed. Protocols in UC are sets of interactive Turing machines behaving according a highly specific execution model. On one hand, Turing machines are a standard and well-understood model of classical computation, and the flexibility of the notion of environment allows for encoding

a wide variety of security properties. On the other hand, the technical specificity of the interaction model makes it difficult to formally apply the model; in practice papers tend not to work with the precise model, and it is not clear to the author that the model is sufficiently well-understood to justify the common level of hand-waving. Furthermore, this machine model makes it difficult to apply UC to other models of computation. For instance, to deal with quantum cryptography, we need to use a mathematically separate framework based on quantum which redoes much of the work of the UC paper—in fact, there are at least three competing frameworks for doing so [Unr04; BM04; Unr10], each with their own ad-hoc notion of "quantum machine."

In Questions 2 and 3, we asked about the definition and scope of composition. The subroutine substitution operation from UC is extremely general. Canetti argues, and the last twenty years have demonstrated, that together with control functions which model adversarial network conditions, UC composition can handle sequential, parallel, concurrent, and asynchronous composition; that it can handle compositions with variable numbers of rounds and subroutine calls; with coordinated or uncoordinated timings; with adaptively chosen inputs and adversaries; composition with shared and independent state; and more. One good source of examples of these claims is [Can06].

In Question 4, we asked how the system derives security definitions for composite protocols. UC works similarly to the oracle model: there is one "outermost" protocol, and we substitute real protocols for idealized subroutines within the protocol. The key point is that the definition of UC-emulation is strong enough to allow a protocol to be substituted for an ideal functionality virtually anywhere and any time. However, this imposes a high proof burden which makes many UC proofs intractable, as even aside from the low-level details of the machine model, constructing a simulator that is in constant conversation with the environment is quite burdensome. The high proof burden also leads to impossibility results that can require somewhat ad-hoc setup assumptions to overcome [CF01; Bar+04; KL07; JM20].

In Question 5, we asked which kinds of adversaries the system accommodates. UC is again quite flexible in this regard. Adversarial behavior is built into the protocol via backdoor tapes and corruption messages, and then a separate party called the adversary activates (potentially in a controlled, stateful way, so that they cannot for instance just corrupt all the parties) corruption messages during the protcol. In this way, UC conveniently avoids dealing in the formalism with different models of adversaries, since they may be specified on a protocol-by-protocol basis. For instance, a protocol meant to be secure against semi-honest adversaries may just allow backdoor messages which ask the corrupted party to forward its state to the environment, while a protocol meant to be secure against malicious adversaries may allow a party to be completely piloted over via instructions sent to its backdoor tape. In [Can00, Section 7.1.1], Canetti gives examples of a wide variety of adversarial models which can be incorporated in this way. Furthermore, this approach allows UC proofs to compose protocols which are secure against different forms of adversary.

In Question 6, we asked what protocols we can securely compose secure protocols with. UC does not require that the protocols we compose with are secure; in fact there are essentially no requirements on the "outer" protocol into which the substitution will occur. We also saw previously that any framework for composition needs some way to

assert that the protocols being composed are "independent enough" that one does not give away the secret key of the other; UC does this with the notion of "subroutine respecting" protocols, which are one of the only technical limits on composition.

In Question 7, we asked how many times we can compose protocols. We have not discussed nested UC-composition, but the universal composition theorem holds up to polynomially many nested substitutions of subroutines, which as discussed earlier is a tight bound.

Finally, we give two pieces of evidence of the naturality of UC. Technically, [Lin03] proved that, if a protocol for multi-party function computation is secure under parallel composition with even a constant number of (not-necessarily-secure) other protocols, then it is UC-secure. Socially, a precise connection has recently been established between UC and the independently-formulated *robust compilation* (RC) framework from programming language security [PKW22]; while RC is a comparatively new tool, this connection suggests a cross-field applicability of even the technical details of UC.

That said, while UC has clear merits, especially in terms of its incredible flexibility, there are serious disadvantages to carrying around that degree of complexity. In particular, UC cannot handle other models of computation, its proofs are often messy and hard to verify, and it carries several impossibility results that seem somewhat artificial. We will conclude the chapter by surveying some alternatives.

1.3.5 Alternative Approaches

Several authors attempt to simplify UC by reducing the intended scope. This approach is most notably taken by the "simple UC" approach of [CCL15] in the special case of standard multiparty computation, as well by "simplified UC"[10] [Wik16], which fixes the number of parties and does not handle adaptive adversaries. The approach of [CCL15] has been quite successful; simple UC has been widely used for composable proofs of secure multiparty computation in the literature [MR19; HSS20; Lin22; SKM23].

Another approach in the direction of UC is to use proof automation technologies from programming language theory to give constructive UC proofs which can be checked and even implemented by machine. For instance, IPDL [Mor+21], a logic for reasoning about secure probabilistic message-passing computations, has a weak equational theory generated by a UC-inspired observational equivalence relation, while symbolic UC [BU13] and the interactive lambda calculus [LHM19] build programming calculi for UC variants over the π-calculus. There has even been early progress, in the form of EasyUC [CSV19], in implementing domain-specific languages for frameworks of this sort in proof assistants. While these approaches generally trade complex machine models for complex typing disciplines without reducing the complexity off the core framework, the hope is that these typing disciplines will facilitate easier proof automation, removing a lot of the complexity from human view.

Slightly further from UC, several authors give UC-like frameworks with different low-

[10] Confusingly, not only are simple UC and simplified UC unrelated, the abstract of [CCL15] actually refers to their framework as simplified UC, whereas in the paper and the rest of the literature it is called simple UC. The simple UC model of [CCL15] is a better-known framework and references to this term in the literature generally refer to this model.

level machine models. This approach is taken, for instance, by IITM-based UC [Cam+19], which uses "interactive inexhaustable Turing machines", and GNUC [HS11], which uses statically-linked composition rather than the dynamic linking of UC. None of these alternative models have caught on to any significant degree; they generally seem to suffer either from expressiveness issues or from the same overcomplexity as UC.

A more radical approach is to ignore the low-level details entirely, and instead give an algebraic axiomatization of the properties which a machine model ought to satisfy. This approach is taken by both constructive cryptography [Mau12] and abstract cryptography [MR11]. In these closely-related models, there is an abstract notion of a "resource system," which is a partially ordered set of resources and set of reductions between them satisfying some axioms. These algebraic theories can then be instantiated explicitly with resources and reductions obtained from some specific class of cryptographic systems.

As this brief survey demonstrates, the problem of cryptographic composability is an active and important area of research with many different ongoing approaches. An excellent high-level summary of the state of the field (as of 2019) is the report from the Dagstuhl seminar on the subject [Cam+19].

Chapter 2

Category Theory

The notion of a *category*, originally developed as an abstraction for certain ideas in pure mathematics, turns out to be the natural algebraic axiomatization of a collection of strongly typed, composable processes, such as functions in a strongly typed programming language. More philosophically, we can think of a category as an *algebra of composition*, and category theory as the mathematical study of composition. In this chapter, we will develop the basic theory of categories, prioritizing examples from computer science where possible.

Basic texts on category theory include [Mac71] and [Rie17], while the connection to computer science is explored in [Pie91] and [BW90]. A more advanced treatment of the connection, especially applications to programming language theory, is [Jac99].

2.1 Basic Notions

2.1.1 Categories

Definition 2.1 (category)**.** A *category* C consists of the following data:

- a collection[1] of objects, overloadingly also called C;
- for each pair of objects $x, y \in C$, a collection of *morphisms* $C(x, y)$;
- for each object $x \in C$, a designated *identity morphism* $x \xrightarrow{1_x} x$;
- for each pair of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$, a designated *composite morphism* $x \xrightarrow{gf} z$.

This data must satisfy the following axioms:

- *unitality*: for any $x \xrightarrow{f} y$, $1_y f = f = f 1_x$;
- *associativity*: for any $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, $(hg)f = h(gf)$.

Notation. In addition to those used above, many syntaxes are common in the literature for basic categorical notions. For convenience, we survey some here, though we will try to be consistent in our notation.

[1] We use the word *collection* for foundational reasons: in many important examples, the objects and morphisms do not form sets. We ignore such foundational issues here; they are discussed in [Mac71, subsection 1.6].

- A morphism $f \in C(x, y)$ is often written $f : x \to y$ or $x \xrightarrow{f} y$; x is called its *domain* or *source* and y is called it *codomain* or *target*.
- Morphisms may be called maps, arrows, or homomorphisms; the class of morphisms $C(x, y)$ may also be written $\mathrm{Hom}_C(x, y)$ or just $\mathrm{Hom}(x, y)$, and is often called a *hom-set*.
- Composition is written gf or $g \circ f$; in the literature it is sometimes written in the left-to-right order fg; we will never do this.
- Identities are written 1_x, id_x, or just x where the context is clear; we will never do the latter.

Example 2.2 (functional programming languages)**.** Consider some strongly-typed functional programming language L, whose functions are never side-effecting. Then under very modest assumptions about L, we can make a category $\mathcal{L}$, as follows:

- the objects of $\mathcal{L}$ are the types of L;
- the morphisms $\mathcal{L}(A, B)$ are the functions of type $A \to B$;
- the identities 1_A are the identity functions $A \to A$;
- composition of morphisms are the usual function composition.

If L is truly non-side-effecting, then it's straightforward to check that this construction does indeed satisfy the axioms of a category; see for instance [BW90, subsection 2.2] to see the necessary assumptions spelled out rigorously.

Categories are also widespread in mathematics, as the following examples show.

Example 2.3 (concrete categories)**.** The following are all categories:

- Set is the category of sets and functions.
- Grp is the category of groups and group homomorphisms.
- Ring is the category of rings and ring homomorphisms.
- Top is the category of topological spaces and homeomorphisms.
- For any field $\Bbbk$, $\mathrm{Vect}_\Bbbk$ is the category of vector spaces over $\Bbbk$ and linear transformations.

We call such categories, whose objects are structured sets and whose morphisms are structure-preserving set-functions, *concrete*. On the other hand, many categories look quite different.

Example 2.4. The following are also categories:

- The *empty category* has no objects and no morphisms.
- The *trivial category* has a single object and its identity morphism.
- Any group (or, more generally, monoid) can be thought of as a category with a single object, a morphism for every element, and composition given by the monoid multiplication.
- Any poset (or, more generally, preorder) $(P, \leq)$ can be thought of as a category whose objects are the elements of P, with a unique morphism $x \to y$ if and only if $x \leq y$. In this sense, composition is a "higher-dimensional" transitivity, and identities are higher-dimensional reflexivity.

- Associated to any directed graph is the *free category* on the graph, whose objects are nodes and whose morphisms are paths. In particular, the identities are just the empty paths, while composition concatenates two paths.
- Let $M = (Q, \delta)$ be an automaton over an alphabet Σ, so that $\delta : Q \times \Sigma \to Q$ is a transition function (one may replace Q with $\mathcal{P}(Q)$ in the codomain to represent a nondeterministic automaton). There is an associated category $\mathcal{M}$ whose objects are exactly the states and whose morphisms $\mathcal{M}(q_1, q_2)$ are the words $w \in \Sigma^*$ such that, if M is in the state q_1 and receives w as input, it ends in the state q_2. The identity morphism 1_q is the empty word, and composition is concatenation of words[2].
- There is a category whose objects are (roughly) multisets of molecules and whose morphisms are chemical reactions. See [BP17] for a formalization of this notion.

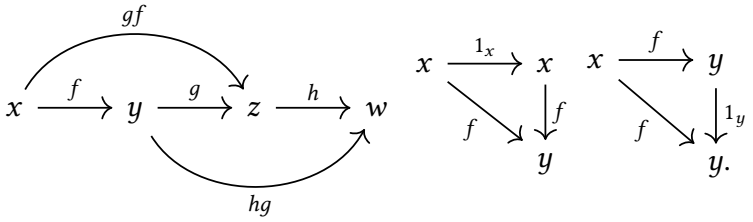
When working with categories, we often want to show that two complex composites of morphisms equate. In this case, we prefer graphical notation to the more traditional symbolic equalities of Definition 2.1. A diagram in a category C looks something like so[3]:

$$\begin{array}{ccc} w & \xrightarrow{f} & x \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ y & \xrightarrow[k]{} & z. \end{array}$$

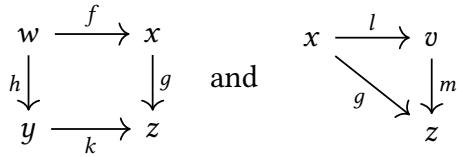
This diagram identifies four objects $w, x, y, z \in C$, and four morphisms $f \in C(w, x)$, $g \in C(x, z)$, $h \in C(w, y)$, and $k \in C(y, z)$.

We say that a diagram *commutes* if, for any pair of paths through the diagram with the same start and end, the composite morphisms are equal. In this language, the previous diagram commutes if and only if $gf = kh$.

Example 2.5. The axioms of Definition 2.1 are expressed by commutativity of the following three diagrams:



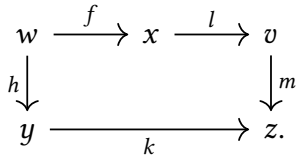
The key idea is that commutative diagrams can be "pasted", allowing us to build up complex equalities from simpler ones. For instance, if



[2] I believe this example is due to [Gog+73, Example 2.2].

[3] The notion of a diagram can be made precise fairly easily; see [Rie17, subsection 1.6].

both commute, then by pasting along the shared morphism g, so does



Note that, in order for these diagrams to be well-defined, we need composition to be associative: otherwise the top-right path of the previous diagram would be ambiguous. In some sense, the algebraic axioms are chosen exactly so that the diagrammatic calculus is coherent. This will be a repeated theme for us.

Regardless, this pasting property is essentially just a re-expression of the transitivity and substitution properties of equality, but gives an extraordinarily useful geometric intuition to categorical arguments.

2.1.2 (Iso)morphisms

The philosophy of category theory is that

to study an object, one should study its morphisms.

Indeed, in every category, morphisms give enough information to recover the data of an object.

Example 2.6. In the following categories, we can reconstruct an object by "probing" it with morphisms from suitable choices of other objects.

- Let X be a set. A function $f : \{*\} \to X$ is exactly a choice of $f(*) \in X$, so the morphisms $\textsc{Set}(\{*\}, X)$ identify exactly the elements of X, i.e. the entire data of a set.
- Let X be a topological space. A continuous map $f : \{*\} \to X$ picks out the points of X, as before. Let $S = \{0, 1\}$, with $\{1\}$ open; this is the *Sierpinski space*. Then a continuous map $f : X \to S$ consists of a choice of open set $f^{-1}(1) \subseteq X$, so the morphisms $\textsc{Top}(X, S)$ identify exactly the open sets of X. Together with the points, this is the entire data of a topological space.
- Let G be a group. A group homomorphism $f : \mathbb{Z} \to G$ is determined by a choice of $f(1) \in G$, so these pick out the elements. Letting $-$ be the group homomorphism $\mathbb{Z} \to \mathbb{Z}$ which takes z to $-z$, the composite $f-$ picks out the inverse of the element identified by f. To recover the multiplicative structure, we consider the *free product group* $G \bullet H$, whose elements are words $g_1 h_1 g_2 h_2 \cdots g_n h_n$ modulo the relations of G and H, and whose multiplication is concatenation. There is a canonical map $\varphi : \mathbb{Z} \to \mathbb{Z} \bullet \mathbb{Z}$ given by $1 \mapsto 11'$ (we represent elements in the second copy of $\mathbb{Z}$ with $'$s). Given two morphisms $f, g : \mathbb{Z} \to G$, we can define a map $f \bullet g : \mathbb{Z} \bullet \mathbb{Z} \to G$ by $z_1 z_1' \cdots z_n z_n' \mapsto f(z_1)g(z_1') \cdots f(z_n)g(z_n')$. Because the map $(f \bullet g)\varphi : \mathbb{Z} \to G$ picks out exactly the element $f(1)g(1)$, we have recovered the entire structure of G purely by studying $\textsc{Grp}(\mathbb{Z}, G)$.

These examples are instances of a much more general theory, which we begin to develop here. We first need to formalize what we mean by "recovering the data" of an object.

Definition 2.7 (isomorphism)**.** A morphism $f : x \to y$ in a category C is an *isomorphism* if there exists an *inverse morphism* $g : y \to x$ such that $gf = 1_x$ and $fg = 1_y$. Two objects x and y are *isomorphic*, written $x \cong y$, if there exists an isomorphism between them.

Example 2.8. The general notion of isomorphism recovers the familiar notions in virtually every common setting.

- Every identity morphism is an isomorphism with itself as the inverse.
- Isomorphisms in Set are bijections; in Grp are group isomorphisms; in $\text{Vect}_{\mathbb{k}}$ are vector space isomorphisms; and in Top are homeomorphisms.
- Let G be a group with associated category $\mathcal{G}$. Then since composition is group multiplication, every morphism in $\mathcal{G}$ is an isomorphism. (In fact, we can take this as a definition: a *monoid* is a category with one object, while a *group* is a monoid in which every morphism is an isomorphism. A *groupoid* is then a category in which every morphism is an isomorphism; groupoids, which generalize groups, are a very interesting algebraic object in their own right.)
- Let P be a poset with associated category $\mathcal{P}$. Then antisymmetry of a poset implies that the only isomorphisms in are the identities. (A *preorder* is a category in which every hom-set has at most one element; a *poset* is a preorder in which the only isomorphisms are the identities.)
- Let $M = (Q, \delta)$ be a non-deterministic automaton over the alphabet Σ, so that $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is the transition function. Recall that the identities in $\mathcal{M}$ are the empty words. As such, an isomorphism between two states q_1 and q_2 is a word w which takes q_1 to q_2, together with a word w' which takes q_2 to q_1, such that the concatenate ww' is the empty string. In other words, w and w' are both empty—so two states are isomorphic if and only if the machine can freely move between them at any point.

Isomorphisms satisfy the basic properties we expect.

Proposition 2.9. *Inverses are unique. Explicitly, if $f : x \to y$ is an isomorphism with inverses $g, h : y \to x$, then $g = h$.*

Proof. We have

$$g = 1_x g = (hf)g = h(fg) = h1_y = h. \qquad \square$$

Notation. We are now justified in unambiguously writing the inverse of an isomorphism f as f^{-1}.

Proposition 2.10. *Being isomorphic is an equivalence relation on the class of objects in a category C.*

Proof. We need to show:

- Reflexivity. The identity 1_x is an isomorphism $x \cong x$.

- Symmetry. Given an isomorphism $f : x \to y$, f^{-1} is an isomorphism $y \to x$ with inverse f.
- Transitivity. Given isomorphisms $f : x \to y$ and $g : y \to z$, gf is an isomorphism $x \to z$ with inverse $f^{-1}g^{-1}$. □

We now take a first step towards justifying the assertion as the beginning of the section.

Proposition 2.11. *Let $x \cong y$ in a category C. Then:*

1. *for every $z \in C$, $C(z, x) \cong C(z, y)$;*
2. *for every $z \in C$, $C(x, z) \cong C(y, z)$.*

Proof. Let $f : x \to y$ be an isomorphism.

