

Categories for Cryptographic Composability

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Advised by Angélica and Adam

Cryptography

The Simulation Paradigm

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We will treat the more limited setting of N -party computation.

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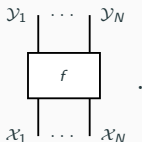
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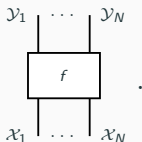


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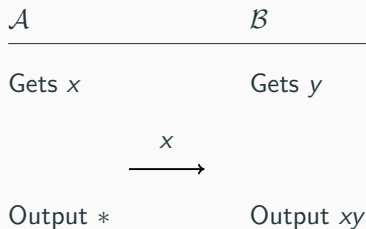


Running example: $f(x, y) = (*, xy)$.

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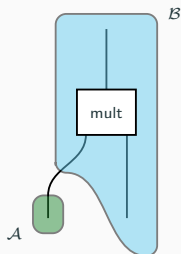
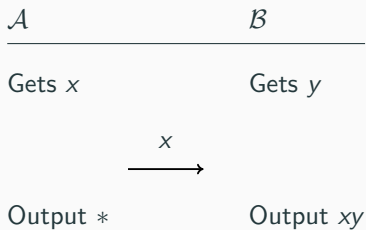
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We will treat *honest* adversaries. Usually you want to quantify over adversarial machines \mathcal{A}' .

Computational Indistinguishability

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Definition

Two probability ensembles $\{X_n\}$ and $\{Y_n\}$ (over the set A) are *computationally indistinguishable* if for any distinguisher \mathcal{D} ,

$$|\Pr[\mathcal{D}(X_n) = 1] - \Pr[\mathcal{D}(Y_n) = 1]| = \text{negl}(n).$$

We write $\{X_n\} \stackrel{c}{\equiv} \{Y_n\}$.

Definition

A protocol $\langle \mathcal{A}_1, \dots, \mathcal{A}_N \rangle$ is *secure* if for each choice of inputs x_1, \dots, x_N and each party i , there exists a simulator \mathcal{S} such that

$$\mathcal{S}(x_i, f_i(x_1, \dots, x_N)) \stackrel{c}{=} \text{view}_i^{\langle \mathcal{A}_1, \dots, \mathcal{A}_N \rangle}(x_1, \dots, x_N).$$

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Our running example is secure: the simulator can compute xy/y .

Composition

At least two ways to compose ideal functionalities:



sequential composition

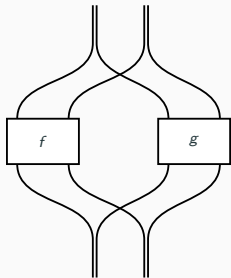
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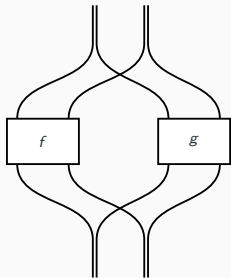
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Many more things to consider.

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- High proof burdens

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Category Theory

Categories

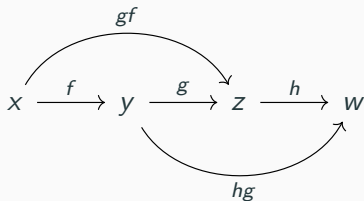
A category has:

- objects x, y, z, \dots ;
- morphisms $x \xrightarrow{f} y, \dots$;
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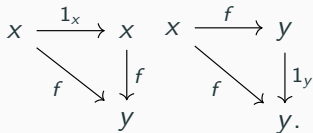
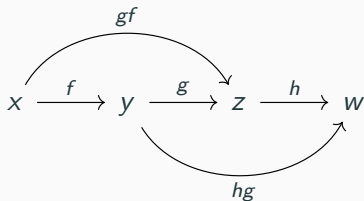
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- functional programming languages & logics.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ assigns:

- to each $x \in \mathcal{C}$, an $Fx \in \mathcal{D}$;
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- the free group, vector space, etc.