First, define a map $f_* : C(z, x) \to C(z, y)$ by post-composition, i.e. $f_*(g) = fg$. We claim that f_*^{-1}, defined similarly, is an inverse of f_*. Letting $h \in C(z, x)$, we have

$$f_*^{-1}(f_*(h)) = f^{-1}(fh) = (f^{-1}f)h = 1_x h = h,$$

and the same on the other side.

Similarly, define a map $f^* : C(y, z) \to C(x, z)$ by pre-composition, i.e. $f^*(g) = gf$. Then an identical check shows that $(f^{-1})^*$, defined similarly, is an inverse of f^*. □

To show the other direction, we will need a little bit more machinery. Once shown, this result will indeed imply that the entire structure of an object can be identified by studying its morphisms. We will finally do this in the form of Theorem 2.27.

2.1.3 Functors

Enmeshed in the categorical mindset, we understand that morphisms—relationships—between objects are of crucial importance. Since we now want to study categories, we ask the natural question: what is the right notion of morphism between categories? The answer is a *functor*, which is just a structure-preserving map between categories.

Definition 2.12 (functor)**.** A *functor* $F : C \to \mathcal{D}$ consists of the following data:

- for each object $x \in C$, an object $Fx \in \mathcal{D}$;
- for each morphism $f \in C(x, y)$, a morphism $Ff \in \mathcal{D}(Fx, Fy)$.

This data must preserve the structure of the category, namely identities and composites, meaning:

- for each object $x \in C$, $F1_x = 1_{Fx}$;
- for each pair of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in C, $F(gf) = (Fg)(Ff)$.

Example 2.13. In mathematics, functors are ubiquitous as representations of procedures for producing structures of one sort from structures of another. For instance, the following are all functors:

- On any category C, there is an *identity functor* $1_C : C \to C$ which takes each object and morphism to itself.
- There is a functor $\mathcal{P}_\exists : \textsc{Set} \to \textsc{Set}$ which takes a set X to its powerset, and a set-function $f : X \to Y$ to the direct image map given by

$$f_\exists(A) = \{y \in Y : \exists a \in A \text{ such that } y = f(a)\}.$$

- There is a distinct functor $\mathcal{P}_\forall : \textsc{Set} \to \textsc{Set}$ which takes a set X to its powerset, and a set-function $f : X \to Y$ to the map given by

$$f_\forall(A) = \{y \in Y : \forall x \in X, f(x) = y \text{ implies } x \in A\}.$$

  As these examples show, the action of a functor on morphisms is not determined by its action on objects. (In fact, as usual in category theory, it is the action on morphisms—in particular, on the identities—which determines the action on objects.)
- There is a functor $\mathrm{GL}_n : \textsc{Ring} \to \textsc{Grp}$ which takes a ring R to the multiplicative group $\mathrm{GL}_n(R)$ of invertible n-by-n matrices with coefficients in R, with entry-wise action of homomorphisms. The functor GL_1 has a special interpretation as the functor which takes a ring R to the multiplicative group of units in R. We write $(-)^\times : \textsc{Ring} \to \textsc{Grp}$.
- There is a functor $\texttt{Maybe} : \textsc{Set} \to \textsc{Set}$ which takes a set X to the set $X \sqcup \{\bot\}$, where $\bot$ is a new element, and a function f to its extension by $f(\bot) = \bot$.
- Similarly, there is a functor $\texttt{List} : \textsc{Set} \to \textsc{Set}$ which takes a set X to the set of all finite lists of elements in X, and a set-function f to its mapping over lists, i.e.

$$(\texttt{List}f)([x_1, \ldots, x_n]) = [f(x_1), \ldots, f(x_n)].$$

  In other contexts, this functor is also called the *free monoid* or the *Kleene star*.
- For any field $\Bbbk$, there is a functor $\textsc{Set} \to \textsc{Vect}_\Bbbk$ which takes a set X to the $\Bbbk$-span of X, and a set-function f to its linear extension. This is also called the *free vector space*. More generally, any free construction—such as the free group, free ring, etc.—forms a functor.
- Let C be a concrete category, such as those of Example 2.3. Then the *forgetful functor* $U : C \to \textsc{Set}$ takes each object to its underlying set, and each morphism to its underlying set-function, "forgetting" the additional structure.
- There is also a forgetful functor $\textsc{Ring} \to \textsc{Grp}$ which takes each ring to its underlying additive group, and each ring homomorphism to its underlying group homomorphism.

Example 2.14. As the following examples show, whenever we can think of each instance of a certain mathematical structure as a category, functors reproduce the right notion of structure-preserving transformation between those structures.

- Let P and Q be posets with associated categories $\mathcal{P}$ and $\mathcal{Q}$, and let $F : \mathcal{P} \to \mathcal{Q}$ be a functor. Let $p_1 \leq_P p_2$, so that there is a unique morphism $p_1 \to p_2$ in $\mathcal{P}$. Since F must take this morphism to a morphism $Fp_1 \to Fp_2$, it must hold that $Fp_1 \leq_Q Fp_2$.

Furthermore, this is the only requirement on functors, as the statements about identities and composites assert equalities between morphisms, but any two morphisms with the same domain and codomain are equal in a poset. As such, functors between posets are exactly monotone maps.

- Let G and H be groups with associated categories $\mathcal{G}$ and $\mathcal{H}$. A functor $F : \mathcal{G} \to \mathcal{H}$ assigns the single object of $\mathcal{G}$ to the single object of $\mathcal{H}$, and each morphism in $\mathcal{G}$, which is an element $g \in G$, to a morphism (element) $Fg \in H$. That this preserves composites tells us that it preserves group multiplication, and hence it is a homomorphism. The fact that F preserves identities is extraneous, since every group homomorphism preserves identities. As such, functors between groups are exactly group homomorphisms.
- Let L_1 and L_2 be functional programming languages with associated categories $\mathcal{L}_1$ and $\mathcal{L}_2$. We think of a functor $F : \mathcal{L}_1 \to \mathcal{L}_2$ as an embedding—or, more technically, a *model*—of $\mathcal{L}_1$ in $\mathcal{L}_2$. Specifically, for any function in $\mathcal{L}_1$, F identifies a corresponding function in $\mathcal{L}_2$, and so F allows us to think of computations in L_2 as "simulating" computations in L_1.

The following class of functors are especially important.

Definition 2.15 (hom-functors)**.** Let $x \in C$. There is a functor

$$C(x, -) : C \to \textsc{Set},$$

the *covariant hom-functor at* x, which takes an object y to the hom-set $C(x, y)$, and a morphism $f : y \to z$ to its action by post-composition, $f_*(g) = fg$[4].

Since isomorphic objects are meant to look identical to all the machinery of category theory, we should expect the following result.

Proposition 2.16. *Let $F : C \to \mathcal{D}$ be a functor and let $f : x \to y$ be an isomorphism in C. Then $Ff : Fx \to Fy$ is an isomorphism.*

Proof. We have that

$$FfFf^{-1} = F(ff^{-1}) = F1_x = 1_{Fx},$$

and the same works on the other side. □

Notice that both functorality axioms are exactly what is required to prove this result.

If functors are morphisms between categories, then we should expect that there is a category of categories. This is indeed the case, but we first need to show that functors can be composed.

Proposition 2.17. *Let $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. Then there is a* composite functor *$GF : C \to \mathcal{E}$, defined by $(GF)x = G(Fx)$ and $(GF)f = G(Ff)$. Furthermore, this composition is associative and unital, with identities 1_C.*

[4]The analogous functor $C(-, x)$ requires a little bit of machinery—the notions of *opposite categories* and *contravariant functors*—which are outside our scope. It is defined in any introductory text on category theory.

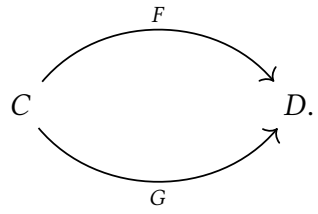
Example 2.18. The composite of the forgetful functors Ring → Grp and Grp → Set is exactly the forgetful functor Ring → Set.

Definition 2.19. The *category of categories* Cat has categories as objects and functors as morphisms.

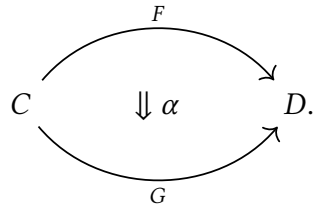
The foundationally-inclined reader will correctly object to this definition, which implies that Cat should be an object of itself, leading to issues involving Russell's paradox. There are several resolutions to this—for instance, letting Cat be the category of so-called *locally small* categories, whose hom-sets $C(x, y)$ each form sets. We ignore these issues here.

2.1.4 Natural Transformations

The notion of a *natural transformation* can be somewhat mysterious, but is ultimately a workhorse of categorical machinery. We can think of a category C geometrically as a single point, in which case a functor $F : C \to \mathcal{D}$ is an oriented line—an arrow. Two functors $F, G : C \to \mathcal{D}$ look like



A natural transformation is a square—or, if you prefer, a disk—which "fills in the hole":



In other words, a natural transformation is a morphism between functors.

More concretely, recall that functors can be thought of as tools which, given a structure of one kind, produce one of another. In this sense, natural transformations are a mechanism for converting between such constructions. For each object $x \in C$, we have two ways to construct an object of $\mathcal{D}$, i.e. Fx and Gx. Of course, objects of $\mathcal{D}$ are related by morphisms, so a natural transformation $\alpha : F \Rightarrow G$ should identify a morphism $\alpha_x : Fx \to Gx$ for each $x \in C$.

This is not quite enough. We want to ensure that the morphisms α_x are somehow "consistent" with the morphisms of C. We formalize that intuition now.

Definition 2.20 (natural transformation)**.** Let $F, G : C \to \mathcal{D}$ be functors. A *natural transformation* $\alpha : F \Rightarrow G$ consists of, for every object $x \in C$, a *component* $\alpha_x : Fx \to Gx$

such that, for every morphism $f : x \to y$ in C, the following diagram (a *naturality square*) commutes:

$$\begin{array}{ccc} Fx & \xrightarrow{\alpha_x} & Gx \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fy & \xrightarrow[\alpha_y]{} & Gy. \end{array}$$

The idea is that it does not matter whether we first move from x to y via *any morphism* f, or first move from F to G via α; natural transformations commute with any morphism. This is the sense in which natural transformations are natural.

Example 2.21. There are many important examples of natural transformations.

- For any functor F, there is an *identity natural transformation* $1_F : F \Rightarrow F$, whose components are each the identities $(1_F)_x = 1_{Fx}$.
- There is a natural transformation $\alpha : 1_{\text{SET}} \Rightarrow \mathcal{P}_\exists$ with components $\alpha_X : x \mapsto \{x\}$.
- The *dual* of a vector space V over $\Bbbk$ is the vector space of linear maps into $\Bbbk$, i.e. $V^* = \text{VECT}_\Bbbk(V, \Bbbk)$. There is a natural transformation $\alpha : 1_{\text{VECT}} \Rightarrow (-)^{**}$ whose components α_V take any $v \in V$ to the map $\text{ev}_v : \mathbb{V}^* \to \Bbbk$ given by $T \mapsto Tv$.
- There is a natural transformation $\det : \text{GL}_n \Rightarrow (-)^\times$, where R^* is the ring of units from Example 2.13, which takes the determinant of an invertible matrix.
- Recall from Example 2.14 that functors between posets are exactly monotone maps. A natural transformation $\alpha : F \Rightarrow G$ between two monotone maps $\mathcal{P} \to \mathcal{Q}$ consists of, for each $p \in \mathcal{P}$, a morphism $Fp \to Gp$. Since $\mathcal{Q}$ is a poset, there is at most one such morphism, and it exists if and only if $Fp \leq Gp$. As such, there can only be one such natural transformation, and it exists if and only if $F \leq G$ in the pointwise ordering.
- Let $F, G : \mathcal{L}_1 \to \mathcal{L}_2$ be models of a programming language L_1 in L_2. A natural transformation $\alpha : F \Rightarrow G$ is a *transpilation* between the models: it tells us how to convert programs written in the model F into programs written in the model G. The naturality squares assert exactly that this transpilation is *sound*, i.e. that it preserves the meaning of programs.

Example 2.22. Here are three natural transformations common in functional programming.

- Let $\texttt{reverse}_X$ be the function which reverses lists of elements in X, i.e.

  $$[x_1, \ldots, x_n] \mapsto [x_n, \ldots, x_1].$$

  Then `reverse` is a natural transformation `List` $\Rightarrow$ `List`.
- Let $\texttt{head}_X$ be the function which gets the first element of a list if it exists, i.e.

  $$[x_1, \ldots, x_n] \mapsto x_1, \qquad [] \mapsto \bot.$$

  Then `head` is a natural transformation `List` $\Rightarrow$ `Maybe`.

- Let $\texttt{toList}_X$ be the function $\texttt{Maybe}X \to \texttt{List}X$ given by

$$x \mapsto [x], \qquad \bot \mapsto [].$$

  Then `toList` is a natural transformation $\texttt{Maybe} \Rightarrow \texttt{List}$.

Each of these are special cases of the so-called *Reynolds abstraction theorem* from programming language theory, which says that (parametrically) polymorphic functions are natural [Rey83]. This theorem is explored in great detail by [Wad89].

If we think of natural transformations as morphisms between functors $\mathcal{C} \to \mathcal{D}$, then following the category-theoretic philosophy, there should be a category of functors. Indeed, natural transformations can be composed, as follows.

Proposition 2.23. *Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors and let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be natural transformations. Then there is a* vertical composite natural transformation *$\beta\alpha : F \Rightarrow H$, whose components are $(\beta\alpha)_x = \beta_x \alpha_x$. Furthermore, this composition is associative and unital, with identities 1_F.*

The name *vertical composite* comes from the following picture:

As the name implies, there is a horizontal composite, defined in e.g. [Rie17, Lemma 1.7.4].

Definition 2.24 (functor category)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *functor category* $[\mathcal{C}, \mathcal{D}]$ has functors $\mathcal{C} \to \mathcal{D}$ as objects and natural transformations as morphisms.

Here is one example of the advantage of working with categorical structure: we already know what the notion of an isomorphism of functors has to be.

Definition 2.25 (natural isomorphism)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\alpha : F \Rightarrow G$ is a *natural isomorphism* if it is an isomorphism in the category $[\mathcal{C}, \mathcal{D}]$.

Proposition 2.26. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\alpha : F \Rightarrow G$ is a natural isomorphism if and only each of its components $\alpha_x : Fx \to Gx$ are isomorphisms in $\mathcal{D}$.*

We can now state the correct form of the converse to Proposition 2.11.

Theorem 2.27. *Let x and y be objects in a category $\mathcal{C}$ such that $\mathcal{C}(x, -) \cong \mathcal{C}(y, -)$. Then $x \cong y$.*

Proof. Let $\eta : C(x, -) \Rightarrow C(y, -)$ be a natural isomorphism. Define

$$t = \eta_x(1_x),$$

which is a morphism $y \to x$, and

$$u = \eta_y^{-1}(1_y),$$

which is a morphism $x \to y$. We claim these are inverses.

Naturality of η applied to u asserts that

$$\begin{array}{ccc} C(x,x) & \xrightarrow{\eta_x} & C(y,x) \\ {\scriptstyle u_*}\downarrow & & \downarrow{\scriptstyle u_*} \\ C(x,y) & \xrightarrow[\eta_y]{} & C(y,y) \end{array}$$

commutes. Following 1_x around the top and right, we get

$$u_*(\eta_x(1_x)) = u_*(t) = ut,$$

while on the left and bottom we get

$$\eta_y(u_*(1_x)) = \eta_y(u) = 1_y,$$

so commutativity implies $ut = 1_y$.

Similarly, naturality of η^{-1} applied to t asserts that

$$\begin{array}{ccc} C(y,y) & \xrightarrow{\eta_y^{-1}} & C(x,y) \\ {\scriptstyle t_*}\downarrow & & \downarrow{\scriptstyle t_*} \\ C(y,x) & \xrightarrow[\eta_x^{-1}]{} & C(x,x) \end{array}$$

commutes. Following 1_y around the top and right, we get

$$t_*(\eta_y^{-1}(1_y)) = t_*(u) = tu,$$

while on the left and bottom we get

$$\eta_x^{-1}(t_*(1_y)) = \eta_x^{-1}(t) = 1_x,$$

so again $tu = 1_x$. This completes the proof. □

This theorem is a special case of the *Yoneda lemma*, arguably the most important theorem in category theory. The contravariant result, with the functors $C(-, x)$, is also true, but outside our scope. Together, these theorems tell us that objects in a category are indeed determined by their morphisms.

2.2 Monoidal Categories

In ordinary categories, composition is sequential: if morphisms are interpreted as computational processes, the composite gf means roughly "first do f, then do g." In many settings, we want to consider both sequential and parallel (or concurrent) composition. The categorical axiomatization of this idea is *monoidal categories*.

2.2.1 The Definition

To model parallel composition, we want a binary operation $\otimes$ which assigns, to each pair of processes (morphisms) $f : x \to y$ and $g : w \to z$, their parallel composite $f \otimes g$. If we think of objects as types, this parallel composite can only run given inputs of both types x and w, to feed to f and g respectively, and should produce two outputs of types y and z. To represent this notion, we also need a way to pair types (objects), which means a binary operation also called $\otimes$ on objects. This dual assignment on both objects and morphisms suggests functorality: we will ask that $\otimes$ is a functor $C \times C \to C$.

What axioms should this data satisfy? As in most well-behaved algebraic structures, there should be an identity for $\otimes$ on objects, which we will write I. Computationally, we may think of I as a "trivial resource," which may freely be created and has no uses. This I induces an identity, the morphism 1_I, for $\otimes$ on morphisms, so we do not need to add an identity on morphisms as an extra axiom. We would also like parallel composition to associate, so that we can sensibly talk about performing n processes in parallel. It is therefore tempting to list the following axioms:

$$I \otimes x = x = x \otimes I; \qquad (x \otimes y) \otimes z = x \otimes (y \otimes z).$$

While this notion, called a *strict monoidal category*, is useful, it is not the most natural axiomatization. For instance, even the category Set, with the ordinary Cartesian product, is not strictly monoidal: the identity is $\{*\}$, but $\{*\} \times X$ is not equal to X, instead merely isomorphic. The point is that there is interesting structure in the way that even isomorphic objects relate to each other; we do not want to lose it by forcing strict equality.

However, we do not want to allow the structure of these natural isomorphisms to be too strange. For instance, one can imagine two ways to convert from $I \otimes (x \otimes y)$ to $x \otimes y$:

$$I \otimes (x \otimes y) \cong x \otimes y \quad \text{and} \quad I \otimes (x \otimes y) \cong (I \otimes x) \otimes y \cong x \otimes y.$$

The first directly uses unitality, while the second associates and then uses unitality. A *coherence axiom* asserts that choices like this do not matter: every pair of composites of our canonical isomorphisms with the same domain and codomain should commute.

We are not quite ready; there is one remaining technical issue, though this paragraph may be safely skipped. It may happen that two domains equate "accidentally", so that, for instance,

$$((x \otimes y) \otimes z) \otimes w = x \otimes (y \otimes (z \otimes w)). \tag{2.1}$$

In this case, the version of the coherence axiom stated above implies that the isomorphisms

$$((x \otimes y) \otimes z) \otimes w \cong (x \otimes y) \otimes (z \otimes w) \quad \text{and} \quad x \otimes (y \otimes (z \otimes w)) \cong (x \otimes y) \otimes (z \otimes w)$$

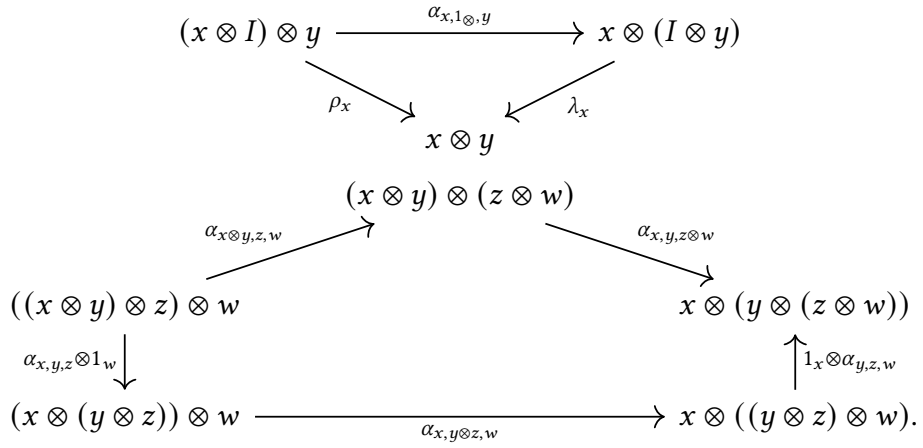
should commute; they do, after all, have the same domain and codomain. But the first re-associates from the left to the right, and the second re-associates from the right to the left: these are structurally different actions, which only "look the same" because of the accident of Equation 2.1, so our theory should not require them to commute. There is a way to formalize a correct abstract notion of coherence—see for instance [Mac71, subsection VII.2]—but fortunately, Mac Lane's *coherence theorem* enables an easier axiomatization.

We are finally now ready to state the definition of a monoidal category.

Definition 2.28 (monoidal category)**.** A *monoidal category* C consists of the following data:

- an underlying category C;
- a functor $\otimes : C \times C \to C$, called the *monoidal product*;
- an object $I \in C$, called the *monoidal unit*;
- a natural isomorphism $\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$, called the *associator*;
- a natural isomorphism $\lambda_x : I \otimes x \to x$, called the *left unitor*[5];
- a natural isomorphism $\rho_x : x \otimes I \to x$, called the *right unitor*.

This data must make the following diagrams, called the *triangle* and *pentagon* identities, commute:



The above diagrams look arbitrary, but as mentioned, they are exactly what is required for the correct notion of coherence. On first exposure to these ideas, it is safe to ignore the exact statement of the identities and work with the intuition that any two ways of associating or unitalizing should be the same.

In the above definition, the natural isomorphisms α, λ, and ρ feel in some sense more like axioms than data. This is another key component of the category-theoretic philosophy, one which should feel comfortable to computer scientists, who often assume the existence of concrete objects which structure our models:

structure is a kind of data.

[5]The letters λ and ρ are chosen for their association with L and R, respectively.

If we think of categories as algebras of structure, it is natural that we should think of axiomatic structure as an algebraic object which may be manipulated[6].

2.2.2 Examples

The notion of a monoidal category is quite general; we survey some important examples here.

Example 2.29. Let us very explicitly construct the required data to show that Set is a monoidal category under the Cartesian product. The monoidal unit is the singleton $\{*\}$. The associator is the natural isomorphism with components

$$\begin{aligned} \alpha_{X,Y,Z} \colon (X \times Y) \times Z &\to X \times (Y \times Z) \\ ((x,y),z) &\mapsto (x,(y,z)). \end{aligned}$$

The left and right unitors are the natural isomorphism with components

$$\begin{aligned} \lambda_X \colon \{*\} \times X &\to X \\ (*,x) &\mapsto x, \end{aligned} \qquad \begin{aligned} \rho_X \colon X \times \{*\} &\to X \\ (x,*) &\mapsto x. \end{aligned}$$

A common complaint about category theory is at play here: we now have a large number of relationships to demonstrate, including functorality of $\times$, naturality of α, λ, and ρ, and the pentagon and triangle identities. The author's opinion is that this work will ultimately save effort, by allowing us to use a powerful abstract theory across any structure we have shown to be monoidal, but if the reader is not convinced, one solution is to work even more generally. For instance, by showing that the Cartesian product satisfies a simple property called the *universal property of the product*, we could automatically conclude on the grounds of a general theorem that it is monoidal. Abstraction of this sort ultimately saves effort, but it is not always comfortable at first. Regardless, in order to exemplify the definition in all its detail, we continue with the explicit demonstration.

To show functorality of $\times$, we need to determine its action on morphisms. Letting $f : X \to Y$ and $g : W \to Z$, we define

$$\begin{aligned} f \times g \colon X \times W &\to Y \times Z \\ (x,y) &\mapsto (f(x), g(y)). \end{aligned}$$

This is functorial: it takes an identity $1_{(X,W)} = (1_X, 1_W)$ to $1_{X \times W}$, and the composite of two pairs of morphisms to composite of their action on pairs.

[6] Of course, Definition 2.28 still carries a traditional-looking equational theory in the form of the triangle and pentagon identities. The key difference is that this theory is an assumption about the "two-dimensional" structure of the natural transformations, whereas associativity and unitality are assumptions about the "one-dimensional" structure of the functor $\otimes$. We could continue to generalize, instead asking that these diagrams are themselves witnessed by "three-dimensional" isomorphisms between the natural isomorphisms α, λ, and ρ. Repeating this process *ad infinitum*, the natural endpoint of the structure-as-data philosophy is so-called ∞*-category theory*.

To show naturality of α, let $(f, g, h) : (X, Y, Z) \to (X', Y', Z')$ be a morphism in $\textsc{Set}^3$. We need to show that the following diagram commutes:

$$\begin{array}{ccc} (X \times Y) \times Z & \xrightarrow{\alpha_{X,Y,Z}} & X \times (Y \times Z) \\ \downarrow{\scriptstyle (f\times g)\times h} & & \downarrow{\scriptstyle f\times(g\times h)} \\ (X' \times Y') \times Z' & \xrightarrow[\alpha_{X',Y',Z'}]{} & X' \times (Y' \times Z'). \end{array}$$

Tracking the action of a triple $((x, y), z)$ through both paths, we see the needed equality:

$$\begin{array}{ccc} ((x, y), z) & \xmapsto{\alpha_{X,Y,Z}} & (x, (y, z)) \\ \downarrow{\scriptstyle (f\times g)\times h} & & \downarrow{\scriptstyle f\times(g\times h)} \\ ((f(x), g(y)), h(z)) & \xmapsto[\alpha_{X',Y',Z'}]{} & (f(x), (g(y), h(z))). \end{array}$$

To show naturality of λ, let $f : X \to Y$. Since the only morphism $\{*\} \to \{*\}$ is $1_{\{*\}}$, naturality is entailed by commutativity of the following diagram:

$$\begin{array}{ccc} \{*\} \times X & \xrightarrow{\lambda_X} & X \\ \downarrow{\scriptstyle 1_{\{*\}}\times f} & & \downarrow{\scriptstyle f} \\ \{*\} \times Y & \xrightarrow[\lambda_Y]{} & Y, \end{array} \quad \text{i.e.} \quad \begin{array}{ccc} (*, x) & \xmapsto{\lambda_X} & x \\ \downarrow{\scriptstyle 1_{\{*\}}\times f} & & \downarrow{\scriptstyle f} \\ (*, f(x)) & \xmapsto[\lambda_Y]{} & f(x). \end{array}$$

Naturality of ρ is similar. We show the pentagon identity by its action on $(((x, y), z), w)$:

$$\begin{array}{ccccc} & & ((x, y), (z, w)) & & \\ & \overset{\alpha_{X\times Y,Z,W}}{\nearrow} & & \overset{\alpha_{X,Y,Z\times W}}{\searrow} & \\ (((x, y), z), w) & & & & (x, (y, (z, w))) \\ \downarrow{\scriptstyle \alpha_{X,Y,Z}\times 1_W} & & & & \uparrow{\scriptstyle 1_X\times\alpha_{Y,Z,W}} \\ ((x, (y, z)), w) & & \xmapsto[\alpha_{X,Y\times Z,W}]{} & & (x, ((y, z), w)). \end{array}$$

The triangle identity is similar.

While we will never again be so explicit, we hope the previous example makes the axioms of a monoidal category more concrete.

Example 2.30. There are many more examples of monoidal categories throughout mathematics.

- $\textsc{Vect}_{\mathbb{k}}$ is monoidal with the tensor product of vector spaces.
- $\textsc{Cat}$ is monoidal with the product category.

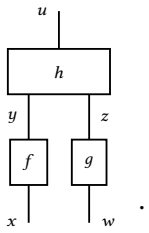
- Let L be a strongly-typed functional programming language with *product types* $A \times B$, for instance the simply-typed lambda calculus. Then the category $\mathcal{L}$ is monoidal with forming product types as the monoidal product and the unit type as the monoidal unit.

Example 2.31 (concurrent programming [MM90])**.** Returning to our motivation of parallelism, here is a very different example. Let L be a strongly-typed functional *concurrent* programming language, by which we mean that it can run computations concurrently on different machine threads. Then again under reasonable assumptions, $\mathcal{L}$ is monoidal, with concurrent branching as the monoidal product and the do-nothing program as the monoidal unit.

2.2.3 String Diagrams

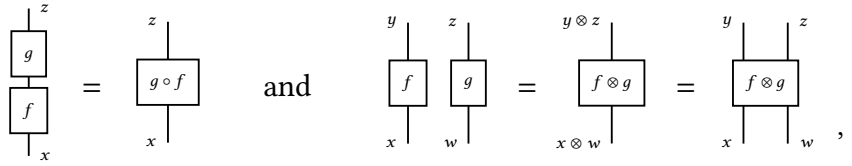
In monoidal categories, there are two "formal mechanisms" for building morphisms: sequential composition $\circ$ and parallel composition $\otimes$. String diagrams are a graphical calculus for morphisms using these mechanisms. String diagrams and related calculi are explored in great detail by [Sel11]; we give a basic outline here.

Consider a monoidal category C with three morphisms $f : x \to y$, $g : w \to z$, and $h : y \otimes z \to u$. We can can form a new morphism $h \circ (f \otimes g) : x \otimes w \to u$. We encode this new morphism in the following *string diagram*, written bottom-up:



.

Explicitly, the idea is as follows. A morphism is a labelled box, with "wires"[7] coming into and out of labelled with the domain and codomain. We can hook up two wires representing the same object—this is sequential composition. We can also place boxes or wires side-by-side—this is parallel composition. Accordingly:



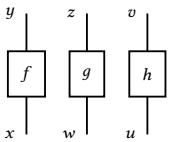
,

where in the first equality we have assumed $y = w$, so that the composition makes sense. As the left hand side of the first equality suggests, we often suppress the label of "intermediate" wires, as they are implicit from the types of the morphisms; in fact, we may even

[7] As the word "wire" suggests, a string diagram can be thought of as a circuit, where the morphisms/boxes are thought of as gates. This correspondence has recently been made precise by [BS22], but the analogy is much older, and it is a useful intuition even without any rigor. This analogy and many others are discussed in [BS11].

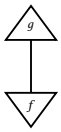
at times suppress the labels of the input and output wires. Finally, if there is no box, then a wire may be read as the identity for its type.

Consider the following diagram:



Do we read this as $(f \otimes g) \otimes h$ or $f \otimes (g \otimes h)$? There is not an unambiguous choice, but fortunately the coherence theorem, discussed in Section 2.2.1, means that there is a unique natural isomorphism equating these morphisms. As such, the general rule is that *string diagrams define morphisms up to unique natural isomorphism.*

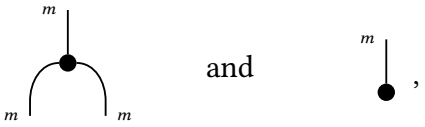
Similarly, since wires of type I can be created or destroyed at will using λ and ρ, we just do not draw such wires. A morphism with domain or codomain I is represented with a triangle, so that for instance if $f : I \to x$ and $g : x \to I$, then



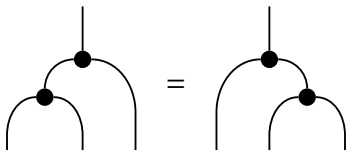
is the morphism $gf : I \to I$.

Sometimes, we work in settings which have some "distinguished" morphisms, in which case we will often write them merely with dots. For instance, recall that a *classical monoid* is a set X together with an associative unital binary operation. Recalling from Example 2.6 that the distinguished unit element $e \in X$ can be associated with the unique set-function $\{*\} \to X$ defined by $* \mapsto e$, we generalize the notion of a monoid as follows.

Definition 2.32 (monoid object)**.** Let C be a monoidal category. A *monoid object* in C is an object m together with distinguished morphisms $\mu : m \otimes m \to m$ and $\eta : I \to m$, depicted as

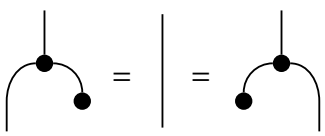


called the *multiplication* and *unit*. This data must make the equalities



(2.2)

and



(2.3)

hold.

Let us be very explicit about what these equalities say. Equation 2.2 takes in three wires of type m. On the left, it associates them to the left, so we start with $(m \otimes m) \otimes m$. We first multiply on the left while doing nothing on the right, and then multiply the product with the thing on the right: this is the composite morphism

$$(m \otimes m) \otimes m \xrightarrow{\mu \otimes 1_m} m \otimes m \xrightarrow{\mu} m.$$

On the right, the ms are associated to the right, so we have the composite morphism

$$m \otimes (m \otimes m) \xrightarrow{1_m \otimes \mu} m \otimes m \xrightarrow{\mu} m.$$

It may be worrying that these morphisms have different domains, but as discussed, string diagrams merely identify morphisms up to coherence isomorphism. As such, for the axiom to make sense, there should be a canonical natural isomorphism making the domains and codomains of these morphisms equate, and indeed there is: $\alpha_{m,m,m}$ for the domains, and just the identity for the codomains. Thus, Equation 2.2 asserts commutativity of the diagram

$$\begin{array}{ccc} (m \otimes m) \otimes m & \xrightarrow{\alpha_{m,m,m}} & m \otimes (m \otimes m) \\ \downarrow{\scriptstyle \mu \otimes 1_m} & & \downarrow{\scriptstyle 1_m \otimes \mu} \\ m \otimes m & & m \otimes m \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & m. & \end{array}$$

Meanwhile, Equation 2.3 features three morphisms. On the left, we have

$$m \otimes I \xrightarrow{1_m \otimes \eta} m \otimes m \xrightarrow{\mu} m,$$

in the middle we have the identity $1_m : m \to m$, while on the right we have

$$I \otimes m \xrightarrow{\eta \otimes 1_m} m \otimes m \xrightarrow{\mu} m.$$

Again, the domains are related by the canonical isomorphisms λ and ρ. We can write this equality as commutativity of the diagram

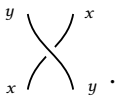
$$\begin{array}{ccccc} I \otimes m & \xrightarrow{\eta \otimes 1_m} & m \otimes m & \xleftarrow{1_m \otimes \eta} & m \otimes I \\ & {\scriptstyle \lambda_m} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle \rho_m} & \\ & & m, & & \end{array}$$

where we suppress the identity 1_m, which could appear at the bottom of the diagram.

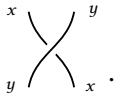
In the following two sections, we will give several examples of definitions—in particular *braided monoidal categories*, *symmetric monoidal categories*, and *monoidal functors*—whose coherence axioms are better understood diagramatically than symbolically. While the axioms themselves are useful to understand, for our purposes it is more important to understand the intuition of the structures in question and their relationship to the graphical calculi. If the reader understands how the diagrams relate to each other, it is generally safe to move on even without a complete understanding of how they are translated into symbolic equalities. The interested reader may find a symbolic statement of the coherence laws in [Mac71, Chapter XI].

2.2.4 Symmetry

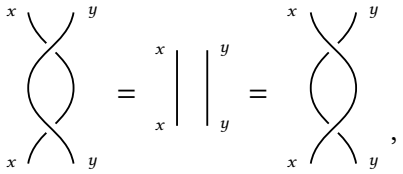
While monoidal categories are necessarily associative, nothing in the definition guarantees that the monoidal product is commutative. As usual, it is too strict to ask for commutativity $x \otimes y = y \otimes x$ as an equational axiom. When we want commutativity, we instead add a natural isomorphism $\gamma_{x,y} : x \otimes y \to y \otimes x$, called the *braiding*, to the data, so named because of its string-diagrammatic representation:



This notation suggests a nice graphical representation of the inverse $\gamma^{-1}_{x,y} : y \otimes x \to x \otimes y$:

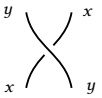
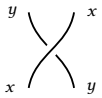


In particular, $\gamma^{-1}_{x,y}$ is indeed an inverse asserts that

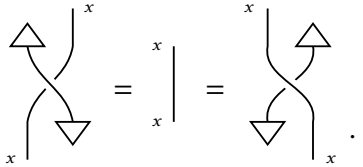


as is suggested by our geometric intuitions[8].

Note that there are two possible braids we could draw $x \otimes y \to y \otimes x$, each of which is *a priori* a different morphism:

$\gamma_{x,y} =$  $\quad \gamma^{-1}_{y,x} =$ 

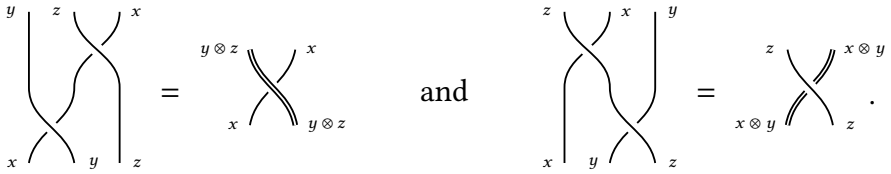
What coherence axioms should this satisfy—in other words, what manipulations should we be allowed to make to the our diagrams? It should certainly be coherent with the identity:



(2.4)

[8] It is, in fact, possible to formalize string diagrams geometrically, using the technology of knot theory; this is due to [JS91].

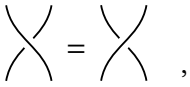
It should also not matter if we braid twice, or braid once with a product, in the sense that:



(2.5)

The previous two axioms define a *braided monoidal category*.

We will care primarily about the stronger case in which



(2.6)

i.e. that $\gamma_{x,y} = \gamma_{y,x}^{-1}$. We may then unambiguously write

we call this map the *symmetry*.

Definition 2.33 (symmetric monoidal category)**.** A *symmetric monoidal category* is a monoidal category C, together with a natural isomorphism $\gamma_{x,y} : x \otimes y \to y \otimes x$, called the *braiding* or *symmetry*, satisfying the coherence laws of Equations 2.4 to 2.6.

Example 2.34. The categories Set, $\text{Vect}_{\mathbb{k}}$, and Cat, with the monoidal structure defined in Section 2.2.2, are all symmetric monoidal.