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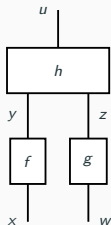
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- \mathbf{CHEM} with the union of multisets of molecules;
- Concurrent languages with concurrent joining and the do-nothing program.

String Diagrams

Given $f : x \rightarrow y$, $g : w \rightarrow z$, and $h : y \otimes z \rightarrow u$, interpret



as $h(f \otimes g)$.

Categorical Cryptography

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- A security definition which works over any SMC.

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Final try (Pavlovic 2014): `COMP` (binary-encoded sets and lifts of computable functions).

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Efficient and Effectful Computation

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Name	Computational Bound	Effect
ENC	none	none
COMP	computability	none
COMPSTOCH	computability	probability
POLY	poly-time computability	none
PPT	poly-time computability	probability

Non-interactive “Protocols”

The category $\mathcal{C} \times \mathcal{D}$ has:

- for objects, pairs (c, d) from \mathcal{C} and \mathcal{D} ;
- for morphisms, pairs (f, g) from \mathcal{C} and \mathcal{D} ;
- composition and identities componentwise.

The category $\text{st}(\mathcal{C} \xrightarrow{F} \mathcal{D})$ has:

- for objects, maps $I \rightarrow Fx$ in \mathcal{D} ;
- for morphisms $(I \xrightarrow{s} Fx) \rightarrow (I \xrightarrow{t} Fy)$, maps $x \xrightarrow{f} y$ in \mathcal{C} such that $(Ff)s = t$;
- composition and identities as in \mathcal{C} .

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We think of \mathcal{C} as “free” processes in \mathcal{D} .

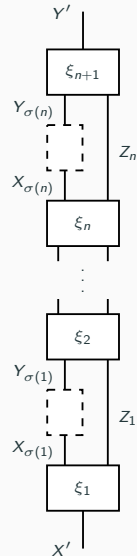
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Assigning Resources

In $n\text{-comb}(\mathcal{C})$, there are two ways we control the assignment of resources:

- a function α assigns each pair in the codomain some resources from the domain;
- a permutation σ orders the resources assigned to each comb.

Let's play with this!

The category of protocols







The category

$$\text{prot}_N(\mathcal{C}) := \text{st}(\text{n-comb}(\mathcal{C}^N) \xrightarrow{\text{n-comb}(\otimes^{N-1})} \text{n-comb}(\mathcal{C}))$$

has:

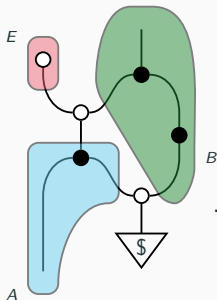
- for objects, objects maps $(X_1 \otimes \cdots \otimes X_N) \rightarrow (Y_1 \otimes \cdots \otimes Y_N)$ in \mathcal{C} ;
- for morphisms, combs drawn from \mathcal{C}^N which preserve the chosen maps.

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1-4) ...

- 5) If $h \in \mathbb{A}(f \otimes g)$ such that $\text{dom } h = x \otimes y$ for some objects x and y , then there is some $h' \in \mathbb{A}1_{\text{cod } f \otimes \text{cod } g}$, $f' \in \mathbb{A}f$, and $g' \in \mathbb{A}g$ such that $\text{dom } f' = x$, $\text{dom } g' = y$, and $h = h' \circ (f' \otimes g')$.

The Security Definition

Definition

Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be a string of symmetric monoidal functors so that F is strong monoidal. Let \mathbb{A} be an attack model on \mathcal{D} . Let $f : (x, s) \rightarrow (y, t)$ be a map in $\text{st}(GF)$ and let a be a map in \mathcal{D} with $\text{dom } a = Fx$.

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Further, f is \mathbb{A} -*secure* if it is secure against all attacks in $\mathbb{A}Ff$ with domain Fx .

The Composition Theorem

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What about *bicategories*?

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Thanks for your time!