2.2.5 Monoidal Functors

Let us work out what a "monoidal functor" between monoidal categories should be. Let C and $\mathcal{D}$ be monoidal categories, and annotate their respective data with subscripts, so that for instance $\otimes_C$ is the monoidal product of C. Let $F : C \to \mathcal{D}$ be a functor.

As usual, we do not want to ask that $Fx \otimes_{\mathcal{D}} Fy = F(x \otimes_C y)$, because categorical axioms tend only to hold up to natural transformations. A sensible choice is thus to ask for a natural isomorphism $\phi_{x,y} : Fx \otimes_{\mathcal{D}} Fy \to F(x \otimes_C y)$, satisfying certain coherence identities. However, even this is often too strong. For instance, `Maybe` does not satisfy this definition: while there is a map

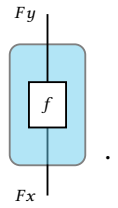
$$\texttt{Maybe}X \otimes \texttt{Maybe}Y \to \texttt{Maybe}(X \otimes Y)$$
$$(x, y) \mapsto (x, y) \quad (x, \bot) \mapsto \bot \quad (\bot, y) \mapsto \bot,$$

and so `Maybe` respects the monoidal structure in some weaker sense, this map is not an isomorphism. Instead, we will often just ask $\phi_{x,y}$ to be a morphism, which tells us how to "convert" monoidal products in $\mathcal{D}$ into monoidal products in the model of $\mathcal{D}$[9]. We

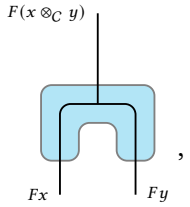
[9] A careful reader may wonder why the morphism goes from $\otimes_{\mathcal{D}}$ to $\otimes_C$, rather than the other way. We do sometimes study the latter under the name *colax monoidal functors*, but the former is far more common. One way to understand this is that the former direction says that product in $\mathcal{D}$ is in some sense "more precise" than product in C, which tends to be why the functor is interesting in the first place.

also need F to be compatible with the monoidal unit, for which we ask for an morphism $\phi : I_{\mathcal{D}} \to FI_{\mathcal{C}}$.

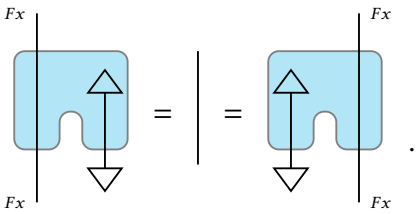
There is a graphical calculus for monoidal functors due to [CS99]; we give a presentation following [Mel06]. The idea is to represent functors as colored boxes which separate the "inside world" of C from the "outside world" of $\mathcal{D}$, so that we may depict the morphism $Ff : Fx \to Fy$ as



If F is monoidal, we write the morphism $\phi_{x,y}$ as

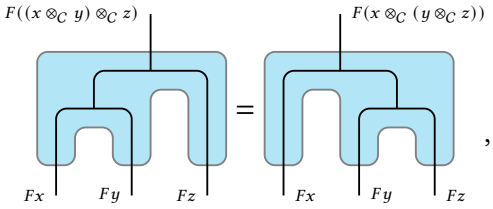


the idea being that at first the blue-shaded wires x and y are connected by white space representing the product $\otimes_{\mathcal{D}}$, and then it becomes blue space representing the product $\otimes_{\mathcal{C}}$. We often don't write the top part of this morphism, instead doing manipulation inside C, which happens in the blue shading. For instance, coherence with the identity states that



(2.7)

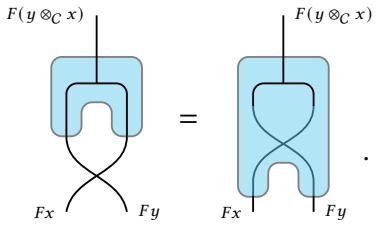
Similarly, compatibility with the associator asserts that



(2.8)

while when C and $\mathcal{D}$ are symmetric monoidal, we might also want compatibility with the

symmetry:



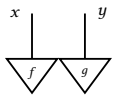
(2.9)

Definition 2.35 (monoidal functor)**.** A functor $F : C \to \mathcal{D}$ between monoidal categories is *lax monoidal*, or just *monoidal*, if there is a natural transformation $\phi_{x,y} : Fx \otimes_{\mathcal{D}} F_y \to F(x \otimes_C y)$ and a morphism $\phi : I_{\mathcal{D}} \to FI_C$ satisfying Equations 2.7 and 2.8. It is further *symmetric* if C and $\mathcal{D}$ are symmetric monoidal categories and the data satisfies Equation 2.9. Finally, it is *strong monoidal* if $\phi_{x,y}$ and ϕ are isomorphisms and *strict monoidal* if they are identities.

Example 2.36. Again, there are many familiar monoidal functors.

- `Maybe` is (lax) symmetric monoidal, but not strong monoidal.
- For any monoidal category C, $C(I, -)$ is monoidal, with the coherence

$$\phi_{x,y} : C(I, x) \times C(I, y) \to C(I, x \otimes y)$$
$$(f, g) \mapsto$$



  If C is symmetric, then so too is $C(I, -)$.
- The $\Bbbk$-span functor is strong symmetric monoidal. In fact, this is one definition of the tensor of vector spaces:

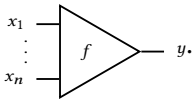
$$\mathrm{span}_{\Bbbk} X \otimes \mathrm{span}_{\Bbbk} Y = \mathrm{span}_{\Bbbk}(X \times Y).$$

- The forgetful functor $U : \textsc{Vect}_{\Bbbk} \to \textsc{Set}$ is monoidal, with $\phi_{X,Y} : U(X) \times U(Y) \to U(X \otimes Y)$ given by the universal property of the tensor product of vector spaces, or explicitly by the map $(v, w) \mapsto v \otimes w$.

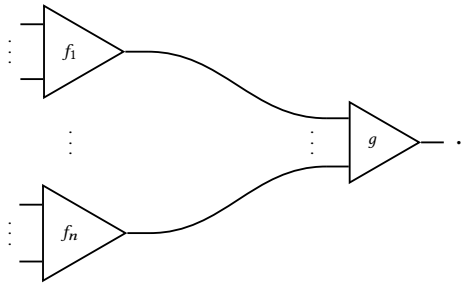
2.2.6 Multicategories

While monoidal categories are extremely elegant structures, as seen in Example 2.29 it can be tedious to construct all the required data. In this section, we give a useful tool for constructing monoidal categories via a related structure called a *multicategory*, which is just a category whose morphisms may take multiple inputs.

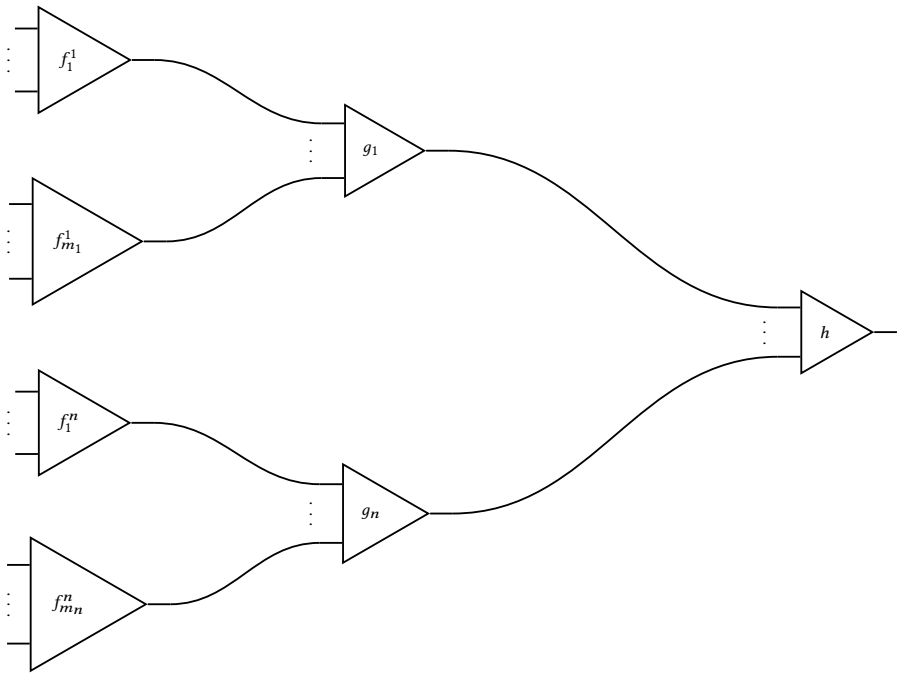
As usual, it is often easier to understand the algebraic structure diagramatically than symbolically. A morphism in a multicategory looks like



Such morphisms can be composed when the domains and codomains line up, as in



This composition is (strictly) associative, which just means that the composite



is well-defined.

We now state the explicit definition.

Definition 2.37 (multicategory)**.** A *multicategory* consists of the following data:

- a collection of objects C;
- for any (possibly empty) finite list of objects $x_1, \ldots, x_n \in C$ and any object $y \in C$, a collection of morphisms $C(x_1, \ldots, x_n; y)$;
- for any finite list of morphisms $f_1, \ldots, f_n$, and any morphism g such that the domain of g has length n and the ith object in its domain is the codomain of f_i, a composite morphism $g \circ (f_1, \ldots, f_n)$ whose domain is the concatenation of the domains of the f_is and whose codomain is the codomain of g;
- for any object $x \in C$, a morphism $1_x \in C(x; x)$.

This data must satisfy the following associativity and unitality axioms:

- for any morphism $(x_1, \ldots, x_n) \xrightarrow{f} y$,
$$1_y \circ f = f = f \circ (1_{x_1}, \ldots, 1_{x_n});$$
- for any morphisms $h, g_1, \ldots, g_n, f_1^1, \ldots, f_1^{m_1}, \ldots, f_n^1, \ldots, f_n^{m_n}$ where the composites are defined,
$$\begin{aligned} &h \circ (g_1 \circ (f_1^1, \ldots, f_1^{m_1}), \ldots, g_n \circ (f_n^1, \ldots, f_n^{m_n})) \\ &= (h \circ (g_1, \ldots, g_n)) \circ (f_1^1, \ldots, f_1^{m_1}, \ldots, f_n^1, \ldots, f_n^{m_n}). \end{aligned}$$

Notation. The *arity* of a morphism in a multicategory is the length of its domain list.

Example 2.38. Any category is a multicategory, with no morphisms of arity other than one. Such multicategories are called *unary*.

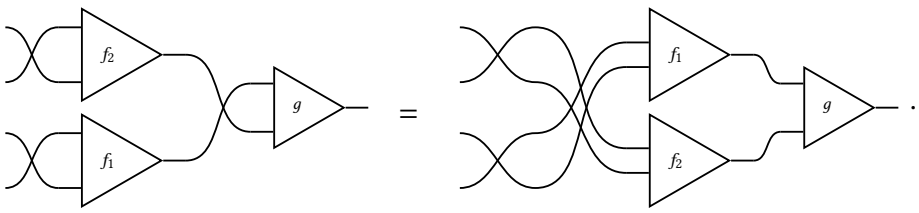
Example 2.39. Any strict monoidal category has an *underlying multicategory*, where the maps $x_1, \ldots, x_n \to y$ are exactly the maps $x_1 \otimes \cdots \otimes x_n \to y$.

We also want to consider a kind of commutativity of domain-forming in multicategories. The basic idea is that we should be able to permute the domain of a hom-set and receive a canonically isomorphic hom-set.

Definition 2.40. A *symmetric multicategory* is a multicategory equipped with, for each n, a right-action of the symmetric group S^n on hom-sets of arity n, i.e. for $\sigma \in S^n$, a map
$$(- \cdot \sigma) : C(x_1, \ldots, x_n; y) \to C(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; y)$$
which commutes with the group structure of S^n, in that
$$(f \cdot \sigma) \cdot \tau = f \cdot (\sigma\tau), \qquad f = f \cdot 1.$$
This action must respect composition, in that, whenever the types line up,
$$(g \cdot \sigma) \circ (f_{\sigma(1)} \cdot \tau_{\sigma(1)}, \ldots, f_{\sigma(n)} \cdot \tau_{\sigma(n)}) = (g \circ (f_1, \ldots, f_n)) \cdot (\sigma \circ (\tau_{\sigma(1)}, \ldots, \tau_{\sigma(n)}).$$

This last equality requires some explanation. We have a map g and a composition-compatible list of maps $f_1, \ldots, f_n$. If we permute the inputs to g by σ, then we need to permute the f_is in the same way. In the most general case, we could also have further permutations τ_i to the inputs of each of the f_is; we need to permute those by σ as well. This gives us the left hand side.

Of the right hand side, we first take the composite of g and the fs, and then want to use the data of σ and the τ_is to permute the inputs to this composite morphisms. The point is that it's sufficient to first permute each bundle of inputs by σ, and then permute each input to each bundle by the appropriate τ. In the case where all the maps have arity 2 and all the permutations are the transposition (12), this equality is as follows:



Bibliographic Notes

There are many different ways to visually present string diagrams. In Section 2.2.3, we have followed the style of [BK22] very closely. We program our diagrams in part using the Ti*k*Z code of [BK22], and code due to Zajj Daugherty.

Chapter 3

Categorical Cryptography

A theory of cryptography should define at least four things: computation, protocols, adversarial behavior, and security. A major advantage of categorical models of cryptography is that they conveniently separate these issues. In particular, we have some underlying category of computations, while we represent categories of protocols with certain constructions on categories; as such, our notions of interaction and security are completely independent of the underlying model of computation.

[todo: lots more intro, cite [BK22]; categorical crypto theories [Hin20; Pav12; Pav14; SV13], categories for specific crypto protocols [BKM19; BMR19]; categorical qm [AC04; CP12; HV19; CK17; CG19]]

3.1 Computation

The categorical theory of computation is well-developed, going back at least to the work of Jim Lambek and several contemporaries around the 1970s [Lam74; Lam80; Law69; See84]. The essential idea generalizes Example 2.2: objects are types and morphisms are typed computations. The most disciplined approach is to consider the categorical structure needed to model certain forms of computation, so that for instance models of simply typed computation are *bicartesian closed categories* [Lam74], of linear computation are *star-autonomous categories* [See89], of quantum computation are *compact-closed categories* [AC04], and of probabilistic computation are *Markov categories* [Fri20]. We will not review this approach here. Instead, our focus will be on constructing specific categorical models of forms of computations of interest to cryptographers.

3.1.1 Deterministic Computation

We would like a category of deterministic computations to have computable functions as morphisms. However, the natural choice, taking sets as objects and computable functions as morphisms, is actually not yet precise. The first issue is that there are several distinct notions of computability on uncountable sets. Each such notion forms a category, but formal definitions are outside our scope, as cryptographers tend not to care about

uncountable sets anyway[1].

We can resolve this issue simply, by limiting ourselves to finite sets, in which case every function is computable (simply by a lookup table):

Definition 3.1 (category of finite sets)**.** The *category of finite sets*, FinSet, has finite sets as objects and functions as morphisms.

However, we often want to work with larger input spaces. The natural guess is to take countable sets and computable functions. The issue here is one of encoding: there is a canonical notion of computability on the set of finite binary strings, but elements of arbitrary sets do not generally have canonical encodings as binary strings. We could solve this issue by limiting ourselves to working only with binary strings:

Definition 3.2 (category of computable binary functions)**.** The *category of computable binary functions* BinComp has sets of binary strings $A \subseteq \{0,1\}^*$ as objects and computable functions as morphisms.

In practice, however, we like to think of computations as working over arbitrary sets, which in particular may have more algebraic structure than sets of binary strings. Our strategy, following [Pav14], will be to work over sets with fixed binary encodings.

Definition 3.3 (binary-encoded set)**.** A *binary-encoded set* is a set X together with an injection $[\![-]\!]_X : X \to \{0,1\}^*$, called the *encoding*.

Note that every binary-encoded set is finite or countable; as such, we avoid the issues with uncountable sets mentioned above.

Notation. When the context is clear, we will generally drop the subscript of $[\![-]\!]$. We write $[\![X]\!]$ for the image of $[\![-]\!]_X$, i.e. $[\![X]\!] = \{s \in \{0,1\}^* : s = [\![x]\!] \text{ for some } x \in X\}$.

Given a function $f : X \to Y$ of binary-encoded sets, we can define a function

$$\begin{aligned} [\![f]\!] : [\![X]\!] &\to [\![Y]\!] \\ [\![x]\!] &\mapsto [\![f(x)]\!]. \end{aligned}$$

This is well-defined exactly because $[\![-]\!]_X$ is injective.

Definition 3.4 (category of computable functions)**.** A function $f : X \to Y$ of binary-encoded sets is *computable* if $[\![f]\!]$ is computable. The *category of computable functions*, Comp, has binary-encoded sets as objects and computable functions as morphisms.

It needs to be shown that this is a category. First, the identities 1_X are computable, as $[\![1_X]\!] = 1_{[\![X]\!]}$ is computable. Second, the composite of computable functions is computable, as the composition of computable binary functions is computable, and composition is preserved by $[\![-]\!]$. As this argument indicates, there is a functor $[\![-]\!] : \text{Comp} \to \text{BinComp}$; in fact this functor is an *equivalence of categories*. Nevertheless, the expanded perspective provided by Comp will be convenient.

[1] These issues are studied in the field of *computable analysis*; see for instance the PhD thesis of Andre Bauer [Bau00].

Finally, we now define a symmetric monoidal structure on Comp. In particular, for two binary-encoded sets X and Y, we would like to define the product $X \otimes Y$ as the set $X \times Y$, but it is unclear what $[\![-]\!]_{X\times Y}$ should be. We first fix an injective pairing map $\langle -,-\rangle : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ which is efficiently computable[2]. We can then define $[\![(x,y)]\!] = \langle [\![x]\!], [\![y]\!]\rangle$; it is a standard check that this defines a symmetric monoidal structure inherited from Set[3].

3.1.2 Probabilistic Computation

Again, there is some subtlety with probabilistic computation. Even in the case of finite sets, not every stochastic function is computable by algorithms with access to fair coin flips[4]. However, there is again a standard notion of computable stochastic function of binary strings, so we can proceed much as before, defining:

Definition 3.5 (category of computable stochastic functions)**.** The category BinCompStoch has sets of binary strings as objects and computable stochastic functions as morphisms.

A stochastic function $f : X \to Y$ between binary-encoded sets is *computable* if $[\![f]\!]$ is computable. The *category of computable stochastic functions* CompStoch has binary-encoded sets as objects and computable stochastic functions as morphisms.

Again, it needs to be shown that this is a category. The identities are computable (and stochastic, since every deterministic function is stochastic), and composition commutes with $[\![-]\!]$, so the composite of computable functions is computable. Furthermore, this category is symmetric monoidal, with pairing of encodings as in Comp.

We give a more abstract characterization of this category. There are only countably many computable probability distributions on $\{0,1\}^*$, since there are only countably many Turing machines. Fix a choice φ of bijection witnessing this fact. Note further that any probability distribution P on a binary-encoded set X induces a probability distribution $[\![P]\!]$ on $\{0,1\}^*$ by

$$\Pr_{s\leftarrow[\![P]\!]}[s = s_0] = \Pr_{x\leftarrow P}[[\![x]\!] = s_0].$$

There is now a monad $G_c : \text{Comp} \to \text{Comp}$, which we call the *computable Giry monad*, which takes any binary-encoded set X to the set of computable probability distributions on X, i.e. those such that $[\![P]\!]$ is a computable probability distribution on $\{0,1\}^*$, with

[2] One such map is computed as follows: given inputs (m,n), start by encoding the length of m in $2\log\log m$ bits: first write a bit of the length, then write a 1 if the length continues and a 0 if it doesn't. Now knowing the length of m, we can append the binary representation of m and then n, which takes $O(\log m + \log n) = O(\log(mn))$ bits. Since $\log\log m = O(\log m)$, in total this algorithm takes $O(\log(mn))$ bits, and just writes across the tape, hence is computable in linear time.

[3] Here is a more abstract way to see this. A suitable pairing function $\langle -,-\rangle$ turns $\{0,1\}$ into an internal commutative monoid in Set. In other work, we show that the category of subobjects of any internal monoid is a monoidal category [SZ24]. The construction here is approximately an application of that general theorem.

[4] We believe this is a slight conceptual problem with the strategy of [BK22, Section 6], which models unbounded probabilistic computation in the category of finite sets and stochastic functions: this category is too powerful to reasonably model computation. This does not pose a technical issue in their specific example.

encoding given by $[\![P]\!]_{G_cX} = \varphi([\![P]\!]_X)$. Given $f : X \to Y$ and $P \in G_cX$, we define the probability distribution $G_cf(P)$ on Y by

$$\Pr_{y \leftarrow G_cf(P)}[y = y_0] = \Pr_{x \leftarrow P}[f(x) = y_0].$$

The unit of G_c is the function $X \to G_cX$ taking x to the point distribution at x. The multiplication is the function $\mu_X : G_cG_cX \to G_cX$ acting by summation: given a probability distribution Q on G_cX, we define a distribution $\mu_X(Q)$ on X by

$$\Pr_{x \leftarrow \mu_X(Q)}[x = x_0] = \sum_{P_0 \in G_cX} \Pr_{P \leftarrow Q}[P = P_0] \Pr_{x \leftarrow P}[x = x_0],$$

which converges because Q is a probability distribution. The proofs of functorality and the monad laws as exactly as for the ordinary Giry monad [Gir82], so we do not give them here. Now CompStoch is in fact (isomorphic to) the Kleisi category of G_c.

3.1.3 Efficient and Effectful Computation

Suppose that we are given some wide subcategory EffBin of BinComp, for instance that of poly-time computable maps. We can define the category EffComp of efficient computations as the wide subcategory of Comp consisting of morphisms f whose encodings $[\![f]\!]$ are in EffBin: this is the *preimage* of EffBin under the functor $[\![-]\!]$.

Definition 3.6. The category P of poly-time computable maps is the wide subcategory of Comp consisting of those morphisms f such that $[\![f]\!]$ is poly-time computable.

Similarly, suppose that we are give some wide subcategory of BinCompStoch, for instance that of poly-time computable stochastic maps. We can similarly define the category EffCompStoch.

Definition 3.7. The category PPT of poly-time computable stochastic maps is the wide is the wide subcategory of CompStoch consisting of those morphisms f such that $[\![f]\!]$ is probabilistic poly-time-computable.

In general, we can perform this construction for any complexity class C which is closed under composition.

Even more generally, let Bin be the category of sets of binary strings and (maybe uncomputable) set-functions between them. Let Enc be the category of binary-encoded sets and (maybe uncomputable) set-functions between them. Then $[\![-]\!]$ is an equivalence of categories Enc $\simeq$ Bin.

Now let EffBin be any subcategory of Bin. Then the *category of efficient computations* Eff is the subcategory of Enc consisting of morphisms f such that $[\![f]\!]$ is in EffBin, i.e. the preimage of EffBin under $[\![-]\!]$. Finally, let T be any monad on Enc which restricts to a monad on Eff. Then the *category of efficient T-computations* is the Kleisi category of the restriction of T to Eff. When T is symmetric lax monoidal, this category is symmetric monoidal.

Example 3.8. Each example in the previous three sections is a special case of this construction.

- When EffBin consists of computable functions and T is the identity monad, we recover Comp.
- When EffBin consists of computable functions and T is the computable Giry monad, we recover CompStoch.
- When EffBin consists of poly-time computable functions and T is the identity monad, we recover P.
- When EffBin consists of poly-time computable functions and T is the *poly-time Giry monad*, which sends a set X to the set of poly-time computable probability distributions on X, we recover PPT.

The point is that for any notion of efficient computation, and any notion of computational effect (since effects are generally monadic [WT03]), as long as the effect can be efficiently represented, we can use the machinery of binary-encoded sets to define a category of efficient computations carrying the given effect.

3.1.4 Quantum Computation

While an complete introduction to quantum computation is outside our scope, we can sketch a categorical perspective; a standard introduction is [NC10]. The *category of quantum computations* FinHilb is the category of finite-dimensional Hilbert spaces over $\mathbb{C}$ and linear maps. Since nontrivial complex Hilbert spaces have uncountably many vectors, we cannot directly model this category using the machinery of the previous section, as there is no way to encode a complex Hilbert space as an object of Bin. If we had developed a more general theory relying on a notion of computability over uncountable sets, then we could now unify these perspectives; indeed, there have been several attempts to monadically embed quantum computation into classical calculi [AG09; Abr+17]. As we have chosen not to develop such a general theory, in this section we will treat Hilbert spaces as our primitive object.

A *quantum computation* is a sequence of unitary transformations and *measurements*. There are several ways to provide categorical semantics to quantum measurement; we follow [HV19].

Let I be the one-dimensional Hilbert space. Note that the maps $I \to I$ correspond to choices of scalars $\lambda \in \mathbb{C}$; as such, we say that a *scalar* is a map $I \to I$. Given a Hilbert space V, a *state* is a map $I \to V$, so that a state is determined by a choice of vector in V. If $I \xrightarrow{a} V$ and $I \xrightarrow{b} V$ are states, then the projection of a onto b has amplitude

$$I \xrightarrow{a} V \xrightarrow{b^\dagger} I,$$

where $(-)^\dagger$ denotes the adjoint; the corresponding element of $\mathbb{C}$ is the inner product $\langle b, a\rangle$. Now the *Born rule* of quantum mechanics asserts that the probability of measuring the outcome b from the state a is $|\langle b, a\rangle|^2$. Categorically, if $I \xrightarrow{a,b} V$ are states, then the *probability of measuring b from the state a* is the scalar

$$I \xrightarrow{a} V \xrightarrow{b^\dagger} I \xrightarrow{b} V \xrightarrow{a^\dagger} I.$$

This is just a brief sketch of a very rich theory; see especially the work of Bob Coecke such as [AC04; CP12; CK17], or the book by Heunen and Vicary [HV19]. The key point is that any fully categorical treatment of cryptography should obtain quantum cryptography as a special case.

3.2 Protocols

The categorical semantics of interactive computation—in particular, of protocols—originates from the study of quantum cryptography, especially of so-called *resource theories* [CFS16]. The idea is to start with some underlying SMC C of computations—fixed throughout this section—and to construct a category of "protocols built from computations in C." Exactly which such construction we choose depends on what we want our protocols to look like.

In all these categories, the basic idea is that we will think of objects as resources and morphisms as protocols, which convert some resources into others. For instance, in the category n-comb(C), morphisms will be "protocols with holes"—when instantiated with specific implementations of the resources they are waiting for, they provide some new resource.

3.2.1 Products

While the product category is used implicitly in the definition of monoidal categories, it is worth exploring it explicitly. Given two categories C and $\mathcal{D}$, the *product category* $C \times \mathcal{D}$ has:

- as objects, pairs (X, Y) of objects in C and $\mathcal{D}$;
- as morphisms $(X, Y) \to (X', Y')$, pairs (f, g) of morphisms so that $f : X \to X'$ and $g : Y \to Y'$;
- composition and identities defined componentwise.

When C and $\mathcal{D}$ are categories of computations, we think of $C \times \mathcal{D}$ as a category of non-interfering parallel computations: a computation in $C \times \mathcal{D}$ is a computation in C and a computation in $\mathcal{D}$, but they cannot interact.

3.2.2 States

In the symmetric monoidal category Set, the morphisms $\{*\} \to X$ are in natural correspondence with the elements of X, by the bijection

$$(* \mapsto x) \mapsto x.$$

Similarly, in the symmetric monoidal category $\mathrm{Vect}_{\Bbbk}$, the morphisms $\Bbbk \to V$ are in natural correspondence with vectors in V, since such maps are determined by their action on the vector $1 \in \Bbbk$. This pattern holds more generally, motivating the following definition.

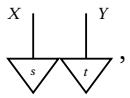
Definition 3.9 (state)**.** A *state* or *generalized element* of C is a morphism $I \to X$ for some object X.

As we know, it is easy to recognize states string-diagramatically: they are downward-pointing triangles.

Definition 3.10 (resource theory of states)**.** The *resource theory of states* $\mathrm{st}(C)$ is the category whose objects are states in C and whose morphisms $(I \xrightarrow{s} X) \to (I \xrightarrow{t} Y)$ are maps $X \xrightarrow{f} Y$ such that $fs = y$. Composition is as in C.

When C is interpreted a category of types of resources and conversions between them, we can think of $\mathrm{st}(C)$ states as a category of specific resources and of conversions between them, forgetting the type information.

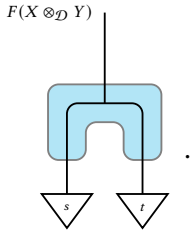
The resource theory of states has a canonical symmetric monoidal structure induced by that of C: the monoidal product of states $I \xrightarrow{s} X$ and $I \xrightarrow{t} Y$ is just the state $I \to X \otimes Y$ given by[5]



while the monoidal product of morphisms is just their product in X. The unit is the state 1_I, while the associator and unitor are inherited from C.

It will be useful to be a little more general. Let $F : \mathcal{D} \to C$ be a lax monoidal functor. Then an *F-state* is a pair of $X \in \mathcal{D}$ and a map $I \to FX$ in C. The *resource theory of F-states* $\mathrm{st}(F)$ is the category whose objects are F-states and whose morphisms $(I \xrightarrow{s} FX) \to (I \xrightarrow{t} FY)$ are maps $X \xrightarrow{f} Y$ in $\mathcal{D}$ such that $(Ff)s = t$. Note what then $C = \mathcal{D}$ and $F = 1_C$, we recover $\mathrm{st}(C)$.

Since F is lax monoidal, there is again general recipe for taking the monoidal product in this category: given $I \xrightarrow{s} FX$ and $I \xrightarrow{t} FY$, the product of s and t is the state



With a little more machinery: $\mathrm{st}(F)$ is the *category of elements* of the functor $\mathcal{D} \xrightarrow{F} C \xrightarrow{C(I,-)} \textsc{Set}$. In general, the category of elements of a functor $F : C \to \textsc{Set}$ has as objects pairs (c, x) where $c \in Fx$, and as morphisms $(c, x) \to (c', x')$, maps $f : c \to c'$ such

[5] It may worry the careful reader that there are two seemingly distinct, albeit coherently naturally isomorphic, morphisms this diagram could represent:

$$I \xrightarrow{\lambda_I^{-1}} I \otimes I \xrightarrow{s \otimes t} X \otimes Y \quad \text{and} \quad I \xrightarrow{\rho_I^{-1}} I \otimes I \xrightarrow{s \otimes t} X \otimes Y.$$

Fortunately, it is a non-obvious but standard result of Kelly that $\lambda_I = \rho_I$ in any SMC [Kel64], so these morphisms agree.

that $Ff(x) = x'$. The category of elements of any lax monoidal functor has a canonical monoidal structure on it induced by that of the codomain; this is the monoidal structure with which we endow $\mathrm{st}(F)$. Observe the similarity of the monoidal product in $\mathrm{st}(F)$ with the coherence map for the functor $C(I, -)$ from Example 2.36.

Following [BK22], we are especially interested in the category $\mathrm{st}(C^2 \xrightarrow{\otimes} C)$. Objects in this category are morphisms $I \to X \otimes Y$ in C, which we can think of as *joint states*. When $C = \textsc{Set}$, every joint state is *independent*, in that it splits into the product of two morphisms $I \to X$ and $I \to Y$, but in more complicated categories like PPT or Hilb this may fail, representing a kind of *entanglement*. In this way, we can express the idea that two parties A and B have a shared uniform random key by the map $I \to X \otimes X$ in PPT that sends $*$ to a uniform random choice of pairs (k, k) for $k \in X$. This map does not split into a pair of stochastic maps $I \to X$ and $I \to X$.

Morphisms $(I \xrightarrow{s} X \otimes Y) \to (I \xrightarrow{t} X' \otimes Y')$ in this category are maps (f, g) in C^2 satisfying $(f \otimes g)s = t$. The maps f and g prescribe the computations undertaken by the respective parties in order to transform the joint state s into the joint state t. Note that there is a kind of locality to morphisms in this category, since they are morphisms in the product category; all of the interaction between the two parties is encoded in the initial joint state. This separation is actually desirable: it will make it easier for us to reason about the security of protocols. However, it means we need a way to describe objects which represent more complicated forms of interaction; we will do so in the next two sections.

It is worth remarking that more generally, we can model computations on n-party states via the category

$$\mathrm{st}(C^N \xrightarrow{\otimes^{N-1}} C),$$

where the ith copy of C represents computations taken by the ith party in the computation[6].

Finally, there is a forgetful functor $\Pi : \mathrm{st}(\mathcal{D} \xrightarrow{F} C) \to \mathcal{D}$, which sends a state $I \xrightarrow{F} X$ to the object X and a map to its underlying map in $\mathcal{D}$. This functor is monoidal, since the monoidal structure on $\mathrm{st}(\mathcal{D} \xrightarrow{F} C)$ is induced by that of $\mathcal{D}$. This functor composes with F to get a forgetful functor with codomain C. We will use these functors to help organize information about the relationship between these categories.

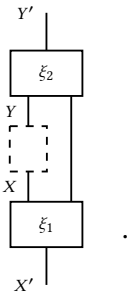
3.2.3 Flat Process Conversions

Recall that we can think of morphisms in $X \xrightarrow{f} Y$ in C as processes converting X to Y. Before arriving at the more general strategy of [BK22], we first treat a simpler case. We will construct a category 1-comb(C) of *flat process conversions*, whose objects are "type signatures" of processes and whose morphisms are recipes for converting between processes with the appropriate signatures.

[6] There is a choice of associativity to be made, but any choice yields a coherently isomorphic category, so we will not worry about it here. A standard assumption—justified by the *strictification theorem for monoidal categories*, which says that every monoidal category is equivalent to strict one—is that the underlying category C is strict.

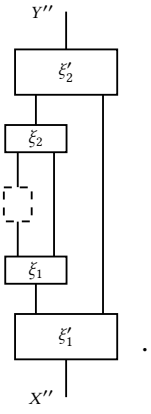
By type signature, we mean a pair of objects (X, Y) in C. The idea is that the resource (X, Y) should be "inhabited" by the morphisms $X \to Y$ in C. To make this work out, whatever notion of morphism in 1-comb(C) we end up with, it should be the case that 1-comb$(C)(I, (X, Y)) \cong C(X, Y)$, i.e. that the morphisms $I \to (X, Y)$ in 1-comb(C) should be in (natural) correspondence with the morphisms $X \to Y$ in C.

To make this work out, a morphism $(X, Y) \to (X', Y')$ in 1-comb(C) consists of the following structure, called a *1-comb*:



Explicitly, a 1-comb consists of an object Z and two morphisms $\xi_1 : X' \to X \otimes Z$ and $\xi_2 : Y \otimes Z \to Y'$. The point is that, if we "plug in" a morphism $X \to Y$ for the hole, we obtain a morphism $X' \to Y'$; Z represents some auxiliary data that isn't needed by the plugged-in morphism. We often call Z the *residual* of the comb. It is a theorem of [CFS16] that in an SMC, any morphism $X' \to Y'$ obtainable as a string diagram which uses exactly one occurrence of a morphism $f : X \to Y$ may be obtained as a 1-comb with f filled in the hole.

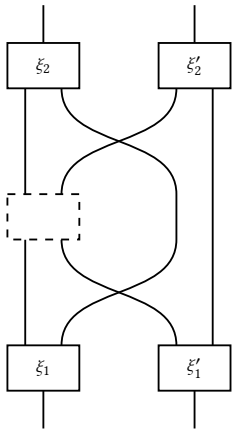
Composition of 1-combs is defined by "nesting": given 1-combs $(Z, \xi_1, \xi_2) : (X, Y) \to (X', Y')$ and $(Z', \xi_1', \xi_2') : (X', Y') \to (X'', Y'')$, we have a composite 1-comb $(X, Y) \to (X'', Y'')$ defined by:



This does indeed form a 1-comb: explicitly, the composite 1-comb is the tuple

$$((Z \otimes Z'), (\xi_1 \otimes 1_{Z'}) \circ \xi_1', (1_Z \otimes \xi_2') \circ \xi_2).$$

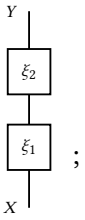
Meanwhile, the monoidal product of 1-combs is as follows:



.

Again, this forms a 1-comb.

We now return to the assertion from the beginning of the section: since a map $I \to I$ carries no data, a 1-comb $I \to (X, Y)$ looks like



;

these are morphisms $X \to Y$, but not bijectively so. To resolve this, we take equivalence classes of 1-combs, where two 1-combs are equivalent if they yield the same morphism when any morphism $W \otimes X \to W \otimes Y$ is plugged into their hole[7] Finally, we can formally define the category.

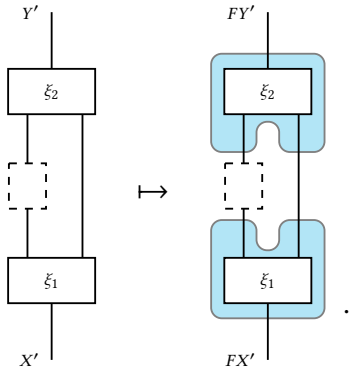
Definition 3.11 (category of flat process conversions)**.** The *category of flat process conversions* 1-comb(C) has as objects pairs (X, Y) of objects in C and as morphisms equivalence classes of 1-combs in C.

Example 3.12. The construction st(1-comb(C)) is the *category of parallel-combinable processes* of [CFS16].

Now suppose that $F : C \to \mathcal{D}$ is a strong monoidal functor; recall that this means there is a natural isomorphism $\phi_{X,Y} : FX \otimes_{\mathcal{D}} FY \to F(X \otimes_C Y)$. Now there is an induced

[7] This is an extensional notion of equality of combs. With significantly more machinery, it is also possible to define combs intensionally, via so-called *coend optics* [Ril18; HC23]: a 1-comb $(X, Y) \to (X', Y')$ is precisely an element of the set $\int^{M \in C} C(X', X \otimes M) \times C(Y \otimes M, Y')$.

functor 1-comb(F) : 1-comb$(C) \to$ 1-comb$(\mathcal{D})$ which acts on 1-combs by



Symbolically, the action of 1-comb(F) is

$$(Z, \xi_1, \xi_2) \mapsto (FZ, \phi^{-1}_{X,Z} F\xi_1, (F\xi_2)\phi_{Y,Z}).$$

We have that 1-comb(F) preserves identities and composition because F does. Furthermore, this construction turns 1-comb into an endofunctor on the category of SMCs and strong monoidal functors.

Cryptographically, we are primarily interested in the category

$$\text{st}(\text{1-comb}(C^2) \xrightarrow{\text{1-comb}(\otimes)} \text{1-comb}(C)).$$

Objects in this category are maps $I \to (X \otimes Y, X' \otimes Y')$ in 1-comb(C), hence maps $X \otimes Y \to X' \otimes Y'$ in C. Morphisms $(X \otimes Y \xrightarrow{f} X' \otimes Y') \to (A \otimes B \xrightarrow{g} A' \otimes B')$ are 1-combs $(X \otimes Y, X' \otimes Y') \to (A \otimes B, A' \otimes B')$, which, when the hole is "filled in" with f, yield the morphism g. The idea is that f represents some shared cryptographic resources which the two parties already have access to; the one-comb is a protocol the parties can use to transform it into the resource g.

We emphasize that these 1-combs "live in C^2: the morphisms ξ_1 and ξ_2 must be in the image of $\otimes$. As before, this means that they satisfy a kind of disjointness: they are really two separate computations running in parallel but independently. All of the interaction between the two parties is encapsulated in the resource f, which is shared between them. As such, it is a general rule that *protocols cannot create extra interactivity on their own; they need input resources enabling interaction.*

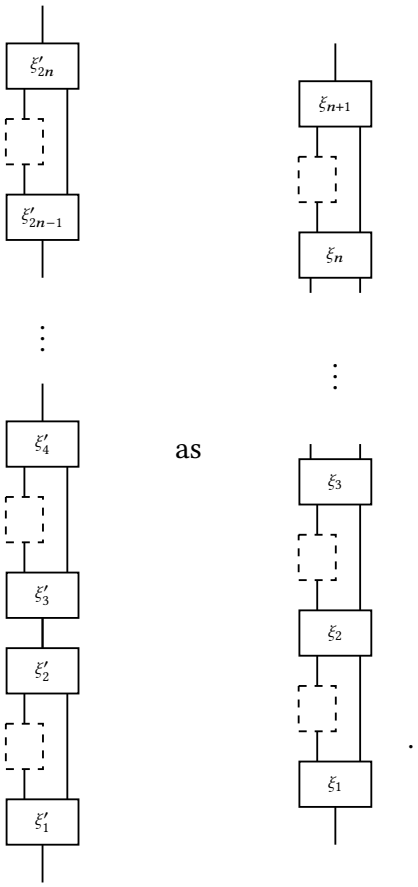
Finally, we note that to model n-party computation we can work in the category

$$\text{st}(\text{1-comb}(C^n) \xrightarrow{\text{1-comb}(\otimes^{n-1})} \text{1-comb}(C)).$$

As an example, let $C = \textsc{Set}$, and let f be the map $X \times * \to * \times X$ given by $(x, *) \mapsto (*, x)$. Then f is a *one-shot channel* from the first party to the second party. However, we have no way to represent protocols which use multiple input resources: our combs only have one hole. We fix this by working with *n-combs*.

3.2.4 Linear Process Conversions

The extension from 1-combs to the n-combs of [BK22] is conceptually straightforward, but technically somewhat messy. An n-comb is just a stack of 1-combs; we can combine the top of one comb with the bottom of the next, so we may as well write[8]



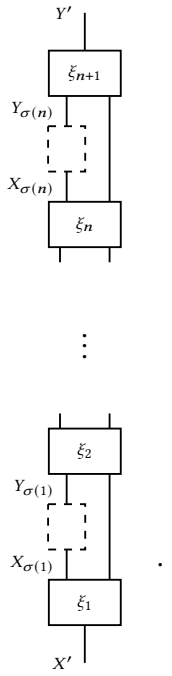
Again by a theorem of [CFS16], any circuit which is obtainable as a string diagram using exactly one occurance of each of a list of n morphisms can also be obtained as the result of plugging those morphisms in to an appropriate n-comb.

Generalizing the case of 1-combs, the objects in the category of n-combs should be finite lists of pairs of objects: a resource of type $[(X_1, Y_1), \ldots, (X_n, Y_n)]$ is a list of maps $X_i \to Y_i$ in C. We can proceed directly to defining the category, but we find it easier to first define a symmetric multicategory of n-combs. The advantage is that this requires us only to define morphisms with one pair in the codomain—what [BK22] call the "basic morphisms"—and then construct a full SMC via a general procedure.

[8] As with 1-combs, there is a more abstract definition of n-combs using coend optics: an n-comb $[(X_1, Y_1), \ldots, (X_n, Y_n)] \to (X', Y')$ is an element of the set

$$\int^{M_1,\ldots,M_n \in C} C(X', X_1 \otimes M_1) \times \prod_{i=1}^{n-1} C(Y_i \otimes M_i, X_{i+1} \otimes M_{i+1}) \times C(Y_n \otimes M_n, Y').$$

Objects in this multicategory are pairs of objects in C. Morphisms $[(X_1, Y_1), \ldots, (X_n, Y_n)] \to (X', Y')$ consist of a permutation σ and an n-comb



The idea is that σ encodes the order in which the protocol uses the input resources; we do not have to use them in the order specified by the domain list. Composition of general combs is as with 1-combs: given an n-comb and n m_k-combs such that the types line up, we nest each of the combs into the outer n-comb. This indeed gives us a symmetric multicategory.

We can now use the following general construction to obtain the category of n-combs:

Definition 3.13. Let C be a (symmetric) multicategory. Then there is an associated (symmetric) monoidal category $C^{\otimes}$ defined as follows:

- an object in $C^{\otimes}$ is a finite list of objects in C, written $x_1 \otimes \cdots \otimes x_n$;
- a morphism $x_1 \otimes \cdots \otimes x_n \xrightarrow{f} y_1 \otimes \cdots \otimes y_m$ consists of a *partition function* $\alpha_f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ and for each $i \in \{1, \ldots, m\}$, a morphism

  $$x_{k_1}, \ldots, x_{k_l} \xrightarrow{f_i} y_i,$$

  called the y_i *component*, where the k_js range over $\alpha_f^{-1}(i)$;
- the composite morphism $x_1 \otimes \cdots \otimes x_n \xrightarrow{f} y_1 \otimes \cdots \otimes y_m \xrightarrow{g} z_1 \otimes \cdots \otimes z_p$ is given by the partition function $\alpha_{gf} = \alpha_g \alpha_f$ and for each z_i, the component

  $$g_i \circ (f_{\sigma_i(1)}, \ldots, f_{\sigma_i(k)})$$

  in C, where σ_i is from g;

- the identity on $x_1 \otimes \cdots \otimes x_n$ is given by the identity partition and the identities 1_{x_i} from C;
- the monoidal product of $x_1 \otimes \cdots \otimes x_n$ and $y_1 \otimes \cdots \otimes y_m$ is given by the concatenation $x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m$;
- the monoidal product of $x_1 \otimes \cdots \otimes x_n \xrightarrow{f} y_1 \otimes \cdots \otimes y_m$ and $z_1 \otimes \cdots \otimes z_p \xrightarrow{g} w_1 \otimes \cdots \otimes w_q$ is given by lifting the partitions to the disjoint union, so the y_i component is just f_i while the w_i component is just g_i.

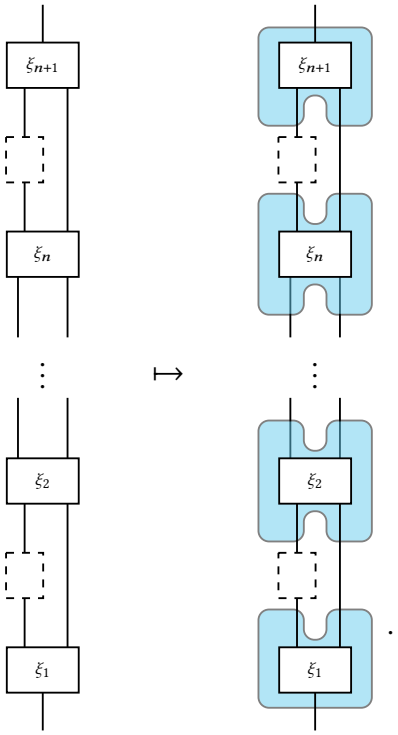
The idea is that each object in $x_1 \otimes \cdots \otimes x_n \in C^{\otimes}$ represents the presence of the "resources" represented by the objects $x_1, \ldots, x_n \in C$. A morphism must consume precisely one "copy" of each resource in that list and produce one copy of each resource in its codomain. The partition α is an allocation of input resources to output resources: $\alpha^{-1}(i)$ is exactly the input resources used to produce the resource y_i.

We can be explicit about what this construction looks like in the case of n-combs.

Definition 3.14. Objects in the category n-comb(C) are finite lists of pairs of objects in C. A morphism $[(X_1, Y_1), \ldots, (X_n, Y_n)] \to [(X'_1, Y'_1), \ldots, (X'_m, Y'_m)]$ is a list of m combs, one for each pair of objects in the codomain, and each of which has holes typed by the pairs in the domain, such that each pair in the domain is used in *exactly one* n-comb and *exactly once* in that n-comb. Composition is by nesting, while the monoidal product is by concatenation of lists.

Example 3.15. The category st(n-comb(C)) is the *category of universally combinable processes* of [CFS16].

Furthermore, n-comb is functorial in the same way as 1-comb. Given a strong monoidal functor $F : C \to \mathcal{D}$, we can define a functor n-comb(F) : n-comb$(C) \to$ n-comb$(\mathcal{D})$ by acting on each n-comb over C by



Cryptographically, still following [BK22] we are interested in the category

$$\mathrm{prot}_N(C) := \mathrm{st}(\text{n-comb}(C^N) \xrightarrow{\text{n-comb}(\otimes^{N-1})} \text{n-comb}(C)),$$

which is a category of N-party protocols with computations from C. Objects in this category are finite lists of maps

$$X_1^i \otimes \cdots \otimes X_N^i \xrightarrow{f_i} Y_1^i \otimes \cdots \otimes Y_N^i$$

in C, which represent shared access to the resources $\{f_i\}$, themselves processes for transforming joint states in C. Morphisms are lists of M combs, so that each map f_i in the domain list is allocated to exactly one comb, and so that plugging in all the f_is into the appropriate combs yields exactly the list of maps in the codomain.

Here is a convenient way to think about the situation, to our knowledge original to us. A map in n-comb(C) is a "schema for a protocol". We can look its fiber[9] under the functor

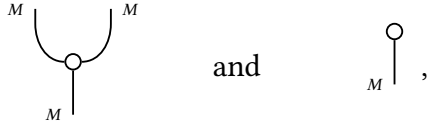
$$\text{n-comb}(\otimes^{N-1})\Pi : \mathrm{prot}_N(C) \to \text{n-comb}(C)$$

to determine what the schema does when instantiated with a specific kind of resource. In some cases, as in the next section, we only care about the maps in $\mathrm{prot}_n(C)$ with some specific domain and codomain, and are interested in verifying that there is an element in the fiber with that correct type: this is a *correctness property* of a protocol. However, there are more complicated situations where we want to verify that the same protocol behaves in one way given an input of some type, and in another way given another input; in this case the expanded perspective provided by the forgetful functor can be useful for organizing the data.

3.2.5 The One-Time Pad

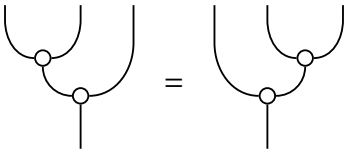
As a first example, we work out in full detail the categorical description of the one-time pad due to [BK22]. For now, let $C = \textsc{CompStoch}$ and $N = 3$; we label the three parties A, B, and E. We pick a message space $M \in C$; we could just say $M = \{0,1\}^*$, but instead let us figure out what "local" structure, by which we mean structure in C which is hence usable by each of the parties on their own, M needs to have.

First, we should be able to copy and delete messages from M; they're just classical information. We represent this with a pair of maps



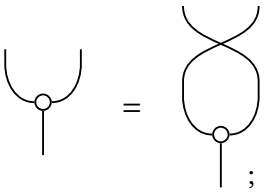
[9] The *fiber* $F^{-1}f$ of a functor $F : C \to \mathcal{D}$ over a morphism $f \in \mathcal{D}$ is collection of morphisms $g \in C$ such that $Fg = f$.

called the *copy* and *deletion* maps. Copying is associative (category theorists call this *coassociativity*, because it is opposite to the direction of normal associativity):



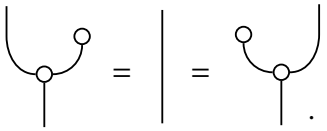
(3.1)

and commutative (*cocommutativity*):



(3.2)

deletion is the inverse of copying (*counitality*)[10]:

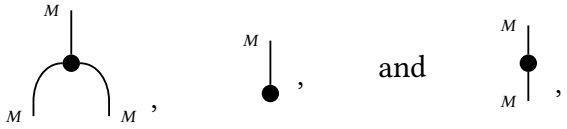


(3.3)

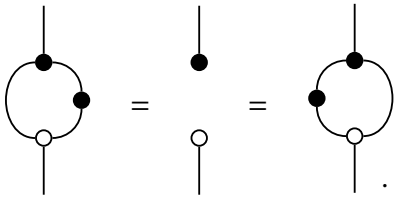
These three equations give M the structure of a *cocommutative comonoid* in C.

We often want to work over categories in which *every* object has such copy and delete maps. We say that an SMC C *supplies cocommutative comonoids* if every object in C is a cocommutative comonoid in such a way that the comonoidal structure on any object $X \otimes Y$ is induced from that on X and Y by the tensor. We are now very close to the definition of a *Markov category*, which is a natural categorical axiomatization of stochastic computation [Fri20].

Recall from Example 1.21 the one-time pad works over an arbitrary group G. As such, we separately need that M looks like a group in C: it should have multiplication, unit, and inverse maps



which are associative and unital in the sense of Definition 2.32 and satisfy the additional *inverse law*:

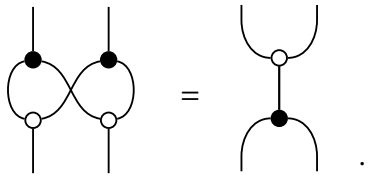


(3.4)

[10] We do not need to axiomatize both sides of this equality when we have commutativity.

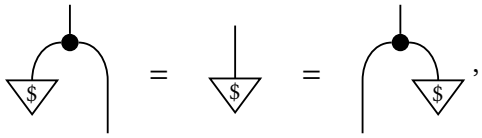
Notice that this law relies on the existence of the copy and delete maps; indeed, it is not possible to define a group without some way to talk about copying.

We need one more compatibility law, which says essentially that multiplication is deterministic: performing the same multiplication twice is the same as performing it once and then copying the result:



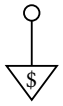
(3.5)

Finally, for the one-time pad we need some way to model randomness; in CompStoch this is the map $I \xrightarrow{\$} M$ which draws a uniform random value from M. Categorically, rather than the specific construction of the map, we care about its properties[11]: it is invariant under multiplication, in the sense that



(3.6)

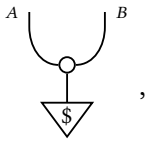
and it is independent, in the sense that



equals the empty diagram. (3.7)

We interpret (3.6) as saying that the product of any group element with a uniform random value is uniform random, while (3.7) says that creating and then deleting a uniform random value does nothing.

All this is just the local structure. To implement the one-time pad, we need two shared resources. First, A and B should have a shared random key drawn from M. In CompStoch, this is the map $I \to M \otimes M \otimes I$ which draws uniformly at random from the set $\{(k, k, *) : k \in M\}$. Diagramatically, we can build this map as

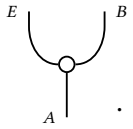


[11] The laws (3.1) to (3.5) give M the structure of a *Hopf object* in C. It turns out that these objects are well-known, and in particular have important applications in quantum computation [Fel17]; for instance, a Hopf object in $\mathrm{Vect}_{\Bbbk}$ is just an ordinary Hopf algebra. This is a major advantage of the categorical machinery: we discover unexpected connections between different kinds of computation and mathematics.

Somewhat surprisingly, elements of Hopf algebras satisfying (3.6) and (3.7) have been well studied under the name *integrals* [Swe69; Lom04; Sul71]. As such, the one-time pad can be instantiated over any Hopf algebra H with an integral, by replacing the uniform random choice of key with the map $\Bbbk \to H$ which sends 1 to the chosen integral. This translation is purely syntactic; everything we will say about the one-time pad applies to this construction as well. We can now start doing cryptography inside a Hopf algebra, or using the theory of Hopf algebras to say things about cryptography.

where we now label the wires with the party in possession of the data, so that for instance a wire labeled A has type $M \otimes I \otimes I$, while the two parallel wires labeled A and B have type $M \otimes M \otimes I$. This is a map in C which cannot be written in C^3, because it does not factor into a product of three separate maps in C.

We also need a way for A to send the encoded message to B and E. In CompStoch, this is the map $M \otimes I \otimes I \to I \otimes M \otimes M$ given by $(c, *, *) \mapsto (*, c, c)$. Again, this can be represented using the structure defined above, as the map

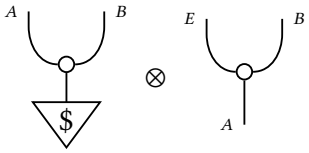


It is a good exercise in string diagram comprehension to spell out this map symbolically. Assuming we chose to left-associate the functor $\otimes^2$ in the definition of $\mathrm{prot}_3(C)$, one way to write it is as the map

$$(M \otimes I) \otimes I \xrightarrow{\rho_{M \otimes I}} M \otimes I \xrightarrow{\gamma_{M,I}} I \otimes M \xrightarrow{1_I \otimes \mathrm{copy}} I \otimes (M \otimes M) \xrightarrow{\alpha^{-1}_{I,M,M}} (I \otimes M) \otimes M.$$

The magic of the coherence theorem is that, once we agree on a choice of associativity, any way of writing this map is the same, and so we can work with the far simpler diagrammatic notation.

From all this, we learn that the domain of the one-time pad should be the object



(3.8)

of $\mathrm{prot}_3(C)$. The goal of the one-time pad is to produce a channel from A to B, so the codomain should be the object

(3.9)

The reader may now object that the one-time pad does not give the eavesdropped no information, as they learn that a message was sent. However, we are not yet attempting to deal with adversarial behavior, so any protocol can simply have Eve forget that information. We will discuss this issue at length in Section 3.3.

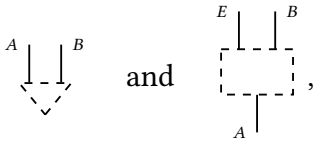
For a second, let us not worry about preserving states, and just think in the category n-comb(C). Recall that the objects in this category are finite lists of pairs of objects in C. The domain of the one-time pad should be

$$[(I \otimes I \otimes I, M \otimes M \otimes I), (M \otimes I \otimes I, I \otimes M \otimes M)],$$

while the codomain should be

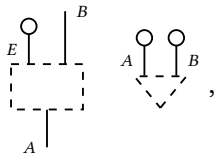
$$[(M \otimes I \otimes I, I \otimes M \otimes I)].$$

A morphism between these should be a 2-comb which takes morphisms of the domain types and produces a morphism of the codomain type. In other words, given two "black box" maps



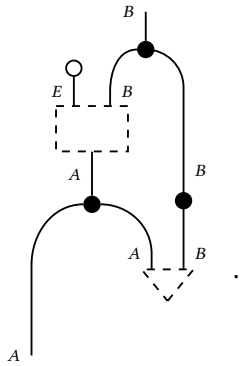
the 2-comb must produce a map

The easiest such 2-comb to write,

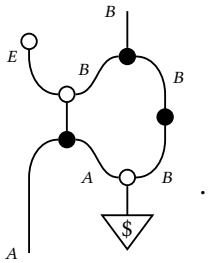


represents simply sending the message unencrypted, without use of the key. Note that the theory does require us to explicitly forget the key, as n-combs must consume all their input resources. We will alleviate this requirement soon.

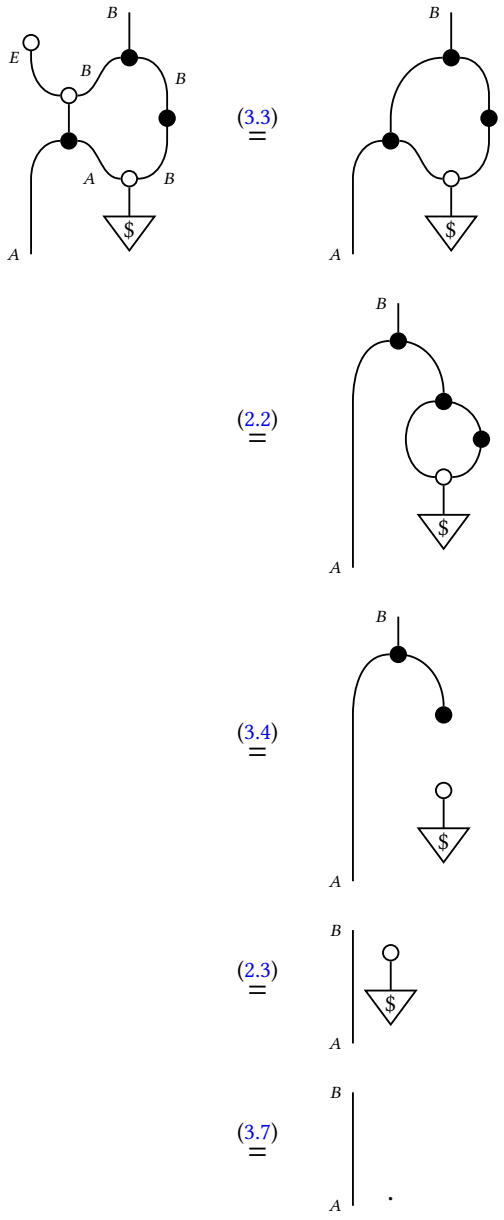
Now the schema for the one-time pad is the 2-comb



This is a morphism in n-comb(C). To check that this protocol is correct, we need to check that it sends the state (3.8) to the state (3.9), i.e. that it is a morphism with the right type in $\text{prot}_3(C)$. Substituting the actual resources in for their generic counterparts, we get the protocol



Now we compute, using counitality, associativity, the inverse law, unitality, and the independence of random choice:



This computation proves that the one-time pad is a morphism (3.8) $\longrightarrow$ (3.9) in $\mathrm{prot}_3(C)$; this is the categorical statement of the correctness of the one-time pad.

We reiterate that the entire preceding discussion relies only on the existence of an object satisfying the axioms (3.1) to (3.7). The one-time pad can be correctly implemented in any category over any object with this structure.

Of course, this entire discussion assumes that Eve does as the protocol instructs and simply deletes the message they read. If they do not, we need another layer of analysis, dealing with adversarial behavior. That will be the subject of Section 3.3.

3.2.6 Extensions to the Framework

Reusable Resources

In the title of Section 3.2.4, we called n-comb(C) a category of *linear* process conversions. This is because each input resource is used exactly once in the list of n-combs. While it is often valuable to have this restriction enforced by the syntax, there are cases where we want to model reusable resources. As suggested by [BK22], we can straightforwardly modify the construction to account for this by making two changes.

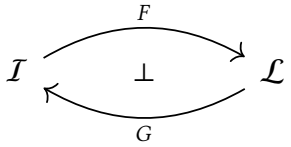
First, recall that in the definition of an n-comb at the beginning of Section 3.2.4, the permutation σ determines the order in which the input resources are used. For an n-comb with m input resources, we can replace this permutation with a function $n \to m$ which assigns to each comb the type of the resource which will fill it. In this way, we can use each resource as many times as we want, including not at all.

However, this is not enough, as in the category n-comb(C) morphisms are lists of n-combs, and the current definition allocates each input resource to exactly one of these combs. One advantage of our choice to use the intermediate step of a multicategory is that we can make the required change directly to the construction in Definition 3.13. When defining morphisms in this category, we used a partition function α to assign input resources to outputs resources. By allowing this function to be a relation, we can allow each input resource to be assigned to multiple output resources. When C is a symmetric multicategory, we call this category $C^{\otimes !}$. Combining these two modifications yields the category n-comb!(C) of [BK22].

We suspect that their choice of notation ! is not accidental: in many ways, this category behaves like the exponential modality ! of linear logic. In linear logic [Gir87], hypotheses must be used once and only once. The ! modality allows a hypothesis to be used any number of times; this allows controlled intuitionistic reasoning with linear frameworks, hence allowing linear logics to be both as expressive as intuitionistic logic, and to have fine-grained control over resource usage. In the case of n-combs, however, we currently have two separate categories n-comb(C) and n-comb!(C), so if we want to model cryptosystems that have some multi-use resources and some single-use resources, we need some way to relate them. We give a solution in Definition 3.16, but first we digress to discuss the categorical semantics of linear logic, which motivate our construction. This explanation uses some categorical terminology we have not introduced, but the reader may safely skip directly to the definition.

Any symmetric monoidal category $\mathcal{L}$ forms a model of the multiplicative-intuitionistic fragment of linear logic [Mel09]. In such settings, ! can be modeled by a lax monoidal comonad $[\![!]\!]$ together with natural transformations $[\![!]\!]x \to I$, and $[\![!]\!]x \to [\![!]\!]x \otimes [\![!]\!]x$. This data is subject to a coherence axiom given in [Mel09, Equation 72].

The modern perspective, motivated by [Ben95], is to focus on resolutions of this comonad, i.e. monoidal adjunctions



such that $FG = [\![!]\!]$, and in particular on resolutions such that the monoidal structure on $\mathcal{I}$ is cartesian, hence a model of conjunctive intuitionistic logic[12]. It turns out that any monoidal adjunction between categories with this structure gives the necessary structure on the comonad GF; as a consequence, such adjunctions are called *linear-non-linear*. In this way, F is an embedding of intuitionistic terms as linear terms, while G forgets the linearity of a term. A standard example of this structure is the free-forgetful adjunction between Set and $\mathrm{Vect}_{\mathbb{k}}$ [VZ14]. Linear-non-linear adjunctions have been widely used for designing resource-aware programming languages [MAF05; KPB15; Pay18; LMZ19].

All this machinery suggests that, to give a system which allows simultaneous reasoning about both single- and multi-use resources, we should look for such an adjunction. There is indeed a forgetful functor $G : \text{n-comb}(C) \to \text{n-comb}!(C)$, which we may think of as forgetting the linearity of an n-comb. Furthermore, the category n-comb$!(C)$ is cartesian monoidal, meaning that the concatenation of two lists is a cartesian product; the projections simply do not use the extra resource, while the universal property is witnessed by concatenating lists of n-combs. As such, n-comb$!(C)$ is a model of conjunctive intuitionistic logic; it represents an intuitionistic, rather than linear, calculus of resources.

However, the functor G seems unlikely to be a right adjoint[13]. The issue is that combs between the same two objects n-comb$!(C)$ may use different numbers of resources, and so ought to be sent to different objects in n-comb(C), but this is impossible for any functor. To resolve this, we need a notion of intuitionistic resource internal to n-comb(C). A solution may perhaps be along the lines of the ∞-combs of [Rom20], which are used there to model stream-like data, but these have slightly differently-structured domains and codomains than n-combs, so the translation is not obvious.

A more direct solution is to extend n-comb(C) with objects $!(A, B)$, representing reusable resources of type $A \to B$, and which are used to build n-combs as in n-comb$!(C)$.

Definition 3.16. Objects in the category n-comb$^*(C)$ are finite multisets[14] of pairs (A, B) and/or $!(A, B)$ of objects in C, called *linear* and *reusable* resources respectively. Given four finite disjoint index sets I, J, K, and L, we now describe morphisms

$$\{!(A_i, B_i), (C_j, B_j) : i \in I, j \in J\} \to \{!(X_k, Y_k), (Z_l, W_l) : k \in K, l \in L\}.$$

To construct such a morphism, we first give a relation $\alpha : K \sqcup L \to I \sqcup J$ such that:

1. each $j \in J$ α-relates to exactly one element;
2. each $k \in K$ α-relates only to elements in I.

Next, for each $k \in K$, we give morphism from n-comb$!(C)$, whose domain is the multiset $\{(A_i, B_i) : i \in \alpha(k)\}$ and whose codomain is (X_k, Y_k). Finally, for each $l \in L$, we first give for some n a function $\sigma : n \to \alpha(l)$, such that for each $j \in J \cap \alpha(l)$, $\sigma^{-1}(j)$ is a singleton.

[12] Normally we consider such situations with significantly more structure than just the multiplicatives, but the situation is the same even in this case.

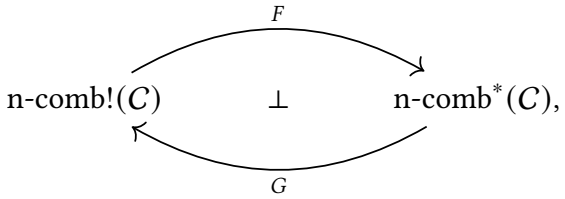
[13] I would like, but do not have, an explicit construction of a limit which G does not preserve; the issue is that it seems hard for n-comb(C) to have very many limits in the first place.

[14] We use multisets instead of lists to simplify the monoidal structure; since all our categories and multicategories are symmetric, the distinction is not important.

We then give an n-comb which uses the resources in $\alpha(j)$ according to the order assigned to them by σ.

The definition is justified as follows. The first condition on α ensures that each linear resource must be used in exactly one comb, while the second ensures that we can only build reusable resources out of reusable resources. We build reusable resources as in n-comb!(C), which was constructed specifically for that purpose. To build linear resources, we can use reusable resources as many times as we want, but must use linear resources exactly once, hence the condition on σ.

Another semantic digression: this category is strict symmetric monoidal with the union of multisets as the product and the empty set as the identity. Furthermore, there is a linear-non-linear adjunction



where G forgets the difference between linear and reusable resources, and F sends all resources to reusable ones. To show this is an adjunction, observe that all the restrictions on the construction of the combs are on the use of linear resources in the domain. As such, when all the resources in the domain are reusable, a morphism in n-comb$^*(C)$ is exactly a morphism in n-comb!(C); thus the identities give a natural isomorphism between adjoint hom-sets.

This may seem like abstract nonsense, but the point is that the theory guided us in constructing a category which models protocols relying on both linear and multi-use resources; this is likely of independent interest to other uses of n-combs. We conjecture that this style of construction extends to graded linear logic (in which resources can have a bounded number possible uses) [GSS92], affine logic (in which resources must be used at most once) [Tro92], relevance logic (in which resources must be used at least once) [DR83], ordered logic (in which resources must be used in a specific order) [Lam58], and to other such substructural resource logics. The general paradigm of adjoint logic [Pru+18] provides categorical semantics for embedding many of these logics in each other; giving "comb-like" constructions of such categorical structures would allow reasoning about resource-bounded protocols with fairly sophisticated resource-usage constraints.

As a final notational point, we modify $\text{prot}_N(C)$ as $\text{prot!}_N(C)$ and $\text{prot}^*_N(C)$ by replacing all the invocations of n-comb with n-comb! and n-comb*, respectively. In particular, these constructions are both functorial in the same way as n-comb.

Shading Diagrams

While [BK22] choose to label the wires with the identities of the parties in possession of that data, we worry that this approach does not easily scale to protocols where multiple objects are relevant. We now give an alternative approach using the shaded boxes of Section 2.2.5. In addition to being less cluttered, this approach has a fairly pleasant abstract justification, though we emphasize that it is merely a syntactic distinction.

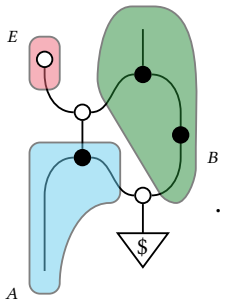
We begin by noting that, in addition to the tensor

$$C^N \xrightarrow{\otimes^{N-1}} C,$$

there are also strong monoidal projection functors

$$C^N \xrightarrow{\pi_i} C.$$

Instead of labelling each wire with the party, we can shade the wires according to the projection functors that they live in the image of. For instance, if Alice is blue, Bob is green, and Eve is red, then the one-time pad can be depicted as



We like the visual clarity provided by this approach; it emphasizes the flow of information and control between the parties in the protocol. As mentioned, it also has a nice justification in terms of the functorial boxes studied previously; unifying analogous notations is always valuable. However, we see two potential issues. First, as usual with color in diagrams, there are accessibility concerns; while we attempt to ameliorate these by labelling the regions and using an accessible colorscheme due to [Tol21], such measures can only go so far. Second, this approach is hard to scale to settings with many parties, as there are only so many visually distinct colors. As such, we think that both approaches have their place.

Parties With Differing Capabilities

It is quite common in cryptography to consider settings where different parties have different capabilities. For instance, we may want to analyze classical protocols which are secure against quantum attackers or zero-knowledge proofs with polynomial verifiers and unbounded provers. While the attack models of [BK22], to be studied in Section 3.3.1, allow treating adversaries with different capabilities from the honest parties, the paper does not directly address honest parties with different capabilities. This can be done within their framework using the categories we constructed in Section 3.1.

As an example, we construct the category of protocols with one unbounded but deterministic party and one PPT party. Recall that unbounded deterministic computation is modeled in the category Comp, while PPT computation is modeled in the category PPT. Both of these include as subcategories into $C = \text{CompStoch}$; call these inclusion functors i and j. As such, we can construct the category

$$\text{st}(\text{n-comb}(\text{Comp} \times \text{PPT}) \xrightarrow{\text{n-comb}(i \times j)} \text{n-comb}(C^2) \xrightarrow{\text{n-comb}(\otimes)} \text{n-comb}(C)).$$

In general, assuming there is a clear ambient category C of into which all the relevant categories include, we write

$$\mathrm{prot}_N(C_1, \ldots, C_N) := \mathrm{st}(\text{n-comb}(\prod_{i=1}^{N} C_i) \hookrightarrow \text{n-comb}(C^N) \xrightarrow{\text{n-comb}(\otimes^{N-1})} \text{n-comb}(C)).$$

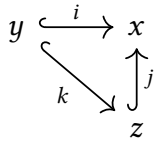
We define $\mathrm{prot}!_N(C_1, \ldots, C_N)$ and $\mathrm{prot}^*_N(C_1, \ldots, C_N)$ similarly.

Joint Input

While the objects in the category $\mathrm{prot}_N(C)$ are morphisms representing joint computations in C, the domains and codomains of these computations are N-fold tensor products of objects in C, and so cannot themselves be entangled. However, it is extremely common in cryptography to want to represent joint or otherwise correlated input. For instance, if $C = \textsc{Set}$, we may only care about inputs from a subset $\{(x, x)\} \subseteq X \times X$. In the framework as described, it is impossible to restrict inputs in such a way that does not decompose into a product.

There are various ad-hoc low-tech solutions to this problem, such as giving the parties an oracle which rejects bad inputs, but we can also modify the construction of our categories to allow for this kind of entangled input to resources. Our goal will be to define a *category of refinements on joint states* which will allow us to refine the domains of our resources.

What is the categorical notion of subset? Every subset $A \subseteq X$ comes with an *inclusion function* $i : A \hookrightarrow X$, which is always an injection. As usual, the categorical approach is to forefront the role of the morphism, in this case the injection. It turns out there is a categorical generalization of injections which makes no reference to objects having elements: a morphism $f : x \to y$ is a *monomorphism* if for all objects z and maps $g, h : z \to x$, if $fg = fh$, then $g = h$. In the category of sets, monomorphisms are exactly injections. Given two monomorphisms $i : y \hookrightarrow x$ and $j : z \hookrightarrow x$, we say that $i \leq j$ if there is a (necessarily unique) morphism $k : y \hookrightarrow z$ such that



commutes. A *subobject* of x is an equivalence class of monomorphisms into x under the relation $i \sim j$ if $i \leq j$ and $j \leq i$.

To make a category of subobjects, we need a way to talk about morphisms between subobjects of different objects. It turns out that in many categories there is a way to talk about images of subobjects under morphisms, though we have not given the background to go into detail here[15]. We write $f_* i$ for the image of i under the morphism f; in all our examples, this is the familiar image of a subset.

[15] For the categorically inclined: we have in mind a factorization system whose right class is the monomorphisms. The direct image of a subobject $i : z \hookrightarrow x$ under a map $f : x \to y$ is the monic part of the factorization of fi.

Now we define the category $\mathrm{pred}(C)$ of predicates on objects in C. Objects in this category are pairs (x, i) where $x \in C$ and i is a subobject of x. Morphisms $(x, i) \to (y, j)$ are maps $f : x \to y$ in C such that $f_* i \leq j$, i.e. so that the image of i under f is contained in j. Composition and identities are as in C.

More generally, as with state we define $\mathrm{pred}(C \xrightarrow{F} \mathcal{D})$ for any functor. Objects are pairs (x, i) where $x \in C$ and i is a subobject of Fx, while morphisms are maps $f : x \to y$ in C such that $(Ff)_* i \leq j$.

If F is strong monoidal and $\otimes_{\mathcal{D}}$ preserves monomorphisms (as it does in all our categories of interest), then this category is also monoidal, with the structure induced by the respective monoidal structures:

$$(x, i) \otimes (y, j) = (x \otimes_C y, (i \otimes_{\mathcal{D}} j)\phi^{-1}_{x,y}).$$

Given a (strong monoidal) functor $F : C \to \mathcal{D}$, there is a (strong monoidal) functor $\mathrm{pred}(F) \to \mathrm{pred}(\mathcal{D})$, which due to the notational ambiguity[16] we call $\overline{F}$. This functor sends a predicate (x, i) to the predicate (Fx, i), and a map f to the map Ff.

Given an SMC C, consider the category

$$\overline{\mathrm{prot}}_N(C) := \mathrm{st}(\text{n-comb}(\mathrm{pred}(\otimes^{N-1})) \xrightarrow{\text{n-comb}(\overline{\otimes^{N-1}})} \text{n-comb}(\mathrm{pred}(C))).$$

Basic objects in this category consists of a subobject i of $X_1 \otimes \cdots \otimes X_N$, a subobject j of $Y_1 \otimes \cdots \otimes Y_N$, and a morphism $f : X_1 \otimes \cdots \otimes X_N \to Y_1 \otimes \cdots \otimes Y_N$ in C such that $f_* i \leq j$. The domain and codomain of this map are refinements of a product state, which specify the allowable input and output states; the map must send allowable inputs to allowable outputs. Objects in this category are finite lists of basic objects. Morphisms are lists of n-combs in C^N whose domains and codomains may be augmented with predicates, and whose maps must respect those predicates. We define $\overline{\mathrm{prot}}!$ and $\overline{\mathrm{prot}}^*$ similarly.

We expect that most readers are somewhat overwhelmed by the proliferation of constructions on categories; this is quite understandable. However, once this initial conceptual barrier is overcome, this proliferation of constructions is actually very helpful; they allow us to fine-tune our base category for any specific use-case.

3.2.7 Interactive Proof

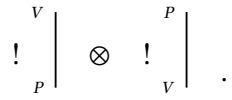
We now give an original representation of interactive proofs within the framework. Recall from Section 1.2.3 that an *interactive proof* for a language $\mathcal{L}$ consists of a prover and a verifier, both given an input x, such that the verifier accepts if $x \in \mathcal{L}$ and does not if $x \notin \mathcal{L}$, even if the prover is behaving maliciously. While we do not yet know how to model malicious behavior, we are already able to model the honest case. Fix a universe of strings A and a decidable[17] language $\mathcal{L}$. This can be made significantly more general, but

[16] Observe that pred is not functorial. Given a functor $F : C \to \mathcal{D}$, the natural thing is to define a functor $\mathrm{pred}(C) \to \mathrm{pred}(\mathcal{D})$ which sends (x, i) to (Fx, Fi), but Fi is not necessarily a monomorphism.

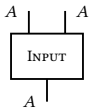
[17] Note that $\mathcal{L}$ needs to be decidable so that its characteristic function is in CompStoch, which allows us to represent it as a resource. This restriction can technically be relaxed if we choose a bigger ambient category of computations, even while still requiring our prover to be computable.

for simplicity we will work over the category CompStoch, and let the verifier be bounded in PPT. We will work in the category $\overline{\text{prot}}^*_2(\text{CompStoch}, \text{PPT})$.

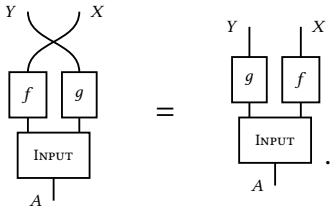
We need to know what the input and output resources of an interactive proof should be. Certainly we need a two-way channel between the prover and verifier. Since our proofs are interactive, this needs to be multi-use, so it should be the resource



As output, the protocol should give a resource such that, on an input $(x, x) : x \in A$, the verifier outputs $\mathbb{1}_{\mathcal{L}}(x)$. Thanks to our work in the previous section, we know how to encode this: the input to this resource should be the subset $\{(x, x)\} \subseteq A \otimes A$. To describe this input constraint pictorially, we want a box

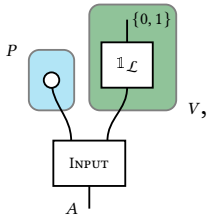


such that doing some separate computations on each of the outputs, and then swapping the results, is the same as doing the same computations on the other side, i.e. for all $f : A \to X$ and $g : A \to Y$,



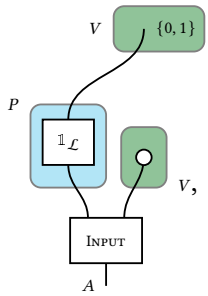
(3.10)

Given such correlated inputs, our interactive proof should output a value of $* \otimes \{0, 1\}$; the prover has no output, while the verifier outputs either to accept or reject. In other words, the n-comb should have codomain $(\text{Input}, 1_{*\otimes\{0,1\}})$; these are both monomorphisms into products, hence objects in pred($\otimes$). The actual resource we want to produce is the map



where the prover is blue and the verifier is green. The point is that a correct interactive proof should amount to the prover doing nothing with its input, while the verifier outputs the characteristic function of the language under proof.

We emphasize that this definition has no security properties; it does not even guarantee completeness. For instance, directly from (3.10) we can see that



the protocol where the prover just sends the answer to the verifier, is correct.

The reader may wonder which type the channels carry; this is a good question, and we have actually been imprecise about it. If we wanted to model any specific interactive proof, we could simply let it carry the type of messages that that proof needs to communicate. For instance, in the previous protocol, we just need a single-use channel which carries a message of type $\{0,1\}$. However, to reason about the existence or non-existence of interactive proofs, we would need a way to model more general channels which can carry any data; we do not currently have a way to do so[18].

3.3 Security

We set aside all these proliferating functors and return to a familiar setting. Let $F : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor, which as in the previous section we interpret as including a class of free or local processes into a broader class of processes. We are interested in the category $\mathrm{st}(F)$; we want to know when a morphism in this category is secure. All of the constructions in the previous section fit this paradigm.

The main issue with defining security in the categorical setting is modelling adversarial behavior. Recall from Section 1.3.4 that Universal Composability avoids dealing with this issue by having the behavior of corrupted parties baked into protocols via backdoor tapes. As this issue relies on a fairly low-level understanding of the machine model, it is hard to adapt to the categorical setting. Furthermore, cryptographic approaches to computation in some sense fundamentally rely on the computations respecting some kind of type system—this is how we interpret the objects in the category.

For expository purposes, in this section we will quite closely follow the technical approach of [BK22]. However, the machinery they develop can be made significantly more general.

[18] Since all our computations are binary-encoded, one option is to let this be a channel over the object $\{0,1\}^*$, but this requires forgetting type information that may be useful. With more categorical machinery, we could instead make this a *polymorphic* channel. Again, we would need to adapt the standard categorical semantics of the polymorphic lambda calculus [See87] to the comb framework. As the type theories involved get increasingly sophisticated, so too do the categorical requirements: we would need to give a kind of indexed cartesian closed category called a *hyperdoctrine* satisfying a fairly intricate equational theory.

3.3.1 Attack Models

The primary tool of [BK22] is the notion of *attack model*, which constrains the possible behavior of the adversary. The definition is chosen specifically so that we can prove a composition theorem.

Definition 3.17. An *attack model* $\mathbb{A}$ on a category C consists of, for each morphism f in C, a collection of morphisms $\mathbb{A}f$ such that:

1. $f \in \mathbb{A}f$;
2. if $f' \in \mathbb{A}f$ and $g' \in \mathbb{A}g$ so that f and g compose and f' and g' compose, then $g'f' \in \mathbb{A}(gf)$;
3. if $f' \in \mathbb{A}f$ and $g' \in \mathbb{A}g$, then $f' \otimes g' \in \mathbb{A}(f \otimes g)$;
4. if $h \in \mathbb{A}(gf)$, then there is some $g' \in \mathbb{A}g$ and $f' \in \mathbb{A}f$ such that $h = g'f'$;
5. if $h \in \mathbb{A}(f \otimes g)$, then there is some $h' \in \mathbb{A}1_{\mathrm{cod}\, f \otimes \mathrm{cod}\, g}$, $f' \in \mathbb{A}f$, and $g' \in \mathbb{A}g$ such that $h = h' \circ (f' \otimes g')$.

The definition is motivated as follows. The collection $\mathbb{A}f$ represents all the possible actions that the adversary could force to occur, if the protocol specifies that the morphism f is supposed to occur. The first condition says that any adversary is allowed to act as an honest party. The second and third say that, if the adversary has a pair of attacks on two separate computations, then they can compose those attacks to get an attack on the composite computation. These all seem very natural in any threat model.

The fourth axiom says that any attack on a sequential composite protocol gf factors into attacks on each of its subprotocols. This point is somewhat subtler; it seems at first that this should rule out attacks where the adversary against the first protocol forwards its view to the adversary against the second, hence allowing the adversary against the second to do something it cannot do on its own. Such attacks are extremely common, and so certainly need to be included. However, the point of the definition is that the morphisms in $\mathbb{A}f$ do not have to have the same domain as f. As such, the composite attack can model the forwarded view by representing the adversary on the second protocol with an attack whose codomain includes that extra input.

The fifth axiom says something similar, but about attacks on parallel processes. It is a little less clear to me that we should believe this axiom.

3.4 Conclusion

3.4.1 Evaluation

3.4.2 Paths Not Taken

Appendix A

Computer Scientific Foundations

In the main body, we have assumed standard material from a course in computability and complexity, including function asymptotics, the notion of an algorithm, and the complexity class P. We briefly overview these ideas here; a standard text is [Sip13].

A.1 Asymptotics

Function asymptotics formalize the notion of a function approximating another function. In particular, for a pair of functions $f, g : \mathbb{N} \to \mathbb{R}$, we often want to compare f and g on large inputs and only up to a constant factor. This is most common in runtime analysis, the idea being that the running time of algorithms on small inputs is less important to their overall performance than their running time on large inputs. We formalize this notion as follows:

Definition A.1 (Function Asymptotics)**.** Let $f, g : \mathbb{N} \to \mathbb{R}$ be a pair of functions which are both non-negative for sufficiently large inputs. We say that f is *big-Oh of* g, written $f = O(g)$, if there exists a constant $c > 0$ such that for all n sufficiently large,

$$f(n) \leq cg(n).$$

In this case, we also say that g is *big-Omega of* f, written $g = \Omega(f)$.

If $f = O(g)$ and $g = O(f)$, we say that f is *big-Theta of* g, written $f = \Theta(g)$. Explicitly, this means that there exist constants $c_1, c_2 > 0$ such that for all n sufficiently large,

$$c_1 f(n) \leq g(n) \leq c_2 f(n).$$

If $f = O(g)$ but $f \neq \Theta(g)$, we say that f is *little-oh of* g, written $f = o(g)$, and g is *little-omega of* f, written $g = \omega(f)$. Explicitly, this means that for all constants $\epsilon > 0$ and all n sufficiently large,

$$f(n) \leq \epsilon g(n).$$

Notation. By abuse of notation, we often write $f(n) = O(g(n))$ to mean that f is $O(g)$; for example, the statement that n^2 is $O(n^3)$ means that the function $f(n) = n^2$ is $O(g)$, where g is the function $n \mapsto n^3$.

Example A.2. We have that:

- $17n^2$ is $o(n^3)$, $\omega(n)$, and $\Theta(n^2)$;
- $\log n$ is $o(n)$, $\omega(1)$, and $\Theta(\ln n)$;
- e^n is $\omega(n^k)$ for any exponent k;
- e^{-n} is $o(n^{-k})$ for any exponent k.

These last two examples are especially important. No matter how big the power, an exponential will always dominate a polynomial for sufficiently big n. Because of the importance of polynomials in theoretical computer science, we say a function f is *negligible* if $f = o(n^k)$ for all k. In this case, we write $f = \mathsf{negl}(n)$ or just $f = \mathsf{negl}$.

Proposition A.3. *Big-Oh is a preorder on the set of functions $\mathbb{N} \to \mathbb{N}$. The induced equivalence relation is exactly big-Theta.*

In the partial order of equivalence classes under Θ, O behaves like $\leq$, o like $<$, and Θ like $=$. As suggested by the notation $f = O(g)$, it is common in some contexts to treat functions as identical with their asymptotic equivalence class.

Proposition A.4. *Let $f_1 = O(g_1)$ and $f_2 = O(g_2)$. Let c be any positive constant. Then,*

$$f_1 + f_2 = O(\max\{g_1, g_2\}), \quad cf_1 = O(g_1), \quad \textit{and} \quad f_1 f_2 = O(g_1 g_2).$$

In other words,

$$O(g_1) + O(g_2) = O(\max\{g_1, g_2\}), \quad cO(g) = O(g), \quad \textit{and} \quad O(g_1)O(g_2) = O(g_1 g_2).$$

Identical results hold for o and Θ.

Proposition A.4 justifies the universal practice of dropping constants and small additive terms from asymptotics, so that for instance $n^2 + n + \ln n = \Theta(n^2)$.

A.2 Algorithms and Determinism

Our basic notion is of an *algorithm* over a finite alphabet Σ, usually $\mathbb{Z}_2$. An algorithm $\mathcal{A}$ is intuitively some set of steps which take an input word x over Σ, perform some transformations, and output another word $\mathcal{A}(x)$ over Σ. An algorithm may have certain *side effects*, such as sending a message or logging a string, and its behavior may not be deterministic. There are several ways to formalize the notion of algorithm—most common in cryptography are Turing machines—but we will not need to be so precise here.

Algorithms may have multiple possible "branches" in their instructions. Consider the following:

Algorithm A.5. On input x, either output 0 or 1.

We say that algorithms of this sort are *nondeterministic*; in contrast, an algorithm is *deterministic* if its instructions do not include such choices. In particular, we say that an algorithm $\mathcal{A}$ *deterministically computes* a function f if it is deterministic and, for any input

$x \in \Sigma^*$, $\mathcal{A}$ outputs the value $f(x) \in \Sigma^*$. In contrast, $\mathcal{A}$ *nondeterministically computes* f if, for any input x, there exists a particular choice of branches such that $\mathcal{A}$ outputs $f(x)$. Thus Algorithm A.5 nondeterministically computes both the functions $x \mapsto 0$ and $x \mapsto 1$. We sometimes view nondeterministic algorithms as computing functions into the power set of Σ^*, so that Algorithm A.5 computes the function $x \mapsto \{0, 1\}$, and we similarly sometimes write $\mathcal{A}(x) = \{0, 1\}$.

An important middle ground is *probabilistic* algorithms. Again, there are many possible models, but the basic idea is that a probabilistic algorithm has access to some source of randomness—say, an arbitrarily long string of independent and uniform coin tosses—which it can use to choose between branches. In this case, it is not enough for there to be some branch which computes a specific function. Instead, we say that an algorithm *computes* f *with bounded probability* if for any input x,

$$\Pr[\mathcal{A}(x) = f(x)] > \frac{2}{3},$$

where the probability is taken over the randomness of $\mathcal{A}$[1]. In this case, we often think of $\mathcal{A}(x)$ as a probability distribution on Σ^*.

Instead of thinking of algorithms operating directly on binary strings, we usually think of them as operating on encodings of mathematical objects. For example:

Algorithm A.6. On input x a natural number, output the number $2x$.

We say that Algorithm A.6 deterministically computes $x \mapsto 2x$, even though it technically operates on encodings of naturals. While there are many possible encodings, we assume that a reasonable encoding is chosen, so that for instance numbers are encoded in binary, rather than unary. Such details will not be relevant for us.

One more subtlety is important. In general, we require that the description of any algorithm $\mathcal{A}$ is finite. However, we may also consider *non-uniform* algorithms, which are sequences of algorithms $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots)$ such that, on an input of length n, $\mathcal{A}$ delegates to $\mathcal{A}_n$. Non-uniform computation is generally stronger than uniform computation, as non-uniform algorithms may encode nonfinite information, as long as they only use finitely much of this information for each input length and hence for each computation[2].

A.3 Complexity Theory

Each algorithm has an associated *running time*, which is informally the number of steps the algorithm takes on a given input. In particular, for an algorithm $\mathcal{A}$, we say that its running time is the function $T_{\mathcal{A}} : \mathbb{N} \to \mathbb{N}$ which takes any natural number n to the maximum number of steps $\mathcal{A}$ takes to terminate on any input of length n. Of course, this

[1] The choice of $\frac{2}{3}$ is not particularly important here—generally any constant $c > \frac{1}{2}$ works.

[2] For instance, non-uniform algorithms may solve the halting problem (which asks whether an input algorithm $\mathcal{M}$ eventually terminates), which is uniformly undecidable. In particular, since there are only finitely many Turing machines of a given size, a non-uniform algorithm may simply encode in $\mathcal{A}_n$ the answer to the halting problem for each Turing machine of length n.

notion is not yet precise, as we don't know what a "step" is, but it is easy to make precise in any standard model of computation.

In general, the running time may depend on the formal model of computation in which the algorithm is constructed, but the *complexity-theoretic Church-Turing thesis* states that "reasonable" models of classical computation recover the same inhabitants of sufficiently robust complexity classes, in particular of those we are about to define. This hypothesis is a heuristic, but has been born out in practice.

Definition A.7 (polynomial-time; P, NP)**.** An algorithm $\mathcal{A}$ is *polynomial-time* if $T_{\mathcal{A}} = O(n^k)$ for some constant k. The class P consists of all functions which are deterministically computable by polynomial-time algorithms. The class NP consists of all functions which are nondeterministically computable by polynomial-time algorithms[3].

The general idea is that polynomial-time algorithms are "efficient in practice." It may sometimes occur that the constant factors or the exponent are so large as to render the algorithm practically useless, but in most cases functions in P are efficiently solvable for practical applications, including cryptography. We can now state the most important open problem in computer science:

Conjecture A.8. *We have that $P \neq NP$.*

While a proof seems completely out of reach, this conjecture is widely believed, and as we will see is necessary for all of modern cryptography; we will assume it here. An introduction to the modern state of P vs. NP is [And17].

Formalizing probabilistic complexity classes is slightly more subtle. Consider the following case:

Algorithm A.9. On input x, output 1 with probability $1 - 2^{-|x|}$; otherwise count from 0 to $2^{|x|}$ and then output 1.

While this algorithm is almost always polynomial-time, it is not polynomial-time when it takes the second branch. The point is that for probabilistic algorithms, $T_{\mathcal{A}}(n)$ is a probability distribution, not just a fixed number. For our purposes, we require that the algorithm *always* runs in polynomial time. As such:

Definition A.10 (probabilistic polynomial-time; BPP)**.** A probabilistic algorithm is *probabilistic polynomial-time* if, for any choice of random bits, $T_{\mathcal{A}} = O(n^k)$ for some constant k. The class BPP consists of all functions which are computable with bounded probability by a probabilistic polynomial-time algorithm.

For non-uniform algorithms, the situation is also slightly more complicated. In particular, it is too much to allow the machines to be arbitrarily large, as they could simply encode lookup tables for every possible input. As such, we ask that the size of each machine is polynomially bounded.

[3] In fact, we have defined here the classes FP and FNP of polynomially- and nondeterministically-polynomially-computable *function problems*. Formally, P and NP are classes of *decision problems*, which are just subsets L of Σ^*—the algorithm must output 1 if its input is in L, and 0 otherwise. Function and decision problems are extremely closely related—for instance, P = NP if and only if FP = FNP—and we will not distinguish between them here.

Definition A.11 (non-uniform polynomial-time; P/poly)**.** A non-uniform algorithm $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ is *polynomial-time* if $T_{\mathcal{A}} = O(n^k)$ for some constant k and the size of each $\mathcal{A}_n$ is $O(n^k)$ for some constant k independent of n. The class P/poly consists of all functions which are computable by non-uniform polynomial-time algorithms.

Non-uniform probabilistic algorithms are similarly defined.

Theorem A.12 (Adelman's theorem)**.** *We have that BPP $\subseteq$ P/poly.*

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